14

Modeling Change with Latent Variables Measured by Continuous Indicators

In this chapter we discuss the common factor model as it is utilized in longitudinal data analysis and describe how changes in the common factors can be examined with growth models. This discussion focuses on common factors that are measured by continuously scored observed variables. We have reserved the discussion of common factors measured by ordinal variables for the next chapter. Modeling change in common factors is a logical extension of our previous discussions in the structural equation modeling framework where the data are arranged in a multivariate (wide) form, but not in the multilevel modeling framework where a single outcome variable is modeled (data are in the long form with a single outcome variable). There is nothing about the multilevel modeling framework that prohibits this type of model from being formulated and estimated, but limits in multilevel modeling software. `NLMIXED` and `nlme` require the outcomes to be organized into a single variable, and so these programs won't be discussed in this chapter (however, see Codd & Cudeck, 2014). `WinBUGS`, on the other hand, can fit these models and is programmed and written as a flexible multilevel modeling program. Thus, we present the mathematical specification based on both the multilevel and structural equation modeling frameworks, but only present programming scripts for the structural equation modeling programs.

A key concept when studying change in latent variables is establishing a common metric for the latent variables across time. (Unlike the observed variables analyzed in previous chapters, latent variables do not have an inherent scale, and longitudinally, it is necessary for that scale to be the same across time.) This is typically done by testing for factorial invariance (Meredith, 1964a, 1964b; Meredith & Horn, 2001; Widaman & Reise, 1997). Factorial invariance, or measurement equivalence, is the idea that the latent variable is measuring the same construct over time and measuring that construct in the same metric over time, such that a one-unit difference in the factor score at one time point means the same thing at all time points. If a single manifest variable is utilized, as was done throughout the book, measurement equivalence is assumed because we treat observed changes in the variable as changes in the construct giving rise to the scores. Only with a measurement model can we examine factorial invariance, and we begin our discussion here.

COMMON FACTOR MODEL

The common factor model (Spearman, 1904) relates a series of observed variables to a series of unmeasured factors thought to underlie the scores on the observed variables. The common factor model can be written as

$$y_{pi} = \tau_p + \lambda_{p1}\eta_{1i} + \lambda_{p2}\eta_{2i} + \ldots + \lambda_{pQ}\eta_{Qi} + u_{pi} \tag{14.1}$$

where y_{pi} is the p^{th} variable measured for individual i, τ_p is the intercept for variable p, λ_{p1} is the factor loading for variable p on the first common factor, η_{1i} is the factor score for individual i on the first latent variable (factor), and so on for Q common factors, and u_{pi} is the unique factor score for p^{th} variable for individual i. This presentation of the common factor model mirrors the specification of a multilevel model where the observed data are thought of as distinct variables. In the structural equation modeling framework, factor models are more often thought about in terms of vectors and matrices. The same common factor model can be written as

$$\mathbf{y}_i = \tau + \mathbf{\Lambda}\eta_i + \mathbf{u}_i \tag{14.2}$$

where $\mathbf{y}_i$ is a $P \times 1$ vector of observed scores for individual i, τ is a $P \times 1$ vector of observed variable intercepts, $\mathbf{\Lambda}$ is a $P \times Q$ matrix of factor loadings, where Q is the number of common factors, η_i is a $Q \times 1$ vector of common factors for individual i, and $\mathbf{u}_i$ is a $P \times 1$ vector of unique factors. The models presented equivalently in Equations 14.1 and 14.2 lead to the same expected mean and covariance structures for the data, which are

$$\mu = \tau + \mathbf{\Lambda}\alpha \tag{14.3}$$

$$\mathbf{\Sigma} = \mathbf{\Lambda\Psi\Lambda}' + \mathbf{\Theta} \tag{14.4}$$

where μ is a $P \times 1$ vector of expected means, $\mathbf{\Sigma}$ is a $P \times P$ matrix of expected variances and covariances, α is a $Q \times 1$ vector of factor means, $\mathbf{\Psi}$ is a $Q \times Q$ covariance matrix for the common factors η_i, and $\mathbf{\Theta}$ is a $P \times P$ covariance matrix for the unique factors.

Regardless of how the model is written, the parameters of a factor model have the same meaning. Factor loadings ($\mathbf{\Lambda}$ or λ) denote regressions of observed measurements on common factors, indicating the strength of the association between observed and latent variables. Like any other regression, a factor loading can be interpreted as the expected change in an observed variable per unit change in the factor. Intercepts (τ) are the means

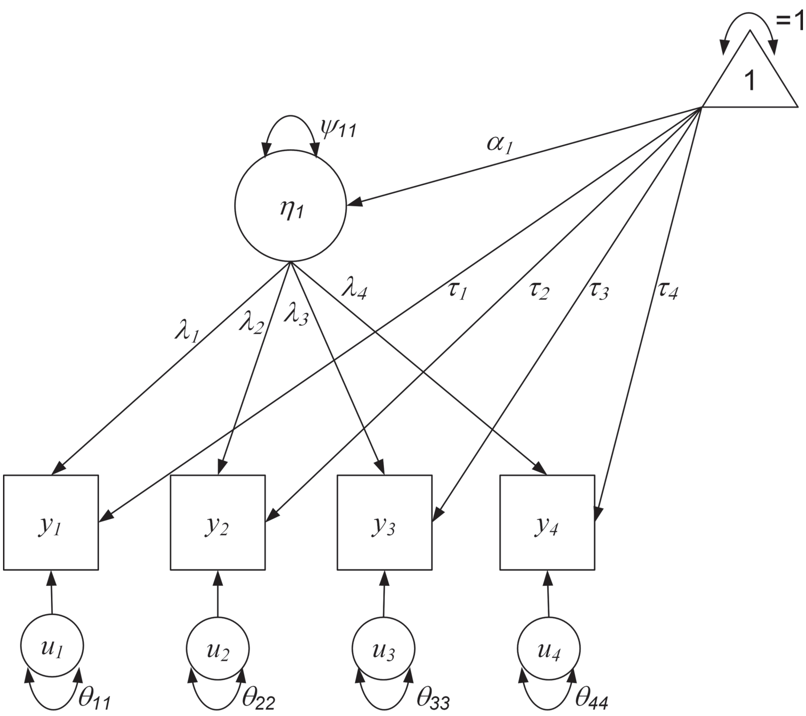

FIGURE 14.1. Path diagram of a common factor model.

of observed variables conditional on the common factors, just as intercepts in a regression reflect the mean of the dependent variable conditional on the independent variables. Finally, unique factors represent the part of the observed variables that is not accounted for by the factors, taking on the same function as residual terms in regression. Figure 14.1 is a path diagram of a common factor model with four observed variables (y_1 through y_4), one common factor (η_1) and four unique factors (u_1 through u_4). Factor loadings (λ_1 through λ_4) originate at the common factor and go to each of the observed variables. The common factor has a mean (α_1) and variance (ψ_{11}), the unique factors have variances (θ_{11} through θ_{44}), and the observed variables have intercepts (τ_1 through τ_4).

Identification

Latent variables (common factors) do not have an inherent scale, which creates both flexibility and causes problems for researchers. The same factor model will fit equally well if the latent variable has a variance of one or 100, provided all other model parameters are free to change along with the latent variable variance, meaning that there are an infinite number of solutions to any factor analytic model (i.e., the value of λ_1 in Figure 14.1 will vary depending on ψ_{11}). Researchers must place constraints on model parameters in such a way that only one of those solutions is possible—a process known as *model identification.* The most common method of identification is to constrain the factor means and variances directly, most often to zero and one, respectively. This method is straightforward, as it requires no manipulation of factor loadings, observed variable intercepts, or residual variances, and a one-unit change in the latent variable is a one standard deviation change, which aids interpretation. Alternative sets of identification criteria involve constraints on a reference indicator, typically constraining the intercept and factor loading for one observed variable to values of zero and one, respectively, and allowing the factor mean and variance to be freely estimated.

It is important to note that identification criteria may either be implicit in or interact with other constraints (e.g., equality constraints) placed on the model. Factorial invariance constraints (discussed in the next section of this chapter) typically constrain the scale of a latent variable to be the same for all time points, so only one time point will require identification constraints. Measuring change in latent variables is not as simple as using the same identification criteria at each time point—constraining the factor mean and variance to be equal at every measurement occasion is not an appropriate way to create an invariant scale. Creating an invariant common factor requires the structure of the common factor model to remain the same at each time point, which involves constraining some model parameters to be equal across time points. Specific tests and constraints are detailed next.

FACTORIAL INVARIANCE OVER TIME

To begin our discussion of factorial invariance over time, we must discuss the longitudinal common factor model. Extending the common factor model of Equation 14.2 to repeatedly measured variables, we can write

$$\mathbf{y}_{ti} = \tau_t + \mathbf{\Lambda}_t \eta_{ti} + \mathbf{u}_{ti} \tag{14.5}$$

where $\mathbf{y}_{ti}$ is a $P \times 1$ vector of observed variables at time t for individual i, τ_t is a $P \times 1$ vector of intercepts at time t, $\mathbf{\Lambda}_t$ is a $P \times Q$ matrix of factor loadings at time t, η_{ti} is a $Q \times 1$ vector of common factor scores at time t for individual i, and $\mathbf{u}_{ti}$ is a $P \times 1$ vector of unique factor scores at time t for individual i.

Longitudinal factorial invariance refers to the equivalence of some parameters of the factor model across time. Having an invariant factor structure is an important aspect of using factor analysis as a scientific tool. Factors have no inherent scale or meaning; their interpretation is derived from their relationships with the set of observed variables. If a factor is to be treated as *the same* across different measurement occasions, that factor's association with other variables must remain constant over some set of variables. Because all of the parameters involved in factor analysis have their own inherent meaning, which parameters are held constant across time affects how we interpret change in common factors.

Methods for testing factorial invariance have a rich history in psychology. Early work on factorial invariance related exclusively to exploratory factor analysis and relied on the comparison and rotation of factor models fit separately to multiple groups. Cattell's (1944) parallel proportional profiles approach and Ahmavaara's (1954) work on the effects of group-based selection on simple structure are examples of early work on this topic. More modern treatments of factorial invariance come from Meredith (1964a, 1964b), who showed that the invariance of factor loadings across subgroups could be achieved when neither factors nor their loadings were standardized. Meredith's work persisted through the use of confirmatory factor analysis, where invariance is commonly tested by comparing multiple models with different parameter constraints. Subsequent work by Meredith (1993) established several nested *levels* of factorial invariance, where each level includes additional constraints on the model meant to attribute additional meaning to the factors. In the least constrained models, differences across time can be due either to changes in the factors or differences in the structure or characteristics of the observed variables. In the most constrained models, all of the changes in observed variable distributions over time can be attributed to the common factors, allowing researchers to study change at the factor level. Testing for factorial invariance is a multistep procedure that often involves fitting four models with an increasing number of constraints across time. The four models are the (1) *configural* invariance model, (2) *weak* invariance model, (3) *strong* invariance model, and (4) *strict* invariance model. Each of these models and their interpretations are discussed in turn.

The *configural* invariance model (Thurstone, 1947; Horn, McArdle, & Mason, 1983) constrains the number of factors and pattern of zero and nonzero loadings to be identical across measurement occasions. Factor variances, factor means, observed variable intercepts, and unique factor variances are free to vary across occasions, as are the magnitude and sign of any factor loadings not explicitly constrained to zero. While the factors extracted at each occasion may be interpreted similarly, these factors cannot be assumed to measure identical constructs at each occasion, nor can they be assumed to lie on the same scale. Longitudinal differences in observed means, variances, and covariances aren't necessarily due to changes in the factors and instead could be due to changes in the observed variables independent of the factors.

Configural invariance models are most commonly used as the initial model fit to serve as a comparison for the more constrained models. Longitudinal configural invariance models assume that the same number of factors are present at each measurement occasion, but this assumption may not be true. Factors may merge or split into different arrangements (e.g., the differentiation–dedifferentiation hypothesis in cognitive aging) or show a significant drop in variance such that certain factors aren't detectable at certain measurement occasions. While configural invariance is typically a starting point for subsequent invariance testing, researchers should pay close attention to both model fit and parameter values from this model to check that these assumptions (e.g., number of factors, same location of zero factor loadings) are met.

Weak (metric) factorial invariance is the least constrained of the three metric invariance models described by Meredith (1993). This model requires the full factor loading matrix ($\mathbf{\Lambda}$) to be equal across all occasions but places no other restrictions on the model. As the factor loading matrix defines the covariances between the observed variables, weak factorial invariance creates proportional covariance structures across time. The magnitude of the covariances may increase or decrease with the variance of the common factors, but this increase or decrease affects all observed variable covariances equally. Observed variable intercepts remain free to vary across occasions, so the longitudinal changes in the observed variable means are not explained by the common factor(s). While the common factor(s) are sufficiently invariant to test regressions or covariances between the factors and additional variables, tests of longitudinal change in the factors demand a stronger form of invariance.

Strong invariance provides a higher level of measurement invariance by additionally constraining observed variable intercepts (τ) to be equal across measurement occasions and, at the same time, allows elements of the latent variable mean vector (α; minus any identification constraints) to vary over occasions. In doing so, strong factorial invariance restricts all longitudinal changes in the observed variable means and covariances to depend on the means, variances, and covariances among the common factors. Thus, changes in the observed variables are caused by changes in the common factors, which allows for the analysis of change in the common factor (i.e., second-order growth models described subsequently). As all mean-level change in the observed variables is attributable to the factors, strong invariance is sufficient to assume that the scale of the latent variable does not differ across measurement occasions.

Lastly, the *strict* factorial invariance places the strongest constraints on the longitudinal factor model. In addition to the constraints on factor loadings ($\mathbf{\Lambda}$) and observed variable intercepts (τ), strict invariance models constrain unique variances (diagonal elements of $\mathbf{\Theta}$) to be equal across measurement occasions. In the strict factorial invariance model, all longitudinal changes in the observed means, variances, and covariances are attributed to the changes in the common factors over time.

Examining factorial invariance for a particular dataset and proposed measurement model requires the fitting of all four levels of factorial invariance. These four models are sequentially nested, such that any model may be compared with any other model using a likelihood ratio test. While weak factorial invariance establishes some degree of common measurement across occasions, either strong or strict invariance is required to attribute changes in the observed variables to changes in the common factors. Thus, these levels of factorial invariance are required to make the move to fitting growth models of common factors. A thorough discussion of measurement invariance and the appropriate tests of measurement invariance can be found in Millsap (2011).

Testing Factorial Invariance with Longitudinal Data

Testing factorial invariance with longitudinal data comes with additional modeling choices. While testing factorial invariance across groups with cross-sectional data is done in the multiple-group modeling framework, *longitudinal* factorial invariance is conducted in a single-group model with the data in wide form. The repeatedly measured variables are represented as separate variables, and the across-time factors are specified in the same model—like a multidimensional factor model along the time dimension. Figure 14.2 is a path diagram of a longitudinal factor model with a repeatedly measured factor (η_{11} through η_{T1}) and strict factorial invariance imposed because there is one set of factor loadings (λ_1 through λ_4), intercepts (τ_1 through τ_4), and unique variances (θ_{11} through θ_{44}). The mean and variance of the common factor at time 1 (η_{11}) is fixed at 0 and 1, which identifies the model and scales the latent variables. The latent variables at times 2 through T have estimated variances and means. As seen in Figure 14.2, the common factors are allowed to covary over time. These covariances provide information regarding the level of relative stability of the factors across time (e.g., high correlations indicate that the between-person differences in the factor are relatively stable across time).

Additionally, and not specified in Figure 14.2, we often allow for unique covariances across time. For example, the unique factors at time 1 are allowed to covary with the unique factors from time 2. That is, the unique factors for y_{11} and y_{12}, y_{21} and y_{22}, y_{31} and y_{32}, and y_{41} and y_{42} are allowed to covary. Similar covariances are typically specified for all observed variables that are the same across time (e.g., y_{11}, y_{12}, and y_{1T}). We do not often impose any equality constraints on these unique covariances; however, it may be reasonable to impose such constraints. For example, assuming a constant time lag between measurement occasions, the lag-1 unique covariances can be constrained to be equal, the lag-2 unique covariances can be constrained to be equal, and so on and so forth. These

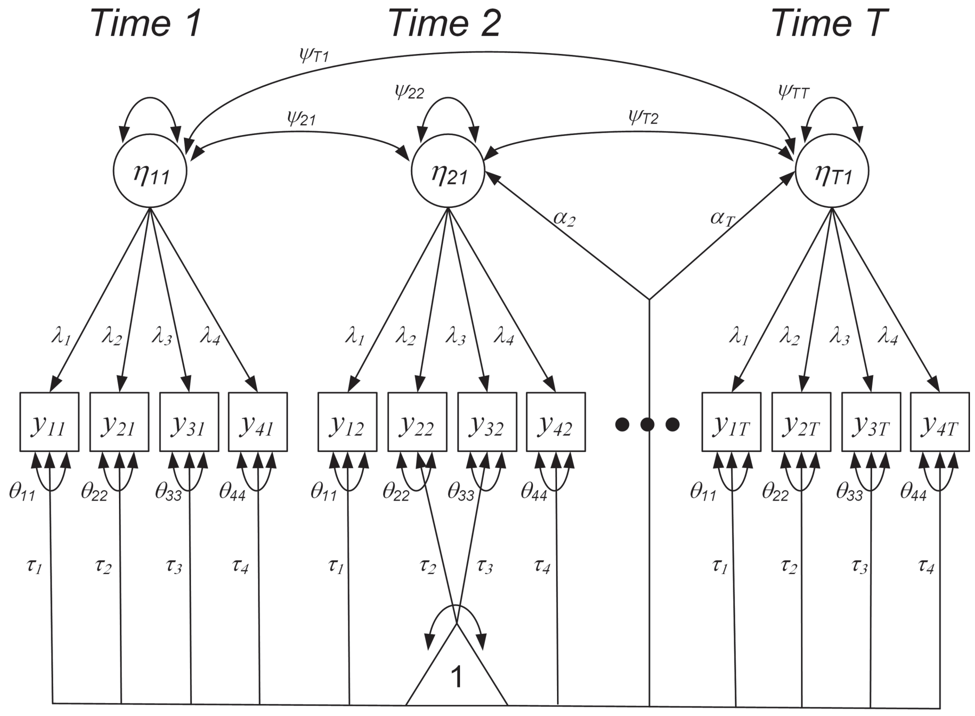

FIGURE 14.2. Path diagram of a longitudinal factor model with strict invariance.

unique covariances are often appropriate because the same observed variable correlates more strongly with itself over time than it does with the other observed variables over time. Failure to account for these autocorrelations can bias other parameters in a factor model and negatively affect model fit.

Strong invariance is needed to examine change at the factor level over time. If strong invariance does not hold across time, it indicates that the meaning of the factor changed over time (if *weak* invariance is not supported) or that the scale of the factor changed over time (if *strong* invariance is not supported). If weak invariance doesn't hold, then the interpretation of at least one (and possibly more) observed variable changed. This can happen when studying change because certain variables may no longer be appropriate. For example, a variable about biting behavior has a different meaning when asked about infants compared to school-age children. That is, biting is fairly common among infants and does not necessarily indicate a problem behavior, whereas biting is rare among school-age children and is indicative of externalizing behavior. Similar assessments can be made for questions related to crying in infants and school-age children (Flora, Curran, Hussong, & Edwards, 2008).

Another example comes from Widaman, Ferrer, and Conger (2010), who detailed a situation in which *numerical facility* had a noninvariant factor loading from second grade through college. Widaman et al. (2010) noted that while performance on the set of items exhibited a structure consistent with a common factor, elementary students relied on counting strategies whereas college students relied on memorization. Thus, the items were not measuring the same underlying skill at different ages. If *strong* invariance doesn't hold, then the factor is unable to account for all of the mean changes. This can happen in longitudinal research when the observed variables show different change patterns across time. For example, crystallized and fluid intelligence are two constructs that are often highly related to one another at any point in time; however, their change patterns are expected to differ, with crystallized intelligence peaking in late adulthood and showing minimal decline and fluid intelligence peaking in the early twenties and showing a strong subsequent decline (see McArdle et al., 2002). Similar discrepancies are apparent when examining changes in academic abilities measured by standardized tests (e.g., Woodcock–Johnson Tests of Achievement) and teacher rating scales (e.g., Academic Rating Scale) because the teacher ratings and standardized tests do not show the same change patterns (i.e., teacher ratings tend to show small changes, whereas standardized tests show large changes).

If strong invariance is not supported, then change models of the factors should not be fit because aspects of the measurement of the construct have changed and it is therefore difficult to distinguish between changes taking place in the individual and changes taking place in the measurement of the construct. At times, it is reasonable to investigate the cause of noninvariance. For example, if there are five observed variables at each measurement occasion and one of these variables has noninvariant parameters, it may be reasonable to simply remove this variable from the measurement model or allow this variable to have noninvariant measurement parameters. In the latter case, changes at the factor level reflect changes taking place in the observed variables that have invariant measurement parameters. If the lack of measurement invariance is due to multiple observed variables, it may not be reasonable to isolate a set of observed variables with invariant measurement parameters.

SECOND-ORDER GROWTH MODEL

Once the factors have been established at each occasion of measurement and *strong* or *strict* invariance is supported, we can begin to model changes at the factor level. The *second-order growth model* (Hancock et al., 2001; McArdle, 1988) is simply the merger of the longitudinal common factor model with the growth model. In this model, the factors for the measurement model (i.e., longitudinal common factor models) are considered *first-order* factors because they are immediately above the measured variables, and the growth factors (i.e., intercept and slope) are considered *second-order* factors because they are two levels above the measured variables. In this model, the second-order growth factors attempt to account for the changes in the mean, variance, and covariances of the first-order factors.

Working from the longitudinal factor model of Equation 14.5, Figure 14.2, and assuming a single common factor at each measurement occasion, the second-order growth model can be written as

$$\eta_i = \Gamma\xi_i + \upsilon_i \tag{14.6}$$

where η_i is a $T \times 1$ vector of common factor variables for individual i, Γ is a $T \times R$ factor loading matrix defining the latent growth factors (e.g., intercept and linear slope in the linear growth model), ξ_i is an $R \times 1$ vector of second-order latent growth factors, and υ_i is a $T \times 1$ vector of latent variable disturbance terms. The second-order factors are assumed to follow a multivariate normal distribution, such that $\xi_i \sim MVN(\kappa, \Phi)$, where κ is an $R \times 1$ vector of latent variable means and Φ is an $R \times R$ covariance matrix. The first-order factor disturbances (υ_i) are assumed to follow a normal distribution, such that $\upsilon_i \sim N(0, \psi)$ where ψ is a $T \times T$ diagonal covariance matrix of disturbances (e.g., latent variable residuals).

Identification, which was discussed for the factor model and the longitudinal factor model, is also an important consideration with second-order growth models. Typically, one factor loading for each first-order factor is fixed to 1, and the mean of the second-order latent variable intercept is fixed at 0. Alternatively, the disturbance variance can be set to a fixed value instead of fixing a factor loading, and an observed variable intercept can be fixed to 0 instead of fixing the second-order latent variable intercept mean to 0. The choice of identifying constraints does not affect model fit but does affect parameter estimates and their interpretation. Ideally, identification constraints are employed that provide the simplest interpretation of model parameters. One approach is to fix one factor loading per latent variable to a value, such that the total variance of the first first-order latent variable is expected to be 1, and fix the mean of the second-order latent variable intercept to 0 (K. Widaman, personal communication, October 12, 2014). With these identification constraints, the first first-order latent variable is essentially in a standardized metric, which makes the change parameters more interpretable. We adopt this approach here.

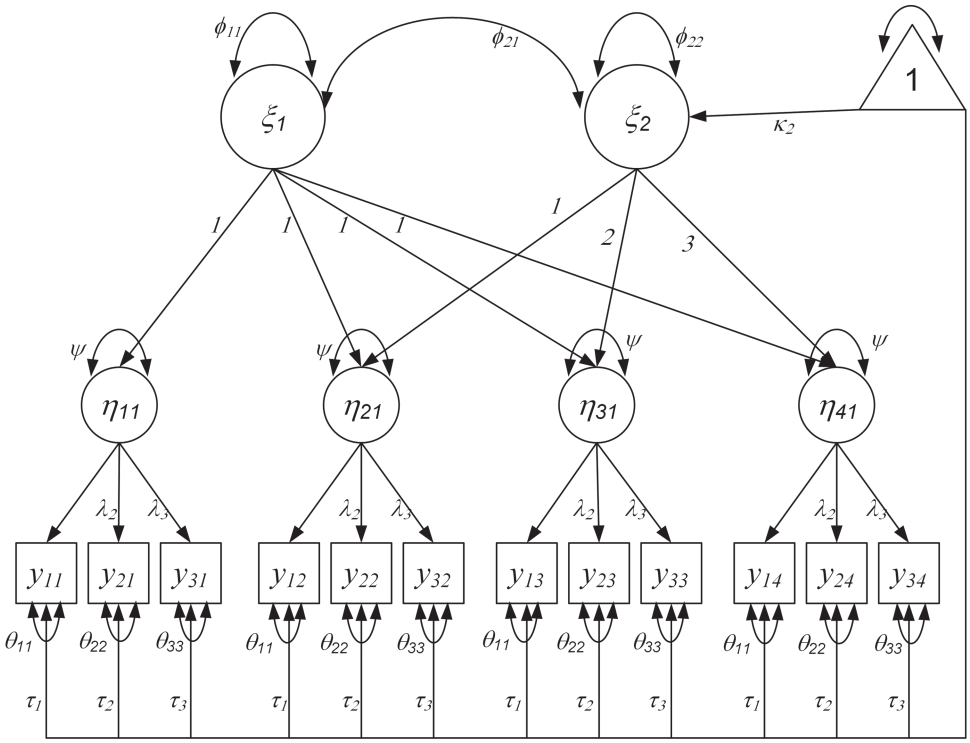

FIGURE 14.3. Path diagram of a second-order growth model.

Figure 14.3 is a path diagram of a second-order growth model following Equations 14.5 and 14.6. In this path diagram, there are three observed variables measured at each occasion (y_{11} through y_{34}), which indicate a single first-order factor at each time point (η_{11} through η_{41}). The changes in the first-order factors are modeled with second-order growth factors ξ_1 and ξ_2, which are the latent variable intercept and slope. The first factor loading for each latent variable is fixed at 1, and the mean of ξ_1 is fixed at 0 for identification; *strict* factorial invariance is imposed.

ILLUSTRATION

To illustrate the use of the longitudinal common factor model and the second-order growth model, we use a random sample of observations from the Early Childhood Longitudinal Study—Kindergarten Cohort (ECLS-K). Reading, mathematics, and science tests were administered in third, fifth, and eighth grades, and these three tests are considered indicators of an academic achievement common factor. The achievement tests in the ECLS-K were adaptive, and the scores used in this analysis are the estimated number of correct responses if the entire test was administered.

TABLE 14.1. Estimated Sample Statistics from the ECLS-K

	1.	2.	3.	4.	5.	6.	7.	8.	9.
Mean	51.01	127.27	99.75	64.81	150.30	123.78	83.35	169.20	140.57
Standard deviation	15.61	29.52	25.54	16.40	27.57	25.40	16.99	28.46	22.92
1. Science grade 3	1.00								
2. Reading grade 3	0.76	1.00							
3. Math grade 3	0.71	0.76	1.00						
4. Science grade 5	0.85	0.74	0.71	1.00					
5. Reading grade 5	0.74	0.86	0.73	0.78	1.00				
6. Math grade 5	0.69	0.72	0.88	0.75	0.76	1.00			
7. Science grade 8	0.76	0.72	0.71	0.82	0.75	0.75	1.00		
8. Reading grade 8	0.69	0.77	0.68	0.74	0.81	0.70	0.79	1.00	
9. Math grade 8	0.67	0.70	0.82	0.71	0.73	0.86	0.79	0.76	1.00

Estimated sample statistics for these scores from the three measurement occasions are presented in Table 14.1. The sample means increase over grade for all three tests, and the sample correlations are all strong—one indication that the reading, science, and math scores may be represented by a common underlying construct. The correlations within time tend to be stronger than the correlations across time, except for the correlations involving the same test over time (e.g., science in grade 3 and science in grade 5, $r = .85$).

STRUCTURAL EQUATION MODELING IMPLEMENTATION

Longitudinal Common Factor Model

`Mplus`

The `Mplus` script for fitting a longitudinal common factor model with *strict* factorial invariance to the ECLS-K data is presented in Script 14.1. We begin by defining the first-order factors (η_i) named `eta_1`, `eta_2`, and `eta_3` for the academic achievement latent variable measured in third, fifth, and eighth grade. These latent variables are defined using the keyword `BY`, and labels are used to constrain the factor loadings to be equal over time. That is, the factor loadings to the science, reading, and mathematics variables are labeled `lambda_S`, `lambda_R`, and `lambda_M`, respectively. We note that an asterisk is placed after the first observed variable when defining each latent variable, which is needed to override the default that the first factor loading is fixed at 1. Constraining the loadings to be equal across time is in line with the specification of the strict invariance model. Next, latent variable variances and covariances are specified by listing the names of the latent variables to specify their variances and then using the `WITH` statement to specify each covariance. The variance of `eta_1` is set to 1 as part of the model identification process. Latent variable means are then specified by listing the names of the latent variables in square brackets. The mean of `eta_1` is fixed at 0 as the

Script 14.1. *Mplus* Script for the Longitudinal Factor Model with Strict Factorial Invariance

```
MODEL:
! Common Factors
    eta_1 BY s_g3 * (lambda_S)
             r_g3  (lambda_R)
             m_g3  (lambda_M);
    eta_2 BY s_g5 * (lambda_S)
             r_g5  (lambda_R)
             m_g5  (lambda_M);
    eta_3 BY s_g8 * (lambda_S)
             r_g8  (lambda_R)
             m_g8  (lambda_M);
! Latent Variable Variances & Covariances
    eta_1@1 eta_2 eta_3;
    eta_1 WITH eta_2 eta_3;
    eta_2 WITH eta_3;
! Latent Variable Means
    [eta_1@0 eta_2 *0 eta_3 *0];
! Unique Variances
    s_g3 s_g5 s_g8 (theta_S);
    r_g3 r_g5 r_g8 (theta_R);
    m_g3 m_g5 m_g8 (theta_M);
! Observed Variable Intercepts
    [s_g3 s_g5 s_g8] (tau_S);
    [r_g3 r_g5 r_g8] (tau_R);
    [m_g3 m_g5 m_g8] (tau_M);
! Unique Factor Covariances
    s_g3 WITH s_g5 s_g8;
    s_g5 WITH s_g8;
    r_g3 WITH r_g5 r_g8;
    r_g5 WITH r_g8;
    m_g3 WITH m_g5 m_g8;
    m_g5 WITH m_g8;
```

second part of the model identification process, and the means of `eta_2` and `eta_3` are freely estimated and given starting values of 0 (`*0`).

Next, unique variances are specified. These variances are specified for the observed variables and represent variability that is unique to the observed variable and not common with the other indicators of the common factor. These unique variances are given labels to constrain them to be equal across time. For example, the unique variances of the science variables are labeled `theta_S`. Again, this constraint is in line with the *strict* factorial invariance model. The observed variable intercepts are specified next. The intercepts are specified by listing the names of the observed variables in square brackets, and these parameters are constrained to be equal over time. For example, the intercepts for the science variables are specified together and labeled `tau_S`. Again, these constraints are in line with the strict invariance model. Lastly, the unique covariances are specified.

These covariances are specified between the same variable measured at different occasions. These covariances are specified using the `WITH` statement. For example, the science variable at third grade is allowed to covary with the science variables measured in fifth and eighth grades (`s_g3 WITH s_g5 s_g8`).

From the strict factorial invariance model specified in Script 14.1, we can remove equality constraints to specify the three other measurement invariance models. Working backwards, the strong invariance model is specified by removing the labels for the unique variances (`theta_S`, `theta_R`, and `theta_M`). The weak invariance model is specified by additionally removing the labels for the observed variable intercepts (`tau_S`, `tau_R`, and `tau_M`) and then fixing the means of `eta_2` and `eta_3` to 0 (for identification). Alternatively, two of the labels for the observed variable intercepts can be removed, and the means of `eta_2` and `eta_3` can remain freely estimated. The configural invariance model is specified by additionally removing the constraints on the factor loadings (`lambda_S`, `lambda_R`, and `lambda_M`) and fixing the variances of `eta_2` and `eta_3` to 1 (for identification). Alternatively, two of the labels for the factor loadings can be removed, and the variances of `eta_2` and `eta_3` can remain freely estimated.

OpenMx

The `OpenMx` script for the longitudinal factor model with *strict* factorial invariance is contained in Script 14.2. The script begins by stating the dataset and listing the manifest and latent variables included in the model. The model is then specified with a series of `mxPath` statements and begins with the unique variances. These are for the observed variables and are specified as two-headed arrows that originate from each observed variable. These paths are freely estimated, given a starting value of 100, and labeled appropriately. The labels for the unique variances of the science, reading, and math variables are constrained to be equal across time using a series of common labels. Thus, the labels for the science variables are `th_s`, the labels for the reading variables are `th_r`, and the labels for the mathematics variables are `th_m`. Next, the unique covariances are specified. In this `mxPath` statement, two-headed paths are specified for every pair of science variables, reading variables, and mathematics variables. These paths are freely estimated, given starting values of 50, and labeled appropriately. The latent variable variances and covariances are then specified. These paths originate from `eta_1`, `eta_2`, and `eta_3`, and we use `connect='unique.pairs'` to specify all the variances and covariances in a single `mxPath` statement. The factor variance for `eta_1` is fixed at 1 (for identification purposes), but the remaining paths contained in this statement are freely estimated. Starting values of 1 are given to the variances of `eta_2` and `eta_3`, and starting values of 0 are given to all covariances.

The factor loadings for the three latent variables are then specified in three `mxPath` statements. All factor loadings originate from a latent variable and go to the appropriate observed variable. All factor loadings are freely estimated, given starting values of 20 and common labels to constrain the factor loadings to be equal across time. That

Script 14.2. OpenMx Script for the Longitudinal Factor Model with Strict Factorial Invariance

```
strict.acad.omx <- mxModel('Strict Model, Path Specification',
  type='RAM', mxData(observed=eclsk, type='raw'),
  manifestVars=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
  latentVars=c('eta_1','eta_2','eta_3'),

# Unique Variances
mxPath(from=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
  arrows=2, free=TRUE, values=100,
  labels=c('th_s','th_r','th_m','th_s','th_r','th_m','th_s','th_r','th_m')),

# Unique Covariances
mxPath(from=c('s_g3','s_g3','s_g5','r_g3','r_g3','r_g5','m_g3','m_g3','m_g5'),
  to=c('s_g5','s_g8','s_g8','r_g5','r_g8','r_g8','m_g5','m_g8','m_g8'),
  arrows=2, free=TRUE, values=50,
  labels=c('th_s53','th_s83','th_s85','th_r53','th_r83','th_r85','th_m53','th_m83','th_m85')),

# Latent Variable Covariances
mxPath(from=c('eta_1','eta_2','eta_3'), connect='unique.pairs',
  arrows=2, free=c(FALSE,TRUE,TRUE,TRUE,TRUE, TRUE), values=c(1,0,0,1 ,0,1),
  labels=c(NA,'psi_21','psi_31','psi_22','psi_32','psi_33')),

# Factor Loadings
mxPath(from='eta_1', to=c('s_g3','r_g3','m_g3'),
  arrows=1, free=TRUE, values=20, labels=c('lambda_s','lambda_r','lambda_m')),

mxPath(from='eta_2', to=c('s_g5','r_g5','m_g5'),
  arrows=1, free =TRUE, values=20, labels=c('lambda_s','lambda_r','lambda_m')),

mxPath(from ='eta_3', to=c('s_g8','r_g8','m_g8'),
  arrows=1, free=TRUE, values=20, labels=c('lambda_s','lambda_r','lambda_m')),

# Latent Variable Means
mxPath(from='one', to=c('eta_2','eta_3'),
  arrows=1, free=TRUE, values=0, labels=c('alpha_2','alpha_3')),

# Observed Variable Intercepts
mxPath(from='one', to=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
  arrows=1, free=TRUE, values=70,
  labels=c('tau_s','tau_r','tau_m','tau_s','tau_r','tau_m','tau_s','tau_r','tau_m'))

) # Close Model
```

is, the factor loadings for the science variables are all labeled `lambda_s`, the factor loadings for the reading variables are labeled `lambda_r`, and the factor loadings for the mathematics variables are labeled `lambda_m`. Next, the latent variable means are specified for `eta_2` and `eta_3`—one-headed arrows originating at the constant and going to the two latent variables. These paths are given starting values of 0 and labeled `alpha_2` and `alpha_3`. Lastly, we specify the intercepts of the observed variables as one-headed arrows originating from the constant and going to each observed variable. These paths are given starting values of 70, and common labels are given to the science (`tau_s`), reading (`tau_r`), and mathematics (`tau_m`) variables to place an equality constraint on these parameters.

The `OpenMx` code in Script 14.2 is easily altered to specify the configural, weak, and strong invariance models. For the strong invariance model, different labels are given to the unique variances in the first `mxPath` statement. The weak invariance model can be specified by additionally providing different labels to the observed variable intercepts and removing the `mxPath` statement for the latent variable means. Alternatively, distinct labels can be provided for the intercepts of the reading and mathematics (or reading and science or mathematics and science) variables. The configural invariance model can be specified by providing different labels either to the factor loadings to the reading and mathematics (or reading and science or mathematics and science) variables, or to all factor loadings and constraining the latent variable variances to one.

Output

Before examining and interpreting parameter estimates from the longitudinal factor model with strict invariance, we report on the fit of the four invariance models. Fit statistics for the four models are contained in Table 14.2. Beginning with the configural invariance model, we see that the model fits the data well with an RMSEA of 0.030, a CFI of 0.998, and a TLI of 0.996. The fit of this model supports the notion of having a single common factor representing academic ability in third, fifth, and eighth grades. Moving to the weak invariance model, we see a significant increase in the χ^2 statistic ($\Delta\chi^2(4)$ = 81.31, $p < .001$), a noticeable jump in the RMSEA, and smaller decreases in the CFI and TLI. Thus, there is a question of whether we are measuring the same construct in all

TABLE 14.2. Fit Statistics for the Measurement Invariance Model

Fit statistic	Configural invariance	Weak invariance	Strong invariance	Strict invariance
$\chi^2(df)$	35.52 (15)	116.83 (19)	540.75 (23)	600.61 (29)
AIC	83,914	83,987	84,403	84,451
BIC	84,120	84,173	84,567	84,584
RMSEA	0.030	0.059	0.123	0.115
CFI	0.998	0.992	0.955	0.951
TLI	0.996	0.984	0.930	0.939

grades; however, the overall fit of the model remains viable with a CFI of 0.992 and a TLI of 0.984. We therefore move onto the strong invariance model. Model fit was further degraded by the constraints imposed on the mean structure ($\Delta\chi^2(4) = 423.92$, $p < .001$) suggesting that the common factors may be unable to adequately capture the changes taking place in the observed variables. At the same time, the overall model fit is justifiable with a CFI of 0.955 and a TLI of 0.930, suggesting that the model adequately captures the observed data. We therefore move to the strict invariance model and find that the strict invariance model fit significantly worse than the strong invariance model ($\Delta\chi^2(6) = 59.86$, $p < .001$). However, again the strict invariance model shows adequate model fit with a CFI of 0.951 and a TLI of 0.939. Thus, we feel it is appropriate to interpret the parameters of the strict invariance model. We note that the power of likelihood ratio tests, as all statistical tests, is affected by sample size. Thus, having a large sample gives us a lot of power to detect noninvariance—even small effects. Researchers have advocated the examination of changes in global fit indices because of the power of likelihood ratio tests; at the same time, researchers should also consider absolute model fit as we did here.

Parameter estimates from `Mplus` and `OpenMx` are contained in Output 14.1 and 14.2, respectively. The factor loadings (`BY` statements in `Mplus`; `lambda_s`, `lambda_r`, and `lambda_m` in `OpenMx`) were significantly different from zero; however, in the current metric it is difficult to determine their strength. Thus, we look at the standardized output for this information. In `Mplus`, standardized output can be obtained by requesting `STANDARDIZED` in the `OUTPUT:` command, and `mxStandardizeRAMpaths()` can be used in `OpenMx` to obtain the standardized parameter estimates. We note that the standardized factor loadings can vary with time even when equality constraints are imposed on the raw (unstandardized) factor loadings. Examining the standardized factor loadings, we find that they were strong (> 0.84), indicating that the common factors were strongly indicated by their respective observed variables.

Focusing on the estimates associated with the common factors, we find that the mean of the latent variable (`Means` of `eta_2` and `eta_3`; `alpha_2` and `alpha_3`) was 1.02 in fifth grade and 1.95 in eighth grade. To give background to these estimates, we can use the distribution of the latent variable at third grade, which had a mean of 0 and a standard deviation of 1. Thus, from third to fifth grade the mean changed by a little more than a standard deviation of the third-grade distribution. Similarly, from fifth to eighth grade, the mean changes by a little less than a standard deviation based on the third-grade distribution. The variance (`Variances` of `eta_2` and `eta_3`; `psi_22` and `psi_33`) of the common factor in fifth and eighth grade was 1.01 and 0.97, respectively, indicating that the magnitude of between-person differences in academic achievement did not change much from third through eighth grade. Next, we examine the correlations among the common factors across time and find that the common factors were strongly correlated over time (`WITH` statements involving `eta_1`, `eta_2`, and `eta_3`; `psi_21`, `psi_31`, and `psi_32`). The latent variable covariances were reported in the outputs, but again, we can review the standardized output and see that the common factor correlations were all greater than 0.91, indicating that the between-person differences in the academic factor were very stable from third through eighth grades.

Output 14.1. *Mplus* Output for the Longitudinal Factor Model with Strict Factorial Invariance

```
MODEL RESULTS
                                                    Two-Tailed
              Estimate       S.E.   Est./S.E.         P-Value
ETA_1    BY
   S_G3         15.176      0.319      47.527           0.000
   R_G3         22.982      0.520      44.171           0.000
   M_G3         21.236      0.473      44.917           0.000
ETA_2    BY
   S_G5         15.176      0.319      47.527           0.000
   R_G5         22.982      0.520      44.171           0.000
   M_G5         21.236      0.473      44.917           0.000
ETA_3    BY
   S_G8         15.176      0.319      47.527           0.000
   R_G8         22.982      0.520      44.171           0.000
   M_G8         21.236      0.473      44.917           0.000
ETA_1    WITH
   ETA_2         0.968      0.013      72.280           0.000
   ETA_3         0.904      0.019      48.240           0.000
ETA_2    WITH
   ETA_3         0.949      0.027      35.095           0.000
S_G3     WITH
   S_G5         28.853      3.118       9.254           0.000
   S_G8          6.616      3.301       2.004           0.045
S_G5     WITH
   S_G8          5.353      3.311       1.617           0.106
R_G3     WITH
   R_G5        112.069      8.673      12.921           0.000
   R_G8         69.528      9.345       7.440           0.000
R_G5     WITH
   R_G8         79.187      9.640       8.214           0.000
M_G3     WITH
   M_G5        112.591      7.308      15.407           0.000
   M_G8         93.469      7.990      11.698           0.000
M_G5     WITH
   M_G8        102.540      8.014      12.795           0.000
Means
   ETA_1         0.000      0.000     999.000         999.000
   ETA_2         1.020      0.025      41.016           0.000
   ETA_3         1.947      0.045      43.248           0.000
Intercepts
   S_G3         51.433      0.448     114.744           0.000
   R_G3        126.363      0.702     179.898           0.000
   M_G3        100.193      0.651     153.912           0.000
   S_G5         51.433      0.448     114.744           0.000
   R_G5        126.363      0.702     179.898           0.000
   M_G5        100.193      0.651     153.912           0.000
   S_G8         51.433      0.448     114.744           0.000
   R_G8        126.363      0.702     179.898           0.000
   M_G8        100.193      0.651     153.912           0.000
```

(*continued*)

Output 14.1. (*Continued*)

Variances				
ETA_1	1.000	0.000	999.000	999.000
ETA_2	1.011	0.026	38.332	0.000
ETA_3	0.971	0.036	27.168	0.000
Residual Variances				
S_G3	60.274	2.939	20.506	0.000
R_G3	208.700	8.172	25.537	0.000
M_G3	173.322	7.236	23.952	0.000
S_G5	60.274	2.939	20.506	0.000
R_G5	208.700	8.172	25.537	0.000
M_G5	173.322	7.236	23.952	0.000
S_G8	60.274	2.939	20.506	0.000
R_G8	208.700	8.172	25.537	0.000
M_G8	173.322	7.236	23.952	0.000

Output 14.2. `OpenMx` Output for the Longitudinal Factor Model with Strict Factorial Invariance

free parameters:

	name	matrix	row	col	Estimate	Std.Error
1	lambda_s	A	s_g3	eta_1	15.1749088	0.31829496
2	lambda_r	A	r_g3	eta_1	22.9806572	0.51768575
3	lambda_m	A	m_g3	eta_1	21.2362864	0.47104498
4	th_s	S	s_g3	s_g3	60.2698421	2.75251768
5	th_r	S	r_g3	r_g3	208.7606975	7.91909668
6	th_m	S	m_g3	m_g3	173.2977095	7.00338013
7	th_s53	S	s_g3	s_g5	28.8421738	2.94524110
8	th_r53	S	r_g3	r_g5	112.1324486	8.44106632
9	th_m53	S	m_g3	m_g5	112.5824844	7.07780122
10	th_s83	S	s_g3	s_g8	6.5929310	2.98503902
11	th_s85	S	s_g5	s_g8	5.3481884	2.63275428
12	th_r83	S	r_g3	r_g8	69.4982349	9.03710638
13	th_r85	S	r_g5	r_g8	79.1903571	9.34634205
14	th_m83	S	m_g3	m_g8	93.4654618	7.74374739
15	th_m85	S	m_g5	m_g8	102.4878349	7.76500982
16	psi_21	S	eta_1	eta_2	0.9679592	0.01338906
17	psi_22	S	eta_2	eta_2	1.0105512	0.02635231
18	psi_31	S	eta_1	eta_3	0.9038030	0.01875611
19	psi_32	S	eta_2	eta_3	0.9492382	0.02705506
20	psi_33	S	eta_3	eta_3	0.9707644	0.03580693
21	tau_s	M	1	s_g3	51.4331658	0.44774023
22	tau_r	M	1	r_g3	126.3627211	0.70221039
23	tau_m	M	1	m_g3	100.1927482	0.65088353
24	alpha_2	M	1	eta_2	1.0202422	0.02477327
25	alpha_3	M	1	eta_3	1.9470769	0.04487849

Next, we briefly review the estimates associated with the observed variables. The observed variable intercepts (`Intercepts` of `s_g3`, `r_g3`, `m_g3`; `tau_s`, `tau_r`, and `tau_m`) provide some information about the scale of the observed variables. These estimates can be used to help describe the expected means of the observed variables at the first occasion (when the mean of the common factor was 0). Since these variables are in their own unique metric, it is difficult to attribute much meaning to them outside the context of being part of the estimated model. The unique variances (`Residual Variances` of `s_g3`, `r_g3`, `m_g3`; `th_s`, `th_r`, and `th_m`) represent the amount of variability in the observed variable that was not accounted for by the common factor. Individually, these parameters are difficult to interpret, but turning to the standardized output, we can determine how much of the variance in the observed variables was accounted for by the common factors and we see that the common factor accounted for between 71 and 79% of the variability in the observed scores. This information is reflective of the magnitude of the standardized factor loadings. The unique covariances (`WITH` statement involving observed variables; `th_s53`, `th_s83`, and `th_s85` for the science variables) were mostly significantly different from zero, suggesting that the unique aspects of these variables were associated over time. This highlights the importance of accounting for these unique aspects of covariation over time. We now build upon this strict invariance model and examine the changes in the first-order factors with a second-order growth model.

Second-Order Growth Model

M*plus*

The M*plus* script for a second-order latent basis growth model is contained in Script 14.3. The first part of the script follows the specification of the strict factorial invariance model with a few important changes that we highlight here. Beginning with the factor loadings for the first-order factors, we have fixed the factor loadings to the science variables to 15.176, which may seem like an arbitrary value. First, we note that this is an identification constraint. Second, this value was the estimate of the factor loading from the strict invariance model. In the strict invariance model, the variance of the `eta_1` was fixed at 1. Thus, fixing this factor loading to the estimate from the strict invariance model would suggest that the expected variance of `eta_1` will be approximately 1, and this can provide greater interpretation of the remaining estimated parameters from the second-order growth model. The first-order latent variable *disturbance* variances are now constrained to be equal across time and labeled `psi`. We note that they are disturbance variances because they represent variability in the common factors that is not accounted for by the second-order growth model. The first-order covariances are all fixed at 0 because the second-order growth model is expected to account for the covariances among the first-order factors. Similarly, the means of the first-order factors are fixed at 0 because the second-order growth model will attempt to account for the changes in the first-order means across time. The remainder of the common factor part of the script is unchanged.

Script 14.3. `Mplus` Script for the Second-Order Latent Basis Growth Model

```
MODEL:
! Common Factors
    eta_1 BY s_g3@15.176
             r_g3 (lambda_R)
             m_g3 (lambda_M);
    eta_2 BY s_g5@15.176
             r_g5 (lambda_R)
             m_g5 (lambda_M);
    eta_3 BY s_g8@15.176
             r_g8 (lambda_R)
             m_g8 (lambda_M);

! Latent Variable Variances & Covariances
    eta_1 eta_2 eta_3 (psi);
    eta_1 WITH eta_2@0 eta_3@0;
    eta_2 WITH eta_3@0;

! Latent Variable Means
    [eta_1@0 eta_2@0 eta_3@0];

! Unique Variances
    s_g3 s_g5 s_g8 (theta_S);
    r_g3 r_g5 r_g8 (theta_R);
    m_g3 m_g5 m_g8 (theta_M);

! Observed Variable Intercepts
    [s_g3 s_g5 s_g8] (tau_S);
    [r_g3 r_g5 r_g8] (tau_R);
    [m_g3 m_g5 m_g8] (tau_M);

! Unique Factor Covariances
    s_g3 WITH s_g5 s_g8;
    s_g5 WITH s_g8;
    r_g3 WITH r_g5 r_g8;
    r_g5 WITH r_g8;
    m_g3 WITH m_g5 m_g8;
    m_g5 WITH m_g8;

! Latent Basis Growth Model
    xi_1 BY eta_1@1 eta_2@1 eta_3@1;
    xi_2 BY eta_1@0 eta_2*.5 eta_3@1;
    xi_1 xi_2;
    xi_1 WITH xi_2;
    [xi_1@0 xi_2*2];
```

The second-order growth model is specified at the end of Script 14.3. First, we define `xi_1`, and `xi_2`. `xi_1` is the latent variable intercept and has factor loadings that are fixed at 1, and `xi_2` is the shape factor with two fixed and one estimated factor loading. The first and last factor loadings are fixed at 0 and 1, respectively, and the second factor loading will be estimated from the data following the unstructured or latent basis growth model specification. Next, latent variable variances and covariance are specified by listing `xi_1` and `xi_2` and then stating `xi_1 WITH xi_2`. Finally, the means of `xi_1` and `xi_2` are specified in square brackets. The mean of `xi_1` is fixed at 0 for identification, and the mean of `xi_2` is given a starting value of 2.

OpenMx

The `OpenMx` script for the second-order latent basis growth model is contained in Script 14.4. The beginning of the script closely follows the specification of the strict invariance model. Here we note the changes and additions. The first change is in the specification of the latent variable variance and covariance matrix. Instead of specifying a full covariance matrix for the first-order factors and placing an identification constraint on the variance of `eta_1`, we specify only the latent variable variances, give them a starting value of 0.20, and constrain them to be equal using the common label `psi`. Next, the factor loadings are specified. In the strict invariance script, these parameters were freely estimated but constrained to be equal across time. Now, we place an identification constraint here as we fix the factor loading from each latent variable to the respective science variable to 15.1749088, which was the estimated factor loading for this variable in the strict invariance model. As in the strict invariance model, the factor loadings for reading and mathematics are constrained to be equal across time and labeled `lambda_r` and `lambda_m`.

The additions to the script appear after `## Second-Order Growth Specification ##`. First, we define `xi_1` and `xi_2`. The factor loadings for `xi_1`, the second-order intercept, go to the first-order factors and are fixed at 1. The factor loadings for `xi_2`, the second-order shape factor, also go to the first-order factors. The factor loading to `eta_1` is fixed at 0, the factor loading to `eta_2` is freely estimated and labeled `gamma_22`, and the factor loading to `eta_3` is fixed at 1 following our typical specification for an unstructured or latent basis growth model. Next, second-order latent variable variances and covariances are specified for `xi_1` and `xi_2` using `connect='unique.pairs'`. These parameters are given starting values and labels beginning with `phi`. Lastly, we specify the mean of the second-order shape factor. This path originates from the constant and goes to `xi_2`, given a starting value of 2, and labeled `kappa_2`. We note that the mean of `xi_1` is fixed at 0 for identification purposes and was not specified.

Output

Before reporting on the parameter estimates and their interpretation, we discuss model fit. The second-order latent basis growth model showed good fit ($\chi^2(31) = 606.99$, CFI = 0.950, TLI = 0.943, RMSEA = 0.112) and the fit of this model can be compared against the strict invariance model. Comparing these two models, we find that the change in fit was significant ($\chi^2(2) = 6.38$, $p < .05$); however, the change in fit was minimal given the sample size and lack of change in the global and absolute fit indices. This suggests that the growth model was able to adequately capture the mean changes in the first-order factors as well as their variances and covariances.

Script 14.4. OpenMx Script for the Second-Order Latent Basis Growth Model

```
lgm.acad.omx <- mxModel ('Second Order Latent Basis Growth Model, Path Specification',
 type='RAM', mxData (observed=eclsk, type='raw'),
 manifestVars=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
 latentVars=c('eta_1','eta_2','eta_3','xi_1','xi_2'),

# Unique Variances
mxPath(from=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
 arrows=2, free=TRUE, values=100,
 labels=c('th_s','th_r','th_m','th_s','th_r','th_m','th_s','th_r','th_m')),

# Unique Covariances
mxPath(from=c('s_g3','s_g3','s_g5','r_g3','r_g3','r_g5','m_g3','m_g3','m_g5'),
 to=c('s_g5','s_g8','s_g8','r_g5','r_g8','r_g8','m_g5','m_g8','m_g8'),
 arrows=2, free=TRUE, values=50,
 labels=c('th_s53','th_s83','th_s85','th_r53','th_r83','th_r85','th_m53','th_m83','th_m85')),

# Latent Variable Disturbance Variances
mxPath(from=c('eta_1','eta_2','eta_3'),
 arrows=2, free=TRUE, values=.2, labels='psi'),

# Factor Loadings
mxPath(from='eta_1', to=c('s_g3','r_g3','m_g3'),
 arrows=1, free=c(FALSE,TRUE, TRUE), values=15.1749088,
 labels=c(NA,'lambda_r','lambda_m')),

mxPath(from='eta_2', to=c('s_g5','r_g5','m_g5'),
 arrows=1, free=c(FALSE,TRUE, TRUE), values=15.1749088,
 labels=c(NA,'lambda_r','lambda_m')),
```

```
mxPath(from='eta_3', to=c('s_g8','r_g8','m_g8'),
 arrows=1, free=c(FALSE,TRUE, TRUE), values=15.1749088,
 labels=c(NA,'lambda_r','lambda_m')),

# Observed Variable Intercepts
mxPath(from='one', to=c('s_g3','r_g3','m_g3','s_g5','r_g5','m_g5','s_g8','r_g8','m_g8'),
 arrows=1, free=TRUE, values=70,
 labels=c('tau_s','tau_r','tau_m','tau_s','tau_r','tau_m','tau_s','tau_r','tau_m')),

## Second - Order Growth Specification ##
# Factor Loadings
mxPath(from='xi_1', to=c('eta_1','eta_2','eta_3'),
 arrows=1, free=FALSE, values=1),

mxPath(from='xi_2', to=c('eta_1','eta_2','eta_3'),
 arrows=1, free=c(FALSE, TRUE, FALSE), values=c(0,.5,1),
 labels=c(NA,'gamma_22',NA)),

# Latent Variable Variances & Covariances
mxPath(from=c('xi_1','xi_2'), connect='unique.pairs',
 arrows=2, free=TRUE, values=c(.8,0,.5),
 labels=c('phi_11','phi_21','phi_22')),

# Latent Variable Means
mxPath(from='one', to=c('xi_2'),
 arrows=1, free=TRUE, values=2, labels='kappa_2')

) # Close Model
```

Parameter estimates from Mplus and OpenMx are contained in Output 14.3 and 14.4, respectively. For Mplus, we only list the parameter estimates associated with the second-order growth model because there should not be much change in the parameter estimates associated with the first-order factors. Beginning with the second-order growth factors, the mean of the shape factor (Means of xi_2; kappa_2) was 1.95 and represents the expected mean change in the academic achievement factor from third through eighth grade. Since the scale of the first-order factors is arbitrary, it can be difficult to interpret the magnitude of this parameter. The identification constraints imposed in the model have specified the total variance of the first-order factor in third grade to be approximately one. Thus, the mean change from third through eighth grade represents almost a two standard deviation increase when compared to the amount of between-person differences in academic achievement at third grade. The variance of the second-order intercept and shape factors (Variances of xi_1 and xi_2; psi_11 and psi_22) were both significant, indicating that children significantly varied in their level of academic achievement in third grade and their rate of growth. The covariance between the second-order intercept and shape factors was negative (xi_1 WITH xi_2; psi_21) and transformed into a correlation equaled –0.19. Thus, academic achievement in third grade was slightly negatively associated with the rate of growth from third through eighth grade.

To determine the within-person rate of change at different points in time, we need to consider the factor loadings of xi_2 (xi_2 BY eta_2; gamma_2). The factor loadings

Output 14.3. Mplus Output for the Second-Order Latent Basis Growth Model

```
MODEL RESULTS
                                                        Two-Tailed
              Estimate    S.E.    Est./S.E.      P-Value
XI_1     BY
  ETA_1       1.000   0.000      999.000       999.000
  ETA_2       1.000   0.000      999.000       999.000
  ETA_3       1.000   0.000      999.000       999.000
XI_2     BY
  ETA_1       0.000   0.000      999.000       999.000
  ETA_2       0.524   0.006       85.607         0.000
  ETA_3       1.000   0.000      999.000       999.000
XI_1     WITH
  XI_2       -0.061   0.019       -3.114         0.002
Means
  XI_1        0.000   0.000      999.000       999.000
  XI_2        1.947   0.024       82.219         0.000
Variances
  XI_1        0.982   0.042       23.184         0.000
  XI_2        0.110   0.018        6.277         0.000
Residual Variances
  ETA_1       0.024   0.004        6.026         0.000
  ETA_2       0.024   0.004        6.026         0.000
  ETA_3       0.024   0.004        6.026         0.000
```

Output 14.4. OpenMx Output for the Second-Order Latent Basis Growth Model

```
free parameters:
        name    matrix      row     col        Estimate      Std.Error
1    lambda_r        A     r_g3   eta_1     22.98897356    0.296771614
2    lambda_m        A     m_g3   eta_1     21.23548090    0.250448952
3    gamma_22        A    eta_2    xi_2      0.52391537    0.006237537
4        th_s        S     s_g3    s_g3     60.27110458    4.423763569
5        th_r        S     r_g3    r_g3    208.60749540   10.791908950
6        th_m        S     m_g3    m_g3    173.82419834    9.925948880
7      th_s53        S     s_g3    s_g5     28.70353095    4.524295110
8      th_r53        S     r_g3    r_g5    113.25278838   11.188649411
9      th_m53        S     m_g3    m_g5    113.49470209   10.015172838
10     th_s83        S     s_g3    s_g8      6.80230565    6.341068127
11     th_s85        S     s_g5    s_g8      5.73944175    6.344437494
12     th_r83        S     r_g3    r_g8     67.86753635   12.757414077
13     th_r85        S     r_g5    r_g8     78.02129251   12.907379093
14     th_m83        S     m_g3    m_g8     92.86851524   10.894410307
15     th_m85        S     m_g5    m_g8    100.70326575   10.861819735
16        psi        S    eta_1   eta_1      0.02426940    0.004031375
17     phi_11        S     xi_1    xi_1      0.98255711    0.042414332
18     phi_21        S     xi_1    xi_2     -0.06073336    0.019537961
19     phi_22        S     xi_2    xi_2      0.11001589    0.017562896
20      tau_s        M        1    s_g3     51.42363666    0.451225126
21      tau_r        M        1    r_g3    126.33967208    0.708471750
22      tau_m        M        1    m_g3    100.18119852    0.653793246
23    kappa_2        M        1    xi_2      1.94712088    0.023734060
```

begin at 0.00 in third grade, go to 0.52 in fifth grade, and to 1.00 in eighth grade. Thus, 52% of the total predicted changes from third through eighth grade took place between third and fifth grades. Using these factor loadings and the mean of `xi_2`, we find that the mean rate of change from third to fifth grade was 0.51 units $((0.52 \cdot 1.95)/2)$ per year (unit is approximately the third grade standard deviation), whereas the mean rate of change from fifth to eighth grade was 0.31 units $(((1 - 0.52) \cdot 1.95)/3)$ per year. Knowing these mean rates of change, it is easy to visualize how the expected rate of change decreased from third through eighth grade.

The final parameter estimate that we discuss here is the disturbance variance (`Residual Variance` of `eta_1`; `psi`), which was estimated to be 0.02 and represents true variability in academic achievement that was not accounted for by the second-order growth model. To reiterate, our scaling of the first-order latent variables, we can see that adding this disturbance variance to the variance of `xi_1` yields 1.01, which is the total expected variance of `eta_1`.

IMPORTANT CONSIDERATIONS

We cannot stress enough the importance of measurement invariance when studying change over time. Measurement invariance is often assumed because single-indicator models (all models discussed up to this chapter) are typically fit and testing measurement invariance is impossible with such data. Even with multivariate data, this assumption is often untested with longitudinal data because of select and small samples, lack of power, and unestimable models, but when possible, it is necessary to evaluate the degree of measurement invariance present in the observed variables over time. Researchers often make a yes or no decision regarding measurement invariance—either the parameters are invariant or they are not—and we note that this approach may not be appropriate without putting the *degree* of noninvariance in context. That is, it is important for the field to consider effect sizes for measurement noninvariance. One way to think about this is that measurement parameters are never invariant (never exactly identical) and we should consider what degree of noninvariance should prevent further analysis (e.g., change modeling) and what degree of noninvariance we can live with and continue to move forward with our analysis. There has been some research in item response modeling on effect sizes for measurement noninvariance (i.e., differential item functioning), and similar work should be done with continuous indicator factor models.

Beyond tests for measurement invariance, second-order growth models, compared to first-order growth models, have multiple benefits. First, we note that the power to detect variance in change and covariances among changes is greater with second-order models (Hertzog, Lindenberger, Ghisletta, & von Oertzen, 2006; Hertzog, von Oertzen, Ghisletta, & Lindenberger, 2008). That is, it should be easier to detect significant associations involving second-order intercepts and slopes compared with first-order intercepts and slopes. Second, second-order growth models can be fit when there are changes in the measurement of the construct over time. McArdle and Hamagami (2004) fit a second-order growth model to longitudinal intelligence data where the Stanford–Binet was administered when the participants were children, the Stanford–Binet and Wechsler Adult Intelligence Scale were administered when the participants were young adults, and then the Wechsler Adult Intelligence Scale was administered when the participants were adults. Using first-order models would only enable change modeling for specific spans of age (childhood to young adulthood and then young adulthood to adulthood). The second-order growth model enabled the study of changes from childhood through adulthood.

MOVING FORWARD

In the next chapter, we conclude our discussion of modeling change in latent entities with a discussion on using item response models when studying change. This chapter follows the current chapter closely but specifically deals with dichotomous and polytomous observed variables. As in this chapter, we first discuss the importance of establishing measurement invariance with item response models before modeling change.