Quantifier Elimination in tame algebraic extensions of $\mathbb{F}_p((\mathbb{Z}))$

Blaise Boissonneau

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1 Introduction

Tame fields lie at the border of our understanding of henselian valued fields; they are gentle enough to be understandable, but they can be complex enough to give examples of IP & NTP2 fields.

Definition 1.1 (Tame fields)**.** A valued field (K, v) is said to be tame iff it is henselian, of residue characteritic $p > 0$ and defectless, it has value group p-divisible, and it has residue perfect.

In 2001 Brigandt gave QE results for tame algebraic extensions of $\mathbb{Q}_p$ [Brigandt-01]. We build on their results to give QE results in tame algebraic extensions of $\mathbb{F}_p((\mathbb{Z}))$. We first observe that any such field can be written as $K = k((\Gamma))$, with $k/\mathbb{F}_p$ algebraic, and with $\Gamma \subset \mathbb{Q}$ p-divisible.

Our main result is:

Theorem 1.2. *Let K be a tame algebraic extension of $\mathbb{F}_p((\mathbb{Z}))$, then K has QE down to $1\exists$-formulas in $\mathcal{L}_v$.*

It can be refined in several ways, as we will later see.

2 Brigandt's method

Brigandt first finds the right language for tame algebraic extensions of $\mathbb{Q}_p$: extend the one-sorted valued fields language with enough positive existential polynomial predicates such that one can assume common substructures are relatively algebraically closed [Brigandt-01, thm. 3 & cor. 4]. Which predicates one has to include depends on how close the field is to being algebraically closed.

Once we have the language, the core result of the quantifier elemination proof is that given a tame alg. ext. of $\mathbb{Q}_p$, 2 structures elementary equivalent to it and a common (rel. alg. closed) substructure, value groups and residue fields are elementary equivalent over it [Brigandt-01, prop. 5 & 6]. This requires strong properties of residue fields, thus works in the case where the residue is elementary equivalent to an algebraic extension of $\mathbb{F}_p$, but not in general.

Once we have elementary equivalence of residue fields and value groups, we are done. This is thanks to relative subcompleteness results which have been extensively studied by Kuhlmann. Brigandt shows a version working in their specific case and conclude.

3 Our setting

We work in the one-sorted language of valued fields. All the QE results will be achieved in this language augmented with some predicates for positive existential formulas, thus all the isomorphism results will continue to hold.

Proposition 3.1 (prop. 5 & 6 in [Brigandt-01]). *Let T be the complete theory of a given fixed tame algebraic extension of $\mathbb{F}_p((\mathbb{Z}))$. Let $K, L \vDash T$ and let N be a common relatively algebraically closed substructure. Then we have:*

- *$vK \equiv_{vN} vL$ as pure ordered abelian groups,*
- *$Kv \equiv_{Nv} Lv$ as pure fields.*

These are done by Brigandt and their proof works in our setting, since the residue and value group considered are identical.

Proposition 3.2 (thm. 7.3 in [Kuhlmann-14]). *Tame fields are relatively subcomplete, that is, if K, L are tame fields and N is a common defectless subfield such that vL/vN is torsion free and Lv/Nv is separable, then we have:*

$$vK \equiv_{vN} vL \,\&\, Kv \equiv_{Nv} Lv \Rightarrow K \equiv_N L,$$

where the equivalences on the left are in the language of ordered groups and of pure fields, the eqiuvalence on the right is in the language of valued fields.

Wrapping everything together, we get:

Theorem 3.3. *Let T be a complete theory of tame fields. Let $K, L \vDash T$ and let N be a common substructure. Suppose:*

- *$vK \equiv_{vN} vL \,\&\, Kv \equiv_{Nv} Lv$,*

- *The language is augmented with enough positive existential predicates to insure we can take N relatively algebraically closed.*

Then T has QE in this augmented language.

Let $\mathcal{L}_v = \{0, 1, +, -, \cdot, \div, \mathcal{O}\}$ be the language of valued fields – with a predicate for the val ring. Let P_n, R_n and W_n be predicate symbols, to be interpreted as:

- $P_n(x) : \exists y\, x = y^n$,
- $R_n(a_0,\cdots, a_{n-1}) : (\bigwedge_i a_i \in \mathcal{O}) \to \exists y \in \mathcal{O}\, \overline{y^n + a_{n-1}y^{n-1} + \cdots + a_0} = 0$,
- $W_n(a_0,\cdots, a_{n-1}) : \exists y\, y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0$.

Corollary 3.4. *Any tame algebraic extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((\mathbb{Z}))$ have QE in the language $\mathcal{L}_v \cup \{W_n \mid n \in \mathbb{N}\}$.*

If we have furthermore that the residue is p-closed, then we have QE in the language $\mathcal{L}_v \cup \{P_n, R_n \mid n \in \mathbb{N}\}$.

If we have that the residue is algebraically closed, then we have QE in the language $\mathcal{L}_v \cup \{P_n \mid n \in \mathbb{N}\}$.

This follows from [Brigandt-01, thm. 3 & cor. 4].
what if the value group is divisible? Can we get rid of P_n?

4 Next questions

Which of these fields are NIP/NTP2/whatever else?

References

[Brigandt-01] Ingo Brigandt, *Quantifier Elimination in Tame Infinite p-Adic Fields*, The Journal of Symbolic Logic, 2001.

[Kuhlmann-14] Franz-Viktor Kuhlmann, *THE ALGEBRA AND MODEL THEORY OF TAME VALUED FIELDS*, 2014.