# EXTENDED STATE OBSERVERS FOR A CLASS OF UNCERTAIN LINEAR SYSTEMS

## 1. ESO design

In this note, we consider a class of finite-dimensional systems described by the differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Hw(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $x(.) \in \mathbb{R}^{n_x}$ is the state vector, $u(.) \in \mathbb{R}^{n_u}$ is the control vector, $w(.) \in \mathbb{R}^{n_w}$ is the vector of external disturbances, and $y(.) \in \mathbb{R}^{n_y}$ is the measured output vector of the system. The real matrices $A$, $B$, $H$, $C$, and $D$ are of compatible dimensions.

**Assumption 1.** The system (1) satisfies the following two conditions:

1. The matrix $H$ is full rank.
2. The pair $(C, A)$ is observable.

In the framework of extended state observation, our goal is to not only estimate the state vector $x$, but the disturbance vector $w$ as well. In that respect, conventional ESOs append the disturbance to the system state vector by an integral action, which transfers the uncertainty about the disturbance to its time derivative [1]. However, this approach does not provide suitable estimation convergence for fast time-varying disturbances and often, requires high observer gains for a satisfactory performance [2, 3]. Besides, according to the internal model principle of the output regulation theory, an integral action guarantees zero-error steady-state convergence only for step-type disturbances [4]. To remedy this issue, we incorporate a more general disturbance model into the ESO design. Following the terminology of the output regulation theory, we assume that the disturbance signal $w(.)$ belongs to the solution space of an *exosystem*, which is defined by a fixed differential equation. In this note, we consider an exosystem of the form

$$\begin{aligned} \dot{v}(t) &= Sv(t) + Fh(t), \\ w(t) &= Ev(t) \end{aligned} \tag{2}$$

where $v(.) \in \mathbb{R}^{n_v}$ is the state vector, $h(.) \in \mathbb{R}^{n_h}$ is an unknown input signal, and the real matrices $S$, $F$, and $E$ are of appropriate dimensions.

**Assumption 2.** The exosystem (2) satisfies the following conditions:

1. The input signal $h(.)$ is bounded.
2. Any solution $v(.)$ of the exosystem is globally bounded.
3. The pair $(E, S)$ is observable.

**Remark 1.** By setting $S = O$, and $F = E = I$, under Assumption 2, the exosystem (2) reduces to the conventional integral action.

The exosystem (2) together with the original system (1), form the following extended state-space system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Hw(t), \\ \dot{v}(t) &= Sv(t) + Fh(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3}$$

To investigate the feasibility of the observer design, the observability of the extended system (3) should be examined.

**Proposition 1.** *Under Assumption 2 and 3, the extended system* (3) *is observable.*

*Proof.* By the Popov-Belevitch-Hauth (PBH) rank test, the following matrix pencil should have full column rank for all $\lambda \in \mathbb{C}$;

$$\begin{pmatrix} \lambda I - A & -HE \\ O & \lambda I - S \\ C & O \end{pmatrix}. \tag{4}$$

By the observability of $(C, A)$, the first block column of (4) has the rank $n_x$ over $\mathbb{C}$. Similarly, since $(E, S)$ is observable and $H$ is full rank, the second block column has the rank $n_v$ over $\mathbb{C}$. Therefore, the matrix (4) has full column rank for all $\lambda \in \mathbb{C}$. $\square$

For the extended system (3), we propose the following ESO;

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + H\hat{w}(t) + L_1\phi_1(y(t) - \hat{y}(t)), \\ \dot{\hat{v}}(t) &= S\hat{v}(t) + L_2\phi_2(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C\hat{x}(t) + Du(t), \\ \hat{w}(t) &= E\hat{v}(t). \end{aligned} \tag{5}$$

where $\hat{x}(.) \in \mathbb{R}^{n_x}$, $\hat{v}(.) \in \mathbb{R}^{n_v}$, $\hat{y}(.) \in \mathbb{R}^{n_y}$, and $\hat{w}(.) \in \mathbb{R}^{n_w}$ are the estimates of $x(.)$, $v(.)$, $y(.)$, and $w(.)$, respectively, $L_1$ and $L_2$ are design gain matrices of appropriate dimension, and the functions $\phi_1, \phi_2 \colon \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ define the ESO error feedback. We consider the class of function that satisfy the following assumption.

**Assumption 3.** The functions $\phi_i(.)$, satisfy the following conditions for $i = 1, 2$:

1. $\phi_i(.)$ is locally Lipschitz continuous. This conditions ensures that the solutions of the ESO differential equations (5) are well-defined in the sense of Caratheodory.
2. The function $\phi_i(\chi)$, $\chi \in \mathbb{R}^n$ is decentralized in the sense that its $j$-th component $\phi_{ij}$ depends only on $\chi_j$, the $j$-th component of $\chi$.
3. For some known numbers $\beta_i > \alpha_i \geq 0$, $\phi_i(.)$ belongs to the sector $[a_i, b_i]$ globally. Mathematically speaking, the following inequality holds for all $\chi \in \mathbb{R}^{n_y}$ [5, chapter 5];

$$\left(\phi_i(\chi) - \alpha_i\chi\right)^\top \left(\phi_i(\chi) - \beta_i\chi\right) \leq 0. \tag{6}$$

To study the convergence of the ESO (5), we introduce the following error variables:

$$
\begin{aligned}
e_1(t) &:= x(t) - \hat{x}(t), \\
e_2(t) &:= v(t) - \hat{v}(t)
\end{aligned}
\tag{7}
$$

Differentiating $e_1(.)$ and $e_2(.)$ with respect to time, we obtain the following estimation error dynamics:

$$
\begin{aligned}
\dot{e}(t) &= A_a e(t) - L_{1a}\phi_1\left(z(t)\right) - L_{2a}\phi_2\left(z(t)\right) + B_a h(t), \\
z(t) &= C_a e(t),
\end{aligned}
\tag{8}
$$

where $e(.) = \mathrm{col}(e_1(.), e_2(.)) \in \mathbb{R}^{n_x+n_v}$ is the estimation error state vector, $z(.) := y(.) - \hat{y}(.)$ and

$$
A_a = \begin{pmatrix} A & HE \\ O & S \end{pmatrix},\ L_{1a} = \begin{pmatrix} L_1 \\ O \end{pmatrix},\ L_{2a} = \begin{pmatrix} O \\ L_2 \end{pmatrix},\ B_a = \begin{pmatrix} O \\ H \end{pmatrix},\ C_a = \begin{pmatrix} C & O \end{pmatrix}.
$$

To examine the stability of the error dynamics (8), we consider a candidate Lyapunov-Lurie function of the form

$$
V(e) = e^\top P e + \sum_{i=1}^{2}\sum_{j=1}^{n_y} \gamma_{ij} \int_0^{z_j} \phi_{ij}(\sigma) - (\beta_i - \alpha_i)\sigma \, d\sigma,
\tag{9}
$$

where $P \succ 0$, and $\gamma_{ij} \geq 0$ [6]. For the estimation error dynamics, we consider a desired performance output $\zeta(.) \in \mathbb{R}^{n_\zeta}$ of the form

$$
\zeta(t) := C_d e(t),
\tag{10}
$$

with the real matrix $C_d$ of suitable dimensions. Now, we investigate the conditions under which, the differential inequality

$$
\dot{V}(e(t)) + \zeta^\top(t)\zeta(t) - \mu^2 h^\top(t)h(t) \leq -\kappa_0 e^\top(t)e(t)
\tag{11}
$$

holds for some positives $\mu$ and $\kappa_0$. We note that in addition to stability, the inequality (11) also entails an $\mathcal{L}_2$ disturbance attenuation from $h(.)$ to $\zeta(.)$.

**Theorem 1.** *Consider the estimation error dynamics* (8) *with the desired output* (10)*, and assume that Assumptions 1-3 hold. For a given $\kappa_0 > 0$, assume there exist $P \succ 0$, diagonal matrices $\Gamma_1, \Gamma_2 \succeq 0$, and numbers $\bar{\mu}, \tau_1, \tau_2 > 0$ that satisfy the following LMI;*

$$
\begin{pmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\
\star & \Pi_{22} & \Pi_{23} & \Pi_{24} \\
\star & \star & \Pi_{33} & \Pi_{34} \\
\star & \star & \star & \Pi_{44}
\end{pmatrix} \preceq 0,
\tag{12}
$$

*where*

$$\Pi_{11} = PA_a + A_a^\top P + \kappa_0 I + C_d^\top C_d - \sum_{i=1}^{2}(\beta_i - \alpha_i)\Gamma_i A_a^\top C_a^\top C_a - \sum_{i=1}^{2}\tau_i\alpha_i\beta_i C_a^\top C_a,$$

$$\Pi_{12} = -PL_{1a} + \frac{1}{2}\Gamma_1 A_a^\top C_a^\top + \frac{1}{2}\sum_{i=1}^{2}(\beta_i - \alpha_i)C_a^\top C_a L_1 + \frac{1}{2}(\beta_1 + \alpha_1)\tau_1 C_a^\top,$$

$$\Pi_{13} = -PL_{2a} + \frac{1}{2}\Gamma_2 A_a^\top C_a^\top + \frac{1}{2}\sum_{i=1}^{2}(\beta_i - \alpha_i)C_a^\top C_a L_2 + \frac{1}{2}(\beta_2 + \alpha_2)\tau_2 C_a^\top,$$

$$\Pi_{14} = PB_a - \frac{1}{2}\sum_{i=1}^{2}(\beta_i - \alpha_i)C_a^\top C_a B_a,$$

$$\Pi_{22} = -\Gamma_1 L_1^\top C_a^\top - \tau_1 I,$$

$$\Pi_{23} = -\frac{1}{2}\Gamma_1 C_a L_{2a} - \frac{1}{2}\Gamma_2 L_{1a}^\top C_a^\top,$$

$$\Pi_{24} = \frac{1}{2}\Gamma_1 C_a B_a,$$

$$\Pi_{33} = -\Gamma_2 L_2^\top C_a^\top - \tau_2 I,$$

$$\Pi_{34} = \frac{1}{2}\Gamma_2 C_a B_a,$$

$$\Pi_{44} = -\bar{\mu} I.$$

*Then,*

1. *Under $h(t) \equiv 0$, the error dynamics* (8) *has a globally exponentially stable equilibrium at the origin.*
2. $\|\zeta(.)\|_{\mathcal{L}_2} < \mu\|h(.)\|_{\mathcal{L}_2}$*, with* $\mu^2 = \bar{\mu}$.

*Proof.* Both results of the theorem hold if the candidate Lurie-Lyapunov function (9) satisfies the differential inequality (11) along the trajectories of the error dynamics (8) [7]. Thereby, we aim to phrase (11) as a quadratic inequality whose feasibility can be verified by solving an LMI convex optimization. However, one should note that the class of the functions $\phi_i(.)$ is confined to those satisfying the sector-bounded property (6). Therefore, according to the S-procedure [8], we consider the feasibility of the following modified inequality;

$$\begin{aligned} &\dot{V}(e(t)) + \kappa_0 e^\top(t)e(t) + \zeta^\top(t)\zeta(t) - \mu^2 h^\top(t)h(t) \\ &- \sum_{i=1}^{2}\tau_i\left(\phi_i(z) - \alpha_i z\right)^\top\left(\phi_i(z) - \beta_i z\right) \leq 0, \end{aligned} \tag{13}$$

for some numbers $\tau_i > 0$, $i = 1, 2$. Substituting the time derivative of $V(e)$ along the the trajectories of the error dynamics (8), and defining $\Gamma_i := \mathrm{diag}(\gamma_{ij})_{j=1}^{n_y}$, $i = 1, 2$, we obtain the LMI (12) from the ineqality (13). □

**Remark 2.** The optimal value of the $\mathcal{L}_2$ performance index $\mu$ can be obtained by minimizing $\bar{\mu}$ subject to the LMI (12).

1.1. **LMI-based ESO synthesis.** Theorem 1 provides an LMI-based analysis method to examine the convergence of the ESO (5) in terms of stability of its error dynamics (8). By using the results of Theorem 1, we derive an LMI-based synthesis method to obtain the ESO gain matrices. We note that by considering $L_1$ and $L_2$ as the design variables, the matrix inequality (12) is no longer linear and besides, the structural constraints of the matrices $L_{1a}$ and $L_{2a}$ should be taken into account. Therefore, to enable an LMI-based synthesis, we make the following simplification:

- The function $V(e)$ is quadratic, that is, $\gamma_{ij} = 0$.
- The ESO (5), uses the same error feedback functions in both $\hat{x}$ and $\hat{v}$ channels, that is, $\phi_1(.) \equiv \phi_2(.) =: \phi(.)$. Accordingly, the error dynamics reduces to

$$\begin{aligned} \dot{e}(t) &= A_a e(t) - L_a \phi\left(z(t)\right) + B_a h(t), \\ z(t) &= C_a e(t), \end{aligned} \tag{14}$$

  where $L_a = \begin{pmatrix} L_1^\top & L_2^\top \end{pmatrix}^\top$. Moreover, the sector bounded property (6) holds for $\phi(.)$ with $\alpha_1 = \alpha_2 =: \alpha$, and $\beta_1 = \beta_2 =: \beta$.

In the ESO synthesis, we pursue the following main specifications as the objectives:

**S1**. Under $h(t) \equiv 0$, the error dynamics (14) has a globally exponentially stable equilibrium at the origin.

**S2**. The $\mathcal{L}_2$ disturbances attenuation $\|\zeta(.)\|_{\mathcal{L}_2} < \mu \|h(.)\|_{\mathcal{L}_2}$ in achieved for a given $\mu > 0$.

**S3**. To prevent measurement noise amplification, $\|L_a\|$ is equal or less than a given upper bound $c > 0$.

**Theorem 2.** *Consider the estimation error dynamics* (14) *with the desired output* (10) *and assume that Assumptions 1-3 hold with $\phi_1(.) \equiv \phi_2(.) =: \phi(.)$, $\alpha_1 = \alpha_2 =: \alpha$, and $\beta_1 = \beta_2 =: \beta$. For given numbers $c_1, c_2, \kappa_0, \mu > 0$, assume there exist $P \succ 0$, a matrix $Y$, and numbers $\bar{\mu}, \tau > 0$ that satisfy the following LMIs;*

$$\begin{pmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} \\ \star & \bar{\Pi}_{22} & \bar{\Pi}_{23} \\ \star & \star & \bar{\Pi}_{33} \end{pmatrix} \preceq 0, \tag{15a}$$

$$P - c_1^{-1} I \succeq 0, \tag{15b}$$

$$\begin{pmatrix} -c_2^2 I & Y^\top \\ -Y & -I \end{pmatrix} \preceq 0 \tag{15c}$$

*where*

$$\begin{aligned} \bar{\Pi}_{11} &= PA_a + A_a^\top P + \kappa_0 I + C_d^\top C_d - \tau\alpha\beta C_a^\top C_a, \\ \bar{\Pi}_{12} &= -Y + \frac{1}{2}(\beta + \alpha)\tau C_a^\top, \\ \bar{\Pi}_{13} &= PB_a, \\ \bar{\Pi}_{22} &= -\tau_1 I, \\ \bar{\Pi}_{23} &= O, \\ \bar{\Pi}_{33} &= -\mu^2 I. \end{aligned}$$

*Then, the error dynamics* (14) *satisfies the specifications* ***S1-S3*** *with $L_a = P^{-1}Y$, and $c = c_1 c_2$.*

*Proof.* The specification **S1** and **S2** corresponds to the inequality (11), which resulted in the LMI (12). Accordingly, the LMI (15a) is obtained from (12) by applying the simplification $\Gamma_1 = \Gamma_2 = O$, $\phi_1(.) = \phi_2(.) = \phi(.)$ and defining $Y := PL_a$.

Since $\|L\| \leq \|P^{-1}\|.\|Y\|$, the specification **S3** holds with $c = c_1 c_2$ if $\|P^{-1}\| \leq c_1$, and $\|Y\| \leq c_2$. Note that $\|P^{-1}\| \leq c_1$ is equivalent to $\|P\| \geq c_1$, which gives the LMI (15b). Besides, $\|Y\| \leq c_2$ is equivalent to $Y^\top Y - c_2^2 I \preceq 0$, which by the Schure complement, corresponds to the LMI (15c). □

**Remark 3.** By considering $\bar{\mu} = \mu^2$ as a design variable, one can obtain the optimal value of the $\mathcal{L}_2$ disturbance attenuation index by minimizing $\bar{\mu}$ subject to the LMIs (15a)-(15c).

## References


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