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MLP Algebraic Expansion and Simplification:

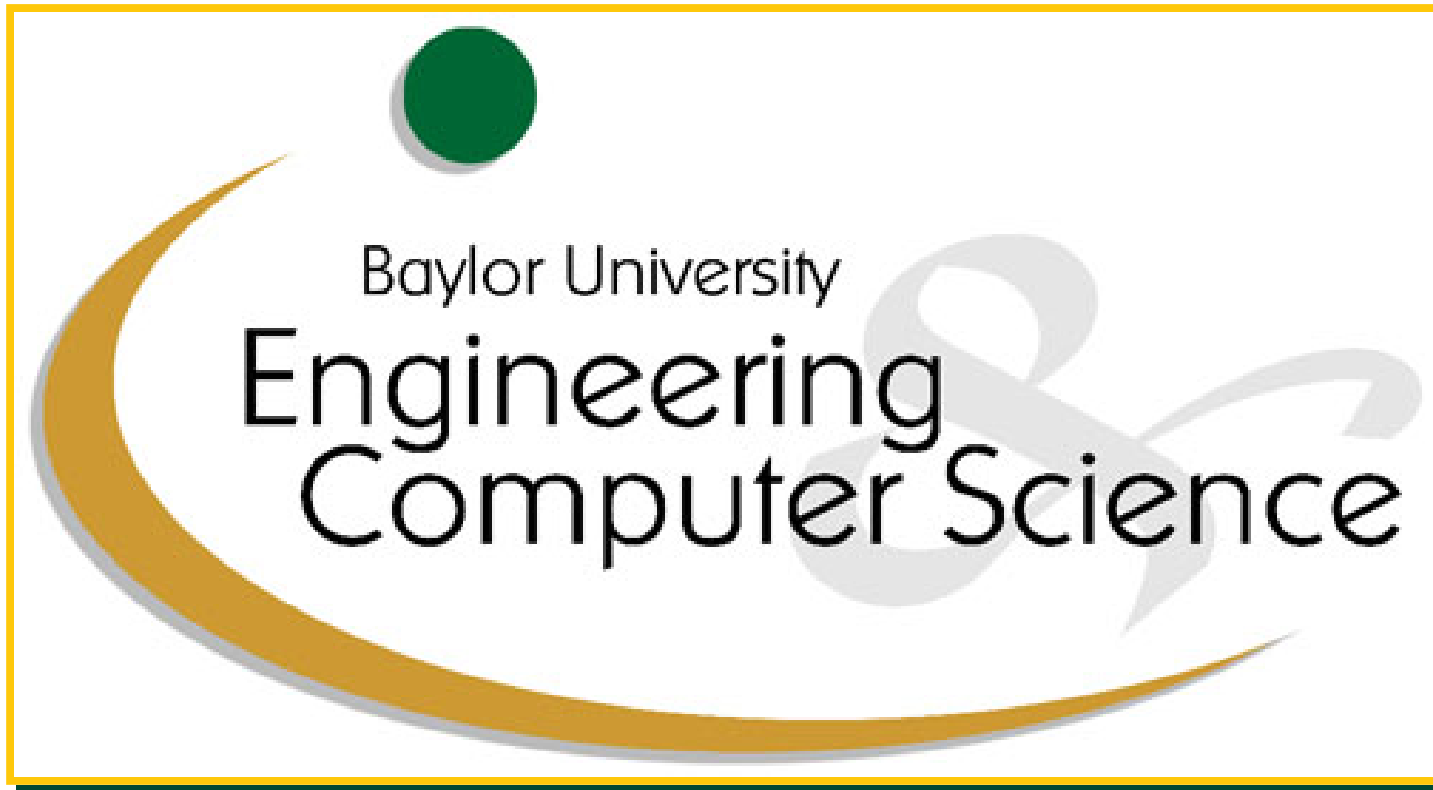
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Reducing the computational error and increasing computational efficiency,

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$$\begin{aligned}
\frac{1}{\beta^2(u)p^2(u)} &= a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5 \\
\int_{u_0}^{u_1} \frac{(u_1 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_0}^{u_1} (u_1 - u)^2 (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) du \\
&= \frac{1}{X_0} \int_{u_0}^{u_1} (u_1^2 - 2u_1u + u^2) (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) du \\
&= \frac{1}{X_0} \int_{u_0}^{u_1} \left[ u_1^2 (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) - 2u_1u (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) + u^2 (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) \right] du \\
&= \frac{1}{X_0} \int_{u_0}^{u_1} \left[ u_1^2 (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) - 2u_1 (a_0u + a_1u^2 + a_2u^3 + a_3u^4 + a_4u^5 + a_5u^6) + (a_0u^2 + a_1u^3 + a_2u^4 + a_3u^5 + a_4u^6 + a_5u^7) \right] du \\
&= \frac{1}{X_0} \left[ u_1^2 \left( a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6} \right) - 2u_1 \left( \frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7} \right) + \left( \frac{a_0u^3}{3} + \frac{a_1u^4}{4} + \frac{a_2u^5}{5} + \frac{a_3u^6}{6} + \frac{a_4u^7}{7} + \frac{a_5u^8}{8} \right) \right] \Big|_{u=u_0}^{u_1} \\
&= \frac{1}{X_0} \left[ \left( a_0u_1^3 + \frac{a_1u_1^4}{2} + \frac{a_2u_1^5}{3} + \frac{a_3u_1^6}{4} + \frac{a_4u_1^7}{5} + \frac{a_5u_1^8}{6} \right) - u_1^2 \left( a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6} \right) - 2 \left( \frac{a_0u_1^3}{2} + \frac{a_1u_1^4}{3} + \frac{a_2u_1^5}{4} + \frac{a_3u_1^6}{5} + \frac{a_4u_1^7}{6} + \frac{a_5u_1^8}{7} \right) \right. \\
&\quad \left. + 2u_1 \left( \frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7} \right) + \left( \frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8} \right) - \left( \frac{a_0u_0^3}{3} + \frac{a_1u_0^4}{4} + \frac{a_2u_0^5}{5} + \frac{a_3u_0^6}{6} + \frac{a_4u_0^7}{7} + \frac{a_5u_0^8}{8} \right) \right] \\
&= \frac{1}{X_0} \left\{ a_0u_1^3 \left( 1 - 1 + \frac{1}{3} \right) + a_1u_1^4 \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + a_2u_1^5 \left( \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + a_3u_1^6 \left( \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + a_4u_1^7 \left( \frac{1}{5} - \frac{2}{6} + \frac{1}{7} \right) + a_5u_1^8 \left( \frac{1}{6} - \frac{2}{7} + \frac{1}{8} \right) \right. \\
&\quad \left. - u_1^2 \left( a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6} \right) + 2u_1 \left( \frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7} \right) - \left( \frac{a_0u_0^3}{3} + \frac{a_1u_0^4}{4} + \frac{a_2u_0^5}{5} + \frac{a_3u_0^6}{6} + \frac{a_4u_0^7}{7} + \frac{a_5u_0^8}{8} \right) \right\}
\end{aligned}$$

$$\int_{u_0}^{u_1} \frac{(u_1 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{12} + \frac{a_2u_1^5}{30} + \frac{a_3u_1^6}{60} + \frac{a_4u_1^7}{105} + \frac{a_5u_1^8}{168} - u_1^2 \left( a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6} \right) \right. \\
\left. + 2u_1 \left( \frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7} \right) - \left( \frac{a_0u_0^3}{3} + \frac{a_1u_0^4}{4} + \frac{a_2u_0^5}{5} + \frac{a_3u_0^6}{6} + \frac{a_4u_0^7}{7} + \frac{a_5u_0^8}{8} \right) \right\}$$

$$\begin{aligned}
\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_0}^{u_1} (u_1 - u) (a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5) du \\
&= \frac{1}{X_0} \left[ u_1 \left( a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6} \right) - \left( \frac{a_0 u^2}{2} + \frac{a_1 u^3}{3} + \frac{a_2 u^4}{4} + \frac{a_3 u^5}{5} + \frac{a_4 u^6}{6} + \frac{a_5 u^7}{7} \right) \right]_{u=u_0}^{u_1} \\
&= \frac{1}{X_0} \left[ \left( a_0 u_1^2 + \frac{a_1 u_1^3}{2} + \frac{a_2 u_1^4}{3} + \frac{a_3 u_1^5}{4} + \frac{a_4 u_1^6}{5} + \frac{a_5 u_1^7}{6} \right) - u_1 \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) \right. \\
&\quad \left. - \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{3} + \frac{a_2 u_1^4}{4} + \frac{a_3 u_1^5}{5} + \frac{a_4 u_1^6}{6} + \frac{a_5 u_1^7}{7} \right) + \left( \frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7} \right) \right] \\
&= \frac{1}{X_0} \left\{ \left[ a_0 u_1^2 \left( \frac{1}{1} - \frac{1}{2} \right) + a_1 u_1^3 \left( \frac{1}{2} - \frac{1}{3} \right) + a_2 u_1^4 \left( \frac{1}{3} - \frac{1}{4} \right) + a_3 u_1^5 \left( \frac{1}{4} - \frac{1}{5} \right) + a_4 u_1^6 \left( \frac{1}{5} - \frac{1}{6} \right) + a_5 u_1^7 \left( \frac{1}{6} - \frac{1}{7} \right) \right] \right. \\
&\quad \left. - u_1 \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) + \left( \frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7} \right) \right\}
\end{aligned}$$

$$\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{6} + \frac{a_2 u_1^4}{12} + \frac{a_3 u_1^5}{20} + \frac{a_4 u_1^6}{30} + \frac{a_5 u_1^7}{42} \right) - u_1 \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) + \left( \frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7} \right) \right]$$

$$\begin{aligned}
\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_0}^{u_1} (a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5) du \\
&= \frac{1}{X_0} \left[ a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6} \right]_{u=u_0}^{u_1}
\end{aligned}$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left( a_0 u_1 + \frac{a_1 u_1^2}{2} + \frac{a_2 u_1^3}{3} + \frac{a_3 u_1^4}{4} + \frac{a_4 u_1^5}{5} + \frac{a_5 u_1^6}{6} \right) - \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) \right]$$

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_1}^{u_2} (u_2 - u)^2 (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) du = \frac{1}{X_0} \int_{u_1}^{u_2} (u_2^2 - 2u_2u + u^2) (a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5) du \\
&= \frac{1}{X_0} \left[ u_2^2 \left( a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6} \right) - 2u_2 \left( \frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7} \right) + \left( \frac{a_0u^3}{3} + \frac{a_1u^4}{4} + \frac{a_2u^5}{5} + \frac{a_3u^6}{6} + \frac{a_4u^7}{7} + \frac{a_5u^8}{8} \right) \right]_{u=u_1}^{u_2} \\
&= \frac{1}{X_0} \left[ \left( a_0u_2^3 + \frac{a_1u_2^4}{2} + \frac{a_2u_2^5}{3} + \frac{a_3u_2^6}{4} + \frac{a_4u_2^7}{5} + \frac{a_5u_2^8}{6} \right) - u_2^2 \left( a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6} \right) - 2 \left( \frac{a_0u_2^3}{2} + \frac{a_1u_2^4}{3} + \frac{a_2u_2^5}{4} + \frac{a_3u_2^6}{5} + \frac{a_4u_2^7}{6} + \frac{a_5u_2^8}{7} \right) \right. \\
&\quad \left. + 2u_2 \left( \frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7} \right) + \left( \frac{a_0u_2^2}{3} + \frac{a_1u_2^4}{4} + \frac{a_2u_2^5}{5} + \frac{a_3u_2^6}{6} + \frac{a_4u_2^7}{7} + \frac{a_5u_2^8}{8} \right) - \left( \frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8} \right) \right] \\
&= \frac{1}{X_0} \left\{ a_0u_2^3 \left( 1 - 1 + \frac{1}{3} \right) + a_1u_2^4 \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + a_2u_2^5 \left( \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + a_3u_2^6 \left( \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + a_4u_2^7 \left( \frac{1}{5} - \frac{2}{6} + \frac{1}{7} \right) + a_5u_2^8 \left( \frac{1}{6} - \frac{2}{7} + \frac{1}{8} \right) \right. \\
&\quad \left. - u_2^2 \left( a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6} \right) + 2u_2 \left( \frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7} \right) - \left( \frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \left\{ \left[ \frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60} + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168} \right] - u_2^2 \left( a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6} \right) \right. \\
&\quad \left. + 2u_2 \left( \frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7} \right) - \left( \frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8} \right) \right\} \\
&= \frac{1}{X_0} \left\{ \left( \frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60} + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168} \right) + u_1 (-a_0u_2^2) + u_1^2 \left( \frac{-a_1u_2^2}{2} + \frac{2a_0u_2}{2} \right) + u_1^3 \left( \frac{-a_2u_2^2}{3} + \frac{2a_1u_2}{3} - \frac{a_0}{3} \right) + u_1^4 \left( \frac{-a_3u_2^2}{4} + \frac{2a_2u_2}{4} - \frac{a_1}{4} \right) \right. \\
&\quad \left. + u_1^5 \left( \frac{-a_4u_2^2}{5} + \frac{2a_3u_2}{5} - \frac{a_2}{5} \right) + u_1^6 \left( \frac{-a_5u_2^2}{6} + \frac{2a_4u_2}{6} - \frac{a_3}{6} \right) + u_1^7 \left( \frac{2a_5u_2}{7} - \frac{a_4}{7} \right) + u_1^8 \left( \frac{a_5}{8} \right) \right\} \\
&= \frac{1}{X_0} \left\{ \left( \frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60} + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168} \right) + u_1 (-a_0u_2^2) + u_1^2 \left( \frac{-a_1u_2^2 + 2a_0u_2}{2} \right) + u_1^3 \left( \frac{-a_2u_2^2 + 2a_1u_2 - a_0}{3} \right) + u_1^4 \left( \frac{-a_3u_2^2 + 2a_2u_2 - a_1}{4} \right) \right. \\
&\quad \left. + u_1^5 \left( \frac{-a_4u_2^2 + 2a_3u_2 - a_2}{5} \right) + u_1^6 \left( \frac{-a_5u_2^2 + 2a_4u_2 - a_3}{6} \right) + u_1^7 \left( \frac{2a_5u_2 - a_4}{7} \right) + u_1^8 \left( \frac{-a_5}{8} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \left\{ \left( \frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60} + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168} \right) + u_1 (-a_0u_2^2) + u_1^2 \left[ -\left( \frac{a_1}{2} \right) u_2^2 + 2 \left( \frac{a_0}{2} \right) u_2 \right] + u_1^3 \left[ -\left( \frac{a_2}{3} \right) u_2^2 + 2 \left( \frac{a_1}{3} \right) u_2 - \frac{a_0}{3} \right] \right. \\
&\quad \left. + u_1^4 \left[ -\left( \frac{a_3}{4} \right) u_2^2 + 2 \left( \frac{a_2}{4} \right) u_2 - \frac{a_1}{4} \right] + u_1^5 \left[ -\left( \frac{a_4}{5} \right) u_2^2 + 2 \left( \frac{a_3}{5} \right) u_2 - \frac{a_2}{5} \right] + u_1^6 \left[ -\left( \frac{a_5}{6} \right) u_2^2 + 2 \left( \frac{a_4}{6} \right) u_2 - \frac{a_3}{6} \right] + u_1^7 \left[ 2 \left( \frac{a_5}{7} \right) u_2 - \frac{a_4}{7} \right] + u_1^8 \left[ -\frac{a_5}{8} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_1}^{u_2} (u_2 - u) (a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5) du \\
&= \frac{1}{X_0} \left[ u_2 \left( a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6} \right) - \left( \frac{a_0 u^2}{2} + \frac{a_1 u^3}{3} + \frac{a_2 u^4}{4} + \frac{a_3 u^5}{5} + \frac{a_4 u^6}{6} + \frac{a_5 u^7}{7} \right) \right]_{u=u_1}^{u_2} \\
&= \frac{1}{X_0} \left[ \left( a_0 u_2^2 + \frac{a_1 u_2^3}{2} + \frac{a_2 u_2^4}{3} + \frac{a_3 u_2^5}{4} + \frac{a_4 u_2^6}{5} + \frac{a_5 u_2^7}{6} \right) - u_2 \left( a_0 u_1 + \frac{a_1 u_1^2}{2} + \frac{a_2 u_1^3}{3} + \frac{a_3 u_1^4}{4} + \frac{a_4 u_1^5}{5} + \frac{a_5 u_1^6}{6} \right) \right. \\
&\quad \left. - \left( \frac{a_0 u_2^2}{2} + \frac{a_1 u_2^3}{3} + \frac{a_2 u_2^4}{4} + \frac{a_3 u_2^5}{5} + \frac{a_4 u_2^6}{6} + \frac{a_5 u_2^7}{7} \right) + \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{3} + \frac{a_2 u_1^4}{4} + \frac{a_3 u_1^5}{5} + \frac{a_4 u_1^6}{6} + \frac{a_5 u_1^7}{7} \right) \right] \\
&= \frac{1}{X_0} \left\{ \left[ a_0 u_2^2 \left( \frac{1}{1} - \frac{1}{2} \right) + a_1 u_2^3 \left( \frac{1}{2} - \frac{1}{3} \right) + a_2 u_2^4 \left( \frac{1}{3} - \frac{1}{4} \right) + a_3 u_2^5 \left( \frac{1}{4} - \frac{1}{5} \right) + a_4 u_2^6 \left( \frac{1}{5} - \frac{1}{6} \right) + a_5 u_2^7 \left( \frac{1}{6} - \frac{1}{7} \right) \right] \right. \\
&\quad \left. - u_2 \left( a_0 u_1 + \frac{a_1 u_1^2}{2} + \frac{a_2 u_1^3}{3} + \frac{a_3 u_1^4}{4} + \frac{a_4 u_1^5}{5} + \frac{a_5 u_1^6}{6} \right) + \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{3} + \frac{a_2 u_1^4}{4} + \frac{a_3 u_1^5}{5} + \frac{a_4 u_1^6}{6} + \frac{a_5 u_1^7}{7} \right) \right\}
\end{aligned}$$

$$\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left( \frac{a_0 u_2^2}{2} + \frac{a_1 u_2^3}{6} + \frac{a_2 u_2^4}{12} + \frac{a_3 u_2^5}{20} + \frac{a_4 u_2^6}{30} + \frac{a_5 u_2^7}{42} \right) - u_2 \left( a_0 u_1 + \frac{a_1 u_1^2}{2} + \frac{a_2 u_1^3}{3} + \frac{a_3 u_1^4}{4} + \frac{a_4 u_1^5}{5} + \frac{a_5 u_1^6}{6} \right) + \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{3} + \frac{a_2 u_1^4}{4} + \frac{a_3 u_1^5}{5} + \frac{a_4 u_1^6}{6} + \frac{a_5 u_1^7}{7} \right) \right]$$

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_1}^{u_2} (a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5) du \\
&= \frac{1}{X_0} \left[ a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6} \right]_{u=u_1}^{u_2}
\end{aligned}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left( a_0 u_2 + \frac{a_1 u_2^2}{2} + \frac{a_2 u_2^3}{3} + \frac{a_3 u_2^4}{4} + \frac{a_4 u_2^5}{5} + \frac{a_5 u_2^6}{6} \right) - \left( a_0 u_1 + \frac{a_1 u_1^2}{2} + \frac{a_2 u_1^3}{3} + \frac{a_3 u_1^4}{4} + \frac{a_4 u_1^5}{5} + \frac{a_5 u_1^6}{6} \right) \right]$$

$$\int_{u_0}^{u_1} \frac{(u_1 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left( \frac{a_0}{3} \right) u_1^3 + \left( \frac{a_1}{12} \right) u_1^4 + \left( \frac{a_2}{30} \right) u_1^5 + \left( \frac{a_3}{60} \right) u_1^6 + \left( \frac{a_4}{105} \right) u_1^7 + \left( \frac{a_5}{168} \right) u_1^8 - u_1^2 \left[ (a_0)u_0 + \left( \frac{a_1}{2} \right) u_0^2 + \left( \frac{a_2}{3} \right) u_0^3 + \left( \frac{a_3}{4} \right) u_0^4 + \left( \frac{a_4}{5} \right) u_0^5 + \left( \frac{a_5}{6} \right) u_0^6 \right] \right. \\ \left. + 2u_1 \left[ \left( \frac{a_0}{2} \right) u_0^2 + \left( \frac{a_1}{3} \right) u_0^3 + \left( \frac{a_2}{4} \right) u_0^4 + \left( \frac{a_3}{5} \right) u_0^5 + \left( \frac{a_4}{6} \right) u_0^6 + \left( \frac{a_5}{7} \right) u_0^7 \right] - \left[ \left( \frac{a_0}{3} \right) u_0^3 + \left( \frac{a_1}{4} \right) u_0^4 + \left( \frac{a_2}{5} \right) u_0^5 + \left( \frac{a_3}{6} \right) u_0^6 + \left( \frac{a_4}{7} \right) u_0^7 + \left( \frac{a_5}{8} \right) u_0^8 \right] \right\}$$

$$\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left( \frac{a_0}{2} \right) u_1^2 + \left( \frac{a_1}{6} \right) u_1^3 + \left( \frac{a_2}{12} \right) u_1^4 + \left( \frac{a_3}{20} \right) u_1^5 + \left( \frac{a_4}{30} \right) u_1^6 + \left( \frac{a_5}{42} \right) u_1^7 \right] - u_1 \left[ (a_0)u_0 + \left( \frac{a_1}{2} \right) u_0^2 + \left( \frac{a_2}{3} \right) u_0^3 + \left( \frac{a_3}{4} \right) u_0^4 + \left( \frac{a_4}{5} \right) u_0^5 + \left( \frac{a_5}{6} \right) u_0^6 \right] \right. \\ \left. + \left[ \left( \frac{a_0}{2} \right) u_0^2 + \left( \frac{a_1}{3} \right) u_0^3 + \left( \frac{a_2}{4} \right) u_0^4 + \left( \frac{a_3}{5} \right) u_0^5 + \left( \frac{a_4}{6} \right) u_0^6 + \left( \frac{a_5}{7} \right) u_0^7 \right] \right\}$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ (a_0)u_1 + \left( \frac{a_1}{2} \right) u_1^2 + \left( \frac{a_2}{3} \right) u_1^3 + \left( \frac{a_3}{4} \right) u_1^4 + \left( \frac{a_4}{5} \right) u_1^5 + \left( \frac{a_5}{6} \right) u_1^6 \right] - \left[ (a_0)u_0 + \left( \frac{a_1}{2} \right) u_0^2 + \left( \frac{a_2}{3} \right) u_0^3 + \left( \frac{a_3}{4} \right) u_0^4 + \left( \frac{a_4}{5} \right) u_0^5 + \left( \frac{a_5}{6} \right) u_0^6 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left( \frac{a_0}{3} \right) u_2^3 + \left( \frac{a_1}{12} \right) u_2^4 + \left( \frac{a_2}{30} \right) u_2^5 + \left( \frac{a_3}{60} \right) u_2^6 + \left( \frac{a_4}{105} \right) u_2^7 + \left( \frac{a_5}{168} \right) u_2^8 \right] - u_2^2 \left[ (a_0)u_1 + \left( \frac{a_1}{2} \right) u_1^2 + \left( \frac{a_2}{3} \right) u_1^3 + \left( \frac{a_3}{4} \right) u_1^4 + \left( \frac{a_4}{5} \right) u_1^5 + \left( \frac{a_5}{6} \right) u_1^6 \right] \right. \\ \left. + 2u_2 \left[ \left( \frac{a_0}{2} \right) u_1^2 + \left( \frac{a_1}{3} \right) u_1^3 + \left( \frac{a_2}{4} \right) u_1^4 + \left( \frac{a_3}{5} \right) u_1^5 + \left( \frac{a_4}{6} \right) u_1^6 + \left( \frac{a_5}{7} \right) u_1^7 \right] - \left[ \left( \frac{a_0}{3} \right) u_1^3 + \left( \frac{a_1}{4} \right) u_1^4 + \left( \frac{a_2}{5} \right) u_1^5 + \left( \frac{a_3}{6} \right) u_1^6 + \left( \frac{a_4}{7} \right) u_1^7 + \left( \frac{a_5}{8} \right) u_1^8 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left( \frac{a_0}{2} \right) u_2^2 + \left( \frac{a_1}{6} \right) u_2^3 + \left( \frac{a_2}{12} \right) u_2^4 + \left( \frac{a_3}{20} \right) u_2^5 + \left( \frac{a_4}{30} \right) u_2^6 + \left( \frac{a_5}{42} \right) u_2^7 \right] - u_2 \left[ (a_0)u_1 + \left( \frac{a_1}{2} \right) u_1^2 + \left( \frac{a_2}{3} \right) u_1^3 + \left( \frac{a_3}{4} \right) u_1^4 + \left( \frac{a_4}{5} \right) u_1^5 + \left( \frac{a_5}{6} \right) u_1^6 \right] \right. \\ \left. + \left[ \left( \frac{a_0}{2} \right) u_1^2 + \left( \frac{a_1}{3} \right) u_1^3 + \left( \frac{a_2}{4} \right) u_1^4 + \left( \frac{a_3}{5} \right) u_1^5 + \left( \frac{a_4}{6} \right) u_1^6 + \left( \frac{a_5}{7} \right) u_1^7 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ (a_0)u_2 + \left( \frac{a_1}{2} \right) u_2^2 + \left( \frac{a_2}{3} \right) u_2^3 + \left( \frac{a_3}{4} \right) u_2^4 + \left( \frac{a_4}{5} \right) u_2^5 + \left( \frac{a_5}{6} \right) u_2^6 \right] - \left[ (a_0)u_1 + \left( \frac{a_1}{2} \right) u_1^2 + \left( \frac{a_2}{3} \right) u_1^3 + \left( \frac{a_3}{4} \right) u_1^4 + \left( \frac{a_4}{5} \right) u_1^5 + \left( \frac{a_5}{6} \right) u_1^6 \right] \right\}$$

Note that in the above equations, terms in bold denote those that can be precalculated, either before compilation (as is the case for the  $a_i/c_j$  terms, which are constants) or once per iteration (as is the case with the  $u_0/u_2$  terms). Note also that the terms in green, blue, cyan, and brown highlight the terms that are exactly the same and appear in multiple expressions.

Since the only thing that really matters in our calculations is the depth  $u_1 - u_0$  and the remaining depth  $u_2 - u_1$ , we see that these are relative distances, which are unaffected by a translation applied to each term (i.e.  $u_0$ ,  $u_1$ , and  $u_2$ ). Thus, we can simplify our calculations considerably by simply defining the entry coordinate  $u_0 = 0$

so that the depth inside the object is  $u_1 - u_0 = u_1$ . Therefore, the above calculations reduce to

$$\int_{u_0}^{u_1} \frac{(u_1 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{12}\right)u_1^4 + \left(\frac{a_2}{30}\right)u_1^5 + \left(\frac{a_3}{60}\right)u_1^6 + \left(\frac{a_4}{105}\right)u_1^7 + \left(\frac{a_5}{168}\right)u_1^8 \right]$$

$$\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{6}\right)u_1^3 + \left(\frac{a_2}{12}\right)u_1^4 + \left(\frac{a_3}{20}\right)u_1^5 + \left(\frac{a_4}{30}\right)u_1^6 + \left(\frac{a_5}{42}\right)u_1^7 \right]$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right]$$

$$\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left(\frac{a_0}{3}\right)u_2^3 + \left(\frac{a_1}{12}\right)u_2^4 + \left(\frac{a_2}{30}\right)u_2^5 + \left(\frac{a_3}{60}\right)u_2^6 + \left(\frac{a_4}{105}\right)u_2^7 + \left(\frac{a_5}{168}\right)u_2^8 \right] - u_2^2 \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right. \\ \left. + 2u_2 \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7 \right] - \left[ \left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{4}\right)u_1^4 + \left(\frac{a_2}{5}\right)u_1^5 + \left(\frac{a_3}{6}\right)u_1^6 + \left(\frac{a_4}{7}\right)u_1^7 + \left(\frac{a_5}{8}\right)u_1^8 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left(\frac{a_0}{2}\right)u_2^2 + \left(\frac{a_1}{6}\right)u_2^3 + \left(\frac{a_2}{12}\right)u_2^4 + \left(\frac{a_3}{20}\right)u_2^5 + \left(\frac{a_4}{30}\right)u_2^6 + \left(\frac{a_5}{42}\right)u_2^7 \right] - u_2 \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right. \\ \left. + \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ (a_0)u_2 + \left(\frac{a_1}{2}\right)u_2^2 + \left(\frac{a_2}{3}\right)u_2^3 + \left(\frac{a_3}{4}\right)u_2^4 + \left(\frac{a_4}{5}\right)u_2^5 + \left(\frac{a_5}{6}\right)u_2^6 \right] - \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right\}$$

Notice that several other terms involve the coordinate  $u_2$ , which remains constant throughout the MLP calculations, so this can be calculated once and stored so it does not need to be calculated again. Also note that several polynomial terms (those in green and blue) are common to several terms and need only be calculated once each iteration. Then since there are only 3 polynomial terms in  $u_1$  that are not common to other terms, there are only a total of 5 polynomial terms in  $u_1$  that need to be calculated each iteration. However, notice that the coefficients on many of these terms, and indeed on many of the polynomial terms in general, involve only a small number of different ratios of the  $a_i$  coefficients. Therefore, not only do we define the  $a_i$  coefficients prior to compilation, but we also calculate the  $a_i/c_j$  terms prior to compilation since these remain constant through the entire MLP process, not just for a single history. For each  $a_i$ , there are only a handful of corresponding  $c_j$  that arise in these polynomial expansions (there are a total of 26 total values of  $a_i/c_j$  that appear in these expressions) and precalculating these and storing them as constant saves a considerable number of computations and time. Hence, there are 3 levels of computational simplification that can be exploited here: (1) the various  $a_i/c_j$  are calculated before compilation and stored as constant, (2) the terms involving  $u_0$  and  $u_2$  are calculated once per history, where the definition  $u_0 = 0$  reduces this to just those terms involving  $u_2$ , and (3) the polynomial terms in  $u_1$  which appear in multiple expressions are calculated once per iteration (i.e. for each depth) and inserted into each expression where it appears. We then see that inserting the 5<sup>th</sup> order approximation of the  $1/\beta^2(x)p^2(x)$  term into the integrals, performing the integration, manipulating the result algebraically to gather like terms, and simplifying the final expression has made it possible to identify unnecessary and redundant calculations and allowed us



to reduce the computational load significantly (by perhaps an order of magnitude or more). In addition, the term

$$\int_{u_0}^{u_1} \frac{(u_1 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{12}\right)u_1^4 + \left(\frac{a_2}{30}\right)u_1^5 + \left(\frac{a_3}{60}\right)u_1^6 + \left(\frac{a_4}{105}\right)u_1^7 + \left(\frac{a_5}{168}\right)u_1^8 \right]$$

$$\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{6}\right)u_1^3 + \left(\frac{a_2}{12}\right)u_1^4 + \left(\frac{a_3}{20}\right)u_1^5 + \left(\frac{a_4}{30}\right)u_1^6 + \left(\frac{a_5}{42}\right)u_1^7 \right]$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right]$$

$$\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left(\frac{a_0}{3}\right)u_2^3 + \left(\frac{a_1}{12}\right)u_2^4 + \left(\frac{a_2}{30}\right)u_2^5 + \left(\frac{a_3}{60}\right)u_2^6 + \left(\frac{a_4}{105}\right)u_2^7 + \left(\frac{a_5}{168}\right)u_2^8 \right] - u_2^2 \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right. \\ \left. + 2u_2 \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7 \right] - \left[ \left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{4}\right)u_1^4 + \left(\frac{a_2}{5}\right)u_1^5 + \left(\frac{a_3}{6}\right)u_1^6 + \left(\frac{a_4}{7}\right)u_1^7 + \left(\frac{a_5}{8}\right)u_1^8 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ \left(\frac{a_0}{2}\right)u_2^2 + \left(\frac{a_1}{6}\right)u_2^3 + \left(\frac{a_2}{12}\right)u_2^4 + \left(\frac{a_3}{20}\right)u_2^5 + \left(\frac{a_4}{30}\right)u_2^6 + \left(\frac{a_5}{42}\right)u_2^7 \right] - u_2 \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right. \\ \left. + \left[ \left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7 \right] \right\}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0} \left\{ \left[ (a_0)u_2 + \left(\frac{a_1}{2}\right)u_2^2 + \left(\frac{a_2}{3}\right)u_2^3 + \left(\frac{a_3}{4}\right)u_2^4 + \left(\frac{a_4}{5}\right)u_2^5 + \left(\frac{a_5}{6}\right)u_2^6 \right] - \left[ (a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6 \right] \right\}$$

## 1 MLP Calculations

$$\begin{aligned}
y_{MLP} &= \begin{bmatrix} t_1 \\ \theta_1 \end{bmatrix} = (\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1)^{-1} (\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2) \\
R_0 y_0 &= \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} t_0 + (u_1 - u_0) \theta_0 \\ \theta_0 \end{bmatrix} \\
&= (\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{t_1} & \sigma_{t_1 \theta_1} \\ \sigma_{t_1 \theta_1} & \sigma_{\theta_1} \end{bmatrix}^{-1} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{t_2} & \sigma_{t_2 \theta_2} \\ \sigma_{t_2 \theta_2} & \sigma_{\theta_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \frac{1}{\sigma_{t_1} \sigma_{\theta_1} - (\sigma_{t_1 \theta_1})^2} \begin{bmatrix} \sigma_{\theta_1} & -\sigma_{t_1 \theta_1} \\ -\sigma_{t_1 \theta_1} & \sigma_{t_1} \end{bmatrix} + \frac{1}{\sigma_{t_2} \sigma_{\theta_2} - (\sigma_{t_2 \theta_2})^2} \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2} & -\sigma_{t_2 \theta_2} \\ -\sigma_{t_2 \theta_2} & \sigma_{t_2} \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1 \theta_1}' \\ -\sigma_{t_1 \theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & -\sigma_{t_2 \theta_2}' \\ -\sigma_{t_2 \theta_2}' & \sigma_{t_2}' \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1 \theta_1}' \\ -\sigma_{t_1 \theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_2 \theta_2}' \\ -\sigma_{t_2 \theta_2}' & \sigma_{t_2}' - (u_2 - u_1) \sigma_{t_2 \theta_2}' \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1 \theta_1}' \\ -\sigma_{t_1 \theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_2 \theta_2}' \\ (u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_2 \theta_2}' & (u_2 - u_1)^2 \sigma_{\theta_2}' - 2(u_2 - u_1) \sigma_{t_2 \theta_2}' + \sigma_{t_2}' \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \sigma_{\theta_1}' + \sigma_{\theta_2}' & (u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_1 \theta_1}' - \sigma_{t_2 \theta_2}' \\ (u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_1 \theta_1}' - \sigma_{t_2 \theta_2}' & (u_2 - u_1)^2 \sigma_{\theta_2}' - 2(u_2 - u_1) \sigma_{t_2 \theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' \end{bmatrix}^{-1} \\
&= \frac{\begin{bmatrix} (u_2 - u_1)^2 \sigma_{\theta_2}' - 2(u_2 - u_1) \sigma_{t_2 \theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2) \sigma_{\theta_2}' + \sigma_{t_1 \theta_1}' + \sigma_{t_2 \theta_2}' \\ (u_1 - u_2) \sigma_{\theta_2}' + \sigma_{t_1 \theta_1}' + \sigma_{t_2 \theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{(\sigma_{\theta_1}' + \sigma_{\theta_2}') [(u_2 - u_1)^2 \sigma_{\theta_2}' - 2(u_2 - u_1) \sigma_{t_2 \theta_2}' + \sigma_{t_1}' + \sigma_{t_2}] - [(u_2 - u_1) \sigma_{\theta_2}' - \sigma_{t_1 \theta_1}' - \sigma_{t_2 \theta_2}']^2} \\
&= \frac{\begin{bmatrix} (u_2 - u_1)^2 \sigma_{\theta_2}' - 2(u_2 - u_1) \sigma_{t_2 \theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2) \sigma_{\theta_2}' + \sigma_{t_1 \theta_1}' + \sigma_{t_2 \theta_2}' \\ (u_1 - u_2) \sigma_{\theta_2}' + \sigma_{t_1 \theta_1}' + \sigma_{t_2 \theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{\sigma_{\theta_1}' \sigma_{\theta_2}' (u_2 - u_1)^2 + 2(\sigma_{\theta_2}' \sigma_{t_1 \theta_1}' - \sigma_{\theta_1}' \sigma_{t_2 \theta_2}') (u_2 - u_1) - (\sigma_{t_1 \theta_1}' + \sigma_{t_2 \theta_2}')^2 + (\sigma_{\theta_1}' + \sigma_{\theta_2}') (\sigma_{t_1}' + \sigma_{t_2}')}
\end{aligned}$$

$$\begin{aligned}
& (\Sigma_1^{-1}R_0y_0 + R_1^T\Sigma_2^{-1}y_2) \\
&= \begin{bmatrix} \sigma_{t_1} & \sigma_{t_1\theta_1} \\ \sigma_{t_1\theta_1} & \sigma_{\theta_1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{t_2} & \sigma_{t_2\theta_2} \\ \sigma_{t_2\theta_2} & \sigma_{\theta_2} \end{bmatrix}^{-1} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix} \\
&= \frac{1}{\sigma_{t_1}\sigma_{\theta_1} - \sigma_{t_1\theta_1}^2} \begin{bmatrix} \sigma_{\theta_1} & -\sigma_{t_1\theta_1} \\ -\sigma_{t_1\theta_1} & \sigma_{t_1} \end{bmatrix} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \frac{1}{\sigma_{t_2}\sigma_{\theta_2} - \sigma_{t_2\theta_2}^2} \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2} & -\sigma_{t_2\theta_2} \\ -\sigma_{t_2\theta_2} & \sigma_{t_2} \end{bmatrix} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & -\sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' \end{bmatrix} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} \begin{bmatrix} t_0 + (u_1 - u_0)\theta_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}'t_2 - \sigma_{t_2\theta_2}'\theta_2 \\ \sigma_{t_2}'\theta_2 - \sigma_{t_2\theta_2}'t_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{\theta_1}'[t_0 + (u_1 - u_0)\theta_0] - \sigma_{t_1\theta_1}'\theta_0 \\ \sigma_{t_1}'\theta_0 - \sigma_{t_1\theta_1}'[t_0 + (u_1 - u_0)\theta_0] \end{bmatrix} + \begin{bmatrix} \sigma_{\theta_2}'t_2 - \sigma_{t_2\theta_2}'\theta_2 \\ (u_2 - u_1)(\sigma_{\theta_2}'t_2 - \sigma_{t_2\theta_2}'\theta_2) + \sigma_{t_2}'\theta_2 - \sigma_{t_2\theta_2}'t_2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \sigma_{\theta_1}' [t_0 + (u_1 - u_0)\theta_0] - \sigma_{t_1\theta_1}' \theta_0 + \sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2 \\ \sigma_{t_1}' \theta_0 - \sigma_{t_1\theta_1}' [t_0 + (u_1 - u_0)\theta_0] + (u_2 - u_1) (\sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2) + \sigma_{t_2}' \theta_2 - \sigma_{t_2\theta_2}' t_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{\theta_1}' t_0 + \sigma_{\theta_1}' (u_1 - u_0)\theta_0 - \sigma_{t_1\theta_1}' \theta_0 + \sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2 \\ \sigma_{t_1}' \theta_0 - \sigma_{t_1\theta_1}' [t_0 + (u_1 - u_0)\theta_0] + (u_2 - u_1) (\sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2) + \sigma_{t_2}' \theta_2 - \sigma_{t_2\theta_2}' t_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{\theta_1}' t_0 + [\sigma_{\theta_1}' (u_1 - u_0) - \sigma_{t_1\theta_1}'] \theta_0 + \sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2 \\ \sigma_{t_1}' \theta_0 - \sigma_{t_1\theta_1}' [t_0 + (u_1 - u_0)\theta_0] + (u_2 - u_1) (\sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2) + \sigma_{t_2}' \theta_2 - \sigma_{t_2\theta_2}' t_2 \end{bmatrix}
\end{aligned}$$

Now we need to matrix multiply the  $2 \times 2$  matrix from the left hand term by the  $2 \times 1$  vector from the right hand term:

$$\begin{aligned}
& (\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1)^{-1} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} R_1[0] & R_1[1] \\ R_1[2] & R_1[3] \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2] & \Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3] \\ \Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2] & \Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3] \end{pmatrix} \\
&= \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2] & \Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3] \\ \Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2] & \Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3] \end{pmatrix} \right\}^{-1} \\
&= \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] + R_1^T[0](\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]) + R_1^T[1](\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]) & \Sigma_1^{-1}[1] + R_1^T[0](\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]) + R_1^T[1](\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]) \\ \Sigma_1^{-1}[2] + R_1^T[2](\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]) + R_1^T[3](\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]) & \Sigma_1^{-1}[3] + R_1^T[2](\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]) + R_1^T[3](\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]) \end{pmatrix} \right\}^{-1} \\
&= (\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2) \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0] & R_0[1] \\ R_0[2] & R_0[3] \end{pmatrix} \begin{pmatrix} y_0[0] \\ y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} y_2[0] \\ y_2[1] \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0]y_0[0] + R_0[1]y_0[1] \\ R_0[2]y_0[0] + R_0[3]y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1] \\ \Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1] \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[1] (R_0[2]y_0[0] + R_0[3]y_0[1]) \\ \Sigma_1^{-1}[2] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[3] (R_0[2]y_0[0] + R_0[3]y_0[1]) \end{pmatrix} + \begin{pmatrix} R_1^T[0] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[1] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \\ R_1^T[2] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[3] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[1] (R_0[2]y_0[0] + R_0[3]y_0[1]) \\ \Sigma_1^{-1}[2] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[3] (R_0[2]y_0[0] + R_0[3]y_0[1]) \end{pmatrix} + \begin{pmatrix} R_1^T[0] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[1] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \\ R_1^T[2] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[3] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_1^{-1}[0] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[1] (R_0[2]y_0[0] + R_0[3]y_0[1]) + R_1^T[0] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[1] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \\ \Sigma_1^{-1}[2] (R_0[0]y_0[0] + R_0[1]y_0[1]) + \Sigma_1^{-1}[3] (R_0[2]y_0[0] + R_0[3]y_0[1]) + R_1^T[2] (\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]) + R_1^T[3] (\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]) \end{pmatrix}
\end{aligned}$$