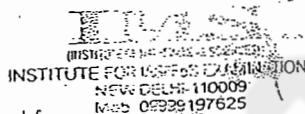


IMS
MATHS
BOOK - 02

Set-IPARTIAL DIFFERENTIAL EQUATIONS

Partial diff. eqn: An eqn involving the derivatives of a dependent variable w.r.t more than one independent variable, is called a PDE.

$$\text{Ex: (1)} \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2$$

$$(2) \quad \frac{\partial^2 z}{\partial x^2} \neq k \left(\frac{\partial^3 z}{\partial x^3} \right)^2$$

$$(3) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Order of PDE: The order of the highest order derivative involving in a differential eqn is called the order of the PDE.

The examples (1), (2) and (3) orders are one, three & two respectively.

Degree of PDE: The degree (i.e., power) of the highest order derivative involving in the diff. eqn is called the degree of PDE.

The above examples (1), (2) & (3) degrees are one, two and one.

Linear partial diff. eqn: A partial diff. eqn is said to be linear if (i) the dependent variable say z and all its partial derivatives occur in first degree only and (ii) no product of dependent variable (or) partial derivatives occur.

$$\text{Ex: (1)} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{are linear.}$$

$$(2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$(3) \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{are not linear.}$$

$$(4) \quad -\frac{\partial^2 z}{\partial x^2} = k \left(\frac{\partial^3 z}{\partial x^3} \right)^2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{are not linear.}$$

- An eqn which is not linear is called non-linear PDE.
- In the case of two independent variables x and y , z will usually be taken as the independent and z as the dependent variable.
- The partial diff. coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are denoted by p & q .
i.e., $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.
- The second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by r, s, t .
i.e., $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, and $t = \frac{\partial^2 z}{\partial y^2}$.

Note In the case of n independent variables, we take them to be x_1, x_2, \dots, x_n and z as the dependent variable. In this case we use the following notations.

$$P_1 = \frac{\partial z}{\partial x_1}, P_2 = \frac{\partial z}{\partial x_2}, P_3 = \frac{\partial z}{\partial x_3}, \dots, P_n = \frac{\partial z}{\partial x_n}.$$

- (2) Sometimes the partial derivatives are also denoted by suffixes.

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \text{ and so on.}$$

Formation (Derivation) of PDE:

— Partial diff. eqns can derived in two ways.

- (I) By the elimination of arbitrary constants from a relation b/w x, y and z
and (II) By the elimination of arbitrary functions of three variables.

I. By the elimination of arbitrary constants:

Let z be a function of x and y such that

$$f(x, y, z, a, b) = 0 \quad \text{where } a \text{ & } b \text{ are arbitrary constants.}$$

Differentiating (1) partially w.r.t x & y we get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\text{i.e., } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} P = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} Q = 0 \quad \leftarrow (2)$$

Now eliminating 'a' and 'b' from (2) & (3)
we obtain an eqn of the form

$$f(x, y, z, P, Q) = 0 \quad \leftarrow (4)$$

which is the required PDE of first order

Note: If the number of arbitrary constants to be eliminated is equal to the number of independent variables then the derived partial differential eqn is of the first order.

But if the number of arbitrary constants to be eliminated is greater than number of independent variables then the derived partial diff. eqns will be of the second order (or) higher orders.

II. By the elimination of arbitrary functions.

Suppose we have a relation between x, y and z of the type $f(u, v) = 0 \leftarrow (1$

where u and v are known as functions of x, y & z and f is arbitrary function of u & v .

Now we treat z dependent variable and x & y are independent variables.

Differentiating (1) w.r.t x we get,

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} \frac{\partial z}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial z}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} \frac{\partial z}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial z}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + 0 + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + 0 + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0 \quad (\because p = \frac{\partial z}{\partial x})$$

$$\Rightarrow \frac{\partial f}{\partial u} / \frac{\partial f}{\partial v} = - \frac{\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right)}{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right)} \quad \text{--- (2)}$$

Similarly differentiating ① w.r.t y we get

$$\frac{\partial f}{\partial u} / \frac{\partial f}{\partial q} = - \frac{\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right)}{\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right)} \quad \text{--- (3)} \quad (\because q = \frac{\partial z}{\partial y})$$

Now eliminating f from ② & ③ we get

$$\frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot p}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot p} = \frac{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q}{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q}$$

$$\Rightarrow \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) = \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial z} \right) p + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right) q \\ = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

$$\Rightarrow P.p + Q.q = R \quad \text{--- (4)}$$

$$\text{where } P = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial z} = \frac{\partial(u, v)}{\partial(y, z)}$$

$$Q = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} = \frac{\partial(u, v)}{\partial(x, z)}$$

$$R = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}$$

The eqn ④ is a PDE of the first order.

Note: ① If the given relation between x, y, z contains two arbitrary functions then the derived partial diff. eqn will contain partial derivatives of an order higher than two except in particular cases.

② The PDE ④ derived in ① is a linear i.e., powers of p & q are both unity while the PDE ④ derived in ② need not be linear.

Type-II

(1) → Form a PDE by elimination of arbitrary constants a & b from the eqn $z = ax + by + a^2 + b^2$

Sol: Given eqn is $z = ax + by + a^2 + b^2 \quad \text{--- (1)}$

Diff. (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = a \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = b \quad \text{--- (3)}$$

Now eliminating a, b from (2), (2) & (3) we get

$$z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required PDE.

(2) → Eliminate a and b from $z = axe^x + \frac{1}{2}a^2e^{2x} + b$.

Sol: Given eqn is $z = axe^x + \frac{1}{2}a^2e^{2x} + b \quad \text{--- (1)}$

Diff (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = ae^x \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = axe^x + a^2e^{2x}$$

$$= a(ae^x) + (ae^x)^2 \quad \text{--- (3)}$$

Now sub (2) in eqn (3)

$$(3) \Rightarrow \frac{\partial z}{\partial y} = a\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required PDE.

→ Form a PDE by eliminating arbitrary constants from the following relations.

$$(3) z = ax + (1-a)y + b ; a, b \quad (9) z = (x+a)(y+b); a, b$$

$$(4) az + b = a^2x + y ; a, b \quad (10) z = Ae^{pt} \sin(pyt)$$

$$(5) z = (x-a)^2 + (y-b)^2 ; a, b$$

$$(6) z = a(x+y) + b ; a, b$$

$$(7) z = ax + by + ab ; a, b$$

$$(8) z = ax + a^2y + b ; a, b$$

→ Form a PDE by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol:

$$\text{Given } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots \quad (1)$$

Differentiating (1) w.r.t x & y , we get

$$\frac{\partial x}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{x}{a^2} + \frac{z}{c^2} \frac{\partial z}{\partial x} = 0 \quad \dots \quad (2)$$

$$\text{and } \frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{y}{b^2} + \frac{z}{c^2} \frac{\partial z}{\partial y} = 0 \quad \dots \quad (3)$$

Differentiating (2) w.r.t x and (3) w.r.t y , we find

$$\frac{1}{a^2} \left(\frac{\partial z}{\partial x} \right)^2 + \frac{z}{c^2} \frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow c^2 + a^2 \left(\frac{\partial z}{\partial x} \right)^2 + a^2 \frac{\partial^2 z}{\partial x^2} = 0 \quad \dots \quad (4)$$

$$\text{and } \frac{1}{b^2} + \frac{1}{c^2} \left(\frac{\partial z}{\partial y} \right)^2 + \frac{z}{c^2} \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow c^2 + b^2 \left(\frac{\partial z}{\partial y} \right)^2 + b^2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \dots \quad (5)$$

$$\text{from (2)} \quad \tilde{z} = - \frac{y \frac{\partial z}{\partial x}}{x} \quad \dots \quad (6)$$

Sub (6) in (4), we get-

$$(4) \Rightarrow - \frac{a^2}{x} \frac{\partial z}{\partial x} + a^2 \left(\frac{\partial z}{\partial x} \right)^2 + \frac{a^2}{x} \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow a^2 \left[- \frac{z}{x} \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x} \right)^2 + \frac{z}{x} \frac{\partial^2 z}{\partial x^2} \right] = 0$$

$$(6) \Rightarrow -z \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x} \right)^2 + z x \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow x z \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x} \right)^2 - z \frac{\partial z}{\partial x} = 0 \quad \dots \quad (7)$$

Similarly, from (3) & (5),

$$z y \frac{\partial^2 z}{\partial y^2} + y \left(\frac{\partial z}{\partial y} \right)^2 - z \frac{\partial z}{\partial y} = 0 \quad \dots \quad (8)$$

Thus (7) & (8) are two possible forms of
the required equations of order 2.

Q.1 Find the differential eqn of all spheres of radius r having centre in the xy -plane. (IAS-96) (1)

Sol> The eqn of any sphere of radius r , having centre $(h, k, 0)$ in the xy -plane is given by

$$(x-h)^2 + (y-k)^2 + (z-0)^2 = r^2 \text{ where } h \text{ and } k \text{ are}$$

$$\rightarrow (x-a)^2 + (y-b)^2 + z^2 = r^2 \quad \text{(1) arbitrary constants}$$

Differentiating eqn (1) partially w.r.t x & y

$$(x-h) + z \frac{\partial z}{\partial x} = 0 \Rightarrow (x-h) = -z^p \quad (2)$$

$$\left(\because \frac{\partial z}{\partial x} = p\right)$$

$$(y-b) + z \frac{\partial z}{\partial y} = 0 \Rightarrow (y-b) = -z^q \quad (3) \quad \left(\because \frac{\partial z}{\partial y} = q\right)$$

Sub. (2) & (3) in eqn (1)

$$x^p + z^q + z^r = r^2$$

$$z^r (p+q+r) = r^2.$$

which is the required partial differential equation

Q.2 Form the differential eqn by eliminating a and b

$$\text{from } z = (x+a)(y+b)$$

Given $z = (x+a)(y+b)$. (1)
Differentiating (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = 2x(y+b)$$

$$P = 2x(y+b)$$

$$\Rightarrow y+b = P/x \quad (2)$$

$$\frac{\partial z}{\partial y} = 2y(x+a)$$

$$Q = 2y(x+a) \quad (3)$$

$$\Rightarrow x+a = Q/2y$$

Sub (2) and (3) in eqn (1)

$$z = \frac{P}{2x} - \frac{Q}{2y}$$

$$\Rightarrow z = \frac{P}{2x} \quad \rightarrow 4xyz = Pz$$

which is the required partial differential equation

(3)

find the differential equation of the set of all right circular cones whose axes coincide with z-axis.

Sol: The general equation of the set of all right circular cones whose axes coincide with z-axis having semivertical angle α and vertex at $(0, 0, c)$ is given by

$$x^2 + y^2 + (z-c)^2 \tan^2 \alpha = 1 \quad (1)$$

where α and c are arbitrary constants.

Differentiating eq(1) partially w.r.t $x, y,$

$$2x = 2(z-c) \tan \alpha \frac{\partial z}{\partial x}$$

$$\Rightarrow x = p(z-c) \tan \alpha \quad (2)$$

$$2y = 2(z-c) \frac{\partial z}{\partial y} \tan \alpha$$

$$\Rightarrow y = q(z-c) \tan \alpha \quad (3)$$

$$\text{from (3)} \quad z-c = \frac{y}{q \tan \alpha} \quad (4)$$

Sub (4) in (2)

$$x = p \frac{y}{q \tan \alpha} \tan \alpha$$

$$\Rightarrow x = py/q$$

$\Rightarrow qz = py$ Which is the required equation.

(4)

Eliminate a, b and c from $z = a(x+y) + b(x-y) + abt + c$

$$\text{Given } z = a(x+y) + b(x-y) + abt + c \quad (1)$$

Differentiating eq(1) partially w.r.t a, b, t

$$\frac{\partial z}{\partial a} = x + y \quad (2)$$

$$\frac{\partial z}{\partial y} = a+b \quad \text{--- (3)}$$

$$\frac{\partial z}{\partial x} = ab \quad \text{--- (4)}$$

$$\text{w.t. } Ha b = (a+b)^2 \rightarrow (ab)^2$$

from (2), (3) & (4)

$$A \frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2$$

IAS 2009

Show that the differential equation of all cones which have their vertex at the origin is $Px + qy = z$ verify that $y^2 + 2xz + xy = 0$ is a surface satisfying the above equation.

The eqn of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{--- (1)}$$

where a, b, c, f, g, h are parameters.

Differentiating eqn (1) partially w.r.t. x & y :

$$2ax + 2by \frac{\partial z}{\partial x} + 2g(x \frac{\partial z}{\partial x} + z) + 2hy = 0$$

$$ax + gy + hy + P(cz + fy + gx) = 0 \quad \text{--- (2)}$$

$$2by + 2cz \frac{\partial z}{\partial y} + 2f(y \frac{\partial z}{\partial y} + z) + 2gx \frac{\partial z}{\partial y} + 2g(x \frac{\partial z}{\partial y} + z) = 0$$

$$-by + fz + bx + q((cz + fy + gx) = 0 \quad \text{--- (3)}$$

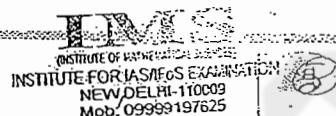
Multiplying (2) by x and (3) by y and adding, we have

$$ax^2 + gy^2 + hy^2 + P(cz^2 + fy^2 + gx^2) + by^2 + fz^2 + hy^2 + q((cz^2 + fy^2 + gx^2) = 0$$

$$\Rightarrow (ax^2 + by^2 + gz^2 + fy^2 + 2hxy) + P(cz^2 + fy^2 + gx^2) + 2yz(cz + fy + gx) = 0$$

$$\Rightarrow (ax^2 + by^2 + gz^2 + fy^2 + 2hxy) + (P + qy)(cz + fy + gx) = 0 \quad \text{--- (4)}$$

$$\text{from eqn (1)} \quad ax^2 + by^2 + gz^2 + fy^2 = -cz^2 - fz^2 - gy^2 \quad \text{--- (5)}$$



sub eqn ⑦ in eqn ④

$$-(cz + fy + gz) + (cz + fy + gz)(px + qy) = 0$$

$$\Rightarrow -z(cz + fy + gz) + (cz + fy + gz)(px + qy) = 0$$

$$\Rightarrow (cz + fy + gz)(px + qy - z) = 0$$

$$\Rightarrow px + qy - z = 0 \quad \text{--- (A)}$$

which is the required differential equation

$$\text{Given surface } \text{is } yz + zx + xy = 0 \quad \text{--- (6)}$$

Differentiating eqn ⑥ partially w.r.t
x and y

$$y \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} x + z + y = 0 \quad \text{and} \quad y \frac{\partial z}{\partial y} + z + x \frac{\partial z}{\partial y} + x = 0$$

$$\Rightarrow y p + px + z + y = 0 \quad \Rightarrow yq + z + xq + x = 0$$

$$\Rightarrow p(x+y) + z + y = 0 \quad \Rightarrow (x+q)z + (x+z) = 0$$

$$\Rightarrow p = \frac{(z+y)}{x+y} \quad \Rightarrow z = -\frac{(x+z)}{x+y}$$

sub ⑧ and ⑨ in eqn (A)

$$px + qy - z = \frac{-(z+y)x}{x+y} + \frac{-(x+z)y}{x+y} - z$$

$$= \frac{-xz - xy - zy - xz - yz}{x+y}$$

$$= \frac{-2xz - 2xy - 2yz}{x+y}$$

$$= \frac{-2(xz + yz + xy)}{x+y}$$

$$= 0$$

$\therefore xz + yz + xy = 0$
by eqn (6)

∴ eqn ⑥ is a surface

satisfying eqn (A)

Type-II

\checkmark form a PDE by eliminating the arbitrary function ϕ from $z = e^{ny} \phi(x-y)$. — (1)

\checkmark : Differentiating (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = e^{ny} \phi'(x-y)$$

$$\Rightarrow P = e^{ny} \phi'(x-y) \quad (\because \frac{\partial z}{\partial x} = P) \quad (2)$$

$$\text{and } Q = n e^{ny} \phi(x-y) + e^{ny} \phi'(x-y) \quad (1)$$

$$Q = n e^{ny} \phi(x-y) - e^{ny} \phi'(x-y) \quad (3)$$

Sub (1) & (2) in eqn (3)

$$Q = nZ - P$$

$$\Rightarrow P + Q = nZ$$

which is the required PDE of order one.

\checkmark form a PDE by eliminating the arbitrary functions f and F from $z = f(x+ay) + F(x-ay)$

Sol: Given $z = f(x+ay) + F(x-ay)$ — (1)

: Diff (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + F'(x-ay) \quad (2)$$

$$\text{and } \frac{\partial z}{\partial y} = af'(x+ay) - af'(x-ay) \quad (3)$$

Diff (2) & (3) partially w.r.t x & y respectively, we get.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay) \quad (4)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 F''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2 [f''(x+ay) + F''(x-ay)] \quad (5)$$

now sub (4) in (5)

$$\text{we get } \left[\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2} \right] \text{ which is the required PDE}$$

~~(3)~~ $\Rightarrow z = y + 2f\left(\frac{1}{x} + \log y\right); f$ is an arbitrary function.

~~(3)~~ Given $z = y + 2f\left(\frac{1}{x} + \log y\right) \quad \text{--- (1)}$

Diff (1) partially w.r.t x & y , we get

$$P = 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(-\frac{1}{x^2}\right) \quad \text{--- (2)}$$

$$\text{and } Q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y} \quad \text{--- (3)}$$

$$\text{②} \quad 2f'\left(\frac{1}{x} + \log y\right) = -Px^2 \quad \text{--- (4)}$$

Sub (4) in (3), we get

$$Q = 2y - Px^2 \cdot \frac{1}{y}$$

$$\Rightarrow Qy = 2y - Px^2$$

$$\Rightarrow \boxed{Px^2 + Qy = 2y}$$

which is the required PDE of order one

~~(4)~~ $\checkmark z = x^n f(y/x); f$ is an arbitrary function

~~(1)~~ Given $z = x^n f(y/x) \quad \text{--- (1)}$

Diff (1) partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = nx^{n-1}f(y/x) + x^n f'(y/x) \cdot \left(-\frac{y}{x^2}\right) \quad \text{--- (2)}$$

$$\text{and } \frac{\partial z}{\partial y} = x^n f'(y/x) \cdot \left(\frac{1}{x}\right)$$

$$\Rightarrow x \frac{\partial z}{\partial y} = x^n f'(y/x) \quad \text{--- (3)}$$

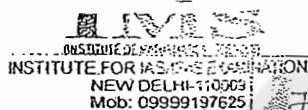
$$\text{②} \quad \frac{\partial z}{\partial x} = n \frac{x^n}{x} f(y/x) + x^n f'(y/x) \cdot \left(-\frac{y}{x^2}\right) \quad \text{--- (4)}$$

Sub (3) & (4) in (4), we get

$$\frac{\partial z}{\partial x} = n z + x \left(\frac{\partial z}{\partial y} \right) \left(-\frac{y}{x^2} \right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = n z - \frac{y}{x} \left(\frac{\partial z}{\partial y} \right)$$

$$\Rightarrow \boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz}, \text{ which is the required PDE}$$



→ form partial diff. eqns by eliminating the arbitrary functions from the following equations.

$$(5) z = f(x+iy) + f(x-iy) \quad (8) z = ff(x-y)$$

$$(6) z = e^{ax+by} f(ax+by) \quad (9) z = f(x+y)$$

$$(7) x^a y^b + n z = \phi(x^a y^b + z^2) \quad (10) z = f(y/x)$$

$$(11) \rightarrow \phi(x+y+z, x^a y^b - z^2) = 0$$

$$\text{SOP: Given } \phi(x+y+z, x^a y^b - z^2) = 0$$

$$\text{let } u = x+y+z, v = x^a y^b - z^2$$

$$\text{Then the given eqn is } \phi(u, v) = 0 \quad \text{--- (1)}$$

Diff (1) w.r.t x partially, we get

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + P \frac{\partial v}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + P \frac{\partial v}{\partial z} \right) = 0$$

$$\frac{\partial \phi}{\partial u} (1+P) + \frac{\partial \phi}{\partial v} (2x - 2zP) = 0 \quad \text{--- (2)}$$

$$\left(\because \frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial z} = -2z \right)$$

Diff (1) w.r.t y partially, we get

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + Q \frac{\partial v}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + Q \frac{\partial v}{\partial z} \right) = 0$$

$$\frac{\partial \phi}{\partial u} (1+Q) + \frac{\partial \phi}{\partial v} (2y - 2xz) = 0 \quad \text{--- (3)}$$

$$\text{from (2)} \quad \frac{\partial \phi}{\partial u} / \frac{\partial \phi}{\partial v} = -\frac{2(x-zP)}{1+P} \quad \text{--- (4)}$$

$$\text{and from (3)} \quad \frac{\partial \phi}{\partial u} / \frac{\partial \phi}{\partial v} = -\frac{2(y-xz)}{1+Q} \quad \text{--- (5)}$$

from (4) & (5)

$$\frac{-x(z-P)}{(1+P)} = \frac{-y(z-Q)}{(1+Q)}$$

$$\Rightarrow (1+Q)(x-zP) = (1+P)(y-zQ)$$

$$\Rightarrow x + xz - zP - zPQ = y + yz - zQ - zQ$$

$$\Rightarrow |P(y+z) - (x+z)Q = x - y|$$

which is the required p.d.e of order one

$$\Rightarrow z = f(x-y) + g(x+y)$$

$$\text{Given } z = f(x-y) + g(x+y) \quad \dots \textcircled{1}$$

Diff \textcircled{1} partially w.r.t x & y , we get

$$\frac{\partial z}{\partial x} = f'(x-y) 2x + g'(x+y) 2x.$$

$$= 2x [f'(x-y) + g'(x+y)] \quad \dots \textcircled{2}$$

$$\text{and } \frac{\partial z}{\partial y} = f'(x-y)(-1) + g'(x+y) \quad \dots \textcircled{3}$$

Diff \textcircled{2} & \textcircled{3} partially w.r.t x & y respectively

$$\frac{\partial^2 z}{\partial x^2} = 2x [f''(x-y)(2x) + g''(x+y) \cdot 2x]$$

$$+ 2[f'(x-y) + g'(x+y)]$$

$$= 4x^2 [f''(x-y) + g''(x+y)] + 2[f'(x-y) + g'(x+y)] \quad \dots \textcircled{4}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x-y) + g''(x+y) \quad \dots \textcircled{5}$$

$$\text{from eqn } \textcircled{2}, f'(x-y) + g'(x+y) = \frac{1}{2x} \left(\frac{\partial z}{\partial x} \right) \quad \dots \textcircled{6}$$

Sub \textcircled{5} & \textcircled{6} in \textcircled{4}

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 \left(\frac{\partial z}{\partial y^2} \right) + 2 \cdot \left(\frac{1}{2x} \right) \left(\frac{\partial z}{\partial x} \right)$$

$$\Rightarrow \boxed{x \frac{\partial^2 z}{\partial x^2} = 4x^3 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x}}$$

Which is the required PDE

Equations Solvable by direct integrations

8

We now consider the PDE's which can be solved by direct integration. In place of the usual constants of integration, we must use arbitrary functions of the variable held fixed.

(a) solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18x^2 y^2 + \sin(2x-y) = 0$

Solve:

Integrating twice w.r.t x. and keeping y fixed,

we get

$$\frac{\partial^2 z}{\partial x^2 \partial y} + 9x^2 y^2 - \frac{1}{2} \cos(2x-y) = f(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^2 y^2 - \frac{1}{4} \sin(2x-y) = x f(y) + g(y)$$

Now integrating w.r.t y & keeping x fixed,
we get

$$z + x^2 y^3 - \frac{1}{4} \cos(2x-y) = x \int f(y) dy + \int g(y) dy + h(x)$$

Taking $\int f(y) dy = u(y)$

$\int g(y) dy = v(y)$

$$z + x^2 y^3 - \frac{1}{4} \cos(2x-y) = x u(y) + v(y) + w(x)$$

where u, v, w are arbitrary function

(b) Solve $\frac{\partial z}{\partial x} + z = 0$; given that when $x=0, z=e^y$

and $\frac{\partial z}{\partial x} = f$

Ans: $z = e^y + e^y \cos x$

C) Solve the following eqns:

(1) $\frac{\partial z}{\partial x} = \frac{x}{y} + a$

(2) $\frac{\partial z}{\partial x^2} = xy$

(3) $\frac{\partial^2 u}{\partial x \partial t} = e^t \cos x$

(4) $\frac{\partial z}{\partial x} = a^2 z$; given that when $x=0, \frac{\partial z}{\partial x} = \sin x$ and $\frac{\partial z}{\partial y} = 0$.

PDE of order one:-

Classification of first order partial diff. eqns are: (1) linear, (2) semi-linear (3) quasilinear and (4) non-linear eqns.

- (1) Linear eqn: A first-order eqn $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q , and z .
- i.e, if the given eqn is of the form

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

Eg: (1) $y^2p + x^2y^2q = xyz + x^2y^3$

(2) $p + q = z + xy$.

- (2) Semi-linear: A first order partial diff. eqn $f(x, y, z, p, q) = 0$ is known as semi-linear eqn if it is linear in p and q , and the coefficients of p & q are functions of x & y only.
- i.e, if the given eqn is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

Eg: (1) $xyp + x^2y^2q = x^2y^2z$

(2) $yzp + xq = \frac{x^2}{yz}$.

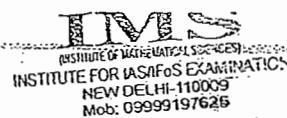
- (3) Quasi-linear eqn: A first order PDE $f(x, y, z, p, q) = 0$ is known as quasi-linear eqn, if it is linear p & q .
- i.e, if the given eqn of the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

Eg: (1) $x^2p + y^2q = xy$

(2) $(x^2 - y^2)p + (y^2 - x^2)q = z - xy$.

- (4) Non-linear: A first order PDE $f(x, y, z, p, q) = 0$ which doesn't come under above three types, is known as a



non-linear eqn.

$$\text{Ex: } (1) \quad p^2 + q^2 = 1$$

$$(2) \quad pq = z$$

$$(3) \quad x^2 p^2 + y^2 q^2 = z^2.$$

Defn: A linear PDE of the first order is known as Lagrange's linear eqn, if it is of the form $Pp + Qq = R$. — (1)

where P, Q, R are functions of x, y, z .

This eqn is called a quasi-linear equation.

This eqn (1) is obtained by eliminating an arbitrary function f from $f(u, v) = 0$ — (2)

where u, v are functions of x, y, z .

Theorem: The general solution of the linear PDE.

1987 $Pp + Qq = R$ — (1) is $f(u, v) = 0$ — (2) where f is arbitrary function

and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution

of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — (3)

where P, Q, R are functions of x, y, z .

Proof:

Now diff. (2) partially w.r.t. x & y , we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) = 0$$

$$\text{and } \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) = 0.$$

Now eliminating $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} P & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} P \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} Q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} Q \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} \right) P + \left(\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} \right) Q = 0$$

$$= \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} \right)$$

$$\Rightarrow Pp + Qq = R$$

- where $P = \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \right)$

$$Q = \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

which is of the same form of eqn①

∴ ② is g.s. of ①.

Now consider $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$

where c_1 & c_2 are arbitrary constants.

By differentiating, we get

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \text{--- (i)}$$

$$\text{and } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

$$\Rightarrow \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \quad \text{--- (ii)}$$

By cross multiplication we get

$$\frac{dx}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y}}$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which is known as the eqn ④

$u(x, y, z) = c_1$ & $v(x, y, z) = c_2$ are solutions

of ④.

Note: Equations ④ are called Lagrange's auxiliary eqns (or) subsidiary eqns for ①.

Working rule for solving of Lagrange's eqn

$$Pp + Qq = R$$

Step 1: Write the given eqn in standard form $Pp + Qq = R$ ①

Step 2: Write the Lagrange's auxiliary eqn for ①.

namely $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (2)}$

Step 3: Solve these simultaneous eqns. (2) by using the well known methods.

Let $u(x, y, z) = C_1$, $v(x, y, z) = C_2$ be two independent solutions of (2).

Step 4: Write the g.s. of (1) as $f(u, v) = 0$ (say)

$$u = \phi(v) \text{ or } v = \psi(u)$$

Methods to solve the simultaneous eqns $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Given eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (1)}$

where P, Q, R are functions of x, y, z .

It can be solved in three methods.

Consider the three sets of eqns.

$$\frac{dx}{P} = \frac{dy}{Q}, \quad \frac{dx}{P} = \frac{dz}{R}, \quad \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (2)}$$

Method [1]: If any two eqns of (2) are integrable by the method of variables separable, we find their general solutions and that pair of solutions form the complete solution of the system (1).

Method [2]: If one eqn of (2) only integrable, by the method of variables separable, we can find its g.s. and this solution may be used to find the solution of another set of eqn (2).

The pair of these solutions give the g.s. of the given equation (1).

Method [3]: If no eqn of (2) is integrable, then we

$$\text{write } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = \frac{l_2 dx + m_2 dy + n_2 dz}{l_2 P + m_2 Q + n_2 R}$$

where l_1, m_1, n_1 ; l_2, m_2, n_2 are real numbers or functions of x, y, z .

Case(i) If we choose l_1, m_1, n_1 and l_2, m_2, n_2 such that $l_1P + m_1Q + n_1R = 0$ and $l_2P + m_2Q + n_2R = 0$ then $l_1dx + m_1dy + n_1dz = 0$ and $l_2dx + m_2dy + n_2dz = 0$ which on integration gives two eqns.
∴ These eqns. together give the complete solution.

Case(ii): If we choose l_1, m_1, n_1 & l_2, m_2, n_2 such that, $l_1P + m_1Q + n_1R \neq 0$; $\frac{l_1dx + m_1dy + n_1dz}{l_1P + m_1Q + n_1R} = d\phi$ and $l_2P + m_2Q + n_2R \neq 0$; $\frac{l_2dx + m_2dy + n_2dz}{l_2P + m_2Q + n_2R} = d\psi$
then $\phi(x, y, z) = C_1$, $\psi(x, y, z) = C_2$ will become the g-e. of system (i).

Note: l_1, m_1, n_1 and l_2, m_2, n_2 are called multipliers.

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problems based on Method II

(1) Solve $x^2P + y^2Q = z^2$.

Sol: Given $x^2P + y^2Q = z^2 \quad \text{--- (1)}$

Clearly which is in the form of $Px^2 + Qy^2 = R$

Here $P = x^2$; $Q = y^2$; $R = z^2$

Now the Lagrange's auxiliary eqns of (1) are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{--- (2)}$$

Now taking the first two fractions of (2), we get

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \left[-\frac{1}{x} + \frac{1}{y} = C_1 \right] \quad \text{--- (3)}$$

Now taking the first and the last fractions of (2), we get

$$\frac{dx}{x^2} = \frac{dz}{z^2} \Rightarrow \left[-\frac{1}{x} + \frac{1}{z} = C_2 \right] \quad \text{--- (4)}$$

∴ from (3) & (4) the required G.S. of (1) is

$$f\left(-\frac{1}{x} + \frac{1}{y}, -\frac{1}{x} + \frac{1}{z}\right) = 0 \quad \text{where } f \text{ is an arbitrary function.}$$

(2) Solve $\left(\frac{y^2}{x}\right)P + x^2Q = y^2$

Sol: Given that $\left(\frac{y^2}{x}\right)P + x^2Q = y^2 \quad \text{--- (1)}$

Clearly which is in the form of $Px^2 + Qy^2 = R$

Here $P = \frac{y^2}{x}$; $Q = x^2$ and $R = y^2$.

Now the Lagrange's auxiliary eqns of (1) are

$$\frac{dx}{\frac{y^2}{x}} = \frac{dy}{x^2} = \frac{dz}{y^2} \quad \text{--- (2)}$$

Taking the first two fractions of (2), we get

$$\frac{x^2 dx}{y^2} = \frac{dy}{x^2} \Rightarrow \frac{x^2 dz}{y^2} = \frac{dy}{x}$$

$$\Rightarrow x^2 dz = y^2 dy$$

$$\Rightarrow x^3 - y^3 = C_1 \quad \text{--- (3)}$$

Taking the first and last fractions of (2), we get

$$\frac{xdz}{y^2} = \frac{dz}{y^2} \Rightarrow x dz = z dz$$

$$\Rightarrow x^2 - z^2 = 2C_2 \quad \text{--- (4)}$$

The g.s. of ① is $f(x^2+y^2, z^2-2xy) = 0$

where f is arbitrary function

(3) Solve $a(p+q) = z$

(4) Solve $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x$

(5) Solve $zp = -x$.

(6) Solve $p \tan x + q \sec y = \tan z$.

(7) Solve $y^2 p - xy q = x(z - 2y)$

Sol: Given that $y^2 p - xy q = x(z - 2y) \quad \text{--- ①}$
which is in the form of $Pp + Qq = R$

$P = y^2$; $Q = -xy$, $R = x(z - 2y)$.

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dp}{y^2} = \frac{dq}{-xy} = \frac{dz}{x(z-2y)} \quad \text{--- ②}$$

Taking the first two fractions of ②

$$\frac{dy}{y^2} = \frac{dx}{-xy} \Rightarrow -xdx = ydy$$

$$\Rightarrow [x^2 + y^2 = c_1] \quad \text{--- ③}$$

Taking the last two fractions of ②, we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dz}{dy} = \frac{2y-z}{y}$$

$$\Rightarrow \frac{dz}{dy} + \left(\frac{1}{y}\right)z = 2 \quad \text{--- ④}$$

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

G.S. of ④ is

$$z-y = \int 2y dy + C_2$$

$$zy = y^2 + C_2$$

$$\Rightarrow [zy - y^2 = C_2] \quad \text{--- ⑤}$$

The required g.s. of ① is $f(x^2+y^2, zy-y^2) = 0$

where f is an arbitrary function

problems based on Method (2):

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(1) → Solve $P + 3Q = 5z + \tan(y-3x)$.

Sol: Given that $P + 3Q = 5z + \tan(y-3x)$ — (1)

Comparing (1) with $Pp + Qq = R$

$$P=1, Q=3, R=5z+\tan(y-3x)$$

Now the Lagrange's A-Eqns of (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \quad (2)$$

Now taking first two fractions of (2), we get

$$\therefore \frac{dx}{1} = \frac{dy}{3} \Rightarrow \frac{dy}{3} = dx \Rightarrow dy = 3dx \Rightarrow y - 3x = c_1 \quad (3)$$

Now taking last two fractions of (2), we get

$$\frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow \frac{dy}{3} = \frac{dz}{5z+\tan c_1} \quad (\text{from (3)})$$

$$\Rightarrow \frac{1}{3}y = \frac{1}{5}\log(5z+\tan c_1) + c_2$$

$$\Rightarrow \frac{1}{3}y - \frac{1}{5}\log(5z+\tan c_1) = c_2$$

$$\Rightarrow \frac{1}{3}y - \frac{1}{5}\log[5z+\tan(y-3x)] = c_2 \quad (4)$$

G.S. of (4) is

$$f\left(y-3x, \frac{1}{3}y - \frac{1}{5}\log(5z+\tan(y-3x))\right) = 0$$

where f is an arbitrary function.

(2) → Solve $z(z^2+xy)(px-qy)=x^4$.

Sol: Given that $z(z^2+xy)(px-qy)=x^4$

$$\Rightarrow z(z^2+xy)px - z(z^2+xy)qy = x^4$$

$$\Rightarrow z^2(z^2+xy)p + [y - (z^2+xy)]q = x^4 \quad (1)$$

Comparing (1) with $Pp + Qq = R$

$$P = xz(z^2 + xy) ; Q = -yz(z^2 + xy)$$

Now Lagrange's A.E's of (1) are

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{z^4} \quad (2)$$

Taking first two fractions of (2), we get

$$[xy = c_1] \quad (3)$$

Taking first and last fractions of (2), we get

$$\frac{dx}{xz(z^2 + xy)} = \frac{dz}{z^4}$$

$$\Rightarrow \frac{dx}{x(z^2 + c_1)} = \frac{dz}{z^4} \quad (\text{from } 3)$$

$$\Rightarrow \frac{dx}{z(z^2 + c_1)} = \frac{dz}{z^3} \Rightarrow z^3 dz = (z^3 + c_1 z) dz$$

$$\Rightarrow \frac{z^4}{4} = \frac{z^4}{4} + c_1 z^2 + c_2$$

$$\Rightarrow [z^4 - z^4 - 2z^2(xy)] = 4c_2 \quad (4)$$

\therefore Q.E of (1) is

$$f(xy, z^4 - z^4 - 2z^2xy) = 0$$

where f is an arbitrary function.

(3) Solve $xzP + yzQ = xy$

(4) Solve $P - 2Q = 3x^2 \sin(y + 2x)$

Problems based on Method (3): (Casey)

(1) Solve $(mz - ny)P + (nx - lz)Q = ly - mx$

Sol: Given that $(mz - ny)P + (nx - lz)Q = ly - mx \quad (1)$

Comparing (1) with $Pp + Qq = R$

$$P = mz - ny ; Q = nx - lz ; R = ly - mx$$

Now the Lagrange's A.E. of (1) are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad (2)$$

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NOW using the multipliers $x, y \& z$:

$$\text{each fraction of } (2) = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating, we get -

$$[x^2 + y^2 + z^2 = 2c_1] \quad (3)$$

Again using the multipliers l, m, n :

$$\text{each fraction of } (2) = \frac{l dx + m dy + n dz}{0}$$

$$\Rightarrow l dx + m dy + n dz = 0$$

$$\Rightarrow [lx+my+nz = c_2] \quad (4)$$

∴ from (3) & (4), the required g.e. of (1) is

$$f(x, y) = 0$$

$$(2) \rightarrow x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2) \quad \text{Multiples are } x, y, z \& \frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$$

$$(3) \rightarrow x(y-z)p + y(z-x)q = z(x-y) \quad \text{Multiples are } 1, 1 \& \frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$$

$$(4) \rightarrow x(y^2-z^2)p - y(z^2-x^2)q = z(x^2+y^2) \quad \text{Multiples are } x, y, z \& \frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$$

$$(5) \rightarrow (y^2+z^2)p - xyq = -zx \quad \text{Multiples are } 1, 1, 1 \& \frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$$

$$2004 (6) \rightarrow x(y^2+z^2)p - y(x^2+z^2)q = z(x^2-y^2) \quad \text{Multiples are } 1, 1, 1 \& \frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$$

$$(7) \rightarrow \text{Solve } (x-y)p + (x+y)q = 2xz \quad (1)$$

$$\text{Sol: } \frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \quad (2)$$

Taking first two fractions of (2), we get

$$\frac{dx}{x-y} = \frac{dy}{x+y} \Rightarrow (x+y)dx + (y-x)dy = 0$$

$$\Rightarrow (xdx+ydy) + (ydx-xdy) = 0$$

$$\Rightarrow \frac{xdx+ydy}{x^2+y^2} + \frac{ydx-xdy}{x^2+y^2} = 0$$

$$\frac{1}{2} d \log(x+y) + d(\tan^{-1}\left(\frac{y}{x}\right)) = 0$$

$$\frac{1}{2} \log(x+y) + \tan^{-1}\left(\frac{y}{x}\right) = c_1 \quad (3)$$

using the multipliers 1, 1, $-\frac{1}{2}$

each fraction of (3) = $\frac{dx+dy-\frac{1}{2}dz}{(x-y)+(x+y)-\frac{1}{2}(2xz)}$

$$= \frac{dx+dy-\frac{1}{2}dz}{2x-2x} \\ \geq \frac{dx+dy-\frac{1}{2}dz}{0}$$

$$\Rightarrow x+y-\log z = c_2 \quad (4)$$

∴ from (3) & (4) - the required g.s. of (1) is

$$f(u, v) = 0.$$

Case 2

1992 solve $(y+z)p + (z+x)q = x+y \quad (1)$

2012 $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad (2)$

using the multipliers 1, -1, 0

each fraction of (2) = $\frac{dx-dy}{y-x} -$

$$= \frac{d(x-y)}{(x-y)} \quad (3)$$

Again using the multipliers 0, 1, -1.

each fraction of (1) = $\frac{dy-dz}{x-y} -$

$$= \frac{dy-dz}{-(y-z)} \quad (4)$$

Finally, using multipliers 1, 1, 1.

each fraction of (1) = $\frac{dx-dy+dz}{2(x+y+z)} -$

$$= \frac{d(x+y+z)}{2(x+y+z)} \quad (5)$$

from (3), (4) & (5), we have

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad (6)$$

Taking first two fractions of (6)

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)}$$

$$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$$

$$\Rightarrow \boxed{\frac{x-y}{y-z} = c_1} \quad (7)$$

Now taking the last two fractions of (6), we get

$$\log(x+y+z) + \log(y-z) = \log c_2$$

$$\Rightarrow \boxed{(x+y+z)(y-z)^2 = c_2} \quad (8)$$

∴ from (7) & (8),

the required g.s. of (1) is

$$f\left(\frac{x-y}{y-z}, (y-z)^2(x+y+z)\right) = 0$$

1996. Solve $y^z(x-y)^p + x^z(y-x)^q = z(x^z+y^z) \quad (1)$

$$\text{Soln: } \frac{dx}{y^z(x-y)} = \frac{dy}{x^z(y-x)} = \frac{dz}{z(x^z+y^z)} \quad (2)$$

$$\text{from (2)} \quad \frac{dx}{y^z(x-y)} = \frac{dy}{x^z(y-x)}$$

$$\Rightarrow \frac{dx}{y^z} = \frac{dy}{x^z}$$

$$\Rightarrow x^z dx + y^z dy = 0$$

$$\Rightarrow \boxed{x^3 + y^3 = 3z} \quad (3)$$

Choosing the multipliers 1, -1, 0; we get

$$\text{each fraction of (2)} = \frac{dx/dy}{(x-y)(x^z+y^z)} \quad (4)$$

Now creating third fraction of (2) & the fraction (4)

$$\text{we get, } \frac{dx}{z(x^z+y^z)} = \frac{d(x-y)}{(x-y)(x^z+y^z)}$$

$$\log z = \log(x-y) + \log c_1$$

$$\Rightarrow \boxed{\frac{z}{x-y} = c_2} \quad (5)$$

From (4) & (5) the required g.s. of (1) is $\boxed{f(x,y) = 0}$

→ Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ multipliers; x, y, z

→ $(1+y)p + (1+z)q = z$; multipliers; 1, 1, 0

→ $xz p + yz q = xy$; multipliers; $\frac{1}{x}, \frac{1}{y}, 0$

→ Solve $(x^2 - yz)p + (y^2 - zx)q = x^2 - xy$.

Sol: Given that $(x^2 - yz)p + (y^2 - zx)q = x^2 - xy$ — ①

Comparing with $P_1 + Q_1 = R$.

$P = x^2 - yz$; $Q = y^2 - zx$; $R = x^2 - xy$

Now the Lagrange's A.E's are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{x^2 - xy} \quad \text{--- ②}$$

Now using the multipliers 1, -1, 0 and 0, 1, -1 we get, each fraction of ② =

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - z^2 - zx + xy}$$

$$\Rightarrow \frac{dx - dy}{x^2 - y^2 + z(x-y)} = \frac{dy - dz}{y^2 - z^2 + x(y-z)}$$

$$\Rightarrow \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)}$$

$$\Rightarrow \frac{dx - dy}{x^2 - y^2} = \frac{dy - dz}{y^2 - z^2} \quad \text{on integration}$$

$$\Rightarrow \boxed{\frac{x-y}{y^2 - z^2} = c_1} \quad \text{--- ③}$$

Using the multipliers 1, 1, 1; we get

$$\text{each fraction of ②} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

Again using the multipliers x, y, z ; we get

$$\text{each fraction of ②} = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} \\ = \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} \quad \text{--- ④}$$

from ④ & ⑤ we have

$$\frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} = \frac{2dx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\Rightarrow (x+y+z)(dx + dy + dz) = 2dx + ydy + zdz$$

$$\Rightarrow \frac{(x+y+z)^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C$$

$$\Rightarrow (x+y+z)^2 - (x^2 + y^2 + z^2) = 2C$$

$$\Rightarrow \sqrt{xy + yz + zx} = \sqrt{C} \quad \text{--- ⑥}$$

\therefore from ③ & ⑥, the required g.e. of ①

is $f\left(\frac{x-y}{y-z}, \frac{xy+yz+zx}{x^2+y^2+z^2}\right) = 0$
where f is an arbitrary function.

Now solve $\cos(x+y)p + \sin(x+y)q = z \quad \text{--- ⑦}$

$$\text{Com: } \frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \quad \text{--- ⑧}$$

Now using the multipliers 1, 1, 0 and 1, -1, 0.

$$\text{each fraction of ⑧} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{da - dy}{\cos(x+y) - \sin(x+y)} \quad \text{--- ⑨}$$

From ⑧ & ⑨ we have

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{da - dy}{\cos(x+y) - \sin(x+y)} \quad \text{--- ⑩}$$

Now taking last two fractions of ⑩

$$\frac{(dx + dy)}{\cos(x+y) + \sin(x+y)} = \frac{da - dy}{\cos(x+y) - \sin(x+y)}$$

$$\frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} da(x+y) = dx(x+y)$$

$$\Rightarrow \log [\cos(x+y) + \sin(x+y)] = (x-y) + \log a$$

$$\Rightarrow [\cos(x+y) + \sin(x+y)] e^{(x-y)} = c_1 \quad \text{--- ⑪}$$

NOW taking first two fractions of (4), we get

$$\frac{dx}{z} = \frac{dx+dy}{\cos(x+y) + \sin(x+y)}$$

$$\Rightarrow \frac{dz}{z} = \frac{\frac{1}{2}(dx+dy)}{\sin(x+y+\frac{\pi}{4})}$$

$$\Rightarrow \frac{dz}{z} = \frac{1}{2} \cosec(x+y+\frac{\pi}{4}) dx + dy$$

$$\Rightarrow \sqrt{2} \log z = \log \left| \tan \left(x+y+\frac{\pi}{4} \right) \right| + \log c_2$$

$$\Rightarrow \log z^{\sqrt{2}} = \log \tan \left(\frac{x+y}{2} + \frac{\pi}{8} \right) + \log c_2$$

$$\Rightarrow z^{\sqrt{2}} \cot \left(\frac{x+y}{2} + \frac{\pi}{8} \right) = c_2 \quad \text{--- (6)}$$

From (5) & (6)
the required g.s. of (1) is

$$f \left[[\cos(x+y) + \sin(x+y)] e^{\frac{y-x}{2}}, z^{\frac{\sqrt{2}}{2}} \cot \left(\frac{x+y}{2} + \frac{\pi}{8} \right) \right] = 0$$

where f is an arbitrary function.

$$93 \rightarrow \text{Col}re(x^2 + 2xy)P + y^2 + 2x^2yQ = 2x(x+y)$$

multiples
1, 1, 0 & 1, -1, 0

$\frac{1}{x}, \frac{1}{y}, 0$

$$\rightarrow P+Q = x+y+2$$

multiples = 1, 1, 1

$$94 \rightarrow (2x^2 + y^2 + z^2 - 2y^2 - 2x - 2y)P + (x^2 + 2y^2 + z^2 - y^2 - 2x - 2xy)Q = 0$$

$$= x^2 + y^2 + 2z^2 - y^2 - 2x - 2xy$$

multiples = 1, -1, 0; 0, 1, -1; -1, 0, 1

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* The linear eqn containing more than
two independent variables:

The generalisation of Lagrange
method is as follows:

Let the linear eqn with 'n' independent
variables a_1, a_2, \dots, a_n be

$$P_1 P_1 + P_2 P_2 + \dots + P_n P_n = R \quad \text{--- (1)}$$

where P_1, P_2, \dots, P_n and R are fun of
 x_1, x_2, \dots, x_n and Z

Here P_i denotes $\frac{\partial Z}{\partial x_i}$, $i = 1, 2, \dots, n$.

Then g.s of (1) is given by

$$f(x_1, x_2, \dots, x_n) = 0$$

where $f(x_1, x_2, \dots, x_n, Z) = C_1 P_1 + C_2 P_2 + \dots + C_n P_n$
are independent solns of the auxiliary eqn

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dZ}{R}$$

Solve

$$x_2 x_3 P_1 + x_3 x_1 P_2 + x_1 x_2 P_3 + x_1 x_2 x_3 = 0$$

$$\text{Given eqn } B x_1 x_2 P_1 + x_2 x_3 P_2 + x_1 x_3 P_3 = -x_1 x_2 x_3$$

Dividing (1) w/t

$$P_1 P_1 + P_2 P_2 + P_3 P_3 = R$$

$$\therefore P_1 = x_2 x_3, P_2 = x_3 x_1, P_3 = x_1 x_2 \& R = x_1 x_2 x_3$$

Now the Lagrange auxiliary eqns of (1) are

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} = \frac{dx_3}{x_1 x_2} = \frac{dZ}{x_1 x_2 x_3} \quad \text{--- (2)}$$

Taking first two fractions of (2),

$$\frac{dx_1}{dx_2} = \frac{dx_2}{dx_3} \Rightarrow \frac{dx_1}{x_2} = \frac{dx_3}{x_1} \Rightarrow x_1 dx_1 = x_3 dx_3 \Rightarrow [x_1^2 - x_3^2 = C] \quad \text{--- (3)}$$

Taking 2nd & 3rd fractions of (2) (24)

$$\frac{dx}{2x^2y} = \frac{dy}{2y^2} \Rightarrow 2x^2y = dy \Rightarrow 2x^2 = \frac{dy}{y}$$

Taking the first & fourth fractions of (2)

$$wedges \quad 2x^2 + 2y = 8 \quad (5)$$

∴ from (3), (4) & (5), the g.c. of (2) is
 $f(x^2 - w^2, w^2 - z^2, z^2 - x^2) = 0$

where f is arbitrary fn.

$$\rightarrow x_2 x_3 z p_1 + x_3 x_1 z p_2 + x_1 x_2 z p_3 = x_1 x_2 x_3$$

$$99. \rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \left(\frac{\partial u}{\partial z} \right) = 2xyz$$

$$A-E: \frac{\partial u}{x} = \frac{\partial u}{y} = \frac{\partial u}{z} = \frac{\partial u}{xyz}$$

$$95. \rightarrow (y+z+w) \frac{\partial u}{\partial x} + (x+z+w) \frac{\partial u}{\partial y} + (x+y+w) \frac{\partial u}{\partial z} = 2xyz$$

$$A-E: \frac{\partial u}{y+z+w} = \frac{\partial u}{x+z+w} = \frac{\partial u}{x+y+w} = \frac{\partial u}{xyz} \quad (2)$$

each fraction of (2) =

$$-\frac{\partial u}{x+z+w} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} = \frac{dx-dy}{x-y} = \frac{\partial u}{xyz}$$

$$\rightarrow (x_3 - x_1) p_1 + x_1 p_2 - x_2 p_3 + x_2 - (x_1 + x_2 + x_3) = 0$$

X →

\rightarrow P.T if $x_1^3 + x_2^3 + x_3^3 = 1$ when $z=0$, 17

the soln of the eqn $(s-x_1)p_1 + (s-x_2)p_2 + (s-x_3)p_3 = s-z$

can be given in the form

$$s^3 \{ (x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3 \} = (x_1+x_2+x_3-3z)^3$$

where $s = x_1+x_2+x_3+z$ and $p_i = \frac{\partial z}{\partial x_i}$

sol? Given that $(s-x_1)p_1 + (s-x_2)p_2 + (s-x_3)p_3 = s-z$ ①

where $s = x_1+x_2+x_3+z$ ②

∴ the Lagrange's Multipliers of ① are

$$\frac{dx_1}{s-x_1} = \frac{dx_2}{s-x_2} = \frac{dx_3}{s-x_3} = \frac{dz}{s-z}$$

$$\Rightarrow \frac{dx_1}{x_1+x_2+x_3+z} = \frac{dx_2}{x_2+x_3+z} = \frac{dx_3}{x_3+x_1+z} = \frac{dz}{x_1+x_2+x_3+z} \quad \text{(by using ②)}$$

each fraction of ③ is equal to

$$= \frac{dx_1+dx_2+dx_3-3dz}{(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}$$

$$= \frac{dx_1+dx_2+dx_3-3dz}{-(x_1+x_2+x_3)+3z}$$

$$= \frac{dx_1+dx_2+dx_3-3dz}{(x_1+x_2+x_3)-3z} = \frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} \quad \text{④}$$

$$\text{Again, each fraction of ③} = \frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)} \\ = \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \quad \text{⑤}$$

from ④ and ⑤

$$\frac{d(x_1+x_2+x_3+3z)}{(x_1+x_2+x_3-3z)} = \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \Rightarrow \frac{d(x_1+x_2+x_3+3z)}{x_1+x_2+x_3-3z} = \frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z} = 0$$

integrating

$$\log(x_1+x_2+x_3+3z) + 3 \log(x_1+x_2+x_3-3z) = \log a$$

$$\Rightarrow (x_1 + x_2 + x_3 + z) (x_1 + x_2 + x_3 - 3z)^3 = a \quad (6)$$

Given That $x_1^3 + x_2^3 + x_3^3 = 1$ when $z = 0$

From (6) Given $(x_1 + x_2 + x_3)^3 (x_1 + x_2 + x_3 - 3z) = a$

$$\Rightarrow a = (x_1 + x_2 + x_3)^4 \quad (7)$$

from (6) & (7)

$$(x_1 + x_2 + x_3 + z) (x_1 + x_2 + x_3 - 3z)^3 = (x_1 + x_2 + x_3)^4 \quad (8)$$

Now each fraction of (8) $= \frac{d(x_1 + x_2)}{(x_1 - z)}$

$$= \frac{3(x_1 - z)^2 d}{-3(x_1 - z)^3}$$

$$= \frac{d(x_1 - z)^3}{-3(x_1 - z)^3} \quad (9)$$

by symmetry, each fraction of (8) is also

$$= \frac{d(x_2 - z)^3}{-3(x_2 - z)^3} = \frac{d(x_3 - z)^3}{-3(x_3 - z)^3} \quad (10)$$

Using (9) and (10)

each fraction of (8) is

$$= \frac{d(x_1 - z)^3}{-3(x_1 - z)^3} = \frac{d(x_2 - z)^3}{-3(x_2 - z)^3} = \frac{d(x_3 - z)^3}{-3(x_3 - z)^3}$$

$$= \frac{d[(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]}{-3[(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]} \quad (11)$$

From (8) and (11), we have

$$3d(x_1 + x_2 + x_3 - 3z) = d[(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]$$

$(x_1 + x_2 + x_3 - 3z)$

Integrating $\frac{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3}{(x_1 + x_2 + x_3 - 3z)}$

$$3 \log(x_1 + x_2 + x_3 - 3z) + \log b = \log[(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]$$

$$(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 = b(x_1 + x_2 + x_3 - 3z)^3 \quad (12)$$

where b is an arbitrary constant.

Given $x_1^2 + x_2^2 + x_3^2 = 1$ when $z = 0$
from eqn (12)

$$x_1^2 + x_2^2 + x_3^2 = b(x_1 + x_2 + x_3)^3$$

$$\Rightarrow 1 = b(x_1 + x_2 + x_3)^3$$

$$\Rightarrow b = \frac{1}{(x_1 + x_2 + x_3)^3}$$

$$(12) \Rightarrow (x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 = \frac{(x_1 + x_2 + x_3 - 3z)^3}{(x_1 + x_2 + x_3)^3} \quad \text{Taking both sides to power } 4, \text{ we get}$$

$$\Rightarrow [(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]^4 = \frac{(x_1 + x_2 + x_3 - 3z)^12}{(x_1 + x_2 + x_3)^12} \quad (13)$$

Raising both sides of eqn (8) to power 3,
we have

$$(x_1 + x_2 + x_3 + z)^3 (x_1 + x_2 + x_3 - 3z)^9 = (x_1 + x_2 + x_3)^{12} \quad (14)$$

Multiplying the corresponding sides of (13) and (14),
we have

$$[(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3] (x_1 + x_2 + x_3 + z)^3 = (x_1 + x_2 + x_3 - 3z)^3$$

$$S^3 [(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3]^4 = (x_1 + x_2 + x_3 - 3z)^3$$

Since $x_1 + x_2 + x_3 + z \neq 0$



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19

Integral surfaces passing through a given curve

To find the integral surface of the general solution of the linear partial differential eqn $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$ which passes through a given curve.

Let $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \quad \text{--- (1)}$

be the given eqn.

Let its auxiliary eqns give the following two independent solutions:

$$u(x, y, z) = c_1 \quad \& \quad v(x, y, z) = c_2 \quad \text{--- (2)}$$

then g.s. of (1) is $f(u, v) = 0$

where f is arbitrary function arising from a relation $f(c_1, c_2) = 0$ between the constants c_1 & c_2 --- (3)

we have to consider the problem of determining the function f in special cases.

Method(1): If we want to find integral surface passing through the given curve whose eqn in parametric form is given by $x = x(t)$, $y = y(t)$, $z = z(t)$ where t is parameter.

then (2) may be expressed as

$$u(x(t), y(t), z(t)) = c_1 \quad \text{and} \quad v(x(t), y(t), z(t)) = c_2 \quad \text{--- (4)}$$

Now eliminating the parameter t from (4), we get a relation involving c_1 & c_2 .

Finally we replace c_1 & c_2 with the help of (3) and obtain the required integral surface.

Method(2): We want to find the integral surface passing through the given curve which is determined by the following eqns $\phi(x, y, z) = 0$ & $\psi(x, y, z) = 0$ --- (5)

Now we eliminate x, y, z from the four eqns (2) & (5) and obtain a relation between c_1 & c_2 .

finally, replace C_1 by $u(x, y, z)$ & C_2 by $v(x, y, z)$ in that relation and obtain the required integral surface.

Problem (Based on second method).

→ find the integral of the PDE $(x-y)p + (y-x-z)q = z$ through the circle $z=1$, $x^2+y^2=1$.

Sol: Given that $(x-y)p + (y-x-z)q = z \quad \text{--- (1)}$

Lagrange's A-Es are

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} \quad \text{--- (2)}$$

: Using the multipliers 1, 1, L.

$$\text{each fraction of (2)} = \frac{dx+dy+dz}{0}$$

$$\Rightarrow dx+dy+dz=0$$

$$\Rightarrow [x+y+z = C_1] \quad \text{--- (3)}$$

Taking last two eqns of (2)

$$\frac{dy}{y-x-C_1} = \frac{dz}{z} \quad \left[\begin{array}{l} \text{from (3)} \\ x+y+z = C_1 \\ \Rightarrow y-C_1 = -x-z \end{array} \right]$$

$$\frac{dy}{y-C_1} = \frac{dz}{z}$$

$$\frac{1}{2} \log(2y-C_1) = \log z + \log c$$

$$\log(2y-C_1) = \log z^2 C^2$$

$$2y-C_1 = z^2 C \quad \text{where } C=c$$

$$\Rightarrow \frac{2y-(x+y+z)}{z^2} = C \quad (\text{from (3)})$$

$$\Rightarrow \frac{y-x-z}{z^2} = C \quad \text{--- (4)}$$

The curve is given by $z=1$, $x+y=1$.

Taking $z=1$ in (3) & (4), we get

$$x+y = C_1 - 1 \quad \text{--- (5)} \quad y-x = C_2 + 1 \quad \text{--- (6)}$$

$$\text{But } 2(x^2+y^2) = (x+y)^2 + (y-x)^2 \quad \text{--- (1)}$$

now using (5), (6) & (7) in (1), we get

$$2(c_1) = (c_1-1)^2 + (c_1+1)^2$$

$$\Rightarrow 2 = c_1^2 + c_2^2 - 2c_1 + 2c_2 + 2$$

$$\Rightarrow c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0 \quad \text{--- (8)}$$

putting the values c_1 & c_2 in (8), we get

$$(x+y+z)^2 + (y-z-x)^2 - 2(x+y+z) + 2(y-z-x) = 0$$

$$\Rightarrow z^4(x+y+z)^2 + (y-z-x)^2 - 2z^2(x+y+z) + 2z^2(y-z-x) = 0$$

→ find the eqn of the integral surface of the diff.

$$(x-yz)p + (yz-xz)q = z^2 - xy \quad \text{--- (1)}$$

which passes through the line $x=1, y=0$

Soln:

$$\frac{x-y}{y-z} = c_1 \quad \text{--- (2)}$$

$$xy + yz + zx = c_2 \quad \text{--- (3)}$$

$$\text{The given curve is } x=1, y=0 \quad \text{--- (4)}$$

using (4) in (2) & (3) we get

$$\left[-\frac{1}{z} \right] = c_1 \quad \boxed{z = c_2} \quad \text{--- (5)}$$

$$\text{From (5)} \left(-\frac{1}{z} \right)(z) = c_1 c_2$$

$$\Rightarrow [c_1 c_2] = -1 \quad \text{--- (6)}$$

Using (2) & (3) in (6), we get

$$\left(\frac{x-y}{y-z} \right) (xy + yz + zx) = -1$$

→ find the eqn of surface satisfying $xy + yz + zx = 0$
and passing through $y^2 + z^2 = 1 ; x + z = 2$

(Method)

→ find the integral surface of the linear PDE

$$x(y+z)p + y(x+z)q = (x-yz)z \quad \text{--- (1)}$$

which contains
the straight line $x+y=0, z=1$

Soln:

$$xyz = c_1 \quad x^2 + y^2 - 2z = c_2 \quad \text{--- (3)}$$

Method-2

$$\text{The given curve } x+y=0 \text{ & } x=1 \quad \text{--- (4)}$$

Taking t as a parameterput $x=t$ in (4), we get

$$y = -t \quad \text{and} \quad z = 1$$

$$\therefore x = t; \quad y = -t; \quad z = 1. \quad \text{--- (5)}$$

Using (5) in (2) & (3), we get

$$\begin{aligned} t(-t)(1) &= c_1 \quad \& \quad t^2 + t^2 - 2 = c_2 \\ \Rightarrow -t^2 &= c_1 \quad \& \quad 2t^2 - 2 = c_2 \\ \Rightarrow t^2 &= -c_1 \quad \& \quad \Rightarrow 2(-c_1) - 2 = c_2 \\ & \quad \quad \quad \Rightarrow 2c_1 + c_2 + 2 = 0 \end{aligned}$$

Using (2) and (3) in (6), we get

$$\begin{aligned} 2(xy) &+ x^2 + y^2 - 2z + 2 = 0 \\ \Rightarrow x^2 + y^2 + 2xy - 2z + 2 &= 0 \end{aligned}$$

(6)

2nd MethodNow eliminating x, y, z from (2), (3) & (4)

$$\begin{aligned} \text{we get } xy &= c_1 \quad \& \quad x^2 + y^2 - 2 = c_2 \\ \Rightarrow (x+y)^2 - 2xy - 2 &= c_2 \\ \Rightarrow 0 - 2(c_1) - 2 &= c_2 \\ \Rightarrow 2c_1 + c_2 + 2 &= 0 \end{aligned}$$

From (2) & (1), we get

$$x^2 + y^2 - 2z + 2xy - 2z + 2 = 0$$

2nd
Ques

find the general solution of P.D.E

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$$

and also find the particular solution
which passes through the lines $x=1, y=0$.

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2005

Ques find the particular integral of : 21

$$x(y-z)P + y(z-x)Q = z(z-y) \text{ which}$$

represents a surface passing through
 $x=y=z$.

Sol Given eqn is

$$x(y-z)P + y(z-x)Q = z(z-y) \quad (1)$$

Lagrange's A.E.'s are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(z-y)} \quad (2)$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get
each fraction of (2) = $\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z)}$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log(yz) = \log c_1$$

$$\Rightarrow xyz = c_1 \quad (3)$$

Again taking the multipliers as x, y, z , we get
each fraction of (2) = $\frac{xdy + ydz}{x(y-z)}$

$$xdy + ydz = 0$$

$$\Rightarrow (xyz)^2 = c_2 \quad (4)$$

But curve is $x=y=z \quad (5)$

Now we eliminate x, y, z from (3), (4) & (5) we get

$$c_1^2 = c_2 \quad \& \quad 27 = c_2$$

$$\Rightarrow \left(\frac{c_2}{7}\right)^2 = c_1 \neq$$

$$\Rightarrow c_2^2 - 27c_1 = 0$$

$$\Rightarrow (xyz)^2 - 27(xyz) = 0$$

which is reqd surface of (1)



Charpit's Method:

We now give a general method due to Charpit for finding the complete integral of a non-linear differential eqn of the first order.

Let the given eqn be $f(x, y, z, p, q) = 0 \quad \text{--- (1)}$

since z depends on x & y , we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = pdx + qdy \quad \text{--- (2)}$$

The fundamental idea in Charpit's method is the introduction of another PDE of the first order

$$g(x, y, z, p, q, a) = 0 \quad \text{--- (3)}$$

which contains arbitrary constant 'a' and

(i) we can solve the eqns (1) & (3) for

$$p = p(x, y, z, a) \quad \text{&} \quad q = q(x, y, z, a)$$

(ii) substituting these values of p & q in (2),
the eqn (2) becomes

$$dx = p(x, y, z, a) dx + q(x, y, z, a) dy \quad \text{--- (4)}$$

This gives the solution, provided (4) is integrable.

If such a relation (3) has been found, the
solution of the eqn (4)

$$\phi(x, y, z, a, b) = 0 \quad \text{--- (5)}$$

containing two arbitrary constants a & b will
be a solution of eqn (1).

Also it is a complete integral of the eqn (1).

How to determine g

Differentiating (1) & (3) w.r.t x , we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0$$

$$\text{and } \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial g}{\partial q} \cdot \frac{\partial q}{\partial x} = 0$$

Set-IINon-linear Eqns:-

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The integrals or solutions of the non-linear
partial differentiable eqns of the first order:

With the relation of the type $f(x, y, z, a, b) = 0$ (1)
gives rise to a PDE of the first order of the form

$$F(x, y, z, p, q) = 0. \quad (2)$$

On the elimination of arbitrary constants a & b , where
 x, y are independent variables and z is dependent variable.

- If (1) has been derived from (2) then (1) is a solution of (2).
- Any such relation (1) which contains as many
arbitrary constants as there are independent variables,
is called the complete integral or complete solution of (2).
- Any particular integral of (2) is obtained by giving
particular values to a & b in (1).

Singular Integral (SI):

The singular integral is obtained by eliminating
 a & b from the three eqns $f(x, y, z, a, b) = 0$, $\frac{\partial f}{\partial a} = 0$
and $\frac{\partial f}{\partial b} = 0$.

General Integral (GI):

If in the eqn (1), one of the constants is
a function of the other say $b = \phi(a)$ then (1) becomes

$$f(x, y, z, a, \phi(a)) = 0. \quad (3)$$

If (3) is a one-parameter subfamily of the family (1).
The eqn. of the envelope of the family of surfaces
represented by (3) is also a solution of the eqn (2).

It is called the general integral of (2) corresponding
to the complete integral (1).

The eqn of the envelope of the surfaces represented
by (3) is obtained by eliminating ' a ' between the eqns
 $f(x, y, z, a, \phi(a)) = 0$ and $\frac{\partial f}{\partial a} = 0$.

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \quad \text{--- (6)}$$

$$\text{and } \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} p + \frac{\partial g}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial g}{\partial q} \cdot \frac{\partial q}{\partial x} = 0$$

Again diff (1) & (3) w.r.t. y , we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \quad \text{--- (7)}$$

$$\text{and } \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} q + \frac{\partial g}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial g}{\partial q} \cdot \frac{\partial q}{\partial y} = 0$$

Now eliminating $\frac{\partial p}{\partial x}$ from the eqns in (6) &

$\frac{\partial q}{\partial y}$ from the eqns in (7), we get

$$\left(\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) + p \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right) + \frac{\partial g}{\partial x} \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right) = 0 \quad \text{--- (8)}$$

$$\text{and } \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial y} \right) + q \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right) + \frac{\partial g}{\partial y} \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right) = 0 \quad \text{--- (9)}$$

$$\text{since } \frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

$$= \frac{\partial p}{\partial y}$$

$$\frac{\partial g}{\partial x} = \frac{\partial p}{\partial y}$$

$$(8) + (9) =$$

$$\left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) + p \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right) +$$

$$\left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial y} \right) + q \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right) = 0 \quad \left(\because \frac{\partial p}{\partial y} = \frac{\partial z}{\partial x} \right)$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial g}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial g}{\partial p} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial p} \right) \frac{\partial g}{\partial z} +$$

$$+ \left(-\frac{\partial f}{\partial p} \right) \frac{\partial g}{\partial x} + \left(-\frac{\partial f}{\partial p} \right) \frac{\partial g}{\partial y} = 0 \quad \text{--- (10)}$$

Clearly it is a linear PDE of the first order with x, y, z, p, q as independent variables and f as a dependent variable.

The Lagrange's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dx}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dy}{-q \frac{\partial f}{\partial p}} = \frac{dz}{-p \frac{\partial f}{\partial q}} = 0 \quad (1)$$

These eqns are known as Charpit's auxiliary eqns.

Any of integrals of (1) satisfies (0). If such an integral contains p or q (or both) it can be taken the required second PDE (3).

Note: It should be noted that not all of Charpit's eqns (1) need be used, but that p or q must occur in the solution obtained.

Working Rule of Charpit's Method:

Step 1: Transfer terms of the given eqn to LHS and denote the entire expression by f.

Step 2: write down the Charpit's auxiliary eqns (1).

Step 3: using the value of f in step 1, write down the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc. occurring in step (2) and put these values in Charpit's auxiliary eqns (1).

Step 4: After simplifying step (3) select two proper fractions so that the resulting integral may come out to be the simplest relation involving atleast one of p and q.

Step 5: The simplest relation of step (4) is solved along with the given eqn to determine p & q.

Step 6: put these values of p & q in

$dz = pdx + qdy$
which on integration give the complete integral of the given eqn.

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problems:

→ find the complete integral of $px + qy = pq$.

Sol: Given that $px + qy = pq$

$$\Rightarrow px + qy - pq = 0$$

$$\text{Let } f(x, y, z, p, q) = px + qy - pq = 0 \quad \textcircled{1}$$

Charpit's auxiliary eqns are

$$\frac{dx}{-pf+pq} = \frac{dy}{-qf+qr} = \frac{dz}{-p^2t-pq^2r} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-x+q} = \frac{dy}{-y+p} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+P(0)} = \frac{dq}{q} \quad \textcircled{2}$$

Taking last two fractions of $\textcircled{2}$, we get

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a$$

$$\Rightarrow [p/q = a] \Rightarrow [p = qa] \quad \textcircled{3}$$

$$\textcircled{1} \equiv qx + qy - qa = 0$$

$$\Rightarrow q[x + y - a] = 0$$

$$\Rightarrow x + y - a = 0 \quad (\because q \neq 0)$$

$$\Rightarrow x + y = a$$

$$\Rightarrow [q = \frac{x+y}{a}] \quad \textcircled{4}$$

$$\textcircled{3} \equiv P = \left(\frac{x+y}{a}\right)a$$

$$\Rightarrow [P = x+y] \quad \textcircled{5}$$

putting these values of p and q in $dz = pdx + qdy$,

$$\text{we get } dz = (x+y)dx + \left(\frac{x+y}{a}\right)dy$$

$$\Rightarrow adz = (x+y)(adx + dy)$$

$$\Rightarrow adz = (x+y)d(x+y)$$

$$\Rightarrow az = (x+y)^2 + b$$

which is the complete integral of $\textcircled{1}$.

²⁰⁰⁰ → solve by Charpit's method eqn

$$P^2x(x-1) + 2Pqxy + q^2y(y-1) - 2Pxz - 2qyz + z^2 = 0$$

Sol Let $f(x, y, z, P, Q) =$

$$P^2x(x-1) + 2Pqxy + q^2y(y-1) - 2Pxz - 2qyz + z^2 = 0 \quad (1)$$

The Charpit's auxiliary eqns are

$$\frac{dx}{-fp} = \frac{dy}{-fq} = \frac{dz}{-Pf_p - qf_q} = \frac{dp}{f_x + Pf_z} = \frac{dq}{fy + Qf_z} \quad (2)$$

From (1), $f_x = P^2(x-1) + 2Pqy - 2Pz$

$$f_y = 2Pqx + q^2(y-1) - 2qz$$

$$f_z = -2Px - 2qy + 2z$$

$$f_p = 2Px(x-1) + 2qxy - 2xz$$

$$f_q = 2Pqy + 2qy(y-1) - 2yz$$

and $f_x + Pf_z = -P^2$; $fy + Qf_z = -q^2$

$$\begin{aligned} \therefore (2) &\equiv \frac{dx}{-(2Px^2 - 2Px + 2qxy - 2xz)} = \frac{dy}{-(2Pxy + 2qy^2 - 2qy - 2z)} \\ &= \frac{dz}{-P[2Px(x-1) + 2qxy - 2xz] - q[2Pxy + 2qy(y-1) - 2yz]} \\ &= \frac{dp}{-P^2} = \frac{dq}{-q^2} \quad (3) \end{aligned}$$

$$\text{each fraction of (3)} = \frac{\frac{1}{P}dp}{-\frac{1}{P}} = \frac{\frac{1}{2}da}{-q} = \frac{\frac{1}{P}dp - \frac{1}{q}dq}{-P+q} \quad (4)$$

Also each fraction of (3) = $\frac{1}{2}dx - \frac{1}{q}dy$

$$-2Px + 2P - 2qy + 2z + 2qx + 2qy - 2q - 2z \quad (5)$$

$$\begin{aligned} \therefore (4) \&\& (5) \\ \Rightarrow \frac{\frac{1}{P}dp - \frac{1}{q}dq}{-(P-q)} &= \frac{\frac{1}{2}dx - \frac{1}{q}dy}{2(P-q)} \end{aligned}$$

$$\Rightarrow \frac{1}{2}(\frac{1}{P}dp - \frac{1}{q}dq) = \frac{1}{q}dx - \frac{1}{P}dp$$

Integrating, we get

$$\frac{1}{2}(\log x - \log y) = \log q - \log p + \log a.$$

$$\Rightarrow \left(\frac{p}{y}\right)^{Y_2} = \frac{q \cdot a}{p}$$

$$\therefore p = \frac{ay^{Y_2}q}{x^{Y_2}} ; a \text{ is arbitrary constant.} \quad (6)$$

$$\therefore (p_x + q_y - z) = p^2 x + q^2 y$$

$$\Rightarrow p_x + q_y - z = \pm \sqrt{p^2 x + q^2 y} \quad (7)$$

taking +ve sign in (7)

$$p_x + q_y - z = \sqrt{p^2 x + q^2 y} \quad (8)$$

$$\Rightarrow \left(\frac{ay^{Y_2}q}{x^{Y_2}}\right)x + qy - z = \sqrt{\left(\frac{ay^{Y_2}q}{x^{Y_2}}\right)^2 x + q^2 y} \quad (by 6)$$

$$\Rightarrow aq(xy)^{Y_2} + qy - z = \sqrt{ya^2 q^2 + q^2 y} = qy^{Y_2} (1+a^2)^{Y_2} -$$

$$\Rightarrow q[y + a(xy)^{Y_2} - (1+a^2)y^{Y_2}] = z$$

$$\Rightarrow q = \frac{z}{y + a(xy)^{Y_2} - (1+a^2)y^{Y_2}} \quad (9)$$

Putting these values in $dz = pdx + qdy$

$$dz = \frac{az \, dx}{x^{Y_2} [y^{Y_2} + a^2 y^{Y_2} - (1+a^2)^{Y_2}]} + \frac{z \, dy}{y^{Y_2} [y^{Y_2} + a^2 y^{Y_2} - (1+a^2)^{Y_2}]}$$

$$\Rightarrow \frac{dz}{z} = \frac{ay^{Y_2} dx + x^{Y_2} dy}{(ay)^{Y_2} [y^{Y_2} + a^2 y^{Y_2} - (1+a^2)^{Y_2}]}$$

$$\Rightarrow \log z = 2 \log \left[\frac{y^{Y_2} + a^2 y^{Y_2} - (1+a^2)^{Y_2}}{(ay)^{Y_2}} \right] + \log b$$

$$\Rightarrow z = b \left[\frac{y^{Y_2} + a^2 y^{Y_2} - (1+a^2)^{Y_2}}{(ay)^{Y_2}} \right]^2, b \text{ is an arbitrary constant.}$$

2002 Solve $z = \frac{1}{2}(p+q^2) + (p-q)(q-y)$

2002 Find two complete integrals of the PDE $x^2 p^2 + y^2 q^2 - 4 = 0$

Soln Given that $x^2 p^2 + y^2 q^2 - 4 = 0$

$$\text{Let } f(x, y, z, p, q) = x^2 p + y^2 q - 4 = 0 \quad \textcircled{1}$$

i.e. Charpit's A.Es are

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{dz}{f_z} = \frac{dp}{f_p} = \frac{dq}{f_q}$$

$$\Rightarrow \frac{dx}{-2x^2 p} = \frac{dy}{-2y^2 q} = \frac{dz}{-4} = \frac{dp}{x^2 p} = \frac{dq}{y^2 q} \quad \textcircled{2}$$

To find first complete integral:

Taking the first & fourth fraction of \textcircled{2}, we get

$$\frac{dx}{-2x^2 p} = \frac{dp}{x^2 p} \Rightarrow \frac{dx}{-x} = \frac{dp}{p}$$

$$\Rightarrow \log z = -\log p + \log c$$

$$\Rightarrow \log(pz) = \log c$$

$$\Rightarrow pz = c \Rightarrow p = \frac{c}{z}$$

$$\textcircled{1} = x^2 p^2 + y^2 q^2 - 4$$

$$\Rightarrow x^2 p^2 + y^2 q^2 = 4 \Rightarrow q^2 = \frac{4 - x^2 p^2}{y^2}$$

$$\Rightarrow q = \frac{\sqrt{4 - x^2 p^2}}{y}$$

Substituting the values of p & q in

$$dz = pdx + qdy, \text{ we get}$$

$$\Rightarrow dz = \frac{c}{z} dx + \frac{\sqrt{4 - x^2 p^2}}{y} dy$$

Integrating

$$z = c \log z + \sqrt{4 - x^2 p^2} \log y + \log b$$

Taking 2nd & last fractions of \textcircled{2}, we get

$$\frac{dy}{-2y^2 q} = \frac{dq}{y^2 q^2} \Rightarrow \frac{dy}{-y} = \frac{dq}{q^2}$$

$$\Rightarrow \log y = -\log q + \log d$$

$$\Rightarrow y q = d$$

$$\Rightarrow q = \frac{d}{y}$$

$$\textcircled{1} = x^2 p^2 + d^2 = 4$$

$$\Rightarrow p^2 = \frac{4 - d^2}{x^2} \Rightarrow p = \frac{\sqrt{4 - d^2}}{x}$$

substituting the values of p & q in
 $dz = pdx + qdy$.

$$\Rightarrow dz = \frac{\sqrt{4-d^2}}{x} dx + \frac{d}{y} dy$$

$$\Rightarrow z = \int \frac{\sqrt{4-d^2}}{x} \log x + d \log y$$

→ find three complete integrals of $pq = px + qy$.

→ find a complete, singular, and general integrals of $(p^2+q^2)y = qx$.

Sol: Given that $(p^2+q^2)y - qx = 0$

Let $f(x, y, z, p, q) = (p^2+q^2)y - qx = 0$... (1)

A Eqns are

$$\frac{dx}{dp} = \frac{dy}{dq} = \frac{dz}{dq} = \frac{dp}{dq} = \frac{dz}{dp}$$

$$\Rightarrow \frac{dp}{-2pq} = \frac{dy}{-2qy+z} = \frac{dx}{-2py+qz-2q^2y} = \frac{dz}{-qz} = \frac{dq}{p^2} \quad (2)$$

Taking last two fractions of (2)

$$\frac{dp}{-qz} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$\Rightarrow pdp + qdq = 0$$

$$\Rightarrow p^2 + q^2 = a^2 \quad (3)$$

$$(1) \Rightarrow dy = qz$$

$$\Rightarrow z = \frac{ad}{F}$$

$$(3) \Rightarrow p^2 + \frac{a^2y^2}{z^2} = a^2$$

$$\Rightarrow p^2 = a^2 - \left(\frac{ay}{z}\right)^2$$

$$\Rightarrow p = \sqrt{a^2 - \left(\frac{ay}{z}\right)^2}$$

$$\Rightarrow P = \frac{a}{z} \sqrt{z^2 - a^2y^2}$$

Putting these p and q in $dz = pdx + qdy$

$$\Rightarrow dz = \frac{a}{z} \sqrt{z^2 - a^2 y^2} dx + a \frac{y}{z} dy$$

$$\Rightarrow \frac{z dz - a^2 y dy}{\sqrt{z^2 - a^2 y^2}} = adx$$

$$\Rightarrow (z^2 - a^2 y^2)^{1/2} = ax + b$$

$$\Rightarrow z^2 - a^2 y^2 = (ax + b)^2 \quad \text{--- (4)}$$

which is the required complete integral.

Singular integral:

Diff (4) w.r.t. a & b, we get

$$-2ay^2 = 2(ax + b)x$$

$$\Rightarrow [ax^2 + bx + ay^2 = 0] \quad \text{--- (5)}$$

$$\text{and } 2(ax + b) = 0$$

$$\Rightarrow [ax + b = 0] \quad \text{--- (6)}$$

Now eliminating a & b from (5), (6) & (6), we get

$$(5) \equiv x^2 + ay^2 = 0 \quad (\text{by (6)})$$

$$\Rightarrow [a = 0] \quad (\because y \neq 0)$$

$$(6) \equiv [b = 0]$$

$$\therefore (4) \equiv z = 0 \quad \text{which clearly satisfies (1)}$$

\therefore It is the required singular solution of (1).

General Integral:

Let $b = \phi(a)$ in (4), then $z^2 - a^2 y^2 = [ax + \phi(a)]^2$ (7)

Diff (7) partially w.r.t. a, we get

$$-2ay^2 = 2(ax + \phi(a))(x + \phi'(a)) \quad \text{--- (8)}$$

G.S. is obtained by eliminating 'a' from (5) & (8).

1997 → find a Complete integral of $z(p^2 z^2 + q^2) = 1$

1996 → find a complete integral of $z = p^2 z^2 + q^2 + p^2 + q^2$

1994 → find a complete integral of $16 p^2 z^2 + q^2 + 4z^2 - 4 = 0$

1995 → " " " " $2x(z^2 q^2 + 1) = p^2 z$

1993 → " " " " $p^2 + q^2 - 2px - 2qy + 1 = 0$

Special Types of equations:

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We shall consider some special types of first-order partial differential eqns whose solutions may be obtained easily by Charpit's method.

Type 1: Equations involving only P & q :

for eone of the type $f(P, q) = 0$ — ①

Charpit's auxiliary eqns are

$$\frac{du}{-2f/\partial P} = \frac{dy}{-2f/\partial q} = \frac{dz}{-P \frac{\partial f}{\partial P} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + P \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{du}{-2f/\partial P} = \frac{dy}{-2f/\partial q} = \frac{dz}{-P \frac{\partial f}{\partial P} - q \frac{\partial f}{\partial q}} = \frac{dp}{0} = \frac{dq}{0}$$

Taking third & fourth fractions, we get

$$dp = 0 \Rightarrow [P = a] \text{ (constant)} \quad ②$$

$$① \Rightarrow f(a; q) = 0$$

$$\Rightarrow q = \text{constant} \quad ③$$

$$= \phi(a) \text{ (say)}$$

putting these values in $dz = P dx + q dy$

$$\Rightarrow dz = adx + \phi(a)dy$$

Integrating

$$z = ax + \phi(a)y + b \quad ④$$

where b is constant

which is a complete integral of ①

It contains two arbitrary constants a & b .

General Integral:

putting $b = \psi(a)$ in ④

where ψ is arbitrary function

we get,

$$z = ax + \phi(a)y + \psi(a) \quad ⑤$$

diff ⑤ partially w.r.t 'a', we get

$$0 = x + \phi'(a)y + \psi'(a) \quad \text{--- (6)}$$

eliminating a b/w (5) & (6)

Singular Integral: The singular integral, if it exists, is obtained by eliminating a & b between the complete integral (1) and the eqns. formed by differentiating (1) partially w.r.t a & b .

i.e., b/w the eqns

$$z = ax + \phi(a)y + b$$

$$\partial z = a + \phi'(a)y \text{ and } 0 = 1$$

Since $1=0$ is inconsistent (means less)

∴ In this case there is no singular solution.

→ Solve. $pq=k$, where k is constant.

Eqn: The given eqn is $pq=k$ --- (1)

where k is constant.

Clearly the eqn (1) is of the form $f(p, q)=0$

∴ Its complete integral is

$$z = ax + \phi(a)y + b \quad \text{--- (2)}$$

$$\text{Taking } a=p; \phi(a)=q$$

$$\therefore (1) \equiv a\phi(a)=k$$

$$\Rightarrow \phi(a) = \frac{k}{a}$$

$$\therefore (2) \equiv z = ax + \frac{k}{a}y + b \quad \text{--- (3)}$$

Where a & b are arbitrary constants.

? It is the required complete integral.

To find singular integral:

Diff (3) partially w.r.t a & b , we get

$$0 = x - \frac{k}{a^2}y$$

$0=1$ which is meaningless

∴ The given eqn ① has no singular integral

General integral:

putting $b=\phi(a)$. In ③, we get

$$z = ax + \frac{k}{a}y + \phi(a) \quad ④$$

diff ④ partially w.r.t. a, we get

$$0 = z - \frac{ky}{a} + \phi'(a) \quad ⑤$$

∴ The required G.I is obtained by eliminating a between ④ & ⑤.

- Solve $q = 3p^2$
- Solve $q = e^{-p/a}$ where a is a constant.
- Find the complete integral of $\sqrt{p} + \sqrt{q} = 1$

Equations reducible to type 1:

→ Find the complete integral of

$$x^2y + 6xz^2y + 2z^2x^2 + 4x^2y = 0 \quad ①$$

$$\text{Soln } ① = x^2y \left(\frac{\partial z}{\partial x}\right)^2 + 6xz^2y \left(\frac{\partial z}{\partial x}\right) + 2z^2x^2 \left(\frac{\partial z}{\partial y}\right) + 4x^2y = 0$$

Dividing by x^2y , we get

$$\Rightarrow \frac{z^2}{x^2} \left(\frac{\partial z}{\partial x}\right)^2 + \frac{6z^2}{x^2} \left(\frac{\partial z}{\partial x}\right) + \frac{2z^2}{y} \left(\frac{\partial z}{\partial y}\right) + 4 = 0$$

$$\Rightarrow \left(\frac{z}{x} \frac{\partial z}{\partial x}\right)^2 + 6 \left(\frac{z}{x} \frac{\partial z}{\partial x}\right) + 2 \left(\frac{z}{y} \frac{\partial z}{\partial y}\right) + 4 = 0 \quad ②$$

Putting $2dx = dx$; $ydy = dy$; $\frac{z}{x}dz = dz$

$$\Rightarrow \frac{x^2}{2} = x; \frac{y^2}{2} = y; \frac{z^2}{2} = z$$

$$② = \left(\frac{\partial z}{\partial x}\right)^2 + 6 \left(\frac{\partial z}{\partial x}\right) + 2 \left(\frac{\partial z}{\partial y}\right) + 4 = 0$$

$$\Rightarrow P_z^2 + 6P_z + 2Q_z + 4 = 0$$

$$\text{where } P_z = \frac{\partial z}{\partial x}, Q_z = \frac{\partial z}{\partial y}$$

Clearly ③ is in the form of $f(p, q) = 0$

∴ The complete integral is of the form

$$Z = ax + \phi(a)y + b \quad \text{--- (4)}$$

where $a = p$ & $\phi(a) = q$.

$$\textcircled{3} \equiv a^2 + 6a + 2\phi(a) + 4 = 0$$

$$\Rightarrow \phi(a) = -\frac{(a^2 + 6a + 4)}{2}$$

$$\textcircled{4} \equiv Z = ax - \left(\frac{a^2 + 6a + 4}{2}\right)y + b$$

where a & b are arbitrary constants.

$$\therefore \frac{Z^2}{2} = a^2 \left(\frac{x^2}{2}\right) - \left(\frac{a^2 + 6a + 4}{2}\right) \left(\frac{y^2}{2}\right) + b^2 =$$

which is the required complete integral.
of ①

$$\rightarrow \text{Solve } x^2 p^2 + y^2 q^2 = z^2 \quad \text{--- (5)}$$

$$\Rightarrow \frac{x^2}{z^2} \left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2} \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{x}{z} \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{\frac{1}{z} \frac{\partial z}{\partial x}}{\frac{1}{z} \frac{\partial z}{\partial x}}\right)^2 + \left(\frac{\frac{1}{z} \frac{\partial z}{\partial y}}{\frac{1}{z} \frac{\partial z}{\partial y}}\right)^2 = 1 \quad \text{--- (6)}$$

putting $\frac{1}{z} \frac{\partial z}{\partial x} = dx$; $\frac{1}{z} \frac{\partial z}{\partial y} = dy$; $\frac{1}{z} dz = dz$

$$\Rightarrow \boxed{\log x = X}; \quad \boxed{\log y = Y}; \quad \boxed{\log z = Z}$$

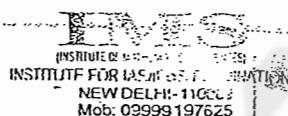
$$\therefore \textcircled{6} \equiv \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow p^2 + q^2 = 1 \quad \text{--- (7)}$$

It is of the form $f(p, q) = 0$

The complete integral is of the form

$$Z = ax + \phi(a)y + b \quad \text{--- (8)}$$



Taking $a = P$ & $\phi(a) = Q$

$$\therefore \textcircled{3} \equiv a^2 + [\phi(a)]^2 = 1$$

$$\Rightarrow [\phi(a)]^2 = 1 - a^2$$

$$\Rightarrow \phi(a) = \sqrt{1 - a^2}$$

$$\therefore \textcircled{4} \equiv Z = ax + (\sqrt{1 - a^2})y + b$$

where a & b are arbitrary constants.

$$\Rightarrow \log z = a \log x + (\sqrt{1 - a^2}) \log y + b$$

which is the required complete integral.

If we take

$$a = \cos \alpha, b = \log c$$

then complete integral is

$$\log z = \cos \alpha \log x + \sin \alpha \log y + \log c.$$

$$\Rightarrow Z = x \cdot y \cdot c \quad \text{(P)}$$

where α & c are arbitrary constants.

Singular Integrals

diff (1) partially w.r.t α & c , we get

$$0 = c \left[x \cdot \left(y \frac{\sin \alpha}{\log y} \right) \cos \alpha + y \frac{\sin \alpha}{x} \cos \alpha - \log x \left(\sin \alpha \right) \right]$$

$$\Rightarrow x \cdot y \cdot \left(\cos \alpha \log y - \sin \alpha \log x \right) = 0 \quad \text{(6)}$$

$$\text{and } 0 = x \cdot y \cdot \text{(c)} \quad \text{(7)}$$

Eliminating α, c from (1), (5), (6), (7), we get

$Z = 0$
which is the required singular solution.

G.E: putting $c = \phi(\alpha)$

$$\textcircled{3} \equiv Z = x \cdot y \cdot \sin \alpha \phi(\alpha) \quad \text{(8)}$$

diff (8) partially w.r.t α .

$$0 = x^m - \phi'(x) \left[y^{\sin x} \log y \cdot \cos x \right] + x^m y^{\sin x} \phi'(x)$$

Eliminating α from ⑧ & ⑨, we get the required G.I. of ①.

Find a complete integral of

$$\text{⑩) } pq = x^m y^n z^l \quad \text{⑪) } pq = x^m y^n z^l$$

Sol' (1) Given that

$$pq = x^m y^n z^l \quad \text{⑫}$$

$$\Rightarrow \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = x^m y^n z^l$$

$$\Rightarrow \left(\frac{\partial z}{x^m y^n} \right) \left(\frac{\partial z}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) = 1$$

$$\Rightarrow z^k dz = dz ; \quad x^m dx = dx ; \quad y^n dy = dy$$

$$\Rightarrow \boxed{z = \frac{x^{m+1}}{m+1}} ; \quad \boxed{x = \frac{z^{n+1}}{n+1}} ; \quad \boxed{y = \frac{z^{n+1}}{n+1}}$$

Type 2nd Eqns not involving the independent variables

If the partial diff. eqn is of the type $f(z, p, q) = 0$ ⑬

Charpit's auxiliary eqns reduce to

$$\frac{dx}{df/p} = \frac{dq}{df/q} = \frac{dz}{df} = \frac{dp}{-q \frac{\partial f}{\partial z}} = \frac{dq}{-q \frac{\partial f}{\partial z}} \quad \text{⑭}$$

Taking last two fractions, we get

$$\frac{dq}{q} = \frac{dp}{p} \Rightarrow \frac{q}{p} = a$$

$$\Rightarrow \boxed{q = pa} \quad \text{⑮}$$

where a is arbitrary constant.

From $dz = pdx + qdy$, we get

$$dz = p(dx + qdy)$$

$$\Rightarrow dz = P dx + a dy$$

$$\Rightarrow dz = P dx \quad \text{where } x = x + ay$$

$$\Rightarrow \boxed{\frac{dz}{dx} = P}$$

$$(3) \Rightarrow \boxed{q = a \frac{dz}{dx}}$$

$$\therefore (1) \Leftrightarrow f(z, \frac{dz}{dx}, a \frac{dz}{dx}) = 0$$

which is an ordinary diff. eqn of the first order and solving it, a complete integral can be obtained.

Working rule:

Step (1): Write down the given eqn $f(p, q, z) = 0$

Step (2): put $p = \frac{dz}{dx}$ & $q = a \frac{dz}{dx}$ where $x = x + ay$

Step (3): Solving the resulting ODE in the variables z & x then substitute $x = x + ay$. This gives the complete integral.

Note: Some times using transformations convert to the form of type (II).

→ find a complete integral of $q(pz + q^2) = 4$

Sol? Given that $q(pz + q^2) = 4 \quad \text{--- (1)}$

Clearly it is of the form $f(p, q, z) = 0$

where $P = \frac{dz}{dx}$, $q = a \frac{dz}{dx}$.

$$(1) \Leftrightarrow q \left[\left(\frac{dz}{dx} \right)^2 z + a^2 \left(\frac{dz}{dx} \right)^2 \right] = 4$$

where $x = x + ay \quad \text{--- (2)}$

$$\Rightarrow q \left(z + a^2 \right) \left(\frac{dz}{dx} \right)^2 = 4$$

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$$\Rightarrow \frac{dz}{dx} = \frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \frac{3}{2}\sqrt{z+a^2} dz = dx$$

$$\Rightarrow \frac{3}{2} \cdot \frac{(z+a^2)^{3/2}}{3/2} = x+b \quad \text{where } b \text{ is arbitrary constant}$$

$$\Rightarrow (z+a^2)^{3/2} = (x+By) + b \quad (\text{by } \textcircled{1})$$

$$\Rightarrow (z+a^2)^3 = (x+ay+b)^2$$

where a & b are arbitrary constants.
which is the required complete integral.

→ find the complete integral of the eqn $Pz^2 + q^2 = 1$

1991. find a complete integral of $P^3 + q^3 - 3Pqz = 0$

1993. find a complete integral of $z^2(Pz^2 + q^2 + 1) = k^2$ where k is constant.

→ find a complete integral of $P^2y = z(z - Px)$ (1)

Hint:

$$(y \cdot \frac{\partial z}{\partial y})^2 = z(z - x \cdot \frac{\partial z}{\partial x})$$

$$\Rightarrow \left(\frac{\partial z}{\frac{1}{y} \partial y}\right)^2 = z(z - x \cdot \frac{\partial z}{\partial x}) \quad \text{--- (2)}$$

taking $\frac{1}{y} dy = dx \quad || \quad \int y dy = dy$
 $\log x = x \quad || \quad \log y = y$

$$\therefore (2) \equiv \left(\frac{\partial z}{\partial y}\right)^2 = z(z - \frac{\partial z}{\partial x})$$

$$\Rightarrow z(z - P) = Q^2 \quad \text{where } P = \frac{\partial z}{\partial x}; \quad Q = \frac{\partial z}{\partial y}$$

and proceed

Type (III)Separable equations

Eqns not involving $\frac{dx}{dy}$ and it happens the terms containing P & x can be separated from those containing q & y .

i.e., they have the form

$$\therefore f_1(x, P) = f_2(y, q) \quad \text{--- (1)}$$

corresponding Charpit's auxiliary eqns will be

$$\frac{dx}{\frac{\partial f_1}{\partial P}} = \frac{dy}{-\frac{\partial f_2}{\partial q}} = \frac{dp}{-\frac{\partial f_1}{\partial x}} = \frac{dq}{-\frac{\partial f_2}{\partial y}} \quad \text{--- (2)}$$

$$\frac{\partial f_1}{\partial P} dp + \frac{\partial f_1}{\partial x} dx = 0$$

$$\Rightarrow df_1 = 0 \quad (\because df_1 = \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial P} dp)$$

$$\Rightarrow f_1 = \text{constant}$$

$$\Rightarrow f_1(x, P) = a \quad (\text{say}) \quad \text{--- (3)}$$

$$\therefore \text{--- (1)} \Rightarrow f_2(y, q) = f_1(x, P)$$

$$\Rightarrow f_2(y, q) = a \quad \text{--- (4)}$$

solving (3) & (4) for P & q , we get

$$P = F_1(x, a), \quad q = F_2(y, a)$$

putting these values of P & q in $dx = P dx + q dy$, we get

$$dx = f_1(x, a) dx + f_2(y, a) dy$$

Integrating we get

$$x = \int f_1(x, a) dx + \int f_2(y, a) dy + b$$

where b is an arbitrary constant.

which is the required complete integral.

Working rule:

Step 1: Write the given eqn in the form $f_1(x, P) = f_2(y, q)$

Step 2: putting both sides of the above eqn equal

to an arbitrary constant we get the two eqns.

Step 3: Solving them for $p \& q$.

Substitute the values of $p \& q$ in

$$dz = pdx + qdy \dots$$

Integrate, we get the complete integral of ①

Note: Some times using transformations eqns

reduce to the form of type III

→ find a complete integral of $p^2 + q^2 = x + y$

Sol: Given that $p^2 + q^2 = x + y \dots$ ①

$$\Rightarrow p^2 - x = y - q^2 \dots$$
 ②

$$\Rightarrow p - x = y - q^2 = a \text{ (say).}$$

$$\Rightarrow p - x = a \dots$$
 ③

$$\& y - q^2 = a \dots$$
 ④

$$\text{⑤} \equiv p = \sqrt{x+a} \& \text{⑥} \equiv y = \sqrt{y-a}$$

∴ putting these values of $p \& q$

in $dz = pdx + qdy$

$$\Rightarrow dz = \sqrt{x+a} dx + \sqrt{y-a} dy$$

Integrating

$$z = \frac{2}{3} (x+a)^{3/2} + \frac{2}{3} (y-a)^{3/2} + S$$

which is the required complete integral.

1989 → Find a complete integral of $z^2(p^2 + q^2) = x + y$

Sol: i.e. $z^2 \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = x + y \dots$ ①

$$\Rightarrow \left(z \frac{\partial z}{\partial x} \right)^2 + \left(z \frac{\partial z}{\partial y} \right)^2 = x + y \dots$$
 ②

Taking $z dz = dz$

$$\textcircled{2} \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 -$$

$$\Rightarrow P^2 + Q^2 = x^2 + y^2 \quad \text{where } P = \frac{\partial z}{\partial x}, Q = \frac{\partial z}{\partial y}$$

and proceed.

89) Find a complete integral of $z(P^2 - Q^2) = x - y$

200) Find the complete integral of the PDE

$$2P^2Q^2 + 3x^2y^2 = 8x^2y^2(x^2 + y^2)$$

Sol: Given that

$$2P^2Q^2 + 3x^2y^2 = 8x^2y^2(x^2 + y^2) \quad \textcircled{1}$$

$$\Rightarrow 2Q^2(P^2 - 4x^4) = x^2y^2(8x^2 - 3)$$

$$\Rightarrow \frac{P^2 - 4x^4}{x^2} = \frac{y^2(8x^2 - 3)}{2Q^2} = 4x^2 \quad (\text{say})$$

$$\Rightarrow \frac{P^2 - 4x^4}{x^2} = 4x^2; \quad \frac{y^2(8x^2 - 3)}{2Q^2} = 4x^2$$

$$\Rightarrow P^2 = 4x^2(x^2 + y^2); \quad 8Q^2(y^2 - x^2) = 3y^2$$

$$\Rightarrow P = 2x(x^2 + y^2)^{1/2}; \quad Q^2 = \frac{3y^2}{8(y^2 - x^2)} = \left(\frac{1}{4}\right)\left(\frac{3}{2}\right)\frac{y^2}{y^2 - x^2}$$

Substituting the values of P & Q in,

$$dz = Pdx + Qdy$$

$$\Rightarrow dz = 2x(x^2 + y^2)^{1/2}dx + \left(\frac{3}{2}\right)\left(\frac{y}{2}\right)(y^2 - x^2)$$

$$\Rightarrow dz = \int dx + \left(\frac{3}{2}\right)\left(\frac{y}{2}\right)(y^2 - x^2)^{1/2} dy$$

$$\Rightarrow z = \frac{4}{3}(x^2 + y^2)^{3/2} + \left(\frac{3}{2}\right)\left(\frac{y}{2}\right)(y^2 - x^2)^{1/2} + C$$

Type (iv) Clairaut eqn:-

A first order PDE is said to be Clairaut form if it is in the form $z = px + qy + f(p, q)$ — (1)

The corresponding Charpit's auxiliary eqns are

$$\frac{dx}{x+fp} = \frac{dy}{y+fq} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$$\Rightarrow p=a \quad \& \quad q=b$$

$$\therefore (1) \Rightarrow z = ax + by + f(a, b) \quad (2)$$

which is the required complete integral.

→ To find the general integral

put $b = \phi(a)$ in (2), where ϕ is an arbitrary function.

$$\text{then } z = ax + y\phi(a) + f\{a, \phi(a)\} \quad (3)$$

Diff (3) partially w.r.t. a , we get

$$0 = x + y\phi'(a) + f'(a) \quad (4)$$

Eliminating a b/w (3) & (4), we get G.I. of (1).

→ To find the S.I., eliminate a & b b/w the three eqns $z = ax + by + f(a, b)$

$$x + \frac{\partial f}{\partial a} = 0 \quad \text{and} \quad y + \frac{\partial f}{\partial b} = 0$$

Note: Some times, using the transformations eqns reduce to the form of type IV.

Problem: Find the singular integral of

$$z = px + qy + c\sqrt{1+p^2+q^2}$$

Sol: Given $z = px + qy + c\sqrt{1+p^2+q^2}$

It is of the Clairaut's form.

Its complete integral is
$$z = ax + by + c\sqrt{1+a^2+b^2}$$
 — (1)

Singular integral:

Diff ① partially w.r.t a and b,

we get $a = x + \frac{ae}{\sqrt{1+a^2+b^2}}, b = y + \frac{be}{\sqrt{1+a^2+b^2}}$ ②

from ② & ③

$$x^2+y^2 = \frac{a^2+c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-x^2-y^2 = \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow 1+a^2+b^2 = \frac{c^2}{c^2-x^2-y^2} \quad \text{--- ④}$$

from ② $a = -x \frac{\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-x}{\sqrt{c^2-x^2-y^2}} \quad (\text{by ④}) \quad \text{--- ⑤}$$

and from ③ $b = -y \frac{\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-y}{\sqrt{c^2-x^2-y^2}} \quad \text{--- ⑥}$$

putting the values from ④, ⑤ & ⑥ in ①,
the singular solution is

$$z = -\frac{x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

$$\Rightarrow z^2 = c^2 - x^2 - y^2$$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = c^2}$$

→ Find a complete and singular integral of
 $u_2 y z = p_2 + 2p^2 x y + 2q^2 x y^2$.

Sol: Given that $u_2 y z = p_2 + 2p^2 x y + 2q^2 x y^2 \quad \text{--- (1)}$

$$\begin{aligned} \Rightarrow z &= \frac{1}{2xy} \left(\frac{\partial z}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) + \frac{1}{2xy} \left(\frac{\partial z}{\partial x} \right) xy + \frac{1}{2xy} \left(\frac{\partial z}{\partial y} \right) y^2 \\ \Rightarrow z &= \left(\frac{1}{2x} \cdot \frac{\partial z}{\partial x} \right) \left(\frac{1}{2y} \frac{\partial z}{\partial y} \right) + \left(\frac{1}{2x} \frac{\partial z}{\partial x} \right) y^2 + \left(\frac{1}{2y} \frac{\partial z}{\partial y} \right) y^2 \end{aligned}$$

Taking $2x dx = dx$; $2y dy = dy$
 $\Rightarrow \boxed{x^2 = x}$; $\Rightarrow \boxed{y^2 = y}$

$$\text{Q.E. } z = \left(\frac{\partial z}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) + \left(\frac{\partial z}{\partial x} \right) x + \left(\frac{\partial z}{\partial y} \right) y$$

$$\Rightarrow z = P x + Q y + P & Q \quad \text{--- (2)} \\ \text{where } P = \frac{\partial z}{\partial x}; Q = \frac{\partial z}{\partial y}.$$

Clearly which is in Clairaut's form.

∴ The complete integral of (3) is

$$z = a x + b y + a b. \quad (\text{by putting } P=a \text{ & } Q=b)$$

$$\Rightarrow \boxed{z = ax + by + ab} \quad \text{--- (4)}$$

which is the required complete integral of (1).

Singular integral:

Differentiating (4) w.r.t. a & b , we get

$$0 = x^2 + b \Rightarrow \boxed{b = -x^2} \quad \text{--- (5)}$$

$$\text{and } 0 = y^2 + a \Rightarrow \boxed{a = -y^2} \quad \text{--- (6)}$$

$$\text{Q.E. } z = -y^2 x^2 - x^2 y^2 + x^2 y^2$$

$$\Rightarrow \boxed{z = -x^2 y^2}$$

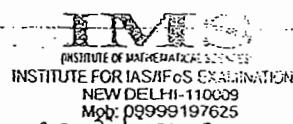
which is the required singular integral of (1).

2008

Find complete and singular integrals of

$$2xz - p_2 x^2 - 2q_2 y + p_2 = 0$$

using Charpit's method



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Solutions satisfying given conditions

We shall consider the determination of surfaces which satisfy the PDE $f(x, y, z, p, q)=0$ and which satisfy some other condition as passing through given curve (or) circumscribing a given surface. we shall also consider how to derive the complete integral from another.

First of all, we shall discuss how to determine the solution of (1) which passes through a given curve 'c' which has parametric eqns

$$x=x(t), y=y(t), z=z(t) \quad (2)$$

where t is parameter.

If there is an integral surface of the eqn (1) through the curve 'c', then it is;

(a) A particular case of the complete integral

$$f(x, y, z, a, b)=0 \quad (3)$$

obtained by giving particular values to a or b .

(or)

(b) A particular case of the general integral

corresponding to (3).

i.e., the envelope of a one-parameter subfamily of (3).

(or)

(c) The envelope of the two-parameter system (3).

Now the points of intersection of the surface (3) and the curve 'c' are determined in terms of the parameter 't' by the eqn $f(x(t), y(t), z(t), a, b)=0 \quad (4)$

and the condition that the curve 'c' should touch the

Surface (3) is that the eqn (4) must have two equal roots. (i.e., $b^2 - 4ac = 0$)

(or) the eqn (4) can

$$\text{the eqn } \frac{d}{dt} f(x(t), y(t), z(t), a, b) = 0 \quad (3)$$

should have a common root.

Now eliminating 't' from (1) & (3), we get the relation b/w a & b of the type $q(a, b) = 0 \quad (4)$

The eqn (4) may be factorised into a set of alternative eqns.

$$- b = \phi_1(a), \quad b = \phi_2(a), \quad \dots \quad (5)$$

Each of which defines a subsystem of one-parameter.

The envelope of each of these one-parameter subsystems is a solution of the problem.

problems

2004) Find a complete integral of the PDE

$(p^2 + q^2)x = pz$ and deduce the solution which passes through the curve $x=0, z^2 = 4y$.

Sol: Given that $(p^2 + q^2)x = pz$

$$\text{let } f(x, y, z, p, q) = (p^2 + q^2)x - pz = 0 \quad (1)$$

By Charpit's method its complete integral

$$z^2 = a^2 x^2 + (ay + b)^2 \quad (2)$$

and the given curve is $x=0, z^2 = 4y \quad (3)$

now taking 't' as parameter in (3)

$$\text{we get } x=0, \quad y=t^2, \quad z=2t \quad (4)$$

The intersection of (2) & (4) is

$$(2t)^2 = a^2(0)^2 + (at^2 + b)^2$$

$$\Rightarrow 4t^2 = a^2 t^4 + b^2 + 2abt^2$$

$$\Rightarrow a^2(t^2)^2 + (2ab - 4)t^2 + b^2 = 0 \quad (5)$$

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This has equal roots

$$\begin{aligned} \text{if } & (2ab - 4) = 4a^2b^2 = 0 \\ \Rightarrow & 4 - 4ab = 0 \\ \Rightarrow & ab = 1 \quad \text{--- (6)} \\ \therefore & b = \frac{1}{a} \end{aligned}$$

$$\therefore (7) \equiv z^2 = a^2x^2 + (ay + \frac{1}{a})^2 \quad \text{--- (7)}$$

∴ which is the one parameter subsystem of (2)

$$\begin{aligned} (8) & \equiv x^2a^2 + y^2a^2 + \frac{1}{a^2} + 2y - z^2 = 0 \\ & \Rightarrow (x^2 + y^2)a^4 + (2y - z^2)a^2 + 1 = 0 \quad \text{--- (8)} \end{aligned}$$

This has equal roots

$$\text{if } (2y - z^2)^2 - 4(x^2 + y^2) = 0 \quad \text{--- (9)}$$

which required envelope of (8)

$$\begin{aligned} (9) & \equiv 2y - z^2 = 2\sqrt{x^2 + y^2} \\ & \Rightarrow z^2 = 2y - 2\sqrt{x^2 + y^2} \end{aligned}$$

which is the required solution
of the eqn (1).

- find a complete integral of the eqn $p^2x + q^2y = z$ and hence derive the eqn of an integral surface of which the line $y=1, x+z=0$ is a generator
- Show that Integral surface of the eqn $z(1-q^2) = 2(p^2x + q^2y)$ which pass through the line $x=1, y=bz+k$ has the eqn $(y-Kx)^2 = z^2 \{(1+b^2)x - 1\}$.

III The problem of deriving one complete integral from another may be treated in a very similar way.

Suppose we know that $f(x, y, z, a, b) = 0 \quad \text{--- (1)}$

is complete integral of $F(x, y, z, p, q) = 0 \quad \text{--- (2)}$

and we want to show that another relation

$g(x, y, z, h, k) = 0 \quad \text{--- (3)}$
where h & k are arbitrary constants

is also complete integral of ②.

We choose on the surface ③ a curve C in whose ears the constants b, k appear as independent parameters and then find the envelope of the one-parameter subfamily of ① touching the curve C .

Since this solution contains two arbitrary constants, it is a complete integral;

→ show that the equation $xpq + yq^2 = 1$ has complete integrals (a) $(x+5)^2 = 4(ax+y)$
(b) $kx(x+b) = K^2y+1x^2$

and deduce (b) from (a)

Sol: Given that $xpq + yq^2 = 1$ — ①

Charpit's auxiliary eqns are:

$$\begin{aligned}\frac{dx}{-fp} &= \frac{dy}{fq} = \frac{dz}{-pf_p - 2fq} = \frac{dp}{f_x + pf_2} = \frac{dq}{fy + qf_2} \\ \Rightarrow \frac{dx}{-xq} &= \frac{dy}{-[xp + 2yq]} = \frac{dz}{p(xq) + q(xp + 2yq)} = \frac{dp}{fq} = \frac{dq}{q^2} \quad \text{--- ②}\end{aligned}$$

Now taking last two fractions from ②, we have

$$\therefore p = qa$$

$$\text{②} \quad xq^2a + yq^2 = 1$$

$$\Rightarrow q^2(xa + y) = 1$$

$$\Rightarrow \boxed{q = \frac{1}{\sqrt{xa+y}}}$$

$$\boxed{\therefore p = \frac{a}{\sqrt{xa+y}}}$$

Substituting the values of p and q in $dz = pdx + qdy$

$$\Rightarrow dz = \frac{1}{\sqrt{xa+y}} [adx + dy]$$

$$\Rightarrow dz = \frac{d(ax+y)}{\sqrt{ax+y}}$$

$$\Rightarrow z = 2(ax+y)^{1/2} + b$$

$$\Rightarrow (z+b)^2 = 4(ax+y) \quad \text{--- (2)}$$

Taking first & last fractions of (2), we get

$$\frac{dz}{dx} = \frac{da}{a}$$

$$\Rightarrow da = k \Rightarrow a = \frac{k}{x}$$

$$\therefore (2) \Rightarrow PK + Y \frac{k^2}{x^2} = 1$$

$$\Rightarrow P = \frac{1}{k} \left(\frac{x^2 - YK^2}{x^2} \right)$$

$$\therefore dz = \left(\frac{x^2 - YK^2}{x^2 k} \right) dx + \frac{k}{x} dy$$

$$dz = \frac{1}{k} dx - \frac{YK}{x^2} dx + \frac{k}{x} dy$$

$$\Rightarrow dz = \frac{1}{k} dx + d\left(Y \frac{k}{x}\right)$$

$$\Rightarrow z = \frac{x}{k} + \frac{YK}{x} + h$$

$$\Rightarrow (z+h) = -\frac{x^2 + YK^2}{Kx}$$

$$\Rightarrow Kx(z+h) = x^2 + YK^2$$

$$\Rightarrow Kx(z+h) = Ky + x^2 \quad \text{--- (3)}$$

Now consider the curve from (3)

$$y=0; z = K(z+h) \quad \text{--- (3)}$$

where h, k are independent parameters

Now taking 't' as parameter in (3)

$$\text{we get } z=t; x=K(t+h); y=0 \quad \text{--- (4)}$$

The intersection of (2) & (4) is

$$(t+h)^2 = 4[ak(t+h)]$$

$$\Rightarrow t^2 + (2b - 4ak)t + b^2 - 4akh = 0$$

This has equal roots if $(2b-4ak)^2 - 4(1)(b^2-4ak^2) = 0$

$$\Rightarrow b^2 + 4a^2k^2 - 4abk - b^2 + 4ak^2 = 0$$

$$\Rightarrow a^2k^2 - abk + ak^2 = 0$$

$$\Rightarrow ak[ak - b + k] = 0$$

$$\Rightarrow \boxed{b = h + ak} \quad (\because ak \neq 0)$$

(a) $[z + (h + ak)]^2 = 4(ax + y) \quad (1)$
 which is the one-parameter subsystem of (1)

(5) $z^2 + (h + ak)^2 + 2z(h + ak) = 4ax + 4y$

$$\Rightarrow z^2 + h^2 + a^2k^2 + 2hk + 2hz + 2ka - 4ax - 4y = 0$$

$$\Rightarrow k^2a^2 + (2hk + 2zk - 4x)a + \underline{z^2 + 2hz + h^2} - 4y = 0$$

This has equal roots

$$\text{if } (2hk + 2zk - 4x)^2 - 4k^2(z^2 + h^2 - 4y) = 0$$

$$\Rightarrow (hk + zk - 2x)^2 = k^2(z^2 + h^2 - 4y)$$

$$\Rightarrow h^2k^2 + z^2k^2 + 4x^2 + 2hkzk - 4xz - 4x^2k^2 - 4xh^2k = 0$$

$$= 2h^2k^2 + k^2z^2 + k^2h^2 - 4y^2$$

$$\Rightarrow 4x^2 - 4xz - 4x^2k^2 - 4xh^2k = -4y^2$$

$$\Rightarrow x^2 + y^2 = x^2k^2 + xh^2k$$

$$\Rightarrow kx(z + y) = k^2y + x^2$$

→ Show that the diff. eqn $2xz + y^2 = x(x^2 + y^2)$ has a complete integral $z + a^2x = axy + bx^2$ and deduce that $x(y + hz)^2 = 4(z - kx^2)$ is also a complete integral

→ Find the complete integral of diff. eqn $\frac{dp}{dx}(1+q) = (y+z)q$ corresponding to the integral of char. eqn which involving only y & x , and deduce that $(z + hz + k)^2 = 4hx(k - y)$

III

The determination of surfaces which satisfy
the PDE $F(x, y, z, p, q) = 0$ — (1)

and which satisfy some other condition such as
circumscribing a given surface.

→ Two surfaces are said to be circumscribe each
other if they touch along curve.

→ Now we shall suppose that $f(x, y, z, a, b) = 0$ — (2)
is a complete integral of (1).

Now we wish to find, by using (2), an integral
surface of (1), which circumscribes the surface
 Σ whose eqn is $\psi(x, y, z) = 0$ — (3)

If we have a surface E ; $u(x, y, z) = 0$ — (4)

of the required kind then it will be one of
the three kinds:

(a) A particular case of the complete integral
 $f(x, y, z, a, b) = 0$

obtained by giving particular values to
a or b.

(b) A particular case of the general integral
corresponding to (2)

i.e., the envelope of a one-parameter
subfamily of (2)

(c) The envelope of two-parameter system (2).

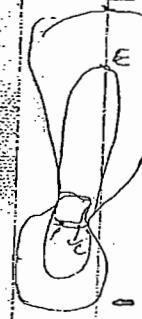
→ We now to find the surface (2) which touch Σ
and see if they provide a solution of the problem.

→ The surface (2) touches the surface (4) iff
the eqns (1), (3) and $\frac{f_x}{P_2} = \frac{f_y}{Q_2} = \frac{f_z}{R_2}$ — (5)
are consistent

NOW eliminating x, y and z from these eqns, we get the relation b/w $a \& b$ of the form $\xi(a,b)=0$

This relation may be factorized into set of alternative eqns $b = \phi_1(a), b = \phi_2(a), \dots$ (6)

each of which defines a subsystem of (2) whose members touch (3).



The points of contact lie on the surface whose eqn is obtained by eliminating $a \& b$ from the eqns (6) & (7).

The curve C is the intersection of this surface with Σ . Each of the relations (6) defines a subsystem whose envelope E touches Σ along C .

Show that the only integral surface of the eqn $2q(z-px-qy) = 1+q^2$ which is circumscribed about the paraboloid $zx=y^2+z^2$ is the enveloping cylinder which touches it along its section by the plane $y+1=0$.

Solⁿ: Given that $2q(z-px-qy) = 1+q^2$ (1)

$$\Rightarrow z - px - qy = \frac{1+q^2}{2q}$$

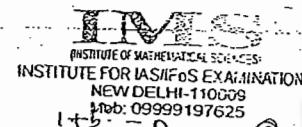
$$\Rightarrow z = px + qy + \frac{1+q^2}{2q} \quad (2)$$

Clearly which is in Clairaut's form

$$z = px + qy + f(p, q)$$

The required complete integral of (1) is

$$z = ax + by + \frac{1+b^2}{2b} \quad (\text{by putting } p=a, q=b)$$



$$\text{Let } f(x, y, z, a, b) = 2 - ax - by - \frac{1+b}{2b} = 0 \quad (3)$$

and given that integral surface of (3) which circumscribes about the paraboloid $2x = y + z^2$

$$\text{Let } \varphi(x, y, z) = 2x - y - z^2 \quad (4)$$

$$\text{Now } \frac{f_x}{4x} = \frac{f_y}{4y} = \frac{f_z}{4z}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{-2y} = \frac{1}{-z^2} \quad (5)$$

$$\Rightarrow \frac{a}{2} = \frac{b}{-2y} \text{ & } \frac{a}{2} = \frac{1}{-z^2}$$

$$\Rightarrow \boxed{y = -\frac{b}{a}} \text{ & } \boxed{z = \pm \sqrt{\frac{1}{a}}} \quad (6)$$

Now eliminating x b/w (3) & (4), we get

$$2x = a(y + z^2) + 2by + \frac{2(b^2+1)}{2b}$$

$$\Rightarrow 2bz = aby + abz^2 + 2b^2y + b^2 + 1 \quad (7)$$

and eliminating y & z from (7) by using (6)

we get

$$(ab-a)(b^2+1) = 0$$

$$\Rightarrow \boxed{b=a} \quad (\because b^2+1 \neq 0)$$

which defines a subsystem of (3) whose envelope is a surface of the required kind.

The envelope of the subsystem

$$[2(x+y)+1]a^2 - 2az + 1 = 0 \quad (8)$$

$$4z^2 - 4[2(x+y)+1]a^2 = 0$$

$$\Rightarrow z^2 = 2(x+y) + 1 \quad (8)$$

Since the surface (4) touches the surface (8)

$$2x - y^2 = 2(x+y) + 1$$

$$\Rightarrow -y^2 = 2y + 1$$

$$\Rightarrow y^2 + 2y + 1 = 0$$

$$\Rightarrow (y+1)^2 = 0$$

$$\Rightarrow \underline{y+1 = 0}$$

→ find the integral surface of the PDE

$(y+p^2)^2 = z(1+p^2+q^2)$ circumscribed
about the surface $x^2-z^2=2y$

→ show that the integral surface of the eqn

$2y(1+p^2) = pq$ which is circumscribed
about the cone $x^2+y^2=z^2$ has eqn

$$\underline{z^2 = y^2(4y^2+4x+1)}$$

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Jacobi's Method

Working rule for solving PDE's with three (or) more than three independent variables

Step 1: Suppose the given eqn with three independent variables is $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ — (1) in which the dependent variable does not appear; x_1, x_2, x_3 are independent variables and $p_i = \frac{\partial f}{\partial x_i}$; $i=1, 2, 3$.

Step 2: Write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{\frac{\partial f}{\partial p_1}} = \frac{dx_2}{\frac{\partial f}{\partial p_2}} = \frac{dx_3}{\frac{\partial f}{\partial p_3}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}}$$

Solving these eqns, we obtain two additional

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad (2)$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad (3)$$

where a_1 & a_2 arbitrary constants.

Step 3: Verify that relations (2) & (3) satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0$$

$$\Rightarrow (F_1, F_2) = \sum_{r=1}^3 \frac{\partial (F_1, F_2)}{\partial (x_r, p_r)} = 0 \quad (4)$$

If (4) is satisfied then solve (1), (2) & (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3 .

Substitute these values in

$$dx = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

It gives the complete integral of the given eqn and containing three arbitrary constants.

Note: While solving a PDE with four independent variables,

Step 1: The given eqn is of the form

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0 \quad \text{--- (1)}$$

Step 2: Write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{\partial f / \partial p_1} = \frac{dx_2}{\partial f / \partial p_2} = \frac{dx_3}{\partial f / \partial p_3} = \frac{dx_4}{\partial f / \partial p_4} = \frac{dp_1}{\partial f / \partial x_1} = \frac{dp_2}{\partial f / \partial x_2} = \frac{dp_3}{\partial f / \partial x_3} = \frac{dp_4}{\partial f / \partial x_4}$$

Solving these eqns, we obtain three additional eqns

$$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1 \quad \text{--- (2)}$$

$$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2 \quad \text{--- (3)}$$

$$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3 \quad \text{--- (4)}$$

where a_1, a_2, a_3 are arbitrary constants.

Step 3: Verify the relations (2), (3), & (4) satisfy the following three conditions—

$$(F_1, F_2) = \sum_{r=1}^4 \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0 \quad \text{--- (5)}$$

$$(F_2, F_3) = \sum_{r=1}^4 \frac{\partial(F_2, F_3)}{\partial(x_r, p_r)} = 0 \quad \text{--- (6)}$$

$$\text{and } (F_3, F_1) = \sum_{r=1}^4 \frac{\partial(F_3, F_1)}{\partial(x_r, p_r)} = 0 \quad \text{--- (7)}$$

If (5), (6) & (7) are satisfied they solve (1), (2)

(3) & (4) for p_1, p_2, p_3 & p_4 in terms of x_1, x_2, x_3, x_4

and substitute these values in $dx = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$

which gives the complete integral of (1)
and containing four arbitrary constant.

Q7 Find a complete integral of $p_1^3 + p_2^2 + p_3 = 1$

Sol: Let the given can be

$$f(x_1, x_2, x_3; p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0 \quad \text{--- (1)}$$

Now Jacobi's A.E's are

$$\frac{dx_1}{\partial f / \partial p_1} = \frac{dx_2}{\partial f / \partial p_2} = \frac{dx_3}{\partial f / \partial p_3} = \frac{dp_1}{\partial f / \partial x_1} = \frac{dp_2}{\partial f / \partial x_2} = \frac{dp_3}{\partial f / \partial x_3}$$

$$\Rightarrow \frac{dx_1}{-3P_1} = \frac{dx_2}{-P_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$$

from the first 5 fractions, we get

$$\therefore dp_1 = 0 \text{ and } dp_2 = 0$$

$$\Rightarrow P_1 = a_1 \text{ and } P_2 = a_2$$

$$\text{Here } F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 - a_1 = 0 \quad \textcircled{2}$$

$$P_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - a_2 = 0 \quad \textcircled{3}$$

$$\text{Now } (F_1, F_2) = \sum_{r=1}^3 \frac{\partial(F_1, F_2)}{\partial(x_r, P_r)}$$

$$= \frac{\partial(F_1, F_2)}{\partial(x_1, P_1)} + \frac{\partial(F_1, F_2)}{\partial(x_2, P_2)} + \frac{\partial(F_1, F_2)}{\partial(x_3, P_3)}$$

$$= \frac{\partial F_1}{\partial x_1} \frac{\partial F_2}{\partial P_1} - \frac{\partial F_1}{\partial P_1} \frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2} \frac{\partial F_2}{\partial P_2} - \frac{\partial F_1}{\partial P_2} \frac{\partial F_2}{\partial x_2} \\ + \frac{\partial F_1}{\partial x_3} \frac{\partial F_2}{\partial P_3} - \frac{\partial F_1}{\partial P_3} \frac{\partial F_2}{\partial x_3}$$

$$= 0$$

$$\therefore (F_1, F_2) = 0$$

\therefore The eqns (2) & (3) taken as additional eqns

\therefore solving 0, ② & ③ for P_1, P_2 & P_3

$$\text{we have } P_1 = a_1, P_2 = a_2 \text{ & } P_3 = 1 - a_1^3 - a_2^2.$$

$$\therefore \text{putting these values in } dz = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$$

$$\text{we have } dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$$

Integrating, we get

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$$

where a_1, a_2, a_3 are arbitrary constants

which is the required complete integral.

1998 → Find a complete integral of $2P_1 x_1 x_2 + 3P_2 x_3 + b_2^2 P_3 = 0$

→ find a complete integral of $P_1 P_2 P_3 = z^3 x_1 x_2 x_3$

$$\therefore \frac{\partial z}{\partial x_1} \frac{\partial z}{\partial x_2} \frac{\partial z}{\partial x_3} = z^3 x_1 x_2 x_3$$

$$\text{Soln: } \left(\frac{1}{z} \frac{dz}{dx_1} \right) \left(\frac{1}{z} \frac{dz}{dx_2} \right) \left(\frac{1}{z} \frac{dz}{dx_3} \right) = x_1 x_2 x_3 \quad \text{--- (1)}$$

Taking $\frac{1}{z} dz = dZ$

$$\Rightarrow \log z = Z$$

$$\therefore (1) \equiv \left(\frac{\partial Z}{\partial x_1} \right) \left(\frac{\partial Z}{\partial x_2} \right) \left(\frac{\partial Z}{\partial x_3} \right) = x_1^2 x_2 x_3$$

$$\text{Let } f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 P_2 P_3 - x_1 x_2 x_3 = 0 \quad \text{--- (2)}$$

$$\text{where } P_1 = \frac{\partial Z}{\partial x_1}, \quad P_2 = \frac{\partial Z}{\partial x_2}, \quad P_3 = \frac{\partial Z}{\partial x_3}$$

NOW the Jacobi's auxiliary cans are

$$\frac{dx_1}{-P_2 P_3} = \frac{dx_2}{-P_1 P_3} = \frac{dx_3}{-P_1 P_2} = \frac{dP_1}{x_2 x_3} = \frac{dP_2}{-x_1 x_3} = \frac{dP_3}{-x_1 x_2} \quad \text{--- (3)}$$

$$(2) \equiv P_2 P_3 = \frac{x_1^2 x_2 x_3}{P_1}$$

\therefore first and fourth fractions of (3) give

$$\frac{dx_1}{-x_1 x_2 x_3} = \frac{dP_1}{-x_2 x_3} \Rightarrow \frac{dP_1}{P_1} = \frac{dx_1}{x_1}$$

Integrating, we get

$$\therefore P_1 = a_1 x_1$$

$$\text{Let } f_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 - a_1 x_1 \quad \text{--- (4)}$$

$$\text{Similarly we have } f_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - a_2 x_2 \quad \text{--- (5)}$$

$$\begin{aligned} (F_1, P_2) &= \sum_{r=1}^3 \frac{\partial (F_1, F_2)}{\partial (x_r, P_2)} = \frac{\partial (F_1, F_2)}{\partial (x_1, P_2)} + \frac{\partial (F_1, F_2)}{\partial (x_2, P_2)} + \frac{\partial (F_1, F_2)}{\partial (x_3, P_2)} \\ &= \frac{\partial F_1}{\partial x_1} \frac{\partial P_2}{\partial P_1} - \frac{\partial F_1}{\partial P_1} \frac{\partial P_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2} \frac{\partial P_2}{\partial P_2} - \frac{\partial F_1}{\partial P_2} \frac{\partial P_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3} \frac{\partial P_2}{\partial P_3} - \frac{\partial F_1}{\partial P_3} \frac{\partial P_2}{\partial x_3} \\ &= 0 \end{aligned}$$

\therefore The cans (4) & (5) taken as additional cans.

$$\text{Solving (2), (4) & (5), we get } P_1 = a_1 x_1, \quad P_2 = a_2 x_2,$$

$$P_3 = \frac{a_3}{a_1 a_2}$$

Putting these values in $dZ = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$

$$dZ = a_1 x_1 dx_1 + a_2 x_2 dx_2 + \frac{a_3}{a_1 a_2} dx_3$$

Integrating, we get

$$Z = \frac{1}{2} a_1 x_1^2 + \frac{1}{2} a_2 x_2^2 + \frac{1}{2 a_1 a_2} x_3^2 + a_3$$

Taking $Z = 16 gZ$

$$2 \log Z = a_1 x_1^2 + a_2 x_2^2 + \frac{a_3}{a_1 a_2} x_3^2 + a_3,$$

which is the required integral.

Cauchy's Method of Characteristics:

for solving non-linear differential eqns.

Working rule:

Let us consider the non-linear PDE

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

Suppose we wish to find the solution of (1) which passes through a given curve whose parametric eqns are

$$x = f_1(\lambda), \quad y = f_2(\lambda), \quad z = f_3(\lambda) \quad \text{--- (2)}$$

where λ is a parameter.

Then in the solution

$$x = x(P_0, q_0, x_0, y_0, z_0, t_0, t)$$

$$y = y(P_0, q_0, x_0, y_0, z_0, t_0, t) \quad \text{and}$$

$$z = z(P_0, q_0, x_0, y_0, z_0, t_0, t)$$

of the characteristic eqns. of (1) are

$$x'(t) = f_p, \quad y'(t) = f_q, \quad z'(t) = pf_p + qf_q$$

$$p'(t) = -f_x = pf_z \quad \text{and} \quad q'(t) = -f_y - qf_z \quad \text{--- (4)}$$

where $x'(t) = \frac{dx}{dt}$ etc

and $f_p = \frac{\partial f}{\partial p}$ etc.

We shall assume that

$$x_0 = f_1(\lambda), \quad y_0 = f_2(\lambda), \quad z_0 = f_3(\lambda) \quad \text{as the}$$

initial values of x, y, z respectively; then the corresponding initial values of P_0, q_0 are determined by the following relations

$$f'_3(\lambda) = P_0 f_1(\lambda) + q_0 f_2(\lambda) \quad \&$$

$$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), P_0, q_0) = 0$$

If these values of x_0, y_0, z_0, P_0, q_0 and the

appropriate value of 't' to substitute in the eqn ③.

we find that x, y, z can be expressed in terms of the two parameters t & λ of the form

$$x = \phi_1(t, \lambda), \quad y = \phi_2(t, \lambda) \quad \& \quad z = \phi_3(t, \lambda) \quad \text{--- (4)}$$

which are known as characteristic strips of ①.

Finally by eliminating λ & t from ④,

we get the relation of the form $\psi(x, y, z) = 0$

which is the required integral surface of

① passing through the given curve ②.

2002 Find the solution of the eqns

$z = \frac{1}{2}(p^x + q^y) + (p-x)(q-y)$ which
passes through the x -axis.

Soln: Given that $z = \frac{1}{2}(p^x + q^y) + (p-x)(q-y)$

$$\text{Let } f(x, y, z, p, q) = \frac{1}{2}(p^x + q^y) + (p-x)(q-y) - z. \quad \text{--- (1)}$$

we are to find the integral surface of ①
which passes through x -axis whose parametric
eqns are $x = \lambda, y = 0, z = 0$

where λ is the parameter.

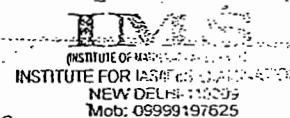
$$\text{i.e., } x = f_1(\lambda) = \lambda, \quad y = f_2(\lambda) = 0, \quad z = f_3(\lambda) = 0$$

set the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q

be taken as $x_0 = f_1(\lambda) = \lambda; \quad y_0 = f_2(\lambda) = 0, \quad z_0 = f_3(\lambda) = 0$

now we find the initial values p_0 & q_0 by

the following relations



$$f_3(\lambda) = p_0 f_1(\lambda) + q_0 f_2(\lambda) \quad \&$$

$$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$$

$$\therefore f(x_0, y_0, z_0, p_0, q_0) = 0$$

$$\Rightarrow 0 = p_0(1) + q_0(0) \quad \&$$

$$\frac{1}{2}(p_0 + q_0) + (p_0 - x_0)(z_0 - y_0) - z_0 = 0$$

$$\Rightarrow \boxed{p_0 = 0} \quad \& \quad \frac{1}{2}q_0 - x_0(z_0 - y_0) - z_0 = 0 \quad (\because x_0 = 0)$$

$$\Rightarrow \frac{1}{2}q_0 - z_0(z_0 - 0) - 0 = 0$$

$$\Rightarrow q_0(\frac{1}{2}q_0 - z_0) = 0$$

$$\Rightarrow \frac{1}{2}q_0 = z_0 \quad (\because q_0 \neq 0)$$

$$\Rightarrow \boxed{q_0 = 2z_0}$$

$$\therefore x_0 = \lambda, y_0 = 0, z_0 = 0, p_0 = 0, \& q_0 = 2\lambda. \& t = t_0$$

now the characteristic eqns of ① are ③

$$x'(t) = \frac{\partial f}{\partial p} = p + (q - y) \quad \text{--- ④}$$

$$y'(t) = \frac{\partial f}{\partial q} = q + (p - x) \quad \text{--- ⑤}$$

$$z'(t) = p + [p + q - y] + q [q + p - x] \quad \text{--- ⑥}$$

$$p'(t) = -\frac{\partial f}{\partial x} - p \frac{\partial f}{\partial z}$$

$$= (q - y) - p(-1) \\ = q - y + p \quad \text{--- ⑦}$$

$$q'(t) = -\frac{\partial f}{\partial y} - q \frac{\partial f}{\partial z}$$

$$= (p - x) - q(-1) = p - x + q \quad \text{--- ⑧}$$

from ④ & ⑧, we have $x'(t) = p'(t)$

$$\Rightarrow \frac{dx}{dt} = \frac{dp}{dt}$$

$$\Rightarrow dx = dp$$

$$\Rightarrow \boxed{x = p + C_1} \quad \text{--- ⑨}$$

From ⑤ & ⑧, we get

$$y'(t) = q'(t)$$

$$\begin{aligned} \Rightarrow dy &= dq \\ \Rightarrow y &= q + c_2 \end{aligned} \quad (10)$$

Using the initial values in ⑨ & ⑩

$$\begin{aligned} ⑨ \quad x = 0 + c_1 &\quad ; \quad ⑩ \quad 0 = 2\lambda + c_2 \\ \Rightarrow c_1 &= \lambda \quad \Rightarrow c_2 = -2\lambda \end{aligned}$$

∴ from ⑨ & ⑩, we have

$$x = p + \lambda \quad \& \quad y = q - 2\lambda \quad (11)$$

from ④, ⑦ & ⑧, we get

$$\begin{aligned} \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} &= p + q - x \\ \Rightarrow \frac{d}{dt}(p+q-x) &= p+q-x \\ \Rightarrow \frac{d(p+q-x)}{p+q-x} &= dt \\ \Rightarrow \log(p+q-x) &= t + \log C_3 \\ \Rightarrow p+q-x &= C_3 e^t \end{aligned} \quad (12)$$

From ⑤, ⑦ & ⑧, we get

$$\begin{aligned} \frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} &= p + q - y \\ \Rightarrow \frac{d(p+q-y)}{p+q-y} &= dt \\ \Rightarrow p+q-y &= C_4 e^t \end{aligned} \quad (13)$$

Using the initial values in ⑦ & ⑬, we get

$$p_0 + q_0 - x_0 = C_3 e^{t_0} \quad \& \quad p_0 + q_0 - y_0 = C_4 e^{t_0}$$

$$\text{Taking } t = t_0 = 0.$$

$$\Rightarrow p_0 + q_0 - x_0 = C_3 \quad \& \quad 0 + 0 - 0 = C_4$$

$$\Rightarrow \boxed{\lambda = c_3} \quad \& \quad \boxed{c_4 = 2\lambda}$$

from (12) & (13), we get

$$p+q+\lambda = \lambda e^t \quad \& \quad p+q-\lambda = 2\lambda e^t \quad (15)$$

now from (1), (14) & (15), we get

$$p+q-(p+\lambda) = \lambda e^t ; \quad p+q-(q-2\lambda) = 2\lambda e^t$$

$$\Rightarrow q-\lambda = \lambda e^t \\ \Rightarrow \boxed{q = \lambda(1+e^t)} \quad (16)$$

∴ from (11), we have

$$x = \lambda(2e^t - 1) \quad \& \quad y = \lambda(e^t - 1) \quad (17)$$

$$(6) \equiv \frac{dz}{dt} = p[p+q-y] + q[p+q-x]$$

$$= 2\lambda(e^t - 1)(2\lambda e^t) + \lambda(1+e^t)(\lambda e^t) \\ = 5\lambda^2 e^{2t} - 3\lambda^2 e^t \quad (\text{from } (14) \& (15))$$

$$dz = \lambda^2(5e^{2t} - 3e^t) dt$$

$$z = \lambda^2 \left(\frac{5}{2}e^{2t} - 3e^t \right) + C_5$$

By using the initial values,
 $z_0 = \lambda^2 \left(\frac{5}{2}e^{2t_0} - 3e^{t_0} \right) + C_5$

$$\Rightarrow 0 = \lambda^2 \left(\frac{5}{2} - 3 \right) + C_5$$

$$\Rightarrow C_5 = \frac{\lambda^2}{2}$$

$$\boxed{z = \lambda^2 \left(\frac{5}{2}e^{2t} - 3e^t \right) + \frac{\lambda^2}{2}} \quad (18)$$

The required characteristic strips of (1)

are given by

$$x = \lambda(2e^t - 1), \quad y = \lambda(e^t - 1) \quad \& \quad z = \lambda \left(\frac{5}{2}e^{2t} - 3e^t \right) + \frac{\lambda^2}{2}$$

(19)

(19)

Now eliminating $t \& \lambda$ from (19)

now solving (1) & (2) from (19) for $\lambda \& e^t$, we get

$$\lambda = 2\lambda \left(\frac{y+\delta}{\lambda} \right) - \lambda \quad (\because \text{from (2)} \quad e^t = \frac{y+\delta}{\lambda})$$

$$\Rightarrow \lambda = 2y + 2\lambda - \lambda$$

$$\Rightarrow \lambda = 2y + \lambda$$

$$\Rightarrow \boxed{\lambda = x - 2y}$$

$$(2) \Rightarrow y = (x - 2y)(e^t - 1)$$

$$\Rightarrow e^t - 1 = \frac{y}{x - 2y}$$

$$\Rightarrow \boxed{e^t = \frac{x-y}{x-2y}} \Rightarrow \boxed{e^t = \frac{x-y}{x-2y}}$$

$$(1) \Rightarrow z = (x - 2y) \left[\frac{5}{2} \left(\frac{x-y}{x-2y} \right)^2 - 3 \left(\frac{x-y}{x-2y} \right) \right] + \frac{1}{2} \quad (3)$$

which is the required solution of (1)

passing through the given curve.

1999 find characteristic of the eqn $pq=z$ and the integral surface which passes through the parabola $x=0, y^2=z$.

Sol: Given that $pq=z$

$$\Rightarrow f(x, y, z, p, q) = pq - z = 0 \quad (1)$$

now we are to find the integral surface of (1) which is passing through the parabola.

$$x=0, y^2=z$$

whose parametric eqns are

$$x=0, y=\lambda \& z=\lambda^2$$

$$\text{i.e., } x=f_1(\lambda), y=f_2(\lambda) \& z=f_3(\lambda) \quad (2)$$

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Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q
be taken as

$$x_0 = f_1(\lambda) = 0, \quad y_0 = f_2(\lambda) = \lambda, \quad z_0 = f_3(\lambda) = \lambda^2$$

Now we find the initial values p_0 & q_0 by
the following relations

$$f'_3(\lambda) = p_0 f'_1(\lambda) + q_0 f'_2(\lambda) \quad 8$$

$$f\{f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0\} = 0$$

$$\text{i.e., } f(0, \lambda, \lambda^2, p_0, q_0) = 0$$

$$\Rightarrow 2\lambda = p_0(0) + q_0(1) \quad 8, \quad p_0 q_0 - \lambda^2 = 0$$

$$\Rightarrow 2\lambda = q_0 \quad \Rightarrow p_0(2\lambda) - \lambda^2 = 0$$

$$\Rightarrow [q_0 = 2\lambda] \quad \Rightarrow p_0 = \frac{\lambda}{2}$$

$$\therefore q_0 = 0, \quad y_0 = \lambda, \quad z_0 = \lambda^2, \quad q_0 = 2\lambda, \quad p_0 = \lambda/2$$

Now the characteristic eqns of (1) are

$$x'(t) = \frac{\partial f}{\partial p} = q, \quad \text{--- (4)}$$

$$y'(t) = \frac{\partial f}{\partial q} = p, \quad \text{--- (5)}$$

$$z'(t) = pq + p q = 2pq \quad \text{--- (6)}$$

$$p'(t) = -\frac{\partial f}{\partial x} - p \frac{\partial f}{\partial z} = -p(-i) = p \quad \text{--- (7)}$$

$$q'(t) = -\frac{\partial f}{\partial y} - q \frac{\partial f}{\partial z} = -q(-i) = q \quad \text{--- (8)}$$

Now from (4) & (8), we have

$$x'(t) = q'(t)$$

$$\Rightarrow dx = dq$$

$$\Rightarrow [x = q + c_1] \quad \text{--- (9)}$$

from ⑤ & ⑦, we have

$$\begin{aligned} -y'(t) = p'(t) &\Rightarrow dy = dp \\ \Rightarrow y = p + C_2 & \end{aligned} \quad (16)$$

now using the initial values in ④ & ⑯
we get $x_0 = q_0 e^{t_0}$ & $y_0 = p_0 + C_2$

$$\begin{aligned} \Rightarrow 0 = p_0 + C_2 & \quad \& \quad x = 2\lambda + C_1 \\ \Rightarrow C_2 = -p_0 & \quad \Rightarrow C_1 = \frac{\lambda}{2} + C_1 \\ C_1 = -2\lambda & \end{aligned}$$

∴ from ⑨ & ⑯, we have

$$x = q - 2\lambda \quad (17) \quad y = p + \frac{\lambda}{2} \quad (18)$$

$$\begin{aligned} \text{⑩} \equiv \frac{dp}{dt} &= p \\ \Rightarrow \frac{dp}{p} &= dt \Rightarrow \log p = t + \log C_3 \\ \Rightarrow p &= C_3 e^t \end{aligned} \quad (19)$$

$$\begin{aligned} \text{⑪} \equiv q'(t) = q &\Rightarrow \frac{dq}{q} = dt \\ \Rightarrow \log q &= t + \log C_4 \\ \Rightarrow q &= C_4 e^t \end{aligned} \quad (20)$$

using the initial values in ⑯ & ⑪, we get

$$p_0 = C_3 e^{t_0} \quad \& \quad q_0 = C_4 e^{t_0}$$

$$\Rightarrow \frac{\lambda}{2} = C_3 \quad \& \quad 2\lambda = C_4 \quad (\because t_0 = 0)$$

from ⑯ & ⑪, we have

$$p = \frac{\lambda}{2} e^t \quad \& \quad q = 2\lambda e^t \quad (21)$$

$$\text{⑫} \equiv x = 2\lambda e^t - 2\lambda \Rightarrow x = 2\lambda(1 - e^{-t}) \quad (22)$$

$$\text{⑬} \equiv y = \frac{\lambda}{2} e^t + \frac{\lambda}{2} \Rightarrow y = \frac{\lambda}{2}(e^t + 1) \quad (23)$$

$$\text{⑭} \equiv z' = 4pq$$

$$\Rightarrow \frac{dz}{dt} = 2\left(\frac{\lambda}{2}\right)(2\lambda e^t)$$

$$\Rightarrow dz = 2\lambda^2 e^{2t} dt$$

$$\Rightarrow z = \lambda^2 e^{2t} + c_5 \quad \text{--- (18)}$$

Using the initial values in (18), we get

$$z_0 = \lambda^2 e^{2t_0} + c_5$$

$$\Rightarrow \lambda^2 = \lambda^2(1) + c_5$$

$$\Rightarrow c_5 = 0$$

$$\therefore (18) \equiv z = \lambda^2 e^{2t}$$

∴ The required characteristics of ① are given by

$$x = 2\lambda(e^t - 1), \quad y = \frac{\lambda}{2}(e^t + 1), \quad z = \lambda^2 e^{2t} \quad \text{--- (21)}$$

Now eliminating e^t & λ from (19), (20) & (21)

$$(19) \equiv x = \frac{4y}{e^t+1} (e^t - 1)$$

$$\Rightarrow x e^t + x = 4y e^t - 4y$$

$$\Rightarrow e^t(x - 4y) = -(x + 4y)$$

$$\Rightarrow e^t = \frac{-(x + 4y)}{x - 4y}$$

$$(20) \equiv \lambda = \frac{2y}{e^t + 1} = \frac{2y}{\frac{-(x + 4y)}{x - 4y} + 1} = \frac{2y(x - 4y)}{2x - 8y} = \frac{-2y(x + 4y)}{8y} = -\frac{x + 4y}{4}$$

$$\therefore \lambda = \frac{x + 4y}{-4}$$

$$\therefore z = \frac{(x + 4y)^2}{(-4)} \cdot \frac{(x + 4y)^2}{(x - 4y)^2}$$

$$\therefore z = \frac{(x + 4y)^2}{16}$$

which is the required integral surface of ①.

2005 P.T for the eqn $z + px + qy - 1 - pqx^2y^2 = 0$

the characteristic steps are given by

$$x = \frac{1}{B + ce^t}, \quad y = \frac{1}{A + de^t}, \quad z = E - (Ae^t + Be^t)e^{-t}$$

$$P = A(B + ce^t)^2, \quad q = B(A + de^t)^2$$

where A, B, C, D and E are arbitrary constants.
Hence find the integral surface which passes through the line $z=0, xy=1$.

Soln: Given that $z + px + qy - 1 - pqx^2y^2 = 0$

$$\text{Let } f(x, y, z, p, q) = z + px + qy - 1 - pqx^2y^2 = 0 \quad \text{--- (1)}$$

We are to find integral surface of (1) which passes through the line $x=y, z=0$ ---

whose parametric eqns are

$$x = \lambda, \quad y = \lambda \quad \& \quad z = 0$$

$$\text{i.e., } x = f_1(\lambda) = \lambda \quad ; \quad y = f_2(\lambda) = \lambda \quad \& \quad z = f_3(\lambda) = 0$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = f_1(\lambda) = \lambda, \quad y_0 = f_2(\lambda) = \lambda, \quad z_0 = f_3(\lambda) = 0$$

Now we find the initial values p_0 & q_0 by the relations

$$f'_3(\lambda) = p_0 f'_1(\lambda) + q_0 f'_2(\lambda) \quad \&$$

$$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$$

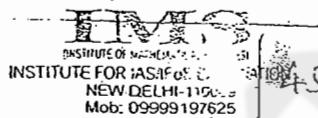
$$\Rightarrow f(\lambda, \lambda, 0, p_0, q_0) = 0$$

$$\Rightarrow 0 = p_0(1) + q_0(1) \quad \& \quad 0 + p_0(\lambda) + q_0(\lambda) - 1 - p_0q_0\lambda^4 = 0$$

$$\Rightarrow \boxed{p_0 + q_0 = 0} \quad \Rightarrow \lambda(p_0 + q_0) - 1 - p_0q_0\lambda^4 = 0$$

$$\Rightarrow \lambda(0) - p_0q_0\lambda^4 = 0$$

$$\Rightarrow \boxed{p_0q_0 = -\frac{1}{\lambda^4}}$$



$$\text{Now } (P_0 - q_0)^2 = (P_0 + q_0)^2 - 4P_0q_0$$

$$= 0 + \frac{4}{\lambda^4}$$

$$\therefore P_0 - q_0 = \frac{2}{\lambda^2} \quad \text{(i)}$$

$$\text{from (5) & (i)} \quad 2P_0 = \frac{2}{\lambda^2}$$

$$\Rightarrow P_0 = \frac{1}{\lambda^2} \quad \& \quad q_0 = -\frac{1}{\lambda^2}$$

$$\therefore x_0 = \lambda, y_0 = \lambda, z_0 = 0, P_0 = \frac{1}{\lambda^2} \& q_0 = -\frac{1}{\lambda^2} \quad \text{(ii)}$$

NOW the characteristic eqns of (1) are

$$x'(t) = fp = x - qx^2y^2 \quad \text{(1)}$$

$$y'(t) = fq = y - px^2y^2 \quad \text{(2)}$$

$$\begin{aligned} z'(t) &= p[2 - qx^2y^2] + q[y - px^2y^2] \\ &= px^2y - pqx^2y^2 \end{aligned} \quad \text{(3)}$$

$$\begin{aligned} p'(t) &= -[pP - 2pqxy^2] - p(t) \\ &= -2p[1 - qxy^2] \end{aligned} \quad \text{(4)}$$

$$\begin{aligned} q'(t) &= -[q - 2pqx^2y] - q(t) \\ &= -2q[1 - px^2y] \end{aligned} \quad \text{(5)}$$

from (4) & (5), we have

$$x'(t) = \frac{p'(t)}{-2p}$$

$$\Rightarrow -\frac{2}{x} dx = \frac{1}{p} dp$$

$$\Rightarrow -2\log x = \log p + \log C_1$$

$$\Rightarrow x^2 = pc \quad \text{(6)}$$

from (5) & (6), we have

$$y'(t) = y \left(\frac{-q'(t)}{2q} \right)$$

$$\Rightarrow -\frac{2}{y} dy = \frac{1}{q} dq$$

$$\Rightarrow -2\log y = \log q + \log C_2$$

$$\Rightarrow \boxed{y^{-2} = qC_2} \quad \text{--- (10)}$$

Using the initial values in (9) & (10), we get

$$y_0^{-2} = P_0 C_1 \quad \& \quad y_0^{-2} = q_0 C_2$$

$$\Rightarrow \lambda^2 = \frac{1}{n^2} C_1 \quad \& \quad \lambda^2 = -\frac{1}{n^2} C_2$$

$$\Rightarrow \boxed{C_1 = 1} \quad \& \quad \boxed{C_2 = -1}$$

\therefore from (9) & (10), we have

$$\boxed{\frac{-1}{y^2} = P} \quad \& \quad \boxed{\frac{-1}{y^2} = q} \quad \text{--- (11)}$$

Continuing in this way.

Now find the characteristic strips of the eqn
 $xp + yq - pq = 0$ and then find the eqn of

Integral surface through curve $x = \frac{3}{2}, y = 0$

\rightarrow write down and integrate completely, the
 equations for the characteristics of $(1+q^2)z = px$.

Expressing x, y, z and p in terms of ϕ , where

$q = \tan \phi$ and determine the integral

surface which passes through parabola

$$x^2 = 2t, y = 0$$

\rightarrow Determine the characteristics of the equation -
 $z = p^2 q^2$ and find the integral surface which

passes through the parabola $4z + x^2 = 0, y = 0$.

\rightarrow Integrate the eqns of the characteristics of
 the eqn $p^2 + q^2 = 4z$.

Expressing x, y, z and p in terms of q and then
 find the solutions of this equation which
 reduce to $z = x^2 + 1, y = 0$

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SET-III

Linear partial Differential Equations

with constant coefficients:

The general linear partial differential equation of n order higher than the first:

A PDE for which the dependent

variable and its derivatives appear only in the first degree and are not multiplied together, the coefficients all being constants (or the λ 's) of a set.

is called a linear PDE.

The general form of such an eqn can be

$$\text{written in the form } \left(\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^{n-1} z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^{n-2} z}{\partial x^{n-2} \partial y^2} + \dots + A_{n-1} \frac{\partial z}{\partial y^{n-1}} \right)$$

$$+ \left(B_0 \frac{\partial^n z}{\partial y^n} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-1} \partial y} + B_2 \frac{\partial^{n-2} z}{\partial x^{n-2} \partial y^2} + \dots + B_{n-1} \frac{\partial z}{\partial x^{n-1}} \right)$$

$$+ \dots + \left(H_0 \frac{\partial z}{\partial x} + H_1 \frac{\partial z}{\partial y} \right) + N_0 z = f(x, y) \quad (1)$$

where the co-efficients A_{ij} , B_{ij} , H_i , N_0 are constants.

function of x & y .

If the co-efficients of various terms are constants, then it is called a linear PDE with constant coefficients.

If all the derivatives appearing in it are of the same order, then the resulting eqn

is called a linear homogeneous PDE with constant coefficients and, if it is of the form

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^{n-1} z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^{n-2} z}{\partial x^{n-2} \partial y^2} + \dots + A_{n-1} \frac{\partial z}{\partial y^{n-1}} + N_0 z = 0$$

where A_i , N_0 are constants.

Denoting the operators $-\frac{\partial}{\partial y} D$; $\frac{\partial^2}{\partial y^2} D^2$, (2)

$$\therefore (D_y^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D^0) f = f(y)$$

$$\Rightarrow F(D, D') z = f(y) \quad (3)$$

$$\text{where } F(D, D') = D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D^0$$

Note:- $F(D, D')$ is a homogeneous

function in D, D' of degree n .

Solution of a linear homogeneous partial differential eqn w.r.t constant coefficients:

If W is the complementary function (C.F.)
and Z a particular integral of a linear
P.D.E $F(D, D') z = f(y)$ then $w + Z$ is a
g.s of the linear P.D.E.

If w_1, w_2, \dots, w_n are solns of the homogeneous
linear P.D.E $F(D, D') z = 0$ then $\sum c_i w_i$ is
also a soln where c_1, c_2, \dots, c_n are arbitrary
constants.

Determination of the C.F. of the linear P.D.E
with constant coefficients $F(D, D') z = f(y)$:

Let $F(D, D') z = f(y)$ be the given linear
homogeneous P.D.E with constant coeff.
then $[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D^0] z = f(y)$
the terms A_1, A_2, \dots, A_n are constants.

The complementary function (C.F) of (1) is (3)

The g.e of $F(D, D') z = 0$

$$\text{i.e. } [D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D^n D'^n] z = 0,$$

$$\Rightarrow [(D-m_1 D') (D-m_2 D') (D-m_3 D') \dots (D-m_n D')] z = 0 \quad \text{--- (4)}$$

where m_1, m_2, \dots, m_n are some constants.

The soln of any one of the eqns

$$(D-m_1 D') z = 0, (D-m_2 D') z = 0, \dots (D-m_n D') z = 0 \quad \text{--- (4)}$$

is also a soln of (1).

We now show that the general soln of

$$(D-m D') z = 0 \text{ if } z = \phi(y+m),$$

where ϕ is an arbitrary fun.

$$\text{now } (D-m D') z = 0 \Rightarrow \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0.$$

$$\Rightarrow P(x)z = 0$$

clearly which is in lagrange's form

$$P_1 P_2 + Q_1 Q_2 = 0$$

\therefore the lagrange's auxiliary eqns of (1) are

$$\frac{dx}{P_1} = \frac{dy}{Q_1} = \frac{dz}{P_2 + Q_2} \quad \text{--- (5)}$$

now taking first two fractions of (5)

$$\text{we get } \frac{dx}{P_1} = \frac{dy}{Q_1} \Rightarrow dy = m dx$$

integrating, we get

now taking third fraction \Rightarrow

$$dz = 0$$

$$\Rightarrow z = C_1$$

! from (7) & (8), the g.s
of (5) is $Z = \phi(y+mx)$ (4)

where ϕ is arbitrary function
we assume that a soln of (2) is
of the form $Z = \phi(y+mx)$ (5)
where ϕ is arbitrary function w.r.t. const.

now from (3),

$$DZ = \frac{\partial Z}{\partial x} = \frac{\partial}{\partial x} [\phi(y+mx)] \\ = m\phi'(y+mx)$$

$$D^2Z = \frac{\partial^2 Z}{\partial x^2} = m^2\phi''(y+mx)$$

$$D^3Z = \frac{\partial^3 Z}{\partial x^3} = m^3\phi'''(y+mx)$$

and $D^4Z = \frac{\partial^4 Z}{\partial x^4} = m^4\phi''''(y+mx)$

$$D^5Z = \frac{\partial^5 Z}{\partial x^5} = m^5\phi''''''(y+mx)$$

$$D^6Z = \frac{\partial^6 Z}{\partial x^6} = m^6\phi^{(6)}(y+mx)$$

Now, by general $D^m D^n Z = \frac{d^m}{dx^m} \frac{d^n}{dy^n} Z = m^m n^m \phi^{(m+n)}(y+mx)$

This is true if $m+n$ is a root of the
eqn $m^m + n^m + m^{m-1}n + \dots + m = 0$

The eqn (10) is called the auxiliary eqn (10).
It is obtained by putting $D=m$; $D'=1$ in (2) $F(D)=0$

But in general, the eqn (1) can give 'n' roots, say m_1, m_2, \dots, m_n .

Each value of m_i will give a soln of (2).

→ If all the roots of the auxiliary eqn (1) are distinct, the solns of (2) is the C.F. of (1) is

$$\begin{aligned} z &= \phi_1(y+m_1) + \phi_2(y+m_2) + \dots + \phi_n(y+m_n) \\ \text{i.e. } z &= \sum \phi_r(y+m_r); \quad r=1, 2, \dots, n \end{aligned} \quad (15)$$

Case of equal roots:

If the auxiliary eqn (1) has two equal roots i.e. $m_1 = m_2$ i.e. m_1, m_1, m_3, \dots

Now consider the eqn

$$(D-m^2)(D-n^2)z = 0 \quad (\text{from (1)}) \quad (16)$$

Putting $(D-m^2)z = 0$ we get

$$(D-m^2)z = 0$$

the soln of which is $w = \phi_1(y+m)$.

$$\therefore (D-n^2)z = \phi_1(y+m) \quad (\text{from (1)})$$

$$D-n^2 = \phi_2(y+m)$$

Lagrange's auxiliary eqns of (1) are

$$\frac{dy}{1} = \frac{dy}{m} = \frac{dt}{\phi_1(y+m)} \quad (17)$$

$$\text{Taking the first two fractions, } \frac{dy}{1} = \frac{dy}{m} \Rightarrow \int dy = \int \frac{dy}{m} \quad (18) \text{ i.e.}$$

arity and
w = c₁ e^{my}

Taking first & last forcing (6)

$$\frac{dx}{t} = \frac{dz}{\phi(z)} \quad (from (1))$$

$$\Rightarrow dz = \phi(z) dt$$

$$\Rightarrow z = \lambda \phi(z) + b$$

where b is arbitrary

$$\Rightarrow z = \lambda \phi(y_{avg}) + b$$

Since b is arbitrary,

$$taking b = \phi_1(a)$$

the soln. of (1) is

$$z = \lambda \phi(y_{avg}) + \phi_1(y_{avg})$$

$$Tz = \lambda \phi(y_{avg}) + \phi_1(y_{avg})$$

proceeding in the same way,
where ϕ & ϕ_1 are arbitrary

If the auxiliary eqn (1) has repeated roots

either the cff of (1) is

$$z = \phi_1(y_{avg}) + \lambda \phi_2(y_{avg}) + \lambda^2 \phi_3(y_{avg}) + \dots + \phi_r(y_{avg})$$

Working rule for finding c.c.

Step 1: Write down the given eqn in
standard form

$$(D^m + A_1 D^{m-1} + A_2 D^{m-2} + \dots + A_{m-1} D + A_m) z = f(x, y)$$

Step 2: replacing D by m and D^2 by y_2 as the
coefficient of z , we obtain the A.E for (1)
as $m^m + A_1 m^{m-1} + A_2 m^{m-2} + \dots + A_{m-1} m + A_m = 0$ (2)

Step(1): solve (2) for 'm'. (7)

Some cases will arise:

case(i): Let $m = m_1, m_2, \dots, m_n$ (distinct roots)

then C.F. of ①
 $C.F. = \phi_1(y+m_1) + \phi_2(y+m_2) + \dots + \phi_n(y+m_n)$

where $\phi_1, \phi_2, \dots, \phi_n$ are
arbitrary fun.

case(ii): Let $m = m'$ (repeated n times)

then C.F. of ① =
 $\phi_1(y+m') + \phi_2(y+m') + \dots + \phi_n(y+m')$

case(iii): corresponding to a non-repeated
factor D on LHS of ①, the part of
C.F. is taken as $\phi(y)$.

case(iv): corresponding to a repeated factor
D on LHS of ①, the part of
C.F. is taken as $\phi(y) + \phi_2(y) + \dots + \phi_n(y)$.

case(v): corresponding to a non-repeated
factor D' on LHS of ①, the part
of C.F. is taken as $\phi(y)$.

case(vi): corresponding to a repeated
factor D' on LHS of ①,
the part of C.F. is taken as
 $\phi_1(y) + \phi_2(y) + \dots + \phi_n(y)$.

Alternative working rule for finding C.F.

Let the given diff. eqn be $F(D, D') z = f(x)$.

Factorise $F(D, D')$ into linear factors
of the form $(bD - aD')$.

Then we use the following results:

(i) Corresponding to each non-repeated factor
 $(bD - aD')$, the part of C.F. is taken as
 $\phi(bx - ax)$.

(ii) Corresponding to a repeated factor
 $(bD - aD)^m$, the part of C.F. is taken
as $\phi_1(bx - ax) + x\phi_2(bx - ax) + x^2\phi_3(bx - ax)$
 $+ \dots + x^{m-1}\phi_m(bx - ax)$.

(iii) Corresponding to a non-repeated factor D' ,
the part of C.F. is taken as $\phi(y)$.

(iv) Corresponding to an repeated factor D'^m ,
the part of C.F. is taken as
 $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \dots + x^{m-1}\phi_m(y)$.

(v) Corresponding to a non-repeated factor D' ,
the part of C.F. is taken as $\phi(u)$.

(vi) Corresponding to a repeated factor
 D'^m , the part of C.F. is taken as
 $\phi_1(u) + x\phi_2(u) + x^2\phi_3(u) + \dots + x^{m-1}\phi_m(u)$.

Note: $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $t = \frac{\partial^2 z}{\partial x^2}$, $b = \frac{\partial^2 z}{\partial x \partial y}$,
 $c = \frac{\partial^2 z}{\partial y^2}$.

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$$\rightarrow \text{solve } 2x + 5y + 2t = 0$$

Sol: Given that $\begin{array}{l} 2x + 5y + 2t = 0 \\ \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial t^2} \end{array}$

$$\begin{aligned} W.K.T. \quad & x = \frac{\partial^2 z}{\partial x^2}, \quad S = \frac{\partial^2 z}{\partial y^2}, \quad T = \frac{\partial^2 z}{\partial t^2} \\ & = Dz \quad = DDz \quad = D^2 z \end{aligned}$$

$$\therefore \textcircled{1} =$$

$$\left[2D^2 + 5DD' + 2D'' \right] z = 0 \quad \textcircled{2}$$

$$\text{A/c of } \textcircled{2} \text{ is } \quad 2m^2 + 5m + 2 = 0 \quad (\because D = m, D' = 1)$$

$$\Rightarrow (2m+1)(m+2) = 0$$

$$\Rightarrow m = -\frac{1}{2}, -2$$

\therefore the g.s of $\textcircled{1}$ is

$$z = \phi_1(y - \ln x) + \phi_2(y + \ln x).$$

where
 ϕ_1 & ϕ_2 are arbitrary
fn.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \textcircled{1}$$

$$(D^2 + D')z = 0 \quad \textcircled{2}$$

$$\text{A/c if } \textcircled{2} \text{ is } m+1 = 0$$

$$\Rightarrow m = -1$$

$$\Rightarrow m = \pm 1$$

\therefore the g.s of $\textcircled{1}$ is

$$z = \phi_1(y + \ln x) + \phi_2(y - \ln x).$$

$$\rightarrow \text{solve } x + t + 2s = 0$$

Given that $x = \frac{\partial^2 z}{\partial x^2}, \quad t = \frac{\partial^2 z}{\partial y^2}, \quad s = \frac{\partial^2 z}{\partial t^2}$ $\textcircled{1}$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial^2 z}{\partial t^2} = 0$$

$$\Rightarrow (D^2 + 2DD' + D'')z = 0$$

A.E of (2) is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1.$$

G.S.f of (1) is $Z = \phi_1(y+x) + \phi_2(y-x)$

1987 → solve $r = e^{rt}$

$$\rightarrow \text{solve } \frac{\partial^2 Z}{\partial x^2} - \frac{\partial^2 Z}{\partial y^2} - 6 \frac{\partial^2 Z}{\partial xy} = 0$$

$$\rightarrow (D^2 - D D^T) Z = 0$$

Given $D^2 - (D^2 - D D^T) Z = 0$

A.E of (1) is

$$m^2 - m = 0$$

$$\Rightarrow m(m-1) = 0$$

$$\Rightarrow m=0; m=1.$$

G.S.f of (1) is

$$Z = \phi_1(y) + \phi_2(y+x)$$

where ϕ_1 & ϕ_2 are arbitrary fun.

Solve

$$D D^T Z = 0$$

$$\Rightarrow \frac{\partial^2 Z}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial^2 Z}{\partial x \partial y} = 0$$

$$\Rightarrow \frac{\partial^2 Z}{\partial y^2} = 0$$

$$\Rightarrow Z = \phi_1(y) \quad \& \quad Z = \phi_2(y)$$

G.S.f of (1) is $Z = \phi_1(y) + \phi_2(y)$

Solve $D^T Z = 0$

A.E is

$m^2 - m = 0$... G.S.f of (1) is $Z = \phi_1(y) + \phi_2(y)$

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Particular Integral

Let us consider an equation

$$f(D, D') z = f(xy) \quad \text{--- (1)}$$

then P.I. of (1) is denoted by:

$$\frac{1}{D} f(xy)$$

Note: (1) $\frac{1}{D}$ means integration partially w.r.t. x.
 $\frac{1}{D'}$ means integration partially w.r.t. y.

$$(2) D [\phi(xy)] = \frac{\partial}{\partial x} [\phi(xy)] \\ = a \phi(xy)$$

$$D' [\phi(xy)] = \frac{\partial}{\partial y} [\phi(xy)] \\ = b [\phi(xy)]$$

By general

$$D^r [\phi(xy)] = a^r \phi^{(r)}(xy)$$

$$D'^s [\phi(xy)] = b^s \phi^{(s)}(xy)$$

$$\text{and } D D' [\phi(xy)] = a^r b^s \phi^{(r+s)}$$



Working rule

To find P.I. of an eqn. $f(D, D') z = f(xy)$
where $f(D, D')$ is a homogeneous
function of D, D' of degree
proceed as follows

(i) When $F(a+b) \neq 0$,

$$\text{we have } P.D = \frac{1}{F(D, D')} \phi(a+b)$$

$$= \frac{1}{F(a+b)} \int \int \dots \int \phi(v) dv dv \dots dv$$

$$\text{where } P.D = \frac{1}{F(D, D')} \phi(ax+by)$$

$$= \frac{1}{F(a,b)} \times \text{sum of } \phi(v) \text{ with } v = ax+by$$

where $v = ax+by$ (ii) when $F(a+b) = 0$,now $f(a+b) = 0$ iff $(bD-aD')$ is a factor of $F(D, D')$.

$$\text{we have } P.D = \frac{1}{F(D, D')} \phi(a+b)$$

$$= \frac{1}{(bD-aD')^n} \phi(a+b)$$

$$= \frac{1}{b^n n!} \phi(a+b)$$

To find p.d. of $\phi(a+b)$ we need to evaluate it at $a=0, b=0$ (as $\phi(a+b) = \phi(a) + \phi(b)$)
arbitrary integer
additive function
of $a+b$)

We have $P.D = \frac{1}{F(D, D')} v$ where $v = ax+by$ It evaluated by expanding
the symbolic function $\frac{1}{F(D, D')}$ in an
infinite series of ascending powers of D and

Note:-

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If $n \in \mathbb{N}$, $\frac{1}{F(DD!)}^n$ should beexpanded in powers of $\frac{D^1}{D}$,whereas if $n \in \mathbb{Z}$, $\frac{1}{F(DD!)}$ should beexpanded in powers of $\frac{D^1}{D^1}$.problems:→ solve $4r - 4s + t = 16 \log(x+iy)$.sol Given that $4r - 4s + t = 16 \log(x+iy)$.

$$\Rightarrow 4 \frac{\partial r}{\partial x} - 4 \frac{\partial s}{\partial x} + \frac{\partial t}{\partial x} = 16 \log(x+iy)$$

$$\Rightarrow (4D)^x - 4(0)^x + D^1 x^2 = 16 \log(x+iy) \quad (1)$$

A.E of (1) is

$$4m^x - 4m + 1 = 0.$$

$$\Rightarrow (2m-1)^x = 0$$

$$\Rightarrow m = \frac{1}{2}$$

$$C.F = \phi_1(x^{1/2}) + n\phi_1'(x^{1/2})$$

$$= \phi_1\left[\frac{1}{2}(x+iy)\right] + 2\phi_1'\left[\frac{1}{2}(x+iy)\right]$$

$$C.F = \phi_1(x+iy) + 2\phi_1'(x+iy)$$

$$\text{Now P.I.} = \frac{1}{4D^x - 4(0)^x + 0^1} = \frac{1}{16 \log(x+iy)}$$

$$= 16 \left[\frac{1}{(2D-D^1)^x} \log(x+iy) \right]$$

$$= 16 \left[\frac{1}{2 \cdot 2^x} \log(x+iy) \right] = \left(\frac{1}{2^x} \right) \log(x+iy)$$

\rightarrow solve $(D^2 + 3DD' + 2D'^2) z = x+y$ 14

$$\text{sol} \quad [C.F. = \phi_1(y-x) + \phi_2(y-x)]$$

where ϕ_1 & ϕ_2 are arbitrary

$$P.I. = \frac{1}{D^2 + 3DD' + 2D'^2} (x+y)$$

$$= \frac{1}{-1^2 + 2D(D+2D')} \int \int x e^{2D} de$$

$$= \frac{1}{-b} \int \frac{ve^v}{v} dv$$

$$= \frac{1}{-b} \frac{v^2}{2}$$

$$= \frac{1}{2b} (x+y)^2$$

\therefore G.S. of (1) is $z = C.F. + P.I.$

$$\rightarrow z = \phi_1(y-x) + \phi_2(y-x) + \frac{1}{2b} (x+y)^2$$

\rightarrow solve $x - 2y + t = \sin(m+n)t$

to solve $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \cos(m+n)y$

sol

Given that

$$(D^2 + D'^2) z = \cos(m+n)y$$

A.P. S. $m+n=0$

$\Rightarrow m=-n$

$$[C.F. = \phi_1(y+ix) + \phi_2(y-ix)]$$

$$P.I. = \frac{1}{D^2 + D'^2} \cos(m+n)y$$

$$= \frac{1}{D^2 + D'^2} \frac{1}{2} [\cos(m+n)y + \cos(-m-n)y]$$

$$= \left[\frac{1}{D^2 + D'^2} \cos(m+n)y + \frac{1}{D^2 + D'^2} \cos(m-n)y \right] \quad ②$$

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$$\text{Now } \frac{1}{D+D'} \cos(\text{main}) = \frac{1}{\text{main}} \int \omega \text{ due}$$

$$= \frac{1}{\text{main}} \text{ force due}$$

$$= \frac{-1}{\text{main}} \omega \text{ due}$$

$$= \frac{-1}{\text{main}} \omega \text{ (main)}$$

$$\text{and } \frac{1}{D+D'} \cos(\text{main}) = \frac{-1}{\text{main}} \cos(\text{main})$$

$$\textcircled{2} \ Z^{\text{PI}} = \frac{1}{D+D'} \cos(\text{main}) \cos(\text{main})$$

$$= \frac{-1}{2} \left[\frac{-1}{\text{main}} (\cos(\text{main}) + \cos(\text{main})) \right]$$

$$\cancel{\cos(\text{main})}$$

$$\text{LHS of } \textcircled{1} \ Z = e^{st} + P \cdot 0$$

$$= P_1(C_1 s) + P_2(C_2 s)$$

$$+ \frac{-1}{\text{main}} \cos(\text{main}) \cos(\text{main})$$

1981 → solved $r+5s+6t = (y-2z)^{-1}$
as usual for c

$$\underline{P \cdot 0} \ Z = \frac{1}{D+5D'+6D''} (y-2z)$$

$$= \frac{-1}{(D+2D')(D+3D')}$$

$$= \frac{1}{D+3D'} \left\{ \frac{1}{D+2D'} (y)$$

$$= \frac{1}{D+2D!} \left[\frac{1}{-2+3D!} \int e^{-1} de \right] \quad (16)$$

here we ignore

$$= \frac{1}{D+2D!} [\log e]$$

$$= \frac{1}{D+2D!} \log (y^m)$$

$$= \frac{1}{(1)!!} \log (y^m)$$

$$= \underline{\underline{\log (y^m)}}$$

$$\therefore Q.S. f(1) \text{ if } z = C e^x + F e^{2x}$$

→ solve $(D^2 + 2D! + D^m)z = e^{mx}$.

→ solve $\log e = mx$

$$\text{ie. } z = e^{mx}$$

1994, solve $(D^2 + 2D! + 2D^m)z = e^{mx}$

by expanding the particular integral (P.I.)
in ascending powers of D as well as by
ascending powers of $D!$

$$\text{Sol} \quad \text{Given } (D^2 + 2D! + 2D^m)z = e^{mx}$$

$$\rightarrow A \in f(0) \text{ if } m+2m+2=0$$

$$\Rightarrow m = -1, -1$$

$$\therefore [G.F. = \phi_1(y^m) + \phi_2(y^{-1})]$$

$$P \cdot I = \frac{1}{D^m + 2D^l + 2D^k} (x+yz)$$

(81)

$$\begin{aligned} &= \frac{1}{2D^k \left[1 + \left(\frac{D^m}{2D^k} + \frac{1}{2} \cdot \frac{D^l}{D^k} \right) \right]} (x+yz) \\ &= \frac{1}{2D^k} \left[1 + \left(\frac{D^m}{2D^k} + \frac{1}{2} \cdot \frac{D^l}{D^k} \right) \right]^{-1} (x+yz) \\ &= \frac{1}{2D^k} \left[1 - \left(\frac{D^m}{2D^k} + \frac{1}{2} \cdot \frac{D^l}{D^k} \right) + \dots \right] (x+yz) \\ &\approx \frac{1}{2D^k} \left(\cancel{\text{(higher terms)}} (x+yz) \right) \approx \frac{1}{2D^k} (x+yz) \\ &= \frac{1}{2} \cdot \frac{1}{D^k} \left[xy - \frac{y^2}{4} \right] \\ &= \frac{1}{2} \left[\frac{xy}{D^k} - \frac{y^2}{4D^k} \right] = \frac{xy}{4} - \frac{y^2}{16} \end{aligned}$$

$$\therefore Q \cdot f \circ P \cdot D \cdot Z = C \in \mathbb{R}$$

$$\Rightarrow Z = \phi_1(yz) + \phi_2(yz) + \frac{xy}{4} - \frac{y^2}{16}$$

Again, by expanding in ascending powers
of D^k , we have

$$\begin{aligned} P \cdot I &= \frac{1}{D^m + 2D^l + 2D^k} (x+yz) = \frac{1}{D^k \left[1 + \frac{2D^l}{D^k} + \frac{2D^m}{D^k} \right]} (x+yz) \\ &= \frac{xy}{2} - \frac{y^2}{12} \end{aligned}$$

$$\therefore Q \cdot f \circ P \cdot D \cdot Z = C \in \mathbb{R}$$

$$\Rightarrow Z = \phi_1(yz) + \phi_2(yz) + \frac{xy}{2} -$$

\rightarrow solve $(D^2 - 6DD' + 9D'^2)Z = 12x^2 + 36xy$. (18)

sol

$$C.F. = \phi_1(y+2x) + x\phi_2(y+2x),$$

$$P.D. = \frac{1}{D^2 - 6DD' + 9D'^2} (12x^2 + 36xy)$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{6D'}{D} + 9 \left(\frac{D'}{D} \right)^2 \right) \right] (12x^2 + 36xy)$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{6D'}{D} - 9 \left(\frac{D'}{D} \right)^2 \right) + \left(\frac{6D'}{D} - 9 \left(\frac{D'}{D} \right)^2 \right)^2 \right] (12x^2 + 36xy)$$

$$\frac{1}{D^2} \left[(12x^2 + 36xy) + \frac{6}{D} (76x) \right]$$

$$\frac{1}{D^2} \left[12x^2 + 36xy + 6 \left(26 \right) \frac{x^2}{x} \right]$$

$$\frac{1}{D^2} \left[4x^2 + 18xy + 26x^2 \right]$$

$$24x^2 + 6x^2y + 9x^2$$

$$10x^4 + 6x^2y$$

Ans of (1) P.S.C.F. exp. 6

A general method finding the P.E.

consider the eqn $(D - mD')Z = \phi(x,y)$

$$\Rightarrow P - mQ = \phi(x,y) \quad (1)$$

Lagrange's rule are -

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{\phi(x,y)} \quad (2)$$

Taking first two brackets of (1), we get
 $dy \text{ term} \Rightarrow [y + m_1 x] \text{ (a constant).}$

(19)

Again taking first & last brackets of (1), we get

$$dz = \phi(x, y) dx.$$

$$\Rightarrow dx = \phi(x, a-m_1 z) dz \quad (\because y+m_1 x=a)$$

Integrating we get

$$z = \int \phi(x, a-m_1 z) dz.$$

$$(19) \quad \frac{1}{(a-m_1 z)} \phi(x, y) dz = 0$$

$$= \int \phi(x, a-m_1 z) dz \quad (3)$$

where after integration the constant a is to be replaced by $y+m_1 x$ (since the L.H.S. does not contain any arbitrary constant).

Now if the given eqn. is $F(D, D') z = \phi(x, y)$
where $F(D, D') = (D-m_1 D')(D-m_2 D) = (D-m_1 D^2)$.

$$\text{Then P.C.} = \frac{1}{F(D, D')} \phi(x, y)$$

$$\Rightarrow \text{P.C.} = \frac{1}{(D-m_1 D^2)(D-m_2 D)} \phi(x, y)$$

P.C. can be evaluated by the repeated application of the above method.

(19)

Problem:

$$\text{Solve } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \sin x$$

$$\text{The given eqn. is } (D+D')^2 z = \sin x$$

$$\text{A.E. is } m+1=0 \Rightarrow m=0$$

i.e., P.C. is zero.

$$\text{Now } P.E. = \frac{1}{D+D!} \sin y$$

$$= \frac{1}{D+D!} \sin(y+\alpha)$$

$$= \int \sin[(y+\alpha)(\omega t)] dy$$

$$\int \sin y dy$$

$$= -\cos y$$

$$\therefore \text{Q.F. } Z = C.F + P.E.$$

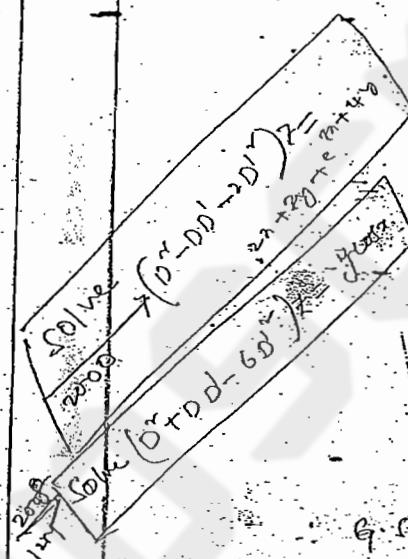
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{6 \partial^2 z}{\partial xy} = y \sin y$$

$$\text{Given eqn is } (\partial^2 z + \partial D^2 - 6 \partial^2) z = y \sin y$$

$$\text{m.e. is } \omega^2 + m^2 - 6 = 0 \Rightarrow m = \pm 3$$

$$\therefore C.F. = \phi_1(y-\omega x) + \phi_2(y+\omega x)$$

$$P.E. = \frac{1}{D+D!-6D!} \sin y = \frac{1}{(D-2D!) (D+6D!)} (y \sin y)$$



$$\frac{1}{D-2D!} \left[\frac{1}{D-(\omega^2)D!} G(y^2) \right]$$

$$\frac{1}{D-2D!} \left[\int (6+3x) \sin y dy \right]$$

$$= \frac{1}{D-2D!} [y(6 \cos y + 3 \sin y)]$$

$$= \int [-(\omega^2 - 6)y \cos y + 3 \sin y] dy$$

$$= -y \sin y - \cos y$$

$$\therefore Q.F. \text{ if } Z = C.F + P.E.$$

solve $y-t = \tan^3 y - \tan y - \tan^3 y$.

$$\text{solve } \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{\partial^2 z}{\partial y^2} = (y-1) \cdot c^3$$

Non-homogeneous linear partial differential eqns w/ constant coefficients: (21)

A linear partial differential eqn which is not homogeneous is called a non-homogeneous linear eqn.

Consider the diff. eqn $F(D, D')z = f(x, y)$

when $F(D, D')$ is a homogeneous function w.r.t. D, D' , it can always be resolved into linear factors. But the result is not always true when $F(D, D')$ is non-homogeneous.

Now we classify linear differential operators $F(D, D')$ into two types.

These are: (i) $F(D, D')$ is reducible if it can be written as the product of linear factors of the form $D + ad/x + b$ with a, b constants.

(ii) $F(D, D')$ is irreducible if it cannot be so written.

→ (i) C.F. of non-homogeneous linear eqn

when $F(D, D')$ can be resolved into linear factors:

The C.F. of non-homogeneous linear eqn (1) is the g.s. of the eqn $F(D, D')z = 0$ (2).

Let us consider a simple non-hom.

$$\text{eqn } (D - mD' - k)z = 0$$

$$\Rightarrow D - mD' = k \quad (3)$$

Lagrange's rule is

$$\frac{dx}{1} = \frac{dy}{m} = \frac{dz}{k}$$

Taking the first two fractions of (1), we get
 $dy + m_1 dz \Rightarrow [y + m_1 z] = c_1 \text{ (const.)}$

Again taking the first & the last fraction of (1), we get

$$\frac{dz}{z} = k dy$$

$$\Rightarrow \log z = k y + \log b$$

$$\Rightarrow z = b e^{ky}$$

$$\Rightarrow z = e^{ky} \phi(y) \quad (b = \phi(0))$$

$$\Rightarrow [z = e^{ky} \phi(y + m_1)]$$

which is the soln of (1)

7 If $f(D, D')$ can be factored into non-repeated linear factors $(D - m_1 D' - n_1), (D - m_2 D' - n_2), \dots, (D - m_n D' - n_n)$ then the eqn (2)

becomes $(D - m_1 D' - n_1)(D - m_2 D' - n_2) \dots (D - m_n D' - n_n) z = 0$

∴ g.s of (2) is

$$z = e^{ky} [\phi_1(y + m_1) + \phi_2(y + m_2) + \dots + \phi_n(y + m_n)]$$

Note:- If the eqn is $f(D + f_1 D') z = 0$ then
 $z = e^{f_1 y} [\phi_1(y) + \phi_2(y)]$

7 If $f(D, D')$ has repeated factors

(i) If $(D - m_1 D' - n)$ occurs twice then the g.s of (2) is

$$z = e^{ky} [\phi_1(y + m_1) + A \phi_2(y + m_1)]$$

(ii) If $(D - m_1 D' - n)$ occurs 3 times then we use

g.s of (2) is

$$z = e^{ky} [\phi_1(y + m_1) + A \phi_2(y + m_1) + B \phi_3(y + m_1)]$$

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(ii)

When linear factors of $R(D)$
are not possible:

In case $R(D)$ is irreducible,
i.e. it can not be resolved into linear
factor (in D & D'), the above methods of finding
complementary function fail.
In such cases a trial method
is used to find same.

$$\text{Ex:- solve } (2D^4 - 2D^3 D' + D'^2) Z = 0$$

$$\Rightarrow (2D^2 - D') \cdot (D^2 - D') Z = 0 \quad \text{--- (1)}$$

Let $Z = A e^{kx}$ be the soln

$$\text{corresponding to } (D^2 - D') Z = 0 \quad \text{--- (2)}$$

$$\Rightarrow D^2 [A e^{kx}] - D[A e^{kx}] = 0$$

$$\Rightarrow A k^2 e^{kx} - A k e^{kx} = 0$$

$$\Rightarrow A f(k) e^{kx} = 0$$

$$\Rightarrow f(k) = 0 \quad (\text{As } A \neq 0)$$

$$\boxed{K^2 = 0}$$

Putting $K^2 = 0$ in (2), we get
 $Z = A e^{kx}$ where
some all values of k can be found
the more general soln of

$$Z = \boxed{A e^{kx}}$$

By the g.s of $(2D^2 - D^1)Z = 0$ is

$$Z = \sum a e^{(4t^2 + 2t)}$$

the most g.s of the given eqn. ①

$$\text{is } Z = \sum a e^{(4t^2 + 2t)}$$

solve $(D - D^2)Z = 0$

solve $(D^2 - D^1 + D - D^0)Z = 0$

80) Given $\rightarrow (D^2 - D^1 + D - D^0)Z = 0$ ①
 $\Rightarrow (D - D^0)(D + D^1 + 1)Z = 0$.

There are distinct factors.

G.S of ① is

$$Z = e^{0t} \phi_1(yt) + e^{-t} \phi_2(y^{-1}t)$$

solve $D(D^1)(D - D^0 - 1)Z = 0$

80) There are three distinct factors.

The g.s of ② is

$$Z = \phi_1(y) + \phi_2(y^{-1}) + e^{-yt} \phi_3(y^{-1}t)$$

Particular Integral:

The complete soln of $F(D, D')z = f(x)$

$$\text{is } z = C.F + P.I.$$

$$\text{where } P.I. = \frac{1}{F(D, D')} f(x) \quad (1)$$

The methods of obtaining particular integrals of non-homogeneous linear eqns with constant coefficients are very similar to those of ordinary linear eqns with constant coefficients.

We now give some rules of finding the particular integrals.

case (i) If $f(x)$ = e^{ax} and $F(a)$ ≠ 0.

$$\text{then } P.I. = \frac{e^{ax}}{F(D, D')} \quad \text{or} \quad \frac{e^{ax}}{F(a)}$$

case (ii) If $f(x)$ = $\sin(bx)$ or $\cos(bx)$.

$$\text{then } P.I. = \frac{1}{F(D, D')} \sin(bx) \quad \text{is evaluated}$$

by putting $D = b$, $D' = -b^2$

provided the denominator is not 0.

case (iii) If $f(x)$ = x^n then

$$P.I. = \frac{1}{F(D, D')} (x^n) = [F(D, D')]^{-1} x^n$$

which can be evaluated as
 $[F(D, D')]^{-1}$ after taking powers of

case(v) $f(x,y) = e^{ax+by}$

$$\text{Then } P.D = \frac{1}{F(D,D')} e^{ax+by}$$

$$= e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)}$$

1991 solve $D + P - 2 = Z + ay$

$$\text{Given } D + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - D^2 = Z + ay$$

$$\Rightarrow (D^2 + D - D^2)Z = ay$$

$$\Rightarrow (D-1)(D+1)Z = ay$$

\therefore There are two distinct linear factors
eg $(D-1)(D+1)Z = 0$.

GF of ① is

$$e^x \phi_1(y) + e^{-y} \phi_2(x)$$

$$\text{as per } P.D = \frac{1}{(D-1)(D+1)} (ay)$$

$$= -(-D)^{-1} (1+D)^{-1} (ay)$$

$$= -[(1+D+D^2+D^3+\dots)(1+D+D^2+\dots)]$$

$$= -[1+D+D^2+\dots](ay)$$

$$= -[ay - a(0) + 0(0) - 0]$$

$$= -[ay - a + y - 1]$$

The reqd. g.f. is $Z = C \phi + P.F.$

$$\text{i.e. } Z = e^x \phi_1(y) + e^{-y} \phi_2(x) - (ay + a - y + 1)$$

1997
solve $(D^2 - D \cdot D' + D' - 1) Z = w \sin(y)$ (17)

1998
solve $r - 1 + 2r - 2 = 2^r y^r$

1998
solve $(D - 2D' + 2) Z = 2^r \sin(y + \pi)$.

2007
solve $(D^2 - D' - 3D + 3D')Z = 2^y + e^{2y}$

solve $r - 5 + p = 1$.

solve $(D - 2D' - 2) Z = 2e^{2x} \tan(y + \pi)$ (1)

$$C.F. = e^{2x} [\phi_1(y + \pi) + i\phi_2(y + \pi)]$$

$$P.G. = \frac{1}{(D - 2D' - 2)^2} e^{2x} \tan(y + \pi).$$

$$= 2e^{2x} \cdot \frac{1}{[D+2 - 3(D'+0) - 2]} \tan(y + \pi)$$

$$= 2e^{2x} \cdot \frac{1}{(D - 3D')^2} \tan(y + \pi)$$

$$= 2e^{2x} \cdot \frac{1}{(D - 3D')^2} \tan(y + \pi) \cdot \left(\frac{1}{(D - D')} \right)$$

$$= 2e^{2x} \tan(y + \pi)$$

As reqd g.s of Q. $\underline{\underline{Z = e^{2x} \tan(y + \pi)}}$

Eqs reducible to linear form
with constant coefficients:

A PDE having variable co-efficients
can sometimes be reduced to an eqn
with constant coefficients by suitable
substitutions.

Reduce an eqn of the form

$$\text{Ans} \quad \frac{\partial z}{\partial x} + A_1 \frac{\partial z}{\partial y} + A_2 \frac{\partial z}{\partial y^2} + \dots = f(x, y) \quad \text{linear}$$

linear eqn with constant coefficients.

Note: In the eqn (1), the term $\frac{\partial z}{\partial y^2}$ is
multiplied by the variable expression

To transform the eqn (1),

putting $x = t^x$, $y = t^y$

$$\Rightarrow [x = \log t], [y = \log t]$$

$$\text{Now } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} = \frac{1}{t} \cdot \frac{\partial z}{\partial t}$$

$$\therefore \boxed{x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}}$$

$$\therefore x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} = D(z)$$

$$\text{Now } x \frac{\partial}{\partial x} (x \frac{\partial z}{\partial x}) = x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial t} \right) + x \frac{\partial}{\partial x} z$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} = (x \frac{\partial}{\partial x} - 1) x \frac{\partial z}{\partial x}$$

$$= (D-1) D z$$

$$= D(D-1) z \quad \boxed{(2)}$$

29

Engeneral $\frac{\partial^m}{\partial x^m} z = D(D-1)(D-2) \dots (D-m+1)z$

Now $\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y} \cdot \frac{\partial y}{\partial y} = \frac{1}{y} \frac{\partial^2}{\partial y}$

$$\Rightarrow \boxed{y \frac{\partial^2}{\partial y} = \frac{\partial^2}{\partial y}}$$

$$\therefore y \frac{\partial^2}{\partial y} = \frac{\partial^2}{\partial y} = D'(D'-1)z$$

say
so $y \frac{\partial^2}{\partial y} = D'(D'-1)z$

Engeneral $y \frac{\partial^2}{\partial y} = D'(D'-1) \dots (D'-m+1)z$

Also $y \frac{\partial^2}{\partial y} = DD'z$

and $y^m \frac{\partial^m}{\partial x^m} z = D(D-1) \dots (D-m+1) D'(D'-1)z$

After substitution reduce the eqn to eqn
having constant coefficients and now it
can be easily solved by the methods of
hom & non-hom linear eqns with constant
coefficients.

~~problems~~ solve $\frac{\partial^2}{\partial x^2} + xy \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial y^2}$

so putting $x = e^x$, $y = e^y$

and denoting the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

$$\therefore \textcircled{1} \in [D(D-1) + 2D]D' +$$

$$\Rightarrow [D^2 - D + 2D]D' + D'$$

$$\Rightarrow (D+D')(D+D')$$

$$\therefore \textcircled{1} \in D^2 z = \phi_1(y)$$

$$\begin{aligned} z &= \phi_1(\log x + \log y) + \phi_2(\log y - \log x) \\ &= \phi_1(\log(\frac{x}{y})) + \phi_2(\log(\frac{y}{x})) \quad (\because \log e^x = x) \\ &= f_1(\frac{x}{y}) + \alpha f_2(\frac{y}{x}) \end{aligned}$$

which is eqn 3. 10

1987

$$\text{Solve } x^2 \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial y^2} = xy$$

$$\text{putting } x = e^x; y = e^y \quad \text{eqn 1.}$$

and applying the operators $\frac{\partial}{\partial x}$ & $\frac{\partial}{\partial y}$

$$\text{① } \left[D(D-1) - D'(D'-1) \right] z = e^{x+y} \quad D \text{ & } D'$$

$$\Rightarrow [D^2 - D^2 - D + D'] z = e^{x+y}$$

$$\Rightarrow (D - D') (D + D' - 1) z = e^{x+y} \quad \text{②.}$$

$$\therefore \bar{C}F = \phi_1(y+x) + e^x \phi_2(y-x)$$

$$= \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$$

$$= \phi_1(\log x) + \alpha \phi_2(\log \frac{y}{x}) \quad (\because \frac{y}{x} = e^y)$$

$$= f_1(\log x) + \alpha f_2(\frac{y}{x})$$

$$\text{Now P.F. } = \frac{e^{x+y}}{(D - D')(D + D' - 1)}$$

$$= \frac{1}{(D - D')(1 + 1 - 1)} e^{x+y}$$

$$= \frac{1}{D - D'} e^{x+y}$$

$$= \frac{x}{1} e^{x+y}$$

$$= (\log x) \cdot xy$$

\therefore Ans of ① is $x = c e^x + p e^y$

1993
1997

$$\text{Solve } x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + p_1 - 2y = \log x.$$

Cauchy's Problem for Second Order Partial Differential equation. Characteristic equation and Characteristic Curves (or simply characteristics) of the second-order Partial Differential Equations.

Cauchy's Problem: Consider the second order Partial differential equation

$$Rx + Sy + Tt + f(x, y, z, p, q) = 0 \quad (1)$$

in which R, S and T are functions of x and y only. The Cauchy's Problem consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed values of z and $\frac{dz}{dn}$, where n is the distance measured along the normal to the curve.

As an example of Cauchy's Problem for the second order Partial differential equation, consider the following Problem:

To determine solution of $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ with the following data prescribed on the x-axis: $z(x, 0) = f(x)$, $z_y(x, 0) = g(x)$. Observe that y-axis is the normal to the given curve (x-axis here).

Characteristic equations and Characteristic Curves.

Corresponding to (1), consider the A-quadratic

$$Rx^2 + Sy^2 + Tt^2 = 0 \quad (2)$$

when $S^2 - 4RT > 0$, (2) has real roots. Then

differential equation $(\frac{dy}{dx}) + \lambda(x,y) = 0$ ————— (2)

are called the characteristic equations.

The solutions of (3) are known as characteristic curves or simply the characteristics of the second order Partial differential equation (1).

Now, Consider the following three cases.

Case(i): If $S^2 - 4RT > 0$ (i.e if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 , say so that we have two characteristic equations

$$\left(\frac{dy}{dx}\right) + \lambda_1(x,y) = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right) + \lambda_2(x,y) = 0$$

solving these we get two distinct families of characteristics.

Case(ii): If $S^2 - 4RT = 0$ (i.e (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). solving it, we get only one family of characteristics.

Case(iii): If $S^2 - 4RT < 0$ (i.e. (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic.

→ Find the characteristics of $y^2x - x^2t = 0$

Sol'n: Given $y^2x - x^2t = 0$ ————— (1)

Comparing (1) with $Rx^2 + Sxy + Tx^2 + f(x,y,z,P,Q) = 0$

here $R=y^2$, $S=0$ and $T=-x^2$

Then $S^2 - 4RT = 0 - 4 \cdot y^2(-x^2)$
 $= 4x^2y^2 > 0$

and hence (1) is hyperbolic everywhere except on the coordinate axes $x=0$ and $y=0$.

The λ -quadratic is $Rx^2 + Sx + T = 0$ (or)

$$y^2\lambda^2 - x^2 = 0 \quad \text{--- (2)}$$

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots)

corresponding characteristic equations are.

$$(dy/dx) + (\lambda/y) = 0 \quad \text{and} \quad (dy/dx) - (\alpha/y) = 0$$

$$x dx + y dy = 0 \quad \text{and} \quad x dy - y dx = 0$$

Integrating, $x^2 + y^2 = C_1$ and $x^2 - y^2 = C_2$.

which are the required families of characteristics.

Here these are families of Circles and hyperbolae respectively.

Find the characteristics of $x^2r + 2xy^2s + y^2t = 0$.

Sol'n : Given $x^2r + 2xy^2s + y^2t = 0 \quad \text{--- (1)}$

Comparing (1) with $Rx^2 + Sx + T + f(x, y, z, p, q) = 0$,

here $R = x^2$, $S = 2xy^2$ and $T = y^2$.

Then, $S^2 - 4RT = 4x^2y^4 - 4x^2y^2 = 0$

and hence (1) is parabolic everywhere.

The λ -quadratic is $Rx^2 + Sx + T = 0$ or x^2

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$
(equal roots)

The characteristic equation is $(dy/dx) - (y/x) = 0$

(or) $(\frac{1}{y}) dy - (\frac{1}{x}) dx = 0$ giving $y/x = c_1$, (or) $y = c_1 x$.

which is the required family of characteristic.

Here it represents a family of straight lines passing through the origin.

H.W

→ Find the characteristics of $4r+5s+t+p+q-2=0$

[Ans: $y-x=c_1$, and $y-(y_0)=c_2$]

→ Find the characteristics of $(\sin^2 x) r + (2 \cos x) s - t = 0$

[Ans: $y+\cot x - \operatorname{Cot} x = c_1$, $y+\operatorname{Cosec} x + \operatorname{cot} x = c_2$]

(5)

IVMS Applications of Differential Equations

Mod: 03/09/15/3/5

In physical problems, we always seek a solution of the differential equation, which satisfies some specified conditions known as the boundary conditions.

The differential equation together with these boundary conditions, constitute a boundary value problem.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions.

Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables.

Method of Separation of variables (or product method)

- It involves a solution which breaks up into a product of functions each of which contains only one of the variables.

The following example explains this method.

→ Solve C by the method of separation of variables:

$$\frac{dy}{dx} = 2x^2 \cdot 2x^2 \cdot 2x^2$$

Ans: Assume the trial solution $x^m y^n$.

where x is a function of x alone

and y that of y alone.

Substituting this value of x in the given equation.

we have

$$x^m y - 2x^m y + x^m y = 0$$

where $\frac{dy}{dx} = m x^{m-1}$, $y' = \frac{dy}{dx}$ etc.

Separating the variables,

we get

$$(x^m - 2x^m) y + x^m y = 0$$

$$\Rightarrow \frac{x^m - 2x^m}{x^m} = \frac{y'}{y}$$

Since x and y are independent

variables, therefore it can only true if each side is equal to the same constant. i.e.,

$$\frac{x'' - 2x'}{x} = \frac{-y'}{y} = a$$

$$\Rightarrow \frac{x'' - 2x'}{x} = a \quad \text{and} \quad \frac{-y'}{y} = a$$

i.e., $x'' - 2x' - ax = 0$ ie. $y'' - 2y' - ay = 0$

To solve the equation (i),

the auxiliary equation is

$$m^2 - 2m - a = 0$$

$$\Rightarrow m = 1 \pm \sqrt{1+a}$$

∴ the solution of (i) is

$$x = C_1 e^{(1+\sqrt{1+a})x} + C_2 e^{(1-\sqrt{1+a})x}$$

and the solution of (ii) is $y = C_3 e^{-ax}$

Substituting these values of x and y

in (1), we get

$$z = \left\{ C_1 e^{(1+\sqrt{1+a})x} + C_2 e^{(1-\sqrt{1+a})x} \right\} C_3 e^{-ax}$$

$$\text{i.e., } z = k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x}$$

where $k = C_1 C_3$ and

which is the required complete sol.

→ Using the method of separation of

$$\text{value } \frac{dy}{dt} = 2 \frac{du}{dt} + u. \text{ where } u =$$

SOL: Assume the solution $u(t) =$

Substituting in the given equation, we have

$$x^T = 2x \cdot T + xT$$

$$\Rightarrow (x' - x)T = 2xT$$

$$\Rightarrow \frac{x' - x}{2x} = \frac{T}{T} = k \text{ (say)}$$

$$\therefore \frac{x' - x}{2x} = k \quad \text{and} \quad \frac{T}{T} = k \leftarrow (\text{iii})$$

$$\Rightarrow x' - x - 2kx = 0$$

$$\Rightarrow x' = (1+2k)x$$

$$\Rightarrow \frac{x'}{x} = (1+2k) \quad (\text{ii})$$

coming (ii) $\log x = (1+2k)x + \log c$
 $\Rightarrow x = ce^{(1+2k)x}$

From (iii) $\log T = kt + \log c'$

$$\Rightarrow T = c'e^{kt}$$

Thus from (i) we have

$$u(x, t) = xT = cc'e^{(1+2k)x+kt}$$

Given that $u(x, 0) = 6e^{-3x}$

from (i) $u(x, 0) = 6e^{-3x} = cc'e^{(1+2k)x}$

$$\Rightarrow cc' = 6 \quad \text{and} \quad 1+2k = -3$$

Substituting these values in (i), we get

$$cc'e^{(1+2k)x+kt} \quad i.e. \quad u = 6e^{-(3x+kt)}$$

which is the required solution.

Note: Suppose the given partial differential equation involves n independent variables,

x_1, x_2, \dots, x_n and one dependent variable 'u'. Then assuming the equation possesses product solution of the form

$$u(x_1, x_2, \dots, x_n) = x_1(x_1) \cdot x_2(x_2) \cdots x_n(x_n) \quad (1)$$

where x_i is a function of x_i only
 $(i=1, 2, \dots, n)$

On substitution of (1) into the given equation, we shall obtain 'n' ordinary differential equations one in each of the unknown functions x_i ($i=1, 2, \dots, n$)

* Solve the following equations by the method of separation of variables:

(1) $-py^3 + qx^2 = 0$

(2) $\frac{4\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{5y}$ when $x=0$

(3) $\frac{3\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$, $u(0,0) = 4e^3$

(4) find a solution of the equation $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + xu$

in the form $u = f(x, y)$.
solve the equation subject to the conditions $u=0$ and $2u = 1+e^{-2y}$, when $x=0$ for all values of y

The following are the well-known partial differential equations:

(i) wave equation : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2}$

(ii) One dimensional heat flow equation :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(iii) two dimensional heat flow equation which in steady state becomes the two dimensional.

Laplace's equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) vibrating membrane : Two dimensional wave equation i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$.

(v) Laplace's equation in three dimensions

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. etc.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions, and the combination of such solutions gives the desired solution. Quite often a certain condition is not applicable. In such cases the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

Some basic definitions

Rest: A body is said to be at rest if it does not change its position with time with respect to its surroundings.

Motion: A body is said to be in motion if it changes its position with time with respect to its surroundings.

Terms rest and motion are relative to each other.

For example when a train is running, two passengers sitting in the train beside each other are at rest with respect to each other but they are in motion with respect to the person standing outside the train.

Displacement: the shortest distance between the starting point to the ending point is called 'displacement'. It is a vector.

(ii) Displacement is a vector quantity representing a change of position.

Deflection: A sudden change in the direction that something is moving in.

Distance: Total length of the path covered by a body.

Velocity: Displacement of a body per unit time is said to be its velocity.

If s is displacement that takes place in time t then velocity of the body is given as

$$\text{velocity} = \frac{\text{displacement}}{\text{time taken}}$$

Acceleration: If velocity of a body changes with time either due to change in magnitude or direction or both, it is said to have acceleration.

i.e., the rate of change of velocity is called acceleration.

Equilibrium: A system of forces acting on a particle is said to be in equilibrium if it is either at rest or moves with uniform motion in a straight line.

Mass: mass of a body is the quantity of matter it contains.

Force: force is an external agency which changes or tends to change the state of rest or of uniform motion in a straight line.

The effect of force acting on a rigid body depends not only on magnitude but also on its direction and point of application.

weight of body:

The force with which a body is attracted towards the centre of the earth due to the gravitational attraction is called the weight of the body on the earth.

$$W = mg$$

where m is mass of body and g is acceleration due to gravity for earth.

Mass of the body remains constant at any place but weight of the body varies with changes in g .

Gravitational force:

Gravitational force is a long range force and is responsible for the attraction between particles of different masses in universe.

Nature of gravity:

The gravitational force between two bodies is always attractive. It depends upon the masses of the bodies and the distance between them. The greater the masses the greater is the force, the greater the separation the lesser is the force. Independent of the nature of the bodies.

Tension is a pulling force which is exerted on a body by means of a string or rod.

Thrust is a pushing force which is exerted on a body by means of a rod and not by means of string, because a string is flexible.

Newton's Laws of Motion

Newton's First Law:

Every body continues to be in its state of rest or of uniform motion along a straight line unless it is acted on by an external force to change its state.

Newton's Second Law:

The rate of change of momentum of a body is directly proportional to the external force applied and it takes place in the same direction in which the external force is acting.

$$F = ma$$

Momentum: The momentum (P) of a body is defined as the product of its mass (m) and velocity (v).

$$P = mv$$

Newton's Third Law:

To every action there is always an equal and opposite reaction.

$$\text{Action} = \text{Reaction}$$

- Action and reaction are equal in magnitude and opposite in direction.
- They always occur in pairs.

principle of superposition of waves;

principle of superposition of wave states that when two or more waves are simultaneously impressed on the particles of the medium, the resultant displacement of any particle is equal to the algebraic sum of displacements of all the waves.

If y_1, y_2, y_3 etc. are the displacements due to the overlapping waves, the resultant displacement of any particle is given by

$y = y_1 + y_2 + y_3 + \dots$
Thus the resultant wave form can be obtained by the principle of superposition of waves.

The general linear homogeneous partial differential equation of the second order

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial xy} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0 \quad (1)$$

Suppose that (i) $u_1, u_2, u_3, \dots, u_n$ is an infinite set of solutions of (1) as x varies in \mathbb{R} in xy -plane.

(ii) The infinite series $u = u_1 + u_2 + \dots$ converges and its differential term is R . Then, by principle the function u , defined by -

a solution of ① in \mathbb{R} . Here \mathbb{R} denotes the set of all real numbers.

Fourier sine series: If it be required to expand $f(x)$ as a sine series in $0 < x < l$, then its expansion will give the Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

Also known as half-range sine series.
Fourier cosine series: If it be required to expand $f(x)$ as a cosine series in $0 < x < l$, then its expansion will give the Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \text{ where } a_0 = \frac{2}{l} \int_0^l f(x) dx,$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

Also known as half-range cosine series.

Double Fourier sine series:

If it be required to expand $f(x, y)$ as a

sine series in rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, then its expansion will give double Fourier sine series.

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

$$\text{where } A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx, \quad m = 1, 2, 3, \dots, n = 1, 2, 3, \dots$$

Triple Fourier sine series: If it be required to expand

$f(x, y, z)$ as a sine series in parallelopiped $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, then its expansion will give triple Fourier sine

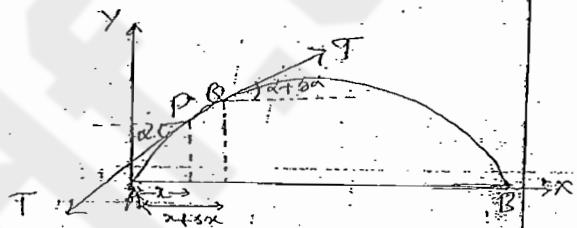
$$f(x, y, z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{lmn} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c},$$

$$\text{where } A_{lmn} = \frac{8}{abc} \int_0^a \int_0^b \int_0^c f(x, y, z) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} dz dy dx, \quad l = 1, 2, 3, \dots, m = 1, 2, 3, \dots, n = 1, 2, 3, \dots$$

vibrations of stretched elastic string - wave

Consider a tightly stretched elastic string of length l and fixed ends A and B subjected to constant tension T as shown in the figure.

The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.



Let the string be released from rest and allowed to vibrate.

We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB of the string entirely in one plane.

Taking the end A as the origin, AB as the x-axis and Ay perpendicular as the y-axis, to feel the effect place entirely in the xy -plane. If the string is in the position APB

consider the motion of an element between two points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$, the tangents makes angles α and $\alpha + \delta\alpha$.

Since the element is moving upwards with acceleration $\frac{dy}{dt}$.

Since there is no motion in horizontal direction, we have

$$T \cos(\alpha + \delta\alpha) = T \cos\alpha = Q$$
$$\Rightarrow T \cos(\alpha + \delta\alpha) = T \cos\alpha = T \text{ (constant)}$$

Also the vertical component of the force acting on this element $pQ = T \sin(\alpha + \delta\alpha) - T \sin\alpha$

$$= T \{ \sin(\alpha + \delta\alpha) - \sin\alpha \}$$
$$= T \{ \tan(\alpha + \delta\alpha) - \tan\alpha \}$$
$$= T \left\{ \frac{\sin(\alpha + \delta\alpha)}{\cos(\alpha + \delta\alpha)} - \frac{\sin\alpha}{\cos\alpha} \right\}$$

If m be the mass per unit length of the string, then mass of the element

$$pQ = m \delta x.$$

Then by Newton's second law of motion we have

$$m \frac{dy}{dx} = T \left\{ \frac{(dy)}{(\cos\alpha + \delta\alpha)} - \frac{(dy)}{\cos\alpha} \right\}$$
$$\text{i.e. } \frac{dy}{dx} = \frac{T}{m} \left\{ \frac{(dy)}{(\cos\alpha + \delta\alpha)} - \frac{(dy)}{\cos\alpha} \right\}$$

Taking limits as $\delta\alpha \rightarrow 0$, i.e. $\delta x \rightarrow 0$,

we have $\frac{dy}{dx} = \frac{T}{m} \frac{dy}{dx}$

$$\frac{dy}{dx} = C \frac{dy}{dx} \quad \text{where } C = \frac{T}{m}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional wave equation.

Solution of one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Soln: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ — (1)

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Let solution of (1) be of the form

$$y(x, t) = X(x)T(t)$$
 — (2)

where x is a function of x and
 T is a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = X''T$ and $\frac{\partial^2 y}{\partial x^2} = X''T$.

Substituting these values in (1),
we get

$$X''T = c^2 X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}$$
 — (3)

Clearly the left side of (3) is a function
of x only and the right side is a
function of t only.

Since x and t are independent variables
(3) can hold good if each side is equal
to a constant (k say).

Then (3) leads to the ordinary differ-
ential equations

$$\frac{d^2 X}{dx^2} - k^2 X = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} - k^2 T = 0$$
 — (4)

Now we solve (4) and (5); Three cases arise

(i) when $k=0$. Then $x = a_1x + a_2$; $T = a_3t + a_4$

(ii) when k is positive.

$$\text{Let } k = p^2 \text{ (say)}$$

$$\text{Given } x = b_1e^{pt} + b_2e^{-pt}; T = b_3e^{cpt} + b_4e^{-cpt}$$

(iii) when k is negative.

$$\text{Let } k = -p^2 \text{ (say)}$$

$$\text{Then } x = c_1 \cos px + c_2 \sin px$$

$$T = c_3 \cos cpt + c_4 \sin cpt$$

Thus the various possible solutions of wave equation (1) are

$$y(x, t) = (a_1x + a_2)(a_3t + a_4) \quad (1)$$

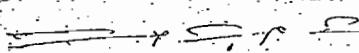
$$y(x, t) = (b_1e^{px} + b_2e^{-px})(b_3e^{cpt} + b_4e^{-cpt}) \quad (2)$$

$$\text{and } y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad (3)$$

of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t . Hence (2) must involve trigonometric terms. Accordingly the solution given by (3) i.e.

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

is the only suitable solution of the wave equation.



→ Boundary conditions and initial conditions of one dimensional wave equation:

→ The boundary conditions which the solution has to satisfy are

(i) $y(x, t) = 0$ when $x=0$

(ii) $y(x, t) = 0$ when $x=l$ (where l is the length of the stretched string).

These should satisfy for every value of t .
i.e., At the end points of the string are

fixed, for all time :-

$$y(0, t) = 0 \text{ and } y(l, t) = 0$$

→ If the string is made to vibrate by pulling it in a curve $y=f(x)$ and then releasing it, the initial conditions are

$$y(x, t=0) = f(x) \text{ i.e., } y(x, 0) = 0 \text{ and}$$

$$\frac{\partial y}{\partial t}(x, t=0) = 0 \text{ i.e., } \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

Here the initial velocity of the string is zero i.e., the string starts from the position of rest.

→ If the string is made to vibrate by giving its each point v when in equilibrium position a specified velocity, the initial conditions are of the form

$y(x, t=0)$ when $t=0$ i.e. $y(x_0) = 0$

$$\frac{dy(x, t)}{dt} = g_{0x} \text{ when } t=0 \text{ i.e. } \left(\frac{dy}{dt}\right)_{t=0} = g_{0x}$$

If the string is given both a displacement and velocity initially, then the initial conditions are of the form

$$y(x, t) = f_{0x} \text{ when } t=0 \text{ i.e. } y(x_0) = f_{0x}$$

$$\frac{dy(x, t)}{dt} = g_{0x} \text{ when } t=0 \text{ i.e. } \left(\frac{dy}{dt}\right)_{t=0} = g_{0x}$$

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General solution of one-dimensional wave equation satisfying the given boundary and initial conditions.

Ex(1)

Show that the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

under conditions $y(0, t) = 0$, $y(l, t) = 0$ has

$$y(x, 0) = f(x), (y)_t|_{t=0} = g(x)$$

has solution of the form:

$$y(x, t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$$E_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$F_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$$

Sol: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (1)

where $y(x, t)$ is the deflection of the string. Let it be stretched between fixed points $(0, 0)$ and $(l, 0)$. Then we are to find $y(x, t)$ under the following boundary conditions (B.C.) and initial conditions (I.C.)

I.C. $y(0, t) = 0, y(l, t) = 0$ for all t (2)

I.C. $y(x, 0) = f(x)$ (Initial deflection) (3)

$(y)_t|_{t=0} = g(x)$ (Initial velocity) (4)

Suppose that (1) has the solution of the form $y(x, t) = X(x) T(t)$

Substituting this value of x in ①, we have

$$XT^H = C^2 X^H T$$

$$\Rightarrow \frac{X^H}{X} = \frac{1}{C^2} \frac{T^H}{T} = \mu \quad (\text{say})$$

$$\Rightarrow X^H - \mu X = 0 \quad \text{--- (6)}$$

$$\text{and } T^H - \mu C^2 T = 0 \quad \text{--- (7)}$$

Using ⑥, ⑦ gives

$$X(0) T^H = 0 \quad \text{and} \quad X C^2 T(0) = 0 \quad \text{--- (8)}$$

Since $T^H \neq 0$ leads to $X \neq 0$.

so suppose that $T(0) \neq 0$.

Then ⑧ gives

$$X(0) = 0 \quad \text{and} \quad X(0) = 0 \quad \text{which are boundary conditions} \quad \text{--- (9)}$$

we now solve ⑥ under B.C. ⑨.

Three cases arise-

case ①: let $\mu = 0$. Then solution of ⑥ is

$$X(0) = A + B \quad \text{--- (10)}$$

Using B.C. ⑨, ⑩ gives

$$X(0) = A(0) + B \Rightarrow 0 = B$$

$$\text{and } X(0) = A(0) + B \Rightarrow 0 = A(0) + 0 \Rightarrow A = 0$$

$$\Rightarrow X(0) = 0$$

This leads to $X \neq 0$, which does not

satisfy B.C. ⑨ and ⑩.

so we reject $\mu = 0$.

case(2): Let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (6) is
(i.e., positive)

$$x(t) = Ae^{\lambda t} + Be^{-\lambda t} \quad \text{--- (11)}$$

Using B.C. (9), (11) gives

$$x(0) = 0 = A + B \quad \text{i.e., } A + B = 0 \quad \text{--- (12)}$$

and $x(l) = 0 = Ae^{\lambda l} + Be^{-\lambda l}$

Solving above we get

$$Ae^{\lambda l} - Ae^{-\lambda l} = 0$$

$$\Rightarrow A(e^{\lambda l} - e^{-\lambda l}) = 0 \quad (\because e^{\lambda l} \neq e^{-\lambda l} + 0)$$

$$\Rightarrow A = 0$$

from (12) $B = 0$

$$\Rightarrow x(t) = 0$$

This leads to $y \equiv 0$ which does not

satisfy (3) and (4).

So we reject $\mu = \lambda^2$.

case(3): Let $\mu = -\lambda^2$, $\lambda \neq 0$.
(i.e., μ is negative)

Then solution of (6)

$$x(t) = A \cos \lambda t + B \sin \lambda t \quad \text{--- (13)}$$

Using B.C. (9), (13) gives

$$x(0) = 0 = A(0) + B(0) \Rightarrow A = 0$$

and $x(l) = 0 = 0 + B \sin \lambda l$

$$\Rightarrow B \sin \lambda l = 0$$

$$\Rightarrow \sin \lambda l = 0$$

Here we take $B \neq 0$

Since otherwise $x \equiv 0$

which does not $\neq 0$

$$\text{Now } \sin \lambda l = 0$$

$$\Rightarrow \lambda l = n\pi ; \quad n=1, 2, \dots$$

$$\Rightarrow \boxed{\lambda = \frac{n\pi}{l}}, \quad n=1, 2, \dots$$

From (3), we have

$$x(x) = B \sin \frac{n\pi x}{l} ; \quad n=1, 2, \dots$$

Hence non-zero solutions $x_n(x)$ of (6)

are given by

$$\boxed{x_n(x) = B_n \sin \left(\frac{n\pi x}{l} \right)} \quad (14)$$

From (4)

$$T'' - \mu C^2 T = 0$$

$$\Rightarrow T'' + \lambda^2 C^2 T = 0 \quad (C = \mu = -\lambda^2)$$

$$\Rightarrow T'' + \frac{n^2 \pi^2 C^2}{l^2} T = 0 \quad (\because \lambda = \frac{n\pi}{l})$$

whose general solution is

$$T_n(t) = C_n \cos \left(\frac{n\pi ct}{l} \right) + D_n \sin \left(\frac{n\pi ct}{l} \right)$$

$$y_n(x, t) = x_n(x) T_n(t)$$

$$= B_n \sin \frac{n\pi x}{l} \left[C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right]$$

$$= \boxed{\left[E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l} \right] \sin \frac{n\pi x}{l}} \quad (15)$$

are solutions of (1) satisfying (3).

Here $E_n = B_n C_n$ and $F_n = B_n D_n$ are new arbitrary constant.

In order to obtain a solution also satisfying

(2) and (4), we consider more general

$$\text{solution } y(x, t) = \sum_{n=1}^{\infty} y_n(x, t)$$

$$\text{ie, } y(x,t) = \sum_{n=1}^{\infty} \left\{ B_n \cos \frac{n\pi ct}{l} + f_n \sin \frac{n\pi ct}{l} \right\} \sin \frac{n\pi x}{l} \quad (15)$$

differentiating (15) partially w.r.t "t", we get

$$\begin{aligned} \frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} \left\{ B_n \left(-\sin \frac{n\pi ct}{l} \right) \cdot \frac{n\pi c}{l} + f_n \cos \frac{n\pi ct}{l} \cdot \frac{n\pi c}{l} \right\} \sin \frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} \left\{ B_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l} + f_n \cos \frac{n\pi ct}{l} \cdot \frac{n\pi c}{l} \right\} \sin \frac{n\pi x}{l}. \end{aligned}$$

putting $t=0$ in (15) and (16)

and using the I.C. (3) and (4), we get

$$(15) \quad f(x) = \sum_{n=1}^{\infty} \left\{ B_n \cos \frac{n\pi c t_0}{l} + 0 \right\} \sin \frac{n\pi x}{l}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$\text{and} \quad (16) \quad g(x) = \sum_{n=1}^{\infty} \left\{ 0 + \frac{n\pi c}{l} f_n \cos \frac{n\pi c t_0}{l} \right\} \sin \frac{n\pi x}{l} \quad (\because \frac{\partial y}{\partial t} = g(x))$$

$\Rightarrow g(x) = \sum_{n=1}^{\infty} \frac{n\pi c f_n}{l} \sin \frac{n\pi x}{l}$

which are Fourier sine series expansion
for $f(x)$ and $g(x)$ respectively

according we get,

$$f_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad (17)$$

$$\text{and } \frac{n\pi c f_n}{l} = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \Rightarrow f_n = \frac{2}{n\pi c l} \int_0^l g(x) dx$$

Hence the required solution is given by (15)

$$\text{ie, } y(x,t) = \sum_{n=1}^{\infty} \left\{ B_n \cos \frac{n\pi ct}{l} + f_n \sin \frac{n\pi ct}{l} \right\} \sin \frac{n\pi x}{l}$$

$$\text{where } B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Note:particular case I.: If initial velocity

$$y_t(x,0) = g(x) = 0 \text{ then } f_n = 0 \text{ from (18)}$$

∴ in this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} E_n \cos n\pi t \sin \frac{n\pi x}{l}$$

where E_n is given by (17)

particular case II.: If initial displacement

$$y_t(x,0) = f(x) = 0, \text{ then } E_n = 0 \text{ by (16)}$$

∴ in this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} f_n \sin n\pi t \sin \frac{n\pi x}{l}$$

where f_n is given by (18)

A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = y_0 \sin \frac{n\pi x}{l}$ from which it is released at time $t=0$. Find the displacement at any point at a distance x from one end at time t .

Solⁿ: The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (1)$$

As the end points of the string are fixed,

for all time

B.C. $y(0,t) = 0$ and $y(l,t) = 0$ (2)

S.C. Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 \leq x \leq l$ (3)

and initial displacement $y(x, 0) = y_0 \sin \frac{n\pi x}{l}$ 13
 $0 \leq x \leq l$ 4

proceeding like as in Ex-① tell

equation ⑤.

$$\text{i.e., } y(x, t) = \sum_{n=1}^{\infty} \left\{ E_n \cos n\pi c t + F_n \sin n\pi c t \right\} \sin \frac{n\pi x}{l} \quad 5$$

Differentiating ⑤ partially w.r.t t, we get

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} \left\{ -E_n n\pi c \sin \frac{n\pi c t}{l} + F_n n\pi c \cos \frac{n\pi c t}{l} \right\} \sin \frac{n\pi x}{l} \quad 6$$

putting $t=0$ in ⑥ and ⑦

and using initial conditions ③ & ④

we get

$$\text{⑤} \quad y(0, t) = y_0 \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l} \quad 7$$

$$\text{⑥} \quad \left(\frac{dy}{dt} \right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c \cdot F_n}{l} \sin \frac{n\pi x}{l}$$

$$\text{where } F_n = \frac{2}{n\pi c} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

i.e., $F_n \neq 0$

From ⑦, we have

$$y_0 \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l}$$

Comparing the coefficients of like terms

on both sides, we have

$$F_1 = y_0 \text{ and } F_n = 0 \text{ for } n \neq 1$$

Equation ⑤ reduces to

$$y(x, t) = y_0 \sin \frac{n\pi x}{l} \cos n\pi c t$$

for the vibrating string of length l , if it is given by $y(x, 0) = C \sin x$, C being constant from net, then find the displacement $y(x, t)$ that: taking another example $l=2\pi$ and $y_0=1$

A string of length l has its ends $x=0$ and $x=l$ fixed. It is released from rest in the position

$y = \{4x(l-x)\}/l^2$. Find an expression for the displacement of the string at any subsequent time.

Sol:

The displacement function $y(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{(1)}$$

Subject to boundary conditions:

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t \geq 0 \quad \text{(2)}$$

and initial conditions namely

$$\text{Initial velocity} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{(3)}$$

$$\text{Initial displacement (zero effect)} = \frac{4}{l^2} \quad \text{(4)}$$

proceeding like as in Ex-①. Full solution

$$y(x,t) = \sum_{n=1}^{\infty} [F_n \cos \frac{n\pi x}{l} + f_n \sin \frac{n\pi x}{l}] \sin nt \quad \text{(5)}$$

differentiating (5) partially w.r.t. t , we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} [f_n \sin \frac{n\pi x}{l} + n\omega F_n \cos \frac{n\pi x}{l}] \sin nt \quad \text{(6)}$$

putting $t=0$ in eqn (5) and (6).

and using initial conditions (3) and (4)

we get

$$y(0,0) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi 0}{l} = 0 \quad \text{(by (3))} \quad \text{(7)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} n\omega F_n \sin \frac{n\pi 0}{l} = 0 \quad ; \quad \text{(by (4))}$$

where for $\frac{2}{n\pi c} \int_0^l (\sin n\pi x) dx = 0$

$$\text{and } E_1 = \frac{2}{l} \int_0^l \frac{4\pi^2 c(l-x)}{l^2} \sin \frac{n\pi x}{l} dx$$

$$\Rightarrow E_1 = \frac{8\lambda}{l^3} \int_0^l (l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{8\lambda}{l^3} \left[(l-x) \left(-\frac{1}{n\pi} \cos \frac{n\pi x}{l} \right) \right]_0^l - \left[-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{8\lambda}{l^3} \left[(l-x) \left(-\frac{1}{n\pi} \cos \frac{n\pi x}{l} \right) + (l-x) \frac{l}{n\pi} \sin \frac{n\pi x}{l} \right]$$

$$+ \left[\frac{l}{n\pi} \sin \frac{n\pi x}{l} \right]_0^l + \frac{2l^2}{n^2\pi^2} \sin \frac{n\pi l}{l}$$

$$= \frac{8\lambda}{l^3} \left[(l-x) \left(-\frac{1}{n\pi} \cos \frac{n\pi x}{l} \right) + (l-x) \frac{l}{n\pi} \sin \frac{n\pi x}{l} \right]$$

$$= \frac{2l^3}{n^2\pi^2} \cos \frac{n\pi x}{l} \Big|_0^l$$

$$= \frac{8\lambda}{l^3} \left[(0-0) + \left(\frac{2l^3}{n^2\pi^2} \sin n\pi - 0 \right) \frac{-l^2}{n^3\pi^2} (\cos n\pi - 1) \right]$$

$$= \frac{8\lambda}{l^3} \left[\frac{2l^2}{n^2\pi^2} \sin n\pi - \frac{2l^3}{n^3\pi^2} (\cos n\pi - 1) \right]$$

$$E_1 = \begin{cases} \frac{8\lambda}{l^3} \left[0 - \frac{2l^3}{n^3\pi^2} (-1) \right] = 0 & \text{if } n \text{ is even} \\ \frac{8\lambda}{l^3} \left[0 - \frac{2l^3}{n^3\pi^2} (1-1) \right] = 0 & \text{if } n \text{ is odd} \end{cases}$$

Substituting the values of E_1 and E_2

$$\text{expression 8} \\ y(x,t) = \frac{32\lambda}{l^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{l}$$

Ques. A taut string of length l has its ends at $x=0$ and $x=l$ fixed. The mid point is taken to a small height h and released from rest at time $t=0$. Find the displacement function $y(x,t)$.

Hint: B.C. $y(0,t) = y(l,t) = 0$

Initial position of the string at $t=0$ is

made up of two straight line segments OB and BA as shown in the figure

and string is released from rest.

The equation of OB is given by

$$y=0 = \frac{h}{l}x \quad (x=0) \text{ for } 0 \leq x \leq \frac{l}{2}$$

$$\Rightarrow y = \frac{2h}{l}x \quad \text{for } 0 \leq x \leq \frac{l}{2}$$

The equation of BA is given by

$$y=0 = \frac{h}{l}(l-x) \quad \text{for } \frac{l}{2} \leq x \leq l$$

$$\Rightarrow y = \frac{2h(l-x)}{l} \quad \text{for } \frac{l}{2} \leq x \leq l$$

Hence, the initial displacement is given by

$$y(x,0) = \begin{cases} \frac{2h}{l}x, & 0 \leq x \leq \frac{l}{2} \\ \frac{2h(l-x)}{l}, & \frac{l}{2} \leq x \leq l. \end{cases}$$

and the initial velocity $\frac{\partial y}{\partial t}|_{t=0} = 0$

Ans. $y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{T}$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$,

$$= \frac{2}{l} \left[\int_0^{\frac{l}{2}} \frac{2h}{l}x \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l \frac{2h(l-x)}{l} \sin \frac{n\pi x}{l} dx \right]$$

$$= \begin{cases} \frac{8Gh^{n+1}}{(2n+1)^2 \pi^2}, & \text{if } n = 2m+1 \text{ (odd)} \\ 0, & \text{if } n = 2m \text{ (even)} \end{cases}$$

$$\partial_y(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\sin((2m-1)\pi x)}{l} \frac{\cos((2m-1)\pi ct)}{l}$$

→ A tightly stretched elastic string of length l , with fixed end points $x=0$ and $x=l$. Initially its free position is given by $y = y_0 \sin \frac{3\pi x}{l}$, y_0 being constant. Find the displacement $y(x,t)$.

Hint: B.C. $y(0,t) = y(l,t) = 0$, $\partial_t y(0,t) = 0$.

I.C. Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 < x < l$

Initial displacement $= y(x,0) = y_0 \sin \frac{3\pi x}{l}$

proceeding like as in ex-①, till eqn ③,

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ F_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l} \right\} \frac{8h \sin \frac{n\pi x}{l}}{l} \quad ③$$

Differentiating ③ partially w.r.t. t , we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -\frac{n\pi c}{l} F_n \sin \frac{n\pi ct}{l} + F_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \right\} \frac{8h \sin \frac{n\pi x}{l}}{l} \quad ④$$

putting $t=0$ in ③ and ④

and using the I.C. ① and ②, we get

$$⑤ y(x,0) = y_0 \sin \frac{3\pi x}{l} = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l}$$

$$⑥ \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c F_n}{l} \sin \frac{n\pi x}{l} \quad ⑥$$

$$\text{where } F_n = \frac{2}{n\pi c} \int_0^l (y_0 \sin \frac{3\pi x}{l}) \sin \frac{n\pi x}{l} dx = 0$$

now from ⑤

$$y_0 \sin \frac{3\pi x}{l} = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow y_0 \left[\frac{3\sin \frac{3\pi x}{l}}{l} - \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l} \right] = 0$$

Comparing the coefficients

$$\text{we have } F_1 = \frac{3y_0}{4}, F_2 = 0, F_3 = 0$$

Substituting these values in ①,
the required displacement is given by

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi c t}{l}$$

- A tightly stretched elastic string of length π , with fixed end points $x=0$ and $x=\pi$ is initially in the position y_0 given by $y = y_0 \sin^3 x$; y_0 being constant. Find the displacement $y(x,t)$.

Ans: $y(x,t) = \frac{2y_0}{\pi} \sin x \cos ct - \frac{y_0}{\pi} \sin 3x \cos 3ct$

[putting in the above problem]

- Solve the one dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$, $0 \leq x \leq \pi$, $t > 0$ subject to the following initial and boundary conditions.
- (i) $y(x, 0) = \sin^3 x$, $0 \leq x \leq \pi$ (ii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, $0 \leq x \leq \pi$
 - (iii) $y(0, t) = y(\pi, t) = 0$, for $t > 0$.
- find the deflection $y(x,t)$ of the vibrating string (length π , and $c^2 = 1$) corresponding to zero initial velocity and initial deflection $y(x) = k(\sin x - \sin^3 x)$.

Ans: $y(x, t) = \sin x \cos t - \cos x \sin^3 t$

2003

The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

(or)

Find the deflection $y(x,t)$ of a vibrating string, stretched between fixed points $(0,0)$ and $(3l,0)$, corresponding to zero initial velocity and following initial deflection:

$$f(x) = \begin{cases} hx & \text{when } 0 \leq x \leq l \\ h(3l-x) & \text{when } l \leq x \leq 2l \\ h(x-2l) & \text{when } 2l \leq x \leq 3l \end{cases} \quad (A)$$

where h is constant

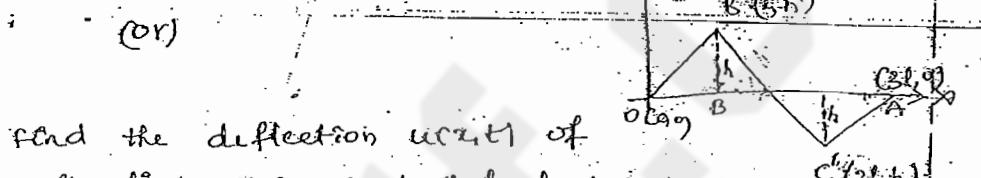
Sol: The displacement $y(x,t)$ of any point of the string is given by

$$\text{B.C. } y(0,t) = y(3l,t) = 0 \quad (1)$$

$$\text{I.C. } y(x,0) = f(x) \quad (\text{since free initially}) \quad (2)$$

$$\text{and } \frac{\partial y}{\partial t}|_{t=0} = 0 \quad (3)$$

proceeding like as in Ex-① till eqn (14)
by replacing l by $3l$,



we have,

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ F_n \cos \frac{n\pi x}{3l} + f_n \sin \frac{n\pi x}{3l} \right\} \frac{\sin \frac{n\pi t}{2l}}{2l} \quad (4)$$

Differentiating (4) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -F_n \frac{n\pi C}{3l} \sin \frac{n\pi x}{3l} + \frac{n\pi C}{3l} f_n \cos \frac{n\pi x}{3l} \right\} \frac{\sin \frac{n\pi t}{2l}}{2l} \quad (5)$$

Putting $t=0$ in (4) and (5)

- subtracting the L.C. (4) and (5), we get

$$(6) \sum_{n=1}^{\infty} \left(\frac{\partial y}{\partial t} \right)_{t=0} = \sum_{n=1}^{\infty} f_n \frac{n\pi C}{3l} \sin \frac{n\pi x}{3l} = 0 \quad (\text{Cf } (4))$$

$$\text{where } f_n = \frac{2}{\pi} \int_0^{3l} f(x) \sin \frac{n\pi x}{3l} dx = 0$$

$$(7) \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{3l} = 0$$

$$\text{where } F_n = \frac{2}{3l} \int_0^{3l} f(x) \sin \frac{n\pi x}{3l} dx$$

Now

$$F_n = \frac{2}{3l} \int_0^{3l} f(x) \sin \frac{n\pi x}{3l} dx$$

$$= \frac{2}{3l} \left[\int_0^l f(x) \sin \frac{n\pi x}{3l} dx + \int_l^{3l} f(x) \sin \frac{n\pi x}{3l} dx + \int_{3l}^{5l} f(x) \sin \frac{n\pi x}{3l} dx \right]$$

$$= \frac{2}{3l} \left[\int_0^l f(x) \sin \frac{n\pi x}{3l} dx + \int_1^3 f(x) \sin \frac{n\pi x}{3l} dx + \int_3^5 f(x) \sin \frac{n\pi x}{3l} dx \right]$$

continuing in this way we get

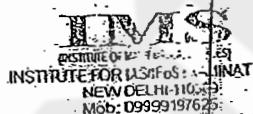
$$F_n = \frac{18b}{n\pi a} \left[\sin \frac{n\pi a}{3l} - \left\{ \sin \left(n\pi \frac{l}{3} \right) - \sin \left(n\pi \frac{5l}{3} \right) \right\} \right]$$

$$= \frac{18b}{n\pi a} \left[\sin \frac{n\pi l}{3} \left(\sin \frac{n\pi x}{3l} - \cos \frac{n\pi x}{3l} \right) \right]$$

$$= \frac{18h}{n^2\pi^2} \left[\sin \frac{n\pi}{3} - 0 + \cos n \sin \frac{n\pi}{3} \right] \quad (25 \text{ min } 20)$$

$$= \frac{18h}{n^2\pi^2} [1 + \cos n\pi] \sin \frac{n\pi}{3}$$

$$= \frac{18h}{n^2\pi^2} [1 + (-1)^n] \sin \frac{n\pi}{3}$$



Thus $E_n = 0$ if n is odd.

$$E_m = \frac{36h}{m^2\pi^2} \sin \frac{m\pi}{3} \text{ if } n \text{ is even}$$

$$\text{i.e., } \frac{86h}{4m^2\pi^2} \sin \frac{2m\pi}{3}, m \neq 2$$

$$= \frac{9h}{m^2\pi^2} \sin \frac{2m\pi}{3}$$

Putting the values of E_n and E_m in (4),
the required deflection is given by

$$y(x, t) = \sum_{m=1}^{\infty} \frac{9h}{m^2\pi^2} \sin \frac{2m\pi}{3} \cos \frac{m\pi x}{3l} \cos \omega t$$

$$\Rightarrow y(x, t) = \frac{9h}{\pi^2} \sum_{m=1}^{\infty} \sin \frac{2m\pi}{3} \cos \frac{m\pi x}{3l} \cos \frac{m\pi t}{3l}$$

Putting $x = \frac{3l}{2}$ in (7), we find that the displacement of the mid-point of the string
is $y(\frac{3l}{2}, t) = 0$.

because $\sin m\pi = 0$ for all integral values of m .

This shows that the mid-point of the
string always rest.

A tightly stretched string of length l
ends is initially in equilibrium position
vibrating by giving each point a vel.
Find displacement. Ans: $y(x, t) = \frac{180}{12\pi^2} \{$

2004 A uniform string of length l held tightly between $x=0$ and $x=l$ with no initial displacement, is struck at $x=0$, $0 < a < l$ with velocity v_0 . Find the displacement of the string at any time $t \geq 0$.

Hint: B.C. $y(0, t) = y(l, t) = 0 \quad \forall t$

I.C. ^{initial displacement} $y(x, 0) = 0, 0 \leq x \leq l$.

Initial velocity $= y_t(x, 0) = v_0 ; 0 \leq x \leq l$.

$$\text{Ans: } y(x, t) = \frac{4v_0l}{ct^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}$$

2005 A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l-x)$, find its displacement.

$$\text{Ans: } y(x, t) = \frac{8kl^3}{ct^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^5} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}$$

2006 A deflection of a vibrating string of length l , is governed by the partial differential equation $y_{tt} = c^2 y_{xx}$. The ends of the string are fixed at $y_{tt} = 0$ at $x=0$ and l . The initial velocity $y_t(x, 0) = 0$. The initial displacement is given by

$$y(x, 0) = \begin{cases} 0, & 0 < x < l \\ 1, & l < x < 1 \end{cases}$$

Find the deflection of the string at any instant of time.

$$\text{Ans: } y(x, t) = \frac{4l}{ct^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}$$

~~2009~~
~~09/2009~~

A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x) = \mu x(l-x)$, where μ is constant and then released. Find the displacement of any point x of the string at any time $t > 0$.

$$\text{Ans: } y(x,t) = \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi ct}{l}$$

A string of length l is initially at rest in its equilibrium position and motion is started giving each of its points a velocity v given by

$$v = kx \text{ if } 0 \leq x \leq \frac{l}{2} \text{ and } v = k(l-x) \text{ if } \frac{l}{2} \leq x \leq l$$

Find the displacement function $y(x,t)$.

$$\text{Ans: } \frac{4kl^2}{\pi^3} \sum_{m=1}^{\infty} \frac{(m+1)}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}$$

If the string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $v_0 \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$ where $0 \leq x \leq l$ at $t=0$. Find the displacement function.

$$\text{Ans: } y(x,t) = \frac{lv_0}{2\pi c} \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l} + \frac{lv_0}{5\pi c} \sin \frac{5\pi x}{l} \sin \frac{5\pi ct}{l}$$

A string is stretched between the fixed points $(0,0)$ and $(l,0)$ and released at rest from the given by $f(x) = \begin{cases} \frac{2\pi x}{l}, & \text{when } 0 \leq x \leq l_2 \\ \frac{2k(l-x)}{l}, & \text{when } l_2 < x \leq l \end{cases}$

Find the deflection of the string at

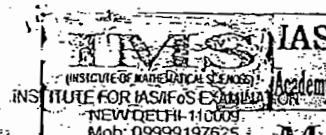
$$\text{Ans: } \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{(2n+1)}{(2n-1)^2} \cos \frac{(2n-1)\pi ct}{l} \sin \frac{(2n-1)\pi x}{l}$$

A taut string of length 20 cm fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity is given by

$$\dot{v} = x \cdot u \text{ ms}^{-1}$$

$= 20x \text{ ms}^{-1}$ for $0 \leq x < 20$, x being the distance from one end. Determine the displacement at any subsequent time.

Set-11



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MATHEMATICS

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* Numerical Analysis *

Solution of Algebraic and Transcendental equations:

Introduction: In this chapter, we shall discuss some numerical methods for solving algebraic and transcendental equations.

The equation $f(x)=0$ is said to be algebraic if $f(x)$ is purely a polynomial in x . If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then equation $f(x)=0$ is called a transcendental equation.

The equations $x^3 - 7x + 8 = 0$, $x^4 + 4x^3 + 7x^2 + 6x + 3 = 0$ are algebraic.

The equations $\sin x = 3x$, $x - \sin x = 0$ and $e^x = 4x$ are transcendental.

Algebraically, the real number x is called the real root (or zero) of the function $f(x)$ of the equation $f(x)=0$ if and only if $f(x)=0$ is the real root of an eq.

is the value of x where the graph of $f(x)$ meets the x -axis in rectangular co-ordinate system.

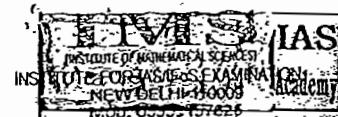
We shall assume that the equation $f(x) = 0$ (1) has only isolated roots, that is for each root of the equation there is a neighbourhood which does not contain any other roots of the equation.

Approximately the isolated roots of the equation (1) has two stages:

- (1) Isolating the roots that is finding the smallest possible interval (a, b) containing one and only one root of the equation (1).
- (2) Improving the values of the approximate roots to the specified degree of accuracy. Now we state a very useful theorem of mathematical analysis. Without proof.

Theorem: Intermediate value property.

If $f(x)$ is a real valued continuous function on the closed interval $a \leq x \leq b$. If $f(a)$ and $f(b)$ have opposite signs,

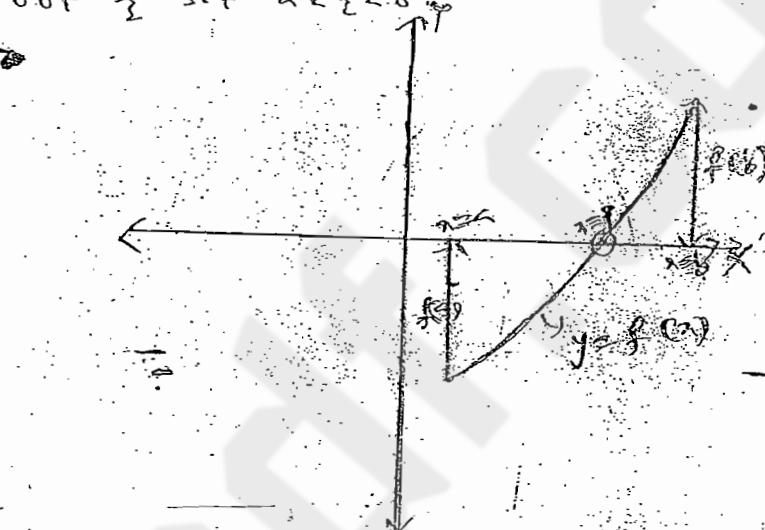


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then the graph of the function $y = f(x)$
crosses the x -axis at least once,
that is $f(x) = 0$, has at least one
root $\{ \text{s.t. } a < b \}$



✓ Broadly speaking, all the known numerical methods - for solving either a transcendental equation (or) an algebraic equation can be classified into two groups:

direct methods → iterative methods

Direct methods require no knowledge of the initial approximation of a root of the equation $f(x) = 0$.

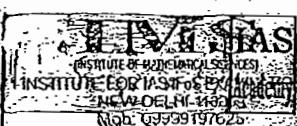
Iterative methods do req. approximation to roots
(Iteration means repeated application process or a pattern of action).

How to get the first approximation?
We can find the approximate value of the root of $f(x) = 0$ either by a graphical method (or) by an analytical method as explained below:

Graphical method:

The real root of the equation $f(x) = 0$ (1) can be determined approximately as the abscissas of the points of intersection of the graph of the function $y = f(x)$ with the x -axis. If $f(x)$ is simple, we shall draw the graph of $y = f(x)$ w.r.t. a rectangular axis x on and y by. The points at which the graph meets the x -axis are the location of the roots of (1).

If $f(x)$ is not simple we replace equation (1) by an equivalent equation say $\phi(x) = \psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ are simpler than $f(x)$. Then the x -co-ordinate of the point of intersection of the graphs gives ~~gives~~ the crude approximation of the real roots of the equation (1).



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problem 1)

Solve the equation $\log_{10} x = 1$, graphically.

Sol. The given equation (1)

$\log_{10} x = 1$ can be written

$$\text{as } \log_{10} x = 1 \quad (2)$$

where $\log_{10} x$ is equal to

simpler than $\log_{10} x = 1$, constructing

the curves $y = \log_{10} x$ and $y = 1/x$

we get x-coordinates of the point of intersection as

2.5

The approximate value of the root of $\log_{10} x = 1$

$$\therefore \text{PS } \{ = 2.5$$

Ques → solve $x^2 + x - 1 = 0$ graphically

H.W. → solve $e^x + x + 1 = 0$ graphically

→ solve $x - \sin x + 2 = 0$

Sol. Let the given equa

$$f(x) = x - \sin x$$

It can be written as

$$x-1 = \sin x$$

where $x-1$ and $\sin x$ simpler than

constructing the curve $y = x-1$ & $y = \sin x$

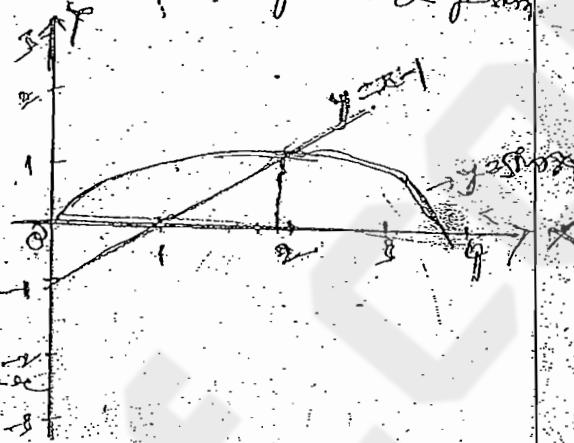
we get

in co-ordinates

of the point

of the hyper-

section as fig -



The approximate
value of the
root of ① is $x = 1.9$.

Analytical method

This method is based on
intermediate value property. We
shall illustrate it through an
example.

$$\text{Let } f(x) = x - \sqrt{1 + \sin x} \quad \text{--- ②}$$

$$\text{Now } f(0) = -1$$

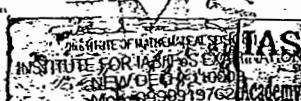
$$f(1) = 1 - \sqrt{1 + \sin(1)}$$

$$= 1 - \sqrt{1 + 0.84147}$$

$$\approx 1.64295$$

$$f(0) < 0 \text{ & } f(1) > 0$$

i.e. $f(0)$ and $f(1)$ are opposite signs



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(4)

... By intermediate value property
there is atleast one root b/w
 $x=0$ and $x=1$.

This method is often used to find the
first approximation to a root of either
transcendental equation or algebraic equation.
Hence, in analytical method, we must
always start with an initial interval
(a, b), so that $f(a)$ and $f(b)$ have
opposite signs.

Bisection Method

This method is due to Bolzano.
Suppose, we wish to locate the root
of the equation $f(x) = 0$ in an interval
say, (x_0, x_1) . Let $f(x_0)$ and $f(x_1)$ have opposite
signs, such that $f(x_0) \cdot f(x_1) < 0$.

Then the graph of the function
crosses the x -axis between x_0 and x_1 ,
which guarantees the existence of
at least one root in the interval.
The desired root is approximated
defined by the midpoint

If $f(x_1) = 0$, then x_1 is the desired root of $f(x) = 0$. However, if $f(x_1) \neq 0$, then the root may be between x_0 and x_1 or it may not.

Now, we define the next approximation by
 $x_3 = \frac{x_0+x_2}{2}$ provided $f(x_0) \cdot f(x_2) < 0$, then the root may be found b/w x_0 and x_2
 (or) by $x_3 = \frac{x_1+x_2}{2}$ provided $f(x_1) \cdot f(x_2) < 0$, then the root lies b/w x_1 and x_2 etc.

Thus, at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in the given figure. This process of halving the intervals is continued to determine a small and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root.

(This method is known as an Iteration method)



Geometrical illustration of bisection method.

→ solve $x^3 - 9x + 1 = 0$, for the root b/w $x=2$ and $x=4$ by the bisection method.

Sol Let $f(x) = x^3 - 9x + 1$.

Since $f(2) = -9.0$, $f(4) = 29.0$

$\therefore f(2), f(4) < 0$

Hence the root lies b/w 2 and 4.

Let $x_0 = 2$, $x_1 = 4$. Then the first approximation to the root is $x_2 = \frac{x_0+x_1}{2}$

$$= \frac{2+4}{2} = 3 \Rightarrow x_2 = 3$$

Since $f(x_2) = f(3) = 1 > 0$

$\therefore f(2), f(3) < 0$ i.e. $f(x_0), f(x_2) < 0$

Hence the root lies b/w 2 and 3.

The second approximation to the

root is $x_3 = \frac{x_0+x_2}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$\therefore x_3 = 2.5$

Since $f(x_3) = f(2.5) < 0$

$\therefore f(3), f(2.5) < 0$

i.e. $f(3), f(2.5) < 0$.

Hence the root lies b/w 3 & 2.5.

The third approximation to the root

$$\therefore x_4 = \frac{x_2+x_3}{2} = \frac{3+2.5}{2} = 2.75$$

i.e. $x_4 = 2.75$

By, we can find the

$$[x_5 = 2.875] \text{ and } [x_6 =]$$

and the process can be continued until the root is obtained to the desired accuracy.

Now we can write in

x_n	$f(x_n)$
2	3
3	-5.875
4	-2.9531
5	-1.1113
6	-0.0901

solve the equation $x^3 = 9x + 1$ for the root lying b/w 2 and 3, correct to three significant figures.

Sol. Let $f(x) = x^3 - 9x + 1$

Since $f(2) = 8 - 18 + 1$
 $= -9 < 0$ and

$f(3) = 27 - 27 + 1$
 $= 1 > 0$

$\therefore f(2) \cdot f(3) < 0$

Hence the root lies b/w 2 & 3.

Let $a_0 = 2$, $b_0 = 3$

n	a_n (\approx)	b_n (\approx)	$f(a_n)$ (\approx)	$f(b_n)$ (\approx)
0	2	3	2.5	-5.8 (<0)
1	2.5	3	2.75	-2.9 (<0)
2	2.75	3	2.88	-1.03 (<0)
3	2.88	3	2.94	-0.05 (<0)
4	2.94	3	2.97	0.42 (>0)
5	2.94	2.97	2.955	0.21 (>0)
6	2.94	2.955	2.9425	0.08 (>0)
7	2.94	2.9425	2.9438	0.017 (>0)
8	2.94	2.9425	2.9419	-0.016 (<0)
9	2.94	2.9419	2.9428	0.001 (>0)

In the 8th step, a_1, b_1 and a_{11} are equal upto three significant figures.

we can take 2.94 as a root upto three significant figures.

$$\therefore \text{the root of } x^3 - 9x + 1 = 0 \text{ is } 2.94$$

→ find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method in four stages.

Sol Let $f(x) = x^3 - 4x - 9$

since $f(2)$ is -ve and

$f(3)$ is +ve

$$\therefore f(2) \cdot f(3) < 0$$

∴ hence the root lies b/w 2 and 3.

∴ first approximation to the root

$$\text{P.S. } x_1 = \frac{2+3}{2} = [2.5]$$

NOW $f(x_1) = f(2.5)$

$$= (2.5)^3 - 4(2.5) - 9$$

$$= -3.375$$

= -ve

$$\therefore f(3) \cdot f(x_1) < 0$$

∴ hence the root lies b/w x_1 and 3.

∴ second approximation to the

$$\text{root P.S. } x_2 = \frac{x_1+3}{2}$$

$$= \frac{2.5+3}{2}$$

$$= [2.25]$$

NOW $f(x_2) = f(2.25)$

$$= (2.25)^3 - 4(2.25) - 9$$

$$= 0.3969$$

i.e. +ve

$$f(x_2) \cdot f(x_1) < 0$$

∴ hence the root lies b/w x_1 and x_2 .

(b)

The third approximation to the

$$\text{root is } x_3 = \frac{x_1 + x_2}{2} \\ = 1.2625$$

$$\text{Now } f(x_3) = (2.625)^3 - 4(2.625)^2 - 1 \\ = -1.4121 \\ \text{i.e. -ve}$$

$$\therefore f(x_2) f(x_3) < 0$$

Hence the root lies b/w x_2 and x_3 .

∴ The fourth approximation to the

$$\text{root is } x_4 = \frac{1}{2}(x_2 + x_3) \\ = 1.26875$$

Hence the root is 1.26875 approximately.

→ Find the real root to four decimals
of the equation $x^6 - x^4 - x^2 - 1 = 0$ which lies
b/w 1 and 2.

Sol) Let $f(x) = x^6 - x^4 - x^2 - 1$
Since $f(1) = -2 < 0$ &
 $f(2) = 39 > 0$

$$\therefore f(1) \cdot f(2) < 0.$$

Hence the root lies b/w 1 & 2.

The first approximation to the

$$\text{root is } x_1 = \frac{1+2}{2} = 1.5$$

$$\text{Now } f(x_1) = f(1.5)$$

$$= +ve$$

$$\therefore f(1) \cdot f(x_1) < 0$$

Hence the root lies b/w 1 & x_1 .

The second approximation to the

$$\text{root is } x_2 = \frac{1+1.5}{2} = 1.25$$

$$\text{Now } f(x_2) = f(1.25) \\ = \text{+ve.}$$

$$\therefore f(x_1) \cdot f(x_2) < 0.$$

Hence the root lies b/w x_1 & x_2 .
The third approximation to the root

$$\text{if } x_3 = \frac{x_1 + x_2}{2} \\ = \frac{1.25 + 1.5}{2} = 1.375$$

$$\text{Now } f(1.375) \text{ is +ve.}$$

$$\therefore f(x_3) \cdot f(x_1) < 0.$$

Hence the root lies b/w x_2 & x_3 .
The fourth approximation to the

$$\text{root } x_4 = x_3 + \frac{x_3 - x_2}{2} \\ = 1.375 + \frac{1.375 - 1.25}{2} \\ = 1.4375$$

$$\text{Now } f(x_4) = f(1.4375)$$

$$= \text{+ve.}$$

$\therefore f(x_3) \cdot f(x_4) < 0$
Hence the root lies b/w
 x_3 & x_4 .

Now the fifth approximation

$$\text{to the root is } x_5 = \frac{x_2 + x_4}{2} \\ = 1.375 + \frac{1.4375}{2} \\ = 1.40625$$

$$\text{Now } f(x_5) = \text{+ve.}$$

$$\therefore f(x_4) \cdot f(x_5) < 0$$

Hence the root lies b/w
 x_4 & x_5 .

$$\begin{aligned} \therefore x_6 &= \frac{x_5 + x_4}{2} \\ &= \frac{1.375 + 1.40625}{2} \\ &= \boxed{1.390625} \end{aligned}$$

NOW $f(x_6) = -ve$

$$\therefore f(x_5) \cdot f(x_6) < 0$$

Hence the root lies b/w x_6 & x_5
the 7th approximation to the

$$\begin{aligned} \text{root is } x_7 &= \frac{x_6 + x_5}{2} \\ &= \frac{1.390625 + 1.40625}{2} \\ &= \boxed{1.3984375} \end{aligned}$$

$$\text{NOM } f(x_7) = f(1.3984375)$$

= -ve.

$$\therefore f(x_6) \cdot f(x_7) < 0$$

Hence the root lies b/w x_7 & x_6

The 8th approximation to

$$\text{the root is } x_8 = \frac{x_7 + x_6}{2}$$

$$\begin{aligned} &= \frac{1.3984375 + 1.40625}{2} \\ &= \boxed{1.40234375} \end{aligned}$$

NOW $f(x_8) < 0$

$$\therefore x_9 = \frac{x_8 + x_7}{2}$$

$$= \frac{1.40234375 + 1.40625}{2}$$

$$= 1.4043 \text{ (nearly)}$$

Now $f(x_9) > 0$

$$\begin{aligned}x_{10} &= \frac{x_8 + x_9}{2} \\&= \frac{1.402343734 + 1.4043}{2} \\&= \boxed{1.4033}\end{aligned}$$

(8)

Now $f(x_{10}) < 0$

$$\begin{aligned}\therefore x_{11} &= \frac{x_{10} + x_9}{2} \\&= \frac{1.4033 + 1.4043}{2} \\&= \boxed{1.4038}\end{aligned}$$

Now $f(x_{11}) = \text{pre.}$

$$\begin{aligned}\therefore x_{12} &= \frac{x_{10} + x_{11}}{2} \\&= \frac{1.4033 + 1.4043}{2} \\&= \boxed{1.40355}\end{aligned}$$

Hence the root to four
decimals of $x^6 - 2x^3 - 1 = 0$ lying
between 1 and 2 is 1.4036 (approximately).

Find to three decimals
root of the equation $3x - e^{2x} - 1 = 0$.
Ans: 0.607

Find one root of $e^x - 3x = 0$
correct to two decimal places.

Ans: 1.51

H.W. → find the root of $\tan x + x = 0$
upto two decimal places which
lies b/w 2 and 2.1

[Ans: 2.03]

H.W. → find a root of the equation
 $x^3 - 4x - 9 = 0$ correct to three decimal
places by using bisection method.

[Ans: 0.7065]

H.W. → compute one the root of $2x - 3 \sin x - 5 = 0$,
by bisection method, correct to three
significant figures.

[Ans: 2.8f]

H.W. → compute one root of $x + \log x - 2 = 0$
correct to two decimal places which
lies b/w 1 and 2

[Ans: 1.56]

Note(0) While applying bisection method we must be careful to check that $f(x)$ is continuous.

(9)

For example, we may come across functions like $f(x) = \frac{1}{x-1}$. If we consider the interval $(0.5, 1.5)$, then $f(0.5), f(1.5) < 0$. In this case we may be tempted to use bisection method. But we cannot use the method here because $f(x)$ is not defined at middle point $x=1$, where can overcome these difficulties by taking $f(x)$ to be continuous throughout the partial bisection interval. (Note that, if f is continuous function on $[a, b]$ and $f(a) \neq f(b)$ then f assumes every value b/w $f(a)$ and $f(b)$.)

Therefore we should always examine the continuity of the function in the initial interval before attempting the bisection method.

Note(2): It may happen that a function has more than one root in an interval. The bisection method helps us for determining one root only. We can determine the other roots by properly choosing the initial intervals.

A numerical process starts with an initial approximation and iteration this approximation until we get the accurate value of the root.

Let us consider another iterative method now:

* Regula Falsi Method:

— This method is also known as the method of false position.

— The Latin word Regula falsi means rule of falsehood. It does not mean that the rule is a false statement. But it conveys that the roots that we get according to the rule are approximate roots and not necessarily exact roots. This method is similar to the bisection method.

— The bisection method for finding approximate roots has a drawback that it makes use of only the signs of $f(a)$ and $f(b)$. It does not use the values $f(a)$, $f(b)$ in the computations.

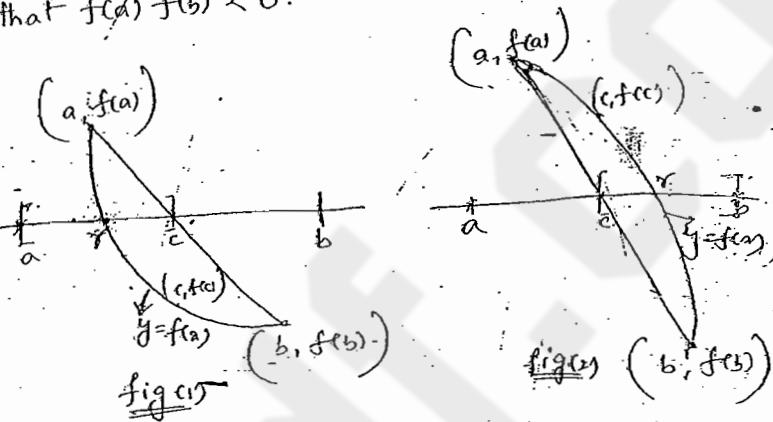
for example-

If $f(a) = 100$ and $f(b) = -0.1$, then by the bisection method the first approximate value of a root of $f(x)$ is the mid value x_0 of the interval (a, b) . But at x_0 , $f(x_0)$ is nowhere near 0.

∴ In this case it makes more sense to take a value near to -0.1 than the middle value as the approximation to the root.

This drawback is to some extent overcome by the Regula-falsi method.

Geometrically, suppose we want to find a root of the eqn $f(x)=0$, where $f(x)$ is a continuous function. As in the bisection method, we first find an interval (a, b) such that $f(a) f(b) < 0$.



The condition $f(a) f(b) < 0$ means that the points $(a, f(a))$ and $(b, f(b))$ lie on the opposite sides of the x -axis.

If the line joining $(a, f(a))$ and $(b, f(b))$ crosses the x -axis at some point $(c, 0)$, then we take the x -coordinate of that point as

the first approximation.

If $f(c)=0$, then $x=c$ is the required root.

If $f(a) f(c) < 0$, then the root lies in (a, c) (fig(1)).

If $f(c) f(b) < 0$, then the root lies in (c, b) (fig(2)).

In this case the graph of $y=f(x)$ is concave near the root x . Otherwise if $f(a) f(c) > 0$,

the root lies in (a, c) (fig(2)). In this case

the graph of $y=f(x)$ is convex.

Having fixed the interval in which the root lies, we repeat the above pro-

In mathematical form,,

The formula for the line joining the two points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b-a} (x-a)$$

$$\Rightarrow \frac{y - f(a)}{f(b) - f(a)} = \frac{x-a}{b-a} \quad \text{--- (1)}$$

Since the straight line intersects the x -axis at $(c, 0)$, the point $(c, 0)$ lies on the straight-line. putting $x=c$, $y=0$ in eqn(1), we get

$$\Rightarrow \frac{-f(a)}{f(b)-f(a)} = \frac{c-a}{b-a}$$

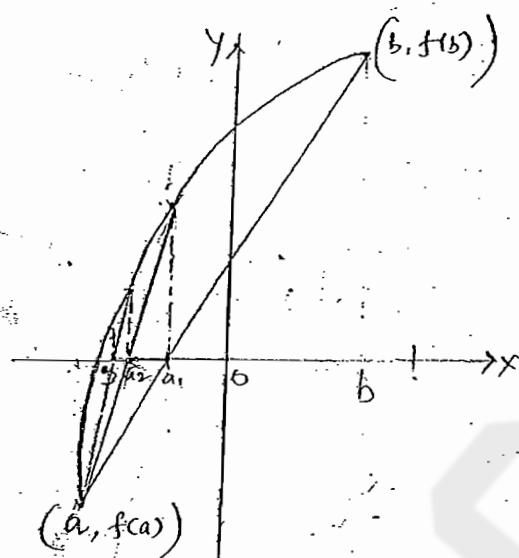
$$\Rightarrow \frac{c-a}{b-a} = \frac{-f(a)}{f(b)-f(a)}$$

$$\Rightarrow c = a - \frac{f(a)}{f(b)-f(a)} (b-a)$$

$$\Rightarrow c = a \frac{f(b) - b f(a)}{f(b) - f(a)} \quad \text{--- (2)}$$

This expression for 'c' gives an approximate value of a root of $f(x)$.

NOW, examine the sign of $f(c)$ and decide in which interval (a,c) or (c,b) the root lies. we thus obtain a new interval such that $f(x)$ is of opposite signs at the end points of this interval. By repeating this process, we get a sequence of intervals (a,b) , (a,a_1) , (a,a_2) , ...



(11)

We stop the process when either of the following holds.

- (i) The interval containing the zero of $f(x)$ is of sufficiently small length.
- (ii) The difference between two successive approximations is negligible.

In the iteration format, the method is usually written as

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \boxed{I}$$

where (x_0, x_1) is the interval in which the root lies.

We now summarise this method

- steps:

Step 1: find numbers x_0 and x_1 such

Step 2: Set $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

This gives the first appr

Step 3: If $f(x_3) = 0$ then x_3 is the required root. If $f(x_3) \neq 0$ and $f(x_0)f(x_3) < 0$, then the next approximation lies in (x_0, x_3) . Otherwise it lies in (x_2, x_1) .

Step 4: Repeat the process till the magnitude of the difference between two successive iterated values x_i and x_{i+1} is less than the accuracy required.

Note: $|x_{i+1} - x_i|$ gives the error after i^{th} iteration.

Ques: Find a real root of the eqn $x^3 - 2x - 5 = 0$ by the method of false position to three decimal places.

Sol: Let $f(x) = x^3 - 2x - 5$.

so that $f(2) = -1$ and $f(3) = 16$

$$\therefore f(2) f(3) < 0$$

Hence the root lies b/w 2 and 3.

Take $x_0 = 2$, $x_1 = 3$:

$$f(x_0) = -1, f(x_1) = 16$$

By the method of false position, we get

$$x_2 = x_0 - \frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)} \quad \text{①}$$

$$= 2 - \frac{3-2}{16+1} (-1)$$

$$= 2 + \frac{1}{17} = \frac{35}{17} = 2.0588$$

Now $f(x_2) = -0.3908$.

$$\therefore f(2.0588), f(3) < 0$$

Hence the root lies between 2.0588 and 3.

Take $x_0 = 2.0588$, $n_i = 3$

$$\therefore f(x_0) = -0.3908, f'(x_0) = 16.$$

$$\text{from } ① \\ x_1 = 2.0588 - \frac{3 - 2.0588}{16 + 0.3908} (-0.3908) \\ = 2.0813$$

now repeating this process, the successive approximations are given by

$$x_2 = 2.0862, x_3 = 2.0915, x_4 = 2.0934,$$

$$x_5 = 2.0941, x_6 = 2.0943 \text{ etc.}$$

The approximate root is 2.094, correct to 3 decimal places.

→ The equation $x^3 + 7x^2 + 9 = 0$ has a root b/w -8 and -7. Use the Regular iteration method, too obtain the root rounded to 3 decimal places. Stop the iteration when $|x_i| < 10^{-4}$.

Soln, Let $f(x) = x^3 + 7x^2 + 9$.

$$\text{Take } x_0 = -8 \text{ and } x_1 = -7.$$

$$f(x_0) = f(-8) = -55 \neq 0$$

$$f(x_1) = f(-7) = 9 \neq 0.$$

By method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad ①$$

$$= \frac{(-8)(9) - (-7)(-55)}{9 + 55}$$

$$x_2 = -7.1606.$$

∴ The first approximation to

Now $f(x_2) = 1.862856 > 0$

and $f(x_0)f(x_2) = f(-8)f(-7.1406) < 0$

Hence the root lies between -8 and -7.1406 .

Take $x_0 = -8$ and $x_1 = -7.1406$.

$\therefore f(x_0) = -55$ and $f(x_1) = 1.862856$

\therefore from ①

$$x_2 = \frac{(-8)(1.862856) + (-7.1406)(-55)}{1.862856 + 55}$$

$$= -7.168174.$$

\therefore The second approximation to the root

$$\therefore x_2 = -7.168174.$$

NOW repeating this process, the successive approximations are given by

$$x_4 = -7.1735649, x_5 = -7.1745906$$

$$x_6 = -7.1747855, x_7 = -7.1748226.$$

The absolute value of the difference between the 6th and 7th iterated values

$$\approx |7.1748226 - 7.1747855| = 0.0000371 < 10^{-4}$$

\therefore we stop the iteration here.

Further, the value of $f(x)$ at 6th

iterated value is $0.00046978 = 4.6978 \times 10^{-4}$

which is close to zero.

Hence -7.175 is an approximate root

of $x^3 + 7x^2 + 9 = 0$ rounded off to

3 decimal places.

ANSWER

Determine an approximate root of the equation (13)

2008 $\cos x - x^2 = 0$ using Regula Falsi method, correct to 4 decimal place

Sol: $f(x) = \cos x - x^2$

so that $f(0) = 1$ and $f(1) = \cos 1 - 1$
 $= -2.17798$

$\therefore f(0) f(1) < 0$

Hence, the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$.

$f(x_0) = 1$ and $f(x_1) = -2.17798$

By the method of false position, we get

$$\begin{aligned}x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad (1) \\&= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1} \\&= 0.31467\end{aligned}$$

\therefore The first approximation to the root is

$x_2 = 0.31467$

Now $f(x_2) = 0.51987 \neq 0$

$\therefore f(x_2) f(x_1) < 0$

\therefore The root lies b/w 0.31467 and 1 .

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (1),

$$x_3 = \frac{(0.3146)(-2.17798)}{-2.17798 - 0.51987} = 0.446$$

$x_3 = 0.44673$

The 2nd approximation to

$x_3 = 0.44673$

Now repeating this process, the successive approximations are

$$x_4 = 0.51502, x_5 = 0.50995, \\ x_6 = 0.51520, x_7 = 0.51692, x_8 = 0.51748, \\ x_9 = 0.51767, x_{10} = 0.51775, \text{ etc.}$$

∴ The approximate root is 0.5177
Correct to 4 decimal places

→ Find a real root of the eqn $\log_{10} x = 1.2$
by regula-falsi method correct to four decimal places.

Ans: 2.7406

→ Use the method of false position to find the fourth root of 32 correct to three decimal places.

$$\text{Soln: Let } x^4 = 32 \text{ then } x^4 - 32 = 0 \\ \text{Let } f(x) = x^4 - 32. \quad \text{Ans: } 12.378.$$

2007
12p
→ Use the method of false position to find a real root of $x^3 - 5x + 7 = 0$ lying between 2 and 3 correct to 3 places of decimal.

→ Use the Regula-falsi method to compute a real root of the eqn $x^3 - 9x + 1 = 0$

(i) if the root lies b/w 2 and 4

(ii) if the root lies b/w 2 and 3.

Comment on the results.

→ Use Regula-falsi method to find a real root of the eqn $\log x - \cos x = 0$ accurate to four decimal places after three successive approximations.

Ans: 1.3030

→ Use Regula-falsi method to show that the real root of $\log_{10} x = 1.2$ lies b/w 3 and 2.740646.

Note: In regula-falsi method, at each stage we find an interval (x_0, x_1) which contains a root and then apply iteration formula [I]. This procedure has a disadvantage. To overcome this, regula-falsi method is modified.

The modified method is known as Secant method.

In this method we choose x_0 and x_1 as any two approximations of the root. The interval (x_0, x_1) need not contain the root. Then we apply formula [I] with $x_0, x_1, f(x_0)$ and $f(x_1)$.

The iterations are now defined as:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

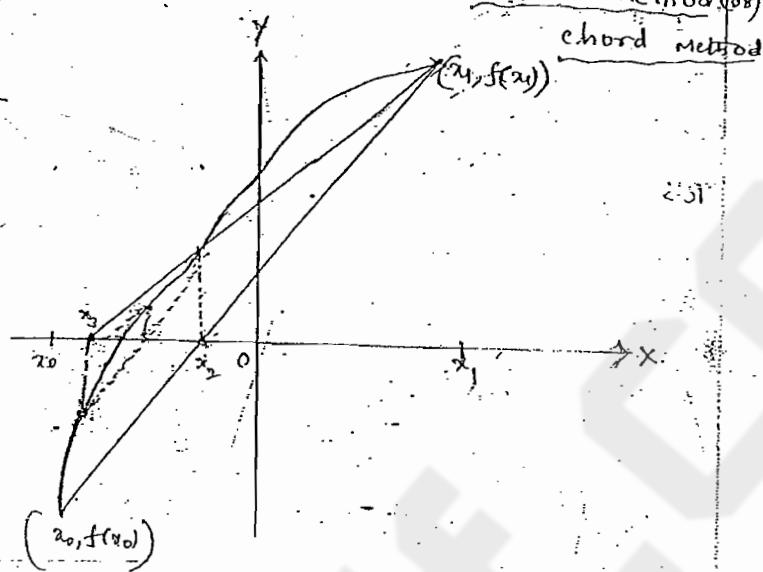
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_{n+1} = \frac{x_n f(x_n) - x_{n-1} f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Geometrically, in Secant method, we replace the graph of $f(x)$ in the interval (x_n, x_{n-1}) by a straight line joining two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the curve and take the point of intersection with x -axis as approximate value of the root.

Any line joining two points on the

is called a secant line. That is why this method is known as Secant Method (or) chord method.



Determine an approximate root of the eqn $x^2 - 2x + 1 = 0$ using secant method starting with $x_0 = 2.6$ and $x_1 = 2.5$, rounded-off to 5 decimal places. Compare the result with the exact root.

$$1 + \sqrt{2}$$

Soln : Let $f(x) = x^2 - 2x + 1$

starting with $x_0 = 2.6$ and $x_1 = 2.5$ the successive approximations are

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{2.6 f(2.5) - 2.5 f(2.6)}{f(2.5) - f(2.6)} \\ &= \frac{2.6 (0.18) - (2.5) (0.56)}{0.25 - 0.56} = 2.41935484 \end{aligned}$$

and $f(x_3) = 0.0145682$

(15)

To find the next approximation, we compute

$$\begin{aligned}x_4 &= \frac{x_3 f(x_3) - x_2 f(x_2)}{f(x_3) - f(x_2)} \\&= \frac{(2.5)(0.0145682) - (2.41935484)(0.01)}{(0.0145682) - (0.01)} \\&= 2.41436464\end{aligned}$$

proceeding similarly, we get

$$x_5 = 2.41421384 \text{ and } x_6 = 2.41421356$$

Since x_4 and x_5 rounded off to 5 decimal places are the same, we stop the process.

The required root rounded off to 5 decimal places is 2.41421.

The exact value of the root $1 + \sqrt{2} = 2.414213562373095$, which is rounded off to 5 decimal places. Hence the computed root and exact root are the same when we round off to five decimal places.

Ques. Determine an approximate root of the eqn $\cos x - x^2 = 0$ using second method starting with the two initial approximations as $x_0 = 0.7$ and $x_1 = 0.72$ correct to 4 decimal places.

→ find an approximate root of the eqn $x^2 + x^2 - 3x - 3 = 0$ using
 (a) regula-falsi method correct to 3
 (b) secant method starting with $x_0 = 1.5$ and $x_1 = 1.6$ correct to 3 decimal places.

b) Compare the results obtained by (i) & (ii) in part (a).

Sol: Let $f(x) = x^3 + x^2 - 3x - 3$

Take $x_0 = 1$ and $x_1 = 2$.

$$f(x_0) = -4 < 0 \text{ and } f(x_1) = 3 > 0$$

∴ The root lies between 1 and 2.

By the method of false position,
the 1st approximation is given by

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{①} \\ &= \frac{1(3) - 2(-4)}{3 - (-4)} = \frac{11}{7} = 1.57142 \end{aligned}$$

Now $f(x_2) = -1.36449 < 0$ and $f(x_1)f(x_2) < 0$

∴ The root lies between 1.57142 and 2.

Take $x_0 = 1.57142$ and $x_1 = 2$.

$$f(x_0) = -1.36449 \text{ and } f(x_1) = 3$$

From ①

$$x_3 = \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449}$$

$$x_3 = 1.70540$$

Now repeating this process, the successive approximations are given by
 $x_4 = 1.7188$, $x_5 = 1.73140$, and $x_6 = 1.73194$.
 Since x_5 and x_6 are correct to 3 decimal places are same.

we stop the process here.

Hence the root correct to 3 decimal places is 1.731.

(i) secant method:

Starting with $x_0 = 1, x_1 = 2$ the successive approximations are

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(3) - 2(-4)}{3 - (-4)} = \frac{11}{7} = 1.57142$$

To calculate the next approximation, take $x_1 = 2$ and $x_2 = 1.57142$. We get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449} = 1.70540$$

To find the 3rd approximation

let $x_2 = 1.57142$ and $x_3 = 1.70540$

$$x_4 = \frac{(1.57142)f(1.70540) - (1.70540)f(1.57142)}{f(1.70540) - f(1.57142)}$$

$$= \frac{(1.57142)(-0.29784) - (1.70540)(-1.36449)}{-0.29784 + 1.36449}$$

$$= 1.73578$$

Repeating this process,

we get $x_5 = 1.73199, x_6 = 1.73205$ Since x_5 and x_6 rounded-off to 3 decimal places are the same, we stop here.Hence the root is 1.732, rounded-off 3 decimal places.(ii) $|x_{i+1} - x_i|$ gives the error after the i th iteration.

In Regula-falsi method, the error after 15th iteration is

$$|x_6 - x_5| = |1.73194 - 1.73140| \\ = 0.00011$$

Whereas in Secant method, the error after 5th iteration is

$$|x_6 - x_5| = |1.73205 - 1.73199| \\ = 0.00006$$

This shows that the error in the case of Secant method is smaller than that in Regula-falsi method for the same number of iterations.

Newton-Raphson Method:

- This method is one of the most useful method for finding roots of an algebraic equation.
- Suppose we want to find an approximate root of the eqn $f(x) = 0$.
- If $f(x)$ is continuous, then we can apply either bisection method or regula-falsi method to find approximate roots.
- Now if $f(x)$ and $f'(x)$ are continuous, then we can use a new iteration method called Newton-Raphson method. This method gives the result more faster than bisection or regula-falsi methods.
- The underlying idea of the method is due to mathematician Isaac Newton. But the method as now used is due to the mathematician Raphson.
- Suppose we want to find a root of the equation $f(x) = 0$ where $f(x)$ and $f'(x)$ are continuous.
- Let x_0 be an initial approximation and assume that x_0 is close to the exact root x and $f'(x_0) \neq 0$.
- Let $x = x_0 + h$ where h is a small quantity.
- Hence $f(x) = f(x_0 + h) = 0$

Now, expanding $f(x_0+h)$ by Taylor's theorem, we get

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

Since h is small, neglecting the terms containing h^2 and higher powers, we get

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

This gives a new approximation to α as

$$x_1 = x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now the iteration can be defined by -

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

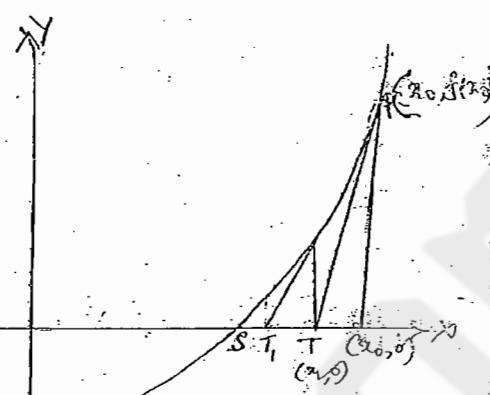
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{--- ①}$$

which is the Newton Raphson formula.

Geometrical Interpretation

Suppose the graph of the function

$y = f(x)$ crosses the x -axis at α . Then $x = \alpha$ is the root of the eqn. $f(x) = 0$.



If x_0 is an initial approximation to the root a , then the corresponding point on the graph is $P(x_0, f(x_0))$. We draw a tangent to the curve at P , it intersects the x -axis at T . Let x_1 be the coordinate of T . Let $S(x_1)$ denote the point on the x -axis where the curve cuts the x -axis, where a is the root of the equation $f(a)=0$.

We take x_1 as the new approximation which may be closer to a than x_0 .

Now let us find the tangent at $P(x_0, f(x_0))$.

The slope of the tangent at P is given by $f'(x_0)$.

By the point-slope form of the expression for a tangent to a curve

$$18 \quad y - f(x_0) = f'(x_0)(x_1 - x_0)$$

The tangent passes through the point

$$T(x_1, 0)$$

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This x_1 is the first iterated value.

To get the second iterated value we again consider a tangent at the point $p(x_1, f(x_1))$ on the curve and repeat the process. Then we get $T_2(x_2, 0)$ on the x-axis.

From the figure, we observe that T_1 is more

closer to $S(1, 0)$ than T . Therefore after each iteration the approximation is coming closer and closer to the actual root.

5. Find a real root of the eqn $x^3 - 4x + 1 = 0$ using Newton-Raphson method, starting with $x_0 = 0$ rounded off to 4 decimal places.

Sol: Let $f(x) = x^3 - 4x + 1$.

$$f'(x) = 3x^2 - 4$$

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

The initial approximation is $x_0 = 0$.

The Newton's iteration formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (1)$$

Putting $n=0$, in (1)

the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{1}{4} = \frac{1}{4} = 0.25$$

Putting $n=1$ in ①,

the second approximation is

$$\begin{aligned}x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\&= 0.25 - \frac{f(0.25)}{f'(0.25)} \\&= 0.25 - \frac{0.015625}{(-3.8125)} \\&= 0.254098\end{aligned}$$

Similarly we get

$$x_3 = 0.25401.$$

Since x_2 and x_3 rounded off to four decimal places are the same, we stop the iteration here.

Hence the root is 0.2541

Ques

Using Newton-Raphson method find the real root of the eqn $x^3 - 6x + 4 = 0$ lying between 0 and 1 correct to 4 decimal places.

Soln: We have $f(x) = x^3 - 6x + 4$.

$$f'(x) = 3x^2 - 6.$$

Clearly $f(x)$ and $f'(x)$ are continuous.

We have $f(0) = 4$ and $f(1) = -1$

$$f(0)f(1) < 0$$

The root lies b/w 0 & 1.

The value of the root is nearer to

Let $x_0 = 0.7$ be the approximation to the root.

$$\text{Now } f(x_0) = f(0.7) = 0.143$$

$$\text{and } f'(x_0) = f'(0.7) = -4.62$$

(19)

Then by Newton's iteration formula,

We get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.7 - \frac{0.143}{(-4.39428)}$$

$$= 0.7316$$

Now $f(x_1) = f(0.7316) = 0.0019805$

and $f'(x_1) = f'(0.7316) = -4.39428$

∴ The second approximation of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7316 + \frac{0.0019805}{-4.39428}$$

$$= 0.73250699.$$

Q) Find the smallest positive root of $2x - \tan x = 0$
by Newton-Raphson method, correct to 5 decimal places. [Ans: 1.16556, let $x_0 = 1$]

By using the Newton-Raphson method,
find an approximate root of $2x - 2 - \sin x = 0$, by starting from $x_0 = 1$,
in the interval $[0, \pi/2]$ with error less than 10^{-5} . [Ans: 1.498701]

→ Find a real root of the eqn $x^2 - x - 1 = 0$
using Newton-Raphson method, correct to four decimal places. [Hint: $f(0) < 0$ & $f(1.3) > 0$] [Ans: 1.3247]

→ find the real root of the $\sin 3x = \cos x$
by using Newton-Raphson method [Ans: 0.60713] (root lies b/w 0 & 1)

→ find the real root of the $\sin x \log x = 1.2$ correct to five decimal places.

$$\text{Ans: } 2.74065 \quad [\text{root lies b/w } 2 \text{ & } 3]$$

→ Apply Newton-Raphson's method to determine a root of the eqn $f(x) = \cos x - x^2 = 0$ such that $|f(x^k)| < 10^{-8}$, where x^k is the approximation to the root.

$$\begin{aligned} \text{Ans: } & 0.51775736 \quad [\text{Let } x^0 = 1] \\ \text{Here } f'(x^k) &= -0.2910 \times 10^{-10} \\ \therefore |f(x^k)| &< 10^{-8}. \end{aligned}$$

→ We shall now consider an application of Newton-Raphson formula.

W.K.T finding the square root of a number is not easy unless we use a calculator. Calculators use some algorithm to obtain this value.

We shall now illustrate how Newton-Raphson method enables us to obtain such an algorithm for calculating square roots.

Ex: Find an approximate value of $\sqrt{2}$ using the Newton-Raphson formula.

Ques: Let $x = \sqrt{2}$.

$$\Rightarrow x^2 - 2 = 0$$

$$\text{let } f(x) = x^2 - 2$$

$$\text{then } f'(x) = 2x$$

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

$f(x)$ satisfies all the conditions for Newton-Raphson method.

Choose $x_0 = 1$ be the initial approximation to the root. $(\because \sqrt{1} < \sqrt{2} < \sqrt{4})$

The iteration formulae [The root is nearest to 1]

$$\text{is } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2} \left[x_n + \frac{2}{x_n} \right] \quad \text{--- (1)}$$

Putting $n = 0, 1, 2, 3, \dots$, we get

$$x_1 = \frac{1}{2} \left[x_0 + \frac{2}{x_0} \right]$$

$$\Rightarrow x_1 = \frac{1}{2} \left[1 + \frac{2}{1} \right]$$
$$= \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2} \left[1.5 + \frac{2}{1.5} \right] = 1.4166667$$

$$x_3 = \frac{1}{2} \left[1.4166667 + \frac{2}{1.4166667} \right]$$
$$= 1.41242157$$

Similarly, we get

$$x_4 = 1.4142136$$

$$x_5 = 1.4142136$$

Hence the value of $\sqrt{2}$ correct to seven decimal places is 1.4142136.

Note The method used in the above example is applicable for finding

Square root of any +ve real number.

For example. we want to find an approximate value of \sqrt{N} where N is a positive real number. Then we consider eqn $x^2 = N$.
The iterated formula is

$$x_{n+1} = \frac{1}{2} \left[x_n + \frac{N}{x_n} \right]$$

[2]. from the above example and examples (198)(2) we find that Newton-Raphson method gives the root very fast.

One reason for this is that the derivative $|f'(x)|$ is large compared to $|f(x)|$ for any $x = x_n$. The quantity $\frac{|f(x)|}{|f'(x)|}$ which is the difference between two iterated values is small in this case.

→ In general we can say that if $|f'(x_n)|$ is large compared to $|f(x_n)|$, then we can obtain the desired root very fast by this method.

→ The Newton-Raphson method has some limitations. Some of the difficulties are as given below:

(i) Suppose $f(x)$ is zero in a neighbourhood of the root, then it may happen that $f'(x_n) = 0$ for some x_n . In this case we cannot apply Newton-Raphson formula, since division by zero is not allowed.

(2) Another difficulty is that it may happen that $f'(x)$ is zero only at the roots.

This happens in either of the situations:

(i) $f(x)$ has multiple root at a i.e., a polynomial function $f(x)$ has a multiple root a of order p , then $f(x)$ can be written as

$$\therefore f(x) = (x-a)^p h(x)$$

where $h(x)$ is a function such that $h(a) \neq 0$.

— for a general function $f(x)$, this means

$$f(a) = f'(a) = f''(a) = \dots = f^{p-1}(a) = 0$$

and $f^p(a) \neq 0$.

(ii) $f(x)$ has a stationary point (point of maximum or minimum) at the root.
i.e., $f'(x) = 0$ at some point $x = x_0$.

H.W. → Using Newton-Raphson method find the

(i) square root of 8.

Ans : 2.828427

(ii) cube of $\sqrt[3]{28}$

Ans : 5.2915

→ Using Newton-Raphson method prove that

(i) Iterative formula for $\frac{1}{N}$ is $x_{n+1} = x_n(2-Nx_n)$

(ii) Iterative formula for $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{Nx_n})$

(iii) Iterative formula for $\sqrt[K]{N}$ is $x_{n+1} = \frac{1}{K}[(K-1)x_n + \frac{N}{x_n^{K-1}}]$

Sol: (i) Let $\frac{1}{N} = \frac{x}{N} \Rightarrow N = \frac{1}{x}$

$$\Rightarrow \frac{1}{x} - N = 0$$

$$\text{Let } f(x) = \frac{1}{x} - N.$$

then $f(x) = \frac{1}{x^2} = \bar{x}^2$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\frac{1}{x_n^2} - N\right)}{\left(-\frac{2}{x_n^3}\right)}$$

$$= x_n + \left(\frac{1}{2x_n} - N\right)x_n^2$$

$$= x_n + x_n - Nx_n^2$$

$$\boxed{x_{n+1} = x_n(2 - Nx_n)}$$

which is the required result.

(ii)

Let $x = \frac{1}{\sqrt{n}}$

$$\Rightarrow x^2 = \frac{1}{n}$$

$$\Rightarrow x^2 - \frac{1}{n} = 0$$

Let $f(x) = x^2 - \frac{1}{n}$

then $f'(x) = 2x$.

By Newton-Raphson iteration formula,
if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(x_n^2 - \frac{1}{n}\right)$$

$$= \frac{(2x_n - x_n^2 + \frac{1}{n})}{2x_n}$$

$$= \frac{x_n^2 + \frac{1}{n}}{2x_n} = \frac{1}{2} \left[x_n + \frac{1}{nx_n} \right]$$

(iii)

Let $x = \sqrt[n]{N} \Rightarrow x^k = N$

$$\Rightarrow x^k - N = 0$$

Let $f(x) = x^k - N$.

$$\text{Then } f'(x) = kx^{k-1}$$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^k - N}{kx_n^{k-1}} \\ &= \frac{kx_n^k - x_n^k + N}{kx_n^{k-1}} = \frac{1}{k} \left[(k-1)x_n^k + \frac{N}{x_n^{k-1}} \right] \end{aligned}$$

→ Evaluate the following (correct to four decimal places) by Newton-Raphson method.

(i) $\sqrt[3]{31}$ (ii) $\sqrt[4]{14}$ (iii) $\sqrt[3]{24}$ (iv) $(30)^{-\frac{1}{5}}$ [Hint: put $k=-5$ in formula (iii)]

[Ans: 3.1423]

[Ans: 3.7413]

[Ans: 2.8848]

[Ans: 0.5065]

Using Newton-Raphson's method, show that the iteration formula for finding the reciprocal of the p^{th} root of N is

$$x_{n+1} = x_n \cdot \frac{(p+1 - Nx_n^p)}{p}$$

Sol:

$$\begin{aligned} x &= \frac{1}{\sqrt[p]{N}} \Rightarrow x^p = \frac{1}{N} \Rightarrow x = N^{-\frac{1}{p}} \\ &\Rightarrow x^{-p} = N \\ &\Rightarrow x^{-p} - N = 0. \end{aligned}$$

$$\text{Let } f(x) = x^{-p} - N$$

$$f'(x) = -p x^{-p-1}$$

By Newton-Raphson iteration formula,

if x_n denotes the n^{th} -iterate

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{-p} - N}{-p x_n^{-p-1}} \\ &= \frac{p x_n^{-p} + (N - x_n^{-p}) x_n^{-p}}{p} \\ &= \frac{p x_n^{-p} + (N - x_n^{-p}) x_n^{-p}}{p} \\ &= \frac{p x_n^{-p} + (N - x_n^{-p}) x_n^{-p}}{p} \\ &= \frac{p x_n^{-p} + (N - x_n^{-p}) x_n^{-p}}{p} \\ &= \frac{p x_n^{-p} + (N - x_n^{-p}) x_n^{-p}}{p} \end{aligned}$$

which is the required formula

Convergence criterion

(23)

We shall now introduce a new concept called convergence criterion related to an iteration process. This criterion gives us an idea of ~~how many successive~~ how many successive iterations have to be carried out to obtain the desired accuracy.

Definition: Let $x_0, x_1, \dots, x_{n-1}, \dots$ be the successive approximations of an iteration process, we denote the sequence of these approximations as $\{x_n\}_{n=0}^{\infty}$. We say that $\{x_n\}_{n=0}^{\infty}$ converges to a root α with order $p \geq 1$ if

$$|x_{n+1} - \alpha| \leq \lambda |x_n - \alpha|^p \quad \textcircled{1}$$

for some number $\lambda > 0$. p is called the order of convergence and λ is called the asymptotic error constant.

For each n , we denote by e_n $|x_n - \alpha|$. Then the eqn (1) be written as

$$|e_{n+1}| \leq \lambda |e_n|^p \quad \textcircled{2}$$

This inequality shows the relationship between the error in successive approximations.

For example:

- Suppose $p = 2$ and $|e_n| \approx 10^{-2}$ for $n \geq 10$, we can expect that $|e_{n+1}| \approx 10^{-4}$.

Thus if p is large, the iteration is rapid.

When p takes the values $1, 2, 3$ then we say that the convergence is linear, quadratic and cubic respectively.

By the case of linear convergence ($p=1$), then we receive that $\lambda < 1$.

∴ Eqn(1) becomes

$$|x_{n+1} - \alpha| \leq \lambda |x_n - \alpha| \text{ for all } n \geq 0. \quad (3)$$

If this condition is satisfied for an iteration process then we say that the iteration process converges linearly.

Setting $n=0$ in the inequality (3), we get

$$|x_1 - \alpha| \leq \lambda |x_0 - \alpha|.$$

for $n=1$, we get

$$\begin{aligned} |x_2 - \alpha| &\leq \lambda |x_1 - \alpha| \\ &\leq \lambda^2 |x_0 - \alpha| \end{aligned}$$

for $n=2$

$$\begin{aligned} |x_3 - \alpha| &\leq \lambda |x_2 - \alpha| \\ &\leq \lambda^3 |x_1 - \alpha| \\ &\leq \lambda^3 |x_0 - \alpha|. \end{aligned}$$

Using induction on n , we get

$$|x_n - \alpha| \leq \lambda^n |x_0 - \alpha| \quad (4)$$

If either of the inequalities (3) or (4) is

satisfied, then we conclude that $\{x_n\}_{n=0}^\infty$

converges to the root.

Convergence of bisection method:

Suppose that we apply the bisection method on the interval $[a_0, b_0]$ for the eqn $f(x) = 0$. In this method we construct intervals $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ each of which contains the required root of the given eqn. In each step the interval width is reduced by $\frac{1}{2}$.

$$\text{i.e. } b_1 - a_1 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

$$b_n - a_n = \frac{b_0 - a_0}{2^n} \quad (3)$$

Clearly the eqn $f(x) = 0$ has a root in $[a_0, b_0]$. Let a be the root of the eqn. Then a lies in all the intervals $[a_i, b_i]$, $i = 0, 1, 2, \dots$ for any n . Let $c_n = \frac{a_n + b_n}{2}$ denote the middle point of the interval $[a_n, b_n]$. Then c_0, c_1, c_2, \dots are taken as successive approximations to the root a .

Let us check the inequality (3) for $\{c_n\}$.

For each n , a lies in the interval $[a_n, b_n]$ we have

$$|c_n - a| \leq \frac{|b_n - a_n|}{2}$$

Thus $\{c_n\}_{n=0}^{\infty}$ converges to the root a . Hence

Say that the bisection method always e.g.
for practical purposes, we should be able to decide at what stage we can stop the iteration to have an acceptably good approximate value of a . The number of iterations required to achieve a given accuracy for the bisection method can be obtained.

- Suppose that we want an approximate solution with an error bound of 10^{-M} .

Taking Logarithms on both sides of eqn ① we find the number of iterations required,

say n , approximately given by

$$n = \text{int} \left[\frac{\log(b_0 - a_0) - \log 10^{-M}}{\log 2} \right]$$

$$\frac{b_0 - a_0}{2^n} \leq 10^{-M}$$

(6) $\log \frac{b_0 - a_0}{2^n} \leq \log 10^{-M}$

where the symbol 'int' stands for the integral part of the number in the bracket and $[a_0, b_0]$ is the initial interval in which a root lies.

Since $b_0 - a_0$ is large, we take n as the nearest integer.

Ex: Suppose that the bisection method is used to find a zero of $f(x)$ in the interval $[0, 1]$. How many times this interval be bisected to guarantee that we have an approximate root with absolute error less than or equal to 10^{-5} ?

Sol: Let n denote the required number.

Given $a_0 = 0, b_0 = 1$ and $M = 5$.

From eqn ⑥

$$n = \text{int} \left[\frac{\log(b_0 - a_0) - \log 10^{-M}}{\log 2} \right]$$

$$\begin{aligned} n &= \text{int} \left[\frac{\log 1 - \log 10^{-5}}{\log 2} \right] \\ &= \text{int} \left[\frac{11.51292542}{0.693146778} \right] \\ &= \text{int} [16.60964047] \\ n &= 17. \text{(approximately).} \end{aligned}$$

25

- The following table gives the minimum no. of iterations required to find an approximate root in the interval $[0, 1]$ for various acceptable errors.

E	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24

This table shows that for getting an approximate value with an absolute error bounded by 10^{-5} , we have to perform 17 iterations.

→ Thus even though the bisection method is simple to use, it requires a large no. of iterations to obtain a reasonably good approximate root. This is one of the disadvantages of the bisection method.

Convergence criteria for Secant Method:

Let $f(x)=0$ be the given eqn. Let a denote a simple root of the eqn $f(x)=0$. Then we have

$$f'(a) \neq 0.$$

The iteration formula for the Secant method is

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} f(x_i) \quad (i)$$

for each i , set $\epsilon_i = x_i - a$.

$$\Rightarrow x_i = \epsilon_i + a.$$

Substituting $x_i = \epsilon_i + a$ in eqn (i)

$$x_{i+1} = \epsilon_i + a - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i + a) - f(\epsilon_{i-1} + a)} f(\epsilon_i + a) \quad (ii)$$

$$\epsilon_{i+1} = \epsilon_i - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i + a) - f(\epsilon_{i-1} + a)} f(\epsilon_i + a) \quad (iii)$$

Now expanding $f(\epsilon_i + a)$ and $f(\epsilon_{i-1} + a)$ using Taylor's theorem about the point a

we get

$$\begin{aligned} f(\epsilon_i + a) &= f(a) + \frac{f'(a)}{1!} \epsilon_i + \frac{f''(a)}{2!} \epsilon_i^2 + \dots \\ &= f'(a) \epsilon_i + \frac{f''(a)}{2!} \epsilon_i^2 + \dots \quad (iv) \\ &= f'(a) \left[\epsilon_i + \frac{f''(a)}{2f'(a)} \epsilon_i^2 + \dots \right] \quad (v) \end{aligned}$$

Similarly

$$f(\epsilon_{i-1} + a) = f(a) \left[\epsilon_{i-1} + \frac{f''(a)}{2f'(a)} \epsilon_{i-1}^2 + \dots \right] \quad (vi)$$

$$\begin{aligned} f(\epsilon_i + a) - f(\epsilon_{i-1} + a) &= f'(a) \left[(\epsilon_i - \epsilon_{i-1}) + (\epsilon_i - \epsilon_{i-1}) \frac{f''(a)}{2f'(a)} \right] \\ &= f'(a)(\epsilon_i - \epsilon_{i-1}) \left[1 + (\epsilon_i - \epsilon_{i-1}) \frac{f''(a)}{2f'(a)} \right] \quad (vii) \end{aligned}$$

Substituting eqns (iii) & (v) in eqn (iv), we get

$$\begin{aligned} \epsilon_{i+1} &= \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1})}{f'(x)(\epsilon_i - \epsilon_{i-1}) \left[1 + (\epsilon_i + \epsilon_{i-1}) \frac{f''(x)}{2f'(x)} + \dots \right]} \\ &= \epsilon_i - \left[\epsilon_i + \frac{f''(x)}{2f'(x)} \epsilon_i^2 + \dots \right] \left[1 + \frac{1}{2} (\epsilon_i + \epsilon_{i-1}) \frac{f''(x)}{f'(x)} + \dots \right] \\ &= \epsilon_i - \left[\epsilon_i + \frac{1}{2} \epsilon_i^2 \frac{f''(x)}{f'(x)} + \dots \right] \left[1 - \frac{1}{2} (\epsilon_i + \epsilon_{i-1}) \frac{f''(x)}{f'(x)} + \dots \right] \\ &= \epsilon_i - \left[\epsilon_i + \frac{1}{2} \frac{f''(x)}{f'(x)} (\epsilon_i^2 - \epsilon_i^2 - \epsilon_i \epsilon_{i-1}) + \dots \right] \end{aligned}$$

By neglecting the terms involving $\epsilon_i \epsilon_{i-1}$ in the above expression, we get

$$\epsilon_{i+1} \approx \epsilon_i \left[\frac{f''(x)}{2f'(x)} \right] \quad \text{--- (vi)}$$

This relationship between the errors is called error eqn. This relationship holds only if a is a simple root.

Now using eqn (vi) we will find the numbers p and λ such that

$$\epsilon_{i+1} = \lambda \epsilon_i^p ; \quad i = 0, 1, 2, \dots \quad \text{--- (vii)}$$

Setting $i = j-1$, we obtain

$$\epsilon_j = \lambda \epsilon_{j-1}^p$$

$$(v) \quad \epsilon_j = \lambda \epsilon_{j-1}^p \quad \text{--- (viii)}$$

Taking p^{th} root on both sides of (viii), we get

$$\epsilon_j^{1/p} = \lambda^{1/p} \epsilon_{j-1}^{1/p}$$

$$\Rightarrow \epsilon_{j-1} = \lambda^{-1/p} \epsilon_j^{1/p} \quad \text{--- (ix)}$$

from eqns (vi) & (vii); we have

$$\lambda e_i^p = e_i e_{i-1} \frac{f''(x)}{2f'(x)}$$

$$\Rightarrow \lambda e_i^p = \frac{f''(x)}{2f'(x)} e_i e_{i-1} \lambda e_i^{p-1} \quad (\text{by eqn(vi)})$$

$$\Rightarrow \lambda e_i^p = \frac{f''(x)}{2f'(x)} e_i e_{i-1}^{p-1} \quad \dots \quad (viii)$$

equating the powers of e_i on both sides of eqn(vii)

we get

$$p = 1 + \frac{1}{\rho}$$

$$\Rightarrow p - \rho - 1 = 0$$

which gives $\rho = \frac{1+\sqrt{5}}{2}$. ($\because \rho$ cannot be $-ve$).
neglecting the minus sign

$$\rho = \frac{1+\sqrt{5}}{2} = 1.618$$

Now, to get the number λ , we equate the constant terms on both sides of eqn(vii),

$$\text{we get } \lambda = \frac{f''(x)}{2f'(x)} e_i^{p-1}$$

$$\Rightarrow e_i^{1+p} = \frac{f''(x)}{2f'(x)}$$

$$\Rightarrow \boxed{\lambda = \left[\frac{f''(x)}{2f'(x)} \right]^{p/p+1}}$$

Hence the order of convergence of the Secant method is $p = 1.62$ and the asymptotic error constant is $\left[\frac{f''(x)}{2f'(x)} \right]^{p+1}$

Ex: The following are the five successive iterations obtained by Secant method to find the root $x=2$ of the eqn $x^2 - 3x + 2 = 0$.

$$x_2 = -2.6, x_3 = -2.4, x_3 = -2.106598985$$

$$x_4 = -2.022641412 \text{ and } x_5 = -2.000022537$$

Compute the asymptotic error constant
and show that $\epsilon_5 = \lambda G_4$.

Sol: Let $f(x) = x^3 - 3x + 2$

$$f(x) = 3x^2 - 3 \quad \text{and} \quad f'(x) = 6x$$

$$\therefore f(-2) = 9 \quad \text{and} \quad f''(-2) = -12$$

$$\text{we have } \lambda = \left[\frac{f''(x)}{2f'(x)} \right]^{1/18}$$

$$\lambda = \left[\frac{-12}{18} \right]^{\frac{1}{1+1.62}} = \left(\frac{-2}{3} \right)^{\frac{1.62}{2.62}} = \left(\frac{-2}{3} \right)^{0.618}$$

$$\lambda = -0.778351205$$

$$\text{Now } \epsilon_5 = |x_5 - \alpha|$$

$$= |-2.000022537 + 2|$$

$$= 0.000022537$$

$$\text{and } G_4 = |x_4 - \alpha|$$

$$= |-2.022641412 + 2|$$

$$= 0.022641412$$

$$\text{Then } \lambda G_4 = 0.778351205 \times 0.022641412$$

$$= 0.00021246$$

$$\approx 0.00002253$$

$$\approx \epsilon_5$$

$$\lambda G_4 \approx \epsilon_5$$

Convergence of Newton-Raphson Method.

Newton-Raphson iteration formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow 0$$

To obtain the order of convergence, assume that α is a simple root of $f(x)=0$.

- Let $x_i = \alpha + e_i$, $i=0, 1, 2, \dots$

Also $x_{i+1} - \alpha = \epsilon_{i+1}$

∴ from ①

$$\epsilon_{i+1} = e_i + \frac{f(\alpha + e_i)}{f'(\alpha + e_i)}$$

$$\Rightarrow \epsilon_{i+1} = e_i - \frac{f(\alpha + e_i)}{f'(\alpha + e_i)} = \frac{e_i f'(\alpha + e_i) - f(\alpha + e_i)}{f'(\alpha + e_i)}$$

Now expanding $f(\alpha + e_i)$ and $f'(\alpha + e_i)$, using Taylor's theorem, about the point α ,

we obtain

$$\epsilon_{i+1} = \frac{e_i}{2} \left[f''(\alpha) + e_i f'''(\alpha) + \frac{e_i^2 f''(\alpha)}{2} + \dots \right] - \left[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2 f''(\alpha)}{2} + \dots \right]$$

$$f(\alpha) + e_i f'(\alpha) + e_i^2 f''(\alpha) + \dots$$

But $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

$$\therefore \epsilon_{i+1} = \frac{\frac{e_i^2 f''(\alpha)}{2} + \dots}{f'(\alpha) \left[1 + e_i \frac{f'''(\alpha)}{f'(\alpha)} + \dots \right]}$$

$$= \frac{\left[\frac{e_i^2 f''(\alpha)}{2} + \dots \right]}{f'(\alpha)} \cdot \frac{1}{\left[1 + e_i \frac{f'''(\alpha)}{f'(\alpha)} + \dots \right]}$$

$$= \frac{\left[\frac{e_i^2 f''(\alpha)}{2} + \dots \right]}{f(\alpha)} \left[1 - e_i \frac{f'''(\alpha)}{f'(\alpha)} + \dots \right]$$

On neglecting e_i^3 and higher power of e_i ,

(28)

$$\text{we get } e_{i+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_i^2$$

This shows that the errors satisfy ~~the~~ -

$$\text{the inequality } |e_{i+1}| \leq \lambda |e_i|^2 \text{ with } \lambda = \frac{|f''(\alpha)|}{2|f'(\alpha)|}$$

Hence Newton Raphson method is of order 2
i.e., the Newton Raphson method has second
order convergence.

and the error is proportional to the
square of the previous error in each
step.

Note: If α is a multiple root i.e., $f''(\alpha)=0$, then
the convergence is not quadratic, but only linear.

For example:

Let $f(x) = (x-2)^4 = 0$. Starting with the initial
approximation $x_0 = 2.1$, compute the iterations
 x_1, x_2, x_3 and x_4 using Newton-Raphson method.
Is the sequence converging quadratically or
linearly?

Sol: Let $f(x) = (x-2)^4$.
The given function has multiple
roots at $x=2$ of order 4.
Newton Raphson iteration formula for
the given equation is

$$\begin{aligned} x_{i+1} &= x_i - \frac{(x_i-2)^4}{4(x_i-2)^3} \\ &= x_i - \frac{1}{4}(x_i-2) = \frac{1}{4}(3x_i + 2) \end{aligned}$$

Starting with $x_0 = 2.1$, the iterations are given by

$$x_1 = \frac{1}{4}(6.8 + 2) = \frac{8.8}{4} = 2.075$$

$$\text{Similarly, } x_2 = 2.05625$$

$$x_3 = 2.0421875$$

$$x_4 = 2.031640625$$

$$\text{Now } G_0 = x_0 - 2 = 0.1$$

$$G_1 = x_1 - 2 = 0.075$$

$$G_2 = x_2 - 2 = 0.05625$$

$$G_3 = x_3 - 2 = 0.0421875$$

$$G_4 = x_4 - 2 = 0.031640625$$

$$\text{we have } G_1 = 0.075 =$$

$$= \frac{3}{4} \times 0.1$$

$$= \frac{3}{4} G_0$$

$$\therefore G_1 = \frac{3}{4} G_0$$

$$\text{and } G_2 = \frac{3}{4} G_1$$

$$G_3 = \frac{3}{4} G_2$$

$$G_4 = \frac{3}{4} G_3$$

i.e., the convergence is linear in this case.

Also, the error is reduced by a factor of $\frac{3}{4}$ with each iteration.

Hence, the quadratic equation $x^2 - 4x - 20 = 0$ has a double root at $x = \sqrt{20}$. Starting with $x_0 = 1.5$, compute three successive iterations to the root by Newton-Raphson method. Does the result converge quadratically or linearly?

Set-III

Solution of system of linear equations

Introduction:

System of linear equations arise in a large number of areas, both directly in modelling physical situations and indirectly in the numerical solution of other mathematical models.

These applications occur in all areas of the physical, biological, and engineering sciences.

For instance, in physics, the problem of steady state temperature in a plate is reduced to solving linear equations.

Linear algebraic systems are also involved in the optimization theory, least squares fitting of data, numerical solution of boundary value problems for ordinary and partial differential eqns, statistical inference etc. In the numerical solution of systems of linear algebraic eqns play a very important

Numerical methods for solving linear algebraic systems may be divided into two types: direct and iterative.

Direct methods are those which, in the absence of round-off or other errors, yield the exact solution in a finite number of elementary operations.

Iterative methods start with an initial approximation and by applying a suitably chosen lead to successively better approx.

The general form of a system of m linear eqns in n unknowns x_1, x_2, \dots, x_n can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{--- (1)}$$

$$\Rightarrow Ax = B \quad \text{--- (2)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- The solution of the system of eqns (2) gives n unknown values x_1, x_2, \dots, x_n which satisfy the system simultaneously.
- A system of eqns (2) is said to be consistent if it has at least one solution. If no solution exists, then the system is said to be inconsistent.
- The system of eqns (2) is said to be homogeneous if $b=0$, that is all the elements b_1, b_2, \dots, b_m are zero, otherwise the system is called non-homogeneous.
- In this lesson, we consider only non-homogeneous, and we restrict $m=n$ (i.e., the number of eqns = the no. of unknowns).
- A non-homogeneous system of n linear eqns in n unknowns has a unique solution iff the coefficient matrix A is non-singular. (i.e., $|A| \neq 0$)

If A is non-singular, then A^{-1} exists and the solution of System (2) can be expressed as $\bar{x} = A^{-1}b$. 2

In case the matrix A is singular, then the system (2) has no solution if $b \neq 0$ or has an infinite number of solutions if $b=0$. Here we assume that, A is non-singular matrix.

The methods of solution of the system (2) may be classified into two types:

(i) Direct Methods: which in the absence of round-off errors give the exact solution in a finite number of steps.

(ii) Iterative Methods: Starting with an approximate solution vector $x^{(0)}$, these methods generate a sequence of approximate solution vectors $\{x^{(k)}\}$ which converge to the exact solution vector x at the number of iterations $k \rightarrow \infty$.

Thus iterative methods are infinite processes. Since we perform only a finite number of iterations, these methods can only find some approximation to the solution vector x .

Direct Methods for Special Matrices:

We now discuss three special forms of matrix A in eqn (2) for which the solution vector x can be obtained directly.

Case (i): $A = D$, where D is a diagonal matrix.

In this case, the system of eqn (2) are in the form

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ \vdots & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \textcircled{3}$$

and $|A| = \det(A) = a_{11}a_{22}a_{33}\dots a_{nn}$

Since the matrix A is non-singular, $a_{ii} \neq 0$ for $i=1, 2, 3, \dots, n$ and we obtain the solution as $x_i = \frac{b_i}{a_{ii}}$, $i=1, 2, 3, \dots, n$.

Case ii: $A=L$, where L is a lower triangular matrix ($a_{ij}=0$, $j>i$). The system of eqns (2) is now of the form

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ \vdots & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \textcircled{4}$$

and $|A| = a_{11}a_{22}\dots a_{nn}$

Since the coefficient matrix A is non-singular, $a_{ii} \neq 0$, $i=1, 2, \dots, n$.

Solving the first eqn and then successively solving second, third and so on, we obtain

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}}$$

$$x_3 = \frac{(b_3 - a_{31}x_1 - a_{32}x_2)}{a_{33}}$$

$$x_n = \frac{(b_n - \sum_{j=1}^{n-1} a_{nj}x_j)}{a_{nn}}$$

In general, we have for any i , $x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j}{a_{ii}}$ (3)
 $i = 1, 2, \dots, n$.

Since the unknowns in this method are solved by forward substitution, this method is called the forward substitution method.

Case (ii): $A = U$, where U is an upper triangular matrix ($a_{ij} = 0, j > i$). The system (2) is now of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{nn}x_n = b_n$$

$$\bar{a}_{nn}x_n = \bar{b}_n$$

$$\text{and } |A| = a_{11}a_{22} \dots a_{nn}$$

Since the coefficient matrix A is non-singular

$$a_{ii} \neq 0, i = 1, 2, \dots, n$$

Solving unknowns in the order x_n, x_{n-1}, \dots, x_1 ,

we get

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{(b_{n-1} - a_{(n-1)n}x_n)}{a_{(n-1)(n-1)}}$$

$$x_1 = \left(b_1 - \sum_{j=2}^n a_{1j}x_j \right) / a_{11}$$

In general we have for any i , $x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij}x_j}{a_{ii}}$

The unknowns are solved by back substitution and this method is called the back substitution method. Thus, the equations (2) are exactly solvable, i.e., (2) can be transformed into any

Direct Method :-

Gaussian Elimination Method :-

In the Gaussian elimination method, the solution to the system of eqns (2) is obtained in two stages.

In the first stage, the given system of eqns is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n, x_{n-1}, x_{n-2}, \dots, x_2, x_1$.

This method is explained by considering a system of 'n' eqns in 'n' unknowns in the form as follows.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \left. \right\} \quad (6)$$

Stage I: we divide the first eqn by a_{11} and then

Subtract this eqn multiplied by $a_{21}, a_{31}, \dots, a_{n1}$ from the 2nd, 3rd, ..., n^{th} eqn. Then

the system (6) reduces to the following form:

$$\begin{aligned} x_1 + a_{12}'x_2 + \dots + a_{1n}'x_n &= b_1' \\ a_{22}'x_2 + \dots + a_{2n}'x_n &= b_2' \\ \vdots & \quad \vdots \\ a_{n2}'x_2 + \dots + a_{nn}'x_n &= b_n' \end{aligned} \quad \left. \right\} \quad (7)$$

Here, we can observe that the last $(n-1)$ eqns are independent of x_1 , i.e. x_1 is eliminated from the last $(n-1)$ eqns.

This procedure is repeated with the second eqn of (F) i.e., we divide the second eqn by a_{22}' and then x_2 is eliminated from 3rd, 4th, ..., nth eqns of (F). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$\left. \begin{array}{l} c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = d_1 \\ c_{22}x_2 + \dots + c_{2n}x_n = d_2 \\ \vdots \\ c_{nn}x_n = d_n \end{array} \right\} \quad (8)$$

Stage 2:

Now, the values of the unknowns are determined by back substitution procedure, in which we obtain x_n from the last eqn of (8) and then substituting this value of x_n in the preceding eqn, we get the value of x_{n-1} . Continuing this way, we can find values of all other unknowns in the order x_n, x_{n-1}, \dots, x_1 .

In this method, we observe that the determinant of the coefficient matrix is obtained as a by-product, i.e.,

$$|A| = c_{11}c_{22} \dots c_{nn}$$

Example: Solve the following system of eqns using Gaussian elimination method.

$$\left. \begin{array}{l} 2x + 3y - z = 5 \\ 4x + 4y - 3z = 3 \\ -2x + 3y - z = 1 \end{array} \right\} \quad \text{(i)}$$

Sol: The given system of eqns (i) is solved in two stages.

Stage I (Reduction to upper-triangular form)

We divide the first eqn by 2 and then subtract the resulting eqn (multiplied by 4 and -2) from the second eqn. and third eqn respectively. Thus, we eliminate x from the 2nd and 3rd eqns.

The resulting new system is given by

$$\left. \begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ -2y - z = -7 \\ 6y - 2z = 6 \end{array} \right\} \quad \text{(ii)}$$

Now, we divide the second eqn of (ii) by -2 and eliminate y from the last eqn and the modified system is given by

$$\left. \begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ y + \frac{z}{2} = \frac{7}{2} \\ -5z = -15 \end{array} \right\} \quad \text{(iii)}$$

Stage II (Back substitution):

From the last eqn of (iii)

we get $\boxed{z = 3}$

Using this value of z, the second eqn of (iii) gives,

$$y = \frac{7}{2} - \frac{3}{2} = 2$$

$\Rightarrow [y = 2]$

Using these values of y and z in the first eqn of (ii), we get -

$$x = 1.$$

Thus, the solution of the given system is $x = 1, y = 2, z = 3.$

Note: We can write the above procedure more conveniently in matrix form. Since the arithmetic operations we have performed here affect only the elements of the matrix A and the matrix B, we consider the augmented matrix i.e., $[A|B]$ and perform the elementary row operations on the augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{22} & a_{23} & b_2 \\ a_{32} & a_{33} & b_3 \end{array} \right] \quad R_2 \rightarrow R_2 - \frac{a_{21}}{a_{11}} R_1$$

$$R_3 \rightarrow R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{22} & a_{23} & b_2 \\ a_{32} & a_{33} & b_3 \end{array} \right] \quad R_3 \rightarrow R_3 - \frac{a_{32}}{a_{22}} R_2$$

i.e., $[A|B] \xrightarrow{\text{Gaussian elimination}} [U|C]$

which is in desired form.

$$\text{where } a'_{22}, a'_{23}, a'_{32}, b'_3, b'_2, b'_1, a''_{33}, b''_3$$

are given by eqns (7) & (8).

- The diagonal elements a_{11} , a_{22} and a_{33} which have been assumed to be non-zero are called pivot elements.
- we observe that for a linear system of order 3, the elimination was performed in $3-1 = 2$ stages. In general for a system of 'n' eqns given by eqns (2) the elimination is performed in $(n-1)$ stages.

At the i^{th} stage of elimination, we eliminate x_i starting from $(i+1)^{\text{th}}$ row upto the n^{th} row. Sometimes, it may happen that the elimination process stops in less than $(n-1)$ stages.

But this is possible only when no eqns containing the unknowns are left or when the coefficients of all the unknowns in remaining eqns become zero. Thus if the process stops at the r^{th} stage of elimination then we get a derived system of the form,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \dots + a_{2n}x_n = b_2$$

$$\begin{matrix} & & (r-1) \\ a_{rr}x_r + \dots + a_{rn}x_n & = b_r \\ 0 & = b_{r+1}^{(r-1)} \end{matrix}$$

$$0 = b_n^{(r-1)}$$

Where $r \leq n$ and $a_{11} \neq 0$, $a_{22} \neq 0$, ..., $a_{rr} \neq 0$.

In the solution of system of linear eqns we can expect two different situations.
(i) $r=n$ (ii) $r < n$

Ex(1) Solve the following system of eqns by using Gaussian elimination method. (6)

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 4 \\ x_1 + 4x_2 - 2x_3 &= 4 \\ -x_1 + 2x_2 + 4x_3 &= 2 \end{aligned}$$

Soln: we have

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 1 & 4 & -2 & 4 \\ -1 & 2 & -4 & 2 \end{array} \right] \xrightarrow{\text{R}_2 \rightarrow R_2 - 4R_1} \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15 & -9 & 0 \\ 0 & 9 & 15 & 3 \end{array} \right] \xrightarrow{\text{R}_3 \rightarrow R_3 + \frac{R_2}{3}} \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15 & -9 & 0 \\ 0 & 0 & 16 & 3 \end{array} \right] \xrightarrow{\text{R}_3 \rightarrow R_3 - \frac{3R_2}{16}} \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15 & -9 & 0 \\ 0 & 0 & \frac{16}{5} & \frac{3}{5} \end{array} \right]$$

Using back substitution method, we get

$$x_3 = \frac{3}{5}, x_2 = \frac{9}{5}, x_1 = 1$$

$$\text{Also } |A| = -36.$$

Thus in this case we observe that $r=n=3$ and the given system of eqn has a unique solution. Also the coefficient matrix is non-singular.

Ex(2): Solve the system of eqns

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

Using Gauss elimination method.

Does the solution exist?

Soln: we have

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3}x_3 & x_3 & -2 \\ 0 & -2 & 2 & 0 \end{array} \right] \quad R_2 \rightarrow R_2 - \frac{2}{3}R_1 \\ R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3}x_3 & x_3 & -2 \\ 0 & 0 & 0 & 12 \end{array} \right] \quad R_3 \rightarrow R_3 - 6R_2$$

In this Case $r < n$ and elements b_1, b_2 and b_3' are all non-zero. Since we cannot determine x_3 from the last eqn, the system has no solution.

In such situation we say that the eqns are inconsistent. Also $|A|=0$.

i.e., the coefficient matrix is singular.

Ex(3):

Solve the system of eqns -

$$16x_1 + 22x_2 + 4x_3 = -2$$

$$4x_1 - 3x_2 + 2x_3 = 9$$

$$12x_1 + 25x_2 + 2x_3 = -11$$

Using Gauss elimination method.

Soln: we have

$$\left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 4 & -3 & 2 & 9 \\ 12 & 25 & 2 & -11 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & 19/2 \\ 0 & \frac{17}{2} & -1 & -19/2 \end{array} \right] \quad R_2 \rightarrow R_2 - \frac{1}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & \frac{19}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

NOW in this Case $r < n$ and elements b_1, b_2 are non-zero, but $b_3^{(2)}$ is zero.

Also the last eqn is satisfied for any value of x_3 .

Thus we get $x_3 = \text{any value}$

$$x_2 = -\frac{2}{17} \left(\frac{19}{2} - x_3 \right)$$

$$x_1 = \frac{1}{16} (-2 - 22x_2 - 4x_3).$$

Hence the system of eqns has infinitely many solutions.

Also $|A| = 0$.

→ We now summarise these conclusions as follows:

(i) If $r=n$ then the system of eqns (2) has unique solution which can be obtained by using the back substitution method. Moreover the coefficient matrix A in this case is non-singular.

(ii) If $r < n$ and all the elements $b_{r+1}^{(r)}, b_{r+2}^{(r)}, \dots, b_n^{(r)}$ are not zero then the system has no solution.

In this case we say that the system of eqns is inconsistent.

(iii) If $r < n$ and all the elements $b_{r+1}^{(r)}, b_{r+2}^{(r)}, \dots, b_n^{(r)}$, if present, are zero, then the system has infinite number of solutions.

In this case the system has only r linearly independent rows.

In both the cases (ii) and (iii), the matrix A is singular.

→ Use the Gaussian elimination method solve the following system of eqns.

$$(1) \quad x_1 + 2x_2 + x_3 = 3$$

$$3x_1 - 2x_2 - 4x_3 = -2$$

$$2x_1 + 3x_2 - x_3 = -6$$

$$\boxed{\text{Ans: } x_3 = 5, x_2 = -3}$$

$$\begin{array}{l} \textcircled{2} \quad 3x_1 + 18x_2 + 9x_3 = 18 \\ \quad 2x_1 + 3x_2 + 3x_3 = 117 \\ \quad 4x_1 + 9x_2 + 2x_3 = 283 \end{array}$$

Ans: $x_3 = 4, x_2 = -13, x_1 = 72$

$$\textcircled{3} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 5 \\ 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4 & 6 \\ 1 & 0 & 1 & 1 \end{array} \right]$$

Ans: $x_4 = 1, x_3 = 2, x_2 = 1, x_1 = 0$

$$\textcircled{4} \quad \left[\begin{array}{cccc|c} 3 & 2 & -1 & -4 & 21 \\ 1 & 1 & 3 & -1 & 10 \\ 2 & 1 & -3 & 0 & -4 \\ 0 & -1 & 8 & -5 & 16 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -4 & 21 \\ 0 & 1 & 3 & -1 & 10 \\ 0 & 1 & 7 & 5 & -3 \\ 0 & 0 & 7 & -5 & 16 \end{array} \right]$$

Ans: Inconsistent.
we cannot determine
 x_4 from the
last eqn.

$$\textcircled{5} \quad \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 21 \\ -1 & 2 & -1 & 0 & 0 & 10 \\ 0 & 1 & 2 & -1 & 0 & -4 \\ 0 & 0 & -1 & 2 & -1 & 16 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array} \right]$$

Ans: $x_5 = x_4 = x_3 = x_2 = x_1 = 1$

- we can apply Gaussian elimination method to a system of eqns. of any order. However, what happens if any one of the diagonal elements i.e. the pivots in the triangularization process vanishes. Then the method will fail. In such situations we modify the Gaussian elimination method and this procedure is called pivoting.
- In the elimination process, if any one of the pivot elements $a_{11}, a_{22}, \dots, a_{nn}$ vanishes or becomes very small compared to other elements in that row, then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This strategy is called pivoting.
- The pivoting is of the following two types:

- (i) partial pivoting: In the first stage of elimination, the first column is searched for the largest

(8)

element in magnitude and the largest element is then brought at the position of the first pivot by interchanging the first row with the row having the larger element in magnitude in the first column. In the second stage of elimination, the second column is searched for the largest element in magnitude among the $(n-1)$ elements leaving the first element and then this largest element in magnitude is brought at the position of the second pivot by interchanging the second row with rows having the largest element in the second column. This searching and interchanging of rows is repeated in all the $(n-1)$ stages of the elimination.

Complete pivoting:

We search the matrix A for the largest element in magnitude and bring it as the first pivot. This requires not only an interchanging of equations but also an interchange of the position of the variables.

Complete pivoting is much more complicated and is not often used.

Solve the system of equations.

$$x_1 + x_2 + x_3 = 6$$

$$3x_1 + 3x_2 + 4x_3 = 20$$

$$2x_1 + x_2 + 3x_3 = 13$$
 Using Gauss elimination method with partial pivoting

Now let us try first to solve the system

without pivoting

we have $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array} \right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array} \right] \quad R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1$$

See the above matrix the second pivot has the value zero and the elimination procedure cannot be continued further unless, pivoting is used.

Now let us use the partial pivoting:

In the first column 3 is the largest element

Interchanging the rows 1^{st} & 2^{nd} ,

we get

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array} \right]$$

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & -1 & 1 & -\frac{1}{3} \end{array} \right] \quad R_2 \rightarrow R_2 - \frac{1}{3}R_1 \\ R_3 \rightarrow R_3 - \frac{2}{3}R_1$$

In the second column, 1 is the largest element in magnitude leaving the first element.

Interchange the second and third rows,

we have

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array} \right]$$

Clearly the resultant matrix is in triangular form and no further elimination is required.

Using the back substitution method, we obtain the solution: $x_3 = 2, x_2 = 1, x_1 = 3$.

→ solve the system of eqns

$$0.0003x_1 + 1.566x_2 = 1.569$$

$$0.3454x_1 - 0.436x_2 = 3.018$$

Using Gauss elimination method with and without pivoting.

Sol: without pivoting -

We have

$$[A|B] = \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -0.436 & 3.018 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1803.0 & -1803.0 \end{array} \right]$$

Clearly which is in triangular form and no further elimination is required.

Using back substitution method

we obtain the solution $x_2 = 1.001$
 $x_1 = 34.333$

which is highly inaccurate compared to the exact solution.

With pivoting:

We interchange the 1st and 2nd rows

we get

$$[A|B] = \left[\begin{array}{cc|c} 0.3454 & -0.436 & 3.018 \\ 0.0003 & 1.566 & 1.569 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 0.345 & -0.436 & 3.018 \\ 0 & 1.566 & 1.566 \end{array} \right]$$

Clearly which is in triangular form and no further elimination is required.

Using back substitution method we obtain

the solution $x_2 = 1.8$ & $x_1 = 10$

which is the exact solution.

(9)

HW → Solve the system of eqns

$$x+y+z=7$$

$$3x+3y+4z=24$$

$2x+y+3z=16$ by Gaussian elimination
method with partial pivoting

$$\boxed{\text{Ans: } x=3, y=1, z=3}$$

→ Solve the Gaussian elimination method with partial pivoting. The following system of eqns

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

$$\boxed{\text{Ans: } x_1=1, x_2=1, x_3=2, x_4=2}$$

Gauss-Jordan elimination method:

This method is a variation of the Gauss elimination method.

In the Gauss elimination method, using elementary row operations, we transform the matrix A to an upper triangular matrix U and obtain the solution by back substitution method.

In Gauss-Jordan elimination method not only the elements below the diagonal but also the elements above the diagonal of A are made zero at the same time.

In other words, we transform the matrix A to a diagonal matrix D. This diagonal matrix may then be reduced to an identity matrix by dividing each row

by its pivot element.

Alternatively, the diagonal elements can also be made unity at the same time when the reduction is performed.

This transforms the coefficient matrix into an identity matrix; on completion of the Gauss-Jordan method, we have

$$[A|B] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|d]$$

The solution is given by

$$x_i = d_i, i = 1, 2, \dots, n.$$

Pivoting can be used to make the pivot non-zero or to make it the largest element in magnitude in that column, as discussed earlier.

Generally, the Gauss-Jordan elimination method requires more number of operations compared to the Gaussian elimination method.

Compared to the Gaussian elimination method, we therefore, do not use this method for solving system of eqns. but is very commonly used for finding the inverse matrix.

This is done by augmenting the matrix A by the identity matrix I of the order same as that of A. Using elementary row operations on the augmented matrix $[A|I]$, we reduce the matrix A to the form I and in the process the matrix I is transformed to A^{-1} .

$$i.e., [A|I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|A^{-1}]$$

→ solve the system of eqns

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 2x_2 - x_3 = 6$$

$3x_1 + 5x_2 + 3x_3 = 4$ by using the Gauss-Jordan method with pivoting.

Pmt: we have

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 3 & 4 \end{array} \right] \quad (\text{Interchanging first and second row})$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 0 & \frac{1}{4} & \frac{5}{4} & \frac{1}{2} \\ 0 & \frac{11}{4} & \frac{15}{4} & \frac{1}{2} \end{array} \right] \quad R_2 \rightarrow R_2 - \frac{1}{4}R_1 \\ R_3 \rightarrow R_3 - \frac{11}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 0 & \frac{11}{4} & \frac{15}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{5}{4} & \frac{1}{2} \end{array} \right] \quad (\text{Interchanging 2nd and 3rd row})$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & -\frac{56}{11} & \frac{72}{11} \\ 0 & \frac{11}{4} & \frac{15}{4} & \frac{1}{2} \\ 0 & 0 & \frac{10}{11} & \frac{-5}{11} \end{array} \right] \quad R_3 \rightarrow R_3 - \frac{1}{11}R_2 \\ R_1 \rightarrow R_1 - \frac{12}{11}R_2$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & \frac{11}{4} & 0 & \frac{4}{8} \\ 0 & 0 & \frac{10}{11} & \frac{5}{11} \end{array} \right] \quad R_1 \rightarrow R_1 + \frac{56}{10}R_3 \\ R_2 \rightarrow R_2 - \frac{33}{8}R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \quad R_1 \rightarrow R_1 \\ R_2 \rightarrow \frac{4}{11}R_2 \\ R_3 \rightarrow \frac{11}{10}R_3$$

which is the desired form.

$$\therefore x_1 = 1; x_2 = \frac{1}{2}; x_3 = -\frac{1}{2}$$

→ find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \text{ using Gauss-Jordan method}$$

Sol:

Using the augmented matrix $[A|I]$,

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \quad R_1 \rightarrow \frac{1}{3}R_1$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{14}{3} & -\frac{7}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array} \right] \quad R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array} \right] \quad R_2 \rightarrow \frac{3}{11}R_2$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{11} & \frac{3}{11} & x_{11} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 20/11 & 2/11 & x_{11} & 1 \end{array} \right] \quad R_1 \rightarrow R_1 - \frac{1}{3}R_2 \\ R_3 \rightarrow R_3 + \frac{7}{3}R_2$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -5/11 & 3/11 & x_{11} & 0 \\ 0 & 1 & 1/3 & 1/11 & -3/11 & 0 \\ 0 & 0 & 1 & 2/20 & -7/20 & 1/20 \end{array} \right] \quad R_3 \rightarrow \frac{11}{20}R_3$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{array} \right] \quad R_1 \rightarrow R_1 - \frac{5}{11}R_2 \\ R_2 \rightarrow R_2 - \frac{1}{11}R_3$$

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{20} & -\frac{1}{20} & -\frac{7}{20} \\ \frac{3}{20} & -\frac{7}{20} & -\frac{3}{20} \\ \frac{1}{20} & -\frac{7}{20} & \frac{11}{20} \end{bmatrix}$$

→ find the inverse of the coefficient matrix of the system
 $x_1 + 2x_2 + 3x_3 = 1$
 $4x_1 + 3x_2 - x_3 = 6$
 $3x_1 + 5x_2 + 3x_3 = 4$ by the Gauss-Jordan method with partial pivoting and hence solve the system.

Soln: we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Using the augmented matrix $[A|I]$, we obtain

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] R_2 \rightarrow \frac{1}{4}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 1 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{1}{4} & 1 \end{array} \right] R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 4R_1$$

continuing in this way, we get-

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{15} & \frac{8}{15} & -\frac{2}{15} \\ 0 & 1 & 0 & -\frac{3}{10} & 0 & \frac{1}{10} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{15} & -\frac{1}{10} \end{array} \right] = [I|A^{-1}]$$

∴ solution of the system is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}B$$

$$= \begin{bmatrix} \frac{7}{15} & \frac{8}{15} & -\frac{2}{15} \\ -\frac{3}{10} & 0 & \frac{1}{10} \\ \frac{11}{10} & -\frac{1}{15} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

→ find the inverse of the following matrices by using Gauss-Jordan method.

$$(1) A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & x_2 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ -1 & -7x_2 & -17 & 5x_2 \end{bmatrix}$$

$$(2) A = \begin{bmatrix} 1 & \frac{3}{2} & 2 & x_2 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 2x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(3) A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$(4) A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 7 \end{bmatrix}$$

→ Using Gauss-elimination method, find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

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3.8.1 Gaussian Elimination Method

In this method, if A is a given matrix, for which we have to find the inverse, at first, we place an identity matrix, whose order is same as that of A , adjacent to A which we call an *augmented matrix*. Then the inverse of A is computed in two stages. In the first stage, A is converted into an upper triangular form, using Gaussian elimination method as discussed in Section 3.2. In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix. Finally, when A is transformed into an identity matrix, the adjacent matrix gives the inverse of A . In order to increase the accuracy of the result, it is essential to employ partial pivoting. To understand the sequence of the steps involved, we consider an example.

Example 3.9 Use the Gaussian elimination method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution At first, we place an identity matrix of the same order adjacent to the given matrix. Thus, the augmented matrix can be written as

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (1)$$

Stage I (Reduction to upper triangular form): Let R_1 , R_2 and R_3 denote the first, second and third rows of a matrix. In the first column of Eq. (1), 4 is the largest element, thus interchanging R_1 and R_2 to bring the pivot element 4 to the place of a_{11} , we have the augmented matrix in the form

$$\left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (2)$$

Divide R_1 by 4 to get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (3)$$

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Perform $R_2 - R_1 \rightarrow R_2$, which gives

$$\left[\begin{array}{cccc|ccc} 1 & 3 & -1 & 0 & \frac{1}{4} & 0 \\ 1 & 4 & -4 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 5 & 1 & -1 & 0 \\ 0 & 4 & 4 & 4 & 0 & 1 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (4)$$

Perform $R_3 - 3R_1 \rightarrow R_3$ in Eq. (4), which yields

$$\left[\begin{array}{cccc|ccc} 1 & 3 & -1 & 0 & \frac{1}{4} & 0 \\ 1 & 4 & -4 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 5 & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array} \right] \quad (5)$$

Now, looking at the second column for the pivot, the max ($1/4, 11/4$) is $11/4$. Therefore, we interchange R_2 and R_3 in Eq. (5) and get

$$\left[\begin{array}{cccc|ccc} 1 & 3 & -1 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 1 & 4 & -4 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 5 & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 5 & 1 & -\frac{1}{4} & 0 \end{array} \right] \quad (6)$$

Now, divide R_2 by the pivot $a_{22} = 11/4$, and obtain

$$\left[\begin{array}{cccc|ccc} 1 & 3 & -1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 1 & 4 & -4 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 5 & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 5 & 1 & -\frac{1}{4} & 0 \end{array} \right] \quad (7)$$

Performing $R_3 - (1/4)R_2 \rightarrow R_3$ in (7) yields

$$\left[\begin{array}{cccc|ccc} 1 & 3 & -1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & \frac{1}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & \frac{1}{11} \end{array} \right] \quad (8)$$

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Finally, we divide R_3 by $(10/11)$, thus getting an upper triangular form

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right] \quad (9)$$

Stage II (Reduction to an identity matrix): Multiply R_3 by $-1/4$ and $-15/11$ respectively and subtract it from R_1 and R_2 of Eq. (9), we get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right] \quad (10)$$

Finally, performing $R_1 - (3/4) R_2 \rightarrow R_1$ in Eq. (10), we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Thus, we have

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix} \quad (11)$$

We can easily check $[A][A^{-1}] = [I]$.

3.8.2 Gauss-Jordan Method

This method is similar to Gaussian elimination method, with the essential difference that the stage I of reducing the given matrix to an upper triangular form is not needed. However, the given matrix can be directly reduced to an identity matrix using elementary row transformations. This technique is illustrated in the following example.

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Example 3.10 Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by Gauss-Jordan method.

Solution Let R_1 , R_2 and R_3 denote the first, second and third rows of a matrix. We place an identity matrix adjacent to the given matrix as a first step and the resulting augmented matrix is given by

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (1)$$

Performing $R_2 - 4R_1 \rightarrow R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \quad (2)$$

Now, performing $R_3 - 3R_1 \rightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array} \right] \quad (3)$$

Carrying out further operations $R_2 + R_1 \rightarrow R_1$ and $R_3 + 2R_2 \rightarrow R_3$, we arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array} \right] \quad (4)$$

Now, dividing the third row by -10 , we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & 11 & -1 & -1 \end{array} \right] \quad (5)$$

Proceeding in this way
we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 2 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array} \right] = \left[\begin{array}{cc|cc} I & & A^{-1} \end{array} \right]$$

Indirect methods:

Iteration Method:

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Direct methods provide the exact solution in a finite number of steps provided exact arithmetic is used and there is no round-off error. Also, direct methods are generally used when the matrix A is having few zero elements and the order of the matrix is not very large say $n \leq 50$.

Iterative methods, on the other hand, start with an initial approximation and by applying a suitably chosen algorithm, lead to successively better approximations. Even if the process converges, it gives only an approximate solution. These methods are generally used when the matrix A is sparse (many elements are zero) and the order of the matrix A is very large say $n > 50$. Sparse matrices have very few non-zero elements. In most cases these non-zero elements lie on or near the main diagonal giving rise to tridiagonal, or five diagonal matrix systems.

It may be noted that there are no fixed rules to decide when to use direct methods and when to use iterative methods.

However, when the coefficient matrix is sparse or large, the use of iterative methods is ideally suited to find the solution which take advantage of the sparse nature of the matrix involved.

The General Iteration Method

We start with some initial approximate solution vector $x^{(0)}$ and generate a sequence of approximations $\{x^{(k)}\}$ which converge to the exact solution vector x as $k \rightarrow \infty$. If the method is convergent, each iteration produces a better approximation to the exact solution, we repeat the iterations till the required accuracy is obtained.

Therefore, in an iterative method the amount of computation depends on the desired accuracy whereas in direct methods the amount of computation is fixed. The number of iterations needed to obtain the desired accuracy also depends on the initial approximation, closer the initial approximation to the exact solution, faster will be the convergence.

Now consider the system of eqns

$$Ax = B \quad \textcircled{1}$$

where A is $n \times n$ non-singular matrix.

$$\begin{aligned} \Rightarrow a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \textcircled{2}$$

we assume the diagonal coefficients $a_{ii} \neq 0$ ($i = 1, 2, \dots, n$)

If some $a_{ii} = 0$ then we rearrange the eqns,
so that this condition holds

NOW we rewrite the system ② as

$$\begin{aligned}x_1 &= \frac{-1}{a_{11}}(a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n) + \frac{b_1}{a_{11}} \\x_2 &= \frac{-1}{a_{22}}(a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n) + \frac{b_2}{a_{22}} \\&\vdots \\x_n &= \frac{-1}{a_{nn}}(a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn-1}x_{n-1}) + \frac{b_n}{a_{nn}}\end{aligned}\quad (3)$$

In matrix form, system ③ can be written as

$$\underline{x} = H\underline{x} + \underline{c}$$

where $H =$

$$\begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \dots & -\frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \dots & -\frac{a_{nn-1}}{a_{nn}} & 0 \end{bmatrix} \quad (4)$$

and the elements of \underline{c} are $c_i = \frac{b_i}{a_{ii}}$ ($i=1,2,\dots,n$)

To solve ③, we make an initial guess $\underline{x}^{(0)}$ of the solution vector and substitute into the RHS of eqn ③. The solution of equation ③ will then yield a vector $\underline{x}^{(1)}$, which hopefully is a better approximation to the solution than $\underline{x}^{(0)}$. We then substitute $\underline{x}^{(1)}$ into the RHS of eqn ③ and get another approximation $\underline{x}^{(2)}$. We continue in this manner until the successive iterations $\underline{x}^{(k)}$ have converged to the required number of significant figures.

In general we can write the iteration method for solving the linear system

of eqns (1) in the form $x^{(k+1)} = Hx^{(k)} + c$, $k=0, 1, 2, \dots$
 where $x^{(k)}$ and $x^{(k+1)}$ are the approximations
 for x at the k^{th} and $(k+1)^{\text{th}}$ iterations
 respectively.

This is called the iteration matrix and depends
 on A and c is a column vector and depends
 on both A and B .

The matrix H is generally a constant matrix
 when the method (3) is CGT, then

$$\lim_{k \rightarrow \infty} x^{(k)} = \lim_{k \rightarrow \infty} Hx^{(k+1)} = x$$

and we obtain from eqn (5),

$$x = Hx + c \quad (6)$$

If we define the error vector at the k^{th} iteration
 as $e^{(k)} = x^{(k)} - x$

then subtracting eqn (6) from eqn (5) (i.e. (5) - (6))

we obtain

$$\begin{aligned} x^{(k+1)} - x &= H[x^{(k)} - x] \\ \Rightarrow x^{(k+1)} - x &= He^{(k)} \\ \Rightarrow e^{(k+1)} &= He^{(k)} \quad (\because e^{(k)} = x^{(k)} - x) \end{aligned}$$

$$\begin{aligned} \text{Q.E.D. } e^{(k)} &= He^{(k-1)} \\ &= H(He^{(k-2)}) \\ &= H^2e^{(k-2)} \\ &= H^3e^{(k-3)} \\ &= H^4e^{(k-4)} = \dots = H^ke^{(0)} \end{aligned}$$

where $e^{(0)}$ is the error in the initial approximate
 vector. Thus, for the convergence of the
 iterative method, we must have $H^ke^{(0)} = 0$
 independent of $e^{(0)}$.

Gauss-Seidel iteration method:

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Consider the system of eqns (2) written in form (3).
for this system of eqns, we define the

Gauss-Seidel method as :

$$x_1^{(k+1)} = -\frac{1}{a_{11}} (a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)} - b_1) \quad (k)$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}} (a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \dots + a_{2n}x_n^{(k)} - b_2) \quad (k)$$

$$x_3^{(k+1)} = -\frac{1}{a_{33}} (a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} + \dots + a_{3n}x_n^{(k)} - b_3) \quad (k)$$

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \left[\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^n a_{ij}x_j^{(k)} - b_i \right] \quad (k)$$

$i = 1, 2, 3, \dots, n$

Note that, in the first eqn of system (1),
we substitute the initial approximation

$(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ on RHS.

In the second eqn, we substitute $(x_1^{(1)}, x_2^{(0)}, \dots, x_n^{(0)})$
on RHS.

In third eqn, we substitute $(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, \dots, x_n^{(0)})$.

We continue in this manner until all the
components have been improved. At the end
of this first iteration, we will have an
improved vector $(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$.

The entire process is then repeated. In other
words, the method uses an improved component
as soon as it becomes available. It is

for this reason the method is also called the method of successive displacements.

Q) Perform four iterations (rounded to four decimal places) using the Gauss-Seidel method for solving the system of eqns

$$\begin{bmatrix} 8 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 7 \end{bmatrix} \text{ with } x^{(0)} = 0$$

The exact solution is $x = (1, -6, -1)$.

Soln:

Given system is

$$\begin{aligned} & \left. \begin{aligned} & 8x_1 + x_2 + x_3 = 1 \\ & x_1 + 5x_2 + x_3 = 16 \\ & x_1 + x_2 - 4x_3 = 7 \end{aligned} \right\} \quad (i) \\ & \left. \begin{aligned} & x_1 = -\frac{1}{8}(1 - x_2 - x_3) \\ & x_2 = -\frac{1}{5}(16 - x_1 - x_3) \\ & x_3 = -\frac{1}{4}(7 - x_1 - x_2) \end{aligned} \right\} \quad (ii) \end{aligned}$$

By the Gauss-Seidel method, system (ii)

can be written as

$$x_1^{(k+1)} = -\frac{1}{8}(1 - x_2^{(k)} - x_3^{(k)})$$

$$x_2^{(k+1)} = -\frac{1}{5}(16 - x_1^{(k+1)} - x_3^{(k)})$$

$$x_3^{(k+1)} = -\frac{1}{4}(7 - x_1^{(k+1)} - x_2^{(k+1)})$$

where $k = 0, 1, 2, \dots$

Now taking $x^{(0)} = 0$, we obtain the following iterations -

$$\begin{aligned} k=0: \quad x_1^{(1)} &= -\frac{1}{8}(1 - x_2^{(0)} - x_3^{(0)}) = -\frac{1}{8}(1 - 0 - 0) \\ &= -\frac{1}{8} = -0.125 \end{aligned}$$

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$$\begin{aligned}x_2^{(1)} &= -\frac{1}{5} [16 - x_1^{(1)} - x_3^{(1)}] \\&= -\frac{1}{5} [16 + 0.125 - 0] \\&= -3.225\end{aligned}$$

$$\begin{aligned}x_3^{(1)} &= -\frac{1}{4} [7 - x_1^{(1)} - x_2^{(1)}] \\&= -\frac{1}{4} [7 + 0.125 + 3.225] \\&= -2.5875\end{aligned}$$

K=1:

$$\begin{aligned}x_1^{(2)} &= -\frac{1}{8} [1 - x_2^{(1)} - x_3^{(1)}] \\&= -\frac{1}{8} [1 + 3.225 + 2.5875] \\&= -0.8516\end{aligned}$$

$$\begin{aligned}x_2^{(2)} &= -\frac{1}{5} [16 - x_1^{(2)} - x_3^{(1)}] \\&= -\frac{1}{5} [16 + 0.8516 + 2.5875] \\&= -2.8878 \\x_3^{(2)} &= -\frac{1}{4} [7 - x_1^{(2)} - x_2^{(2)}] \\&= -\frac{1}{4} [7 + 0.8516 + 3.8878] \\&= -2.9349\end{aligned}$$

K=2:

$$\begin{aligned}x_1^{(3)} &= -\frac{1}{8} [1 - x_2^{(2)} - x_3^{(2)}] \\&= -\frac{1}{8} [1 + 3.8878 + 2.9349] \\&= -0.9778\end{aligned}$$

$$\begin{aligned}x_2^{(3)} &= -\frac{1}{5} [16 - x_1^{(3)} - x_3^{(2)}] \\&= -\frac{1}{5} [16 + 0.9778 + 2.9349] \\&= -3.9825\end{aligned}$$

$$\begin{aligned}x_3^{(3)} &= -\frac{1}{4} [7 - x_1^{(3)} - x_2^{(3)}] \\&= -\frac{1}{4} [7 + 0.9778 + 3.9825] \\&= -2.9901\end{aligned}$$

K=3:

$$x_1^{(4)} = -\frac{1}{8}[1+3 \cdot 9825 + 2 \cdot 9901] \\ = -0.9966$$

$$x_2^{(4)} = -\frac{1}{5}[16 + 0.9966 + 2 \cdot 9901] \\ = -3.9973$$

$$x_3^{(4)} = -\frac{1}{4}[7 + 0.9966 + 3 \cdot 9973] \\ = -2.9985$$

which is a good approximation to the exact solution $x = (-1 - 4 - 3)^T$ with maximum error 0.0034

→ Solve the following eqns

$$2x_1 - x_2 + 0x_3 = 7$$

$$-x_1 + 2x_2 - x_3 = 1$$

$$0x_1 - x_2 + 2x_3 = 1$$

Using Gauss-Seidel method of iteration and perform three iterations.

Sol: The given system of eqns can be written as

$$\left. \begin{array}{l} x_1 = \frac{1}{2}(7+x_2) \\ x_2 = \frac{1}{2}(1+x_1+x_3) \\ x_3 = \frac{1}{2}(1+x_2) \end{array} \right\} \quad \text{①}$$

By the Gauss-Seidel method, system ①

can be written as

$$x_1^{(k+1)} = \frac{1}{2}(7+x_2^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{2}(1+x_1^{(k+1)} + x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{2}(1+x_2^{(k+1)})$$

where $k=0, 1, 2, \dots$

Now taking $x^{(0)} = 0$, we obtain the following iterations

(18)

$$\underline{k=0} : x_1^{(1)} = \frac{1}{2}(7+0) = \frac{7}{2} = 3.5$$

$$x_2^{(1)} = \frac{1}{2}(1+x_1^{(1)}+x_3^{(0)}) \\ = \frac{1}{2}(1+3.5+0) \\ = \frac{4.5}{2} = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1+x_2^{(1)}) \\ = \frac{1}{2}(1+2.25) = \frac{1}{2}(3.25) \\ = 1.625$$

$$\underline{k=1} : x_1^{(2)} = \frac{1}{2}(7+x_2^{(1)}) \\ = \frac{1}{2}(7+2.25) = \frac{9.25}{2} = 4.625$$

$$x_2^{(2)} = \frac{1}{2}(1+x_1^{(2)}+x_3^{(1)}) \\ = \frac{1}{2}(1+4.625+1.625) \\ = \frac{1}{2}(7.25) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}(1+x_2^{(2)}) \\ = \frac{1}{2}(1+3.625) = \frac{4.625}{2} = 2.3125$$

$$\underline{k=2} : x_1^{(3)} = \frac{1}{2}(7+x_2^{(2)}) \\ = \frac{1}{2}(7+3.625) = \frac{10.625}{2} = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}(1+x_1^{(3)}+x_3^{(2)}) \\ = \frac{1}{2}(1+5.3125+2.3125) \\ = \frac{8.625}{2} = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}(1+x_2^{(3)}) \\ = \frac{1}{2}(1+4.3125) = 2.6563$$

→ use the Gauss-Seidel method for solving the following system of eqns

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{with } \bar{x}^{(0)} = [0.5 \ 0.5 \ 0.5 \ 0.5]^T$$

Compare the result with the exact solution is

$$x = [1 \ 1 \ 1 \ 1]^T$$

~~2001
2004~~ → Using the Gauss-Seidel method and starting solution $x_1 = x_2 = x_3 = 0$, determine the solution of the following system of eqns in two iterations

$$10x_1 - x_2 - x_3 = 8$$

$$x_1 + 10x_2 + x_3 = 12$$

$$x_1 - x_2 + 10x_3 = 10$$

Compare the approximate solution with the exact solution

~~2004
1994~~ → Using Gauss-Seidel iterative method, find the solution of the following system:

$$4x - y + 8z = 26$$

$$5x + 2y - z = 6$$

$$x - 10y + 2z = -13 \text{ up to three iterations}$$

→ Find the solution of the following system of eqns

$$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_2 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4} \text{ using Gauss Seidel and perform the first four iterations}$$

→ Solve the system eqns

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$2x - 3y + 2z = 25$ by Gauss-Seidel iterative method and perform the first four iterations

