CAVITY QED

(intro).

INTRODUCTION TO CAVITY QED

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$$(i\hbar \not{\partial} - m_e c)\psi = 0$$

DIRAC EQ (Q MECHANICS + RELATIVITY)
(among others) Light-matter interaction

$$\Downarrow$$

$$H = \frac{1}{2m_e}(\bar{p} + q\bar{A})^2 - e\phi + \frac{\mu_B}{\hbar}\vec{\sigma}\cdot\vec{B}$$

Pauli Hamiltonian (non-relativistic limit)

$$\Downarrow$$

dipole $\leftrightarrow$ E.M

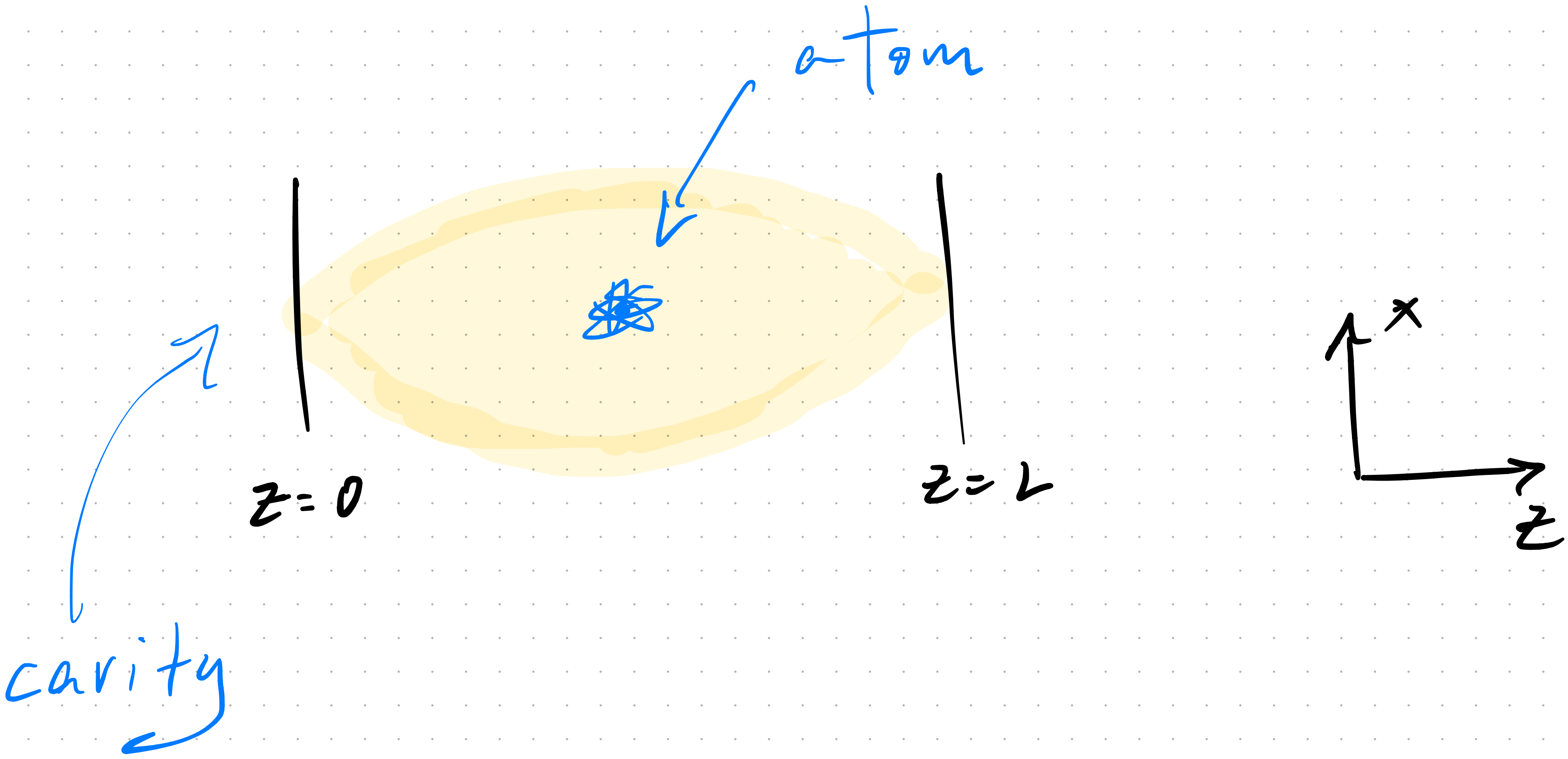
$$H = \hat{d}\cdot\hat{E}$$

Exercise(s): write the Pauli-matrices

show $\sigma^x = \sigma^+ + \sigma^-$

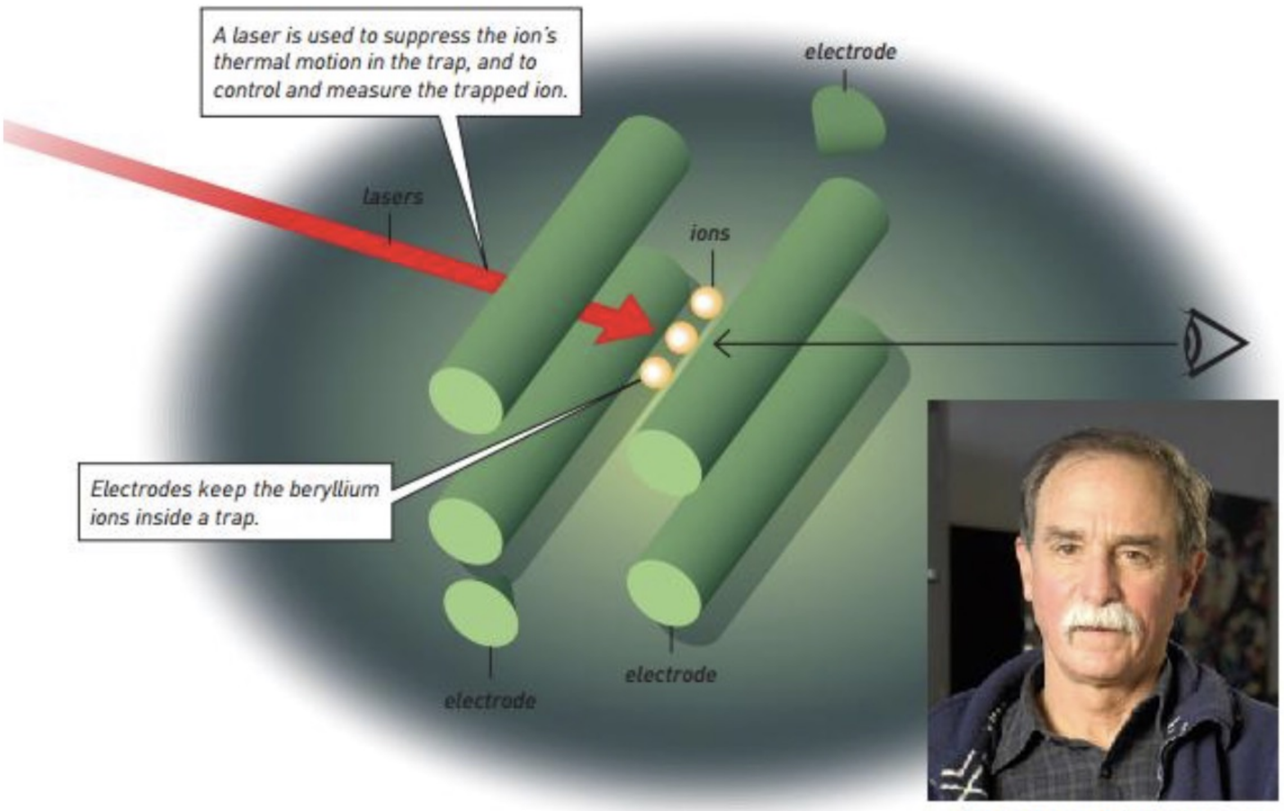
Relate σ^z & $\sigma^+\sigma^-$

CAVITY QED



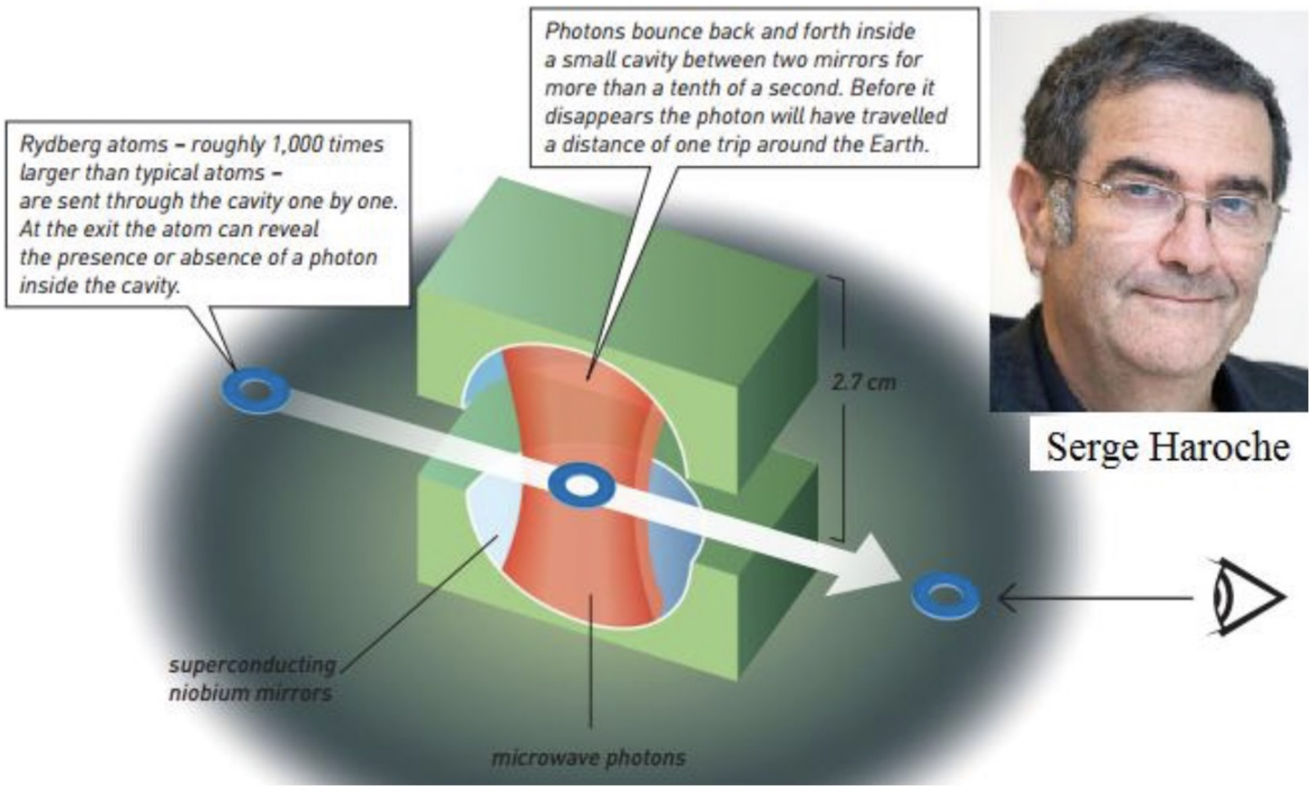
- light-matter coupling weak.
  → use a cavity
- light-matter entanglement
- Qubit ↔ Qubit interaction
  → quantum architecture.

2012 Nobel Prize



David J. Wineland

Wineland

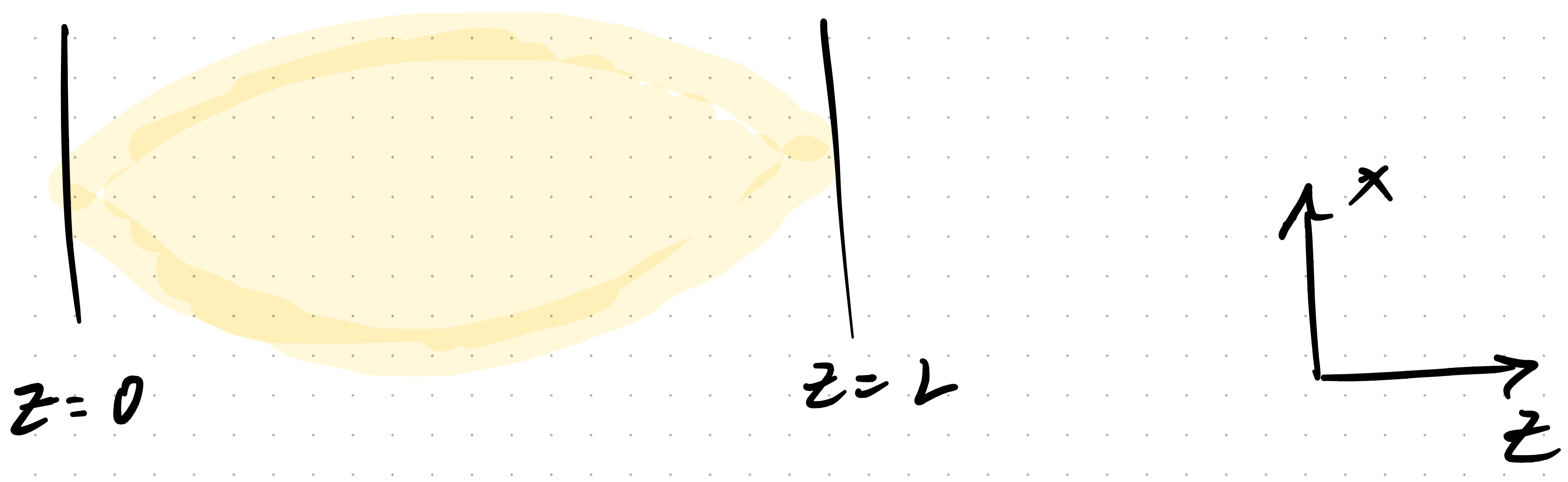


Serge Haroche

Haroche.

QUANTUM THEORY OF A CAVITY

cQED 1



* Maxwell EQs (coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$)

solution.

$$(\partial_t^2 - c^2\nabla^2) A = 0 \longrightarrow A \sim f(z)\, q(t) \qquad (1\text{-}1)$$

* cavity $\leftrightarrow$ MIRRORS $\leftrightarrow$ BOUNDARY CONDITIONS.

$$E(0) = E(L) = 0 \qquad (1\text{-}2)$$

* $E = -\dfrac{\partial A}{\partial t}$; $B = \vec{\nabla}\times\vec{A}$

set $\vec{A} = N\, f(z)\, q(t)\, \hat{x}$

$$\vec{E} = -N\, f(z)\, \dot{q}(t)\, \hat{x}$$

$$\vec{B} = N\, q(t) \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & 0 & 0 \end{vmatrix} = N\, q(t)\, f'(z)\, \hat{y}$$

$\hat{x}, \hat{y}, \hat{z} \equiv$ unit vectors.

* With the Boundary conditions.

$$f(z) = \sin kz \qquad k = \frac{\pi}{L} n \qquad n \in \mathbb{Z}$$

$$f'(z) = k \cos kz.$$

EXERCISE

SHOW THAT $A \sim f \cdot q$ with $f = \sin kz$, $q = e^{i\omega t}$

IS SOLUTION OF (1-1) AND SATISFIES THE BOUNDARY CONDITIONS (1-2)

THEREFORE ($N \equiv$ normalization cte.)

$$E = -N \sum_n \sin k_n z \, \dot{q}_n(t) \, \hat{x}$$

$$B = N \sum_n k_n \cos k_n z \, q_n \, \hat{y}$$

ENERGY

$$E = \frac{1}{2} \int \varepsilon_0 E^2 + \frac{1}{\mu} B^2 \, dz$$

$$= \frac{N^2}{2} \sum_n \varepsilon_0 \frac{L}{2} \dot{q}_n^2 + \frac{L}{2\mu_0} k^2 q_n^2$$

$$= N^2 \frac{\varepsilon_0 L}{2} \sum_n \dot{q}_n^2 + \underbrace{\frac{1}{\varepsilon_0 \mu_0}}_{c^2} k^2 q_n^2$$

Sum of HARMONIC OSCILATORS

with

$$\boxed{\omega_n = c k = \frac{c \cdot \pi}{L} n}$$

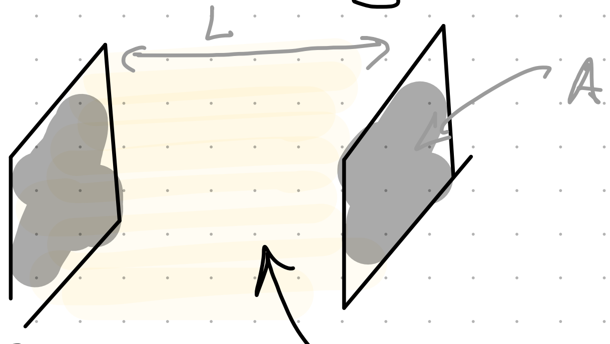
THEREFORE, A CAVITY

$$H = \sum \omega_n a_n^\dagger a_n$$

with $\omega_n = \frac{c \pi}{L} n \qquad n = 1, 2, 3, \ldots$

AN ATOM IN A CAVITY

This is a rather general argument:



Volumen $= L \times A$

field inside the cavity

SINGLE MODE CAVITY

$H = \omega a^{+} a$ with $\omega_c = c \frac{\pi}{L}$

$$\vec{E} = \vec{E}_{rms}(\bar{r})\,(a^{+} + a)$$

EXERCISE: check that

$$\langle 0 | \vec{E} \cdot \vec{E} | 0 \rangle = |\vec{E}_{rms}|^2$$

* ZERO POINT ENERGY:

$$\frac{1}{2}\hbar\omega = \frac{1}{2}\int_V dr^3\, \varepsilon_0 E_{rms}^2 + \frac{1}{\mu_0} B_{rms}^2$$

$$= \int dr^3\, \varepsilon_0 E_{rms}^2 = \frac{1}{2} V \varepsilon_0 E_{rms}^2$$

$V = A \cdot L = c\pi/\omega_c$

$$E_{rms}^2 = \frac{\hbar \omega_c}{V \varepsilon_0} = \frac{\hbar \omega_c^2}{A c \varepsilon_0 \pi}$$

* COUPLING STRENGHT:

$$H = \hat{d} \cdot \hat{E} \qquad \text{(see LECTURE NOTES)}$$

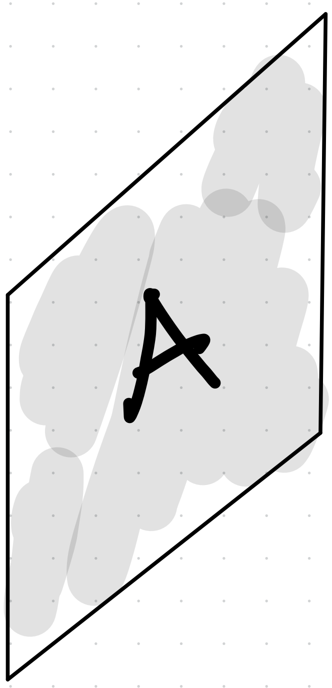
$$= \underbrace{e\,|d|\,E_{rms}}_{\text{light-matter coupling.}}\ \sigma^x \left(a^\dagger + a\right)$$

$$H = \hbar g\, \sigma^x \left(a^\dagger + a\right)$$

with

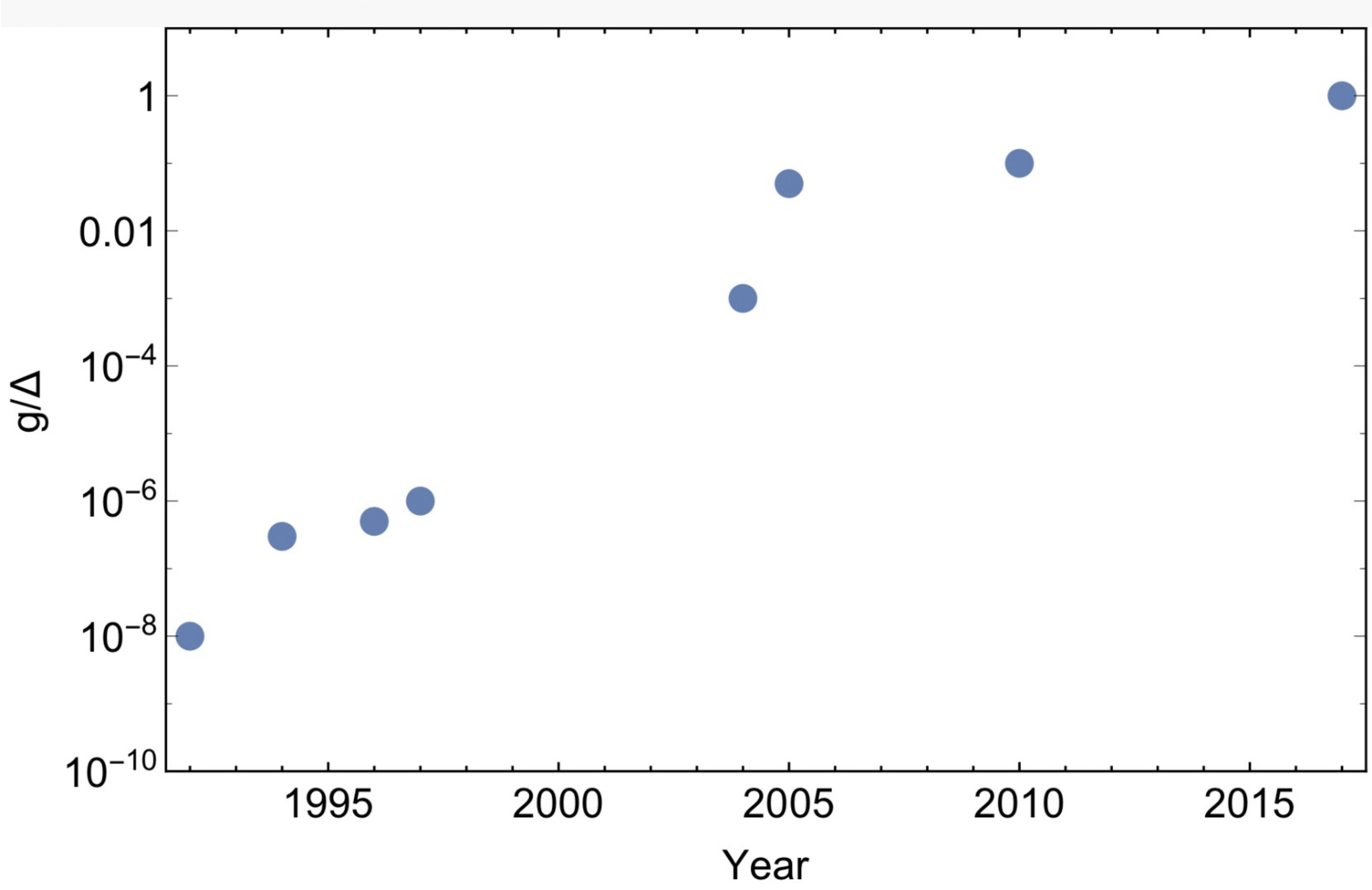
$$g = e\,|d|\,E_{rms} = \frac{|d|}{\sqrt{A}} \sqrt{\frac{e^2}{\hbar c\, \varepsilon_0 \pi}}\ \omega_c$$

THEREFORE



versus r_{atom} $\Rightarrow$ $\frac{g}{\omega_c}$ "SMALL"

* SOME EXAMPLES:



CQED 7

THE QUANTUM RABI MODEL

$$H_{qR} = \underbrace{-\frac{\Delta}{2}\sigma^z}_{\Delta\sigma^+\sigma^- = \Delta\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}} + \omega_c a^\dagger a + g\,\sigma^x (a^\dagger + a)$$

understanding the q-Rabi:

set $g = 0$ & $\Delta = \omega_c$

@ RESONANCE

$\#$excitations $= n_{at} + n_{ph}$

$\vdots$

$\#2$ $\{|0;2\rangle, |1;1\rangle\}$

$\#1$ $\{|0;1\rangle, |1;0\rangle\}$

$\#0$ $|0;0\rangle$

Energy levels

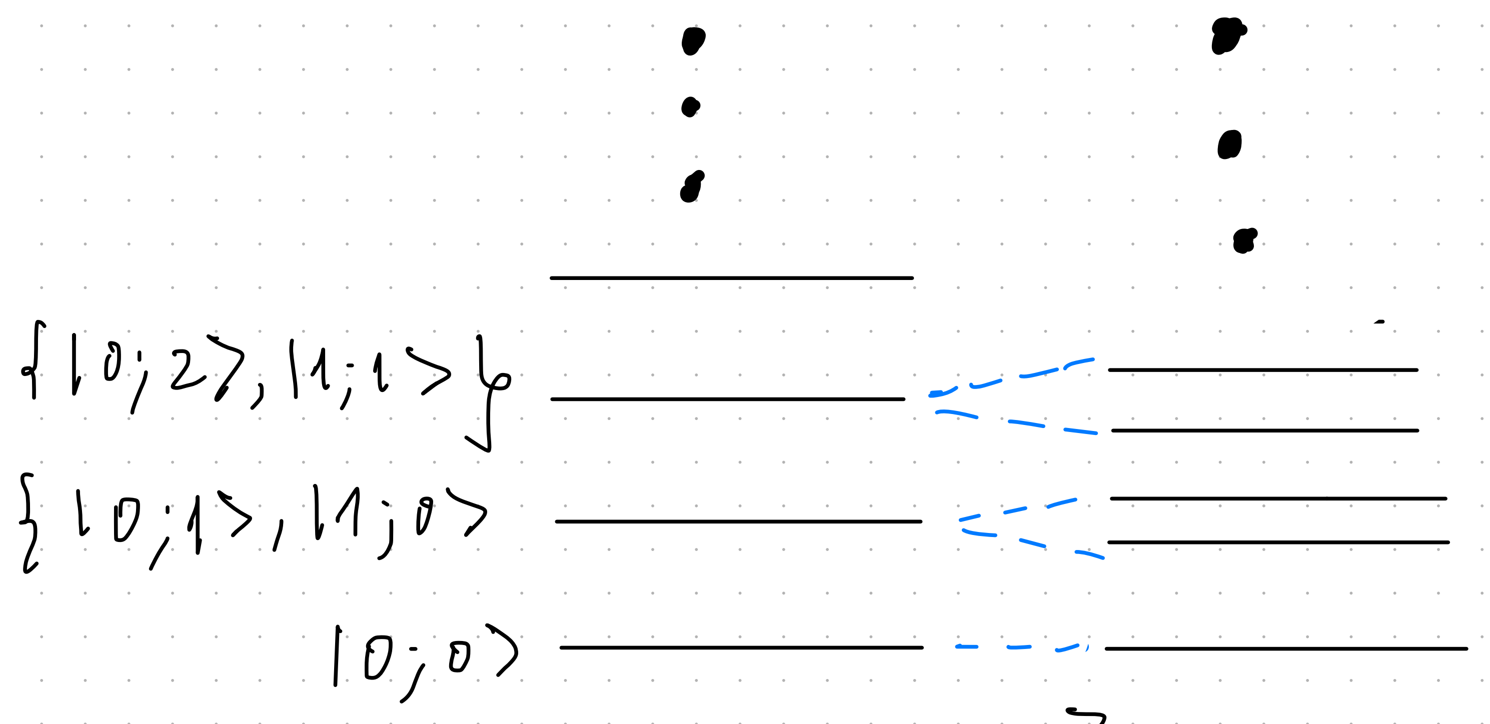
"local" basis

$|n_{atom} ; n_{photons}\rangle$

$n_{atom} = 0, 1$; $n_{photons} = 0, 1, 2, 3 \ldots$

EXERCISE: Relate σ^z & $\sigma^+\sigma^-$

$g=0$ & RESONANCE $\qquad$ $g \neq 0$ (small)



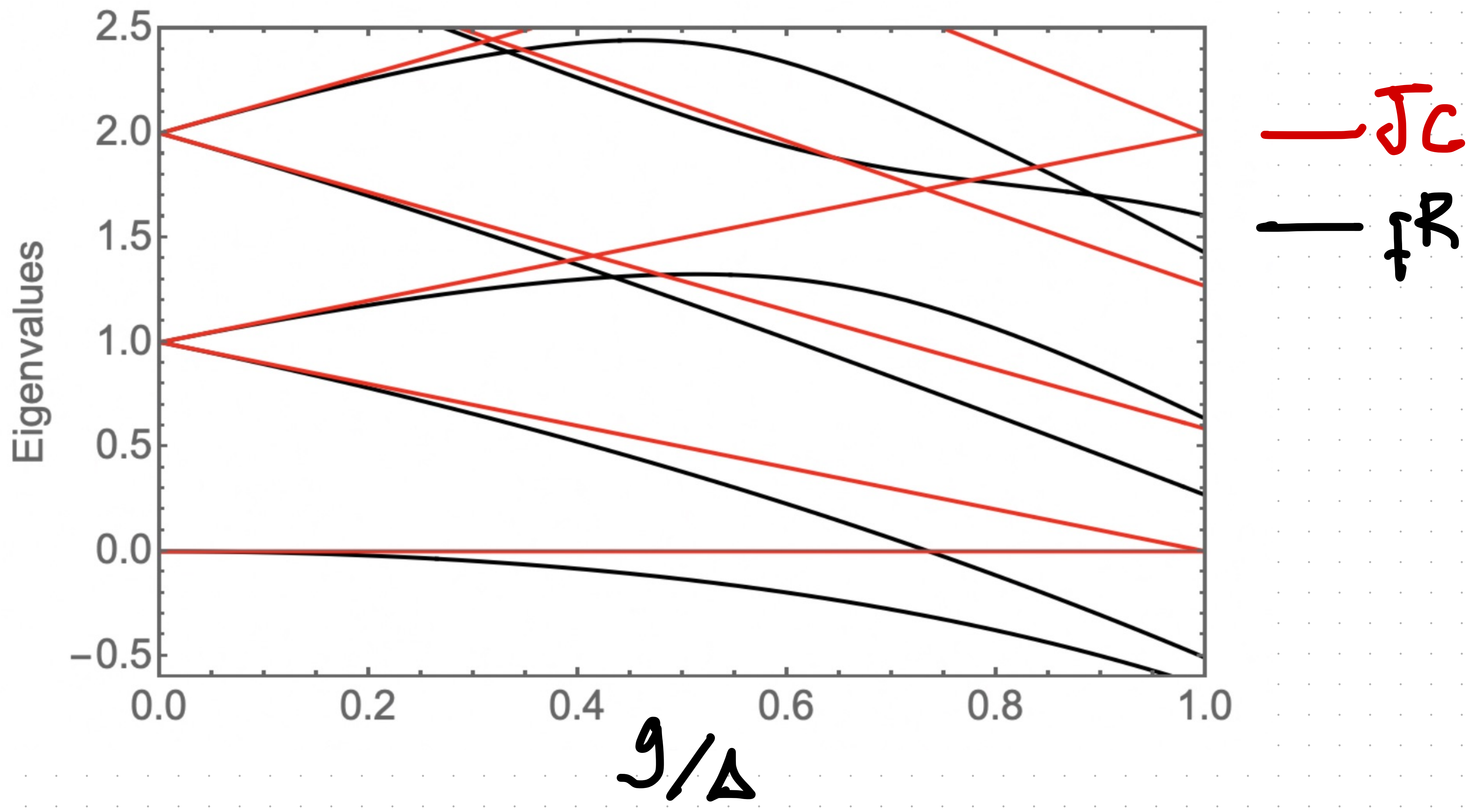
theory of perturbations

ONLY NUMBER CONSERVING

$\Rightarrow$ THUS, USING $\sigma^x = \sigma^+ + \sigma^-$

$$H_{JC} = -\frac{\Delta}{2}\sigma^z + \omega_c a^+ a + g\left(\sigma^+ a + \sigma^- a^+ + \cancel{\sigma^+ a^+ + \sigma^- a}\right)$$

JC model $\qquad$ RWA APPROX.



JC PHYSICS

The RWA brings number conservation. This allows to solve the H_{JC} exactly:

1) Consider the basis

$$\mathcal{H} = \mathcal{H}_{atom} \otimes \mathcal{H}_{phot}$$

$$\{ |0, n\rangle , |1, n\rangle \} \rightarrow dim = 2 \times N_{ph.}$$

atom number   photon number

2) split in a direct sum

$$\mathcal{H} = \bigoplus_{n=1} H_n$$

$$H_n = span\{ |0, n\rangle , |1, n-1\rangle \}$$

$$\hat{n} := \sigma^+ \sigma^- + a^+ a \Rightarrow \hat{n}|0,n\rangle = n|0,n\rangle$$

$$\hat{n}|1,n-1\rangle = n|1,n-1\rangle$$

3) EXERCISE, show that

$$H_{JC} = \begin{pmatrix} \boxed{\cdot} & & 0 \\ & \boxed{\cdot\ \cdot} & \\ 0 & & \boxed{\cdot\ \cdot} \end{pmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ \vdots \end{array}$$

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4) Each of this 2×2 block

$$
\begin{array}{c}
|0\, n\rangle \\ |1, n-1\rangle
\end{array}
\begin{pmatrix}
\omega_c n & g\sqrt{n} \\
g\sqrt{n} & \omega_c n + \underbrace{\Delta - \omega_c}_{\delta}
\end{pmatrix}
$$

* Eigenvalues: $E_{n\pm} = \omega_c \cdot n + \frac{1}{2}\left(\delta \pm \underbrace{\sqrt{4g^2 n + \delta^2}}_{=\Omega_n}\right)$

* Eigenvectors: $|\psi_-\rangle = (\delta + \Omega, -2g\sqrt{n})$

$|\psi_+\rangle = (\delta - \Omega, -2g\sqrt{n})$

therefore $|\psi_-\rangle = \cos\theta_{n/2}|0\, n\rangle + \sin\theta_{n/2}|1\; n-1\rangle$

$|\psi_+\rangle = -\sin\theta_{n/2}|0\, n\rangle + \cos\theta_{n/2}|1\; n-1\rangle$

with $\theta_n = \tan^{-1}\left(\frac{g\sqrt{n}}{\delta}\right)$

EXERCISE: Do the calculations of this page.

$|\psi_{n\mp}\rangle$ POLARITONS
!!!
entangled light-matter
states. A LOT OF PHYSICS!

• EXAMPLE

i) consider the special case $\delta = 0$ (RESONANCE)

ii) initial state $|\psi(0)\rangle = |1,0\rangle$ (atom excited)

↓ dynamics.

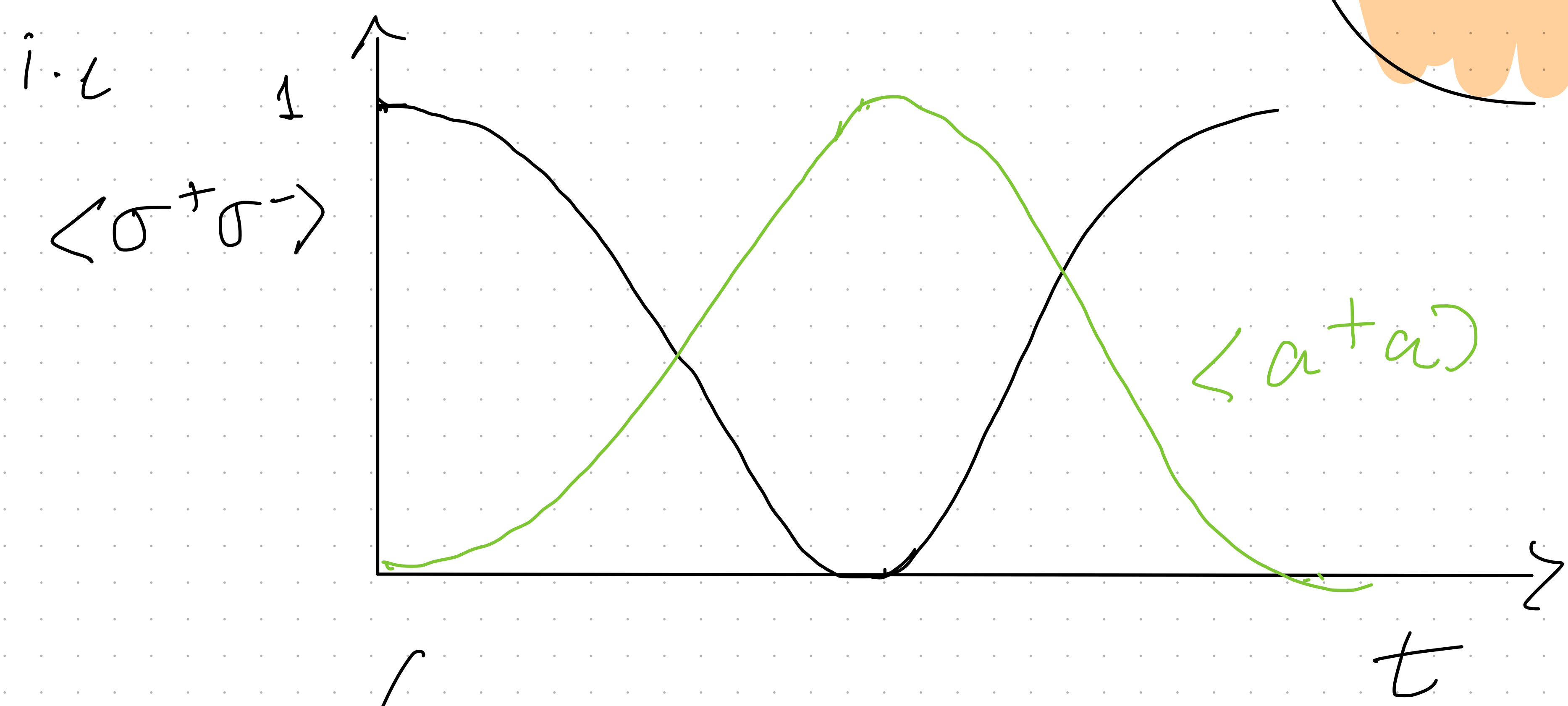
$$|\psi_{1\mp}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle \pm |01\rangle\right) \Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(\psi_+ + \psi_-\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_+t}\Big(\underbrace{\psi_-}_{\frac{1}{\sqrt{2}}|10\rangle+|01\rangle} + \underbrace{\psi_+ e^{i(E_+-E_-)t}}_{\frac{1}{\sqrt{2}}|10\rangle-|01\rangle}\Big)$$

$$= \frac{1}{2}\left(|10\rangle\left(1+e^{i\delta E t}\right) + |01\rangle\left(1-e^{-i\delta E t}\right)\right)$$

$$\Rightarrow \langle\sigma^+\sigma^-\rangle = \frac{1}{4}\left|1+e^{i\delta E t}\right|^2$$

$$= \frac{1}{4}\left(1+1+2\cos\delta E t\right)$$



quantum Rabi oscillations, freq:

$$\Omega = \sqrt{4g^2} = 2g$$

EXERCISE.

start with $|01\rangle$

and $|20\rangle$

How does the Rabi-freq. scales with n?

BEYOND THIS INTRO

* Losses (next class)

* more than 1 atom / 1 cavity.

  $\downarrow$

  QUBIT $\leftrightarrow$ QUBIT GATES.

* BEYOND TLS.

EXERCISES

* Polariton eigenvectors.

The 2x2 block can be rewritten as:

$$\begin{pmatrix} 0 & g\sqrt{n} \\ g\sqrt{n} & \delta \end{pmatrix} + \omega_c n \mathbb{1}$$

$$E_{\mp} = \frac{1}{2}\delta \mp \underbrace{\sqrt{\delta^2 + 4g^2 N}}_{\Omega}$$

Eigenvector: $N(\delta+\Omega, 2g\sqrt{N}) =$
(minimum energy)

$= (A, B) \longrightarrow (\cos\theta/2, \sin\theta/2)$

$\tan\theta/2 = A/B$

use

$$\tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2} = \frac{2A/B}{1-A^2/B^2} = \frac{2AB}{B^2-A^2} = \frac{2(\delta+\Omega)\,2g\sqrt{N}}{\cancel{4g^2N} - \cancel{4g^2N} - \delta^2 - \delta^2 - 2\delta\Omega}$$

$$= \frac{\cancel{2}\cancel{(\delta+\Omega)}\,2g\sqrt{N}}{\cancel{2}\delta\cancel{(\delta+\Omega)}}$$

$$\dot{a} = -i\omega_c a + ig\sigma^-$$

$$\dot{\sigma}^- = -i\Delta\sigma^- +$$

$$\left\{ |0\,0\rangle \xrightarrow{a^+} |0,1\rangle,\ |1,0\rangle \right\}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^+ a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho_{23}$$

$$\langle \sigma^+ a \rangle = \Bigg(\begin{matrix} 0 & 0 & 0 \end{matrix}$$

$$\boxed{\approx \langle a \rangle}$$

$$\langle \sigma^+ a \rangle = \mathrm{Tr}\, \rho\, \sigma^+ a = \rho_{ij} (\sigma^+ a)_{ji} \quad \text{"}\ \rho_{32}$$

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho_{21}$$

$$\sigma^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dot{a} = -i\omega_c a - ig\tau - \kappa a$$

$$\dot{\tau} = -i\Delta\tau - iga - \gamma\tau$$

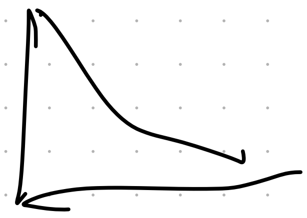
$$\begin{pmatrix} \omega_c - i\kappa & g \\ g & \Delta - i\gamma \end{pmatrix}$$

Set $\omega_c = \Delta$

$$\boxed{4g^2 - (\kappa - \gamma)^2 > 0}$$

$$4c^2 + a^2 + a^2 + 4c^2 - 2a\sqrt{a^2+4c^2}$$

$$2(a^2 + 4c^2) - 2a\sqrt{a^2+4c^2}$$



$$\cancel{2\left(R_n \sqrt{\delta^2 + 4c^2} -\right)}$$

$$2R_n\,(R_n - \delta)$$

define $\Omega = \sqrt{\delta^2 + 4g^2}$

$$(\delta + \Omega\,,\ -2g) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\Omega+\delta}{\Omega}}\ ,\ \right)$$

$$(\delta - \Omega\,,\ -2g)$$

$$N = (\delta+\Omega)^2 + 4g^2 = \delta^2 + 4g^2 + \Omega^2 + 2\delta\Omega$$
$$= 2\Omega^2 + 2\delta\Omega$$
$$= 2\Omega(\Omega+\delta)$$

(A, B) $\quad$ $(\delta + \Omega, -2g)$

$A = \cos \theta/2$

$B = \sin \theta/2$

$$\frac{A}{B} = \tan \theta/2 = \frac{1-\cos\theta}{\sin\theta} =$$

$$= \frac{1}{\sin\theta} - \cot\theta$$

$$\tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2} = \frac{A/B}{1-(A/B)^2} = \frac{A/\cancel{B}\, B^{\cancel{2}}}{B^2-A^2} = \frac{AB}{B^2-A^2}$$

$$4g^2 - (\delta+\Omega)^2 = \cancel{4g^2} - \delta^2 - \cancel{4g^2} - \delta^2 - 2\delta\Omega$$

$$= -\cancel{2}\delta(\delta+\Omega)$$

$$AB = -2g(\delta+\Omega)$$