Propagation in an optical trimer

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We study electromagnetic field propagation through a triangular array of identical waveguides using a symmetry based approach to take advantage of the underlying $SU(3)$ symmetry.

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1. INTRODUCTION

Some random ideas that may help write a coherent introduction, in no particular order

In chemistry, a trimer is composed of three identical building blocks, we can borrow such an idea and think of three coupled identical photonic waveguides as an optical trimer.

Passive planar three-waveguide couplers have proved a reliable platform for fast, robust beam coupling based on adiabatic coupling or Ermakov-Lewis-Riesenfeld invariants [? ? ? ? ? ?].

The planar platform has been used in three-waveguide nonlinear directional couplers, where the waveguides present an active Kerr nonlinearity, to produce all-optical spatial switching [? ? ? ? ? ? ?] and logic gates [?].

Triangular three-waveguide nonlinear directional couplers have been used to design all-optical logic gates [? ?].

Light propagating through a three-waveguide coupler with identical waveguides can be described by coupled mode theory, c.f. [?] and references therein,

$$-i\partial_z \mathcal{E}_0(z) = g_{01}(z)\mathcal{E}_1(z)(z) + g_{02}(z)\mathcal{E}_2(z), \tag{1}$$
$$-i\partial_z \mathcal{E}_1(z) = g_{01}(z)\mathcal{E}_0(z)(z) + g_{12}(z)\mathcal{E}_2(z), \tag{2}$$
$$-i\partial_z \mathcal{E}_2(z) = g_{02}(z)\mathcal{E}_0(z)(z) + g_{12}(z)\mathcal{E}_1(z), \tag{3}$$

where the complex field amplitude at the jth waveguide is given by $\mathcal{E}_j(z)$ and the coupling between the jth and kth waveguides is $g_{jk}(z)$; we have obviated the common effective refractive index that just provides an overall phase. These complex field equations can be cast in a Schrödinger-like form [?],

$$-i\partial_z |\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle, \tag{4}$$

where kets from Dirac notation represent column vectors,

$$|\mathcal{E}(z)\rangle = (\mathcal{E}_0(z), \mathcal{E}_1(z), \mathcal{E}_2(z))^T, \tag{5}$$

and operators represent square matrices, in this particular case, the mode-coupling matrix,

$$\hat{H}(z) = \begin{pmatrix} 0 & g_{01}(z) & g_{02}(z) \\ g_{01}(z) & 0 & g_{12}(z) \\ g_{02}(z) & g_{12}(z) & 0 \end{pmatrix}, \tag{6}$$

points to an underlying $SU(3)$ symmetry.

2. $SU(3)$ BREVIARY AND SOLUTION TO THE GENERAL MODEL

Group theory, as an instrument to explore the underlying structure of mathematical models describing the physical world, brings a layer of abstraction into physics that allows deeper insight. For example, unitary matrices of rank three, just like our mode-coupling matrix in Eq.(6), are commonly characterized by the special unitary group $SU(3)$. This group is a household name in physics that is often related to the work of Gell-Mann [?]. Typically, the fundamental building blocks of this group are given by the well known Gell-Mann matrices, which provide a punctual description of the underlying algebraic structure. Here, however, we will choose a different representation for the group

[?],

$$\hat{I}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{Y}_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\hat{I}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{I}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{U}_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{V}_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{V}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{7}$$

due to the fact that we can understand matrices $\hat{I}_\pm$, $\hat{U}_\pm$ and $\hat{V}_\pm$ as those describing the coupling of the electromagnetic field between waveguides zero and one, one and two, and zero and two, in that order. Thus, our mode-coupling matrix, in terms of the $SU(3)$ group, is given by the following expression,

$$\begin{aligned} \hat{H} &= g_{01}(z)\left(\hat{I}_+ + \hat{I}_-\right) + g_{12}(z)\left(\hat{U}_+ + \hat{U}_-\right) \\ &\quad + g_{02}(z)\left(\hat{V}_+ + \hat{V}_-\right). \end{aligned} \tag{8}$$

Now, let us revisit the power of using underlying symmetries to evaluate propagation in arrays of coupled waveguides [?]. Any given normalized field vector that solves the Schrödinger-like mode coupling equation, Eq.(4),

$$|\mathcal{E}(z)\rangle = \hat{U}(z)|\mathcal{E}(0)\rangle, \tag{9}$$

can be written in terms of a $su(3)$ Lie algebra, in other words, the propagator is given by the following expression,

$$\hat{U}(z) = \prod_{j=1}^{8} e^{i\theta_j(z)\hat{X}_j}. \tag{10}$$

where the algebra elements, $e^{i\theta_j(z)\hat{X}_j}$, are just the exponential map of the group generators, Eq.(7), the functions $\theta_j(z)$ are complex functions ruled by the dynamics provided by the mode-coupling matrix, and the impinging field amplitudes are collected in the normalized initial field vector $|\mathcal{E}(0)\rangle$.

Note, there is no apriori ordering of $su(3)$ elements to write the propagator. However, the values of the prefixed $\theta_j(z)$ functions do depend on the chosen order. We will choose a particular ordering,

$$\begin{aligned} \hat{U}(z) &= e^{i\iota_+(z)\hat{I}_+} e^{i\mu_+(z)\hat{U}_+} e^{i\nu_+(z)\hat{V}_+} e^{\iota_0(z)\hat{I}_0} \\ &\quad \times e^{iy_0(z)\hat{Y}} e^{i\nu_-(z)\hat{V}_-} e^{i\mu_-(z)\hat{U}_-} e^{i\iota_-(z)\hat{I}_-}, \end{aligned} \tag{11}$$

that keeps us in line with the idea of understanding propagation through waveguide lattices as generalized Gilmore-Perelomov coherent states [?].

Substituting the propagated field vector, Eq.(9) considering as propagator Eq.(11), into the mode coupling equation, Eq.(4), is a cumbersome but straightforward operation that yields a set of eight coupled differential equations,

$$\iota'_+ = g_{01}\left(\iota_+^2 + 1\right) + g_{12}\iota_+\nu_+ - ig_{03}\nu_+, \tag{12}$$

$$\iota'_0 = i\left[\left(-2g_{01} + ig_{02}\mu_+\right)\iota_+ - g_{02}\nu_+ + g_{12}\mu_+\right], \tag{13}$$

$$\iota'_- = e^{i\iota_0}\left(g_1 - ig_{02}\mu_+\right), \tag{14}$$

$$\begin{aligned} \mu'_+ &= \left(-g_{01} + ig_{02}\mu_+\right)\iota_+\mu_+ + g_{12}\left(\mu_+^2 + 1\right) \\ &\quad + \left(ig_{01} + g_{02}\mu_+\right)\nu_+, \end{aligned} \tag{15}$$

$$\nu'_+ = g_{01}\iota_+\nu_+ + g_{02}(\nu_+^2 + 1) - ig_{12}\iota_+, \tag{16}$$

$$y'_0 = -i\frac{3}{2}[g_{02}\nu_+ + \mu_+(g_{12} + ig_{02}\iota_+)], \tag{17}$$

$$\nu'_- = e^{\frac{1}{2}i(2y_0+\iota_0)}g_{02} + ie^{i\iota_0}\mu_-(ig_{02}\mu_+ - g_{01}), \tag{18}$$

$$\mu'_- = e^{iy_0 - \frac{1}{2}i\iota_0}(g_{12} + ig_{02}\iota_+), \tag{19}$$

where, for the sake of space, we have used $f \equiv f(z)$ and $f' \equiv \partial_z f(z)$ for all propagation dependent auxiliary functions and couplings.

Non-linear differential equations are known to be hard to solve and finding a solution often requires intuition and knowledge of the system being analyzed. Before delving into details, we would like to point out a key feature of the present model, $\hat{H}^T(z) = \hat{H}(z)$, that is, the mode-coupling matrix is symmetric, and, as a direct consequence, the propagator shares the same property,

$$\hat{U}^T(z) = \hat{U}(z). \tag{20}$$

This feature allows us to conclude that the propagator functions are symmetric,

$$\iota_+(z) = \iota_-(z) \tag{21}$$

$$\mu_+(z) = \mu_-(z) \tag{22}$$

$$\nu_+(z) = \nu_-(z). \tag{23}$$

Furthermore, we observe that two equations, namely Eq.(12) and Eq.(16) only include terms of $\iota_+(z)$ and $\nu_+(z)$ and their derivatives. Therefore, they are decoupled from the rest. Nonetheless, these two equations prove intractable and we will pursue a different route to finding a solution. For reasons that will become apparent in a moment, we introduce a set of five auxiliary functions,

$$\Gamma(z) = e^{-\frac{3}{2}iy_0(z)} \tag{24}$$

$$\Delta(z) = ie^{-\frac{3}{2}iy_0(z)}\mu_\pm(z) \tag{25}$$

$$\Theta(z) = e^{-\frac{2}{3}iy_0(z)}(-\iota_\pm(z)\mu_\pm(z) + i\nu_\pm(z)) \tag{26}$$

$$\Pi(z) = e^{-\frac{2}{3}iy_0(z)}(e^{iy_0(z)-\frac{1}{2}i\iota_0(z)} - \mu_\pm(z)^2) \tag{27}$$

$$\Sigma(z) = e^{-\frac{2}{3}iy_0(z)}(i\iota_\pm(z)\Pi(z) - \mu_\pm(z)\nu_\pm(z)) \tag{28}$$

These equations can be further decoupled,

$$\begin{pmatrix} \Theta'(z) \\ \Delta'(z) \\ \Gamma'(z) \end{pmatrix} = iH \begin{pmatrix} \Theta(z) \\ \Delta(z) \\ \Gamma(z) \end{pmatrix} \tag{29}$$

$$\begin{pmatrix} \Sigma'(z) \\ \Pi'(z) \\ \Delta'(z) \end{pmatrix} = iH \begin{pmatrix} \Sigma(z) \\ \Pi(z) \\ \Delta(z) \end{pmatrix} \tag{30}$$

via the identity,

$$g_{01}(z)\Theta(z) + g_{12}(z)\Gamma(z) = g_{02}(z)\Sigma(z) + g_{12}(z)\Pi(z), \quad (31)$$

which is a direct consequence of $\mu_+(z) = \mu_-(z)$. These two sets of differential equations can be readily solved with an additional set of initial values,

$$\Theta(z) = 0, \quad \Delta(z) = 0, \quad \Sigma(z) = 0, \quad (32)$$
$$\Gamma(z) = 1, \quad \Pi(z) = 1. \quad (33)$$

The two coupled differential sets, Eq. (29) and Eq.(30), in conjunction the initial values, Eq. (32) and Eq.(33), unequivocally determine the five auxiliary functions that return,

$$\mu_\pm(z) = \frac{-i\Delta(z)}{\Gamma(z)} \quad (34)$$
$$\iota_\pm(z) = \frac{i(-\Gamma(z)\Sigma(z) + \Delta(z)\Theta(z))}{\Delta(z)^2 - \Gamma(z)\Pi(z)} \quad (35)$$
$$\nu_\pm(z) = \frac{-i(\Delta(z)\Sigma(\zeta) - \Pi(z)\Theta(z))}{\Delta(z)^2 - \Gamma(z)\Pi(z)} \quad (36)$$
$$y_0(z) = \frac{3}{2} i \log(\Gamma(z)) \quad (37)$$
$$\iota_0(z) = 2i \log(\frac{\Gamma(z)\Pi(z) - \Delta(z)^2}{\sqrt{\Gamma(z)}}) \quad (38)$$

Note that the phase functions $y_0(z)$ and $i_0(z)$ are of logarithmical nature and the rest are quotients of the products of the solution basis.

3. TRIMER WITH CONSTANT COUPLINGS

While we have provided a formal solution to propagation through the optical trimer, considering a specific solution may help build further intuition. Thus, in this section, we will discuss the special case where all the couplings are independent of the propagation distance. For the sake of simplicity, we will introduce the dimensionless propagation parameter $\zeta = g_{01}z$, such that the mode-coupling differential equation becomes

$$-i\partial_\zeta |\mathcal{E}(\zeta)\rangle = \hat{H}|\mathcal{E}(\zeta)\rangle, \quad (39)$$

with the mode-coupling matrix,

$$\hat{H} = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}, \quad (40)$$

given in terms of the dimensionless parameters,

$$\alpha = \frac{g_{02}}{g_{01}}, \quad \beta = \frac{g_{12}}{g_{01}}. \quad (41)$$

Under this constant mode-coupling matrix, we could use the results presented in the past section to build a particular solution, but it is well known that a set of linear first order differential equations is equivalent to a single linear differential equation of higher order. It seems worthwhile deriving such an higher order differential equation for $\Delta(\zeta)$, which is the auxiliary function that connects the two sets of differential equations, Eq.(29) and Eq.(30). After some algebra, it is possible to write,

$$\Delta'''(\zeta) + i(1 + \alpha^2 + \beta^2)\Delta'(\zeta) - 2\alpha\beta\Delta(\zeta) = 0, \quad (42)$$

with initial values,

$$\Delta(0) = 0, \quad (43)$$
$$\Delta'(0) = i\beta, \quad (44)$$
$$\Delta''(0) = -\alpha. \quad (45)$$

It is easy to see that $\Delta(\zeta)$ has the following solution,

$$\Delta(\zeta) = \delta_1 e^{i\gamma_1\zeta} + \delta_2 e^{i\gamma_2\zeta} + \delta_3 e^{i\gamma_3\zeta}, \quad (46)$$

where constant parameters γ_j are the eigenvalues of the mode-coupling matrix determined by the characteristic polynomial, a reduced cubic,

$$\gamma_j^3 - (1 + \alpha^2 + \beta^2)\gamma_j - 2\alpha\beta = 0. \quad (47)$$

It is straightforward to notice that there are three different real eigenvalues for real, positive, non-zero coupling parameters, $\alpha, \beta > 0$. These proper values can be writen in a closed but non-compact form, so we will not write them explicitly. Furthermore, the coefficients are given by

$$\delta_1 = \frac{\alpha - \beta(\gamma_2 + \gamma_3)}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad (48)$$
$$\delta_2 = \frac{\alpha - \beta(\gamma_1 + \gamma_3)}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}, \quad (49)$$
$$\delta_3 = \frac{\alpha - \beta(\gamma_1 + \gamma_2)}{(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}. \quad (50)$$

The remaining auxiliary functions are straightforward to calculate,

$$\Theta(\zeta) = \frac{\beta(1+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha(1-\beta^2)} \quad (51)$$
$$\Gamma(\zeta) = \frac{-(1+\beta^2)\Delta(\zeta) + i\alpha\beta\Delta'(\zeta) - \Delta''(\zeta)}{\alpha(1-\beta^2)} \quad (52)$$
$$\Sigma(\zeta) = \frac{\beta(\alpha^2+\beta^2)\Delta(\zeta) - i\alpha\Delta'(\zeta) + \beta\Delta''(\zeta)}{\alpha^2 - \beta^2} \quad (53)$$
$$\Pi(\zeta) = \frac{-\alpha(\alpha^2+\beta^2)\Delta(\zeta) + i\beta\Delta'(\zeta) - \alpha\Delta''(\zeta)}{\alpha^2 - \beta^2}. \quad (54)$$

Thus, the propagator functions, $\iota_\pm(z)$, $\mu_\pm(z)$, $\nu_\pm(z)$, $\iota_0(z)$ and $y_0(z)$, will effectively contain terms involving the three eigenvalues as well as sums and differences thereof.

4. APPLICATIONS

Let us present some practical examples of optical trimers that may have a feasible experimental realization.

A. Identical couplings and the discrete Fourier transform

The mode-coupling matrix for three-identical waveguides distributed in an equilateral triangle configuration, $\alpha = \beta = 1$,

$$\hat{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (55)$$

is related to the cyclic group in dimension three,

$$\hat{H} = \hat{Z}_3 + \hat{Z}_3^2, \quad (56)$$

where the generator of the cyclic group,

$$\hat{Z}_3 = \hat{I}_+ + \hat{U}_+ + \hat{V}_-, = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (57)$$

It is well known that the cyclic group is diagonalized by the discrete Fourier transform, $\hat{\Lambda} = \hat{F}_n \hat{Z}_n \hat{F}_n^\dagger$, where the discrete Fourier transform of rank n is given by the operator $\hat{F}_n$, in the case of $n = 3$,

$$\hat{F}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}, \quad (58)$$

and $\hat{\Lambda}$ is a diagonal rank n matrix containing the roots of unity, $\hat{\Lambda}_{mn} = \delta_{m,n} e^{i\frac{2\pi}{n}i}$ with $m, n = 0, 1, 2$. In this particular case, it is possible to compose a propagator,

$$U(\zeta) = \hat{F}_3^\dagger e^{i\hat{\Lambda}_3 \zeta} e^{i\hat{\Lambda}_3^2 \zeta} \hat{F}_3, = \frac{1}{3} \begin{pmatrix} 2+e^{3i\zeta} & -1+e^{3i\zeta} & -1+e^{3i\zeta} \\ -1+e^{3i\zeta} & 2+e^{3i\zeta} & -1+e^{3i\zeta} \\ -1+e^{3i\zeta} & -1+e^{3i\zeta} & 2+e^{3i\zeta} \end{pmatrix} e^{-i\zeta} \quad (59)$$

where we have used the fact that the elements of the cyclic group of rank 3 commute between them, $[\hat{Z}_3, \hat{Z}_3^2] = 0$ because $\hat{Z}_3^3 = \mathbb{1}_3$.

Figure 1 shows the trajectories described by the absolute value of the field amplitudes, $|\mathcal{E}_j(z)|$, as they propagate. All of trajectories will lie over the surface of an octant of the sphere due to unitary propagation. Figure 1(a) shows the response to impulses, $\mathcal{E}_j = \delta_{j,k}$ with $j = 0, 1, 2$ and a fixed $k = 0, 1, 2$. Figure 1(b) shows the trajectories given by initial field superpositions of the more general form: $\mathcal{E}_j = \alpha_j e^{i\phi t}$ with $\alpha \in \mathbb{R}$ and $\sum_j |\alpha_j|^2 = 1$. From the propagator, it is possible to see that only two commensurate frequencies are involved in the propagation of initial fields, thus, the trajectories will be closed and well defined.

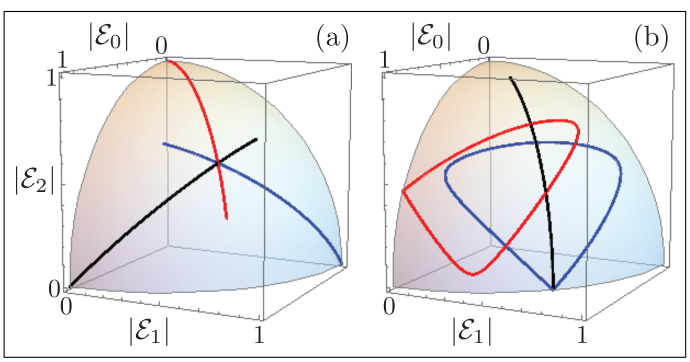

Fig. 1. (Color online) Absolute amplitude trajectories, $(|E_0(z)|, |E_1(z)|, |E_2(z)|)$, for initial fields, $(|E_0(0)|, |E_1(0)|, |E_2(0)|)$, impinging (a) only the zeroth (black), $(1, 0, 0)$, first (blue), $(0, 1, 0)$, and second (red), $(0, 0, 1)$, waveguides and (b) initial fields impinging two waveguides at a time with and without a relative phase, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ (black), $(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0)$ (blue), and $(\frac{2}{\sqrt{5}}, 0, \frac{i}{\sqrt{5}})$ (red).

B. Two identical couplings and the golden ratio

In the case of two equal coupling parameters the unitless Hamiltonian becomes

$$H = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix} \quad (60)$$

Note that the Hamiltonian is $\hat{Z}_2$-invariante, i.e. it is invariant under xchanging the first and the second waveguide. This symmetry also is reflected by the eigenvalues which are $\{-1, \bar{\varphi}, \varphi\}$ with

$$\bar{\varphi} = \frac{1}{2}\left(1 - \sqrt{8\alpha^2 + 1}\right), \quad (61)$$

$$\varphi = \frac{1}{2}\left(1 + \sqrt{8\alpha^2 + 1}\right). \quad (62)$$

It is simple to see that $\bar{\varphi} + \varphi = 1$ and the second and third eigenvalue do transform into each other upon applying the complex conjugate. Note that the latter eigenvalue assumes the golden ratio for $\alpha = \sqrt{1/2}$. Due to the $\hat{Z}_2$-symmetry the propagator can be calculated directly from the Hamiltonian and can be written as:

$$U(\zeta) = \frac{1}{\varphi - \bar{\varphi}} \begin{pmatrix} f(\zeta) & g(\zeta) & h(\zeta) \\ g(\zeta) & f(\zeta) & h(\zeta) \\ h(\zeta) & h(\zeta) & i(\zeta) \end{pmatrix} \quad (63)$$

with

$$f(\zeta) = \frac{\varphi}{2} e^{i\zeta\varphi} - \frac{\bar{\varphi}}{2} e^{i\zeta\bar{\varphi}} + \frac{1}{2}(\varphi - \bar{\varphi}) e^{-i\zeta}, \quad (64)$$

$$g(\zeta) = \frac{\varphi}{2} e^{i\zeta\varphi} - \frac{\bar{\varphi}}{2} e^{i\zeta\bar{\varphi}} - \frac{1}{2}(\varphi - \bar{\varphi}) e^{-i\zeta}, \quad (65)$$

$$h(\zeta) = \alpha\left(e^{i\zeta\varphi} - e^{i\zeta\bar{\varphi}}\right), \quad (66)$$

$$i(\zeta) = \varphi e^{i\zeta\bar{\varphi}} - \bar{\varphi} e^{i\zeta\varphi} \quad (67)$$

In operator form the propagator yields

$$U(\zeta) = \frac{g(\zeta)}{\varphi - \bar{\varphi}}(I_+ + I_-) + \frac{h(\zeta)}{\varphi - \bar{\varphi}}(U_+ + U_- + V_+ + V_-) + \frac{2}{3}\frac{f(\zeta) - i(\zeta)}{\varphi - \bar{\varphi}}(U_0 + V_0) + \frac{1}{3}\frac{i(\zeta) + 2f(\zeta)}{\varphi - \bar{\varphi}}\mathbb{1} \quad (68)$$

Obviously the three eigenvalues are stationary point of the system. However,the particular symmetry of the system might allow for further interesting states.
Looking at the propagator functions (64) to (67) reveals that two of the equations, namely $h(\zeta)$ and $i(\zeta)$ depend on only two of the three eigenfrequencies, φ and $\bar{\varphi}$. Equation (63) further reveals that the evolution of the third component only depends on those two equations $h(\zeta)$ and $i(\zeta)$. Now one can hope that if we chose the initial eigenstate accordingly one of the two eigenfrequencies in the evolution of the third component will cancel out. That is, if the evolution only depends on one frequency, it will only change in phase and the amplitude will remain immutable. Hence, lets look for such a state where no energy transfer into or out of the third component occurs. Without loss of generality we can assume that the state-vector at a given time bears following form:

$$\mathcal{E}(\zeta = 0) = \begin{pmatrix} 0 \\ \mathcal{E}_2(0) \\ \mathcal{E}_3(0) \end{pmatrix} \quad (69)$$

Which implies that the third component evolves as:

$$(\varphi - \bar{\varphi})\mathcal{E}_3(\zeta) = h(\zeta)\mathcal{E}_2(0) + i(\zeta)\mathcal{E}_3(0) \tag{70}$$

As argued above the third waveguide will not change in amplitude if one of the two eigenfrequencies cancel out. This is fulfilled if either of the two following conditions is met:

$$\mathcal{E}_2(0) = \mathcal{E}_3(0)\frac{\varphi}{\alpha} \quad or \quad \mathcal{E}_2(0) = \mathcal{E}_3(0)\frac{\bar{\varphi}}{\alpha} \tag{71}$$

Consequently we obtain two solutions for a state with stationary amplitude in the third waveguide. Without loss of generality we can normalise the initial state such that the amplitude of the third component is one, ie $\mathcal{E}_3(0) = 1$. Then the time evolution of the system is given by:

$$\mathcal{E}_{\varphi}(\zeta) = \begin{pmatrix} \frac{g(\zeta)}{\varphi-\bar{\varphi}}\frac{\varphi}{\alpha} + \frac{h(\zeta)}{\varphi-\bar{\varphi}} \\ \frac{f(\zeta)}{\varphi-\bar{\varphi}}\frac{\varphi}{\alpha} + \frac{h(\zeta)}{\varphi-\bar{\varphi}} \\ e^{i\zeta\varphi} \end{pmatrix} \tag{72}$$

and

$$\mathcal{E}_{\bar{\varphi}}(\zeta) = \begin{pmatrix} \frac{g(\zeta)}{\varphi-\bar{\varphi}}\frac{\bar{\varphi}}{\alpha} + \frac{h(\zeta)}{\varphi-\bar{\varphi}} \\ \frac{f(\zeta)}{\varphi-\bar{\varphi}}\frac{\bar{\varphi}}{\alpha} + \frac{h(\zeta)}{\varphi-\bar{\varphi}} \\ e^{i\zeta\bar{\varphi}} \end{pmatrix} \tag{73}$$

5. CONCLUSION

We have shown that it is possible to solve the light evolution equations in a coupled three-core waveguide, in terms of the Lie group generators of su(3). We focused our attention on a reduced class of structures where the coupling constants are constant in position......... As an example we consider the case of equal coupling constants. we found that the dynamics of such a system i governed by a linear third order differential equation and the corresponding five auxiliary functions can be expressed in term of its solution. we also established a connection between a waveguide cluster with equal couplings and the well known Fourier transform, possibly opening a path to realise quantum Fourier transformations. furthermore we studies an isosceles waveguide triangle. We showed that two equal couplings corresponds to a Z2 symmetry of the Hamiltonian. We also showed that the symmetry allows for two interesting stated characterised by absence of energy transfer into the third core.