On the Runge Example

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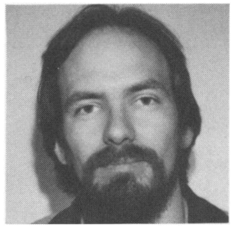

1. Introduction. The "Runge phenomenon," or Runge example, is the classic illustration of polynomial interpolation *non*convergence. Stated briefly, if $p_n(x)$ is the polynomial interpolating

$$f(x) = (1 + x^2)^{-1}, \qquad x \in [-5, 5] \tag{1}$$

at the equidistant nodes $x_j^{(n)} = -5 + 10(j/n)$ $(j = 0, 1, \ldots, n)$, then $p_n \to f$ uniformly only if $|x| < x_c \approx 3.63$. If $x_c < |x| < 5$, then divergence occurs.

The example is given at the end of Runge's paper [9], in which he discusses the general theory of convergence for interpolation at equidistant nodes. Essentially, the distribution of nodes defines (in the limit, as $n \to \infty$), a family of curves $C(\rho)$—Runge called them *U*-curves—which are centered about the origin. For a fixed $\rho > 0$, convergence occurs if f is analytic inside $C(\rho)$. In the case of Runge's example, the contour $C(\rho^*)$ which passes through the singularities of f at $\pm i$ also crosses the axis at $x_c = \pm 3.63\ldots$, which explains the result given above.

Unfortunately a complete and rigorous development of these results (see below) requires some subtle analysis as well as the evaluation of complex integrals via the theory of residues. This yields some very elegant and aesthetically pleasing mathematics, but it also places the material beyond the comprehension of most students in undergraduate numerical analysis courses. On the other hand, the *fact* of the Runge example is important enough that one should present it to such a class. But to present it without any justification is difficult, at best, since the Runge example is not the least bit intuitive—and few texts provide any help.

The present paper attempts to fill this gap by bringing together several explanations/developments of the Runge example which do not require extensive complex analysis. Not surprisingly, these are not as sharp as the complete (complex) analysis but they do provide some insight to the essential points.

For completeness, we also outline the complex error analysis, and provide some discussion of the role played by the Chebyshev nodes. Our style is informal, and, hence, many of the proofs are done by reference or merely in outline. The primary goal is to provide an adequate basis for explaining polynomial interpolation nonconvergence. (We might also note in passing that the complex remainder theory is an excellent application of residue theory, and as such could well be presented to a beginning class in complex variables.)

2. A Selected Review of Interpolation Remainder Theory. Let $n > 0$ and $f \in C^{n+1}(I)$, $I = [a, b]$, be given. Denote the interpolation nodes by $\{x_j^{(n)}\}$, for $0 \leqslant j \leqslant n$. Then, if p_n is the polynomial of degree n which interpolates f at those nodes, the usual error estimate is [**1**, p. 56]:

$$f(x) - p_n(x) = \frac{1}{(n+1)!}\left\{\prod_{j=0}^{n}\left(x - x_j^{(n)}\right)\right\} f^{(n+1)}(\xi), \tag{2}$$

where ξ is a point on the interval containing x and the nodes. Convergence proofs based on (2) require some estimate of $|f^{(k)}(x)|$ as $k \to \infty$; for example, if $|f^{(k)}(x)| \leqslant M$ for all k and all $x \in I$, then it is easy to show $p_n \to f$ uniformly on I. (It is worth noting that this holds for an arbitrary distribution of the nodes $\{x_j^{(n)}\}$.) Because it is so difficult to estimate the derivative term, (2) has been of little use in explaining the Runge example. There is a related estimate [**4**]

$$f(x) - p_n(x) = \left\{\prod_{j=0}^{n}\left(x - x_j^{(n)}\right)\right\} f[x_0, \dots, x_n, x], \tag{3}$$

where $f[x_0, \dots, x_n, x]$ is the $(n+1)$st Newton divided difference of f [**4**]. This form of the remainder has been used by Isaacson and Keller [**4**] to treat the Runge example, and we expand upon this approach in §3d, below.

From now on, the polynomial $w_n(x)$ will be defined as

$$w_n(x) = \prod_{j=0}^{n}\left(x - x_j^{(n)}\right). \tag{4}$$

The complex analogue of (2) and (3) is a contour integral [**1**, p. 67]:

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_T} \frac{w_n(z)}{w_n(\zeta)}\,\frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{5}$$

Here C_T is the boundary of a domain T, f is analytic in T, and z and all the nodes are contained in T. Note that T is allowed to contain "holes."

A key point in the analysis of the error is the behavior of $|w_n(z)|$ as $n \to \infty$. From [**1**, p. 84] we have the following:

LEMMA 1. *Assume the* $\{x_j^{(n)}\}$ *are equidistant nodes on* $[a, b]$, *and define*

$$\sigma_n(z) = |w_n(z)|^{1/(n+1)}.$$

Then

$$\lim_{n\to\infty} \sigma_n(z) = \sigma(z) \tag{6}$$

exists for all z. *In particular*,

$$\sigma(z) = \exp\left\{\frac{1}{b-a}\int_a^b \log|z - s|\,ds\right\}. \tag{7}$$

Note that (7) says that $\sigma(z)$ is the geometric mean of $|z - s|$, which is not surprising, since $|w_n(z)|^{1/(n+1)}$ is the geometric mean of the $|z - x_j^{(n)}|$.

For $\rho > 0$, consider now the family of curves

$$C(\rho) = \{z \in \mathbb{C} | \sigma(z) = \rho\}.$$

These are smooth concentric curves about the midpoint of $[a, b]$; in fact, their level curves in terms of (x, y) coordinates can be computed by integrating in (7). These curves and the placement of z relative to them are the key to convergence: if $\zeta \in C(\rho)$ and $z \in C(\rho')$, $\rho' < \rho$, we can show

$$\lim_{n\to\infty} \left| \frac{w_n(z)}{w_n(\zeta)} \right| = 0,$$

which can be used to prove convergence of p_n to f.

Proposition. *Let the interpolation nodes* $\{x_j^{(n)}\}$ *be contained in a contour* $C(\rho)$ *and suppose* f *is analytic inside* $C(\rho)$. *Then* $p_n \to f$ *uniformly on* $C(\rho')$, $\rho' < \rho$.

Proof. Let z be in $C(\rho')$; then (5) implies

$$|f(z) - p_n(z)| \leqslant C \max_{\zeta \in C(\rho)} \left| \frac{w_n(z)}{w_n(\zeta)} \right|.$$

This follows from the maximum modulus theorem and the fact that $|z - \zeta| > 0$ (since the two curves $C(\rho)$ and $C(\rho')$ do not intersect). But, for ε arbitrarily small and n sufficiently large,

$$\left| \frac{w_n(z)}{w_n(\zeta)} \right|^{1/(n+1)} = \left(\frac{\sigma_n(z)}{\sigma_n(\zeta)} \right) \leqslant \left(\frac{\rho' + \varepsilon}{\rho - \varepsilon} \right) < 1;$$

thus

$$|f(z) - p_n(z)| \leqslant C\theta^{n+1}$$

for $0 < \theta < 1$, and convergence follows.

Suppose now that f is not analytic inside $C(\rho)$; suppose in fact, that there is a single simple pole at z^*, and, in order to enclose z in a contour $C(\rho)$, we must also enclose z^*. Then, for the error representation (5) to be valid, the contour C_T must consist of the union of $C(\rho)$ and a small path around z^*, say C^*. Then

$$\frac{1}{2\pi i} \int_{C(\rho)} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{C_T} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{C^*} \frac{w_n(z)}{w_n(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Since z is inside $C(\rho)$, the integral on the left goes to zero as $n \to \infty$, exactly as in the previous case; the first integral on the right is exactly the error $f(z) - p_n(z)$; and the last integral can be quickly evaluated by a residue. Thus we have

$$f(z) - p_n(z) = \left(\frac{w_n(z)}{w_n(z^*)} \right) \left(\frac{f(z^*)}{z^* - z} \right) + \delta_n, \tag{8}$$

where $|\delta_n| \to 0$ as $n \to \infty$. Now, since $z \in C(\rho)$ and $z^* \in C(\rho^*)$, $\rho^* < \rho'$, it

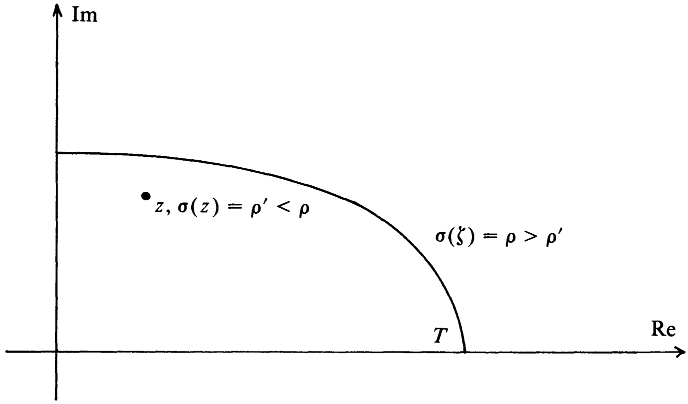

FIG. 1. Convergence occurs in the region T so long as f is analytic there.

follows that, as before,

$$\left|\frac{w_n(z)}{w_n(z^*)}\right|^{1/(n+1)} = \left(\frac{\sigma_n(z)}{\sigma_n(z^*)}\right) \geqslant \frac{\rho' - \varepsilon}{\rho^* + \varepsilon} > 1.$$

Thus the error grows without bound. (See Figures 1 and 2.) If we enclose more than one pole, then the *-terms in (8) are replaced by an appropriate summation over all the poles; divergence still occurs.

In the case of Runge's example (1), the contour $C(\rho_c)$ which passes through the singularities $\pm i$ crosses the x-axis at $x_c = 3.6333843024$. Thus, for $|x| < x_c$ we can draw C_T such that each $\zeta \in C_T$ is on a contour $C(\rho_\zeta)$, $\rho_\zeta > \rho_c - \varepsilon$, and C_T encloses

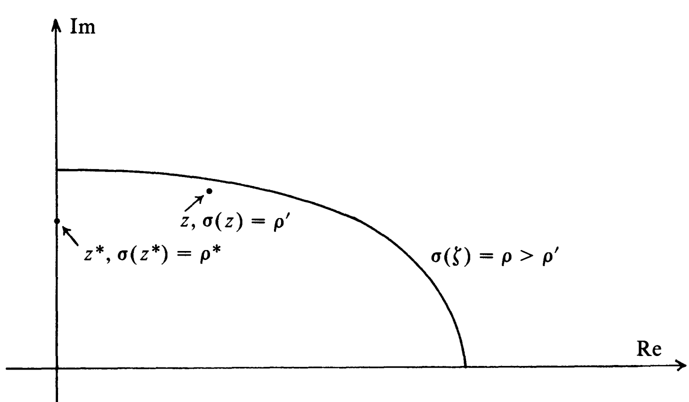

FIG. 2. Divergence occurs at z because we must enclose the pole z^* in order to also enclose z.

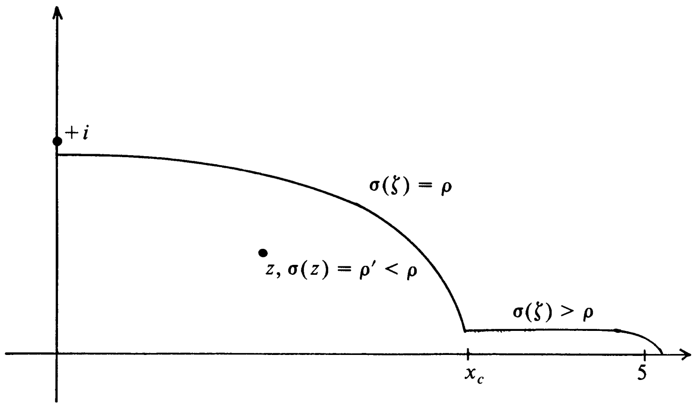

FIG. 3. The Runge Example: convergence occurs for z because $\sigma(z) < \rho$ and the contour C_T (solid curve) consists of points ζ satisfying $\sigma(\zeta) \geq \rho$.

all the interpolation nodes as well as x, but not the poles $\pm i$. (See Figure 3.) Since $|x| < x_c$ it follows that $x \in C(\rho)$, $\rho < \rho_c - \varepsilon$; hence

$$\min_{\zeta \in C_T} \left| \frac{w_n(x)}{w_n(\zeta)} \right|^{1/(n+1)} \to \theta < 1$$

and convergence follows.

If x is outside x_c then any contour enclosing x must also enclose the poles and divergence occurs.

We summarize the foregoing discussion in the following theorem:

THEOREM 1. *Let* $\{p_n\}$ *be a sequence of polynomials interpolating* f *at the equidistant points* $\{x_k^{(n)}\}$, $0 \leqslant k < n$, *with each* $x_k^{(n)} \in [a, b]$. *Assume* f *is analytic on* $[a, b]$. *Let* $\sigma(z)$ *be as defined in* (6). *Then*:

i) If f *is analytic for all* z *such that* $\sigma(z) < \rho$, *then* $p_n \to f$ *for each* z^* *such that* $\sigma(z^*) < \rho$. *The convergence is uniform for all* z^* *such that* $\sigma(z^*) \leqslant \rho^* < \rho$.

ii) If f *has a pole* z^*, *and* z *is such that* $\sigma(z) > \sigma(z^*)$, *then* $p_n(z) \nrightarrow f(z)$.

If we switch from uniform nodes to the Chebyshev nodes (these are the roots of the $(n + 1)$st Chebyshev polynomial—see §4), then the contours $C(\rho)$ become true ellipses having foci at $x = \pm 5$ [**1**, p. 83]. Thus, it is easy to choose a contour which completely encloses the interval without hitting the singularities at $\pm i$. In fact, the following theorem holds.

THEOREM 2. *If* f *is analytic in an open domain containing* $I = [a, b]$, *and* p_n *interpolates* f *at the Chebyshev nodes on* $[a, b]$, *then*

$$p_n \to f$$

uniformly on $[a, b]$. (*Note that if* $[a, b] \neq [-1, 1]$ *the Chebyshev nodes must be transformed*.)

3. The Runge Example Without Residues. In this section we present several discussions aimed at justifying (if only partially) the Runge example without having to use residues or the complex remainder (5).

3a. *Interpolation is Ill-Conditioned* [**3**]. Here we show that small errors in computing the $\{f(x_k^{(n)})\}$—round-off error, or experimental "noise"—can be greatly magnified by interpolation at equidistant nodes, especially for x "near the edge of the interval."

Consider the Lagrange form of the interpolating polynomial [**1**, p. 33]:

$$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x),$$

where

$$l_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{x - x_k^{(n)}}{x_j^{(n)} - x_k^{(n)}} \right).$$

If the nodes are equally spaced then

$$x_j^{(n)} = a + jh, \qquad h = \left(\frac{b-a}{n} \right).$$

Suppose now that a small error is made in computing each $f(x_j)$. That is, we actually form

$$\bar{p}_n(x) = \sum_{j=0}^{n} \bar{f}(x_j) l_j(x),$$

where $\bar{f}(x_j) = f(x_j) + \varepsilon_j$. Then the error due to roundoff is

$$E_n = \sum_{j=0}^{n} l_j(x) \varepsilon_j.$$

Let $x = a + h/2$ so that x is near the edge of the interval. Then

$$l_j\left(a + \frac{h}{2}\right) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{\frac{1}{2} - k}{j - k} \right),$$

and, after a page or so of calculation, this becomes

$$l_j\left(a + \frac{h}{2}\right) = \left(\frac{-1}{2j-1} \right) \binom{n}{j} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right].$$

Using Stirling's formula the bracketed term can be estimated as

$$\left[\frac{(2n)!}{2^{2n}(n!)^2}\right] \sim \frac{1}{\sqrt{n}}.$$

Thus,

$$l_j\left(a+\frac{h}{2}\right) \sim \frac{1}{\sqrt{n}}\left(\frac{-1}{2j-1}\right)\binom{n}{j}.$$

If $n = 2m$ and $j = m$, i.e., even for j near the middle of the interval, we have

$$\binom{n}{j} = \binom{2m}{m} = \frac{(2m)!}{(m!)^2} \sim \frac{2^n}{\sqrt{m}},$$

so that

$$l_m\left(a+\frac{1}{2}\right) \sim \left(\frac{-1}{n-1}\right)m^{-1}(2^n).$$

Hence the small error $\varepsilon_{n/2}$ is multiplied by a factor which grows exponentially. Unless we are extremely lucky with cancellation, the amplified error will eventually dominate the calculation. Unfortunately, this approach to the Runge phenomenon ignores the role of f and leaves the impression that the divergence is perhaps due to machine error, which is not at all the case. On the other hand, it is a valuable demonstration of why polynomial interpolation (with equidistant nodes) at high degree is not, in general, a good approximation technique, even for analytic functions.

3b. *Growth of the Interpolate* [**2**], [**7**]. Obviously if $\|p_n\| \to \infty$ then p_n isn't a good approximation to "nice" functions f. If $\|\ \ \|$ is the sup-norm, then we have

$$\begin{aligned}\|p_n\| &= \left\|\sum_{j=0}^{n} l_j f(x_j)\right\| \\ &\leqslant \|f\| \sum_{j=0}^{n} \|l_j\| \\ &= \|f\|\Lambda_n,\end{aligned}$$

and there are functions f for which this bound is sharp [**2**], i.e., equality holds. For equally spaced nodes $x_j^{(n)}$ it can be shown [**8**, pp. 87–99] that

$$\Lambda_n \geqslant Cn^{-3/2}(2^{n-1}), \tag{9}$$

thus the norm of the interpolate is unbounded. (The argument leading to (9) is very similar to what we used in the preceding subsection.) Again, we have a result showing that the l_j's can grow rapidly, but it is not clear how this worst case estimate would apply to a very smooth function like $(1 + x^2)^{-1}$. The functions for which $\|p_n\| = \|f\|\Lambda_n$ are not, in general, very smooth.

3c. *Derivative Bounds*. The previous two discussions made no hypotheses on the function being interpolated. Here we consider what sort of behavior for $f(x)$ would lead to convergence or divergence.

The simplest estimate (2) says

$$f(x) - p(x) = \frac{1}{(n+1)!} w_n(x) f^{(n+1)}(\xi_x).$$

Let $x = \bar{x}_j = x_j + \frac{1}{2}h$, i.e., x is halfway between two nodes. Then

$$\begin{aligned} w_n(\bar{x}_j) &= \prod_{k=0}^{n} \left(x_j + \frac{1}{2}h - x_k \right), \\ &= h^{n+1} \prod_{k=0}^{n} \left(j - k + \frac{1}{2} \right). \end{aligned}$$

For $j = 0$, $\bar{x}_j$ is a point near the edge of $[a, b]$, and in this case

$$\begin{aligned} |w(\bar{x}_0)| &= h^{n+1} \prod_{k=0}^{n} \left(k - \frac{1}{2} \right), \\ &= h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \frac{(2n)!}{n!}, \end{aligned}$$

so that

$$\begin{aligned} |f(\bar{x}_0) - p(\bar{x}_0)| &= h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \frac{(2n)!}{n!(n+1)!} |f^{(n+1)}(\xi_x)| \\ &= \frac{1}{n+1} h^{n+1} \left(\frac{1}{2} \right)^{2n+1} \left[\frac{(2n)!}{(n!)^2} \right] |f^{(n+1)}(\xi_x)| \end{aligned}$$

and Stirling's formula yields

$$|f(\bar{x}_0) - p_n(\bar{x}_0)| \sim h^{n+1} n^{-3/2} |f^{(n+1)}(\xi_x)|. \tag{10}$$

Now computational experience as well as the preceding sections lead us to believe that the error at the ends of the interval should go to infinity as n does, i.e., (10) can be considered a worst case estimate. Yet the leading factors in (10) behave like $n^{-3/2} h^{n+1}$ so that the derivative term must grow quite rapidly to force divergence. A quasi-uniform estimate of this type can be had by noting (from Stirling's formula)

$$\frac{1}{(n+1)!} |w_n(x)| \leqslant \frac{(b-a)^{n+1}}{(n+1)!} \leqslant \frac{C}{\sqrt{n+1}} \left(\frac{e(b-a)}{n+1} \right)^{n+1}.$$

Thus, for any x,

$$|f(x) - p_n(x)| \leqslant \frac{C}{\sqrt{n+1}} \left(\frac{e(b-a)}{n+1} \right)^{n+1} |f^{(n+1)}(\xi_x)|.$$

Unfortunately this approach does not yield a lower bound, and so divergence does not necessarily follow for $|f^{(n+1)}(\xi_x)|$ large. But it does show that $\|f^{(n+1)}\| \sim \sqrt{n+1}\,(n+1)/e(b-a)^{n+1}$ is necessary for there to be any chance of divergence.

In summary, then, we have:

THEOREM 3. *Let $\{p_n\}$ interpolate f at the equidistant nodes $\{x_j^{(n)}\}$, $0 \leqslant j \leqslant n$. If*

$$\lim_{n\to\infty}\left\{\frac{\|f^{(n+1)}\|}{\sqrt{n+1}}\left(\frac{e(b-a)}{n+1}\right)^{n+1}\right\} = 0,$$

then

$$p_n \to f,$$

uniformly on $[a, b]$.

The above discussion (and theorem) also appears to contradict the oft-read dictum "divergence of interpolation polynomials is due to the growth of the $l_k(x)$ functions." (See [**2**, p. 25] and [**7**, p. 35] for statements to this effect.) While the importance of the growth of the l_k should not be underestimated, (10) clearly indicates that the interpolation will converge for functions whose derivatives behave "moderately well" as $n \to \infty$. (Note, however, that the results of §3a still show that polynomial interpolation is *ill-conditioned* as $n \to \infty$.)

3d. *A Real Variable Estimate*. Consider a generalization of Runge's example

$$f(x) = (x^2 + s^2)^{-1}, \qquad x \in [-a, a].$$

In this subsection we will develop a series of estimates which show that polynomial interpolation (with equidistant nodes) to f converges only for $|x|$ sufficiently small and s sufficiently large. We do this with the estimate (3), for which we need an expression for the divided difference.

LEMMA 2.

$$f[x_0, \ldots, x_n, x] = \left(\frac{1}{iw_n(si)}\right)\left(\frac{r_n}{x^2 + s^2}\right), \qquad r_n = \begin{cases} x, & n \text{ even} \\ si, & n \text{ odd} \end{cases}.$$

Proof. We write $f(x)$ as

$$f(x) = \frac{1}{2si}\left\{\frac{1}{x - si} - \frac{1}{x + si}\right\}$$

and note that the divided difference operator is linear. Further, letting

$$g_1(x) = \frac{1}{x - si}, \qquad g_2(x) = \frac{1}{x + si},$$

we have (by induction on m)

$$g_1[\xi_0,\dots,\xi_m] = (-1)^m \frac{1}{\prod_{j=0}^{m}(\xi_j - si)},$$

$$g_2[\xi_0,\dots,\xi_m] = (-1)^m \frac{1}{\prod_{j=0}^{m}(\xi_j + si)}.$$

Thus, using (4),

$$f\left[x_0^{(n)},\dots,x_n^{(n)},x\right] = \frac{1}{2si}\left\{\frac{1}{(x-si)w_n(si)} - \frac{1}{(x+si)w_n(-si)}\right\}$$

But the symmetric distribution of the nodes implies that $x_j^{(n)} = -x_{n-j}^{(n)}$, so $w_n(-si) = (-1)^{n+1}w_n(si)$. Hence

$$\begin{aligned} f\left[x_0^{(n)},\dots,x_n^{(n)},x\right] &= \frac{1}{2si}\left\{\frac{1}{(x-si)w_n(si)} - \frac{(-1)^{n+1}}{(x+si)w_n(si)}\right\} \\ &= \left\{\frac{1}{iw_n(si)}\right\}\left\{\frac{r_n}{x^2+s^2}\right\}, \end{aligned}$$

which completes the proof.

It follows immediately from the lemma and (3) that

$$|f(x) - p_n(x)| = \frac{|r_n|}{s^2+x^2}\left|\frac{w_n(x)}{w_n(si)}\right|. \tag{11}$$

Thus (just as in the complex analysis) convergence depends entirely on the limiting value of $|w_n(x)/w_n(si)|$. In particular, $p_n(x) \to f(x)$ if and only if $\sigma(x)$ (as defined in (6)) is less than $\sigma(si)$. Further, $p_n \to f$ uniformly whenever

$$\max_{x\in[-a,a]} \sigma(x) < \sigma(si).$$

Thus uniform convergence is a function of the two parameters a (half-length of interval of interpolation) and s (distance from poles to real-axis). An ordinary calculation shows that

$$\max_{x\in[-a,a]} \sigma(x) = \sigma(a) = 2a/e.$$

Further, we find that $\sigma(si) \leqslant \sigma(a)$ if and only if

$$\ln(1+\xi^2) + 2\xi\arctan(1/\xi) - \ln 4 > 0,$$

where $\xi = s/a$. This function is monotone increasing for $\xi \geqslant 0$ and has a unique root $\xi^* \approx .5255$. Thus we get uniform convergence for $s > \xi^* a$. Note that for

Runge's original function this is not achieved until the interval shrinks to (approximately) $[-1.9,\ 1.9]$, whereas on $[-5,5]$ we must move the singularities to (approximately) ± 2.63. This leads to the following result.

THEOREM 4. *Let f be of the form*

$$f(x) = (x^2 + s^2)^{-1}$$

for $x \in [-a, a]$. If p_n interpolates f at the equidistant nodes $\{x_j^{(n)}\}$, $0 \leqslant j \leqslant n$, then $p_n \to f$ uniformly on $[-a, a]$ if and only if

$$s > \xi^* a,$$

for $\xi^* \approx .5255$.

This last theorem is the crux of the matter, for it clearly indicates the separate roles played by the singularities and by the interval length.

4. Chebyshev Nodes. Consider now what happens if the Chebyshev nodes are used instead of equidistant nodes. Define

$$t_j^{(n)} = \cos\left[\left(j + \frac{1}{2}\right)\frac{\pi}{n+1}\right], \qquad 0 \leqslant j \leqslant n$$

as the Chebyshev nodes, i.e., the roots of the $(n+1)$st Chebyshev polynomial $T_{n+1}(x)$. (In what follows we assume the interval is now $[-1,1]$.) Then we have [**1**, p. 61]

$$\prod_{j=0}^{n}\left(x - t_j^{(n)}\right) = \tilde{T}_{n+1}(x) = \left(\frac{1}{2}\right)^n T_{n+1}(x),$$

and $|T_{n+1}(x)| \leqslant 1$ for all $x \in [-1,1]$. It is a well-known (but by no means trivial) result that, for this choice of nodes [**8**, p. 94],

$$\Lambda_n = \sum_{j=0}^{n} \|l_j\| \leqslant \frac{2}{\pi}\log n + 4.$$

Thus, the growth of the l_k functions is much less than for equidistant nodes. Moreover, we see that the effects of round off error are much less:

$$\|E_n\| \leqslant \varepsilon_{\max}\Lambda_n \leqslant \left(\frac{2}{\pi}\log n + 4\right)\varepsilon_{\max}.$$

In terms of the derivative bounds of §3c, we have

$$|f(x) - p_n(x)| \leqslant \frac{1}{(n+1)!}|\overline{T}_{n+1}(x)||f^{(n+1)}(\xi_x)|,$$

$$\leqslant \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n |f^{(n+1)}(\xi_x)|.$$

Since the Chebyshev polynomials are, in an appropriate sense, minimal over $[-1,1]$, we can in fact assert that this choice of nodes minimizes the error by minimizing the upper bound for $|w_n(x)|/(n+1)!$. Divergence is still possible if $|f^{(n+1)}(\xi_x)|$ grows too rapidly.

Finally, note that the analysis of §3d carries through for any set of (symmetric) nodes as far as (11). Thus, for the Chebyshev nodes in $[-1,1]$, we have that

$$|f(x)-p_n(x)| \sim \left|\frac{w_n(x)}{w_n(si)}\right|.$$

But $w_n(x) = \tilde{T}_{n+1}(x)$ for any x, hence

$$\left|\frac{w_n(x)}{w_n(si)}\right| = \left|\frac{T_{n+1}(x)}{T_{n+1}(si)}\right| \leqslant \frac{1}{|T_{n+1}(si)|} = \frac{1}{|\cos(n+1)z|},$$

where $z = \arccos(si)$. Standard properties of the elementary functions on $\mathbb{C}$ then imply that

$$z = \frac{\pi}{2} - i\log\left(s+\sqrt{s^2+1}\right).$$

Thus

$$\begin{aligned}|\cos(n+1)z| &= \left|\cos\left((n+1)\frac{\pi}{2} - i(n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right| \\ &= \begin{cases}\left|\cosh\left((n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right|, & n \text{ odd},\\ \left|\sinh\left((n+1)\log\left(s+\sqrt{s^2+1}\right)\right)\right|, & n \text{ even},\end{cases} \\ &\geqslant C(1+s)^{n+1},\end{aligned}$$

and so

$$|f(x)-p_n(x)| \leqslant \frac{C}{(1+s)^{n+1}}, \quad \text{for all } x,$$

which implies uniform convergence for all $s > 0$.

Historical Comment. Although most references use only Runge's name it appears that Meray ([**5**], [**6**]) also contributed to understanding this phenomenon.

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