Let G be a ~~definable~~ ~~group~~?
No: Let G be!

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1 Vendémiaire 234

Summary This paper is somewhere in between a note and an erratum. We state several chain conditions for definable families of groups in the most general way that we could prove, correcting a mistake that was present in some of our previous articles.

Conventions We write lowercase letters for single elements or finite tuples alike. We write "$\subset$" for subsets – proper or not. We write *we* for *all the mistakes I have done in my life, my terrible regrets and I.*

Context

يا دلع يا دلع
قول للقلوب الحيارى
والناس السهرى
اسمع

Today is the fall equinox. It is therefore the first day of the year in the french revolutionnary calendar, so, the 1st Vendémiaire 234. It is also, by quite a coïncidence, the first day of the year in the jewish calendar, so today is also the 1st Tishrei 5786.

Today is also the day of the official recognition of the Palestinian state by France, after years of empty promisses, and while still arming war criminals. It's the bare minimum, but still something to rejoice about. Today is the first day of all of this, and many more.

Today is not the first day we discuss chain conditions, we already wrote a paper on those. But today, we start by stating the classical result establishing that, under assumption of stability, a decreasing chain of sets cannot shrink forever. We do not know whether the Palestinian territory will shrink forever, or whether a stable situation can be reached. We can only hope.

تمشّيتوا على القمر
طيب شو بدكم بعد بتتمشّوا بحيّنا
شو بدكم منا
شوي انسونا
منشان الله، فلتونا!

Chain conditions and dividing lines

The most famous chain condition on definable families of groups is the Baldwin-Saxl condition, usually stated as follow:

Let G be a definable group in a monster model M of a complete theory T, and let $(H_a)_{a\in M}$ be a uniformly definable family of subgroups. If T is NIP, Then there is a finite N such that for any finite set $B \subset M$, there is $B_0 \subset B$ of cardinality $\leqslant N$ such that $\bigcap_{a\in B} H_a = \bigcap_{a\in B_0} H_a$.

Furthermore, if T is stable, it also works for infinite B.

As a quick history: this stems from [1], the 1973 paper by Baldwin and Saxl, where it doesn't appear. It is however stated there (Theorem 3.2) that "If a locally nilpotent group is stable then it is solvable". The techniques of the proof can easily be extended to prove what is now called "Baldwin-Saxl's chain condition".

I believe that Poizat is the first one to phrase this condition as a story of intersections of definable families in stable groups, see [5, Proposition 4] but I also believe it must have been stated in that way beforehand, since Poizat quotes this as if it was obviously the statement.

Anyway, back to our discussion. Let's contrapose Baldwin-Saxl: suppose we find some definable family which fails this chain condition. Then T has IP. So, there is a formula shattering an infinite set. Where is this formula? It's not a very hard question to answer: it's the formula defining the family failing the chain condition. This means that we can localize this argument, that is, instead of assuming that T is NIP, we can assume only one formula is NIP. On another hand, we can also dissect the proof and strip all the non-necessary assumptions on G. It is often claimed that G just needs to be a type-definable group for the condition to hold. But it doesn't need to be. To be precise, it doesn't have to be type-definable nor to a be a group.

While we're here in the limbo between the blabering and the maths (also called more blaering), let's fix the setup: $\mathcal{L}$ is a first-order language, and M is an $\mathcal{L}$-structure. Formulas will always be partitionned, so let's get that out of the way: When we write $\varphi(z_1, \ldots, z_n)$, we mean that each variable appearing as a free variable in φ belongs to exactly one of the tuples z_i. As a caveat for multisorted-structures, if x is a tuple of variables from various sorts, $M^{|x|}$ will be the set of tuples "of the same mould" as x, that is, $b \in M^{|x|}$ iff for each i, b_i belongs to the same sort as x_i.

Stability

We first prove a weak thing for stable theories. We claim that we could strengthen the argument to some form of local stability, as in, "I don't order a subset of A of size $N+1$ with witnesses from G iff I staisfy Baldwin-Saxl for N", but we are too lazy to do this analysis in detail, and we don't really care about stable theories. Sorry! You can prove it yourself.

Proposition 1 (Baldwin-Stable)**.** *Let $\varphi(x, y)$ be a stable formula and fix a set $A \subset M^{|x|}$. Consider the set $H_a = \varphi(a, G)$ for $a \in A$. Let $B \subset A$, potentially infinite; then there is a finite set B_0 such that $\bigcap_{a\in B} H_a = \bigcap_{a\in B_0}$.*

Proof. Suppose not. Take $a_0 \in B$, set $X_0 = H_{a_0}$. Then there must exists $a_1 \in B$ such that $X_0 \cap H_{a_1} \subsetneq X_0$. Take $X_1 = X_0 \cap H_{a_1}$. Construct similarly $(X_i)_{i<\omega}$. Now take $b_i \in X_i \setminus X_{i+1}$. Now $\varphi(a_i, b_j)$ holds iff $i < j$. Oops! $\square$

Isn't this just Baldwin-Saxl? Kindof. First, we did not require G or A to be definable, and second, G is not even a magma! In fact! For this direction, we do not need anything.

A neat consequence of this remark is the existence, in a stable theory (we could even loosen the assumption to some form of locally stable theory where only the formulas defining subgroups of G are required to be stable), of the connected component G^0 for any set G, whether they are groups or not, and whether they are type-definable or not; though you need to specify what kind of formula you consider when you take intersections. Only those defining groups? Only those defining some finite quotient? Underwhich equivalence relation? etc. But given a setup, you can talk about connected components.

NIPity

We say that a partitionned $\mathcal{L}$-formula $\varphi(x,y)$ shatters a set $A \subset M^{|x|}$ with witnesses from $B \subset M^{|y|}$ iff for any $A_0 \subset A$, there is $b \in B$ with $\varphi(A,b) = A_0$.

Let $(G,\cdot)$ be a magma with $G \subset M^{|y|}$. Let $A \subset M^{|x|}$ be a set of parameters such that $H_a = \varphi(a,G)$ is a group under the composition law of the magma $(G,\cdot)$, for any $a \in A$.

We say that $(H_a)_{a\in A}$ has the BS-property with respect to N iff for any $B \subset A$ with cardinality $> N$, there is a $b \in B$ such that:

$$\bigcap_{a\in B} H_a = \bigcap_{a\in B\setminus\{b\}} H_a.$$

An easy iterative argument reduces this condition to "any finite intersection is an intersection of (at most) N sets".

Proposition 2 (Bald-NIP-Saxl)**.** *The formula φ shatters a subset $A_0 \subset A$ of cardinality $N+1$ with witnesses from G iff the family $(H_a)_{a\in A}$ doesn't have the BS-property with respect to N.*

Proof.

⇒: Suppose that the family $(H_a)_{a\in A}$ doesn't have the BS-property with respect to N, so we can find a $B \subset A$ witnessing this failure, that is, $\#B > N$, and we have:

$$\bigcap_{a\in B} H_a \subsetneq \bigcap_{a\in B\setminus\{b\}} H_a$$

for all $b \in B$. We now fix $A_0 \subset B$ of cardinality $N+1$, and prove that φ shatters A_0 with witnesses from G. For any $x \in A_0$, take $b_x \in \bigcap_{a\in B\setminus\{b\}} H_a \setminus H_x$, this is possible. For any $B_0 \subset A_0$, we define $b_{B_0} = \prod_{x\in A_0\setminus B_0} b_x$, where the product denotes the composition law of G (which might not commute; so let us point out that the order doesn't matter). We have $M \vDash \varphi(a, b_{B_0})$ iff $x \in B_0$, so φ indeed shatters A_0 with witnesses in G – in fact, with witnesses in $\bigcup_{a\in A_0} H_a$.

⇐: Suppose that $(H_a)_{a\in A}$ has the BS-property with respect to N, and suppose we can find $a_0,\cdot\cdot,a_N \in A$ and $(b_I)_{I\subset\{0,\cdot\cdot,N\}} \in G$ such that $M \vDash \varphi(b_I, a_i)$ iff $i \in I$. Now by BS, $\bigcap_{0\leqslant i\leqslant N} H_i = \bigcap_{0\leqslant i<N} H_i$ (maybe reindexing it). But now, let $b = b_{\{0,\cdot\cdot,N-1\}}$, and we know that $M \vDash \varphi(b, a_i)$ for $i < N$, which means that $b \in \bigcap_{0\leqslant i<N} H_i$, thus $b \in H_N$, and thus $M \vDash \varphi(b, a_N)$, which contradicts the choice of a and b. □

This means that if T is NIP, any definable family of groups have the BS-property; but it's much weaker.

Another neat consequence is the existence of G^{00} in a (sortof locally) NIP theory, for any magma G, possibly not definable. The existence of G^0 for any magma G is also established in stable theories. I'm afraid however that G^∞ does require type-definability.

Finally, let's fix the aforementioned mistake in our previous work, namely, [2, Proposition 2.4]. We claimed, incorrectly, that not shattering arbitrarily large subsets of A with witnesses in G meant that the formula φ was NIP. But φ could shatter some other set with some other witnesses!

Fortunately, this small erratum doesn't break much. The only time we apply this fact is in a setting where all this discussion doesn't matter:

Corollary 3 ([2, cor. 2.5])**.** *In a field K of characteristic $p > 0$, the formula $\varphi(x,y) : \exists t\, x = y(t^p - t)$ is NIP iff K has no AS-extension.*

So, in this context, $(G,\cdot) = (K,+)$, and $A = K$. So saying that "φ doesn't shatter a subset of A with witnesses in G" is exaclty like saying "φ doesn't shatter shit" and is thusfore NIP. Phew! We dodged a bullet there.

NIP$_n$ity

We say that an $\mathcal{L}$-formula $\varphi(x, y_1, \dots, y_n)$ n-shatters the sets $A_1 \subset M^{|y_1|}, \dots, A_n \subset M^{|y_n|}$ with witnesses from $B \subset M^{|x|}$ iff for any $A \subset A_1 \times \cdots \times A_n$, there is $b \in B$ with $\varphi(b, A_1, \dots, A_n) = A$.

Let $(G, \cdot)$ be a magma with $G \subset M^{|x|}$. Let $A_1 \subset M^{|y_1|}, \dots, A_n \subset M^{|y_n|}$ be sets of parameters such that $H_{a_1,\dots,a_n} = \varphi(G, a_1, \dots, a_n)$ is a group under the composition law of the magma $(G, \cdot)$, for any $a_1, \dots, a_n \in A \times \cdots \times A_n$.

We say that $(H_{\overline{a}})_{\overline{a} \in \overline{A}}$ has the BSH$_n$-property with respect to $\overline{N} = (N_1, \dots, N_n)$ iff for any subsets $B_1 \subset A_1, \dots, B_n \subset A_n$ with $\#B_i > N_i$, there is a $\overline{b} \in \overline{B}$ such that:

$$\bigcap_{\overline{a} \in \overline{B}} H_{\overline{a}} = \bigcap_{\overline{a} \in \overline{B} \setminus \{\overline{b}\}} H_{\overline{a}}.$$

This does not reduce very well to an simpler statement like we had in the NIP case.

Proposition 4 (Bald-NIP$_n$-Pel)**.** *The formula φ n-shatters some subsets $A_1^0 \subset A_1, \dots, A_n^0 \subset A_n$ with $\#A_i^0 = N_i + 1$ with witnesses from G iff the family $(H_{\overline{a}})_{\overline{a} \in \overline{A}}$ doesn't have the BSH$_n$-property with respect to $\overline{N}$.*

Before we jump into the proof, let us state a conjecture strengthening, in some sense, the previous proposition.

Conjecture 5. *There is no better pun combining the names of Baldwin, Saxl, Hempel, and NIP$_n$.*

Proof (of Bald-NIP$_n$-Pel).

$\Leftarrow$: Suppose that the family $(H_{\overline{a}})_{\overline{a} \in \overline{A}}$ doesn't have the BSH$_n$-property with respect to N, so there exists subsets $B_1 \subset A_1, \dots, B_n \subset A_n$ with $\#B_i > N_i$, such that

$$\bigcap_{\overline{a} \in \overline{B}} H_{\overline{a}} \subsetneq \bigcap_{\overline{a} \in \overline{B} \setminus \{\overline{b}\}} H_{\overline{a}}.$$

for any $\overline{b} \in \overline{B}$.

We now fix for each i sets $A_i^0 \subset B_i$ with $\#A_i^0 = N_i + 1$. We prove that φ n-shatters $\overline{A}^0$ with witnesses from G. For any $\overline{x} \in \overline{A}^0$, take $b_{\overline{x}} \in \bigcap_{\overline{a} \in \overline{A}^0 \setminus \{\overline{x}\}} H_{\overline{a}} \setminus H_{\overline{x}}$, this is possible. For any $B_0 \subset \overline{A}^0$, we define $b_{B_0} = \prod_{\overline{x} \in \overline{A}^0 \setminus B_0} b_{\overline{x}}$, where the product denotes the composition law of G (again the order doesn't matter). We have $M \vDash \varphi(b_{B_0}, \overline{a})$ iff $\overline{a} \in B_0$, so φ indeed n-shatters $\overline{A}^0$ with witnesses in G.

$\Rightarrow$: Suppose that φ checks the BSH$_n$ condition with respect to $\overline{N}$, and suppose we can find $(a_j^i)_{j \leqslant N_i}^{1 \leqslant i \leqslant n} \in \overline{A}$ and $(b_J)_{J \subset \{0,\cdots,N_1\} \times \cdots \times \{0,\cdots,N_n\}} \in G$ such that $M \vDash \varphi(b_J, a_{j_1}^1, \dots, a_{j_n}^n)$ iff $\overline{j} \in J$. Now by assumption, there is $\overline{k}$ such that $\bigcap_{\overline{j}} H_{a_{\overline{j}}} = \bigcap_{\overline{j} \neq \overline{k}} H_{a_{\overline{j}}}$. But now, let $b = b_{(\{0,\cdots,N_1\} \times \cdots \times \{0,\cdots,N_n\}) \setminus \{\overline{k}\}}$; we know that $M \vDash \varphi(b, a_{j_1}^1, \dots, a_{j_n}^n)$ iff $\overline{j} \neq \overline{k}$, which means that $b \in \bigcap_{\overline{j} \neq \overline{k}} H_{a_{\overline{j}}}$. But this means $b \in H_{a_{\overline{k}}}$, which yields $\mathbb{M} \vDash \varphi(b, a_{k_1}^1, \dots, a_{k_n}^n)$ and contradicts the choice of b. $\square$

Again, the same erratum as for the NIP case hold: in my previous work, namely, [2, Proposition 3.6 and Lemma 3.8], I was sloppy and didn't properly consider the possibilities of A or G to be non-definable, and didn't consider the possibility of φ to witness IP$_n$ outisde of those. Consider this mistake fixed.

We also remark that, similar to the NIP case, some connected components exist in NIP$_n$ theories. We are reaching the very edge of my understanding and I'm almost not very confident in claiming things anymore, but I'll claim them nonetheless: [3, Theorem 4.9] holds for $(G, \cdot)$ a (non-definable) magma in theories which are "locally" NIP$_n$.

NTP2ity

Things are, as expected, slightly more complicated in the NTP2 world. Chernikov-Kaplan-Simon established a suitable condition for definable families of subgroups; namely, instead of saying that the intersection of $N+1$ of them is the same as just N of them, [4, Lemma 2.1] is saying that the intersection of all but one of them is not quite the whole intersection, but the later is of finite index in the former.

Then, in my aforementionned paper, I proceeded to localize this argument. The proof is long and tedious, so let's just state it:

Proposition 6 (Baldnikov-Kaplan-Simple)**.** *Let T be an $\mathcal{L}$-theory, $M \vDash T$ a monster, $G \subset M^{|x|}$ a set, $\cdot$ an operation on G, and $A \subset M^{|y|}$. Let $\varphi(x, y)$ be an $\mathcal{L}$-formula such that for any $a \in A$, $H_a = \varphi(G, a)$ is a group under the composition law of G. Furthermore, we assume that H_a is a normal subgroup of $\bigcup_{a \in A} H_a$. Let $\psi(x; y, z)$ be the formula $\exists w\, (\varphi(w, y) \wedge x = w \cdot z)$. Then $\psi(x; yz)$ does not admit a TP2-pattern over $G \times A$ iff the CKS-condition holds: for any $(a_i)_{i \in \omega} \in A$, there is i such that $[H_{\neq i} : H]$ is finite.*

See [2, Porism 4.6] for a proof of a slightly different version which you can then adapt to fit this one. And on this note,

ضجرت البشر
حان وقت الوداع.

References

[1] J. T. Baldwin and J. Saxl. Logical stability in group theory. *Journal of the Australian Mathematical Society*, 21(3):267–276, 1976.

[2] B. Boissonneau. Artin–Schreier extensions and combinatorial complexity in henselian valued fields. *The Journal of Symbolic Logic*, 89(4):1747–1767, 2024.

[3] A. Chernikov and N. Hempel. On n-dependent groups and fields II. *Forum of Mathematics, Sigma*, 9:e38, 2021.

[4] A. Chernikov, I. Kaplan, and P. Simon. Groups and fields with NTP2. *Proceedings of the American Mathematical Society*, 143, 12 2012.

[5] B. Poizat. Sous-groupes définissables d'un groupe stable. *Journal of Symbolic Logic*, 46(1):137–146, 1981.