Optical Flow
Horn and Schunck

The goal of this exercise is to estimate the Optical Flow thanks to the Horn and Shunck method[1]. We have to find the optical flow (u, v) which minimizes :

$$E(u,v) = \iint \left\{ (I_x u + I_y v + I_t)^2 + \lambda^2 (u_x^2 + u_y^2 + v_x^2 + v_y^2) \right\} dxdy \tag{1}$$

1. If we pose $F(u, v, u_x, u_y, v_x, v_y) = (I_x u + I_y v + I_t)^2 + \lambda^2(u_x^2 + u_y^2 + v_x^2 + v_y^2)$, the fundamental lemma of calculus of variations[2] shows that (u, v) is solution of the Euler-Lagrange equation :

$$\begin{aligned} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} = 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_y} = 0 \end{aligned} \tag{2}$$

   Show that (u, v) is solution of :

$$\begin{aligned} I_x(I_x u + I_y v + I_t) - \lambda^2 \triangle u = 0 \\ I_y(I_x u + I_y v + I_t) - \lambda^2 \triangle v = 0 \end{aligned} \tag{3}$$

2. As shown in the section 8 of Horn and Shunck's paper, we can approximate the discrete Laplacian operator by :

$$\triangle u(x,y) = \bar{u}(x,y) - u(x,y) \tag{4}$$

   where $\bar{u}(x, y)$ is a weighted average of u in a neighborhood around (x, y). Deduce the iterative scheme for the estimation of (u, v) :

$$\begin{aligned} u^{k+1} = \bar{u}^k - \tfrac{I_x(I_x\bar{u}^k + I_y\bar{v}^k + I_t)}{\lambda^2 + I_x^2 + I_y^2} \\ v^{k+1} = \bar{v}^k - \tfrac{I_y(I_x\bar{u}^k + I_y\bar{v}^k + I_t)}{\lambda^2 + I_x^2 + I_y^2} \end{aligned} \tag{5}$$

1. Horn B.K.P and Shunck B.G. Determining optical flow. Artificial Intelligence. Vol 17 pp 185-203
2. `https://en.wikipedia.org/wiki/Calculus_of_variations#Euler.E2.80.93Lagrange_equation`