Ejercicios tema 14:

① Probar que para $a \in]0,1[$ se tiene:

$$\int_0^{2\pi} \frac{\cos^2(3t)\,dt}{1+a^2-2a\cos(2t)} = \pi \frac{a^2-a+1}{1-a}$$

Debemos encontrar una función adecuada que al integrarla sobre la cfa. unidad $\gamma : [0, 2\pi] \to \mathbb{C}$ nos de la integral que queremos.

$\gamma(t) = e^{it}$

$\gamma'(t) = ie^{it}$

$$\int_\gamma f(z)dz = \int_0^{2\pi} f(e^{it})\,ie^{it}\,dt$$ Buscamos entonces f de modo que

$f(e^{it})ie^{it}$ esté relacionado con $\dfrac{\cos^2(3t)}{1+a^2-2a\cos(2t)}$

$$1+a^2-2a\cos(2t) = |1-ae^{i2t}|^2 = |1-a\cos(2t) - ia\,\text{sen}(2t)|^2 = (1-a\cos(2t))^2 + a^2\,\text{sen}^2(2t)$$

$$= 1-2a\cos(2t) + \underbrace{a^2\cos^2(2t)+a^2\,\text{sen}^2(2t)}_{a^2} = 1+a^2-2a\cos(2t)$$

$$|1-ae^{i2t}|^2 = |1-a(e^{it})^2|^2 = (1-a(e^{it})^2)\overline{(1-a(e^{it})^2)} =$$

$$= (1-a(e^{it})^2)(1-a(e^{-it})^2)$$

Llamando $g(z) = (1-az^2)(1-a\cdot\frac{1}{z^2})$ tenemos que

$$g(e^{it}) = (1-a(e^{it})^2)(1-a(e^{-it})^2) = 1+a^2-2a\cos(2t)$$

Trabajamos ahora con el numerador:

$$\cos^2(3t) = \frac{1+\cos(6t)}{2} = \text{Re}\,h(e^{it})$$

$\cos(6t) + i\,\text{sen}(6t) = e^{i6t} = (e^{it})^6$ tomando $h(z) = \dfrac{1+z^6}{2}$ obtenemos

$$\text{Re}\,h(e^{it}) = \text{Re}\frac{1+(e^{it})^6}{2} = \frac{1+\cos(6t)}{2}$$

$$f(z)=\frac{h(z)}{z\,g(z)} \qquad f(e^{it})=\frac{h(e^{it})}{e^{it}g(e^{it})}$$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{\frac{1+(e^{it})^6}{2}\cdot i\cancel{e^{it}}}{\cancel{e^{it}}(1+a^2-2a\cos(2t))}\,dt = \int_0^{2\pi}\frac{\left[\overset{\cos^2(3t)}{\overset{\shortparallel}{\frac{1+\cos(6t)}{2}}} + \frac{i\,\text{sen}(6t)}{2}\right]i}{1+a^2-2a\cos(2t)}\,dt =$$

$$= -\int \frac{\frac{\text{sen}(6t)}{2}}{1+a^2-2a\cos(2t)} + i\int_0^{2\pi}\frac{\cos^2(3t)}{1+a^2-2a\cos(2t)}\,dt$$

$$\rightsquigarrow \int_0^{2\pi}\frac{\cos^2(3t)}{1+a^2-2a\cos(2t)}\,dt = \text{Im}\left(\int_\gamma f(z)\,dz\right)$$

$$f(z)=\frac{h(z)}{z\,g(z)} = \frac{1}{2}\,\frac{1+z^6}{z(1-az^2)\left(1-\frac{a}{z^2}\right)} = \frac{1}{2}\,\frac{(1+z^6)\,z}{(1-az^2)(z^2-a)}$$

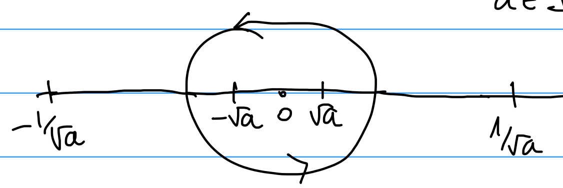
$f:\mathbb{C}\setminus\{\pm\sqrt{a}, \pm\frac{1}{\sqrt{a}}\}\to\mathbb{C}$ Llamamos $\Omega=\mathbb{C}$ y $A=\{\pm\sqrt{a}, \pm\frac{1}{\sqrt{a}}\}$

$f\in\mathcal{H}(\Omega\setminus A)$
Ω abto.
$A\subset\Omega$ y $A'\cap\Omega=\emptyset$
γ es un ciclo en $\Omega\setminus A$
γ es nul-homólogo con respecto a Ω (trivialmente)

Teorema $\Longrightarrow$ de los residuos

$$\int_\gamma f(z)\,dz = 2\pi i\left[\overset{1}{\overset{\shortparallel}{\text{Ind}_\gamma(-\sqrt{a})}}\,\text{Res}(f(z),-\sqrt{a}) + \underset{\underset{1}{\shortparallel}}{\text{Ind}_\gamma(\sqrt{a})}\cdot\text{Res}(f(z),\sqrt{a})\right]$$

$a\in\,]0,1[\ \rightsquigarrow \sqrt{a}\in\,]0,1[$
$\Downarrow$
$\frac{1}{\sqrt{a}}>1$



$\lim_{z\to\sqrt{a}}(z-\sqrt{a})f(z) \rightsquigarrow \exists$ y es distinto de cero luego f tiene un polo de orden 1 en y además
$\lim_{z\to-\sqrt{a}}(z+\sqrt{a})f(a) \nearrow$

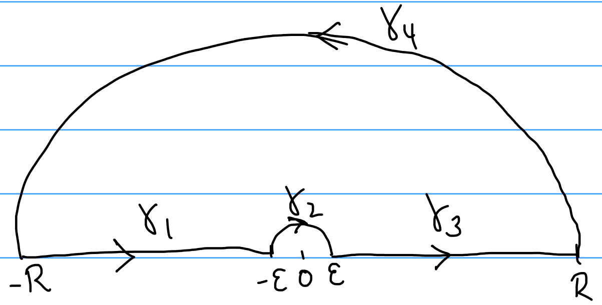
$\text{Res}(f(z),\sqrt{a}) = \lim_{z\to\sqrt{a}}(z-\sqrt{a})f(z)$ Ejercicio

$\text{Res}(f(z),-\sqrt{a}) = \lim_{z\to-\sqrt{a}}(z+\sqrt{a})f(z)$

—— o ——

$x^4+a^4=0 \iff x^4=-a^4$ las soluciones son $a\cdot$ raíces cuartas de -1

(9) $f: \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$ $\quad f(z) = \dfrac{1-e^{2iz}}{z^2}$ $\quad f \in \mathcal{H}(\mathbb{C}\setminus\{0\})$



Para cada $0<\varepsilon<R$ consideramos el camino cerrado

$$\Gamma_{R,\varepsilon} = \gamma_1+\gamma_2+\gamma_3+\gamma_4$$

$\gamma_1 : [-R,-\varepsilon] \longrightarrow \mathbb{C}$ $\quad \gamma_2 : [0,\pi] \longrightarrow \mathbb{C}$ $\quad \gamma_3 : [\varepsilon, R] \longrightarrow \mathbb{C}$ $\quad \gamma_4 : [0,\pi] \longrightarrow \mathbb{C}$

$\gamma_1(x) = x$ $\quad \gamma_2(x) = \varepsilon e^{i(\pi - x)}$ $\quad \gamma_3(x) = x$ $\quad \gamma_4(x) = Re^{ix}$

Llamemos $\Omega = \mathbb{C}$, $A=\{0\}$ $\quad f \in \mathcal{H}(\Omega\setminus A)$

Para cada $0<\varepsilon<R$

$f \in \mathcal{H}(\Omega\setminus A)$
Ω abierto
$A \subset \Omega$ y $A' \cap \Omega = \phi$
$\Gamma_{R,\varepsilon}$ es un ciclo en $\Omega\setminus A$
nul homólogo con respecto a Ω

$\xRightarrow[\text{Residuos}]{\text{Teorema}}$ $\displaystyle\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = 0 \quad \forall\, 0<\varepsilon<R$

Por tanto tenemos

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 0 \qquad (1)$$

$$\int_{\gamma_3} f(z)\,dz = \int_{\varepsilon}^{R} \frac{1-e^{2ix}}{x^2}\,dx = \int_{\varepsilon}^{R} \frac{1-(\cos(2x)+i\,\mathrm{sen}(2x))}{x^2}\,dx$$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{1-e^{2ix}}{x^2}\,dx = \int_{-R}^{-\varepsilon} \frac{(1-\cos(2x))-i\,\mathrm{sen}(2x)}{x^2}\,dx$$

Como $\dfrac{\mathrm{sen}(2x)}{x^2}$ es impar $\displaystyle\int_{-R}^{-\varepsilon} \frac{\mathrm{sen}(2x)}{x^2}\,dx = -\int_{\varepsilon}^{R} \frac{\mathrm{sen}(2x)}{x^2}\,dx$

Como $\dfrac{1-\cos(2x)}{x^2}$ es par $\displaystyle\int_{-R}^{-\varepsilon} \frac{1-\cos(2x)}{x^2}\,dx = \int_{\varepsilon}^{R} \frac{1-\cos(2x)}{x^2}\,dx$

Por tanto, $\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = 2\int_{\varepsilon}^{R} \frac{1-\cos(2x)}{x^2}dx = 4\int_{\varepsilon}^{R} \frac{\operatorname{sen}^2(x)}{x^2}dx$

$1-\cos(2x) = 2\operatorname{sen}^2(x)$

Entonces $\lim\limits_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\gamma_1} f(z) + \int_{\gamma_3} f(z)dz = 4\int_0^{+\infty} \frac{\operatorname{sen}^2(x)}{x^2}dx \quad (2)$

Calculamos ahora $\lim\limits_{\varepsilon\to 0} \int_{\gamma_2} f(z)dz = \lim\limits_{\varepsilon\to 0} -\int_{-\gamma_2} f(z)dz = (*)$

$-\gamma_2$ es una semicircunferencia centrada en cero recorrida en sentido positivo

Si probamos que f tiene un polo de orden 1 en cero, la proposición de ayer nos dice que

$(*) = -i\pi \operatorname{Res}(f(z),0) = -i\pi(-2i) = -2\pi$

$\lim\limits_{z\to 0} \cancel{z} \frac{1-e^{2iz}}{z^{\cancel{2}}} \overset{L'H}{=} \lim\limits_{z\to 0} \frac{-2i\,e^{2iz}}{1} = -2i \neq 0 \Rightarrow f$ tiene tiene un polo de orden 1 en cero y además $\operatorname{Res}(f(z),0) = -2i$

Por tanto, $\lim\limits_{\varepsilon\to 0} \int_{\gamma_2} f(z)dz = -2\pi \quad (3)$

Por último calculamos $\lim\limits_{R\to\infty} \int_{\gamma_4} f(z)dz$:

$\left| \int_{\gamma_4} f(z)dz \right| \leq \underbrace{l(\gamma_4)}_{\pi R} \cdot \frac{2}{R^2} = \frac{2\pi}{R} \xrightarrow{R\to\infty} 0$

Si $z \in \gamma_4^*$ tenemos que $z = R(\cos x + i \operatorname{sen} x) \quad x \in [0,\pi] \quad |z| = R$

$|f(z)| = \frac{|1-e^{2iz}|}{|z|^2} \leq \frac{1+|e^{2iz}|}{R^2} \leq \frac{2}{R^2}$

$|e^{2iz}| = |e^{2iR(\cos x + i\operatorname{sen} x)}| = |e^{2iR\cos x - 2R\operatorname{sen} x}| = e^{-2R\operatorname{sen} x} \leq 1$

$|e^z| = e^{\operatorname{Re} z}$

$x \in [0,\pi] \Rightarrow \operatorname{sen} x \geq 0 \Rightarrow -2R\operatorname{sen} x \leq 0 \rightsquigarrow e^{-2R\operatorname{sen} x} \leq 1$

Por tanto $\lim\limits_{R\to\infty} \int_{\gamma_4} f(z)dz = 0 \quad (4)$

Tomamos límite con $R \to \infty$ y $\varepsilon \to 0$ en (1) y utilizamos (2), (3) y (4) para deducir que

$$4\int_0^{+\infty} \frac{\operatorname{sen}^2(x)}{x^2}\,dx - 2\pi = 0 \Rightarrow \underline{\underline{\int_0^{+\infty} \frac{\operatorname{sen}^2(x)}{x^2}\,dx = \frac{\pi}{2}}}$$