# MLP Algebraic Expansion and Simplification:

Reducing the computational error and increasing computational efficiency,

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\begin{aligned}
\frac{1}{\beta^2(u)p^2(u)} &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 \\
\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_0}^{u_1} (u_1-u)^2 \left(a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5\right) du \\
&= \frac{1}{X_0} \int_{u_0}^{u_1} \left(u_1^2 - 2u_1 u + u^2\right)\left(a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5\right) du \\
&= \frac{1}{X_0} \left[u_1^2\left(a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6}\right) - 2u_1\left(\frac{a_0 u^2}{2} + \frac{a_1 u^3}{3} + \frac{a_2 u^4}{4} + \frac{a_3 u^5}{5} + \frac{a_4 u^6}{6} + \frac{a_5 u^7}{7}\right) + \left(\frac{a_0 u^3}{3} + \frac{a_1 u^4}{4} + \frac{a_2 u^5}{5} + \frac{a_3 u^6}{6} + \frac{a_4 u^7}{7} + \frac{a_5 u^8}{8}\right)\right]_{u=u_0}^{u_1} \\
&= \frac{1}{X_0} \left[\left(a_0 u_1^3 + \frac{a_1 u_1^4}{2} + \frac{a_2 u_1^5}{3} + \frac{a_3 u_1^6}{4} + \frac{a_4 u_1^7}{5} + \frac{a_5 u_1^8}{6}\right)\right. \\
&- u_1^2\left(a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6}\right) \\
&\quad - 2\left(\frac{a_0 u_1^3}{2} + \frac{a_1 u_1^4}{3} + \frac{a_2 u_1^5}{4} + \frac{a_3 u_1^6}{5} + \frac{a_4 u_1^7}{6} + \frac{a_5 u_1^8}{7}\right) \\
&+ 2u_1\left(\frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7}\right) \\
&\quad + \left(\frac{a_0 u_1^3}{3} + \frac{a_1 u_1^4}{4} + \frac{a_2 u_1^5}{5} + \frac{a_3 u_1^6}{6} + \frac{a_4 u_1^7}{7} + \frac{a_5 u_1^8}{8}\right) \\
&\left. - \left(\frac{a_0 u_0^3}{3} + \frac{a_1 u_0^4}{4} + \frac{a_2 u_0^5}{5} + \frac{a_3 u_0^6}{6} + \frac{a_4 u_0^7}{7} + \frac{a_5 u_0^8}{8}\right)\right] \\
&= \frac{1}{X_0} \left\{a_0 u_1^3\left(1 - 1 + \frac{1}{3}\right) + a_1 u_1^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + a_2 u_1^5\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + a_3 u_1^6\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)\right. \\
&+ a_4 u_1^7\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + a_5 u_1^8\left(\frac{1}{6} - \frac{2}{7} + \frac{1}{8}\right) \\
&- u_1^2\left(a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6}\right) \\
&\quad + 2u_1\left(\frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7}\right) \\
&\quad \left. - \left(\frac{a_0 u_0^3}{3} + \frac{a_1 u_0^4}{4} + \frac{a_2 u_0^5}{5} + \frac{a_3 u_0^6}{6} + \frac{a_4 u_0^7}{7} + \frac{a_5 u_0^8}{8}\right)\right\}
\end{aligned}
$$

$$\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \begin{aligned}[t] &\frac{1}{X_0}\left\{\frac{\boldsymbol{a_0}u_1^3}{\mathbf{3}} + \frac{\boldsymbol{a_1}u_1^4}{\mathbf{12}} + \frac{\boldsymbol{a_2}u_1^5}{\mathbf{30}} + \frac{\boldsymbol{a_3}u_1^6}{\mathbf{60}} + \frac{\boldsymbol{a_4}u_1^7}{\mathbf{105}} + \frac{\boldsymbol{a_5}u_1^8}{\mathbf{168}}\right. \\ &-u_1^2\left(\boldsymbol{a_0u_0} + \frac{\boldsymbol{a_1u_0^2}}{\mathbf{2}} + \frac{\boldsymbol{a_2u_0^3}}{\mathbf{3}} + \frac{\boldsymbol{a_3u_0^4}}{\mathbf{4}} + \frac{\boldsymbol{a_4u_0^5}}{\mathbf{5}} + \frac{\boldsymbol{a_5u_0^6}}{\mathbf{6}}\right) \\ &+2u_1\left(\frac{\boldsymbol{a_0u_0^2}}{\mathbf{2}} + \frac{\boldsymbol{a_1u_0^3}}{\mathbf{3}} + \frac{\boldsymbol{a_2u_0^4}}{\mathbf{4}} + \frac{\boldsymbol{a_3u_0^5}}{\mathbf{5}} + \frac{\boldsymbol{a_4u_0^6}}{\mathbf{6}} + \frac{\boldsymbol{a_5u_0^7}}{\mathbf{7}}\right) \\ &\left.-\left(\frac{\boldsymbol{a_0u_0^3}}{\mathbf{3}} + \frac{\boldsymbol{a_1u_0^4}}{\mathbf{4}} + \frac{\boldsymbol{a_2u_0^5}}{\mathbf{5}} + \frac{\boldsymbol{a_3u_0^6}}{\mathbf{6}} + \frac{\boldsymbol{a_4u_0^7}}{\mathbf{7}} + \frac{\boldsymbol{a_5u_0^8}}{\mathbf{8}}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u) p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0} \int_{u_0}^{u_1} (u_1 - u) \left( a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 \right) du \\
&= \frac{1}{X_0} \left[ u_1 \left( a_0 u + \frac{a_1 u^2}{2} + \frac{a_2 u^3}{3} + \frac{a_3 u^4}{4} + \frac{a_4 u^5}{5} + \frac{a_5 u^6}{6} \right) \right. \\
&\quad \left. - \left( \frac{a_0 u^2}{2} + \frac{a_1 u^3}{3} + \frac{a_2 u^4}{4} + \frac{a_3 u^5}{5} + \frac{a_4 u^6}{6} + \frac{a_5 u^7}{7} \right) \right]_{u=u_0}^{u_1} \\
&= \frac{1}{X_0} \left[ \left( a_0 u_1^2 + \frac{a_1 u_1^3}{2} + \frac{a_2 u_1^4}{3} + \frac{a_3 u_1^5}{4} + \frac{a_4 u_1^6}{5} + \frac{a_5 u_1^7}{6} \right) \right. \\
&\quad - u_1 \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) \\
&\quad - \left( \frac{a_0 u_1^2}{2} + \frac{a_1 u_1^3}{3} + \frac{a_2 u_1^4}{4} + \frac{a_3 u_1^5}{5} + \frac{a_4 u_1^6}{6} + \frac{a_5 u_1^7}{7} \right) \\
&\quad \left. + \left( \frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7} \right) \right] \\
&= \frac{1}{X_0} \left\{ \left[ a_0 u_1^2 \left( \frac{1}{1} - \frac{1}{2} \right) + a_1 u_1^3 \left( \frac{1}{2} - \frac{1}{3} \right) + a_2 u_1^4 \left( \frac{1}{3} - \frac{1}{4} \right) \right. \right. \\
&\quad \left. + a_3 u_1^5 \left( \frac{1}{4} - \frac{1}{5} \right) + a_4 u_1^6 \left( \frac{1}{5} - \frac{1}{6} \right) + a_5 u_1^7 \left( \frac{1}{6} - \frac{1}{7} \right) \right] \\
&\quad - u_1 \left( a_0 u_0 + \frac{a_1 u_0^2}{2} + \frac{a_2 u_0^3}{3} + \frac{a_3 u_0^4}{4} + \frac{a_4 u_0^5}{5} + \frac{a_5 u_0^6}{6} \right) \\
&\quad \left. + \left( \frac{a_0 u_0^2}{2} + \frac{a_1 u_0^3}{3} + \frac{a_2 u_0^4}{4} + \frac{a_3 u_0^5}{5} + \frac{a_4 u_0^6}{6} + \frac{a_5 u_0^7}{7} \right) \right\}
\end{aligned}
$$

$$\boxed{\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \begin{aligned} &\frac{1}{X_0}\left[\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{6} + \frac{a_2u_1^4}{12} + \frac{a_3u_1^5}{20} + \frac{a_4u_1^6}{30} + \frac{a_5u_1^7}{42}\right)\right. \\ &-u_1\left(\boldsymbol{a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}}\right) \\ &\left.+\left(\boldsymbol{\frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7}}\right)\right] \end{aligned}}$$

$$\begin{aligned} \int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} &= \frac{1}{X_0}\int_{u_0}^{u_1}\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\ &= \frac{1}{X_0}\left[a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right]_{u=u_0}^{u_1} \end{aligned}$$

$$\boxed{\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \begin{aligned} &\frac{1}{X_0}\left[\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\right. \\ &\left.-\left(\boldsymbol{a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}}\right)\right] \end{aligned}}$$

$$\begin{aligned} \int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} &= \frac{1}{X_0}\int_{u_1}^{u_2}(u_2 - u)^2\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\ &= \frac{1}{X_0}\int_{u_1}^{u_2}\left(u_2^2 - 2u_2u + u^2\right)\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\ &= \frac{1}{X_0}\left[u_2^2\left(a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right)\right. \\ &\quad - 2u_2\left(\frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7}\right) \\ &\quad \left.+\left(\frac{a_0u^3}{3} + \frac{a_1u^4}{4} + \frac{a_2u^5}{5} + \frac{a_3u^6}{6} + \frac{a_4u^7}{7} + \frac{a_5u^8}{8}\right)\right]_{u=u_1}^{u_2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{X_0}\left[\left(a_0u_2^3 + \frac{a_1u_2^4}{2} + \frac{a_2u_2^5}{3} + \frac{a_3u_2^6}{4} + \frac{a_4u_2^7}{5} + \frac{a_5u_2^8}{6}\right)\right.\\
&- u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\\
&\quad - 2\left(\frac{a_0u_2^3}{2} + \frac{a_1u_2^4}{3} + \frac{a_2u_2^5}{4} + \frac{a_3u_2^6}{5} + \frac{a_4u_2^7}{6} + \frac{a_5u_2^8}{7}\right)\\
&+ 2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\\
&\quad + \left(\frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{4} + \frac{a_2u_2^5}{5} + \frac{a_3u_2^6}{6} + \frac{a_4u_2^7}{7} + \frac{a_5u_2^8}{8}\right)\\
&\left.-\left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right]\\
&= \frac{1}{X_0}\left\{a_0u_2^3\left(1 - 1 + \frac{1}{3}\right) + a_1u_2^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + a_2u_2^5\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + a_3u_2^6\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)\right.\\
&+ a_4u_2^7\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + a_5u_2^8\left(\frac{1}{6} - \frac{2}{7} + \frac{1}{8}\right)\\
&- u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\\
&\quad + 2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\\
&\quad \left.-\left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right\}
\end{aligned}
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$$
\boxed{\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} =
\begin{aligned}
&\frac{1}{X_0}\left\{\left[\boldsymbol{\frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60} + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168}}\right]\right.\\
&-u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\\
&\quad +2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\\
&\quad \left.-\left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right\}
\end{aligned}}
$$

$$
= \frac{1}{X_0}\left\{\left(\boldsymbol{\frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{12} + \frac{a_2u_2^5}{30} + \frac{a_3u_2^6}{60}6 + \frac{a_4u_2^7}{105} + \frac{a_5u_2^8}{168}}\right)\right.
$$

$$= \frac{1}{X_0}\left\{\left(\frac{a_0u_2^3}{3}+\frac{a_1u_2^4}{12}+\frac{a_2u_2^5}{30}+\frac{a_3u_2^6}{60}+\frac{a_4u_2^7}{105}+\frac{a_5u_2^8}{168}\right)\right.$$

$$+u_1\left(-a_0u_2^2\right)+u_1^2\left(\frac{-a_1u_2^2+2a_0u_2}{2}\right)+u_1^3\left(\frac{-a_2u_2^2+2a_1u_2-a_0}{3}\right)+u_1^4\left(\frac{-a_3u_2^2+2a_2u_2-a_1}{4}\right)$$

$$\left.+u_1^5\left(\frac{-a_4u_2^2+2a_3u_2-a_2}{5}\right)+u_1^6\left(\frac{-a_5u_2^2+2a_4u_2-a_3}{6}\right)+u_1^7\left(\frac{2a_5u_2-a_4}{7}\right)+u_1^8\left(\frac{-a_5}{8}\right)\right\}$$

$$\boxed{\begin{aligned}\int_{u_1}^{u_2}\frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = {} & \frac{1}{X_0}\left\{\left(\frac{a_0u_2^3}{3}+\frac{a_1u_2^4}{12}+\frac{a_2u_2^5}{30}+\frac{a_3u_2^6}{60}+\frac{a_4u_2^7}{105}+\frac{a_5u_2^8}{168}\right)\right.\\ & +u_1\left(-a_0u_2^2\right)+u_1^2\left[-\left(\frac{a_1}{2}\right)u_2^2+2\left(\frac{a_0}{2}\right)u_2\right]+u_1^3\left[-\left(\frac{a_2}{3}\right)u_2^2+2\left(\frac{a_1}{3}\right)u_2-\frac{a_0}{3}\right]\\ & +u_1^4\left[-\left(\frac{a_3}{4}\right)u_2^2+2\left(\frac{a_2}{4}\right)u_2-\frac{a_1}{4}\right]+u_1^5\left[-\left(\frac{a_4}{5}\right)u_2^2+2\left(\frac{a_3}{5}\right)u_2-\frac{a_2}{5}\right]\\ & \left.+u_1^6\left[-\left(\frac{a_5}{6}\right)u_2^2+2\left(\frac{a_4}{6}\right)u_2-\frac{a_3}{6}\right]+u_1^7\left[2\left(\frac{a_5}{7}\right)u_2-\frac{a_4}{7}\right]+u_1^8\left[-\frac{a_5}{8}\right]\right\}\end{aligned}}$$

$$
\begin{aligned}
\int_{u_1}^{u_2} \frac{u_2-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} &= \frac{1}{X_0}\int_{u_1}^{u_2}(u_2-u)\left(a_0+a_1u+a_2u^2+a_3u^3+a_4u^4+a_5u^5\right)du \\
&= \frac{1}{X_0}\left[u_2\left(a_0u+\frac{a_1u^2}{2}+\frac{a_2u^3}{3}+\frac{a_3u^4}{4}+\frac{a_4u^5}{5}+\frac{a_5u^6}{6}\right)\right. \\
&\quad \left.-\left(\frac{a_0u^2}{2}+\frac{a_1u^3}{3}+\frac{a_2u^4}{4}+\frac{a_3u^5}{5}+\frac{a_4u^6}{6}+\frac{a_5u^7}{7}\right)\right]_{u=u_1}^{u_2} \\
&= \frac{1}{X_0}\left[\left(a_0u_2^2+\frac{a_1u_2^3}{2}+\frac{a_2u_2^4}{3}+\frac{a_3u_2^5}{4}+\frac{a_4u_2^6}{5}+\frac{a_5u_2^7}{6}\right)\right. \\
&\quad -u_2\left(a_0u_1+\frac{a_1u_1^2}{2}+\frac{a_2u_1^3}{3}+\frac{a_3u_1^4}{4}+\frac{a_4u_1^5}{5}+\frac{a_5u_1^6}{6}\right) \\
&\quad -\left(\frac{a_0u_2^2}{2}+\frac{a_1u_2^3}{3}+\frac{a_2u_2^4}{4}+\frac{a_3u_2^5}{5}+\frac{a_4u_2^6}{6}+\frac{a_5u_2^7}{7}\right) \\
&\quad \left.+\left(\frac{a_0u_1^2}{2}+\frac{a_1u_1^3}{3}+\frac{a_2u_1^4}{4}+\frac{a_3u_1^5}{5}+\frac{a_4u_1^6}{6}+\frac{a_5u_1^7}{7}\right)\right] \\
&= \frac{1}{X_0}\left\{\left[a_0u_2^2\left(\frac{1}{1}-\frac{1}{2}\right)+a_1u_2^3\left(\frac{1}{2}-\frac{1}{3}\right)+a_2u_2^4\left(\frac{1}{3}-\frac{1}{4}\right)+a_3u_2^5\left(\frac{1}{4}-\frac{1}{5}\right)+a_4u_2^6\left(\frac{1}{5}-\frac{1}{6}\right)+a_5u_2^7\left(\frac{1}{6}-\frac{1}{7}\right)\right]\right. \\
&\quad -u_2\left(a_0u_1+\frac{a_1u_1^2}{2}+\frac{a_2u_1^3}{3}+\frac{a_3u_1^4}{4}+\frac{a_4u_1^5}{5}+\frac{a_5u_1^6}{6}\right) \\
&\quad \left.+\left(\frac{a_0u_1^2}{2}+\frac{a_1u_1^3}{3}+\frac{a_2u_1^4}{4}+\frac{a_3u_1^5}{5}+\frac{a_4u_1^6}{6}+\frac{a_5u_1^7}{7}\right)\right\}
\end{aligned}
$$

$$\boxed{\begin{aligned}\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \; & \frac{1}{X_0}\Bigg[\left(\boldsymbol{\frac{a_0u_2^2}{2} + \frac{a_1u_2^3}{6} + \frac{a_2u_2^4}{12} + \frac{a_3u_2^5}{20} + \frac{a_4u_2^6}{30} + \frac{a_5u_2^7}{42}}\right) \\ & -u_2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right) \\ & +\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\Bigg]\end{aligned}}$$

$$\begin{aligned}\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} &= \frac{1}{X_0}\int_{u_1}^{u_2}\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\ &= \frac{1}{X_0}\left[a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right]_{u=u_1}^{u_2}\end{aligned}$$

$$\boxed{\begin{aligned}\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \; & \frac{1}{X_0}\Bigg[\left(\boldsymbol{a_0u_2 + \frac{a_1u_2^2}{2} + \frac{a_2u_2^3}{3} + \frac{a_3u_2^4}{4} + \frac{a_4u_2^5}{5} + \frac{a_5u_2^6}{6}}\right) \\ & -\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\Bigg]\end{aligned}}$$

$$\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{12}\right)u_1^4 + \left(\frac{a_2}{30}\right)u_1^5 + \left(\frac{a_3}{60}\right)u_1^6 + \left(\frac{a_4}{105}\right)u_1^7 + \left(\frac{a_5}{168}\right)u_1^8 - u_1^2\left[(a_0)u_0 + \left(\frac{a_1}{2}\right)u_0^2 + \left(\frac{a_2}{3}\right)u_0^3 + \left(\frac{a_3}{4}\right)u_0^4 + \left(\frac{a_4}{5}\right)u_0^5 + \left(\frac{a_5}{6}\right)u_0^6\right]$$
$$+2u_1\left[\left(\frac{a_0}{2}\right)u_0^2 + \left(\frac{a_1}{3}\right)u_0^3 + \left(\frac{a_2}{4}\right)u_0^4 + \left(\frac{a_3}{5}\right)u_0^5 + \left(\frac{a_4}{6}\right)u_0^6 + \left(\frac{a_5}{7}\right)u_0^7\right] - \left[\left(\frac{a_0}{3}\right)u_0^3 + \left(\frac{a_1}{4}\right)u_0^4 + \left(\frac{a_2}{5}\right)u_0^5 + \left(\frac{a_3}{6}\right)u_0^6 + \left(\frac{a_4}{7}\right)u_0^7 + \left(\frac{a_5}{8}\right)u_0^8\right]\Bigg\}$$

$$\int_{u_0}^{u_1} \frac{u_1-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{6}\right)u_1^3 + \left(\frac{a_2}{12}\right)u_1^4 + \left(\frac{a_3}{20}\right)u_1^5 + \left(\frac{a_4}{30}\right)u_1^6 + \left(\frac{a_5}{42}\right)u_1^7\right] - u_1\left[(a_0)u_0 + \left(\frac{a_1}{2}\right)u_0^2 + \left(\frac{a_2}{3}\right)u_0^3 + \left(\frac{a_3}{4}\right)u_0^4 + \left(\frac{a_4}{5}\right)u_0^5 + \left(\frac{a_5}{6}\right)u_0^6\right]$$
$$+\left[\left(\frac{a_0}{2}\right)u_0^2 + \left(\frac{a_1}{3}\right)u_0^3 + \left(\frac{a_2}{4}\right)u_0^4 + \left(\frac{a_3}{5}\right)u_0^5 + \left(\frac{a_4}{6}\right)u_0^6 + \left(\frac{a_5}{7}\right)u_0^7\right]\Bigg\}$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right] - \left[(a_0)u_0 + \left(\frac{a_1}{2}\right)u_0^2 + \left(\frac{a_2}{3}\right)u_0^3 + \left(\frac{a_3}{4}\right)u_0^4 + \left(\frac{a_4}{5}\right)u_0^5 + \left(\frac{a_5}{6}\right)u_0^6\right]\Bigg\}$$

$$\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left[\left(\frac{a_0}{3}\right)u_2^3 + \left(\frac{a_1}{12}\right)u_2^4 + \left(\frac{a_2}{30}\right)u_2^5 + \left(\frac{a_3}{60}\right)u_2^6 + \left(\frac{a_4}{105}\right)u_2^7 + \left(\frac{a_5}{168}\right)u_2^8\right] - u_2^2\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]$$
$$+2u_2\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7\right] - \left[\left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{4}\right)u_1^4 + \left(\frac{a_2}{5}\right)u_1^5 + \left(\frac{a_3}{6}\right)u_1^6 + \left(\frac{a_4}{7}\right)u_1^7 + \left(\frac{a_5}{8}\right)u_1^8\right]\Bigg\}$$

$$\int_{u_1}^{u_2} \frac{u_2-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left[\left(\frac{a_0}{2}\right)u_2^2 + \left(\frac{a_1}{6}\right)u_2^3 + \left(\frac{a_2}{12}\right)u_2^4 + \left(\frac{a_3}{20}\right)u_2^5 + \left(\frac{a_4}{30}\right)u_2^6 + \left(\frac{a_5}{42}\right)u_2^7\right] - u_2\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]$$
$$+\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7\right]\Bigg\}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\Bigg\{\left[(a_0)u_2 + \left(\frac{a_1}{2}\right)u_2^2 + \left(\frac{a_2}{3}\right)u_2^3 + \left(\frac{a_3}{4}\right)u_2^4 + \left(\frac{a_4}{5}\right)u_2^5 + \left(\frac{a_5}{6}\right)u_2^6\right] - \left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]\Bigg\}$$

Note that in the above equations, terms in bold denote those that can be precalculated, either before compilation (as is the case for the $^{a_i}/_{c_j}$ terms, which are constants) or once per iteration (as is the case with the $u_0/u_2$ terms. Note also that the terms in green, blue, cyan, and brown highlight the terms that are exactly the same and appear in multiple expressions.

Since the only thing that really matters in our calculations is the depth $u_1 - u_0$ and the remaining depth $u_2 - u_1$, we see that these are relative distances, which are unaffected by a translation applied to each term (i.e. $u_0$, $u_1$, and $u_2$). Thus, we can simplify our calculations considerably by simply defining the entry coordinate $u_0 = 0$

so that the depth inside the object is $u_1 - u_0 = u_1$. Therefore, the above calculations reduce to

$$\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left[\left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{12}\right)u_1^4 + \left(\frac{a_2}{30}\right)u_1^5 + \left(\frac{a_3}{60}\right)u_1^6 + \left(\frac{a_4}{105}\right)u_1^7 + \left(\frac{a_5}{168}\right)u_1^8\right]$$

$$\int_{u_0}^{u_1} \frac{u_1-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{6}\right)u_1^3 + \left(\frac{a_2}{12}\right)u_1^4 + \left(\frac{a_3}{20}\right)u_1^5 + \left(\frac{a_4}{30}\right)u_1^6 + \left(\frac{a_5}{42}\right)u_1^7\right]$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]$$

$$\begin{aligned}\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = & \frac{1}{X_0}\left\{\left[\left(\frac{a_0}{3}\right)u_2^3 + \left(\frac{a_1}{12}\right)u_2^4 + \left(\frac{a_2}{30}\right)u_2^5 + \left(\frac{a_3}{60}\right)u_2^6 + \left(\frac{a_4}{105}\right)u_2^7 + \left(\frac{a_5}{168}\right)u_2^8\right] - u_2^2\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]\right. \\ & \left. +2u_2\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7\right] - \left[\left(\frac{a_0}{3}\right)u_1^3 + \left(\frac{a_1}{4}\right)u_1^4 + \left(\frac{a_2}{5}\right)u_1^5 + \left(\frac{a_3}{6}\right)u_1^6 + \left(\frac{a_4}{7}\right)u_1^7 + \left(\frac{a_5}{8}\right)u_1^8\right]\right\}\end{aligned}$$

$$\begin{aligned}\int_{u_1}^{u_2} \frac{u_2-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = & \frac{1}{X_0}\left\{\left[\left(\frac{a_0}{2}\right)u_2^2 + \left(\frac{a_1}{6}\right)u_2^3 + \left(\frac{a_2}{12}\right)u_2^4 + \left(\frac{a_3}{20}\right)u_2^5 + \left(\frac{a_4}{30}\right)u_2^6 + \left(\frac{a_5}{42}\right)u_2^7\right] - u_2\left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]\right. \\ & \left. +\left[\left(\frac{a_0}{2}\right)u_1^2 + \left(\frac{a_1}{3}\right)u_1^3 + \left(\frac{a_2}{4}\right)u_1^4 + \left(\frac{a_3}{5}\right)u_1^5 + \left(\frac{a_4}{6}\right)u_1^6 + \left(\frac{a_5}{7}\right)u_1^7\right]\right\}\end{aligned}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[(a_0)u_2 + \left(\frac{a_1}{2}\right)u_2^2 + \left(\frac{a_2}{3}\right)u_2^3 + \left(\frac{a_3}{4}\right)u_2^4 + \left(\frac{a_4}{5}\right)u_2^5 + \left(\frac{a_5}{6}\right)u_2^6\right] - \left[(a_0)u_1 + \left(\frac{a_1}{2}\right)u_1^2 + \left(\frac{a_2}{3}\right)u_1^3 + \left(\frac{a_3}{4}\right)u_1^4 + \left(\frac{a_4}{5}\right)u_1^5 + \left(\frac{a_5}{6}\right)u_1^6\right]\right\}$$

Notice that several other terms involve the coordinate $u_2$, which remains constant throughout the MLP calculations, so this can be calculated once and stored so it does not need to be calculated again. Also note that several polynomial terms (those in green and blue) are common to several terms and need only be calculated once each iteration. Then since there are only 3 polynomial terms in $u_1$ that are not common to other terms, there are only a total of 5 polynomial terms in $u_1$ that need to be calculated each iteration. However, notice that the coefficients on many of these terms, and indeed on many of the polynomial terms in general, involve only a small number of different ratios of the $a_i$ coefficients. Therefore, not only do we define the $a_i$ coefficients prior to compilation, but we also calculate the $a_i/c_j$ terms prior to compilation since these remain constant through the entire MLP process, not just for a single history. For each $a_i$, there are only a handful of corresponding $c_j$ that arise in these polynomial expansions (there are a total of 26 total values of $a_i/c_j$ that appear in these expressions) and precalculating these and storing them as constant saves a considerable number of computations and time. Hence, there are 3 levels of computational simplification that can be exploited here: (1) the various $a_i/c_j$ are calculated before compilation and stored as constant, (2) the terms involving $u_0$ and $u_2$ are calculated once per history, where the definition $u_0 = 0$ reduces this to just those terms involving $u_2$, and (3) the polynomial terms in $u_1$ which appear in multiple expressions are calculated once per iteration (i.e. for each depth) and inserted into each expression where it appears. We then see that inserting the 5th order approximation of the $1/\beta^2(x)p^2(x)$ term into the integrals, performing the integration, manipulating the result algebraically to gather like terms, and simplifying the final expression has made it possible to identify unnecessary and redundant calculations and allowed us to reduce the computational load significantly (by perhaps an order of magnitude or more).

## 1 MLP Calculations

$$y_{MLP} = \begin{bmatrix} t_1 \\ \theta_1 \end{bmatrix} = \left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1} \left(\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2\right)$$

$$R_0 y_0 = \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} t_0 + (u_1 - u_0)\,\theta_0 \\ \theta_0 \end{bmatrix}$$

$$\begin{aligned}
&\left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{t_1} & \sigma_{t_1\theta_1} \\ \sigma_{t_1\theta_1} & \sigma_{\theta_1} \end{bmatrix}^{-1} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{t_2} & \sigma_{t_2\theta_2} \\ \sigma_{t_2\theta_2} & \sigma_{\theta_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \frac{1}{\sigma_{t_1}\sigma_{\theta_1} - (\sigma_{t_1\theta_1})^2} \begin{bmatrix} \sigma_{\theta_1} & -\sigma_{t_1\theta_1} \\ -\sigma_{t_1\theta_1} & \sigma_{t_1} \end{bmatrix} + \frac{1}{\sigma_{t_2}\sigma_{\theta_2} - (\sigma_{t_2\theta_2})^2} \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2} & -\sigma_{t_2\theta_2} \\ -\sigma_{t_2\theta_2} & \sigma_{t_2} \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & -\sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' - (u_2 - u_1)\sigma_{t_2\theta_2}' \end{bmatrix} \right)^{-1} \\
&= \left( \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' \\ (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' & (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_2}' \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \sigma_{\theta_1}' + \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}' \\ (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}' & (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' \end{bmatrix}^{-1} \\
&= \frac{\begin{bmatrix} (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' \\ (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{(\sigma_{\theta_1}' + \sigma_{\theta_2}')\left[(u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}'\right] - \left[(u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}'\right]^2} \\
&= \frac{\begin{bmatrix} (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' \\ (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{\sigma_{\theta_1}'\sigma_{\theta_2}'(u_2 - u_1)^2 + 2(\sigma_{\theta_2}'\sigma_{t_1\theta_1}' - \sigma_{\theta_1}'\sigma_{t_2\theta_2}')(u_2 - u_1) - (\sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}')^2 + (\sigma_{\theta_1}' + \sigma_{\theta_2}')(\sigma_{t_1}' + \sigma_{t_2}')}
\end{aligned}$$

$$
\begin{aligned}
&\left(\Sigma_1^{-1}R_0y_0 + R_1^T\Sigma_2^{-1}y_2\right)\\
&= \begin{bmatrix}\sigma_{t_1} & \sigma_{t_1\theta_1}\\ \sigma_{t_1\theta_1} & \sigma_{\theta_1}\end{bmatrix}^{-1}\begin{bmatrix}1 & u_1-u_0\\ 0 & 1\end{bmatrix}\begin{bmatrix}t_0\\ \theta_0\end{bmatrix} + \begin{bmatrix}1 & 0\\ u_2-u_1 & 1\end{bmatrix}\begin{bmatrix}\sigma_{t_2} & \sigma_{t_2\theta_2}\\ \sigma_{t_2\theta_2} & \sigma_{\theta_2}\end{bmatrix}^{-1}\begin{bmatrix}t_2\\ \theta_2\end{bmatrix}\\
&= \frac{1}{\sigma_{t_1}\sigma_{\theta_1}-\sigma_{t_1\theta_1}{}^2}\begin{bmatrix}\sigma_{\theta_1} & -\sigma_{t_1\theta_1}\\ -\sigma_{t_1\theta_1} & \sigma_{t_1}\end{bmatrix}\begin{bmatrix}1 & u_1-u_0\\ 0 & 1\end{bmatrix}\begin{bmatrix}t_0\\ \theta_0\end{bmatrix} + \frac{1}{\sigma_{t_2}\sigma_{\theta_2}-\sigma_{t_2\theta_2}{}^2}\begin{bmatrix}1 & 0\\ u_2-u_1 & 1\end{bmatrix}\begin{bmatrix}\sigma_{\theta_2} & -\sigma_{t_2\theta_2}\\ -\sigma_{t_2\theta_2} & \sigma_{t_2}\end{bmatrix}\begin{bmatrix}t_2\\ \theta_2\end{bmatrix}\\
&= \begin{bmatrix}\sigma_{\theta_1}{}' & -\sigma_{t_1\theta_1}{}'\\ -\sigma_{t_1\theta_1}{}' & \sigma_{t_1}{}'\end{bmatrix}\begin{bmatrix}1 & u_1-u_0\\ 0 & 1\end{bmatrix}\begin{bmatrix}t_0\\ \theta_0\end{bmatrix} + \begin{bmatrix}1 & 0\\ u_2-u_1 & 1\end{bmatrix}\begin{bmatrix}\sigma_{\theta_2}{}' & -\sigma_{t_2\theta_2}{}'\\ -\sigma_{t_2\theta_2}{}' & \sigma_{t_2}{}'\end{bmatrix}\begin{bmatrix}t_2\\ \theta_2\end{bmatrix}\\
&= \begin{bmatrix}\sigma_{\theta_1}{}' & -\sigma_{t_1\theta_1}{}'\\ -\sigma_{t_1\theta_1}{}' & \sigma_{t_1}{}'\end{bmatrix}\begin{bmatrix}t_0+(u_1-u_0)\theta_0\\ \theta_0\end{bmatrix} + \begin{bmatrix}1 & 0\\ u_2-u_1 & 1\end{bmatrix}\begin{bmatrix}\sigma_{\theta_2}{}'t_2-\sigma_{t_2\theta_2}{}'\theta_2\\ \sigma_{t_2}{}'\theta_2-\sigma_{t_2\theta_2}{}'t_2\end{bmatrix}\\
&= \begin{bmatrix}\sigma_{\theta_1}{}'\,[t_0+(u_1-u_0)\theta_0]-\sigma_{t_1\theta_1}{}'\theta_0\\ \sigma_{t_1}{}'\theta_0-\sigma_{t_1\theta_1}{}'\,[t_0+(u_1-u_0)\theta_0]\end{bmatrix} + \begin{bmatrix}\sigma_{\theta_2}{}'t_2-\sigma_{t_2\theta_2}{}'\theta_2\\ (u_2-u_1)\left(\sigma_{\theta_2}{}'t_2-\sigma_{t_2\theta_2}{}'\theta_2\right)+\sigma_{t_2}{}'\theta_2-\sigma_{t_2\theta_2}{}'t_2\end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} \sigma_{\theta_1}{}' \left[t_0 + (u_1 - u_0)\theta_0\right] - \sigma_{t_1\theta_1}{}'\theta_0 + \sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2 \\ \sigma_{t_1}{}'\theta_0 - \sigma_{t_1\theta_1}{}' \left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left(\sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2\right) + \sigma_{t_2}{}'\theta_2 - \sigma_{t_2\theta_2}{}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\theta_1}{}' t_0 + \sigma_{\theta_1}{}'(u_1 - u_0)\theta_0 - \sigma_{t_1\theta_1}{}'\theta_0 + \sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2 \\ \sigma_{t_1}{}'\theta_0 - \sigma_{t_1\theta_1}{}' \left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left(\sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2\right) + \sigma_{t_2}{}'\theta_2 - \sigma_{t_2\theta_2}{}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\theta_1}{}' t_0 + \left[\sigma_{\theta_1}{}'(u_1 - u_0) - \sigma_{t_1\theta_1}{}'\right]\theta_0 + \sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2 \\ \sigma_{t_1}{}'\theta_0 - \sigma_{t_1\theta_1}{}' \left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left(\sigma_{\theta_2}{}' t_2 - \sigma_{t_2\theta_2}{}'\theta_2\right) + \sigma_{t_2}{}'\theta_2 - \sigma_{t_2\theta_2}{}' t_2 \end{bmatrix}$$

Now we need to matrix multiply the $2 \times 2$ matrix from the left hand term by the $2 \times 1$ vector from the right hand term:

$$\begin{aligned}
&\left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} R_1[0] & R_1[1] \\ R_1[2] & R_1[3] \end{pmatrix} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2] & \Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3] \\ \Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2] & \Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3] \end{pmatrix} \\
= \;& \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]\right) & R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]\right) & R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]\right) \end{pmatrix} \right\}^{-1} \\
= \;& \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] + R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]\right) & \Sigma_1^{-1}[1] + R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]\right) \\ \Sigma_1^{-1}[2] + R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2]\right) & \Sigma_1^{-1}[3] + R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3]\right) \end{pmatrix} \right\}^{-1}
\end{aligned}$$

$$\begin{aligned}
&\left(\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2\right) \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0] & R_0[1] \\ R_0[2] & R_0[3] \end{pmatrix} \begin{pmatrix} y_0[0] \\ y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} y_2[0] \\ y_2[1] \end{pmatrix} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0]y_0[0] + R_0[1]y_0[1] \\ R_0[2]y_0[0] + R_0[3]y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1] \\ \Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1] \end{pmatrix} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix} \\
= \;& \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) + R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) + R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix}
\end{aligned}$$