Congrats, you're now the manager of the bubbly water factory. In the biz they call it soda water, but I'm sure you know that already. Currently the factory is running at 0% capacity. It is your job to get it up and running!

Question 1: Factory Set Up

The factory has machines that do three tasks. The three tasks are bottling, bubbling and capping, defined by the functions below. The input, x, for each function is the volume of plastic inputted into the machine. The output is the final volume of plastic once that step has been completed.

$bottle(x) = \sqrt{x-1}$

$bubble(x) = -x^2 + 10$

$cap(x) = x + 1$

a) Draw basic diagram showing how the factory works. The order of functions to be completed is $bottle \implies bubble \implies cap$.

b) Annotate the diagram to include the domain and range of each function. Remember that both the input and output have to be positive numbers as they represent volume.

c) Convert the diagram to function notation.
e.g. In the form of $f(g(h(x)))$

d) Convert your previous function to composite notation.
e.g. In the form of $(f \circ g \circ h)(x)$

e) To check that the factory is working correctly, make sure that an input of 2 outputs 10.

Question 2: A Terrible Mistake

For some inputs, the bottling works as normal, but it fails when it comes to bubbling. This is very concerning.

a) $bottle(101) = 10$ works, but when we feed 10 to *bubble* function we get -90 which is not allowed as it is a negative number.
Find another input that is not allowed even though it can be fed into the bottle function.

b) Restrict the domain of $bottle(x)$ so that all inputs in the set are allowed.

Question 3: The Inverse Factory

With the factory running at full capacity a cunning rival has decided to flip your business model, creating an inverse factory. This inverse factory is the inverse function of the composite function you created in question 1 and 2.

a) Without any math definitions, what do you think that this new factory does?

b) What is the domain (inputs) of this inverse factory?
Find the set of inputs and the real world equivalent of that set.

c) What is the range (outputs) of this inverse factory?
Find the set of outputs and the real world equivalent of that set.

d) Let the composite factory function equal $F(x)$.
If $F(7.5) = 4.5$, what does $F^{-1}(4.5)$ equal?

e) The two factories decide to work together.
What does $F(F(F^{-1}(5)))$ equal?

f) The unity didn't last and the original factory decides to make a factory that feeds into the next factory. The two factories look like this $F(F(x))$. This is quite long so it can be written as $F^2(x)$.
What does $F^2(7)$ equal?

g) What does $F^{10}(7)$ equal?