

Exercise 7

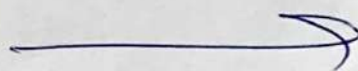
↳ Function $f(x, y) = (x-a)^2 + (y-b)^2 + xy$ is C^∞

↳ The restriction in this case is a compact square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$.

↳ Given that this is a compact square, a minimum (and maximum) must be found. They will be global and inside this square. That's what the theorem of Weierstrass tells us.

↳ Those extremes could be:

1. Inside the square. Then ∇f would be 0 on that local(s) extreme(s).
2. On the edges of the square. Then we can use Lagrange or design an auxiliary function with one variable and compute its derivative and $= 0$.
3. On the vertices of the square.



continues

↳ Case 1:

$$\nabla F(p_0) = (2(x-a)+y, 2(y-b)+x)^T$$

$$\nabla F(p_0) = 0 \Rightarrow p_1 = \left(\frac{2}{3}(2a-b), \frac{2}{3}(2b-a) \right)$$

Depending on a and b , it might be inside or outside our square.

↳ Case 2:

i) On the edge $x=0$, we take $g_1(z) = f(0,z) = a^2 + (z-b)^2$; $g_1(z) = 2(z-b)$ which is zero when $z=b$. Then $p_2 = (0, b)$

ii) On the edge $x=1$, we take $g_2(z) = f(1,z) = (1-a)^2 + (z-b)^2 + y$; $g_2(z) = 2(z-b)+1$
 $g_2'(z) = 0 \Leftrightarrow z = \frac{2b-1}{2} \Rightarrow p_4 = \left(1, \frac{2b-1}{2} \right)$

iii) On the edge $y=0$, for symmetry,
 $p_3 = (a, 0)$

iv) " " " $y=1$, " " $p_5 = \left(\frac{2a-1}{2}, 1 \right)$

→
continues

L₃ (case 3):

$$P_6 = (0, 0), \quad P_7 = (1, 0), \quad P_8 = (0, 1), \quad P_9 = (1, 1)$$

→ Let's evaluate this points on f . The one will be the minimum.

$$f(P_1) = \frac{1}{3} (4ab - a^2 - b^2)$$

$$f(P_2) = a^2 \quad f(P_3) = b^2$$

$$f(P_4) = (1-a)^2 + \frac{1}{4} + \frac{2b-1}{2}$$

$$f(P_5) = (1-b)^2 + \frac{1}{4} + \frac{2a-1}{2}$$

$$f(P_6) = a^2 + b^2$$

P_6 will never be minimum because $f(P_6) > f(P_2)$

and $> f(P_3)$. If $a=b=0$ then $P_6 = P_2 = P_3$;

in this case this point will be the minimum

because $f(x, y) \geq 0$ always.

$$f(P_7) = (1-a)^2 + b^2$$

P_7 won't be minimum, $f(P_7) > f(P_3)$. If $a=1$;

when $P_7 = P_3$

→
continue

$$f(p_8) = a^2 + (1-b)^2$$

↳ Same, $f(p_8) > f(p_2)$. When $b = 1$ then

$$p_8 = p_2.$$

$$f(p_9) = (1-a)^2 + (1-b)^2 + 1$$

Example

↳ So we take points 1, 2, 3, 4, 5 and 9.

$$\text{↳ } a = b = 1$$

$p_1 = (2/3, 2/3)$ which is inside our square

$$f(p_1) = 2/3; \quad f(p_2) = f(p_3) = f(p_8) = 1 > 2/3;$$

$$f(p_4) = f(p_5) = 3/4 > 2/3.$$

$$\text{↳ } a = 0$$

$$p_2 = (0, 1); \quad f(p_2) = 0 \quad \text{and} \quad f(x, y) > 0 \quad \forall (x, y)$$

↳ Symmetry, $b = 0$ then p_3 is the minimum.

↳ $a = b = 2$; p_4 and p_5 are the minimum:

$$f(p_4) = f(p_5) = 11/4; \quad f(p_2) = f(p_3) = 4 > 11/4 \quad f(p_9) = 3 > 11/4$$

$f(p_1) \notin \text{square}$

↳ $a = b = 3$; p_9 is the minimum.

$f(p_9) = 9$; p_1, p_2, p_3, p_4, p_5 not in the square.