Numerical Discretization in High Accuracy MPC Applications

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Motivation

MPC on continuous-time systems:

- Continuous-time system $\xrightarrow{\text{numerical discretization}}$ Discrete-time model
- Zero-order hold control inputs in the discrete-time model
- Higher accuracy vs. increased computational load

Goal:

- Trade-off between prediction accuracy and computational complexity
- Zero-order hold vs. Higher-order hold control inputs

Introduction/Agenda

- Runge Kutta (RK) method
- Gauss collocation method
- RK: Simulation setup and results
- Gauss collocation and FOH: Simulation setup and results
- Conclusion

Runge Kutta method

$$\dot{x} = f(t, x, u) = A_{\text{con}} x + B_{\text{con}} u \xrightarrow{\text{RK method}} x_{k+1} = A_{\text{rk}} x_k + B_{\text{rk}} u_k$$

The family of explicit RK methods is given by

$$x_{k+1} = x_k + \sum_{i=1}^{s} b_i \kappa_i, \tag{1}$$

where

$$\begin{aligned}
\kappa_1 &= \Delta t f(t_k, x_k, u_k), \\
\kappa_2 &= \Delta t f(t_k + c_2 \Delta t, x_k + \Delta t(a_{21}\kappa_1), u_k), \\
\kappa_3 &= \Delta t f(t_k + c_3 \Delta t, x_k + \Delta t(a_{31}\kappa_1 + a_{32}\kappa_2), u_k) \\
&\vdots \\
\kappa_s &= \Delta t f(t_k + c_s \Delta t, x_k + \Delta t(a_{s1}\kappa_1 + a_{s2}\kappa_2 + \cdots + a_{s,s-1}\kappa_{s-1}), u_k).
\end{aligned} \tag{2}$$

Runge Kutta method

Butcher tableau [Grüne+ 2017]

$$
\begin{array}{c|ccccc}
c_1 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

RK1 (Euler)

$$
\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}
$$

RK2 (Heun)

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

RK4 (Runge Kutta)

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6}
\end{array}
$$

Runge Kutta method

Applying RK2 to continuous-time system $\dot{x} = A_{\text{con}}x + B_{\text{con}}u$ yields

$$\begin{aligned} x_{k+1} &= x_k + \frac{1}{2}(\kappa_1 + \kappa_2), \\ \kappa_1 &= \Delta t(A_{\text{con}}x_k + B_{\text{con}}u_k), \\ \kappa_2 &= \Delta t(A_{\text{con}}(x_k + \kappa_1) + B_{\text{con}}u_k). \end{aligned} \tag{3}$$

Substituting these equations into each other gives

$$x_{k+1} = \underbrace{\left(I + \Delta t A_{\text{con}} + \frac{\Delta t^2}{2!}A_{\text{con}}^2\right)}_{\mathrm{A_{rk2}}} x_k + \underbrace{\left(\Delta t B_{\text{con}} + \frac{\Delta t^2}{2!}A_{\text{con}}B_{\text{con}}\right)}_{\mathrm{B_{rk2}}} u_k. \tag{4}$$

Runge Kutta method

$$\dot{x} = A_{\mathrm{con}}x + B_{\mathrm{con}}u \xrightarrow{\text{RK method}} x_{k+1} = A_{\mathrm{rk}}x_k + B_{\mathrm{rk}}u_k,$$

$$\text{RK1}: \begin{cases} A_{\mathrm{rk1}} = I + \Delta t A_{\mathrm{con}}, \\ B_{\mathrm{rk1}} = \Delta t B_{\mathrm{con}}, \end{cases}$$

$$\text{RK2}: \begin{cases} A_{\mathrm{rk2}} = I + \Delta t A_{\mathrm{con}} + \frac{\Delta t^2}{2!}A_{\mathrm{con}}^2, \\ B_{\mathrm{rk2}} = (\Delta t I + \frac{\Delta t^2}{2!}A_{\mathrm{con}})B_{\mathrm{con}}. \end{cases}$$

$$\text{RK4}: \begin{cases} A_{\mathrm{rk4}} = I + \Delta t A_{\mathrm{con}} + \frac{\Delta t^2}{2!}A_{\mathrm{con}}^2 + \frac{\Delta t^3}{3!}A_{\mathrm{con}}^3 + \frac{\Delta t^4}{4!}A_{\mathrm{con}}^4, \\ B_{\mathrm{rk4}} = (\Delta t I + \frac{\Delta t^2}{2!}A_{\mathrm{con}} + \frac{\Delta t^3}{3!}A_{\mathrm{con}}^2 + \frac{\Delta t^4}{4!}A_{\mathrm{con}}^3)B_{\mathrm{con}}. \end{cases}$$

Gauss collocation method

$$\dot{x} = f(x) + gu = A_{\text{con}}x + B_{\text{con}}u \xrightarrow{\text{Gauss collocation}} x_{k+1} = A_{\text{d}}x_k + B_{\text{d}}u_k,$$

State dynamics with state stage values x_k^i, i = 1,...,s :

$$\begin{aligned} x_{k+1} &= x_k + \Delta t \sum_{i=1}^{s} b_i^s f\left(x_k^i\right) + \Delta t g \sum_{i=1}^{s} b_i^s u_k^i \\ x_k^i &= x_k + \Delta t \sum_{j=1}^{s} a_{ij}^s f\left(x_k^i\right) + \Delta t g \sum_{j=1}^{s} a_{ij}^s u_k^j \end{aligned} \tag{5}$$

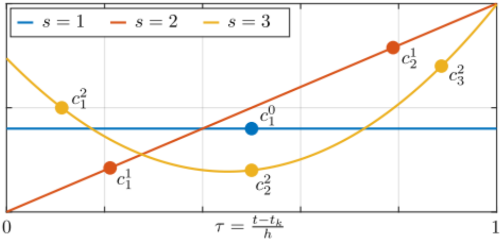
Control input for $t \in [t_k, t_{k+1})$ generated via an $s-1$ order hold element using Lagrange interpolation polynomials [Kotyczka+ 2021]:

$$u\left(t_k + \tau \Delta t\right) = \sum_{i=1}^{s} u_k^i \ell_i^{s-1}(\tau), \quad \ell_i^{s-1}(\tau) = \prod_{\substack{l=1,\\ l \neq i}} \frac{\tau - c_l}{c_i - c_l}$$

Gauss collocation method

Control stage values in the collocation points (weights of Lagrange polynomials):

$$u(t_k^i) = u_k^i, \quad t_k^i = t_k + c_i \Delta t, \quad i = 1, \dots, s.$$



$c_1^0 = \frac{1}{2}$

$c_{1/2}^1 = \frac{1}{2} \mp \frac{\sqrt{3}}{6}$

$c_{1/3}^2 = \frac{1}{2} \mp \frac{\sqrt{15}}{10}, \quad c_2^2 = \frac{1}{2}.$

Figure: Control signal shapes on the unit interval based on Gauss collocation points.

- Zero-order hold control input: $u(t) = u_k^1$
- First-order hold control input: $u(t) = u_k^1 l^1(t) + u_k^2 l^2(t) = u_k^1 \frac{t-c_2^1}{c_1^1-c_2^1} + u_k^2 \frac{t-c_1^1}{c_2^1-c_1^1}$

Gauss collocation method

Numerical solution with state stage values x_k^i and control stage values u_k^i, $i = 1, \dots, s$:

a) ZOH:

$$
\begin{aligned}
x_{k+1} &= x_k + \Delta t b_1 A_{\text{con}} x_k^1 + \Delta t b_1 B_{\text{con}} u_k^1, \\
x_k^1 &= x_k + \Delta t a_{11} A_{\text{con}} x_k^1 + \Delta t a_{11} B_{\text{con}} u_k^1.
\end{aligned}
\tag{6}
$$

b) FOH:

$$
\begin{aligned}
x_{k+1} &= x_k + \Delta t b_1 A_{\text{con}} x_k^1 + \Delta t b_2 A_{\text{con}} x_k^2 \\
&\quad + \Delta t b_1 B_{\text{con}} u_k^1 + \Delta t b_2 B_{\text{con}} u_k^2, \\
x_k^1 &= x_k + \Delta t a_{11} A_{\text{con}} x_k^1 + \Delta t a_{12} A_{\text{con}} x_k^2 \\
&\quad + \Delta t a_{11} B_{\text{con}} u_k^1 + \Delta t a_{12} B_{\text{con}} u_k^2, \\
x_k^2 &= x_k + \Delta t a_{21} A_{\text{con}} x_k^1 + \Delta t a_{22} A_{\text{con}} x_k^2 \\
&\quad + \Delta t a_{21} B_{\text{con}} u_k^1 + \Delta t a_{22} B_{\text{con}} u_k^2.
\end{aligned}
\tag{7}
$$

Simulation setup

$$A_{\text{con}} = \begin{bmatrix} 0 & 7.5 \\ -1.5 & -0.1 \end{bmatrix}, \quad B_{\text{con}} = \begin{bmatrix} 0.145 \\ 0.5 \end{bmatrix}. \tag{8}$$

The initial condition is $x(0) = [2, 2]^\top$ and the state and control input is subject to the following constraints

$$\begin{aligned} -1 \leq x_1 \leq 2.8 \\ -1 \leq x_2 \leq 2.1 \\ -26 \leq u \leq 26 \end{aligned} \tag{9}$$

Simulation setup

The MPC solves the optimization problem:

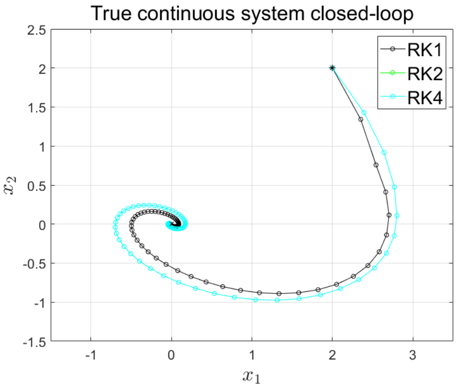
$$\min_{(x,u)} \sum_{k=i}^{i+N-1} l_k\left(x_k, u_k\right) + l_N\left(x_{k+N}\right), \forall k \in [i, \cdots, i+N-1]$$

where

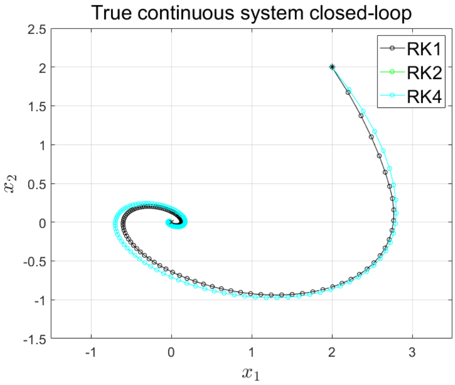
$$l_k = x_k^T Q x_k + u_k^T R u_k,$$
$$l_N = x_{k+N}^T Q_N x_{k+N}$$

We set the prediction horizon $N = 10$, the weight matrices $Q = Q_N = \text{diag}([10, 1])$ and $R = 1$.

Results of RK

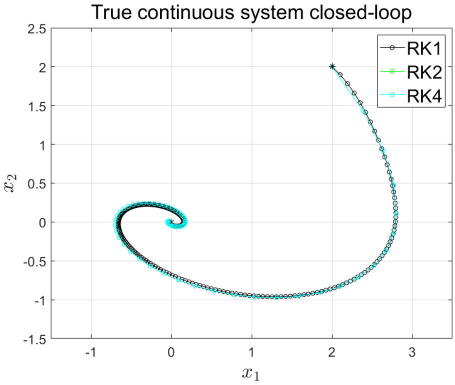


(a) Sampling time $\Delta t = 0.04$

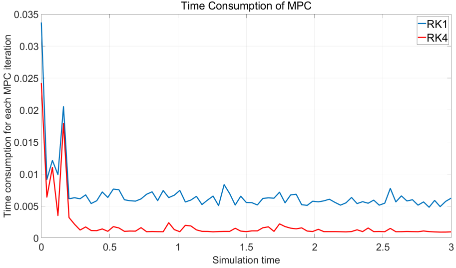


(b) Sampling time $\Delta t = 0.02$

Results of RK

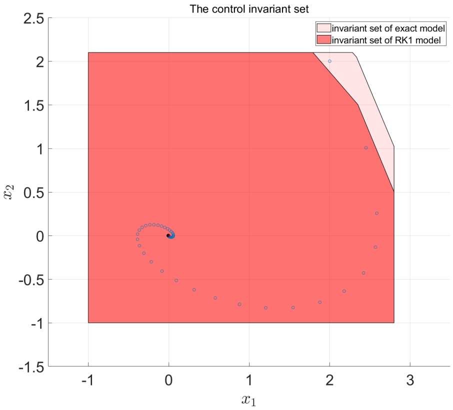


(a) Sampling time $\Delta t = 0.008$ for RK1
Sampling time $\Delta t = 0.04$ for RK2 and RK4

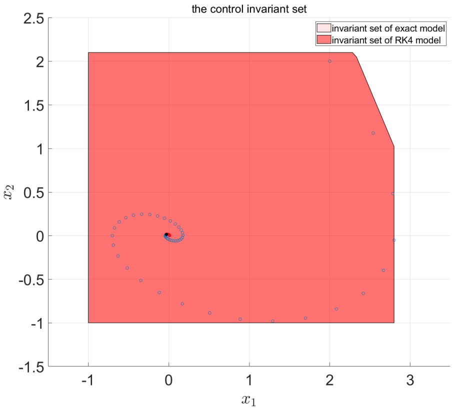


(b) Time consumption of MPC iteration

Results of RK

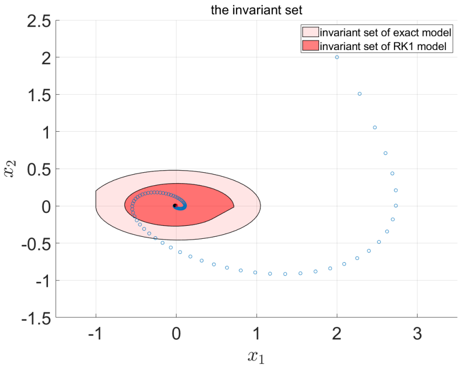


(a) The control invariant set for RK1 model in the case of $\Delta t = 0.06$

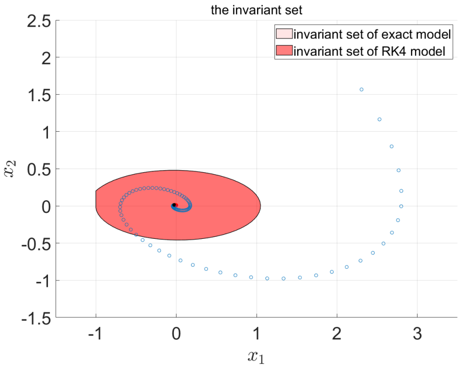


(b) The control invariant set for RK4 model in the case of $\Delta t = 0.06$

Results of RK

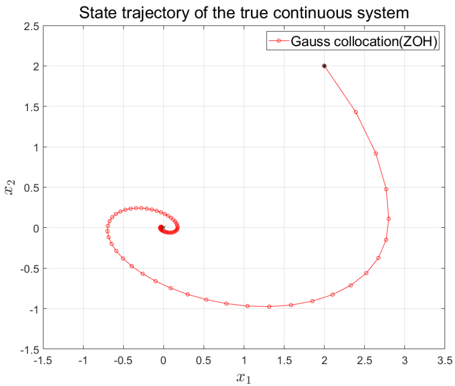


(a) The invariant set of RK1 model in the case of $\Delta t = 0.06$

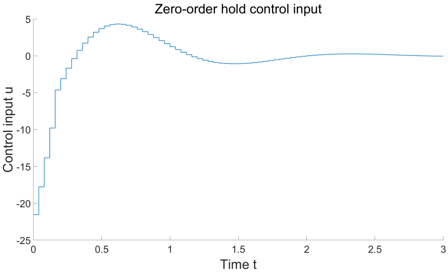


(b) The invariant set of RK4 model in the case of $\Delta t = 0.06$

Results of Gauss collocation

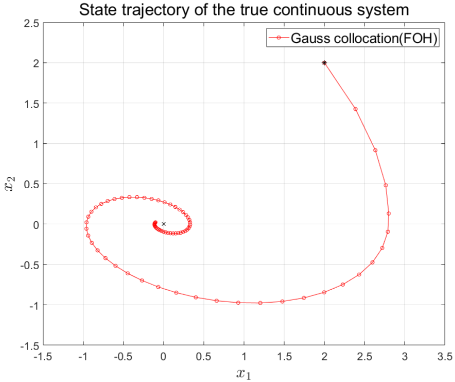


(a) State trajectory of the true continuous system for ZOH

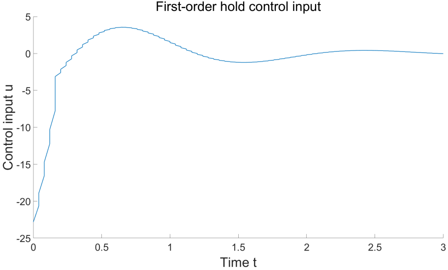


(b) ZOH control input

Results of Gauss collocation

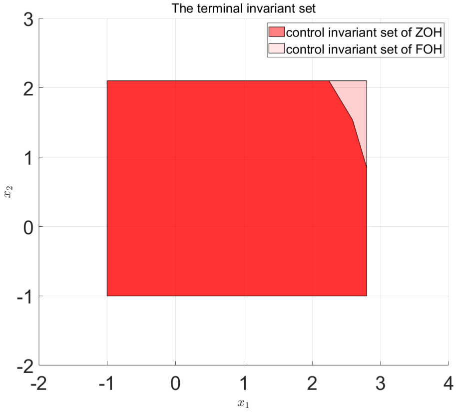


(a) State trajectory of the true continuous system for FOH



(b) FOH control input

Results of Gauss collocation



Conclusion

- Increasing order of RK method $\rightarrow$ improved prediction accuracy & closed-loop performance
- RK1 with small $\Delta t \approx$ RK4 with large Δt, but increased computational load.
- Increased control invariant set for higher-order hold input

Outlook:

- Nonlinear systems
- Higher-order hold control inputs
- Continuity constraint for control jumps

References

Lars Grüne and Jürgen Pannek. **Nonlinear model predictive control**. In: *Nonlinear model predictive control*. Springer, 2017, pp. 45–69.

Paul Kotyczka, Christian J Martens and Laurent Lefèvre. **High Order Discrete-Time Control Based on Gauss-Legendre Collocation**. In: *IFAC-PapersOnLine* 54.19 (2021), pp. 237–242.