Optical trimer, A theoretical physics approach to waveguide couplers

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We study electromagnetic field propagation through an ideal, passive, triangular three-waveguide coupler using a symmetry based approach to take advantage of the underlying $SU(3)$ symmetry. The planar version of this platform has proven valuable in photonic circuit design providing optical sampling, filtering, modulating, multiplexing, and switching. We show that a group-theory approach can readily provide a starting point for design optimization of these devices. We also try to present our analysis as a practical tutorial on the use of group theory to study photonic lattices for those not familiar with abstract algebra methods.

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1. INTRODUCTION

The planar three-waveguide coupler [1] has proven a reliable platform for optical devices. It has been shown to provide tunable sampling, filtering [2], modulation [3] and power coupling [4] in voltage driven systems, as well as power dividers and combiners in passive devices [5–8] that have allowed efficient signal referencing for integrated optical biosensors [9].

In most of the reported literature, optimization seems the standard approach favored by the optics community to design waveguide couplers [10, 11] but, recently, analogies with quantum mechanical systems have provided an alternative complementary approach [12, 13]. This has also impacted the design of planar three-waveguide couplers that, for example, have provided fast, robust directional beam coupling designed either by standard optimization [14–16] or by quantum analogies [17–20].

Here, our aim is to motivate photonic designers to go beyond analogies between photonic lattices and quantum systems. We will try our best to bridge the gap between theoretical physics and optics to show how the underlying symmetries of a photonic lattice can shed light into the design process. For this, we will use a general version of the three-waveguide coupler. In the next section, we will introduce the mode-coupling model and expose its underlying $SU(3)$ symmetry. Then, we will show how to construct a propagator for any given physical configuration using a Gilmore-Perelomov coherent state approach [21]. In order to provide practical examples, we will focus on arrays of identical waveguides with three identical couplings, which are related to the discrete Fourier transform, an two identical couplings, which are related to the golden ratio and allow devices with a single stable output. Finally, we will present a summary with possible extensions allowed by linear and nonlinear three-waveguide couplers.

2. THREE-WAVEGUIDE COUPLER

Light propagating through an ideal, general three-waveguide coupler can be described by coupled mode theory, c.f. [22] and references therein,

$$-i\partial_z \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix} = \begin{pmatrix} \omega_0(z) & g_{01}(z) & g_{02}(z) \\ g_{01}(z) & \omega_1(z) & g_{12}(z) \\ g_{02}(z) & g_{12}(z) & \omega_2(z) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix}. \quad \textbf{(1)}$$

Here, the complex field amplitude at the jth waveguide is given by $\mathcal{E}_j(z)$, the effective refractive index at the jth waveguide is $\omega_j(z)$, and the effective coupling between the jth and kth waveguides is $g_{jk}(z)$. These complex field equations can be cast in a Schrödinger-like form [22],

$$-i\partial_z|\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle, \quad \textbf{(2)}$$

where kets and operators in Dirac notation represent column vectors and square matrices, in that order. We can normalize the intensity, $\sum_j |\mathcal{E}_j(z)|^2 = 1$, as we are dealing with an ideal lossless device. Experimental realization of this model include, but are not limited, to laser inscribed photonic waveguides [23] and multicore optical fibers [], Fig. **??**(a), whispering-mode cavities [], Fig. **??**(b), or microwave resonators [24] and, outside the field of optics, standard RLC-circuits [].

The formal solution,

$$|\mathcal{E}(z)\rangle = \hat{U}(z)|\mathcal{E}(0)\rangle, \quad (3)$$

to this ordinary matrix linear differential equation is provided by an ordered exponential [25, 26],

$$\hat{U}(z) = \text{Texp}\left[\int_0^z \hat{H}(x)dx\right]. \quad (4)$$

Usually, it is not straightforward to calculate this propagator, but underlying symmetries simplify this endeavor [27? , 28]. While group theory is extensively used in mathematical optics [29, 30], it may be possible that the standard Lie algebra approach may look more complicated than it actually is for those outside that field. We hope that the following can help vanquish that feeling.

3. GROUP THEORY APPROACH

Group theory, as an instrument to explore the underlying structure of mathematical models describing the physical world, brings a layer of abstraction into physics that allows deeper insight. As such, it has become an essential tool in quantum mechanics. Coupled mode theory delivers a Schrödinger-like form describing light propagating through arrays of coupled waveguides, thus, the use of group theory to calculate propagation in these systems seems like a natural step.

The mode-coupling matrix $\hat{H}$ for our triangular three-waveguide coupler is a unitary matrix of rank three with trace equal to $\sum_{j=0}^{3}\omega_j(z)$. It is useful to decomposed it into a unit matrix part and a traceless part,

$$\hat{H}(z) = \frac{1}{3}\sum_{j=0}^{3}\omega_j(z)\hat{\mathbb{1}} + \hat{\mathcal{H}}(z). \quad (5)$$

The traceless part can be written in terms of the special unitary group $SU(3)$ which is a household name in physics often related to the work of Gell-Mann [31] and Ne'eman [?],

$$\begin{aligned}\hat{\mathcal{H}}(z) =& \frac{1}{2}\omega_y(z)\hat{Y} + \omega_i(z)\hat{I}_0 + g_{01}(z)\left(\hat{I}_+ + \hat{I}_-\right) \\ &+ g_{12}(z)\left(\hat{U}_+ + \hat{U}_-\right) + g_{02}(z)\left(\hat{V}_+ + \hat{V}_-\right). \quad (6)\end{aligned}$$

Here, we have defined the auxiliary effective refractive index $\omega_y(z) = [\omega_0(z) + \omega_1(z) - 2\omega_2(z)]/2$, $\omega_i(z) = \omega_0(z) - \omega_1(z)$, and used the following representation for the $SU(3)$ group [32],

$$\hat{Y} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \hat{I}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{I}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{I}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{U}_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{V}_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{V}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (7)$$

due to the fact that we can understand matrices $\hat{I}_\pm$, $\hat{U}_\pm$ and $\hat{V}_\pm$ as those describing the coupling of the electromagnetic field between waveguides zero and one, one and two, and zero and two, in that order.

At this point, our original Schrödinger-like equation is written in terms of the identity matrix and a linear combination of Lie group generators for $SU(3)$. The identity part only induces and overall phase,

$$e^{i\phi(z)\hat{\mathbb{1}}} = e^{\frac{i}{3}\int_0^z(\omega_0(\zeta)+\omega_1(\zeta)+\omega_2(\zeta))d\zeta}\,\hat{\mathbb{1}}. \quad (8)$$

such that,

$$\hat{U}(z) = e^{\frac{i}{3}\int_0^z(\omega_0(\zeta)+\omega_1(\zeta)+\omega_2(\zeta))d\zeta}\,\hat{\mathcal{U}}(z). \quad (9)$$

Now, for the $SU(3)$ part $\hat{\mathcal{U}}(z)$, Wei and Norman demonstrated that any such equation can be treated by an algebraic method providing the following propagator [?],

$$\hat{\mathcal{U}}(z) = \prod_{j=1}^{8} e^{i\theta_j(z)\hat{X}_j}, \quad (10)$$

where the $su(3)$ algebra elements, $e^{i\theta_j(z)\hat{X}_j}$, are just the exponential map of the group generators, $\{\hat{Y}, \hat{I}_0, \hat{I}_\pm, \hat{U}_\pm, \hat{V}_\pm\}$, and the functions $\theta_j(z)$ are complex functions ruled by the dynamics provided by the mode-coupling matrix, $\hat{H}$. Note, there is no apriori ordering of $su(3)$ elements to write the propagator. However, the values of the $\theta_j(z)$ functions do depend on the chosen order Different orderings have been studied in the quantum optics literature [34? ? ?]. We will choose a particular ordering,

$$\begin{aligned}\hat{\mathcal{U}}(z) =& e^{i\iota_+(z)\hat{I}_+}e^{i\mu_+(z)\hat{U}_+}e^{i\nu_+(z)\hat{V}_+}e^{\iota_0(z)\hat{I}_0} \\ &\times e^{iy_0(z)\hat{Y}}e^{i\nu_-(z)\hat{V}_-}e^{i\mu_-(z)\hat{U}_-}e^{i\iota_-(z)\hat{I}_-}, \quad (11)\end{aligned}$$

that keeps us in line with the idea of understanding propagation through waveguide lattices as generalized Gilmore-Perelomov coherent states [21].

The next step is straightforward but cumbersome, we substitute the formal solution $|\mathcal{E}(z)\rangle$, using the propagator above, being careful in keeping the ordering through the derivation process. Then, we use the actions of elements of the $su(3)$ algebra on elements of the $SU(3)$ group [33] to find the differential equation set for the auxiliary functions,

$$\begin{aligned}
\iota_+' &= g_{01}\iota_+^2 + (g_{02}\nu_+ + i\omega_i)\,\iota_+ - ig_{12}\nu_+ + g_{01}, & (12)\\
\mu_+' &= (g_{12} + ig_{02}\iota_+)\,\mu_+^2 + \left[g_{02}\nu_+ - g_{01}\iota_+ + \frac{i}{2}\left(2\omega_y - \omega_i\right)\right]\mu_+ & \\
&\quad + ig_{01}\nu_+ + g_{12}, & (13)\\
\nu_+' &= g_{02}\nu_+^2 + \left[g_{01}\iota_+ + \frac{i}{2}\left(2\omega_y + \omega_i\right)\right]\nu_+ - ig_{12}\iota_+ + g_{02}, & (14)\\
\iota_0' &= \omega_i - i2g_{01}\iota_+ + ig_{12}\mu_+ - g_{02}\left(\iota_+\mu_+ + i\nu_+\right), & (15)\\
y_0' &= \omega_y - i\frac{3}{2}[g_{02}\nu_+ + (g_{12} + ig_{02}\iota_+)\mu_+], & (16)\\
\nu_-' &= g_{02}e^{i\left(y_0 + \frac{1}{2}\iota_0\right)} - e^{i\iota_0}\left(g_{02}\mu_+ + ig_{01}\right)\mu_-, & (17)\\
\mu_-' &= e^{i\left(y_0 - \frac{1}{2}\iota_0\right)}(g_{12} + ig_{02}\iota_+), & (18)\\
\iota_-' &= e^{i\iota_0}\left(g_{01} - ig_{02}\mu_+\right), & (19)
\end{aligned}$$

where, for the sake of space, we have used $f \equiv f(z)$ and $f' \equiv \partial_z f(z)$ for all propagation dependent auxiliary functions and couplings.

Non-linear differential equations are known to be hard to solve and finding a solution often requires intuition and knowledge of the system being analyzed. Before delving into details, we would like to point out a key feature of passive, lossless optical models, their mode-coupling matrices are real symmetric, $\hat{H}^T(z) = \hat{H}(z)$ where the operation O^T stands for transposition, and, as a direct consequence, the propagator shares the same property,

$$\hat{U}^T(z) = \hat{U}(z). \quad \textbf{(20)}$$

This feature allows us to conclude that the propagator functions are symmetric,

$$\xi_+(z) = \xi_-(z), \quad \xi = \iota, \nu, \mu. \quad \textbf{(21)}$$

Furthermore, we observe that two equations, namely Eq.(12) and Eq.(14) only include terms of $\iota_+(z)$ and $\nu_+(z)$ and their derivatives. Therefore, they are decoupled from the rest. Nonetheless, these two equations prove intractable and we will pursue a different route to finding a solution.

Note, that the propagator can be written as a matrix,

$$\hat{\mathcal{U}}(z) = \begin{pmatrix} \Xi(z) & \Sigma(z) & \Theta(z) \\ \Sigma(z) & \Pi(z) & \Delta(z) \\ \Theta(z) & \Delta(z) & \Gamma(z) \end{pmatrix}. \quad \textbf{(22)}$$

For reasons that will become apparent in a moment, we introduce a set of five auxiliary functions,

$$\begin{aligned}
\Gamma(z) &= e^{-i\frac{2}{3}y_0(z)}, & \textbf{(23)}\\
\Delta(z) &= i\Gamma(z)\mu_+(z), & \textbf{(24)}\\
\Theta(z) &= \Gamma(z)\left[-\iota_+(z)\mu_+(z) + i\nu_+(z)\right], & \textbf{(25)}\\
\Pi(z) &= \Gamma(z)\left[e^{iy_0(z)}e^{-i\frac{1}{2}\iota_0(z)} - \mu_+(z)^2\right], & \textbf{(26)}\\
\Sigma(z) &= i\iota_+(z)\Pi(z) - \Gamma(z)\mu_+(z)\nu_+(z)), & \textbf{(27)}
\end{aligned}$$

and the sixth can be written in terms of all others,

$$\Xi(z) = \frac{1 + \Pi(z)\Theta^2(z) + \Gamma(z)\Sigma^2(z) - 2\Delta(z)\Theta(z)\Sigma(z)}{\Pi(z)\Gamma(z) - \Delta^2(z)}. \quad \textbf{(28)}$$

The original functions can be put in terms of these auxiliary,

$$\begin{aligned}
\iota_+(z) &= i\frac{\Gamma(z)\Sigma(z) - \Delta(z)\Theta(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, & \textbf{(29)}\\
\mu_+(z) &= -i\frac{\Delta(z)}{\Gamma(z)}, & \textbf{(30)}\\
\nu_+(z) &= i\frac{\Pi(z)\Theta(z) - \Delta(z)\Sigma(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, & \textbf{(31)}\\
\iota_0(z) &= i2\log\frac{\Gamma(z)\Pi(z) - \Delta^2(z)}{\Gamma^{\frac{1}{2}}(z)}, & \textbf{(32)}\\
y_0(z) &= i\frac{3}{2}\log\Gamma(z). & \textbf{(33)}
\end{aligned}$$

Note that the phase functions $y_0(z)$ and $i_0(z)$ are of logarithmic nature and the rest are quotients of the products of the solution basis. We now have the simplest matrix differential equation for the propagator [? ?],

$$\partial_z\hat{\mathcal{U}}(z) = i\mathcal{H}\hat{\mathcal{U}}(z), \quad \textbf{(34)}$$

with the initial conditions,

$$\Theta(z) = 0, \quad \Delta(z) = 0, \quad \Sigma(z) = 0, \quad \Gamma(z) = 1, \quad \Pi(z) = 1. \quad \textbf{(35)}$$

This differential equation system is overdetermined due to the characteristics of the original matrix and provides the following identities,

$$\begin{aligned}
g_{01}(\Pi - \Xi) + g_{02}\Delta - g_{12}\Theta + \omega_i\Sigma &= 0, & \textbf{(36)}\\
g_{01}\Delta + g_{02}(\Gamma - \Xi) - g_{12}\Sigma + \left(\omega_y + \frac{1}{2}\omega_i\right)\Theta &= 0, & \textbf{(37)}\\
g_{01}\Theta - g_{02}\Sigma(z) + g_{12}(\Gamma - \Pi) + \left(\omega_y - \frac{1}{2}\omega_i\right)\Delta &= 0. & \textbf{(38)}
\end{aligned}$$

A. Optical trimer

While we have provided a formal solution to propagation through the optical trimer, considering a specific solution may help build further intuition. For the sake of simplicity, let us now consider the triangular three-waveguide array with constant couplings and identical waveguides and call it an optical trimer. We will introduce the dimensionless propagation parameter $\zeta = g_{01}z$, such that the mode-coupling differential equation becomes

$$-i\partial_\zeta|\mathcal{E}(\zeta)\rangle = \hat{H}|\mathcal{E}(\zeta)\rangle, \quad \textbf{(39)}$$

with the mode-coupling matrix,

$$\hat{H} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}, \quad \textbf{(40)}$$

given in terms of the dimensionless parameters,

$$\alpha = \frac{g_{02}}{g_{01}}, \quad \beta = \frac{g_{12}}{g_{01}}. \quad \textbf{(41)}$$

Now, we can use the results above to build a particular solution, but it is well known that a set of linear first order differential equations is equivalent to a single linear differential equation of higher order. After some algebra, we can derive a higher order differential equation for $\Delta(\zeta)$,

$$\Delta'''(\zeta) + i(1 + \alpha^2 + \beta^2)\Delta'(\zeta) - 2\alpha\beta\Delta(\zeta) = 0, \quad \textbf{(42)}$$

with boundary conditions,

$$\Delta(0) = 0, \quad \Delta'(0) = i\beta, \quad \Delta''(0) = -\alpha. \quad \textbf{(43)}$$

Note that the remaining auxiliary functions are straightforward to calculate,

$$\begin{aligned}
\Theta(\zeta) &= \frac{\beta(1+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha(1-\beta^2)} & \textbf{(44)}\\
\Gamma(\zeta) &= \frac{-(1+\beta^2)\Delta(\zeta) + i\alpha\beta\Delta'(\zeta) - \Delta''(\zeta)}{\alpha(1-\beta^2)} & \textbf{(45)}\\
\Sigma(\zeta) &= \frac{\beta(\alpha^2+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha^2 - \beta^2} & \textbf{(46)}\\
\Pi(\zeta) &= \frac{-\alpha(\alpha^2+\beta^2)\Delta(\zeta) + i\beta\Delta'(\zeta) - \alpha\Delta''(\zeta)}{\alpha^2 - \beta^2}. & \textbf{(47)}
\end{aligned}$$

It is simple to see that $\Delta(\zeta)$ has the following solution,

$$\Delta(\zeta) = \delta_1\, e^{i\gamma_1\zeta} + \delta_2\, e^{i\gamma_2\zeta} + \delta_3\, e^{i\gamma_3\zeta}, \quad \textbf{(48)}$$

where constant parameters γ_j are the eigenvalues of the mode-coupling matrix determined by the characteristic polynomial, a reduced cubic,

$$\gamma_j^3 - (1 + \alpha^2 + \beta^2)\gamma_j - 2\alpha\beta = 0. \quad \textbf{(49)}$$

It is straightforward to notice that there are three different real eigenvalues for real, positive, non-zero coupling parameters, $\alpha, \beta > 0$. These proper values can be writen in a closed but non-compact form, so we will not write them explicitly. Furthermore, the coefficients are given by

$$\delta_1 = \frac{\alpha - \beta(\gamma_2 + \gamma_3)}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad (50)$$

$$\delta_2 = \frac{\alpha - \beta(\gamma_1 + \gamma_3)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}, \quad (51)$$

$$\delta_3 = \frac{\alpha - \beta(\gamma_1 + \gamma_2)}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}. \quad (52)$$

Thus, the propagator functions, $\iota_\pm(z)$, $\mu_\pm(z)$, $\nu_\pm(z)$, $\iota_0(z)$ and $y_0(z)$, will effectively contain terms involving the three eigenvalues as well as sums and differences thereof.

4. APPLICATIONS

As we just saw, the optical trimer is simple enough to allow us the construction of a closed form solution and, to our advantage, it is experimentally feasible to realize it. Now the obvious question is if there is a use for it. In the following, we will show that a judicious choice of coupling parameters provides different types of well-defined trajectories that can be used for the design of integrated photonic circuits.

A. Identical couplings and the discrete Fourier transform

The mode-coupling matrix for three-identical waveguides distributed in an equilateral triangle configuration, $\alpha = \beta = 1$,

$$\hat{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (53)$$

is related to the cyclic group in dimension three,

$$\hat{H} = \hat{Z}_3 + \hat{Z}_3^2, \quad (54)$$

where the generator of the cyclic group are the following,

$$\hat{Z}_3 = \hat{I}_+ + \hat{U}_+ + \hat{V}_-, = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (55)$$

It is well known that the cyclic group is diagonalized by the discrete Fourier transform,

$$\hat{\Lambda} = \hat{F}_n \hat{Z}_n \hat{F}_n^\dagger, \quad (56)$$

where the discrete Fourier transform of rank n is given by the operator $\hat{F}_n$, in the case of $n = 3$,

$$\hat{F}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}, \quad (57)$$

and $\hat{\Lambda}$ is a diagonal rank n matrix with the roots of the unit,

$$\hat{\Lambda}_{mn} = \delta_{m,n} e^{i\frac{2\pi}{n} i}, \quad m, n = 0, 1, 2, \quad (58)$$

on the diagonal. In this particular case, it is possible to compose a propagator,

$$U(\zeta) = \hat{F}_3^\dagger e^{i\hat{\Lambda}_3 \zeta} e^{i\hat{\Lambda}_3^2 \zeta} \hat{F}_3, = \frac{1}{3} \begin{pmatrix} 2 + e^{3i\zeta} & -1 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & 2 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & -1 + e^{3i\zeta} & 2 + e^{3i\zeta} \end{pmatrix} e^{-i\zeta} \quad (59)$$

where we have used the fact that the elements of the cyclic group of rank 3 commute between them, $\left[\hat{Z}_3, \hat{Z}_3^2\right] = 0$ because $\hat{Z}_3^3 = \mathbb{1}_3$.

Figure 1 shows the trajectories described by the absolute value of the field amplitudes, $|\mathcal{E}_j(z)|$, as they propagate. All of trajectories will lie over the surface of an octant of the sphere due to unitary propagation. Figure 1(a) shows the response to impulses, $\mathcal{E}_j = \delta_{j,k}$ with $j = 0, 1, 2$ and a fixed $k = 0, 1, 2$. Figure 1(b) shows the trajectories given by initial field superpositions of the more general form: $\mathcal{E}_j = \alpha_j e^{i\phi t}$ with $\alpha \in \mathbb{R}$ and $\sum_j |\alpha_j|^2 = 1$. From the propagator, it is possible to see that only two commensurate frequencies are involved in the propagation of initial fields, thus, the trajectories will be closed and well defined.

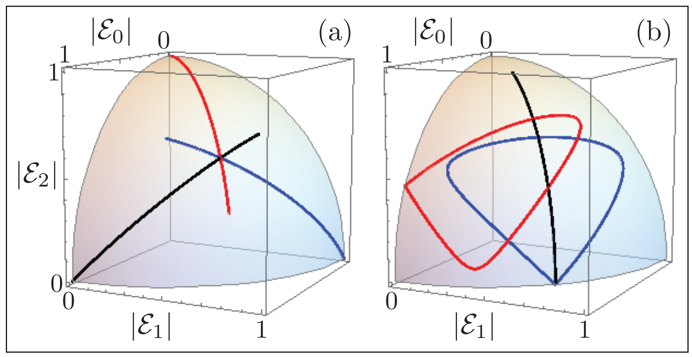

Fig. 1. (Color online) Absolute amplitude trajectories, $(|\mathcal{E}_0(z)|, |\mathcal{E}_1(z)|, |\mathcal{E}_2(z)|)$, for initial fields, $(|\mathcal{E}_0(0)|, |\mathcal{E}_1(0)|, |\mathcal{E}_2(0)|)$, impinging (a) only the zeroth (solid black), $(1, 0, 0)$, first (dashed blue), $(0, 1, 0)$, and second (dotted red), $(0, 0, 1)$, waveguides and (b) initial fields impinging two waveguides at a time with and without a relative phase, $(1, 1, 0)/\sqrt{2}$ (solid black), $(1, i, 0)/\sqrt{2}$ (dashed blue), and $(2, 0, i)/\sqrt{5}$ (dotted red).

B. Two identical couplings and the golden ratio

In the case of two equal coupling parameters the unitless Hamiltonian becomes

$$H = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix} \quad (60)$$

Note that the Hamiltonian is $\hat{Z}_2$-invariante, i.e. it is invariant under exchanging the first and the second waveguide. This symmetry also is reflected by the eigenvalues which are $\{-1, \bar{\varphi}, \varphi\}$ with

$$\bar{\varphi} = \frac{1}{2}\left(1 - \sqrt{8\alpha^2 + 1}\right), \quad (61)$$

$$\varphi = \frac{1}{2}\left(1 + \sqrt{8\alpha^2 + 1}\right). \quad (62)$$

It is simple to see that $\bar{\varphi} + \varphi = 1$ and that the latter eigenvalue becomes the golden ratio for $\alpha = \sqrt{1/2}$. Due to the $\hat{Z}_2$-symmetry the propagator can be calculated directly using the group theory approach,

$$\begin{aligned} \Delta(\zeta) &= \Theta(z), && (63)\\ &= \frac{\alpha}{\varphi - \bar{\varphi}} \left(e^{i\zeta\varphi} - e^{i\zeta\bar{\varphi}} \right), && (64)\\ \Gamma(\zeta) &= \frac{1}{\varphi - \bar{\varphi}} \left(\varphi e^{i\zeta\bar{\varphi}} - \bar{\varphi} e^{i\zeta\varphi} \right), && (65)\\ \Sigma(\zeta) &= \frac{1}{2\left(\varphi - \bar{\varphi}\right)} \left(\varphi e^{i\zeta\varphi} - \bar{\varphi} e^{i\zeta\bar{\varphi}} \right) - \frac{1}{2} e^{-i\zeta}, && (66)\\ \Pi(\zeta) &= \Xi(\zeta), && (67)\\ &= \frac{1}{2\left(\varphi - \bar{\varphi}\right)} \left(\varphi e^{i\zeta\varphi} - \bar{\varphi} e^{i\zeta\bar{\varphi}} \right) + \frac{1}{2} e^{-i\zeta}. && (68) \end{aligned}$$

Obviously the three eigenvalues are stationary points of the system. However, this particular symmetry might allow further interesting states. Note that the auxiliary functions $\Delta(\zeta)$ and $\Gamma(\zeta)$ depend on only two of the three eigenfrequencies, φ and $\bar{\varphi}$. Furthermore, a little bit of algebra considering a initial state without any field impinging the zeroth waveguide, $\mathcal{E}_0 = 0$, reveals that propagation in the second waveguide, $\mathcal{E}_2(z)$, only depends on those two auxiliary functions, $\Delta(\zeta)$ and $\Gamma(\zeta)$,

$$\mathcal{E}_2(\zeta) = \Delta(\zeta)\mathcal{E}_1(0) + \Gamma(\zeta)\mathcal{E}_2(0). \tag{69}$$

Now, we can hope that choosing a proper input will cancel out one of the two eigenvalues that appear in the evolution of the field amplitude at the second waveguide. That is, if the evolution only depends on a single eigenvalue, it will only change in phase and the field amplitude will remain immutable. Hence, lets look for such a state where no energy transfer into or out of the second waveguide occurs. This is fulfilled if either of the two following conditions is met,

$$\mathcal{E}_1(0) = \frac{\varphi}{\alpha}\mathcal{E}_2(0) \quad \text{or} \quad \mathcal{E}_1(0) = \frac{\bar{\varphi}}{\alpha}\mathcal{E}_2(0). \tag{70}$$

Consequently we obtain two solutions for a state with stationary amplitude in the second waveguide. Without loss of generality, we can give the propagated amplitude vector up to a normalization constant,

$$|\mathcal{E}_x(\zeta)\rangle = \begin{pmatrix} \frac{\varphi}{\alpha}\Sigma(\zeta) + \Delta(\zeta) \\ \frac{\varphi}{\alpha}\Pi(z) + \Delta(\zeta) \\ e^{i\zeta\varphi} \end{pmatrix} \quad x = \varphi, \bar{\varphi}. \tag{71}$$

5. CLASSICAL MECHANICS AND ENVELOPES

An lossless waveguide is an optical cavity, where something missing here ?!

In quantum mechanics, the Hamiltonian for three coupled harmonic oscillators is given by:

$$\hat{H}(t) = \sum_{j=0}^{2} \omega_j(t)\hat{a}_j^\dagger \hat{a}_j + \sum_{j\neq k=0}^{2} g_{jk}(t)\hat{a}_j^\dagger \hat{a}_k, \tag{72}$$

where, by analogy with equation (1), $\omega_j(t)$ are the time dependent free oscillator frequencies and $g_{ij}(t)$ refer to the time dependent coupling constants. The Hamiltonian above has been studied elsewhere and here we will only consider two limiting

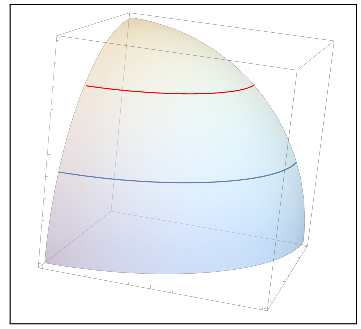

Fig. 2. (Color online) Absolute amplitude trajectories, $(|\mathcal{E}_0(z)|, |\mathcal{E}_1(z)|, |\mathcal{E}_2(z)|)$, for the states with stationary field amplitude in the second waveguide as obtained in Eq.(71).

cases. The classical limit is usually associated with the coherent state solution and involves an infinite number of photons. The opposite end of the spectrum is marked by the single photon limit. At any given time only a single photon occupies the system and consequently the ansatz is as follows,

$$|\psi(t)\rangle = \mathcal{E}_0(t)\,|1,0,0\rangle + \mathcal{E}_1(t)\,|0,1,0\rangle + \mathcal{E}_2(t)\,|0,0,1\rangle\,, \tag{73}$$

with $\sum_j |\mathcal{E}_j(z)|^2 = 1$. Applying the single photon state, equation (73), to the quantum Hamiltonian, equation (72), yields a differential equation set equivalent to the mode coupling theory result, Eq.(42) up to substitution of backwards time by propagation distance, or, better,

$$i\partial_t \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix} = \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix}. \tag{74}$$

The result above is a differential equation for the solution of the three coupled wave guides with respect to the complex amplitudes of each wave-guide. In certain circumstances and and for reason that will become apparent shortly it is feasible to derive equations of motions for other sets of variables.

We can use quadratures,

$$\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \; \hat{p} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \tag{75}$$

and consider classical variables, where we have accounted for the change in the time propagation,

$$i\partial_t \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix}, \tag{76}$$

$$i\partial_t \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix} = -\begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}, \tag{77}$$

This can also be done with the idea of Euler angles for $SU(3)$ in [33]

In the classical limit, the canonical pair provided by the creation and annihilation operators can be replaced by the classical canonical pair of intensity and phase, $\left\{n_j, \phi_j\right\}$, $\hat{a}_j \to \sqrt{n_j}e^{i\phi_j}$. This delivers a classical Hamiltonian,

$$H(t) = \sum_{j=0}^{2} \omega_j(t) n_j + \sum_{j\neq k=0}^{2} g_{jk}\sqrt{n_j n_k}\cos\left(\phi_j - \phi_k\right), \tag{78}$$

the equations of motion for the canonical pairs are

$$\partial_t n_j = \frac{\partial H}{\partial \phi_j} \tag{79}$$

$$\partial_t \phi_j = -\frac{\partial H}{\partial n_j} \tag{80}$$

and these are the coupled mode equations describing the evolution of intensity and phase.

Classicam mechanics: if the three normal modes frequencies, e.g. eigenvalues of the mode-coupling matrix in Eq.(9), are commesurate then the propagated complex fields will be periodic [35].

In order to visualize the dynamics of the system we will choose a (pseudo?) Poincaré phase-space given by the square roots of the three waveguide intensities $(|\mathcal{E}_0|, |\mathcal{E}_1|, |\mathcal{E}_2|)$. Here normal mode frequencies that are rational multiples of each other, commensurate, will translate into well-defined closed trajectories. For incommensurate normal frequencies the trajectories will be ergodic and fill a region of phase space defined by the energy of the motion [35].

Let us turn to the vast class of incommensurate trajectories. In many situations the exact route of the trajectory is not of primary importance and it will suffice to have gained some information on the region that will be traversed. The region of a given trajectory is characterised by the boundary which encompasses the very region. Therefore in the remainder of this section we will attend to finding the corresponding boundary which we also refer to as the envelope of the trajectory.

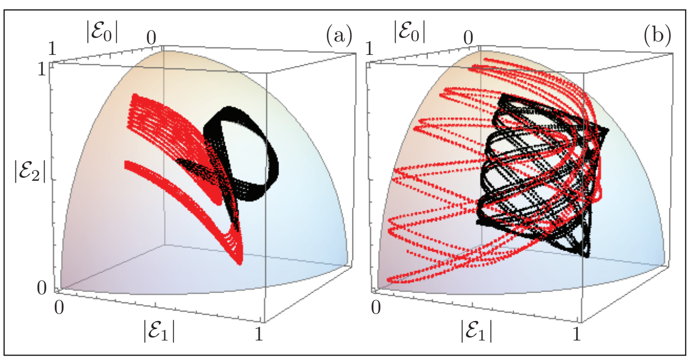

Fig. 3. (Color online) Absolute amplitude trajectories, $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, for initial random fields impinging random optical trimmers with constant parameters.

It proves viable to notice that for a given energy and a given set of parameters in most cases an infinity number of distinct trajectories prevails. The conjunction of all these trajectories pertaining to a certain energy span a path-connected surface. Firstly we will be concerned with finding the outer boundary, or envelope, of the union of all trajectories pertaining to Energy E. Equation (78) readily reveals that for given intensities n_1, n_2 and n_3 the energy is maximised when all three phases are zero. Consequently the envelope is defined by all triplets n_1, n_2 and n_3 which fulfil following condition

$$H_{Bndry}(n_1, n_2, n_3) := \sum_{j=0}^{2} \omega_j(t) n_j + \sum_{j\neq k=0}^{2} g_{jk}\sqrt{n_j n_k} = E, \tag{81}$$

Due to energy conservation, the three intensities are not independent and we can safely omit the third intensity in the following discussion, ie. $H_{Bndry}(n_1, n_2, n_3) \to H_{Bndry}(n_1, n_2)$. Numerically, once a given set of initial intensities and/or energy was chosen $H_{Bndry}(n_1, n_2) \to E$ the trajectory of the envelope is characterised by the following differential equation:

$$\frac{\partial H_{Bndry}}{\partial n_1}\delta n_1 + \frac{\partial H_{Bndry}}{\partial n_2}\delta n_2 = 0 \tag{82}$$

Having obtained the envelope of the union of all trajectories we can turn to the area of a single trajectory. We recall that we focus our interest on parameters such that the corresponding trajectories are incommensurate, i.e. they fill all the area inside a closed loop. To find the envelope of a certain trajectory, that is the aforementioned loop, we assume we already have obtained the explicit solution for the trajectory. Our solution is most easily expressed as solution to equation (1) which is given by:

$$\mathcal{E}(t) = a_1 e_1 e^{i\omega_1 t} + a_2 e_2 e^{i\omega_2 t} + a_3 e_3 e^{i\omega_3 t} \tag{83}$$

where e_1, e_2 and e_3 are the eigenvectors with corresponding eigenfrequencies ω_1, ω_2 and ω_3. Parameters $a1, a2$ and $a3$ are chosen in accordance with the initial values n_1 and n_2. Note that $n_j = \mathcal{E}_j \mathcal{E}^*{}_j$ The following is a bit handwaving, we probably need to explain it better Incommensurate trajectories are characterised by broken periodicity and we can introduce a phase relevant parameter which controls/mimics the incommensurability:

$$\mathcal{E}(t,\phi) = a_1 e_1 e^{i\omega_1 t + \phi} + a_2 e_2 e^{i\omega_2 t} + a_3 e_3 e^{i\omega_3 t} \tag{84}$$

The parameter ϕ acts such that a given trajectory or more precisely a section of the trajectory is offset in a direction perpendicular to the trajectory. This however can not hold true at the boundary or envelope and hence we establish following condition for the envelope:

$$\frac{\left(\frac{\partial n_1}{\partial t}\right)}{\left(\frac{\partial n_2}{\partial t}\right)} = \frac{\left(\frac{\partial n_1}{\partial \phi}\right)}{\left(\frac{\partial n_2}{\partial \phi}\right)} \tag{85}$$

which is equal to:

$$D(t,\phi) := Det[\begin{pmatrix} \frac{\partial n_1}{\partial t} & \frac{\partial n_2}{\partial t} \\ \frac{\partial n_1}{\partial \phi} & \frac{\partial n_2}{\partial \phi} \end{pmatrix}] = 0 \tag{86}$$

Accordingly, the envelope can be calculated numerically from following differential equation:

$$\frac{\partial D(t,\phi)}{\partial t}\delta t + \frac{\partial D(t,\phi)}{\partial \phi}\delta\phi = 0 \tag{87}$$

6. CONCLUSION

We have shown that it is possible to solve the light evolution equations in a coupled three-core waveguide, in terms of the Lie group generators of su(3). We focused our attention on a reduced class of structures where the coupling constants are constant in position......... As an example we consider the case of equal coupling constants. we found that the dynamics of such a

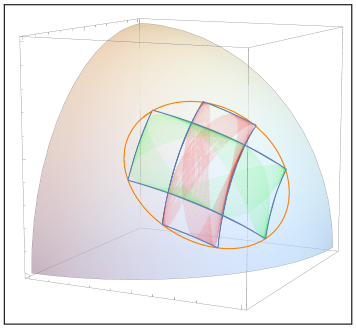

Fig. 4. (Color online) Absolute amplitude trajectories, $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, for two distinct trajectories pertaining to the energy E. The boundary of all possible trajectories corresponding to energy E. The envelopes of the indivudual trajectories are are calculated according to equation ().

system i governed by a linear third order differential equation and the corresponding five auxiliary functions can be expressed in term of its solution. we also established a connection between a waveguide cluster with equal couplings and the well known Fourier transform, possibly opening a path to realise quantum Fourier transformations. furthermore we studies an isosceles waveguide triangle. We showed that two equal couplings corresponds to a Z2 symmetry of the Hamiltonian. We also showed that the symmetry allows for two interesting stated characterised by absence of energy transfer into the third core.

REFERENCES

1. K. Iwasaki, S. Kurazono, and K. Itakura, "The coupling of modes in three dielectric slab waveguides," Electron. Comm. Jpn. **58**, 100 – 108 (1975).
2. H. H. Haus and J. C. G. Fonstad, "Three-waveguide couplers for improved sampling and filtering," IEEE J. Quantum Elect. **12**, 2321 – 2325 (1981).
3. J. P. Donelly, N. L. Demeo, G. A. Ferrante, and K. B. Nichols, "A high-frequency gaas optical guided-wave electrooptic interferometric modulator," IEEE J. Quantum Elect. **21**, 18 – 21 (1985).
4. W. Charczenko and A. R. Mickelson, "Symmetric and asymmetric pertubations of the index of refraction in three-waveguide optical planar couplers," J. Opt. Soc. Am. A **6**, 202 – 212 (1989).
5. J. P. Donelly, J. N. L. DeMeo, and G. A. Ferrante, "Three-guide optical couplers in gaas," J. Lightwave Technol. **1**, 417 – 424 (1983).
6. J. P. Donelly, "Limitations on power-transfer efficiency in three-guide optical couplers," IEEE J. Quantum Elect. **22**, 610 – 616 (1986).
7. J. P. Donelly, H. A. Haus, and N. Whitaker, "Symmetric three-guide optical coupler with nonidentical center and outside guides," IEEE J. Quantum Elect. **23**, 401 – 406 (1987).
8. H. Kubo and K. Yasumoto, "Numerical analysis of three-parallel embedded optical waveguides," J. Lightwave Technol. **7**, 1924 – 1930 (1989).
9. B. J. Luff, J. S. Wilkinson, J. Piehler, U. Hollenback, J. Ingenhoff, and N. Fabricius, "Integrated optical mach-zehnder biosensor," J. Lightwave Technol. **16**, 583 – 592 (1998).
10. C.-C. Su and S.-E. Shih, "Optimization of power transfer in multiple-waveguide couplers," IEEE J. Quantum Elect. **25**, 1666 – 1670 (1989).
11. J. Petrovic, "Multiport waveguide couplers with periodic energy exchange," Opt. Lett. **40**, 139 – 142 (2015).
12. A. Perez-Leija, R. Keil, A. Kay, H. Moya-Cessa, S. Nolte, L.-C. Kwek, B. Rodríguez-Lara, A. Szameit, and D. Christodoulides, "Coherent quantum transport in photonic lattices," Phys. Rev. A **87**, 012309 (2013).
13. A. Perez-Leija, R. Keil, H. Moya-Cessa, A. Szameit, and D. N. Christodoulides, "Perfect transfer of path-entangled photons in j_x photonic lattices," Phys. Rev. A **87**, 022303 (2013).
14. W.-C. Ng, M. S. Stern, and S.-J. Chua, "The design of triple rib waveguide couplers by the discrete spectral index method," J. Lightwave Technol. **17**, 475 – 482 (1999).
15. V. M. Schneider and H. T. Hattori, "Wavelength insensitive asymmetric triple mode evolution couplers," Opt. Commun. **187**, 129 – 133 (2001).
16. E. Narevicius, R. Narevich, Y. Berlatzky, I. Shtrichman, G. Rosenblum, and I. Vorobeichik, "Adiabatic mode multiplexer for evanescentcoupling-insensitive optical switching," Opt. Lett. **30**, 3362 – 3364 (2005).
17. E. Paspalakis, "Adiabatic three-waveguide directional coupler," Opt. Commun. **258**, 30 – 34 (2006).
18. A. Salandrino, K. Makris, D. N. Christodoulides, Y. Lahini, Y. Silberberg, and R. Morandotti, "Analysis of a three-core adiabatic directional coupler," Opt. Commun. **282**, 4524 – 4526 (2009).
19. S.-Y. Tseng and Y.-W. Jhang, "Fast and robust beam coupling in a three waveguide directional coupler," IEEE Photonics Technology Letters **25**, 2478–2481 (2013).
20. B. M. Rodríguez-Lara, H. M. Moya-Cessa, and D. N. Christodoulides, "Propagation and perfect transmission in three-waveguide axially varying couplers," Phys. Rev. A **89**, 013802 (2014).
21. L. V. Vergara and B. M. Rodríguez-Lara, "Gilmore-perelomov symmetry based approach to photonic lattices," Opt. Express **23**, 22836–22846 (2015).
22. B. M. Rodríguez-Lara, F. Soto-Eguibar, and D. N. Christodoulides, "Quantum optics as a tool for photonic lattice design," Phys. Scr. **90**, 068014 (2015).
23. A. Szameit and S. Nolte, "Discrete optics in femtosecond-laser-written photonic structures," J. Phys. B: At. Mol. Opt. Phys. **43**, 163001 (2010).
24. J. A. Franco-Villafane, E. Sadurní, S. Barkhofen, U. Kuhl, F. Mortessagne, and T. H. Selligman, "First experimental realization of the dirac oscillator," Phys. Rev. Lett. **111**, 170405 (2013).
25. W. Magnus, "On the exponential solution of differential equations with a linear operator," Comm. Pure Appl. Math. **7**, 649 – 673 (1954).
26. S. Blanes, F. Casas, J. A. Oteo, and J. Ros, "The magnus expansion and some of its applications," Physics Reports **470**, 151 – 238 (2009).
27. S. Lie, "Theorie der transformationsgruppen i," Math. Ann. **16**, 441 – 528 (1880).
28. A. Neumaier and D. Westra, "Classical and quantum mechanics via lie algebras," ArXiv: 0810.1019.
29. K. B. Wolf, *Geometric optics on phase space* (Springer, Germany, 2004).
30. V. Lakshminarayanan, M. L. Calvo, and T. Alieva, eds., *Mathematical Optics* (CRC Press, Boca Raton, 2012).
31. M. Gell-Mann, "The eightfold way: A theory of strong interaction symmetry," Tech. Rep. CTSL20, California Institute of Technology (1961).
32. R. Ticciati, *Quantum field theory for mathematicians* (Cambridge University Press, Cambridge, U. K., 1999).
33. T. J. Nelson, "A set of harmonic functions for the group su(3) as specialized matrix elements of a general finite transformation," J. Math. Phys. **8**, 857 – 863 (1967).
34. G. Dattoli, P. D. Lazzaro, and A. Torre, "Su(1,1), su(2) and su(3) coherence-preserving hamiltonians and time-ordering techniques," Phys. Rev. A **35**, 1582 – 1589 (1987).
35. H. Goldstein, *Classical mechanics* (Addison-Wesley, U.S.A., 1980).