Math 2010
A2

Blaise Delfora

48021191

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MATH2010 MATH2100 MATH7100
Assignment 2

Use the **Gradescope assignment submission link on Blackboard** to submit this assignment. The submission must consist of **a single PDF file** containing a printout of your Mathematica notebook and your handwritten/typed solutions. It is important that your name and student number appear on the first page of your submission. Your pdf file should be legible and not too large. Files that are poorly scanned and/or illegible may not be marked. **Once you have uploaded your file, you must assign page number(s) to each question!**

Assignments submitted after the due date will attract a penalty, as outlined in the course profile.

In this assignment, you may only **use Mathematica where indicated**.

Total: **40 marks**, allocated as indicated on each problem.

Problem 1. Solve for the phase curves of the following system

$$\begin{aligned}\dot{x} &= y^2\\ \dot{y} &= (2x+1)y.\end{aligned}$$

Sketch the phase curves and indicate the direction of flow by an arrow. Additionally, determine the critical points of the system. **(6 marks)**

Problem 2. Consider the non-linear system of ODEs given by

$$\begin{aligned}y_1' &= f_1(y_1, y_2) = -2y_2(3 - y_2 - y_1),\\ y_2' &= f_2(y_1, y_2) = y_1(-4 + y_2 + 2y_1).\end{aligned}$$

1. Using Mathematica, find all the critical points of the system.**(1 mark)**
2. Using Mathematica, calculate the linearised system about each of the critical points, and classify the critical points, including type and stability. **(2 marks)**
3. Using Mathematica, find the nullclines of the system. **(1 mark)**
4. Using Mathematica, sketch a phase portrait for the non-linear system. Clearly identify each of the critical points and nullclines. **(2 marks)**

Problem 3. The Duffing's equation $\ddot{x} - 4x + c\dot{x} + x^3 = 0$, for some damping constant c, describes the motion of a mechanical system in a twin-well potential field.

1. Express this second order ODE as a system of coupled first order ODEs. (**2 marks**)
2. For $c = 0$, use the chain rule to solve for the phase curves $y_2(y_1)$. (**2 marks**)

3. For $c = 2$, calculate the linearised system about each of the critical points, and classify the critical points, including type (saddle, proper/improper/degenerate node, spiral, center, etc) and stability. If a critical point is a saddle or a node, identify the relevant eigenvalues and eigenvectors of the linearised system. If a critical point is a spiral or a centre, identify the direction of rotation and stability type of the linearised system. (**3 marks**)

4. For each of the critical points found in part 3, does the linearised system determine the general type and stability of the trajectories around the critical point? Justify your answer. (**1 marks**)

5. Sketch a phase portrait for the non-linear system, by hand. Clearly identify each of the critical points and include the nullclines to obtain a more acurate phase portrait. (**3 marks**)

Problem 4. 1. Find the Laplace transform of the following functions.

(a) $f(t) = e^{2t}\sin(\pi t)$. **(1 marks)**

(b) $f(t) = t^2 u(t-3)$ **(3 marks)**

2. Consider the functions $f(t)$ and $g(t)$, where $g(t) = f(\frac{t}{\pi})$. Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Use the definition of Laplace transform to show that $G(s) = \pi F(\pi s)$. **(2 marks)**

3. Find the inverse Laplace transform of the function

$$F(s) = \frac{s+8}{s^2-16}.$$

(2 marks)

Problem 5. Consider the following system of differential equations:

$$\frac{dx}{dt} = -2x + 2y + e^{-t},$$
$$\frac{dy}{dt} = x - y.$$

Use Laplace transforms to solve the given system of ODEs, subject to the initial conditions $x(0) = 0$ and $y(0) = 0$. (**5 marks**)

Problem 6. Write the following function in terms of step functions. Then, use the second shifting theorem to find its Laplace transform.

$$f(t) = \begin{cases} -\sin(\pi t) & 1 < t < 2 \\ 0 & t \leq 1 \text{ or } t \geq 2 \end{cases}.$$

(4 marks)

Problem 1. Solve for the phase curves of the following system

$$\begin{aligned} \dot{x} &= y^2 \\ \dot{y} &= (2x+1)y. \end{aligned}$$

Sketch the phase curves and indicate the direction of flow by an arrow. Additionally, determine the critical points of the system. **(6 marks)**

① Setup ODEs

$f(x) = \dot{x} = y^2$

$g(x) = \dot{y} = (2x+1)y$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

② Find crit points

$(2x+1)y = 0$

$2xy + y = 0$

$y(2x+1) = 0$

$y = 0 \Rightarrow \begin{pmatrix} x=0 \\ x=-\frac{1}{2} \end{pmatrix}$

$x = -\frac{1}{2} \Rightarrow (y=0)$

Crit Points

$(-\frac{1}{2}, 0)$

$(x, 0)$

$x \in \mathbb{R}$

③ Sub $y=0$, to let x represent any value including $-\frac{1}{2}$ or 0

④ Phase portrait

④.1 Solve from $\dfrac{dy_2}{dy_1} = \dfrac{f_2(y_1, y_2)}{f_1(y_1, y_2)}$

$$\frac{dy}{dx} = \frac{(2x+1)y}{y^2} = \frac{2x+1}{y}$$

④.2 Integrate Expression

$$\int y\,dy = \int 2x+1\,dx$$

$$\frac{1}{2}y^2 = x^2 + x + c$$

$$y^2 = 2(x^2+x+c)$$

$$y = \pm\sqrt{2(x^2+x+c)}$$

$y = \pm\sqrt{2x^2+2x+2\cdot\frac{1}{4}}$

$y = \pm\sqrt{2(x^2+x+\frac{1}{4})}$

$y = \pm\sqrt{2} \times (x+\frac{1}{2})$

Equations of Asymptote Lines

[Hyperbola! So put in general form].

⑤ Gen. Form

$$y^2 = 2\left[(x+\tfrac{1}{2})^2 - \tfrac{1}{4} + c\right]$$

$$y^2 = 2(x+\tfrac{1}{2})^2 - \tfrac{1}{2} + 2c$$

$$0 = 2(x+\tfrac{1}{2})^2 - \tfrac{1}{2} - y^2 + 2c$$

$$2(x+\tfrac{1}{2})^2 - y^2 = \tfrac{1}{2} - 2c$$

$$\frac{2(x+\frac{1}{2})^2}{(\frac{1}{2}-2c)} - \frac{(y^2-0)}{(\frac{1}{2}-2c)} = \pm 1$$

$$\boxed{\frac{(x+\frac{1}{2})^2}{\frac{1}{2}(\frac{1}{2}-2c)} - \frac{(y^2-0)}{\frac{1}{2}-2c} = 1}$$

⑤.1 Solve a, b, k

$a^2 = \frac{1}{2}(\frac{1}{2}-2c)$

$a^2 = \frac{1}{4} - c$

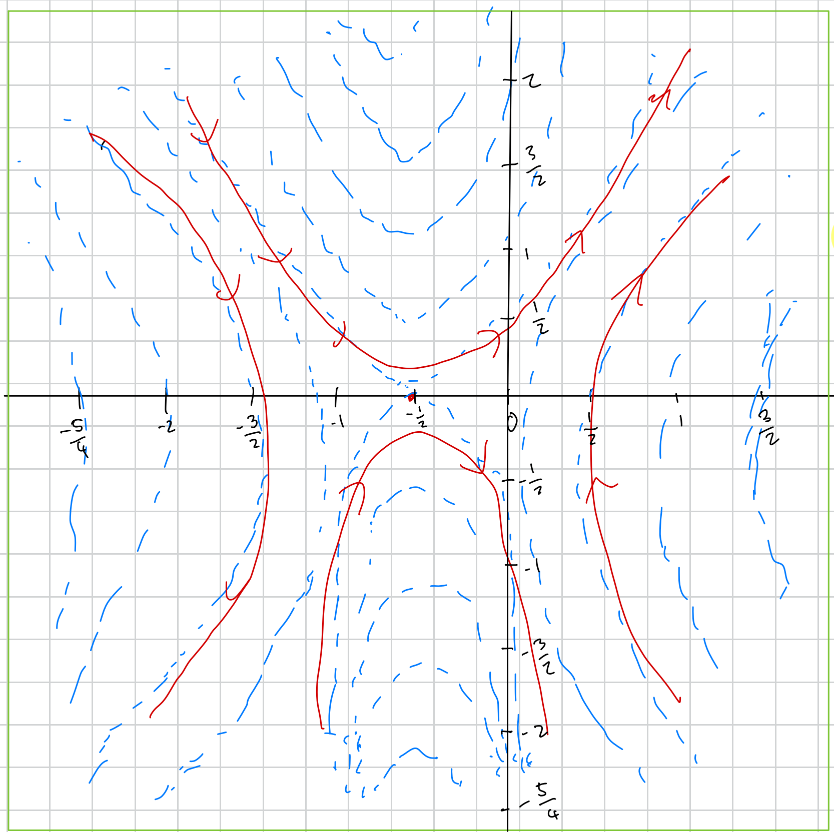
$a = (\sqrt{\frac{1}{4}-c})^2$

$b^2 = \frac{1}{2} - 2c$

$b = (\sqrt{\frac{1}{2}-2c})^2$

Set $(\sqrt{\frac{1}{2}-2c})^2 = k$

$a = \frac{k}{2}$ $\quad$ $b = k$



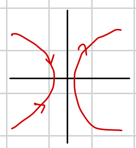
⑤.2 k value implication

→ So inserting values for a, b

$$\frac{(x+\frac{1}{2})^2}{(\frac{k}{2})} - \frac{(y^2-0)}{k} = \pm 1$$

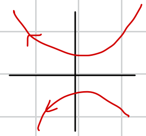
→ k can take on values

For $k>0$ $\quad$ $\frac{1}{2} - 2c > 0 \Rightarrow 2c > -\frac{1}{2} \Rightarrow c > \frac{1}{4}$



For $k=0$ $\quad$ Undefined

For $k<0$ $\quad$ $\frac{1}{2} - 2c < 0 \Rightarrow c < \frac{1}{4}$



Problem 2. Consider the non-linear system of ODEs given by

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = -2y_2(3 - y_2 - y_1), \\ y_2' &= f_2(y_1, y_2) = y_1(-4 + y_2 + 2y_1). \end{aligned}$$

1. Using Mathematica, find all the critical points of the system. **(1 mark)**
2. Using Mathematica, calculate the linearised system about each of the critical points, and classify the critical points, including type and stability. **(2 marks)**
3. Using Mathematica, find the nullclines of the system. **(1 mark)**
4. Using Mathematica, sketch a phase portrait for the non-linear system. Clearly identify each of the critical points and nullclines. **(2 marks)**

$x = 3 - y$
$x + y = 3$
$y = 3 - x$

$y = -2(-2 + x)$
$y = 4 + -2x$

1

```
In[◦]:= (*Step 1*)
        (*Define the system*)
        xPrime1[x_, y_] := -2 y (3 - y - x)
        yPrime1[x_, y_] := x (-4 + y + 2 x)


        critpoints = Solve[{xPrime1[x, y] == 0, yPrime1[x, y] == 0}, {x, y}]

        (*Done*)
Out[◦]=
        {{x → 0, y → 3}, {x → 1, y → 2}, {x → 2, y → 0}, {x → 0, y → 0}}
```

2

```
In[◦]:= (*Step 2 - Linear System*)
        jacobian = D[{xPrime1[x, y], yPrime1[x, y]}, {{x, y}}]

Out[◦]=
        {{2 y, -2 (3 - x - y) + 2 y}, {-4 + 4 x + y, x}}

In[◦]:= jacobianatcritpoints = jacobian /. critpoints
Out[◦]=
        {{{6, 6}, {-1, 0}}, {{4, 4}, {2, 1}}, {{0, -2}, {4, 2}}, {{0, -6}, {-4, 0}}}

In[◦]:= eigenvalues = Eigenvalues[jacobian] /. critpoints

Out[◦]=
```

$$\left\{\left\{\frac{1}{2}\left(6-2\sqrt{3}\right), \frac{1}{2}\left(6+2\sqrt{3}\right)\right\}, \left\{\frac{1}{2}\left(5-\sqrt{41}\right), \frac{1}{2}\left(5+\sqrt{41}\right)\right\}, \left\{\frac{1}{2}\left(2-2\,i\sqrt{7}\right), \frac{1}{2}\left(2+2\,i\sqrt{7}\right)\right\}, \left\{-2\sqrt{6}, 2\sqrt{6}\right\}\right\}$$

```
In[◦]:= classification[eigenvalues_] :=
         Module[{realParts, imagParts}, realParts = Re[eigenvalues];
          imagParts = Im[eigenvalues];
          If[AllTrue[imagParts, # == 0 &], If[AllTrue[realParts, # > 0 &], "Unstable Node",
            If[AllTrue[realParts, # < 0 &], "Stable Node", "Saddle Point"]],
           If[AllTrue[realParts, # < 0 &], "Stable Spiral",
            If[AllTrue[realParts, # > 0 &], "Unstable Spiral", "Center"]]]]

        fulldisplay = classification /@ eigenvalues

Out[◦]=
        {Unstable Node, Saddle Point, Unstable Spiral, Saddle Point}
```

(3)

In[∘]:=
```
(*Step 3 - Nullclines*)
xNullclines = Solve[xPrime[x, y] == 0, x]
yNullclines = Solve[yPrime[x, y] == 0, y]
```
Out[∘]=
```
{{x → 3 - y}}
```
Out[∘]=
```
{{y → -2 (-2 + x)}}
```

In[∘]:=
```
(*Output the results*)
{xNullclines, yNullclines}
```
Out[∘]=
```
{{{x → 3 - y}}, {{y → -2 (-2 + x)}}}
```

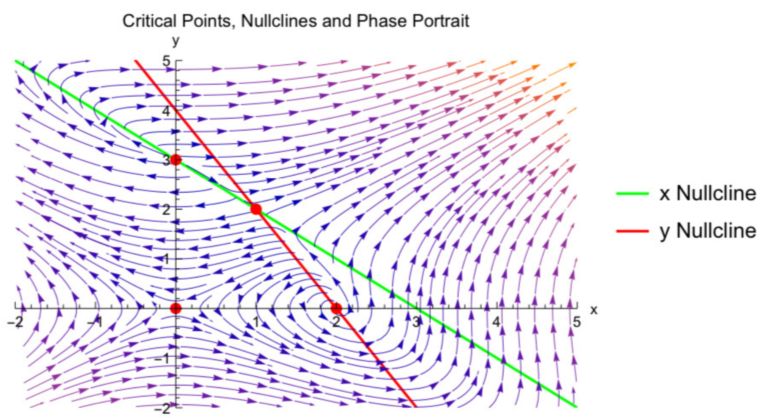
(4)

In[∘]:=
```
(*Step 4 - Phase Portrait*)
plotRange = {{-2, 5}, {-2, 5}};

(*Plot nullclines*)
nullclinePlot =
  Plot[{3 - x, 4 - 2 x}, {x, -2, 5}, PlotStyle → {Green, Red}, PlotRange → plotRange,
   AxesLabel → {"x", "y"}, PlotLegends → {"x Nullcline", "y Nullcline"}, Mesh → None];

(*Plot the phase portrait*)
phasePortrait =
  StreamPlot[{xPrime[x, y], yPrime[x, y]}, {x, -2, 5}, {y, -2, 5}, StreamStyle → Black,
   StreamPoints → Fine, PlotRange → plotRange, AxesLabel → {"x", "y"}];

(*Combine plots*)
Show[nullclinePlot, phasePortrait,
 Graphics[{Red, PointSize[Large], Point[{x, y} /. critpoints]}], PlotRange → plotRange,
 AxesLabel → {"x", "y"}, PlotLabel → "Critical Points, Nullclines and Phase Portrait"]
```
Out[∘]=



Problem 3. The Duffing's equation $\ddot{x} - 4x + c\dot{x} + x^3 = 0$, for some damping constant c, describes the motion of a mechanical system in a twin-well potential field.

1. Express this second order ODE as a system of coupled first order ODEs. (**2 marks**)
2. For $c = 0$, use the chain rule to solve for the phase curves $y_2(y_1)$. (**2 marks**)
3. For $c = 2$, calculate the linearised system about each of the critical points, and classify the critical points, including type (saddle, proper/improper/degenerate node, spiral, center, etc) and stability. If a critical point is a saddle or a node, identify the relevant eigenvalues and eigenvectors of the linearised system. If a critical point is a spiral or a centre, identify the direction of rotation and stability type of the linearised system. (**3 marks**)
4. For each of the critical points found in part 3, does the linearised system determine the general type and stability of the trajectories around the critical point? Justify your answer. (**1 marks**)
5. Sketch a phase portrait for the non-linear system, by hand. Clearly identify each of the critical points and include the nullclines to obtain a more acurate phase portrait. (**3 marks**)

① $\ddot{x} - 4x + c\dot{x} + x^3 = 0$

$$\begin{cases} u = x, \\ v = \dot{x} \end{cases} \qquad \begin{cases} \dot{u} = \dot{x} = v \\ \dot{v} = \ddot{x} = 4x - x^3 - c\dot{x} = 4u - u^3 - cv \end{cases}$$

② $$\frac{\dot{v}}{\dot{u}} = \frac{4x - x^3 \cancel{- c\dot{x}}}{\dot{x}} = \frac{4u - u^3}{v}$$

$c=0$

Set as u, v variables

Set as: $\int v\dot{v} = \int 4u - u^3 \dot{u}$

So $$\frac{v^2}{2} = \frac{4}{2}u^2 - \frac{u^4}{4} + c$$

$$v^2 = 2\left(2u^2 - \frac{u^4}{4} + c\right)$$

$$v = \pm\sqrt{2\left(2u^2 - \frac{u^4}{4} + c\right)}$$

③ $c = 2$

$$\begin{cases} f(x) = \dot{u} = \dot{x} = v \\ g(x) = \dot{v} = \ddot{x} = 4x - x^3 - c\dot{x} = 4u - u^3 - cv \end{cases}$$

Set $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ →

For $v=0$, $c=2$

$$\begin{cases} 4x - x^3 - 2\dot{x} = 0 \\ 4u - u^3 - 2v = 0 \end{cases}$$

$\therefore (u=0, v=0)$

For $v=0$, $c=2$

$4u - u^3 = 0$

$u(4 - u^2) = 0$

$u = 0$ OR $(4 - u^2) = 0$

$4 = u^2$

$u = \sqrt{4} = \pm 2$

$\therefore (u = 0, -2, +2)$ →

Crit Points

$(0,0)$

$(2,0)$

$(-2,0)$

Jacobian

$$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 4-3u & -2 \end{vmatrix}$$ →

Test at critical points

Next Page

(0,0)

$$J = \begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} \quad \left[\begin{matrix} \det(A) = 0 - 4 = -4 < 0 \\ Tr(A) = -2 \qquad Tr(A) < 0 \end{matrix} \right]$$

$$\rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) - 4 = \lambda^2 + 2\lambda - 4$$

$$\rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (4)}}{2} = \frac{-2 \pm \sqrt{4+16}}{2} = \frac{-2 \pm \sqrt{20}}{2}$$

$$= \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$$\lambda_1 = -1 - \sqrt{5} \quad \lambda_2 = -1 + \sqrt{5}$$

$\lambda_1 = -1 - \sqrt{5}$

$\det(A - \lambda_1 I) \rightarrow$

$$\left| \begin{matrix} 1+\sqrt{5} & 1 \\ 4 & -2+1+\sqrt{5} \end{matrix} \right| \begin{matrix} 0 \\ 0 \end{matrix} \left| \begin{matrix} (1+\sqrt{5})x_1 + 1x_2 = 0 \Rightarrow (1+\sqrt{5})x_1 = -x_2 \quad (1) \\ 4x_1 + (-1+\sqrt{5})x_2 = 0 \Rightarrow -4x_1 = (-1+\sqrt{5})x_2 \quad (2) \end{matrix} \right.$$

From (1) we have

$$x_2 = (-1-\sqrt{5})x_1 \quad \bar{x}_1 = \begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix}$$

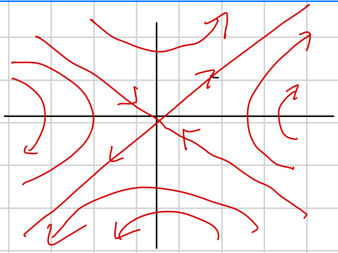
$\lambda_2 = -1 + \sqrt{5}$

$\det(A - \lambda_2 I) =$

$$\left| \begin{matrix} 1-\sqrt{5} & 1 \\ 4 & -2+1-\sqrt{5} \end{matrix} \right| \begin{matrix} 0 \\ 0 \end{matrix} \left| \begin{matrix} (1-\sqrt{5})x_1 + 1x_2 = 0 \quad (3) \\ 4x_1 + (-1-\sqrt{5})x_2 = 0 \quad (4) \end{matrix} \right.$$

From (3)

$$(-1+\sqrt{5})x_1 = x_2 \quad \bar{x}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}$$



NOT DRAWN EXACT TO FUNCTION!

$$\left(\begin{matrix} \lambda_1 < 0 \\ \lambda_2 > 0 \end{matrix} \right) - \begin{matrix} \bar{x}_1 = \begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix} \\ \bar{x}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix} \end{matrix}$$

(0,0) → Saddle Point

(2,0)

(1) $J = \begin{vmatrix} 0 & 1 \\ -8 & -2 \end{vmatrix} = 8 \quad \left[\begin{matrix} \det(A) > 0 \\ Tr(A) = -2 < 0 \end{matrix} \right]$

(2) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -8 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 8$ (A)

(3) $\lambda = \dfrac{-2 \pm \sqrt{4 - 4(1 \cdot 8)}}{2} = \dfrac{-2 \pm \sqrt{4-32}}{2} = \dfrac{-2 \pm 2\sqrt{-7}}{2} = -1 \pm \sqrt{7} \Rightarrow \lambda_1 = -1 + \sqrt{7}i$, $\lambda_2 = -1 - \sqrt{7}i$

Since $Re(\lambda) < 0 \rightarrow$ Spiral

$Re(\lambda_1) < 0$

$Re(\lambda_2) < 0$

↓

Stable Spiral → Since stable, must spiral inwards

↓

Test for Dir.

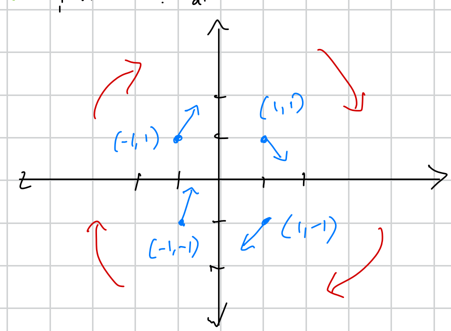
For Stable Spiral, it is pointed towards origin, AKA travelling towards origin

Let us test w/ easy points

(1,1)	$\frac{dx}{dt} = 1$	$\frac{dx^2}{dt} = -8-2 = -10$
(-1,1)	$\frac{dx}{dt} = 1$	$\frac{dx^2}{dt} = 8-2 = 6$
(-1,-1)	$\frac{dx}{dt} = -1$	$\frac{dx^2}{dt} = 8+2 = 10$
(1,-1)	$\frac{dx}{dt} = 1$	$\frac{dx^2}{dt} = -8+2 = -6$

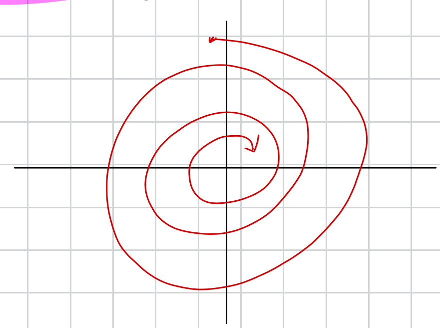
$$\frac{dx_1}{dt} = 0x_1 + x_2$$

$$\frac{dx_2}{dt} = -8x_1 - 2x_2$$



∴ Clockwise

(2,0) → Stable Clockwise Spiral



$(-2,0)$

$$J = \begin{vmatrix} 0 & 1 \\ 8 & -2 \end{vmatrix} = 8 \qquad \begin{bmatrix} \det(A) > 0 \\ Tr(A) = -2 < 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -8 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) + 8 = \lambda^2 + 2\lambda + 8$$

$\lambda^2 + 2\lambda + 8 = 0$

From Eq. A, from prev. crit point, we recognise:

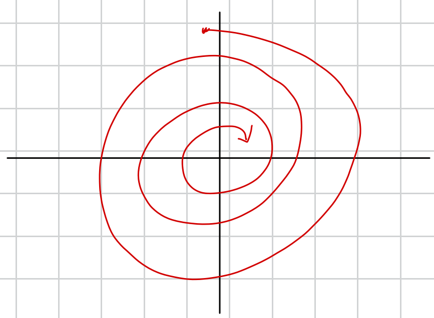
$\lambda_1 = -1 + \sqrt{7}i$

$\lambda_2 = -1 - \sqrt{7}i$

Since Eigenvalues same as $(2,0)$, we recognise that the function will have the same shape & direction, & behavior (from B From $(2,0)$)

$\therefore$ Stable Clockwise Spiral

$(-2,0) \rightarrow$ Stable Clockwise Spiral



SUMMARY

Coordinates	Nature	Dir.
$(0,0)$	Saddle	N/A
$(2,0)$	Stable Spiral	
$(-2,0)$	Stable Spiral	

(4) The Critical Points

$\left.\begin{array}{l}(0,0)\\(-2,0)\\(2,0)\end{array}\right\}$

2 special cases for linearized systems

resultant flows $\neq$ Non-linear flow

(1) $\det(A)=0$

(2) $\text{trace}(A)=0 \quad \det(A)>0$

There are no cases that have these conditions in this circuit,

Therefore, the linearized system determines the general type & stability of trajectories around the crit point.

Nullclines

(A) $\dot{u} = \dot{x} = v = 0$

$[x = u]$

horizontal nullcline

(B) $(\dot{v}=0)$

$\dot{v} = \ddot{x} = 4x - x^3 - 2x$

$0 = 4u - u^3 - 2v$

$-2v = u^3 - 4u$

$2v = 4u - u^3$

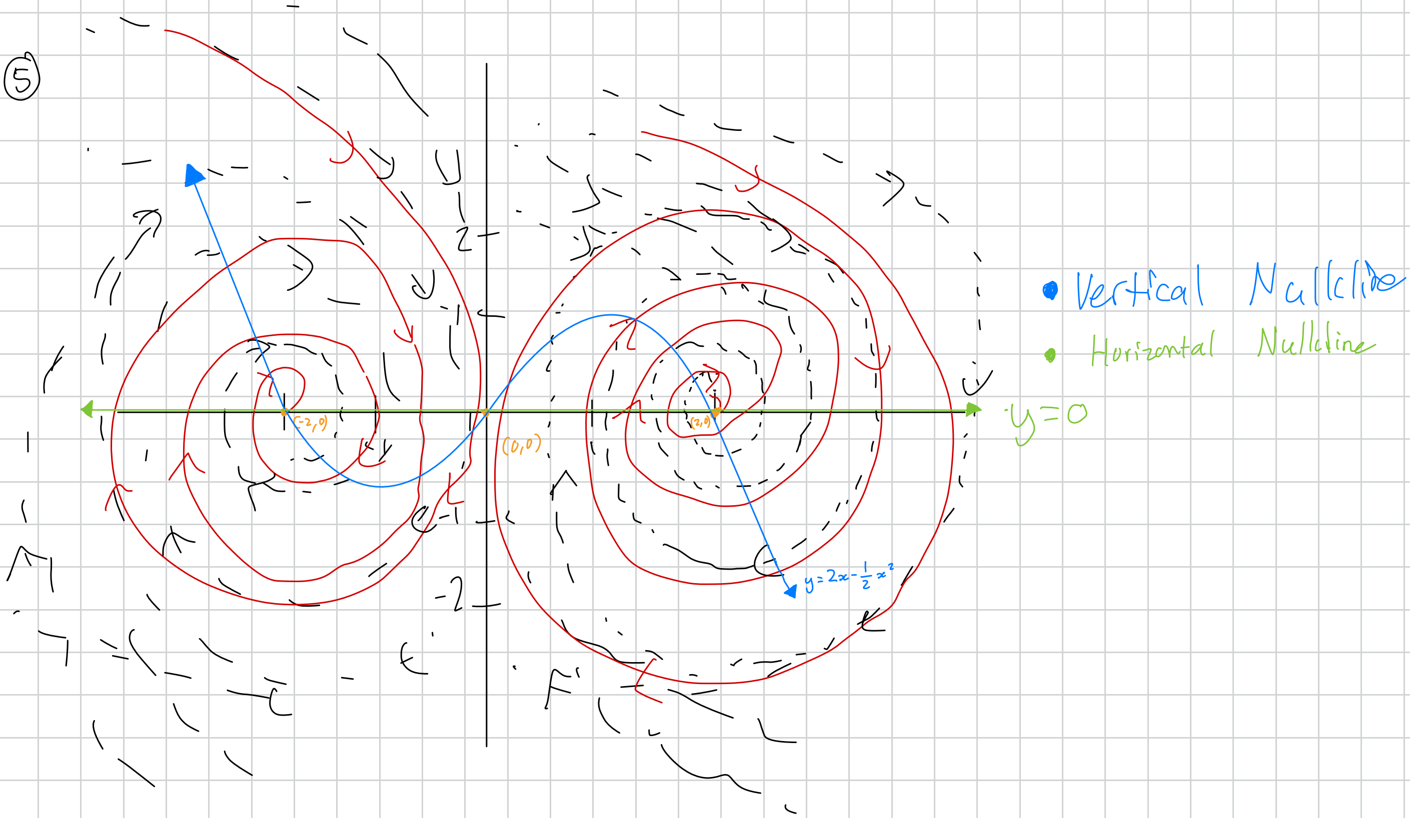
$v = \frac{4u - u^3}{2}$

$v = 2u - \frac{1}{2}u^3$

$[y = v]$

vertical nullcline

(5)



Problem 4. 1. Find the Laplace transform of the following functions.

(a) $f(t) = e^{2t}\sin(\pi t)$. **(1 marks)**

(b) $f(t) = t^2 u(t-3)$ **(3 marks)**

2. Consider the functions $f(t)$ and $g(t)$, where $g(t) = f(\frac{t}{\pi})$. Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Use the definition of Laplace transform to show that $G(s) = \pi F(\pi s)$. **(2 marks)**

3. Find the inverse Laplace transform of the function

$$F(s) = \frac{s+8}{s^2-16}.$$

(2 marks)

① a) $f(t) = e^{2t}\sin(\pi t)$

$\mathcal{L}[e^{at}\sin(\alpha t)] = \frac{\alpha}{(s-a)^2+\alpha^2}$

$\mathcal{L}(f(t)) = \frac{\pi}{(s-2)^2+\pi^2}$

b) $f(t) = t^2 u(t-3)$ $\quad (g(t) = t^2)$

$f(t) = g(t)\,u(t-3)$

So, $\mathcal{L}(f(t)) = \mathcal{L}(g(t)\,u(t-3))$

$F(s) = e^{-3s}\cdot\mathcal{L}(g(t+3))$

$= e^{-3s}\cdot\mathcal{L}(t+3)^2$

$= e^{-3s}\cdot\mathcal{L}(t^2+6t+9)$

$= e^{-3s}\cdot[\mathcal{L}(t^2)+\mathcal{L}(6t)+\mathcal{L}(u)]$

$= e^{-3s}\left(\frac{2!}{s^3}+6\cdot\frac{1!}{s^2}+\frac{9}{s}\right)$

② $g(t) = f\left(\frac{t}{\pi}\right)$

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \lim_{h\to\infty}\left[\int_0^h e^{-st} f(t)\,dt\right]$

So we have

$G(s) = \pi F(\pi s)$

$\mathcal{L}\left(f\left(\frac{t}{\pi}\right)\right) = \int_0^\infty f\left(\frac{t}{\pi}\right) e^{-s\left(\frac{t}{\pi}\right)}\,dt = \int_0^\infty f\left(\frac{t}{\pi}\right) e^{-\frac{st}{\pi}}\,dt$

$G(s) = \int_0^\infty e^{-s(\pi u)} f(u)\cdot\pi\,du = \pi\int_0^\infty e^{-\pi s u} f(u)\,du = \pi F(\pi s)$

Find dt, du, t

$u = \frac{t}{\pi} \Rightarrow t = \pi u$

$\frac{du}{dt} = \frac{1}{\pi}$

$du = \frac{1}{\pi}dt$

$dt = \pi du$

③ $F(s) = \frac{s+8}{s^2-16} = \frac{s+8}{(s-4)(s+4)} = \overbrace{\frac{A}{(s-4)} + \frac{B}{(s+4)}}^{A} = \frac{(s+4)A+(s-4)B}{(s-4)(s+4)}$

$s+8 = (s+4)A + (s-4)B =$

$s=-4$ ① $\qquad s=4$ ②

① $4 = 0A - 8B \Rightarrow B = -\frac{1}{2}$

② $12 = 8A + 0B \Rightarrow A = \frac{12}{8} = \frac{3}{2}$

$B, A \to$ A

$F(s) = \frac{\left(\frac{3}{2}\right)}{(s-4)} - \frac{\left(\frac{1}{2}\right)}{(s+4)} = \frac{3}{2}\cdot\left(\frac{1}{s-4}\right) - \frac{1}{2}\left(\frac{1}{s+4}\right)$

$$\boxed{\mathcal{L}^{-1} F(s) = \frac{3}{2}e^{4t} - \frac{1}{2}e^{-4t}}$$

Problem 5. Consider the following system of differential equations:

$$① \quad \frac{dx}{dt} = -2x + 2y + e^{-t},$$

$$② \quad \frac{dy}{dt} = x - y.$$

Use Laplace transforms to solve the given system of ODEs, subject to the initial conditions $x(0) = 0$ and $y(0) = 0$. (**5 marks**)

$① \quad \frac{dx}{dt} = -2x + 2y + e^{-t}$ $\qquad x(0)=0$

$② \quad \frac{dy}{dt} = \frac{dy}{dt} = x - y$ $\qquad y(0)=0$

$\mathcal{L}(\dot{x}) = sX(s) - x_0$ $\qquad (x(0)=0,\ y(0)=0)$

$\mathcal{L}(\dot{y}) = sY(s) - y_0$

For ①

$sX(s) = -2X(s) + 2Y(s) + \frac{1}{s+1}$

$s[Y(s)(s+1)] = -2[Y(s)(s+1)] + 2Y(s) + \frac{1}{s+1} - 2[Y(s)(s+1)] + 2Y(s) - s[Y(s)(s+1)] + \frac{1}{s+1} = 0$

$Y(s)[-2(s+1) + 2 - s(s+1)] + \frac{1}{s+1} = 0$

$Y(s)[-2s - \cancel{2+2} - s^2 - s] + \frac{1}{s+1} = 0$

$Y(s)[-s^2 - 3s] + \frac{1}{s+1} = 0$

$Y(s) = \frac{-\left(\frac{1}{s+1}\right)}{-s^2-3s} = \frac{1}{(-s^2-3s)(s+1)} = \frac{1}{-s^3-s^2-3s-3s} = \frac{1}{s^3+2s^2+3s} = \frac{1}{s(s+3)(s+1)}$

$Y(s) = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1} = \underbrace{\frac{A(s+3)(s+1)}{s} + \frac{B(s)(s+1)}{s+3} + \frac{C(s)(s+3)}{s+1}}_{} = \frac{1}{s(s+3)(s+1)}$

$\begin{bmatrix} s = -3 \\ s = -1 \\ s = 0 \end{bmatrix}$

For $s = -3$

$0A + 6B + 0C = 1$

$\Rightarrow B = \frac{1}{6}$

For $s = -1$

$0A + 0B + 2C = 1$

$-2C = 1$

$\Rightarrow C = -\frac{1}{2}$

For $s = 0$

$3A + 0B + 0C$

$3A = 1$

$\Rightarrow A = \frac{1}{3}$

So we obtain $Y(s) = \left(\frac{1}{3} \cdot \frac{1}{s}\right) + \left(\frac{1}{6} \cdot \frac{1}{s+3}\right) - \left(\frac{1}{2} \cdot \frac{1}{s+1}\right)$

Thus $y(t) = \mathcal{L}^{-1}\, Y(s) = \frac{1}{3} + \frac{1}{6} e^{-3t} - \frac{1}{2} e^{-t}$

Now to $Y(s) \rightarrow$ ① $Y(s)(s+1) = X(s)$

$\frac{\cancel{(s+1)}}{s(s+3)\cancel{(s+1)}} = X(s)$

$X(s) = \frac{A}{s} + \frac{B}{s+3} = \frac{1}{s(s+3)}$

$A(s+3) + Bs = 1$

$s = -3 \qquad s = 0$

For $s = 0$

$3A = 1$

$A = \frac{1}{3}$

For $s = -3$

$-3B = 1$

$B = -\frac{1}{3}$

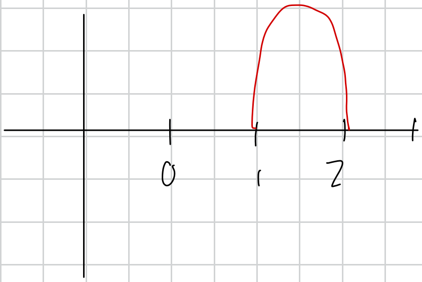
$X(s) = \frac{1}{3} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{1}{s+3}$

$x(t) = \mathcal{L}^{-1}\, X(s) = \frac{1}{3} - \frac{1}{3} e^{-3t}$

Problem 6. Write the following function in terms of step functions. Then, use the second shifting theorem to find its Laplace transform.

$$f(t) = \begin{cases} -\sin(\pi t) & 1 < t < 2 \\ 0 & t \leq 1 \text{ or } t \geq 2 \end{cases}.$$

$$f(t) = \begin{cases} -\sin(\pi t) & 1 \leq t < 2 \\ 0 & \text{elsewhere} \end{cases}$$



$u(t-a) - u(t-b)$

Gen Form

$\mathcal{L}\{f(t-h)\cdot u(t-h)\} = e^{-sh} F(s)$

$\sin(\pi(t-1)) = \sin(\pi t - \pi) = -\sin(\pi t)$

$\Rightarrow \sin(\pi t + 2\pi) = \sin(\pi t)$

$f(t) = -\sin(\pi t)[u(t-1) - u(t-2)] = -\sin(\pi t)u(t-1) + \sin(\pi t)u(t-2)$

$F(s) = \mathcal{L}\{-\sin(\pi t)u(t-1)\} + \mathcal{L}\{\sin(\pi t)u(t-2)\}$

$= \mathcal{L}\{-\sin(\pi(t+1))\}e^{-s} + \mathcal{L}\{\sin(\pi t)\}e^{-2s}$

$= e^{-s}\left(\frac{\pi}{s^2+\pi^2}\right) + e^{-2s}\left(\frac{\pi}{s^2+\pi^2}\right)$

$$\boxed{F(s) = \frac{\pi}{s^2+\pi^2}\left(e^{-s} + e^{-2s}\right)}$$