Participatory Budgeting With Multiple Resources
Final Report

George Blagden - Third Year
Supervised by Markus Brill

April 28, 2025

Contents

Abstract

Participatory budgeting is a type of election used in communities around the world. Community projects are proposed, each with a cost. Voters select the projects they like. A voting mechanism is used to pick a subset of projects that does not exceed the budget. A simple mechanism is a greedy rule. Projects get selected in decreasing order of vote count and get added to the outcome if they can fit. The method of equal shares (MES) is another mechanism that can be used. Each voter receives an equal share of the budget. Projects get selected and are funded by the supporters. Usually, participatory budgeting elections only have one resource – cost. Extending the election to support additional resources creates a richer model and allows for more complex projects to be modelled. This paper takes mechanisms from single resource participatory budgeting and extends them to work when there are multiple resources. This project has shown experimentally that the two extensions of MES create fairer elections, use up more of the budget, and exclude fewer voters than the greedy rule. This paper has also shown that, under appropriate conditions, one of the extensions satisfies a strong proportionality axiom – EJR+. This is a property of a voting rule that discusses how fair the outcome is. To sum up, this paper introduces some mechanisms for participatory budgeting with multiple resources, proves some properties about them, and discusses the results of experiments on them.

Keywords: Participatory budgeting, social choice, voting, elections, algorithms, proportional representation, axioms, innovative democracy.

Acknowledgements

I would like to thank my supervisor Markus Brill for his constant support throughout the project. His expertise in the field has helped me learn about the topic and talk through new ideas during the project. I would also like to express thanks to Anton Baychkov for providing code to test EJR+ violations, insights into the field, and helping to inspire some new directions to take the project in.

1 Introduction

Participatory budgeting (PB) is a type of election that allows the community to engage with projects and new developments in the local area. Typically, councils will propose a selection of different projects. Each project will have a cost, and the election will have an overall budget. The voters are then able to select the projects that they would like to see completed. This is often done with approval voting, where each voter chooses the projects they like.

PB has been in use since its creation in Porto Alegre, Brazil in 1989. It was introduced to help spread funding among neighbourhoods and help poorer communities receive a fair share of the funding. Since the first uses in Brazil, participatory budgeting has grown in popularity with over one hundred municipalities having implemented it [Wampler, 2000].

Its popularity has extended outside of Brazil, with many countries across the globe now using it to engage voters in the democratic process. In Europe in 2019, there were around 4,500 elections. Across the globe there were 11,600 elections. Poland has made PB compulsory for many of its cities [De Vries et al., 2022]. In the UK, PB was introduced to Scotland in 2014. Since then, there have been many elections across the country [https://pbscotland.scot/, 2025]. For the 2020/21 fiscal year, local governments in Scotland allocated £78,302,012 of funding to PB projects. With this, over 40,000 people participated in making these decisions [https://www.cosla.gov.uk/about-cosla/our-teams/communities/participatory-budgeting, 2025].

An evaluation of Scottish PB by O'Hagan et al. [2019] found that it can engage local councillors in communities; bring communities together at PB events; and allow residents to engage in allocation of service budgets. Additionally, PB makes the decision-making process more transparent and holds local governments more accountable [Sgueo, 2016]. However, reports have commented on the lack of diversity of voters and poor accessibility at the events [O'Hagan et al., 2019]. Another potential issue with PB is that it could cause over-promising from the government. This might happen if, for example, the projects selected do not meet environmental restrictions imposed on the council. Due to this, not all of them would be funded. This could reduce trust in the local government, and in the democratic process. Another issue is that conflicting projects could be chosen, one of them will not be built.

Budget: £300,000	1	2	3	4	5	6	7	8	9	10
Park: £160,000	x	x	x	x	x	x	x	x		
Theatre: £90,000	x	x	x	x			x	x		x
Social Club: £60,000				x	x	x			x	x

Table 1: A simple PB election

A simple election might look like the one shown in Table 1. Each voter (1 to 10) approves of the projects marked with an x in their column. There are a number of different sets of projects which are feasible. Feasible sets are collections of projects that do not exceed the budget. $\varnothing$, $\{\text{Park}, \text{Theatre}\}$, $\{\text{Park}, \text{Social Club}\}$, and $\{\text{Theatre}, \text{Social Club}\}$ are all possible sets. How can we design an algorithm to compute a set of winners that is feasible and represents the voters' opinions? Obviously, the empty set is not a good outcome since some projects can be funded. The park should be selected as it has the most votes. Both the theatre and social club could be viable. Selecting $\{\text{Park}, \text{Theatre}\}$ would leave voter 9 out, whereas $\{\text{Park}, \text{Social Club}\}$ would give at least one project to each voter. In this example, choosing $\{\text{Park}, \text{Social Club}\}$ is the best option.

The obvious first choice of rule would be a greedy one. This rule would select projects with the most votes. However, this rule has been shown to not satisfy any fairness properties. A recent development in the field has been the method of equal shares (MES). This rule shares the budget evenly among voters and then allows voters to spend on their supported projects. Brill and Peters [2023] proved that MES satisfies a strong fairness property called EJR+. This property is a proportionality axiom; voting rules that satisfy this create outcomes that are representative of the population. If a large enough group of voters agree on a project, it should be in the outcome. Papasotiropoulos et al. [2024] developed a version of MES called BOS, which solves some issues with MES while approximately satisfying this axiom.

Often, projects will have multiple costs that are separate from monetary. For example: land usage, environmental impact, planning restrictions, and building time. These costs may also be part of the local government's budget. Extending PB to have multiple different resource budgets would allow councils to budget for these too. Another, more abstract use of a multi-resource model, is modelling project constraints and category limits. By adding a resource with a budget of 1 and the conflicting projects having cost of 1, at most one of these will be chosen in the outcome. Having these additional resources could increase trust in the local government as the outcome is significantly more likely to be fully funded. Table 2 shows an example of an election with multiple resources. These are: money, land usage, and a resource to model the conflict between the park and the theatre.

Budget: (£300,000, 900m^2,1)	1	2	3	4	5	6	7	8	9	10
Park: (£160,000, 750m^2,1)	x	x	x	x	x	x	x	x		
Theatre: (£90,000, 80m^2,1)	x	x	x	x			x	x		x
Social Club: (£60,000, 80m^2,0)				x	x	x			x	x

Table 2: A simple PB election with three resources

This project focuses on extending the idea of PB into multiple resources and how to design some algorithms that determine the winners fairly and proportionally. It will introduce ideas from recent PB research by introducing some extensions of MES and BOS. It will also experimentally show some fairness results. Rey et al. [2023] set up a general framework for multi-resource elections and discussed some potential methods based on standard voting rules. However, they did not discuss any proportionality properties. Motamed et al. [2022] discussed a few new rules for multi-resource PB and discussed fairness properties.

Adding extra resources could have several benefits for PB elections. Since councils must provide more information to the community before the vote takes place, a greater amount of planning would have to occur in the initial stages. This would be needed to determine the resources needed to budget for outside of cost. This would lead to better planning and potentially reduce problems later in the development process. Adding additional resources means that the government can guarantee that they can fund all the projects that get selected. It would increase trust in both the local government and PB as voters will have the confidence that the selected projects will all be funded.

Outside of typical PB, the multi-resource case has other applications. Consider a dinner party with n guests who can vote on the dishes they want to see. Each dish will require different ingredients, each with a limit on the quantity available to the chef. This could be turned into a PB instance with the projects being dishes and the resources being the amounts of different ingredients (e.g. chicken, bread, pasta, etc.). It could also be used to furnish a communal space. The resources could be the budget to spend, the limited space allowed, and limits on the number of a single type of furniture (e.g. at most 2 coffee tables).

1.1 Outline of the project

This project begins by introducing the mathematical background and notation from single resource PB. A model is set up, followed by the definition of some properties and the mechanisms that inspired this project. A model for PB with multiple resources is introduced, using the notation defined, based on the models by Rey et al. [2023] and Motamed et al. [2022]. The main contribution is defining extensions of MES and BOS to multiple resources by introducing three new mechanisms: multi-MES, EES, and multi-BOS. For completeness, an extension of the greedy rule will also be discussed.

Multi-MES shares the budget of the project evenly between the voters. The projects get selected based on how fairly the cost can be split among its supporters, who then pay for it. This is a direct extension of MES. EES takes this idea but allows voters to swap their budget around to better suit their needs. For example, a voter could spend some money to gain extra land area. This provides more flexibility when funding projects. Finally, multi-BOS is a direct extension of BOS. Voters each receive an even share of the budget. Projects get selected in the same way as multi-MES, but voters can overspend by a small amount to spread the cost more fairly.

For multi-MES, we will prove runtime and correctness for an algorithm, followed by showing that it does not use a maximal amount of the budget. The latter will be fixed by introducing completion mechanisms. For EES, it will be shown that, under certain circumstances, the outcome will be the same as the outcome when single resource MES is used. For these methods, we will experimentally show their benefits over the greedy rule, and conjecture that they both approximately satisfy a proportionality axiom.

2 Background

As mentioned, mechanisms from the single resource case have inspired many of the algorithms discussed in this project. Hence, a good place to start is by discussing single resource PB. This section introduces the current research and introduces some properties that will be used in our discussion of the new methods.

The first thing to introduce is the mathematical formulation of a PB election. There are many different equivalent formulations of PB. The one provided in this project is based on the definitions by Peters et al. [2021] and Brill et al. [2023]. It balances being easy to understand with the common notation for such elections.

Definition 2.1 (Participatory Budgeting (PB))**.** A PB election is a tuple $(N, C, \text{cost}, b, A)$ where:

- N is the set voters.
- C is the set of projects.
- $\text{cost} : C \to \mathbb{R}_+$ is the cost of each project.
- $b \in \mathbb{R}_+$ is the budget.
- $A = (A_1, \ldots, A_n)$. For each voter $i \in N$, A_i is the set of projects voter i approves of.

The number of voters will be denoted n and the number of projects will be denoted m. From this definition, the set of voters who support a project c can be defined as $N_c = \{i \in N : c \in A_i\}$.

The goal is to find a set of projects whose total cost does not exceed the budget and represents the views of the voters.

Example 2.2. The example in Table 1 would be defined as the election $(N, C, b, \text{cost}, A)$ where:

- $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $C = \{\text{Park}, \text{Theatre}, \text{Social Club}\}$
- $b = 300,000$
- $\text{cost}(\text{Park}) = 160,000$; $\text{cost}(\text{Theatre}) = 90,000$; $\text{cost}(\text{Social Club}) = 60,000$
- $A = (A_1, A_2, \ldots, A_{10})$ where $A_2 = \{\text{Park}, \text{Theatre}\}$ for example.

Furthermore, the set $N_{Park} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Feasible sets are all subsets of projects that fit within the budget constraint $\text{Feas}(E) = \{W \subseteq C : \text{cost}(W) \leq b\}$. This is the collection of all feasible sets, the sets that can be the outcome of the election. A maximal feasible set is one that is maximal in $\text{Feas}(E)$ with respect to the subset relation.

To pick the winners, a voting rule (or voting mechanism) is used. This is a function that maps a PB election onto a feasible set of outcomes.

2.1 Properties

There are several properties that measure how well a voting mechanism performs. A common property is a proportionality axiom. This is a property that quantifies how evenly among the voters the budget is spread. However, this is not the only way to measure the fairness of mechanisms.

Voting mechanisms should not leave any budget left over. If there is still sufficient budget remaining to fund another project, the outcome would not seem fair. The outcome should maximally spend on the budget. Hence, this is a basic property that mechanisms should satisfy.

Definition 2.3 (Exhaustiveness)**.** A voting mechanism F is exhaustive if the output, $F(E)$, is a maximal element of $\text{Feas}(E)$ for any election E. That is, no more projects can be added to the set and keep its feasibility [Motamed et al., 2022].

$$\forall c \in C \setminus F(E),\ F(E) \cup \{c\} \notin \text{Feas}(E)$$

If there are projects that can still be funded with the remaining budget, the voting rule should have selected them. If no extra projects can be funded, then this is fine since a maximal number of projects has been chosen. Voters should not expect the remaining budget to be spent since none of the projects they voted on can be afforded.

A mechanism that can easily be controlled by a small number of voters would not be fair. Voting rules should exclude as few voters as possible from the outcome. That is, not too many voters have none of their supported projects in the outcome.

Definition 2.4 (Exclusion Ratio)**.** The exclusion ratio is the proportion of voters who have none of their supported projects in the output set [Papasotiropoulos et al., 2024].

$$ER(W) = \frac{|\{i \in N : A_i \cap W = \varnothing\}|}{n}$$

We want mechanisms to minimise this as much as possible. Elections should represent the full population so if a large group is left out, the election would not be fair.

For the last property, satisfaction measures need to be introduced. This is a function that measures how much a voter likes a set of projects. It has many uses including the definitions of proportionality axioms and even in some more complicated voting rules [Faliszewski et al., 2023].

Definition 2.5 (Satisfaction Measure)**.** A satisfaction measure quantifies how good a voter finds the outcome of a project. It is a function $\mu : \mathcal{P}(C) \to \mathbb{R}_{\geq 0}$ such that:

- $\mu(W) = 0 \iff W = \varnothing$
- If $W \subseteq W'$ then $\mu(W) \leq \mu(W')$

For a vote $i \in N$, the satisfaction they receive is $\mu_i(W) = \mu(A_i \cap W)$ [Brill et al., 2023].

There are many different types of satisfaction measure, although in this paper only cost satisfaction is considered as many properties are defined in terms of this. Cost satisfaction assumes that the amount of utility a voter gains from a project being selected is equal to the cost of the project $\mu_i(W) = \text{cost}(W \cap A_i)$. These measures are important when discussing how much benefit different voters get out of the outcome. Furthermore, this measure is used implicitly in the formulation of an important proportionality axiom – EJR+.

Definition 2.6. Given a single resource PB election, a feasible outcome W satisfies extended justified representation up to any project (EJR+) if, for every group $N' \subseteq N$ such that $\bigcap_{i \in N'} A_i \setminus W \neq \varnothing$, and $p \in \bigcap_{i \in N'} A_i \setminus W$, there is a voter $i \in N'$ with:

$$\text{cost}(A_i \cap W) + \text{cost}(p) > \frac{|N'|b}{n}$$

The intuition behind this definition is that: for any group of voters who agree on a set of projects that have not been selected, each of the projects provides a voter with too much satisfaction. The group's total amount they should spend is $\frac{|N'|b}{n}$ and adding this extra project would mean the group would have spent too much.

Each voting rule for PB with multiple resources has been inspired by a pre-existing mechanism for PB. These mechanisms are:

- Greedy.
- Method of Equal Shares.
- Method of Equal Shares with Bounded Overspending.

2.2 Greedy Rule

The first, and simplest, rule is a greedy rule. Projects are selected in decreasing order of number of votes. Starting off with a simple rule provides a baseline to compare how well more complicated rules perform. Furthermore, the greedy rule is commonly used in elections, and it is fast to compute the winners [Peters et al., 2021]. It has similarities to voting systems such as plurality where the winner is the candidate with the most votes.

Definition 2.7 (Greedy Rule)**.** The greedy rule runs in two steps:

1. Order the projects $c \in C$ by $|N_c|$.
2. For each project, if it can be added to the outcome without exceeding the budget, add it. If not, move on to the next project.

This method is very efficient, but the outcome does not satisfy any nice properties in general [Papasotiropoulos et al., 2024].

2.3 Method Of Equal Shares

Instead of greedily selecting projects, there are other ways that will produce better, more proportional results. The method of equal shares (MES) is a voting rule that was introduced by Peters et al. [2021] as an alternate way to select and fund projects. They showed how MES satisfies a proportionality axiom known as EJR up to one project. This is not as strong as EJR+ and is NP hard to check for violations. It was also shown to be not exhaustive. However, they defined completion mechanisms that generate an outcome that exhausts the budget. Brill and Peters [2023] showed that MES satisfies EJR+. Satisfying this strong proportionality axiom motivates creating a rule for multiple resources based off MES in the aim to approximately satisfy EJR+.

Definition 2.8 (Method of Equal Shares)**.** Consider a PB election $(N, C, b, \text{cost}, A)$.

1. Split the budget evenly among all voters. $b_i = \frac{b}{n}$
2. Select the ρ-affordable project with the smallest value of ρ. See Definition 2.9.
3. Each supporter of the project spends $\min(b_i, \text{cost}(c) \cdot \rho)$.
4. Repeat steps 2 and 3 until the remaining projects all have a ρ value of $+\infty$.

Intuitively, MES gives each voter an equal share of the budget for them to spend on projects. The project that gets picked is the one with the lowest fairness score ρ. This project is the one whose cost can be split most evenly among its supporters. Each supporter tries to pay as close to an equal share of the project as possible. The value ρ here is the maximum proportion of the cost that a voter must spend. Smaller values of ρ suggest that the project can be split more evenly and among more voters. For example, a score of 1/3 would suggest each voter spends at most a third of the cost. This would be a more even split that 1/2 which suggests at least one voter must cover half of the cost.

Definition 2.9 (ρ-affordability)**.** A project c is ρ-affordable if the following holds:

$$\text{cost}(c) = \sum_{i \in N_c} \min(b_i, \text{cost}(c) \cdot \rho)$$

It has been shown that the minimum value for ρ can be found by finding the minimum number of voters who must spend their entire budget, with the rest of them splitting the cost evenly [Peters et al., 2021].

Example 2.10. Table 3 shows the setup of a PB election.

Budget: £300,000	1	2	3	4	5	6	7	8	9	10
Park: £160,000	x	x	x	x	x	x	x	x		
Theatre: £90,000	x	x	x			x	x	x		x
Social Club: £60,000			x	x	x				x	x

Table 3: Example election

Each voter receives a budget of $b_i = £30,000, \quad i = 1, \ldots, 10$. This graph shows the initial spread of the budget among the voters.

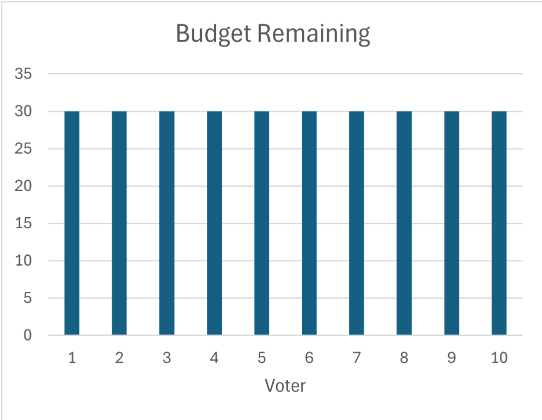
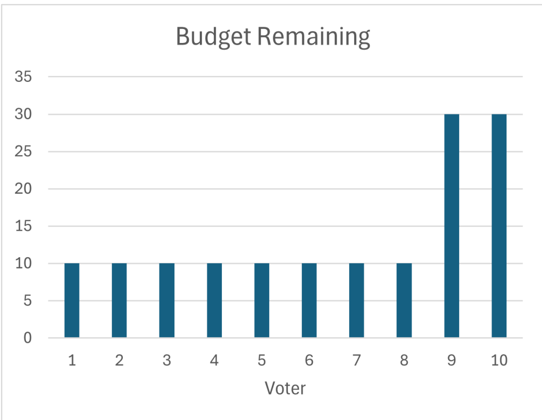


Figure 1: Initial Budgets

The project that gets chosen will be the one with the lowest value such that it is ρ-affordable. Begin by calculating the value for each voter.

Park: Cost can be split evenly among all 8 voters. So $\rho = \frac{1}{8}$
Theatre: Cost can be split evenly among all 7 voters. So $\rho = \frac{1}{7}$
Social Club: Cost can be split evenly among all 5 voters. So $\rho = \frac{1}{5}$

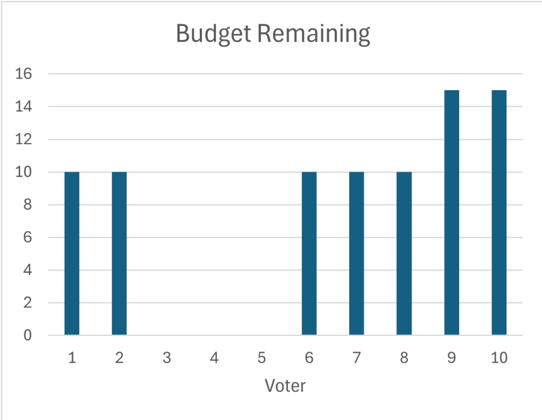
Hence, the park is funded with each supporter paying £$20,000$. Their remaining budgets are shown in the figure.



Like with the first step, we need to find the ρ values for each project so that the one that can be funded most evenly is chosen.

Theatre: Only way to fund is if all voters spend their entire budget. Voters $1, 2, 3, 6, 7, 8$ spend £$10,000$ and voter 10 pays £$30,000$ which is $1/3$ of the cost. $\rho = \frac{1}{3}$
Social Club: Only way to fund is if all voters spend their entire budget. Voters $3, 4$ and 5 spend £$10,000$ and voters 9 and 10 pay £$15,000$ which is $1/4$ of the cost. $\rho = \frac{1}{4}$

Hence the social club is funded. Each supporter of the social club spends their entire budget. The remaining budget for each voter after this can be seen in the figure.



At this point, no more projects can be afforded. The algorithm ends. It outputs the set $W = \{\text{Park}, \text{Social Club}\}$.

2.3.1 The Helenka Paradox

One major flaw of the method of equal shares is known as the Helenka paradox [Papasotiropoulos et al., 2024]. Projects which use up nearly all of the budget and have a large vote share may not be chosen over unpopular projects. The name refers to an election held in Zabrze in the district of Helenka, Poland. Two projects (A and B) were proposed with project A costing the whole budget and B costing significantly less.

Example 2.11. Table 4 details the PB election in Zabrze.

Budget: \$310k	403 Voters	11 Voters
A: $\$310k$	x	
B: $\$6k$		x

Table 4: Helenka Paradox

Each voter receives $\$\frac{310}{414}k = \$\frac{155}{207}k$ to spend. Hence, the 403 supporters of project A can only spend $\approx \$301k$ so cannot afford to buy it. Project B will get selected.

2.3.2 Completion Mechanisms

Peters et al. [2021] showed that, in general, MES does not exhaust the budget. This can produce some unsatisfactory results. Voters would not agree with an outcome where projects can still be funded. Hence, a completion mechanism should be used. These mechanisms turn an outcome that does not use the full budget and turns it into one that does. Not all mechanisms find a maximal set containing our original outcome though. One popular rule reruns MES with small increases to the budget until the outcome exhausts the original budget [Peters et al., 2021]. These rules are more complicated to extend to higher dimensions so in this paper only simplest one will be considered. The greedy completion rule sequentially adds projects to the outcome until no more can be added. These are considered in descending order of $|N_c|$.

2.4 Method of Equal Shares with Bounded Overspending

To fix the Helenka paradox, Papasotiropoulos et al. [2024] introduced a variant of MES known as the method of equal shares with bounded overspending (BOS). The method allows voters to overspend on projects by a small amount to fund the project more fairly.

Definition 2.12 (BOS)**.** Consider a PB election $(N, C, b, \text{cost}, A)$.

1. Split the budgets evenly among all voters. $b_i = \frac{b}{n}$
2. Select the (α, ρ)-affordable project with the smallest value of ρ/α (See Definition 2.13).
3. Each supporter of the project ends up with $\max(0, b_i - \rho \cdot \text{cost}(c))$.
4. Remove any projects that no longer fit into the budget and go back to step 2.

The pair (α, ρ) quantifies the proportion of the project α that is ρ-affordable.

Definition 2.13. A project c is (α, ρ)-affordable if:

$$\alpha \cdot \text{cost}(c) = \sum_{i \in N_c} \min(b_i, \rho \cdot \alpha \cdot \text{cost}(c))$$

If a project cannot be afforded by its voters, this algorithm may still fund it. If such project has sufficiently many voters, it can be funded only with a small amount of overspending by each voter. Obviously, overspending should be penalised so that is why the quantity ρ/α is used. The smaller amount of the project that can be afforded, the more this quantity increases.

Example 2.14. Table 5 shows the set up for the election.

Budget: \$200	20 Voters	5 Voters
A: \$96	x	
B: \$10		x
C: \$100	x	x

Table 5: A PB election

$b_i = \$8$
A: All of the project cost can be funded with each voter spending $\frac{1}{20}$ of the cost. So $\alpha = 1,\ \rho = \frac{1}{20}$.
B: All of the project cost can be funded with each voter spending $\frac{1}{5}$ of the cost. So $\alpha = 1,\ \rho = \frac{1}{5}$.
C: All of the project cost can be funded with each voter spending $\frac{1}{25}$ of the cost. So $\alpha = 1,\ \rho = \frac{1}{25}$.
So ρ/α for projects A, B, and C are 0.05, 0.2, and 0.04, respectively. Project C will get funded. Each voter spends \$4.

The next step is to decide between projects A and B.
$b_i = \$4$
A: $\frac{5}{6}$ cost can be funded with each voter spending $\frac{1}{20}$ of the cost. So $\alpha = \frac{5}{6},\ \rho = \frac{1}{20}$.
B: All of the project cost can be funded with each voter spending $\frac{1}{5}$ of the cost. So $\alpha = 1,\ \rho = \frac{1}{5}$.
The values of ρ/α for these projects are 0.06 for A and 0.2 for B. A gets funded with each voter spending \$4.80. They each overspend by \$0.80.

Therefore, the outcome of this election is $\{A, C\}$.

3 Model

Moving away from a single resource election, the first step is to define a PB election with multiple resources. It is based on definitions by Motamed et al. [2022] and Rey et al. [2023], but it has been slightly adjusted to agree with recent papers discussing PB.

To keep the notation consistent with the model for single resource PB, the new model will be defined as a direct extension to this. The model should be consistent with the single resource definition when the number of resources is one. This will mean that it will be easy to create extensions of single resource mechanisms. The properties will also transfer across nicely. The model defined here is inspired by the definitions introduced by Motamed et al. [2022] and Rey et al. [2023].

Definition 3.1 (Participatory Budgeting with Multiple Resources)**.** A PB election is a tuple $(N, C, \mathbf{b}, \text{cost}, A)$ where:

- $N = [n]$ is the set of voters.
- $C = \{c_1, \ldots, c_m\}$ is the set of projects (candidates).
- $\mathbf{b} \in \mathbb{R}^r_+$ is the budget, where $\mathbf{b} = (\mathbf{b}[1], \ldots, \mathbf{b}[r])$.
- $\text{cost} : C \to \mathbb{R}^r_{\geq 0}$ is the cost function and $\text{cost}[j]$ is the jth component function. Assume that no project has a cost of $\mathbf{0}$.
- $A = (A_1, \ldots, A_n)$ is the approval profile where $A_i \subseteq C$ is the set of projects voter i approves of.

The goal is to find a set of projects $W \subseteq C$ such that $\text{cost}[j](W) \leq \mathbf{b}[j]$ for all resources $j \in \{1, \ldots, r\}$.

This is the natural extension from the single resource definition. The only change is that the cost and budget are now r-dimensional vectors. Assume that no project's cost is $\mathbf{0}$ since otherwise it would be trivially funded and should not be in the election. The number of voters will be denoted n, the number of projects denoted m, and the number of resources denoted r. To discriminate between the index of the resource and the voter, square brackets will be used for resources and subscripts for voters. For example, $\mathbf{b}[3]$ is the budget in the third resource and $\mathbf{b}_3$ is the budget that voter 3 has in MES.

Example 3.2. Consider a PB election with three resources: money, land use and entertainment. Additionally, suppose the voters are residents of a small town, each numbered 1 to 8000. The council have £1 million to spend, $100km^2$ of land, and can fund at most 2 entertainment projects. Suppose the projects are: Park, Theatre, New signage, Benches, and Tennis courts.

- $N = \{1, \ldots, 8000\}$
- $C = \{\text{Park},\ \text{Theatre},\ \text{New Signage},\ \text{Benches}, \text{Tennis Courts}\}$
- $\mathbf{b} = (\pounds 1000000, 100km^2, 2)$
- For example $\text{cost}(\text{Park}) = (400000, 20, 1)$, and $\text{cost}(\text{Benches}) = (10000, 0, 0)$
- If voter 324 supports the park, benches and theatre: $A_{324} = \{\text{Park}, \text{Benches}, \text{Theatre}\}$
- Suppose voter 23 does not vote: $A_{23} = \varnothing$.

Extending the model in this way allows for a richer and more expansive model. As well as allowing councils to budget for more than one resource, they can also impose limits on types of project and ensure that conflicting ones do not get selected together. Furthermore, the setup is near identical to Definition 2.1 which would make it easy to change from single to multiple resources. The similarity means that new rules are simple to implement because the framework for the elections is the same. Additionally, if a council were to implement PB with multiple resources, it would be cheap and easy to make the swap from standard PB.

4 Greedy Rule for Multiple Resources

A simple rule to use when selecting projects is an extension of the greedy rule defined in Section 2. It is very efficient to compute and would be the natural choice for anyone setting up an election. Motamed et al. [2022] showed however, that the greedy rule does not satisfy even weak fairness axioms. It sits as a nice benchmark for the performance of the other mechanisms proposed but it should not be used in practice.

Definition 4.1. The greedy rule runs in two steps:

1. Order projects in decreasing order by $|N_c|$.
2. For each project, if it can be added to the outcome without exceeding the budget, add it. If not, move on to the next project.

The greedy rule can often produce results that are undesirable. Example 4.2 shows one way in which the greedy rule produces a poor result.

Example 4.2. Consider the setup in Table 6.

$\mathbf{b} = (100, 100, 100)$	50 voters	1 voter	50 voters
$c_1 : (52, 0, 0)$	x	x	
$c_2 : (50, 50, 50)$	x		
$c_3 : (50, 50, 50)$			x

Table 6: Election where the greedy rule produces a poor result.

Since c_1 has 51 supporters, it will get selected. The greedy rule will then finish as no more projects can be funded.

The set $\{c_2, c_3\}$ is also feasible and uses up the full budget. Additionally, 100 out of 101 voters receive a project with which they are happy. These two reasons make this more desirable than the greedy result.

4.1 Algorithm for the Greedy Rule

Algorithm 1 Compute the winners of greedy.

Input: $E = (N, C, \mathbf{b}, \text{cost}, A)$.
Output The set of winners of the election W.

$W \leftarrow \varnothing$
$n_c \leftarrow |\{i \in N : c \in A_i\}|$
Let $c_1, \ldots, c_m$ be projects in descending order of n_c
for $k = 1$ to m **do**
  if $\text{cost}[j](c_k) \leq \mathbf{b}[j]\ \forall j \in \{1, \ldots, r\}$ **then**
    $W \leftarrow W \cup \{c_k\}$
    $\mathbf{b} \leftarrow \mathbf{b} - \text{cost}(c_k)$
Return W

Theorem 4.3. Algorithm 1 computes a feasible set of projects in $O(n \log n + m(n + r))$ steps.

Proof. Suppose the output of the greedy rule is not feasible. Let $c_t \in W$ be the first project added that makes the set not feasible. Hence, at step $t - 1$, the cost of project c_t is more than the remaining budget in at least one resource. Thus, the greedy rule does not add c_t to W. $\square$

4.2 Properties

Although the greedy rule does produce some good outcomes, in general they will not satisfy stronger properties. This is directly inherited from the single resource case.

Lemma 4.4. For $r = 1$, the greedy rule operates the same as the greedy rule defined in Section 2.2.

Proof. The definition is identical; the proof is trivial $\square$

Proposition 4.5. The greedy rule is exhaustive [Motamed et al., 2022].

Proof. Suppose for a contradiction the output of greedy W is not a maximal feasible set. Then, there exists project $c \notin W$ such that it can still be funded with the remaining budget. Therefore, greedy has not finished. $\square$

We may want to know if there is a bound on the number of voters who may be excluded from the outcome. Disappointingly, we can always find an election where the greedy rule produces an outcome with an exclusion ratio arbitrarily close to 1.

Theorem 4.6. A tight upper bound on the exclusion ratio is 1.

Proof. Any election must have an exclusion ratio of at most 1. Consider the election $E_k = (\{0, \ldots, k\}, \{c_1, \ldots, c_k\}, \mathbf{b} = (k, 1), \text{cost}, A)$ defined by:

- 0 approves of c_k only.
- i approves of c_k only.
- $\text{cost}(c_1) = (k, 0)$, $\text{cost}(c_i) = (1, 0)\ \forall i > 1$.

Let W_k be the output of greedy on election E_k. Since greedy will only fund project c_1, all voters $2, \ldots, k$ are excluded. So, $\text{ER}(W_k) = \frac{k-1}{k+1}$

$$\begin{aligned}
\sup\{\text{ER}(W_k) : k \in \mathbb{N}\} &= \lim_{k\to\infty} \frac{k-1}{k+1} \\
&= \lim_{k\to\infty} \frac{1-\frac{1}{k}}{1+\frac{1}{k}} \\
&= 1
\end{aligned}$$

So, for any $0 < \varepsilon \leq 1$ there is an election with the exclusion ratio $1 - \varepsilon$. Therefore, 1 is a tight upper bound for the exclusion ratio of the greedy rule. $\square$

This means that, there is always an election where the greedy rule produces a result that leaves out nearly everybody. In fact, this is common with multi-winner mechanisms even when they produce proportional results – like MES with one resource.

5 Multi-Method of Equal Shares

The first main contribution of this paper is to extend MES to multiple resources. Multi-method of equal shares (multi-MES) is a direct extension of the MES, and we will show that, for one resource, they are the same. Since MES satisfies the strong proportionality property EJR+, and experimentally has a lower exclusion ratio than greedy [Papasotiropoulos et al., 2024], it is a natural place to start looking for new mechanisms. This simple extension will lead onto more complex mechanisms later. This section introduces multi-MES, describes an algorithm to efficiently compute the winners and finally proves simple properties about the mechanism.

Definition 5.1. Given a PB election $(N, C, \mathbf{b}, \text{cost}, A)$, and some ρ-aggregation function f (Definition 5.2), multi-MES conducts the following steps:

1. Split the budget equally among voters $\mathbf{b}_i = \frac{\mathbf{b}}{n} = \left(\frac{\mathbf{b}[1]}{n}, \ldots, \frac{\mathbf{b}[r]}{n}\right)$.
2. Choose the ρ-affordable project with the lowest value of $\rho = f(\rho[1], \ldots, \rho[r])$ (See Definition 5.3).
3. If this value of $\rho = +\infty$ then return the set W
4. Each supporter spends $\min(\mathbf{b}_i[j], \text{cost}[j](c) \cdot \rho[j])$ for each resource $j = 1, \ldots, r$ and c gets added to W.
5. Return to step 2.

The mechanism runs in a similar way to MES. Each voter receives an even share of the budget, the project which can be most fairly funded is selected. This time, the $\rho[j]$ values for each resource are calculated individually and then combined using some aggregation function. Next, each voter spends the equal share of the cost of each resource, if possible, if not they spend their entire remaining budget for that resource.

Since ρ-affordability is initially defined with only one resource, this notion must be extended. To do this, each resource will be considered individually and then combined together to create an overall fairness score. This aggregation function needs to create a value ρ that represents how well a project can be funded. Unfortunately, standard aggregation functions will not work as the min function suggests. Suppose one resource could be afforded with fairness score x and another could not be afforded (fairness score would be $+\infty$). The minimum function would combine these and return x, suggesting the project can be afforded. This would cause an issue since the project cannot actually be afforded. Furthermore, an aggregation function should not produce a result that is completely different to the iput vector x. Thus, two conditions are needed on the types of aggregation function: it should be bounded by the maximum and minimum values of x and, if one of the x_is is infinity then the output should also be infinity.

Definition 5.2. A ρ-aggregation function, is a function $f : (\mathbb{R} \cup \{+\infty\})^r \to (\mathbb{R} \cup \{+\infty\})$ such that:

1. $f(x) \in [\min_i\{x_i\}, \max_i\{x_i\}]$
2. $f(x) = +\infty$ iff $\exists i = 1, \ldots, r : x_i = +\infty$

If a project cannot be afforded by voters for some resource, this would mean that they cannot fund the entire project. Thus, the resulting value for ρ should be $+\infty$ since this means the project is unaffordable. To use functions such as min, a small adjustment should be made so that it satisfies this condition. Some examples of such aggregation functions are:

- $\max(\rho[1], \ldots, \rho[r])$.
- $\min'(\rho[1], \ldots, \rho[r]) = \begin{cases} +\infty & \exists j : \rho[j] = \infty \\ \min(\rho[1], \ldots, \rho[r]) & \text{otherwise} \end{cases}$.
- $\text{mean}(\rho[1], \ldots, \rho[r])$.
- $\text{median}'(\rho[1], \ldots, \rho[r]) = \begin{cases} +\infty & \exists j : \rho[j] = \infty \\ \text{median}(\rho[1], \ldots, \rho[r]) & \text{otherwise} \end{cases}$.

For functions where an extension to a ρ-aggregation function is natural (like min' and median') their normal names will be used. The choice of f can impact the project which gets chosen first. Using these it is easy to define what it now means for a project to be ρ-affordable.

Definition 5.3 (ρ-affordability)**.** Given a ρ-aggregation function f, a project is ρ-affordable if $\exists(\rho[1], \ldots, \rho[r])$ such that for every $j = 1 \ldots r$:

$$\text{cost}[j](c) = \sum_{i \in N_c} \min(b_i[j], \text{cost}[j](c) \cdot \rho[j])$$

and $\rho = f((\rho[1], \ldots, \rho[r]))$. If no such $\rho[i]$ exists, $\rho[i] = +\infty$.

The definition considers each resource individually and finds the best way to fund each resource. To get the overall fairness score, these get aggregated together by the function f. The most natural function to take for this would be max since this would take the score of the resource that gets split least fairly. Unless otherwise specified this will be the function used for the remainder of the paper.

Example 5.4 shows how using the aggregation functions max, min and median will affect the project chosen.

Example 5.4. Table 7 shows the costs of two projects, and their approval votes from four voters.

	1	2	3	4
$c_1 : \text{cost}(c_1) = (10, 10)$	x	x		
$c_2 : \text{cost}(c_2) = (10, 10)$			x	x

Table 7: Approval votes for each project

At this point in the election the voters' budgets are shown in Table 8.

	1	2	3	4
$\mathbf{b}_i$	(5,8)	(5,2)	(6,3)	(4,7)

Table 8: Remaining budgets

Using this information, we can calculate the values for which the two projects are ρ-affordable.
Project c_1:

- Resource 1 can be shared equally; so $\rho[1] = \frac{1}{2}$.
- Resource 2 cannot be shared equally; the only way is for voter 1 to spend 8/10 of the budget and voter 2 to spend 2/10. So $\rho[2] = \frac{4}{5}$.

Project c_2:

- Resource 1 cannot be shared equally; voter 1 must spend 6/10 of the budget and voter 2 spends 4/10. So $\rho[1] = \frac{3}{5}$
- Resource 2 cannot be shared equally; the only way is for voter 1 to spend 7/10 of the budget and voter 2 to spend 3/10. So $\rho[2] = \frac{7}{10}$.

Table 9 shows the resulting ρ-values for different aggregation functions and which project then gets selected.

f	c_1	c_2	Project Selected in multi-MES
max	$4/5$	$7/10$	c_2
min	$1/2$	$3/5$	c_1
mean	$13/20$	$13/20$	Tie

Table 9: The projects that get picked when different aggregation functions are used.

This example shows how different aggregation functions will create different results. Overall, some may produce better results than others. In general, the best aggregation function would depend on the specific circumstances.

Before showing an example, we need to check that this definition of ρ-affordability makes sense. A project should be affordable if and only if the value of ρ is not infinity.

Lemma 5.5. Voters cannot fund a project if and only if $\rho = +\infty$

Proof. ($\implies$) Suppose project c cannot be funded by its supporters. Then, for some resource j:

$$\begin{aligned}\sum_{i \in N_c} \min(\mathbf{b}_i[j], \mathrm{cost}[j](c) \cdot \rho[j]) &\leq \sum_{i \in N_c} \mathbf{b}_i[j] \\ &< \mathrm{cost}[j](c)\end{aligned}$$

For any $\rho[j]$. Hence $\rho[j] = +\infty \implies \rho = +\infty$.

($\impliedby$) Since $\rho = +\infty$ there is a resource j such that $\rho[j] = +\infty$. Then, there does not exist some $\rho[j] \in \mathbb{R}$ such that:

$$\mathrm{cost}[j](c) = \sum_{j \in N_c} \min(\mathbf{b}_i[j], \mathrm{cost}[j](c) \cdot \rho[j])$$

Fix $\rho[j] = M$ arbitrary such that $\mathbf{b}_i[j] < M \cdot \mathrm{cost}[j](c)\ \forall i \in N_c$.

$$\begin{aligned}\sum_{i \in N_c} \mathbf{b}_i[j] &= \sum_{i \in N_c} \min(\mathbf{b}_i[j], \rho[j] \cdot \mathrm{cost}[j](c)) \\ &< \mathrm{cost}[j](c)\end{aligned}$$

So, the project cannot be afforded in resource j. Hence it cannot be afforded. □

To better understand how multi-MES works, it is best to look at an example.

Example 5.6. (Running multi-MES) For the following example, the aggregation function max will be used. Table 10 shows the projects and their approval votes. The resources being spent could be thought of as money, land space, and entertainment.

$\mathbf{b} = (200, 100, 8)$	1	2	3	4	5	6	7	8	9	10
Park P: $(100, 60, 2)$	x		x	x	x			x		x
Public toilets L: $(20, 14, 0)$									x	x
Theatre T: $(150, 40, 5)$		x	x	x	x	x	x	x		x

Table 10: Project costs and approval votes

Let W be the set of funded projects. To begin with each voter receives $(20, 10, 0.8)$ to spend.

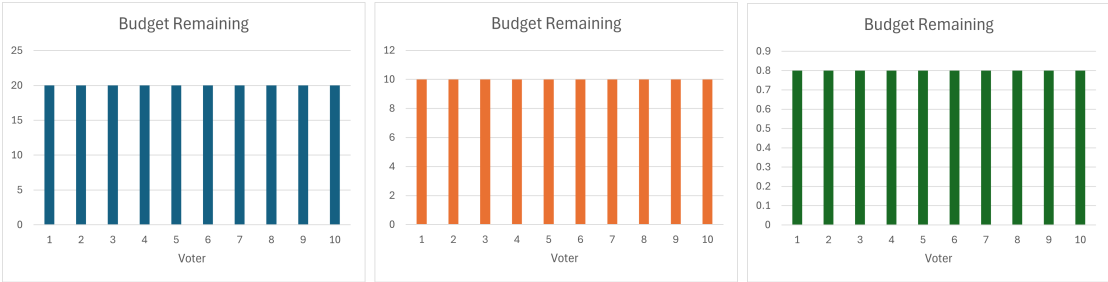


Figure 2: Initial Budgets.

The next step is to calculate the minimum ρ values for the project and resource.

Park:

- Resource 1 can be split evenly so $\rho[1] = \frac{1}{6}$
- Resource 2 can be split evenly so $\rho[2] = \frac{1}{6}$
- Resource 3 can be split evenly so $\rho[3] = \frac{1}{6}$

Public toilets:

- Resource 1 can be split evenly so $\rho[1] = \frac{1}{2}$
- Resource 2 can be split evenly so $\rho[2] = \frac{1}{2}$
- Resource 3 has cost 0 so we assume this is split evenly (even though any value for ρ works). So $\rho[3] = \frac{1}{2}$

Theatre:

- Resource 1 can be split evenly so $\rho[1] = \frac{1}{8}$
- Resource 2 can be split evenly so $\rho[2] = \frac{1}{8}$
- Resource 3 can be split evenly so $\rho[3] = \frac{1}{8}$

Hence, by aggregating these values together using the max function, the theatre will have the lowest ρ value. So, the theatre gets added to W. The supporters of the theatre each spend $(18.75, 5, 0.625)$, leaving them with $(1.25, 5, 0.175)$. Voters 1 and 9 still have $(20, 10, 0.8)$.

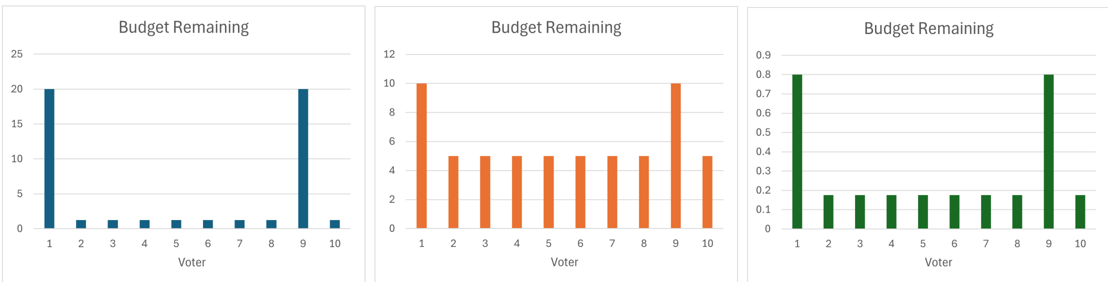


Figure 3: New Budgets.

Using the new budgets, the values for ρ are recalculated for the park and public toilets.
Park:

- Resource 1 cannot be funded so $\rho[1] = +\infty$.
- Resource 2 cannot be funded $\rho[2] = +\infty$.

- Resource 3 can be split evenly so $\rho[3] = \frac{1}{6}$.

Public toilets:

- Since voter 10 only has 1.25 of resource 1 remaining, voter 9 must spend 18.75 to pay for it. Hence $\rho[1] = \frac{18.75}{20} = \frac{15}{16}$
- Resource 2 cannot be split evenly. Voter 10 can only spend 5, hence voter 9 must spend 9. So $\rho[2] = \frac{9}{14}$
- Resource 3 can be split evenly so $\rho[3] = \frac{1}{2}$.

The minimum ρ values for the park and public toilets are $+\infty$ and $\frac{15}{16}$ respectively. Choosing the smallest of the two funds the public toilets. Since the park was already unaffordable, it will not be able to be purchased in the next step; the mechanism stops. The output is $W = \{\text{Public Toilets}, \text{Theatre}\}$.

5.1 An Algorithm for multi-MES

To calculate the winners of an election using multi-MES, the first step is to calculate the ρ values for each project. Algorithm 2 uses the method provided by Peters et al. [2021] to compute the values $\rho[j]$ for each project. This can be done because, for each resource, the value of ρ[j] can be calculated in the same way as with a single resource. Therefore, it is just as easy to compute these fairness scores when there are multiple resources, providing further motivation for using multi-MES.

Algorithm 2 Calculating $\rho[j]$ for a given resource j.

Input: Project c, resource j, Supporters N_c, budgets b_i
Output $\rho[j]$ for resource j and project c

```
S ← Sort N_c by budget b_i[j] i ∈ N_c
p ← cost[j](c)
v = |S|
for i ∈ S do
    if b_i[j] < p/v then
        p ← p − b_i[j]
        v ← v − 1
    else
        Break
return p / (v·cost[j](c))
```

The minimum value of $\rho[j]$ is the maximum proportion of the cost a voter must pay. This can be computed by sorting the voters in increasing order of $\mathbf{b}_i[j]$ and checking if they can split the remaining cost evenly with everyone with more budget. If not, this voter must spend their entire budget; this amount is removed from the cost of the project. This computes the correct value of $\rho[j]$ since the algorithm is the same as for single resource PB.

Lemma 5.7. Algorithm 2 computes the minimum value for which a project is ρ-affordable in dimension j in time $O(n \log n)$.

Proof. Sorting the set of voters takes $O(n \log n)$ time, the rest of the algorithm takes linear time. ☐

Proposition 5.8. To compute the values of ρ for each project. $O(mr)$ applications of the algorithm must be computed. Hence, finding the values of ρ for all the projects takes $O(nmr \log n)$ time.

Proof. Each of the m projects needs $(\rho[1], \ldots, \rho[r])$ to be calculated. Hence, r runs of the algorithm. $\square$

Using this algorithm (named "ComputeRho") the winners of multi-MES can be calculated using Algorithm 3.

Algorithm 3 Compute the winners of multi-MES.

Input: $E = (N, C, b, \text{cost}, A)$, aggregation function f.
Output The set of winners of the election W.

$W \leftarrow \varnothing$
$b_i \leftarrow \frac{b}{n}$ for each voter $i = 1, \ldots, n$
$N_c \leftarrow \{i \in N : c \in A_i\}$ for each project c
while $C \neq \varnothing$ **do**
  for $c \in C$ **do**
    $\rho_c \leftarrow (\text{ComputeRho}(c, j, N_c) \text{ for } j = 1, \ldots, r\ )$
  $c^* \leftarrow \text{argmin}(f(\rho_c))$
  if $f(\rho_{c^*}) = +\infty$ **then**
    Break
  for $i \in N_{c^*}, j = 1, \ldots r$ **do**
    $b_i[j] \leftarrow b_i[j] - \min(b_i[j], \text{cost}(c^*)\rho_{c^*}[j])$
  $W \leftarrow W \cup \{c^*\}$
  $C \leftarrow C \setminus \{c^*\}$
return W

Now we know how to compute the ρ-values for each project and resource. The rest of the mechanism is easy to compute. The project with the lowest value of $f(\rho[1], \ldots, \rho[r])$ is picked and each voter spends the amount that they owe. For each resource, this will either be $\mathbf{b}_i[j]$ or, if they can afford it, $\rho[j] \cdot \text{cost}[j](c)$.

Theorem 5.9. The winner of multi-MES can be computed in time $O(nmr \log n)$.

Proof. First, by following the steps of the algorithm, on each step, the project with the lowest aggregated value of ρ is selected and funded. Hence the algorithm outputs the multi-MES winners. On each pass through the loop, it takes $O(nmr \log n)$ steps to find the values of ρ for the projects. Funding takes $O(nmr)$ steps. Thus, each loop takes $O(nmr(\log n + 1) = O(nmr \log n)$ steps. This loop happens at most m times. Hence the runtime of Algorithm 3 is $O(nm^2 r \log n)$. $\square$

5.2 Properties of Multi-MES

Although this paper does not theoretically discuss stronger proportionality properties, we can still see some simpler properties of multi-MES. One main result is that multi-MES and MES are the same for one resource. This means that stronger properties may still hold for multi-MES. This is an area for further study and will not be discussed theoretically here.

Lemma 5.10. Multi-MES is equivalent to MES when $r = 1$.

Proof. Any aggregation function f would set $\rho = \rho[1]$ in the multidimensional case. Hence in both mechanisms, the same project will get chosen to be funded in each step. The method of funding is the same so the voters will end up with the same budgets after each step. $\square$

As mentioned in Section 3, outputting an exhaustive set is a desirable property of mechanisms. Voters would be unhappy if other projects can be afforded, even if the outcome is proportional.

Proposition 5.11. Multi-MES is not exhaustive.

Proof. Let $r = 1$ and choose a PB scenario where MES does not exhaust the budget. Since multi-MES produces the same output, the output of multi-MES will also not be exhaustive. $\square$

Similarly to the single resource case, since MES is not exhaustive, a completion mechanism can be used to generate an outcome which maximally uses the budget. A greedy rule can be used to add the projects with the most votes until no more projects can be afforded.

Example 5.12. Table 11 shows the setup of an election E.

$\mathbf{b} = (20, 15)$	6 voters	4 voters
A: $(15, 10)$	x	
B: $(5, 10)$		x

Table 11: An election where MES does not produce an outcome which is exhaustive.

Each voter receives a budget of $(2, 1.5)$. The supporters of project A only have $(12, 9)$, so they cannot afford the project. Similarly, the supporters of B only have $(8, 6)$ to spend. Therefore, multi-MES will output the feasible set $\varnothing$. This is not a maximal feasible set since $\varnothing \subseteq \{A\} \in$ Feas(E). Greedy completion will add the project A to the outcome since it has the most votes. The outcome is $\{A\}$ which is now a maximal feasible set.

5.2.1 Helenka Paradox Style Problems

Since, by Lemma 5.10, multi-MES and MES coincide for one resource, problems like the Helenka paradox will still occur when multi-MES is used. In fact, the Helenka paradox can occur for any number of resources.

Example 5.13. Table 12 takes the election from Example 2.11 and introduces a second resource.

$\mathbf{b} = (\$310k, 200m^2)$	403 Voters	11 Voters
A: $\$310k, 10m^2$	x	
B: $\$6k, 5m^2$		x

Table 12: The Helenka Paradox but with an extra resource

The issue arises from the first resource. For resource 1, each voter receives $\$\frac{310}{414}k = \$\frac{155}{207}k$. So, the 403 supporters of project A can only spend $\approx \$301k$ so cannot afford to buy it. Hence project B will get selected. The second resource is very cheap so can be funded by both sets of voters. This does not affect the outcome since the supporters of A cannot fund the first resource.

Theorem 5.14. A project c will not be funded if it has fewer than $\frac{\text{cost}[j](c)}{\mathbf{b}[j]} \cdot n$ votes for some resource $j = 1, \ldots, r$.

Proof. Consider a project c which has less than $\frac{\text{cost}[j](c)}{\mathbf{b}[j]} \cdot n$ votes. Then:

$$\begin{aligned}\sum_{i \in N_c} \mathbf{b}_i[j] &< \sum_{i \in N_c} \frac{\mathbf{b}[j]}{n} \\ &< \frac{\text{cost}[j](c)}{\mathbf{b}[j]} \cdot n \cdot \frac{\mathbf{b}[j]}{n} \\ &< \text{cost}[j](c)\end{aligned}$$

This means that, as a group, the voters cannot afford c in some resource. Hence, it cannot get selected by multi-MES at any point. □

Corollary 5.15. Fix $r \geq 1$, Suppose $a \in C$ with $\text{cost}[j](a) = \mathbf{b}[j]$ for some j. Then multi-MES will not select project a unless $\forall i \in N : a \in A_i$.

Proof. Since $\frac{\text{cost}[j](c)}{\mathbf{b}[j]} = \frac{\mathbf{b}[j]}{\mathbf{b}[j]} = 1$, Theorem 5.14 implies that the project needs full support to be selected. □

This means that we can easily encounter elections that produce Helenka style results. In fact, this is much more common with more resources because it is more likely that a popular project will have a resource costing close to the maximum budget. This is especially problematic with project conflict resources as these can often only cost the entire budget or zero. Thus, none of the conflicting projects would get picked unless one receives full support.

6 Exchanged Equal Shares

Keeping with extensions of MES, one may wonder whether voters can exchange between resources. Here, exchanged equal shares (EES) will be introduced, which allows voters to exchange between their resources using an appropriate exchange rate. For three resources, which ones are used for the exchanges becomes more complicated. Therefore, to start, just two resources will be considered. Afterwards, a more general funding scheme will be introduced that works with any number of resources. Using this, we will show how, under appropriate conditions, the result will be the same as running MES on one resource.

As Theorem 5.14 suggests, even projects which are expensive in one resource are difficult to select. Voters should have the flexibility to exchange some surplus resource for others. Example 6.1 shows one example where allowing this would produce a better result.

Example 6.1. Table 13 shows a PB election.

$\mathbf{b} = (100, 100)$	1	2	3	4	5	6	7	8	9	10
P: $(10, 90)$	x	x	x	x	x	x				
L: $(90, 10)$							x	x	x	x
T: $(11, 11)$	x									x

Table 13: Project costs and approval votes

For multi-MES, the budget is split evenly among voters: $b_i = (10, 10)$. Then it is clear that projects P and L cannot be afforded in one dimension. So, MES will return project T only.

This example shows that it is easy to find instances where popular projects cannot be picked due to being expensive in one resource. However, in the second resource, the voters will have lots of budget left over. What would happen if we allowed voters to spend some of one resource to get the other? This is the motivation for allowing voters to exchange between their budgets to better suit their needs. Voters of projects that conflict may want to spend some of their other resource to get enough to collectively afford the conflict resource for example. This section introduces the notion of an exchange rate and how multi-MES can be extended to allow for exchanges of resource. With 3 or more resources, it becomes harder to determine how a voter is making these exchanges so we will start with two resources before moving onto a more general framework.

Definition 6.2. For a PB scenario $(N, C, \mathbf{b}, \text{cost}, A)$, with $b = (£x_1, \$y_1)$, exchanged equal shares (EES) works as follows:

1. Split the budget evenly between voters, $b_i = (£\frac{x_1}{n}, \$\frac{y_1}{n})$ for each voter $i \in N$.
2. Remove any projects that do not fit within the remaining budget.
3. Choose a project as described in Section 6.2.
4. Fund as described in Section 6.3.
5. Go back to step 2 with budget (x_t, y_t).

6.1 Exchange Rates

The main difference between multi-MES and EES is the ability for voters to convert their resources to other ones. This extra ability allows for projects which may not be able to be funded by the voters, get funded. For example, this allows for projects which are popular but too expensive along one dimension to be funded. Hence, creating a fairer outcome.

Definition 6.3. Consider a cost or budget, an exchange rate between two resources i and j is a value $\eta[i \to j]$ such that y amount of resource i creates $\eta[i \to j] \cdot y$ amount of resource j.

This is the general idea behind an exchange rate. In currency, the exchange rate between GBP and EUR determines the number of euros that £1 is worth. Hence, in the same way as these, the number of euros is the number of pounds multiplied by the exchange rate. This could also be extended to other resources. For example, if $1m^2$ of land cost £20, the exchange rate from land to pounds would be 20. It could be seen as selling land to get extra money. This would be an operation that a supporter of small projects may want to perform.

To support future definitions, the exchange rates can be used to create other quantities. These will be used to simplify other definitions and suggest similarities to single resource PB.

Definition 6.4 (Price of a project)**.** The price of a project is the cost of the project all converted into the first resource.

$$\text{price}(c) = \sum_{j=1}^{r} \eta[j \to 1] \cdot \text{cost}[j](c)$$

Definition 6.5. The value β is the budget of the election converted into the first resource.

$$\beta = \sum_{j=1}^{r} \eta[j \to 1] \cdot b[j]$$

All of the methods in this paper are independent of the order of the resources. This is because none of the methods conduct different steps depending on the resource. Hence, the definitions below can use any resource. For consistency, the first resource will always be the one converted to, and money will always be the first resource.

There is, however, a specific type of exchange rate that will be used in our method. It should capture the relative rarity between the resource. It should be more expensive to convert to a resource with a smaller budget than to go the other direction. To keep this property throughout, the rates are recalculated with the remaining budget every time. When there are two resources, for clarity, the units £ and \$ will be used to differentiate between them.

Definition 6.6 (Exchange Rate)**.** For a budget of $(\pounds x_t, \$ y_t)$ the exchange rates are:

- From \$ to £: $\eta[\$ \to \pounds] = \frac{x_t}{y_t}$
- From £ to \$: $\eta[\pounds \to \$] = \frac{y_t}{x_t}$

This definition gives equal weight to different resources. When one is rarer, the cost of 1 unit of this resource increases. After each project is funded, these rates can be recalculated to maintain the relative rarity between the resources.

Example 6.7. Budget $= (200, 300)$

1. Voter has (100,8) and wants to convert £20 into dollars. They will receive $20 \cdot \frac{300}{200} = \$30$.
2. Voter needs £30 to fund the project. They will need to spend $30 \cdot \frac{300}{200} = \$45$.
3. Voter needs an additional \$10 for the project. They will need to convert $10 \cdot \frac{200}{300} = \pounds\frac{20}{3}$ into dollars.

Example 6.8 shows how allowing voters to convert this way can create better overall outcomes than multi-MES would. This is a simpler version of Example 6.1 to show how popular projects could get funded.

Example 6.8. Consider a PB scenario given in Table 14.

$\mathbf{b} = (300, 400)$	18 voters	2 voters
$c_1 : (20, 400)$	x	
$c_2 : (20, 30)$		x

Table 14: Table showing the voters and what projects they approve.

Each voter receives a budget of $(15, 20)$. Hence, the 18 voters of project c_1 cannot afford to purchase it. Therefore, multi-MES only funds project c_2 which has 2 voters.

Whereas, if the voters could exchange as above, they would be able to afford c_1 by converting 5/3 of resource 1 into 20/9 of resource 2. This means that project c_1 can be funded evenly between all voters. This creates a better outcome since the number of voters for project c_1 is significantly more than those who voted for c_2.

6.2 Project Selection

To select the project, it seems natural to use some kind of ρ-affordability, like the other mechanisms described here. However, since exchanging resources is undesirable, this should be considered when selecting projects. Projects that require more exchanging should be penalised and receive a higher fairness score. To begin, a simpler notion is defined which will be used to create (ρ, ε)-converted affordability.

Definition 6.9. A project ρ-converted-affordable if the following holds:

$$\text{price}(c) = \sum_{i \in N} \min(\beta_i, \text{price}(c) \cdot \rho)$$

Where $\beta_i = b_i[0] + b_i[1] \cdot \frac{x}{y}$ (The converted budget of voter i).

Theorem 6.10. A project can be afforded by its voters if and only if there exists a value $\rho \in \mathbb{R}$ such that it is ρ-converted affordable.

Proof. Suppose a project c can be funded by its voters. This means that, for all resources j, $\text{cost}[j]$ can be afforded.

$$\begin{aligned} \text{cost}[j](c) &\leq \sum_{i \in N_c} \mathbf{b}_i[j] \\ \implies \text{price}(c) &= \sum_{j=1}^{r} \eta[j \to 1] \cdot \text{cost}[j](c) \\ &\leq \sum_{j=1}^{r} \eta[j \to 1] \sum_{i \in N_c} \mathbf{b}_i[j] \\ &= \sum_{i \in N_c} \sum_{j=1}^{r} \eta[j \to 1] \mathbf{b}_i[j] \\ &= \sum_{i \in N_c} \beta_i \end{aligned}$$

So, the project's price can be afforded by the converted budgets of the voters. Therefore, there is a value ρ such that the project is ρ-converted affordable.

Now suppose a project cannot be funded by its voters. Following the same sequence of inequalities but with $>$, we can see that for any $\rho \in \mathbb{R}$:

$$\text{price}(c) > \sum_{i \in N_c} \beta_i \geq \sum_{i \in N_c} \min(\beta_i, \rho \cdot \text{price}(c))$$

Hence, for any value ρ, the project is not ρ-converted affordable. $\square$

6.2.1 Dealing With Exchanges of Resource

Since the mechanism allows for voters to exchange between resources, the amount exchanged should be considered when selecting projects. More exchanges during funding is a less desirable property than a project with fewer. A second quantity ε is introduced that quantifies these exchanges.

Definition 6.11. For some value ρ, a project $c \in C$ is (ρ, ε)-converted affordable if the following properties hold:

- c is ρ-converted affordable.
- ε quantifies the amount of funds that will need to be exchanged to fund the project.

There are many different options for what ε could be. A simple example would be an indicator function (1 if exchanging happens and 0 if not). Let $\eta = \eta[\$ \to £]$ be the \$ to £ exchange rate. Fix project $c \in C$. Let $T = \sum_{i \in N_c} \mathbf{b}_i$. The following could be some more examples of how ε could be calculated.

- 0. In elections where the amount of exchanging is of no issue, set $\varepsilon = 0$.
- Gross conversion. ε is the total amount (in £) that voters need to convert from £ into \$ and from \$ into £.

$$\varepsilon = \max(0, \text{cost}[1](c) - T[1]) + \eta \cdot \max(0, \text{cost}[2](c) - T[2])$$

- Net conversion. ε is the net converting that will need to be done.

$$\varepsilon = |\max(0, \text{cost}[1](c) - T[1]) - \eta \cdot \max(0, \text{cost}[2](c) - T[2])|$$

- Indicator.

$$\varepsilon = \begin{cases} 1 & \max(0, \text{cost}[1](c) - T[1]) + \eta \cdot \max(0, \text{cost}[2](c) - T[2]) > 0 \\ 0 & \text{Otherwise} \end{cases}$$

- Gross initial conversion. The sum of the amounts that each voter would need to convert to reach their target of $\text{cost}(c)/|N_c|$.

$$\varepsilon = \sum_{i \in N_c} \max\left(0, \frac{\text{cost}[1](c)}{|N_c|} - \mathbf{b}_i[1]\right) + \eta \cdot \sum_{i \in N_c} \max\left(0, \frac{\text{cost}[2](c)}{|N_c|} - b_i[2]\right)$$

- Net initial conversion. The sum of the net amounts that each voter would need to convert to reach their target of $\text{cost}(c)/|N_c|$.

$$\varepsilon = \sum_{i \in N_c} \left|\max\left(0, \frac{\text{cost}[1](c)}{|N_c|} - \mathbf{b}_i[1]\right) - \eta \cdot \max\left(0, \frac{\text{cost}[2](c)}{|N_c|} - \mathbf{b}_i[2]\right)\right|$$

- Number of conversions. The number of voters who need to convert to reach the target of $\text{cost}(c)/|N_c|$.

$$\varepsilon = \left|\left\{i \in N_c : \frac{\text{cost}[1](c)}{|N_c|} - \mathbf{b}_i[1] > 0 \text{ or } \frac{\text{cost}[2](c)}{|N_c|} - \mathbf{b}_i[2] > 0\right\}\right|$$

Furthermore, all the examples could be given as a proportion of cost/number of voters. The benefit of having this would mean that all projects get treated equally.

Example 6.12. Consider a PB scenario with current exchange rate 1.
Consider two projects a and b, with their costs, total of voter budgets, and the gross conversion shown in Table 15.

Project	Cost	$\sum_{i \in N_c} \mathbf{b}_i$	Gross Conversion
a	$(10, 0)$	$(0, 10)$	10
b	$(100, 0)$	$(80, 20)$	20

Table 15: Details of two projects

Project a requires a smaller amount of gross conversion in comparison to project b. However, this makes up 100% of project a's budget, whereas the gross conversion is only 20% of project b's budget. It is easy to see that project b would be preferred as a smaller proportion of the cost needs to be converted.

6.2.2 Selecting projects

In BOS, the project is selected with the smallest value of ρ/α. Since $\alpha < 1$, this increases the fairness score for smaller fractions of the project. This formula penalises smaller values of α, much like how larger values of ε should be penalised. Taking inspiration from this, an initial idea would be to consider projects with the lowest value of $\rho\varepsilon$. Projects which require more exchanging of resources would have a higher fairness score. However, this would mean that projects which do not need resource exchanges cannot be differentiated. Both projects would have $\varepsilon = 0$ and hence $\rho\varepsilon = 0$. Instead, calculate the fairness score by increasing the value of ρ by a factor of ε. Formally, the fairness score would be $\rho(1+\varepsilon)$. The project with the lowest value of $\rho(1+\varepsilon)$ for which it is (ρ,ε)-affordable would be chosen.

Example 6.13. Consider an election with budget $(200, 100)$. Furthermore, suppose that ε is the gross cost that needs to be exchanged as a proportion of the project cost.

Consider the voters' budgets described in Example 6.13. All of whom are voting for a project with cost $(100, 50)$.

1	2	3	4	5
$(25, 8)$	$(25, 8)$	$(30, 8)$	$(20, 10)$	$(20, 6)$

To calculate the minimum value for ρ it is convenient to start by converting everything to the first resource. The exchange rate will be 2.

- price(c)=200

1	2	3	4	5
41	41	30	40	32

The value for ρ will be $\frac{41}{200}$ since all voters will have to spend their entire budget.

To find ε (The proportion of the cost to be exchanged in the first round), calculate the sum of the additional resource needed and convert to resource 1. Each voter needs to try to spend $\frac{1}{5}(100, 50) = (20, 10)$. Voter 1 needs an extra 2 units of the second resource, so would exchange 4 of the first resource. This can be calculated for each of the other voters. So the total amount converted will be:

$$4 + 4 + 4 + 8 = 20$$

So, this project is $(\frac{41}{200}, \frac{1}{10})$-converted-affordable. This project will be selected unless other projects have a fairness score less than $\frac{41}{200}(1 + \frac{1}{10}) = \frac{451}{2000}$.

6.3 Sequential Funding

The exchange rates described above become vital when determining how to fund the project. Initially, the funding will be completed in "funding rounds", where each round the remaining amount that needs to be funded will be shared between voters with non-zero funds. Exchanging between resources is only used as a necessity. This is when the voter cannot afford the project in one of the dimensions. Therefore, voters cannot convert too much during the process.

Consider project selection round t, with budget $(\pounds x_t, \$y_t)$, and project $c \in C$ is selected with voters N_c. Let the number of voters for c be $n_c = |N_c|$. Each voter is aiming to fund $\frac{\text{cost}(c)}{n_c} = (\pounds u, \$v)$.
For each voter $i \in N_c$ the following is done:

(1) (Voter i has enough funds in £ and \$): Voter i spends (u, v) of their budget.

(2) (Voter i cannot afford either £ or \$): Voter i spends b_i.

(3) (Voter i can afford £ but not \$ (or vice versa)): Voter i needs $\$k$ so spends $\pounds\left(\frac{x_t}{y_t} \cdot k\right)$ to get this much.

a. If they can now afford (u, v) spend (u, v).

b. If not, return to original budget and convert as many £s as possible into \$ using the exchange rate $\frac{y_t}{x_t}$ and spend this amount.

If the entire cost of the project has not yet been spent, repeat with further rounds of funding conducted in the same way. This time, the remaining cost that needs to be spent by voters is split evenly between the voters with non-zero budgets. If the new target is not fit, this continues until the project has been funded. Once funded, the new budget is: $(x_{t+1}, y_{t+1}) = (x_t, y_t) - \text{cost}(c)$. It should be verified that the project gets paid for.

Theorem 6.14. Sequential funding terminates having funded the selected project.

Proof. In each funding round, either every voter pays the target amount or at least one voter cannot pay the target amount. In the first case, this means that the project has been fully funded since the target amount is the remaining cost divided by the number of voters still with funds.

Therefore, after at most $|N_c|$ funding rounds, there will be no voters left. If the project has not been fully funded. This means that the voters could not afford to pay for the project, implying that it would not have been selected. Thus, after at most $|N_c|$ rounds, the project has been funded. $\square$

This theorem means that sequential funding is a valid way for voters to pay for the project. To understand how this works, here is example of how a project gets paid for by its supporters.

Example 6.15 (Funding)**.** Consider an election with budget $(£200, \$320)$ and a project whose cost is $(£100, \$160)$. Suppose the voters of the project have the budgets in Table 16.

1	2	3	4	5
$(£15, \$16)$	$(£15, \$10)$	$(£20, \$20)$	$(£8, \$25)$	$(£20, \$25)$
6	7	8	9	10
$(£10, \$16)$	$(£20, \$10)$	$(£10, \$15)$	$(£5, \$25)$	$(£0, \$30)$

Table 16: Voter budgets

To fund this project, first calculate the exchange rates:

- \$ to £: 0.625
- £ to \$: 1.6

In the first round, each voter needs to attempt to spend $(£10, \$16)$ (This is the budget divided by the number of voters). Iterating through each voter gives:

- Voters 1, 3, 5, and 6 have sufficient budget so they each spend $(£10, \$16)$
- Voters 2 and 7 needs \$6 so uses $6 \cdot 0.625 = £3.75$. They can now each spend $(£10, \$16)$.
- Voter 4 needs $£2$ so spends $£2 \cdot 1.6 = \$3.2$. They can now spend $(£10, \$16)$.
- Voter 9 needs $£5$ so spends $£5 \cdot 1.6 = \$8$. They can now spend $(£10, \$16)$.
- Voter 8 needs \$1 so spends $\$1 \cdot 0.625 = £0.625$. They now have $(£9.375, \$16)$ which is not sufficient to spend an even amount so instead they only convert $£0$ into $\$0$ to spend $(£10, \$15)$.

- Voter 10 needs £10 so spends $£10 \cdot 1.6 = \$16$. They now have $(£10, \$14)$ which is not sufficient to spend an even amount so instead they only convert \$14 into £8.75 to spend $(£8.75, \$14)$.

After this round of funding, there is $(£1.25, \$1)$ left. This must be funded by the remaining voters with non-zero budget. Table 17 shows the new budgets of each voter.

1	2	3	4	5
$(£5, \$0)$	$(£1.25, \$0)$	$(£10, \$4)$	$(£0, \$5.8)$	$(£10, \$9)$
6	7	8	9	10
$(£0, \$0)$	$(£6.25, \$0)$	$(£0, \$0)$	$(£0, \$1)$	$(£0, \$0)$

Table 17: Voter budgets

Since voters 6,8 and 10 have zero budget left, the remaining seven voters must evenly fund the remaining cost of the project. Each one needs to fund $(£5/28, \$1/7)$.

To summarise, EES selects the project that can be most evenly shared among its voters weighted by the amount exchanged. Different measures of ε can be used depending on the situation and use of the method. To fund, each voter attempts to afford an even share of the cost. If they cannot do this, they exchange until either they can or get as close to the target as possible. This method only works for two resources because, for three, there is the additional task of working out which resources the voter should exchange. To fix this, a more general method of funding will be defined. Before this, Example 6.16 shows how EES can be used in an election.

Example 6.16. Table 18 shows a PB election.

$\mathbf{b} = (£100k, 1000m^2)$	1	2	3	4	5	6	7	8	9	10
Park: $(£10k, 900m^2)$	x	x	x	x	x	x				
Theatre: $(£90k, 100m^2)$							x	x	x	x
Social Club: $(£11k, 110m^2)$	x						x			

Table 18: Example election to show EES

Each voter receives and equal share of the budget. They each get $(£10k, 100m^2)$. Using MES, the supporters of the park cannot afford the area, and the supporters of the theatre cannot afford the cost. Hence, MES will select the social club only. This outcome does not use much of the budget and excludes all but two voters. The setup of the election suggests that supporters of the park could buy extra land to fund it. This will be what happens using EES.

Assume for simplicity that $\varepsilon \equiv 0$. The exchange rates are:

- $\eta[£ \to m^2] = \frac{1000}{100k} = \frac{1}{100}$.
- $\eta[m^2 \to £] = \frac{100k}{1000} = 100$.

To compute the ρ values, consider the converted election.

$$\begin{aligned}
\beta &= 100,000 + 100 \cdot 1000 = 200,000 \\
\text{price(Park)} &= 10,000 + 100 \cdot 900 = 100,000 \\
\text{price(Theatre)} &= 90,000 + 100 \cdot 100 = 100,000 \\
\text{price(Social Club)} &= 11,000 + 100 \cdot 110 = 22,000
\end{aligned}$$

Each voter has $20,000$ to spend. All three projects can be funded evenly. Hence, the project that gets selected is the park.

To fund evenly, each voter needs to spend $(\frac{10k}{6}, 150)$. Since each voter only has $100m^2$ they each need to convert £$5,000$ into $50m^2$. So, each voter now has (£$5k, 150m^2$) which is sufficient to fund the park evenly.

The new budget for the remaining projects is (£$90k, 100m^2$). The new exchange rates are:

- $\eta[£ \to m^2] = \frac{100}{90k} = \frac{1}{900}$.
- $\eta[m^2 \to £] = \frac{90k}{100} = 900$.

To compute the ρ values, consider the converted election.

$$\begin{aligned}\beta &= 90,000 + 900 \cdot 100 = 180,000\\ \text{price(Theatre)} &= 90,000 + 900 \cdot 100 = 180,000\\ \text{price(Social Club)} &= 11,000 + 900 \cdot 110 = 110,000\end{aligned}$$

Each supporter of the theatre has $100,000$ to spend. It can be funded evenly. Since the social club has fewer votes, it cannot be funded better than the theatre. Hence, the theatre is the one that gets funded. Each voter needs to spend $(22,500, 25)$. They can each convert $\frac{125}{9}m^2$ into $\frac{125}{9} \cdot 900 =$ £$12,500$. SO each voter now has (£$22.5k, \frac{775}{9}m^2$) which is sufficient for each voter to spend the required amount.

The entire budget has now been used up so no more projects can be funded. Hence, the outcome of this election is $\{\text{Park}, \text{Theatre}\}$. This outcome is much better than the MES outcome since it uses the full budget, and every voter has a project they like in the outcome.

6.4 General Funding

Sequential funding is a verbose way to calculate the amount each voter spends. However, the way projects are funded in standard MES could be thought of happening in an analogous way. In each round, a voter tries to spend an even share of the project cost. Thus, this motivates an attempt to convert the verbose method into a simpler one using ρ.

Definition 6.17 (General Funding)**.** To fund a project, the following process can be conducted:

1. Convert all voters' budgets into the first resource to compute β_i.
2. Each supporter of c spends $\min(\beta_i, \text{price}(c) \cdot \rho)$.
3. The budget of each voter is converted back to a vector in $\mathbb{R}^r_{\geq 0}$ using a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^r_{\geq 0}$ such that the following holds:

$$\beta_i = \sum_{j=1}^{r} \eta[j \to 1] \cdot f(\beta_i)[j]$$

As well as being simpler and more efficient to implement, general funding would allow for an extension past two resources. The condition on the function means that it preserves the spending power of the voter. When converting the amount back to multiple resources, they should not gain any extra funds. In MES, a project gets funded by each voter spending $\min(b_i, \rho \cdot \text{cost}(c))$. This is similar to how general funding works, suggesting that this way will be much quicker and may only depend on how easy it is to compute f.

Conjecture 6.18. General funding defines a class of funding methods, each one determined by the function f.

Conjecture 6.19. Sequential funding is a member of this class.

Example 6.20. Let $\beta_i \in \mathbb{R}$ be the converted budgets, $\mathbf{b}_i$ be the budget before funding. Two functions could be:

- $f(\beta_i) = (\beta_i, 0, \ldots, 0)$ - keep converted.
- Let η be the exchange rate. For voter i let $s_i = \max(0, b_i[1] - \rho[1]\text{cost}[1](c))$. So

$$f(\beta_i) = (s_i, \eta(\beta_i - s_i))$$

  This converts the remaining budget to the second resource.

In the example, the function $f(x_i) = (x_i, 0, \ldots, 0)$ never converts the voters' budgets back to multiple resources. Supposing the budget could be converted to resource 1, how does this impact the outcome of the election? This specific case only makes a small, but realistic, change from the previous setup. In reality, some of the resources could be used to get extra of others. For example, spare money could be used to buy extra land. Hence, we would like to see how this affects the outcomes, and whether these resources should be included separately to money.

Theorem 6.21. Let E be a PB election with two resources. Suppose the budget can be converted using the exchange rates $\eta[i \to j]$. Furthermore, let $\varepsilon \equiv 0$. Then, EES computes the same output as MES on input $(N, C, \beta, \text{price}, A)$.

Proof. (Sketch)
This case is where the rule is slightly loosened where projects that cannot be afforded do not get removed before each step.

Also, the function described above is $f(x) = (x, 0, \ldots, 0)$, and the exchange rate between resources and resource 1 is fixed. Since the budgets remain in the first resource, and exchanging is not penalised, the values for ρ will be same in each step as MES and the same projects will be funded. □

This would mean that, under appropriate conditions, there is no benefit to consider a PB election with multiple resources.

Example 6.22. Consider a PB election with resources: money and land usage. Suppose each unit of land costs £η and we are in the scenario above where you can freely convert between resources.

Then, for a project c with $\text{cost}(c) = (£100,000, 3km^2)$ and having land costing $£20,000/km^2$. This would be equivalent to buying the land as part of the project cost. Hence, the project cost could just be £160,000. Single resource MES could be run on such instances.

7 Multi-Bounded Overspending

As defined previously, the method of equal shares with bounded overspending (BOS), is an extension to MES which allows voters some freedom on how much they can overspend on their budget. One problem that this solves is the Helenka paradox [Papasotiropoulos et al., 2024]. As seen in Corollary 5.15 and Example 5.13, Helenka style examples still appear in multi-MES. Since BOS solves it in one dimension, this motivates the interest to look at how it works in higher dimensions. Like multi-MES, the first step is to define a higher dimensional analogue of (α, ρ)-affordability.

Definition 7.1. Let $c \in C$ be a project in a PB election with r resources. Then c is (α, ρ)-affordable if, for each resource i, c is $(\alpha[i], \rho[i])$-affordable in that resource and $(\alpha, \rho) = \underset{(\alpha[i],\rho[i])}{\operatorname{argmin}}(\alpha[i])$.

This definition is similar in setup to Definition 5.3. The value $(\alpha[i], \rho[i])$ is calculated for each resource individually and then an aggregation function is used. Here, the aggregation function is fixed as the α should be the smallest fraction of the project that can be bought by the voters.

Definition 7.2. Given a PB election, the winners using multi-BOS are computed as follows:

1. Split the budget evenly among all voters $\mathbf{b}_i = \frac{\mathbf{b}}{n} = \left(\frac{\mathbf{b}[1]}{n}, \ldots, \frac{\mathbf{b}[r]}{n}\right)$.
2. Calculate $((\alpha[1], \rho[1]), \ldots, (\alpha[r], \rho[r]))$ for each project. Let $(\alpha, \rho) = \underset{(\alpha[i],\rho[i])}{\operatorname{argmin}}(\alpha[i])$.
3. Pick the project with the smallest value of ρ/α.
4. For each resource, each supporter of c spends $\text{cost}[j](c) \cdot \rho[i]$. So has $\max(0, \mathbf{b}_i[j] - \text{cost}[j](c) \cdot \rho[j])$ remaining.
5. Remove unaffordable projects and return to step 2 until no more projects can be afforded.

Example 7.3. Example of using multi-BOS.

Table 19 shows the voters and costs for each project. Also describing how they are distributed.

$\mathbf{b} = (200, 40, 150)$	9 voters	41 voters	50 voters
$c_1 : \text{cost}(c_1) = (50, 10, 90)$	x	x	
$c_2 : \text{cost}(c_2) = (40, 10, 0)$		x	
$c_3 : \text{cost}(c_3) = (50, 30, 60)$			x

Table 19: The setup of the election

The first step of BOS is to evenly split the budget between all the voters. Each voter receives $\mathbf{b}_i = (2, 0.4, 1.5)$. Next, for each project calculate the values $(\alpha[j], \rho[j])$ for each resource.

Project c_1:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$
- Same with the second resource. $\alpha[2] = 1,\ \rho[2] = \frac{1}{50}$
- The 50 voters can only afford 75 where 90 is needed. So, 5/6 of the project can be funded whilst being split evenly. $\alpha[3] = \frac{5}{6},\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the last resource to get:

$$\rho/\alpha = \rho[3]/\alpha[3] = \frac{3}{125} = 0.024$$

Project c_2:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{41}$
- Same with the second resource. $\alpha[2] = 1,\ \rho[2] = \frac{1}{41}$
- Same with the third resource. $\alpha[3] = 1,\ \rho[3] = \frac{1}{41}$

Hence, the value of ρ/α is $\frac{1}{41} \approx 0.2439$

Project c_3:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$

- The 50 voters can only afford 20 where 30 is needed. So, 2/3 of the project can be funded whilst being split evenly. $\alpha[2] = \frac{2}{3},\ \rho[2] = \frac{1}{50}$
- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{3}{100} = 0.03$$

The project with the lowest value is c_1 so each voter funds the project by spending $(1, 0.2, 1.8)$. They each overspend by $(0, 0, 0.3)$.

The first 50 voters now have $(1, 0.2, 0)$ remaining. The same process runs for a second round.

Project c_2:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{41}$
- The voters can only afford 8.2 out of the 10 needed. $\alpha[2] = \frac{41}{50},\ \rho[2] = \frac{1}{41}$
- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{41}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{50}{1681} \approx 0.02974$$

Project c_3:

- The first resource can be split evenly among the supporters. So $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$
- The 50 voters can only afford 20 where 30 is needed. So, 2/3 of the project can be funded whilst being split evenly. $\alpha[2] = \frac{2}{3},\ \rho[2] = \frac{1}{50}$
- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{3}{100} = 0.03$$

The project with the lowest value is c_2 so it gets funded. The algorithm finishes as no more projects can be afforded. The set of winners is $\{c_1, c_2\}$.

The outcome in this example is better than how multi-MES would perform as only c_2 is affordable by its supporters. However, it still isolates half of the voters. When there is a single resource, the voters who overspend no longer have any say in which projects can be selected. However, with multiple resources, if they overspend on one, these voters may still be able to contribute for other resources. Because overspending violates the equal shares, voters should be penalised.

7.1 Penalising Overspending

One way to penalise voters who overspend is to charge them a penalty for doing such. This has been done by reducing the voter's budget by the percentage of the cost they owed, which they overspent on.

Definition 7.4. Split the budgets equally between the voters $\mathbf{b_i} = \mathbf{b}/n$. Pick the project with the lowest value of ρ/α. Now fund the project $c \in C$ as follows.
Let $s_i[j] = b_i[j] - \text{cost}[j](c) \cdot \rho[j]$ be voter i's remaining balance in resource j. If it is negative, then they have overspent. Do one of the following:

- If $s_i[j] = b_i[j]$ then do nothing (voter did not vote for the project).
- Otherwise let D_i be the set of projects voter i overspent on. Change their budget for each $j \in [r]$ as follows:

$$b_i[j] = \max(0, s_i[j]) \cdot \prod_{k \in D_i} \left(1 - \frac{-s_i[k]}{\text{cost}[k](c) \cdot \rho[k]}\right)$$

Repeat until no more projects can be afforded.

This works in the same way as BOS but charges voters who overspend. Each voter has their budget reduced by the percentage of the cost that they overspent on. Hence, a voter who overspends on a large proportion of the cost will end up being charged higher than other voters who have not overspent.

Here, voters who do not overspend on a project are not penalised (since $D_i = \varnothing$) and voters who are get penalised proportional to the fraction of the bill they borrowed for each resource. The term $\frac{-s_i[k]}{\text{cost}[k](c) \cdot \rho[k]}$ is the proportion of the project cost they owed that they overspent on. Each resource is decreased by this percentage.

Example 7.5. Suppose voter i has budget $(10, 10)$ and they fund a project c with the following values:

- $\text{cost}(c) = (30, 40)$
- $\rho[1] = \frac{1}{2}$
- $\rho[2] = \frac{1}{10}$

Then calculate the values of s_i. The voter has overspent on resource 1 and not on resource 2.

- $s_i[1] = 10 - 1 \cdot 30 \cdot \frac{1}{2} = -5$. (Overspent by 5)
- $s_i[2] = 10 - 1 \cdot 40 \cdot \frac{1}{10} = 6$

Hence, $D_i = \{1\}$ and the new budgets can be computed. Since the voter overspent on resource 1, they no longer have any funds of that resource. For resource 2, first deduct the project cost, then remove the fraction that they overspent in resource 1.

- $\mathbf{b}_i[1] = 0$
- $\mathbf{b}_i[2] = 6 \cdot \left(1 - \frac{5}{30 \cdot \frac{1}{2}}\right) = 4$

That is, since they borrowed 1/3 of their share of the project cost, the other budgets get reduced by 1/3.

Example 7.6. Example of using multi-BOS with penalising overspending.

Table 20 shows the voters and costs for each project. Also describing how they are distributed.

$\mathbf{b} = (200, 40, 150)$	9 voters	41 voters	50 voters
$c_1 : \text{cost}(c_1) = (50, 10, 90)$	x	x	
$c_2 : \text{cost}(c_2) = (40, 10, 0)$		x	
$c_3 : \text{cost}(c_3) = (50, 30, 60)$			x

Table 20: The cost and votes for each project

In the same way that multi-BOS runs, first split the budget evenly among the voters. Each voter receives $\mathbf{b}_i = (2, 0.4, 1.5)$. Next, find the (α, ρ) values for each project.

Project c_1:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$
- Same with the second resource. $\alpha[2] = 1,\ \rho[2] = \frac{1}{50}$
- The 50 voters can only afford 75 where 90 is needed. So, 5/6 of the project can be funded whilst being split evenly. $\alpha[3] = \frac{5}{6},\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the last resource to get:

$$\rho/\alpha = \rho[3]/\alpha[3] = \frac{3}{125} = 0.024$$

Project c_2:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{41}$
- Same with the second resource. $\alpha[2] = 1,\ \rho[2] = \frac{1}{41}$
- Same with the third resource. $\alpha[3] = 1,\ \rho[3] = \frac{1}{41}$

Hence, the value of ρ/α is $\frac{1}{41} \approx 0.2439$

Project c_3:

- The first resource can be split evenly among the supporters. $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$
- The 50 voters can only afford 20 where 30 is needed. So, 2/3 of the project can be funded whilst being split evenly. $\alpha[2] = \frac{2}{3},\ \rho[2] = \frac{1}{50}$
- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{3}{100} = 0.03$$

The project with the lowest value is c_1 so each voter funds the project by spending $(1, 0.2, 1.8)$. They each overspend by $(0, 0, 0.3)$.

The first 50 voters now have $(1, 0.2, 0)$ remaining. Since they each overspent on the third resource, each voter gets penalised by a factor of $(1 - \frac{0.3}{1.8}) = \frac{5}{6}$. Therefore, these voters end up with $(1, 0.2, 0) \cdot \frac{5}{6} = \left(\frac{5}{6}, \frac{1}{6}, 0\right)$. The same process runs for a second round.

Project c_2:

- The voters can only afford $\frac{41}{48}$ of the cost in the first resource. $\alpha[1] = \frac{41}{48},\ \rho[1] = \frac{1}{41}$
- The voters can only afford $\frac{41}{6}$ out of the 10 needed. $\alpha[2] = \frac{41}{60},\ \rho[2] = \frac{1}{41}$
- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{41}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{60}{1681} \approx 0.035693$$

Project c_3:

- The first resource can be split evenly among the supporters. So $\alpha[1] = 1,\ \rho[1] = \frac{1}{50}$
- The 50 voters can only afford 20 where 30 is needed. So, 2/3 of the project can be funded whilst being split evenly. $\alpha[2] = \frac{2}{3},\ \rho[2] = \frac{1}{50}$

- The third resource can be split evenly. $\alpha[3] = 1,\ \rho[3] = \frac{1}{50}$

Hence, the value of ρ/α is found by taking the values from the second resource to get:

$$\rho/\alpha = \rho[2]/\alpha[2] = \frac{3}{100} = 0.03$$

The project with the lowest value is c_3 so it gets funded. The algorithm finishes as no more projects can be afforded. The set of winners is $\{c_1, c_3\}$. We can see that this is a better outcome than before since each voter has one project in the outcome.

To summarise, multi-BOS is a direct extension of BOS where the project that gets selected is the one with the lowest value of ρ/α; these are determined by computing $(\rho[j], \alpha[j])$ and picking the one with the smallest value of α. We also saw how voters can still have influence when they overspend, this motivated penalising voters who overspend. Future research could be done to further explore this idea and find a better heuristic to determine how much a voter should be fined for overspending.

8 Proportionality Axioms

One way to evaluate mechanisms is with proportionality axioms. If a rule satisfies a proportionality axiom, that means that the outcome will always be fairly distributed among the voters. In EJR+ (Definition 2.6), if a group of voters agree on a project, then one voter would gain too much utility from it being added. Because MES satisfies EJR+, and our mechanisms are extensions of MES, this is the axiom that we will focus on here by defining four extensions to multiple resources. These will all be the same as EJR+ when there is one resource but will use different properties of the multi-resource model. We will show the differences between the axioms, prove that greedy does not satisfy them, and EES satisfies one when it is in the setting of Theorem 6.21.

Naturally, EJR+ could be extended to multiple dimensions in two ways. First by considering the converted budget and cost of the projects. Alternatively, considering each resource in turn and requiring one or all of the inequalities to be satisfied.

Definition 8.1. Given a PB election, a feasible outcome W satisfies Converted EJR+ (CEJR+) if, for every group $N' \subseteq N$, and $p \in \bigcap_{i \in N'} A_i \setminus W$, there is a voter $i \in N'$ with:

$$\text{price}(A_i \cap W) + \text{price}(p) > \frac{|N'|\beta}{n}$$

Intuitively, consider the PB election where all resources have been converted to the first one and then check if the outcome satisfies EJR+. This is one natural way of extending EJR+ since the cost satisfaction a voter gets with multiple resources could be seen as this converted cost, or price. This axiom then states that, for any cohesive group and project they agree on, adding it to the outcome would give one voter too much satisfaction. Thus, adding that project would not be fair.

Another way to extend EJR+ into higher dimensions is to consider each resource individually.Either all resources should satisfy the inequality or just one resource should.

Definition 8.2. Given a PB election, a feasible outcome W satisfies All-EJR+ if, for all resources $j \in \{1, \ldots, r\}$, for every group $N' \subseteq N$ and $p \in \bigcap_{i \in N'} A_i \setminus W$, there is a voter $i = i(j) \in N'$ (Voter i depends on the resource) with:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|\mathbf{b}[j]}{n}$$

Definition 8.3. Given a PB election, a feasible outcome W satisfies 1-EJR+ if, for some resource j, for every group $N' \subseteq N$ and $p \in \bigcap_{i \in N'} A_i \setminus W$, there is a voter $i \in N'$ with:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|\mathbf{b}[j]}{n}$$

These axioms now consider the satisfaction a voter gains from a project for each resource. Again, it is clear to see how they reduce to EJR+ for one resource. Intuitively, for any resource, we need to check that EJR+ holds. That means, for every cohesive group and project, some voter gets too much satisfaction for the chosen resource. This needs to hold for each resource for All-EJR+ and just one resource for 1-EJR+. It is important to note that the voter gets picked for each resource, so the voter used for group N' in resource 1 may be different to the one used for the same group in resource 2.

The intuition behind these axioms is to check each part using a single resource election and EJR+. Hence, this intuition should be confirmed. Furthermore, this will create a simple way to check if an election satisfies the properties.

Lemma 8.4. Fix an election $E = (N, C, \mathbf{b}, \text{cost}, A)$. The following three equivalences hold for an outcome W:

(1) W satisfies CEJR+ iff the election $E' = (N, C, \beta, \text{price}, A)$ satisfies EJR+.

(2) W satisfies All-EJR+ iff the elections $E_j = (N, C, \mathbf{b}[j], \text{cost}[j], A)$ all satisfy EJR+.

(3) W satisfies 1-EJR+ iff one of the elections $E_j = (N, C, \mathbf{b}[j], \text{cost}[j], A)$ satisfies EJR+.

Proof. The proofs of these statements are all the same so here we will just prove (1). The other two proofs can be found in Lemma A.3 in the appendix.
$(\implies)$ E satisfies CEJR+ means:

$$\text{price}(A_i \cap W) + \text{price}(p) > \frac{|N'|\beta}{n}$$

Since price is the cost function for E' and β is the budget. E' satisfies EJR+.

$(\impliedby)$ E' satisfies EJR+ means:

$$\text{price}(A_i \cap W) + \text{price}(p) > \frac{|N'|\beta}{n}$$

Which is the definition of CEJR+ for election E. □

Corollary 8.5. Checking CEJR+, All-EJR+ and 1-EJR+ can be done in polynomial time.

Proof. Since checking EJR+ is polynomial [Brill and Peters, 2023], we can analyse the runtime of each of these algorithms. Converting an instance to a single resource one can be done in polynomial time followed by a single check of EJR+, so CEJR+ is easy to check.

All-EJR+ and 1-EJR+ both require $O(r)$ uses of an algorithm to check EJR+ so can also be computed in polynomial time. □

This theorem provides easy ways to check all three of these axioms. An oracle to check EJR+ can be used; each of the axioms can be checked with at most r calls to to it.

CEJR+ is unrelated to both All-EJR+ and 1-EJR+ as the following theorems show.

Theorem 8.6. An outcome W satisfying CEJR+ is not a sufficient condition for W to satisfy All-EJR+.

Proof. The proof is by the counterexample given in Table 21 with output set $W = \{b\}$.

$\mathbf{b} = (50, 50)$	98 voters	2 voters
$a : \text{cost}(a) = (50, 49)$	x	
$b : \text{cost}(b) = (1, 0)$		x

Table 21: Counterexample

We need to show that W does satisfy CEJR+ but does not satisfy All-EJR+. This will show that CEJR+ is not sufficient for All-EJR+. To show W satisfies CEJR+, by Lemma 8.4, we need to show that the converted election satisfies EJR+.

$\beta = 100$	98 voters	2 voters
$a : \text{price}(a) = 99$	x	
$b : \text{price}(b) = 1$		x

The MES output on this election is $\{b\}$, since MES satisfies EJR+, this converted election satisfies EJR+. Therefore, it satisfies CEJR+.

To show that W does not satisfy All-EJR+, we need to find a resource, set of voters, and project, such that the inequality does not hold. Consider resource 2. Let N' be the set of supporters of a. For any voter in N' the following holds:

$$\begin{aligned} \text{cost}[2](\{a\} \cap W) + \text{cost}[2](a) &= 0 + 49 \\ &\leq 49 \\ &= \frac{|N'|\mathbf{b}[2]}{n} \end{aligned}$$

Hence, the election does not satisfy All-EJR+. □

Theorem 8.7. An outcome W satisfying CEJR+ is not a sufficient condition for W to satisfy 1-EJR+.

Proof. The proof is by the counterexample given in Table 22 with output set $W = \varnothing$.

$\mathbf{b} = (6, 6)$	3 voters	3 voters
$a : \text{cost}(a) = (3, 5)$	x	
$b : \text{cost}(b) = (5, 3)$		x

Table 22: Counterexample

To show that this is a counterexample, we need to prove that the outcome W satisfies CEJR+ and not 1-EJR+. To show it satisfies CEJR+, we can show that the outcome satisfies EJR+ in the converted instance.

$\beta = 12$	3 voters	3 voters
$a : \text{price}(a) = 8$	x	
$b : \text{price}(b) = 8$		x

The MES output on this election is $\varnothing$, since MES satisfies EJR+, this converted election satisfies EJR+. Therefore, it satisfies CEJR+.

Next, to show that the outcome does not satisfy 1-EJR+, for each resource we need to find a set of voters and a project that fails to satisfy the inequality. Consider each resource in turn and find a set where the inequality fails. Starting with the first resource, let $N' = N_a, p = a$ then $\forall\, i \in N'$:

$$\text{cost}[1](A_i \cap W) + \text{cost}[1](a) = 0 + 3 = 3 \leq \frac{3 \cdot 6}{6}$$

Next, for the second resource, let $N' = N_b, p = b$ then $\forall\, i \in N'$:

$$\text{cost}[2](A_i \cap W) + \text{cost}[2](b) = 0 + 3 = 3 \leq \frac{3 \cdot 6}{6}$$

Hence the outcome does not satisfy 1-EJR+. ☐

Theorem 8.8. An outcome W satisfying All-EJR+ is not a sufficient condition for W to satisfy CEJR+

Proof. The proof is by the counterexample given in Table 23.

$\mathbf{b} = (6, 6)$	1	2	3	4	5	6
$a : \text{cost}(a) = (2, 2)$	x	x	x			
$c_1 : \text{cost}(c_1) = (2, 0)$	x					
$c_2 : \text{cost}(c_2) = (1, 1)$		x				
$c_3 : \text{cost}(c_3) = (0, 2)$			x			

Table 23: When the outcome is $W = \{c_1, c_2, c_3\}$ this is a counterexample.

To fail CEJR+ we need to find a set of voters and a project they all like such that the inequality fails for all voters in the set. Let $N' = \{1, 2, 3\}$ and $p = a \in \bigcap_{i \in N'} A_i \setminus W$.

$$\begin{aligned}
\text{Voter 1: } \text{price}(W \cap A_1) + \text{price}(a) &= \text{price}(c_1) + \text{price}(a) = 2 + 4 \\
&= 6 \leq \frac{3 \cdot 12}{6} \\
\text{Voter 2: } \text{price}(W \cap A_2) + \text{price}(a) &= \text{price}(c_2) + \text{price}(a) = 2 + 4 \\
&= 6 \leq \frac{3 \cdot 12}{6} \\
\text{Voter 3: } \text{price}(W \cap A_3) + \text{price}(a) &= \text{price}(c_2) + \text{price}(a) = 2 + 4 \\
&= 6 \leq \frac{3 \cdot 12}{6}
\end{aligned}$$

Hence, the election fails to satisfy CEJR+.

To satisfy All-EJR+, for every resource, set of voters, and project they agree on we need to find a voter where the inequality holds. There are 6 subsets where $\bigcap A_i \setminus W \neq \varnothing$ and all 6 need to be checked for each resource. These are: $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. For all of these sets, the only project in the intersection of A_is that is not in the outcome is a.

For the singletons, the sum needs to be more than $\frac{1 \cdot 6}{6} = 1$, this is achieved instantly as $\text{cost}[j](a) = 2 > 1$.

Now we consider each set in turn: $N' = \{1, 2\}$

$$\begin{aligned}\text{Resource 1: let } i = 1 \implies \text{cost}[1](A_1 \cap W) + \text{cost}[1](a) &= \text{cost}[1](c_1) + \text{cost}[1](a)\\ &= 2 + 2 = 4 > 2 = \frac{2 \cdot 6}{6}\\ \text{Resource 2: let } i = 2 \implies \text{cost}[2](A_2 \cap W) + \text{cost}[2](a) &= \text{cost}[2](c_2) + \text{cost}[2](a)\\ &= 2 + 1 = 3 > 2 = \frac{2 \cdot 6}{6}\end{aligned}$$

$N' = \{1, 3\}$

$$\begin{aligned}\text{Resource 1: let } i = 1 \implies \text{cost}[1](A_1 \cap W) + \text{cost}[1](a) &= \text{cost}[1](c_1) + \text{cost}[1](a)\\ &= 2 + 2 = 4 > 2 = \frac{2 \cdot 6}{6}\\ \text{Resource 2: let } i = 3 \implies \text{cost}[2](A_3 \cap W) + \text{cost}[2](a) &= \text{cost}[2](c_3) + \text{cost}[2](a)\\ &= 2 + 2 = 4 > 2 = \frac{2 \cdot 6}{6}\end{aligned}$$

$N' = \{2, 3\}$

$$\begin{aligned}\text{Resource 1: let } i = 2 \implies \text{cost}[1](A_2 \cap W) + \text{cost}[1](a) &= \text{cost}[1](c_2) + \text{cost}[1](a)\\ &= 2 + 1 = 3 > 2 = \frac{2 \cdot 6}{6}\\ \text{Resource 2: let } i = 3 \implies \text{cost}[2](A_3 \cap W) + \text{cost}[2](a) &= \text{cost}[2](c_3) + \text{cost}[2](a)\\ &= 2 + 2 = 4 > 2 = \frac{2 \cdot 6}{6}\end{aligned}$$

$N' = \{1, 2, 3\}$

$$\begin{aligned}\text{Resource 1: let } i = 1 \implies \text{cost}[1](A_1 \cap W) + \text{cost}[1](a) &= \text{cost}[1](c_1) + \text{cost}[1](a)\\ &= 2 + 2 = 4 > 3 = \frac{3 \cdot 6}{6}\\ \text{Resource 2: let } i = 3 \implies \text{cost}[2](A_3 \cap W) + \text{cost}[2](a) &= \text{cost}[2](c_3) + \text{cost}[2](a)\\ &= 2 + 2 = 4 > 3 = \frac{3 \cdot 6}{6}\end{aligned}$$

Therefore, for all resources, for all subsets of N' where they agree on at least one project and for all projects p in this intersection that have not been selected, there is a voter $i \in N'$ such that:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|\mathbf{b}[j]}{n}$$

Hence, this outcome satisfies All-EJR+.

Therefore, by means of a counterexample we have shown that satisfying All-EJR+ is not a sufficient condition for satisfying CEJR+. $\square$

These theorems have shown that CEJR+ requires separate conditions to All-EJR+ and 1-EJR+; they do not depend on each other. Hence, they should be tested separately. It also follows that 1-EJR+ does not imply CEJR+ since, otherwise, this would mean that All-EJR+ $\implies$ 1-EJR+ $\implies$ CEJR+ which is a contradiction. This difference between All-EJR+ and CEJR+ motivates the search for a parent of the two. This axiom will join the two together. However, it will be too strong to use since All-EJR+ is difficult to satisfy.

Definition 8.9. Given a PB election, a feasible outcome W satisfies Strong All-EJR+ if, for every group $N' \subseteq N$ and $p \in \bigcap_{i \in N'} A_i \setminus W$, there is a voter $i \in N'$ such that for any resource j:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|\mathbf{b}[j]}{n}$$

All-EJR+ allows for a different voter to satisfy the inequality for each resource. On the other hand, strong All-EJR+ requires the same voter to satisfy the inequality for all resources. This is stronger since the voter must always be the same.

Proposition 8.10. For $r = 1$ EJR+ = CEJR+ = All-EJR+ = 1-EJR+ = Strong All-EJR+

Proof. For one resource, the statements of all the properties collapses to the statement for EJR+. For CEJR+ price = cost when there is a single resource. □

This means that when showing properties do not hold, we can use examples from the single resource mechanism that the rule has been derived from. The motivation for introducing Strong All-EJR+ was as a common ancestor of CEJR+ and All-EJR+. Hence, we need to show that this axiom implies both.

Proposition 8.11. Strong All-EJR+ implies CEJR+. That is, for any election $(N, C, \mathbf{b}, \text{cost}, A)$, if an outcome W satisfies strong All-EJR+ then W satisfies CEJR+.

Proof. Suppose outcome W of election $(N, C, \mathbf{b}, \text{cost}, A)$ satisfies Strong All-EJR+. Fix $N' \subseteq N$, any project $p \in \bigcap_{i \in N'} A_i \setminus W$, let candidate $i \in N'$ such that the following holds for any resource j:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|\mathbf{b}[j]}{n}$$

$$\begin{aligned}
\text{price}(A_i \cap W) + \text{price}(p) &= \sum_{j=1}^{r} \eta[j \to 1]\text{cost}[j](A_i \cap W) + \eta[j \to 1]\text{cost}[j](p) \\
&> \sum_{j=1}^{r} \eta[j \to 1]\frac{|N'|\mathbf{b}[j]}{n} \\
&= \frac{|N'| \sum_{j=1}^{r} \eta[j \to 1]\mathbf{b}[j]}{n} \\
&= \frac{|N'|\beta}{n}
\end{aligned}$$

So, this outcome satisfies CEJR+. □

Proposition 8.12. Strong All-EJR+ implies All-EJR+. That is, for any election $(N, C, \mathbf{b}, \text{cost}, A)$, if an outcome W satisfies strong All-EJR+ then W satisfies All-EJR+.

Proof. Suppose outcome W of election $(N, C, \mathbf{b}, \text{cost}, A)$ satisfies strong All-EJR+. Fix $N' \subseteq N$ and any project $p \in \bigcap_{i \in N'} A_i \setminus W$. Let the candidate $i \in N'$ be such that the following holds for any resource j:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Then, for any resource j and set $N' \subseteq N$, project p and voter i satisfy:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Hence, W satisfies All-EJR+. □

Therefore, these properties sit in a hierarchy:

$$
\begin{tikzcd}
& \text{Strong All-EJR+} \arrow[dl] \arrow[dr] & \\
\text{CEJR+} & & \text{All-EJR+} \arrow[d] \\
& & \text{One-EJR+}
\end{tikzcd}
$$

Figure 4: The relationships between the EJR+ extensions.

Theorem 8.13. There exists an election such that no feasible sets satisfy All-EJR+.

Proof. Table 24 outlines the construction of such an election.

$\mathbf{b} = (100, 100)$	50 voters	50 voters
$c_1 : (100, 1)$	x	
$c_2 : (1, 100)$		x

Table 24: Given this election E, no outcome can satisfy All-EJR+.

The feasible outcomes of this election are $\text{Feas}(E) = \{\varnothing, \{c_1\}, \{c_2\}\}$. An outcome does not satisfy All-EJR+ if:

$$\exists r \; \exists N' \; \exists p \in \bigcap_{i \in N'} A_i \setminus W \quad \forall i \in N' \text{ The inequality holds.}$$

For the outcome $\varnothing$: let $r = 1$ $N' = N_{c_2}$, $p = c_2$ and any voter $i \in N_{c_2}$.

$$\begin{aligned} \text{cost}[1](\varnothing \cap \{c_2\}) + \text{cost}[1](c_2) &= 0 + 1 \\ &\leq \frac{50 \cdot b[1]}{100} \end{aligned}$$

For the outcome $\{c_1\}$: let $r = 1$ $N' = N_{c_2}$, $p = c_2$ and any voter $i \in N_{c_2}$.

$$\begin{aligned} \text{cost}[1](\{c_1\} \cap \{c_2\}) + \text{cost}[1](c_2) &= 0 + 1 \\ &\leq \frac{50 \cdot b[1]}{100} \end{aligned}$$

For the outcome $\{c_2\}$: let $r = 2$ $N' = N_{c_1}$, $p = c_1$ and any voter $i \in N_{c_1}$.

$$\begin{aligned} \text{cost}[2](\{c_2\} \cap \{c_1\}) + \text{cost}[2](c_1) &= 0 + 1 \\ &\leq \frac{50 \cdot b[2]}{100} \end{aligned}$$

Hence, no feasible set satisfies All-EJR+. We have constructed an election that cannot satisfy this axiom. $\square$

Since All-EJR+ cannot be satisfied in general, strong All-EJR+ is too strong to expect it to be able to be satisfied. Apart from tying All-EJR+ with CEJR+, there does not seem to be any additional need to consider this property.

With these different axioms, we can now check whether our mechanisms satisfy these. It should not be expected that a rule satisfies All-EJR+ since not all elections can satisfy it. Because of the strength of CEJR+, it also seems quite difficult to satisfy. However, it will be shown that the relaxation of EES discussed in Section 6.4 does satisfy it.

A voting mechanism satisfies a proportionality axiom if, for any election, the outcome W satisfies the property. To begin, it is easy to show that greedy does not satisfy any of the axioms.

Theorem 8.14. The greedy rule does not satisfy any of these extensions to EJR+.

Proof. Since greedy does not satisfy EJR+ for $r = 1$, by Proposition 8.10 greedy will not satisfy any of these EJR+ extensions. □

This is not a surprise since the performance of the greedy algorithm is the motivation for creating more elaborate mechanisms.

Theorem 8.15. In the setting described by Theorem 6.21, the result of EES satisfies CEJR+.

Proof. CEJR+ states that the selection satisfies EJR+ after being converted to a single resource. Since, in the setting of Theorem 6.21, the output is the same as MES, and MES satisfies EJR+, The result of EES in this setting will satisfy CEJR+. □

As discussed, this relaxation is subtle and may suggest that EES approximately satisfies this. However, this will not be discussed theoretically and is an area for further study.

9 Experimental Results

To understand how these rules work on real life instances, some experiments were run. These experiments measure how well the different rules perform using different metrics. The full results are given in appendix Section B.

9.1 Data

The data has been taken from the participatory budgeting library PabuLib [Faliszewski et al., 2023]. Since only approval instances have been considered, all 745 approval PB elections were used to run experiments on. However, for longer experiments, a smaller set of elections with up to 50 projects was used. To convert a single resource model into one with multiple, the budget was evenly split, and each project's cost was split randomly.

Definition 9.1. Let $(N, C, b, \text{cost}, A)$ be a single resource PB election. Fix $r \in \mathbb{N} \setminus \{0\}$. Let the budget $b' = (\frac{b}{r})_{i=1}^{r}$. For each project, let $p_1 \leq p_2 \leq \cdots \leq p_{r-1}$ be drawn from Uniform$(0, 1)$ and set $\text{cost}'(c) = (p_1 - 0, p_2 - p_1, \ldots, p_{r-1} - p_{r-2}, 1 - p_{r-1}) \cdot \text{cost}(c)$. Then the corresponding PB election with r resources is $(N, C, b', \text{cost}', A)$.

By corollary A.5, the expected cost of each resource for a project is $\frac{\text{cost}(c)}{r}$.

When comparing the metrics against number of projects, to simplify the graphs, the data was split into 5 chunks. Each one containing as close to 20% of the instances as possible. The splits were: 1-8, 9-13, 14-21, 22-38 and 39+. The number of projects ranged from 1 to 163. For this, the number of resources was fixed at $r = 2$ since this allowed EES to be evaluated.

9.2 Code

In Python there is a Participatory budgeting library PabuTools [Faliszewski et al., 2023]. The library contains classes to set up election instances and approval profiles. It also includes functionality to run methods on these instances and analyse the results. Due to restriction on the data type of the budget, a new instance class had to be created. It was made close to the original so that it could be used in tandem with the profile class. All the required multi-resource mechanisms were coded using these classes and a testing framework to run tests on large datasets was made.

9.3 Mechanisms

Three mechanisms were evaluated against several different metrics.

- Greedy.
- Multi-MES.
- EES (2 resources only).

Extensions of BOS were not evaluated due to being significantly slower than the others and time constraints with the project. Furthermore, as an additional goal, it was not necessary to evaluate them. However, this could be an avenue for further research to understand how these extensions perform.

9.4 Runtime

Computing the winners of a PB election should be efficient to compute. One way to compare rules is to compare their runtime. The average runtime is the average time it takes for the rule to compute an outcome for the collection of instances.

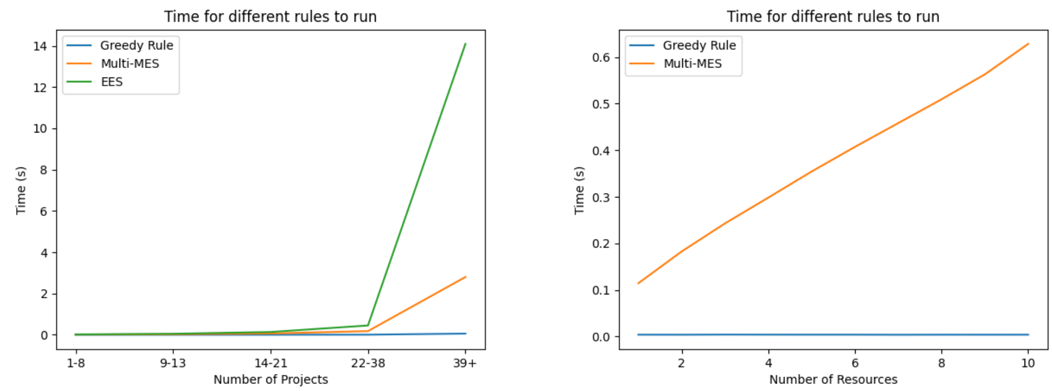


Figure 5: Average runtime of the mechanisms

When comparing the number of projects to runtime, EES runs much slower than the others. This is likely because of the longer way of funding in comparison to MES. EES takes $O(n^2)$ operations to fund the project since, in the worst case, only one voter drops out in each round. Multi-MES, on the other hand, uses $O(n \log n)$ steps to fund the projects. For large numbers of voters, this is a large time saving. The elections with the most voters are also the ones with the most projects. This further exacerbates the time difference since more projects get selected.

As expected, multi-MES is linear in the number of resources whilst greedy looks almost constant. The greedy rule is quicker since it is much simpler than the others.

9.5 Budget Usage

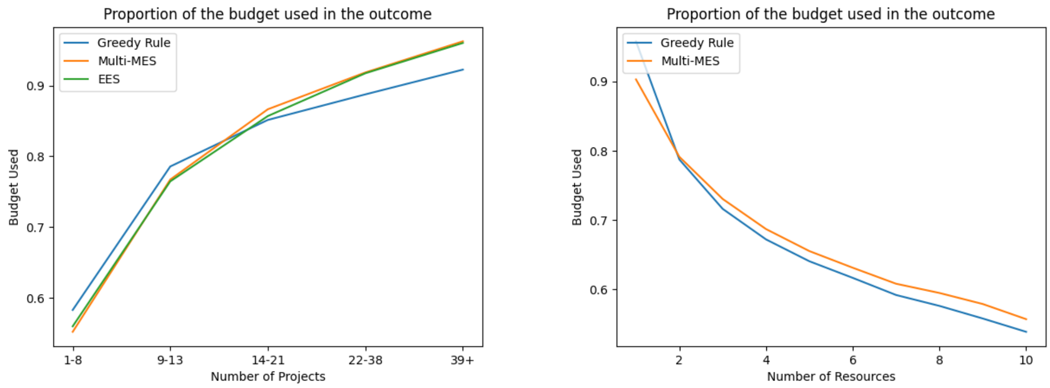


Figure 6: Average proportion of the budget used.

For small numbers of projects, the greedy rule tends to use up more budget than multi-MES and EES. This is likely due to projects being unable to be funded by their supporters. EES and multi-MES use up almost the same amount of budget. They only differ when a project cannot be afforded, and this has a lower value of $\rho(1+\varepsilon)$ than a project that can be afforded. In this graph, MES uses more of the budget but in other tests EES used more of the budget. This discrepancy could be down to the way the project costs are randomly distributed.

For one and two resources, the greedy rule uses more budget on average than multi-MES. This is likely because expensive projects in MES are much less likely to be picked. For more than two resources, it is difficult to fund projects anyway, so the greedy rule starts to pick cheap projects. Here, multi-MES will be picking projects more fairly, so it is unsurprising that more of the budget it used up.

9.6 Exclusion Ratio

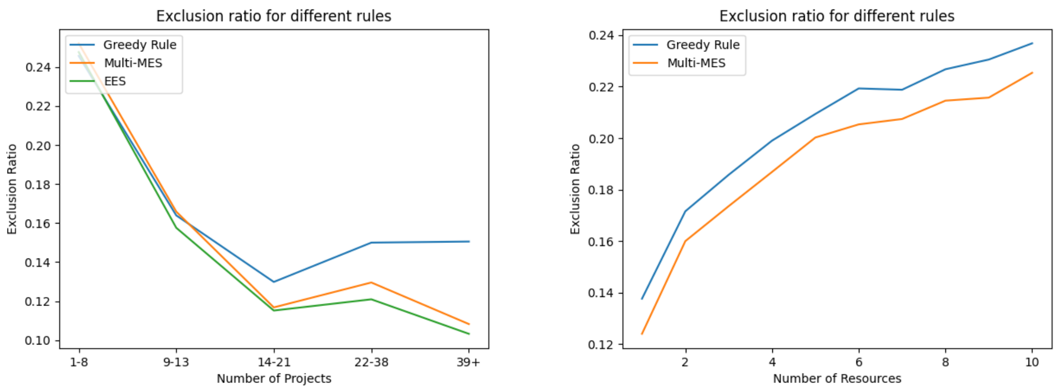


Figure 7: Average exclusion ratio

Across all projects, EES has the lowest average exclusion ratio. Due to the way projects get selected, voters who like projects that will not be selected by multi-MES still have a chance of being included. The greedy rule performs poorly since it can easily only select projects from half of the voters. For the same reason, the greedy rule always has a higher average exclusion ratio than multi-MES for any number of resources.

9.7 EJR+ Violations

An EJR+ violation occurs when a cohesive group of voters does not satisfy the conditions for EJR+. The number of sets of voters which violates the condition effectively quantifies how close mechanisms are to satisfying the axiom. With fewer violations, the better the outcome is despite not satisfying the axiom.

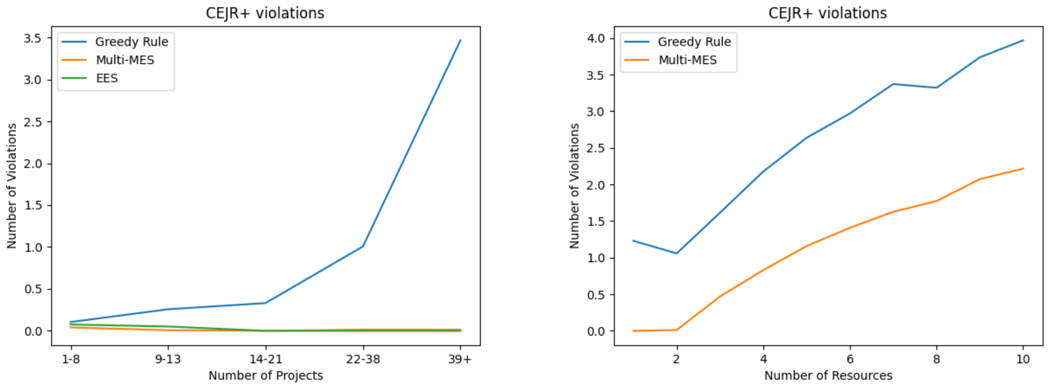


Figure 8: CEJR+ Violations

As expected, the greedy rule does not produce proportional results, no matter number of projects or resources. However, both multi-MES and EES produce almost proportional results. EES does better overall because for 14+ projects there are no violations, whereas multi-MES only has zero violations for the 14-21 category.

Concerning resources, multi-MES stops being almost proportional for more than two resources. This could be because of the way the projects' cost is calculated or because more constraints means projects are harder to fund. It is likely that this will be the same for EES. These two observations have suggested some conjectures about the proportionality of multi-MES and EES. Approximately satisfying an axiom will not be defined here; this could be an area for future research.

Conjecture 9.2. For two resources, multi-MES approximately satisfies CEJR+.

Conjecture 9.3. For two resources, EES approximately satisfies CEJR+.

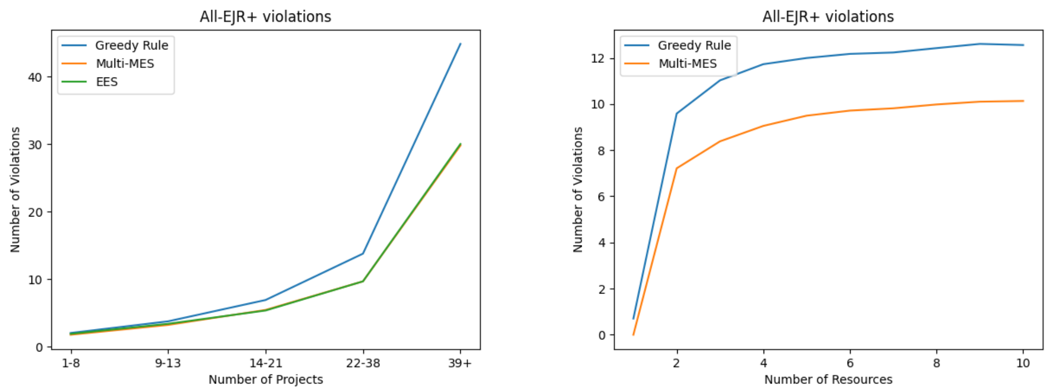


Figure 9: All-EJR+ Violations

None of the three rules satisfy All-EJR+ for any number of projects or resources. This is because All-EJR+ cannot be satisfied in general. The number of violations seems to grow logarithmically with the number of resources. This suggests that the number of violations cannot grow too large for the outcomes of these rules.

9.8 Aggregation Functions

Multi-MES can be parametrised by the ρ-aggregation function discussed in Definition 5.2. One part of deciding how to use multi-MES is to choose such a function to use. The most natural one is to use max, since it would quantify the worst resource that gets shared. However, it is interesting to see how others perform. These tests were done using 5 resources and the aggregation functions max, min, median, and mean.

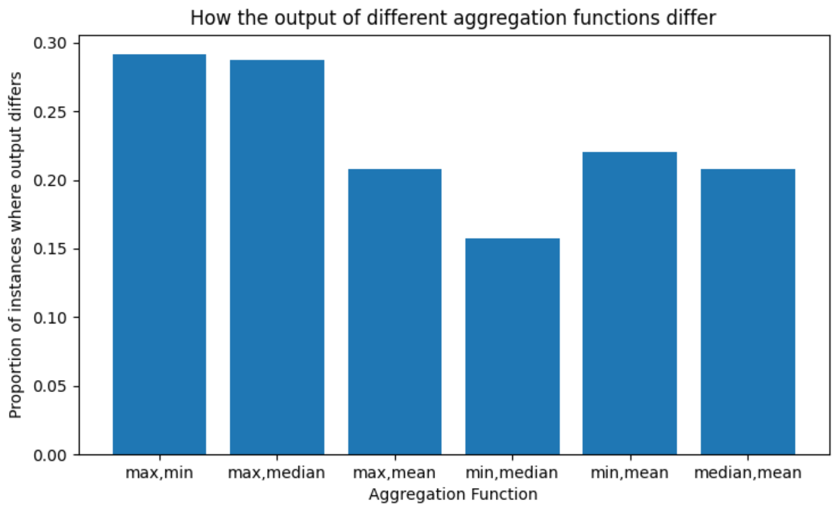


Figure 10: Number of times the aggregation functions differ.

Figure 10 shows the proportion of instances where the outcome differs between the two aggregation functions. Max has the most differences with other functions. Unsurprisingly, most differences occurred with median and min. However, this was only in 30% of cases. The data suggests that changing the aggregation function may not have much of an impact on the outcome of the election. Figure 11 further supports this since all of the aggregation functions have the same exclusion ratio.

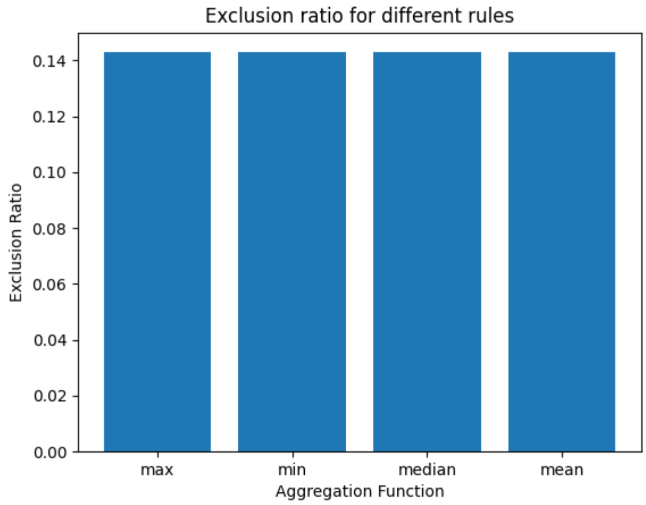


Figure 11: Average exclusion ratio for different aggregation functions.

10 Conclusion

To summarise, this project has explored how participatory budgeting could involve multiple resources. We introduced a model for multi-resource PB elections which has many similarities to the setup in current literature; explored four methods to compute winners, which are all extended from a single resource mechanism; defined some proportionality axioms based on EJR+; and ran a number of experiments to determine the quality of the different mechanisms.

The greedy rule is efficient to compute and is easy to understand. Despite these benefits, it was shown that the outcomes do not satisfy any of the proportionality axioms. Furthermore, in the experiments, greedy rule often uses up less budget and excludes more voters than the other rules defined.

As a first extension of MES, multi-MES is also easy to understand but slower to compute than greedy. Experimentally, multi-MES performed as well as other methods when considering proportionality axioms.

After moving to a setting where voters can exchange between resources, we saw some small improvements in comparison to multi-MES. This setting also produced an equivalence between a relaxation of the multi-resource and the single resource model. This has prompted part of the discussion of Section 10.1.

Motivated by the Helenka paradox, the project briefly explored extensions of MES with bounded overspending (BOS). Multi-BOS was developed along with an extension that charges voters for overspending. This was needed since voters who overspend should be penalised. A topic for additional study could be to further explore this mechanism.

Overall, these mechanisms all have benefits when comparing between complexity, speed, and satisfaction of nice properties as shown in Table 25.

Mechanism	Speed	Complexity	Fairness
Greedy	Fast	Simple	Poor
Multi-MES	Medium	Simple	Good
EES	Slow	Moderately Complex	Very Good
Multi-BOS	Slow	Complicated	Untested

Table 25: A qualitative summary of the different mechanisms. Speed = How fast the outcome can be computed, Complexity = How easy the mechanism is to understand, Fairness = How well the method performed in the tests other than runtime.

One way of looking at how fairly the projects get assigned was to consider proportionality axioms. Since MES satisfies EJR+, this was a good starting point to develop some axioms for multiple resources. We showed that CEJR+ and All-EJR+ are separate but are connected by strong All-EJR+. We also showed that All-EJR+ cannot always be satisfied in an election, implying that strong All-EJR+ is too difficult to satisfy. Hence, we only tested whether elections satisfied CEJR+ and All-EJR+. These strong axioms give an insight into how fair the different rules are. They may satisfy weaker axioms, but this project has conjectured that multi-MES and EES both approximately satisfy CEJR+.

In the future, this research could be used to influence further study into PB mechanisms for multiple resources. The experiments provide some conjectures that could be further investigated. It could also be used to build a piece of software to run PB elections when multiple resources are involved.

10.1 Setting Up An Election

There are a few steps one should take when deciding to set up a PB election with multiple resources. First, decide on the projects. These projects should all be similar in style and benefit the community. Having too many projects or different types could increase the number of constraints too much. Furthermore, the more projects there are, the less likely people will want to partake in the election.

Once the projects have been decided, the next step is to scope them out and decide on physical resource constraints. These resources should only be ones where money cannot be spent to increase the supply. For example, if the council can spend money to buy additional land, then the price per unit should just be included in the monetary resource. However, if planning constraints restrict the land available to build on, then this would be a valid choice of resource. Some other examples could be:

- Cost.
- CO_2 emissions.
- Construction time.

Aside from the physical resources, we can also consider some extra resource constraints. For example, there could be limits on a certain type of project. If there can only be 3 leisure projects, set the budget to be 3 and each leisure project costing 1. Alternatively, two projects may be incompatible. This could be modelled by adding a resource with budget 1. The two projects would each have a cost of 1.

Example 10.1. Consider a set of projects $C = \{a, b, c, d, e, f\}$ where projects a, f cannot be built together; only 2 leisure projects can be built, where a, b, d, e are such projects; and there is a limit on cost and CO_2 emissions. Then the budget of the project could be:

$$(£100,000, 320kg, 2, 1)$$

The resources are: (cost, CO_2, No. Leisure projects, Conflict of a and f).

The costs of the projects:

- $\mathbf{cost}(a) = (£30,000, 40kg, 1, 1)$
- $\mathbf{cost}(b) = (£50,000, 100kg, 1, 0)$
- $\mathbf{cost}(c) = (£18,000, 120kg, 0, 0)$
- $\mathbf{cost}(d) = (£23,000, 45kg, 1, 0)$
- $\mathbf{cost}(e) = (£46,000, 80kg, 1, 0)$
- $\mathbf{cost}(f) = (£10,000, 60kg, 0, 1)$

Next, decide on the voting rule to use. This paper has proposed a few possible mechanisms that could be used to decide the winners. In practice, mechanisms should balance fairness and simplicity. More complicated mechanisms may seem less fair to the average voter, even when they are the fairest. Multi-MES with some kind of completion mechanism should be considered as it is fairly easy and intuitive to understand whilst performing well in many experiments.

Multi-MES could cause an issue much like the Helenka paradox if there is a project conflict resource. This follows from Corollary 5.15 since each project could have cost equal to the budget. To attempt to fix this, the cost in this resource could be set to $0.5 + \delta$ for some small $\delta > 0$. Theorem 5.14 guarantees that the project would need at least half of the voters to support it. Instead of having an extra resource, to further improve the chances of one of the projects being picked, the ballot could restrict voters from selecting both of the conflicting projects. If both end up in the outcome, the one with the fewest votes would be discarded.

Single resource PB elections are often held at community events. Projects are discussed and voters get to understand which projects to vote on. These events have been shown to increase engagement and bring local communities together [O'Hagan et al., 2019]. Such events should be used to run multi-resource elections too. One of the issues mentioned by O'Hagan et al. [2019] was that a lack of accessibility was preventing people from taking part in the events. Hence, an online voting option and stream of the event should also take place for people who are unable to attend.

10.2 Further Study

10.2.1 Negative Project Costs

At this point, most types of constraint have been covered with resources being non-negative. The model currently allows for project conflicts and category limits. However, there are still a few constraints that we cannot model. Namely, dependencies on other projects and targets to hit. These could be satisfied by allowing negative budgets and costs. Since the rules defined here only pick projects whose cost is less than the budget, negative budgets (allowing for targets to be hit) would not work. However, we can consider negative costs.

Definition 10.2 (Participatory Budgeting with Multiple Resources)**.** A PB election is a tuple $(N, C, \mathbf{b}, \text{cost}, A)$ where:

- $N = [n]$ is the set of voters.
- $C = \{c_1, \ldots, c_m\}$ is the set of projects (candidates).
- $\mathbf{b} \in \mathbb{R}^r$ is the budget, where $\mathbf{b} = (\mathbf{b}[1], \ldots, \mathbf{b}[r])$.
- $\text{cost} : C \to \mathbb{R}^r$ is the cost function and $\text{cost}[j]$ is the jth component function. We can assume that no project has a cost of $\mathbf{0}$.

- $A = (A_1, \ldots, A_n)$ is the approval profile where $A_i \subseteq C$ is the set of projects voter i approves of.

The goal is to find a set of projects $W \subseteq C$ such that $\text{cost}[j](W) \leq b[j]$ for all resources $j \in \{1, \ldots, r\}$

Example 10.3. Suppose that, for a tennis court to be built, the park needs building first. This could be modelled with a dependency resource d such that:

- $\mathbf{b}[d] = 0$
- $\text{cost}[d](\text{Park}) = -1$
- $\text{cost}[d](\text{Tennis Court}) = 1$

Then, the only way that the tennis court gets picked is if the park has already been selected, increasing the budget to 1 to allow it to be afforded.

Example 10.4. Suppose there is a resource constraining total CO_2 emissions. A project such as planting trees may contribute some decrease in such resource. In this new model the project would have a negative cost in this resource.

The greedy rule would work in the same way as before. However, every time a project with negative cost is chosen, it should return to the start of the list and fund projects again. This is because the projects that are now affordable have changed, one that could not be funded before may now be affordable.

Example 10.5. Table 26 shows a PB election with negative costs.

$\mathbf{b} = (100, 0)$	1	2	3	4	5	6	7	8	9	10
Park: $(80, -1)$	x	x		x	x		x	x		
Tennis Court: $(20, 1)$	x	x	x	x	x	x			x	x
Cafe: $(20, 0)$	x	x	x	x						x

Table 26: Resources: (Cost, Tennis court-Park dependency)

The greedy rule would order the projects: Tennis Courts, Park, Cafe. The tennis courts cannot be funded so it moves to the park. This can be funded, and the new budget is $(20, 1)$. The algorithm will then start at the beginning of the list again and fund the tennis courts. The budget is used up, so the algorithm ends with $W = \{\text{Park}, \text{Tennis Courts}\}$.

Theorem 10.6. This algorithm looks at $O(n^2)$ projects in the worst case.

Proof. The worst case would occur when selecting projects in reverse order of vote count. Consider an election with n voters, projects, and resources such that:

- $\mathbf{b} = (0, 0, \ldots, 0)$
- $\text{cost}(c_1) = (1, 0, \ldots, 0),\ |N_{c_1}| = n$
- $\text{cost}(c_2) = (-1, 1, 0 \ldots, 0),\ |N_{c_2}| = n - 1$
- $\text{cost}(c_3) = (0, -1, 1, 0, \ldots, 0),\ |N_{c_3}| = n - 2$
- ...
- $\text{cost}(c_{n-1}) = (0, \ldots, -1, 1),\ N_{c_{n-1}} = 1$
- $\text{cost}(c_n) = (0, \ldots, 0, -1),\ N_{c_n} = 0$

The projects will be considered in the order: $c_1, c_2, c_3, \ldots, c_n$. The algorithm will scan the list until reaching the only affordable project c_n. This then repeats as the only affordable project is c_{n-1}. This repeats until all projects get funded after looking at $O(n^2)$ elements in the list. $\square$

Multi-MES would work in the same way as original MES, no changes are needed. Since, when a project is funded, the negative cost would be split evenly and distributed among each voter. These supporters would then have enough budget to fund other projects with positive values of this resource.

Further study can be done in this area to understand how introducing negative costs impacts the properties of the mechanisms. This would be a useful piece of research since it would enrich the PB model further by allowing for more types of resources.

10.2.2 Satisfying Proportionality Axioms

Due to time constraints and the challenging proofs, this project has not proven that any of the mechanisms satisfy CEJR+. Experimentally, we have shown that neither do in general. However, can it be proven that these mechanisms approximately satisfy CEJR+?

10.2.3 Software For Running Elections

A piece of further work could be to develop a framework for easily setting up these elections, gathering results, and calculating the winner using multi-MES or EES.

10.2.4 Testing Extensions of BOS

As mentioned previously, the experiments did not evaluate any extensions of BOS. This could be an area for further study to understand how they perform when extended to multiple resources.

10.3 Evaluation

10.3.1 Completion of Main Objectives

The main objectives for the project were to set up the model, develop some mechanisms, and run experiments on these. Overall, these have been fully completed, however some parts I would have liked to have done differently.

First, setting up the model was smooth. It required the bulk of the research to understand the different notations to use and what decisions to take on how to formulate the model. As I was going to be following many of the results from the paper that introduced MES [Peters et al., 2021], it seemed natural to follow their lead. However, towards the end of the project, it became apparent that using utilities instead of approval sets made defining some of the proportionality axioms tricky. Looking back, I would have started with my current setup as it is easy to understand and more common than using utilities.

Developing different mechanisms was the main aim of the project and has been completed successfully. This was the longest section of the project. It took up most of term 1 and the first part of term 2. This part of the project was successful, however spending so much time looking at BOS may have hampered my progress. This was because extending BOS was an additional objective. Despite this, I was able to create multi-MES and EES. Both were well motivated and with some interesting results and conjectures.

Finally, the second main part of the project was to code the mechanisms and run tests on them. This was successful with many results showing the improvement in performance of the mechanisms in comparison to the naive greedy rule. The PB library PabuTools was used extensively to build the mechanisms and run tests. However, it became apparent quickly that some of the classes defined in the library were not compatible with multiple resource constraints (namely projects and instances). Therefore, significantly more work was required to create the mechanisms as these classes needed to be redefined. Despite this, the code was still finished in time to generate some results for the presentation. Another source of delay came from an unforeseen problem with the DCS machines. The code to test EJR+ violations seemed to create a large memory leak, causing the code to fail when running on the batch compute system. However, after adjusting the code to make it more efficient, it was able to run on a windows PC.

10.3.2 Completion of Additional Objectives

As part of the project, I set out some additional objectives to complete. These were:

- Explore abstract project costs.
- Extend MES with bounded overspending.
- Time-constrained PB.

The first additional objective was completed successfully, and I was able to explore different ways extra resources could be used for different constraints. The ones discussed were to model conflicts between projects and limits on the types of project. This objective inspired asking whether negative project costs could be used to model dependencies. Overall, this objective was quick to complete and could be easily done during the research. The additional objective should have discussed researching a model with negative costs.

The second was started and some research has been done on extending BOS. This objective was interesting to work on as it has some different nuances to consider compared to MES. These have been discussed however not much work was done to see how well these perform. The extensions were coded but they did not end up being used because of time constraints and how much slower they were than MES. In the future, it would be interesting to learn whether these extensions perform better or worse than extensions of MES.

The final extra objective (to explore time constrained PB) was an extension proposed based on the previous title – "Extensions of PB". Hence, after changing the scope of the project to just consider PB with multiple resources, this objective was no longer in the scope. Due to the length of this project, a third additional objective was not needed.

10.3.3 Project Completion

Overall, the project has been completed successfully and has produced some interesting results. Despite the small delays throughout, the project has still been completed within the time. Furthermore, all of the main objectives were achieved alongside two of the additional objectives. The project has been well scoped with sufficient content and challenge. Looking back, creating a small piece of software to run these elections may have been a good fit for a fourth objective. Adding this would have brought the motivation for the project back from theoretical research and into real life application. Despite this, the project has still provided some motivation of why the multi-resource model may be better in certain scenarios; this was important to understand the main goal of the project.

10.4 Project Management and Ethical Considerations

An iterative approach to the research was a good fit as, due to the nature of the project, a lot of time was spent iterating on previous ideas to generate new ideas and build knowledge. This methodology allowed me to make small steps and learn from previous examples. Furthermore, previous work is what inspired many of the experiments to run and the theoretical results. Deciding to keep this methodology for the experiments was a benefit as the work style was able to stay consistent, keeping me on track with the project. Despite the benefits, an agile approach to development would have fit better. This would have allowed me to make lots of smaller gains throughout the project instead of larger sections of the development during term two.

Storing my code on the cloud allowed for easy access between machines in the DCS and my own machine. This was particularly important for running it because most tests were run using the DCS batch system. Due to a memory leak, some of the tests had to be run on my own machine. The same applies for the deliverables. Keeping these on the cloud has kept them safe was easy to access in different places. From a project management perspective, keeping all my work in the same place kept all of my progress close together and easy to access when compiling the final report. Additionally, once sections were completed, they were placed into a first draft of the report. This meant, when writing, most of the technical content had already been completed.

Using the Gantt chart (see Section C) to keep track of my progress was particularly useful to see where I should be at different stages of the project. This kept me on track and allowed for timely completion. Small changes were made towards the end of term two because of the initial stages of the project taking longer than expected. However, since then the project has kept on track with only the coding taking longer than expected.

10.4.1 Legal, Social and Ethical Issues

Due to being a theoretical and research heavy piece of work, no such issues were encountered.

10.4.2 Generative AI Declaration

No artificial intelligence has been used to aid the completion of the project. ChatGPT was used as a final proofread. Suggested changes were limited to spelling, punctuation and some small comments on sentence structure. All changes were made at my own discretion and no changes were made that would alter the project.

10.5 Author's Assessment of the Project

This project contributes new research in PB with multiple resources. It has introduced three new mechanisms (multi-MES, EES, and multi-BOS) and proportionality axioms based off EJR+. Computational social choice is an important and growing field in computer science. This project expands on current ideas in the field and can be used to further understanding of PB with multiple resources. This project could also be used by councils who wish to run PB elections when they are choosing a mechanism to use. Overall, the project has been a success because it has introduced new ideas and proved theorems about the mechanisms. However, the project has been limited by the theoretical gains due to the complexity of mechanisms that work with multiple resources.

Appendix A Additional Proofs

Proposition A.1. Cost satisfaction is a satisfaction measure.

Proof. Recall that the cost satisfaction measure is defined as $\mu(W) = \sum_{c \in W} \text{cost}(c)$, and to be a satisfaction measure the following needs to hold:

- $\mu(W) = 0 \iff W = \varnothing$
- $W \subseteq W' \implies \mu(W) \leq \mu(W')$

$\mu(W) = 0$ iff $\sum_{c \in W} \text{cost}(c) = 0$ iff $\forall c \in W$ $\text{cost}(c) = 0 \vee W = \varnothing$. Since $\text{cost}(c) > 0$ $\forall c$ we have $W = \varnothing$.

Suppose $W \subseteq W'$ then:

$$\begin{aligned}
\mu(W') &= \sum_{c \in W'} \text{cost}(c) \\
&= \sum_{c \in W} \text{cost}(c) + \sum_{c \in W' \setminus W} \text{cost}(c) \\
&\geq \sum_{c \in W} \text{cost}(c) \\
&= \mu(W)
\end{aligned}$$

□

Theorem A.2. Algorithm 2 returns the minimum value ρ for which project c is ρ-affordable in resource j. The algorithm is based on a method to compute ρ values for 1D instances by Peters et al. [2021].

Proof. Let $1, \dots, k$ be the voters who approve of project c such that $b_1[j] \leq b_2[j] \leq \dots \leq b_k[j]$. Let $\rho[j]$ be the output of Algorithm 2, t be the first voter such that $b_t[j] \geq \rho[j]\text{cost}[j](c)$. First, it can easily be verified that the project is ρ-affordable in resource j:

$$\sum_{i \in N_c} \min(b_i[j], \rho[j]\text{cost}[j](c)) = \sum_{s=1}^{t-1} b_s[j] + \sum_{s=t}^{k} \rho \text{cost}[j](c)$$

Since $\rho = \frac{p}{v \cdot \text{cost}[j](c)}$, $p = \text{cost}[j](c) - \sum_{s=1}^{t-1} b_s[j]$, and $v = k - (t-1)$:

$$\begin{aligned}
(1) &= (\text{cost}[j](c) - p) + (k - t + 1)\frac{p}{(k-t+1)\text{cost}[j](c)}\text{cost}[j](c) \\
&= \text{cost}[j](c)
\end{aligned}$$

Now to show this is the minimum one, suppose there exists some $\rho' < \rho$ such that c is ρ'-affordable in resource j. Then the number of voters remaining must have increased. Let $t' < t$ be the first voter who can afford $\text{cost}[j](c)\rho'$. However, once the algorithm reaches voter t', since they can afford to share a ρ' fraction of the cost, they should be able to spend p/v. This contradicts the output of the algorithm. □

Lemma A.3. Fix an election $E = (N, C, \mathbf{b}, \text{cost}, A)$. The following three equivalences hold for an outcome W:

(1) W satisfies CEJR+ iff the election $E' = (N, C, \beta, \text{price}, A)$ satisfies EJR+.

(2) W satisfies All-EJR+ iff the elections $E_j = (N, C, \mathbf{b}[j], \text{cost}[j], A)$ all satisfy EJR+.

(3) W satisfies 1-EJR+ iff one of the elections $E_j = (N, C, \mathbf{b}[j], \text{cost}[j], A)$ satisfies EJR+.

Proof. **Proof of (1):**
$(\implies)$ E satisfies CEJR+ means:

$$\text{price}(A_i \cap W) + \text{price}(p) > \frac{|N'|\beta}{n}$$

Since price is the cost function for E' and β is the budget. E' satisfies EJR+.

$(\impliedby)$ E' satisfies EJR+ means:

$$\text{price}(A_i \cap W) + \text{price}(p) > \frac{|N'|\beta}{n}$$

Which is the definition of CEJR+ for election E.

Proof of (2):
$(\implies)$ E satisfies All-EJR+ means:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Since cost is the cost function for $E[j]$ and $b[j]$ is the budget. $E[j]$ satisfies EJR+. Since j was arbitrary they all satisfy EJR+.

$(\impliedby)$ $E[j]$ satisfies EJR+ means for all N' and project $p \in \bigcap_{i \in N'} A_i \setminus W$ there is a voter $i \in N'$ such that:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Since this holds for any resource j, this is exactly the definition of All-EJR+.

Proof of (3):
$(\implies)$ E satisfies 1-EJR+ means that for some resource j:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Since cost is the cost function for $E[j]$ and $b[j]$ is the budget. $E[j]$ satisfies EJR+.

$(\impliedby)$ $E[j]$ satisfies EJR+ means for all N' and project $p \in \bigcap_{i \in N'} A_i \setminus W$ there is a voter $i \in N'$ such that:

$$\text{cost}[j](A_i \cap W) + \text{cost}[j](p) > \frac{|N'|b[j]}{n}$$

Since this holds for some resource j, this is exactly the definition of 1-EJR+.

□

Lemma A.4. For n points $X_1, \ldots, X_n$ drawn from Uniform$(0, 1)$. The expected length of the $n + 1$ intervals is $\frac{1}{n+1}$

Proof. Since each segment will be the same, consider the expected distance of the maximum point to 1. $1 - \mathbb{E}[\max(X_i)]$.

$$\begin{aligned}
\mathbb{P}(\max(X_i) < x) &= \mathbb{P}(X_1 < x, \ldots, X_n < x) \\
&= \mathbb{P}(X_1 < x)^n \\
&= x^n
\end{aligned}$$

So, the probability density function is $f(x) = nx^{n-1}$ since $\int_0^x f(t)dt = [t^n]_0^x = \mathbb{P}(X < x)$ by the definition of a PDF. Hence:

$$\begin{aligned}\mathbb{E}[\max(X_i)] &= \int_0^1 xf(x)\,dx \\ &= \int_0^1 nx^n\,dx \\ &= \left[\frac{n}{n+1}x^{n+1}\right]_0^1 \\ &= \frac{n}{n+1}\end{aligned}$$

Hence, the expected length of the interval closest to 1 is $\frac{1}{n+1}$. Since the distribution is uniform, this is the expected length of any interval. □

Corollary A.5. The expected value of $\text{cost}'[i](c)$ is $\frac{\text{cost}(c)}{r}$.

Proof.

$$\begin{aligned}\mathbb{E}\left[\text{cost}'[i](c)\right] &= \mathbb{E}\left[\text{cost}(c)\cdot\lambda([p_{i-1}, p_i])\right] \\ &= \text{cost}(c)\mathbb{E}\left[\lambda([p_{i-1}, p_i])\right] \\ &= \text{cost}(c)\cdot\frac{1}{r}\end{aligned}$$

□

Appendix B Tables of Results

The following tables are the results of the experiments used in the graphs in Section 9 to 4 decimal places.

Runtime

Rule	1-8	9-13	14-21	22-38	39+
Greedy	0.0005	0.0011	0.0018	0.0041	0.0567
Multi-MES	0.0106	0.0244	0.0618	0.1764	2.7980
EES	0.0181	0.0478	0.1331	0.4493	14.0914

Table 27: Projects

Rule	1	2	3	4	5	6	7	8	9	10
Greedy	0.0040	0.0040	0.0042	0.0041	0.0041	0.0041	0.0039	0.0040	0.0040	0.00040
Multi-MES	0.1146	0.1827	0.2430	0.2984	0.3544	0.4073	0.4584	0.5095	0.5630	0.6284

Table 28: Resources

Budget Usage

Rule	1-8	9-13	14-21	22-38	39+
Greedy	0.5832	0.7857	0.8514	0.8876	0.9225
Multi-MES	0.5523	0.7673	0.8665	0.9185	0.9625
EES	0.5601	0.7648	0.8569	0.9174	0.9602

Table 29: Projects

Rule	1	2	3	4	5	6	7	8	9	10
Greedy	0.9581	0.7875	0.7161	0.6720	0.6404	0.6164	0.5917	0.5759	0.5577	0.5384
Multi-MES	0.9031	0.7913	0.7304	0.6869	0.6551	0.6311	0.6079	0.5946	0.5787	0.5568

Table 30: Resources

Exclusion Ratio

Rule	1-8	9-13	14-21	22-38	39+
Greedy	0.2457	0.1640	0.1280	0.1500	0.1506
Multi-MES	0.2521	0.1659	0.1168	0.1296	0.1083
EES	0.2476	0.1576	0.1152	0.1210	0.1033

Table 31: Projects

Rule	1	2	3	4	5	6	7	8	9	10
Greedy	0.1376	0.1716	0.1858	0.1990	0.2094	0.2193	0.2188	0.2267	0.2305	0.2368
Multi-MES	0.1240	0.1600	0.1736	0.1869	0.2002	0.2053	0.2074	0.2146	0.2157	0.2254

Table 32: Resources

CEJR+ Violations

Rule	1-8	9-13	14-21	22-38	39+
Greedy	0.1047	0.2574	0.3311	1.007	3.466
Multi-MES	0.0407	0.0073	0.0000	0.0145	0.0135
EES	0.0756	0.0515	0.0000	0.0000	0.0000

Table 33: Projects

Rule	1	2	3	4	5	6	7	8	9	10
Greedy	1.2295	1.05778	1.6134	2.1772	2.6376	2.9705	3.3718	3.3208	3.7369	3.9691
Multi-MES	0.0000	0.0107	0.4685	0.8295	1.1584	1.4081	1.6255	1.7732	2.0725	2.2161

Table 34: Resources

All-EJR+ Violations

Rule	1-8	9-13	14-21	22-38	39+
Greedy	2.0523	3.7794	6.9338	13.7971	44.8446
Multi-MES	1.7907	3.2426	5.4636	9.6957	29.8041
EES	1.9244	3.4118	5.3775	9.6957	30.0270

Table 35: Projects

Rule	1	2	3	4	5	6	7	8	9	10
Greedy	0.7022	9.5856	11.0245	11.7252	11.9928	12.1712	12.2317	12.4216	12.6043	12.5554
Multi-MES	0.0000	7.2129	8.3813	9.0504	9.4935	9.7137	9.8129	9.9799	10.1007	10.1309

Table 36: Resources

Aggregation Functions

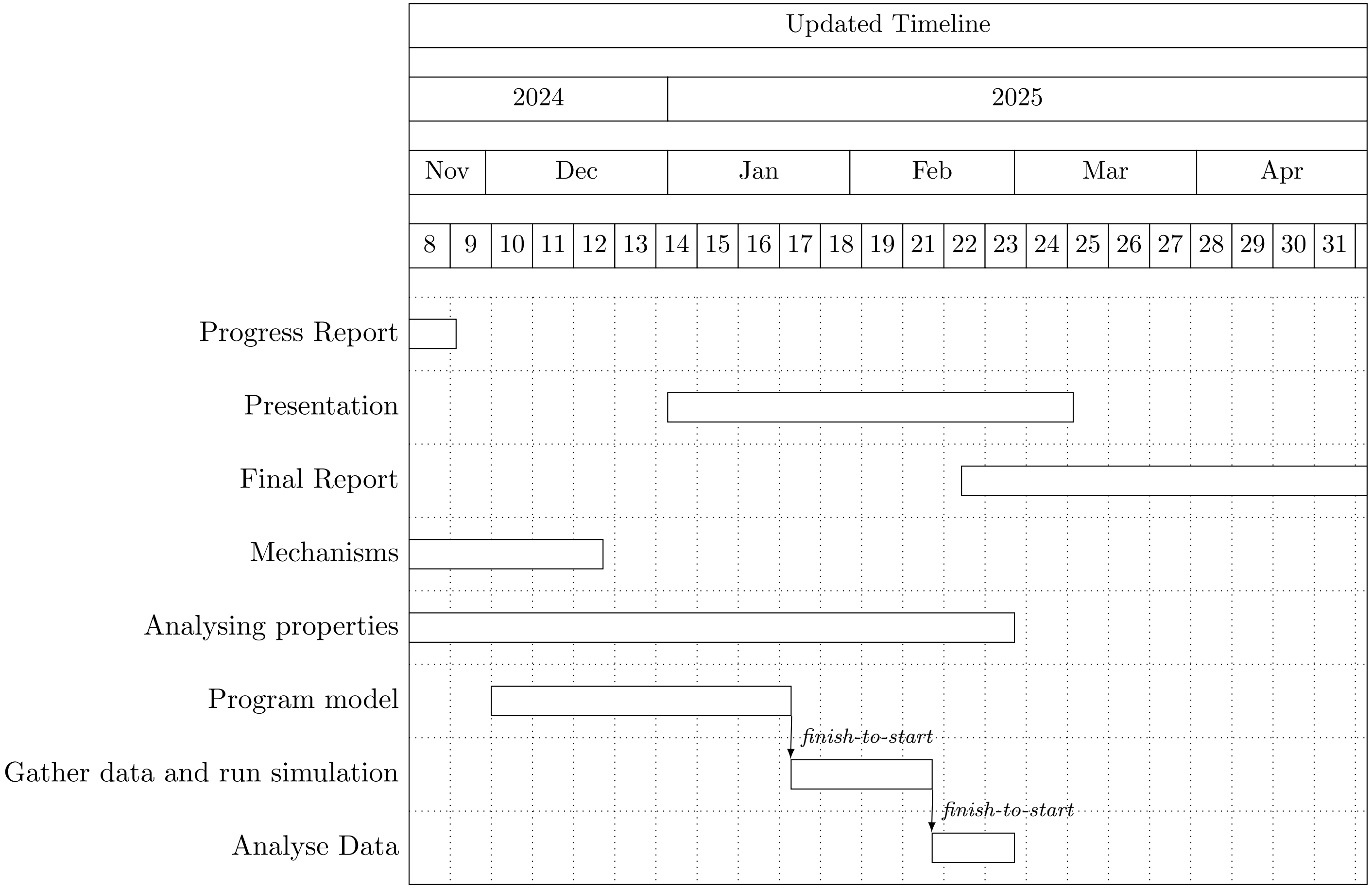
	max	min	median	mean
max	0	-	-	-
min	217	0	-	-
median	214	117	0	-
mean	155	164	155	0

Table 37: Average number of elections where the aggregation functions differ (out of 745)

Function	Exclusion Ratio
max	0.1429
min	0.1429
median	0.1429
mean	0.1429

Table 38: Other results using aggregation functions

Appendix C Project Timeline



References

Brian Wampler. *A guide to participatory budgeting.* International Budget Partnership, 2000.

Michiel S De Vries, Juraj Nemec, and David Špaček. International trends in participatory budgeting. *Cham: Palgrave Macmillan*, 2022.

Angela O'Hagan, Clementine Hill OConnor, Claire MacRae, and Paul Teedon. *Evaluation of Participatory Budgeting in Scotland 2016-2018.* Scottish Government, 2019.

Gianluca Sgueo. Participatory budgeting: An innovative approach. *Available at SSRN 2712213*, 2016.

Markus Brill and Jannik Peters. Robust and verifiable proportionality axioms for multiwinner voting. In *Proceedings of the 24th ACM Conference on Economics and Computation*, page 301, 2023.

Georgios Papasotiropoulos, Seyedeh Zeinab Pishbin, Oskar Skibski, Piotr Skowron, and Tomasz Was. Method of equal shares with bounded overspending. *arXiv preprint arXiv:2409.15005*, 2024.

Simon Rey, Ulle Endriss, and Ronald de Haan. A general framework for participatory budgeting with additional constraints. *Social Choice and Welfare*, pages 1–37, 2023.

Nima Motamed, Arie Soeteman, Simon Rey, and Ulle Endriss. Participatory budgeting with multiple resources. In *European Conference on Multi-Agent Systems*, pages 330–347. Springer, 2022.

Dominik Peters, Grzegorz Pierczyński, and Piotr Skowron. Proportional participatory budgeting with additive utilities. *Advances in Neural Information Processing Systems*, 34:12726–12737, 2021.

Markus Brill, Stefan Forster, Martin Lackner, Jan Maly, and Jannik Peters. Proportionality in approval-based participatory budgeting. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 37, pages 5524–5531, 2023.

Piotr Faliszewski, Jarosław Flis, Dominik Peters, Grzegorz Pierczyński, Piotr Skowron, Dariusz Stolicki, Stanisław Szufa, and Nimrod Talmon. Participatory budgeting: Data, tools and analysis. In *32nd International Joint Conference on Artificial Intelligence (IJCAI-23)*, pages 2667–2674. International Joint Conferences on Artificial Intelligence Organization, 2023.