MLP Algebraic Expansion and Simplification:

Reducing the computational error and increasing computational efficiency

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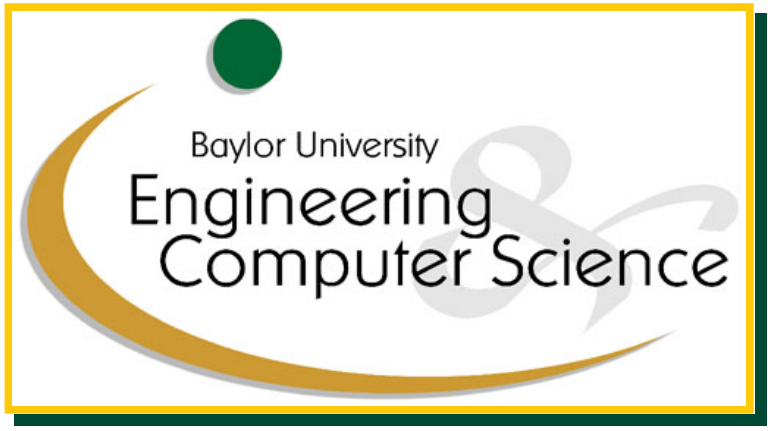

$$\frac{1}{\beta^2(u)p^2(u)} = a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5$$

$$\begin{aligned}
\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} &= \frac{1}{X_0}\int_{u_0}^{u_1}(u_1-u)^2\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\
&= \frac{1}{X_0}\int_{u_0}^{u_1}\left(u_1^2 - 2u_1u + u^2\right)\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right)du \\
&= \frac{1}{X_0}\left[u_1^2\left(a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right)\right. \\
&\quad - 2u_1\left(\frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7}\right) \\
&\quad \left.+\left(\frac{a_0u^3}{3} + \frac{a_1u^4}{4} + \frac{a_2u^5}{5} + \frac{a_3u^6}{6} + \frac{a_4u^7}{7} + \frac{a_5u^8}{8}\right)\right]_{u=u_0}^{u_1} \\
&= \frac{1}{X_0}\left[\left(a_0u_1^3 + \frac{a_1u_1^4}{2} + \frac{a_2u_1^5}{3} + \frac{a_3u_1^6}{4} + \frac{a_4u_1^7}{5} + \frac{a_5u_1^8}{6}\right)\right. \\
&\quad - u_1^2\left(a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}\right) \\
&\quad - 2\left(\frac{a_0u_1^3}{2} + \frac{a_1u_1^4}{3} + \frac{a_2u_1^5}{4} + \frac{a_3u_1^6}{5} + \frac{a_4u_1^7}{6} + \frac{a_5u_1^8}{7}\right) \\
&\quad + 2u_1\left(\frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7}\right) \\
&\quad + \left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right) \\
&\quad \left.-\left(\frac{a_0u_0^3}{3} + \frac{a_1u_0^4}{4} + \frac{a_2u_0^5}{5} + \frac{a_3u_0^6}{6} + \frac{a_4u_0^7}{7} + \frac{a_5u_0^8}{8}\right)\right] \\
&= \frac{1}{X_0}\left\{a_0u_1^3\left(1 - 1 + \frac{1}{3}\right) + a_1u_1^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + a_2u_1^5\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + a_3u_1^6\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)\right. \\
&\quad + a_4u_1^7\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + a_5u_1^8\left(\frac{1}{6} - \frac{2}{7} + \frac{1}{8}\right) \\
&\quad - u_1^2\left(a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}\right) \\
&\quad + 2u_1\left(\frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7}\right) \\
&\quad \left.-\left(\frac{a_0u_0^3}{3} + \frac{a_1u_0^4}{4} + \frac{a_2u_0^5}{5} + \frac{a_3u_0^6}{6} + \frac{a_4u_0^7}{7} + \frac{a_5u_0^8}{8}\right)\right\}
\end{aligned}$$

$$\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\begin{array}{l} \dfrac{\boldsymbol{a_0u_1^3}}{\boldsymbol{3}} + \dfrac{\boldsymbol{a_1u_1^4}}{\boldsymbol{12}} + \dfrac{\boldsymbol{a_2u_1^5}}{\boldsymbol{30}} + \dfrac{\boldsymbol{a_3u_1^6}}{\boldsymbol{60}} + \dfrac{\boldsymbol{a_4u_1^7}}{\boldsymbol{105}} + \dfrac{\boldsymbol{a_5u_1^8}}{\boldsymbol{168}} \\ -u_1^2\left(\boldsymbol{a_0u_0} + \dfrac{\boldsymbol{a_1u_0^2}}{\boldsymbol{2}} + \dfrac{\boldsymbol{a_2u_0^3}}{\boldsymbol{3}} + \dfrac{\boldsymbol{a_3u_0^4}}{\boldsymbol{4}} + \dfrac{\boldsymbol{a_4u_0^5}}{\boldsymbol{5}} + \dfrac{\boldsymbol{a_5u_0^6}}{\boldsymbol{6}}\right) \\ +2u_1\left(\dfrac{\boldsymbol{a_0u_0^2}}{\boldsymbol{2}} + \dfrac{\boldsymbol{a_1u_0^3}}{\boldsymbol{3}} + \dfrac{\boldsymbol{a_2u_0^4}}{\boldsymbol{4}} + \dfrac{\boldsymbol{a_3u_0^5}}{\boldsymbol{5}} + \dfrac{\boldsymbol{a_4u_0^6}}{\boldsymbol{6}} + \dfrac{\boldsymbol{a_5u_0^7}}{\boldsymbol{7}}\right) \\ -\left(\dfrac{\boldsymbol{a_0u_0^3}}{\boldsymbol{3}} + \dfrac{\boldsymbol{a_1u_0^4}}{\boldsymbol{4}} + \dfrac{\boldsymbol{a_2u_0^5}}{\boldsymbol{5}} + \dfrac{\boldsymbol{a_3u_0^6}}{\boldsymbol{6}} + \dfrac{\boldsymbol{a_4u_0^7}}{\boldsymbol{7}} + \dfrac{\boldsymbol{a_5u_0^8}}{\boldsymbol{8}}\right)\end{array}\right\}$$

$$\begin{aligned}
\int_{u_0}^{u_1} \frac{u_1-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} &= \frac{1}{X_0}\int_{u_0}^{u_1}(u_1-u)\left(a_0+a_1u+a_2u^2+a_3u^3+a_4u^4+a_5u^5\right)du \\
&= \frac{1}{X_0}\left[u_1\left(a_0u+\frac{a_1u^2}{2}+\frac{a_2u^3}{3}+\frac{a_3u^4}{4}+\frac{a_4u^5}{5}+\frac{a_5u^6}{6}\right)\right. \\
&\quad \left.-\left(\frac{a_0u^2}{2}+\frac{a_1u^3}{3}+\frac{a_2u^4}{4}+\frac{a_3u^5}{5}+\frac{a_4u^6}{6}+\frac{a_5u^7}{7}\right)\right]_{u=u_0}^{u_1} \\
&= \frac{1}{X_0}\left[\left(a_0u_1^2+\frac{a_1u_1^3}{2}+\frac{a_2u_1^4}{3}+\frac{a_3u_1^5}{4}+\frac{a_4u_1^6}{5}+\frac{a_5u_1^7}{6}\right)\right. \\
&\quad -u_1\left(a_0u_0+\frac{a_1u_0^2}{2}+\frac{a_2u_0^3}{3}+\frac{a_3u_0^4}{4}+\frac{a_4u_0^5}{5}+\frac{a_5u_0^6}{6}\right) \\
&\quad -\left(\frac{a_0u_1^2}{2}+\frac{a_1u_1^3}{3}+\frac{a_2u_1^4}{4}+\frac{a_3u_1^5}{5}+\frac{a_4u_1^6}{6}+\frac{a_5u_1^7}{7}\right) \\
&\quad \left.+\left(\frac{a_0u_0^2}{2}+\frac{a_1u_0^3}{3}+\frac{a_2u_0^4}{4}+\frac{a_3u_0^5}{5}+\frac{a_4u_0^6}{6}+\frac{a_5u_0^7}{7}\right)\right] \\
&= \frac{1}{X_0}\left\{\left[a_0u_1^2\left(\frac{1}{1}-\frac{1}{2}\right)+a_1u_1^3\left(\frac{1}{2}-\frac{1}{3}\right)+a_2u_1^4\left(\frac{1}{3}-\frac{1}{4}\right)\right.\right. \\
&\quad \left.+a_3u_1^5\left(\frac{1}{4}-\frac{1}{5}\right)+a_4u_1^6\left(\frac{1}{5}-\frac{1}{6}\right)+a_5u_1^7\left(\frac{1}{6}-\frac{1}{7}\right)\right] \\
&\quad -u_1\left(a_0u_0+\frac{a_1u_0^2}{2}+\frac{a_2u_0^3}{3}+\frac{a_3u_0^4}{4}+\frac{a_4u_0^5}{5}+\frac{a_5u_0^6}{6}\right) \\
&\quad \left.+\left(\frac{a_0u_0^2}{2}+\frac{a_1u_0^3}{3}+\frac{a_2u_0^4}{4}+\frac{a_3u_0^5}{5}+\frac{a_4u_0^6}{6}+\frac{a_5u_0^7}{7}\right)\right\}
\end{aligned}$$

$$\int_{u_0}^{u_1} \frac{u_1 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\left[\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{6} + \frac{a_2u_1^4}{12} + \frac{a_3u_1^5}{20} + \frac{a_4u_1^6}{30} + \frac{a_5u_1^7}{42}\right) - u_1\left(\boldsymbol{a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}}\right) + \left(\boldsymbol{\frac{a_0u_0^2}{2} + \frac{a_1u_0^3}{3} + \frac{a_2u_0^4}{4} + \frac{a_3u_0^5}{5} + \frac{a_4u_0^6}{6} + \frac{a_5u_0^7}{7}}\right)\right]$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\int_{u_0}^{u_1} \left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right) du$$

$$= \frac{1}{X_0}\left[a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right]_{u=u_0}^{u_1}$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\left[\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right) - \left(\boldsymbol{a_0u_0 + \frac{a_1u_0^2}{2} + \frac{a_2u_0^3}{3} + \frac{a_3u_0^4}{4} + \frac{a_4u_0^5}{5} + \frac{a_5u_0^6}{6}}\right)\right]$$

$$\int_{u_1}^{u_2} \frac{(u_2 - u)^2}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\int_{u_1}^{u_2} (u_2 - u)^2 \left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right) du$$

$$= \frac{1}{X_0}\int_{u_1}^{u_2} \left(u_2^2 - 2u_2u + u^2\right)\left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right) du$$

$$= \frac{1}{X_0}\left[u_2^2\left(a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right) - 2u_2\left(\frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7}\right) + \left(\frac{a_0u^3}{3} + \frac{a_1u^4}{4} + \frac{a_2u^5}{5} + \frac{a_3u^6}{6} + \frac{a_4u^7}{7} + \frac{a_5u^8}{8}\right)\right]_{u=u_1}^{u_2}$$

$$= \frac{1}{X_0}\left[\left(a_0u_2^3 + \frac{a_1u_2^4}{2} + \frac{a_2u_2^5}{3} + \frac{a_3u_2^6}{4} + \frac{a_4u_2^7}{5} + \frac{a_5u_2^8}{6}\right)\right.$$
$$- u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)$$
$$- 2\left(\frac{a_0u_2^3}{2} + \frac{a_1u_2^4}{3} + \frac{a_2u_2^5}{4} + \frac{a_3u_2^6}{5} + \frac{a_4u_2^7}{6} + \frac{a_5u_2^8}{7}\right)$$
$$+ 2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)$$
$$+ \left(\frac{a_0u_2^3}{3} + \frac{a_1u_2^4}{4} + \frac{a_2u_2^5}{5} + \frac{a_3u_2^6}{6} + \frac{a_4u_2^7}{7} + \frac{a_5u_2^8}{8}\right)$$
$$\left.- \left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right]$$
$$= \frac{1}{X_0}\left\{a_0u_2^3\left(1 - 1 + \frac{1}{3}\right) + a_1u_2^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + a_2u_2^5\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + a_3u_2^6\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)\right.$$
$$+ a_4u_2^7\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + a_5u_2^8\left(\frac{1}{6} - \frac{2}{7} + \frac{1}{8}\right)$$
$$- u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)$$
$$+ 2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)$$
$$\left.- \left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right\}$$

$$\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[\frac{\boldsymbol{a_0u_2^3}}{\boldsymbol{3}} + \frac{\boldsymbol{a_1u_2^4}}{\boldsymbol{12}} + \frac{\boldsymbol{a_2u_2^5}}{\boldsymbol{30}} + \frac{\boldsymbol{a_3u_2^6}}{\boldsymbol{60}} + \frac{\boldsymbol{a_4u_2^7}}{\boldsymbol{105}} + \frac{\boldsymbol{a_5u_2^8}}{\boldsymbol{168}}\right]\right.$$
$$- u_2^2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)$$
$$+ 2u_2\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)$$
$$\left.- \left(\frac{a_0u_1^3}{3} + \frac{a_1u_1^4}{4} + \frac{a_2u_1^5}{5} + \frac{a_3u_1^6}{6} + \frac{a_4u_1^7}{7} + \frac{a_5u_1^8}{8}\right)\right\}$$

$$= \frac{1}{X_0}\left\{\left(\frac{\boldsymbol{a_0u_2^3}}{\boldsymbol{3}} + \frac{\boldsymbol{a_1u_2^4}}{\boldsymbol{12}} + \frac{\boldsymbol{a_2u_2^5}}{\boldsymbol{30}} + \frac{\boldsymbol{a_3u_2^6}}{\boldsymbol{60}} + \frac{\boldsymbol{a_4u_2^7}}{\boldsymbol{105}} + \frac{\boldsymbol{a_5u_2^8}}{\boldsymbol{168}}\right)\right.$$
$$+ u_1\left(-a_0u_2^2\right) + u_1^2\left(\frac{\boldsymbol{-a_1u_2^2}}{\boldsymbol{2}} + \frac{\boldsymbol{2a_0u_2}}{\boldsymbol{2}}\right) + u_1^3\left(\frac{\boldsymbol{-a_2u_2^2}}{\boldsymbol{3}} + \frac{\boldsymbol{2a_1u_2}}{\boldsymbol{3}} - \frac{\boldsymbol{a_0}}{\boldsymbol{3}}\right) + u_1^4\left(\frac{\boldsymbol{-a_3u_2^2}}{\boldsymbol{4}} + \frac{\boldsymbol{2a_2u_2}}{\boldsymbol{4}} - \frac{\boldsymbol{a_1}}{\boldsymbol{4}}\right)$$
$$\left.+ u_1^5\left(\frac{\boldsymbol{-a_4u_2^2}}{\boldsymbol{5}} + \frac{\boldsymbol{2a_3u_2}}{\boldsymbol{5}} - \frac{\boldsymbol{a_2}}{\boldsymbol{5}}\right) + u_1^6\left(\frac{\boldsymbol{-a_5u_2^2}}{\boldsymbol{6}} + \frac{\boldsymbol{2a_4u_2}}{\boldsymbol{6}} - \frac{\boldsymbol{a_3}}{\boldsymbol{6}}\right) + u_1^7\left(\frac{\boldsymbol{2a_5u_2}}{\boldsymbol{7}} - \frac{\boldsymbol{a_4}}{\boldsymbol{7}}\right) + u_1^8\left(\frac{\boldsymbol{a_5}}{\boldsymbol{8}}\right)\right\}$$

$$= \frac{1}{X_0}\left\{\left(\frac{\boldsymbol{a_0u_2^3}}{\boldsymbol{3}} + \frac{\boldsymbol{a_1u_2^4}}{\boldsymbol{12}} + \frac{\boldsymbol{a_2u_2^5}}{\boldsymbol{30}} + \frac{\boldsymbol{a_3u_2^6}}{\boldsymbol{60}} + \frac{\boldsymbol{a_4u_2^7}}{\boldsymbol{105}} + \frac{\boldsymbol{a_5u_2^8}}{\boldsymbol{168}}\right)\right.$$
$$+ u_1\left(-a_0u_2^2\right) + u_1^2\left(\frac{\boldsymbol{-a_1u_2^2 + 2a_0u_2}}{\boldsymbol{2}}\right) + u_1^3\left(\frac{\boldsymbol{-a_2u_2^2 + 2a_1u_2 - a_0}}{\boldsymbol{3}}\right) + u_1^4\left(\frac{\boldsymbol{-a_3u_2^2 + 2a_2u_2 - a_1}}{\boldsymbol{4}}\right)$$
$$\left. + u_1^5\left(\frac{\boldsymbol{-a_4u_2^2 + 2a_3u_2 - a_2}}{\boldsymbol{5}}\right) + u_1^6\left(\frac{\boldsymbol{-a_5u_2^2 + 2a_4u_2 - a_3}}{\boldsymbol{6}}\right) + u_1^7\left(\frac{\boldsymbol{2a_5u_2 - a_4}}{\boldsymbol{7}}\right) + u_1^8\left(\frac{\boldsymbol{-a_5}}{\boldsymbol{8}}\right)\right\}$$

$$\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left(\frac{\boldsymbol{a_0u_2^3}}{\boldsymbol{3}} + \frac{\boldsymbol{a_1u_2^4}}{\boldsymbol{12}} + \frac{\boldsymbol{a_2u_2^5}}{\boldsymbol{30}} + \frac{\boldsymbol{a_3u_2^6}}{\boldsymbol{60}} + \frac{\boldsymbol{a_4u_2^7}}{\boldsymbol{105}} + \frac{\boldsymbol{a_5u_2^8}}{\boldsymbol{168}}\right)\right.$$
$$+u_1\left(\boldsymbol{-a_0u_2^2}\right) + u_1^2\left[-\left(\frac{\boldsymbol{a_1}}{\boldsymbol{2}}\right)\boldsymbol{u_2^2} + \boldsymbol{2}\left(\frac{\boldsymbol{a_0}}{\boldsymbol{2}}\right)\boldsymbol{u_2}\right] + u_1^3\left[-\left(\frac{\boldsymbol{a_2}}{\boldsymbol{3}}\right)\boldsymbol{u_2^2} + \boldsymbol{2}\left(\frac{\boldsymbol{a_1}}{\boldsymbol{3}}\right)\boldsymbol{u_2} - \frac{\boldsymbol{a_0}}{\boldsymbol{3}}\right]$$
$$+u_1^4\left[-\left(\frac{\boldsymbol{a_3}}{\boldsymbol{4}}\right)\boldsymbol{u_2^2} + \boldsymbol{2}\left(\frac{\boldsymbol{a_2}}{\boldsymbol{4}}\right)\boldsymbol{u_2} - \frac{\boldsymbol{a_1}}{\boldsymbol{4}}\right] + u_1^5\left[-\left(\frac{\boldsymbol{a_4}}{\boldsymbol{5}}\right)\boldsymbol{u_2^2} + \boldsymbol{2}\left(\frac{\boldsymbol{a_3}}{\boldsymbol{5}}\right)\boldsymbol{u_2} - \frac{\boldsymbol{a_2}}{\boldsymbol{5}}\right]$$
$$\left.+u_1^6\left[-\left(\frac{\boldsymbol{a_5}}{\boldsymbol{6}}\right)\boldsymbol{u_2^2} + \boldsymbol{2}\left(\frac{\boldsymbol{a_4}}{\boldsymbol{6}}\right)\boldsymbol{u_2} - \frac{\boldsymbol{a_3}}{\boldsymbol{6}}\right] + u_1^7\left[\boldsymbol{2}\left(\frac{\boldsymbol{a_5}}{\boldsymbol{7}}\right)\boldsymbol{u_2} - \frac{\boldsymbol{a_4}}{\boldsymbol{7}}\right] + u_1^8\left[-\frac{\boldsymbol{a_5}}{\boldsymbol{8}}\right]\right\}$$

$$\int_{u_1}^{u_2} \frac{u_2-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\int_{u_1}^{u_2}(u_2-u)\left(a_0+a_1u+a_2u^2+a_3u^3+a_4u^4+a_5u^5\right)du$$
$$= \frac{1}{X_0}\left[u_2\left(a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right)\right.$$
$$\left.-\left(\frac{a_0u^2}{2} + \frac{a_1u^3}{3} + \frac{a_2u^4}{4} + \frac{a_3u^5}{5} + \frac{a_4u^6}{6} + \frac{a_5u^7}{7}\right)\right]_{u=u_1}^{u_2}$$
$$= \frac{1}{X_0}\left[\left(a_0u_2^2 + \frac{a_1u_2^3}{2} + \frac{a_2u_2^4}{3} + \frac{a_3u_2^5}{4} + \frac{a_4u_2^6}{5} + \frac{a_5u_2^7}{6}\right)\right.$$
$$-u_2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)$$
$$-\left(\frac{a_0u_2^2}{2} + \frac{a_1u_2^3}{3} + \frac{a_2u_2^4}{4} + \frac{a_3u_2^5}{5} + \frac{a_4u_2^6}{6} + \frac{a_5u_2^7}{7}\right)$$
$$\left.+\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\right]$$
$$= \frac{1}{X_0}\left\{\left[a_0u_2^2\left(\frac{1}{1}-\frac{1}{2}\right) + a_1u_2^3\left(\frac{1}{2}-\frac{1}{3}\right) + a_2u_2^4\left(\frac{1}{3}-\frac{1}{4}\right) + a_3u_2^5\left(\frac{1}{4}-\frac{1}{5}\right) + a_4u_2^6\left(\frac{1}{5}-\frac{1}{6}\right) + a_5u_2^7\left(\frac{1}{6}-\frac{1}{7}\right)\right]\right.$$
$$-u_2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)$$
$$\left.+\left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\right\}$$

$$\int_{u_1}^{u_2} \frac{u_2 - u}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\left[\left(\frac{\boldsymbol{a_0u_2^2}}{\boldsymbol{2}} + \frac{\boldsymbol{a_1u_2^3}}{\boldsymbol{6}} + \frac{\boldsymbol{a_2u_2^4}}{\boldsymbol{12}} + \frac{\boldsymbol{a_3u_2^5}}{\boldsymbol{20}} + \frac{\boldsymbol{a_4u_2^6}}{\boldsymbol{30}} + \frac{\boldsymbol{a_5u_2^7}}{\boldsymbol{42}}\right) - u_2\left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right) + \left(\frac{a_0u_1^2}{2} + \frac{a_1u_1^3}{3} + \frac{a_2u_1^4}{4} + \frac{a_3u_1^5}{5} + \frac{a_4u_1^6}{6} + \frac{a_5u_1^7}{7}\right)\right]$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\int_{u_1}^{u_2} \left(a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5\right) du$$

$$= \frac{1}{X_0}\left[a_0u + \frac{a_1u^2}{2} + \frac{a_2u^3}{3} + \frac{a_3u^4}{4} + \frac{a_4u^5}{5} + \frac{a_5u^6}{6}\right]_{u=u_1}^{u_2}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)} \frac{du}{X_0} = \frac{1}{X_0}\left[\left(\boldsymbol{a_0u_2} + \frac{\boldsymbol{a_1u_2^2}}{\boldsymbol{2}} + \frac{\boldsymbol{a_2u_2^3}}{\boldsymbol{3}} + \frac{\boldsymbol{a_3u_2^4}}{\boldsymbol{4}} + \frac{\boldsymbol{a_4u_2^5}}{\boldsymbol{5}} + \frac{\boldsymbol{a_5u_2^6}}{\boldsymbol{6}}\right) - \left(a_0u_1 + \frac{a_1u_1^2}{2} + \frac{a_2u_1^3}{3} + \frac{a_3u_1^4}{4} + \frac{a_4u_1^5}{5} + \frac{a_5u_1^6}{6}\right)\right]$$

$$\int_{u_0}^{u_1} \frac{(u_1-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left(\frac{a_0}{3}\right)u_1^3+\left(\frac{a_1}{12}\right)u_1^4+\left(\frac{a_2}{30}\right)u_1^5+\left(\frac{a_3}{60}\right)u_1^6+\left(\frac{a_4}{105}\right)u_1^7+\left(\frac{a_5}{168}\right)u_1^8\right.$$
$$-u_1^2\left[(a_0)u_0+\left(\frac{a_1}{2}\right)u_0^2+\left(\frac{a_2}{3}\right)u_0^3+\left(\frac{a_3}{4}\right)u_0^4+\left(\frac{a_4}{5}\right)u_0^5+\left(\frac{a_5}{6}\right)u_0^6\right]$$
$$+2u_1\left[\left(\frac{a_0}{2}\right)u_0^2+\left(\frac{a_1}{3}\right)u_0^3+\left(\frac{a_2}{4}\right)u_0^4+\left(\frac{a_3}{5}\right)u_0^5+\left(\frac{a_4}{6}\right)u_0^6+\left(\frac{a_5}{7}\right)u_0^7\right]$$
$$\left.-\left[\left(\frac{a_0}{3}\right)u_0^3+\left(\frac{a_1}{4}\right)u_0^4+\left(\frac{a_2}{5}\right)u_0^5+\left(\frac{a_3}{6}\right)u_0^6+\left(\frac{a_4}{7}\right)u_0^7+\left(\frac{a_5}{8}\right)u_0^8\right]\right\}$$

$$\int_{u_0}^{u_1} \frac{u_1-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[\left(\frac{a_0}{2}\right)u_1^2+\left(\frac{a_1}{6}\right)u_1^3+\left(\frac{a_2}{12}\right)u_1^4+\left(\frac{a_3}{20}\right)u_1^5+\left(\frac{a_4}{30}\right)u_1^6+\left(\frac{a_5}{42}\right)u_1^7\right]\right.$$
$$-u_1\left[(a_0)u_0+\left(\frac{a_1}{2}\right)u_0^2+\left(\frac{a_2}{3}\right)u_0^3+\left(\frac{a_3}{4}\right)u_0^4+\left(\frac{a_4}{5}\right)u_0^5+\left(\frac{a_5}{6}\right)u_0^6\right]$$
$$\left.+\left[\left(\frac{a_0}{2}\right)u_0^2+\left(\frac{a_1}{3}\right)u_0^3+\left(\frac{a_2}{4}\right)u_0^4+\left(\frac{a_3}{5}\right)u_0^5+\left(\frac{a_4}{6}\right)u_0^6+\left(\frac{a_5}{7}\right)u_0^7\right]\right\}$$

$$\int_{u_0}^{u_1} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[(a_0)u_1+\left(\frac{a_1}{2}\right)u_1^2+\left(\frac{a_2}{3}\right)u_1^3+\left(\frac{a_3}{4}\right)u_1^4+\left(\frac{a_4}{5}\right)u_1^5+\left(\frac{a_5}{6}\right)u_1^6\right]\right.$$
$$\left.-\left[(a_0)u_0+\left(\frac{a_1}{2}\right)u_0^2+\left(\frac{a_2}{3}\right)u_0^3+\left(\frac{a_3}{4}\right)u_0^4+\left(\frac{a_4}{5}\right)u_0^5+\left(\frac{a_5}{6}\right)u_0^6\right]\right\}$$

$$\int_{u_1}^{u_2} \frac{(u_2-u)^2}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[\left(\frac{a_0}{3}\right)u_2^3+\left(\frac{a_1}{12}\right)u_2^4+\left(\frac{a_2}{30}\right)u_2^5+\left(\frac{a_3}{60}\right)u_2^6+\left(\frac{a_4}{105}\right)u_2^7+\left(\frac{a_5}{168}\right)u_2^8\right]\right.$$
$$-u_2^2\left[(a_0)u_1+\left(\frac{a_1}{2}\right)u_1^2+\left(\frac{a_2}{3}\right)u_1^3+\left(\frac{a_3}{4}\right)u_1^4+\left(\frac{a_4}{5}\right)u_1^5+\left(\frac{a_5}{6}\right)u_1^6\right]$$
$$+2u_2\left[\left(\frac{a_0}{2}\right)u_1^2+\left(\frac{a_1}{3}\right)u_1^3+\left(\frac{a_2}{4}\right)u_1^4+\left(\frac{a_3}{5}\right)u_1^5+\left(\frac{a_4}{6}\right)u_1^6+\left(\frac{a_5}{7}\right)u_1^7\right]$$
$$\left.-\left[\left(\frac{a_0}{3}\right)u_1^3+\left(\frac{a_1}{4}\right)u_1^4+\left(\frac{a_2}{5}\right)u_1^5+\left(\frac{a_3}{6}\right)u_1^6+\left(\frac{a_4}{7}\right)u_1^7+\left(\frac{a_5}{8}\right)u_1^8\right]\right\}$$

$$\int_{u_1}^{u_2} \frac{u_2-u}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[\left(\frac{a_0}{2}\right)u_2^2+\left(\frac{a_1}{6}\right)u_2^3+\left(\frac{a_2}{12}\right)u_2^4+\left(\frac{a_3}{20}\right)u_2^5+\left(\frac{a_4}{30}\right)u_2^6+\left(\frac{a_5}{42}\right)u_2^7\right]\right.$$
$$-u_2\left[(a_0)u_1+\left(\frac{a_1}{2}\right)u_1^2+\left(\frac{a_2}{3}\right)u_1^3+\left(\frac{a_3}{4}\right)u_1^4+\left(\frac{a_4}{5}\right)u_1^5+\left(\frac{a_5}{6}\right)u_1^6\right]$$
$$\left.+\left[\left(\frac{a_0}{2}\right)u_1^2+\left(\frac{a_1}{3}\right)u_1^3+\left(\frac{a_2}{4}\right)u_1^4+\left(\frac{a_3}{5}\right)u_1^5+\left(\frac{a_4}{6}\right)u_1^6+\left(\frac{a_5}{7}\right)u_1^7\right]\right\}$$

$$\int_{u_1}^{u_2} \frac{1}{\beta^2(u)p^2(u)}\frac{du}{X_0} = \frac{1}{X_0}\left\{\left[(a_0)u_2+\left(\frac{a_1}{2}\right)u_2^2+\left(\frac{a_2}{3}\right)u_2^3+\left(\frac{a_3}{4}\right)u_2^4+\left(\frac{a_4}{5}\right)u_2^5+\left(\frac{a_5}{6}\right)u_2^6\right]\right.$$
$$\left.-\left[(a_0)u_1+\left(\frac{a_1}{2}\right)u_1^2+\left(\frac{a_2}{3}\right)u_1^3+\left(\frac{a_3}{4}\right)u_1^4+\left(\frac{a_4}{5}\right)u_1^5+\left(\frac{a_5}{6}\right)u_1^6\right]\right\}$$

1 MLP Calculations

$$y_{MLP} = \begin{bmatrix} t_1 \\ \theta_1 \end{bmatrix} = \left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1} \left(\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2\right)$$

$$R_0 y_0 = \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} t_0 + (u_1 - u_0)\,\theta_0 \\ \theta_0 \end{bmatrix}$$

$$\left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1}$$

$$= \left(\begin{bmatrix} \sigma_{t_1} & \sigma_{t_1\theta_1} \\ \sigma_{t_1\theta_1} & \sigma_{\theta_1} \end{bmatrix}^{-1} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{t_2} & \sigma_{t_2\theta_2} \\ \sigma_{t_2\theta_2} & \sigma_{\theta_2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \left(\frac{1}{\sigma_{t_1}\sigma_{\theta_1} - (\sigma_{t_1\theta_1})^2} \begin{bmatrix} \sigma_{\theta_1} & -\sigma_{t_1\theta_1} \\ -\sigma_{t_1\theta_1} & \sigma_{t_1} \end{bmatrix} + \frac{1}{\sigma_{t_2}\sigma_{\theta_2} - (\sigma_{t_2\theta_2})^2} \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2} & -\sigma_{t_2\theta_2} \\ -\sigma_{t_2\theta_2} & \sigma_{t_2} \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & -\sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' \end{bmatrix} \begin{bmatrix} 1 & u_2 - u_1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' - (u_2 - u_1)\sigma_{t_2\theta_2}' \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} + \begin{bmatrix} \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' \\ (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_2\theta_2}' & (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_2}' \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \sigma_{\theta_1}' + \sigma_{\theta_2}' & (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}' \\ (u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}' & (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' \\ (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{(\sigma_{\theta_1}' + \sigma_{\theta_2}')\left[(u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}'\right] - \left[(u_2 - u_1)\sigma_{\theta_2}' - \sigma_{t_1\theta_1}' - \sigma_{t_2\theta_2}'\right]^2}$$

$$= \frac{\begin{bmatrix} (u_2 - u_1)^2\sigma_{\theta_2}' - 2(u_2 - u_1)\sigma_{t_2\theta_2}' + \sigma_{t_1}' + \sigma_{t_2}' & (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' \\ (u_1 - u_2)\sigma_{\theta_2}' + \sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}' & \sigma_{\theta_1}' + \sigma_{\theta_2}' \end{bmatrix}}{\sigma_{\theta_1}'\sigma_{\theta_2}'(u_2 - u_1)^2 + 2(\sigma_{\theta_2}'\sigma_{t_1\theta_1}' - \sigma_{\theta_1}'\sigma_{t_2\theta_2}')(u_2 - u_1) - (\sigma_{t_1\theta_1}' + \sigma_{t_2\theta_2}')^2 + (\sigma_{\theta_1}' + \sigma_{\theta_2}')(\sigma_{t_1}' + \sigma_{t_2}')}$$

$$\left(\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2\right)$$

$$= \begin{bmatrix} \sigma_{t_1} & \sigma_{t_1\theta_1} \\ \sigma_{t_1\theta_1} & \sigma_{\theta_1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{t_2} & \sigma_{t_2\theta_2} \\ \sigma_{t_2\theta_2} & \sigma_{\theta_2} \end{bmatrix}^{-1} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix}$$

$$= \frac{1}{\sigma_{t_1}\sigma_{\theta_1} - \sigma_{t_1\theta_1}{}^2} \begin{bmatrix} \sigma_{\theta_1} & -\sigma_{t_1\theta_1} \\ -\sigma_{t_1\theta_1} & \sigma_{t_1} \end{bmatrix} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \frac{1}{\sigma_{t_2}\sigma_{\theta_2} - \sigma_{t_2\theta_2}{}^2} \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2} & -\sigma_{t_2\theta_2} \\ -\sigma_{t_2\theta_2} & \sigma_{t_2} \end{bmatrix} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} \begin{bmatrix} 1 & u_1 - u_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' & -\sigma_{t_2\theta_2}' \\ -\sigma_{t_2\theta_2}' & \sigma_{t_2}' \end{bmatrix} \begin{bmatrix} t_2 \\ \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\theta_1}' & -\sigma_{t_1\theta_1}' \\ -\sigma_{t_1\theta_1}' & \sigma_{t_1}' \end{bmatrix} \begin{bmatrix} t_0 + (u_1 - u_0)\theta_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ u_2 - u_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2 \\ \sigma_{t_2}' \theta_2 - \sigma_{t_2\theta_2}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\theta_1}' [t_0 + (u_1 - u_0)\theta_0] - \sigma_{t_1\theta_1}' \theta_0 \\ \sigma_{t_1}' \theta_0 - \sigma_{t_1\theta_1}' [t_0 + (u_1 - u_0)\theta_0] \end{bmatrix} + \begin{bmatrix} \sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2 \\ (u_2 - u_1)\left(\sigma_{\theta_2}' t_2 - \sigma_{t_2\theta_2}' \theta_2\right) + \sigma_{t_2}' \theta_2 - \sigma_{t_2\theta_2}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} {\sigma_{\theta_1}}'\left[t_0 + (u_1 - u_0)\theta_0\right] - {\sigma_{t_1\theta_1}}'\theta_0 + {\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2 \\ {\sigma_{t_1}}'\theta_0 - {\sigma_{t_1\theta_1}}'\left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left({\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2\right) + {\sigma_{t_2}}'\theta_2 - {\sigma_{t_2\theta_2}}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} {\sigma_{\theta_1}}' t_0 + {\sigma_{\theta_1}}'(u_1 - u_0)\theta_0 - {\sigma_{t_1\theta_1}}'\theta_0 + {\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2 \\ {\sigma_{t_1}}'\theta_0 - {\sigma_{t_1\theta_1}}'\left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left({\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2\right) + {\sigma_{t_2}}'\theta_2 - {\sigma_{t_2\theta_2}}' t_2 \end{bmatrix}$$

$$= \begin{bmatrix} {\sigma_{\theta_1}}' t_0 + \left[{\sigma_{\theta_1}}'(u_1 - u_0) - {\sigma_{t_1\theta_1}}'\right]\theta_0 + {\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2 \\ {\sigma_{t_1}}'\theta_0 - {\sigma_{t_1\theta_1}}'\left[t_0 + (u_1 - u_0)\theta_0\right] + (u_2 - u_1)\left({\sigma_{\theta_2}}' t_2 - {\sigma_{t_2\theta_2}}'\theta_2\right) + {\sigma_{t_2}}'\theta_2 - {\sigma_{t_2\theta_2}}' t_2 \end{bmatrix}$$

Now we need to matrix multiply the 2×2 matrix from the left hand term by the 2×1 vector from the right hand term:

$$
\begin{aligned}
&\left(\Sigma_1^{-1} + R_1^T \Sigma_2^{-1} R_1\right)^{-1} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} R_1[0] & R_1[1] \\ R_1[2] & R_1[3] \end{pmatrix} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]R_1[0] + \Sigma_2^{-1}[1]R_1[2] & \Sigma_2^{-1}[0]R_1[1] + \Sigma_2^{-1}[1]R_1[3] \\ \Sigma_2^{-1}[2]R_1[0] + \Sigma_2^{-1}[3]R_1[2] & \Sigma_2^{-1}[2]R_1[1] + \Sigma_2^{-1}[3]R_1[3] \end{pmatrix} \\
= & \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[0]+\Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[0]+\Sigma_2^{-1}[3]R_1[2]\right) & R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[1]+\Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[1]+\Sigma_2^{-1}[3]R_1[3]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[0]+\Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[0]+\Sigma_2^{-1}[3]R_1[2]\right) & R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[1]+\Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[1]+\Sigma_2^{-1}[3]R_1[3]\right) \end{pmatrix} \right\}^{-1} \\
= & \left\{ \begin{pmatrix} \Sigma_1^{-1}[0] + R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[0]+\Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[0]+\Sigma_2^{-1}[3]R_1[2]\right) & \Sigma_1^{-1}[1] + R_1^T[0]\left(\Sigma_2^{-1}[0]R_1[1]+\Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]R_1[1]+\Sigma_2^{-1}[3]R_1[3]\right) \\ \Sigma_1^{-1}[2] + R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[0]+\Sigma_2^{-1}[1]R_1[2]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[0]+\Sigma_2^{-1}[3]R_1[2]\right) & \Sigma_1^{-1}[3] + R_1^T[2]\left(\Sigma_2^{-1}[0]R_1[1]+\Sigma_2^{-1}[1]R_1[3]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]R_1[1]+\Sigma_2^{-1}[3]R_1[3]\right) \end{pmatrix} \right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\Sigma_1^{-1} R_0 y_0 + R_1^T \Sigma_2^{-1} y_2\right) \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0] & R_0[1] \\ R_0[2] & R_0[3] \end{pmatrix} \begin{pmatrix} y_0[0] \\ y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0] & \Sigma_2^{-1}[1] \\ \Sigma_2^{-1}[2] & \Sigma_2^{-1}[3] \end{pmatrix} \begin{pmatrix} y_2[0] \\ y_2[1] \end{pmatrix} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0] & \Sigma_1^{-1}[1] \\ \Sigma_1^{-1}[2] & \Sigma_1^{-1}[3] \end{pmatrix} \begin{pmatrix} R_0[0]y_0[0] + R_0[1]y_0[1] \\ R_0[2]y_0[0] + R_0[3]y_0[1] \end{pmatrix} + \begin{pmatrix} R_1^T[0] & R_1^T[1] \\ R_1^T[2] & R_1^T[3] \end{pmatrix} \begin{pmatrix} \Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1] \\ \Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1] \end{pmatrix} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) \end{pmatrix} + \begin{pmatrix} R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix} \\
= \; & \begin{pmatrix} \Sigma_1^{-1}[0]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[1]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) + R_1^T[0]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[1]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \\ \Sigma_1^{-1}[2]\left(R_0[0]y_0[0] + R_0[1]y_0[1]\right) + \Sigma_1^{-1}[3]\left(R_0[2]y_0[0] + R_0[3]y_0[1]\right) + R_1^T[2]\left(\Sigma_2^{-1}[0]y_2[0] + \Sigma_2^{-1}[1]y_2[1]\right) + R_1^T[3]\left(\Sigma_2^{-1}[2]y_2[0] + \Sigma_2^{-1}[3]y_2[1]\right) \end{pmatrix}
\end{aligned}
$$