# Recommender Systems: Latent Factor Models

CS246: Mining Massive Datasets
Jure Leskovec, Stanford University
http://cs246.stanford.edu



## The Netflix Prize

### Training data

- 100 million ratings, 480,000 users, 17,770 movies
- 6 years of data: 2000-2005

#### Test data

- Last few ratings of each user (2.8 million)
- Evaluation criterion: Root Mean Square Error (RMSE)
- Netflix's system RMSE: 0.9514

#### Competition

- 2,700+ teams
- \$1 million prize for 10% improvement on Netflix

# The Netflix Utility Matrix R

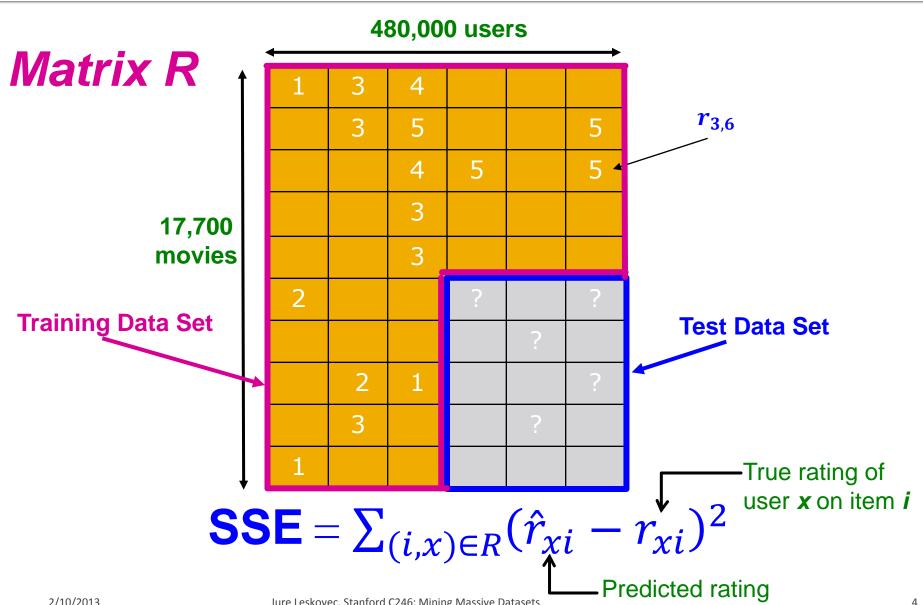
## Matrix R

17,700 movies

←					<del></del>
1	3	4			
	3	5			5
		4	5		5
		3			
		3			
2			2		2
				5	
	2	1			1
	3			3	
1					

480,000 users

# Utility Matrix R: Evaluation



# BellKor Recommender System

The winner of the Netflix Challenge

Multi-scale modeling of the data:

Combine top level, "regional" modeling of the data, with a refined, local view:

Global:

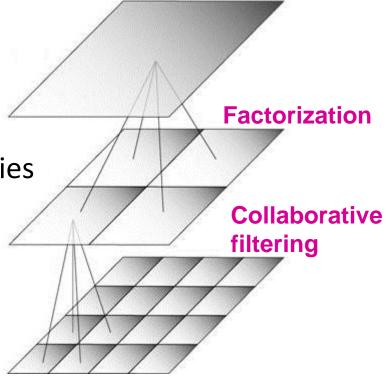
Overall deviations of users/movies

Factorization:

Addressing "regional" effects

Collaborative filtering:

Extract local patterns



**Global effects** 

# **Modeling Local & Global Effects**

#### Global:

- Mean movie rating: 3.7 stars
- The Sixth Sense is 0.5 stars above avg.
- Joe rates 0.2 stars below avg.
  - ⇒ Baseline estimation:

Joe will rate The Sixth Sense 4 stars

- Local neighborhood (CF/NN):
  - Joe didn't like related movie Signs
  - ⇒ Final estimate:
    Joe will rate The Sixth Sense 3.8 stars







# Recap: Collaborative Filtering (CF)

- Earliest and most popular collaborative filtering method
- Derive unknown ratings from those of "similar" movies (item-item variant)
- Define similarity measure  $s_{ij}$  of items i and j
- Select k-nearest neighbors, compute the rating
  - N(i; x): items most similar to i that were rated by x

$$\hat{r}_{xi} = \frac{\sum_{j \in N(i;x)} s_{ij} \cdot r_{xj}}{\sum_{j \in N(i;x)} s_{ij}}$$

 $s_{ij}$ ... similarity of items i and j  $r_{uj}$ ...rating of user x on item j N(i;x)... set of items similar to item i that were rated by x

# **Modeling Local & Global Effects**

In practice we get better estimates if we model deviations:

$$\hat{r}_{xi} = b_{xi} + \frac{\sum_{j \in N(i;x)} s_{ij} \cdot (r_{xj} - b_{xj})}{\sum_{j \in N(i;x)} s_{ij}}$$

baseline estimate for  $r_{xi}$ 

$$b_{xi} = \mu + b_x + b_i$$

 $\mu$  = overall mean rating

 $b_x$  = rating deviation of user x

=  $(avg. rating of user x) - \mu$ 

 $\mathbf{b}_i = (avg. \ rating \ of \ movie \ \mathbf{i}) - \boldsymbol{\mu}$ 

#### **Problems/Issues:**

- 1) Similarity measures are "arbitrary"
- 2) Pairwise similarities neglect interdependencies among users
- **3)** Taking a weighted average can be restricting

Solution: Instead of  $s_{ij}$  use  $w_{ij}$  that we estimate directly from data

# Idea: Interpolation Weights $w_{ij}$

Use a weighted sum rather than weighted avg.:

$$\widehat{r_{xi}} = b_{xi} + \sum_{j \in N(i;x)} w_{ij} (r_{xj} - b_{xj})$$

- A few notes:
  - We sum over all movies j that are similar to i and were rated by x
  - $lackbox{ } lackbox{ } lac$ 
    - We allow:  $\sum_{j \in N(i,x)} w_{ij} \neq 1$
  - $w_{ij}$  models interaction between pairs of movies (it does not depend on user x)
  - N(i; x) ... set of movies rated by user x that are similar to movie i

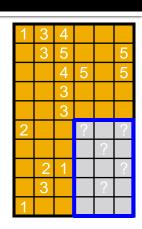
# Idea: Interpolation Weights $w_{ij}$

- $\widehat{r_{xi}} = b_{xi} + \sum_{j \in N(i,x)} w_{ij} (r_{xj} b_{xj})$
- How to set  $w_{ij}$ ?
  - Remember, error metric is SSE:  $\sum_{(i,u)\in R} (\hat{r}_{ui} r_{ui})^2$
  - Find w<sub>ii</sub> that minimize SSE on training data!
    - Models relationships between item i and its neighbors j
  - w<sub>ij</sub> can be learned/estimated based on x and all other users that rated i

## Why is this a good idea?

# Recommendations via Optimization

- Here is what we just did:
  - Goal: Make good recommendations
    - Quantify goodness using SSE:So, Lower SSE means better recommendations



- We want to make good recommendations on items that some user has not yet seen. Can't really do this. Why?
- Let's set values w such that they work well on known (user, item) ratings
  - And **hope** these **w**s will predict well the unknown ratings
- This is the first time in the class that we see
   Optimization methods

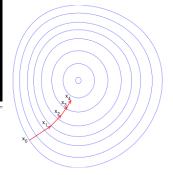
# Recommendations via Optimization

- Idea: Let's set values w such that they work well on known (user, item) ratings
- How to find such values w?
- Idea: Define an objective function and solve the optimization problem
- Find w<sub>ij</sub> that minimize SSE on training data!

$$\min_{w_{ij}} \sum_{x} \left( \left[ b_{xi} + \sum_{j \in N(i;x)} w_{ij} (r_{xj} - b_{xj}) \right] - r_{xi} \right)^{2}$$

Think of w as a vector of numbers

# Interpolation Weights



- We have the optimization problem, now what?
- Gradient decent

- $\min_{w_{ij}} \sum_{x} \left( \left[ b_{xi} + \sum_{j \in N(i;x)} w_{ij} (r_{xj} b_{xj}) \right] r_{xi} \right)^{2}$
- Iterate until convergence:  $w = w \eta \nabla w$

 $\eta$  ... learning rate

• where  $\nabla w$  is gradient (derivative evaluated on data):

$$\nabla w = \left[\frac{\partial}{\partial w_{ij}}\right] = 2\sum_{x} \left( \left[ b_{xi} + \sum_{k \in N(i;x)} w_{ik} (r_{xk} - b_{xk}) \right] - r_{xi} \right) (r_{xj} - b_{xj})$$

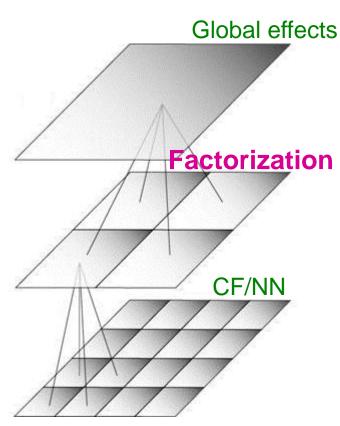
for 
$$j \in \{N(i; x), \forall i, \forall x\}$$
  
else  $\frac{\partial}{\partial w_{ij}} = \mathbf{0}$ 

■ **Note:** we fix movie i, go over all  $r_{xi}$ , for every movie  $j \in N(i; x)$ , we compute  $\frac{\partial}{\partial w_{ii}}$ 

while 
$$|w_{new} - w_{old}| > \varepsilon$$
:  
 $w_{old} = w_{new}$   
 $w_{new} = w_{old} - \eta \cdot \nabla w_{old}$ 

# Interpolation Weights

- So far:  $\widehat{r_{xi}} = b_{xi} + \sum_{j \in N(i;x)} w_{ij} (r_{xj} b_{xj})$ 
  - Weights  $\mathbf{w}_{ij}$  derived based on their role; no use of an arbitrary similarity measure  $(\mathbf{w}_{ij} \neq \mathbf{s}_{ij})$
  - Explicitly account for interrelationships among the neighboring movies
- Next: Latent factor model
  - Extract "regional" correlations



## Performance of Various Methods

Global average: 1.1296

User average: 1.0651

Movie average: 1.0533

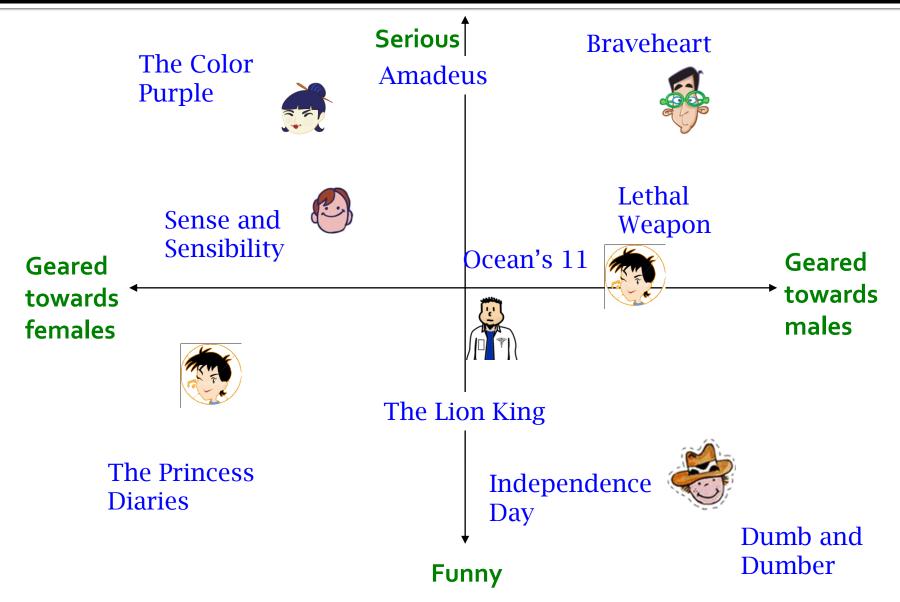
Netflix: 0.9514

**Basic Collaborative filtering: 0.94** 

CF+Biases+learnt weights: 0.91

Grand Prize: 0.8563

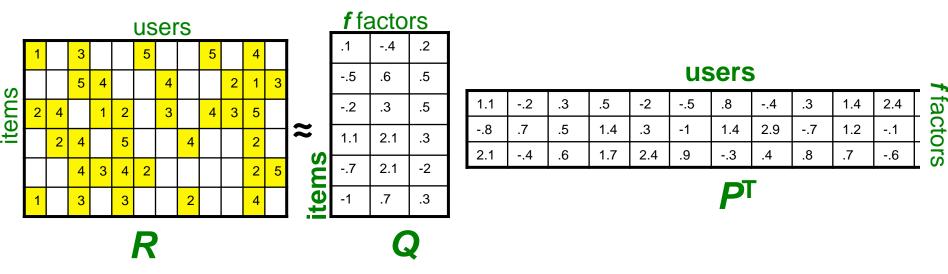
# Latent Factor Models (e.g., SVD)



## **Latent Factor Models**

**SVD:**  $A = U \Sigma V^T$ 

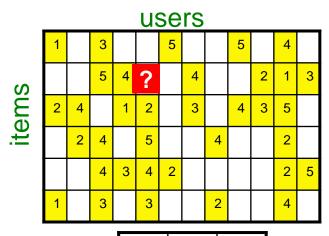
■ "SVD" on Netflix data:  $\mathbf{R} \approx \mathbf{Q} \cdot \mathbf{P}^T$ 



- For now let's assume we can approximate the rating matrix R as a product of "thin"  $Q \cdot P^T$ 
  - R has missing entries but let's ignore that for now!
    - Basically, we will want the reconstruction error to be small on known ratings and we don't care about the values on the missing ones

# Ratings as Products of Factors

■ How to estimate the missing rating of user x for item i?





$\hat{r}_{xi} =$	$q_i$	$p_x^T$
$=\sum$	$q_{if}$	$\cdot p_{xf}$
= <b>-</b>	= row <i>i</i> o = colum	of <b>Q</b> n <b>x</b> of <b>P</b> <sup>T</sup>
$P_X$ -	- coluiti	

items	.1	4	.2
	5	.6	.5
	2	.3	.5
	1.1	2.1	.3
	7	2.1	-2
	-1	.7	.3

users .3 .5 -.5 .3 -.2 -2 1.1 -.4 .7 .5 1.4 1.4 2.9 -1 -.7 -.4 1.7 2.4 -.3 .4

**f** factors

2.4

-.1

-.6

-.9

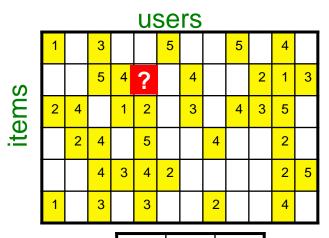
1.3

1.4

1.2

# Ratings as Products of Factors

■ How to estimate the missing rating of user x for item i?





$\hat{r}_{xi} =$	$q_i$	$p_x^T$
$=\sum$	$q_{if}$	$\cdot p_{xf}$
	: row <i>i</i> o = columi	f <b>Q</b> n <b>x</b> of <b>P</b> <sup>T</sup>

items	.1	4	.2	
	5	.6	.5	
	2	.3	.5	
	1.1	2.1	.3	
	7	2.1	-2	
	-1	.7	.3	

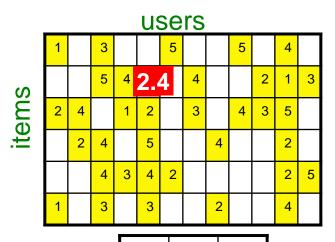
users .3 .5 -.5 .3 2.4 -.2 1.4 -.9 1.1 -2 -.4 .7 .5 1.4 1.4 2.9 1.2 -1 -.7 1.3 -.1 -.4 1.7 2.4 -.3 .4 -.6

PT

f factors

# Ratings as Products of Factors

■ How to estimate the missing rating of user x for item i?





$\hat{r}_{xi} =$	$q_i$	$p_x^T$
$=\sum$	$q_{if}$	$\cdot p_{xf}$
= <b>-</b>	= row <i>i</i> o = colum	of <b>Q</b> n <b>x</b> of <b>P</b> <sup>T</sup>
$P_X$ -	- coluiti	

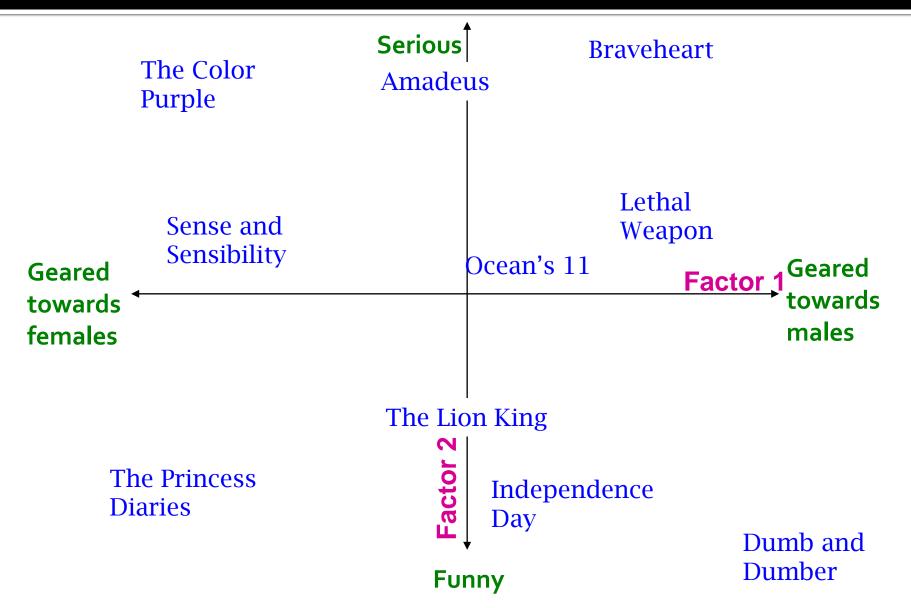
(0	.1	4	.2		
	5	.6	.5		
items	2	.3	.5		
ite	1.1	2.1	.3		
	7	2.1	-2		
	-1	.7	.3		
<b>f</b> factors					

ors	1.1	2	.3	.5	-2	5	.8	4	.3	1.4	2.4	9
acto	8	.7	.5	1.4	.3	-1	1.4	2.9	7	1.2	1	1.3
ff	2.1	4	.6	1.7	2.4	.9	3	.4	.8	.7	6	.1

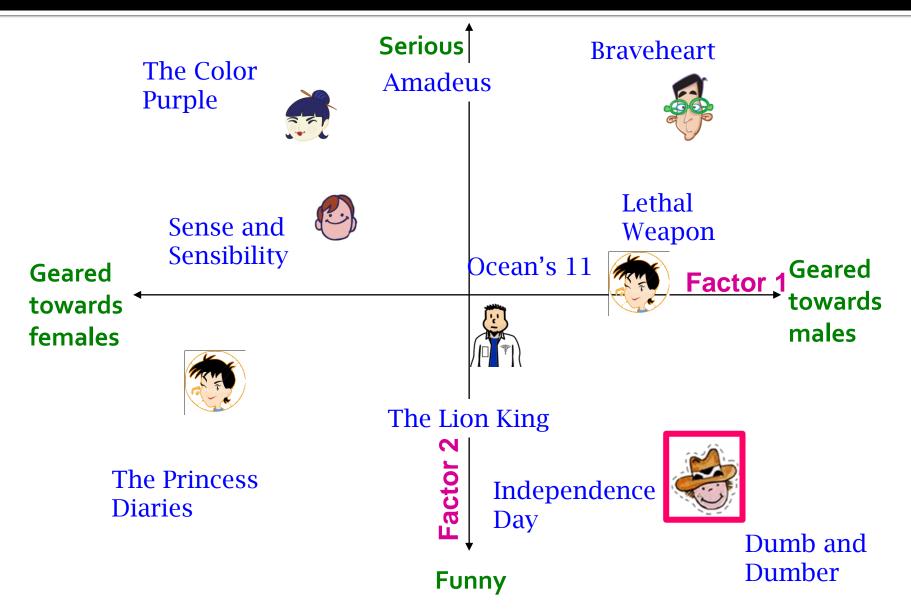
USERS

PT

## **Latent Factor Models**



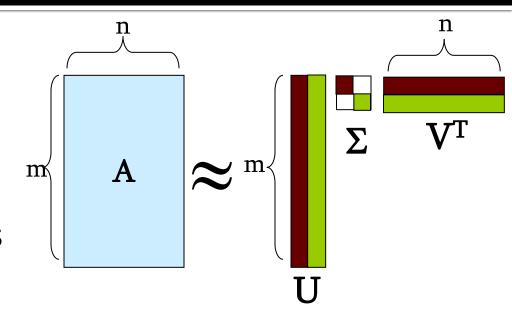
## **Latent Factor Models**



## Recap: SVD

#### Remember SVD:

- A: Input data matrix
- U: Left singular vecs
- V: Right singular vecs
- Σ: Singular values



SVD gives minimum reconstruction error (SSE!)

$$\min_{U,V,\Sigma} \sum_{ij} \left( A_{ij} - \left[ U \Sigma V^{\mathrm{T}} \right]_{ij} \right)^2$$
 The sum goes over all entries. But our  $R$  has missing entries!

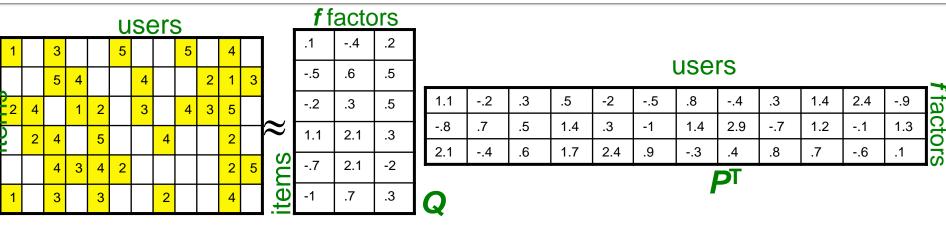
■ So in our case, "SVD" on Netflix data:  $R \approx Q \cdot P^T$ 

$$A = R$$
,  $Q = U$ ,  $P^{T} = \sum V^{T}$ 

$$\hat{\boldsymbol{r}}_{xi} = \boldsymbol{q}_i \cdot \boldsymbol{p}_x^T$$

But, we are not done yet! R has missing entries!

## **Latent Factor Models**



- SVD isn't defined when entries are missing!
- Use specialized methods to find P, Q

$$\min_{P,Q} \sum_{(i,x)\in\mathbb{R}} (r_{xi} - q_i \cdot p_x^T)^2$$

$$\hat{r}_{xi} = q_i \cdot p_x^T$$

- Note:
  - We don't require cols of P, Q to be orthogonal/unit length
  - P, Q map users/movies to a latent space
  - The most popular model among Netflix contestants

# Dealing with Missing Entries

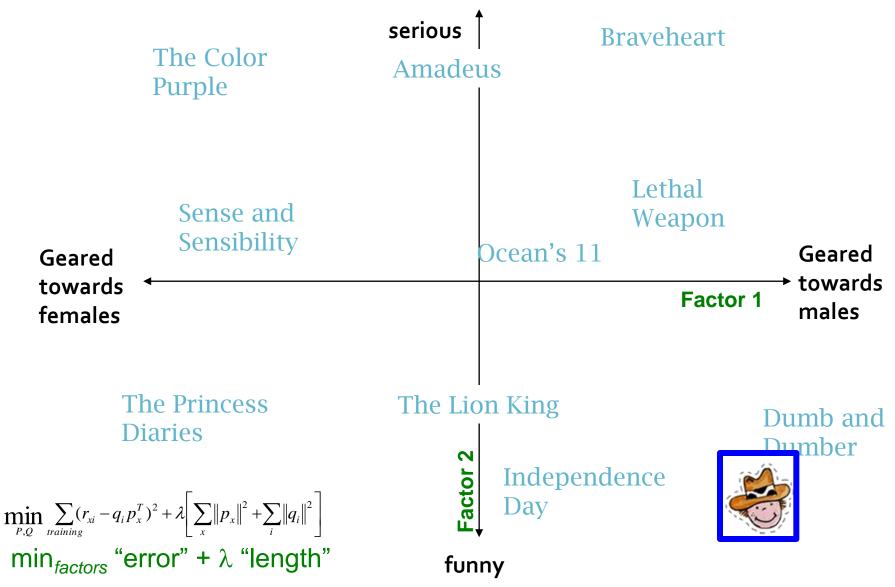
- Want to minimize SSE for unseen test data
- Idea: Minimize SSE on training data

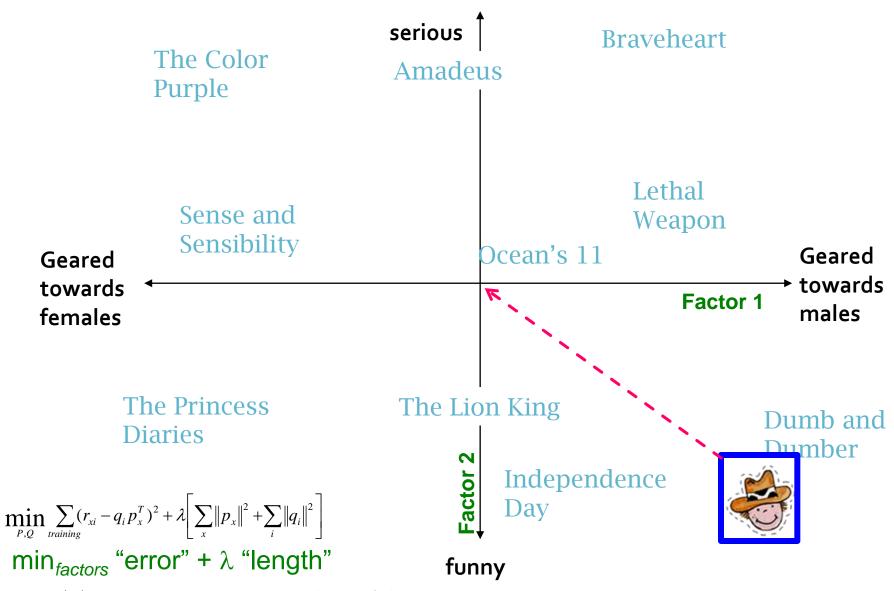


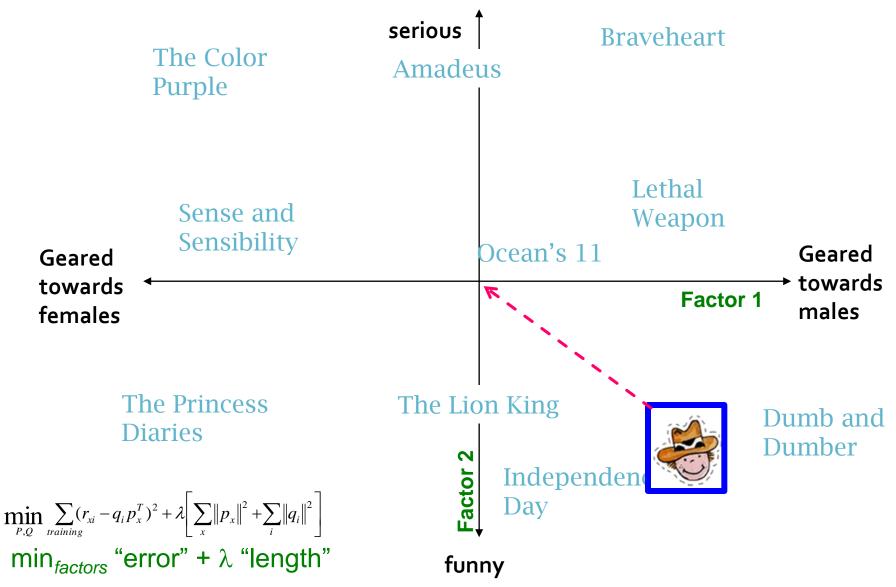
- Want large f (# of factors) to capture all the signals
- But, SSE on test data begins to rise for f > 2
- Regularization is needed!
  - Allow rich model where there are sufficient data
  - Shrink aggressively where data are scarce

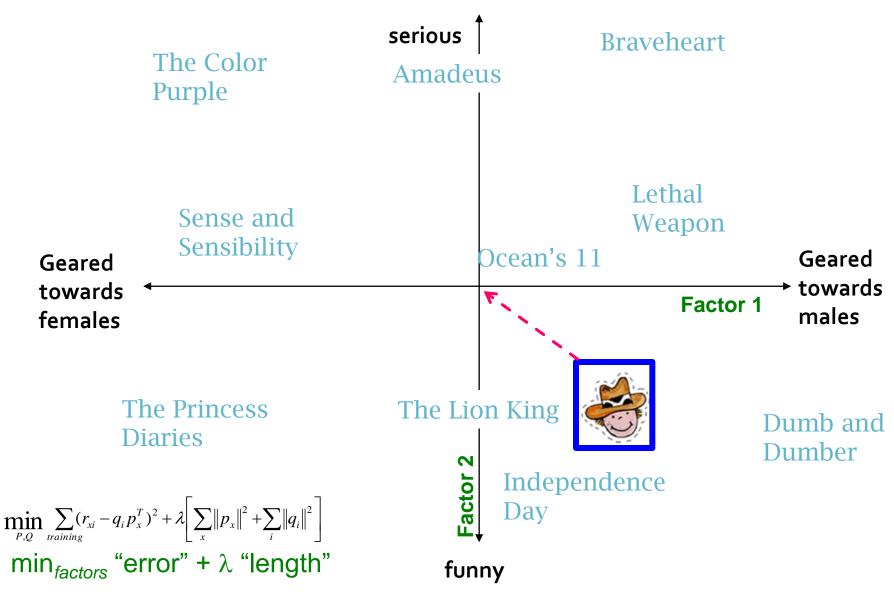
$$\min_{P,Q} \sum_{\text{training}} (r_{xi} - q_i p_x^T)^2 + \lambda \left[ \sum_{x} \|p_x\|^2 + \sum_{i} \|q_i\|^2 \right]$$
"error"
"length"

 $\lambda$ ... regularization parameter

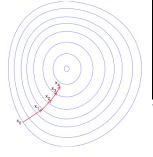








## Stochastic Gradient Descent



Want to find matrices P and Q:

$$\min_{\substack{P,Q \text{ training} \\ \text{Gradient decent:}}} (r_{xi} - q_i \ p_x^T)^2 + \lambda \left[ \sum_{x} \left\| p_x \right\|^2 + \sum_{i} \left\| q_i \right\|^2 \right]$$

- - Initialize P and Q (using SVD, pretend missing ratings are 0)
  - Do gradient descent:

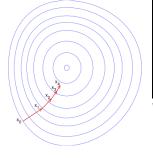
$$\blacksquare$$
 *P* ← *P* -  $\eta$  ·  $\nabla$  P

• 
$$Q \leftarrow Q - \eta \cdot \nabla Q$$

How to compute gradient of a matrix? Compute gradient of every element independently!

- Where  $\nabla Q$  is gradient/derivative of matrix Q:  $\nabla Q = [\nabla q_{if}]$  and  $\nabla q_{if} = \sum_{x,i} -2(r_{xi} - q_i p_x^T)p_{xf} + 2\lambda q_{if}$ 
  - lacktriangle Here  $q_{if}$  is entry f of row  $oldsymbol{q}_i$  of matrix  $oldsymbol{Q}$
- **Observation: Computing gradients is slow!**

## Stochastic Gradient Descent



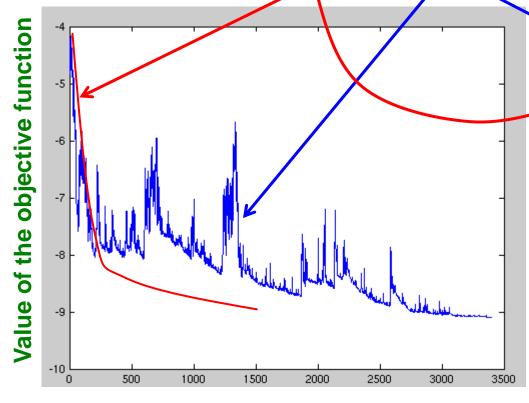
- Gradient Descent (GD) vs. Stochastic GD
  - Observation:  $\nabla Q = [\nabla q_{if}]$  where

$$\nabla q_{if} = \sum_{x,i} -2(r_{xi} - q_{if}p_{xf})p_{xf} + 2\lambda q_{if} = \sum_{x,i} \nabla Q(r_{xi})$$

- Here  $q_{if}$  is entry f of row  $q_i$  of matrix Q
- $Q = Q \eta \nabla Q = Q \eta \left[ \sum_{x,i} \nabla Q (r_{xi}) \right]$
- Idea: Instead of evaluating gradient over all ratings evaluate it for each individual rating and make a step
- GD:  $\mathbf{Q} \leftarrow \mathbf{Q} \eta \left[ \sum_{r_{xi}} \nabla \mathbf{Q}(r_{xi}) \right]$
- SGD:  $Q \leftarrow Q \eta \nabla Q(r_{xi})$ 
  - Faster convergence!
    - Need more steps but each step is computed much faster

## SGD vs. GD

Convergence of GD vs. SGD



Iteration/step

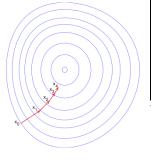
**GD** improves the value of the objective function at every step.

**SGD** improves the value but in a "noisy" way.

**GD** takes fewer steps to converge but each step takes much longer to compute.

In practice, **SGD** is much faster!

## Stochastic Gradient Descent



#### Stochastic gradient decent:

- Initialize P and Q (using SVD, pretend missing ratings are 0)
- Then iterate over the ratings (multiple times if necessary) and update factors:

#### For each $r_{xi}$ :

$$\bullet \ \varepsilon_{xi} = r_{xi} - q_i \cdot p_x^T$$

$$q_i \leftarrow q_i + \eta \left( \varepsilon_{xi} \, p_x - \lambda \, q_i \right)$$

$$p_x \leftarrow p_x + \eta \left( \varepsilon_{xi} \ q_i - \lambda \ p_x \right)$$

### 2 for loops:

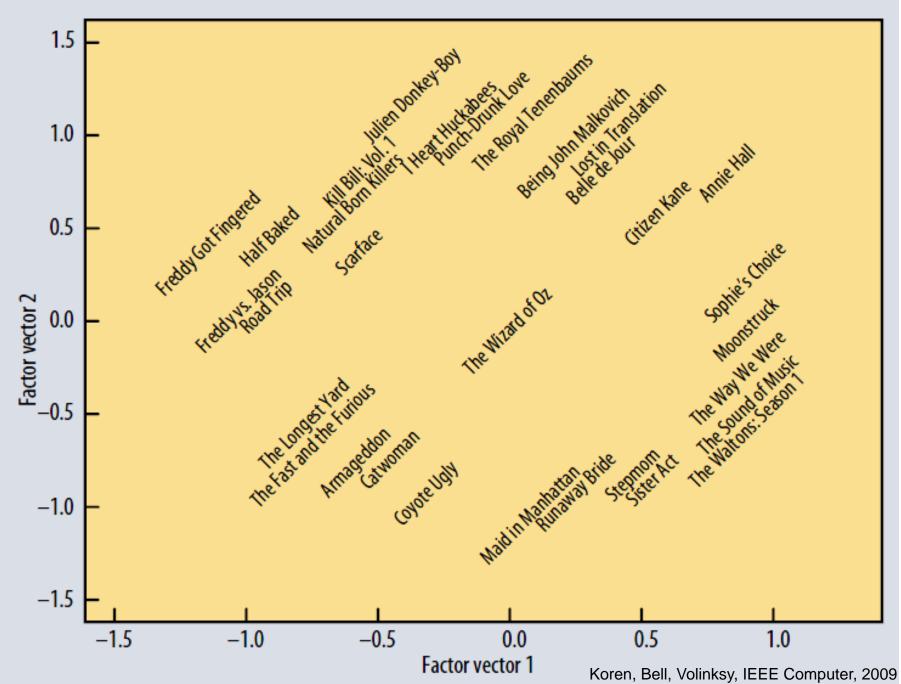
- For until convergence:
  - For each r<sub>xi</sub>
    - Compute gradient, do a "step"

(derivative of the "error")

(update equation)

(update equation)

 $\eta$  ... learning rate



# Extending Latent Factor Model to Include Biases

# **Modeling Biases and Interactions**

#### user bias



#### movie bias



#### user-movie interaction



#### **Baseline predictor**

- Separates users and movies
- Benefits from insights into user's behavior
- Among the main practical contributions of the competition
  - $\mu = \mu$  = overall mean rating
  - $\mathbf{b}_{\mathbf{x}}$  = bias of user  $\mathbf{x}$
  - $\mathbf{b}_{i}^{\hat{}}$  = bias of movie  $\mathbf{i}$

#### **User-Movie interaction**

- Characterizes the matching between users and movies
- Attracts most research in the field
- Benefits from algorithmic and mathematical innovations

#### **Baseline Predictor**

We have expectations on the rating by user x of movie i, even without estimating x's attitude towards movies like i







- Rating scale of user x
- Values of other ratings user gave recently (day-specific mood, anchoring, multi-user accounts)

- (Recent) popularity of movie i
- Selection bias; related to number of ratings user gave on the same day ("frequency")

# Putting It All Together

$$r_{xi} = \mu + b_x + b_i + q_i \cdot p_x^T$$

Overall

mean rating

Overall

user x

Bias for movie i

interaction

#### Example:

- Mean rating:  $\mu = 3.7$
- You are a critical reviewer: your ratings are 1 star lower than the mean:  $b_x = -1$
- Star Wars gets a mean rating of 0.5 higher than average movie:  $b_i = +0.5$
- Predicted rating for you on Star Wars:

$$= 3.7 - 1 + 0.5 = 3.2$$

## Fitting the New Model

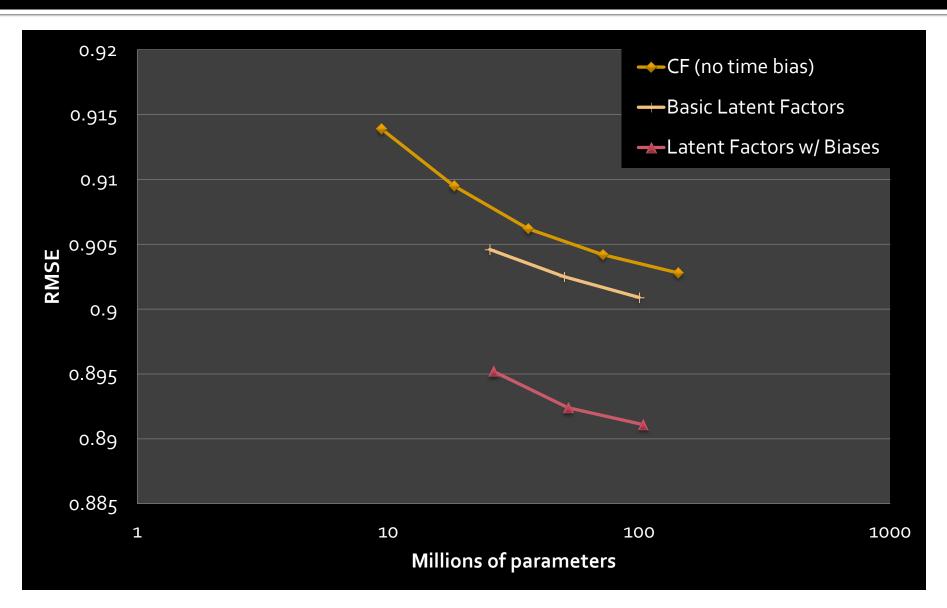
$$\min_{Q,P} \sum_{(x,i)\in R} \left( r_{xi} - (\mu + b_x + b_i + q_i p_x^T) \right)^2$$
goodness of fit

$$+\lambda \left(\sum_{i} \|q_i\|^2 + \sum_{x} \|p_x\|^2 + \sum_{x} \|b_x\|^2 + \sum_{i} \|b_i\|^2\right)$$
regularization

 $\lambda$  is selected via gridsearch on a validation set

- Stochastic gradient decent to find parameters
  - Note: Both biases  $b_{ij}$ ,  $b_{ij}$  as well as interactions  $q_{ij}$ ,  $p_{ij}$ are treated as parameters (we estimate them)

### Performance of Various Methods



### Performance of Various Methods

Global average: 1.1296

User average: 1.0651

Movie average: 1.0533

Netflix: 0.9514

Basic Collaborative filtering: 0.94

Collaborative filtering++: 0.91

Latent factors: 0.90

**Latent factors+Biases: 0.89** 

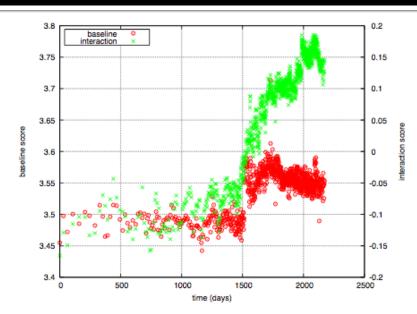
Grand Prize: 0.8563

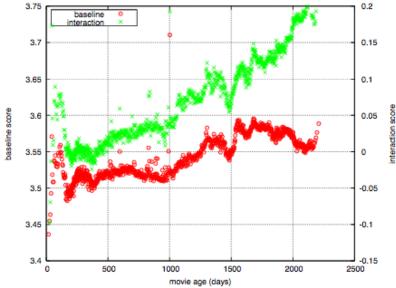
# The Netflix Challenge: 2006-09

### **Temporal Biases Of Users**

- Sudden rise in the average movie rating (early 2004)
  - Improvements in Netflix
  - GUI improvements
  - Meaning of rating changed
- Movie age
  - Users prefer new movies without any reasons
  - Older movies are just inherently better than newer ones

Y. Koren, Collaborative filtering with temporal dynamics, KDD '09





## Temporal Biases & Factors

#### Original model:

$$r_{xi} = \mu + b_x + b_i + q_i \cdot p_x^T$$

Add time dependence to biases:

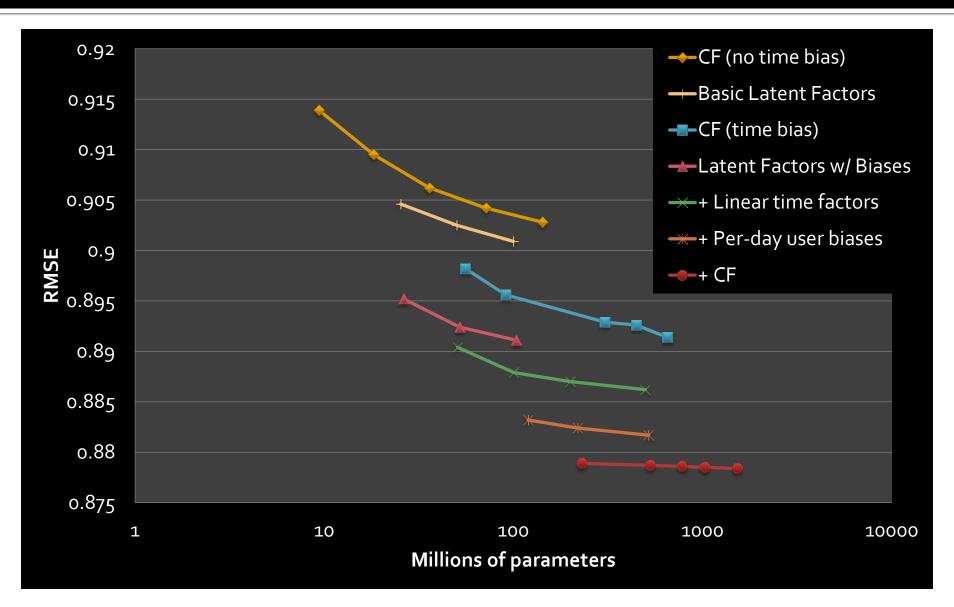
$$r_{xi} = \mu + b_x(t) + b_i(t) + q_i \cdot p_x^T$$

- Make parameters  $b_u$  and  $b_i$  to depend on time
- (1) Parameterize time-dependence by linear trends
  - (2) Each bin corresponds to 10 consecutive weeks

$$b_i(t) = b_i + b_{i,\operatorname{Bin}(t)}$$

- Add temporal dependence to factors
  - $p_x(t)$ ... user preference vector on day t

# **Adding Temporal Effects**



### Performance of Various Methods

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Latent factors+Biases: 0.89

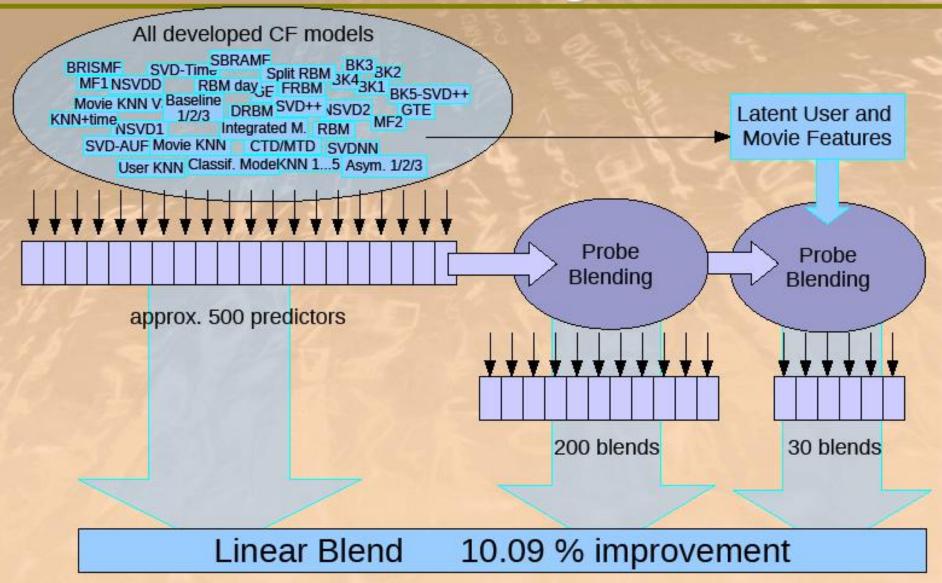
**Latent factors+Biases+Time: 0.876** 

Still no prize! (2)
Getting desperate.
Try a "kitchen sink" approach!

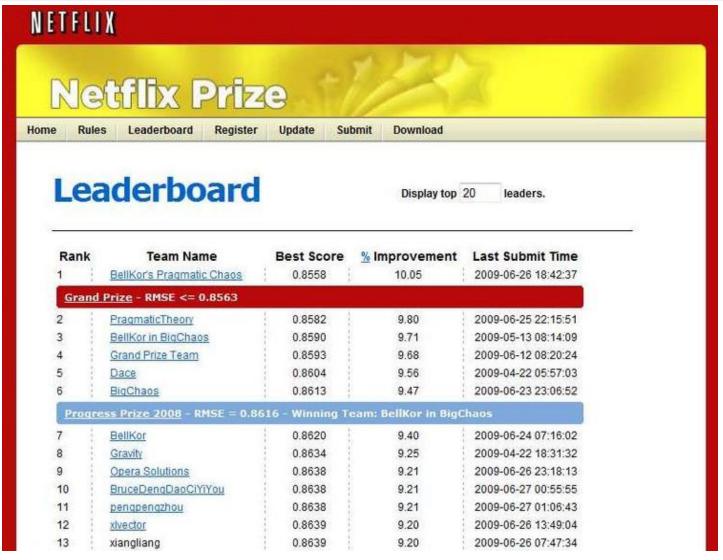
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#### The big picture

### Solution of BellKor's Pragmatic Chaos



## Standing on June 26th 2009



June 26th submission triggers 30-day "last call"

## The Last 30 Days

#### Ensemble team formed

- Group of other teams on leaderboard forms a new team
- Relies on combining their models
- Quickly also get a qualifying score over 10%

#### BellKor

- Continue to get small improvements in their scores
- Realize that they are in direct competition with Ensemble

#### Strategy

- Both teams carefully monitoring the leaderboard
- Only sure way to check for improvement is to submit a set of predictions
  - This alerts the other team of your latest score

## 24 Hours from the Deadline

- Submissions limited to 1 a day
  - Only 1 final submission could be made in the last 24h
- 24 hours before deadline...
  - BellKor team member in Austria notices (by chance) that Ensemble posts a score that is slightly better than BellKor's
- Frantic last 24 hours for both teams
  - Much computer time on final optimization
  - Carefully calibrated to end about an hour before deadline
- Final submissions
  - BellKor submits a little early (on purpose), 40 mins before deadline
  - Ensemble submits their final entry 20 mins later
  - ....and everyone waits....

#### **Netflix Prize**



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#### Leaderboard

Showing Test Score. Click here to show quiz score

Display top 20 ‡ leaders.

Rank	Team Name	Best Test Score	% Improvement	Best Submit Time
Grand	Prize - RMSE = 0.8567 - Winning To	sam: BellKor's Pragn	natic Chans	
1	BellKor's Pragmatic Chaos	0.8567	10.06	2009-07-26 18:18:28
2	The Ensemble	0.8567	10.06	2009-07-26 18:38:22
3	Grand Prize Team	0.8002	J.9	001012:4:4-
4	Opera Solutions and Vandelay United	0.8588	9.84	2009-07-10 01:12:31
5	Vandelay Industries !	0.8591	9.81	2009-07-10 00:32:20
6	PragmaticTheory	0.8594	9.77	2009-06-24 12:06:56
7	BellKor in BigChaos	0.8601	9.70	2009-05-13 08:14:09
В	<u>Dace</u>	0.8612	9.59	2009-07-24 17:18:43
9	Feeds2	0.8622	9.48	2009-07-12 13:11:51
10	BigChaos	0.8623	9.47	2009-04-07 12:33:59
11	Opera Solutions	0.8623	9.47	2009-07-24 00:34:07
12	BellKor	0.8624	9.46	2009-07-26 17:19:11
Progr	<u>ess Prize 2008</u> - RMSE = 0.8627 - W	inning Team: BellKo	r in BigChaos	
13	xiangliang	0.8642	9.27	2009-07-15 14:53:22
14	Gravity	0.8643	9.26	2009-04-22 18:31:32
15	Ces	0.8651	9.18	2009-06-21 19:24:53
16	Invisible Ideas	0.8653	9.15	2009-07-15 15:53:04
17	Just a guy in a garage	0.8662	9.06	2009-05-24 10:02:54
18	J Dennis Su	0.8666	9.02	2009-03-07 17:16:17
9	Craig Carmichael	0.8666	9.02	2009-07-25 16:00:54
20	acmehill	0.8668	9.00	2009-03-21 16:20:50

# Million \$ Awarded Sept 21st 2009



## Acknowledgments

- Some slides and plots borrowed from Yehuda Koren, Robert Bell and Padhraic Smyth
- Further reading:
  - Y. Koren, Collaborative filtering with temporal dynamics, KDD '09
  - http://www2.research.att.com/~volinsky/netflix/bpc.html
  - http://www.the-ensemble.com/