

Optical trimer, A theoretical physics approach to waveguide couplers

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We study electromagnetic field propagation through an ideal, passive, triangular three-waveguide coupler using a symmetry based approach to take advantage of the underlying $SU(3)$ symmetry. The planar version of this platform has proven valuable in photonic circuit design providing optical sampling, filtering, modulating, multiplexing, and switching. We show that a group-theory approach can readily provide a starting point for design optimization of the triangular version. Our analysis is presented as a practical tutorial on the use of group theory to study photonic lattices for those not familiar with abstract algebra methods. In particular, we study the equilateral trimer to show the relation of pearl-necklace arrays with the Discrete Fourier Transform due to their cyclic group symmetry, and the isosceles trimer to show its relation with the golden ratio and its ability to provide stable output at a single waveguide. For the sake of completeness, we go beyond the standard optical-quantum analogy to show that it is possible to derive coupled-mode equations for intensity-phase, and that it is also possible to calculate envelopes for the propagation of all possible inputs belonging to an energy class, as well as individual input field amplitudes, without the need of point-to-point propagation. © 2016 Optical Society of America

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1. INTRODUCTION

The planar three-waveguide coupler [1] has proven a reliable platform for optical devices. It has been shown to provide tunable sampling, filtering [2], modulation [3] and power coupling [4] in voltage driven systems, as well as power dividers and combiners in passive devices [5–8] that have allowed efficient signal referencing for integrated optical biosensors [9].

In most of the reported literature, optimization seems the standard approach favored by the optics community to design waveguide couplers [10, 11] but, recently, analogies with quantum mechanical systems have provided a complementary approach [12, 13]. This has also impacted the design of planar three-waveguide couplers that, for example, have provided fast, robust directional beam coupling designed either by standard optimization [14–16] or by quantum analogies [17–20].

Here, our aim is to motivate photonic designers to go beyond the standard analogies between photonic lattices and quantum systems. We will try our best to bridge the gap between theoretical physics and optics to show how the underlying symmetries of a photonic lattice can shed light onto the design process. For this, we will use a general version of the three-waveguide coupler. In the next section, we will introduce the

mode-coupling model and expose its underlying $SU(3)$ symmetry. Then, we will construct a propagator for any given physical configuration using a Gilmore-Perelomov coherent state approach [21]. In order to provide practical examples, we will focus on arrays of identical waveguides, which we call optical trimers thereon, with three identical couplings and relate them to the discrete Fourier transform through the cyclic group. We will also present results for the optical trimer with two identical couplings, and relate them to the golden ratio and devices allowing a single intensity stable output. Next, for the sake of completeness, we will show that it is possible to recover the coupled mode theory equations for field intensity and phase if we use the optical-quantum analogy in the classical limit. We will also show that a classical mechanical analysis of the optical-quantum analogy can predict boundary limits for all possible propagation classes defined by a device, as well as propagation envelopes for individual initial inputs without the need of actual propagation. Finally, we will present a summary and discuss possible extensions allowed by linear and nonlinear three-waveguide couplers.

2. TRIANGULAR THREE-WAVEGUIDE COUPLER

Light propagating through an ideal, triangular three-waveguide coupler can be described by coupled mode theory, c.f. [22] and references therein,

$$-i\partial_z \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix} = \begin{pmatrix} \omega_0(z) & g_{01}(z) & g_{02}(z) \\ g_{01}(z) & \omega_1(z) & g_{12}(z) \\ g_{02}(z) & g_{12}(z) & \omega_2(z) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(z) \\ \mathcal{E}_1(z) \\ \mathcal{E}_2(z) \end{pmatrix}.$$

Here, the complex field amplitude at the j th waveguide is given by $\mathcal{E}_j(z)$, the effective refractive index at the j th waveguide is $\omega_j(z)$, and the effective coupling between the j th and k th waveguides is $g_{jk}(z)$. These complex field equations can be cast in a Schrödinger-like form [22],

$$-i\partial_z |\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle, \quad (2)$$

where kets and operators in Dirac notation represent column vectors and square matrices, in that order. We can normalize the intensity, $\sum_j |\mathcal{E}_j(z)|^2 = 1$, as we are dealing with an ideal lossless device. Experimental realization of this model include, but are not limited to, laser inscribed photonic waveguides [23] and multicore optical fibers, Fig. 1(a), whispering-mode cavities [?], Fig. 1(b), or microwave resonators [24].

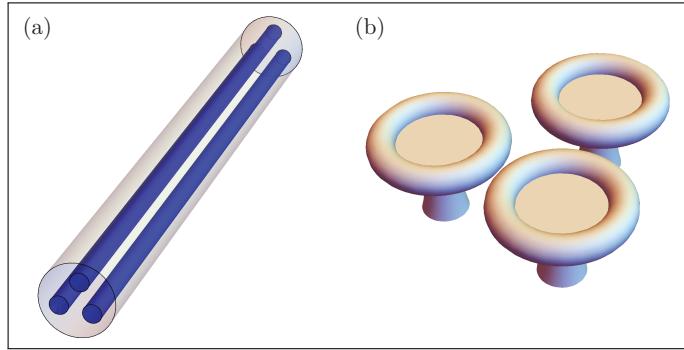


Fig. 1. (Color online) Some experimentally feasible realizations for triangular waveguide couplers, (a) multicore optical fibers or inscribed photonic waveguides, (b) whispering-gallery mode cavities.

The formal solution to this ordinary differential equation,

$$|\mathcal{E}(z)\rangle = \hat{U}(z)|\mathcal{E}(0)\rangle, \quad (3)$$

is provided by an ordered exponential [25, 26],

$$\hat{U}(z) = \text{Texp} \left[\int_0^z \hat{H}(x) dx \right]. \quad (4)$$

Usually, it is not straightforward to calculate this propagator, but underlying symmetries simplify this endeavor [27? , 28]. While group theory is extensively used in mathematical optics [29, 30], it may be possible that the standard Lie algebra approach may look more complicated than it actually is for those outside that field. We hope that the following can help vanquish that feeling.

3. GROUP THEORY APPROACH

Group theory, as an instrument to explore the underlying structure of mathematical models describing the physical world, brings a layer of abstraction into physics that allows deeper

insight. As such, it has become an essential tool in quantum mechanics. Coupled mode theory delivers a Schrödinger-like form describing light propagating through arrays of coupled waveguides, thus, the use of group theory to calculate propagation in these systems seems like a natural step.

The mode-coupling matrix \hat{H} for our triangular three-waveguide coupler is a unitary matrix of rank three with trace (1) equal to $\sum_{j=0}^3 \omega_j(z)$. It is useful to decompose it into a unit matrix part and a traceless part,

$$\hat{H}(z) = \frac{1}{3} \sum_{j=0}^3 \omega_j(z) \hat{\mathbb{1}} + \hat{\mathcal{H}}(z). \quad (5)$$

The traceless part can be written in terms of the special unitary group $SU(3)$ which is a household name in physics often related to the work of Gell-Mann [31] and Ne'eman [?],

$$\begin{aligned} \hat{\mathcal{H}}(z) = & \frac{1}{2} \omega_y(z) \hat{Y} + \omega_i(z) \hat{I}_0 + g_{01}(z) (\hat{I}_+ + \hat{I}_-) \\ & + g_{12}(z) (\hat{U}_+ + \hat{U}_-) + g_{02}(z) (\hat{V}_+ + \hat{V}_-). \end{aligned} \quad (6)$$

Here, we have defined the auxiliary effective refractive index $\omega_y(z) = [\omega_0(z) + \omega_1(z) - 2\omega_2(z)]/2$, $\omega_i(z) = \omega_0(z) - \omega_1(z)$, and used the following representation for the $SU(3)$ group [32],

$$\begin{aligned} \hat{Y} &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \hat{I}_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{I}_+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{I}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{U}_+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \hat{V}_+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{V}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (7)$$

due to the fact that we can understand matrices \hat{I}_{\pm} , \hat{U}_{\pm} and \hat{V}_{\pm} as those describing the coupling of the electromagnetic field between waveguides zero and one, one and two, and zero and two, in that order, Fig. 2.

At this point, our original Schrödinger-like equation is written in terms of the identity matrix and a linear combination of Lie group generators for $SU(3)$. The identity part only induces and overall phase,

$$e^{i\phi(z)\hat{\mathbb{1}}} = e^{\frac{i}{3} \int_0^z (\omega_0(\zeta) + \omega_1(\zeta) + \omega_2(\zeta)) d\zeta} \hat{\mathbb{1}}. \quad (8)$$

such that the propagator can be rewritten,

$$\hat{U}(z) = e^{\frac{i}{3} \int_0^z (\omega_0(\zeta) + \omega_1(\zeta) + \omega_2(\zeta)) d\zeta} \hat{\mathcal{U}}(z). \quad (9)$$

Now, for the $SU(3)$ part, $\hat{\mathcal{U}}(z)$, Wei and Norman demonstrated that any such equation can be treated by an algebraic method providing the following propagator [?],

$$\hat{\mathcal{U}}(z) = \prod_{j=1}^8 e^{i\theta_j(z)\hat{X}_j}, \quad (10)$$

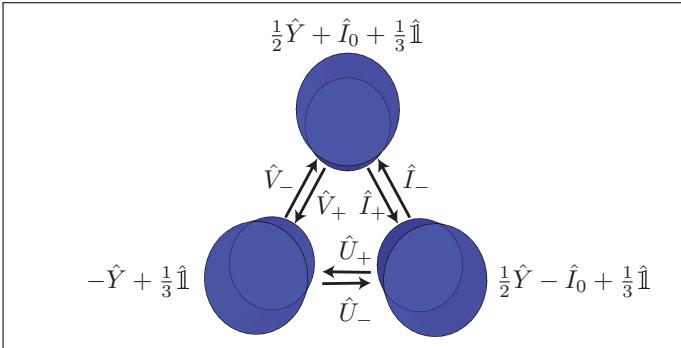


Fig. 2. (Color online) Effective refractive indices and couplings between the waveguides as described by the $SU(3)$ group elements.

where the $su(3)$ algebra elements, $e^{i\theta_j(z)\hat{X}_j}$, are just the exponential map of the $SU(3)$ group generators, $\{\hat{Y}, \hat{I}_0, \hat{I}_{\pm}, \hat{U}_{\pm}, \hat{V}_{\pm}\}$, and the functions $\theta_j(z)$ are complex functions ruled by the dynamics provided by the mode-coupling matrix, $\hat{H}(z)$. There is no a priori ordering of $su(3)$ elements to write the propagator, however, the values of the $\theta_j(z)$ functions do depend on the chosen order and different orderings have been studied in the quantum optics literature [34? ? ?]. We will choose a particular ordering,

$$\hat{U}(z) = e^{i\mu_+(z)\hat{I}_+} e^{i\mu_+(z)\hat{U}_+} e^{i\nu_+(z)\hat{V}_+} e^{i\theta_0(z)\hat{I}_0} e^{i\mu_-(z)\hat{Y}} e^{i\nu_-(z)\hat{V}_-} e^{i\mu_-(z)\hat{U}_-} e^{i\nu_-(z)\hat{I}_-}, \quad (11)$$

that keeps us in line with the idea of understanding propagation through waveguide lattices as generalized Gilmore-Perelomov coherent states [21].

The next step is straightforward but cumbersome, we substitute the formal solution, $|\mathcal{E}(z)\rangle$, using the propagator above and being careful in keeping the ordering through the derivation process. Then, we use the actions of elements of the $su(3)$ algebra on elements of the $SU(3)$ group [33] to find the differential equation set for the auxiliary functions,

$$\iota'_+ = g_{01}\iota_+^2 + (g_{02}\nu_+ + i\omega_i)\iota_+ - ig_{12}\nu_+ + g_{01}, \quad (12)$$

$$\mu'_+ = (g_{12} + ig_{02}\iota_+)\mu_+^2 + \left[g_{02}\nu_+ - g_{01}\iota_+ + \frac{i}{2}(2\omega_y - \omega_i)\right]\mu_+ + ig_{01}\nu_+ + g_{12}, \quad (13)$$

$$\nu'_+ = g_{02}\nu_+^2 + \left[g_{01}\iota_+ + \frac{i}{2}(2\omega_y + \omega_i)\right]\nu_+ - ig_{12}\iota_+ + g_{02}, \quad (14)$$

$$\iota'_0 = \omega_i - i2g_{01}\iota_+ + ig_{12}\mu_+ - g_{02}(\iota_+\mu_+ + iv_+), \quad (15)$$

$$y'_0 = \omega_y - i\frac{3}{2}[g_{02}\nu_+ + (g_{12} + ig_{02}\iota_+)\mu_+], \quad (16)$$

$$\nu'_- = g_{02}e^{i(y_0 + \frac{1}{2}\iota_0)} - e^{i\iota_0}(g_{02}\mu_+ + ig_{01})\mu_-, \quad (17)$$

$$\mu'_- = e^{i(y_0 - \frac{1}{2}\iota_0)}(g_{12} + ig_{02}\iota_+), \quad (18)$$

$$\iota'_- = e^{i\iota_0}(g_{01} - ig_{02}\mu_+), \quad (19)$$

where, for the sake of space, we have used $f \equiv f(z)$ and $f' \equiv \partial_z f(z)$ for auxiliary functions and couplings.

Non-linear differential equations are known to be hard to solve and finding a solution often requires intuition and knowledge of the system being analyzed. Before delving into details, we would like to point out a key feature of passive, lossless optical models that we are analyzing, their mode-coupling matrices are real symmetric, $\hat{H}^T(z) = \hat{H}(z)$ where the operation

O^T stands for transposition, and, as a direct consequence, the propagator shares the same property,

$$\hat{U}^T(z) = \hat{U}(z). \quad (20)$$

This feature allows us to conclude that the propagator functions are symmetric,

$$\xi_+(z) = \xi_-(z), \quad \xi = \iota, \nu, \mu. \quad (21)$$

Furthermore, we observe that two equations, namely Eq.(12) and Eq.(14) only include terms of $\iota_+(z)$ and $\nu_+(z)$ and their derivatives. Therefore, they are decoupled from the rest. Most of the time, numerical methods would be the only way to find these auxiliary functions.

An alternate approach can be followed if we write the propagator as a matrix,

$$\hat{U}(z) = \begin{pmatrix} \Xi(z) & \Sigma(z) & \Theta(z) \\ \Sigma(z) & \Pi(z) & \Delta(z) \\ \Theta(z) & \Delta(z) & \Gamma(z) \end{pmatrix}, \quad (22)$$

where, for reasons that will become apparent in a moment, we introduced a set of five auxiliary functions,

$$\Gamma(z) = e^{-i\frac{2}{3}y_0(z)}, \quad (23)$$

$$\Delta(z) = i\Gamma(z)\mu_+(z), \quad (24)$$

$$\Theta(z) = \Gamma(z)[-i_+(z)\mu_+(z) + iv_+(z)], \quad (25)$$

$$\Pi(z) = \Gamma(z)[e^{iy_0(z)}e^{-i\frac{1}{2}\iota_0(z)} - \mu_+(z)^2], \quad (26)$$

$$\Sigma(z) = i\iota_+(\zeta)\Pi(z) - \Gamma(z)\mu_+(z)v_+(z)), \quad (27)$$

and the sixth can be written in terms of all others,

$$\Xi(z) = \frac{1 + \Pi(z)\Theta^2(z) + \Gamma(z)\Sigma^2(z) - 2\Delta(z)\Theta(z)\Sigma(z)}{\Pi(z)\Gamma(z) - \Delta^2(z)}. \quad (28)$$

The original functions can be put in terms of these auxiliary,

$$\iota_+(z) = i\frac{\Gamma(z)\Sigma(z) - \Delta(z)\Theta(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, \quad (29)$$

$$\mu_+(z) = -i\frac{\Delta(z)}{\Gamma(z)}, \quad (30)$$

$$\nu_+(z) = i\frac{\Pi(z)\Theta(z) - \Delta(z)\Sigma(z)}{\Delta^2(z) - \Gamma(z)\Pi(z)}, \quad (31)$$

$$\iota_0(z) = i2\log\frac{\Gamma(z)\Pi(z) - \Delta^2(z)}{\Gamma^{\frac{1}{2}}(z)}, \quad (32)$$

$$y_0(z) = i\frac{3}{2}\log\Gamma(z). \quad (33)$$

Note that the phase functions $y_0(z)$ and $\iota_0(z)$ are of logarithmic nature and the rest **others?!(maybe better)** are quotients of the products of the solution basis. We now have the simplest matrix differential equation for the propagator [? ?],

$$\partial_z \hat{U}(z) = i\mathcal{H}\hat{U}(z), \quad (34)$$

with the initial condition $\hat{U}(0) = \mathbb{1}$, that translates into,

$$\Theta(z) = 0, \quad \Delta(z) = 0, \quad \Sigma(z) = 0, \quad \Gamma(z) = 1, \quad \Pi(z) = 1. \quad (35)$$

This differential equation system is overdetermined due to the characteristics of the original matrix and provides the following identities,

$$g_{01}(\Pi - \Xi) + g_{02}\Delta - g_{12}\Theta + \omega_i\Sigma = 0, \quad (36)$$

$$g_{01}\Delta + g_{02}(\Gamma - \Xi) - g_{12}\Sigma + \left(\omega_y + \frac{1}{2}\omega_i\right)\Theta = 0, \quad (37)$$

$$g_{01}\Theta - g_{02}\Sigma(z) + g_{12}(\Gamma - \Pi) + \left(\omega_y - \frac{1}{2}\omega_i\right)\Delta = 0. \quad (38)$$

Again, for waveguide couplers that depend on the propagation distance, this formal solution will be analytically tractable only for a handful of cases, for example the planar three-waveguide coupler [20], but it will be always amenable to solution by numerical methods.

A. Optical trimer

While we have provided a formal solution to propagation through the most general triangular waveguide coupler, considering a particular solution may help build further intuition. For the sake of simplicity, let us now consider the constant optical trimer, which is a triangular three-waveguide array with constant couplings and identical waveguides. We will introduce the dimensionless propagation parameter $\zeta = g_{01}z$, such that the mode-coupling differential equation becomes

$$-i\partial_\zeta|\mathcal{E}(\zeta)\rangle = \hat{\mathcal{H}}|\mathcal{E}(\zeta)\rangle, \quad (39)$$

with the mode-coupling matrix,

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}, \quad (40)$$

given in terms of the dimensionless parameters,

$$\alpha = \frac{g_{02}}{g_{01}}, \quad \beta = \frac{g_{12}}{g_{01}}. \quad (41)$$

Now, we can use the results above to build a particular solution, but it is well known that a set of linear first order differential equations is equivalent to a single linear differential equation of higher order. After some algebra, we can derive a third order differential equation for $\Delta(\zeta)$,

$$\Delta'''(\zeta) + i(1 + \alpha^2 + \beta^2)\Delta'(\zeta) - 2\alpha\beta\Delta(\zeta) = 0, \quad (42)$$

with boundary conditions,

$$\Delta(0) = 0, \quad \Delta'(0) = i\beta, \quad \Delta''(0) = -\alpha. \quad (43)$$

Note that the remaining auxiliary functions are straightforward to calculate,

$$\Theta(\zeta) = \frac{\beta(1 + \beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha(1 - \beta^2)} \quad (44)$$

$$\Gamma(\zeta) = \frac{-(1 + \beta^2)\Delta(\zeta) + i\alpha\beta\Delta'(\zeta) - \Delta''(\zeta)}{\alpha(1 - \beta^2)} \quad (45)$$

$$\Sigma(\zeta) = \frac{\beta(\alpha^2 + \beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha^2 - \beta^2} \quad (46)$$

$$\Pi(\zeta) = \frac{-\alpha(\alpha^2 + \beta^2)\Delta(\zeta) + i\beta\Delta'(\zeta) - \alpha\Delta''(\zeta)}{\alpha^2 - \beta^2}. \quad (47)$$

The main auxiliary function $\Delta(\zeta)$ has the following solution,

$$\Delta(\zeta) = \delta_1 e^{i\gamma_1\zeta} + \delta_2 e^{i\gamma_2\zeta} + \delta_3 e^{i\gamma_3\zeta}, \quad (48)$$

where the constant parameters γ_j are the eigenvalues of the mode-coupling matrix determined by the characteristic polynomial, a reduced cubic,

$$\gamma_j^3 - (1 + \alpha^2 + \beta^2)\gamma_j - 2\alpha\beta = 0. \quad (49)$$

It is straightforward to notice that there are three different real eigenvalues for real, positive, non-zero coupling parameters, $\alpha, \beta > 0$. These proper values can be written in a closed but non-compact form, so we will not write them explicitly. Furthermore, the coefficients are given by

$$\delta_1 = \frac{\alpha - \beta(\gamma_2 + \gamma_3)}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad (50)$$

$$\delta_2 = \frac{\alpha - \beta(\gamma_1 + \gamma_3)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}, \quad (51)$$

$$\delta_3 = \frac{\alpha - \beta(\gamma_1 + \gamma_2)}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}. \quad (52)$$

Thus, the propagator functions, $\iota_\pm(z)$, $\mu_\pm(z)$, $\nu_\pm(z)$, $\iota_0(z)$ and $y_0(z)$, will effectively contain terms involving the three eigenvalues as well as sums and differences thereof. Figure 3 shows the trajectories described by the absolute value of the field amplitudes, $|\mathcal{E}_j(z)|$, as they propagate through the constant optical trimer with random parameters and initial field distribution calculated provided by the group theory approach.

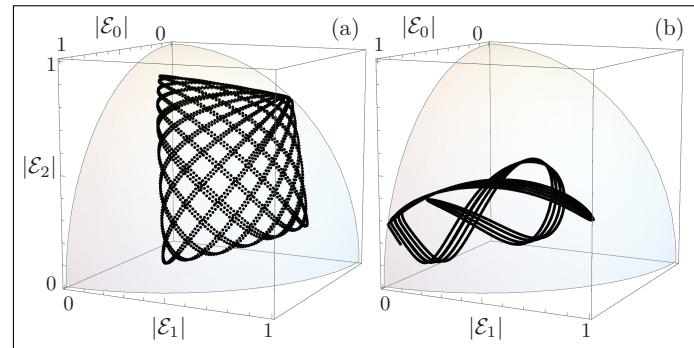


Fig. 3. (Color online) Propagation trajectories of absolute amplitudes, $(|\mathcal{E}_0(z)|, |\mathcal{E}_1(z)|, |\mathcal{E}_2(z)|)$, for two constant optical trimers with random parameters and random normalized initial field distributions.

4. APPLICATIONS

As we just saw, the optical trimer is simple enough to allow us the construction of a closed form solution and, to our advantage, it is experimentally feasible to realize it. Now the obvious question is if there is a use for it. In the following, we will show that a judicious choice of coupling parameters provides different types of well-defined propagation of field amplitudes that can be used for the design of integrated photonic circuits.

A. Identical couplings and the discrete Fourier transform

The mode-coupling matrix for three-identical waveguides distributed in an equilateral triangle configuration, $\alpha = \beta = 1$,

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (53)$$

is related to the cyclic group in dimension three,

$$\hat{\mathcal{H}} = \hat{Z}_3 + \hat{Z}_3^2, \quad (54)$$

where the generator of the cyclic group are the following,

$$\begin{aligned} \hat{Z}_3 &= \hat{I}_+ + \hat{U}_+ + \hat{V}_-, \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (55)$$

It is well known that the cyclic group is diagonalized,

$$\hat{\Lambda} = \hat{F}_n \hat{Z}_n \hat{F}_n^\dagger, \quad (56)$$

by the discrete Fourier transform of rank n given, for $n = 3$,

$$\hat{F}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}, \quad (57)$$

and $\hat{\Lambda}$ is a diagonal rank n matrix showing the n th-roots of the unit,

$$\hat{\Lambda}_{mn} = \delta_{m,n} e^{i \frac{2\pi}{n} i}, \quad m, n = 0, 1, 2, \quad (58)$$

on the main diagonal. In this particular case, it is possible to compose a propagator,

$$\begin{aligned} U(\zeta) &= \hat{F}_3^\dagger e^{i \hat{\Lambda}_3 \zeta} e^{i \hat{\Lambda}_3^2 \zeta} \hat{F}_3, \\ &= \frac{1}{3} \begin{pmatrix} 2 + e^{3i\zeta} & -1 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & 2 + e^{3i\zeta} & -1 + e^{3i\zeta} \\ -1 + e^{3i\zeta} & -1 + e^{3i\zeta} & 2 + e^{3i\zeta} \end{pmatrix} e^{-i\zeta} \end{aligned} \quad (59)$$

where we have used the fact that the elements of the cyclic group of rank 3 commute between them, $[\hat{Z}_3, \hat{Z}_3^2] = 0$ because $\hat{Z}_3^3 = \mathbb{1}_3$.

Figure 4 shows the trajectories described by the absolute value of the field amplitudes, $|\mathcal{E}_j(z)|$, as they propagate. All trajectories will lie on... All of trajectories will lie over the surface of an octant of the sphere due to unitary propagation, $\sum_j |\mathcal{E}_j(z)|^2 = 1$. Figure 4(a) shows the response to impulse input, $\mathcal{E}_k(0) = \delta_{j,k}$ with $j = 0, 1, 2$ and a fixed $k = 0, 1, 2$. Figure 3(b) shows the trajectories given by initial field superpositions of the more general form: $\mathcal{E}_j = \alpha_j e^{i\phi_j t}$ with $\alpha \in \mathbb{R}$ and $\sum_j |\alpha_j|^2 = 1$. Note that the roots of the mode-coupling matrix, $\hat{\mathcal{H}}$, are doubly degenerate and commensurate,

$$\gamma_{1,2} = -1, \quad \gamma_{23} = 2. \quad (60)$$

Thus, propagation involves just two commensurate eigenvalues and all the possible field amplitude propagation trajectories, $(|\mathcal{E}_0|, |\mathcal{E}_1|, |\mathcal{E}_2|)$, will be closed and well defined as those shown in Fig. 3.

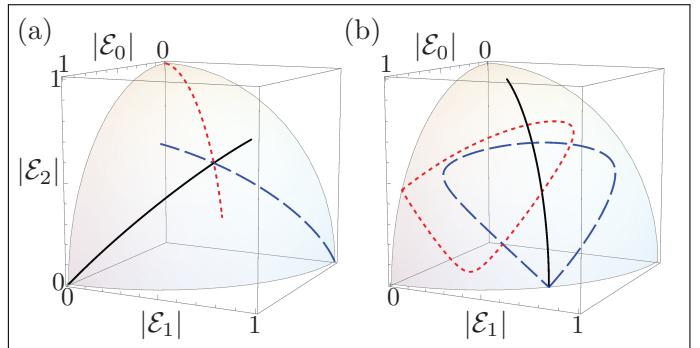


Fig. 4. (Color online) Propagation trajectories of absolute amplitudes, $(|\mathcal{E}_0(z)|, |\mathcal{E}_1(z)|, |\mathcal{E}_2(z)|)$, for initial fields impinging (a) only the zeroth (solid black), $(1, 0, 0)$, first (dashed blue), $(0, 1, 0)$, and second (dotted red), $(0, 0, 1)$, waveguides and (b) initial fields impinging two waveguides at a time with and without a relative phase, $(1, 1, 0)/\sqrt{2}$ (solid black), $(1, i, 0)/\sqrt{2}$ (dashed blue), and $(2, 0, i)/\sqrt{5}$ (dotted red).

B. Two identical couplings and the golden ratio

In the case of two equal coupling parameters the unitless mode-coupling matrix becomes,

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix} \quad (61)$$

Note that exchanging the first and second waveguides returns the same mode-coupling matrix; in other words, it is \hat{Z}_2 -invariant. This symmetry is reflected by the eigenvalues,

$$\gamma_1 = -1, \quad \gamma_2 = \bar{\varphi}, \quad \gamma_3 = \varphi, \quad (62)$$

with

$$\bar{\varphi} = \frac{1}{2} \left(1 - \sqrt{8\alpha^2 + 1} \right), \quad (63)$$

$$\varphi = \frac{1}{2} \left(1 + \sqrt{8\alpha^2 + 1} \right). \quad (64)$$

It is simply to see that $\bar{\varphi} + \varphi = 1$ and that the latter eigenvalue becomes the golden ratio for $\alpha = \sqrt{1/2}$. Again, due to the \hat{Z}_2 -symmetry, the propagator can be calculated directly using the group theory approach,

$$\Delta(\zeta) = \Theta(z), \quad (65)$$

$$= \frac{\alpha}{\varphi - \bar{\varphi}} \left(e^{i\zeta\varphi} - e^{i\zeta\bar{\varphi}} \right), \quad (66)$$

$$\Gamma(\zeta) = \frac{1}{\varphi - \bar{\varphi}} \left(\varphi e^{i\zeta\bar{\varphi}} - \bar{\varphi} e^{i\zeta\varphi} \right), \quad (67)$$

$$\Sigma(\zeta) = \frac{1}{2(\varphi - \bar{\varphi})} \left(\varphi e^{i\zeta\varphi} - \bar{\varphi} e^{i\zeta\bar{\varphi}} \right) - \frac{1}{2} e^{-i\zeta}, \quad (68)$$

$$\Pi(\zeta) = \Xi(\zeta), \quad (69)$$

$$= \frac{1}{2(\varphi - \bar{\varphi})} \left(\varphi e^{i\zeta\varphi} - \bar{\varphi} e^{i\zeta\bar{\varphi}} \right) + \frac{1}{2} e^{-i\zeta}. \quad (70)$$

Obviously, when the eigenvalues are commensurate, the field amplitude propagation trajectories will be closed and well defined. However, this particular symmetry might allow further interesting states. Note that the auxiliary functions $\Delta(\zeta)$ and

$\Gamma(\zeta)$ depend on only two of the three eigenfrequencies, φ and $\bar{\varphi}$. Furthermore, a little bit of algebra considering an initial state without any field impinging at the zeroth waveguide, $\mathcal{E}_0(0) = 0$, reveals that propagation in the second waveguide, $\mathcal{E}_2(z)$, only depends on those two auxiliary functions, $\Delta(\zeta)$ and $\Gamma(\zeta)$,

$$\mathcal{E}_2(\zeta) = \Delta(\zeta)\mathcal{E}_1(0) + \Gamma(\zeta)\mathcal{E}_2(0). \quad (71)$$

Now, we can hope that choosing a proper input will cancel out one of the two eigenvalues that appear in the evolution of the field amplitude at the second waveguide. That is, if the evolution only depends on a single eigenvalue, it will only change in phase and the field amplitude at the second waveguide will remain immutable. This is fulfilled when either

$$\mathcal{E}_1(0) = \frac{\varphi}{\alpha}\mathcal{E}_2(0) \quad \text{or} \quad \mathcal{E}_1(0) = \frac{\bar{\varphi}}{\alpha}\mathcal{E}_2(0), \quad (72)$$

are met as initial conditions. Without loss of generality, we can give the propagated amplitude vector,

$$|\mathcal{E}_x(\zeta)\rangle = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} \frac{\varphi}{\alpha}\Sigma(\zeta) + \Delta(\zeta) \\ \frac{\varphi}{\alpha}\Pi(z) + \Delta(\zeta) \\ e^{i\zeta\varphi} \end{pmatrix} \quad x = \varphi, \bar{\varphi}, \quad (73)$$

up to a normalization constant \mathcal{N} . Figure 5 shows propagation trajectories for the absolute field amplitudes in an isosceles constant optical trimer with parameter $\alpha = 1/\sqrt{2}$ chosen to provide the golden ratio, Fig. 5(a), and $\alpha = 1/2$, Fig. 5(b), with initial field amplitudes provided by Eq.(72) ensuring a stable output at the second waveguide.

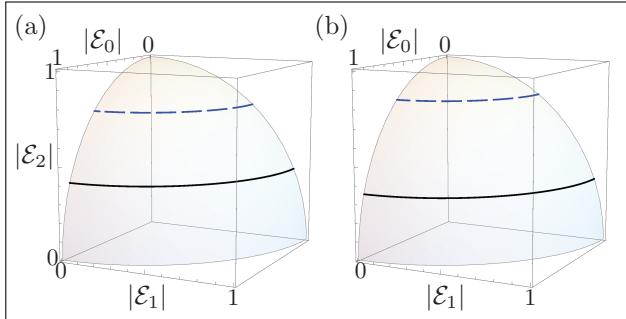


Fig. 5. (Color online) Propagation trajectories of absolute amplitudes, ($|\mathcal{E}_0(z)|$, $|\mathcal{E}_1(z)|$, $|\mathcal{E}_2(z)|$), for the states with stationary field amplitude in the second waveguide related to φ (solid black) and $\bar{\varphi}$ (dashed blue) described by Eq.(73) with (a) $\alpha = 1/\sqrt{2}$ and (b) $\alpha = 1/2$.

5. BEYOND THE STANDARD APPROACH

Finally, we want to show that a theoretical physics treatment of photonic lattices can go beyond the standard approach to the quantum mechanical analogy that considers an array of coupled waveguides as an array of coupled resonators, for the triangular waveguide coupler,

$$\hat{H}(t) = \sum_{j=0}^2 \omega_j(t) \hat{a}_j^\dagger \hat{a}_j + \sum_{j,k=0}^2 g_{jk}(t) \hat{a}_j^\dagger \hat{a}_k, \quad (74)$$

where the operators \hat{a}_j (\hat{a}_j^\dagger) annihilate (create) a photon in the j th resonator, and realizes that, with the single photon limit ansatz,

$$|\psi(t)\rangle = \mathcal{E}_0(t)|1,0,0\rangle + \mathcal{E}_1(t)|0,1,0\rangle + \mathcal{E}_2(t)|0,0,1\rangle, \quad (75)$$

with $\sum_j |\mathcal{E}_j(z)|^2 = 1$, it yields the same system of differential equations,

$$i\partial_t \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix} = \begin{pmatrix} \omega_0(t) & g_{01}(t) & g_{02}(t) \\ g_{01}(t) & \omega_1(t) & g_{12}(t) \\ g_{02}(t) & g_{12}(t) & \omega_2(t) \end{pmatrix} \begin{pmatrix} \mathcal{E}_0(t) \\ \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{pmatrix}, \quad (76)$$

than as coupled-mode theory, Eq.(1), once we make the substitution $t = -z$. This analogy allowed us to pursue a group theory approach in the previous sections for the propagation of the field amplitudes but it can also allows **allow us??** to go further and derive equations of motion for different sets of variables.

A. Classical limit and intensity-phase equations

For example, if we forget about the single-photon limit and take the classical limit, where the states of light can be taken as coherent states with a high number of photons, then, the canonical pair provided by the creation and annihilation operators can be replaced by the canonical pair of classical intensity, n_j , and phase, ϕ_j , such that $\hat{a}_j \rightarrow \sqrt{n_j}e^{i\phi_j}$, we obtain a classical Hamiltonian,

$$H(z) = \sum_{j=0}^2 \omega_j(z)n_j + \sum_{j \neq k=0}^2 g_{jk}(z)\sqrt{n_j n_k} \cos(\phi_j - \phi_k), \quad (77)$$

that provides us with the following equations of motion for the classical canonical pairs,

$$\partial_z n_j(z) = -\frac{\partial H(z)}{\partial \phi_j} \quad (78)$$

$$\partial_z \phi_j(z) = \frac{\partial H(z)}{\partial n_j}, \quad (79)$$

where we have already accounted for the switch from time to propagation distance. These six coupled differential equations for the three degrees of freedom can be reduced to two degrees of freedom if we substitute the intensity $n_2(z) = 1 - n_0(z) - n_1(z)$ and use the phase $\phi_2(z)$ as reference, such that we define the phase differences $\delta_j(z) = \phi_j(z) - \phi_2(z)$ for $j = 0, 1$, ending up with just four coupled differential equations. This result is the coupled mode theory description of the propagation of intensity and phase.

B. Classical mechanics and propagation envelopes

We can also choose the standard classical mechanics canonical pair provided by the field quadratures,

$$\hat{q} = \frac{1}{\sqrt{2}} (\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \frac{i}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}), \quad (80)$$

and, again, consider classical variables to obtain equations of motion for the field quadratures,

$$i\partial_z |p(z)\rangle = H(z)|q(z)\rangle, \quad (81)$$

$$-i\partial_z |q(z)\rangle = H(z)|p(z)\rangle, \quad (82)$$

where, again, we have used kets to represent column vectors, $|x(z)\rangle = (x_0(z), x_1(z), x_2(z))^T$ with $x = p, q$, and made the change from time to propagation distance. Before moving on, let us focus on the absolute value of the field amplitudes that we have been using to plot the dynamics of our optical trimers, ($|\mathcal{E}_0|$, $|\mathcal{E}_1|$, $|\mathcal{E}_2|$). These square roots of the intensities at the three waveguides can be understood as a Poincaré phase space in the

classical mechanics sense. Then, we can follow a standard classical mechanics approach to find the action-phase variables of the system [35]. If the phases are commensurate, the trajectories in phase space will be periodic, and if the phases are incommensurate, the trajectories will be ergodic and fill a region of the phase space defined by the energy of the initial state,

$$\epsilon = \langle \mathcal{E}(0) | \hat{H}(0) | \mathcal{E}(0) \rangle. \quad (83)$$

For mode-coupling matrices that do not depend on the propagation distance, if the eigenvalues are commensurable, the trajectories in phase space are well-defined and closed, otherwise they will fill a region defined by the initial energy. Figure 6 shows propagation trajectories of the absolute field amplitudes for the numerical propagation of random parameter sets and initial conditions yielding incommensurate eigenvalues and, thus, ergodic trajectories.

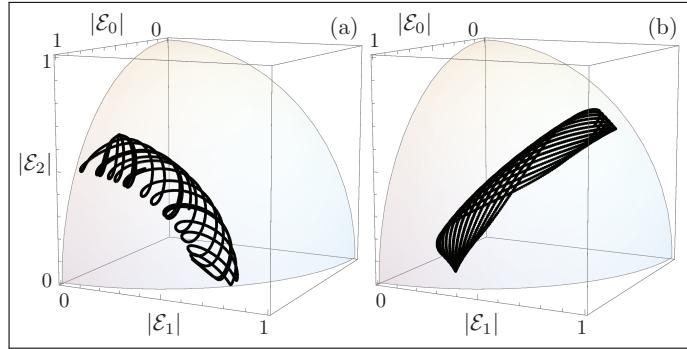


Fig. 6. (Color online) Propagation trajectories of absolute amplitudes, ($|E_0(z)|$, $|E_1(z)|$, $|E_2(z)|$), for initial random fields impinging random triangular three-waveguide couplers with constant parameters yielding ergodic trajectories.

Let us focus our attention now on the vast class of incommensurate trajectories and, for the sake of simplicity, keep using optical devices that do not change with the propagation distance. In many design circumstances, the exact point-to-point evolution of the field amplitudes is not of primary importance and it suffices to gain information on the general region that will be traversed. This is characterized by the boundary which encompasses the very region. It proves viable to notice that for a given energy, ϵ , and a given set of parameters, ω_j and g_{jk} , in most cases, an infinity number of distinct trajectories prevails. The conjunction of all these propagation trajectories pertaining to a given energy span a path-connected surface. We will now find the outer boundary, or envelope, of all propagation trajectories related to an initial energy ϵ . Equation (77) readily reveals that the energy is maximized for given intensities n_0 , n_1 and n_2 when all three phases are zero. Consequently, the boundary is defined by all triplets (n_0, n_1, n_2) that fulfill the following condition,

$$H_B(n_0, n_1, n_2) = \epsilon, \quad (84)$$

with

$$H_B(n_0, n_1, n_2) = \sum_{j=0}^2 \omega_j n_j + \sum_{j \neq k=0}^2 g_{jk} \sqrt{n_j n_k}. \quad (85)$$

Due to our normalization requirements, $\sum_{j=0}^2 n_j = 1$, the three intensities are not independent and we can safely omit the third intensity in the following discussion, $H_B(n_0, n_1, n_2) \rightarrow H_B(n_0, n_1)$.

Numerically, once a given set of initial intensities or energy is chosen, the envelope is characterized by the following differential equation,

$$\frac{\partial H_B}{\partial n_1} \delta n_1 + \frac{\partial H_B}{\partial n_2} \delta n_2 = 0. \quad (86)$$

This is the envelope for all possible trajectories with energy ϵ . Figure 7 shows such envelope as a solid black line.

Now, we can turn to calculate calculating the envelope of a single trajectory. We have already discussed how to find the propagator for any given three-waveguide coupler. In the particular case at hand, constant effective refractive indices and couplings, the propagated field can be written as,

$$|\mathcal{E}(z)\rangle = \sum_{j=1}^3 c_j e^{i\gamma_j z} |e_j\rangle, \quad (87)$$

where the vectors $|e_j\rangle$ are the three normal modes of the coupler with corresponding eigenvalues ω_j and the normal mode decomposition parameters are defined as the inner product between the normal mode and the initial condition, $c_j = \langle e_j | \mathcal{E}(0) \rangle$. We can introduce a phase relevant parameter which distorts the ratio between eigenvalues,

$$|\mathcal{E}(z, \phi)\rangle = c_0 e^{i(\gamma_0 z + \phi)} |e_0\rangle + c_1 e^{i\gamma_1 z} |e_1\rangle + c_2 e^{i\gamma_2 z} |e_2\rangle, \quad (88)$$

The parameter ϕ makes a given trajectory or, more precisely, a section of the trajectory, offset in a direction perpendicular to the trajectory Not sure if the sentence is good. This however cannot hold true at the envelope and, hence, we can establish the following condition for the envelope,

$$\left(\frac{\frac{\partial n_0}{\partial z}}{\frac{\partial n_1}{\partial z}} \right) = \left(\frac{\frac{\partial n_0}{\partial \phi}}{\frac{\partial n_1}{\partial \phi}} \right), \quad (89)$$

which is identical to requiring a null determinant,

$$D(z, \phi) = \left| \begin{pmatrix} \frac{\partial n_0}{\partial z} & \frac{\partial n_1}{\partial z} \\ \frac{\partial n_0}{\partial \phi} & \frac{\partial n_1}{\partial \phi} \end{pmatrix} \right|, \quad (90)$$

$$= 0. \quad (91)$$

Accordingly, the envelope can be calculated from the following differential equation,

$$\frac{\partial D(z, \phi)}{\partial z} \delta z + \frac{\partial D(z, \phi)}{\partial \phi} \delta \phi = 0. \quad (92)$$

Figure 7 shows two examples of envelopes of individual trajectories with particular initial field amplitudes as dashed blue and dotted red lines.

6. CONCLUSION

We have shown that it is possible to approach the propagation of electromagnetic field through triangular three-waveguide couplers, in terms of the Lie algebra $su(3)$. We derived a formal solution à la Wei-Norman in line with the idea of Gilmore-Perelomov coherent states. In order to provide practical examples, we focused our attention on a reduced class of structures where the waveguides are identical and the effective couplings are constant, independent of the propagation distance, and called coined them?? these systems constant optical trimers. We established a connection between the equilateral trimer, actually

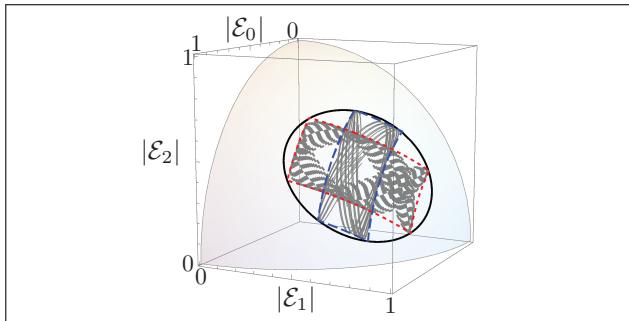


Fig. 7. (Color online) Propagation trajectories of absolute amplitudes, ($|E_0(z)|$, $|E_1(z)|$, $|E_2(z)|$), for two distinct trajectories pertaining to the energy ϵ (solid gray). The boundary of all possible trajectories (solid black) and the envelopes of the individual trajectories (dashed blue and dotted red) are calculated according to equation Eq.(92).

any circular array of waveguides with identical couplings, and the discrete Fourier transform, and between the isosceles trimer and the golden ratio where it is possible to suppress the energy transfer to one of the waveguides.

We went beyond the standard classical-quantum analogy in the single photon limit and showed that, using the classical limit, it is possible to recover the coupled mode differential equations describing the propagation of intensity and phase. Furthermore, we also showed that choosing to work with the classical mechanics canonical pair, which are the equivalent of the electromagnetic field quadratures, allows us to calculate the envelope of all possible trajectories with a given initial energy as well as individual envelopes for specific input field amplitude distributions. All these results may provide a working platform for the design of photonic integrated devices where the main interest is to generate a specific output from a set of particular inputs instead of studying point-to-point propagation through the devices.

We hope this manuscript could also serve as a tutorial for the use of theoretical physics tools in the design of photonic lattices and, viceversa, encourage theoretical physicists to make incursions on the field of photonic circuit design.

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