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Exercise 5.

↳ In the wikipedia I found ~~that~~ a linear system can be expressed like this function:

$$f(x) = \frac{1}{2} \cdot x^T \cdot Ax - x^T b$$

↳ Let's compute it:

$$f(x) = \frac{1}{2} \cdot (x, y, z) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} - (x\ y\ z) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \frac{1}{2} x^2 + y^2 + \frac{3}{2} z^2 - x - y - z$$

↳ Why this system can be expressed like this function?

Because finding the minimum of this function means solving the system:

$$f(x) = (x-1, 2y-1, 3z-1)$$

↳ It's gradient equals my system!

$$\nabla f(x) = Ax - b$$

→

continue

↳ Let's proceed computing z_2:

$$\nabla F(x_1) = \left(0, \, 2/5, \, -3/5\right)^T$$

$$z_2 = -\nabla F(x_1) + \frac{(\nabla F(x_1))^T \cdot \nabla F(x_1)}{(\nabla F(x_0))^T \cdot \nabla F(x_0)} \cdot z_1 =$$

$$= -\begin{pmatrix} 0 \\ 2/5 \\ -3/5 \end{pmatrix} + \frac{13/25}{13} \cdot \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -13/25 \\ 13/25 \end{pmatrix}$$

↳ And my following α:

$$\alpha_2^* = \frac{(0 \;\; -13/25 \;\; 13/25) \cdot \begin{pmatrix} 0 \\ 2/5 \\ -3/5 \end{pmatrix}}{\frac{5 \cdot 13^2}{25^2}} = \frac{5}{13}$$

↳ To find my second x:

$$x_2 = x_1 + \alpha_2 \cdot z_2 = \begin{pmatrix} 1 \\ 7/10 \\ 2/15 \end{pmatrix} + \frac{5}{13} \begin{pmatrix} 0 \\ -13/25 \\ 13/25 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix}$$

$\longrightarrow$

continues

↳ How do I know that I'm finding a minimum?

Because:

$$\mathcal{H} f = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} = A$$

↳ Where each eigen value are > 0. In fact, it's a quadratic function.

↳ Let's start the algorithm:

$$x_0 = (1, 2, 1)^T \qquad \nabla f(x_0) = (1-1,\ 2\cdot 3-1,\ 3\cdot 1-1)^T$$

↳ Then

$$z_1 = -\nabla f(x_0) = (0, -3, -2)^T$$

↳ And α:

$$\alpha_1^* = -\frac{z_1^T \cdot z_1}{z_1^T \cdot A \cdot z_1} = \frac{13}{(0,-3,-2)\cdot\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix}} = \frac{13}{30}$$

↳ Now I can find my x_1:

$$x_1 = x_0 + \alpha_1^* \cdot z_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{13}{30} \cdot \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7/10 \\ 2/15 \end{pmatrix}$$

↳ And repeat the process n times in this case, $\mathbb{R}^3$, $n=3$.

→

continues

↳ looks like we found the solution of the system in just 2 "moves"! We are lucky.
How do we know? Because:

$$\nabla f(x_2) = (0, 0, 0)^T$$

and $z_3 = (0, 0, 0)^T \Rightarrow x_3 = x_2$