

Harvey Mudd College Math Tutorial:

Solving Systems of Linear Equations; Row Reduction

Systems of linear equations arise in all sorts of applications in many different fields of study. The method reviewed here can be implemented to solve a linear system

$$\begin{array}{cccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

of any size. We write this system in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

That is,
 $A\mathbf{x} = \mathbf{b}.$

We can capture all the information contained in the system in the single **augmented matrix**

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

We will solve the original system of linear equations by performing a sequence of the following elementary row operations on the augmented matrix:

Elementary Row Operations

- I. Interchange two rows.
- II. Multiply one row by a nonzero number.
- III. Add a multiple of one row to a different row.

Do you see how we are manipulating the system of linear equations by applying each of these operations?

When a sequence of elementary row operations is performed on an augmented matrix, the linear system that corresponds to the resulting augmented matrix is equivalent to the original system. That is, the resulting system has the same solution set as the original system. Our

strategy in solving linear systems, therefore, is to take an augmented matrix for a system and carry it by means of elementary row operations to an equivalent augmented matrix from which the solutions of the system are easily obtained. In particular, we bring the augmented matrix to Row-Echelon Form:

Row-Echelon Form

A matrix is said to be in **row-echelon** form if

1. All rows consisting entirely of zeros are at the bottom.
2. In each row, the first non-zero entry from the left is a 1, called the **leading 1**.
3. The leading 1 in each row is to the right of all leading 1's in the rows above it.

If, in addition, each leading 1 is the only non-zero entry in its column, then the matrix is in **reduced row-echelon form**.

It can be proven that every matrix can be brought to row-echelon form (and even to reduced row-echelon form) by the use of elementary row operations. At that point, the solutions of the system are easily obtained.

In the following example, suppose that each of the matrices was the result of carrying an augmented matrix to reduced row-echelon form by means of a sequence of row operations.

Example

The augmented matrix

$$A_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array} \right]$$

in reduced row-echelon form, corresponds to the system

$$\begin{array}{rcl} x_1 & = & 2 \\ x_2 & = & 3 \\ x_3 & = & -4 \end{array}$$

which is already fully solved!

The augmented matrix

$$A_2 = \left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

also in reduced row-echelon form, corresponds to the system

$$\begin{array}{rcl} x_1 & -3x_3 & = -5 \\ x_2 & +2x_3 & = 4 \\ & 0 & = 0 \end{array}$$

Letting $x_3 = t$, we find that $x_2 = -2t + 4$ and $x_1 = 3t - 5$. Thus, the system has infinitely many solutions, parametrized for all t as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t - 5 \\ -2t + 4 \\ t \end{bmatrix}$$

Finally, the augmented matrix

$$A_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

again in reduced row-echelon form, corresponds to the system

$$\begin{array}{rcl} x_1 & & = 3 \\ x_2 & & = 2 \\ & 0 & = 1 \end{array}$$

which clearly has no solution. The system is inconsistent.

Notes

- If a matrix is carried to row-echelon form by means of elementary row operations, the number of leading 1's in the resulting matrix is called the **rank** r of the original matrix.
- Suppose that a system of linear equations in n variables has a solution. Then the set of solutions has $n - r$ parameters, where r is the rank of the augmented matrix.
- Suppose that A is an $n \times n$ **invertible** matrix. Then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$. That is, the reduced row-echelon augmented matrix will be of the form

$$\left[I \mid A^{-1}\mathbf{b} \right].$$

Gaussian Elimination

1. If the matrix is already in row-echelon form, then stop.
2. Otherwise, find the first column from the left with a non-zero entry a and move the row containing that entry to the top of the rows being worked on.
3. Multiply that row by $1/a$ to create a leading 1.
4. Subtract multiples of that row from the rows below it to make each entry below the leading 1 zero. We are now done working on that row.
5. Repeat steps 1–4 on the rows still being worked on.

Notes

- In practice, you have some flexibility in the application of the algorithm. For instance, in Step 2 you often have a choice of rows to move to the top.
- A more computationally-intensive algorithm that takes a matrix to **reduced** row-echelon form is given by the Gauss-Jordan Reduction.

Example

We will use Gaussian Elimination to solve the linear system

$$\begin{array}{rrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ 2x_1 & - & x_2 & + & x_3 & = & 8 \\ 3x_1 & & & - & x_3 & = & 3 \end{array} .$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array} \right]$$

The Gaussian Elimination algorithm proceeds as follows:

$$\begin{array}{llll}
\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array} \right] & \begin{array}{l} \text{(Row 1)} \\ \text{(Row 2)} \\ \text{(Row 3)} \end{array} & \longrightarrow & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array} \right] \begin{array}{l} \text{(Row 1)} \\ \text{(Row 2}-2\cdot\text{Row 1)} \\ \text{(Row 3}-3\cdot\text{Row 1)} \end{array} \\
& & \longrightarrow & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array} \right] \begin{array}{l} \text{(Row 1)} \\ (-1/5\cdot\text{Row 2}) \\ \text{(Row 3)} \end{array} \\
& & \longrightarrow & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array} \right] \begin{array}{l} \text{(Row 1)} \\ \text{(Row 2)} \\ \text{(Row 3}+6\cdot\text{Row 2)} \end{array} \\
& & \longrightarrow & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] \begin{array}{l} \text{(Row 1)} \\ \text{(Row 2)} \\ (-1/4\cdot\text{Row 3)} \end{array}
\end{array}$$

We have brought the matrix to row-echelon form. The corresponding system

$$\begin{array}{rclcl}
x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\
& & x_2 & + & x_3 & = & 2 \\
& & & & x_3 & = & 3
\end{array}$$

is easily solved from the bottom up:

$$\begin{array}{l}
x_3 = 3 \\
x_2 + 3 = 2 \longrightarrow x_2 = -1 \\
x_1 + 2(-1) + 3(3) = 9 \longrightarrow x_1 = 2.
\end{array}$$

Thus, the solution of the original system is $x_1 = 2$, $x_2 = -1$, $x_3 = 3$.

In the Exploration, use the Row Reduction Calculator to practice solving systems of linear equations by reducing the augmented matrices to row-echelon form.

Exploration

Key Concepts

To solve a system of linear equations, reduce the corresponding augmented matrix to row-echelon form using the **Elementary Row Operations**:

- I. Interchange two rows.
- II. Multiply one row by a nonzero number.
- III. Add a multiple of one row to a different row.

Gaussian Elimination is one algorithm that reduces matrices to row-echelon form.

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