哈密顿算子

哈密顿算子—矢量微分算子

$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z}$$

[del]

[nabla]

1. 算子∇ 既有微分的性质， 又有矢量的特点;

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[del]

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1. 算子▽ 既有微分的性质， 又有矢量的特点;

$$\text{grad}u = \nabla u$$

$$\nabla u = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \right) u = \frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z$$

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2. 算子∇ 在不同的坐标系中有不同的表达式;

直角坐标系中 $\Longrightarrow$ $$\nabla u = \frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z$$

柱坐标系中 $\Longrightarrow$ $$\nabla u = \frac{\partial u}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\vec{e}_\phi + \frac{\partial u}{\partial z}\vec{e}_z$$

球坐标系中 $\Longrightarrow$ $$\nabla u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\vec{e}_\phi$$

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1. 算子∇ 既有微分的性质，又有矢量的特点；

2. 算子∇ 在不同的坐标系中有不同的表达式；

3. 设 a、b 为任意常数，函数 u_1、u_2、u 为任意标量场，则 $\nabla(au_1 + bu_2) = a\nabla u_1 + b\nabla u_2$

$$\nabla(u_1 u_2) = u_1\nabla u_2 + u_2\nabla u_1$$

若$\nabla u \equiv 0$，则$u =$ 常数

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$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z}$$

4.直角坐标系中几个常用公式:

$$\nabla u = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right)u = \frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z$$

$$\nabla \bullet \vec{A} = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right) \bullet \left(A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

哈密顿算子—矢量微分算子

$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z}$$

4.直角坐标系中几个常用公式:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

注意此项的符号与顺序

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e}_z$$

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$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z}$$

5.拉普拉斯算子

$$\nabla \bullet \nabla = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \right) \bullet \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$$

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

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$$\boxed{\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z}$$

$$\nabla^2 \vec{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z\right)$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\vec{e}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\vec{e}_y$$

$$+ \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\vec{e}_z$$

题3．设 $R=|\vec{r}-\vec{r}'|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ **为源点** $\vec{r}'$ **到场点** $\vec{r}$ **的距离，**$\vec{R}$ **的方向规定为从源点指向场点，证明下列结果:**

$$(1).\ \nabla R=\frac{\vec{R}}{R} \quad \nabla' R=-\frac{\vec{R}}{R}=-\nabla R \qquad (2).\ \nabla\frac{1}{R}=-\nabla'\frac{1}{R}=-\frac{\vec{R}}{R^3}$$

分析：本题要注意算符 ∇ **和算符** ∇' **的区别，其中** ∇ **是对场点作用，而** ∇' **是对源点作用，即**

$$\nabla=\frac{\partial}{\partial x}\vec{e}_x+\frac{\partial}{\partial y}\vec{e}_y+\frac{\partial}{\partial z}\vec{e}_z \qquad \nabla'=\frac{\partial}{\partial x'}\vec{e}_x+\frac{\partial}{\partial y'}\vec{e}_y+\frac{\partial}{\partial z'}\vec{e}_z$$

常用矢量关系式，要记住

题3．设 $R=|\vec{r}-\vec{r}'|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ **为源点** $\vec{r}'$ **到场点** $\vec{r}$ **的距离，**$\vec{R}$ **的方向规定为从源点指向场点，证明下列结果:**

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证明：(1) 因为 $\nabla R=\frac{\partial R}{\partial x}\vec{e}_x+\frac{\partial R}{\partial y}\vec{e}_y+\frac{\partial R}{\partial z}\vec{e}_z$

$$\frac{\partial R}{\partial x}=\frac{1}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}\cdot 2(x-x')=\frac{(x-x')}{R}$$

同理: $\frac{\partial R}{\partial y}=\frac{(y-y')}{R}\qquad \frac{\partial R}{\partial z}=\frac{(z-z')}{R}$

$$\Rightarrow\quad \nabla R=\frac{(x-x')}{R}\vec{e}_x+\frac{(y-y')}{R}\vec{e}_y+\frac{(z-z')}{R}\vec{e}_z=\frac{\vec{R}}{R}$$

题3．设 $R=|\vec{r}-\vec{r}\,'|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ **为源点** $\vec{r}\,'$ **到场点** $\vec{r}$ **的距离，**$\vec{R}$ **的方向规定为从源点指向场点，证明下列结果:**

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证明：(1) 因为 $\nabla' R=\dfrac{\partial R}{\partial x'}\vec{e}_x+\dfrac{\partial R}{\partial y'}\vec{e}_y+\dfrac{\partial R}{\partial z'}\vec{e}_z$

$$\frac{\partial R}{\partial x'}=\frac{1}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}\cdot\left[-2(x-x')\right]=-\frac{(x-x')}{R}$$

同理: $\dfrac{\partial R}{\partial y'}=-\dfrac{(y-y')}{R}\quad \dfrac{\partial R}{\partial z'}=-\dfrac{(z-z')}{R}$

$$\Rightarrow\quad \nabla' R=-\left[\frac{(x-x')}{R}\vec{e}_x+\frac{(y-y')}{R}\vec{e}_y+\frac{(z-z')}{R}\vec{e}_z\right]=-\frac{\vec{R}}{R}=-\nabla R$$

题3．设 $R=|\vec{r}-\vec{r}'|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ **为源点** $\vec{r}'$ **到场点** $\vec{r}$ **的距离，**$\vec{R}$ **的方向规定为从源点指向场点，证明下列结果:**

$$(1).\ \nabla R=\frac{\vec{R}}{R}\quad \nabla' R=-\frac{\vec{R}}{R}=-\nabla R \qquad (2).\ \nabla\frac{1}{R}=-\nabla'\frac{1}{R}=-\frac{\vec{R}}{R^3}$$

证明：(2) 因为

$$\nabla\frac{1}{R}=-\frac{1}{R^2}\nabla R=-\frac{1}{R^2}\frac{\vec{R}}{R}=-\frac{\vec{R}}{R^3}$$

$$\nabla'\frac{1}{R}=-\frac{1}{R^2}\nabla' R=-\frac{1}{R^2}\left(-\frac{\vec{R}}{R}\right)=\frac{\vec{R}}{R^3}=-\nabla\frac{1}{R}$$

本 节 要 点

1. 哈密顿算子与不同物理量的作用关系;

2. 拉普拉斯算子与不同物理量的作用关系;

特别提示：在不同坐标系中 ∇u 、 $\nabla \bullet \vec{A}$ 、$\nabla \times \vec{A}$ 的计算公式也不同。