Time Series Analysis

Autoregressive, MA and ARMA processes

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4. Autoregressive, MA and ARMA processes

4.1 Autoregressive processes

Outline:

- Introduction
- The first-order autoregressive process, AR(1)
- The AR(2) process
- The general autoregressive process AR(p)
- The partial autocorrelation function

Recommended readings:

▷ Chapter 2 of Brockwell and Davis (1996).

▷ Chapter 3 of Hamilton (1994).

▷ Chapter 3 of Peña, Tiao and Tsay (2001).

Introduction

▷ In this section we will begin our study of models for stationary processes which are useful in representing the dependency of the values of a time series on its past.

▷ The simplest family of these models are the autoregressive, which generalize the idea of regression to represent the linear dependence between a dependent variable y (z_t) and an explanatory variable x (z_{t-1}), using the relation:

$$z_t = c + bz_{t-1} + a_t$$

where c and b are constants to be determined and a_t are i.i.d $\mathcal{N}(0, \sigma^2)$. Above relation define the first order autoregressive process.

▷ This linear dependence can be generalized so that the present value of the series, z_t, depends not only on z_{t-1}, but also on the previous p lags, $z_{t-2}, \ldots, z_{t-p}$. Thus, an autoregressive process of order p is obtained.

The first-order autoregressive process, AR(1)

▷ We say that a series z_t follows a **first order autoregressive process**, or AR(1), if it has been generated by:

$$z_t = c + \phi z_{t-1} + a_t \tag{33}$$

where c and $-1 < \phi < 1$ are constants and a_t is a white noise process with variance σ^2. The variables a_t, which represent the new information that is added to the process at each instant, are known as innovations.

Example 35

We will consider z_t as the quantity of water at the end of the month in a reservoir. During the month, $c + a_t$ amount of water comes into the reservoir, where c is the average quantity that enters and a_t is the innovation, a random variable of zero mean and constant variance that causes this quantity to vary from one period to the next.

If a fixed proportion of the initial amount is used up each month, $(1 - \phi)z_{t-1}$, and a proportion, ϕz_{t-1} , is maintained the quantity of water in the reservoir at the end of the month will follow process (33).

The first-order autoregressive process, AR(1)

▷ The condition $-1 < \phi < 1$ is necessary for the process to be stationary. To prove this, let us assume that the process begins with $z_0 = h$, with h being any fixed value. The following value will be $z_1 = c + \phi h + a_1$, the next, $z_2 = c + \phi z_1 + a_2 = c + \phi(c + \phi h + a_1) + a_2$ and, substituting successively, we can write:

$$
\begin{array}{ccl}
z_1 & = & c + \phi h + a_1 \\
z_2 & = & c(1 + \phi) + \phi^2 h + \phi a_1 + a_2 \\
z_3 & = & c(1 + \phi + \phi^2) + \phi^3 h + \phi^2 a_1 + \phi a_2 + a_3 \\
\vdots & \vdots & \qquad\qquad\qquad \vdots \\
z_t & = & c \sum_{i=0}^{t-1} \phi^i + \phi^t h + \sum_{i=0}^{t-1} \phi^i a_{t-i}
\end{array}
$$

If we calculate the expectation of z_t, as $E[a_t] = 0$,

$$E\left[z_t\right] = c \sum_{i=0}^{t-1} \phi^i + \phi^t h.$$

For the process to be stationary it is a necessary condition that this function does not depend on t.

The first-order autoregressive process, AR(1)

▷ The mean is constant if both summands are, which requires that on increasing t the first term converges to a constant and the second is canceled. Both conditions are verified if $|\phi| < 1$, because then $\sum_{i=0}^{t-1} \phi^i$ is the sum of an geometric progression with ratio ϕ and converges to $c/(1- \phi)$, and the term ϕ^t converges to zero, thus the sum converges to the constant $c/(1- \phi)$.

▷ With this condition, after an initial transition period, when $t \to \infty$, all the variables z_t will have the same expectation, $\mu = c/(1 - \phi)$, independent of the initial conditions.

▷ We also observe that in this process the innovation a_t is uncorrelated with the previous values of the process, z_{t-k} for positive k since z_{t-k} depends on the values of the innovations up to that time, $a_1, ..., a_{t-k}$, but not on future values. Since the innovation is a white noise process, its future values are uncorrelated with past ones and, therefore, with previous values of the process, z_{t-k}.

The first-order autoregressive process, AR(1)

▷ The AR(1) process can be written using the notation of the **lag operator**, B, defined by

$$Bz_t = z_{t-1}. \tag{34}$$

Letting $\widetilde{z}_t = z_t - \mu$ and since $B\widetilde{z}_t = \widetilde{z}_{t-1}$ we have:

$$(1 - \phi B)\widetilde{z}_t = a_t. \tag{35}$$

▷ This condition indicates that a series follows an AR(1) process if on applying the operator $(1 - \phi B)$ a white noise process is obtained.

▷ The operator $(1 - \phi B)$ can be interpreted as a filter that when applied to the series converts it into a series with no information, a white noise process.

The first-order autoregressive process, AR(1)

▷ If we consider the operator as an equation, in B the coefficient ϕ is called the factor of the equation.

▷ The stationarity condition is that this factor be less than the unit in absolute value.

▷ Alternatively, we can talk about the root of the equation of the operator, which is obtained by making the operator equal to zero and solving the equation with B as an unknown;

$$1 - \phi B = 0$$

which yields $B = 1/\phi$.

▷ The condition of stationarity is then that the root of the operator be greater than one in absolute value.

The first-order autoregressive process, AR(1)

Expectation

▷ Taking expectations in (33) assuming $|\phi| < 1$, such that $E[z_t] = E[z_{t-1}] = \mu$, we obtain

$$\mu = c + \phi\mu$$

Then, the **expectation** (or mean) is

$$\mu = \frac{c}{1-\phi} \tag{36}$$

Replacing c in (33) with $\mu(1-\phi)$, the process can be written in deviations to the mean:

$$z_t - \mu = \phi(z_{t-1} - \mu) + a_t$$

and letting $\widetilde{z}_t = z_t - \mu$,

$$\widetilde{z}_t = \phi\widetilde{z}_{t-1} + a_t \tag{37}$$

which is the most often used equation of the AR(1).

The first-order autoregressive process, AR(1)

Variance

▷ The variance of the process is obtained by squaring the expression (37) and taking expectations, which gives us:

$$E(\widetilde{z}_t^2) = \phi^2 E(\widetilde{z}_{t-1}^2) + 2\phi E(\widetilde{z}_{t-1} a_t) + E(a_t^2).$$

We let σ_z^2 be the variance of the stationary process. The second term of this expression is zero, since as $\widetilde{z}_{t-1}$ and a_t are independent and both variables have null expectation. The third is the variance of the innovation, σ^2, and we conclude that:

$$\sigma_z^2 = \phi^2 \sigma_z^2 + \sigma^2,$$

from which we find that the **variance** of the process is:

$$\sigma_z^2 = \frac{\sigma^2}{1 - \phi^2}. \tag{38}$$

Note that in this equation the condition $|\phi| < 1$ appears, so that σ_z^2 is finite and positive.

The first-order autoregressive process, AR(1)

▷ It is important to differentiate the marginal distribution of a variable from the conditional distribution of this variable in the previous value. The marginal distribution of each observation is the same, since the process is stationary: it has mean μ and variance σ_z^2. Nevertheless, the conditional distribution of z_t if we know the previous value, z_{t-1}, has a conditional mean:

$$E(z_t|z_{t-1}) = c + \phi z_{t-1}$$

and variance σ^2, which according to (38), is always less than σ_z^2.

▷ If we know z_{t-1} it reduces the uncertainty in the estimation of z_t, and this reduction is greater when ϕ^2 is greater.

▷ If the AR parameter is close to one, the reduction of the variance obtained from knowledge of z_{t-1} can be very important.

The first-order autoregressive process, AR(1)

Autocovariance function

$\triangleright$ Using (37), multiplying by z_{t-k} and taking expectations gives us γ_k, the covariance between observations separated by k periods, or the **autocovariance of order k**:

$$\gamma_k = E\left[(z_{t-k} - \mu)(z_t - \mu)\right] = E\left[\widetilde{z}_{t-k}(\phi\widetilde{z}_{t-1} + a_t)\right]$$

and as $E\left[\widetilde{z}_{t-k} a_t\right] = 0$, since the innovations are uncorrelated with the past values of the series, we have the following recursion:

$$\gamma_k = \phi\gamma_{k-1} \qquad k = 1, 2, \ldots \tag{39}$$

where $\gamma_0 = \sigma_z^2$.

$\triangleright$ This equation shows that since $|\phi| < 1$ the dependence between observations decreases when the lag increases.

$\triangleright$ In particular, using (38):
$$\gamma_1 = \frac{\phi\sigma^2}{1 - \phi^2} \tag{40}$$

Autocorrelation function, ACF

▷ Autocorrelations contain the same information as the autocovariances, with the advantage of not depending on the units of measurement. From here on we will use the term simple autocorrelation function (ACF) to denote the autocorrelation function of the process in order to differentiate it from other functions linked to the autocorrelation that are defined at the end of this section.

▷ Let ρ_k be the **autocorrelation of order k**, defined by: $\rho_k = \gamma_k/\gamma_0$, using (39), we have:

$$\rho_k = \phi\gamma_{k-1}/\gamma_0 = \phi\rho_{k-1}.$$

Since, according to (38) and (40), $\rho_1 = \phi$, we conclude that:

$$\rho_k = \phi^k \qquad (41)$$

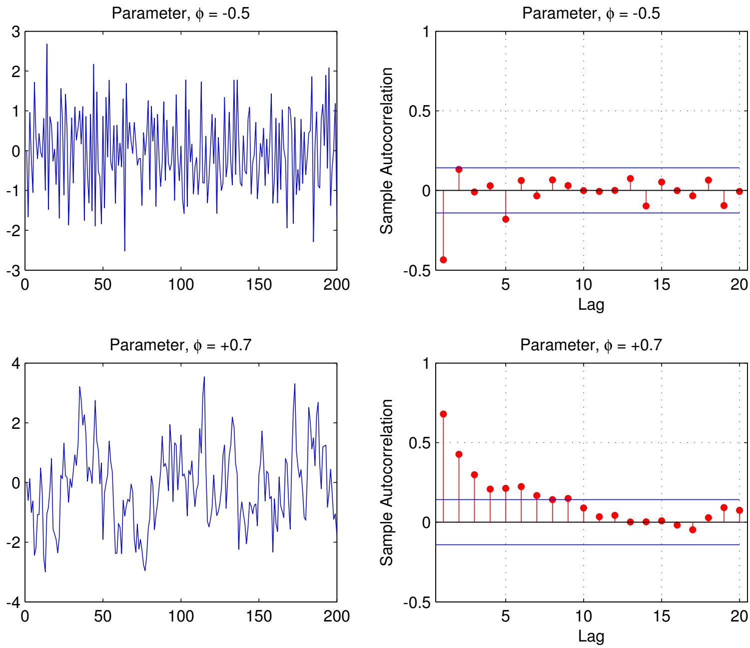
and when k is large, ρ_k goes to zero at a rate that depends on ϕ.

Autocorrelation function, ACF

▷ The expression (41) shows that the autocorrelation function of an AR(1) process is equal to the powers of the AR parameter of the process and decreases geometrically to zero.

▷ If the parameter is positive the linear dependence of the present on past values is always positive, whereas if the parameter is negative this dependence is positive for even lags and negative for odd ones.

▷ When the parameter is positive the value at t is similar to the value at $t-1$, due to the positive dependence, thus the graph of the series evolves smoothly. Whereas, when the parameter is negative the value at t is, in general, the opposite sign of that at $t-1$, thus the graph shows many changes of signs.


Parameter, ϕ = -0.5
Parameter, ϕ = -0.5
Sample Autocorrelation
Lag
Parameter, ϕ = +0.7
Parameter, ϕ = +0.7
Sample Autocorrelation
Lag

Representation of an AR(1) process as a sum of innovations

▷ The AR(1) process can be expressed as a function of the past values of the innovations. This representation is useful because it reveals certain properties of the process. Using $\widetilde{z}_{t-1}$ in the expression (37) as a function of $\widetilde{z}_{t-2}$, we have

$$\widetilde{z}_t = \phi(\phi\widetilde{z}_{t-2} + a_{t-1}) + a_t = a_t + \phi a_{t-1} + \phi^2\widetilde{z}_{t-2}.$$

If we now replace $\widetilde{z}_{t-2}$ with its expression as a function of $\widetilde{z}_{t-3}$, we obtain

$$\widetilde{z}_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3\widetilde{z}_{t-2}$$

and repeatedly applying this substitution gives us:

$$\widetilde{z}_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + ... + \phi^{t-1} a_1 + \phi^t\widetilde{z}_1$$

▷ If we assume t to be large, since ϕ^t will be close to zero we can represent the series as a function of all the past innovations, with weights that decrease geometrically.

Representation of an AR(1) process as a sum of innovations

▷ Other possibility is to assume that the series starts in the infinite past:

$$\widetilde{z}_t = \sum_{j=0}^{\infty} \phi^j a_{t-j}$$

and this representation is denoted as the infinite order moving average, MA(∞), of the process.

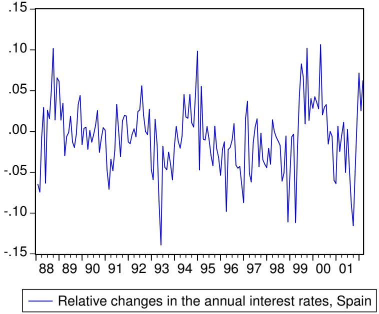
▷ Observe that the coefficients of the innovations are precisely the coefficients of the simple autocorrelation function.

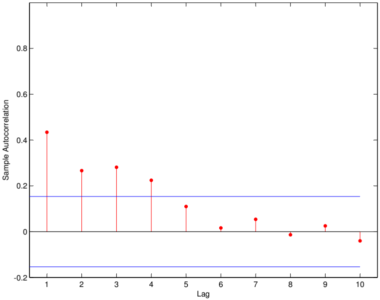
▷ The expression MA(∞) can also be obtained directly by multiplying the equation (35) by the operator $(1 - \phi B)^{-1} = 1 + \phi B + \phi^2 B^2 + \ldots,$ thus obtaining:

$$\widetilde{z}_t = (1 - \phi B)^{-1} a_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \ldots$$

Example 36

The figures show the monthly series of relative changes in the annual interest rate, defined by $z_t = \log(y_t/y_{t-1})$ *and the ACF. The AC coefficients decrease with the lag: the first is of order .4, the second close to* $.4^2 = .16$*, the third is a similar value and the rest are small and not significant.*





Datafile `interestrates.xls`

The AR(2) process

▷ The dependency between present and past values which an AR(1) establishes can be generalized allowing z_t to be linearly dependent not only on z_{t-1} but also on z_{t-2}. Thus the second order autoregressive, or AR(2) is obtained:

$$z_t = c + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \tag{42}$$

where c, ϕ_1 and ϕ_2 are now constants and a_t is a white noise process with variance σ^2.

▷ We are going to find the conditions that must verify the parameters for the process to be stationary. Taking expectations in (42) and imposing that the mean be constant, results in:

$$\mu = c + \phi_1 \mu + \phi_2 \mu$$

which implies

$$\mu = \frac{c}{1 - \phi_1 - \phi_2}, \tag{43}$$

and the condition for the process to have a finite mean is that $1 - \phi_1 - \phi_2 \neq 0$.

The AR(2) process

▷ Replacing c with $\mu(1 - \phi_1 - \phi_2)$ and letting $\widetilde{z}_t = z_t - \mu$ be the process of deviations to the mean, the AR(2) process is:

$$\widetilde{z}_t = \phi_1 \widetilde{z}_{t-1} + \phi_2 \widetilde{z}_{t-2} + a_t. \quad (44)$$

▷ In order to study the properties of the process it is advisable to use the operator notations. Introducing the lag operator, B, the equation of this process is:

$$(1 - \phi_1 B - \phi_2 B^2)\widetilde{z}_t = a_t. \quad (45)$$

▷ The operator $(1 - \phi_1 B - \phi_2 B^2)$ can always be expressed as $(1 - G_1 B)(1 - G_2 B)$, where G_1^{-1} and G_2^{-1} are the roots of the equation of the operator considering B as a variable and solving

$$1 - \phi_1 B - \phi_2 B^2 = 0. \quad (46)$$

The AR(2) process

▷ The equation (46) is called **the characteristic equation** of the operator.

▷ G_1 and G_2 are also said to be factors of the characteristic polynomial of the process. These roots can be real or complex conjugates.

▷ It can be proved that the condition of stationarity is that $|G_i| < 1$, $i = 1, 2$.

▷ This condition is analogous to that studied for the AR(1).

▷ Note that this result is consistent with the condition found for the mean to be finite. If the equation

$$1 - \phi_1 B - \phi_2 B^2 = 0$$

has a unit root it is verified that $1 - \phi_1 - \phi_2 = 0$ and the process is not stationary, since it does not have a finite mean.

Autocovariance function

▷ Squaring expression (44) and taking expectations, we find that the variance must satisfy:

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma^2. \tag{47}$$

▷ In order to calculate the autocovariance, multiplying the equation (44) by $\widetilde{z}_{t-k}$ and taking expectations, we obtain:

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}. \qquad k \geq 1 \tag{48}$$

▷ Specifying this equation for $k = 1$, since $\gamma_{-1} = \gamma_1$, we have

$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1,$$

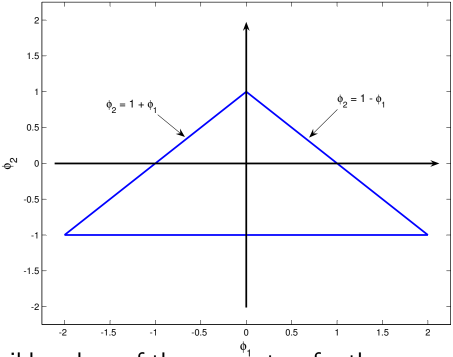
which provides $\gamma_1 = \phi_1\gamma_0/(1-\phi_2)$. Using this expression in (47) results in the formula for the variance:

$$\sigma_z^2 = \gamma_0 = \frac{(1-\phi_2)\,\sigma^2}{(1+\phi_2)\,(1-\phi_1-\phi_2)\,(1+\phi_1-\phi_2)}. \tag{49}$$

Autocovariance function

▷ For the process to be stationary this variance must be positive, which will occur if the numerator and the denominator have the same sign. It can be proved that the values of the parameters that make AR(2) a stationary process are those included in the region:

$$\begin{array}{c} -1 < \phi_2 < 1 \\ \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \end{array}$$



It represents the admissible values of the parameters for the process to be stationary.

The AR(2) process

▷ In this process it is important again to differentiate the marginal and conditional properties. Assuming that the conditions of stationarity are verified, the marginal mean is given by

$$\mu = \frac{c}{1 - \phi_1 - \phi_2}$$

and the marginal variance is

$$\sigma_z^2 = \frac{(1 - \phi_2)\,\sigma^2}{(1 + \phi_2)\,(1 - \phi_1 - \phi_2)\,(1 + \phi_1 - \phi_2)}.$$

▷ Nevertheless, the conditional mean of z_t given the previous values is:

$$E(z_t|z_{t-1}, z_{t-2}) = c + \phi_1 z_{t-1} + \phi_2 z_{t-2}$$

and its variance will be σ^2, the variance of the innovations which will always be less than the marginal variance of the process σ_z^2.

Autocorrelation function

▷ Dividing by the variance in equation (48), we obtain the relationship between the autocorrelation coefficients:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad k \geq 1 \tag{50}$$

specifying (50) for $k = 1$, as in a stationary process $\rho_1 = \rho_{-1}$, we obtain:

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} \tag{51}$$

and specifying (50) for $k = 2$ and using (51):

$$\rho_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2. \tag{52}$$

Autocorrelation function

▷ For $k \geq 3$ the autocorrelation coefficients can be obtained recursively starting from the difference equation (50). It can be proved that the general solution to this equation is:

$$\rho_k = A_1 G_1^k + A_2 G_2^k \tag{53}$$

where G_1 and G_2 are the factors of the characteristic polynomial of the process and A_1 and A_2 are constants to be determined from the initial conditions $\rho_0 = 1,$ (which implies $A_1 + A_2 = 1$) and $\rho_1 = \phi_1 / (1 - \phi_2)$.

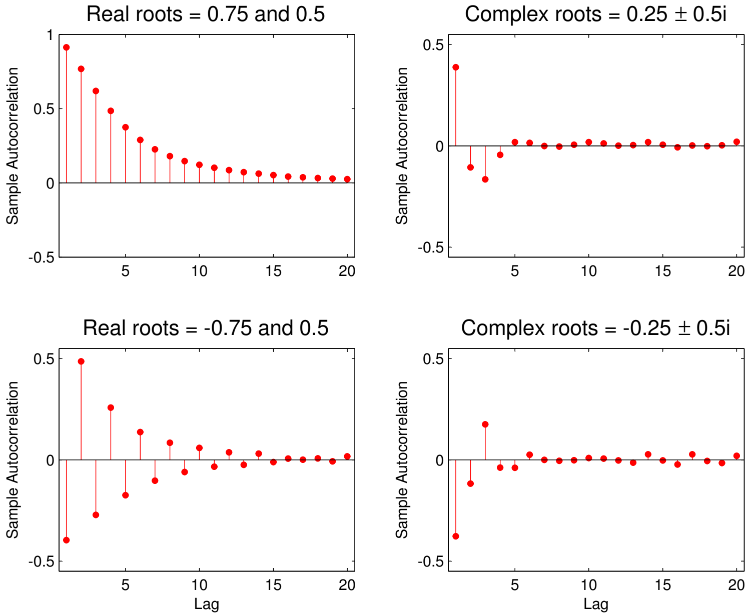
▷ According to (53) the coefficients ρ_k will be less than or equal to the unit if $|G_1| < 1$ and $|G_2| < 1$, which are the conditions of stationarity of the process.

Autocorrelation function

▷ If the factors G_1 and G_2 are complex of type $a \pm bi$, where $i = \sqrt{-1}$, then this condition is $\sqrt{a^2 + b^2} < 1$. We may find ourselves in the following cases:

1. The two factors G_1 and G_2 are real. The decrease of (53) is the sum of the two exponentials and the shape of the autocorrelation function will depend on whether G_1 and G_2 have equal or opposite signs.
2. The two factors G_1 and G_2 are complex conjugates. In this case, the function ρ_k will decrease sinusoidally.

▷ The four types of possible autocorrelation functions for an AR(2) are shown in the next figure.



Real roots = 0.75 and 0.5
Complex roots = 0.25 ± 0.5i
Real roots = -0.75 and 0.5
Complex roots = -0.25 ± 0.5i
Sample Autocorrelation
Lag
1
0.5
0
-0.5
5
10
15
20

Representation of an AR(2) process as a sum of innovations

▷ The AR(2) process can be represented, as with an AR(1), as a linear combination of the innovations. Writing (45) as

$$(1 - G_1 B)(1 - G_2 B)\widetilde{z}_t = a_t$$

and inverting these operators, we have

$$\widetilde{z}_t = (1 + G_1 B + G_1^2 B^2 + \ldots)(1 + G_2 B + G_2^2 B^2 + \ldots)a_t \tag{54}$$

which leads to the MA(∞) expression of the process:

$$\widetilde{z}_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots \tag{55}$$

▷ We can obtain the coefficients ψ_i as a function of the roots equating powers of B in (54) and (55).

Representation of an AR(2) process as a sum of innovations

▷ We can also obtain the coefficients ψ_i as a function of the coefficients ϕ_1 and ϕ_2. Letting $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ since $\psi(B) = (1 - \phi_1 B - \phi_2 B^2)^{-1}$, we have

$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = 1. \tag{56}$$

▷ Imposing the restriction that all the coefficients of the powers of B in (56) are null, the coefficient of B in this equation is $\psi_1 - \phi_1$, which implies $\psi_1 = \phi_1$. The coefficient of B^2 is $\psi_2 - \phi_1\psi_1 - \phi_2$, which implies the equation:

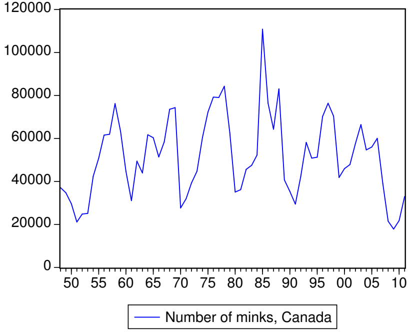
$$\psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} \tag{57}$$

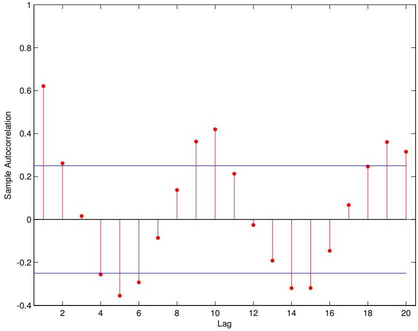
for $k = 2$, since $\psi_0 = 1$. The coefficients of B^k for $k \geq 2$ verify the equation (57) which is similar to the one that must verify the autocorrelation coefficients.

▷ We conclude that the shape of the coefficients ψ_i will be similar to that of the autocorrelation coefficients.

Example 37

The figures show the number of mink sighted yearly in an area of Canada and the ACF. The series shows a cyclical evolution that could be explained by an AR(2) with negative roots corresponding to the sinusoidal structure of the autocorrelation.





Example 38

Write the autocorrelation function of the AR(2) process

$$z_t = 1.2z_{t-1} - 0.32z_{t-2} + a_t$$

▷ The characteristic equation of that process is:

$$0.32X^2 - 1.2X + 1 = 0$$

whose solution is:

$$X = \frac{1.2 \pm \sqrt{1.2^2 - 4 \times 0.32}}{0.64} = \frac{1.2 \pm 0.4}{0.64}$$

▷ The solutions are $G_1^{-1} = 2.5$ and $G_2^{-1} = 1.25$ and the factors are $G_1 = 0.4$ and $G_2 = 0.8$.

▷ The characteristic equation can be written:

$$0.32X^2 - 1.2X + 1 = (1 - 0.4X)(1 - 0.8X).$$

Therefore, the process is stationary with real roots and the autocorrelation coefficients verify:

$$\rho_k = A_1 0.4^k + A_2 0.8^k.$$

▷ To determine A_1 and A_2 we impose the initial conditions $\rho_0 = 1, \rho_1 = 1.2/\left(1.322\right) = 0.91$. Then, for $k = 0$:

$$1 = A_1 + A_2$$

and for $k = 1$,

$$0.91 = 0.4A_1 + 0.8A_2$$

solving these equations we obtain $A_2 = 0.51/0.4$ and $A_1 = -0.11/0.4$.

▷ Therefore, the autocorrelation function is:

$$\rho_k = -\frac{0.11}{0.4}0.4^k + \frac{0.51}{0.4}0.8^k$$

which gives us the following table:

k	0	1	2	3	4	5	6	7	8
ρ_k	1	0.91	0.77	0.63	0.51	0.41	0.33	0.27	0.21

▷ To obtain the representation as a function of the innovations, writing

$$(1 - 0.4B)(1 - 0.8B)z_t = a_t$$

and inverting both operators:

$$z_t = (1 + 0.4B + .16B^2 + .06B^3 + ...)(1 + 0.8B + .64B^2 + ...)a_t$$

yields:

$$z_t = (1 + 1.2B + 1.12B^2 + ...)a_t.$$

The general autoregressive process, AR(p)

▷ We say that a stationary time series z_t follows an **autoregressive process of order** p if:

$$\widetilde{z}_t = \phi_1 \widetilde{z}_{t-1} + \ldots + \phi_p \widetilde{z}_{t-p} + a_t \tag{58}$$

where $\widetilde{z}_t = z_t - \mu$, with μ being the mean of the stationary process z_t and a_t a white noise process.

▷ Utilizing the operator notation, the equation of an AR(p) is:

$$\left(1 - \phi_1 B - \ldots - \phi_p B^p\right) \widetilde{z}_t = a_t \tag{59}$$

and letting $\phi_p(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$ be the polynomial of degree p in the lag operator, whose first term is the unit, we have:

$$\phi_p \left(B\right) \widetilde{z}_t = a_t \tag{60}$$

which is the general expression of an autoregressive process.

The general autoregressive process, AR(p)

▷ The **characteristic equation** of this process is define by:

$$\phi_p(B) = 0 \tag{61}$$

considered as a function of B.

▷ This equation has p roots $G_1^{-1},\ldots,G_p^{-1}$, which are generally different, and we can write:

$$\phi_p(B) = \prod_{i=1}^{p} (1 - G_i B)$$

such that the coefficients G_i are the factors of the characteristic equation.

▷ It can be proved that the process is stationary if $|G_i| < 1$, for all i.

Autocorrelation function

▷ Operating with (58), we find that the autocorrelation coefficients of an AR(p) verify the following difference equation:

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}, \quad k > 0.$$

▷ In the above sections we saw particular cases in this equation for $p = 1$ and $p = 2$. We can conclude that the autocorrelation coefficients satisfy the same equation as the process:

$$\phi_p(B)\,\rho_k = 0 \qquad k > 0. \tag{62}$$

▷ The general solution to this equation is:

$$\rho_k = \sum\nolimits_{i=1}^{p} A_i G_i^k, \tag{63}$$

where the A_i are constants to be determined from the initial conditions and the G_i are the factors of the characteristic equation.

The general autoregressive process, AR(p)

Autocorrelation function

▷ For the process to be stationary the modulus of G_i must be less than one or, the roots of the characteristic equation (61) must be greater than one in modulus, which is the same.

▷ To prove this, we observe that the condition $|\rho_k| < 1$ requires that there not be any G_i greater than the unit in (63), since in that case, when k increases the term G_i^k will increase without limit.

▷ Furthermore, we observe that for the process to be stationary there cannot be a root G_i equal to the unit, since then its component G_i^k would not decrease and the coefficients ρ_k would not tend to zero for any lag.

▷ Equation (63) shows that the autocorrelation function of an AR(p) process is a mixture of exponents, due to the terms with real roots, and sinusoids, due to the complex conjugates. As a result, their structure can be very complex.

Yule-Walker equations

▷ Specifying the equation (62) for $k = 1, ..., p$, a system of p equations is obtained that relate the first p autocorrelations with the parameters of the process. This is called the Yule-Walker system:

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + ... + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + ... + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + ... + \phi_p.
\end{aligned}$$

▷ Defining:

$$\boldsymbol{\phi}' = [\phi_1, ..., \phi_p], \quad \boldsymbol{\rho}' = [\rho_1, ..., \rho_p], \quad \mathbf{R} = \begin{bmatrix} 1 & \rho_1 & ... & \rho_{p-1} \\ \vdots & \vdots & & \vdots \\ \rho_{p-1} & \rho_{p-2} & ... & 1 \end{bmatrix}$$

the above system is written as a matrix:

$$\boldsymbol{\rho} = \mathbf{R}\boldsymbol{\phi} \tag{64}$$

and the parameters can be determined using: $\boldsymbol{\phi} = \mathbf{R}^{-1}\boldsymbol{\rho}$.

Example 39

Obtain the parameters of an AR(3) process whose first autocorrelations are $\rho_1 = 0.9$; $\rho_2 = 0.8$; $\rho_3 = 0.5$. *Is the process stationary?*

▷ The Yule-Walker equation system is:

$$\begin{bmatrix} 0.9 \\ 0.8 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0.9 & 0.8 \\ 0.9 & 1 & 0.9 \\ 0.8 & 0.9 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

whose solution is:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 5.28 & -5 & 0.28 \\ -5 & 10 & -5 \\ 0.28 & -5 & 5.28 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.8 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.89 \\ 1 \\ -1.11 \end{bmatrix}.$$

▷ As a result, the AR(3) process with these correlations is:

$$\left(1 - 0.89B - B^2 + 1.11B^3\right) z_t = a_t.$$

▷To prove that the process is stationary we have to calculate the factors of the characteristic equation. The quickest way to do this is to obtain the solutions to the equation

$$X^3 - 0.89X^2 - X + 1.11 = 0$$

and check that they all have modulus less than the unit.

▷The roots of this equation are -1.0550, $0.9725 + 0.3260i$ and $0.9725 - 0.3260i$.

▷The modulus of the real factor is greater than the unit, thus we conclude that there is no an AR(3) stationary process that has these three autocorrelation coefficients.

Representation of an AR(p) process as a sum of innovations

▷ To obtain the coefficients of the representation MA(∞) form we use:

$$(1 - \phi_1 B - ... - \phi_p B^p)(1 + \psi_1 B + \psi_2 B^2 + ...) = 1$$

and the coefficients ψ_i are obtained by setting the powers of B equal to zero.

▷ It is proved that they must verify the equation

$$\psi_k = \phi_1 \psi_{k-1} + ... + \phi_p \psi_{k-1}$$

which is analogous to that which verifies that autocorrelation coefficients of the process.

▷ As mentioned earlier, the autocorrelation coefficients, ρ_k, and the coefficients of the structure MA(∞) are not identical: although both sequences satisfy the same difference equation and take the form $\sum A_i G_i^k$, the constants A_i depend on the initial conditions and will be different in both sequences.

The partial autocorrelation function

▷ Determining the order of an autoregressive process from its autocorrelation function is difficult. To resolve this problem the partial autocorrelation function is introduced.

▷ If we compare an AR(l) with an AR(2) we see that although in both processes each observation is related to the previous ones, the type of relationship between observations separated by more that one lag is different in both processes:

- In the AR(1) the effect of z_{t-2} on z_t is always through z_{t-1}, and given z_{t-1}, the value of z_{t-2} is irrelevant for predicting z_t.
- Nevertheless, in an AR(2) in addition to the effect of z_{t-2} which is transmitted to z_t through z_{t-1}, there exists a direct effect on z_{t-2} on z_t.

▷ In general, an AR(p) has *direct* effects on observations separated by $1, 2, ..., p$ lags and the *direct* effects of the observations separated by more than p lags are null.

The partial autocorrelation function

▷ The **partial autocorrelation coefficient of order k**, denoted by ρ_k^p, is defined as the correlation coefficient between observations separated by k periods, when we eliminate the linear dependence due to intermediate values.

1. We eliminate from $\widetilde{z}_t$, the effect of $\widetilde{z}_{t-1}, ..., \widetilde{z}_{t-k+1}$ using the regression:

$$\widetilde{z}_t = \beta_1 \widetilde{z}_{t-1} + ... + \beta_{k-1} \widetilde{z}_{t-k+1} + u_t,$$

   where the variable u_t contains the part of $\widetilde{z}_t$ not common to $\widetilde{z}_{t-1}$,...,$\widetilde{z}_{t-k+1}$.
2. We eliminate the effect of $\widetilde{z}_{t-1}, ..., \widetilde{z}_{t-k+1}$from $\widetilde{z}_{t-k}$ using the regression:

$$\widetilde{z}_{t-k} = \gamma_1 \widetilde{z}_{t-1} + ... + \gamma_{k-1} \widetilde{z}_{t-k+1} + v_t,$$

   where, again, v_t contains the part of z_{t-1} not common to the intermediate observations.
3. We calculate the simple correlation coefficient between u_t and v_t which, by definition, is the partial autocorrelation coefficient of order k.

The partial autocorrelation function

▷ This definition is analogous to that of the partial correlation coefficient in regression. It can be proved that the three above steps are equivalent to fitting the multiple regression:

$$\widetilde{z}_t = \alpha_{k1}\widetilde{z}_{t-1} + \ldots + \alpha_{kk}\widetilde{z}_{t-k} + \eta_t$$

and thus $\rho_k^p = \alpha_{kk}$.

▷ The partial autocorrelation coefficient of order k is the coefficient α_{kk} of the variable z_{t-k} after fitting an $AR(k)$ to the data of the series. Therefore, if we fit the family of regressions:

$$\begin{array}{rcl} \widetilde{z}_t & = & \alpha_{11}\widetilde{z}_{t-1} + \eta_{1t} \\ \widetilde{z}_t & = & \alpha_{21}\widetilde{z}_{t-1} + \alpha_{22}\widetilde{z}_{t-2} + \eta_{2t} \\ \vdots & \vdots & \vdots \\ \widetilde{z}_t & = & \alpha_{k1}\widetilde{z}_{t-1} + \ldots + \alpha_{kk}\widetilde{z}_{t-k} + \eta_{kt} \end{array}$$

the sequence of coefficients α_{ii} provides the partial autocorrelation function.

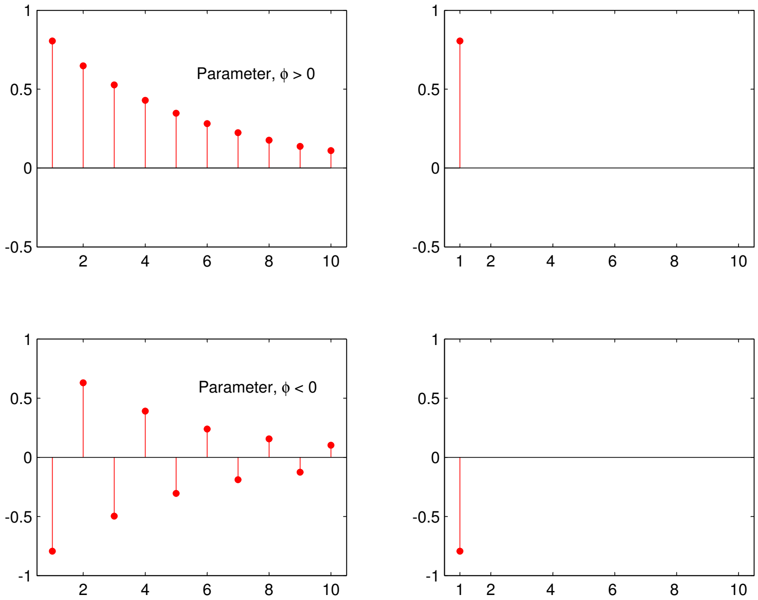
The partial autocorrelation function

▷ From this definition it is clear that an AR(p) process will have the first p nonzero partial autocorrelation coefficients and, therefore, in the partial autocorrelation function (PACF) the number of nonzero coefficients indicates the order of the AR process.

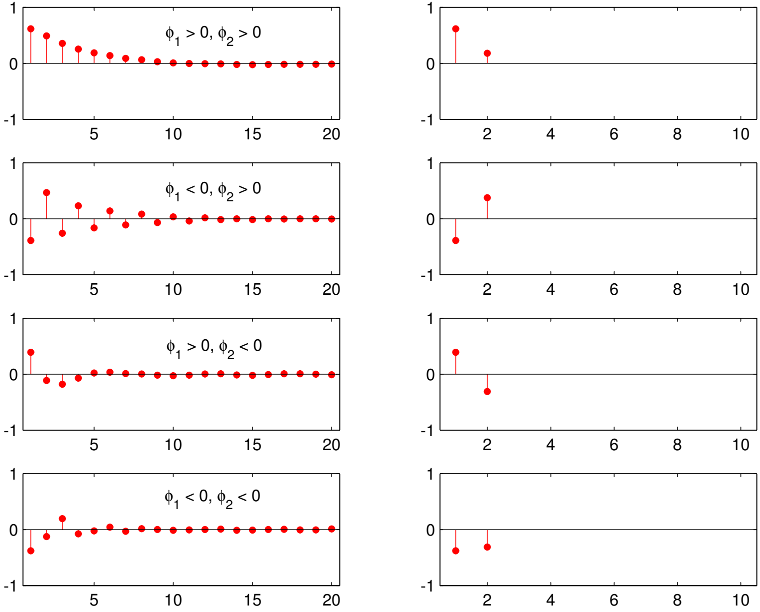
▷ This property will be a key element in identifying the order of an autoregressive process.

▷ Furthermore, the partial correlation coefficient of order p always coincides with the parameter ϕ_p.

▷ The Durbin-Levinson algorithm is an efficient method for estimating the partial correlation coefficients.



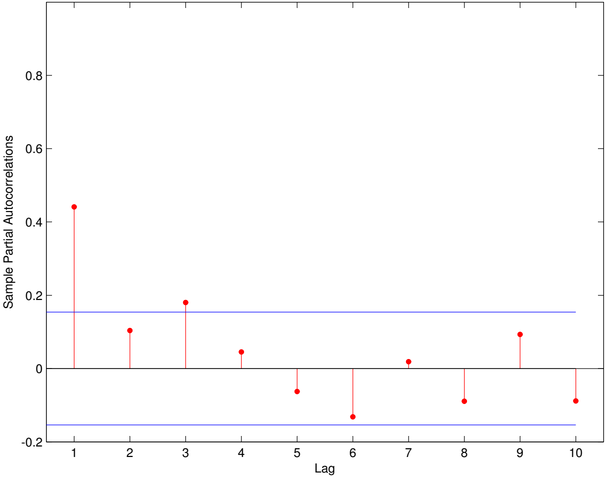
Parameter, ϕ > 0
Parameter, ϕ < 0



$\phi_1 > 0, \phi_2 > 0$
$\phi_1 < 0, \phi_2 > 0$
$\phi_1 > 0, \phi_2 < 0$
$\phi_1 < 0, \phi_2 < 0$

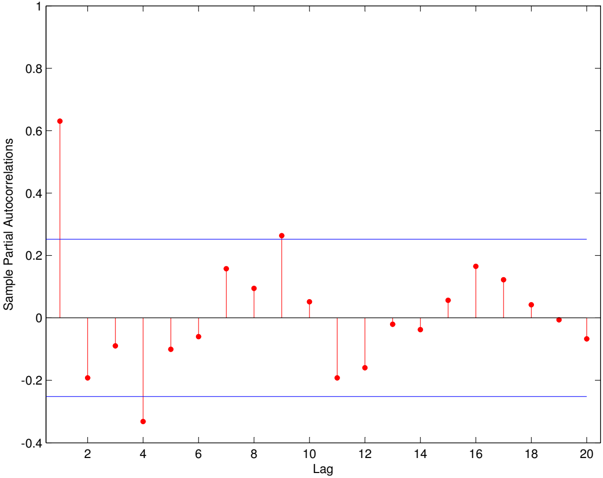
Example 40

The figure shows the partial autocorrelation function for the interest rates series from example 36. We conclude that the variations in interest rates follow an AR(1) process, since there is only one significant coefficient.



Example 41

The figure shows the partial autocorrelation function for the data on mink from example 37. This series presents significant partial autocorrelation coefficients up to the fourth lag, suggesting that the model is an AR(4).



Time Series Analysis

Moving average and ARMA processes

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4.2 Moving average and ARMA processes

Outline:

- Introduction
- The first-order moving average process, MA(1)
- The MA(q) process
- The MA(∞) process and Wold decomposition
- The ARMA(1,1) process
- The ARMA(p,q) processes
- The ARMA processes and the sum of stationary processes

Recommended readings:

▷ Chapters 2 and 3 of Brockwell and Davis (1996).

▷ Chapter 3 of Hamilton (1994).

▷ Chapter 3 of Peña, Tiao and Tsay (2001).

Introduction

▷ The autoregressive processes have, in general, infinite non-zero autocorrelation coefficients that decay with the lag. The AR processes have a relatively "long" memory, since the current value of a series is correlated with all previous ones, although with decreasing coefficients.

▷ This property means that we can write an AR process as a linear function of all its innovations, with weights that tend to zero with the lag. The AR processes cannot represent short memory series, where the current value of the series is only correlated with a small number of previous values.

▷ A family of processes that have this "very short memory" property are the moving average, or MA processes. The MA processes are a function of a finite, and generally small, number of its past innovations.

▷ Later, we will combine the properties of the AR and MA processes to define the ARMA processes, which give us a very broad and flexible family of stationary stochastic processes useful in representing many time series.

The first order moving average, MA(I)

▷ A **first order moving average**, MA(1), is defined by a linear combination of the last two innovations, according to the equation:

$$\widetilde{z}_t = a_t - \theta a_{t-1} \tag{65}$$

where $\widetilde{z}_t = z_t - \mu$, with μ being the mean of the process and a_t a white noise process with variance σ^2.

▷ The MA(1) process can be written with the operator notation:

$$\widetilde{z}_t = (1 - \theta B)\, a_t. \tag{66}$$

▷ This process is the sum of the two stationary processes, a_t and $-\theta a_{t-1}$ and, therefore, will always be stationary for any value of the parameter, unlike the AR processes.

The first order moving average, MA(I)

▷ In these processes we will assume that $|\theta| < 1$, so that the past innovation has less weight than the present. Then, we say that the process is **invertible** and has the property whereby the effect of past values of the series decreases with time.

▷ To justify this property, we substitute a_{t-1} in (65) as a function of z_{t-1}:

$$\widetilde{z}_t = a_t - \theta\left(\widetilde{z}_{t-1} + \theta a_{t-2}\right) = -\theta\widetilde{z}_{t-1} - \theta^2 a_{t-2} + a_t$$

and repeating this operation for a_{t-2} :

$$\widetilde{z}_t = -\theta\widetilde{z}_{t-1} - \theta^2(\widetilde{z}_{t-2} + \theta a_{t-3}) + a_t = -\theta\widetilde{z}_{t-1} - \theta^2\widetilde{z}_{t-2} - \theta^3 a_{t-3} + a_t$$

using successive substitutions of a_{t-3}, a_{t-4}..., etc., we obtain:

$$\widetilde{z}_t = -\sum_{i=1}^{t-1} \theta^i \widetilde{z}_{t-1} - \theta^t a_0 + a_t \qquad (67)$$

The first order moving average, MA(1)

▷ Notice that when $|\theta| < 1$, the effect of $\widetilde{z}_{t-k}$ tends to zero with k and the process is called invertible.

▷ If $|\theta| \geq 1$ it produces the paradoxical situation in which the effect of past observations increases with the distance. From here on, we assume that the process is invertible.

▷ Thus, since $|\theta| < 1$, there exists an inverse operator $(1 - \theta B)^{-1}$ and we can write equation (66) as:

$$\left(1 + \theta B + \theta^2 B^2 + \ldots\right) \widetilde{z}_t = a_t \tag{68}$$

that implies:

$$\widetilde{z}_t = -\sum_{i=1}^{\infty} \theta^i \widetilde{z}_{t-1} + a_t$$

which is equivalent to (67) assuming that the process begins in the infinite past. This equation represents the MA(1) process with $|\theta| < 1$ as an AR(∞) with coefficients that decay in a geometric progression.

The first order moving average, MA(1)

Expectation and variance

▷ The expectation can be derived from relation (65) which implies that $\mathrm{E}[\widetilde{z}_t] = 0$, so

$$\mathrm{E}[z_t] = \mu.$$

▷ The variance of the process is calculated from (65). Squaring and taking expectations, we obtain:

$$E(\widetilde{z}_t^2) = E(a_t^2) + \theta^2 E(a_{t-1}^2) - 2\theta E(a_t a_{t-1})$$

since $E(a_t a_{t-1}) = 0$, a_t is a white noise process and $E(a_t^2) = E(a_{t-1}^2) = \sigma^2$, then we have that:

$$\sigma_z^2 = \sigma^2 \left(1 + \theta^2\right). \tag{69}$$

▷ This equation tells us that the marginal variance of the process, σ_z^2, is always greater than the variance of the innovations, σ^2, and this difference increases with θ^2.

The first order moving average, MA(1)

Simple and partial autocorrelation function

▷ The first order autocovariance is calculated by multiplying equation (65) by $\widetilde{z}_{t-1}$ and taking expectations:

$$\gamma_1 = E(\widetilde{z}_t \widetilde{z}_{t-1}) = E(a_t \widetilde{z}_{t-1}) - \theta E(a_{t-1} \widetilde{z}_{t-1}).$$

▷ In this expression the first term $E(a_t \widetilde{z}_{t-1})$ is zero, since $\widetilde{z}_{t-1}$ depends on a_{t-1}, and a_{t-2}, but not on future innovations, such as a_t.

▷ To calculate the second term, replacing $\widetilde{z}_{t-1}$ with its expression according to (65), gives us

$$E(a_{t-1} \widetilde{z}_{t-1}) = E(a_{t-1}(a_{t-1} - \theta a_{t-2})) = \sigma^2$$

from which we obtain:

$$\gamma_1 = -\theta \sigma^2. \tag{70}$$

Simple and partial autocorrelation function

▷ The second order autocovariance is calculated in the same way:

$$\gamma_2 = E(\widetilde{z}_t \widetilde{z}_{t-2}) = E(a_t \widetilde{z}_{t-2}) - \theta E(a_{t-1} \widetilde{z}_{t-2}) = 0$$

since the series is uncorrelated with its future innovations the two terms are null. The same result is obtained for covariances of orders higher than two.

▷ In conclusion:

$$\gamma_j = 0, \quad j > 1. \tag{71}$$

Dividing the autocovariances (70) and (71) by expression (69) of the variance of the process, we find that the autocorrelation coefficients of an MA(1) process verify:

$$\rho_1 = \frac{-\theta}{1+\theta^2}, \qquad \rho_k = 0 \quad k > 1, \tag{72}$$

and the (ACF) will only have one value different from zero in the first lag.

The first order moving average, MA(1)

Simple and partial autocorrelation function

▷ This result proves that the autocorrelation function (*ACF*) of an MA(I) process has the same properties as the partial autocorrelation function (*PACF*)of an AR(1) process: there is a first coefficient different from zero and the rest are null.

▷ This duality between the AR(1) and the MA(1) is also seen in the partial autocorrelation function, *PACF*.

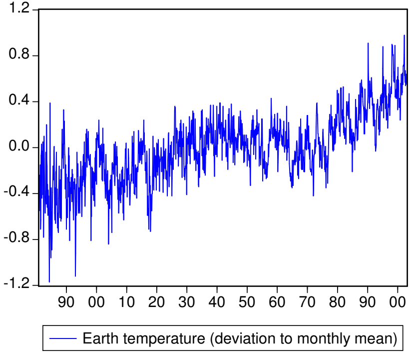
▷ According to (68), when we write an MA(1) process in autoregressive form z_{t-k} has a direct effect on z_t of magnitude θ^k, no matter what k is.

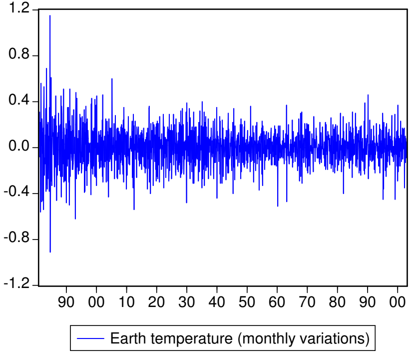
▷ Therefore, the *PACF* have all non-null coefficients and they decay geometrically with k.

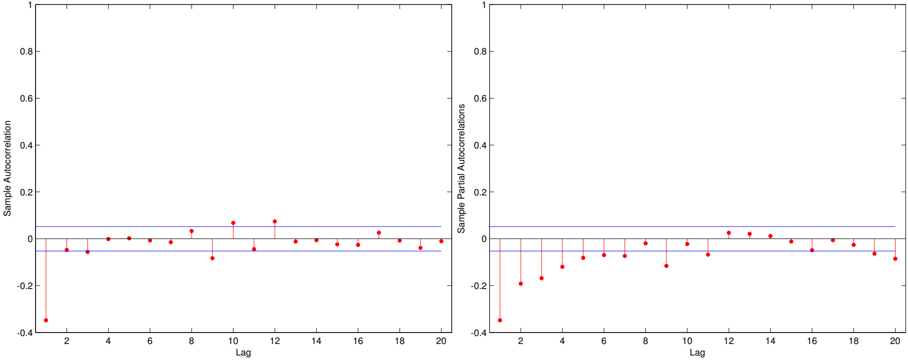
▷ This is the structure of the *ACF* in an AR(I) and, hence, we conclude that the *PACF* of an MA(1) has the same structure as the *ACF* of an AR(1).

Example 42

The left figure show monthly data from the years 1881 - 2002 and represent the deviation between the average temperature of a month and the mean of that month calculated by averaging the temperatures in the 25 years between 1951 and 1975. The right figure show $z_t = y_t - y_{t-1}$, which represents the variations in the Earth's mean temperature from one month to the next.







▷ In the autocorrelation function a single coefficient different from zero is observed, and in the PACF a geometric decay is observed.

▷ Both graphs suggest an MA(1) model for the series of differences between consecutive months, z_t.

The MA(q) process

▷ Generalizing on the idea of an MA(1), we can write processes whose current value depends not only on the last innovation but on the last q innovations. Thus the **MA(q) process** is obtained, with general representation:

$$\widetilde{z}_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - ... - \theta_q a_{t-q}.$$

▷ Introducing the operator notation:

$$\widetilde{z}_t = \left(1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q\right) a_t \tag{73}$$

it can be written more compactly as:

$$\widetilde{z}_t = \theta_q \left(B\right) a_t. \tag{74}$$

▷ An MA(q) is always stationary, as it is a sum of stationary processes. We say that the process is invertible if the roots of the operator $\theta_q \left(B\right) = 0$ are, in modulus, greater than the unit.

The MA(q) process

▷ The properties of this process are obtained with the same method used for the MA(1). Multiplying (73) by $\widetilde{z}_{t-k}$ for $k \geq 0$ and taking expectations, the autocovariances are obtained:

$$\begin{aligned} \gamma_0 &= \left(1+\theta_1^2+\ldots+\theta_q^2\right)\sigma^2 && && (75) \\ \gamma_k &= \left(-\theta_k+\theta_1\theta_{k+1}+\ldots+\theta_{q-k}\theta_q\right)\sigma^2 && k=1,\ldots,q, && (76) \\ \gamma_k &= 0 && k>q, && (77) \end{aligned}$$

showing that an MA(q) process has exactly the first q coefficients of the autocovariance function different from zero.

▷ Dividing the covariances by γ_0 and utilizing a more compact notation, the autocorrelation function is:

$$\begin{aligned} \rho_k &= \frac{\sum_{i=0}^{i=q}\theta_i\theta_{k+i}}{\sum_{i=0}^{i=q}\theta_i^2}, && k=1,\ldots,q && (78) \\ \rho_k &= 0, && k>q, \end{aligned}$$

where $\theta_0=-1$, and $\theta_k=0$ for $k \geq q+1$.

The MA(q) process

▷ To compute the partial autocorrelation function of an MA(q) we express the process as an AR(∞):

$$\theta_q^{-1}(B)\,\widetilde{z}_t = a_t,$$

and letting $\theta_q^{-1}(B) = \pi(B)$, where:

$$\pi(B) = 1 - \pi_1 B - ... - \pi_k B^k - ...$$

and the coefficients of $\pi(B)$ are obtained imposing $\pi(B)\,\theta_q(B) = 1$. We say that the process is invertible if all the roots of $\theta_q(B) = 0$ lie outside the unit circle. Then the series $\pi(B)$ is convergent.

▷ For invertible MA processes, setting the powers of B to zero, we find that the coefficients π_i verify the following equation:

$$\pi_k = \theta_1 \pi_{k-1} + ... + \theta_q \pi_{k-q}$$

where $\pi_0 = -1$ and $\pi_j = 0$ for $j < 0$.

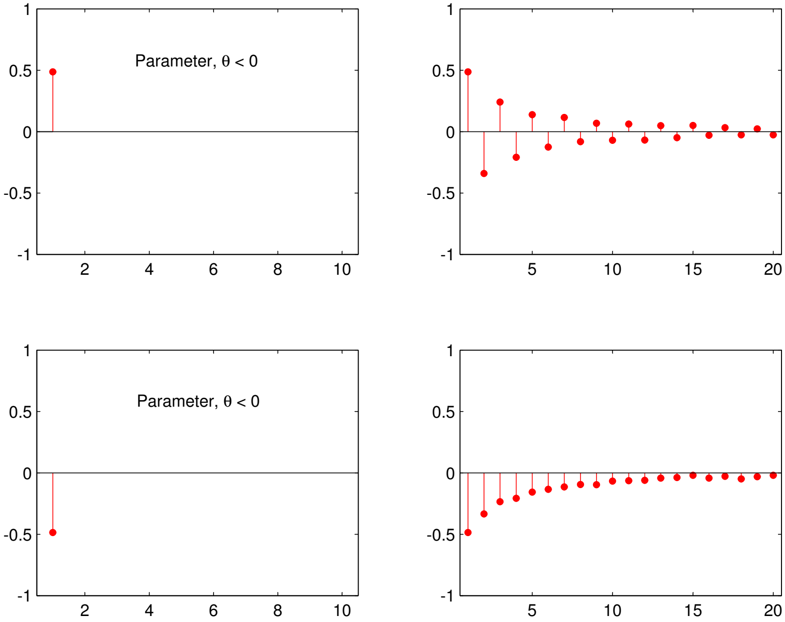
The MA(q) process

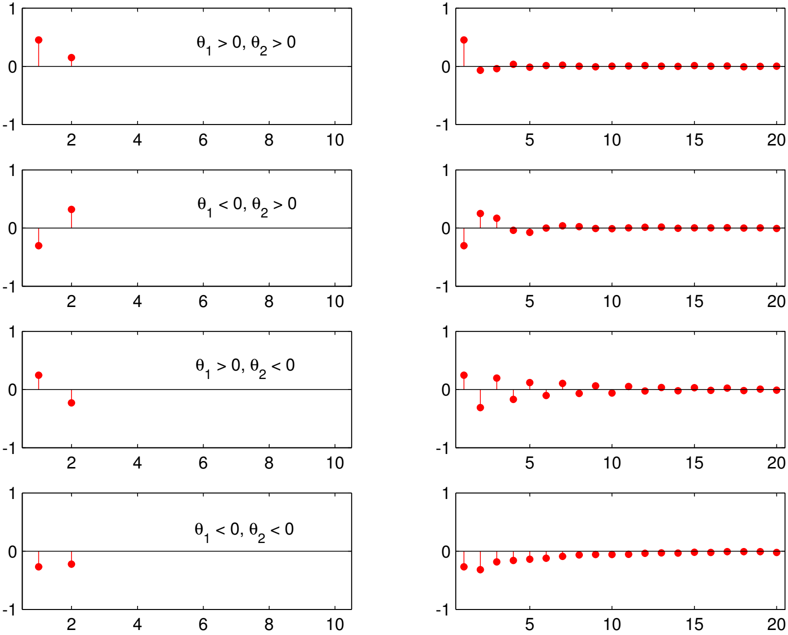
▷ The solution to this difference equation is of the form $\sum A_i G_i^k$, where now the G_i^{-1} are the roots of the moving average operator. Having obtained the coefficients π_i of the representation AR(∞), we can write the MA process as:

$$\widetilde{z}_t = \sum\nolimits_{i=1}^{\infty} \pi_i \widetilde{z}_{t-i} + a_t.$$

▷ From this expression we conclude that the *PACF* of an MA is non-null for all lags, since a direct effect of $\widetilde{z}_{t-i}$ on $\widetilde{z}_t$ exists for all i. The *PACF* of an MA process thus has the same structure as the *ACF* of an AR process of the same order.

▷ We conclude that a duality exists between the AR and MA processes such that the *PACF* of an MA(q) has the structure of the *ACF* of an AR(q) and the *ACF* of an MA(q) has the structure of the *PACF* of an AR(q).


Parameter, θ < 0
Parameter, θ < 0


θ1 > 0, θ2 > 0
θ1 < 0, θ2 > 0
θ1 > 0, θ2 < 0
θ1 < 0, θ2 < 0

The MA(∞) process and Wold decomposition

▷ The autoregressive and moving average processes are specific cases of a general representation of stationary processes obtained by Wold (1938).

▷ Wold proved that any weakly stationary stochastic process, z_t, with finite mean, μ, that does not contain deterministic components, can be written as a linear function of uncorrelated random variables, a_t, as:

$$z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots = \mu + \sum_{i=0}^{\infty} \psi_i a_{t-i} \qquad (\psi_0 = 1) \qquad (79)$$

where $E(z_t) = \mu$, and $E[a_t] = 0;\ Var(a_t) = \sigma^2;\ E[a_t a_{t-k}] = 0,\ k > 1.$

▷ Letting $\widetilde{z}_t = z_t - \mu$, and using the lag operator, we can write:

$$\widetilde{z}_t = \psi(B) a_t, \qquad (80)$$

with $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ being an indefinite polynomial in the lag operator B.

The MA(∞) process and Wold decomposition

$\triangleright$ We denote (80) as the general linear representation of a non-deterministic stationary process.

$\triangleright$ This representation is important because it guarantees that any stationary process admits a linear representation.

$\triangleright$ In general, the variables a_t make up a white noise process, that is, they are uncorrelated with zero mean and constant variance.

$\triangleright$ In certain specific cases the process can be written as a function of normal independent variables $\{a_t\}$. Thus the variable $\widetilde{z}_t$ will have a normal distribution and the weak coincides with strict stationarity.

$\triangleright$ The series $\widetilde{z}_t$, can be considered as the result of passing a process of impulses $\{a_t\}$ of uncorrelated variables through a linear filter $\psi(B)$ that determines the weight of each "impulse" in the response.

The MA(∞) process and Wold decomposition

$\triangleright$ The properties of the process are obtained as in the case of an MA model. The variance of z_t in (79) is:

$$Var\left(z_t\right) = \gamma_0 = \sigma^2 \sum\nolimits_{i=0}^{\infty} \psi_i^2 \tag{81}$$

and for the process to have finite variance the series $\left\{\psi_i^2\right\}$ must be convergent.

$\triangleright$ We observe that if the coefficients ψ_i are zero after lag q the general model is reduced to an MA(q) and formula (81) coincides with (76).

$\triangleright$ The covariances are obtained with

$$\gamma_k = E(\widetilde{z}_t \widetilde{z}_{t-k}) = \sigma^2 \sum\nolimits_{i=0}^{\infty} \psi_i \psi_{i+k},$$

which for $k = 0$ provide, as a particular case, formula (81) for the variance.

$\triangleright$ Furthermore, if the coefficients ψ_i are zero after lag q on, this expression provides the autocovariances of an MA(q) expression.

The MA(∞) process and Wold decomposition

▷ The autocorrelation coefficients are given by:

$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{i=0}^{\infty} \psi_i^2}, \tag{82}$$

which generalizes the expression (78) of the autocorrelations of an MA(q).

▷ A consequence of (79) is that any stationary process also admits an autoregressive representation, which can be of infinite order. This representation is the inverse of that of Wold, and we write

$$\widetilde{z}_t = \pi_1 \widetilde{z}_{t-1} + \pi_2 \widetilde{z}_{t-2} + \ldots + a_t,$$

which in operator notation is reduced to

$$\pi(B)\widetilde{z}_t = a_t.$$

▷ The AR(∞) representation is the dual representation of the MA(∞) and it is shown that: $\pi(B)\psi(B) = 1$ such that by setting the powers of B to zero we can obtain the coefficients of one representation from those of another.

The AR and MA processes and the general process

▷ It is straightforward to prove that an MA process is a particular case of the Wold representation, as are the AR processes.

▷ For example, the AR(1) process

$$(1 - \phi B)\,\widetilde{z}_t = a_t \tag{83}$$

can be written, multiplying by the inverse operator $(1 - \phi B)^{-1}$

$$\widetilde{z}_t = \left(1 + \phi B + \phi^2 B^2 + \ldots\right) a_t$$

which represents the AR(1) process as a particular case of the MA(∞) form of the general linear process, with coefficients ψ_i that decay in geometric progression.

▷ The condition of stationarity and finite variance, convergent series of coefficients ψ_i^2 , is equivalent now to $|\phi| < 1$.

The AR and MA processes and the general process

▷ For higher order AR process to obtain the coefficients of the MA(∞) representation we impose the condition that the product of the AR and MA(∞) operators must be the unit.

▷ For example, for an AR(2) the condition is:

$$\left(1 - \phi_1 B - \phi_2 B^2\right)\left(1 + \psi_1 B + \psi_2 B^2 + ...\right) = 1$$

and imposing the cancelation of powers of B we obtain the coefficients:

$$\begin{aligned} \psi_1 &= \phi_1 \\ \psi_2 &= \phi_1 \psi_1 + \phi_2 \\ \psi_i &= \phi_1 \psi_{i-1} + \phi_2 \psi_{i-2}, \qquad i \geq 2 \end{aligned}$$

where $\psi_0 = 1$.

The AR and MA processes and the general process

▷ Analogously, for an AR(p) the coefficients ψ_i of the general representation are calculated by:

$$\left(1-\phi_1 B - \ldots - \phi_p B^p\right)\left(1+\psi_1 B + \psi_2 B^2 + \ldots\right) = 1$$

and for $i \geq p$ they must verify the condition:

$$\psi_i = \phi_1 \psi_{i-1} + \ldots + \phi_p \psi_{i-p}, \qquad i \geq p.$$

▷ The condition of stationarity implies that the roots of the characteristic equation of the AR(p) process, $\phi_p(B) = 0$, must lie outside the unit circle.

The AR and MA processes and the general process

▷ Writing the operator $\phi_p(B)$ as:

$$\phi_p(B) = \prod_{i=1}^{p} (1 - G_i B)$$

where G_i^{-1} are the roots of $\phi_p(B) = 0$, it is shown that, expanding in partial fractions:

$$\phi_p^{-1}(B) = \sum \frac{k_i}{(1 - G_i B)}$$

will be convergent if $|G_i| < 1$.

▷ Summarizing, the AR processes can be considered as particular cases of the general linear process characterized by the fact that: (1) all the ψ_i are different from zero; (2) there are restrictions on the ψ_i, that depend on the order of the process.

▷ In general they verify the sequence $\psi_i = \phi_1 \psi_{i-1} + ... + \phi_p \psi_{i-p}$, with initial conditions that depend on the order of the process.

The ARMA(1,1) process

▷ One conclusion from the above section is that the AR and MA processes approximate a general linear MA(∞) process from a complementary point of view:

- The AR admit an MA(∞) structure, but they impose restrictions on the decay patterns of the coefficients ψ_i.
- The MA require a number of finite terms, however, they do not impose restrictions on the coefficients.
- From the point of view of the autocorrelation structure, the AR processes allow many coefficients different from zero, but with a fixed decay pattern, whereas the MA permit a few coefficients different from zero with arbitrary values.

▷ The **ARMA processes** try to combine these properties and allow us to represent in a *reduced* form (using few parameters) those processes whose first q coefficients can be any, whereas the following ones decay according to simple rules.

The ARMA(1,1) process

▷ The simplest process, the **ARMA(1,1)** is written as:

$$\widetilde{z}_t = \phi_1 \widetilde{z}_{t-1} + a_t - \theta_1 a_{t-1},$$

or, using operator notations:

$$(1 - \phi_1 B)\, \widetilde{z}_t = (1 - \theta_1 B)\, a_t, \tag{84}$$

where $|\phi_1| < 1$ for the process to be stationary, and $|\theta_1| < 1$ for it to be invertible.

▷ Moreover, we assume that $\phi_1 \neq \theta_1$. If both parameters were identical, multiplying both parts by the operator $(1 - \phi_1 B)^{-1}$, we would have $\widetilde{z}_t = a_t$, and the process would be white noise.

▷ In the formulation of the ARMA models we always assume that there are no common roots in the AR and MA operators.

The ARMA(1,1) process

The autocorrelation function

▷ To obtain the autocorrelation function of an ARMA(1,1), multiplying (84) by $\widetilde{z}_{t-k}$ and taking expectations, results in:

$$\gamma_k = \phi_1 \gamma_{k-1} + E\left(a_t \widetilde{z}_{t-k}\right) - \theta_1 E\left(a_{t-1} \widetilde{z}_{t-k}\right). \tag{85}$$

▷ For $k > 1$, the noise a_t is uncorrelated with the series history. As a result:

$$\gamma_k = \phi_1 \gamma_{k-1}, \qquad k > 1. \tag{86}$$

▷ For $k = 0$, $E[a_t \widetilde{z}_t] = \sigma^2$ and

$$E\left[a_{t-1} \widetilde{z}_t\right] = E\left[a_{t-1}\left(\phi_1 \widetilde{z}_{t-1} + a_t - \theta_1 a_{t-1}\right)\right] = \sigma^2(\phi_1 - \theta_1)$$

replacing these results in (85), for $k = 0$

$$\gamma_0 = \phi \gamma_1 + \sigma^2 - \theta_1 \sigma^2 \left(\phi_1 - \theta_1\right). \tag{87}$$

The autocorrelation function

▷ Taking $k = 1$ in (85), results in $E[a_t \widetilde{z}_{t-1}] = 0$, $E[a_{t-1} \widetilde{z}_{t-1}] = \sigma^2$ and:

$$\gamma_1 = \phi_1 \gamma_0 - \theta_1 \sigma^2, \tag{88}$$

solving for (87) and (88) we obtain:

$$\gamma_0 = \sigma^2 \frac{1 - 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}$$

▷ To compute the first autocorrelation coefficient, we divide (88) by the above expression:

$$\rho_1 = \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1 - 2\phi_1\theta_1 + \theta_1^2} \tag{89}$$

▷ Observe that if $\phi_1 = \theta_1$, this autocorrelation is zero because, as we indicated earlier, then the operators $(1 - \phi_1 B)$ and $(1 - \theta_1 B)$ are cancelled out and it will result in a white noise process.

The autocorrelation function

▷ In the typical case where both coefficients are positive and $\phi_1 > \theta_1$ it is easy to prove that the correlation increases with $(\phi_1 - \theta_1)$.

▷ The rest of the autocorrelation coefficients are obtained dividing (86) by γ_0, which results in:

$$\rho_k = \phi_1 \rho_{k-1} \quad k > 1 \tag{90}$$

which indicates that from the first coefficient on, the *ACF* of an ARMA(1,1) decays exponentially, determined by parameter ϕ_1 of the AR part.

▷ The difference with an AR(1) is that the decay starts at ρ_1, not at $\rho_0 = 1$, and this first value of the first order autocorrelation depends on the relative difference between ϕ_1 and θ_1. We observe that if $\phi_1 \approx 1$ and $\phi_1 - \theta_1 = \varepsilon$ is small, we can have many coefficients different from zero but they will all be small.

The ARMA(1,1) process

The partial autocorrelation function

▷ To calculate the *PACF*, we write the ARMA(1, 1) in the AR(∞) form:

$$(1-\theta_1 B)^{-1}(1-\phi_1 B)\widetilde{z}_t = a_t,$$

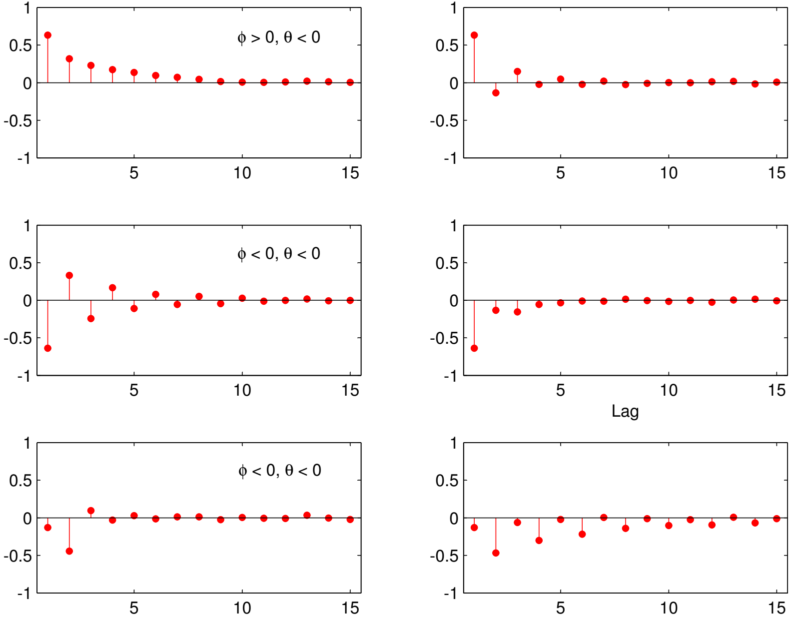
and using $(1-\theta_1 B)^{-1} = 1+\theta_1 B+\theta_1^2 B^2+\ldots$, and operating, we obtain:

$$\widetilde{z}_t = (\phi_1-\theta_1)\widetilde{z}_{t-1}+\theta_1(\phi_1-\theta_1)\widetilde{z}_{t-2}+\theta_1^2(\phi_1-\theta_1)\widetilde{z}_{t-3}+\ldots+a_t.$$

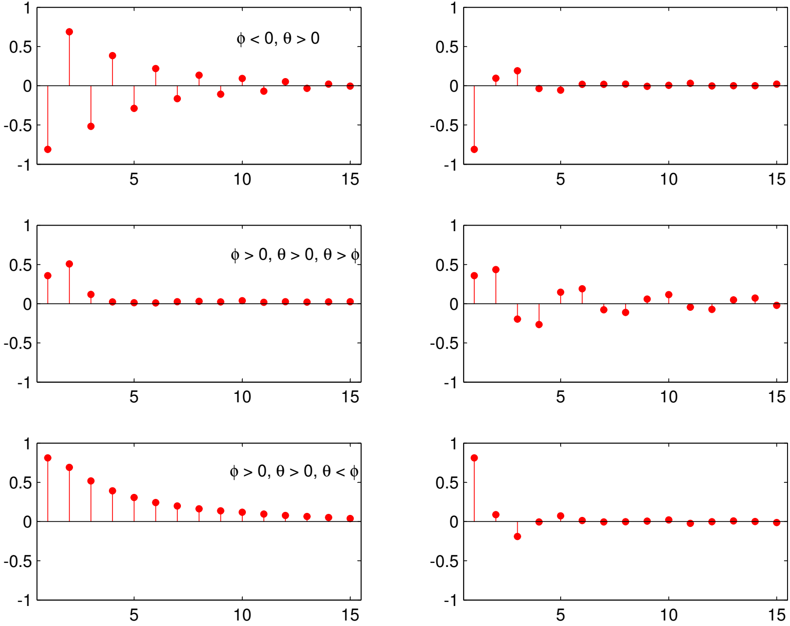
▷ The direct effect of $\widetilde{z}_{t-k}$ on $\widetilde{z}_t$ decays geometrically with θ_1^k and, therefore, the *PACF* will have a geometric decay starting from an initial value.

▷ In conclusion, in an ARMA(1,1) process the *ACF* and the *PACF* have a similar structure: an initial value, whose magnitude depends on $\phi_1-\theta_1$, followed by a geometric decay.

▷ The rate of decay in the *ACF* depends on ϕ_1, whereas in the *PACF* it depends on θ_1.



ϕ > 0, θ < 0
ϕ < 0, θ < 0
ϕ < 0, θ < 0
Lag



ϕ < 0, θ > 0
ϕ > 0, θ > 0, θ > ϕ
ϕ > 0, θ > 0, θ < ϕ

The ARMA(p,q) processes

▷ The **ARMA** (p, q) process is defined by:

$$\left(1-\phi_1 B-\ldots-\phi_p B^p\right) \widetilde{z}_t=\left(1-\theta_1 B-\ldots-\theta_q B^q\right) a_t \tag{91}$$

or, in compact notation,

$$\phi_p(B) \widetilde{z}_t=\theta_q(B) a_t .$$

▷ The process is stationary if the roots of $\phi_p(B)=0$ are outside the unit circle, and invertible if those of $\theta_q(B)=0$ are.

▷ We also assume that there are no common roots that can be cancelled between the AR and MA operators.

The ARMA(p,q) processes

▷ To obtain the coefficients ψ_i of the general representation of the MA(∞) model we write:

$$\widetilde{z}_t = \phi_p(B)^{-1}\theta_q(B)\,a_t = \psi(B)\,a_t$$

and we equate the powers of B in $\psi(B)\,\phi_p(B)$ to those of $\theta_q(B)$.

▷ Analogously, we can represent an ARMA(p,q) as an AR(∞) model making:

$$\theta_q^{-1}(B)\,\phi_p(B)\,\widetilde{z}_t = \pi(B)\,\widetilde{z}_t = a_t$$

and the coefficients π_i will be the result of $\phi_p(B) = \theta_q(B)\,\pi(B)$.

Autocorrelation function

▷ To calculate the autocovariances, we multiply (91) by $\widetilde{z}_{t-k}$ and take expectations,

$$\gamma_k - \phi_1\gamma_{k-1} - \dots - \phi_p\gamma_{k-p} =$$
$$= E\left[a_t\widetilde{z}_{t-k}\right] - \theta_1 E\left[a_{t-1}\widetilde{z}_{t-k}\right] - \dots - \theta_q E\left[a_{t-q}\widetilde{z}_{t-k}\right]$$

▷ For $k > q$ all the terms on the right are cancelled, and dividing by γ_0:

$$\rho_k - \phi_1\rho_{k-1} - \dots - \phi_p\rho_{k-p} = 0,$$

that is:

$$\phi_p(B)\,\rho_k = 0 \qquad k > q, \tag{92}$$

▷ We conclude that the autocorrelation coefficients for $k > q$ follow a decay determined only in the autoregressive part.

The ARMA(p,q) processes

Autocorrelation function

▷ The first q coefficients depend on the MA and AR parameters and of those, p provide the initial values for the later decay (for $k > q$) according to (92). Therefore, if $p > q$ all the *ACF* will show a decay dictated by (92).

▷ To summarize, the *ACF*:

- have $q - p + 1$ initial values with a structure that depends on the AR and MA parameters;
- they decay starting from the coefficient $q - p$ as a mixture of exponentials and sinusoids, determined exclusively by the autoregressive part.

▷ It can be proved that the *PACF* have a similar structure.

Summary

▷ The *ACF* and *PACF* of the ARMA processes are the result of superimposing their AR and MA properties:

- In the *ACF* certain initial coefficients that depend on the order of the MA part and later a decay dictated by the AR part.
- In the *PACF* initial values dependent on the AR order followed by the decay due to the MA part.
- This complex structure makes it difficult in practice to identify the order of an ARMA process.

	ACF	**PACF**
AR(p)	Many non-null coefficients	first p non-null, the rest 0
MA(q)	first q non-null, the rest 0	Many non-null coefficients
ARMA(p,q)	Many non-null coefficients	Many non-null coefficients

ARMA processes and the sum of stationary processes

▷ One reason that explains why the ARMA processes are frequently found in practice is that summing AR processes results in an ARMA process.

▷ To illustrate this idea, we take the simplest case where we add white noise to an AR(1) process. Let

$$z_t = y_t + v_t \tag{93}$$

where $y_t = \phi y_{t-1} + a_t$ follows an AR(1) process of zero mean and v_t is white noise independent of a_t, and thus of y_t.

▷ Process z_t can be interpreted as the result of observing an AR(1) process with a certain measurement error. The variance of this addition process is:

$$\gamma_z(0) = E(z_t^2) = E\left[(y_t^2 + v_t^2 + 2y_t v_t)\right] = \gamma_y(0) + \sigma_v^2, \tag{94}$$

since, as the summands are independent, the variance is the sum of the variance of the components.

ARMA processes and the sum of stationary processes

▷ To calculate the autocovariance we take into account that the autocovariances of process y_t verify $\gamma_y(k) = \phi^k \gamma_y(0)$ and those of process v_t are null. Thus, $k \geq 1$,

$$\gamma_z(k) = E(z_t z_{t-k}) = E\left[(y_t + v_t)(y_{t-k} + v_{t-k})\right] = \gamma_y(k) = \phi^k \gamma_y(0),$$

since, due to the independence of the components, $E\left[y_t v_{t-k}\right] = 0$ for any k and since v_t is white noise $E\left[v_t v_{t-k}\right] = 0$. Specifically, replacing the variance $\gamma_y(0)$ with its expression (94) for $k = 1$, we obtain:

$$\gamma_z(1) = \phi \gamma_z(0) - \phi \sigma_v^2, \tag{95}$$

whereas for $k \geq 2$

$$\gamma_z(k) = \phi \gamma_z(k-1). \tag{96}$$

▷ If we compare equation (95) with (88), and equation (96) with (86) we conclude that process z_t follows an ARMA(1,1) model with an AR parameter equal to ϕ. Parameter θ and the variance of the innovations of the ARMA(1,1) depend on the relationship between the variances of the summands.

ARMA processes and the sum of stationary processes

▷ Indeed, letting $\lambda = \sigma_v^2/\gamma_y(0)$ denote the quotient of variances between the two summands, according to equation (95) the first autocorrelation is:

$$\rho_z(1) = \phi - \phi\frac{\lambda}{1+\lambda}$$

whereas by (96) the remainders verify, for $k \geq 2$,

$$\rho_z(k) = \phi\rho_z(k-1).$$

▷ If λ is very small, which implies that the variance of the additional noise or measurement error is small, the process will be very close to an AR(1), and parameter θ will be very small.

▷ If λ is not very small, we have the ARMA(1,1) and the value of θ depends on λ and on ϕ.

▷ If $\lambda \to \infty$, such that the white noise is dominant, the parameter θ will be equal to the value of ϕ and we have a white noise process.

ARMA processes and the sum of stationary processes

▷ The above results can be generalized for any AR(p) process. It can be proved that:

$$AR(p) + AR(0) = ARMA(p, p),$$

and also that:

$$AR(p) + AR(q) = ARMA(p + q, \max(p, q))$$

▷ For example, if we add two independent AR(1) processes we obtain a new process, ARMA(2,1).

▷ The sum of MA processes is simple: by adding independent MA processes we obtain new MA processes.

ARMA processes and the sum of stationary processes

▷ Let us assume that

$$z_t = x_t + y_t$$

where the two processes x_t, y_t have zero mean and follow independent MA(1) processes with covariances $\gamma_x(k)$, $\gamma_y(k)$, that are zero for $k > 1$.

▷ The variance of the summed process is:

$$\gamma_z(0) = \gamma_x(0) + \gamma_y(0), \tag{97}$$

and the autocovariance of order k

$$E(z_t z_{t-k}) = \gamma_z(k) = E\left[(x_t + y_t)(x_{t-k} + y_{t-k})\right] = \gamma_x(k) + \gamma_y(k).$$

▷ Therefore, all the covariances $\gamma_z(k)$ of order higher than one will be zero because $\gamma_x(k)$ and $\gamma_y(k)$ are zero.

▷ Dividing the equation (44) by $\gamma_z(0)$ and using (97), shows that the autocorrelations verify:

$$\rho_z(k) = \rho_x(k)\lambda + \rho_y(k)(1-\lambda)$$

where:

$$\lambda = \frac{\gamma_x(0)}{\gamma_x(0) + \gamma_y(0)}$$

is the relative variance of the first summand.

▷ In the particular case in which one of the processes is white noise we obtain an MA(1) model whose autocorrelation is smaller than that of the original process. In the same way it is easy to show that:

$$MA(q_1) + MA(q_2) = MA(\max(q_1, q_2)).$$

ARMA processes and the sum of stationary processes

▷ For ARMA processes it is also proved that:

$$ARMA(p_1, q_1) + ARMA(p_2, q_2) = ARMA(a, b)$$

where

$$a \leq p_1 + p_2, \qquad b \leq \max(p_1 + q_1, p_2 + q_2)$$

▷ These results suggest that whenever we observe processes that are the sum of others, and some of them have an AR structure, we expect to observe ARMA processes.

▷ This result may seem surprising at first because the majority of real series can be considered to be the sum of certain components, which would mean that all real processes should be ARMA.

▷ Nevertheless, in practice many real series are approximated well by means of AR or MA series. The explanation for this paradox is that an ARMA$(q + h, q)$ process with q similar roots in the AR and MA parts can be well approximated by an AR(h), due to the near cancellation of similar roots in both members.

Example 43

The figures show the autocorrelation functions of an AR(1) and an AR(0).

Correlogram of AR1

Date: 01/30/08 Time: 17:02
Sample: 1 200
Included observations: 200

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.766	0.766	118.97	0.000
		2	0.563	-0.056	183.56	0.000
		3	0.443	0.075	223.75	0.000
		4	0.369	0.044	251.86	0.000
		5	0.281	-0.059	268.22	0.000
		6	0.226	0.042	278.85	0.000
		7	0.177	-0.022	285.41	0.000
		8	0.123	-0.037	288.60	0.000
		9	0.037	-0.108	288.89	0.000
		10	-0.058	-0.110	289.62	0.000
		11	-0.097	0.030	291.61	0.000
		12	-0.062	0.109	292.43	0.000
		13	-0.022	0.046	292.54	0.000
		14	0.005	0.038	292.54	0.000
		15	-0.009	-0.068	292.56	0.000
		16	-0.030	-0.030	292.75	0.000
		17	-0.045	-0.010	293.21	0.000
		18	-0.046	0.006	293.68	0.000
		19	-0.057	-0.050	294.41	0.000
		20	-0.043	0.012	294.82	0.000

Correlogram of E2

Date: 01/30/08 Time: 17:03
Sample: 1 200
Included observations: 200

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.019	0.019	0.0725	0.788
		2	-0.103	-0.103	2.2310	0.328
		3	-0.076	-0.072	3.4065	0.333
		4	-0.044	-0.053	3.8018	0.433
		5	0.120	0.108	6.8008	0.236
		6	-0.023	-0.042	6.9091	0.329
		7	-0.064	-0.048	7.7612	0.354
		8	0.055	0.066	8.4085	0.395
		9	0.030	0.025	8.5954	0.475
		10	-0.005	-0.019	8.6004	0.570
		11	-0.072	-0.059	9.7150	0.556
		12	0.040	0.065	10.050	0.612
		13	0.132	0.107	13.825	0.386
		14	0.097	0.090	15.852	0.323
		15	-0.024	0.005	15.977	0.384
		16	-0.121	-0.073	19.214	0.258
		17	-0.005	0.004	19.219	0.316
		18	0.042	0.004	19.606	0.355
		19	-0.055	-0.076	20.287	0.378
		20	-0.034	-0.027	20.541	0.425

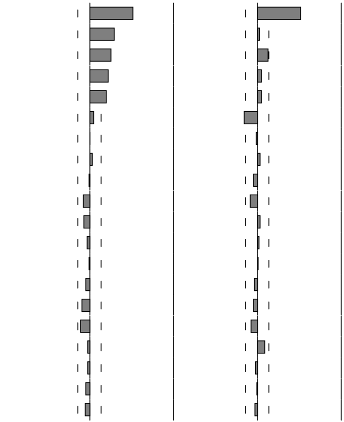
Datafile sumofst.xls

▷ The figure shows the autocorrelation functions of the sum of AR(1)+AR(0).

Correlogram of SUM1

Date: 01/30/08 Time: 17:07
Sample: 1 200
Included observations: 200

Autocorrelation | Partial Correlation



	AC	PAC	Q-Stat	Prob
1	0.536	0.536	58.422	0.000
2	0.307	0.028	77.700	0.000
3	0.270	0.133	92.622	0.000
4	0.234	0.052	103.94	0.000
5	0.208	0.056	112.91	0.000
6	0.053	-0.156	113.49	0.000
7	0.004	-0.009	113.50	0.000
8	0.038	0.033	113.79	0.000
9	-0.000	-0.043	113.79	0.000
10	-0.079	-0.087	115.11	0.000
11	-0.068	0.039	116.10	0.000
12	-0.030	0.019	116.29	0.000
13	-0.008	0.014	116.31	0.000
14	-0.046	-0.038	116.77	0.000
15	-0.090	-0.044	118.55	0.000
16	-0.110	-0.079	121.21	0.000
17	-0.020	0.098	121.29	0.000
18	-0.023	-0.025	121.41	0.000
19	-0.048	-0.003	121.93	0.000
20	-0.052	-0.025	122.53	0.000

Example 44

The figures show the autocorrelation functions of two MA(1).

Correlogram of MA1A

Date: 01/30/08 Time: 17:26
Sample: 1 200
Included observations: 199

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.501	0.501	50.650	0.000
		2	0.072	-0.239	51.700	0.000
		3	0.079	0.218	52.983	0.000
		4	0.075	-0.090	54.146	0.000
		5	0.040	0.067	54.473	0.000
		6	-0.092	-0.204	56.211	0.000
		7	-0.158	0.003	61.412	0.000
		8	-0.059	0.017	62.141	0.000
		9	0.021	0.042	62.234	0.000
		10	-0.005	-0.035	62.240	0.000
		11	-0.024	0.025	62.366	0.000
		12	0.061	0.088	63.153	0.000
		13	0.124	0.024	66.464	0.000
		14	0.047	-0.062	66.938	0.000
		15	0.028	0.084	67.108	0.000
		16	0.001	-0.099	67.109	0.000
		17	0.003	0.075	67.111	0.000
		18	0.031	-0.031	67.324	0.000
		19	-0.037	-0.014	67.621	0.000
		20	-0.052	0.005	68.224	0.000

Correlogram of MA1B

Date: 01/30/08 Time: 17:27
Sample: 1 200
Included observations: 199

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.445	0.445	40.091	0.000
		2	-0.035	-0.292	40.344	0.000
		3	0.033	0.250	40.571	0.000
		4	0.092	-0.082	42.321	0.000
		5	0.043	0.064	42.699	0.000
		6	-0.041	-0.101	43.046	0.000
		7	-0.055	0.024	43.677	0.000
		8	0.002	0.001	43.678	0.000
		9	-0.019	-0.051	43.757	0.000
		10	-0.080	-0.038	45.115	0.000
		11	0.010	0.103	45.137	0.000
		12	0.080	-0.010	46.509	0.000
		13	0.003	-0.024	46.512	0.000
		14	-0.083	-0.064	47.984	0.000
		15	-0.101	-0.055	50.192	0.000
		16	-0.075	-0.043	51.434	0.000
		17	-0.077	-0.055	52.743	0.000
		18	-0.134	-0.085	56.717	0.000
		19	-0.133	-0.029	60.639	0.000
		20	-0.050	0.001	61.199	0.000

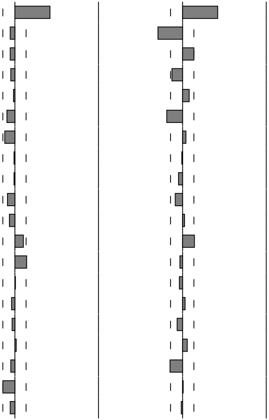
Datafile `sumofst.xls`

▷ The figure shows the autocorrelation functions of the sum of MA(1)+MA(1).

Correlogram of SUM2

Date: 01/30/08 Time: 17:26
Sample: 1 200
Included observations: 199

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.440	0.440	39.065	0.000
		2	-0.051	-0.303	39.586	0.000
		3	-0.055	0.146	40.205	0.000
		4	-0.042	-0.128	40.571	0.000
		5	-0.012	0.084	40.599	0.000
		6	-0.091	-0.191	42.300	0.000
		7	-0.115	0.047	45.078	0.000
		8	-0.007	-0.002	45.086	0.000
		9	-0.002	-0.042	45.088	0.000
		10	-0.083	-0.085	46.532	0.000
		11	-0.063	0.026	47.389	0.000
		12	0.107	0.148	49.853	0.000
		13	0.150	-0.027	54.686	0.000
		14	0.008	-0.040	54.701	0.000
		15	-0.034	0.034	54.945	0.000
		16	-0.031	-0.059	55.158	0.000
		17	0.017	0.065	55.220	0.000
		18	-0.048	-0.152	55.732	0.000
		19	-0.142	0.015	60.189	0.000
		20	-0.053	-0.014	60.820	0.000



Example 45

The figures show the autocorrelation functions of the two sum of AR(1)+MA(1).

Correlogram of SUM3

Date: 01/30/08 Time: 17:31
Sample: 1 200
Included observations: 199

Lag	AC	PAC	Q-Stat	Prob
1	0.612	0.612	75.784	0.000
2	0.275	-0.161	91.114	0.000
3	0.215	0.194	100.54	0.000
4	0.182	-0.024	107.35	0.000
5	0.149	0.061	111.95	0.000
6	0.097	-0.043	113.89	0.000
7	0.022	-0.047	113.99	0.000
8	0.066	0.125	114.91	0.000
9	0.113	0.007	117.59	0.000
10	0.066	-0.030	118.51	0.000
11	0.010	-0.029	118.54	0.000
12	0.059	0.107	119.29	0.000
13	0.130	0.056	122.90	0.000
14	0.165	0.066	128.82	0.000
15	0.150	0.017	133.68	0.000
16	0.119	0.018	136.76	0.000
17	0.075	-0.056	138.00	0.000
18	0.039	-0.032	138.32	0.000
19	-0.000	-0.030	138.32	0.000
20	0.019	0.071	138.41	0.000

Correlogram of SUM4

Date: 01/30/08 Time: 17:32
Sample: 1 200
Included observations: 199

Lag	AC	PAC	Q-Stat	Prob
1	0.682	0.682	93.851	0.000
2	0.412	-0.099	128.27	0.000
3	0.353	0.210	153.63	0.000
4	0.289	-0.039	170.80	0.000
5	0.251	0.094	183.79	0.000
6	0.235	0.022	195.23	0.000
7	0.175	-0.040	201.64	0.000
8	0.093	-0.063	203.45	0.000
9	0.001	-0.109	203.45	0.000
10	-0.064	-0.054	204.32	0.000
11	-0.006	0.136	204.33	0.000
12	0.060	0.046	205.10	0.000
13	0.030	-0.037	205.30	0.000
14	0.011	0.026	205.32	0.000
15	-0.025	-0.074	205.46	0.000
16	-0.086	-0.061	207.09	0.000
17	-0.135	-0.104	211.09	0.000
18	-0.160	-0.071	216.72	0.000
19	-0.148	-0.004	221.60	0.000
20	-0.143	-0.025	226.19	0.000