

# Banach fixed-point theorem

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In mathematics, the Banach fixed-point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922.<sup>[1]</sup>

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## Statement

*Definition.* Let  $(X, d)$  be a metric space. Then a map  $T\colon X\rightarrow X$  is called a contraction mapping on  $X$  if there exists  $q\in [0, 1)$  such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all  $x, y$  in  $X$ .

Banach Fixed Point Theorem. Let  $(X, d)$  be a non-empty complete metric space with a contraction mapping  $T\colon X\rightarrow X$ . Then  $T$  admits a unique fixed-point  $x^*$  in  $X$  (i.e.  $T(x^*) = x^*$ ). Furthermore,  $x^*$  can be found as follows: start with an arbitrary element  $x_0$  in  $X$  and define a sequence  $\{x_n\}$  by  $x_n = T(x_{n-1})$ , then  $x_n \rightarrow x^*$ .

*Remark 1.* The following inequalities are equivalent and describe the speed of convergence:

$$d(x^*, x_n) \leq \frac{q^n}{1 - q} d(x_1, x_0),$$
$$d(x^*, x_{n+1}) \leq \frac{q}{1 - q} d(x_{n+1}, x_n),$$
$$d(x^*, x_{n+1}) \leq qd(x^*, x_n).$$

Any such value of  $q$  is called a *Lipschitz constant* for  $T$ , and the smallest one is sometimes called "the best Lipschitz constant" of  $T$ .

*Remark 2.*  $d(T(x), T(y)) < d(x, y)$  for all  $x \neq y$  is in general not enough to ensure the existence of a fixed point, as is shown by the map  $T\colon [1, \infty) \rightarrow [1, \infty)$ ,  $T(x) = x + 1/x$ , which lacks a fixed point. However, if  $X$  is compact, then this weaker assumption does imply the existence and uniqueness of a fixed point, that can be easily found as a minimizer of  $d(x, T(x))$ , indeed, a minimizer exists by compactness, and has to be a fixed point of  $T$ . It then easily follows that the fixed point is the limit of any sequence of iterations of  $T$ .

*Remark 3.* When using the theorem in practice, the most difficult part is typically to define  $X$  properly so that  $T(X) \subseteq X$ .

## Proofs

### Banach's original proof

Let  $x_0 \in (X, d)$  be arbitrary and define a sequence  $\{x_n\}$  by setting:  $x_n = T(x_{n-1})$ . Banach's original proof can be broken down into several Lemmas:

Lemma 1. For all  $n \in \mathbb{N}$ ,  $d(x_{n+1}, x_n) \leq q^n d(x_1, x_0)$ .

*Proof.* We will proceed using induction, the base of the induction ( $n = 1$ ) holds:

$$d(x_{1+1}, x_1) = d(x_2, x_1) = d(T(x_1), T(x_0)) \leq qd(x_1, x_0).$$

Suppose the statement holds for some  $k \in \mathbb{N}$ . Then we have

$$\begin{aligned} d(x_{(k+1)+1}, x_{k+1}) &= d(x_{k+2}, x_{k+1}) \\ &= d(T(x_{k+1}), T(x_k)) \\ &\leq qd(x_{k+1}, x_k) \\ &\leq qq^k d(x_1, x_0) && \text{induction hypothesis} \\ &= q^{k+1} d(x_1, x_0). \end{aligned}$$

By the principle of mathematical induction, for all  $n \in \mathbb{N}$ , the Lemma is proven.

Lemma 2.  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$  and hence converges to a limit  $x^*$  in  $X$ .

*Proof.* Let  $m, n \in \mathbb{N}$  such that  $m > n$ .

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) && \text{triangle inequality} \\ &\leq q^{m-1} d(x_1, x_0) + q^{m-2} d(x_1, x_0) + \cdots + q^n d(x_1, x_0) && \text{Lemma 1} \\ &= q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k \\ &\leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k \\ &= q^n d(x_1, x_0) \left( \frac{1}{1-q} \right) && \text{geometric series} \end{aligned}$$

Let  $\varepsilon > 0$  be arbitrary, since  $q \in [0, 1)$ , we can find a large  $N \in \mathbb{N}$  so that

$$q^N < \frac{\varepsilon(1-q)}{d(x_1, x_0)}.$$

Therefore, by choosing  $m, n$  large enough we may write:

$$d(x_m, x_n) \leq q^n d(x_1, x_0) \left( \frac{1}{1-q} \right) < \left( \frac{\varepsilon(1-q)}{d(x_1, x_0)} \right) d(x_1, x_0) \left( \frac{1}{1-q} \right) = \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary this proves that sequence is Cauchy. By completeness, then, there must be a point  $x^* \in X$  that the sequence converges to.

Lemma 3.  $x^*$  is a fixed point of  $T$ .

*Proof.* Take the limit of both sides of the recurrence  $x_n = T(x_{n-1})$ ,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} T(x_{n-1})$$

Since  $T$  is a contraction mapping, it is continuous, so we may take the limit inside:

$$\lim_{n \rightarrow \infty} x_n = T \left( \lim_{n \rightarrow \infty} x_{n-1} \right).$$

Thus,  $x^* = T(x^*)$ .

Lemma 4.  $x^*$  is the only fixed point of  $T$  in  $(X, d)$ .

*Proof.* Suppose  $y$  also satisfies  $T(y) = y$ . Then

$$0 \leq d(x^*, y) = d(T(x^*), T(y)) \leq qd(x^*, y).$$

Remembering that  $q \in [0, 1)$ , the above implies that  $0 \leq (1-q)d(x^*, y) \leq 0$ , which shows that  $d(x^*, y) = 0$ . Therefore by positive definiteness,  $x^* = y$ .

### Shorter proof

Now we present a simpler proof that appeared recently in the Journal of Fixed Point Theory and its Application (see reference).

By the triangle inequality, for all  $x, y$  in  $X$ ,

$$\begin{aligned}d(x,y) &\leq d(x,T(x)) + d(T(x),T(y)) + d(T(y),y) \\ &\leq d(x,T(x)) + qd(x,y) + d(T(y),y)\end{aligned}$$

solving for  $d(x,y)$  we get the *Fundamental Contraction Inequality*:

$$d(x,y) \leq \frac{d(T(x),x) + d(T(y),y)}{1-q},$$

and we note that if  $x$  and  $y$  are both fixed points then this implies that  $d(x,y) = 0$ , so  $x = y$ , proving that  $T$  has at most one fixed point. Now define the mapping  $T^n$  by composing  $T$  with itself  $n$  times and note by induction that it satisfies a Lipschitz condition with constant  $q^n$ . It remains to show that for any  $x_0$  in  $X$ , the sequence  $\{T^n(x_0)\}$  is Cauchy and so converges to a point  $x^*$  of  $X$ , which as noted above is clearly a fixed point of  $T$ . If in the Fundamental Inequality we replace  $x$  and  $y$  by  $T^n(x_0)$  and  $T^m(x_0)$ , we find that

$$\begin{aligned}d(T^n(x_0),T^m(x_0)) &\leq \frac{d(T(T^n(x_0)),T^n(x_0)) + d(T(T^m(x_0)),T^m(x_0))}{1-q}, \\ &= \frac{d(T^n(T(x_0)),T^n(x_0)) + d(T^m(T(x_0)),T^m(x_0))}{1-q} \\ &\leq \frac{q^n d(T(x_0),x_0) + q^m d(T(x_0),x_0)}{1-q} \\ &= \frac{q^n + q^m}{1-q} d(T(x_0),x_0)\end{aligned}$$

since  $q < 1$ , the last expression converges to zero as  $n,m \rightarrow \infty$ , proving that  $\{T^n(x_0)\}$  is Cauchy. Note also that as  $m \rightarrow \infty$  gives us

$$d(T^n(x_0),x^*) \leq \frac{q^n}{1-q} d(T(x_0),x_0)$$

derived in the first proof that gives the rate at which  $\{T^n(x_0)\}$  converges to  $x^*$ .

## Applications

- A standard application is the proof of the Picard–Lindelöf theorem about the existence and uniqueness of solutions to certain ordinary differential equations. The sought solution of the differential equation is expressed as a fixed point of a suitable integral operator which transforms continuous functions into continuous functions. The Banach fixed-point theorem is then used to show that this integral operator has a unique fixed point.
- One consequence of the Banach fixed-point theorem is that small Lipschitz perturbations of the identity are bi-lipschitz homeomorphisms. Let  $\Omega$  be an open set of a Banach space  $E$ ; let  $I\colon \Omega \rightarrow E$  denote the identity (inclusion) map and let  $g\colon \Omega \rightarrow E$  be a Lipschitz map of constant  $k < 1$ . Then

1.  $\Omega' := (I+g)(\Omega)$  is an open subset of  $E$ : precisely, for any  $x$  in  $\Omega$  such that  $B(x,r) \subset \Omega$  one has  $B((I+g)(x),r(1-k)) \subset \Omega'$ ;
2.  $I+g\colon \Omega \rightarrow \Omega'$  is a bi-lipschitz homeomorphism;

precisely,  $(I+g)^{-1}$  is still of the form  $I+h\colon \Omega \rightarrow \Omega'$  with  $h$  a Lipschitz map of constant  $k/(1-k)$ . A direct consequence of this result yields the proof of the inverse function theorem.

## Converses

Several converses of the Banach contraction principle exist. The following is due to Czesław Bessaga, from 1959:

Let  $f\colon X \rightarrow X$  be a map of an abstract set such that each iterate  $f^n$  has a unique fixed point. Let  $q \in (0,1)$ , then there exists a complete metric on  $X$  such that  $f$  is contractive, and  $q$  is the contraction constant.

Indeed, very weak assumptions suffice to obtain such a kind of converse. For example if  $f\colon X \rightarrow X$  is a map on a  $T_1$  topological space with a unique fixed point  $a$ , such that for each  $x$  in  $X$  we have  $f^n(x) \rightarrow a$ , then there already exists a metric on  $X$  with respect to which  $f$  satisfies the conditions of the Banach contraction principle with contraction constant  $1/2$ .<sup>[2]</sup> In this case the metric is in fact an ultrametric.

## Generalizations

There are a number of generalizations (some of which are immediate corollaries).<sup>[3]</sup>

Let  $T\colon X \rightarrow X$  be a map on a complete non-empty metric space. Then, for example, some generalizations of the Banach fixed-point theorem are:

- Assume that some iterate  $T^n$  of  $T$  is a contraction. Then  $T$  has a unique fixed point.

- Assume that for each *n*, there exist *c<sub>n</sub>* such that *d(T<sup>n</sup>(x), T<sup>n</sup>(y)) ≤ c<sub>n</sub>d(x, y)* for all *x* and *y*, and that

$$\sum_n c_n < \infty.$$

Then *T* has a unique fixed point.

In applications, the existence and unicity of a fixed point often can be shown directly with the standard Banach fixed point theorem, by a suitable choice of the metric that makes the map *T* a contraction. Indeed, the above result by Bessaga strongly suggests to look for such a metric. See also the article on fixed point theorems in infinite-dimensional spaces for generalizations.

A different class of generalizations arise from suitable generalizations of the notion of metric space, e.g. by weakening the defining axioms for the notion of metric.<sup>[4]</sup> Some of these have applications, e.g., in the theory of programming semantics in theoretical computer science.<sup>[5]</sup>

See also

- Brouwer fixed-point theorem
- Caristi fixed-point theorem
- Fichera's existence principle
- Fixed-point theorems
- Infinite compositions of analytic functions

Notes

1. [http://www.emis.de/journals/BJMA/tex\\_v1\\_n1\\_a1.pdf](http://www.emis.de/journals/BJMA/tex_v1_n1_a1.pdf)

2. Hitzler, Pascal; Seda, Anthony K. (2001). "A ‘Converse’ of the Banach Contraction Mapping Theorem". *Journal of Electrical Engineering*. 52 (10/s): 3–6.

3. Latif, Abdul (2014). "Banach Contraction Principle and its Generalizations" ([https://dx.doi.org/10.1007/978-3-319-01586-6\\_2](https://dx.doi.org/10.1007/978-3-319-01586-6_2)). *Topics in Fixed Point Theory*. Springer.

4. Hitzler, Pascal; Seda, Anthony (2010). *Mathematical Aspects of Logic Programming Semantics*. Chapman and Hall/CRC.

5. Seda, Anthony K.; Hitzler, Pascal (2010). "Generalized Distance Functions in the Theory of Computation". *The Computer Journal*. 53 (4): 443–464. doi:10.1093/comjnl/bxm108 (<https://doi.org/10.1093%2Fcomjnl%2Fbxm108>).

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