Dado $0 < r < 1$ $\quad \displaystyle\int_{C(0,r)} \frac{\log(1+z)}{z}\,dz = 0$

Ej 8.a (rel. 7)

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n+1} \quad \forall z \in D(0,1)$$

$$\frac{\log(1+z)}{z} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n} \quad \forall z \in D(0,1)\setminus\{0\}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n-1} \quad \forall z \in D(0,1)\setminus\{0\} \supset C(0,r)^*$$

admite primitiva en $D(0,1)\setminus\{0\}$ por venir dada como la suma de una serie de potencias $\rightsquigarrow$

$\overset{R-B}{\Longrightarrow}$ $\displaystyle\int_\gamma \frac{\log(1+z)}{z}\,dz = 0 \quad \forall \gamma$ camino cerrado en $D(0,1)\setminus\{0\}$.

$\gamma : C(0,r) \longrightarrow \mathbb{C} \quad \gamma(t) = re^{it} \quad \gamma'(t) = ire^{it}\,dt$

$$= i\int_{-\pi}^{\pi} \log(1+re^{it})\,dt = i\int_{-\pi}^{\pi} \left(\ln|1+re^{it}| + i\arg(1+re^{it})\right) dt \Rightarrow$$

$\log(w) = \ln|w| + i\arg(w)$

$\log(1+re^{it}) = \ln|1+re^{it}| + i\arg(1+re^{it})$

$$\Rightarrow \int_{-\pi}^{\pi} \ln|1+re^{it}|\,dt = 0$$

$$|1+re^{it}| = \left((1+r\cos t)^2 + (r\sin t)^2\right)^{1/2} = \Big(1+2r\cos t\, \underbrace{r^2\cos^2 t + r^2\sin^2 t}_{= r^2}\Big)^{1/2} =$$

$$re^{it} = r\cos t + i\, r\sin t$$

$$= \left(1+r^2+2r\cos t\right)^{1/2}$$

$$\log|1+re^{it}| = \frac{1}{2}\ln\left(1+r^2+2r\cos t\right)$$

$$0 = \int_{-\pi}^{\pi} \ln|1+re^{it}|\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\ln(1+r^2+2r\cos t)}_{\text{par}}\,dt = \int_0^{\pi} \ln(1+r^2+2r\cos t)\,dt$$

$$\Rightarrow \int_{-\pi}^{\pi} \ln|1+re^{it}|\,dt = 0$$