Artin-Schreier extensions & combinatorial complexity

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Stability

A formula $\varphi(x, y)$ is said to have *the order property* (in M) if there is $(a_i)_{i\in\omega}, (b_i)_{i\in\omega} \in M$ such that $M \vDash \varphi(a_i, b_j)$ iff $i < j$.
A first-order theory T is said to be *stable* if no formula has the order property (in any $M \vDash T$). A structure M is said to be stable if $Th(M)$ is.

Stable fields conjecture

A pure field is stable iff it is finite or separably closed.

- "$\Leftarrow$" is known (Wood, 79).
- "$\Rightarrow$" is known for ω-stable (Macintyre, 71), for superstable (Cherlin-Shelah, 80), and for large stable fields (Johnson-Tran-Walsberg-Ye, 2021).
- Infinite stable fields of characteristic p are Artin-Schreier-closed (Scanlon, 99).

NIP – no independence property

A formula $\varphi(x, y)$ is said to have *the independence property* (in M) if there are $(a_i)_{i\in\omega}, (b_J)_{J\subset\omega} \in M$ such that $M \vDash \varphi(b_J, a_i)$ iff $i \in J$.
A first-order theory T is said to be *NIP* if no formula has the independence property (in any $M \vDash T$). A structure M is said to be NIP if $Th(M)$ is.

NIP fields conjecture (Shelah's)

If a pure field is NIP, then it is either finite, separably closed, real closed, or *henselian*.

It is not an equivalence, but it can be made into one by classifying henselian NIP valued fields (Anscombe-Jahnke, 2019).

- It is known for dp-finite fields (Johnson, 2020).
- If a valued field of characteristic p is NIP, then the valuation is henselian (Johnson, 2021).

Henselianity

Let K be a field and $(\Gamma, +, <)$ an ordered abelian group. A valuation is a morphism $v : K^\times \to \Gamma$ such that $v(x+y) \geqslant \min(v(x), v(y))$. We set $v(0) = \infty$.

$\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$ is a valuation ring, $\mathcal{M}_v = \{x \in K \mid v(x) > 0\}$ is a maximal ideal, and $k = \mathcal{O}_v / \mathcal{M}_v$ is the residue field.

A valuation is called *henselian* if simple roots polynomials can be lifted from the residue field.

In the language of rings completed with a binary relation symbol for $v(x) < v(y)$, (henselian) valued fields can be axiomatised in first-order.

Example

On $\mathbb{Q}$, the p-adic valuation v_p is defined by $v_p(p) = 1$ for a fixed prime p and $v_p(q) = 0$ for primes $q \neq p$. The residue field is $\mathbb{F}_p$. Its Cauchy completion $(\mathbb{Q}_p, v_p)$ is henselian.

Model-theory goal: understanding the full structure of a henselian valued field only in terms of the residue field and the value group.

Classification of NIP henselian valued fields

Theorem (Anscombe-Jahnke, 2019)

Let (K, v) be henselian. (K, v) is NIP iff k is NIP and:

1. $\mathrm{ch}(K, k) = (0, 0)$ or (p, p) and (K, v) is SAMK or trivial;
2. $\mathrm{ch}(K, k) = (0, p)$, (K, v_p) is finitely ramified and $(k_p, \overline{v})$ checks 1;
3. $\mathrm{ch}(K, k) = (0, p)$ and $(k_0, \overline{v})$ is AMK.

"$\Leftarrow$" is by Jahnke-Simon's transfer theorem and "$\Rightarrow$" is because:

- k is interpretable and thus NIP;
- Infinite NIP fields of characteristic p are Artin-Schreier closed (Kaplan-Scanlon-Wagner, 2011), so if $\mathrm{ch}(K) = p$ it is SAMK: no separable alebraic immediate extension, Γ p-div, k p-closed.
- In mixed characteristic we decompose around $v(p)$: $K \xrightarrow{\Gamma/\Delta_0} k_0 \xrightarrow{\Delta_0/\Delta_p} k_p \xrightarrow{\Delta_p} k$. Convex subgroups are externally definable, so by Shelah's expansion theorem the structure (K, v, v_0, v_p) remains NIP and we work part by part.

NIP$_n$

A formula $\varphi(x; y_1, \ldots, y_n)$ is said to have IP$_n$ (in M) if there are $(a_i^1)_{i\in\omega}, \ldots, (a_i^n)_{i\in\omega}, (b_J)_{J\subset\omega^n} \in M$ such that $M \vDash \varphi(b_J, a_{i_1}^1, \ldots, a_{i_n}^n)$ iff $(i_1, \ldots, i_n) \in J$.
A first-order theory T is said to be NIP$_n$ if no formula has the independence property (in any $M \vDash T$). A structure M is said to be NIP$_n$ if $Th(M)$ is and strictly NIP$_n$ if it is NIP$_n$ and IP$_{n-1}$.

Examples

- The random n-hypergraph is strictly NIP$_n$.
- $(\mathbb{F}_p^{<\omega}, \mathbb{F}_p, 0, +, \times)$, with $(a_i) \times (b_i) = \sum a_i b_i \in \mathbb{F}_p$, is strictly NIP$_2$.
- Strictly NIP$_n$ pure groups are known (Chernikov-Hempel, 2019).

NIP$_n$ fields

NIP$_n$ fields conjectures

1. A pure field is NIP$_n$ iff it is NIP.
2. A henselian valued field is NIP$_n$ iff it is NIP.

- PAC non-SC fields have IP$_n$ (Duret, 80) (Hempel, 2016)
- $\mathrm{ch}(K) = p$ and (K, v) NIP$_n \Rightarrow$ v henselian (Johnson, 2021) (Chernikov-Hempel, 2021)
- $\mathrm{ch}(K) = p$, K infinite and NIP$_n \Rightarrow$ K AS-closed (Kaplan-Scanlon-Wagner, 2011) (Hempel, 2016)

If we want to prove Anscombe-Jahnke for NIP$_n$:

- We can do one way of the equicharacteristic case, but
- We need a NIP$_n$-transfer theorem,
- For mixed characteristic, when doing the decomposition, we can't use Shelah's expansion theorem.

Localising KSW-H

Proof scheme of KSW-H:

- If K is NIP$_n$, then any definable family of additive subgroups checks Baldwin-Saxl-Hempel's condition,
- $H_{a_1,\ldots,a_n} = \{a_1 \cdots a_n(t^p - t) \mid t \in K\}$ is a such a definable family,
- If $H_{a_1,\ldots,a_n}$ checks Baldwin-Saxl-Hempel's condition, then K is AS-closed.

Baldwin-Saxl-Hempel

Let $H_{a_1,\ldots,a_n} = \{x \in M \mid M \vDash \varphi(x, a_1, \ldots, a_n)\}$ be a definable family of subgroups (of a definable group of M). φ is NIP$_n$ iff there is $N \in \omega$ such that for any $d > N$, for any $(a^i_j)^{i \leqslant n}_{j \leqslant d}$, there is $\overline{k} = (k_1, \ldots k_n)$ with $\bigcap_{\bar{j} \in d^n} H_{a^1_{j_1},\ldots,a^n_{j_n}} = \bigcap_{\bar{j} \neq \overline{k}} H_{a^1_{j_1},\ldots,a^n_{j_n}}$.

Local KSW-H

Let K be infinite and of characteristic p.
$\varphi(x; y_1, \ldots, y_n) : \exists t\, x = y_1 \cdots y_n(t^p - t)$ is NIP$_n$ (in K) iff K is AS-closed.

In mixed characteristic

$\varphi(x; y_1, \dots, y_n)$ is a positive existential formula. If it admits an IP_n-pattern in a residue field, we can lift it by henselianity.

Lifting Artin-Schreier complexity

Let (K, v) be henselian of mixed characteristic, let k be infinite & not AS-closed; then K has IP_n as a pure field.

Thus, when doing the decomposition $K \xrightarrow{\Gamma/\Delta_0} k_0 \xrightarrow{\Delta_0/\Delta_p} k_p \xrightarrow{\Delta_p} k$, we can say that relevant parts are p-closed/p-div/maximal without adding the intermediate valuations in the language.

This allows us to prove one way of Anscombe-Jahnke for NIP_n.

Consequences

- NIP_n henselian valued fields with NIP residue are NIP.
- Conjectures 1 and 2 are equivalent.
- Algebraic extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are NIP_n iff they are NIP.

Work in progress: NTP2 henselian valued fields

- We can do AS-lifting in NTP2 henselian valued fields, this gives us new examples of TP2 algebraic extensions of $\mathbb{Q}_p$.
- We have NTP2 transfer in Anscombe-Jahnke's setting, but we believe it's not the whole picture: $\mathbb{F}_p((\mathbb{Q}))$ would be a counter-example if it is NTP2.
- If a field of characteristic p has QE down to predicates for roots of additive polynomials; those predicates are NTP2 iff the field is AS-finite. $\mathbb{F}_p((\mathbb{Q}))$ has QE down to predicates for all polynomials, maybe this can be refined.