1 Linear Models

1.6 Types of errors
Errors come from: impercise data, mistakes in the model, computational percision,.. We know two types of errors:

- **Absolute** error = approximate value - correct value

$$\Delta x = \overline{x} - x$$

- **Relative** error = $\dfrac{\text{absolute err}}{\text{correct value}}$

$$\delta x = \frac{\Delta x}{x}$$

1.2 Mathematical model is **linear**, when the function F is a linear function of the parameters:

$$F(x, a_1, \dots, a_p) = a_1\phi_1(x) + \dots + a_p\phi_p(x)$$

where $\phi_1, \dots, \phi_p$ are functions of a specific type.

1.3 Least squares method Given points

$$\{(x_1, y_1), \dots, (x_m, y_m)\}, x_i \in R^n, y_i \in R$$

the task is to find a function $F(x, a_1, \dots, a_p)$ that is good fit for the data. The values of the parameters $a_1, \dots, a_p$ should be chosen so that the equations

$$y_i = F(x, a_1, \dots, a_p), i = 1, \dots, m$$

are satisified or, it this is not possible, that the error is as small as possible.
We use **Least squares method** to determine that the sum od squared errors is as small as possible.

$$\sum_{i=1}^m (F(x_i, a_1, \dots, a_p) - y_i)^2$$

1.4 Systems of linear equations
A system of linear equations in the matrix form is given by $A\vec{x} = \vec{b}$, where:

- A is the matrix of coefficients of order $m \times n$ where m is the number of equations and n is the number of unknowns,
- $\vec{x}$ is the vector of unknowns and
- $\vec{b}$ is the right side vector

$$\begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_p(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_p(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \dots & \phi_p(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

1.5 Existance of solutions in linear equations
Let A = $[\vec{a_1}, \dots, \vec{a_n}]$, where $\vec{a_i}$ are vector representing the columns of A. For any vector $\vec{x} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$ the produdct $A\vec{x}$ is a linear combination $A\vec{x} = \sum_i x_i a_i$. The system is **solvable** iff the vector $\vec{b}$ can be expressed as a linear combination of the columns of A, that is it is in the column space of A, $\vec{b} \in C(A)$.
By adding to the columns of A we obtain the extended matrix of the system:

$$[A|\vec{b}] = [\vec{a_1}, \dots, \vec{a_n}|b]$$

The system $A\vec{x} = \vec{b}$ is solvable iff the rank of A equals the rank of the extended matrix $[A|\vec{b}]$, i.e.:

$$\text{rank}A = \text{rank}[A|\vec{b}] =: r$$

The solution is unique if the rank of the two matrices equals num of unknowns (r = n).

1.6 Properties of **squared matrices**
Let $A \in R^{n\times n}$ be a square matrix. The following conditions are equivalent and characterize when a matrix A is **invertible** or **nonsingular**:

- The matrix A has an inverse
- rank A = n
- det(A) $\neq 0$
- The null space $N(A) = \{\vec{x} : A\vec{x} = 0$ is trivial
- All eigenvalues of A are nonzero
- For each $\vec{b}$ the system of equations $A\vec{x} = \vec{b}$ has perciesly one solution

1.7 Generalized inverse of a matrix $A \in R^{n\times m}$ is a matrix $G \in R^{m\times n}$ such that

$$AGA = A$$