

IMS
MATHS
BOOK-15

BMS

Set - I

Bharath



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MATHEMATICS

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The person with 8 yrs of teaching exp.

Differential Equations

Differential eqn: An equation involving derivatives of a dependent variable w.r.t one or more independent variables, is called a differential eqn.

$$\text{Ex: (1)} \quad \frac{dy}{dx} = x \log x$$

$$(2) \quad \frac{dy}{dx} + 3x \left(\frac{dy}{dx} \right)^2 - 5y = \log x$$

$$(3) \quad \frac{dy}{dx} - 4 \frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$$

$$(4) \quad \left(\frac{d^3y}{dx^3} \right)^{2003} + P(x) \frac{dy}{dx} + Q(x) \frac{dy}{dx} + R(x) y = S(x)$$

$$(5) \quad \frac{\partial z}{\partial x} + 2 \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} = 0$$

$$(6) \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$$

Note:

$$\frac{dy}{dx} = y^1 \text{ (or) } y^{(1)} \text{ (or) } y_1; \quad \frac{d^2y}{dx^2} = y^2 \text{ (or) } y^{(2)} \text{ (or) } y_2$$

$$\frac{d^3y}{dx^3} = y^{(3)} \text{ (or) } y_3; \quad \frac{d^n y}{dx^n} = y^{(n)} \text{ (or) } y_n$$

Types of Differential Equations:

(i) Ordinary Diff. eqns: An eqn involving the derivatives of a dependent variable w.r.t a single independent variable, is called an ordinary diff. eqn.

The above examples (1), (2), (3), & (4) are ordinary diff. eqns.

(ii) Partial Diff. eqn: An equation involving the derivatives of a dependent variable w.r.t more than one independent

variable, is called a partial diff-eqn.

The above examples (5) & (6) are partial diff-eqns.

Order of a Diff. egn: The order of the highest order derivative involving in a differential egn is called the order of the diff-eqn.

Ex: (1) $\frac{d^2y}{dx^2} + 4y = e^x$ is of 2nd order.

(2) $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$ is of second order.

(3) $\frac{d^2y}{dx^2} = k \left[1 + \left(\frac{dy}{dx} \right)^3 \right]^{5/3}$ is of 2nd order.

(4) $\log \left(\frac{dy}{dx} \right) = ax + by$ is of 1st order.

(5) $\sin \left(\frac{dy}{dx} \right) = x^{100}$
order = 1

(6) $\cos \left(\frac{dy}{dx} \right) = x^{100}$
order = 1

NOTE 1. A differential egn of order one is of the form $f(x, y, \frac{dy}{dx}) = 0$

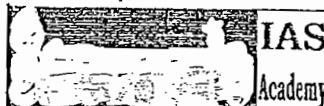
2. A diff-eqn of order two is of the form

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

3. In general, diff-eqn of order 'n' is of the form $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^n y}{dx^n}) = 0$

Degree of a diff-eqn: The degree (i.e., power) of the highest order derivative involving in a diff-eqn, when the derivatives are made free from radicals and fractions, is called the degree of the diff-eqn.

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(2)

Ex: (1) $y \left(\frac{dy}{dx} \right)^3 + y^2 \left(\frac{dy}{dx} \right)^4 + xy = 0$ is of order 2 and degree 3.

$$(2) \frac{dy}{dx^2} = k \left[1 + \left(\frac{dy}{dx} \right)^3 \right]^{5/6} \quad (\text{radical form})$$

Cubing on both sides, we get,

$$\left(\frac{dy}{dx} \right)^3 = k^3 \left(1 + \left(\frac{dy}{dx} \right)^3 \right)^5 \quad \text{order} = 2$$

degree = 3.

$$(3) y \left(\frac{dy}{dx} \right) = \sqrt{x} + \frac{k}{\frac{dy}{dx}} \quad (\text{fractions form})$$

$$\Rightarrow y \left(\frac{dy}{dx} \right)^2 = \sqrt{x} \left(\frac{dy}{dx} \right) + k.$$

\therefore order = 1
degree = 2

$$(4) y = x \frac{dy}{dx} \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\Rightarrow y^2 = x^2 \left(\frac{dy}{dx} \right)^2 \left(1 + \left(\frac{dy}{dx} \right)^2 \right)$$

$$\Rightarrow y^2 = x^2 \left(\frac{dy}{dx} \right)^2 + x^2 \left(\frac{dy}{dx} \right)^4.$$

Order = 1
Degree = 4

$$(5) \frac{d^3y}{dx^3} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\Rightarrow \left(\frac{d^3y}{dx^3} \right)^2 = 1 + \left(\frac{dy}{dx} \right)^2$$

Order = 3
Degree = 2

$$(6) e = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{fractions form}$$

$$\Rightarrow e\left(\frac{d^2y}{dx^2}\right) = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\Rightarrow e^{\left(\frac{d^2y}{dx^2}\right)^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Order = 2 —

Degree = 2

$$(7) \quad (y''')^{4/3} + \sin x \left(\frac{dy}{dx}\right) + xy = x$$

$$\Rightarrow (y''')^{4/3} = x - \sin x \left(\frac{dy}{dx}\right) - xy$$

$$\Rightarrow (y''')^4 = (x - \sin x \left(\frac{dy}{dx}\right)^3 - xy)^3$$

order = 3

degree = 4

$$(8) \quad (y''')^{Y_2} - 2(y')^{Y_4} + xy = 0$$

$$\Rightarrow (y''')^{Y_2} + xy = 2(y')^{Y_4}$$

$$\Rightarrow (y''')^{Y_2} + xy = 2^4 y^4$$

$$\Rightarrow [(y''')^{Y_2} + xy]^2 = 16y^4$$

$$\Rightarrow [y''' + xy + 2xy(y'')^{Y_2}]^2 = 16y^4$$

$$\Rightarrow (y''')^2 + 2^4 y^4 + 4x^2 y^2 y''' + 2x^2 y^2 y''' + 4x^3 y^3 (y'')^{Y_2} + 4xy(y'')^{3/2} = 16y^4$$

$$\Rightarrow 4xy(y'')^{Y_2} [y''' + 2^2 y^2] = [16y^4 - (y''')^2 - 2^4 y^4 - 6x^2 y^2 y''']$$

squaring on both sides

$$(6x^2 y^2 y''')^2 [y''' + 2^2 y^2]^2 = [(16y^4 - (y''')^2 - 2^4 y^4 - 6x^2 y^2 y''')]^2$$

$$\Rightarrow 16x^2 y^2 y''' [(y'')^2 + 4y^4 + 2x^2 y^2 y'''] = (16y^4) + (y''')^4 + \dots$$

∴ order = 3 & degree = 4



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(3)

$$(9) \quad (y''')^{4/3} + (y')^{15} - y = 0$$

Order = 3

Degree = 60

$$(10) \quad (y''')^{3/2} + (y''')^{2/3} = 0$$

Order = 3

Degree = 9

Note:

1. $y = \sin\left(\frac{dy}{dx}\right)$

Order = 1

Degree = not defined

Because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$= \frac{dy}{dx} - \frac{1}{3!} \left(\frac{dy}{dx} \right)^3 + \frac{1}{5!} \left(\frac{dy}{dx} \right)^5 - \frac{1}{7!} \left(\frac{dy}{dx} \right)^7 + \dots$$

Here $x = \frac{dy}{dx}$

Similarly $\cos\left(\frac{dy}{dx}\right)$; $\tan\left(\frac{dy}{dx}\right)$; $\cot\left(\frac{dy}{dx}\right)$, $\sec\left(\frac{dy}{dx}\right)$

and $\operatorname{cosec}\left(\frac{dy}{dx}\right)$ degree does not exist

(or) not defined.

2. $y = x\left(\frac{dy}{dx}\right) + \sin\left(\frac{dy}{dx}\right)$

Order = 1

Degree = not defined.

$$(3) \quad \frac{dy}{dx} + 2e^{\frac{x}{dx}} - 3y = x$$

$$\Rightarrow 2e^{\frac{x}{dx}} = x + 3y - \frac{dy}{dx}$$

$$\Rightarrow x \frac{dy}{dx} = \log \left[\frac{1}{2} \left(x + 3y - \frac{d^2y}{dx^2} \right) \right]$$

\therefore order = 2
degree = not defined.

(4) $3x^2 \frac{d^3y}{dx^3} - \sin \frac{dy}{dx^2} - \cos(xy) = 0$

(5) $(y''')^{1/3} + xy'' = 2005$

$$\Rightarrow (y''')^{1/3} = -xy'' + 2005$$

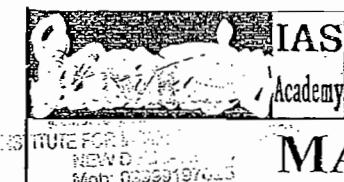
$$\Rightarrow y''' = (2005 - xy'')^3$$

order = 3
degree = 1

(6) $[y'' - 4(y')^2]^{5/2} = ay''$

$$[y'' - 4(y')^2]^5 = a(y'')^2$$

order = 2
degree = 5



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Linear differential eqn:

A differential equation is said to be linear if (i) the dependent variable say 'y' and all its derivatives occur in the first degree only
(ii) no product of dependent variables (or) derivatives occur.

$$\text{Ex: (1)} \quad \frac{dy}{dx} = x + \sin x$$

$$(2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

Non-linear diff. eqn:

A diff. eqn which is not linear is called a non-linear diff. eqn.

$$\text{Ex: (1)} \quad \frac{d^4 x}{dt^4} + \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt} \right)^5 = e^t$$

$$(2) \quad y = f(x) \frac{dy}{dx} + \frac{k}{dy/dx}$$

$$(3) \quad k \frac{dy}{dx} = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}$$

$$(4) \quad \frac{d^2 v}{dx^2} = k \left(\frac{dv}{dx} \right)^2$$

Note: In general, a linear diff. eqn of n^{th} order is of the form

$$\frac{d^n y}{dx^n} + P_1(x) \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \frac{d y}{dx^{n-2}} + \dots + P_{n-1}(x) \frac{dy}{dx} + P_n(x) y = Q(x).$$

Solution of a diff. eqn: Any relation between the dependent and independent variables which when we substitute in the diff. eqn reduces it to an identity is called a solution (or) integral (or) primitive of the diff. eqn.

Ex: $y = ce^{2x}$ is a solⁿ of the diff. eqn $y' = 2y$ (1)

$$\text{because } y = ce^{2x} \Rightarrow y' = 2ce^{2x}$$

$$\therefore (1) \ 2ce^{2x} = 2[ce^{2x}] \text{ is an identity}$$

General solⁿ: The solⁿ of a diff. eqn in which the number of arbitrary constants is equal to the order of the diff. eqn.

Ex: $y = ce^{2x}$ is G.S. of the diff. eqn $y' = 2y$.

Arbitrary constants = Order of the diff. eqn.

Particular solution: A solution obtained by giving particular values to arbitrary constants in the general solution, is called a particular solⁿ (or) particular integral.

Ex: In the above example taking $c=1$

$y = e^{2x}$ is a particular solⁿ of $y' = 2y$

Singular solution:

An eqn. $\Psi(x, y) = 0$ is called singular solution of the diff. eqn $f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{dy}{dx^n}) = 0$

If (i) $\Psi(x, y) = 0$ is a solution of the given diff. eqn.

(ii) $\Psi(x, y) = 0$ does not contain arbitrary constants.

and (iii) $\Psi(x, y) = 0$ is not obtained by giving particular values to arbitrary constants in the general solⁿ.

Arbitrary constants: The complete solⁿ of a diff. eqn of the n^{th} order contains exactly ' n ' arbitrary constants.

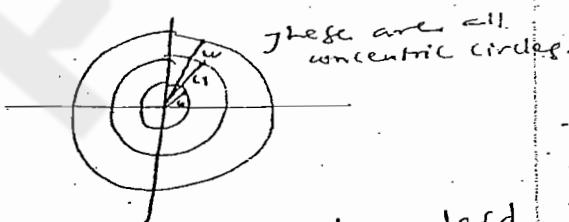
Family of plane curves:

(5)

For each given set of real numbers $c_1, c_2, c_3, \dots, c_n$, the equation $\phi(x, y, c_1, \dots, c_n) = 0$ represents a curve in xy -plane.
For different sets of real values of c_1, c_2, \dots, c_n , the equation $\phi(x, y, c_1, \dots, c_n) = 0$ represents infinitely many curves.

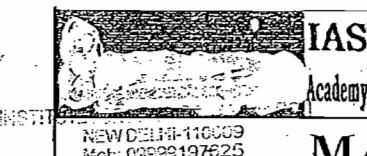
The set of all these curves is called n -parameter family of curves. c_1, c_2, \dots, c_n are called parameters of the family.

Ex:- (1) The set of concentric circles defined by $x^2 + y^2 = c$ is one parameter family if 'c' takes all non-negative values.



These are all concentric circles.

(2) The set of all circles, defd by $(x - c_1)^2 + (y - c_2)^2 = c_3$ is a three-parameter family if c_1, c_2 take all real values and c_3 takes all non-negative real values.



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(6)

Formation of Diff. eqns:-

working rule:To form the diff-eqn from a given eqn in x and y, containing arbitrary constants:Step 1: write down the given eqn.Step 2: Differentiate w.r.t 'x' successively as many as the number of arbitrary constants!Step 3: Eliminate the arbitrary constants from the given eqns of above two steps.

∴ The resulting is the required diff-eqn.

Problems

Q) find the diff-eqn of $y = Ae^{2x} + Be^{-3x}$; (A, B are arbitrary constants)

$$\text{Soln: } y^1 = 2Ae^{2x} - 3Be^{-3x} \quad \text{--- (1)}$$

$$y^2 = 4Ae^{2x} + 9Be^{-3x} \quad \text{--- (2)}$$

$$\begin{aligned} (1) + (2) &\equiv y^1 + y^2 = 6Ae^{2x} + 6Be^{-3x} \\ &= 6(Ae^{2x} + Be^{-3x}) \\ &= 6y. \end{aligned}$$

$$\therefore y^1 + y^2 = 6y$$

$$\Rightarrow y^2 + y^1 - 6y = 0$$

which is the required diff-eqn.

(or)

$$\left| \begin{array}{cc|c} y & e^{2x} & e^{-3x} \\ y^1 & -2e^{2x} & -3e^{-3x} \\ y^2 & -4e^{2x} & -9e^{-3x} \end{array} \right| = 0 \Rightarrow \left| \begin{array}{cc|c} e^{-2x} & e^{-3x} & y-1 & -1 \\ e^{-2x} & e^{-3x} & y^1-2 & 3 \\ e^{-2x} & e^{-3x} & y^2-4 & -9 \end{array} \right| = 0$$

$$\begin{aligned} &\Rightarrow y(18+12) + 1(-9y' - 3y'') - 1(-4y' + 2y'') = 0 \\ &\Rightarrow 30y - 5y' - 5y'' = 0 \quad (\because e^{2x} e^{3x} \neq 0) \\ &\Rightarrow y'' + y' - 6y = 0. \end{aligned}$$

(2) Find the diff-eqn of the family of curves
 $y = a(x-a)^2$, where a is an arbitrary constant.

Soln: Differentiate ① w.r.t x we get,

$$y' = 2a(x-a)$$

$$\text{Now } \frac{\text{①}}{\text{②}} \equiv \frac{1}{2}(x-a) = \frac{y}{y'}$$

$$\Rightarrow 2y = y'(x-a)$$

$$\Rightarrow ay' = xy' - 2y$$

$$\Rightarrow a = \frac{xy' - 2y}{y'}$$

$$\therefore \text{② } y = \left(\frac{xy' - 2y}{y'} \right) \left[x - \left(\frac{xy' - 2y}{y'} \right) \right]^2$$

$$= \frac{(xy' - 2y)}{y'} \left(\frac{2y}{y'} \right)^2$$

$$y = \frac{(xy' - 2y)}{y'} \frac{4y^2}{(y')^3}$$

$$\Rightarrow (y')^3 = (xy' - 2y)4y.$$

which is the required diff-eqn.

(3) Find the diff-eqn of $y = Ae^{ax} + Be^{bx}$; A, B are arbitrary constants.

Ans
$$y'' - (Aa+b) y' + aby = 0$$

(4) Find the diff-eqn of $y = Ae^{-x} + Be^{2x}$; A, B are arbitrary constants.

$$y'' - (-1+2)y' + (-1)(2)y = 0$$

$$\Rightarrow y'' - y' - 2y = 0$$



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(7)

- (5) Find the diff. eqn of $y = Ae^{ax} + Be^{bx} + Ce^{cx}$; (A, B, C)

$$y''' - (a+b+c)y'' + (ab+bc+ca)y' - abc y = 0$$

- (6) find the diff. eqn of $y = ae^x + be^{3x} + ce^{5x}$; (a, b, c)

$$y''' - (1+3+5)y'' + (1 \cdot 3 + 3 \cdot 5 + 5 \cdot 1)y' - (1 \cdot 3 \cdot 5)y = 0$$

$$\Rightarrow y''' - 9y'' + 23y' - 15y = 0$$

- (7) Form the diff. eqn of $y = ae^x + be^{2x} + ce^{3x}$ where a, b, c arbitrary constants.

$$y''' - 6y'' + (2+6+3)y' - 6y = 0$$

$$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$$

- (8) $y = ae^{2x} + be^{-3x} + ce^x$; (a, b, c)

$$y''' - (2-3+1)y'' + (2(-3) + (-3)(1) + 1(2))y' - (2 \cdot -3 \cdot 1)y = 0$$

$$\Rightarrow y''' - 6y'' + (-6-3+2)y' + 6y = 0$$

$$\Rightarrow y''' - 7y'' + 6y = 0$$

- (9) Form the diff. eqn $y = ae^{3x} + be^{5x}$

$$\text{Ans: } y'' - 8y' + 15y = 0$$

- (10) Form the diff. eqn of $y = e^{ax}(C_1 \sin bx + C_2 \cos bx)$

(C_1, C_2 are arbitrary constants)

$$y' = e^{ax} (C_1 b \cos bx - C_2 \sin bx)$$

$$+ ae^{ax} (C_1 \sin bx + C_2 \cos bx)$$

$$\Rightarrow y' = e^{ax} (C_1 b \cos bx - C_2 \sin bx) + ay \quad (\text{for } a \neq 0)$$

$$\Rightarrow y' - ay = e^{ax} (c_1 b \cos bx - c_2 b \sin bx) \quad (1)$$

$$\Rightarrow y'' - ay' = e^{ax} (-c_1 b^2 \sin bx - c_2 b^2 \cos bx) + ae^{ax} (c_1 b \cos bx - c_2 b \sin bx)$$

$$= -b^2 y + a(y' - ay) \quad (\text{by (1) & (2)})$$

$$\Rightarrow y'' - ay' = -b^2 y + ay' - a^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2 + b^2)y = 0}$$

(1) Form the diff-eqn of $y = e^{ax} (C_1 \sin bx - C_2 \cos bx)$ (1)
 where C_1, C_2 are arbitrary constants

$$y = e^{ax} (c_1 b \cos bx + c_2 b \sin bx) + ae^{ax} (c_1 \sin bx - c_2 \cos bx)$$

$$\Rightarrow y' = e^{ax} (c_1 b \cos bx + c_2 b \sin bx) + ay \quad (\text{by (1)})$$

$$\Rightarrow y'' - ay' = e^{ax} (c_1 b \cos bx + c_2 b \sin bx) \quad (2)$$

$$\Rightarrow y'' - ay' = ae^{ax} (c_1 b \cos bx + c_2 b \sin bx) + e^{ax} (-c_1 b^2 \sin bx + c_2 b^2 \cos bx)$$

$$\Rightarrow y'' - ay' = a(y' - ay) - b^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2 + b^2)y = 0}$$

(2) $y = e^{ax} (c_1 \cos x + c_2 \sin x); \quad (c_1, c_2)$

$$y'' - 2(c_1)y' + (1^2 + 1^2)y = 0$$

$$\Rightarrow y'' - 2y' + 2y = 0$$

(3) $y = c_1 \cos 2x + c_2 \sin 2x$

$$y'' - 2(2)y' + (2^2 + 2^2)y = 0$$

$$\Rightarrow y'' - 4y = 0$$



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(8)

- (i) Find the diff. eqn of $Ax^2 + By^2 = 1$; A, B are arbitrary constants.

$$\text{Soln: } \begin{vmatrix} x^2 & y^2 & -1 \\ 2x & 2yy' & 0 \\ 2 & 2(yy'' + (y')^2) & 0 \end{vmatrix} = 0$$

$$\Rightarrow -1 \begin{vmatrix} 2x & 2yy' \\ 2 & 2(yy'' + (y')^2) \end{vmatrix} = 0$$

$$\Rightarrow -1 (4x(yy'' + (y')^2) - 4yy') = 0$$

$$\Rightarrow 4xyy'' + 4x(y')^2 - 4yy' = 0$$

$$\Rightarrow x(yy'' + (y')^2) - yy' = 0$$

(OP)

$$2Ax + 2Byy' = 0$$

$$\Rightarrow \frac{yy'}{x} = -\frac{A}{B}$$

$$\Rightarrow \frac{x((y')^2 + yy'') - yy'}{x^2} = 0$$

$$\Rightarrow x((y')^2 + yy'') - yy' = 0$$

- (ii) Find the diff. eqn of the family of ellipses whose axes coincide with the axes of co-ordinates and centres at the origin.

$$\text{i.e., } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1) \text{ ; } a, b \text{ are arbitrary constants.}$$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$\Rightarrow \frac{yy'}{x} = -\frac{b^2}{a^2} \quad (2)$$

$$\frac{yy' - x((y')^2 + yy'')}{{x}^2} = 0$$

$$\Rightarrow x((y')^2 + yy'') - yy' = 0$$

(6) $xy = ac^x + be^{-x} + x^2 ; (a, b)$

$$\Rightarrow xy - x^2 = ae^x + be^{-x} \quad \text{--- (1)}$$

$$\Rightarrow xy' + y - 2x = ae^x - be^{-x}$$

$$\Rightarrow xy'' + y' + y - 2 = ae^x + be^{-x}$$

$$\Rightarrow xy'' + 2y' - 2 = xy - x^2 \quad (\text{from (1)})$$

(or)

$$(xy - x^2)'' - (0)(xy - x^2)' + (-1)(xy - x^2) = 0$$

$$\Rightarrow (xy' + y - 2x)' - xy + x^2 = 0$$

$$\Rightarrow xy'' + y' - 2 - xy + x^2 = 0$$

~~(7)~~ (7) Form the diff. eqn of the family of circles, given by $x^2 + y^2 + 2ax + 2by + c = 0$; a, b, c arbitrary constants

Soln

$$2x + 2yy' + 2a + 2by' = 0$$

$$x + yy' + a + by' = 0 \quad \text{--- (1)}$$

$$1 + yy'' + (y')^2 + by'' = 0$$

$$\Rightarrow -b = \frac{1 + yy'' + (y')^2}{y''} \quad \text{--- (2)}$$

$$\frac{y''(yy''' + y'y'' + 2y'y'') - y'''(1 + yy'' + (y')^2)}{(y'')^2} = 0$$

$$\Rightarrow y''(yy''' + 3y'y'') - y'''(1 + yy'' + (y')^2) = 0$$

$$\Rightarrow y'''(yy''' - 1 - yy'' + (y')^2) + 3y'(y'')^2 = 0$$

$$\Rightarrow y'''(1 + (y')^2) = \underline{\underline{3y'(y'')^2}}$$

(18) find the diff. eqt family of the curve $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ (9)

where λ is parameter.

$$\text{Soln} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1 \quad \text{--- (1)}$$

$$\frac{2x}{a^2} + \frac{2yy'}{a^2+\lambda} = 0$$

$$\Rightarrow \frac{1}{a^2+\lambda} = \frac{x^2}{a^2(yy')}$$

$$\Rightarrow \left[\frac{1}{a^2+\lambda} = -\frac{x^2}{a^2(yy')} \right]$$

$$\therefore (1) = \frac{x^2}{a^2} \Rightarrow \frac{xy'}{a^2yy'} = 1$$

$$\Rightarrow \left(1 - \frac{y}{y'} \right) = \frac{a^2}{a^2}$$

$$\Rightarrow y' - y = \frac{a^2}{a^2} y'$$

$$\Rightarrow \left(1 - \frac{a^2}{a^2} \right) y' - y = 0$$

which is the required diff. equation.

(19) $y = ax + b \lambda ; (a, b)$

$$\text{Soln} \quad y' = 2ax + b \quad \text{--- (2)}$$

$$y'' = 2a \quad \text{--- (3)}$$

Eliminating the arbitrary constants
from (1), (2) and (3), we have

$$\begin{vmatrix} y - ax & -\lambda \\ y' - 2ax & -1 \\ y'' - 2a & 0 \end{vmatrix} = 0 \Rightarrow a^2 y'' + 2ay' - 2a^2 y'' = 0$$

$$\Rightarrow a^2 y'' - 2ay' + ay = 0$$

$$(20) \quad y = a + b e^x; (a, b)$$

$$\text{so } y' = a + b e^x \quad (1)$$

$$y'' = b e^x \quad (2)$$

Now eliminating, a & b from (1), (2) & (3), we get

$$\begin{vmatrix} y & -1 & -e^x \\ y' & -1 & -e^x \\ y'' & 0 & -e^x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y & -1 & 1 \\ y' & -1 & -1 \\ y'' & 0 & -1 \end{vmatrix} = 0 \quad (\because e^{ax} \neq 0)$$

$$\Rightarrow y(1) + n(-y' + y'') - 1(y'') = 0$$

$$\Rightarrow (a-1)y'' - ay' + y = 0$$

$$(21) \quad y = a e^x + b e^{-x}; (a, b)$$

$$(22) \quad x = A \sin t + B \cos t + t \sin t; (A, B)$$

$$\text{so } x - t \sin t = A \sin t + B \cos t \quad (1)$$

$$\Rightarrow x' - t \cos t - \sin t = A \cos t - B \sin t \quad (2)$$

$$\Rightarrow x'' - \cos t + t \sin t - \cos t = -A \sin t - B \cos t$$

$$\Rightarrow x'' - 2 \cos t + t \sin t = - (A \sin t + B \cos t)$$

$$\Rightarrow x'' - 2 \cos t + t \sin t = x - t \sin t \quad (\text{by (1)})$$

$$\Rightarrow \boxed{x'' - 2 \cos t + 2t \sin t = x}$$

$$(23) \quad y = A \sin(m\alpha + \alpha); (x, \alpha)$$

$$\text{so } y' = [A \cos(m\alpha + \alpha)] m \quad (1)$$

$$y'' = -[A \sin(m\alpha + \alpha)] m^2$$

$$\boxed{y'' = -y m^2} \text{ which is the reqd diff. eqn.}$$

$$\textcircled{2} \Rightarrow y = \underbrace{[1 + (y^1)^2]}_{y''} y'$$

$$\therefore \textcircled{1} \Rightarrow \frac{[1 + (y^1)^2]}{(y'')^2} (y^1)' + \frac{[1 + (y^1)^2]^2}{(y'')^2} = r^2$$

$$\Rightarrow \frac{[1 + (y^1)^2]}{(y'')^2} [(y^1)' + 1] = r^2$$

$$\Rightarrow \boxed{[1 + (y^1)^2]^2 = r^2 (y'')^2}$$

[28] $y = ae^x + be^{-x} + c\cos x + d\sin x$ (a, b, c, d) (1)

Sol $y^1 = ae^x - be^{-x} - c\sin x + d\cos x$ (2)

$$y^2 = ae^x + be^{-x} - c\cos x - d\sin x$$
 (3)

$$y^3 = ae^x + be^{-x} + c\sin x - d\cos x$$
 (4)

$$y^4 = ae^x + be^{-x} + c\cos x + d\sin x$$
 (5)

$$\Rightarrow \boxed{y^4 = y} \quad (\text{from } (1)),$$

which is the required diff. eqn.

HW

[29] Find the diff. eqn of the family of straight lines $y = mx + \frac{a}{m}$

where 'm' is the parameter.

[30] Find the diff. eqn of family of all circles of fixed radius 'r' and centres on y-axis $x^2 + (y-k)^2 = r^2$ (k is parameter, r is constant).

24 $x = A \cos(pt - \lambda) ; (A, \lambda)$

Ans: $\boxed{y'' = -p^2 x}$

(10)

25 $y = A \sin \alpha ; \alpha \text{ is parameter.}$

$yy' = A \alpha$

$\therefore \text{①} \boxed{y'' = (yy') \alpha}$

26 $y = a \cos(\alpha + \beta) ; \alpha \text{ is parameter}$

①

$y' = -a \sin(\alpha + \beta)$

$\frac{\text{①}}{\text{②}} = \frac{y}{y'} = -\cot(\alpha + \beta)$

$\Rightarrow \frac{y'}{y} = -\tan(\alpha + \beta)$

$\Rightarrow \boxed{y' + y \tan(\alpha + \beta) = 0}$

27 Find the diff. equation of the family of circles

$(x-h)^2 + (y-k)^2 = r^2$ where h and k are parameters
as r is fixed constant.

Sol. $(x-h)^2 + (y-k)^2 = r^2$ ①

$(x-h) + (y-k)y' = 0$ ②

$\Rightarrow 1 + (y-k)y'' + (y')^2 = 0$

$\Rightarrow \boxed{(y-k) = -\frac{1+(y')^2}{y''}}$

H.W.

- [31] Find the diff. eqn of all circles of fixed radius 'r' and centres on x-axis.

$$(x-h)^2 + y^2 = r^2, h \text{ is parameter.}$$

H.W.

- [32] The family of all circles touching x-axis at the origin is

$$x^2 + y^2 - ry = 0.$$

H.W.

- [33] The family of all circles touching y-axis at the origin is $x^2 + y^2 - rx = 0$. (r is parameter).

* Solution of Differential equations of the first order and first degree:

Defn: A diff. eqn of first order and first degree is an eqn of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ (or) $M dx + N dy = 0$

where M and N are functions of x & y .

The first order, first degree diff. eqns solving into four methods

(i) Variables separable & (ii) Homogeneous equations (iii) Exact eqns (iv) Linear equations.

(i) Variables separable:

If it is an eqn, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to another side, then the variables are said to be separable.

HW

- [31] Find the diff. eqn of all circles of fixed radius r' and centres on x -axis.

$$(x-h)^2 + y^2 = r'^2, h \text{ is parameter.}$$

HW

- [32] The family of all circles touching x -axis at the origin is

$$x^2 + y^2 - 2ay = 0.$$

HW

- [33] The family of all circles touching y -axis at the origin is $x^2 + y^2 - 2ax = 0$. (a is parameter).

* Solution of Differential equations

of the first order and first degree:

Defn: A diff. eqn of first order and first degree is an eqn of the form $\frac{dy}{dx} = f(x, y)$ (or) $M dx + N dy = 0$

where M and N are functions of x & y .

The first order, first degree diff. eqns solving into four methods

(i) Variables separable & (ii) Homogeneous equations (iii) Exact eqns (iv) Linear equations.

(i) Variables separable:

If it is an eqn, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to another side, then the variables are said to be separated.

Working rule :-

Step(1) : Consider the equation $\frac{dy}{dx} = xy$;

where x is a function -

of x only and

y is a function of x

only.

Step(2) : $\frac{dy}{y} = x dx$; i.e. the variables
have been separated.

Step(3) : Integrating on both sides,

$$\int \frac{dy}{y} = \int x dx + C, \text{ where } C \text{ is}$$

an arbitrary constant

Note :- (1) Never forget to add an
arbitrary constant on one side (only).

A solution without this arbitrary constant
is wrong, for it is not a general
solution.

(2). The nature of the arbitrary
constant depends upon the nature of
the problem.

Problems.

→ solve the following diff. eqns.

$$\text{I} \quad \frac{dy}{dx} = e^{x+y} + x^2 e^y$$

→ ①

$$\frac{dy}{dx} = e^x \cdot e^y + x^2 e^y$$

$$\frac{dy}{dx} = e^y (e^x + x^2)$$

$$\Rightarrow \frac{1}{e^y} dy = (e^x + x^2) dx$$

i.e. the variables have been
separated

Now integrating on both sides, we get

(12)

$$\int \frac{dy}{e^y} = \int (e^x + x^2) dx + C$$

$$\Rightarrow -e^{-y} y = e^x + \frac{x^3}{3} + C$$

$$\Rightarrow \boxed{-e^{-y} y = e^x + \frac{x^3}{3} + C \quad \text{which is the reqd g.s.}}$$

$$\begin{aligned} (2) \quad \frac{dy}{dx} &= e^{x-y} + e^{2x+y} \\ \Rightarrow \frac{dy}{dx} &= e^x \cdot e^{-y} + e^{2x+y} \cdot e^y \\ \Rightarrow \frac{dy}{dx} &= e^{-y} (e^x + e^{2x}) \\ \Rightarrow dy &= (e^x + e^{2x}) dx \end{aligned}$$

Integrating on both sides, we get

$$\boxed{e^y = e^x + \frac{x^3}{3} + C \quad \text{which is the reqd g.s.}}$$

$$(3) \quad y \frac{dy}{dx} = x e^{x+y}$$

$$y \frac{dy}{dx} = \frac{x e^x}{e^{-y}} \Rightarrow y e^{-y} dy = x e^x dx$$

Integrating on both sides, we get

$$\int y e^{-y} dy = \int x e^x dx + C$$

$$\Rightarrow -\frac{e^{-y}}{2} = \frac{e^x}{2} + \frac{C}{2} \quad \text{put } y = t, dy = dt$$

$$\Rightarrow \boxed{e^{-y} + e^x + C = 0}$$

[4] $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{1+y^2}{1+x^2}\right]$$
$$\Rightarrow \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$$
$$\Rightarrow \boxed{\tan^{-1} y + \tan^{-1} x = \tan^{-1} C}$$

[5] $y dx - x dy = xy dx$

$$\Rightarrow \frac{1}{x} dx - \frac{1}{y} dy = dx$$

Integrating on both sides, we get

$$\log y - \log x = x + \log C$$
$$\Rightarrow \log\left(\frac{y}{x}\right) = x$$
$$\Rightarrow \frac{y}{x} = e^x \Rightarrow \boxed{\frac{y}{x} = e^x C}$$

[6] $\sec y \tan y dx + \sec y \tan x dy = 0$

$$\Rightarrow \frac{\sec y}{\tan y} dy = -\frac{\sec x}{\tan x} dx$$
$$\Rightarrow \log(\tan y) = -\log(\tan x) + \log C$$
$$\Rightarrow \log(\tan x \cdot \tan y) = \log C$$
$$\Rightarrow \boxed{\tan x \cdot \tan y = C}$$

[7] $\log\left(\frac{dy}{dx}\right) = ax + by$

$$\Rightarrow \frac{dy}{dx} = e^{ax} \cdot e^{by}$$

$$\Rightarrow \frac{dy}{e^{by}} = e^{ax} dx$$

on integrating on both sides, we get,

$$\left| \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + \frac{e^c}{c} \right|$$

[8] $3e^x + \tan y dx + (1-e^x) \sec^2 y dy = 0.$

$$\Leftrightarrow \frac{3e^x}{1-e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

on integrating on both sides, we get,

$$-3 \log(1-e^x) + \log(\tan y) = \log c.$$

$$\Rightarrow \left| \frac{\tan y}{(1-e^x)^3} = C \right|$$

[9] $(y - n \frac{dy}{da})_n = y$

$$\Rightarrow \frac{dy}{da} - n \frac{dy}{da} = y$$

$$\Rightarrow (n-1)y = n \frac{dy}{da}$$

$$\Rightarrow \frac{n-1}{n} da = \frac{1}{y} dy$$

$$\Rightarrow \frac{1}{y} dy = \left(\frac{1}{n} - \frac{1}{n} \right) da$$

$$\Rightarrow \log y = \log n + \frac{1}{n} + \log c.$$

$$\Rightarrow \boxed{y = n c e^{\frac{1}{n}}}$$

[10] $(a^n - ya^m) dy + (y^m + a y^n) da = 0$

[11] $\frac{dy}{da} = a y + a + y + 1.$

SOL

(13)

$$\Rightarrow \frac{dy}{da} = a(y+1) + (y+1)$$

$$= (y+1)(a+1)$$

$$\Rightarrow \frac{dy}{(y+1)} = (a+1)da$$

$$\boxed{\log(y+1) = \frac{a^2}{2} + a + C}$$

[12] $a(1+y^2)da + y(1+a^2)dy = 0$

$$\Rightarrow \frac{a}{1+a^2}da + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{\log(1+a^2)}{2} + \frac{\log(1+y^2)}{2} = \log C$$

$$\Rightarrow \log[(a^2+1)(y^2+1)] = \log C$$

$$\Rightarrow \boxed{(a^2+1)(y^2+1) = C}$$

[13] $\frac{dy}{da} + \sqrt{\frac{1-y^2}{1-a^2}} = 0$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{da}{\sqrt{1-a^2}} = 0$$

Integrating on both sides, we get

$$\sin^{-1}y + \sin^{-1}a = \sin^{-1}C$$

which is the reqd. eqn.

[14] find the equation of the curve passing through the point (1,1) whose diff. equation is $(y-y^2)da + (a+ay)dy = 0$ ①

sol $y(1-y)da + a(1+y)dy = 0$

$$\Rightarrow \frac{1+y}{y^2}dy + \frac{1-a}{a^2}da = 0$$

integrating, we get

$$\log y + \log x - x + y = C$$

$$\Rightarrow \boxed{\log(xy) + (y-x) = C}$$

which is reqd eqn of the curve.

Since the pt $(1, 1)$, passing through $\textcircled{1}$, we get

$$\log(1 \cdot 1) + (1-1) = C$$

$$\Rightarrow \boxed{C=0}$$

∴ from $\textcircled{2}$, we have

$$\boxed{\log(xy) + (y-x) = 0}$$

H.W $\textcircled{15}$ find the equation of the curve that passes through the point $(1, 2)$ and satisfies the equation $\frac{dy}{dx} = -\frac{2xy}{x^2+1}$.

*SECOND FORM:

Equations reducible to the form $\frac{dy}{dx}$ which variables can be separated:

Equations of the form $\frac{dy}{dx} = f(ax+by+c)$

can be reduced to the form $\textcircled{1}$ which the variables are separable.

$$\text{put } ax+by+c = z$$

diff. w.r.t x , we get,

$$a+by \frac{dy}{dx} = \frac{dz}{da}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b} \left(\frac{dz}{da} - a \right)$$

$$\therefore \textcircled{1} \equiv \frac{1}{b} \left(\frac{dz}{da} - a \right) = f(z)$$

$$\Rightarrow \frac{dz}{(a+bf(z))} = dx$$

The variables have been separated.

Integrating on both sides, we get solution.

problems

→ solve the following diff. equations:

$$\text{① } \frac{dy}{dx} = (3x+y+4)^{-2}$$

$$\text{put } 3x+y+4 = z$$

diff. w.r.t. x, we get

$$3+y' = \frac{dz}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dz}{dx} - 3}$$

∴ from ①, we have,

$$\frac{dz}{dx} - 3 = z^{-2}$$

$$\Rightarrow \frac{dz}{dx} = z^{-2} + 3$$

$$\Rightarrow \frac{dz}{z^{-2} + 3} = dx$$

Integrating on both sides, we get,

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{z}{\sqrt{3}} \right) = x + C$$

$$\begin{aligned} & \int \frac{1}{z^2 + 3} dz \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{z}{\sqrt{3}} \right) + C \end{aligned}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{3x+y+4}{\sqrt{3}} \right) = x + C$$

$$\Rightarrow \tan^{-1} \left(\frac{3x+y+4}{\sqrt{3}} \right) = \sqrt{3}(x+C)$$

[2] $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ (1)

Sol put $x+y = z$

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$$

∴ from (1), we have,

$$\frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{dx} = (1 + \cos z) + \sin z$$

$$= 2 \cos^2 \left(\frac{z}{2}\right) + 2 \sin \frac{z}{2} \cos \frac{z}{2}$$

$$= 2 \cos^2 \left(\frac{z}{2}\right) \left[1 + \tan \left(\frac{z}{2}\right)\right]$$

$$\Rightarrow \frac{dz}{2 \cos^2 \left(\frac{z}{2}\right) \left(1 + \tan \left(\frac{z}{2}\right)\right)} = dx$$

$$\Rightarrow \int \frac{\sec^2 \left(\frac{z}{2}\right)}{2 \left[1 + \tan \left(\frac{z}{2}\right)\right]} dz = \int dx$$

$$\Rightarrow \log \left[1 + \tan \left(\frac{z}{2}\right)\right] = x + C$$

$$\Rightarrow \boxed{\log \left[1 + \tan \left(\frac{x+y}{2}\right)\right] = x + C}$$

[3] $\frac{dy}{dx} + 1 = e^{x+y}$

$$\Rightarrow \frac{dy}{dx} = e^{x+y} - 1$$
(1)

put $x+y = z$

$$\boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$$

(15)

$$\begin{aligned} \text{Q} \quad & \frac{\partial z}{\partial x} - 1 = e^z - 1 \\ \Rightarrow & \frac{\partial z}{\partial x} = e^z \\ \Rightarrow & e^{-z} = x + C \\ \Rightarrow & e^{-(x+y)} = x + C \\ \Rightarrow & \boxed{e^{-(x+y)} + x + C = 0} \end{aligned}$$

$$\begin{aligned} \text{4} \quad & (e^y + 1) \cos x dx + e^y \sin x dy = 0 \\ \Rightarrow & \frac{\cos x}{\sin x} dx + \frac{e^y}{e^y + 1} dy = 0 \\ \Rightarrow & \log(\sin x) + \log(e^y + 1) = \log C \\ \Rightarrow & \boxed{(\sin x)(e^y + 1) = C} \end{aligned}$$

$$\begin{aligned} \text{5} \quad & \frac{dy}{dx} = \sec(x+y) \quad \text{--- Q} \\ \text{put } & x+y = z \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dz}{dx} - 1 \\ \frac{dz}{dx} &= \sec z \end{aligned}$$

$$\begin{aligned} \therefore \text{Q} \quad & \frac{dz}{dx} - 1 = \sec z \\ \frac{\partial z}{\partial x} &= \sec z \\ \Rightarrow & \left(\frac{\cos z}{1 + \cos z} \right) dz = dx \\ \Rightarrow & \left(1 - \frac{1}{1 + \cos z} \right) dz = dx \\ \Rightarrow & \left(1 - \frac{1}{2 \cos^2 z} \right) dz = dx \\ \Rightarrow & \left(1 - \frac{1}{2 \cos^2 z} \right) dz = dx \end{aligned}$$

$$\Rightarrow z + \tan(z) = x + c$$

$$\Rightarrow xy + \tan\left(\frac{xy}{2}\right) = x + c$$

$$\Rightarrow \boxed{y + \tan\left(\frac{xy}{2}\right) = c}$$

(16)

Third Form :-

Differential eqns of the form

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{m(ax+by)+c}{an(by)+c_1}$$

(17)

$$\text{put } ax+by = z$$

$$\Rightarrow a+by = z^1$$

$$\Rightarrow y^1 = \frac{1}{b}(z^1 - a)$$

$$\therefore (1) \Rightarrow \frac{1}{b} \left(\frac{dz}{da} - a \right) = \frac{z+c}{mz+c_1}$$

$$\Rightarrow \frac{dz}{da} = \frac{b(z+c)}{mz+c_1} + a$$

Now separate the variables,
which can be easily
calculated.

Problems

→ solve the following diff. eqns:

(1) $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

Sol $\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \quad (1)$

put $x-y = z$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\therefore \textcircled{1} \equiv 1 - \frac{\delta z}{dz} = \frac{z+3}{2z+5}$$

$$\Rightarrow \frac{\delta z}{dz} = 1 - \frac{z+3}{2z+5}$$

$$\Rightarrow \frac{-\delta z}{dz} = \frac{z+2}{2z+5}$$

$$\Rightarrow \frac{2z+5}{z+2} dz = -\delta z$$

$$\Rightarrow \left(2 + \frac{1}{z+2}\right) dz = -\delta z$$

Q

$$\int \frac{2}{z+2} dz + \int \frac{1}{2z+5} dz = -\int \delta z$$

$$2 + \frac{R}{D}$$

Integrating on both sides, we get.

$$\Rightarrow 2z + \log(z+2) = x + C$$

$$\Rightarrow 2(x-y) + \log(x-y+z) = x + C$$

$$\Rightarrow \boxed{x-2y + \log(x-y+z) = C}$$

(2)

$$\frac{dy}{dx} = \frac{2y-x-3}{2x-4y+5}$$

$$= \frac{2y-x-3}{2(x-2y)+5} \quad \textcircled{1}$$

put $2y-x = z$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{\delta z}{dz} + 1 \right]$$

$$\therefore \textcircled{1} \equiv \frac{1}{2} \left[\frac{\delta z}{dz} + 1 \right] = \frac{z-3}{-2z+5}$$

$$\Rightarrow \frac{dz}{dx} = \frac{4z-11}{-2z+5}$$

$$\Rightarrow \left[\frac{-2z+5}{4z-11} \right] dz = dx$$

$$\Rightarrow \int \left[-\frac{1}{2} - \frac{1}{2} \frac{1}{(4z-11)} \right] dz = \int dx + C$$

(17)

$$\Rightarrow -\frac{1}{2} \int -y_2 \log(4y_2 - 1) = x + C$$

$$\Rightarrow \boxed{-\frac{1}{2} [y_2^2 - 2y_2] - \frac{1}{8} \log[4(y_2 - 1)] = x + C}$$

which is the reqd eqn

~~Homogeneous~~Differential eqns!~~Homogeneous function!~~

A function $f(x,y)$ is said to be a homogeneous function of degree 'n' in x, y , if $f(kx, ky) = k^n f(x,y)$ & n & k is const.

$$\text{Ex: } f(x,y) = \frac{x^2+y^2}{x^2+y^3} \Rightarrow f(kx, ky) = \frac{k^2x^2+k^2y^2}{k^2x^2+k^3y^2} \\ = k^{-1} \frac{(x^2+y^2)}{(x^2+y^3)} \\ = k^{-1} f(x,y).$$

$\therefore f(x,y)$ is a homogeneous function of degree -1 in x, y .

$$(2) f(x,y) = \frac{\sqrt[3]{x} + \sqrt[3]{y}}{x+y} \Rightarrow f(kx, ky) = \frac{\sqrt[3]{kx} + \sqrt[3]{ky}}{kx+ky} \\ = k^{\frac{1}{3}-1} \left(\frac{\sqrt[3]{x} + \sqrt[3]{y}}{x+y} \right) \\ = k^{-\frac{2}{3}} f(x,y).$$

$\therefore f(x,y)$ is homo. function of degree $-\frac{2}{3}$.

$$(3) f(x,y) = \cos x + \tan y$$

$$\Rightarrow f(kx, ky) = \cos kx + \tan ky$$

$$\neq k^n f(x,y).$$

\therefore $f(x,y)$ is not a homogeneous function.

$$(4) f(x,y) = \frac{(fx + gy)}{fx - gy}$$

$$\Rightarrow f(kx, ky) = k^0 f(xy)$$

i.e. $f(xy)$ is a homogeneous fn.
of degree zero for xy

Note:- If $f(xy)$ is a homogeneous fn
of degree zero then $f(xy)$ is a
function of $y/x^{(or)} \frac{y}{x}$.

* Homogeneous Diff. equation:—
A diff. eqn is said to be
homogeneous, if it can be put in
the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$,
where f, g are homogeneous
functions of same degree in x, y .

Working rule:

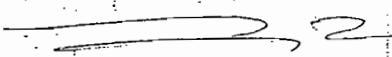
Step(1): Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Step(2): Put the above values
in the given diff. eqn

Step(3): Separate the variables and
integrate.

Step(4): Replace v by y/x to
get the reqd. solution.

Problems:



problems

Solve the following diff. eqn
 $\rightarrow axy dx - (x^3 + y^2) dy = 0$.

$$\Rightarrow \frac{dy}{dx} = \frac{axy}{x^3 + y^2} \quad \text{--- (1)}$$

clearly (1) is homogenous diff. eqn.

$$\text{put } y = vx$$

$$y' = v + x \frac{dv}{dx}$$

$$\text{--- (1)} \equiv v + x \frac{dv}{dx} = \frac{x^2 v}{x^3 + v^2 x^2} = \frac{v}{1 + v^2}$$

$$\therefore x \frac{dv}{dx} = \frac{v - v^2 - v^3}{1 + v^2}$$

$$\Rightarrow \frac{1+v^2}{v^3} dv = \frac{1}{x} dx$$

$$\Rightarrow \int \left(\frac{1}{v^3} + \frac{1}{v} \right) dv = - \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{3} \frac{1}{v^2} + \log v = -\log x + C$$

$$\Rightarrow -\frac{1}{3} \left(\frac{x^3}{y^2} \right) + \log \left(\frac{x}{v} \right) + \log x^2 = C \quad (\because y = vx)$$

$$\Rightarrow \boxed{-\frac{1}{3} \left(\frac{x^3}{y^2} \right) + \log y = C}$$

~~$$(1 + e^{xy}) dx + e^{xy} (1 - y) dy = 0$$~~

$$\text{Sol} \quad \frac{dx}{dy} = -\frac{e^{xy} (1 - y)}{1 + e^{xy}} \quad \text{--- (1)}$$

$$\text{put } z = xy$$

$$\Rightarrow \boxed{z' = v + y \frac{dv}{dy}}$$

$$\text{--- (1)} \equiv v + y \frac{dv}{dy} = -e^{vz} (1 - v)$$

$$\Rightarrow y \frac{dv}{dy} = -e^{vz} + v e^{vz} - v + v$$

$$\Rightarrow y \frac{dv}{dy} = -\frac{[v + e^{vz}]}{1 + e^{vz}} \quad \text{--- (1)}$$

$$\Rightarrow \int \left[\frac{1 + e^{vz}}{v + e^{vz}} \right] dv = \int \frac{1}{y} dy + \log c$$

$$\Rightarrow \log (v + e^{vz}) = -\log y + \log c$$

$$\Rightarrow \log \left(\frac{v}{y} + e^{vz} \right) = -\log y + \log c$$

$$\Rightarrow \log \left[\left(\frac{v}{y} + e^{vz} \right) y \right] = \log c$$

$$\Rightarrow \boxed{\frac{v + y e^{vz}}{y} = c}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{z} + e^{vz} \quad \text{--- (1)}$$

$$\text{put } y = vx$$

$$\Rightarrow y' = v + x \frac{dv}{dx}$$

$$\text{--- (1)} \equiv v + x \frac{dv}{dx} = v + e^{vz}$$

$$\Rightarrow x \frac{dv}{dx} = e^{vz}$$

$$\Rightarrow \int e^{vz} dv = \int \frac{1}{x} dx$$

$$= -e^{-v} = \log x + C$$

$$\Rightarrow \boxed{-e^{-\frac{v}{z}} = \log x + C}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ny}{ny - y} \quad \text{--- (1)}$$

$$\text{put } y = 0 - z \Rightarrow y' = v + x \frac{dv}{dx}$$

$$\text{--- (1)} \equiv v + x \frac{dv}{dx} = \frac{z + vx}{z - vx}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v$$

$$= \frac{1 + v - v^2 + v^2}{1 - v^2}$$

$$\Rightarrow \frac{1-x}{1+x} dx = \frac{1}{2} dx$$

$$\Rightarrow \int \left(\frac{1}{1+x} - \frac{x}{1+x} \right) dx = \int \frac{1}{2} dx$$

$$\Rightarrow \tan^{-1} x - \frac{1}{2} \log(1+x) = \log x + C$$

$$\Rightarrow \tan^{-1}(x) = \log \left[\frac{(1+x)^{1/2}}{x^{1/2}} \right] + C$$

$$\Rightarrow \tan^{-1}(x) = \log \left[\frac{(1+x)^{1/2}}{x^{1/2}} \right] + C$$

Non-Homogeneous Diff. eqns:-

(Or,

eqns Reducible to Hom. form:-

Diff. eqns of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad (1)$$

$$\text{Case (I)} \quad \frac{a}{a_1} \neq \frac{b}{b_1} \quad \text{i.e. } ab_1 - a_1b \neq 0$$

Working rule:

put $x+h$ & $y+k$

where

h & k are constants

$$dx = dx, dy = dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dx}$$

$$(1) \equiv \frac{dy}{dx} = \frac{a(x+h) + b(y+k) + c}{a_1(x+h) + b_1(y+k) + c_1}$$

$$= \frac{ax+by+(ah+bk+c)}{a_1x+b_1y+(a_1h+b_1k+c)}$$

→ (2)

choosing h & k s.t.

$$ah+bk+c=0$$

$$a_1h+b_1k+c_1=0$$

solving these equations

we get

the values of h & k

$$\text{i.e. } \frac{h}{b_1c_1 - b_1c} = \frac{k}{a_1c_1 - a_1c} = \frac{1}{ab_1 - a_1b}$$

$$\Rightarrow h = \frac{b_1c_1 - b_1c}{ab_1 - a_1b} \text{ & } k = \frac{a_1c_1 - a_1c}{ab_1 - a_1b}$$

$$(2) \equiv \frac{dy}{dx} = \frac{ax+by}{a_1x+b_1y}$$

which is clearly homogeneous.

This eqn can be solved

by putting $Y = vx$

finally replacing Y by $Y+k$
and x by $x+h$.

$$\text{Case (II)}: \quad \frac{a}{a_1} = \frac{b}{b_1} \quad \text{i.e. } ab_1 - a_1b = 0$$

∴ h & k both become

infinite.

Hence the method fails.

$$\text{Now } \frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{m} \text{ (say)}$$

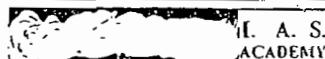
$$\Rightarrow a_1 = am, b_1 = bm$$

$$\therefore (1) \equiv \frac{dy}{dx} = \frac{ax+by+c}{amx+bmby+c_1}$$

$$= \frac{(ax+by)+c}{m(ax+by)+c_1}$$

put $ax+by = z$

this can be easily solved
by variable separable



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MATHEMATICS

By K. VENKANNA

Problems

$$\Rightarrow \frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad \text{--- (1)}$$

$$\frac{1}{1} \neq \frac{1}{-1} \text{ i.e. } \frac{a}{a_1} \neq \frac{b}{b_1}$$

clearly (1) is not homo.

$$\text{put } x = x+h ; y = Y+k$$

$$dx = dx ; dy = dY$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dx}}$$

$$\begin{aligned} \text{--- (1)} &\equiv \frac{dY}{dx} = \frac{Y+k+x+h-2}{Y+k-x-h-4} \\ &= \frac{(x+Y)+(h+k-2)}{(-x+Y)+(h+k-4)} \end{aligned} \quad \text{--- (2)}$$

choosing h, k s.t.

$$\begin{aligned} h+k-2 &= 0 \quad \text{--- (3)} \\ -h+k-4 &= 0 \quad \text{--- (4)} \end{aligned}$$

$$2k = 6$$

$$\Rightarrow \boxed{k=3}$$

$$\boxed{h=-1}$$

$$\therefore \text{--- (2)} \equiv \frac{dY}{dx} = \frac{x+Y}{-x+Y} \quad \text{--- (5)}$$

$$\text{put } Y = v x$$

$$\frac{dY}{dx} = v + x \frac{dv}{dx}$$

$$\therefore \text{--- (5)} \equiv v + x \frac{dv}{dx} = \frac{x(1+v)}{x(-1+v)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+v}{-1+v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+v+V/V-1/V}{-1+v} \\ = \frac{1+v-V^2}{V-1}$$

$$\Rightarrow \frac{V-1}{1+V-V^2} dv = \frac{1}{x} dx$$

$$\Rightarrow \left[\frac{-\frac{1}{2}(-2V+2)}{-V^2+2V+1} \right] dv = \int \frac{1}{x} dx + C$$

$$\Rightarrow -\frac{1}{2} \log [-V^2+2V+1] = \log x + \log C$$

$$\Rightarrow \log \left(\frac{1}{[-V^2+2V+1]^{\frac{1}{2}}} \right) = \log (x C)$$

$$\Rightarrow \frac{1}{[-V^2+2V+1]^{\frac{1}{2}}} = x C$$

$$\Rightarrow \frac{x}{[-V^2+2V+1]^{\frac{1}{2}}} = x C$$

$$\Rightarrow \frac{1}{[-(Y-k)+2(Y-k)(x-h)+(x-h)^2]^{\frac{1}{2}}} = x C$$

$$\Rightarrow \frac{1}{-Y+3+2(Y-3)(x+1)+(x+1)^2} = x C$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y+3}{x^2-2xy+5}$$

13 (6) $\left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + (x + \log x - x \sin y) dy = 0$ ①

Soln $M = y\left(1+\frac{1}{x}\right) + \cos y ; N = x + \log x - x \sin y$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y ; \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - x \sin y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\therefore G.S. of ① is

$$\int [y\left(1+\frac{1}{x}\right) + \cos y] dx + \int 0 dy = 0$$

$$\Rightarrow \boxed{y(x + \log x) + x \cos y = C}$$

(7) $(2ax + by)y dx + (ax + 2by)x dy = 0$

Soln $M = 2axy + by^2 ; N = ax^2 + 2byx$

$$\frac{\partial M}{\partial y} = 2ax + 2by ; \quad \frac{\partial N}{\partial x} = 2ax + 2by$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S. is $\int (2axy + by^2) dx + \int 0 dy = 0$

$$\Rightarrow \boxed{ax^2y + bxy^2 = C}$$

(8) $(x^2 - ay) dx - (ax - y^2) dy = 0$

Soln $M = x^2 - ay ; N = -(ax - y^2)$

$$\frac{\partial M}{\partial y} = -a ; \quad \frac{\partial N}{\partial x} = -a$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S. is $\int (x^2 - ay) dx + \int y^2 dy = 0$

$$\Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = C_1$$

$$\Rightarrow \boxed{x^3 - 3axy + y^3 = C} \text{ where } C = 3C_1$$

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$$\boxed{10} \quad \frac{dy}{dx} = \frac{2x-y}{x+2y-5}$$

$\Rightarrow (2x-y)dx + (x+2y-5)dy = 0 \quad \text{--- (1)}$

$M = 2x-y ; N = x+2y-5$

$$\frac{\partial M}{\partial y} = -1 ; \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S is $\int (2x-y) dx - \int (2y-5) dy = \int 0$

$$\Rightarrow \boxed{x^2 - xy - y^2 + 5y = c}$$

H.W $\boxed{11} \cdot (x^2 + y^2 + a^2)y dy + (x^2 + y^2 - a^2)x dx = 0$

$\boxed{12} \cdot x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0$

$\boxed{13} \cdot (a^2 - axy - y^2)dx - (x+y)^2 dy = 0$

$\boxed{14} \cdot (1 + e^{xy}) dx + e^{xy} \left(1 - \frac{x}{y}\right) dy = 0$

Soln: $M = 1 + e^{xy} ; N = e^{xy} \left(1 - \frac{x}{y}\right)$

$$\begin{aligned} \frac{\partial M}{\partial y} &= e^{xy} \left(-\frac{x}{y^2}\right) ; \frac{\partial N}{\partial x} = e^{xy} \left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{xy} \cdot \frac{1}{y^2} \\ &= -e^{xy} \cdot \frac{x}{y^2} \end{aligned}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\therefore G.S. is

$$\int (1 + e^{xy}) dx + \int 0 dy = \int 0$$

$$\Rightarrow x + e^{xy} = c$$

$$\Rightarrow \boxed{x + y e^{xy} = c}$$

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Exact Differential equations:

Defn. The diff. equation $M(x,y)dx + N(x,y)dy = 0$ where

M & N are functions of x & y , is called an exact diff. equation if $Mdx + Ndy = 0$ is an exact derivative of x & y .

i.e., $Mdx + Ndy = du$, where u is a function of x & y .

$$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

(Ex 1) $xdy + ydx = 0$ is an exact.

$$\text{Because } xdy + ydx = d(xy)$$

$$\Rightarrow xdy + ydx = \frac{\partial}{\partial x}(xy) dx + \frac{\partial}{\partial y}(xy) dy$$

$\frac{1}{x}dy - \frac{y}{x^2}dx = 0$ is an exact.

$$\text{Because } \frac{1}{x}dy - \frac{y}{x^2}dx = \underline{xdy - ydx} = d(y/x)$$

Note: The diff. eqn $Mdx + Ndy = 0$ is an exact

$$\text{if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working rule:

(1) The diff. eqn $Mdx + Ndy = 0$ is an exact

$$\text{if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(2) The G.S. is

$$\int M dx + \int (\text{terms in } N \text{ not containing } x) dy = C$$

y-constant \downarrow Drop the terms containing x

Problems! Solve the following diff-equations.

(1) $(x+2y-2)dx + (2x-y+3)dy = 0$ — ①

Comparing ① with $Mdx + Ndy = 0$

we have $\frac{\partial M}{\partial y} = 2$; $\frac{\partial N}{\partial x} = 2$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S. is $\int (x+2y-2)dx + \int (-y+3)dy = 0$

$$\Rightarrow \frac{x^2}{2} + 2xy - 2x - \frac{y^2}{2} + 3y = C.$$

$$(2) \frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$$

$$\Rightarrow (ax+hy+g)dx + (hx+by+f)dy = 0 \quad \text{--- (1)}$$

$$\frac{\partial M}{\partial y} = h; \quad \frac{\partial N}{\partial x} = h$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S of (1) is

$$\int(ax+hy+g)dx + \int(by+f)dy = \int 0$$

$$\Rightarrow ax^2 + hxy + gx + by^2 + fy = c$$

$$\Rightarrow ax^2 + by^2 + 2gx + 2fy + 2hxy = 2c.$$

Hence (3) $(y^2 e^{xy^2} + 4x^3)dx + (2xy e^{xy^2} - 3y^2)dy = 0 (e^{xy^2} + x^4 - y^3 = c)$.

(4) $(e^y + 1) \cos 2x dx + e^y \sin 2y dy = 0$

Soln: $(e^y + 1) \cos 2x dx + e^y \sin 2y dy = d[(e^y + 1) \sin x]$

\therefore G.S. is $(e^y + 1) \sin x = c$

Ques (5). $y \sin 2x dx - (y^2 + \cos^2 x + 1)dy = 0 \quad \text{--- (1)}$

Soln: $M = y \sin 2x; \quad N = -y^2 - \cos^2 x - 1$

$$\Rightarrow \frac{\partial M}{\partial y} = \sin 2x; \quad \frac{\partial N}{\partial x} = 2 \cos x \sin x \\ = \sin 2x.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

G.S. of (1) is

$$\int y \sin 2x dx - \int (y^2 + 1) dy = \int 0$$

$$\Rightarrow -y \frac{\cos 2x}{2} - \frac{y^3}{3} - y = C_1$$

$$\Rightarrow y \frac{\cos 2x}{2} + \frac{y^3}{3} + y = C \quad \text{where } C = -C_1$$

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Integrating factor:

Sometimes $Mdx + Ndy = 0$ is not exact but it can be made exact by multiplying throughout by a suitable non-zero function $\mu(x, y)$. This multiplier is called the integrating factor.

Note: If the given diff. eqn can be transformed into the following formulas then the equations are exact.

$$(1) \quad d(xy) = ady + y dx$$

$$(2) \quad d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$(3) \quad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(4) \quad d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$$

$$(5) \quad d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$$

$$(6) \quad d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4} = \frac{2xy(ydx - xdy)}{x^4}$$

$$(7) \quad d\left(\frac{x^2}{y^2}\right) = \frac{2y^2x dx - 2x^2y dy}{y^4} = \frac{2xy(ydx - xdy)}{y^4}$$

$$(8) \quad d\left(\frac{e^y}{x}\right) = \frac{x e^y dy - e^y dx}{x^2}$$

$$(9) \quad d\left(\frac{e^x}{y}\right) = \frac{y e^x dx - e^x dy}{y^2}$$

$$(10) \quad d\left[\log|xy|\right] = \frac{x dy + y dx}{xy}$$

$$(11) \quad d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x dy - y dx}{xy}$$

$$(12) \quad d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y dx - x dy}{xy}$$

Ex:- 1 $2y \, dx + x \, dy = 0 \quad \text{--- (1)}$

$$M = 2y ; N = x$$

$$\frac{\partial M}{\partial y} = 2 ; \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying (1) with x , we get

$$2xy \, dx + x^2 \, dy = 0 \quad \text{--- (2)}$$

$$\frac{\partial M}{\partial y} = 2x ; \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\therefore eqn (2) is an exact.

$\therefore x$ is an integrating factor of

$$2y \, dx + x \, dy = 0$$

and $\underline{2y \, dx + x^2 \, dy = d(x^2y)}$.

Ex:- 2

$$y \, dx - x \, dy = 0 \quad \text{--- (1)}$$

$$\frac{\partial M}{\partial y} = 1 ; \frac{\partial N}{\partial x} = -1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

\therefore (1) is not exact.

Multiplying (1) by $\frac{1}{x^2}$, we get,

$$\frac{y}{x^2} \, dx - \frac{1}{x} \, dy = 0 \quad \text{--- (2)}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} ; \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

\therefore (2) is an exact.

$\therefore \frac{1}{x^2}$ is I.F. of (1).

$$\text{and } -\frac{y}{x^2} \, dx + \frac{1}{x} \, dy = 0$$

$$\Rightarrow \underline{-\frac{y}{x^2} \, dx + \frac{1}{x} \, dy = d(y/x)}$$

$$\boxed{4.} \quad ydx - xdy + (1+x^2)dx + x^2 \sin y dy = 0$$

$$\Rightarrow \frac{xdy - ydx}{x^2} - \left(\frac{1}{x^2} + 1\right)dx - \sin y dy = 0$$

$$\Rightarrow d(y/x) - \left(\frac{1}{x^2} + 1\right)dx - \sin y dy = 0$$

Integrating, we get

$$\boxed{y/x - \left(x - \frac{1}{x}\right) + \cos y = C}$$

$$\boxed{5.} \quad y \sin 2x dx = (1+y^2 + \cos^2 x)dy$$

$$\Rightarrow \cos^2 x dy - 2y \sin x \cos x dx + (1+y^2)dy = 0$$

$$\Rightarrow d(y \cos^2 x) + (1+y^2)dy = 0$$

Integrating, we get

$$\boxed{y \cos^2 x + y + \frac{y^3}{3} = C}$$

$$\boxed{6.} \quad y(2x^2 y + e^x)dx - (e^x + y^3)dy = 0$$

$$\Rightarrow e^x dy - e^x dy + 2x^2 y^2 dx - y^3 dy = 0$$

$$\Rightarrow \frac{e^x dy - e^x dy}{y^2} + 2x^2 dx - y dy = 0$$

$$\Rightarrow d(e^x/y) + 2x^2 dx - y dy = 0$$

Integrating, we get

$$\boxed{\frac{e^x}{y} + \frac{2}{3}x^3 - \frac{y^2}{2} = C}$$

$$\boxed{7.} \quad xdy = [y + x \cos^2(y/x)]dx$$

$$\Rightarrow \frac{x dy - y dx}{x^2} = -\frac{1}{x} \cos^2(y/x) dx$$

$$\Rightarrow \frac{x dy - y dx}{x^2 \cos^2(y/x)} = \frac{1}{x} dx$$

$$\Rightarrow \sec^2(y/x) \left[\frac{x dy - y dx}{x^2} \right] = \frac{1}{x} dx$$

$$\Rightarrow d[\tan(y/x)] = \frac{1}{x} dx$$

Integrating, we get

$$\boxed{\tan(y/x) = \log x + C}$$

$$\text{Also } \frac{ydx - ady}{y^2} = d\left(\frac{x}{y}\right);$$

$$\frac{ydx - ady}{xy} = d\left[\log\left(\frac{x}{y}\right)\right];$$

$$\text{and } \frac{ydx - ady}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$\therefore \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2 + y^2}$ are Integrating factors
of $ydx - ady = 0$

From the above example we observe that a diff. eqn has more than one I.F.

Problems

$$\text{1. } xdy - ydx + 2x^3dx = 0$$

$$\text{Soln} \Rightarrow \frac{xdy - ydx}{x^2} + 2x^3dx = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + 2x^3dx = 0$$

Integrating, we get

$$\boxed{\frac{y}{x} + x^2 = c}$$

$$\text{2. } \frac{xdy - ydx}{x^2 + y^2} = xdx$$

$$\text{Soln: } \Rightarrow d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = xdx$$

Integrating, we get

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) = \frac{x^2}{2} + c}$$

$$\text{3. } xdy - ydx = xy^2dx$$

$$\text{Soln: } \Rightarrow \frac{ydx - xdy}{y^2} + xdx = 0 \Rightarrow d\left(\frac{x}{y}\right) + xdx = 0$$

Integrating, we get

$$\boxed{\frac{x}{y} + \frac{x^2}{2} = c}$$

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$$[8]. \left(y + \cos y + \frac{1}{2\sqrt{x}} \right) dx + (x - x \sin y - 1) dy = 0$$

$$\Rightarrow (y dx + x dy) + (\cos y dx - x \sin y dy) + \frac{1}{2\sqrt{x}} dx - dy = 0$$

$$\Rightarrow d(xy) + d(x \cos y) + \frac{1}{2\sqrt{x}} - dy = 0$$

Integrating

$$[xy + x \cos y + \sqrt{x} - y = C]$$

$$[9]. (xy^2 + 2x^2y^3) dx + (x^2y - x^3y^2) dy = 0$$

$$\Rightarrow xy^2(1+2xy) dx + x^2y(1-xy) dy = 0$$

$$\Rightarrow y(1+2xy) dx + x(1-xy) dy = 0$$

$$\Rightarrow (y dx + x dy) + 2xy^2 dx - x^2y dy = 0$$

$$\Rightarrow \frac{y dx + x dy}{x^2y^2} + \frac{2}{x} dx - \frac{1}{y} dy = 0$$

$$\Rightarrow \frac{d(xy)}{x^2y^2} + \frac{2}{x} dx - \frac{1}{y} dy = 0 \quad \text{--- (2)}$$

Clearly (2) is an exact.

Integrating (2), we get

$$-\frac{1}{xy} + 2\log x - \log y = \log C$$

$$\Rightarrow -\frac{1}{xy} + \log\left(\frac{x^2}{yC}\right) = 0$$

$$\Rightarrow \log\left(\frac{x^2}{yC}\right) = \frac{1}{xy}$$

$$\Rightarrow \boxed{\frac{x^2}{yC} = e^{\frac{1}{xy}}}$$

$$[10]. (x^2 + y^2 - a^2) y dy + x(x^2 + y^2 - b^2) dx = 0 \quad \text{--- (1)}$$

$$\Rightarrow (x^2 + y^2) [2y dy + 2x dx] - 2a^2 y dy - 2ab^2 dx = 0$$

$$\Rightarrow (x^2 + y^2) d(x^2 + y^2) - 2a^2 y dy - 2ab^2 dx = 0$$

Integrating.

$$\int z dz - 2a^2 y dy - 2b^2 \int 2x dx = 0 ; \text{ where } x^2 + y^2 = z$$

$$\Rightarrow \frac{x^2}{2} - a^2y^2 - b^2x^2 = C_1$$

$$\Rightarrow [(x^2 + y^2)^2 - 2a^2y^2 - 2b^2x^2] = C \quad \text{where } C = 2C_1$$

(11). $xdy - (y-x)dx = 0 \quad \dots \quad (1)$

$$\Rightarrow xdy - ydx + xdx = 0$$

$$\Rightarrow \frac{xdy - ydx}{x^2} + \frac{1}{x}dx = 0$$

$$\Rightarrow d(\frac{y}{x}) + \frac{1}{x}dx = 0 \quad \dots \quad (2)$$

$$\Rightarrow \int d(\frac{y}{x}) + \int \frac{1}{x}dx = 0$$

$$\Rightarrow \frac{y}{x} + \log x = C$$

H.W.

(12). $xdy - ydx = xy^2dx$.

(13). $xdx + ydy + (x^2 + y^2)dy = 0$

$$\Rightarrow \frac{xdx + ydy}{x^2 + y^2} + dy = 0$$

$$\Rightarrow d(\log \sqrt{x^2 + y^2}) + dy = 0$$

Integrating, we get,

$$\log \sqrt{x^2 + y^2} + y = C$$

(14). $xdy - ydx = (x^2 + y^2)dx$

$$\Rightarrow \frac{xdy - ydx}{x^2 + y^2} = dx$$

$$\Rightarrow d(\tan^{-1}(\frac{y}{x})) = dx$$

Integrating, we get

$$\tan^{-1}(\frac{y}{x}) = x + C$$

(15). $ydx + xdy + \log x dx = 0$

$$\Rightarrow -\frac{ydx + xdy}{x^2} = \frac{\log x}{x^2} \quad \begin{aligned} \log x &= t \Rightarrow \frac{1}{2}dx = dt \\ &\Rightarrow x = e^t \end{aligned}$$

$$\int d(\frac{y}{x}) = \int te^t dt + C$$

$$\Rightarrow y/x = -e^t(t+1) + c$$

$$\Rightarrow y/x = -\frac{1}{x}(\log x + 1) + c.$$

(25)

[16]. $(x^2 + y^2 - 2y)dy = 2x dx$.

$$\Rightarrow (x^2 + y^2)dy = 2xdx + 2y dy$$

$$\Rightarrow \frac{2xdx + 2y dy}{x^2 + y^2} = dy$$

$$\Rightarrow d(\log(x^2 + y^2)) = dy$$

Integrating, we get

$$\boxed{\log(x^2 + y^2) = y + C}$$

[17]. $y dx - x dy = 3x^2 e^{x^2} y^2 dx$.

[18]. $(y - xy^2)dx - (x + x^2 y)dy = 0$

$$\Rightarrow y dx - x dy - xy(dy + x dx) = 0$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy - d(xy) = 0$$

Integrating, we get

$$\boxed{\log x - \log y - xy = C}$$

[19]. $y(2xy + e^x)dx + e^x dy = 0$

$$\Rightarrow y e^x dx - e^x dy + 2xy^2 dx = 0$$

$$\Rightarrow d(e^x/y) + 2x dx = 0$$

Integrating, we get

$$\boxed{e^x/y + x^2 = C}$$

[20]. Solve $x^2 \left(\frac{dy}{dx} \right) + 2xy = \sqrt{1-x^2 y^2}$

$$\Rightarrow x \left[\frac{xdy + ydx}{dx} \right] = \sqrt{1-(xy)^2}$$

$$\Rightarrow -\frac{xdy + ydx}{\sqrt{1-(xy)^2}} = \frac{dx}{x}$$

$$\Rightarrow \frac{d(xy)}{\sqrt{1-(xy)^2}} = \frac{dx}{x}$$

Integrating, we get

$$\boxed{\sin^{-1}(xy) - \log x = C}$$

Methods for finding integrating factors:

Method 1: If $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an I.F.

Problems

Find I.F. and solve the following equations.

$$\text{II} \quad x^2y dx - (x^3 + y^3) dy = 0 \quad \textcircled{1}$$

Sol: Comparing \textcircled{1} with $Mdx + Ndy = 0$

$$M = x^2y; N = -(x^3 + y^3).$$

$$\frac{\partial M}{\partial y} = x^2; \frac{\partial N}{\partial x} = -3x^2.$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

\textcircled{1} is not exact.

$$Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0.$$

$$\Rightarrow \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

Multiplying \textcircled{1} by $-\frac{1}{y^4}$, we get:

$$-\frac{x^2}{y^3} dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0 \quad \textcircled{2}$$

Comparing \textcircled{2} with $Pdx + Qdy = 0$

$$P = -\frac{x^2}{y^3}; Q = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial P}{\partial y} = \frac{3x^2}{y^4}; \frac{\partial Q}{\partial x} = \frac{3x^2}{y^4}.$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

\textcircled{2} is an exact

its solution is given by

$$\int \left(-\frac{x^2}{y^3}\right) dx + \int \frac{1}{y} dy = \int 0$$

$$\Rightarrow -\frac{x^3}{3y^3} + \log y = \log c.$$

$$\Rightarrow \log \left(\frac{y}{c}\right) = \frac{x^3}{3y^3}$$

$$\Rightarrow \boxed{y = c e^{x^3/3y^3}}$$

- (2) $y^2 dx + (x^2 - xy - y^2) dy = 0 \rightarrow \frac{1}{y^2} \frac{dy}{dx} = \frac{x^2 - xy - y^2}{y^2} \rightarrow \tanh^{-1}(x/y) + \log y = c$ (26)
- (3) $(x^2 + y^2) dx - 2xy dy = 0 \rightarrow \frac{1}{x^2 + y^2} \frac{dx}{dy} = 2x/y$
- (4) $2y dx - (x^2 + 2y^2) dy = 0$
- (5) $(x^2 y - 2xy^2) dx - (x^3 - 3x^2 y) dy = 0$
- (6) ~~$(3x^2 - y^3) dx - (2xy^2 - xy^2) dy = 0$~~
- (7) $(y^3 - 2x^2) dx + (2xy^2 - x^3) dy = 0$

Method 2: If $M dx + N dy = 0$ is such that $M = y f_1(x, y)$ and $N = x f_2(x, y)$ i.e., $y f_1(x, y) dx + x f_2(x, y) dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx - Ny}$ is an Integrating factor.

problem:

find I.F & solve.

$$(1) y(1+2xy) dx + x(1-2xy) dy = 0 \quad \text{--- (1)}$$

Sol: Comparing (1) with $M dx + N dy = 0$.

$$M = y(1+2xy); N = x(1-2xy)$$

clearly (1) is of the form $y f_1(x, y) dx + x f_2(x, y) dy = 0$

$$\therefore Mx - Ny = xy + 2x^2y^2 - xy + 2x^2y^2 \\ = 4x^2y^2 \neq 0$$

$$\text{I.F} = \frac{1}{Mx - Ny} = \frac{1}{4x^2y^2}$$

Multiplying (1) by $\frac{1}{4x^2y^2}$

$$\left(\frac{1}{4x^2y^2} + \frac{1}{2x}\right) dx + \left(\frac{1}{4x^2y^2} - \frac{1}{2y}\right) dy = 0 \quad \text{--- (2)}$$

Clearly (2) is an exact.

Integrating, we get

$$-\frac{1}{4xy} + \frac{1}{2} \log x + \frac{1}{2} \log y = c_1$$

$$\Rightarrow \boxed{-\frac{1}{4xy} + 2 \log(x/y) = c} \quad \text{where } c = 4c_1$$

2. $(xy \sin xy + \cos xy) y dx + (xy \sin xy - \cos xy) x dy = 0$

Sol: Comparing ① with $M dx + N dy = 0$ — ①

$$M = (xy \sin xy + \cos xy) y, N = (xy \sin xy - \cos xy) x$$

Clearly ① is of the form $y f_1(x, y) dx + x f_2(x, y) dy = 0$

$$\therefore Mx - Ny = 2xy^2 \sin xy + xy \cos xy - 2xy^2 \sin xy + xy \cos xy \\ = 2xy \cos xy \neq 0$$

$$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy \cos xy}$$

Multiplying ① by $\frac{1}{2xy \cos xy}$

$$(y \tan xy + \frac{1}{2}) dx + (x \tan xy - \frac{1}{y}) dy = 0 \quad \text{--- ②}$$

Clearly ② is an exact.

Integrating, we get—

$$\frac{y \log |\sec xy|}{y} + \log x - \log y = \log c$$

$$\Rightarrow \log |\sec xy| + \log \frac{x}{y} = \log c$$

$$\Rightarrow \log \left| \frac{x}{y} \sec xy \right| = \log c$$

$$\Rightarrow \frac{x}{y} \sec xy = c$$

$$\Rightarrow \boxed{\frac{x}{y} \sec xy = cy.}$$

H.W

3. $(x^2 y^2 + 2x^2 y^3) dx + \underline{(x^2 y - x^3 y^2)} dy = 0$

4. $(x^2 y^2 + 2xy + 1) y dx + (x^2 y^2 - 2xy + 1) x dy = 0$

5. $y(1-xy) dx - x(1+xy) dy = 0$

6. $y(x^2 y^2 + 2) dx + x(2 - 2x^2 y^2) dy = 0$

7. $y(1+xy) dx + x(1-xy) dy = 0$

8. $(x^4 y^4 + x^2 y^2 + 2xy) y dx + \underline{(x^4 y^4 - x^2 y^2 + 2xy)} x dy = 0$

Method 3: If $M dx + N dy = 0$ is such that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ or } k \text{ (constant)}$$

$$\text{then I.F.} = e^{\int f(x) dx} \text{ or } e^{\int k dx}$$

problems:

find I.F. & solve the following diff.-eqns.

$$\text{Q. } (x^2 + y^2 + 2x) dx + 2y dy = 0 \quad \textcircled{1}$$

$$\text{Soln. } M = x^2 + y^2 + 2x ; N = 2y$$

$$\frac{\partial M}{\partial y} = 2y ; \frac{\partial N}{\partial x} = 0$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{Now } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2y} \text{ (constant)}$$

$$\text{I.F.} = e^{\int dx} = e^x$$

multiplying $\textcircled{1}$ by e^x .

$$(x^2 e^x + e^x y^2 + 2x e^x) dx + 2y e^x dy = 0 \quad \textcircled{2}$$

clearly $\textcircled{2}$ is an exact

$$\int e^x (x^2 + y^2 + 2x) dx = 0$$

$$\Rightarrow (x^2 + y^2 + 2x)e^x - \int (2x+2)e^x dx = C$$

$$\Rightarrow (x^2 + y^2 + 2x)e^x - 2[e^x(x-1) + e^x] = C$$

$$\Rightarrow (x^2 + y^2 + 2x)e^x - 2xe^x = C$$

$$\boxed{e^x(x^2 + y^2) = C}$$

$$\text{(or)} \frac{2x dx + 2y dy}{x^2 + y^2} + dx = 0$$

$$d[\log(x^2 + y^2)] + dx$$

Integrating.

$$\log(x^2 + y^2) + x = \log C \Rightarrow \frac{x^2 + y^2}{C} = e^{-x}$$

$$\Rightarrow e^x(x^2 + y^2) = C$$

(27)

$$\boxed{2}. \left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2 \right) dx + \frac{1}{4}(x+xy^2) dy = 0 \quad \text{--- (1)}$$

$$M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2; N = \frac{1}{4}(x+xy^2)$$

$$\frac{\partial M}{\partial y} = 1+y^2 \quad \frac{\partial N}{\partial x} = \frac{1}{4}(1+y^2)$$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Note

$$\frac{\frac{\partial M - \partial N}{\partial y}}{\partial x} = \frac{(1+y^2) - \frac{1}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)} = \frac{\frac{3}{4}}{\frac{1}{4}x} = \frac{3}{x}$$

$$\therefore f = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3} = x^3$$

Multiplying (1) by x^3

$$\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2} \right) dx + \frac{1}{4}(x^4 + x^4y^2) dy = 0 \quad \text{--- (2)}$$

Clearly (2) is an exact.

Integrating, we get

$$\boxed{\left[\frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = C \right]}$$

$$\boxed{3}. (x^2 + y^2 + 1) dx - xy dy = 0$$

$$\boxed{4}. (3xy - 2ay^2) dx + (x^2 - 2axy) dy = 0$$

$$\boxed{5}. (x^2 + y^2 + x) dx + xy dy = 0$$

$$\boxed{6}. (2y^3 + x) dx + 3xy^2 dy = 0.$$

Method 4: If $M dx + N dy = 0$ such that $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = f(y)$
 then if $f = e^{\int f(y) dy}$ or $e^{\int k dy}$. (or) $k(\text{const})$

$$\boxed{1}. (xy^3 + y) dx + 2(x^2y^2 + x + y^4) dy = 0$$

$$\boxed{2}. (y^4 + 2y) dx + (2y^3 + 2y^4 + x) dy = 0$$

$$\boxed{3}. (xy^2 - x^2) dx + (3x^2y^2 + x^2y - 2x^3 + y^2) dy = 0$$

$$\boxed{4}. (y + y^2) dx + xy dy = 0.$$

Method 5: If $M dx + N dy = 0$ can be put in the form of

$$x^\alpha y^\beta (my dx + nx dy) + x^\alpha y^{\beta+1} (ny dx + nx dy) = 0 \text{ where}$$

$\alpha, \beta, \alpha', \beta'; m, n'$; m, n are constants then the given eqn

has an I.F. x^{h+k} where $h & k$ must be obtained by applying the

Condition that the given eqn must become exact after multiplying by x^{h+k} .

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad x^\alpha y^\beta (my dx + nx dy) + x^{\alpha+1} y^{\beta+1} (ny dx + q x dy) = 0 \quad (\text{mp} - nq \neq 0)$$

Linear Differential Equations

(28)

The first order diff. eqn of the form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

where $P(x)$ & $Q(x)$ are functions of x only (or) constants

(or) $\frac{dy}{dx} + P(y) \cdot x = Q(y)$ where $P(y)$ & $Q(y)$ are functions of y only (or) constants. is called a linear diff. eqn.

- Note: [1] There are two types of linear diff. eqns.
 [2]. To solve linear diff. eqns, we take a factor
 - called "Integrating factor".

Working rule:

Type 1: (i) $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

(ii) find I.F. = $e^{\int P dx}$

(iii) G.S is $y(I.F.) = \int Q(I.F.) dx + C$

Type 2: (i) $\frac{dy}{dx} + P(y) \cdot x = Q(y)$

(ii) find I.F. = $e^{\int P dy}$

(iii) G.S is $x(I.F.) = \int Q(I.F.) dy + C$.

Problems: [1] Find I.F. of $\sin x \frac{dy}{dx} + 3y = \cos x$ ①

$$\Rightarrow \frac{dy}{dx} + (3 \operatorname{cosec} x) y = \cot x.$$

$$I.F. = e^{\int 3 \operatorname{cosec} x dx} = e^{3 \log \operatorname{tan}(y_1)}$$

$$= \tan^3(x/2).$$

[2]. Solve $\frac{dy}{dx} + y \cot x = 2 \cos x$.

$$I.F. = e^{\int \cot x dx} = e^{\log \sin x}.$$

$$\begin{aligned} \therefore \text{G.S. is } y \sin x &= \int 2 \cos x \cdot \sin x dx + C \\ &= \int \sin 2x dx + C \end{aligned}$$

$$\boxed{y \sin x = -\frac{\cos 2x}{2} + C.}$$

$$\boxed{3.} \cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + \sec^2 y = \tan x \sec^2 x.$$

$$\text{I.F.} = e^{\int \sec^2 x dx} \\ = e^{\tan x}.$$

\therefore G.S is

$$y e^{\tan x} = \int \tan x \sec^2 x e^{\tan x} dx + C$$

$$= \int t e^t dt + C \quad || \tan x = t \\ \Rightarrow \sec^2 x dx = dt$$

$$= e^t (t-1) + C$$

$$= e^{\tan x} (\tan x - 1) + C.$$

$$\boxed{y e^{\tan x} = e^{\tan x} (\tan x - 1) + C}$$

$\boxed{4.}$

$$\frac{dy}{dx} + \frac{y}{x} = x^n$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

G.S is

$$y x = \int x^{n+1} dx + C$$

$$= \frac{x^{n+2}}{n+2} + C$$

$$\boxed{5.} x \frac{dy}{dx} - 2y = x^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{x} y = x$$

$$\text{I.F.} = e^{-\int \frac{2}{x} dx} = e^{-2 \log x} = \frac{1}{x^2}$$

$$\text{G.S is } y \frac{1}{x^2} = \int \frac{1}{x^2} \cdot x dx + C$$

$$= \int \frac{1}{x} dx + C$$

$$\boxed{y/x^2 = \log x + C}$$

$\boxed{6.}$

$$\frac{dy}{dx} + \frac{2y}{x} = \sin x.$$

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$$

G.S is $yx^2 = \int x^2 \sin x dx + C$
 $= -x^2 \cos x + \int 2x \cos x dx + C$
 $= -x^2 \cos x + 2[\sin x + \cos x] + C$

(29)

[7]. $x \log x \frac{dy}{dx} + y = 2 \log x.$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}$$
 $I.F = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x.$

G.S is $y \log x = \int \frac{2}{x} \log x dx + C$
 $= 2 \left(\frac{(\log x)^2}{2} \right) + C$
 $= (\log x)^2 + C.$

[8]. $x \frac{dy}{dx} + 2y = x^2 \log x.$

[9]. $(\sin 2x) \frac{dy}{dx} + y = \tan x.$

$$\Rightarrow \frac{dy}{dx} - (\csc 2x)y = \frac{1}{2} \sec^2 x$$
 $I.F = e^{-\int \csc 2x dx} = e^{-\frac{-\log(\tan x)}{2}}$
 $= e^{\frac{\log \frac{1}{\tan x}}{2}}$
 $= e^{\frac{\log \frac{1}{\tan x}}{2}}$
 $= \frac{1}{\sqrt{\tan x}}.$

G.S is

$$y \frac{1}{\sqrt{\tan x}} = \int \frac{1}{2} \sec^2 x \frac{1}{\sqrt{\tan x}} dx + C$$
 $= \frac{1}{2} \int \frac{1}{\sqrt{t}} dt + C$
 $= \frac{1}{2} \left(\frac{t^{1/2}}{1/2} \right) + C$
 $= \sqrt{\tan x} + C.$

$\tan x = t$
 $\sec^2 x dx = dt$

[10]. $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}.$

$$\boxed{11} \quad \frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x-\sqrt{1-x^2}}{(1-x^2)^2}$$

$$\text{I.F.} = e^{\int \frac{1}{(1-x^2)^{3/2}} dx}$$

$$= e^{\int \sec^2 \theta d\theta}$$

$$= e^{\tan \theta} = e^{\frac{x}{\sqrt{1-x^2}}}$$

let $x = \sin \theta$
 $dx = \cos \theta d\theta$

$$\text{Q.S. is } y \cdot e^{\frac{x}{\sqrt{1-x^2}}} = \int \frac{x-\sqrt{1-x^2}}{(1-x^2)^2} \cdot e^{\frac{x}{\sqrt{1-x^2}}} dx + C$$

$$= \int \frac{\sin \theta - \cos \theta}{\cos^4 \theta} e^{\tan \theta} \cos \theta d\theta + C$$

$$= \int e^{\tan \theta} (\sec^2 \theta \tan \theta + \sec^2 \theta) d\theta$$

$$= \int [t e^{t dt} + e^t] dt \quad \text{where } \tan \theta = t$$

$$= e^t(t-1) + e^t + C$$

$$= e^{\tan \theta} (\tan \theta - 1) + e^{\tan \theta} + C$$

$$\boxed{12} \quad x \cos x \frac{dy}{dx} + (x \sin x + \cos x)y = 1$$

$$\Rightarrow \frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{\sec x}{x}$$

$$\text{I.F.} = e^{\int (\tan x + \frac{1}{x}) dx} = e^{\log(\sec x) + \log x}$$

$$= x \sec x.$$

Q.S. is

$$y x \sec x = \int \sec^2 x dx + C$$

$$= \tan x + C$$

$$\boxed{13} \quad (x+2y^3) \frac{dy}{dx} = y$$

$$y \frac{dx}{dy} = x + 2y^3$$

$$\Rightarrow \frac{dx}{dy} - \left(\frac{1}{y}\right)x = 2y^2$$

$$\text{I.F} = e^{-\int y dy} = e^{-\log y} = \frac{1}{y}$$

$$\therefore \text{G.S is } \frac{x}{y} = \int 2y \cdot \frac{1}{y} dy + C \\ = y^2 + C$$

(30)

$$\boxed{14}. (1+y^2) dx = (\tan^{-1} y - x) dy$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{(1+y^2)} x = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$$\begin{aligned} \text{G.S is } x e^{\tan^{-1} y} &= \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1+y^2} dy + C \\ &= e^{\tan^{-1} y} (\tan^{-1} y - 1) + C \end{aligned}$$

(15)

$$y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$$

$$\Rightarrow y^2 \frac{dx}{dy} + x = \frac{1}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2} x = \frac{1}{y^3}$$

$$\text{I.F} = e^{\int \frac{1}{y^2} dy} = e^{-y}$$

$$\begin{aligned} \text{G.S is } e^{-y} \cdot x &= \int \frac{1}{y^3} e^{-y} dy + C \\ &= e^{-y} \left[\frac{1}{y^2} - 1 \right] + C \end{aligned}$$

$$\begin{aligned} \text{let } y &= t \\ \frac{1}{y^2} dy &= dt \\ \Rightarrow \frac{1}{y^2} dy &= dt \end{aligned}$$

(16)

$$(x+y+1) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dx}{dy} = x + y + 1$$

$$\Rightarrow \frac{dx}{dy} - x = y + 1$$

$$\text{I.F} = e^{-\int 1 dy} = e^{-y}$$

$$\text{G.S is } x e^{-y} = \int (y+1) e^{-y} dy + C$$

$$= \int y e^{-y} dy + \int e^{-y} dy + C$$

$$= t e^t dt + \int e^t (-dt) + C$$

$$\begin{array}{rcl} \text{let } y &=& t \\ dy &=& dt \end{array}$$

$$= +e^t(t-1) - e^{-t} + c$$

$$= +e^{-y}(-y-1) - e^{-y} + c$$

$$= -e^{-y}(y+2) + c$$

17. $(1+x+xy^2) \cdot \frac{dy}{dx} + (y+y^2) = 0$

18. $x \frac{dy}{dx} + \frac{dy}{dx} + 1 = 0 \quad \text{--- } ①$

putting $\frac{dy}{dx} = p$

$$\Rightarrow x \frac{dp}{dx} + p + 1 = 0 \quad \text{--- } ②$$

$$\Rightarrow \frac{dp}{dx} + \left(\frac{1}{x}\right)p = -\frac{1}{x}$$

$$\& p = e^{\int \frac{1}{x} dx} = x.$$

Sols of ② is

$$px = \int -\frac{1}{x} x dx + c$$

$$xp = -x + c$$

$$\Rightarrow x \frac{dy}{dx} = -x + c$$

$$\Rightarrow dy = \left(-1 + \frac{c}{x}\right) dx$$

$$\Rightarrow y = -x + C \log x + C'$$

$$\Rightarrow xy = C \log x + C'$$

Equations reducible to linear form:

(31)

I. $f'(y) \frac{dy}{dx} + p f(y) = Q$
where p & Q are functions of x only.

putting $f(y) = v$

$$\Rightarrow f'(y) \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} + Pv = Q$$

Clearly which is linear.

II. $f'(x) \frac{dx}{dy} + p f(x) = Q$:

where p & Q are functions of y only.

putting $f(x) = v$

$$\Rightarrow f'(x) \frac{dx}{dy} = \frac{dv}{dy}$$

$$\therefore \frac{dv}{dy} + Pv = Q.$$

which is linear.

(II) Bernoulli's equation.

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$

where P & Q are functions of x alone (or) constants
and ' n ' is constant such that $n \neq 0$ & $n \neq 1$; is called
Bernoulli's diff. eqn.

(Or)

$$\frac{dx}{dy} + P(y)x = Q(y)x^n$$

where P & Q are functions of y alone. (or) constants
and ' n ' is constant such that $n \neq 0$ & $n \neq 1$; is called
Bernoulli's diff. eqn.

Working rule:

$$y^{-n} \frac{dy}{dx} + P(x) y^{1-n} = Q(x) \quad (1)$$

$$\text{put } y^{1-n} = z$$

$$y^{-n} (1-n) \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^n \frac{dy}{dx} = \left(\frac{1}{1-n}\right) \frac{dz}{dx}$$

$$\textcircled{1} \equiv \frac{dz}{dx} + (1-n) p(x)z = q(x),$$

clearly which is linear.

Solve the following diff. eqn:

$$(1) \sec y \frac{dy}{dx} + x \tan y = x^3 \quad \text{--- } \textcircled{1}$$

put $\tan y = t$

$$\sec y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\textcircled{1} \equiv \frac{dt}{dx} + 2xt = x^3 \quad \text{--- } \textcircled{2}$$

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

G.S. of $\textcircled{1}$ is

$$t \cdot e^{x^2} = \int x^3 e^{x^2} dx + C$$

$$= \frac{1}{2} \int x e^z dz + C$$

$$= \frac{1}{2} e^z (z - 1) + C$$

$$\tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

$$x^2 \leq z$$

$$x dx = dz$$

$$x dx = \frac{1}{2} dz$$

$$\textcircled{2}. \quad \frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$$

Dividing by e^y we get

$$e^{-y} \frac{dy}{dx} + \frac{1}{x} e^{-y} = \frac{1}{x^2} \quad \text{--- } \textcircled{1}$$

$$\text{But } e^{-y} = t \Rightarrow e^{-y} \frac{dy}{dx} = -\frac{dt}{dx}$$

$$\frac{dt}{dx} - \frac{1}{x} t = -\frac{1}{x^2} \quad \text{--- } \textcircled{2}$$

$$\text{I.F.} = \frac{1}{x}$$

G.S.

$$t \frac{1}{x} = \int \left(-\frac{1}{x^2}\right) \frac{1}{x} dx + C$$

$$= \frac{1}{2x^2} + C$$

$$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + C$$

$$[3] x \frac{dy}{dx} + y = y^2 \log x.$$

$$[4] \frac{dy}{dx} + \frac{y}{1-x^2} = x \sqrt{y}$$

$$y^{-\frac{1}{2}} \frac{dy}{dx} + \frac{1}{1-x^2} y^{\frac{1}{2}} = x \quad \text{--- (1)}$$

$$\text{put } y^{\frac{1}{2}} = t \Rightarrow \frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \boxed{y^{-\frac{1}{2}} \frac{dy}{dx} = 2 \frac{dt}{dx}}$$

$$\text{--- (1)} \equiv \frac{dt}{dx} + \frac{x}{2(1-x^2)} t = \frac{x}{2}$$

$$\begin{aligned} I.F &= e^{\int \frac{x}{2(1-x^2)} dx} \\ &= \frac{1}{e^{\frac{1}{4} \int \frac{-2x}{1-x^2} dx}} \\ &= \frac{1}{e^{\frac{1}{4} \log(1-x^2)}} \\ &= \frac{1}{e^{\frac{1}{4} \log(1-x^2)}} \\ &= \frac{1}{(1-x^2)^{\frac{1}{4}}} \end{aligned}$$

\therefore G.S of (1) is

$$\begin{aligned} t \frac{1}{(1-x^2)^{\frac{1}{4}}} &= \int \left(\frac{1}{(1-x^2)^{\frac{1}{4}}} \right) \left(\frac{x}{2} \right) dx + C \\ &= -\frac{1}{4} \int \frac{-2x}{(1-x^2)^{\frac{1}{4}}} dx + C \\ &= -\frac{1}{4} \frac{(1-x)^{\frac{3}{4}}}{\frac{3}{4}} + C \end{aligned}$$

$$\underline{y^{\frac{1}{2}} (1-x^2)^{\frac{1}{4}}} = -\frac{1}{3} \frac{1}{(1-x^2)^{\frac{1}{4}}} + C$$

$$[5] \frac{dy}{dx} = e^{2x} f (e^x - e^x)$$

$$= e^{2x} e^f - e^x$$

$$\Rightarrow \frac{dy}{dx} + e^x = e^{2x} e^f$$

$$\Rightarrow e^y \frac{dy}{dx} + e^x e^y = e^{2x} \quad \text{--- (1)}$$

$$\text{put } e^y = t \Rightarrow e^y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + e^x = e^{2x}$$

$$\text{let } f = e^x \Rightarrow e^x = e^x$$

(32)

G.S. is
 $t e^{ex} = \int e^{2x} e^x dx + C$

$= \int z e^z dz + C \quad e^x = z \\ e^x dx = dz$

$= e^z (z-1) + C$

$-e^x e^{ex} = e^{ex} (e^x - 1) + C$

~~(7)~~
 $\frac{dy}{dx} (x^2 y^3 + xy) = 1$

$\Rightarrow \frac{dy}{dx} = x^2 y^3 + xy$

$\Rightarrow \frac{dy}{dx} - xy = x^2 y^3$

$\Rightarrow x^2 \frac{dy}{dx} - yx^2 = y^3 \quad \text{--- (1)}$

put $x^2 = t$

$-x^2 \frac{dt}{dy} = \frac{dt}{dy}$

(1) $\Rightarrow -\frac{dt}{dy} - yt = y^3$

$\Rightarrow \frac{dt}{dy} + yt = -y^3$

$IF = e^{\int y dy} = e^{y^2/2}$

G.S. is $y^2/2$
 $+ e^{y^2/2} = - \int y^3 e^{y^2/2} dy + C$

$= - \int 2z e^z dz + C$

$= -2e^z (z-1) + C$

$y^2/2 = z$

$y^2 = 2z$

$2y dy = 2dz$

$\frac{1}{x} e^{y^2/2} = -2e^{y^2/2} (y^2/2 - 1) + C \quad y dy = dz$

~~(8)~~. $\frac{dy}{dx} + \frac{y}{x} = y^2$

~~(9)~~. $\frac{dy}{dx} = x^2 y^3 - xy$

~~(10)~~. $\frac{dy}{dx} + 1 = e^{x-y}$

~~(11)~~. $x^2 y^2 \frac{dy}{dx} - 2y^3 = 2x^3$

~~(12)~~. $x^2 \frac{dy}{dx} + y \log y = 2y e^x$

$$\boxed{13} \cdot 2y \cos^2 y \frac{dy}{dx} - \frac{2 \sin^2 y}{x+1} = (x+1)^3 \quad \text{--- (1)}$$

$$\text{put } \sin^2 y = t$$

$$2y \cos^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

$$(1) \Leftrightarrow \frac{dt}{dx} - \frac{2t}{x+1} = (x+1)^3$$

$$\text{I.F.} = e^{\int \frac{2}{x+1} dx} = e^{-2 \log(x+1)} = \frac{1}{(x+1)^2}$$

$$\text{G.S. } t + \frac{1}{(x+1)^2} = \int (x+1) dx + C$$

$$\boxed{\sin^2 y + \frac{1}{(x+1)^2} = \frac{x^2}{2} + x + C}$$

$$\boxed{14} \cdot \frac{dy}{dx} + 2 \sin y = x^3 \cos y$$

$$\Rightarrow (\cos y)^{-1} \frac{dy}{dx} + x^2 \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow (\cos y)^{-1} \frac{dy}{dx} + 2x \tan y = x^3 \quad \text{--- (1)}$$

$$\text{put } \tan y = t$$

$$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

$$(1) \Leftrightarrow \frac{dt}{dx} + 2xt = x^3$$

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

$$te^{x^2} = \int e^{x^2} x^3 dx + C$$

$$= \frac{1}{2} \int e^{x^2} d(x^2) + C$$

$$= \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

$$\begin{aligned} x^2 &= t \\ 2x dx &= dt \\ x dx &= \frac{dt}{2} \end{aligned}$$

$$\tan e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

15. find the eqn of the curve which passes through the point $(1, -1)$ and satisfies the diff. eqn

$$x \frac{dy}{dx} + y = x^2 y^2$$

$$\text{2001} \rightarrow \frac{dz}{dx} + \frac{1}{x} \log z = \frac{z}{x^2} (\log z)^2 \quad \text{--- (1)}$$

Dividing by $z(\log z)^2$, we get

$$\frac{1}{z(\log z)^2} \frac{dz}{dx} + \frac{1}{x} (\log z)^{-1} = \frac{1}{x^2} \quad \text{--- (2)}$$

$$\text{put } (\log z)^{-1} = t$$

$$\frac{(-1)(\log z)^{-2}}{z} \frac{dz}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{z(\log z)^2} \frac{dz}{dx} = -\frac{dt}{dx}$$

$$\text{--- (2)} \Leftrightarrow -\frac{dt}{dx} + \frac{1}{x} t = \frac{1}{x^2}$$

$$\Rightarrow \frac{dt}{dx} - \frac{1}{x} t = -\frac{1}{x^2} \quad \text{--- (3)}$$

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

\therefore G.S of (3) is

$$\frac{1}{x} t = \int -\frac{1}{x^2} \cdot \frac{1}{x} dx + C$$

$$= -\int x^3 dx + C$$

$$= -\frac{x^2}{2} + C$$

$$= \frac{1}{2} x^2 + C$$

$$\therefore \boxed{\frac{1}{2} (\log z)^{-1} = \frac{1}{2} x^2 + C}$$

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Miscellanea

To find the n.c and s.c let the equation $m dx + n dy = 0$ may be exact.

34(i)

Proof part 1: Let $m dx + n dy = 0$ be an exact.

Then by defn, $m dx + n dy = du$ (1)

where u is a function of x & y .

$$\text{and } m dx + n dy = du \\ = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow m = \frac{\partial u}{\partial x} \quad | \quad n = \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad ; \quad \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$$\text{Hence } \frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y} \quad (\because \frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y})$$

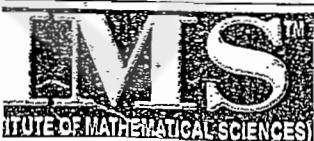
Part 2: ~~(2)~~ Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
To prf redn $n dy$ is exact.

Let $\int m dx = u$ (2)
Where integration has been performed
by treating y as constant.

$$\therefore \frac{\partial}{\partial x} [\int m dx] = \frac{\partial u}{\partial x}$$

$$\Rightarrow m = \frac{\partial u}{\partial x} \quad \text{--- (3)}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial u}{\partial y x} \quad \text{--- (4)}$$



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MATHEMATICS by K. Venkanna

$$\therefore \frac{\partial N}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \frac{\partial N}{\partial y \partial x} = \frac{\partial^2 N}{\partial x \partial y}$$

$$\textcircled{3} \equiv \frac{\partial N}{\partial x} = \frac{\partial^2 N}{\partial x \partial y}$$

Integrating both sides w.r.t. x by treating y as const.

$$\therefore N = \frac{\partial y}{\partial x} + \text{a function of } y$$

$$= \frac{\partial y}{\partial x} + f(y) \text{ say,} \quad \textcircled{5}$$

From $\textcircled{3}$ & $\textcircled{5}$, we have
 $m dx + N dy \equiv \frac{\partial y}{\partial x} dx + \left[\frac{\partial y}{\partial x} + f(y) \right] dy$

$$= \left[\frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial x} dy \right] + f(y) dy$$

$$= du + f(y) dy$$

$$= d[u + \int f(y) dy]. \quad \textcircled{6}$$

which is an exact diff.

Hence $m dx + N dy = 0$ is an exact diff.
com solution of an exact diff. eq. eqn.

If the equation $m dx + N dy = 0$ is exact.

$$\text{Then } m dx + N dy = d[u + \int f(y) dy]$$

$$\Rightarrow m dx + N dy = d[u + \int f(y) dy] = 0 \text{ (by using } \textcircled{5})$$

Integrating both sides, we get

$$\textcircled{3} \equiv u = \int m dx + C \text{ or } \int f(y) dy = C$$

$$\textcircled{5} \equiv f(y) = \text{terms in } N \text{ not containing } y$$

$$\therefore m dx + f(y) (terms in N containing y) dy = C$$



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SYLLABUS FOR IAS/IEGS/CSIR/STATE EXAMINATIONS

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Proofs

34(11)

Method 1 :- If $M dx + N dy = 0$ is a homogeneous and $M+N \neq 0$ then $\frac{1}{M+N}$ is an I.F.

Proof. Given that $M dx + N dy = 0 \quad (1)$
where M & N are homogeneous functions
of the same degree in x & y .

By Euler's theorem on partial differentiation

$$\left. \begin{aligned} x \frac{\partial M}{\partial x} + y \frac{\partial M}{\partial y} &= nM \\ x \frac{\partial N}{\partial x} + y \frac{\partial N}{\partial y} &= nN \end{aligned} \right\} \quad (2)$$

Now $(1) \times \frac{1}{M+N} \equiv \frac{M}{M+N} dx + \frac{N}{M+N} dy = 0$

It will be exact if

$$\frac{\partial}{\partial y} \left(\frac{M}{M+N} \right) = \frac{\partial}{\partial x} \left(\frac{N}{M+N} \right)$$

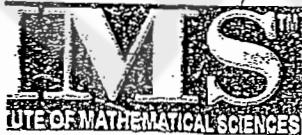
$$\text{i.e. } (M+N) \frac{\partial M}{\partial y} - M \left(x \frac{\partial M}{\partial y} + y \frac{\partial M}{\partial x} + n \right) \quad (M+N)^2$$

$$= (M+N) \frac{\partial N}{\partial x} - N \left(x \frac{\partial N}{\partial x} + y \frac{\partial N}{\partial y} + n \right) \quad (M+N)^2$$

$$\text{i.e. } Mx \frac{\partial M}{\partial y} + Ny \frac{\partial M}{\partial y} - Mx \frac{\partial N}{\partial y} - Ny \frac{\partial N}{\partial y} - MN$$

$$= MN \frac{\partial N}{\partial x} - NY \frac{\partial N}{\partial x} - MN - MN$$

$$\Rightarrow N \left(x \frac{\partial N}{\partial x} + y \frac{\partial N}{\partial y} \right) = M \left(x \frac{\partial M}{\partial x} + y \frac{\partial M}{\partial y} \right)$$



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$$\Rightarrow N \cdot nM = M \cdot nn \quad (\text{by using } ②),$$

which is true. Hence the result.

Alternate Method :-

The given equation is ~~redundant~~ ⁽¹⁾

Where M & N are homogeneous functions of the same degree in x & y .

Now we have

$$\cancel{Md+Nd}y \equiv \frac{1}{2} \left[(Mx+ny) \left(\frac{dx}{x} + \frac{dy}{y} \right) \right]$$

$$Md+Nd \equiv \frac{1}{2} \left[(Mx+ny) \left(\frac{dx}{x} + \frac{dy}{y} \right) + (Mx+ny) \left(\frac{dx}{x} - \frac{dy}{y} \right) \right]$$

$$\Rightarrow Md+Nd \equiv \frac{1}{2} \left[(Mx+ny) d(\log xy) + (Mx+ny) d\left(\log \frac{x}{y}\right) \right]$$

Now dividing by $Mx+ny$ (which is $\neq 0$)

$$\frac{Md+Nd}{Mx+ny} \equiv \frac{1}{2} \left[d\left(\log xy\right) + \frac{Mx+ny}{Mx+ny} d\left(\log \frac{x}{y}\right) \right]$$

Since M & N are homogeneous functions of the same degree in x & y ,

the expression $\frac{Mx+ny}{Mx+ny}$ is homogeneous

and equal to a function of $\frac{x}{y}$ say $f\left(\frac{x}{y}\right)$

$$\therefore \frac{Md+Nd}{Mx+ny} \equiv \frac{1}{2} d\left(\log xy\right) + \frac{1}{2} f\left(\frac{x}{y}\right) d\left(\log \frac{x}{y}\right)$$



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MATERIALS FOR IAS/IFoS/PCS/BAT/RAIL EXAMINATIONS

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$$\Rightarrow \frac{y}{x} = e^{\log(\frac{x}{y})}$$

34(iii)

$$\Rightarrow f(\frac{x}{y}) = f\left(e^{\log(\frac{x}{y})}\right) = F\left(\log(\frac{x}{y})\right)$$

$$\Rightarrow \frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2} d(\log xy) + \frac{1}{2} F'(\log \frac{x}{y}) d(\log \frac{x}{y})$$

which is an exact differential

$$\Rightarrow \frac{Mdx+Ndy}{Mx+Ny} = 0$$

$$\Rightarrow \frac{M}{Mx+Ny} dx + \frac{N}{Mx+Ny} dy = 0$$

is an exact diff. eq.

Method 2: if the equation $Mdx+Ndy=0$ is of the form $yf_1(xy)dx+nf_2(xy)dy=0$

then $\frac{1}{Mx+Ny}$ is an integrating factor (P.F.).

proof The given equation is $Mdx+Ndy=0$ where $M = yf_1(xy)$, $N = nf_2(xy)$. (1)

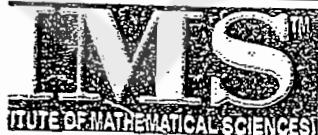
Now we have

$$Mdx+Ndy \equiv \frac{1}{2} [(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right)]$$

$$\Rightarrow Mdx+Ndy \equiv \frac{1}{2} [(Mx+Ny)d(\log xy) + (Mx+Ny)d(\log \frac{x}{y})]$$

Dividing by $Mx+Ny$, ($Mx+Ny \neq 0$) we get,

$$\frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2} \left[\frac{Mx+Ny}{Mx+Ny} d(\log \frac{x}{y}) + d(\log \frac{x}{y}) \right].$$



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$$= \frac{1}{2} \left[\frac{ny f_1(ny) + ny f_2(ny)}{ny f_1(ny) - ny f_2(ny)} d(\log(ny)) + d(\log(\frac{n}{y})) \right]$$

$$= \frac{1}{2} \left[f_1(ny) d(\log(ny)) + d(\log(ny)) \right]$$

Since $ny = e^{\log ny}$

$$\Rightarrow f(ny) = f(e^{\log ny}) = F(\log ny)$$

$$\therefore \text{redn by } \frac{1}{ny} = \frac{1}{2} F(\log ny) d(\log ny) + \frac{1}{2} d(\log ny)$$

which is an exact diff.

$$\Rightarrow \frac{dM}{ny} = 0$$

$$\Rightarrow \frac{M}{ny} dy + \frac{N}{ny} dy = 0 \text{ is an}$$

exact diff. eqn.

$\frac{1}{ny}$ is an ~~exact diff.~~ integrating factor

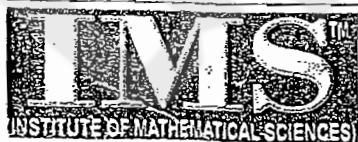
Method 3: If in the equation $dM/dy = 0$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \text{ is a function of } x \text{ only}$$

$= f(x)$ say then $f(x) dx$ is an

K (constant) then e^K is an

Integrating factor.



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Set-IILinear Equations with constant coefficients.

(35)

A linear differential eqn of order 'n' of the form

$$\frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = Q \quad (1)$$

where $a_1, a_2, \dots, a_{n-1}, a_n$ are all constants and Q is any function. It is called a linear diff. eqn with constant coefficients.

for our convenience, the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \dots, \frac{d^{n-1}}{dx^{n-1}}, \frac{d^n}{dx^n}$ are also denoted by D, D^2, D^3, \dots, D^n respectively.

∴ The equation (1) can be written as

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_{n-1} D y + a_n y = Q$$

$$\Rightarrow [D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n] y = Q$$

$$\Rightarrow f(D) y = Q \text{ where } f(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n \quad (2)$$

Homogeneous eqn: If $Q=0$ then (2) is called homogeneous eqn with constant coefficients.

i.e., a linear homogeneous eqn of order 'n' is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0 \quad (3)$$

→ If $y=f(x)$ is the general solution of (3) and $y=p(x)$

is any particular solution of the eqn (2) is not

containing any arbitrary constant. Then $y=f(x)+p(x)$

is called the g.s. of (2).

The method of solving a linear eqn is dividing into two parts:

→ first we find the general solution of the eqn (3).

It is called the Complementary function (C.F.).

It must be contain many arbitrary constants as is the order of the given diff-eqn..

→ Next, we find a particular solution of ② which does not contain arbitrary constants this is called the particular Integral (P.I.).

→ If we add (C.F) and (P.I.) then we get the general solution of ②

i.e., The general solution of ② is

$$y = C.F + P.I \quad (\text{or})$$

$$y = Y_C + Y_p.$$

Auxiliary Eqn (A.E.): Now we consider the diff-eqn

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0 \quad ①$$

$$\text{i.e., } f(D)y = 0$$

$$\text{where } f(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n.$$

The eqn $f(m) = 0$ is called the A.E. of ① where $m=D$.

∴ A.E of ① is given by

$$m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$$

Clearly it will have 'n' roots.

These roots may be real (or) complex or surds.

To find the C.F of $f(D)y = 0$:

$$\text{Consider the eqn } (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0$$

$$\text{i.e., } f(D)y = 0$$

The A.E. of ① is $f(m) = 0$

$$\therefore \text{i.e., } m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \quad ②$$

Case(I): when all the roots of ② are real and distinct

Let $m = m_1, m_2, \dots, m_n$ be the 'n' real and distinct roots of ②.

Then $y = e^{m_1 x}$, $y = e^{m_2 x}$, ..., $y = e^{m_n x}$ are independent solutions of ①.

Hence the g.s. of ① is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

where C_1, C_2, \dots, C_n are constants.

Ex: $(D - m_1) y = 0$ — ①

$$\Rightarrow D y - m_1 y = 0$$

$$\Rightarrow \frac{dy}{dx} = m_1 y$$

$$\Rightarrow \frac{dy}{y} = m_1 dx$$

$$\Rightarrow \log y = m_1 x + \log c$$

$$\Rightarrow \log(y/c) = m_1 x$$

$$\Rightarrow y/c = e^{m_1 x}$$

$$\Rightarrow \boxed{y = c e^{m_1 x}}$$

Case ii: when two roots of ② are equal and other roots are distinct.

Let $m_1 = m_2$ i.e., $m_1, m_1, m_3, m_4, \dots, m_{n-1}, m_n$ be the real and distinct roots of ②.

Then g.s. of ① is

$$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

Ex: $(D - m_1)^2 y = 0$ in which the roots are equal.

$$\Rightarrow (D - m_1)(D - m_1) y = 0$$

$$\Rightarrow (D - m_1)v = 0 \text{ where } v = (D - m_1)y \quad \text{— ②}$$

$$\Rightarrow v = C_1 e^{m_1 x}$$

$$\text{②} \Rightarrow (D - m_1)y = C_1 e^{m_1 x}$$

$$\Rightarrow D y - m_1 y = C_1 e^{m_1 x}$$

$$\Rightarrow \frac{dy}{dx} - m_1 y = C_1 e^{m_1 x} \quad \text{— ③}$$

$$\therefore \boxed{\text{I.F.} = e^{-m_1 x}}$$

$$\therefore \text{G.S. is } y e^{-m_1 x} = \int C_1 dx + C_2 = C_1 x + C_2$$

$$\therefore \boxed{y = (C_1 x + C_2) e^{m_1 x}}$$

Case(iii) when three roots are equal.

∴ G.S of ① is

$$y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_2 x} + \dots + C_n e^{m_n x}$$

Case(iv): when all the roots are equal.

∴ G.S. of ① is

$$y = (C_1 + C_2 x + C_3 x^2 + \dots + C_{n-2} x^{n-2} + C_n x^{n-1}) e^{m_1 x}$$

Case(v): when the A.E. of ① has $\alpha \pm i\beta$ as a pair of complex roots.

$$\text{Let } m_1 = \alpha + i\beta \quad \& \quad m_2 = \alpha - i\beta$$

∴ G.S. of ① is

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

$$= C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{\alpha x} [C_1 (\cos \beta x + i \sin \beta x) + C_2 (\cos \beta x - i \sin \beta x)]$$

$$= e^{\alpha x} [(C_1 + C_2) \cos \beta x + i(C_1 - C_2) \sin \beta x]$$

$$= e^{\alpha x} [A \cos \beta x + B \sin \beta x]$$

$$\text{where } A = C_1 + C_2 ; B = i(C_1 - C_2) ; i = \sqrt{-1}$$

→ If the imaginary roots are repeated, say $\alpha + i\beta$ & $\alpha - i\beta$ occur twice then the solution

will be

$$y = e^{\alpha x} [(A + Bx) \cos \beta x + (C + Dx) \sin \beta x]$$

Note: (i) The expression $e^{\alpha x} (A \cos \beta x + B \sin \beta x)$ can also be written as

$$A e^{\alpha x} \sin(\beta x + B) \text{ or } A e^{\alpha x} \cos(\beta x + B).$$

(ii) If A.E. of ② has $(\alpha \pm \sqrt{\beta})$ a pair of roots

then G.S of ① is

$$y = e^{\alpha x} [C_1 \cosh \sqrt{\beta} x + C_2 \sinh \sqrt{\beta} x]$$

sometimes it may be written as

$$y = C e^{\alpha x} \cosh(\sqrt{\beta} x + C_2)$$

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→ If the root $(\alpha \pm i\beta)$ is repeated then the G.S.

is $y = e^{\alpha x} [(c_1 + c_2 x) \cosh \beta x + (c_3 + c_4 x) \sinh \beta x]$

Problems:

→ find C.F. of $(D^2 - 3D + 2)y = 0$.

Soln: Given that $(D^2 - 3D + 2)y = 0$

$$\Rightarrow f(D)y = 0 \quad \text{where } f(D) = D^2 - 3D + 2$$

A.E. of ① is $f(D) = 0$

$$\text{i.e., } D^2 - 3D + 2 = 0$$

$$\Rightarrow (D-1)(D-2) = 0$$

$$\Rightarrow D=1, 2$$

$$\therefore \text{C.F. of ① is } y = C_1 e^x + C_2 e^{2x}$$

*Y-axis, Y-axis are
independent of D
C.F. of ② is $C_1 e^{2x} + C_2 e^{2x}$*

→ Solve $\frac{d^3y}{dx^3} - 4 \frac{dy}{dx^2} + 5 \frac{dy}{dx} - 2y = 0$

→ solve $(D^4 - 81)y = 0$

→ solve $(D^4 + D^2 + 1)y = 0$

Soln: A.E. is $f(D) = 0$

$$\Rightarrow D^4 + D^2 + 1 = 0$$

$$\Rightarrow (D^2 + 1)^2 - D^2 = 0$$

$$\Rightarrow (D^2 + D + 1)(D^2 + D - 1) = 0$$

$$\Rightarrow D^2 + D + 1 = 0, \quad D^2 - D - 1 = 0$$

$$D = \frac{1 \pm \sqrt{1-4}}{2}, \quad D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$D = \frac{1 \pm \sqrt{3}i}{2}, \quad \frac{-1 \pm \sqrt{3}i}{2}$$

→ find C.F. of $(D^4 + \bar{a})y = 0$

~~$(D^3 + 6D^2 + 12D + 8)y = 0$~~

~~$(D^2 + D + 1)^2 y = 0$~~ (or) ~~$(D^4 + 2D^3 + 3D^2 + 2D + 1)y = 0$~~

~~$(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$~~

~~$(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$~~

~~$\left(\frac{d^3y}{dx^3} + y\right) = 0$~~

$\rightarrow (D^2 - 2D + 5)y = 0$ gives that (i) $y \neq 0$ when $x=0$

$$\text{Ans} : e^x (2\sin 2x) \quad (\text{ii}) \frac{dy}{dx} = 4 \text{ when } x=0.$$

$$\rightarrow (D^2 + 4)y = 0 \quad \in (A \cos 2x + B \sin 2x)$$

$$\cancel{\rightarrow \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = 0 \text{ with } y \neq 0, x=0 \text{ and } \frac{dy}{dx} = 0. \quad (\text{iii})}$$

To find the particular integral:

Let the given diff eqn be

$$(D^n + A_1 D^{n-1} + \dots + A_{n-1} D + A_n) y = Q$$

where $D = \frac{d}{dx}$.

$$\Rightarrow f(D)y = Q, \text{ where } f(D) = D^n + A_1 D^{n-1} + \dots + A_{n-1} D + A_n.$$

$$\text{S.t.g.s. is } \boxed{y = Cf + PI}$$

Inverse Operator:

$$\rightarrow \text{Since } f(D), \frac{1}{f(D)} Q = Q.$$

$\frac{1}{f(D)}$ is the inverse operator of $f(D)$

\rightarrow Since D is the diff operator.

$\Rightarrow \frac{1}{D}$ is an integral operator of D .

$$\frac{1}{D} D = I_D.$$

P.I. of $f(D)y = Q$:

Since $y = \frac{1}{f(D)} Q$ satisfies the eqn $f(D)y = Q$.

\therefore P.I. of $f(D)y = Q$ is $\frac{1}{f(D)} Q$.

METHODS FOR FINDING P.I.:-

Case (i): To find P.I. when $Q = e^{ax}$ when $f(a) \neq 0$.

Since $D e^{ax} = a e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$, \dots , $D^n e^{ax} = a^n e^{ax}$.

$$\Rightarrow f(D)e^{ax} = f(a)e^{ax}; f(a) \neq 0.$$

$$\text{Now } e^{ax} = \frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} \{f(a)e^{ax}\}.$$

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$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad (\because f(a) \neq 0).$$

Working rule:

If $f(D)y = e^{ax}$; $f(a) \neq 0$
where $a = e^{ax}$

$$\text{then P.I.} = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax} \quad (\text{putting } D=a \text{ in } f(D)).$$

Problems:

$$\rightarrow \text{Find P.I. of } \frac{dy}{dx} - 7 \frac{dy}{dx} + 12y = e^{2x}$$

$$\text{Soln: } (D^2 - 7D + 12)y = e^{2x}$$

$$\rightarrow f(D)y = Q \quad \text{where } f(D) = D^2 - 7D + 12 \text{ and } Q = e^{2x}$$

$$\begin{aligned} \therefore P.I. &= \frac{1}{f(D)} Q \\ &= \frac{1}{D^2 - 7D + 12} e^{2x} \\ &= \frac{1}{(D-4)(D-3)} e^{2x} = \frac{1}{2} e^{2x} \end{aligned}$$

$$\rightarrow \text{Find P.I. and solve } (D^2 + D + 1)y = e^{-x}.$$

$$\text{Soln Given that } (D^2 + D + 1)y = e^{-x}.$$

$$\rightarrow f(D)y = e^{-x} \quad \text{--- ①}$$

where $f(D) = D^2 + D + 1$.

A.E. of ① is $f(D) = 0$

$$\Rightarrow D^2 + D + 1 = 0$$

$$\Rightarrow D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2} e^{-x}$$

$$\therefore Y_c = e^{-x} \left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right)$$

$$\text{Now P.I.} = \frac{1}{f(D)} Q = \frac{1}{D^2 + D + 1} (e^{-x})$$

$$= \frac{1}{(-1+1)} (e^{-x}) = e^{-x}$$

$$\therefore Y_p = e^{-x}$$

$$\therefore \text{G.S. of ① is } y = Y_c + Y_p$$

$$= e^{-x} \left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right) + e^{-x}$$

$$\rightarrow \text{solve } (D^2 - 2D - 1)y = \cos bx$$

$$\boxed{\cosh x = \frac{e^{bx} + e^{-bx}}{2}}$$

$$\rightarrow \text{solve } (D^2 + 4D + 6)y = e^{2x}$$

$$\rightarrow \text{solve } \frac{d^2y}{dx^2} - 13 \frac{dy}{dx} + 12y = 2008 \quad Ce^x + C_2 e^{12x} + \frac{2008}{12}$$

$$\rightarrow \text{solve } (D^2 - 5D^2 + 7D - 3)y = e^{2x} \cos bx.$$

$$= e^{2x} \frac{e^x + e^{-x}}{2}$$

Case ii To find P.I when $Q = \sin(ax)$ or $\cos(ax)$ and $f(-a^2) \neq 0$

Since $D(\sin ax) = a \cos ax$; $D^2(\sin ax) = -a^2 \sin ax$

$$D^3(\sin ax) = -a^3 \cos ax; D^4(\sin ax) = (-a^2)^2 \sin ax$$

$$(D^2)^2 \sin ax = (-a^2)^2 \sin ax.$$

$$\therefore (D^2)^n \sin ax = (-a^2)^n \sin ax.$$

$$\therefore f(D^2) \sin ax = f(-a^2) \sin ax$$

where $f(-a^2) \neq 0$.

$$\text{Now } \sin(ax) = \frac{1}{f(D^2)} f(D^2) \sin ax$$

$$= \frac{1}{f(D^2)} \{f(-a^2) \sin ax\}$$

$$\Rightarrow \boxed{\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax.}$$

$$\text{Similarly } \boxed{\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax.}$$

Working rule:-

To find P.I. $Q = \sin ax$ (or) $\cos ax$ and $f(-a^2) \neq 0$.

$$P.I. = \frac{1}{f(D^2)} \sin ax.$$

$$= \frac{1}{f(-a^2)} \sin ax \quad (\text{put } D^2 = -a^2)$$

$$\text{write } D^2 = -a^2; D^3 = -a^2 D, D^4 = (-a^2)^2 = a^4, \dots$$

Note: the linear factor $(D \pm a)$ in the denominator may be removed by multiplying the Nr and Dr with $D \pm a$ and then putting $D^2 = -a^2$

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problems

- solve $(D^2+4)y = \cos 4x$ ($A \cos 2x + B \sin 2x - \frac{1}{2} \cos 4x$).
- solve $(D^2-2D+5)y = \sin 3x$
- solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = \cos 2x \cdot G e^{(Ae^x + Ce^{-x})} e^{\frac{x}{2}} \left(\frac{-2\sin x}{-C \sin x} \right) \dots$
- solve $(D^2-4D+3)y = \sin 3x \cos 2x$.
- $(D^2-4)y = \sin^2 x$.
- $(D^2-9)y = \cos^2 x$.

$$\frac{2\sin A \sin B}{\sin(A+B) + \sin(A-B)}$$

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2}\end{aligned}$$

Case II:

To find P.I. when $Q = x^m$ or polynomial of degree m where m is zero & +ve integer.

$$P.I. = \frac{1}{f(D)} x^m \text{ (polynomial)}$$

Take out common the lowest degree term from $f(D)$. The remaining factor in denominator is of the form $[1 \pm f(D)]$ or $[1-f(D)]$ which is taken in the numerator with negative power.

Now expand $[1 \pm f(D)]^{-1}$ in ascending powers of D by binomial theorem upto D^m and operator upto x^m .

$$\text{(i.e.) } \frac{1}{f(D)} x^m \Rightarrow \frac{1}{[1 \pm f(D)]} x^m \Rightarrow [1 \pm f(D)]^{-1} x^m.$$

The following expansions by binomial theorem

$$(1) (1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5+\dots$$

$$(2) (1-x)^{-1} = 1+x+x^2+x^3+\dots$$

$$(3) (1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$$

$$(4) (1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$

$$\left[\text{Since } (1+x)^n = 1+n x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \right]$$

problems

Solve the following diff-eqns.

- ① $\frac{d^2y}{dx^2} + 4y = x^2$ ③ $(D^2+D+1)y = x^3$ $e^{-\frac{x}{2}} \left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right] + x^3 - 3x^2 + 12$
- ② $\frac{d^2y}{dx^2} - 4y = x^2$ ④ $(D^2-1)y = 2+5x$
 $\Rightarrow (D^2-3D+2)y = x^2$ $\Rightarrow x^2 + x^3 + 3x^2 + 2$

$$\boxed{5} \cdot (D^3 + 8)y = x^4 + 2x + 1$$

$$\boxed{6} \cdot (D^4 - 2D^3 + D^2)y = x^3$$

$$\boxed{7} \cdot (D^2 + 16)y = x^4 + e^{3x} + \cos 3x$$

$$\boxed{8} \cdot (D^2 + D - 2)y = x + \sin x$$

$$\boxed{9} \cdot (D^2 - 5D + 6)y = x + \sin 3x$$

$$\boxed{10} \cdot (D^3 - 4D^2 + 5D)y - 2 = 0$$

Case (iv): To find P.I

$$Ge^{ax} + Ge^{bx} + \frac{2x}{v}$$

where $v = F(x)$

i.e. $v = \sin ax$ or

$\cos ax$ or

etc.

$$P.I = \frac{1}{f(D)} e^{ax} \cdot v$$

Here take e^{ax} outside after replacing

D by $(D+a)$ and operate v by $\frac{1}{f(D+a)}$

$$\therefore P.I = \frac{1}{f(D)} e^{ax} \cdot v$$

$$= e^{ax} \frac{1}{f(D+a)} v$$

Problems

Solve the following diff. equations.

$$\boxed{1} \cdot (D^2 - 4)y = x^2 e^{3x} \quad Gc^{2x} + Ge^{-2x} + \frac{1}{5} [x^2 - 12x + 6^2].$$

$$\boxed{2} \cdot (D^3 - 3D + 2)y = x^2 e^x$$

$$\boxed{3} \cdot (D^2 - 2D + 5)y = e^{2x} \sin x$$

$$\boxed{4} \cdot (D^4 - 1)y = e^x \cos x$$

$$\boxed{5} \cdot (D+1)^3 y = x^2 e^{-x}$$

$$\boxed{6} \cdot D^2 y = e^x \cos x$$

$$\boxed{7} \cdot (D^2 - 4D + 3)y = 2xe^{3x} + ie^x \cos 2x$$

$$\boxed{8} \cdot (D^2 - 3D + 4D - 2)y = e^x + \cos x$$

$$\boxed{9} \cdot (D^2 - 3D + 2D - 1)y = (x + 1)e^x$$

case(v): To find P.I when

$$\text{Q} = e^{an} \text{ and} \\ f(c) = 0$$

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$$\begin{aligned} P.I &= \frac{1}{f(D)} e^{an} \\ &= \frac{1}{f(D)} e^{an} \cdot \frac{1}{1} \\ &= e^{an} \frac{1}{f(D+c)} \cdot 1 \end{aligned}$$

(or)

$$\frac{1}{(D-a)^r} e^{an} = \frac{n^r}{r!} e^{an} \text{ where } r=1, 2, 3, \dots$$

If $f(c) \neq 0$. Then factorize $f(D)$, first
operate on e^{an} by factor which does not
vanish by putting ' a ' for D and
finally the other factor to apply
the above formula).

Problems

Find P.I. of $[D^2 + D^n - D - 1]y = e^n$

$$\begin{aligned} \text{Sol} \quad P.I. &= \frac{1}{D^2 + D^n - D - 1} e^n \\ &= \frac{1}{D^2(D+1) - (D+1)} e^n \\ &= \frac{1}{(D+1)(D^n - 1)} e^n \\ &= \frac{1}{(D+1)^n(D-1)} e^n \\ &= \frac{1}{(D-1)} \left[\frac{1}{(D+1)^n} e^n \right] = \frac{1}{D-1} \left[\frac{1}{4} e^n \right]. \end{aligned}$$

$$= \frac{1}{4} \left[\frac{1}{D-1} e^{\lambda} \right]$$

$$= \frac{1}{4} \left[\frac{\lambda}{\lambda-1} e^{\lambda} \right]$$

$$= \underline{\underline{\frac{1}{4} \lambda e^{\lambda}}}.$$

\rightarrow find P.I of $(D^3 + 3D^2 + 3D + 1) y = e^{\lambda}$

$$\text{Sol. P.I.} = \frac{1}{D^3 + 3D^2 + 3D + 1} e^{\lambda} = \frac{1}{(D+1)^3} e^{\lambda}$$

$$= e^{\lambda} \frac{1}{(D+1+1)^3} \quad (1)$$

$$= e^{\lambda} \frac{1}{D^3} \quad (1)$$

$$= e^{\lambda} \cdot \frac{1}{D^2} [x]$$

$$= e^{\lambda} \frac{1}{D} \left[\frac{x^2}{2} \right]$$

$$= e^{\lambda} \frac{x^3}{6} \quad \underline{\underline{=}}$$

(or)

$$\begin{aligned} \text{P.I.} &= \frac{1}{[D^3 + 3D^2 + 3D + 1]} e^{\lambda} = \frac{1}{3D^2 + 6D + 3} (e^{\lambda}) \\ &= \frac{\lambda \cdot \lambda}{6D + 6} (e^{\lambda}) \\ &= \frac{\lambda \cdot \lambda \cdot \lambda}{6} (e^{\lambda}) \\ &= \frac{\lambda^3}{6} (e^{\lambda}). \end{aligned}$$

* Solve the following diff. eqns:

$$\rightarrow (D+2)(D-1)^3 y = e^{-\lambda}$$

$$\rightarrow (D^2 + D - 6) y = e^{2\lambda}$$

$$\rightarrow (D^2 - 4D + 4) y = e^{2\lambda} + \sin 2\lambda$$

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$$\xrightarrow{\text{H.M.}} (D^2 - 3D + 2)y = \cos \alpha$$

$$\xrightarrow{\text{H.M.}} \frac{d^2y}{dx^2} - 2 \frac{dy^3}{dx^3} + 5 \frac{dy^2}{dx^2} - 8 \frac{dy}{dx} + 4y = e^x$$

\rightarrow find P.I. of

$$\underline{\text{SOL}} \quad (D^2 + \alpha^2)y = \sin \alpha x$$

$$\text{P.I.} = \frac{1}{D^2 + \alpha^2} \sin \alpha x$$

$$= \frac{1}{D^2 + \alpha^2} \text{ I.P. of } (\cos \alpha x + i \sin \alpha x)$$

$$= \frac{1}{D^2 + \alpha^2} \text{ I.P. of } e^{i\alpha x}$$

$$= \text{I.P. of } \left[\frac{1}{D^2 + \alpha^2} e^{i\alpha x} \right] \quad (1)$$

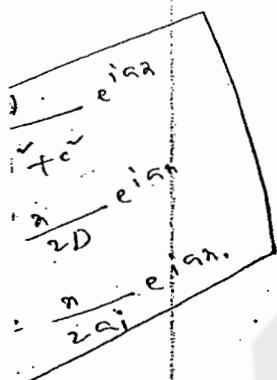
$$\begin{aligned} \text{Now } \frac{1}{D^2 + \alpha^2} e^{i\alpha x} &= e^{i\alpha x} \frac{1}{(D + i\alpha)^2 + \alpha^2} \quad (1) \\ &= e^{i\alpha x} \frac{1}{D^2 - \alpha^2 + 2Di\alpha + \alpha^2} \\ &= e^{i\alpha x} \frac{1}{D^2 + 2\alpha i D} \quad (1) \\ &= e^{i\alpha x} \frac{1}{2\alpha i D \left(1 + \frac{D}{2\alpha i} \right)} \quad (1) \end{aligned}$$

$$= \frac{e^{i\alpha x}}{2\alpha i D} \left(1 + \frac{D}{2\alpha i} \right)^{-1} \quad (1)$$

$$= \frac{e^{i\alpha x}}{2\alpha i D} \quad (1) = \frac{e^{i\alpha x}}{2\alpha i} \quad (2)$$

$$= -\frac{i}{2\alpha} (\cos \alpha x + i \sin \alpha x)$$

$$= \frac{\alpha}{2\alpha} (\sin \alpha x - i \cos \alpha x)$$



$$= \left[\frac{a \sin \alpha}{2\alpha} - i \frac{(a \cos \alpha)}{2\alpha} \right].$$

$$\therefore \text{I.P of } \left[\frac{1}{D^2 + \alpha^2} e^{i\alpha x} \right] = - \frac{a \cos \alpha}{2\alpha}.$$

$$\therefore \boxed{\frac{1}{D^2 + \alpha^2} \sin \alpha = - \frac{a \cos \alpha}{2\alpha}}.$$

$$\text{or } \frac{1}{D^2 + \alpha^2} \sin \alpha = - \frac{a \cos \alpha}{2\alpha}$$

$$= \frac{\alpha}{2} \int \sin \alpha d\alpha$$

$$\text{Sly } \frac{1}{D^2 + \alpha^2} \cos \alpha = \frac{\alpha}{2} \int \cos \alpha d\alpha$$

$$= \frac{\alpha}{2} \left[\frac{\sin \alpha}{\alpha} \right]$$

$$\therefore \boxed{P.I = \frac{1}{D^2 + \alpha^2} \sin \alpha = - \frac{a \cos \alpha}{2\alpha}}$$

if $f(\alpha) \neq 0$

$$\boxed{P.I = \frac{1}{D^2 + \alpha^2} \cos \alpha = \frac{a \sin \alpha}{2\alpha}}$$

if $f(-\alpha) \neq 0$.

To find P.I when $\theta = \sin \alpha$ (or)
 $\cos \alpha$.

$$\therefore P.I = \frac{1}{D^2 + \alpha^2} \sin \alpha = - \frac{\alpha}{2\alpha} \cos \alpha$$

and $f(-\alpha) \neq 0$

$$= \frac{\alpha}{2} \int \sin \alpha d\alpha$$

$$\therefore P.I = \frac{1}{D^2 + \alpha^2} \cos \alpha = \frac{\alpha}{2\alpha} \sin \alpha.$$

$$= \frac{\alpha}{2} \int \cos \alpha d\alpha$$

* Solve the following diff. eqns:

$$\rightarrow (D^2 + 4)y = \sin x. \quad | \sin x = \frac{1 - \cos 2x}{2}$$

$$\rightarrow (D^2 + 1)y = \sin mx, \cos mx \quad | \quad \sin mx + \sin mx \\ \xrightarrow{1991} \left(\frac{d^4}{dx^4} - m^4 y \right) = \sin mx \rightarrow g e^{mx} + g_2 e^{-mx} f(A \cos mx + B \sin mx) \\ + \frac{1}{1-m^4} \sin mx$$

1995 find the solution of

$$\frac{d^2y}{dx^2} + 4y = 8 \cos 2x. \text{ gives that} \\ \Rightarrow \text{real}$$

$$y=0 \text{ and } \frac{dy}{dx}=0 \text{ when } x=0.$$

1992 solve $(D^2 + 4)y = \sin 2x$, gives that

$$\text{when } x=0, y=0 \text{ and } \frac{dy}{dx}=2. \\ \text{by } g_2 \sin 2x - g_1 \cos 2x. \frac{dy}{dx}.$$

$$\xrightarrow{1993} \text{solve } (D^4 + D^2 + 1)y = e^{-\lambda/2} \cos(\frac{\sqrt{3}}{2}x)$$

Sol: Given that

$$(D^4 + D^2 + 1)y = e^{-\lambda/2} \cos(\frac{\sqrt{3}}{2}x). \quad \rightarrow ①$$

$$\text{A.E. of } ① \text{ is } D^4 + D^2 + 1 = 0$$

$$\Rightarrow (D^2 + 1)^2 - D^2 = 0$$

$$\Rightarrow (D^2 + 1 - D)(D^2 + 1 + D) = 0$$

$$\Rightarrow D^2 - D + 1 = 0 \text{ & } D^2 + D + 1 = 0$$

$$\Rightarrow D = \frac{-1 \pm \sqrt{3}i}{2}, \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore y_c = e^{-\lambda/2} \left[C_1 \cos(\frac{\sqrt{3}}{2}x) + C_2 \sin(\frac{\sqrt{3}}{2}x) \right]$$

$$+ e^{-\lambda/2} \left[C_3 \cos(\frac{\sqrt{3}}{2}x) + C_4 \sin(\frac{\sqrt{3}}{2}x) \right]$$

$$\text{P.I} = \frac{1}{D^4 + D^2 + 1} e^{-\lambda/2} \cos(\frac{\sqrt{3}}{2}x)$$

$$= e^{-\lambda/2} \frac{1}{(D - i\omega)^4 + (D + i\omega)^2 + 1} \cos(\frac{\sqrt{3}}{2}x).$$

(42)

$$= e^{-\lambda/2} \frac{1}{D^4 + \frac{1}{16} + D^2 + \frac{D^2}{2} - \frac{D}{2} - 2D^3 + D^2 + \frac{L}{4}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right)$$

$$= e^{-\lambda/2} \frac{1}{D^4 - 2D^3 + \frac{5}{16}D^2 - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right)$$

$$= e^{-\lambda/2} \frac{1}{(D^2 + \frac{3}{4})(D^2 - 2D + \frac{7}{4})} \cos\left(\frac{\sqrt{3}}{2}\alpha\right)$$

$$= e^{-\lambda/2} \frac{1}{(D^2 + \frac{3}{4})} \left[\frac{1}{-\frac{3}{4} - 2D + \frac{7}{4}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$$= e^{-\lambda/2} \frac{1}{D^2 + \frac{3}{4}} \cdot \frac{1}{(-2D)} \cos\left(\frac{\sqrt{3}}{2}\alpha\right)$$

$$= e^{-\lambda/2} \frac{1}{D^2 + \frac{3}{4}} \left[\frac{1+2D}{1-4D} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$$= e^{-\lambda/2} \frac{1}{(D^2 + \frac{3}{4})} \left[\frac{1+2D}{4} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$$= \frac{e^{-\lambda/2}}{4} \left[\cos\left(\frac{\sqrt{3}}{2}\alpha\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$$= \frac{e^{-\lambda/2}}{4} \left[\frac{-\alpha}{2(\sqrt{\frac{3}{2}})} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{\sqrt{3}\alpha}{2(\sqrt{\frac{3}{2}})} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$$= \frac{e^{-\lambda/2}}{4} \left[\frac{\alpha}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) + \alpha \cos\left(\frac{\sqrt{3}}{2}\alpha\right) \right]$$

$\therefore y = y_c + y_p$ is the g.s of
the given diff. eqn.

case(vi)

TO find P.I

(47)

when $Q = \lambda^m v$ where v is a fun of x

$$P.I = \frac{1}{f(D)} (2v) = \lambda \frac{1}{f(D)} v - \frac{f'(0)}{[f(D)]}$$

Note: — By the repeated use of the
above formula $\frac{1}{f(D)} v^m v. (m \neq 1)$

can be determined.

but it will more
tedious.

first
operate on
 D if possible
otherwise
operate on v
then D'

★ Solve the following diff. eqns!

$$\rightarrow (D^2 - 1)y = \lambda \sin 3x + \cos x$$

$$\rightarrow (D^2 - 2D + 1)y = \lambda \sin x$$

$$\rightarrow (D^2 - 2D + 1)y = \lambda e^x \sin x$$

$$\rightarrow (D^2 + 3)y = \lambda \sin x$$

$$\rightarrow (D^2 + 1)y = e^{-x} + \cos x + x^3 + e^{-x} \sin x$$

Note: — If $Q = \lambda^m \sin ax$ (or) $\lambda^m \cos ax$

when the coefficient & of
 $\sin ax$ (or) $\cos ax$ is a^2 or a^3

or higher power of 'a' then

the following method can
also be used.

- * solve the following diff. eqns.

Q) find P.I. of $(D^2 + 1)y = z^2 \sin 2x$

$$\text{Sol} \quad P.I. = \frac{1}{D^2 + 1} z^2 \sin 2x$$

$$= \frac{1}{D^2 + 1} z^2 (\text{I.P. of } e^{2ix})$$

$$= \text{I.P. of } \frac{1}{D^2 + 1} (e^{2ix} z^2) \quad \text{--- (1)}$$

$$\text{Now } \frac{1}{D^2 + 1} (e^{2ix} z^2)$$

$$= e^{2ix} \cdot \frac{1}{(D + 2xi)^2 + 1} z^2$$

$$= e^{2ix} \cdot \left[\frac{1}{D^2 + 4Dx - 3} (z^2) \right]$$

$$= e^{2ix} \left[-\frac{1}{3} \left(1 - \left(\frac{D^2}{3} + \frac{4D}{3} i - \frac{16}{9} \right) \right) (z^2) \right]$$

$$= -\frac{1}{3} e^{2ix} \left[\left(1 + \frac{D^2}{3} + \frac{4D}{3} i - \frac{16}{9} D^2 \right) (z^2) \right]$$

$$= -\frac{1}{3} e^{2ix} \left[z^2 + \frac{2}{3} + \frac{4i}{3} (2x) - \frac{16}{9} (2) \right]$$

$$= -\frac{1}{3} e^{2ix} \left(z^2 + \frac{8}{3} xi - \frac{26}{9} \right).$$

$$= -\frac{1}{3} (\cos 2x + i \sin 2x) \left(z^2 + \frac{8}{3} xi - \frac{26}{9} \right).$$

$$\therefore \text{I.P. of } \frac{1}{D^2 + 1} (e^{2ix} z^2) =$$

$$-\frac{z^2}{3} (\sin 2x) + \frac{26}{27} \sin 2x$$

$$-\frac{8x}{9} \cos 2x.$$

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→ solve $(D^4 + 2D^2 + 1) y = x^n \cos x$
→ solve $(D^2 - 4D + 4) y = 8x^2 e^{2x}$
S.M.D.A.

case (vii)

If ϕ is a function of

then (i) $\frac{1}{D-\alpha} \phi = e^{\alpha x} \int e^{-\alpha x} \phi dx$.

(ii) $\frac{1}{D+\alpha} \phi = e^{-\alpha x} \int e^{+\alpha x} \phi dx$.

problems

→ find P.I. of $\frac{1}{D^2 - 4D + 3} (x e^{4x})$

Sol Now P.I. = $\frac{1}{D^2 - 4D + 3} (x e^{4x})$

$$= \frac{1}{(D-1)(D-3)} (x e^{4x})$$

$$= \frac{1}{(D-1)} \left[\frac{1}{(D-3)} x e^{4x} \right]$$

and so on.

→ $\frac{1}{D^2 - 5D + 6} x e^{4x} = ?$

→ $\frac{1}{6D^2 - D - 2} x e^{4x} = ?$

→ $\frac{1}{D^2 - 7D + 2} \sin(x e^x) = ?$

$$\rightarrow \text{solve } (D^2 + a^2) y = \sec ax$$

$$\underline{\text{SOL}} \quad C.F. = C_1 \cos ax + C_2 \sin ax.$$

$$P.I. = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D-ai} - \frac{1}{D+ai} \right] \sec ax. \quad (1)$$

$$\text{Now } \frac{1}{D-ai} \sec ax = e^{iax} \int e^{-iax} \sec ax da$$

$$= e^{iax} \left[\int (\cos ax - i \sin ax) \frac{da}{\cos ax} \right]$$

$$= e^{iax} \left[\left(a + i \tan a \right) da \right]$$

$$= e^{iax} \left[a + i \log(\cos ax) \right] \quad (2)$$

Replacing i by $-i$ in (2), we get

$$\frac{1}{D+ai} \sec ax = e^{-iia} \left[a - i \log(\cos ax) \right] \quad (3)$$

from (1), (2) & (3) we get

$$\begin{aligned} P.I. &= \frac{1}{2ia} \left[e^{iax} \left(a + i \log(\cos ax) \right) \right. \\ &\quad \left. - e^{-iia} \left(a - i \log(\cos ax) \right) \right] \\ &= \frac{a(e^{iia} - e^{-iia})}{2ia} + \frac{1}{a} \left(\log(\cos ax) \right) \cdot \left(\frac{e^{iia} + e^{-iia}}{2} \right) \end{aligned}$$

(45)

$$= \frac{a}{\omega} \sin \omega t + \frac{1}{\omega} \cos \omega t \log \cos \omega t.$$

\therefore G.S is $y = y_C + y_P$

\rightarrow solve $(D^2 + \omega^2)y = \tan \omega t$

Ans: $y = (c_1 \cos \omega t + c_2 \sin \omega t)$

$$= \left(\frac{1}{\omega}\right) \cos \omega t \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}\omega t\right).$$

\rightarrow solve $(D^2 + \omega^2)y = \csc \omega t$

Ans: P.I = $-\frac{a}{\omega} \cos \omega t + \frac{1}{\omega} \sin \omega t \log(\sin \omega t)$

$$\rightarrow (D^2 + \omega^2)y = \cot \omega t$$

Ans: P.I = $\frac{1}{\omega} \sin \omega t \log |\tan(\frac{\omega t}{2})|$.

~~Find the solution of~~

$$\frac{d^2i}{dt^2} + \left(\frac{R}{L}\right) \frac{di}{dt} + \left(\frac{1}{LC}\right) i = 0$$

where $R^2C \geq 4L$ and

R, C, L are constant.

Sol Given that $\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = 0$ (1)

$$\Rightarrow (D^2 + \frac{R}{L}D + \frac{1}{LC})i = 0$$

A.E of (2) is (2)

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$$

$$\therefore D = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4(0)\left(\frac{1}{LC}\right)}}{2}$$

$$= \frac{1}{2} \left[\frac{-R}{L} \pm \sqrt{\frac{R^2 C - 4 L}{L^2 C}} \right]$$

$$= \frac{1}{2} \left[\frac{-R}{L} \pm \sqrt{\frac{0}{L^2 C}} \right]$$

$$(\because R^2 C = 4 L)$$

$$= \frac{-R}{2L}, \frac{-R}{2L}$$

$$\therefore C.F = (C_1 + C_2 t) e^{-\frac{R}{2L}t}$$

\therefore Q.S of ① is

$$i = (C_1 + C_2 t) e^{-\frac{R}{2L}t}$$

→ solve the eqn $\frac{dy}{dx} = a + b x + c x^2$

given $\frac{dy}{dx} = 0$, when $x=0$
and $y = d$ when $x=0$.

Sol Given that $\frac{dy}{dx} = a + b x + c x^2$

$$\Rightarrow D^2 y = a + b x + c x^2$$

A.E is $D^2 = 0$

$$\Rightarrow D = 0, 0.$$

$$\therefore y_c = C_1 + C_2 x$$

$$P.I = \frac{1}{D^2} (a + b x + c x^2)$$

$$= \frac{1}{D} \int (a + b x + c x^2) dx = \frac{1}{D} \left[a x + \frac{b x^2}{2} + \frac{c x^3}{3} \right]$$

$$= \int \left[c_1 + \frac{bx^n}{2} + \frac{cx^3}{3} \right] \quad (46)$$

$$= \frac{c_1 x}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

$$y_p = \frac{c_1 x}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

\therefore G.S is $y = y_c + y_p$.

$$\Rightarrow y = (c_1 + c_2 x) e^{0x} + \frac{c_1 x}{2} + \frac{bx^3}{6} + \frac{cx^4}{12} \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = c_2 + c_2 x + \frac{bx^n}{2} + \frac{cx^3}{3} \quad (3)$$

Putting $y = d$ and $x = 0$

$$d = c_1 + c_2(0) + 0 + 0 + 0$$

$$\Rightarrow \boxed{c_1 = d}$$

$$\therefore (2) = d + \frac{c_1 x}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

which is reqd soln of (1).

~~$$\text{Q96) If } \frac{d^2x}{dt^2} + \frac{g}{b} (x-a) = 0$$~~

(a, b and g being +ve constns)

and $a = a'$ and $\frac{dx}{da} = 0$ when $t > 0$

then show that $x = a + (a' - a) \cos\left(\sqrt{\frac{g}{b}} t\right)$

Sol Given that $\frac{d^2x}{dt^2} + \frac{g}{b} (x-a) = 0$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{g}{b} y = \frac{g^2}{b}. \quad \text{---(1)}$$

$$\Rightarrow \left(D^2 + \frac{g}{b}\right)y = \frac{g^2}{b} \quad \text{where } D = \frac{d}{dt}$$

$$\text{A.E. of (2) is } D^2 + \frac{g}{b} = 0 \quad \text{---(2)}$$

$$D^2 = -\frac{g}{b} \Rightarrow D = \pm \sqrt{\frac{g}{b}} \quad \text{C}$$

$$\therefore y_c = c_1 \cos \sqrt{\frac{g}{b}} t + c_2 \sin \sqrt{\frac{g}{b}} t \quad \text{to}$$

$$P.I. = \frac{1}{D^2 + \frac{g}{b}} \left(\frac{g^2}{b}\right) \quad (\text{containing } \text{heat})$$

~~Find the solution of the eqn~~

$$(D^2 - 1)y = 1 \quad \text{which vanishes } (y=0)$$

- when $a=0$ and tends to

- a finite limit as $a \rightarrow \infty$ and
D stands for $\frac{d}{dx}$.

$$\underline{\text{SOL}}. \quad (D^2 - 1)y = 1 \quad \text{---(1)}$$

$$\text{A.E. is } D^2 - 1 = 0$$

$$D = \pm 1$$

$$\therefore \boxed{y_c = c_1 e^t + c_2 e^{-t}}$$

$$P.I. = \frac{1}{D^2 - 1} \quad (1) = \frac{1}{D^2 - 1} e^{0x}$$

$$= \frac{1}{-1} = -1$$

$$\therefore \boxed{y_p = -1}$$

G.S is $y = y_c + y_p$ (47)

$$\boxed{y = c_1 e^n + c_2 e^{-n} - 1} \quad (2)$$

Putting $y=0$ and $n=0$ in (1), we get

$$0 = c_1 + c_2 - 1$$

$$\Rightarrow \boxed{c_1 + c_2 = 1} \quad (3)$$

Multiplying both sides of (2) by e^n

we get $y e^n = c_1 (e^n)^n + c_2 - e^n$

Taking limit on both sides of (2)
as $n \rightarrow \infty$ we get

$$\text{L} \lim_{n \rightarrow \infty} y e^n = \text{L} \lim_{n \rightarrow \infty} c_1 (e^n)^n + \text{L} \lim_{n \rightarrow \infty} c_2 - \text{L} \lim_{n \rightarrow \infty} e^n$$

$$\therefore y \times 0 = c_1 \times 0 + c_2 - 0 \\ (\because \text{L} \lim_{n \rightarrow \infty} e^n = 0)$$

$$\Rightarrow \boxed{c_2 = 0}$$

$$\therefore (3) \Rightarrow \boxed{c_1 = 1}$$

$$(2) \Rightarrow \boxed{y = e^n - 1}$$

which is the reqd soln.

^{W.O.I}
₁₉₉₇ Given $(D^2 + 1)y = 24n \cos n$ given
the initial conditions $y=0, y' = 0$,

$$Dy=0, D^2y=0$$

$$D^3y = 0$$

$$\text{Ans: } y = 3n^2 \sin n - n^3 \cos n.$$

1953 solve $\frac{dy}{dx} + w_0 y = \text{const}$
and discuss the nature
of solution as well.

1955 Determine all real valued
solutions of the eq's

$$y''' - iy'' + y' - iy = 0$$

$$y' = \frac{dy}{dx}$$

Sol Given it is $y''' - iy'' + y' - iy = 0$

$$\Rightarrow \frac{d^3y}{dx^3} - i \frac{dy}{dx^2} + \frac{dy}{dx} - iy = 0$$

$$\Rightarrow [D^3 - i D^2 + D - i] y = 0$$

$$D = \frac{d}{dx}$$

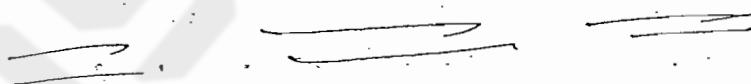
①

$$A.E \text{ is } D^3 - i D^2 + D - i = 0$$

$$\Rightarrow D^2(D - i) + (D - i) = 0$$

$$\Rightarrow D = i, D = \pm i$$

(continuing
next)



EXERCISE FOR IIT Set-III CLASS PLACEMENT TEST

MATHEMATICS by R. VENKATESWARA

CAUCHY-EULER EQUATIONS

48

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = x$$

where a_1, a_2, \dots, a_n are constants, and x is a function of t , is called the Cauchy-Euler homogeneous linear equation of the n^{th} order.

Method of Solution:-

Reduce the linear equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = x$$

into linear equation with constant coefficients.

The given equation is

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = x \quad (1)$$

$$\text{put } x = e^z \Rightarrow z = \log x \quad (x > 0)$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\therefore x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow \boxed{x \frac{dy}{dx} = D_1 y}$$

where $D_1 = \frac{d}{dz}$

$$\text{Now } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right]$$

$$= \frac{d}{dx} \left[\frac{1}{x} \frac{dy}{dz} \right]$$

$$= \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right) + \frac{dy}{dz} \left(\frac{d}{dx} \left(\frac{1}{x} \right) \right)$$



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$$\begin{aligned}
 &= \frac{1}{x} \frac{d}{dz} \left(\frac{dy}{dx} \right) + \frac{dy}{dz} \left(-\frac{1}{x^2} \right) \\
 &= \frac{1}{x} \frac{d}{dz} \left(\frac{1}{x} \frac{dy}{dx} \right) + \frac{dy}{dz} \left(-\frac{1}{x^2} \right) \\
 &= \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz} \\
 &= \frac{1}{x^2} (D_1^2 - D_1)y \quad \text{where } D_1 = \frac{d}{dz} \\
 &= \frac{1}{x^2} D_1(D_1 - 1)y \\
 \therefore \boxed{x^2 \frac{d^2y}{dx^2} = D_1(D_1 - 1)y}
 \end{aligned}$$

$$\text{Sly } x^3 \frac{d^3y}{dx^3} = D_1(D_1 - 1)(D_1 - 2)y$$

$$x^n \frac{d^n y}{dx^n} = D_1(D_1 - 1)(D_1 - 2) \dots (D_1 - (n-1))y$$

i.e. From ①, we have

$$[D_1(D_1 - 1)(D_1 - 2) \dots (D_1 - (n-1))] + a_1 D_1(D_1 - 1)(D_1 - 2) \dots$$

$$(D_1 - (n-2)) + \dots + a_{n-1} D_1 + a_n]y = e^z$$

clearly which is a linear equation with constant coefficients and is therefore solvable for y in terms of z .

→ If $y = f(z)$ is its solution then putting $z = \log x$,
then the required solution is $y = f(\log x)$

Problems

→ Solve the following:

$$(1) (x^2 D^2 - x D + 2)y = x \log x$$

Soln Given that $(x^2 D^2 - x D + 2)y = x \log x \rightarrow ①$

$$\text{Putting } x = e^z \Rightarrow z = \log x$$

$$\text{Let } D_1 = \frac{d}{dz} \text{ then}$$

from ①, We have

$$\begin{aligned}
 &(D_1(D_1 - 1) - D_1 + 2)y = e^z z \\
 \Rightarrow &(D_1^2 - 2D_1 + 2)y = z e^z \rightarrow ②
 \end{aligned}$$

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MATHEMATICS by K. VENKANNA

A.E of (2) is

$$D_1^2 - 2D_1 + 2 = 0 \dots$$

$$D_1 = \frac{2 \pm \sqrt{4 - 4(1)^2}}{2(1)} \dots$$

$$= \frac{2 \pm \sqrt{-4}}{2} \dots$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\therefore Y_C = e^z (C_1 \cos z + C_2 \sin z)$$

P.I of (2) is $\frac{1}{D_1^2 - 2D_1 + 2} (e^z z)$

$$= e^z \left[\frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 2} \right]$$

$$= e^z \left[\frac{1}{D_1^2 + 1 + 2D_1 - 2D_1 - 2 + 2} \right] z$$

$$= e^z \left[\frac{1}{D_1^2 + 1} \right] z$$

$$= e^z \left(1 - \frac{1}{D_1^2 + 1} + \dots \right) z$$

$$= e^z (1 - 0 + \dots) z$$

$$= e^z z$$

$$\therefore P.I. of (2) is Y_p = ze^z$$

$$Y = C_1 (C_1 \cos z + C_2 \sin z) + ze^z$$

$$y = x (C_1 \cos(\log x) + C_2 \sin(\log x)) + x \log x$$

which is the required general solution of (1).



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STUDY MATERIAL FOR IIT-JEE / IIT-JEE / CSIR EXAMINATIONS

MATHEMATICS by K. VENKATESWARA

H.W.

$$(2) \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}$$

Note:- $\frac{x^n dy^n}{dx^n} + a_1 x^{n-1} \frac{dy}{dx} + a_2 x^{n-2} \frac{dy}{dx} + \dots + a_m x^{m-n} y = x$

$$\frac{d^2y}{dx^2} + a_1 x^{n-1} \frac{dy}{dx} + a_2 x^{n-2} \frac{dy}{dx} + \dots + a_m x^{m-n} y = x$$

Let $x = z^m$ then convert only LHS in terms of z by using

$D_1 = \frac{d}{dz}$ but does not change RHS $x = z^m$ in terms of z

∴ The given equation reduces to $f(D_1)$

For this special case, we use the following formula.

directly. $\frac{1}{f(D_1)} z^m = \frac{1}{f(m)} z^m$, provided $f(m) \neq 0$.

Problem :

→ Solve $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x$ — (1)

Sol: Let $x = e^z \Rightarrow z = \log x$ and

$$\text{let } D_1 = \frac{d}{dz}$$

then $[D_1(D_1-1) + 6] y = x$

$$\Rightarrow [D_1^2 - D_1 + 6] y = x$$

$$\Rightarrow [D_1^2 - 5D_1 + 6] y = x \quad \text{--- (2)}$$

A.E of (2) is $D_1^2 - 5D_1 + 6 = 0$

$$\Rightarrow D_1 = 2, 3$$

$$\therefore Y_c = C_1 e^{2z} + C_2 e^{3z}$$

$$Y_p = \frac{1}{D_1^2 - 5D_1 + 6} (x) = \frac{1}{1-5+6} (x) \\ = \frac{1}{2} (x)$$

∴ The g.s (1) is $y = Y_c + Y_p$

$$y = C_1 e^{2z} + C_2 e^{3z} + \frac{1}{2} (x)$$

$$\Rightarrow y = C_1 x^2 + C_2 x^3 + \frac{1}{2} x$$

which is the required g.s of (1) =

Note:- An alternative method for getting P.I of Cauchy-Euler equation without changing the R.H.S in terms of Z.

Let $F(D_1)y = f(x)$ where $D_1 = \frac{d}{dx}$.

$$\text{then } \frac{1}{D_1-a} f(x) = x^a \int x^{a-1} f(x) dx.$$

$$\text{and } \frac{1}{D_1-a} f(x) = x^a \int x^{a-1} f(x) dx.$$

To evaluate P.I, we first factorize $F(D_1)$ into linear factors and then one of the following methods can be used.

Method(I): If the operator $\frac{1}{F(D_1)}$ into partial fractions

then

$$\text{P.I.} = \frac{1}{F(D_1)} f(x) = \left[\frac{A_1}{(D_1-a_1)} + \frac{A_2}{(D_1-a_2)} + \dots + \frac{A_n}{(D_1-a_n)} \right] f(x)$$

Method(II): $P.I. = \frac{1}{(D_1-a_1)(D_1-a_2)\dots(D_1-a_n)} f(x)$

where the operations indicated by factors are to be taken in succession, beginning with the first on the right.

Problem:-

$$\rightarrow \text{Solve } x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x + \sin x$$

$$\text{Sol: Given that } x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x + \sin x \quad \text{--- (1)}$$

$$\text{Let } x = e^z \Rightarrow z = \log x$$

$$\text{then (1)} = [D_1(D_1-1) + 4D_1 + 2] y = x + \sin x$$

$$\Rightarrow [D_1^2 + 3D_1 + 2] y = x + \sin x \quad \text{--- (2)}$$

NOW A.E of (2) is $m^2 + 3m + 2 = 0$

$$\Rightarrow m = -2, -1$$

$$\therefore Y_C = C_1 e^{-2z} + C_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D_1^2 + 3D_1 + 2} (x + \sin x) = \frac{1}{(D_1+2)(D_1+1)} (x + \sin x)$$

$$= \frac{1}{(D_1+2)(D_1+1)} x + \frac{1}{(D_1+2)(D_1+1)} \sin x \quad \text{--- (3)}$$

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MATHEMATICS by M. VERMA

Now $\frac{1}{(D_1+2)(D_1+1)} \sin x = \frac{1}{D_1+2} \left[\frac{1}{D_1+1} \sin x \right]$

$$= \frac{1}{D_1+2} \left[x^1 \int x^{1-1} \sin x dx \right]$$

$$= \frac{1}{D_1+2} \left[\frac{-1}{x} \cos x \right]$$

$$= x^{-2} \int x^{2-1} \left(\frac{-1}{x} \cos x \right) dx$$

$$= -x^2 \int \cos x dx$$

$$= -\frac{1}{x^2} \sin x$$

∴ G.S of ② is $y = y_c + Y_c$

$$y = (C_1 e^{2x} + C_2 e^{-2x}) - \frac{1}{x^2} \sin x + \frac{x}{6}$$

$$\Rightarrow y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{6} + \frac{1}{x^2} \sin x.$$

which is reqd g.s.

* solve the following diff eqns:

$$\rightarrow (x^2 D^2 + xD - 1) y = x^{2x} \text{ Ans: } y = C_1 x + C_2 x^{-1} + \frac{1}{8} (e^{2x}) (x^{-2})$$

$$\rightarrow (x^2 D^2 + xD - 1) y = x^2 e^x \text{ Ans: } y = C_1 x + C_2 x^{-1} + e^x (1-x)$$

$$\rightarrow x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10 (x+1)$$

$$\rightarrow (x^3 D^3 + 2x^2 D^2 + 2x + 1) y = x \log x$$

$$\rightarrow x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = x \log x$$

$$\rightarrow x^3 \frac{d^3 y}{dx^3} + 4x^2 \frac{d^2 y}{dx^2} + 2y = e^x$$

* LEGENDRE'S LINEAR EQUATIONS:

An equation of the form

$$(ax+bx)^n \frac{d^n y}{dx^n} + A_1 (ax+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + A_2 (ax+bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} (ax+bx) \frac{dy}{dx} + A_n y = Q(x)$$



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where A_1, A_2, \dots, A_n are constants and $Q(x)$ is a function of x^1 is called Legendre's linear equation of the n th order.

* Method of Solution:-

Reduce the linear equation

$$(a+bx)^n \frac{d^n y}{dx^n} + A_1(a+bx)^{n-1} \frac{dy}{dx^{n-1}} + A_2(a+bx)^{n-2} \frac{d^2y}{dx^{n-2}} + \dots + A_{n-1}(a+bx) \frac{dy}{dx} + A_n y = Q(x) \quad \text{--- (1)}$$

where $A_1, A_2, \dots, A_{n-1}, A_n$ are constants, into linear equation with constant coefficients.

$$\text{putting } a+bx = e^z \Rightarrow z = \log(a+bx)$$

$$\Rightarrow \frac{dz}{dx} = b \left(\frac{1}{a+bx} \right).$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \left[\frac{b}{a+bx} \right]$$

$$(a+bx) \frac{dy}{dx} = b \frac{dy}{dz}$$

$$\Rightarrow \boxed{(a+bx) \frac{dy}{dx} = b D_1 y} \quad \text{Where } D_1 = \frac{d}{dz}$$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right]$$

$$= \frac{d}{dx} \left[\frac{dy}{dz} \frac{b}{a+bx} \right]$$

$$= \frac{b}{a+bx} \frac{d}{dz} \left(\frac{dy}{dz} \right) + \frac{dy}{dz} \frac{d}{dz} \left(\frac{b}{a+bx} \right).$$

$$= \frac{b}{a+bx} \frac{d}{dz} \left(\frac{dy}{dz} \right) + \frac{dy}{dz} \left(\frac{b^2}{(a+bx)^2} \right)$$

$$= \frac{b}{a+bx} \frac{d}{dz} \left[\frac{dy}{dz} \cdot \frac{b}{a+bx} \right] - \frac{b^2}{(a+bx)^2} \frac{dy}{dz}$$

$$= \frac{b^2}{(a+bx)^2} \frac{d^2y}{dz^2} - \frac{b^2}{(a+bx)^2} \frac{dy}{dz}$$

NOTE FOR IAS / IIT-JEE - CSIR EXAMINATIONS

MATHEMATICS by K. VENKANNA

$$= \frac{b^2}{(a+bx)^2} (D_1^2 - D_1)y$$

$$= \frac{b^2}{(a+bx)^2} D_1(D_1-1)y$$

$$\therefore (a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y$$

$$\text{Sly } (a+bx)^3 \frac{d^3y}{dx^3} = b^3 D_1(D_1-1)(D_1-2)y$$

$$(a+bx)^n \frac{d^ny}{dx^n} = b^n [D_1(D_1-1)(D_1-2)\dots(D_1-(n-1))]y$$

∴ from ①, we have,

$$b^n (D_1(D_1-1)(D_1-2)\dots(D_1-(n-1))) + \\ b^{n-1} A_1 (D_1(D_1-1)(D_1-2)\dots(D_1-(n-2))) + \dots + A_{n-1} b D_1 \\ + A_n y = \alpha (e^z a)$$

Clearly which is a linear equation with constant coefficients and hence is solvable for y in terms of z .

Working rule:

$$\text{putting } a+bx = e^z \Rightarrow z = \log(a+bx)$$

$$\text{let } D_1 = \frac{d}{dz} \text{ then}$$

$$(a+bx) \frac{dy}{dx} = b D_1 y$$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y \text{ etc.}$$

Problems:

→ Solve the following:

$$① [(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1$$

Sol: Given that



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$$\boxed{[(3x+2)^2 D^2 + 3(3x+2)D - 36] Y = 3x^2 + 4x + 1} \quad \text{--- (1)}$$

Putting $3x+2 = e^z \Rightarrow z = \log(3x+2)$

Let $D_1 = \frac{d}{dz}$ then

$$(3x+2) \frac{dy}{dx} = 3Dy$$

$$(3x+2) \frac{d^2y}{dx^2} = 3^2 D_1(D_1 - 1)y$$

From (1), we have

$$\boxed{[3^2 D_1(D_1 - 1) + 3(3)D_1 - 36] Y = \frac{e^{2z} - 1}{3}}$$

$$\Rightarrow 9[D_1^2 - D_1 D_1 - 4] Y = \frac{e^{2z} - 1}{3}$$

$$\Rightarrow [D_1^2 - 4] Y = \frac{e^{2z} - 1}{27} \quad \text{--- (2)}$$

A.E of (2) is $m^2 - 4 = 0$

$$m^2 = 4 \Rightarrow m = \pm 2$$

$$\therefore Y_c = C_1 e^{-2z} + C_2 e^{+2z}$$

$$\begin{aligned} \text{P.I. of (2) is } & \frac{1}{D_1^2 - 4} \left(\frac{e^{2z} - 1}{2z} \right) = \frac{1}{27} \left[\frac{1}{D_1^2 - 4} (e^{2z} - 1) \right] \\ & = \frac{1}{27} \left[\frac{1}{D_1^2 - 4} e^{2z} - \frac{1}{D_1^2 - 4} (1) \right] \\ & = \frac{1}{27} \left[e^{2z} \cdot \frac{1}{(D_1 + 2)(D_1 - 2)} (1) - \frac{1}{4} \right] \\ & = \frac{1}{27} \left[e^{2z} \cdot \frac{1}{D_1^2 + 2D_1} (1) + \frac{1}{4} \right] \\ & = \frac{1}{27} \left[e^{2z} \cdot \frac{z}{2D_1 + 2} (1) + \frac{1}{4} \right] \\ & = \frac{1}{27} \left[e^{2z} \cdot \frac{z}{2} (1) + \frac{1}{4} \right] \\ & = \frac{1}{27} \left[\frac{ze^{2z}}{2} + \frac{1}{4} \right] \\ & = \frac{1}{54} \left[ze^{2z} + \frac{1}{2} \right] \end{aligned}$$

$\therefore \text{G.S of (2) is } Y = Y_c + Y_p$

SOLVED PROBLEMS / QUESTIONS / EXERCISES

MATHEMATICS by K. VENKANNA

$$y = (c_1 e^{-2x} + c_2 x e^{-2x}) + \frac{1}{54} (x e^{2x} + k_2)$$

$$\Rightarrow y = \left[c_1 \frac{1}{(3x+2)^2} + c_2 (3x+2)^2 \right] + \frac{1}{54} \left[\log(3x+2) \cdot (3x+2)^2 + \frac{1}{2} \right]$$

which is the reqd. g.s of ①

$$\text{Hence } (x+1)^2 \frac{d^2y}{dx^2} - 3(x+1) \frac{dy}{dx} + 4y = x^2$$

$$\rightarrow [(5+2x)^2 D^2 - 6(5+2x)D + 8]y = 0$$

$$\text{Hence } (1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)$$

$$y(0) = 0, y'(0) = 2$$

$$\text{Hence } [(x+1)^4 D^3 + 2(x+1)^3 D^2 - (D+1)^2]y = \frac{1}{x+1}$$

* Linear Differential Equations of the second order with variable coefficients

An equation of the form

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = R(x).$$

where, $p(x)$, $q(x)$ and $R(x)$ are functions of x is called linear diff. equation of the second order with variable coefficients.

The linear diff. equation of the second order with variable coefficients can be solved by the following methods.

(1) Change of the dependent variable when a part of the complementary function (C.F.) is known.

(2) Change of the dependent variable and removal of the first derivative (or) reduction to normal form or canonical form.

(3) Change of the independent variable.

(4) Variation of parameters.



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1. To solve $\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x)$ by changing of the dependent variable when a part of the C.F. is known.

Given equation is

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x) \quad (1)$$

and its linear homogenous equation is

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \quad (2)$$

Let $y = u(x)$ be a known solution of the C.F. of (1)

Then $y = uv$ is a solution of (2),

$$\frac{d^2y}{dx^2} + P(x) \frac{du}{dx} + Q(x)u = 0 \quad (3)$$

Let $y = uv$ be the g.s. of (1) where $u = u(x)$
 $v = v(x)$

$$\text{then } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \quad (4)$$

$$\frac{d^2y}{dx^2} = u \frac{d^2v}{dx^2} + 2 \frac{dy}{dx} \frac{dv}{dx} + v \frac{d^2u}{dx^2} \quad (5)$$

$$\therefore (1) \equiv u \frac{d^2v}{dx^2} + 2 \frac{dy}{dx} \frac{dv}{dx} + v \frac{d^2u}{dx^2} + P(x) \left(u \frac{dv}{dx} + v \frac{du}{dx} \right) + Q(x)uv = R(x).$$

$$\Rightarrow v \left[\frac{d^2u}{dx^2} + P(x) \frac{du}{dx} + Q(x)u \right] + u \left[\frac{d^2v}{dx^2} + P(x) \frac{dv}{dx} \right] + 2 \frac{dy}{dx} \frac{dv}{dx} = R(x)$$

$$\Rightarrow v(0) + u \left(\frac{d^2v}{dx^2} + P(x) \frac{dv}{dx} \right) + 2 \frac{dy}{dx} \frac{dv}{dx} = R(x) \quad (\text{by (3)})$$

$$\Rightarrow u \frac{d^2v}{dx^2} + u P(x) \frac{dv}{dx} + 2 \frac{dy}{dx} \frac{dv}{dx} = R(x)$$

$$\Rightarrow u \frac{d^2v}{dx^2} + (u P(x) + 2 \frac{dy}{dx}) \frac{dv}{dx} = R(x)$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P(x) + 2 \frac{dy}{dx} \right) \frac{dv}{dx} = \frac{R(x)}{u} \quad (6)$$

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MATHEMATICS BY K. VERMA

Let $\frac{dv}{dx} = v$ then $\frac{d^2v}{dx^2} = \frac{dv}{dx}$

$$\therefore \textcircled{1} \equiv \frac{dv}{dx} + \left(p(x) + \frac{2}{u} \frac{du}{dx} \right) v = \frac{R(x)}{u}$$

Clearly this is linear equation in v of first order

$$\therefore I.F = e^{\int (p(x) + \frac{2}{u} \frac{du}{dx}) dx}$$

$$= e^{\int p(x) dx + \int \frac{2}{u} du}$$

$$= e^{\int p(x) dx + 2 \log u}$$

$$I.F = u^2 e^{\int p(x) dx}$$

\therefore The g.s of $\textcircled{1}$ is

$$v \cdot u^2 e^{\int p(x) dx} = \int [B u^2 e^{\int p(x) dx}] dx + c$$

$$vu^2 e^{\int p(x) dx} = \int [u^2 e^{\int p(x) dx}] dx + c$$

$$\Rightarrow \frac{dv}{dx} u^2 e^{\int p(x) dx} = \int [R u e^{-\int p(x) dx}] dx + c \quad (\because v = \frac{dv}{dx})$$

$$\Rightarrow \frac{dv}{dx} u^2 e^{\int p(x) dx} = \left[\int (R u e^{-\int p(x) dx}) dx \right] + ce^{-\int p(x) dx}$$

$$\Rightarrow \frac{dv}{dx} = \left[\int \left(\frac{1}{u^2} e^{-\int p(x) dx} \cdot \int (R u e^{-\int p(x) dx}) dx \right) du + \frac{ce^{-\int p(x) dx}}{u^2} \right] dx + c_2$$

\therefore From $\textcircled{1}$, we have

$$y = c_2 u + u \int \left[\frac{1}{u^2} e^{-\int p(x) dx} \int (R u e^{-\int p(x) dx}) dx + \left[\frac{ce^{-\int p(x) dx}}{u^2} \right] dx \right]$$

Since this solution includes the known solution

$y = u(x)$ and it contains two arbitrary constants.

It is the g.s of $\textcircled{1}$



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* Methods for finding one integral (solution) in C.F
by inspection i.e. Solution of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ 54

Method (1):

If $y = e^{ax}$ is a solution of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad \text{--- (i)}$$

$$\text{then } \frac{dy}{dx} = ae^{ax}, \quad \frac{d^2y}{dx^2} = a^2e^{ax}$$

$$\therefore (i) \equiv a^2e^{ax} + p(x)ae^{ax} + q(x)e^{ax} = 0$$

$$\Rightarrow a^2 + ap(x) + q(x) = 0 \quad (\because e^{ax} \neq 0) \quad \text{--- (ii)}$$

∴ If $y = e^{ax}$ is a solution of (i) then $a^2 + ap(x) + q(x) = 0$

putting $a = 1$:

∴ $y = e^x$ is a solution of (i)

$$\text{then } 1 - p + q = 0$$

∴ if $1 - p + q = 0$ then $y = e^x$ is a solution of (i)

if $1 - p + q = 0$ then $y = e^{-x}$ is a solution of (i).

i.e. a part of the C.F. of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$

Method (2):

If $y = x^m$ is a solution of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$

$$\text{then } \frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$\therefore (i) \equiv m(m-1)x^{m-2} + p(x)mx^{m-1} + q(x)x^m = 0$$

$$\Rightarrow m(m-1) + p(x)m + q(x)x^2 = 0$$

∴ If $y = x^m$ is a solution of (i) then

$$m(m-1) + p(x)m + q(x)x^2 = 0$$

putting $m=1$ in the above

∴ $y = x$ is a solution of (i) then

$$0 + xP(x) + x^2Q(x) = 0$$

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MATHEMATICS by K. VENKATESWARA

$$\boxed{P+Qx=0} \quad (\because x \neq 0)$$

Putting $m=2$

$\therefore y=x^2$ is a solution of (i) then
 $2(2-1)+2P.x+Q.x^2=0$

$$\Rightarrow \boxed{2+2Px+Qx^2=0}$$

Conversely, suppose that $P+Qx=0$ then $y=x$ is a solution of (i)
and $2+2Px+Qx^2=0$ then $y=x^2$ is a solution of (i)

Working rule:

Step(1): write the given equation in standard form

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) = R(x)$$

Step(2): find one solution ($u(x)$) of C.F. by using the
following forms

Condition satisfied

one sol'n of C.F. is

(1) $a^2 + Pa + Q = 0$

$$u = e^{ax}$$

(i) $1 + P + Q = 0$

$$u = e^x$$

(ii) $1 - P + Q = 0$

$$u = e^{-x}$$

(2) $m(m-1) + Pm + Qx^2 = 0$

$$u = x^m$$

(i) $P + Q = 0$

$$u = x$$

(ii) $2 + P + Q = 0$

$$u = x^2$$

Step(3): Assume the g.s of given equation is $y = uv$, where
u is obtained by step(2) and v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{1}{u} \frac{dy}{dx}\right) \frac{dv}{dx} = \frac{R}{u}.$$

Problems: solve the following:

$$\rightarrow xy'' - (2x-1)y' + (x-1)y = 0 \quad \dots \textcircled{1}$$

$$\Rightarrow \frac{d^2y}{dx^2} - (2-x)y' + (1-x)y = 0 \quad \dots \textcircled{2}$$



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Comparing ① with

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = R(x)$$

$$\therefore p = -(2 - \frac{1}{x}), q = (1 - \frac{1}{x}) \text{ and } R = 0$$

Now here

$$1 + p + q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} \\ = 0$$

$\therefore y = u e^{cx}$ is a part of C.F of ①

Let $y = uv$ be the g.s of ①

then v is given by

$$\frac{d^2v}{dx^2} + (p + \frac{2}{u} \frac{du}{dx}) \frac{dv}{dx} = \frac{R}{u} \quad \text{--- ③}$$

$$\text{Since } u = e^{cx} \Rightarrow \frac{du}{dx} = e^{cx},$$

$$p + \frac{2}{u} \frac{du}{dx} = -2 + \frac{1}{x} + 2e^{cx} e^{cx} \\ = -2 + \frac{1}{x} + 2 \\ = K_c$$

\therefore From ③, we have

$$\frac{d^2v}{dx^2} + (\frac{1}{x}) \frac{dv}{dx} = 0 \quad \text{--- ④}$$

$$\text{Take } \frac{dv}{dx} = v \Rightarrow \frac{d^2v}{dx^2} = \frac{dv}{dx}$$

$$\therefore ④ \equiv \frac{dv}{dx} + \frac{1}{x} v = 0$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{x} dx$$

$$\Rightarrow \log(vx) = \log c_1$$

$$\Rightarrow vx = c_1$$

$$\Rightarrow \frac{dv}{dx} x = c_1 \quad (\because v = \frac{dv}{dx})$$

$$\Rightarrow dv = \frac{c_1}{x} dx$$

$$\Rightarrow v = c_1 \log x + c_2$$

$\therefore ② = y = e^{cx} (c_1 \log x + c_2)$ is the reqd g.s of ①.

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$y'' + y = \sec x$ given that $\cos x$ is a part of C.F

Sol: Given that $\frac{d^2y}{dx^2} + y = \sec x \quad \dots \textcircled{1}$

and $y = u = \cos x$ is a part of C.F of $\textcircled{1}$

Comparing $\textcircled{1}$ with $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$

$$P = 0, Q = 1, R = \sec x.$$

Let $y = uv$ be the g.s of $\textcircled{1}$

then v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx}\right) \frac{dv}{dx} = \frac{R}{u} \quad \dots \textcircled{2}$$

$$\text{since } u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$\therefore P + \frac{2}{u} \frac{du}{dx} = 0 + \frac{2}{\cos x} \quad (\text{since})$$

Now from $\textcircled{2}$, we have

$$\frac{d^2v}{dx^2} = 2 \tan x \frac{dv}{dx} \quad \frac{\sec x}{\cos x}.$$

$$\Rightarrow \frac{d^2v}{dx^2} = 2 \tan x \frac{dv}{dx} = \sec^2 x \quad \dots \textcircled{3}$$

$$\text{Let } \frac{dv}{dx} = v \Rightarrow \frac{d^2v}{dx^2} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} = -2 \tan x v = \sec^2 x \quad \dots \textcircled{4}$$

$$\text{T.F.} = e^{+2 \log(\sec x)}$$

$$= e^{\cos^2 x}$$

$$\therefore \text{G.S. of } \textcircled{4} \text{ is } v \cos^2 x = \int \sec^2 x \cos^2 x dx + c_1 \\ = x + c_1$$

$$\therefore v \cos^2 x = x + c_1$$

$$\Rightarrow \frac{dv}{dx} \cos^2 x = x + c_1 \quad (\because v = \frac{dv}{dx})$$

$$\Rightarrow dr = (x + c_1) \sec^2 x dx.$$



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$$\begin{aligned}\Rightarrow v &= \int(x+4) \sec^2 x dx + C_2 \\ &= (x+4) \tan x + \log(\cos x) + C_2 \\ &= (x+4) \sin x + \cos x \log(\cos x) + C_2 \cos x\end{aligned}$$

\therefore G.S of ① is $y = uv$

$$\Rightarrow y = (x+4) \sin x + \cos x \log(\cos x) + C_2 \cos x$$

~~$$\begin{aligned}xy'' - (2x+1)y' + (x+1)y &= x^3 e^x \\ (1-x^2)y'' + x^2y' - y &= x(1-x^2)^{3/2} \end{aligned}$$~~

~~\therefore Given equation is a solution if that left hand side is equated to zero.~~

$$\text{Ans: } y = C_1 e^{x^2} + C_2 e^{x^2} + x^2$$

② To solve $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ by changing the dependent variable and removal of the first derivative.

(or) Reduce the diff. equation $y'' + Py' + Qy = R$ where P, Q, R are functions of x , to the form $\frac{d^2u}{dx^2} + Iu = S$ which is known as the normal form of the given equation.

Soln: Given equation is

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

Let $y = uv$. ② be the g.s of ① where u, v are fns of x .

$$\text{Now } \frac{dy}{dx} = u \frac{du}{dx} + v \frac{dv}{dx} \quad \text{--- ③}$$

$$\text{and } \frac{d^2y}{dx^2} = u \frac{d^2v}{dx^2} + 2 \frac{du}{dx} \frac{dv}{dx} + v \frac{d^2u}{dx^2} \quad \text{--- ④}$$

$$\text{①} \Rightarrow u \frac{d^2v}{dx^2} + 2 \frac{du}{dx} \frac{dv}{dx} + v \frac{d^2u}{dx^2} + P(u \frac{dv}{dx} + v \frac{du}{dx}) + Quv = R$$

$$\Rightarrow u \frac{d^2v}{dx^2} + (Pu + 2 \frac{du}{dx}) \frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P \frac{du}{dx} + Qu \right) v = R \quad \text{--- ⑤}$$

To remove the first derivative $\frac{dv}{dx}$ in ⑤

$$\text{choose } u \text{ s.t. } Pu + 2 \frac{du}{dx} = 0 \quad \text{--- ⑥}$$

$$\Rightarrow \frac{du}{dx} = -\frac{P}{2}u$$

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$$\Rightarrow \frac{du}{u} = -\frac{1}{2} P(x) dx$$

$$\Rightarrow \log u = -\frac{1}{2} \int P(x) dx$$

$$\Rightarrow u = e^{-\frac{1}{2} \int P(x) dx}$$

$$(6) \Rightarrow \frac{du}{dx} = -\frac{1}{2} P u \quad (7)$$

$$\frac{d^2 u}{dx^2} = -\frac{1}{2} P \frac{du}{dx} - \frac{1}{2} u \frac{dP}{dx}$$

$$\frac{d^2 u}{dx^2} = -\frac{1}{2} P \left(-\frac{1}{2} P u \right) - \frac{1}{2} u \frac{dP}{dx} \quad (8)$$

$$(9) \Rightarrow u \frac{d^2 v}{dx^2} + \left(Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} \right) v = R$$

$$\Rightarrow \frac{d^2 v}{dx^2} + \left(Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} \right) v = \frac{R}{u}$$

$$\Rightarrow \frac{d^2 v}{dx^2} + I v = S \quad (9)$$

where $I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx}$, $S = \frac{R}{u}$ which is the standard

normal form of (1)

The q.f. of (1) is y^2uv

Where $u = e^{-\frac{1}{2} \int P(x) dx}$ & v is given by (9)

Working rule:

Step(1): Write the given equation in the standard form

$$y'' + P y' + Q y = R$$

Step(2): To remove the first derivative we choose

$$u = e^{-\frac{1}{2} \int P(x) dx}$$

Step(3): Assume the q.f. of the given eqn is y^2uv

then u is - given by step(2) and v is given by

the normal form: $\frac{d^2 v}{dx^2} + I v = S$



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$$\text{Where } I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dp}{dx} \text{ and } S = \frac{R}{u}$$

Problems.

Solve the following:

$$(1) y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} \sin 2x.$$

Soln Comparing (1) with $y'' + P(x)y' + Q(x)y = R(x)$

$$P = -4x; Q = 4x^2 - 1; R = -3e^{x^2} \sin 2x$$

To remove the first derivative we choose

$$u_2 = e^{\int P dx}$$

$$= e^{-\int (-4x) dx}$$

$$= e^{-x^2} \quad (2)$$

Let $y = uv \quad (3)$ be the g.s of (1)

$$\text{then } v \text{ is given by the normal form } \frac{dv}{dx} + Pv = S \quad (4)$$

$$\text{where } I = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dp}{dx}; S = \frac{R}{u}$$

$$\text{Now } I = 4x^2 - 1 - \frac{1}{4} (16x^2) - \frac{1}{2} (-4) \\ = 1$$

$$\text{and } S = -3 \sin 2x$$

$$\therefore (4) \equiv \frac{dv}{dx} + v = -3 \sin 2x$$

$$\Rightarrow (D^2 + 1)v = -3 \sin 2x \quad (5)$$

$$\text{A.E is } D^2 + 1 = 0$$

$$\Rightarrow D = \pm i$$

$$\therefore C.F = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I = \frac{1}{D^2 + 1} (-3 \sin 2x)$$

$$= \frac{-3}{-3} \sin 2x$$

$$= \sin 2x$$

$$\therefore G.S of (5) is v = C_1 \cos 2x + C_2 \sin 2x + \sin 2x.$$

$$\therefore (3) \equiv y = e^{x^2} (C_1 \cos 2x + C_2 \sin 2x) + \sin 2x$$

which is the reqd g.s of (1)

$$\therefore y'' - 2x \tan x \cdot y' + 5y = 0.$$

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→ Make use of the transformation $y(x) = v(x) \sec x$ to obtain the solution of $y'' - 2y' \tan x + 5y = 0$. $y(0) = 0, y'(0) = \sqrt{6}$

③ To solve $\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + qy = R(x)$ by changing the independent variable.

Given eqn is $\frac{d^2y}{dx^2} + p \frac{dy}{dx} + qy = R$

Let the independent variable 'x' be changed into another independent variable z .

where z is a function of
i.e let $z = f(x)$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dz} \cdot \frac{dz}{dx} \right)$$

$$= \frac{dy}{dz} \frac{d^2z}{dx^2} + \frac{dz}{dx} \cdot \frac{d}{dz} \left(\frac{dy}{dz} \right)$$

$$= \frac{dy}{dz} \frac{d^2z}{dx^2} + \frac{dz}{dx} \cdot \frac{d}{dz} \left(\frac{dy}{dz} \right) + \frac{d^2z}{dx^2} \cdot \frac{d}{dz} \left(\frac{dy}{dz} \right) + \frac{d}{dz} \left(\frac{d^2z}{dx^2} \right) \cdot \frac{dy}{dz}$$

$$\stackrel{(1)}{=} \frac{dy}{dz} \frac{d^2z}{dx^2} + \left(\frac{d^2z}{dx^2} \right)^2 \frac{dy}{dz} + p \frac{dy}{dz} \cdot \frac{dz}{dx} + qy = R$$

$$\Rightarrow \left(\frac{dz}{dx} \right)^2 \frac{dy}{dz} + \left(\frac{d^2z}{dx^2} + p \frac{dz}{dx} \right) \frac{dy}{dz} + qy = R$$

$$\Rightarrow \frac{d^2y}{dz^2} + \left[\frac{d^2z}{dx^2} + p \frac{dz}{dx} \right] \frac{dy}{dz} + \left(\frac{d^2z}{dx^2} \right)^2 y = \frac{R}{\left(\frac{dz}{dx} \right)^2}$$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (2)}$$



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$$\text{where } P_1 = \frac{d^2z}{dx^2} + P \frac{dz}{dx} \quad \text{--- (3)}$$

$$\left(\frac{dz}{dx}\right)^2$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \quad \text{--- (4)}$$

$$\text{and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad \text{--- (5)}$$

Here P_1, Q_1, R_1 are functions of x . But these function of z by using $z = f(x)$.

We now choose z s.t. $P_1 = 0$ and $Q_1 = \pm a^2$ (constant)

Case(i) If $P_1 = 0$ then (3) $\equiv \frac{d^2z}{dx^2} + P \frac{dz}{dx} = 0$

$$\Rightarrow \frac{d^2z}{dx^2} / \frac{dz}{dx} = -P$$

$$\Rightarrow \frac{d}{dx} \left(\frac{dz}{dx} \right) = -P$$

$$\therefore \frac{dz}{dx} :$$

$$\Rightarrow \log \left(\frac{dz}{dx} \right) = - \int P dx$$

$$\Rightarrow \frac{dz}{dx} = e^{- \int P dx}$$

$$\Rightarrow z = \int e^{- \int P dx} dx$$

Case(ii): If $Q_1 = \pm a^2$ (real constant) then from (4), we get

$$\frac{Q}{\left(\frac{dz}{dx}\right)^2} = \pm a^2 \Rightarrow \pm Q = a^2 \left(\frac{dz}{dx}\right)^2$$

$$\Rightarrow a \frac{dz}{dx} = \sqrt{\pm Q}$$

$$\Rightarrow a dz = \sqrt{\pm Q} dx$$

(+ve or -ve sign is taken to make the expression under the radical sign +ve).

Problems
→ Solve the following:

$$(1) x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2 \quad \text{--- (1)}$$

$$\text{Sol: } \frac{d^2y}{dx^2} - \frac{1}{x} \frac{dy}{dx} - 4x^2 y = 8x^2 \sin x^2 \quad \text{--- (2)}$$

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Comparing ① with $\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = R(x)$.

$$P = \frac{1}{2}, Q = -4x^2, R = 8x^2 \sin x^2$$

changing the independent variable from x to z

where z is a function of x i.e. $z = f(x)$

∴ The given equation ① transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (2)}$$

$$\text{Where } P_1 = \frac{d^2z}{dx^2} + P \frac{dz}{dx}$$

$$\left(\frac{dz}{dx}\right)^2$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}; R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Choosing z s.t. $\frac{dz}{dx}$ constant

$$\frac{dz}{dx} = -1 \text{ (say)}$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \frac{dz}{dx} = 2x$$

$$\boxed{z = x^2}$$

$$\text{Now } P_1 = 2 + \left(\frac{1}{4}\right)(2x) = 0$$

$$4x^2$$

$$R_1 = \frac{8x^2 \sin x^2}{4x^2} = 2 \sin x^2$$

$$= 2 \sin z$$

$$\therefore ② = \frac{d^2y}{dz^2} + 0 \cdot y = 2 \sin z$$

$$\Rightarrow \frac{d^2y}{dz^2} - y = 2 \sin z \quad \text{--- (3)}$$

$$\Rightarrow \frac{d^2y}{(D^2 - 1)y} = 2 \sin z \text{ where } D = \frac{d}{dz}$$



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$$\therefore CF = c_1 e^z + c_2 e^{-z}$$

$$P.I. = 2 \cdot \frac{1}{D^2} \sin z$$

$$= 2 \cdot \frac{1}{D^2} \sin z = -\sin z$$

$$\therefore G.S. \text{ of } (3) \text{ is } Y = Y_C + Y_P$$

$$Y = c_1 e^z + c_2 e^{-z} - \sin z$$

$$\therefore Y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$

which is the reqd. g.s. of (1)

$$\rightarrow \cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y (\cos^5 x) = \cos^5 x$$

~~$$\text{Solve } \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + (\cos^2 x) y = Q \cos^4 x \quad (1)$$~~

$$P = \tan x; \quad Q = -2 \cos^2 x; \quad R = Q \cos^4 x.$$

changing the independent variable x from 'x' to new independent variable z - where z is a function of x i.e. $z = f(x)$.

\therefore The given equation is transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad (2)$$

$$\text{where } P_1 = \frac{d^2f}{dx^2} + f'' \frac{df}{dx}, \quad Q_1 = \frac{Q}{(f'(x))^2}, \quad R_1 = \frac{R}{(f'(x))^2}$$

$$\text{Let us choose } z \text{ s.t. } -\frac{2 \cos^2 x}{(f'(x))^2} = \text{constant} \quad \left(\text{Grahm + Gambu } n \right) \Rightarrow \frac{f'(x)^2}{2} + \frac{\cos^2 x}{2}$$

$$P_1 = 0, \quad R_1 = 2 \cos^2 x = 2(1 - \sin^2 x)$$



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$$= 2(1-z)$$

$$\therefore \textcircled{2} \equiv \frac{d^2y}{dx^2} + 0 - 2y = 2(1-z)$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1-z)$$

(Continue next soln)

H.W $y'' + (\frac{2}{z})y' + (\frac{a^2}{z^4})y = 0 \quad | \quad \alpha_1 = a^2$

H.W $x^4 y'' + 2x^3 y' + n^2 y = 0 \quad | \quad \alpha_1 = n^2$

H.W $x^6 y'' + 3x^5 y' + a^2 y = \frac{1}{x^2} \quad | \quad \alpha_1 = a^2$

→ Transform the diff. equation

$xy'' - y' + 4x^3 y = x^5$ into z as independent variable
where $z = x^2$ and solve it.

Sol: Given that

$$xy'' - y' + 4x^3 y = x^5 \quad \text{--- } \textcircled{1}$$

$$\text{and } z = x^2 \Rightarrow \frac{dz}{dx} = 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} (2x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dx} \left(\frac{dy}{dz} \cdot 2x \right)$$

$$= \frac{dy}{dz} (2) + 2x \frac{d}{dz} \left(\frac{dy}{dz} \right)$$

$$= 2 \frac{dy}{dz} + 2x \frac{d}{dz} \left(\frac{dy}{dz} (2x) \right)$$

$$= 2 \frac{dy}{dz} + (2x)^2 \frac{d^2y}{dz^2}$$

$$\therefore \textcircled{1} \equiv z \left[2 \frac{dy}{dz} + 4x^2 \frac{d^2y}{dz^2} \right] - 2x \frac{dy}{dz} + 4x^3 y = x^5$$

$$\Rightarrow 4z \frac{d^2y}{dz^2} + 4z = z^2$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = \frac{1}{4} z \quad \text{--- } \textcircled{2}$$

(Continue next soln)

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* An equation of the form

$$\frac{d^n y}{dx^n} + p_1(x) \frac{d^{n-1} y}{dx^{n-1}} + p_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_n(x) y = Q(x)$$

where $p_1(x), p_2(x), \dots, p_n(x)$ & $Q(x)$ functions of x ,

is called a linear diff. equation of order n .

This can be divided into two types:

(i) Homogeneous linear diff. eqn.

(ii) Non-Homogeneous linear diff. eqn.

→ If $Q(x) = 0$ then (i) is called homo.

→ If $Q(x) \neq 0$ then (i) is called non-homo.

The Wronskian Condition

If $y_1 = f_1(x), y_2 = f_2(x), \dots, y_n = f_n(x)$ are

solutions of $f(D)y = 0$ then

$$w(y_1, y_2, \dots, y_n)(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y'_1 & y'_2 & \dots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_{(n-1)} & f_{(n-1)} & \dots & f_{(n-1)} \\ f_1 & f_2 & \dots & f_n \end{vmatrix}$$

→ If $w(y_1, y_2, \dots, y_n)(x) \neq 0$ then the 'n' solutions are L.I. - Linearly Independent

→ If $w(y_1, y_2, \dots, y_n)(x) = 0$ then the 'n' solutions are L.D. - Linearly dependent

Note:- The n th order linear equation $f(D)y = 0$ possesses n distinct solutions which are L.I

Ex:- e^x, e^{2x}, e^{3x}

Let $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$ then



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$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 2e^{6x} \neq 0$$

$$\therefore W \neq 0$$

∴ The functions e^x, e^{2x}, e^{3x} are L.I.

∴ $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$ are L.I. solutions of $f(D)y = 0$

Ex:- (2) e^{2x}, e^{2x}, e^{-x}

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{2x} & e^{2x} & e^{-x} \\ 2e^{2x} & 2e^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} & e^{-x} \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & 4 & 1 \end{vmatrix}$$

$$= e^{3x} [6 - 6 + 0]$$

$$= 0$$

∴ $y = y_1, y_2, y_3$ are not L.I. solns of $f(D)y = 0$.

Note:- The n th order linear equation $f(D)y = 0$ does not possess all solutions are distinct.

which are L.D.

④ Method of Variation of Parameters:-

To solve $\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x)$ by the method of variation of parameters.

Given eqn is $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$

Its homogenous eqn is

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = 0 \quad \dots \textcircled{2}$$

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Let $y_c = c_1 u(x) + c_2 v(x)$ be the g.s of (2) and hence it is c.f of (1)

Since $y_1 = u(x)$, $y_2 = v(x)$ are L.I solutions of (2)

$$\therefore \frac{d^2u}{dx^2} + P \frac{du}{dx} + Qu = 0$$

$$\frac{d^2v}{dx^2} + P \frac{dv}{dx} + Qv = 0 \quad \text{--- (3)}$$

Let $y_p = A u(x) + B v(x)$ be particular integral of (1)

which is obtained from c.f of (1) by replacing c_1 & c_2 by A and B respectively.

which are fns of 'x' Diff w.r.t 'x', we get

$$\frac{dy_p}{dx} = A \frac{du}{dx} + u \frac{dA}{dx} + B \frac{dv}{dx}$$

Now choosing A & B s.t.

$$u \frac{dA}{dx} + v \frac{dB}{dx} = 0 \quad \text{--- (4)}$$

$$\text{Then } \frac{dy_p}{dx} = \frac{du}{dx} + B \frac{dv}{dx} \quad \text{--- (5)}$$

Diff (5) w.r.t 'x', we get,

$$\frac{d^2y_p}{dx^2} = \frac{d^2u}{dx^2} + \frac{da}{dx} \cdot \frac{du}{dx} + B \frac{d^2v}{dx^2} + \frac{dB}{dx} \cdot \frac{dv}{dx} \quad \text{--- (6)}$$

$$\begin{aligned} & \frac{du}{dx^2} + \frac{da}{dx} \frac{du}{dx} + B \frac{d^2v}{dx^2} + \frac{dB}{dx} \frac{dv}{dx} + P(A \frac{du}{dx} + B \frac{dv}{dx}) \\ & + Q(Au + Bv) = R \end{aligned}$$

$$\Rightarrow A \left[\frac{d^2u}{dx^2} + P \frac{du}{dx} + Qu \right] + B \left[\frac{d^2v}{dx^2} + P \frac{dv}{dx} + Qv \right] + \left(\frac{du}{dx} \frac{da}{dx} + \frac{dv}{dx} \frac{dB}{dx} \right) = R$$

$$= A(0) + B(0) + \left(\frac{du}{dx} \frac{da}{dx} + \frac{dv}{dx} \frac{dB}{dx} \right) = R \quad (\text{from (3)})$$

Solving (2) & (4)



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$$\Rightarrow \frac{du}{dx} \frac{dA}{dx} + \frac{dv}{dx} \frac{dB}{dx} = R \quad \text{--- (3)}$$

$$\text{Solving (2) & (3)} \quad u \frac{dA}{dx} + v \frac{dB}{dx} - 0 = 0 \rightarrow (4)$$

$$\frac{dA}{dx} = \frac{\frac{dB}{dx}}{\frac{dv}{dx}} = \frac{-1}{uv' - vu'} \quad \text{where } u = \frac{du}{dx}, \quad v' = \frac{dv}{dx}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-vR}{uv' - vu'}$$

$$\frac{dB}{dx} = \frac{UR}{uv' - vu'}$$

$$\Rightarrow A = - \int \frac{vR}{uv' - vu'} dx,$$

$$B = \int \frac{UR}{uv' - vu'} dx$$

After integration the constant is not added.

(\because A & B are involved in y_p)

i. Substituting the values of A & B in (1)

\therefore The g.s of (1) is $y = y_c + y_p$

$$\Rightarrow y = q u(x) + c_1 v(x) + A u(x) + B v(x)$$

Notes (1) Since the form of $y_c + y_p$ is the same.

But the constants which occur in y_c are changed into functions of the independent variable x in y_p .

For this reason the method of finding the P.I. is called the method of variation of parameters.

(2) The above method can be extended to linear equations of order higher than the two.

(3) The above method is applicable to linear equations with constant coefficients and also variable co-efficients.

(4) W.K.T the given linear eqn of second order can be solved when part of C.F. is known.

$\frac{dy}{dx} + p(x)y = Q(x) \Rightarrow$ complementary $\frac{dy}{dx} + p(x)y = 0 \Rightarrow y_c$ found
 $y_c = u(x)$ & $y_p = A(x)u(x)$ Substituting y_p in $\frac{dy}{dx} + p(x)y = Q(x)$ to find $A(x)$

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$y_c \Rightarrow \frac{dy}{dx} + p(x)u = 0$ K. VENKATESH. $y = y_c + y_p$ &
 Therefore the above method is surely superior to the variation of parameters.

Since this method requires a complete knowledge of the C.F. instead of one solution of it.

Hence the method of variation of parameters should be used only when specifically asked to solve by this method.

Working rule:

(1) Write the given equation in the standard form

$$y'' + Py' + Qy = R.$$

(2) Find the solution of $\frac{dy}{dx} + P\frac{dy}{dx} + Qy = 0$.

$$\text{Let it be } y_c = c_1 u(x) + c_2 v(x)$$

(3) Let the P.I. of the given eqn be $y_p = A(x)u + B(x)v$
where A, B are functions of x .

(4) Find $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$
 $= \frac{du}{dx}v - v \frac{du}{dx}$

(5) Find A and B by using, $A = \int \frac{-VR}{uv' - vu'} dx$ & $B = \int \frac{UR}{uv' - vu'} dx$

(6) The g.s. of the given eqn is $y = y_c + y_p$

$$\Rightarrow y = (c_1 u + c_2 v) + (Au + Bv)$$

→ Solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Given that $(D^2 + a^2)y = \tan ax$ — (1)

A.E of (1) is $D^2 + a^2 = 0$

$$\Rightarrow D^2 = -a^2$$

$$\Rightarrow D = \pm ai$$

$$\therefore y_c = c_1 \cos ax + c_2 \sin ax$$

Let $y_p = A \cos ax + B \sin ax$ be a P.I. of (1)



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where A & B fns of 'x'.

then $u(x) = \cos ax$; $v(x) = \sin ax$; $R(x) = \tan ax$

$$\begin{aligned} \text{Now } uv' - vu' &= \cos ax (\cos ax) \\ &= +\sin ax (\sin ax) \\ &= a(\sin^2 ax + \cos^2 ax) \\ &= a \end{aligned}$$

$$\text{Now } A = \int \frac{-vR}{uv' - vu'} dx$$

$$\begin{aligned} &= \frac{1}{a} \int \left[\frac{1 - \cos^2 ax}{\cos ax} \right] dx \\ &= \frac{-1}{a} \int [\sec ax - \cos ax] dx \\ &= \frac{-1}{a} \left[\log |\sec ax + \tan ax| - \frac{\sin ax}{a} \right] \\ &= -\frac{1}{a^2} [\log |\sec ax + \tan ax| - \sin ax] \end{aligned}$$

$$\text{and } B = \int \frac{UR}{uv' - vu'} dx$$

$$= -\frac{1}{a^2} \cos ax.$$

$$\therefore Y_p = \frac{1}{a^2} [\sin ax - \log |\sec ax + \tan ax - \cos ax|]$$

\therefore The g.s of ① is $y = y_c + Y_p$.

$$\Rightarrow y = (c_1 \cos ax + c_2 \sin ax) + \frac{1}{a^2} [\sin ax - \log |\sec ax + \tan ax - \cos ax|]$$

Solve $[(x-1)^2 - xD + 1]y = (x-1)^2$ by the method of

Variation of parameters.

Sol: Given eqn is

$$(x-1) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = (x-1)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{x}{(x-1)} \frac{dy}{dx} + \frac{1}{(x-1)} y = (x-1)^2 \quad \text{--- ①}$$

Comparing ① with $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$.

$$P = \frac{x}{x-1}; Q = \frac{1}{x-1}; R = (x-1).$$

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Now the homo. eqn of (1) is

$$\frac{d^2y}{dx^2} - \frac{2}{x-1} \frac{dy}{dx} + \frac{1}{(x-1)^2} y = 0$$

$$\text{Now } 1+P+Q = 0$$

$\therefore y = e^x$ is a part of CF of (2).

Let $y_c = uv$ be the g.s of (2), where u, v are

$$\text{then } v \text{ is given by } \frac{d^2v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx}\right) \frac{dv}{dx} = 0 \quad (3)$$

$$\text{Now since } u = e^x \Rightarrow \frac{du}{dx} = e^x$$

$$\therefore P + \frac{2}{u} \frac{du}{dx} = -\frac{2x}{x-1}$$

$$= \frac{x+2}{x-1}$$

$$= \frac{x-2}{x-1}$$

$$\therefore \frac{d^2v}{dx^2} \left(\frac{x-2}{x-1}\right) \frac{dv}{dx} = 0 \quad (4)$$

$$\text{Let } \frac{dv}{dx} = v \text{ then}$$

$$\frac{dv}{dx} + \left(1 - \frac{1}{x-1}\right)v = 0$$

$$\log v = \log(x-1) - x + \log c_1$$

$$\log \left(\frac{v}{(x-1)c_1}\right) = -x$$

$$\Rightarrow v = c_1(x-1)e^{-x}$$

$$\Rightarrow \frac{dv}{dx} = c_1(x-1)e^{-x}$$

$$\Rightarrow v = -c_1 e^{-x} (x+1) + c_1 e^{-x} + c_2$$

$$y_c = e^x [-c_1 e^{-x} (x+1) + c_1 e^{-x} + c_2]$$



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$$= -c_1(x-1) - c_1 + c_2 e^x.$$

$$\therefore \boxed{y_c = c_1 x + c_2 e^x}$$

Let $y_p = au + bv$ be a p. t. of ① where A & B are fns of x
 & $u = x, v = e^x$.

$$\text{Now } \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - vu' =$$

$$= xe^x - e^x$$

$$= e^x(x-1)$$

$$\therefore A = \int \frac{-VR}{uv' - vu'} dx = \int \frac{-e^x(x-1)}{e^x(x-1)} dx$$

$$= -dx$$

$$\text{and } B = \int \frac{UR}{uv' - vu'} dx = \int \frac{x(x-1)}{e^x(x-1)} dx$$

$$= \int xe^x dx$$

$$= -e^x(x+1)$$

$$\therefore y_p = -x(x) - e^x(x+1)e^x$$

$$= -x^2 - (x+1)$$

$$= - (1+x+x^2)$$

\therefore G.S of ① is

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 x + c_2 e^x - (1+x+x^2)}$$

* Apply the method of variation of parameters to solve the following diff. eqns:

1999 $y'' + 4y = \sec nx \quad \rightarrow \cancel{x} \frac{dy}{dx} - y = (x-1) \left(\frac{dy}{dx} - x+1 \right) \rightarrow$

2000 $\cancel{x} \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^{nx} \sin nx \text{ with } C_1 = +2$

2001 $\cancel{x} y'' + 4y = 4 \tan nx \quad \rightarrow \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^{nx} \sin nx \text{ with } C_2 = +4$

2003 $\cancel{x} y'' + 4y = x^2 \tan nx \quad \rightarrow \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^{nx} \sin nx \text{ with } C_3 = +6$

2005 $\cancel{x} y'' + 4y = x^4 \sec^2 nx \quad \rightarrow \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^{nx} \sin nx \text{ with } C_4 = +8$

2008 $\cancel{x} y'' + 4y = -x^4 \sin nx \quad \rightarrow \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^{nx} \sin nx \text{ with } C_5 = +10$

2008-2009 $\cancel{x} y'' + 4y = x^2 e^x \quad \rightarrow \frac{dy}{dx} - 2 \frac{dy}{dx} + y = x e^x \text{ with } C_6 = +12$

DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE, Set-IV

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OF FIRST DEGREE *

If we denote $\frac{dy}{dx}$ by P then the equation of the form $f(x, y, P) = 0$ where P is not of first degree, is called a differential equation of first order but not of first degree.

The most general form of a differential equation of first order and nth degree is $P^n + A_1 P^{n-1} + A_2 P^{n-2} + \dots + A_{n-1} P + A_n Y = 0$
 i.e. $\left(\frac{dy}{dx}\right)^n + A_1 \left(\frac{dy}{dx}\right)^{n-1} + \dots + A_{n-1} \left(\frac{dy}{dx}\right) + A_n Y = 0$

where A_1, A_2, \dots, A_n are functions of x. Such equations can be divided into the following categories.

- (i) Solvable for P
- (ii) Solvable for x
- (iii) Solvable for y
- (iv) Clairaut's equation.

(i) Differential equations solvable for P:

Such equations can be resolved into linear factors of first degree. Let the given equation be

$$P^n + A_1 P^{n-1} + A_2 P^{n-2} + \dots + A_{n-1} P + A_n = 0 \quad \text{--- (1)}$$

Let it be resolved into linear factors to give

$$[P - f_1(x, y)] [P - f_2(x, y)] \dots [P - f_n(x, y)] = 0$$

then $P = f_1(x, y)$, $P = f_2(x, y)$, ..., $P = f_n(x, y)$

These equations on integration give

$$F_1(x, y, C_1) = 0, F_2(x, y, C_2) = 0, \dots, F_n(x, y, C_n) = 0$$

∴ The solution of (1) is

$$F_1(x, y, C_1) : F_2(x, y, C_2) : \dots : F_n(x, y, C_n) = 0$$

But (1) being an equation of first order, its general solution must be contain one arbitrary constant.



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Taking $C_1 = C_2 = C_3 = \dots = C_n = C$

\therefore The general solution (1) is

$$F_1(x, y, C), F_2(x, y, C), \dots, F_n(x, y, C) = 0$$

Problems:

Solve the following differential equations.

$$(i) (P-xy)(P-x^2)(P-y^2) = 0 \quad \text{--- (I)}$$

It is solving for P

$$\begin{aligned} P-xy &= 0 & P = x^2 & ; & P = y^2 \\ \text{--- (1)} & & \text{--- (2)} & & \text{--- (3)} \end{aligned}$$

$$\frac{dy}{dx} = xy ; \quad \frac{dy}{dx} = x^2 ; \quad \frac{dy}{dx} = y^2$$

$$\Rightarrow \log y = \frac{x^2}{2} + C ; \quad y = \frac{x^3}{3} + C ; \quad -\frac{1}{y} = x + C$$

\therefore General solution of (I) is

$$\left(\log y - \frac{x^2}{2} - C \right) \left(y - \frac{x^3}{3} - C \right) \left(-\frac{1}{y} + x + C \right) = 0$$

$$(x+y)(xp+yp+x)(p+2x) = 0$$

$$\Rightarrow P^2 - 5P + 6 = 0 \quad / \Rightarrow (P-3)(P-2) = 0$$

$$\Rightarrow x + yp^2 = (1+xy)P \quad / \Rightarrow x + yp^2 - p - xyp = 0 \Rightarrow (x-p) - yP(x-p) = 0$$

$$\Rightarrow 4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$$

$$\Rightarrow xy^2(p^2+2) = 2py^3 + x^3 \quad / \text{Add and sub } x^2yp$$

$$\Rightarrow y^2p^2 + (x-y)p - x = 0$$

$$\Rightarrow x^2p^2 - 2xyp + 2y^2 - x^2 = 0$$

$$\Rightarrow p^2x^2 = y^2 \quad / (Px)^2 - y^2 = 0 \Rightarrow (Px-y)(Px+y) = 0$$

(ii) Differential Equations solvable for x:

Let the given equation be solvable for 'x' then it can be put in the form

$$x = f(y, P) \quad \text{--- (1)}$$

on diff. (1) w.r.t y, we get

$$\frac{dx}{dy} = F(y, P, \frac{dP}{dy}) \quad \text{or} \quad \frac{1}{P} = f(y, P, \frac{dP}{dy}) \quad \text{--- (2)}$$

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MATHEMATICS by K. VENKATANNA

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Let the solution of ② be $\phi(y, P, C) = 0$ — ③

Eliminating P b/w ① & ③ is given the solution of the given equation.

If it is not possible to eliminate ' P ' then the values of ' x ' and ' y ' in terms of ' P ' in the form $x = f_1(P, C)$ & $y = f_2(P, C)$ together give the solution.

→ Solve the following differential equations

$$(1) \quad xp^3 = a + bp$$

Sol'n: Given that $xp^3 = a + bp$ — ①

It is solving for ' x '

$$\text{we have } x = \frac{a}{P^3} + \frac{bP}{P^3}$$

$$\Rightarrow x = \frac{a}{P^3} + \frac{b}{P^2}$$

Diff. ② w.r.t. ' y ' we get

$$\frac{dx}{dy} = \left(\frac{-3a}{P^4} - \frac{b}{P^3} \right) \frac{dp}{dy}$$

$$-\frac{1}{P} = \left(\frac{-3a}{P^3} - \frac{2b}{P^2} \right) \frac{dp}{dy} \quad (\because \frac{dx}{dy} = \frac{1}{P})$$

$$\Rightarrow \frac{dp}{dy} = \left(\frac{-3a}{P^3} - \frac{2b}{P^2} \right) dp$$

$$y = \frac{3a}{2P^2} + \frac{2b}{P} + C \quad — ③$$

It is not possible to eliminate ' P ' from ② & ③.

General solution of ① is $x = \frac{a}{P^3} + \frac{b}{P^2}$ &

$$y = \frac{3a}{2P^2} + \frac{2b}{P} + C$$

$$\therefore P^3 - 4xyP + 8y^2 = 0 \quad — ①$$

$$\underline{\text{Sol'n}}: 4xyP = P^3 + 8y^2$$

$$x = \frac{P^2}{4y} + \frac{2y}{P} \quad — ②$$



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Diff. w.r.t 'y', we get,

$$\frac{dx}{dy} = \frac{1}{4} \cdot \frac{\left[Y(2P) \frac{dp}{dy} - P^2 \right]}{y^2} + \frac{2 \left[P - y \frac{dp}{dy} \right]}{P^2}$$

$$\Rightarrow \frac{1}{P} = \frac{P}{2y} \frac{dp}{dy} - \frac{P^2}{4y^2} + \frac{2}{P} - \frac{2y}{P^2} \frac{dp}{dy}$$

$$\Rightarrow \frac{1}{P} = \left(\frac{P}{2y} - \frac{2y}{P^2} \right) \frac{dp}{dy} + \left(\frac{2}{P} - \frac{P^2}{4y^2} \right)$$

$$\Rightarrow \left(\frac{P}{2y} - \frac{2y}{P^2} \right) \frac{dp}{dy} + \frac{1}{P} - \frac{P^2}{4y^2} = 0$$

$$\Rightarrow \left(\frac{P}{2y} - \frac{2y}{P^2} \right) \frac{dp}{dy} + \frac{P}{2y} \left(\frac{2y}{P^2} - \frac{P}{2y} \right) = 0$$

$$\Rightarrow \left(\frac{P}{2y} - \frac{2y}{P^2} \right) \left(\frac{dp}{dy} - \frac{P}{2y} \right) = 0$$

Omitting the first factor which leads to a singular solution,

we get.

$$\frac{dp}{dy} - \frac{P}{2y} = 0$$

$$\Rightarrow \frac{dp}{P} = \frac{1}{2} \frac{1}{y} dy$$

$$\Rightarrow \log P = \frac{1}{2} \log y + \log C$$

$$\Rightarrow P = y^{\frac{1}{2}} C \quad \text{--- (3)}$$

Now eliminating P b/w (2) & (3), we get

$$x = \frac{yc^2}{4y} + \frac{2y}{y^{\frac{1}{2}} C}$$

$$\Rightarrow x = \frac{c^2}{4} + \frac{2y^{\frac{1}{2}}}{C}$$

which is the required general solution of (1).

Note: the factor which does not involve a derivative of P w.r.t x or y will be omitted; such factor always lead to singular solutions.

$$\rightarrow x = P^2$$

$$\rightarrow AP^2 + PY - x = 0$$

~~$$\rightarrow y^2 \log y = xyP + P^2$$~~

~~solve~~ solve the equation $y - 2xyP + yP^2 = 0$ where $P = \frac{dy}{dx}$.

EXERCISE FOR IIT-JEE / IIT-JEE / CSIR EXAMINATIONS

MATHEMATICS - 25 by K. VENKATESWARA

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Differential Equations solvable for 'y':

Let the given equation be $y = f(x, P)$ ————— (1)

Diff. w.r.t x and writing $\frac{dy}{dx} = P$,

we get an equation of the form

$$P = F(x, P, \frac{dp}{dx}).$$

This is a differential equation in P & x and we get its solution in the form $\phi(x, P, C) = 0$.

Eliminating P b/w (1) & (2) we get the required solution.

If it is not possible to eliminate P then (1) & (2) can be put in the form $x = f_1(P, C)$,

$$y = f_2(P, C)$$

These two equations together constitute the solution.

Problems:

→ solve the following Differential Equations:

$$(1) \quad y = 3x + \log p$$

It is solving for y

Now diff. w.r.t x we get

$$\frac{dy}{dx} = 3 + \frac{1}{P} \frac{dp}{dx}$$

$$\Rightarrow 3 = 3 + \frac{1}{P} \frac{dp}{dx} \quad (\because \frac{dy}{dx} = P)$$

$$\Rightarrow \frac{1}{P} \frac{dp}{dx} = P - 3$$

$$\Rightarrow \frac{dp}{P(P-3)} = dx$$

$$\Rightarrow \frac{1}{3} \left[\frac{1}{P-3} - \frac{1}{P} \right] dp = dx$$

$$\Rightarrow [\log(P-3) - \log P] = 3x + C$$



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$$\Rightarrow \log\left(\frac{P-3}{P}\right) = 3x + C$$

$$\Rightarrow \frac{P-3}{P} = e^{3x+C}$$

$$\Rightarrow 1 - \frac{3}{P} = e^{3x+C}$$

$$\Rightarrow \frac{1-e^{3x}}{e^C} = \frac{3}{P}$$

$$\Rightarrow P = \frac{3}{1-e^{3x}e^C} \quad \text{--- (2)}$$

Eliminating P from ① & ② we get,

$$Y = 3x + \log\left[\frac{3}{1-e^{3x}e^C}\right]$$

which is required general solution of ①.

$$\rightarrow x^r + p^r x = YP$$

$$\rightarrow x = Y - P^r$$

~~$$\rightarrow P^3 - P(Y+3) + x = 0$$~~

$$\rightarrow Y = 2P + 3P^2$$

Clairaut's Equation:

The differential Equation of the form $Y = xp + f(p)$ is known as Clairaut's equation.

This equation is solved by considering the equation $Y = f(x, P)$, solvable for Y .

Solution of Clairaut's equation:

Let the given equation be

$$Y = xp + f(p) \quad \text{--- (1)}$$

Diff w.r.t x we get

$$\frac{dy}{dx} = x \frac{dp}{dx} + p + f'(p) \frac{dp}{dx}$$

$$\Rightarrow p = x \frac{dp}{dx} + p + f'(p) \frac{dp}{dx}$$

$$\Rightarrow (x + f'(p)) \frac{dp}{dx} = 0.$$

CHAPTER 12 / IIT-JEE / CBSE / COMPETITIONS

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BY K. V. KRISHNA

$$\Rightarrow \frac{dp}{dx} = 0 \quad (\alpha + f'(p) \text{ is discarded})$$

$$\Rightarrow dp = 0 \Rightarrow \boxed{p=c}$$

$$\boxed{\textcircled{1} \Rightarrow y = xc + f(c)}$$

which is the required general solution of $\textcircled{1}$.

Working Rule:

Given equation can be put in the form $y = xp + f(p)$ --- $\textcircled{1}$

In order to find its solution

replace 'p' by 'c'.

\therefore the general solution of $\textcircled{1}$ is $\boxed{y = xc + f(c)}$

Problems:

$$\rightarrow \text{Solve } y = x \left(\frac{dy}{dx} \right) + \left(\frac{dy}{dx} \right)^2$$

$$\text{Sol'n: Given that } y = x \left(\frac{dy}{dx} \right) + \left(\frac{dy}{dx} \right)^2 \quad \textcircled{1}$$

Let $\frac{dy}{dx} = p$ then

$$y = xp + p^2 \quad \textcircled{2}$$

Clearly which is in Clairaut's form

\therefore Replacing 'p' by 'c' in $\textcircled{2}$, we get

$$y = xc + c^2 \text{ which is the required general solution of } \textcircled{1}$$

\rightarrow Solve the following differential Equations:

~~(1)~~ $P = \log(Px - y)$

~~(2)~~ $y = xp + \frac{a}{P}$

~~(3)~~ $\sin(y - xp) = P$ (or) $\sin px \cos y = \cos px \sin y + P$

~~(4)~~ $(y - xp)(P - 1) = P$

~~(5)~~ $(xp - y)^2 = P^2 - 1 \Rightarrow xp - y = \pm \sqrt{P^2 - 1}$



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Equations Reducible to Clairaut's form:Form I :

$$y^2 = pxy + f\left(\frac{py}{x}\right) \quad \text{--- (1)}$$

$$\text{put } x^2 = x; y^2 = y$$

$$2xdx = dx; 2ydy = dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \frac{dy}{dx} \Rightarrow \boxed{P = \frac{y}{x} p}$$

$$\Rightarrow \boxed{p = \frac{x}{y} P}$$

$$\therefore (1) \equiv Y = \frac{x}{y} P \cdot (xy) + f\left(\frac{x}{y} P \frac{y}{x}\right)$$

$$\Rightarrow Y = x^2 P + f(P)$$

$$\Rightarrow \boxed{Y = xP + f(P)} \quad \text{--- (2)}$$

Clearly which is Clairaut's form

The general solution of (2) is $Y = cx + f(c)$

$\Rightarrow Y^2 = cx^2 + f(c)^2$ is the general solution of (1).

→ solve the following differential equations by using the transformations $x^2 = x, y^2 = Y$.

$$(1) \quad x^2(Y - px) = y^2p^2 \Rightarrow x^2y^2 - x^3py = y^2p^2$$

$$\Rightarrow y^2 = pxy + \left(\frac{py}{x}\right)^2$$

~~(2)~~
$$(px - y)(py + x) = b^2p$$

~~(3)~~
$$(px - y)(x - yp) = 2p$$

* Form II :

$$\text{If } e^{by}(a - bp) = f(p) \text{ or } e^{by} = a - bp$$

then put $e^{ax} = x, e^{by} = y$

problems :

→ solve $e^{3x}(p-1) + p^3 e^{2y} = 0 \quad \text{--- (1)}$

$$\text{Solt}: e^{3x}(1-p) = p^3 e^{2y}$$

$$\Rightarrow (1-p) = p^3 e^{2y-3x}$$

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$$\Rightarrow e^y(1-p) = p^3 e^{3y-3x}$$

$$\Rightarrow e^y(1-p) = (pe^{y-x})^3$$

clearly which is in the form of

$$e^{by}(a-bp) = f(p e^{by-ax}), \text{ where } b=1, a=1$$

$$\text{Let } e^x = x; e^y = y$$

$$e^x dx = dx; e^y dy = dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{e^x} \frac{dy}{dx}$$

$$\Rightarrow P = \frac{e^y}{e^x} p$$

$$\Rightarrow \boxed{p = \frac{e^x}{e^y} P}$$

$$\textcircled{1} \equiv x^3 \left(\frac{e^x}{e^y} P - 1 \right) + \left(\frac{e^x}{e^y} P \right)^3 (P)^2 = 0$$

$$\Rightarrow x^3 \left(\frac{x}{y} P - 1 \right) + \left(\frac{x^3}{y} P \right)^3 y^2 = 0$$

$$\Rightarrow xP - y + P^3 = 0$$

$$\Rightarrow \boxed{y = xP + P^3}$$

which is in Clairaut's form.

\therefore It's general solution is $y = xc + c^3$

\Rightarrow $y = xc + c^3$ is general solution of $\textcircled{1}$



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* Solve the following differential equations by using the mentioned transformations.

Differential Equation	Transformation.
$y = 2xp + y^2 p^3$ (or) $y = 2px + y^{n-1} p^n$	$y^2 = y \Rightarrow 2y dy = dy$ $\Rightarrow 2y \frac{dy}{dx} = \frac{dy}{dx}$ $\Rightarrow 2y p = p \text{ where } p = \frac{dy}{dx}, P = \frac{dY}{dx}$
$y + px = x^4 p^2$	$\frac{1}{x} = X$
$y = 3px + 6y^2 p^2$	$y^3 = Y$
$\cos^2 y p^2 + \sin^2 \cos x \cos y p$ $- 8 \sin y \cos^2 x = 0$	$\sin x = x; \sin y = Y$
$(xp - y)^2 = a(1+p^2)(x^2 + y^2)^{3/2}$	$x = r \cos \theta$ $y = r \sin \theta$
$(xp - y)^2 = (x^2 - y^2) \sin^{-1} \left(\frac{y}{x}\right)$	$y = vx$
$(x^2 + y^2)(1+p)^2 - 2(x+y)(1+p) \cdot$ $(x+py) + (x+py)^2 = 0$	$x+y = u$ $x^2 + y^2 = v^2$
$x^2 p^2 + 2xy p + y^2 (1+p) = 0$	$y = Y; xy = Y$ $y = u; ny = v$
$(px^2 + y^2)(px + y) = (p+1)^2$	$x+y = x; (y=Y) \times$ $ny = v$
$xp^2 - 2yp + x + 2y = 0$	$x^2 = x, y-x = Y$
$y^2 + (y-xp) = x^4 p^2$	$\frac{1}{x} = X; \frac{1}{y} = Y$

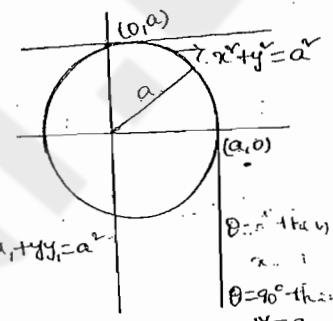
* Singular Solutions:

Def: A solution of a differential equation which is not derived from general solution by giving the particular values to the arbitrary constants, is called a singular solution. The singular solution does not involve any arbitrary constant.

Envelopes:

Consider the equation $x\cos\theta + y\sin\theta = a$, where a is constant. For different values of θ , the equation represents a family of straight lines touching the circle $x^2+y^2=a^2$. Here θ is the parameter of the family of straight lines $x\cos\theta + y\sin\theta = a$.

By the above example, the circle which is touched by a family of straight lines, is called the envelope of the family of straight lines. i.e. the curve E which is touched by a family of curves ' C ' is called the envelope of the family of curves ' C '.



Equation of Tangent:

The equation of tangent at (x_1, y_1) to the parabola $y^2=4ax$ is

$$yy_1 = 2a(x+x_1) \quad \text{--- (i)}$$

* Equation of Tangent in terms of slope (m):

The equation of tangent (i) is

$$y = \left(\frac{2a}{y_1}\right)x + \left(\frac{2ax_1}{y_1}\right) \quad \text{--- (ii)}$$

Let the slope of tangent at (x_1, y_1) be 'm'

Now the eqn of the tangent

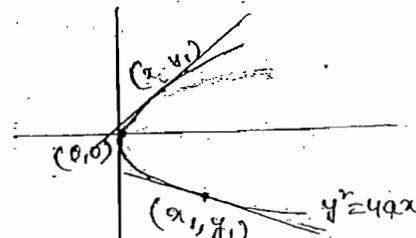
at $P(x_1, y_1)$ on the parabola $y^2=4ax$ --- (1)

Diff. (1) w.r.t 'x'

$$2yy' = 4a \Rightarrow yy' = \frac{4a}{y}$$

From (i), $y_1 = 2a(x+x_1)$

\therefore the eqn of tangent at (x_1, y_1) on the parabola is $y - y_1 = \frac{2a}{y_1}(x - x_1)$



$$\begin{aligned} y^2 &= 4ax; a = 1 \\ yy_1 &= 2(x+x_1) \\ (x_1, y_1) &= (1, 2) \\ \therefore xy &= x(x+1) \\ \Rightarrow x-y+1 &= 0 \\ x &= 1 - y \end{aligned}$$

$$\therefore \frac{2a}{y_1} = m$$

$$\Rightarrow y_1 = \frac{2a}{m}$$

Since (x_1, y_1) lies on parabola $y^2 = 4ax$

$$\Rightarrow y_1^2 = 4ax_1 \Rightarrow \frac{4a^2}{m^2} = 4ax_1$$

$$\Rightarrow x_1 = \frac{a}{m^2}$$

$\therefore (x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m} \right)$ is the point of contact.

\therefore The equation of tangent at $(x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m} \right)$

$$\text{is } y = mx + \frac{a}{m}$$

Note: By the definition of envelope, $y^2 = 4ax$ is the envelope of the straight lines $y = mx + \frac{a}{m}$; m being parameter.

Let us consider the differential equation.

$$y = Px + \frac{a}{P}; P = \frac{dy}{dx}$$

— ①

Clearly which is in Clairaut's form

$$\therefore \text{It's general solution is } y = mx + \frac{a}{m};$$

— ②

m is the parameter.

We know that the equation ② is the tangent to the parabola $y^2 = 4ax$ — ③

Clearly ③ is the envelope of ②

Now we show that $y^2 = 4ax$ is also solution of ①

Diff. ③ w.r.t x , we get

$$2yy' = 4a$$

$$\therefore y' = \frac{2a}{y} \text{ &}$$

the whole parabola ③ covered by ② for different values of m
 \therefore ③ is the sol'n of ①

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from ③,

$$x = \frac{y^2}{4a}$$

$$\therefore ① \equiv y = \left(\frac{2a}{y}\right)\left(\frac{y^2}{4a}\right) + a\left(\frac{y}{2a}\right)$$

 $\Rightarrow y=y$ (Identity)

 $\therefore ③$ is the solution of ①.

→ The equation of the envelope of the family of curves given by the general solution of a differential equation is known as the singular solution. Such a solution does not contain any arbitrary constant and is not a particular case of the general solution.

It is sometimes possible to reduce this solution from the general solution by giving the particular values to the arbitrary constants. In such a case the singular solution is also called particular solution.

→ Let $f(x, y, p)=0$ be the differential equation whose solution is $\phi(x, y, c)=0$, then P-discriminant is obtained by

eliminating P between $f(x, y, p)=0$ & $\frac{\partial f}{\partial p}=0$.

If a part of $f(x, y, p)$ is quadratic then $\Rightarrow b^2-4ac=0$.

C-discriminant is obtained by eliminating C

between $\phi(x, y, c)=0$ & $\frac{\partial \phi}{\partial c}=0$.

→ If $E(x, y)=0$ is a singular solution (envelope) of the differential equation $f(x, y, p)=0$, whose general solution



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is $\phi(x, y, c)=0$ then $E(x, y)$ is a factor of both the discriminants (i.e. P & C) and $E(x, y)=0$ must satisfy the differential equation $f(x, y, P)=0$.

Ex: Show that $x=0$ is the singular solution of

$$4xp^2 = (3x-a)^2$$

Sol: The given differential equation is

$$4xp^2 - (3x-a)^2 = 0 \quad \dots \textcircled{1}$$

$$\therefore \text{general solution is } (y+c)^2 = x(x-a)^2 \quad \dots \textcircled{2}$$

P-discriminants:

$$(0)^2 - 4 \cdot 4x [(3x-a)^2] = 0 \quad (\because b^2 - 4ac = 0)$$
$$\Rightarrow x(3x-a)^2 = 0 \quad \dots \textcircled{3}$$

C-discriminant:

$$\textcircled{2} \equiv c^2 + 2yc + y^2 - x(x-a)^2 = 0$$

$$\Rightarrow (2y)^2 - 4(1)[y^2 - x(x-a)^2] = 0$$

$$\Rightarrow 4y^2 - 4y^2 + x(x-a)^2 = 0$$

$$\Rightarrow x(x-a)^2 = 0 \quad \dots \textcircled{4}$$

$\therefore x=0$ is the factor of both the discriminants.

and it must be satisfy the given differential equation.

since from (1), $4x - \frac{(3x-a)^2}{P^2} = 0$

which is satisfied by $x=0$.

because $\frac{dx}{dy} = 0$ i.e. $\frac{1}{P} = 0$

$\therefore x=0$ is the singular solution of (1).

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→ Each discriminant may have other factor which correspond to other loci associated with the general solution of the given differential equation. Generally the equations of these loci do not satisfy the differential equation, they are known as extraneous loci.
There are three types of Extraneous loci

- (1) Tac-locus (2) Node-locus and (3) Cusp-locus

Methods for finding the singular solution:

→ To find the singular solution of a diff. equation $f(x, y, P) = 0$

- (1) Find its general solution $\Phi(x, y, C) = 0$.
- (2) Find P-discriminant
- (3) Find C-discriminant

* Now P-discriminant equated to zero may include as a factor.

- (1) Envelope i.e. Singular solution once (E)
- (2) Cusp-locus once (C)
- (3) Tac-locus twice (T)
i.e. P-discriminant $= E^2 C^3$

* C-discriminant equated to zero may include as a factor:

- (1) Envelope i.e. singular solution
- (2) Cusp-locus thrice (C³)
- (3) Node-locus twice (N²)
i.e. C-discriminant $= E N^2 C^3$

Problems: Set A

(When equations are solvable for P)

Obtain the complete primitive (i.e. g.s) and singular solution of the following equations. Explaining the geometrical significance of the irrelevant factors that present themselves.



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$$(1) \rightarrow xy^r = (x-a)^r$$

Sol'n: Given that $xy^r = (x-a)^r$ — (1)
It is solvable for 'P'.

$$\therefore [\sqrt{x}P - (x-a)] [\sqrt{x}P + (x-a)] = 0 \quad (2)$$

$$\Rightarrow \sqrt{x}P = x-a; \quad \sqrt{x}P = a-x$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{x} - ax^{\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = ax^{\frac{1}{2}} - \sqrt{x}$$

$$\Rightarrow dy = (\sqrt{x} - ax^{\frac{1}{2}}) dx \quad \Rightarrow dy = (ax^{\frac{1}{2}} - \sqrt{x}) dx$$

$$\Rightarrow y = \frac{2}{3}x^{\frac{3}{2}} - 2ax^{\frac{1}{2}} + C \quad \Rightarrow y = 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C$$

$$\therefore (y - \frac{2}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} - C)(y - 2ax^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} - C) = 0$$

$$\Rightarrow (y-C)^2 - (\frac{2}{3}x^{\frac{3}{2}} - 2ax^{\frac{1}{2}})^2 = 0$$

$$\Rightarrow (y-C)^2 = \frac{4}{9}x(x-3a)^2 \quad (3)$$

which is the required general solution of (1)

Now P-discriminant:

$$0 - 4x[-(x-a)^2] = 0$$

$$\Rightarrow 4x(x-a)^2 = 0$$

$$\Rightarrow x(x-a)^2 = 0 \quad (4)$$

C-discriminant:

$$(3) \equiv y^2 + C^2 - 2yC = \frac{4}{9}x(x-3a)^2$$

$$\Rightarrow C^2 + (-2y)C + y^2 - \frac{4}{9}x(x-3a)^2 = 0$$

$$\Rightarrow (-2y)^2 - 4(1)[y^2 - \frac{4}{9}x(x-3a)^2] = 0 \quad (\because b^2 - 4ac = 0)$$

$$\Rightarrow 4y^2 - 4[y^2 - \frac{4}{9}x(x-3a)^2] = 0$$

$$\Rightarrow \frac{4}{9}x(x-3a)^2 = 0$$

$$\Rightarrow x(x-3a)^2 = 0 \quad (5)$$

$\therefore x=0$ is the factor of both the discriminants (i.e. P&C)

and $x=0$ satisfies the given diff. equation.

$\therefore x=0$ is a singular solution.

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Now $x-a=0$ is a tee-locus because it appears twice in the P-discriminant relation ④,

it does not occur in the C-discriminant relation ⑥ and does not satisfy the differential Equation ①

Now $x-a=0$ is a node-locus because it appears once in the C-discriminant relation ⑥, it does not occur in the P-discriminant relation ④ and does not satisfy the diff. equation ①.

$$(1) \quad 4P^2x(x-a)(x-b) = [3x^2 - 2x(a+b) + ab]^2$$

Given equation is

$$4P^2x(x-a)(x-b)^2 = [3x^2 - 2x(a+b) + ab]^2 \quad \text{--- (1)}$$

Its general solution

$$(y+c)^2 = x(x-a)(x-b)$$

$$\Rightarrow c^2 + 2cy + y^2 - x(x-a)(x-b) = 0 \quad \text{--- (2)}$$

Now P-discriminant:

$$0 - 4(4x)(x-a)(x-b)[-(3x^2 - 2x(a+b) + ab)]^2 = 0$$

$$\Rightarrow x(x-a)(x-b)[3x^2 - 2x(a+b) + ab]^2 = 0 \quad \text{--- (3)}$$

C-discriminant

$$4y^2 - 4 \cdot 1 [y^2 - x(x-a)(x-b)] = 0$$

$$\Rightarrow x(x-a)(x-b) = 0$$

Here $x=0$ appears once in both the discriminants and it satisfies the equation ①.

$$\text{i.e. } 4x(x-a)(x-b) - \frac{[3x^2 - 2x(a+b) + ab]^2}{P^2} = 0 \quad [\because x=0 \\ \Rightarrow \frac{dx}{dy} = 0 \\ = \frac{1}{P} = 0]$$

$\therefore x=0$ is the singular solution.

Similarly $(x-a)=0$ and $(x-b)=0$ are also singular solutions.



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Now $3x^2 - 2x(a+b) + ab = 0$ is a tac-toc because it appears twice in the P-discriminant.

Again solving it for 'x' we get

$$x = \frac{2(a+b) \pm \sqrt{4(a+b)^2 - 12ab}}{6}$$

$$\Rightarrow 3x = (a+b) \pm (a^2 - ab + b^2)^{1/2} \quad \text{--- (5)}$$

The above mentioned 'tac-toc' factors are given by (5)
 \therefore There are two tac-loci given by (5)

Set-B

(When equations are solvable for 'x')

→ Solve and examine for singular solutions.

$$P^3 - 4xyzP + 8y^2 = 0 \quad \text{--- (1)}$$

Sol'n: it is solvable for 'x'

$$4xyzP = P^3 + 8y^2$$

$$\Rightarrow x = \frac{P^2}{4y} + \frac{2y}{P} \quad \text{--- (2)}$$

Diff. w.r.t 'y' we get so on

$$\text{It's g.s. is } Y = C(C-x)^2 \quad \text{--- (3)}$$

P-discriminant:

Diff. (1) partially w.r.t 'P', we get

$$3P^2 + 4xyz = 0$$

$$\Rightarrow P^2 = \frac{4xyz}{3} \quad \text{--- (4)}$$

Now eliminating P from (1) & (4)

$$\text{for this } (1) \equiv 8y^2 = P(4xy - P^2)$$

$$\Rightarrow 64y^4 = P^2(4xy - P^2)^2$$

$$\Rightarrow 64y^4 = \frac{4xyz}{3} \cdot \left(4xy - \frac{4xyz}{3}\right)^2 \quad (\text{from (4)})$$

$$\Rightarrow 64y^4 = \frac{64xyz}{3} \left(\frac{2xy}{3}\right)^2$$

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Set E: Reducible to Clairaut's form)

(Equations Reducible to Clairaut's form)
 → Reduce the differential equation $(Px-y)(x-Py)=2P$. to Clairaut's form by the substitution $x^r=u$ and $y^r=v$ and find its complete primitive and its singular solution, if any.

Sol'n: Given that $(Px-y)(x-Py)=2P$ — (1)

It's g.s is

$$y = cx - \frac{2c}{1-c}$$

(by previous methods)

(or)

$$x^rc = (x^r+y^r-2)c + y^r = 0 \quad (2)$$

$$(1) \equiv xyP^r - (x^r+y^r-2)P + 2 = 0 \quad (3)$$

Now P-discriminant is:

$$(x^r+y^r-2)^2 - (2xy)^2 = 0 \quad (4)$$

$$\Rightarrow (x^r+y^r-2-2xy)(x^r+y^r-2+2xy) = 0$$

$$\Rightarrow [(x-y)^2 - (\sqrt{2})^2][(x+y)^2 - (\sqrt{2})^2] = 0$$

$$\Rightarrow (x-y-\sqrt{2})(x-y+\sqrt{2})(x+y+\sqrt{2})(x+y-\sqrt{2}) = 0$$

Now from (3), C-discriminant is

$$(x^r+y^r-2)^2 - (2xy)^2 = 0$$

which is same as (4).

∴ It also reduces to (4).

∴ P-and C-discriminant relations are coincident here.

$x-y+\sqrt{2}=0$ appears in both the discriminant and satisfies the given diff. equation and hence it is a singular solution.



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Similarly,

$$x-y-\sqrt{2}=0, \quad x+y+\sqrt{2}=0 \quad \text{and}$$

$x+y-\sqrt{2}=0$ are also singular solutions.

W) Reduce the equation $xyP^2 - (x^2 + y^2 - 1)P + xy = 0$ to Clairaut's form by the substitution $x^2 = u$ and $y^2 = v$. Hence show that the equation represents a family of conics touching the four sides of a square.

Sol'n: Given that $xyP^2 - (x^2 + y^2 - 1)P + xy = 0 \quad \text{--- (1)}$

It's gs is $c^2x^2 - c(x^2 + y^2 - 1) + y^2 = 0 \quad \text{--- (2)} \quad (\text{by using previous methods})$
which represents a family of conics.

from (1): the P-discriminant relation is

$$(x^2 + y^2 - 1)^2 - 4x^2y^2 = 0 \quad \text{--- (3)}$$

from (2): the C-discriminant relation is

$$(x^2 + y^2 - 1)^2 - 4x^2y^2 = 0 \quad \text{--- (4)}$$

$$\therefore \text{from (3) \& (4)}, \quad (x^2 + y^2 - 1)^2 - 4x^2y^2 = 0 \quad \text{--- (5)}$$

must be singular solution, because it is present once in both the discriminants.

Again from (5), we have

$$(x+y+1)(x+y-1)(x-y+1)(x-y-1) = 0$$

$$\Rightarrow x+y+1=0, \quad x+y-1=0, \quad x-y+1=0, \quad \text{and} \quad x-y-1=0$$

are four singular solutions (envelopes).

Thus the given diff. equation represents a family of conics given by (2) which are touched by the four lines (envelopes) mentioned above and the four lines form the four sides of a square.

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$$\Rightarrow y^4 = \frac{4x^3y^3}{27}$$

$$\Rightarrow 27y^4 = 4x^3y^3$$

$$\Rightarrow y^3 [27y - 4x^3] = 0$$

$$\Rightarrow y \cdot y [27y - 4x^3] = 0 \quad \text{--- (6)}$$

C-discriminant:

Diff (3) partially w.r.t. 'c' we get

$$0 = (c-x)^2 + 2c(c-x)$$

$$\Rightarrow (c-x)(3c-x) = 0$$

$$\Rightarrow c=x \text{ & } c= \frac{x}{3}$$

If $c=x$ then (3) $\equiv y=0$ (6)If $c=\frac{x}{3}$ then (3) $\equiv y = \frac{10x}{27}$ (7)

from (6) & (7) we have

$$y \left(y - \frac{4x^3}{27} \right) = 0$$

$$\Rightarrow y(27y - 4x^3) = 0 \quad \text{--- (8)}$$

 $\therefore y=0$ & $27y - 4x^3 = 0$ are the singular solutions.

Because they both appear once in (6) & (8) and satisfying the given differential equation.

→ Find the solution of the differential equation $y = 2xp - 4p^2$ Also find the singular solution.Sol:

(when equations are solvable for y)

→ solve the differential equation $(8p^3 - 27)x = 12p^2y$ and investigate whether a singular solution exists.

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Examine the following equations for singular solution and extraneous loci if any

$$000 \rightarrow y = x - 2ap + ap^2 \dots$$

$$001 \rightarrow xp^2 - 2yp + ax = 0$$

$$\rightarrow ap^2 - 2yp + 4x = 0$$

Set-D

(When equations are in Clairaut's form)

→ find the complete solution & singular solution of

$$y = px + \sqrt{b^2 + a^2 p^2}$$

$$\text{Sol'n: Given that } y = px + \sqrt{b^2 + a^2 p^2} \quad \dots \quad (1)$$

clearly which is in Clairaut's form

∴ It's g.s is

$$y = cx + \sqrt{b^2 + a^2 c^2} \quad \dots \quad (2)$$

Now from (1) & (2) both P & C - discriminant!

For this L1

$$(y - px)^2 = b^2 + a^2 p^2$$

$$\Rightarrow (x^2 - a^2)p^2 - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow 4x^2y^2 - 4[x^2 - a^2][y^2 - b^2] = 0$$

$$\Rightarrow x^2y^2 - (x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2) = 0$$

$$\Rightarrow x^2b^2 + a^2y^2 - a^2b^2 = 0$$

which is the P - & C - discriminant.

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differential equation.

Note: In case of Clairaut's equation P-and C-discriminants are always identical.

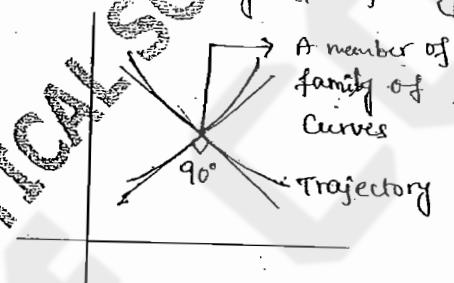
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MATHEMATICS by K. VENKARNA

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Orthogonal TrajectoriesDefinition:

- A curve which cuts every member of a given family of curves according to a given law, is called a trajectory of the given family of curves.
- A trajectory of a family of curves is called an orthogonal trajectory of the family if it cuts every member of the family at right angle.
- A trajectory of a family of curves is called an oblique trajectory of the family if it cuts every member of the given family of curves at an angle $\neq 90^\circ$.



Ex! Consider the two families of curves $y=mx$ & $x^2+y^2=a^2$ where m & a are parameters.

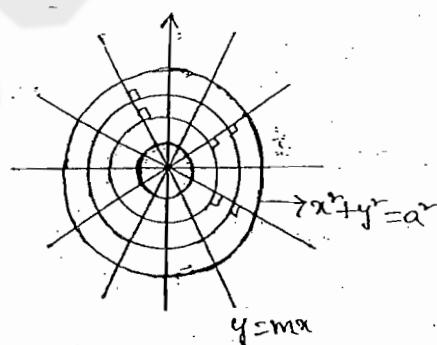
then $y=mx$ is an orthogonal trajectory of the family of circles $x^2+y^2=a^2$.

since every line (i.e. $y=mx$)

passing through the origin of

coordinates is an orthogonal

trajectory of the family of the concentric circles (i.e. $x^2+y^2=a^2$)



Working Rule for finding the orthogonal trajectories of the given family of Curves in Cartesian coordinates (i.e. $f(x,y,c)=0$):

Step(I): form the diff. equation $f\left(x, y, \frac{dy}{dx}\right)=0$ by eliminating ' c ' from the given family of curves $f(x, y, c)=0$.



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Step(2): Replace $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}}$ in $F(x, y, \frac{dy}{dx}) = 0$.

to get the diff. equation $F(x, y, -\frac{dx}{dy}) = 0$ of the family of orthogonal trajectories.

Step(3): solve the diff. equation

$F(x, y, -\frac{dx}{dy}) = 0$ to get the family of orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of the family of curves $x^2y = a^2$; a is parameter.

Sol'n: The given family of curves is

$$x^2y = a^2; \quad a \text{ is parameter.} \quad (1)$$

Diff. w.r.t x , we get

$$2x + 2yy' = 0 \\ \Rightarrow x + yy' = 0 \quad (2)$$

which is the differential equation of the given family of curves (1).

Replacing " y' " by $-\frac{1}{y}$, in (2), we get the diff. equation of the family of orthogonal trajectories

$$\begin{aligned} \therefore x - \frac{y}{y'} &= 0 \\ \Rightarrow xy' &= y \\ \Rightarrow \frac{1}{y} dy &= \frac{1}{x} dx \\ \Rightarrow \log y &= \log x + \log m \\ \Rightarrow y &= mx; \quad m \text{ is parameter.} \end{aligned}$$

which is the required orthogonal trajectories.

→ Find the orthogonal trajectories of the following family of curves.

(i) $y = ax^2$; a is parameter.

(ii) $3xy = x^3 - a^3$; a is parameter.

Now replacing ' y' by $-\frac{1}{y}$ in ③

we get the differential equation of orthogonal trajectories.

$$\therefore y = 2x\left(-\frac{1}{y}\right) + y - \left(-\frac{1}{y}\right)^2$$

$$\Rightarrow y = \frac{-2x}{y} + \frac{y}{y^2}$$

$$\Rightarrow y(y')^2 = -2xy' + y$$

$$\Rightarrow y = 2xy' + y(y')^2 \quad \text{--- (4)}$$

which is the same as the differential equation of the given family of parabolas. i.e., ③ & ④ are same.

\therefore The system of Parabolas ① is self orthogonal.

Q3 \rightarrow show that the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \text{ is self-orthogonal; } \lambda \text{ is parameter.}$$

(or)

Q3 \rightarrow show that the orthogonal trajectory of a system of confocal ellipses is self-orthogonal.

Sol'n: Given equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{--- (1)}$$

Diff ① w.r.t ' x ' we get

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0 \Rightarrow x[b^2+\lambda] + yy'[a^2+\lambda] = 0$$

$$\Rightarrow b^2x + a^2yy' + \lambda(x+yy') = 0$$

$$\Rightarrow \lambda = -\frac{(b^2x + a^2yy')}{x+yy'}$$

$$\text{Now } a^2+\lambda = a^2 - \frac{(b^2x + a^2yy')}{x+yy'}$$

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(iii) $x^2 + y^2 = 2ax$; a is parameter

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$; λ is a parameter.

(v) $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$; λ is a parameter

(vi) $x^2 + y^2 + 2gx + c = 0$; g is parameter

$\rightarrow ax^2 = x^3$; a is parameter.

Self → Orthogonal family of Curves:

A family of curves is said to be self-orthogonal if the diff. eqn of the family of curves is same as the diff. eqn of equation of the orthogonal trajectories of the orthogonal trajectories (Or)

If each member of a given family of curves is self-orthogonal. It intersects all other members orthogonally. Then the given family of curves is said to be self-orthogonal.

Problems:

→ Show that the system of confocal and co-axial parabolas

$$y^2 = 4a(x+a) \text{ is self-orthogonal, } a \text{ being Parameter.}$$

Differentiating w.r.t x we get,

$$2yy' = 4a \quad \dots \textcircled{2}$$

Now eliminating ' a ' b/w. \textcircled{1} & \textcircled{2}, we get

$$y^2 = 2yy' \left(x + \frac{yy'}{2} \right)$$

$$\Rightarrow y^2 = 2xyy' + y^2(y')^2$$

$$\Rightarrow y = 2xy' + y(y')^2 \quad \dots \textcircled{3}$$

which is the differential equation of the given family of Parabolas.



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$$= \frac{a^2x + a^2yy' - b^2x - a^2yy'}{x+yy'}$$

$$\Rightarrow a^2 + \lambda = \frac{(a^2 - b^2)x}{x + yy'}$$

$$\text{and } b^2 + \lambda = b^2 - \frac{(b^2x + a^2yy')}{x + yy'}$$

$$= \frac{b^2x + b^2yy' - b^2x - a^2yy'}{x + yy'}$$

$$= \frac{-(a^2 - b^2)yy'}{x + yy'}$$

$$\textcircled{1} = x^2 \left[\frac{x+yy'}{(a^2 - b^2)x} \right] - y^2 \left[\frac{x+yy'}{(a^2 - b^2)yy'} \right] = 1$$

$$xy'(x+yy') - y(x+yy') = (a^2 - b^2)y'$$

$$\Rightarrow (x+yy')(ay' - y) = (a^2 - b^2)y'$$

$$\Rightarrow (x+yy') \left(x - \frac{y}{y'} \right) = (a^2 - b^2) \quad \textcircled{2}$$

which is the diff eqn of the given system of conics $\textcircled{1}$.

Now replacing y' by $\pm \frac{1}{y}$ in $\textcircled{2}$

we get the differential equation of the family of orthogonally
trajectories.

$$\left(x - \frac{y}{y'} \right) (x+yy') = a^2 - b^2 \quad \textcircled{3}$$

$\textcircled{2}$ & $\textcircled{3}$ are same

The given system of confocal conics is self orthogonal.

Orthogonal trajectories in polar co-ordinates (i.e., $f(r, \theta, c)=0$):

Working Rule:

Step (1): Form the diff equation $F(r, \theta, \frac{dr}{d\theta})=0$ by eliminating 'c' from the given family of curves $f(r, \theta, c)=0$.



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Step (2): Replace $\frac{dr}{d\theta}$ by $\frac{-r}{\frac{dr}{d\theta}}$ (or) $-r^2 \frac{d\theta}{dr}$ in

$F(r, \theta, \frac{dr}{d\theta}) = 0$ to get the diff. equation $F(r, \theta, -r^2 \frac{d\theta}{dr}) = 0$.

of the family of orthogonal trajectories.

Step (3): solve the diff. equation $F(r, \theta, r^2 \frac{d\theta}{dr}) = 0$ to get the orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of family of cardioids

$r = a(1 - \cos\theta)$; where a is the parameter.

Sol'n: Given family of cardioids is $r = a(1 - \cos\theta)$ — (1)

Taking log on both sides we get

$$\log r = \log a + \log(1 - \cos\theta) \quad \text{--- (2)}$$

Diff. (2) w.r.t θ , we get,

$$\frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{1 - \cos\theta} (\sin\theta)$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta} \quad \text{--- (3)}$$

which is the diff. equation of the given family (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (3), we get the differential equation of the orthogonal trajectories.

$$\therefore \frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \frac{\sin\theta}{1 - \cos\theta}$$

$$\Rightarrow -r \frac{d\theta}{dr} = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow -r \frac{d\theta}{dr} = \cot(\theta/2)$$

$$\Rightarrow \frac{dr}{r} = -\tan(\theta/2) d\theta$$

$$\Rightarrow \log r = -\log(\sec(\theta/2)) + C$$

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$$\Rightarrow \log(\delta/c) = \log |\cos(\theta_2)|$$

$$\Rightarrow \delta/c = \cos(\theta_2)$$

$$\Rightarrow \delta/c = \frac{1+\cos\theta}{2}$$

$$\Rightarrow \delta = \frac{c}{2} (1+\cos\theta)$$

$$\Rightarrow \delta = b (1+\cos\theta) \quad (\text{Put } c_2 = b)$$

which is the required orthogonal trajectories of the following family of curves.

$$(1) \delta = \frac{2a}{1+\cos\theta}; a \text{ is parameter}$$

$$(2) r^n \cos n\theta = a^n; a \text{ is parameter}$$

Working rule for finding the oblique trajectories of the given family of curves in Cartesian co-ordinates
(i.e. $f(x, y, C) = 0$):

Step(1): form the diff. equation $F(x, y, \frac{dy}{dx}) = 0$ by eliminating C from the given family of curves. $f(x, y, C) = 0$.

Step(2): Replace $\frac{dy}{dx}$ (i.e. P) by $\frac{P + \tan\alpha}{1 - P \tan\alpha}$ (or) $\frac{1 + P \cot\alpha}{\cot\alpha - P}$

In $F(x, y, \frac{dy}{dx}) = 0$ to get the diff. equation $F\left(x, y, \frac{P + \tan\alpha}{1 - P \tan\alpha}\right) = 0$

where $P = \frac{dy}{dx}$ of the family of trajectories.

Step(3): solve the diff. equation $F\left(x, y, \frac{P + \tan\alpha}{1 - P \tan\alpha}\right) = 0$ to get the family trajectories.

Problems:

→ Find the 45° trajectories of the family of curves $xy = C$. where C is parameter.



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Sol'n! Given family of curves is $xy=c$ — ①

Diff. w.r.t x , we get

$$xy' + y = 0 \quad \text{--- ②}$$

$$\text{(or)} \quad xy + y = 0$$

Replacing y' by $\frac{p + \tan \alpha}{1 - p \tan \alpha}$ in ②.

we get,

$$\frac{x(p + \tan \alpha)}{1 - p \tan \alpha} + y = 0$$

$$\Rightarrow x \left(\frac{p + \tan 45^\circ}{1 - p \tan 45^\circ} \right) + y = 0$$

$$\Rightarrow x \left(\frac{p+1}{1-p} \right) + y = 0$$

$$\Rightarrow x(p+1) + y(1-p) = 0$$

$$\Rightarrow (x-y)p + (x+y) = 0$$

$$\Rightarrow p = \frac{y+x}{y-x}$$

$$\Rightarrow (y+x)dx + (x-y)dy = 0 \quad \text{--- ③}$$

clearly, which is the form of $Mdx + Ndy = 0$.

where $M = y+x$; $N = x-y$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating ③ we get

$$yx + \frac{x^2}{2} - \frac{y^2}{2} = c$$

which is the required trajectory of the given family.

→ Determine the 45° trajectories of the family of concentric

Circles $x^2 + y^2 = a^2$; a is the parameter

$$\Rightarrow (xp-y)^2 = (x^2-y^2) \sin^2(y/x) \quad \text{①}$$

Putting $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow p = v + xp \quad \text{where} \\ p = \frac{dv}{dx}$$

∴ ①

$$[x(v+xp)-y]^2 = [x-xv^2] \sin^2 v$$

$$\Rightarrow (y+xp-y)^2 = x^2(1-v^2) \sin^2 v$$

$$\Rightarrow (xp)^2 = x^2(1-v^2) \sin^2 v$$

$$\Rightarrow p^2 = \frac{1-v^2}{x^2} \sin^2 v$$

$$\Rightarrow p = \pm \sqrt{\frac{1-v^2}{x^2}} \sqrt{\sin^2 v}$$

$$\Rightarrow \frac{dv}{dx} = \pm \sqrt{\frac{1-v^2}{x^2}} \sin v$$

$$\Rightarrow \int \frac{1}{\sqrt{1-v^2} \sqrt{\sin v}} dv = \pm \int \frac{dx}{x} + C$$

$$\sin v = t$$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}} dt = dx$$

$$\Rightarrow \int \frac{dt}{\sqrt{t}} = \pm \log x + C$$

$$\Rightarrow 2t^{1/2} = \pm \log x + C$$

$$\Rightarrow 4t = (\pm \log x + C)^2$$

$$\Rightarrow 4 \sin^2 v = (\pm \log x + C)^2$$

$$\Rightarrow 4 \sin^2(y/x) = (\pm \log x + C)^2$$

which is the required

$$\Rightarrow xp^2 - (y-x)p \cdot y = 1 \quad \text{②}$$

$$\Rightarrow px(p+1) - y(p+1) = 1$$

$$\Rightarrow (px-y)(p+1) = 1$$

$$\Rightarrow px-y = \frac{1}{p+1}$$

$$\Rightarrow y = px - \frac{1}{p+1} \quad \text{③}$$

clearly which is in the form of Clairaut's equation.

put $p=c$, we get

$$y = cx - \frac{1}{c+1} \quad \text{④}$$

which is the G.S. of ①

Due to see Clairaut's form of ②, the p -discriminant and c -discriminant relations are same.

from ①

p -discriminant is given by

$$(1-y)^2 + 4xy = 0$$

$$\Rightarrow (x+y)^2 = 0$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differentiated equation.

$$\rightarrow (xp-y)^2 = a(1+p^2)(x^2+y^2)^{3/2} \quad \textcircled{1}$$

Putting $a = r \cos \theta$; $y = r \sin \theta$

$$\therefore x^2 + y^2 = r^2$$

$$\Rightarrow r dr + r dy = r dr$$

$$\Rightarrow x + y p = r \frac{dr}{d\theta}$$

$$\text{and } \frac{y}{x} = \tan \theta$$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \sec^2 \theta d\theta$$

$$\Rightarrow xp - y = r \frac{d\theta}{d\theta}$$

$$\text{Now } \frac{x+yp}{xp-y} = \frac{r \frac{dr}{d\theta}}{r \frac{d\theta}{d\theta}}$$

$$\Rightarrow \frac{x+yp}{xp-y} = \frac{1}{r} P \text{ where } \frac{dr}{d\theta} = P$$

$$\Rightarrow rx + ry p = Ppx - Py$$

$$\Rightarrow (Px + Py) p = rx + Py$$

$$\Rightarrow p = \frac{rx + Py}{Px - Ry}$$

\textcircled{1}

$$\left[x \left(\frac{rx + Py}{Px - Ry} \right) - y \right]^2 = a \left[1 + \left(\frac{Px + Py}{Px - Ry} \right)^2 \right]^{3/2}$$

$$\Rightarrow (rx^2 + pyx - Px^2 - Ry^2)^2 = a \left[(Px - Ry)^2 + (Px + Py)^2 \right] r^3 \quad \text{(2)}$$

$$\Rightarrow (x^2 + y^2) r^2 = a [(P^2 + r^2)(x^2 + y^2)] r^3$$

$$\Rightarrow (x^2 + y^2) = a (P^2 + r^2) r$$

$$\Rightarrow r^2 = a (P^2 + r^2) r$$

$$\Rightarrow r = a (P^2 + r^2)$$

$$\Rightarrow ar^2 = r - ar^2$$

$$\Rightarrow P = \frac{r - ar^2}{ar}$$

$$\Rightarrow P = \frac{\pm \sqrt{a(1-a^2)}}{\sqrt{a}}$$

$$\Rightarrow \frac{dr}{d\theta} = \pm \sqrt{r} \sqrt{1-ar^2}$$

$$\Rightarrow \int \frac{\sqrt{a}}{\sqrt{r} \sqrt{1-ar^2}} dr = \pm \int d\theta$$

$$\text{put } 1-ar^2 = t \text{ and } \frac{dt}{dr} = \frac{1-2ar^2}{r} \Rightarrow ar^2 dt = -\frac{1}{a} dt$$

$$\Rightarrow dr = -\frac{1}{a} dt$$

$$\int \frac{\sqrt{a} (-\frac{1}{a} dt)}{\sqrt{1-t} \sqrt{t}} = \pm \theta + C$$

$$-\int \frac{dt}{\sqrt{t(1-t)}} = \pm \theta + C$$

$$-\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2 + \frac{1}{4}}} = \pm \theta + C$$

$$\Rightarrow \sin^{-1} \left(\frac{t-1}{2} \right) = \pm \theta + C$$

$$\Rightarrow \sin^{-1} (2t-1) = \pm \theta + C$$

$$\Rightarrow \sin^{-1} [2(1-ar)-1] = \pm \theta + C$$

$$\Rightarrow -i + 2ar = \sin^{-1} (\pm \theta + C)$$

$$\Rightarrow 2ar \sqrt{x^2 + y^2} = 1 - \sin(\pm \tan^{-1} \frac{y}{x} + C)$$

$$\therefore 2a \sqrt{x^2 + y^2} - 1 = \sin^{-1} \left(\pm \tan^{-1} \frac{y}{x} + C \right)$$

which is the required solution

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MATHEMATICS by E. VENKANNAN

* The Laplace Transform *

Introduction:

Laplace transform or Laplace transformation is a method for solving linear differential equations and satisfying given boundary conditions without use of a general solution.

These particular solutions are the ones widely used in physics, mechanics, chemistry, medicine, national defence and many fields of practical research. The knowledge of Laplace transforms in recent years has an essential part of mathematical background required for engineers and scientists.

Integral Transform:

Let $K(P,t)$ be a function of two variables P and t , where P is a parameter (may be real or complex) independent of t . The

function $f(P)$ defined by the integral

(assume to be convergent)

$$f(P) = \int_{-\infty}^{\infty} K(P,t) f(t) dt \text{ is called}$$

the integral transform of the function $f(t)$ and is denoted by

$$T\{f(t)\}$$

The function $K(P,t)$ is called the kernel of the transformation.

Laplace Transform:

If the kernel $K(P,t)$ is defined as

$$K(P,t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-Pt} & \text{for } t \geq 0 \end{cases} \quad f(P) = \int_{-\infty}^{\infty} K(P,t) f(t) dt$$

$$\text{then } f(P) = \int_0^{\infty} e^{-Pt} f(t) dt \quad (1)$$

$$= \int_0^{\infty} e^{-Pt} f(t) dt$$

$$= \int_0^{\infty} e^{-Pt} f(t) dt$$

the function $f(P)$ defined by the integral (1) is called the Laplace transform of the function $f(t)$ and is denoted by

$$L\{f(t)\} \text{ (or) } \bar{F}(P) \text{ (or) } L[F(t)]$$

$$\text{i.e., } L\{f(t)\} = \int_0^{\infty} e^{-Pt} f(t) dt$$

Thus Laplace transform is a



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function of a new variable (or parameter) P given by (1).

Note: The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some values of P , otherwise it does not exist.

* Linearity Property of Laplace Transformation:

A transformation T is said to be linear if for every pair of functions $F_1(t)$ and $F_2(t)$ and for every pair of constants a_1 and a_2 .

we have

$$T\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 T\{F_1(t)\} + a_2 T\{F_2(t)\}$$

→ the Laplace transformation is a linear transformation.

$$\text{i.e., } L\{a_1 F_1(t) + a_2 F_2(t)\} \\ = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

where a_1, a_2 are constants.

Sol'n: we have

$$L\{F(t)\} = \int_0^{\infty} e^{-pt} F(t) dt$$

$$\therefore L\{a_1 F_1(t) + a_2 F_2(t)\} :$$

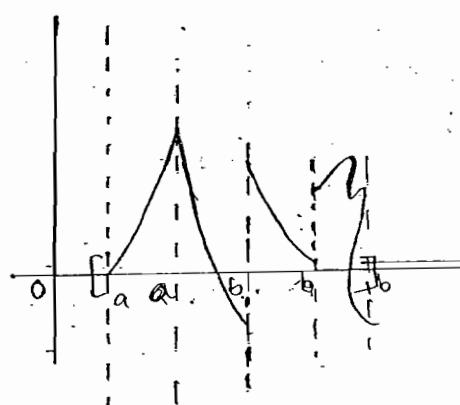
$$= \int_0^{\infty} e^{-pt} \{a_1 F_1(t) + a_2 F_2(t)\} dt$$

$$= a_1 \int_0^{\infty} e^{-pt} F_1(t) dt + a_2 \int_0^{\infty} e^{-pt} F_2(t) dt$$

$$= a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

→ Piecewise (or) Sectionally Continuous Function:

A function $F(t)$ is said to be piecewise (or sectionally) continuous on $t \in [a, b]$, if it defined on that interval and is such that the interval can be subdivided into finite number of intervals, in each of which $F(t)$ is continuous and has finite right and left hand limits.



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* Existence of Laplace Transform:

If $F(t)$ is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$|F(t)| \leq M e^{at}$ for all $t \geq 0$ and for some constants a and M , then the Laplace transform of $F(t)$ exists for all $p > a$.

Proof: we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$$

$$= \int_0^{t_0} e^{-pt} F(t) dt + \int_{t_0}^\infty e^{-pt} F(t) dt \quad \textcircled{1}$$

The integral $\int_0^{t_0} e^{-pt} F(t) dt$ exists since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$.

$$\text{Now } \left| \int_0^\infty e^{-pt} F(t) dt \right| \leq \int_{t_0}^\infty |e^{-pt} F(t)| dt$$

$$\leq \int_{t_0}^\infty e^{-pt} M e^{at} dt$$

Since $|F(t)| \leq M e^{at}$

$$= \int_{t_0}^\infty e^{-(p-a)t} M dt$$

$$= \frac{-e^{-(p-a)t} M}{(p-a)} \Big|_{t_0}^\infty$$

$$= \frac{M e^{-(p-a)t_0}}{p-a}, \quad p > a$$

$$\therefore \int_{t_0}^\infty e^{-pt} F(t) dt \leq \frac{M e^{-(p-a)t_0}}{p-a}, \quad p > a$$

But $\frac{M e^{-(p-a)t_0}}{p-a}$ can be made as small as we please by taking p sufficiently large.

Thus from (1), we conclude that $L\{F(t)\}$ exists for all $p > a$.

Note (1): Above theorem of existence of Laplace transform can also be stated as:

"If $F(t)$ is a function which is piece-wise continuous on every finite interval in the range $t \geq 0$ and is of exponential order 'a' as $t \rightarrow \infty$, the Laplace transform of $F(t)$ exists for all $p > a$ ".

(Or)

"If $F(t)$ is a function of class A, the Laplace transform of $F(t)$ exists



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for all $p > a$ ".

Note(2): Conditions in the above theorem are sufficient but not necessary for the existence of the Laplace transform. If these conditions are satisfied, the Laplace transform must exist.

If these conditions are not satisfied, the Laplace transform may or may not exist.

For eg: Consider the function

$$F(t) = \frac{1}{\sqrt{t}}$$

Here $F(t) \rightarrow \infty$ as $t \rightarrow 0$ from the right. Thus the function $F(t)$ is not piece wise continuous on every finite interval in the range $t \geq 0$.

But $F(t)$ is integrable from 0 to any positive value t_0 .

Also $|F(t)| < M e^{\alpha t}$

for all $t > 1$ with $M=1$ and $\alpha=0$.

$$\begin{aligned} \text{Now } L\{F(t)\} &= \int_0^\infty e^{-pt} F(t) dt \\ &= \int_0^\infty e^{-pt} \frac{1}{\sqrt{t}} dt \quad \text{which} \end{aligned}$$

(i) converge
for $p > 0$.

$$= \frac{2}{\sqrt{p}} \int_0^\infty e^{-x^2} dx \quad \text{putting } pt=x^2 \Rightarrow pt=x^2$$

$$= \frac{2}{\sqrt{p}} \cdot \frac{\sqrt{\pi}}{2} \quad \text{if } \int \frac{1}{2x^2} dx = \frac{1}{2x}$$

$$\left(\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \right) \Rightarrow \int \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{p}} dx$$

Limits of x
if $t=0 \Rightarrow x=0$
 $t=\infty \Rightarrow x=\infty$

$$= \sqrt{\frac{\pi}{p}}, p > 0$$

$\therefore L\left\{ \frac{1}{\sqrt{t}} \right\}$ exists for $p > 0$ even if

$F(t) = \frac{1}{\sqrt{t}}$ is not piece-wise continuous in the range $t \geq 0$.

* Functions of Exponential order:

A function $F(t)$ is said to be of exponential order α as $t \rightarrow \infty$ if there exists a real number M , a number α and a finite number t_0 such that

$$|F(t)| < M e^{\alpha t} \quad \forall t \geq t_0$$

$$(\text{or}) |e^{-\alpha t} F(t)| < M \quad \forall t \geq t_0$$

If a function $F(t)$ is of exponential order α , it is also of β , $\beta > \alpha$ (OR)

A function $F(t)$ is said to be of exponential order α as $t \rightarrow \infty$

if $\lim_{t \rightarrow \infty} e^{-\alpha t} F(t) = \text{finite quantity}$.

* Function of Class A:

A function $F(t)$ is said to be function of class A if (i) it is piecewise (or sectionally) continuous on every finite interval in the range $t \geq 0$.

(ii) $F(t)$ is of exponential order as $t \rightarrow \infty$.

→ Prove that $F(t) = t^n$ is of exponential order as $t \rightarrow \infty$, n being any +ve integer.

$$\text{Sol}': \underset{t \rightarrow \infty}{\text{Lt}} \left\{ e^{-at} F(t) \right\} = \underset{t \rightarrow \infty}{\text{Lt}} e^{-at} t^n, a > 0$$

$$= \underset{t \rightarrow \infty}{\text{Lt}} \frac{t^n}{e^{at}}, (\infty \text{ form})$$

$$= \underset{t \rightarrow \infty}{\text{Lt}} \frac{n!}{a^n t^n} (\text{by L.Hospital rule})$$

$$= \frac{n!}{\infty} = 0$$

$$\therefore \underset{t \rightarrow \infty}{\text{Lt}} e^{-at} t^n = 0 = \text{finite number}$$

$\therefore t^n$ is of exponential order as $t \rightarrow \infty$

$$\text{Note: } |F(t)| = t^n < e^{nt} \forall t > 0.$$

\therefore The given function is of exponential order n.

Ex: Show that t^2 is of exponential order 3.

Sol'': we have

$$\underset{t \rightarrow \infty}{\text{Lt}} \left\{ e^{-at} F(t) \right\} = \underset{t \rightarrow \infty}{\text{Lt}} \left(\frac{t^2}{e^{at}} \right)$$

$$= \underset{t \rightarrow \infty}{\text{Lt}} \frac{2t}{ae^{at}} \quad (\text{By L-Hospital's rule})$$

$$= \underset{t \rightarrow \infty}{\text{Lt}} \frac{2}{a^2 e^{at}} \quad (\text{By L-Hospital's rule})$$

$$= 0, \text{ if } a > 0$$

$\therefore F(t) = t^2$ is of exponential order.

$$\text{Now } |t^2| = t^2 < e^{2t} < e^{3t} \forall t > 0$$

\therefore The given function is of exponential order 3. (\because if $F(t)$ is of exponential order $\alpha = 2$ it is also of $\beta = 3, 3 > 2$)

→ Show that the function e^{t^2} is not of exponential order as $t \rightarrow \infty$

Sol'': we have

$$\underset{t \rightarrow \infty}{\text{Lt}} \left\{ e^{-at} F(t) \right\} = \underset{t \rightarrow \infty}{\text{Lt}} \left\{ e^{-at} e^{t^2} \right\}$$

$$= \underset{t \rightarrow \infty}{\text{Lt}} e^{t(t-a)}$$

$$= \infty \quad \forall a.$$

Hence whatever be the value of a , we cannot find a number M such that $e^{t^2} < M e^{at}$.

\therefore The given function is not of exponential order as $t \rightarrow \infty$.

→ Find the Laplace transform of the function $F(t) = 1$

$$\text{Sol'': we have } L \{ F(t) \} = \int_0^\infty e^{-pt} F(t) dt$$

$$\begin{aligned} \therefore L\{1\} &= \int_0^\infty e^{-pt} \cdot 1 dt \\ &= \left[-\frac{e^{-pt}}{p} \right]_0^\infty \quad (\because e^{-pt} \rightarrow 0 \text{ as } t \rightarrow \infty) \\ &= \frac{1}{p}, \quad p > 0 \end{aligned}$$

Note: Here the condition $p > 0$ is necessary. Since the integral is convergent for $p > 0$ and divergent for $p \leq 0$.

→ Find $L\{t^n\}$, n is +ve integer.

Sol'n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$

$$\begin{aligned} \therefore L\{t^n\} &= \int_0^\infty e^{-pt} t^n dt \\ &\quad \text{Integrating by parts} \\ &= \left[-\frac{1}{p} t^n e^{-pt} \right]_0^\infty + \frac{1}{p} \int_0^\infty n t^{n-1} e^{-pt} dt \\ &= -\frac{1}{p} \underset{t \rightarrow \infty}{\cancel{Lt}} \frac{t^n}{e^{pt}} + 0 + \frac{n}{p} \int_0^\infty e^{-pt} t^{n-1} dt \\ &= 0 + \frac{n}{p} \int_0^\infty e^{-pt} t^{n-1} dt \\ &\quad (\because \underset{t \rightarrow \infty}{\cancel{Lt}} \frac{t^n}{e^{pt}} = 0 \text{ by L-Hospital's rule}) \\ &= \frac{n}{p} \int_0^\infty e^{-pt} t^{n-1} dt \end{aligned}$$

Proceeding similarly, we get

$$\begin{aligned} L\{t^n\} &= \frac{n!}{p^n} \int_0^\infty e^{-pt} dt \\ &= \frac{n!}{p^n} \left[-\frac{e^{-pt}}{p} \right]_0^\infty \\ &= -\frac{n!}{p^n} \left[0 - \frac{1}{p} \right] \\ &= \frac{n!}{p^{n+1}}, \quad p > 0. \end{aligned}$$

↙ show that the Laplace transform of the function

$$F(t) = t^n, \quad -1 < n < 0,$$

exists, although it is not a function

of class A.

Sol'n: Given $F(t) = t^n, -1 < n < 0$

Here $F(t) \rightarrow \infty$ as $t \rightarrow 0$ (for $t > 0$)

i.e., the function is not piecewise continuous on every finite interval

in the range $t > 0$.

we have $\lim_{t \rightarrow \infty} \left\{ e^{-at} F(t) \right\} = \lim_{t \rightarrow \infty} \left(\frac{t^n}{e^{at}} \right)$

$$= \lim_{t \rightarrow \infty} \frac{1}{t^{n-a}}$$

$$= \lim_{t \rightarrow \infty} \frac{1}{t^{m-a}},$$

where $0 < m <$

$$= 0, \text{ if } a > 0. \quad -1 < n < 0$$

$$1 > -n > 0$$

$$1 > m > 0$$

$$\text{Put } m = -n$$

$F(t) = t^n$ is of exponential order.
 Since $F(t) = t^n$ is not sectionally continuous over every finite interval in the range $t \geq 0$.

\therefore It is not a function of class A.
 But t^n is integrable from 0 to any tve number to.

$$\text{Now } L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$$

$$= \int_0^\infty e^{-pt} t^n dt$$

$$= \int_0^\infty e^{-px} \left(\frac{x}{p}\right)^n \frac{1}{p} dx; \quad \text{Putting } pt=x \\ \quad pdt=dx$$

$$\Rightarrow dt = \frac{dx}{p} \\ & \text{& taking } B.D$$

$$= \frac{1}{p} \cdot \frac{1}{p^n} \int_0^\infty e^{-x} x^n dx$$

$$= \frac{1}{p^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$$

(By definition of
Gamma function)

$$= \frac{F(n+1)}{p^{n+1}}, \quad \because F(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

if $n > 0$)

i.e., $n > -1$.

Hence Laplace transform of t^n ,
 $0 > n > -1$ exists, although it is not
 a function of class A.

\rightarrow Find $L\{e^{at}\}$.

$$\underline{\text{Sol'n:}}$$
 Here $L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at} dt$

$$= \int_0^\infty e^{-(p-a)t} dt$$

$$= \left[\frac{-e^{-(p-a)t}}{p-a} \right]_0^\infty, \quad p > a$$

$$= \frac{1}{p-a}$$

\rightarrow Find $L\{\cos at\}$ and hence obtain
 $L(\sin^2 at)$

$$\underline{\text{Sol'n:}}$$
 $L\{\cos at\} = \int_0^\infty e^{-pt} \cos at dt$

$$\begin{aligned} & \left| e^{ax} \sin bx \right. \\ & \left. - e^{ax} (\sin bx - b \cos bx) \right| \\ & \quad a^2 + b^2 \end{aligned}$$

$$= \frac{e^{-pt}}{p^2 + a^2} (p \cos at - a \sin at) \Big|_0^\infty$$

$$= 0 - \left(\frac{-p}{p^2 + a^2} \right), \quad p > 0$$

($\because a \rightarrow \infty$
 $e^{-pt} \rightarrow 0$)

$$= \frac{p}{p^2 + a^2}, \quad p > 0 \quad \text{--- (1)}$$

$$\text{Now } L(\sin^2 at) = L\left\{ \frac{1 - \cos 2at}{2} \right\}$$

$$= \frac{1}{2} L\{1 - \cos 2at\}$$

$$= \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos(2a)t\}$$

$$= \frac{1}{2p} - \frac{1}{2} \frac{p}{p^2 + (2a)^2} \quad (\because \text{by (1)})$$

$$= \frac{1}{2P} - \frac{P}{2(P^2+4a^2)}$$

$$= \frac{2a^2}{P(P^2+4a^2)}$$

→ Find $L\{\cosh at\}$

$$\text{sol'n: } L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2} \cdot \frac{1}{P-a} + \frac{1}{2} \cdot \frac{1}{P+a}$$

$$= \frac{P}{P^2-a^2}, P>a \& P>-a$$

i.e. $P>|a|$
i.e. $|a| < P$.

→ Find (1) $L\{\sin at\}$ (2) $L\{\sinh at\}$

→ Find $L\{\sin t \cos t\}$

$$\text{sol'n: Given } L\{\sin t \cos t\} = L\left\{\frac{1}{2}\sin 2t\right\}$$

$$= \frac{1}{2}L\{\sin 2t\}$$

$$= \frac{1}{2} \cdot \frac{2}{P^2+4}, P>0$$

$$= \frac{1}{P^2+4}, P>0.$$

→ Find (1) $L\{\cosh^2 at\}$

$$(2) L\left\{7e^{2t} + 9e^{3t} + 5\cos t + 7t^3 + 58\sin t + 2\right\}$$

→ Find $L\{F(t)\}$, where $F(t)=\begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

Sol'n: Here $F(t)$ is not defined at

$$t=0, t=1 \& t=2.$$

$$\therefore L\{F(t)\} = \int_0^\infty e^{-Pt} F(t) dt$$

$$= \int_0^1 e^{-Pt} \cdot 0 dt + \int_1^2 e^{-Pt} \cdot t dt + \int_2^\infty e^{-Pt} \cdot 0 dt$$

$$= \int_1^2 e^{-Pt} \cdot t dt$$

$$= \left[-t \frac{e^{-Pt}}{P} \right]_1^2 - \int_1^2 \frac{e^{-Pt}}{-P} dt$$

$$= -\frac{2}{P} e^{-2P} + \frac{e^{-P}}{P} - \left[\frac{e^{-Pt}}{P^2} \right]_1^2$$

$$= -\frac{e^{-2P}}{P} + \frac{e^{-P}}{P} - \frac{e^{-2P}}{P^2} + \frac{e^{-P}}{P^2}$$

$$= \left(\frac{1}{P} + \frac{1}{P^2} \right) e^{-P} - \left(\frac{1}{P^2} + \frac{2}{P} \right) e^{-2P}$$

→ find the L.T. of the function $F(t)$, where $F(t)=\begin{cases} 4, & 0 < t < 1 \\ 3, & t > 1 \end{cases}$

→ Find the L.T. of the function $F(t)$, where $F(t)=\begin{cases} 2t, & 0 \leq t \leq 5 \\ 1, & t > 5 \end{cases}$

→ Find the L.T. of the function $F(t)$, where $F(t)=\begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

→ Find $L\{\sin \sqrt{t}\}$

Sol'n: we have

$$L\{\sin \sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} \dots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} \dots\right\}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$$= L\{t^{\nu_2}\} - \frac{1}{3!} L\{t^{\beta_2}\} + \frac{1}{5!} L\{t^{\gamma_2}\} - \frac{1}{7!} L\{t^{\delta_2}\}$$

+ ...

$$= \frac{T_{3/2}}{p^{3/2}} - \frac{1}{3!} \frac{T_{5/2}}{p^{5/2}} + \frac{1}{5!} \frac{T_{7/2}}{p^{7/2}} - \frac{1}{7!} \frac{T_{9/2}}{p^{9/2}} + \dots$$

$$\left[\because L\{t^n\} = \frac{T_{n+1}}{p^{n+1}} \text{ if } n > -1 \right]$$

$$= \frac{\frac{1}{2}T_2}{p^{3/2}} - \frac{1}{6} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2}T_2}{p^{5/2}} + \frac{1}{120} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}T_2}{p^{7/2}}$$

+ ...

$$\begin{aligned} \therefore T_{n+1} &= nT_n \\ &= n\sqrt{(n-1)+1} \\ &= n(n-1)\sqrt{(n-1)} \\ &= n(n-1)(n-2)\sqrt{(n-2)} \text{ etc} \end{aligned}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{1}{6} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{p^{5/2}} + \frac{1}{120} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{p^{7/2}}$$

+ ...

$$(\because T_2 = \sqrt{\pi})$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} \left[1 - \frac{1}{4p} + \frac{1}{2!} \left(\frac{1}{4p} \right)^2 - \frac{1}{3!} \left(\frac{1}{4p} \right)^3 + \dots \right]$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-\frac{1}{4p}}$$

→ find the LT of the function

$$F(t) = (8\sin t - \cos t)^2$$

→ Find the LT of the function

$$F(t) = \frac{e^{at} - 1}{a}$$

→ Evaluate $L\{F(t)\}$, if
 $F(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$

→ Find $L\{F(t)\}$, if $F(t) = \begin{cases} e^t, & 0 < t \leq 1 \\ 0, & t > 1 \end{cases}$

→ Prove that $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{P}}$.

* → Laplace Transforms of some elementary functions:

$$(1) L\{1\} = \frac{1}{P}, P > 0$$

$$(2) L\{t^n\} = \frac{n!}{P^{n+1}}, P > 0, \text{ where } n \text{ is positive integer.}$$

$$(3) L\{t^n\} = \frac{T_{n+1}}{P^{n+1}}, P > 0, \text{ where } n > -1$$

$$(4) L\{e^{at}\} = \frac{1}{P-a}, P > a$$

$$(5) L\{\sin at\} = \frac{a}{P+a^2}, P > 0$$

$$(6) L\{\cos at\} = \frac{P}{P+a^2}, P > 0$$

$$(7) L\{\sinh at\} = \frac{a}{P^2-a^2}, P > |a| \text{ i.e. } |a| < P$$

$$(8) L\{\cosh at\} = \frac{P}{P^2-a^2}, P > |a| \text{ i.e. } |a| < P$$

Note! If n is positive integer, then

$$T_{n+1} = n!$$

* → First Translation (or) shifting theorem:

$$\text{If } L\{F(t)\} = f(P), \text{ then :}$$

$$L(e^{at} F(t)) = f(P-a).$$

Proof: By definition of Laplace Transform, we have

$$L\{f(t)\} = f(p)$$

$$= \int_0^\infty e^{-pt} F(t) dt$$

$$\therefore f(p-a) = \int_0^\infty e^{-(p-a)t} F(t) dt$$

$$= \int_0^\infty e^{-pt} (e^{at} F(t)) dt$$

$$= L\{e^{at} F(t)\}$$

* Second translation (or shifting)

Theorem:

$$\text{If } L\{F(t)\} = f(p),$$

and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ then

$$L\{G(t)\} = e^{-ap} f(p).$$

Proof: By definition of Laplace transformation, we have.

$$L\{G(t)\} = \int_0^\infty e^{-pt} G(t) dt$$

$$= \int_0^a e^{-pt} G(t) dt + \int_a^\infty e^{-pt} G(t) dt$$

$$= \int_0^a e^{-pt} \cdot 0 dt + \int_a^\infty e^{-pt} F(t-a) dt$$

$$= \int_a^\infty e^{-pt} F(t-a) dt$$

$$= \int_0^\infty e^{-p(a+x)} F(a+x) dx$$

$$= e^{-pa} \int_0^\infty e^{-px} F(x) dx \quad (\because \int_0^\infty e^{-px} F(x) dx)$$

$$= e^{-pa} \int_0^\infty e^{-pt} F(t) dt$$

$$= \int_0^\infty e^{-pt} F(t) dt$$

$$= e^{-pa} L\{F(t)\}$$

$$= e^{-pa} f(p)$$

By Property

of definite

integrals, i.e,

$$\int_a^b f(x) dx = \int_a^b f(t) dt$$

Change of scale Property

Theorem: If $L\{F(t)\} = f(p)$, then

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right).$$

Proof: By definition, we have

$$L\{F(at)\} = \int_0^\infty e^{-pt} F(at) dt$$

Putting $at=x$

$$\Rightarrow t=\frac{x}{a}$$

$$\Rightarrow dt=\frac{1}{a}dx$$

$$= \int_0^\infty e^{-p\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{p}{a}\right)x} F(x) dx$$

$$= \frac{1}{a} \int_a^b e^{-\left(\frac{p}{a}\right)t} F(t) dt$$

$$(\because \int_a^b f(x) dx = \int_a^b f(t) dt)$$

$$= \frac{1}{a} f\left(\frac{p}{a}\right).$$

→ Find $L\{t^3 e^{-3t}\}$ i.e. $L\left\{\frac{e^{-3t} t^3}{e^{at} F(t)}\right\}$

clearly which is in the form $L\{e^{at} F(t)\}$

Sol'n: Now $L\{F(t)\} = L\{t^3\}$

$$= \frac{3!}{p^4} = \frac{6}{p^4} f(p)$$

(say)

from first shifting theorem

$$\mathcal{L}\{e^{at} F(t)\} = f(p-a)$$

$$\mathcal{L}\{t^3 e^{-3t}\} = \frac{6}{(p+3)^4}$$

$$a = -3$$

Find $\mathcal{L}\{e^{-2t} (3\cos 6t - 5\sin 6t)\}$

Sol'n: we have

$$\mathcal{L}\{3\cos 6t - 5\sin 6t\} = 3\mathcal{L}\{\cos 6t\} - 5\mathcal{L}\{\sin 6t\}$$

$$= 3 \cdot \frac{p}{p^2 + 36} - 5 \cdot \frac{6}{p^2 + 36}$$

$$= \frac{3p - 30}{p^2 + 36} = f(p) \text{ say}$$

from first shifting theorem, we have

$$\mathcal{L}\{e^{-2t} (3\cos 6t - 5\sin 6t)\} = f(p+2)$$

$$= \frac{3(p+2) - 30}{(p+2)^2 + 36}$$

$$a = -2$$

$$= \frac{3p - 24}{p^2 + 4p + 40}$$

(OR)

By definition of L.T.

$$\mathcal{L}\{e^{-2t} (3\cos 6t - 5\sin 6t)\}$$

$$= \int_0^\infty e^{-pt} e^{-2t} (3\cos 6t - 5\sin 6t) dt$$

$$= 3 \int_0^\infty e^{-(p+2)t} \cos 6t - 5 \int_0^\infty e^{-(p+2)t} \sin 6t dt$$

$$= \left[3 \frac{e^{-(p+2)t}}{(p+2)^2 + 36} \left[-(p+2)\cos 6t + 6\sin 6t \right] \right]_0^\infty$$

$$= \left[5 \frac{e^{-(p+2)t}}{(p+2)^2 + 36} \left[-(p+2)8\sin 6t - 6\cos 6t \right] \right]_0^\infty$$

$$= 0 + \frac{3(p+2)}{(p+2)^2 + 36} - 5 \left[0 - \frac{(-6)}{(p+2)^2 + 36} \right]$$

$$= \frac{3(p+2)}{(p+2)^2 + 36} - \frac{30}{(p+2)^2 + 36} = \frac{3p - 24}{p^2 + 4p + 40}$$

Find $\mathcal{L}\{e^t (3\sinh 2t - 5\cosh 2t)\}$

Find (1). $\mathcal{L}\{e^t \sinh t\}$ (ii) $\mathcal{L}(e^t (t+3)^2)$

Find $\mathcal{L}\left\{e^{-at} \frac{t^{n-1}}{(n-1)!}\right\}$, where n is +ve integer.

Sol'n: we have

$$\mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\right\} = \frac{(n-1)!}{p^n (n-1)!} \quad (\because \mathcal{L}(t^n) = \frac{n!}{p^{n+1}})$$

$$= \frac{1}{p^n} = f(p), \text{ say}$$

$$\therefore \mathcal{L}\left\{e^{-at} \frac{t^{n-1}}{(n-1)!}\right\} = \frac{1}{(p+a)^n}$$

Applying change of scale Property,

obtain the Laplace transform of

(i) $\sinh 3t$ (ii) $\cosh 3t$

Sol'n: (i) we have $\mathcal{L}\{\sinh t\} = \frac{1}{p^2 - 1}$

$$= f(p), \text{ say}$$

$$\therefore \mathcal{L}\{\sinh 3t\} = \frac{1}{3} f\left(\frac{p}{3}\right) \quad \because \mathcal{L}(F(t)) = f(p)$$

$$\mathcal{L}\{F(at)\} = f\left(\frac{p}{a}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{(P_3)^2 - 1}$$

$$= \frac{3}{P^2 - 9}$$

~~Given~~ $L\{f(t)\} = \frac{P^2 - P + 1}{(2P+1)^2 (P-1)}$

applying the change of scale
property show that

$$L\{F(2t)\} = \frac{P^2 - 2P + 4}{4(P+1)^2 (P-2)}$$

→ Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$

Sol: From Second shifting theorem
we know that if $L\{F(t)\} = f(p)$:

$$\text{and } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{then } L\{G(t)\} = e^{-ap} f(p).$$

Here let $F(t) = e^t$

$$\begin{aligned} \therefore L\{F(t)\} &= L\{e^t\} = \int_0^\infty e^{-pt} e^t dt \\ &= \int_0^\infty e^{-(p-1)t} dt \\ &= \left[\frac{e^{-(p-1)t}}{-(p-1)} \right]_0^\infty \\ &= \frac{1}{p-1}, \quad p > 1 \end{aligned}$$

$$= f(p), \text{ say}$$

$$\text{and } G(t) = \begin{cases} F(t-a) = e^{t-a}, & t > a \\ 0, & t < a \end{cases}$$

$$\therefore L\{G(t)\} = e^{-ap} f(p)$$

$$= \frac{e^{-ap}}{p-1}, \quad p > 1$$

(OR)

$$\text{we have } L\{G(t)\} = \int_0^\infty e^{-pt} G(t) dt$$

$$= \int_0^a e^{-pt} G(t) dt + \int_a^\infty e^{-pt} G(t) dt$$

$$= \int_0^a e^{-pt} (0) dt + \int_a^\infty e^{-pt} e^{t-a} dt$$

$$= e^{-a} \int_a^\infty e^{-(p-1)t} dt$$

$$= e^{-a} \left[\frac{e^{-(p-1)t}}{-(p-1)} \right]_a^\infty$$

$$= e^{-a} \left[0 - \frac{e^{-(p-1)a}}{-(p-1)} \right]$$

$$= \frac{e^{-ap}}{p-1}$$

→ find $L\{F(t)\}$, where

$$F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

→ find $L\{G(t)\}$, where

$$F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$$

Laplace Transform of Derivatives:

Theorem: Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order ' a ' as $t \rightarrow \infty$ and if $F'(t)$ is of class A, then Laplace transform of the derivative $F'(t)$ exists when $p > a$, and $L\{F'(t)\} = PL\{F(t)\} - F(0)$.

Proof:

Case(i) $F'(t)$ is continuous for all $t \geq 0$,

then

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t) dt \quad (1)$$

$$= [e^{-pt} F(t)]_0^\infty + \int_0^\infty p e^{-pt} F(t) dt$$

(Integrating by parts)

$$= \lim_{t \rightarrow \infty} -e^{-pt} F(t) - F(0) + p \int_0^\infty e^{-pt} F(t) dt$$

$$L\{F'(t)\} = \lim_{t \rightarrow \infty} -e^{-pt} F(t) - F(0) + PL\{F(t)\} \quad (2)$$

Since $F(t)$ is continuous for all $t \geq 0$ and is of exponential order ' a ' as $t \rightarrow \infty$.

$|F(t)| \leq M e^{at} \forall t \geq 0$ and for some

constants a and M ,

$$\text{we have } |e^{-pt} F(t)| = e^{-pt} |F(t)|$$

$$\leq e^{-pt} M e^{at}$$

$$= M e^{-(p-a)t} \xrightarrow{t \rightarrow \infty} 0$$

as $t \rightarrow \infty$ if $p > a$

$$\therefore \lim_{t \rightarrow \infty} e^{-pt} F(t) = 0 \text{ for } p > a$$

∴ from (2) we conclude that $L\{F'(t)\}$ exists and $L\{F'(t)\} = PL\{F(t)\} - F(0)$.

Case-2: $F'(t)$ is merely piecewise continuous, the integral (1) may be broken as the sum of integrals in different ranges from 0 to ∞ such that $F'(t)$ is continuous in each of such parts.

Then proceeding as in Case(i), we get

$$L\{F'(t)\} = PL\{F(t)\} - F(0)$$

$$\begin{aligned} L\{F'(t)\} &= \int_0^\infty e^{-pt} F'(t) dt \\ &= \int_0^{t_0} e^{-pt} F'(t) dt + \int_{t_0}^{t_1} e^{-pt} F'(t) dt + \dots \\ &\quad + \int_{t_n}^\infty e^{-pt} F'(t) dt, \quad [\text{In particular}] \\ &\quad L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t) dt \end{aligned}$$

$$\begin{aligned} &= \left[e^{-pt} F(t) \right]_0^{t_0} + p \int_0^{t_0} e^{-pt} F(t) dt + \dots \\ &\quad + \left[e^{-pt} F(t) \right]_{t_0}^\infty + p \int_{t_0}^\infty e^{-pt} F(t) dt \end{aligned}$$

$$= e^{-pt_0} F(t_0) - F(0) + \lim_{t \rightarrow \infty} e^{-pt} F(t) - e^{-pt_0} F(t_0)$$

$$+ p \int_0^{t_0} e^{-pt} F(t) dt + p \int_{t_0}^\infty e^{-pt} F(t) dt$$

$$= -F(0) + p \int_0^\infty e^{-pt} F(t) dt$$

$$= PL\{F(t)\} - F(0)$$

Note 1: If $F(t)$ fails to be continuous at $t=0$ but $\lim_{t \rightarrow 0^+} F(t) = F(0+)$ exists.

$$= F(0+)$$

[i.e., $-F(0+)$ is not equal to $-F(0)$, which may or may not exist]

$$\text{then } L\{F'(t)\} = PL\{F(t)\} - F(0+)$$

Note 2: If $F(t)$ fails to be continuous at $t=a$, then

$$L\{F'(t)\} = PL\{F(t)\} - F(0) - e^{-ap} [F(a+0) - F(a-0)]$$

where $F(a+0)$ and $F(a-0)$ are the limits of F at $t=a$, as t approaches a from the right and from the left respectively.

The quantity $F(a+0) - F(a-0)$ is called the jump discontinuity at $t=a$.

$$\begin{aligned} \text{Proof: } L\{F'(t)\} &= \int_0^\infty e^{-pt} F'(t) dt \\ &= \int_0^a e^{-pt} F'(t) dt + \int_a^\infty e^{-pt} F'(t) dt \\ &= \left[e^{-pt} F(t) \right]_0^a + P \int_a^\infty e^{-pt} F(t) dt + \\ &\quad \left[e^{-pt} F(t) \right]_a^\infty + P \int_a^\infty e^{-pt} F(t) dt \\ &= e^{-pa} F(a-0) - F(0) + P \int_0^\infty e^{-pt} F(t) dt \\ &\quad + \lim_{t \rightarrow \infty} t e^{-pt} F(t) - e^{-pa} F(a+0) + P \int_a^\infty e^{-pt} F(t) dt \\ &\quad (\because \lim_{t \rightarrow \infty} t e^{-pt} F(t) = 0 \text{ by case (1)}) \end{aligned}$$

$$\begin{aligned} &= e^{-ap} F(a-0) - e^{-pa} F(a+0) + 0 \\ &\quad - F(0) + P \int_0^\infty e^{-pt} F(t) dt \\ &= e^{-ap} [F(a-0) - F(a+0)] - F(0) + \\ &\quad PL\{F(t)\} \\ &= PL\{F(t)\} - F(0) + e^{-ap} [F(a-0) - F(a+0)] \end{aligned}$$

Note 3: For more than one discontinuity of the function $F(t)$, appropriate modifications can be made.

→ Laplace Transform of the nth order derivative of $F(t)$:

Let $F(t)$ and its derivatives $F'(t), F''(t), \dots, F^{n-1}(t)$ be continuous functions for all $t \geq 0$ and be of exponential orders as $t \rightarrow \infty$ and if $F^n(t)$ is of class A, then Laplace transform of $F^n(t)$ exists when $p > a$, and is given by

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \dots - F^{n-1}(0)$$

Proof: From the above theorem, we have

$$L\{F'(t)\} = PL\{F(t)\} - F(0) \quad \text{--- (1)}$$

Applying the result (1) to the 2nd order derivative $F''(t)$.

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$$\begin{aligned} L\{F''(t)\} &= PL\{F'(t)\} - F'(0) \\ &= P[PL\{F(t)\} - F(0)] - F'(0) \quad (\because \text{by (1)}) \end{aligned}$$

$$= P^2 L\{F(t)\} - PF(0) - F'(0) \quad \text{--- (2)}$$

Again applying (1) to the 3rd order derivative $F'''(t)$, we have

$$\begin{aligned} L\{F'''(t)\} &= PL\{F''(t)\} - F''(0) \\ &= P^2 L\{F(t)\} - PF(0) - F'(0) \\ &\quad - F''(0) \quad (\because \text{by (2)}) \end{aligned}$$

$$= P^3 L\{F(t)\} - P^2 F(0) - PF'(0) \\ - F''(0)$$

Proceeding similarly, we get

$$\begin{aligned} L\{F^n(t)\} &= P^n L\{F(t)\} - P^{n-1} F(0) - P^{n-2} F'(0) \\ &\quad \dots \dots \dots - F^{n-1}(0). \end{aligned}$$

* Initial-value Theorem:
Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \rightarrow \infty$ and if $F'(t)$ is of class A, then $\lim_{t \rightarrow 0} f(t) = \lim_{P \rightarrow \infty} PL\{F(t)\}$.

Proof: we know that

$$\begin{aligned} L\{F'(t)\} &= PL\{F(t)\} - F(0) \\ \Rightarrow \int_0^\infty e^{-pt} F'(t) dt &= PL\{F(t)\} - F(0) \quad \text{--- (1)} \end{aligned}$$

Taking limit as $P \rightarrow \infty$ in (1)

$$\begin{aligned} \lim_{P \rightarrow \infty} \int_0^\infty e^{-pt} F'(t) dt &= \lim_{P \rightarrow \infty} [PL\{F(t)\} - F(0)] \\ &\quad \text{--- (2)} \end{aligned}$$

Since $F'(t)$ is of class A, i.e., $F'(t)$ is sectionally continuous and of exponential order.

$$\therefore \lim_{P \rightarrow \infty} \int_0^\infty e^{-pt} F'(t) dt = 0$$

∴ from (2)

$$0 = \lim_{P \rightarrow \infty} PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{P \rightarrow \infty} PL\{F(t)\} = F(0)$$

$$\Rightarrow \lim_{P \rightarrow \infty} PL\{F(t)\} = \lim_{t \rightarrow 0} F(t)$$

$$\text{Or } \lim_{t \rightarrow 0} F(t) = \lim_{P \rightarrow \infty} PL\{F(t)\}$$

* Final-value Theorem:

Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \rightarrow \infty$ and if $F'(t)$ is of class A, then $\lim_{t \rightarrow \infty} F(t) = \lim_{P \rightarrow \infty} PL\{F(t)\}$.

Proof: we know that

$$\begin{aligned} L\{F'(t)\} &= PL\{F(t)\} - F(0) \\ \Rightarrow \int_0^\infty e^{-pt} F'(t) dt &= PL\{F(t)\} - F(0) \quad \text{--- (1)} \end{aligned}$$

Taking limit as $P \rightarrow 0$ in (1), we get

$$\lim_{P \rightarrow 0} \int_0^\infty e^{-pt} F'(t) dt = \lim_{P \rightarrow 0} [PL\{F(t)\} - F(0)]$$

$$\Rightarrow \int_0^t F'(t) dt = \int_0^t PL\{F(t)\} - F(0)$$

$$\Rightarrow [F(t)]_0^\infty = \int_0^\infty PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t \rightarrow \infty} F(t) - F(0) = \int_0^\infty PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t \rightarrow \infty} F(t) = \int_0^\infty PL\{F(t)\}$$

* Laplace - Transform of Integrals:

Theorem: If $F(t)$ is piecewise continuous and satisfies $|F(t)| \leq M e^{at}$ for all $t \geq 0$ for some constants 'a' and M, then

$$L\left\{\int_0^t F(x) dx\right\} = \frac{1}{P} L\{F(t)\} = \frac{1}{P} f(P).$$

(P > 0, P > a)

Proof: Let $f(t)$ be piecewise continuous such that
 $|F(t)| \leq M e^{at}$ for some constants
 $\rightarrow \text{①} \quad a < M.$

If ① holds for some negative values of 'a' then it also holds for positive value of 'a'

\therefore Suppose that 'a' is +ve.

$$\text{Let } G(t) = \int_0^t F(x) dx$$

$G(t)$ is continuous and

$$\begin{aligned} |G(t)| &= \left| \int_0^t F(x) dx \right| \\ &\leq \int_0^t |F(x)| dx \\ &\leq \int_0^t M e^{ax} dx \quad (\because \text{by ①}) \\ &= \frac{M(e^{at} - 1)}{a}, \quad a > 0 \end{aligned}$$

$\therefore |G(t)| \leq \frac{M}{a}(e^{at} - 1), \quad a > 0.$

Also $G'(t) = F(t)$,
except for points at which $F(t)$ is discontinuous.
 $\therefore G'(t)$ is piecewise continuous on each finite interval and is of exponential order.

\therefore By existence theorem, Laplace transform of $G'(t)$ exists.

$$\begin{aligned} \therefore L(G'(t)) &= PL\{G'(t)\} - G'(0) \\ &= PL\{G(t)\} \quad (\because G(0) = \int_0^0 F(x) dx) \end{aligned}$$

$$\Rightarrow L\{G(t)\} = \frac{1}{P} L(G'(t)) \Rightarrow G(0) = 0$$

$$\Rightarrow L\left\{\int_0^t F(x) dx\right\} = \frac{1}{P} L\{F(t)\}$$

* Multiplication by powers of t:

Multiplication by t

Theorem: If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$ then

$$L\{tF(t)\} = -f'(p)$$

Proof: We have

$$f(p) = L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$$

$$\therefore f'(p) = \frac{d}{dp} \int_0^\infty e^{-pt} F(t) dt$$

$$= \int_0^\infty \frac{d}{dp} \left\{ e^{-pt} F(t) dt \right\} + 0 \quad (\text{By Leibnitz's rule})$$

$$= - \int_0^\infty te^{-pt} F(t) dt$$

$$= - \int_0^\infty e^{-pt} \{tF(t)\} dt$$

$$= -L\{tF(t)\}$$

$$\therefore L\{tF(t)\} = -f'(p)$$

(or)

$$L\{tF(t)\} = (-1) \frac{d}{dp} f(p)$$

Differentiation
of a function
under an
integral sign.

$$\frac{d}{da} \int_A^B F(x,t) dx$$

$$= \int_A^B \frac{d}{dx} \{F(x,t)\} dx$$

$$+ F(x,t) \Big|_{x=A}^{x=B} \frac{db}{dx}$$

$$- F(x,t) \Big|_{x=A}^{x=B} \frac{da}{dx}$$

where A & B are
functions of x or
constants.

Multiplication by t^n :

If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$ then

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p),$$

where $n = 1, 2, 3, \dots$

Division by t:

Theorem: If $L\{F(t)\} = f(p)$, then

$$L\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(x) dx \quad \text{provided}$$

$\lim_{t \rightarrow 0} \frac{1}{t} F(t)$ exists.

Proof: Let $G(t) = \frac{1}{t} F(t)$
 $\Rightarrow F(t) = t G(t)$

Apply L.T. on both sides

$$L\{F(t)\} = L\{tG(t)\}$$

$$\Rightarrow f(p) = \frac{d}{dp} L\{G(t)\}$$

$$\therefore L\{tF(t)\} = -f'(p)$$

Integrating both sides w.r.t p from

p to ∞ , we get

$$\int_p^\infty f(p) dp = - \left[L\{G(t)\} \right]_p^\infty = - \left[\int_0^\infty e^{-pt} G(t) dt \right]_p^\infty$$

$$\Rightarrow \int_p^\infty f(p) dp = - \cancel{t} \left[\int_0^\infty e^{-pt} G(t) dt \right] +$$

$$+ \cancel{t} \left[\int_p^\infty e^{-pt} G(t) dt \right]$$

$$= 0 + \int_0^\infty e^{-pt} G(t) dt$$

$$- \left[\cancel{t} L\{G(t)\} \right]$$

$$= dt \int_{p \rightarrow \infty}^\infty e^{-pt} G(t) dt = 0$$

$$= L\{G(t)\}$$

$$\Rightarrow L\{G(t)\} = \int_p^{\infty} f(p)dp = \int_p^{\infty} f(x)dx$$

i.e,

$$L\left\{\frac{1}{t} F(t)\right\} = \int_p^{\infty} f(x)dx$$

(By property of definite integral)

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$L\{ae^{at}\} = PL\{e^{at}\} - 1$$

$$\Rightarrow aL\{e^{at}\} = PL\{e^{at}\} - 1$$

$$\Rightarrow (P-a)L\{e^{at}\} = 1$$

$$\Rightarrow L\{e^{at}\} = \frac{1}{P-a}$$

Problems:

→ Using the Laplace transform of derivative of $F(t)$ i.e,

$$L\{F^n(t)\} = P^n L\{F(t)\} - P^{n-1} F(0) - P^{n-2} F'(0) - \dots - P^{n-1} F'(0).$$

Show that (1) $L\{t\} = \frac{1}{P^2}$

(2) $L\{e^{at}\} = \frac{1}{P-a}$ (3) $L\{-as\sin at\} = \frac{a^2}{P+a^2}$

Sol'n: (1) we have

$$L\{F'(t)\} = PL\{F(t)\} - F(0) = ①$$

Here let $F(t) = t$ then $F'(t) = 1$
and $F(0) = 0$

∴ from ①

$$L\{1\} = PL\{t\} = 0.$$

$$L\{t\} = \frac{1}{P} L\{1\}$$

$$= \frac{1}{P} \cdot \frac{1}{P} \quad (\because L\{1\} = \frac{1}{P})$$

$$= \frac{1}{P^2}, \quad P > 0.$$

(2) Let $F(t) = e^{at}$

then $F'(t) = ae^{at}$ and $F(0) = 1$

∴ from (1)

(3) Let $F(t) = -as\sin at$

$$F'(t) = -a^2 s\sin at \text{ and } F''(t) = a^3 s\sin at$$

Also $F(0) = 0$ and $F'(0) = -a^2$

We know that

$$L\{F''(t)\} = P^2 L\{F(t)\} - PF(0) - F'(0)$$

$$\Rightarrow L\{a^3 s\sin at\} = P^2 L\{-as\sin at\} - P(0) - (-a^2)$$

$$\Rightarrow a^2 L\{-as\sin at\} = P^2 L\{-as\sin at\} + a^2$$

$$\Rightarrow P^2 L\{-as\sin at\} - a^2 L\{-as\sin at\} = -a^2$$

$$\Rightarrow P^2 L\{-as\sin at\} + a^2 L\{-as\sin at\} = -a^2$$

$$\Rightarrow (P^2 + a^2) L\{-as\sin at\} = -a^2$$

$$\Rightarrow L\{-as\sin at\} = \frac{-a^2}{P^2 + a^2} \quad ②$$

Note: from equation ②

$$L\{\sin at\} = \frac{a}{P^2 + a^2}$$

→ Find $L\{t\cos at\}$, It is in the form of

$$L\{t F(t)\}.$$

Sol'n: Since $L\{\cos at\} = \frac{P}{P^2 + a^2} = \frac{P}{P^2 + a^2} = (say f(P)),$
 $P > 0$

$$\begin{aligned} \therefore L\{t \cos at\} &= -\frac{d}{dp} L\{\cos at\} \\ &= -\frac{d}{dp} \left(\frac{P}{P^2+a^2} \right) \quad [\because \text{if } L\{f(t)\}=f(p) \\ &\quad \text{then } L\{tf(t)\} = -\frac{d}{dp} f(p)] \\ &= t \frac{P^2+a^2 - P(2P)}{(P^2+a^2)^2} \\ &= \frac{a^2-P^2}{(P^2+a^2)^2} = \frac{P^2-a^2}{(P^2+a^2)^2} \end{aligned}$$

- \rightarrow find (i) $L\{t^2 \sin at\}$ (ii) $L\{t^2 e^{at}\}$
 (iii) $L\{t^3 \cos at\}$ (iv) $L\{t(3 \sin 2t - 2 \cos 2t)\}$
 (v) $L\{\sin at - at \cos at\}$

\rightarrow show that

$$L\{t^n e^{at}\} = \frac{n!}{(P-a)^{n+1}}, \quad P>a$$

Solⁿ: Since $L\{e^{at}\} = \frac{1}{P-a}$ (say $f(P)$),
 $\Rightarrow P>a$

$$\therefore L\{t^n e^{at}\} = (-1)^n \frac{d^n}{dp^n} f(p)$$

$$= (-1)^n \frac{d^n}{dp^n} \left(\frac{1}{P-a} \right)$$

$$= (-1)^n \frac{(-1)^n n!}{(P-a)^{n+1}}$$

$$= \frac{n!}{(P-a)^{n+1}}, \quad P>a$$

\rightarrow show that $L\left\{\frac{\cos \sqrt{P}}{\sqrt{P}}\right\} = \frac{\sqrt{\pi}}{\sqrt{P}} e^{-\frac{1}{4}P}$

Solⁿ: Let $F(t) = \sin \sqrt{P}$

then $F'(t) = \frac{\cos \sqrt{P}}{\sqrt{P}}$ and $F(0)=0$

$$\therefore \text{From } L\{F'(t)\} = PL\{F(t)\} - F(0)$$

$$\Rightarrow L\left\{\frac{\cos \sqrt{P}}{\sqrt{P}}\right\} = PL\{\sin \sqrt{P}\} - 0$$

$$\Rightarrow L\left\{\frac{\cos \sqrt{P}}{\sqrt{P}}\right\} = 2P L\{\sin \sqrt{P}\}$$

$$= 2P \frac{\sqrt{\pi}}{2P^{3/2}} e^{-\frac{1}{4}P} \quad [\text{w.r.t. } L(\sin \sqrt{P})]$$

$$= \frac{\sqrt{\pi}}{\sqrt{P}} e^{-\frac{1}{4}P} \quad = \frac{\sqrt{\pi}}{2P^{3/2}} e^{-\frac{1}{4}P}$$

$$= \frac{\sqrt{\pi}}{P} e^{-\frac{1}{4}P}$$

\rightarrow Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(P)$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Solⁿ: Let $F(t) = \sin t$ [Given problem is in the form of $L\left\{\frac{F(t)}{t}\right\}$]
 Now $L_t \frac{F(t)}{t} = \lim_{t \rightarrow 0} \frac{\sin t}{t}$ check if $\lim_{t \rightarrow 0} \frac{f(t)}{t}$

since $L\{\sin t\} = \frac{1}{P^2+1} = f(P)$, say exist or not.]

\therefore from $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x) dx$, we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{x^2+1} dx \quad \left[\begin{array}{l} \because f(p) = \frac{1}{P^2+1} \\ \Rightarrow f(x) = \frac{1}{x^2+1} \end{array} \right]$$

$$= \left[\tan^{-1} x \right]_p^\infty$$

$$= \left[\frac{\pi}{2} - \tan^{-1} p \right] \quad \left[\begin{array}{l} \because \text{If } x \neq p, x \neq 0 \\ \text{then } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \end{array} \right]$$

$$= \cot^{-1} p$$

$$\begin{aligned}
 &= \tan^{-1} \frac{1}{P} \quad [\because \text{Let } \cot^{-1} P = x] \\
 &= f(P) \text{ say} \\
 &\Rightarrow P = \cot x \\
 &\Rightarrow \tan x = \frac{1}{P} \\
 &\Rightarrow x = \tan^{-1} \left(\frac{1}{P} \right) \\
 \text{Now } L \left\{ \frac{\sin at}{t} \right\} &= aL \left\{ \frac{\sin(at)}{at} \right\} \\
 &= a \cdot \frac{1}{a} \tan^{-1} \frac{1}{(P/a)} \\
 &= \tan^{-1} \frac{a}{P} \quad [\because L\{f(at)\}] \\
 &= \frac{1}{a} f_i(P/a)
 \end{aligned}$$

Since $L\{\cos at\} = \frac{P}{P^2+a^2} = f(P)$, say

we have $L\left\{\frac{\cos at}{t}\right\} = \int_0^\infty \frac{x}{x^2+a^2} dx$

$$= \left[\frac{1}{2} \log(x^2+a^2) \right]_0^\infty$$

$$= \frac{1}{2} \lim_{x \rightarrow \infty} \log(x^2+a^2) - \frac{1}{2} \log(a^2)$$

which does not exist.

($\because \lim_{x \rightarrow \infty} \log(x^2+a^2)$ is infinite)

Hence $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

\therefore If $L\{F(t), t \rightarrow P\} = f(P)$. Show that

$$L\left\{ \int_0^t \frac{F(u)}{u} du, t \rightarrow P \right\} = \frac{1}{P} \int_0^\infty f(y) dy$$

Hence show that

$$L\left\{ \int_0^t \frac{\sin u}{u} du, t \rightarrow P \right\} = \frac{\cot^{-1} P}{P}$$

Sol'n: From the Laplace transform of integrals

we know that $L\left\{ \int_0^t F(u) du \right\} = \frac{1}{P} f(P)$ ①

where $f(P) = L\{F(t)\}$.

$$\begin{aligned}
 &\text{Let } G(t) = \frac{F(t)}{t} \\
 \therefore L\left\{ G(t) \right\} &= L\left\{ \frac{F(t)}{t} \right\} \quad [\because \text{Division by } t] \\
 &= \int_0^\infty f(y) dy \quad \int \left(\frac{F(t)}{t} \right) dt = \int f(x) dx \\
 &= g(P), \text{ say}
 \end{aligned}$$

\therefore from ①

$$L\left\{ \int_0^t G(u) du \right\} = \frac{1}{P} g(P)$$

$$\Rightarrow L\left\{ \int_0^t \frac{F(u)}{u} du \right\} = \frac{1}{P} \int_0^\infty f(y) dy \quad \text{--- ②}$$

Deduction:

Let $F(t) = \sin t$

so that $f(P) = L\{\sin t\} = \frac{1}{P^2+1}$

\therefore from ②

$$\begin{aligned}
 L\left\{ \int_0^t \frac{\sin u}{u} du \right\} &= \frac{1}{P} \int_0^\infty \frac{1}{y^2+1} dy \\
 &= \frac{1}{P} \left[\tan^{-1} y \right]_0^P \\
 &= \frac{1}{P} \left[\frac{\pi}{2} - \tan^{-1} P \right] \\
 &= \frac{1}{P} \cot^{-1} P
 \end{aligned}$$

\rightarrow Prove that $\int_0^\infty \frac{F(t)}{t} dt = \int_0^\infty f(x) dx$,

provided that the integral converges.

Sol'n: we have $L\left\{ \frac{F(t)}{t} \right\} = \int_0^\infty f(x) dx$ (Division by t)

$$\Rightarrow \int_0^\infty e^{-pt} \frac{F(t)}{t} dt = \int_0^p f(x)dx + \int_p^\infty f(x)dx$$

$$= - \int_0^p f(x)dx + \int_0^\infty f(x)dx$$

i.e.

Taking limit as
 $P \rightarrow 0^+$, (assuming the integral converges)

$$\lim_{P \rightarrow 0^+} \int_0^p \frac{e^{-pt} F(t)}{t} dt$$

$$\lim_{P \rightarrow 0^+} \int_0^p \left[-f(x)dx + \int_0^\infty f(x)dx \right]$$

$$\text{i.e. } - \int_0^\infty f(x)dx + \int_0^\infty f(x)dx$$

we have

$$\int_0^\infty \frac{F(t)}{t} dt = \int_0^\infty f(x)dx$$

\Rightarrow Show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Sol'n: Let $F(t) = \sin t$

$$f(p) = L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2 + 1}$$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-pt} \frac{\sin t}{t} dt$$

$$\left[\because L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty f(x)dx \right]$$

$$= \int_0^p f(x)dx$$

$$= \int_p^\infty \frac{1}{1+x^2} dx$$

$$= \left[\tan^{-1}(x) \right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

Taking limit as $p \rightarrow 0$,

We have $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Evaluation of Integrals:

$$\text{If } L\{F(t)\} = f(p)$$

$$\text{i.e., } \int_0^\infty e^{-pt} F(t)dt = f(p)$$

taking limit as $p \rightarrow 0$, we have

$$\int_0^\infty F(t)dt = f(0)$$

assuming that the integral is convergent.

\Rightarrow Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$

Sol'n: Let $F(t) = e^{-at} - e^{-bt}$

$$\therefore f(p) = L\{F(t)\} = L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{p+a} - \frac{1}{p+b}$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(x)dx$$

$$\Rightarrow \int_0^\infty e^{-pt} \left(\frac{e^{-at} - e^{-bt}}{t} \right) dt = \int_0^\infty \left(\frac{1}{x+a} - \frac{1}{x+b} \right) dx$$

$$= \left[\log(x+a) - \log(x+b) \right]_p^\infty$$

$$= \log \left(\frac{1+\frac{a}{x}}{1+\frac{b}{x}} \right)_p^\infty$$

$$= 0 - [\log(p+a) - \log(p+b)]$$

$$= \log \left(\frac{p+b}{p+a} \right)$$

Taking limit as $p \rightarrow 0$, we have

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a}$$

$$\rightarrow \text{show that } \int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt = \log 2$$

$$\rightarrow \text{Evaluate } \int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$$

$$\rightarrow \text{show that } \int_0^\infty t e^{-3t} \sin t dt = \frac{3}{50}$$

Sol'n: Given $\int_0^\infty t e^{-3t} \sin t dt =$

$$= \int_0^\infty e^{-3t} (t \sin t) dt.$$

which is in the form of

$$\int_0^\infty e^{-pt} (t \sin t) dt = L\{t \sin t\} \text{ where } P=3$$

$$\text{Since } L\{t \sin t\} = \frac{1}{P^2+1} = f(P) \text{ say}$$

$$\text{and. } L\{t \sin t\} = (-1) \frac{d}{dp} \frac{1}{P^2+1}$$

$$(\because L\{t F(t)\} = (-1) f'(P))$$

$$(or) \int_0^\infty e^{-pt} t \sin t dt = (-1) \frac{2p}{(P^2+1)^2}$$

Putting $P=3$, we have

$$\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$$

$$\Rightarrow \int_0^\infty t e^{-3t} \sin t dt = \frac{3}{50}$$

$$\rightarrow \text{show that } \int_0^\infty t e^{-t} \cos t dt = \frac{8}{25}$$

$$\underline{\text{Sol'n:}} \text{ we have } L\{t \cos t\} = \frac{d}{dp} L\{\cos t\}$$

$$\Rightarrow \int_0^\infty e^{-pt} t \cos t dt = \frac{d}{dp} \left(\frac{p}{p^2+1} \right)$$

$$= - \frac{[p^2+1-p(2p)]}{(p^2+1)^2}$$

$$= \frac{p^2-1}{(p^2+1)^2}$$

Putting $P=2$, we get

$$\int_0^\infty t e^{-2t} \cos t dt = \frac{3}{25}$$

$$\rightarrow \text{Prove that } \int_0^\infty t^3 e^{-t} \sin t dt = 0.$$

Periodic functions!

Let F be a periodic function with period $T > 0$, that is $F(u+T) = F(u)$, $F(u+2T) = F(u)$.

etc. then

$$L\{F(t)\} = \frac{\int_0^\infty e^{-pt} F(t) dt}{1-e^{-pT}}$$

$$\underline{\text{Proof:}} \text{ we have } L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt \\ = \int_0^T e^{-pt} F(t) dt + \int_0^{2T} e^{-pt} F(t) dt + \int_0^{3T} e^{-pt} F(t) dt + \dots \quad (1)$$

Putting $t=u+T$ in
2nd integral $\Rightarrow dt = du$
3rd integral & so on.

limits: If $t=T \Rightarrow u=0$
 $t=2T \Rightarrow u=T$
 $t=3T \Rightarrow u=2T$

$$\therefore \text{from (1)} \quad L\{F(t)\} = \int_0^T e^{-pt} F(t) dt + \int_0^T e^{-p(u+T)} F(u+T) du \\ + \int_0^T e^{-p(u+2T)} F(u+2T) du + \dots$$

In the'	$\int_0^T e^{-pu} F(u) du + e^{-pT} \int_0^T e^{-pu} F(u) du$
1st integral	$= \int_0^T e^{-pu} F(u) du + e^{-2pT} \int_0^T e^{-pu} F(u) du + \dots$
if $p \neq 0$	$= \left[1 + e^{-pT} + e^{-2pT} + \dots \right] \int_0^T e^{-pu} F(u) du$
	$\downarrow (1-x)^{-1}$

* A real function $f: A \rightarrow \mathbb{R}$ is said to be a periodic function if \exists a positive real no. P such that $f(x+P)=f(x), \forall x \in A$.

The least +ve real number P such that $f(x+P)=f(x), \forall x \in A$ is called period off.

Ex: $f(x) = \sin x$ is a periodic function with Period 2π .
 → for $\cos x$ is 2π , → for $\tan x$ is π .

Some Special Functions:

The Sine and Cosine Integrals:

The sine and cosine integrals, denoted by $S_i(t)$ and $C_i(t)$ respectively are defined by

$$S_i(t) = \int_0^t \frac{\sin u}{u} du$$

$$\text{and } C_i(t) = \int_t^\infty \frac{\cos u}{u} du$$

The Error Function:

The Error function, denoted by $\operatorname{erf}(t)$, is defined by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

The Gamma Function:

If $n > 0$, the Gamma function is defined by

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du$$

The Unit Step Function (also called Heaviside's Unit Function):

The unit step function, denoted by

H(t-a) is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

The Exponential Integral:

The exponential integral is defined by $E_i(t) = \int_t^\infty \frac{e^{-u}}{u} du$

The Bessel Function:

Bessel function of order n is defined by

$$J_n(t) = \frac{4^n}{2^n n! (2n+1)} \left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2 \cdot 4(2n+2)(2n+4)} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! r!(n+r+1)} \left(\frac{t}{2}\right)^{n+2r}$$

$$J_n(t) = \frac{(-1)^0}{1! 1!} \left(\frac{t}{2}\right)^n + \frac{(-1)^1}{1! 2!} \left(\frac{t}{2}\right)^{n+2}$$

$$+ \frac{(-1)^2}{2! 3!} \left(\frac{t}{2}\right)^{n+4} + \dots$$

$$= \frac{1}{1!} \cdot \frac{t^n}{2^n} + \frac{1}{(n+1)1!} \frac{t^{n+2}}{2^2 2^n} +$$

$$\frac{1}{2(n+2)(n+1)1!} \frac{t^{n+4}}{2^4 2^n} + \dots$$

$$= \frac{t^n}{2^n n! (n+1)} \left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2 \cdot 4(2n+2)(2n+4)} \right]$$

Laguerre Polynomial:

Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (e^{-t} t^n), n=0, 1, 2, \dots$$

→ show that

$$(i) L\{\sinh at \cosh at\} = \frac{a(p^2 - 2a^2)}{p^4 + 4a^4}$$

$$\text{and } (ii) L\{\sinh at \sinh at\} = \frac{2a^2 p}{p^4 + 4a^4}$$

$$\underline{\text{Sol'n:}} \quad \text{Since } L\{\sinh at\} = \frac{a}{p^2 - a^2} = f(p), \text{ say}$$

$$\therefore L\{e^{iat} \sinh at\} = f(p - ia) \quad (\text{by first})$$

Shifting Property

$$\rightarrow \text{Prove that } L\{J_0(at)\} = \frac{1}{\sqrt{1+p^2}}$$

$$\text{and hence deduce that } (i) L\{J_0(at)\} = \frac{1}{\sqrt{p^2 + a^2}}$$

$$(i), L\{t J_0(at)\} = \frac{p}{(p^2 + a^2)^{3/2}}$$

$$(ii), L\{e^{at} J_0(at)\} = \frac{1}{p^2 + 2apt + a^2}$$

$$(iv), \int_0^\infty J_0(at) dt = 1$$

Sol'n: we have

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! T^{n+r+1}} \left(\frac{t}{2}\right)^{n+2r}$$

if $n=0$

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! T^{r+1}} \left(\frac{t}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! r!} \left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

Comparing real and imaginary terms on both sides, we get

$$L\{\sinh at \cosh at\} = \frac{a(p^2 - 2a^2)}{p^4 + 4a^4}$$

$$\text{and } L\{\sinh at \sinh at\} = \frac{2a^2 p}{p^4 + 4a^4}$$

$$= \frac{a}{(p-ia)^2 - a^2}$$

$$= \frac{a}{(p^2 - 2a^2) - 2iap} \quad (P-ia)^2 - a^2 = \\ p^2 - 2iap - a^2 \\ = p^2 - 2a^2 - 2iap \\ = \frac{a(p^2 - 2a^2) + 2iap}{(p^2 - 2a^2)^2 + 4a^2 p^2} \quad (\because \text{by dividing} \\ \text{and multiplying by } (p^2 - 2a^2) + 2iap) \\ = \frac{a(p^2 - 2a^2) + 2ia^2 p}{p^4 - 4a^2 p^2 + 4a^4 + 4a^2 p^2}$$

$$= \frac{a(p^2 - 2a^2) + 2ia^2 p}{p^4 + 4a^4}$$

$$\Rightarrow L\{\sinh at (\cosh at + i \sinh at)\}$$

$$= \frac{a(p^2 - 2a^2)}{p^4 + 4a^4} + i \frac{2a^2 p}{p^4 + 4a^4}$$

$$\Rightarrow L\{\sinh at \cos at\} + i L\{\sinh at \sin at\}$$

$$= \frac{a(p^2 - 2a^2)}{p^4 + 4a^4} + i \frac{2a^2 p}{p^4 + 4a^4}$$

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$$L\{J_0(t)\} = L\{1\} - \frac{1}{2^2} L\{t^2\} + \frac{1}{2^2 \cdot 4^2} L\{t^4\} \quad (\text{ii}) L\{t J_0(at)\} = -\frac{d}{dp} L\{J_0(at)\}$$

$$-\frac{1}{2^2 \cdot 4^2 \cdot 6^2} L\{t^6\} + \dots$$

$$= \frac{1}{p} - \frac{1}{2^2} \cdot \frac{2!}{p^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{4!}{p^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \frac{6!}{p^7}$$

$$= -\frac{d}{dp} \left(\frac{1}{\sqrt{p^2+a^2}} \right) = \frac{p}{(p^2+a^2)^{3/2}}$$

+ ...

$$= \frac{1}{p} \left[1 - \frac{1}{2p^2} + \frac{1}{2^2 \cdot 4^2} \frac{4 \times 3 \times 2 \times 1}{p^4} \right]$$

$$\dots - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{p^6}$$

$$= \frac{1}{p} \left[1 - \frac{1}{2} \left(\frac{1}{p^2} \right) + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{p^2} \right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{p^2} \right)^3 + \dots \right]$$

$$= \frac{1}{p} \left[1 + \frac{1}{p^2} \right]^{-\frac{1}{2}}$$

$$= \frac{1}{p \left(1 + \frac{1}{p^2} \right)^{\frac{1}{2}}} = \frac{1}{p \sqrt{1+p^2}}$$

$$= \frac{1}{\sqrt{1+p^2}}$$

(i) we know that by change of scale property if

$$L\{F(t)\} = f(p), \text{ then } L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$$

$$\therefore J_0(at) = \frac{1}{a} \frac{1}{\sqrt{1+\left(\frac{p}{a}\right)^2}} = \frac{1}{\sqrt{p^2+a^2}}$$

(iii) By first shifting theorem

$$L\{e^{at} F(t)\} = f(p-a) \text{ if } L\{F(t)\} = f(p)$$

$$\Rightarrow L\{e^{-at} F(t)\} = f(p+a)$$

$$\therefore L\{e^{-at} J_0(at)\} = \frac{p+a}{\sqrt{(p+a)^2+a^2}}$$

$$= \frac{p+a}{\sqrt{p^2+2ap+a^2}} = \frac{1}{\sqrt{p^2+a^2}} \quad (\because L\{J_0(t)\})$$

(iv) we have $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

$$\Rightarrow \int_0^\infty e^{-pt} J_0(at) dt = \frac{1}{\sqrt{p^2+a^2}}$$

$$\Rightarrow \int_0^\infty e^{-pt} J_0(t) dt = \frac{1}{\sqrt{p^2+1}} \quad (\text{Putting } a=1)$$

\therefore putting $p=0$, we get

$$\int_0^\infty J_0(t) dt = 1$$

→ Prove that $L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}}$ and

hence deduce that $L\{t J_1(t)\} = \frac{1}{(p^2+1)^{3/2}}$

Soln: we know that

Hint: $J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! t^{n+r+1}} \left(\frac{t}{2}\right)^{n+2r}$

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MATHEMATICS by K. VENKATESA

→ Find the Laplace Transform of $\sin t$ Sol'n: we know that

$$S_i(t) = \int_0^t \frac{\sin u}{u} du$$

$$= \int_0^t \left(1 - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots \right) du$$

$$= t - \frac{t^3}{3(3!)} + \frac{t^5}{5(5!)} - \frac{t^7}{7(7!)} + \dots$$

$$\therefore L\{S_i(t)\} = L(t) - \frac{1}{3(3!)} L\{t^3\} + \frac{1}{5(5!)} L\{t^5\}$$

$$= \frac{1}{t} - \frac{1}{7(7!)} L\{t^7\} + \dots$$

$$= \frac{1}{t} - \frac{1}{3(3!)} \cdot \frac{3!}{P^4} + \frac{1}{5(5!)} \cdot \frac{5!}{P^6}$$

$$= \frac{1}{t} - \frac{1}{7(7!)} \cdot \frac{7!}{P^8} + \dots$$

$$= \frac{1}{P} \left[\frac{1}{t} - \frac{1}{3} \frac{1}{P^3} + \frac{1}{5} \frac{1}{P^5} - \frac{1}{7} \frac{1}{P^7} + \dots \right]$$

By Leibnitz's series

→ find $L\{S_i(t)\}$

$$\text{Sol'n: } L\{S_i(t)\} = L\left\{ \int_0^t \frac{\cos u}{u} du \right\}$$

$$\text{Let } F(t) = \int_t^\infty \frac{\cos u}{u} du = - \int_t^\infty \frac{\cos u}{u} du$$

$$\Rightarrow F'(t) = - \frac{d}{dt} \int_t^\infty \frac{\cos u}{u} du$$

By Leibnitz's

$$\text{so that } F'(t) = - \frac{\cos t}{t}$$

$$(or) tF'(t) = - \cos t$$

$$\therefore L\{tF'(t)\} = L\{-\cos t\}$$

$$(or) \frac{d}{dp} L\{F'(t)\} = \frac{P}{P^2+1}$$

$$(or) \frac{d}{dp} [Pf(P) - f(0)] = \frac{P}{P^2+1} \text{ where } f(P) = L\{F(t)\}$$

$$(or) \frac{d}{dp} [Pf(P)] = \frac{P}{P^2+1}, \text{ since } F(0) \text{ is constant.}$$

$$\text{Integrating } Pf(P) = \frac{1}{2} \log(P^2+1) + C \quad (\text{constant})$$

(1)

But from the final value theorem

$$\lim_{P \rightarrow 0} Pf(P) = \lim_{t \rightarrow \infty} F(t) = 0$$

∴ from (1) as $P \rightarrow 0$, we have

$$0 = 0 + C \text{ or } C = 0$$

$$\therefore \text{from (1), } Pf(P) = \frac{1}{2} \log(P^2+1)$$

$$(or) f(P) = L\{F(t)\} = L\{C_i(t)\} = \frac{\log(P^2+1)}{2P}$$

→ If $F(t) = t^2, 0 < t < 2$ and

$$F(t+2) = F(t), \text{ find } L\{F(t)\}$$

Sol'n: Here $F(t)$ is a periodic function with period $T=2$.

∴ from fundamental theorem
(Periodic function)



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we have

$$\begin{aligned}
 L\{F(t)\} &= \frac{\int_0^{\infty} e^{-pt} F(t) dt}{1-e^{-pt}} \\
 &= \frac{\int_0^{\infty} t^2 e^{-pt} dt}{1-e^{-2p}} \\
 &= \frac{\left(\frac{-t^2}{p} e^{-pt}\right)_0^{\infty} + \frac{2}{p} \int_0^{\infty} t e^{-pt} dt}{1-e^{-2p}} \\
 &= \frac{-\frac{4}{p} e^{-2p} + \frac{2}{p} \left\{ \left(\frac{-t}{p} e^{-pt}\right)_0^{\infty} + \frac{1}{p} \int_0^{\infty} e^{-pt} dt \right\}}{1-e^{-2p}} \\
 &= \frac{\frac{4}{p} e^{-2p} - \frac{4}{p^2} e^{-2p} - \frac{2}{p^2} \left(\frac{-e^{-pt}}{p} \right)_0^{\infty}}{1-e^{-2p}} \\
 &= \underline{\underline{-\frac{(4p^2+4p+2)e^{-2p}+2}{p^3(1-e^{-2p})}}}
 \end{aligned}$$

→ find the Laplace transform of Heaviside's unit function $H(t-a)$

sol'n: we have $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

$$\begin{aligned}
 \therefore L\{H(t-a)\} &= \int_0^{\infty} e^{-pt} H(t-a) dt \\
 &= \int_a^{\infty} e^{-pt} \cdot 1 dt \\
 &= \underline{\underline{\frac{e^{-ap}}{p}}}
 \end{aligned}$$

→ show that $L\{E_i(t)\} = \frac{\log(pt)}{p}$

sol'n: we have $E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u} du$

$$\begin{aligned}
 \therefore L\{E_i(t)\} &= L\left\{ \int_t^{\infty} \frac{e^{-u}}{u} du \right\} \\
 &= L\left\{ \int_1^{\infty} \frac{e^{-tv}}{v} dv \right\} dt,
 \end{aligned}$$

Putting $u = tv$ so that $du = t dv$

$$= \int_0^{\infty} e^{-pt} \left\{ \int_1^{\infty} \frac{e^{-tv}}{v} dv \right\} dt$$

(By definition of L.T.)

$$= \int_1^{\infty} \frac{1}{v} \left\{ \int_0^{\infty} e^{-pt} e^{tv} dt \right\} dv,$$

Changing the order of integration.

$$\begin{aligned}
 &= \int_1^{\infty} \frac{1}{v} \frac{1}{p+v} dv \\
 &= \int_1^{\infty} \frac{1}{p} \left(\frac{1}{v} - \frac{1}{p+v} \right) dv \\
 &= \frac{1}{p} \left[\log v - \log(p+v) \right]_1^{\infty} \\
 &= \frac{1}{p} \left[-\log \left(\frac{p}{v} + 1 \right) \right]_1^{\infty} \\
 &= \underline{\underline{\frac{1}{p} \log(p+1)}}
 \end{aligned}$$

→ find $L\{L_n(t)\}$

we have $L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n}(e^t \cdot t^n)$

$$\begin{aligned}
 \therefore L\{L_n(t)\} &= \int_0^{\infty} e^{-pt} \cdot \frac{e^t}{n!} \frac{d^n}{dt^n}(e^t \cdot t^n) dt \\
 &= \frac{1}{n!} \int_0^{\infty} e^{-(p-1)t} \frac{d^n}{dt^n}(e^t \cdot t^n) dt
 \end{aligned}$$

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$$\begin{aligned} &= \frac{1}{n!} \left[e^{-(p-1)t} \cdot \frac{d^{n-1}}{dt^{n-1}} e^{-t} \cdot t^n \right]_0^\infty \\ &= -\frac{1}{n!} \left[-(p-1) \int_0^\infty e^{-(p-1)t} \frac{d^{n-1}}{dt^{n-1}} (e^{-t} \cdot t^n) dt \right] \\ &= 0 + \frac{p-1}{n!} \int_0^\infty e^{-(p-1)t} \frac{d^{n-1}}{dt^{n-1}} (e^{-t} \cdot t^n) dt, \quad p > 1 \\ &[\because \text{each term in } \frac{d^{n-1}}{dt^{n-1}} (e^{-t} \cdot t^n) \text{ contains} \\ &\quad \text{some integral power of } t]. \end{aligned}$$

Proceeding similarly again and again, we have

$$\begin{aligned} L\{L_n(t)\} &= \frac{(p-1)^n}{n!} \int_0^\infty e^{-(p-1)t} (e^{-t} \cdot t^n) dt \\ &= \frac{(p-1)^n}{n!} \int_0^\infty e^{-pt} t^n dt \\ &= \frac{(p-1)^n}{n!} L\{t^n\} \\ &= \frac{(p-1)^n}{n!} \frac{n!}{p^{n+1}} \\ &= \frac{(p-1)^n}{p^{n+1}} \end{aligned}$$

→ show that $L\{(1+te^t)^3\} =$

$$\frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

Sol'n! we have

$$\begin{aligned} L\{(1+te^t)^3\} &= L\{1+t^3e^{-t}+3t^2e^{-2t} \\ &\quad + 3t^3e^{-3t}\} \end{aligned}$$

$$= \frac{1}{p} + 3 \frac{1!}{(p+1)^2} + 3 \frac{2!}{(p+2)^3} + \frac{3!}{(p+3)^4}$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

similarly

$$L\{(1+te^{-t})^3\} = \int_0^\infty (1+te^{-t})^3 \cdot e^{-pt} dt$$

$$= \int_0^\infty [e^{-pt} + 3t^2 e^{-(p+1)t} + 3t^2 e^{-(p+2)t} \\ + 3t^3 e^{-(p+3)t}] dt$$

$$= \int_0^\infty e^{-pt} dt + 3 \int_0^\infty t^2 e^{-(p+1)t} dt \\ + 3 \int_0^\infty t^2 e^{-(p+2)t} dt + \int_0^\infty t^3 e^{-(p+3)t} dt$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}, \quad p > 0$$

$$[\because \int_0^\infty e^{-at} t^{n-1} dt = \frac{T(n)}{a^n},$$

if $a > 0$ and $n > 0.$]



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* The Inverse Laplace Transform

Definition: If $f(p)$ is the Laplace transform of a function $F(t)$, i.e., $L\{F(t)\} = f(p)$, then $F(t)$ is called the Inverse Laplace transform of $f(p)$ and is written as $F(t) = L^{-1}\{f(p)\}$.

$$\text{Ex: } L\{t^n\} = \frac{n!}{p^{n+1}}$$

$$\therefore t^n = L^{-1}\left\{\frac{n!}{p^{n+1}}\right\}$$

Linearity Properties:

Let $f_1(p)$ and $f_2(p)$ be the Laplace transforms of functions $f_1(t)$ and $f_2(t)$ respectively and C_1, C_2 be two constants, then

$$L^{-1}\{C_1f_1(p) + C_2f_2(p)\}$$

$$= C_1L^{-1}\{f_1(p)\} + C_2L^{-1}\{f_2(p)\}$$

$$= C_1F_1(t) + C_2F_2(t)$$

Proof: Given $L\{F_1(t)\} = f_1(p)$ $\Rightarrow F_1(t) = L^{-1}\{f_1(p)\}$

and $L\{F_2(t)\} = f_2(p) \Rightarrow F_2(t) = L^{-1}\{f_2(p)\}$

We have

$$L\{C_1F_1(t) + C_2F_2(t)\} = C_1L\{F_1(t)\} + C_2L\{F_2(t)\}$$

$$= C_1f_1(p) + C_2f_2(p)$$

$$\therefore L^{-1}\{C_1f_1(p) + C_2f_2(p)\} = C_1F_1(t) + C_2F_2(t)$$

$$\therefore C_1L^{-1}\{f_1(p)\} + C_2L^{-1}\{f_2(p)\}$$

$$\rightarrow \text{Find: } L^{-1}\left\{\frac{1}{p}\right\}, p > 0$$

Sol'n: Since $L\{1\} = \frac{1}{p}$

$$\therefore L^{-1}\left\{\frac{1}{p}\right\} = 1$$

$$\rightarrow \text{(ii) } L^{-1}\left\{\frac{1}{p^{n+1}}\right\}, n \text{ is any real number such that } n > -1$$

Sol'n: Since $L\{t^n\} = \frac{n!}{p^{n+1}}, p > 0, n > -1$

$$\therefore L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = t^n$$

$$\rightarrow L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}, n > -1, p > 0$$

If n is +ve integer, then $F(n) = n!$

$$\therefore \text{from (i) } L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}, p > 0.$$

$$\rightarrow \text{(iii) } L^{-1}\left\{\frac{1}{p-a}\right\}$$

Sol'n: Since $L\{e^{at}\} = \frac{1}{p-a}$

$$\therefore L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$$

$$\rightarrow \text{Find: (i) } L^{-1}\left\{\frac{1}{p^2+a^2}\right\}, (ii) L^{-1}\left\{\frac{p}{p^2+a^2}\right\}, p > 0$$

$$(iii) L^{-1}\left\{\frac{1}{p^2-a^2}\right\}, p > |a|$$

$$(iv) L^{-1}\left\{\frac{p}{p^2-a^2}\right\}, p > |a|$$

$$\text{Sol'n: (i) Since } L\{\sin at\} = \frac{a}{p^2 + a^2}$$

$$\therefore L^{-1}\left\{\frac{a}{p^2 + a^2}\right\} = \sin at$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^2 + a^2}\right\} = \frac{1}{a} \sin at$$

$$(ii) \text{ Since } L\{\cos at\} = \frac{p}{p^2 + a^2}$$

$$\therefore L^{-1}\left\{\frac{p}{p^2 + a^2}\right\} = \cos at$$

$$(iii) \text{ Since } L\{\sinh at\} = \frac{a}{p^2 - a^2}$$

$$\therefore L^{-1}\left\{\frac{a}{p^2 - a^2}\right\} = \sinh at$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^2 - a^2}\right\} = \frac{1}{a} \sinh at$$

$$(iv) \text{ Since } L\{\cosh at\} = \frac{p}{p^2 - a^2}$$

$$\therefore L^{-1}\left\{\frac{p}{p^2 - a^2}\right\} = \cosh at$$

\therefore find (i) $L^{-1}\left\{\frac{1}{p^4}\right\}$ & (ii) $L^{-1}\left\{\frac{1}{p^4+4}\right\}$

$$(iii) L^{-1}\left\{\frac{4}{(p^2-2)}\right\} \quad (iv) L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$$

$$\text{Sol'n: (i) } L^{-1}\left\{\frac{1}{p^4}\right\} = L^{-1}\left\{\frac{1}{p^3+1}\right\} = \frac{t^3}{3!}$$

$$\left[\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!} \text{ if } n \text{ is +ve integer} \right]$$

$$(iv) L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{1/2}}\right\} \\ = L^{-1}\left\{\frac{1}{p^{1/2}+1}\right\}$$

$$= \frac{t^{-1/2}}{\Gamma_{1/2}+1} \quad (n = -1/2)$$

$$= \frac{t^{-1/2}}{\Gamma_{1/2}} \quad \left[\because L\{t^n\} = \frac{\Gamma_{n+1}}{p^{n+1}} \right]$$

$$= \frac{t^{1/2}}{\sqrt{\pi}} \quad \Rightarrow L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{\Gamma_{n+1}} \quad (\because \Gamma_{1/2} = \sqrt{\pi})$$

\rightarrow find (i) $L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$

$$(ii) L^{-1}\left\{\frac{p}{p^2+2}\right\} + \frac{6p}{p^2-16} + \frac{3}{p-3}$$

$$\text{Sol'n: (i) } L^{-1}\left\{\frac{1}{p^{5/2}}\right\} = L^{-1}\left\{\frac{1}{p^{5/2}+1}\right\}$$

$$= \frac{t^{5/2}}{\Gamma_{5/2}+1} = \frac{t^{5/2}}{\Gamma_{5/2}}$$

$$= \frac{t^{5/2}}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma_2}$$

$$= \frac{t^{5/2}}{\frac{15}{8} \sqrt{\pi}}$$

$$= \frac{t^{1/2} t^2}{\frac{15}{8} \sqrt{\pi}}$$

$$= \frac{8t^2}{15} \sqrt{\frac{t}{\pi}}$$

$$(ii) L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$$

$$= L^{-1}\left\{\frac{p}{p^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{p}{p^2-4^2}\right\} + 3L^{-1}\left\{\frac{1}{p-3}\right\}$$

$$= \cos \sqrt{2}t + 6 \cosh 4t + 3e^{3t}$$

$$\rightarrow \text{find} (i) L^{-1} \left\{ \frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9} \right\}$$

$$(ii) L^{-1} \left\{ \frac{3}{p^2-3} + \frac{3p+2}{p^3} - \frac{3p-27}{p^2+9} + \frac{6-30p}{p^4} \right\}$$

$$\text{Sol'n: } (i) L^{-1} \left\{ \frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9} \right\}$$

$$= L^{-1} \left\{ \frac{6}{2(p-\frac{3}{2})} \right\} - L^{-1} \left\{ \frac{3}{9p^2-16} \right\} + 4L^{-1} \left\{ \frac{p}{9p^2-16} \right\}$$

$$+ 8L^{-1} \left\{ \frac{1}{16p^2+9} \right\} - 6L^{-1} \left\{ \frac{p}{16p^2+9} \right\}$$

$$= 3L^{-1} \left\{ \frac{1}{p-\frac{3}{2}} \right\} - \frac{3}{9} L^{-1} \left\{ \frac{1}{p^2-(\frac{4}{3})^2} \right\} +$$

$$\frac{4}{9} L^{-1} \left\{ \frac{p}{p^2-(\frac{4}{3})^2} \right\} + \frac{8}{16} L^{-1} \left\{ \frac{1}{p^2+(\frac{3}{4})^2} \right\} - \frac{6}{16} L^{-1} \left\{ \frac{p}{p^2+(\frac{3}{4})^2} \right\}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{3} \frac{1}{4/3} 8 \sinh \frac{4}{3}t + \frac{4}{9} \cdot \cosh \frac{4}{3}t +$$

$$\frac{1}{2} \cdot \frac{1}{(3/4)} 8 \sin 3/4t - \frac{3}{8} \cos 3/4t$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4} \sinh \frac{4}{3}t + \frac{4}{9} \cosh \frac{4}{3}t$$

$$+ \frac{2}{3} \sin 3/4t - \frac{3}{8} \cos 3/4t.$$

\rightarrow Prove that

$$L^{-1} \left\{ \frac{5p}{p^2-7} + \left(\frac{\sqrt{p}-1}{p} \right)^2 - \frac{7}{3p+2} \right\}$$

$$= 1+6t-4\sqrt{\pi} - \frac{7}{3} e^{-\frac{7}{3}t}$$

\rightarrow Show that

$$(i) L^{-1} \left\{ \frac{1}{p} \cos \frac{t}{p} \right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} +$$

$$(ii) L^{-1} \left\{ \frac{1}{p} \sin \frac{t}{p} \right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} +$$

$$(iii) L^{-1} \left\{ \frac{1}{p^3+1} \right\} = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \dots$$

(Cosec $t = (-\frac{x^2}{2!} + \frac{x^4}{4!} + \dots)$)

$$\text{Sol'n: } (i) L^{-1} \left\{ \frac{1}{p} \cos \frac{t}{p} \right\}$$

$$= L^{-1} \left\{ \frac{1}{p} \left[1 - \frac{(tp)^2}{2!} + \frac{(tp)^4}{4!} - \frac{(tp)^6}{6!} + \dots \right] \right\}$$

$$= L^{-1} \left\{ \frac{1}{p} - \frac{1}{p^3} \cdot \frac{1}{2!} + \frac{1}{4! p^5} - \frac{1}{6! p^7} + \dots \right\}$$

$$= L^{-1} \left\{ \frac{1}{p} \right\} - \frac{1}{2!} L^{-1} \left\{ \frac{1}{p^3} \right\} + \frac{1}{4!} L^{-1} \left\{ \frac{1}{p^5} \right\} - \dots$$

$$- \frac{1}{6!} L^{-1} \left\{ \frac{1}{p^7} \right\} + \dots$$

$$= 1 - \frac{1}{2!} \frac{t^2}{2!} + \frac{1}{4!} \frac{t^4}{4!} - \frac{1}{6!} \frac{t^6}{6!} + \dots$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \dots$$

$$(ii) L^{-1} \left\{ \frac{1}{p^3+1} \right\} = L^{-1} \left\{ \frac{1}{p^3[1+\frac{1}{p^3}]} \right\}$$

$$(1+x)^{-1} = 1-x+x^2-x^3+x^4+\dots$$

$$= L^{-1} \left\{ \frac{1}{p^3} \left(1 + \frac{1}{p^3} \right)^{-1} \right\}$$

$$= L^{-1} \left\{ \frac{1}{p^3} \left[1 - \frac{1}{p^3} + \left(\frac{1}{p^3} \right)^2 - \left(\frac{1}{p^3} \right)^3 + \left(\frac{1}{p^3} \right)^4 \dots \right] \right\}$$

$$= L^{-1} \left\{ \frac{1}{p^3} \right\} - L^{-1} \left\{ \frac{1}{p^6} \right\} + L^{-1} \left\{ \frac{1}{p^9} \right\} - L^{-1} \left\{ \frac{1}{p^{12}} \right\} + \dots$$

$$= \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} + \dots$$

$$\rightarrow \text{show that } L^{-1} \left\{ \frac{e^{tp}}{p} \right\} = J_0 2\sqrt{t}$$

where

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2}$$

$$\Rightarrow \text{If } L^{-1}\left\{\frac{P^2-1}{(P^2+1)^2}\right\} = t \cos t, \text{ Prove that}$$

$$L^{-1}\left\{\frac{9P^2-1}{(9P^2+1)^2}\right\} = \frac{1}{9} t \cos \frac{t}{3}$$

$$\underline{\text{Sol'n}}: \text{ Given } L^{-1}\left\{\frac{P^2-1}{(P^2+1)^2}\right\} = t \cos t$$

$$\text{then } L^{-1}\left\{\frac{9P^2-1}{(9P^2+1)^2}\right\} = L^{-1}\left\{\frac{(3P)^2-1}{((3P)^2+1)^2}\right\}$$

$$= \frac{1}{3} t \cos \frac{t}{3} \quad \left[\because L(f(ap)) = \frac{1}{a} F(\frac{t}{a}) \right]$$

$$= \frac{1}{9} t \cos \frac{t}{3}$$

$$\rightarrow \text{If } L^{-1}\left\{\frac{P}{P^2-16}\right\} = \cosh 4t, \text{ then Prove}$$

$$L^{-1}\left\{\frac{P}{2P^2-8}\right\} = \frac{1}{2} \cosh 2t$$

$$\Rightarrow \text{Find } L^{-1}\left\{\frac{e^{-5P}}{(P-2)^4}\right\} \quad \text{It is in the form of } L^{-1}\left\{e^{-ap} f(p)\right\}$$

$$\underline{\text{Sol'n}}: \text{ Let } f(p) = \frac{1}{(P-2)^4}$$

$$\therefore L\{F(t)\} = \frac{1}{(P-2)^4}$$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{(P-2)^4}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{1}{-P^4}\right\}$$

$$= e^{2t} \frac{t^3}{3!} = \frac{1}{6} t^3 e^{2t}$$

Hence by second shifting theorem, we have

$$L^{-1}\left\{e^{-ap} f(p)\right\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\therefore L^{-1}\left\{e^{-5P} f(p)\right\} = \begin{cases} F(t-5), & t > 5 \\ 0, & t < 5 \end{cases}$$

$$\therefore L^{-1}\left\{\frac{e^{-5P}}{(P-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)}, & t > 5 \\ 0, & t < 5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3 e^{2(t-5)}$$

in terms of Heaviside unit step function

$$\rightarrow \text{find } L^{-1}\left\{\frac{e^{4-3P}}{(P+4)^{5/2}}\right\}$$

$$\underline{\text{Sol'n}}: \text{ Let } f(p) = \frac{1}{(P+4)^{5/2}}$$

$$f(t) = L^{-1}\left\{\frac{1}{(P+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{P^{5/2}}\right\}$$

$$= e^{-4t} \frac{t^{5/2-1}}{\sqrt{5/2}} \left| L^{-1}\left\{\frac{1}{P^{n+1}}\right\} = \frac{t^n}{n!} \right.$$

$$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2} \cdot \frac{1}{2} \sqrt{5/2}}$$

$$= e^{-4t} \frac{t^{3/2}}{\sqrt{\pi}}$$

$$\therefore L^{-1}\left\{\frac{e^{4-3P}}{(P+4)^{5/2}}\right\} = e^{4-3P} L^{-1}\left\{\frac{e^{-3P}}{(P+4)^{5/2}}\right\}$$

\therefore by second shifting theorem, we have

$$L^{-1}\left\{e^{-ap} f(p)\right\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\therefore e^{4-3P} L^{-1}\left\{e^{-3P} f(p)\right\} = \begin{cases} e^{4-3P} F(t-3), & t > 3 \\ 0, & t < 3 \end{cases}$$

Hence by the definition of inverse LT we get

$$\mathcal{L}^{-1}\{e^{-ap} f(p)\} = g(t)$$

Note: The result of this theorem can also be expressed in the following two ways

$$1. \mathcal{L}^{-1}\{f(p)\} = F(t), \mathcal{L}^{-1}\{e^{-pa} f(p)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$2. \mathcal{L}^{-1}\{f(p)\} = F(t), \mathcal{L}^{-1}\{e^{-pa} f(p)\} = F(t-a)H(t-a)$$

where $H(t-a)$ is Heaviside unit step function which is defined as follows:

$$H(t-a) = \begin{cases} 1, & \text{when } t > a \\ 0, & \text{when } t < a \end{cases}$$

* Change of scale Property:

Theorem: If $\mathcal{L}^{-1}\{f(p)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ap)\} = \frac{1}{a} F(t/a)$$

Sol'n: Given $\mathcal{L}^{-1}\{f(p)\} = F(t)$

$$\Rightarrow f(p) = \mathcal{L}\{F(t)\}$$

$$\therefore f(p) = \int_0^\infty e^{-pt} F(t) dt$$

$$\therefore f(ap) = \int_0^\infty e^{-apt} F(t) dt$$

$$\text{Put. } at = x \Rightarrow dt = \frac{dx}{a}$$

$$t = x/a$$

$$= \frac{1}{a} \int_0^\infty e^{-px} F\left(\frac{x}{a}\right) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-pt} F\left(\frac{t}{a}\right) dt$$

$$\left(\because \int_a^b f(x) dx \right)$$

$$= \int_a^b F(t) dt$$

$$= \frac{1}{a} \mathcal{L}\{F(t/a)\}$$

$$= \mathcal{L}\left\{\frac{1}{a} F(t/a)\right\}$$

$$\therefore f(ap) = \mathcal{L}\left\{\frac{1}{a} F(t/a)\right\}$$

$$\Rightarrow \mathcal{L}\{f(ap)\} = \frac{1}{a} F(t/a)$$

$$\rightarrow \text{Find } \mathcal{L}\left\{\frac{1}{p^2 - 6p + 10}\right\}$$

$$\text{Sol'n: } \mathcal{L}\left\{\frac{1}{p^2 - 6p + 10}\right\} = \mathcal{L}\left\{\frac{1}{(p-3)^2 + 1}\right\}$$

$$= e^{3t} \mathcal{L}\left\{\frac{1}{p^2 + 1}\right\}$$

$$= e^{3t} \sin t \quad (\text{using first shifting theorem})$$

$$\rightarrow \text{Find (i) } \mathcal{L}\left\{\frac{1}{p^2 + 8p + 16}\right\}.$$

$$(ii) \mathcal{L}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\}$$

$$\text{Sol'n (i) } \mathcal{L}^{-1}\left\{\frac{1}{p^2 + 8p + 16}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(p+4)^2}\right\}$$

$$= e^{-4t} \left\{\frac{1}{p^2}\right\} \quad (\text{by using first shifting theorem})$$

$$= e^{-4t} \frac{t}{1!} = te^{-4t}$$

$$i) L^{-1} \left\{ \frac{P-1}{(P+3)(P^2+2P+2)} \right\}$$

$$\text{Now } \frac{P-1}{(P+3)(P^2+2P+2)} = \frac{A}{P+3} + \frac{BP+C}{P^2+2P+2}$$

$$P-1 = A(P^2+2P+2) + (BP+C)(P+3)$$

$$P-1 = (A+B)P^2 + (2A+3B+C)P + 2A+3C$$

Comparing on both sides

$$A+B=0 \Rightarrow A=-B$$

$$2A+3B+C=1 \Rightarrow 2(-B)+3B+C=1$$

$$\Rightarrow B+C=1$$

$$2A+3C=-1$$

$$\Rightarrow 2(-B)+3C=-1$$

$$\Rightarrow -2B+3C=-1$$

$$2B+2C=2$$

$$-2B+3C=-1$$

$$-5C=1$$

$$C=\frac{1}{5}$$

$$\begin{aligned} B+C &= 1 \\ B &= -\frac{1}{5} \\ A &= -\frac{4}{5} \\ B &= \frac{4}{5} \end{aligned}$$

$$\frac{P-1}{(P+3)(P^2+2P+2)} = -\frac{4}{5(P+3)} + \frac{\frac{4}{5}P+\frac{1}{5}}{P^2+2P+2}$$

$$= -\frac{4}{5(P+3)} + \frac{4P+1}{5(P^2+2P+2)}$$

$$L^{-1} \left\{ \frac{P-1}{(P+3)(P^2+2P+2)} \right\}$$

$$= -\frac{4}{5} L^{-1} \left\{ \frac{1}{P+3} \right\} + \frac{1}{5} L^{-1} \left\{ \frac{4P+1}{P^2+2P+2} \right\}$$

$$= -\frac{4}{5} e^{-3t} L^{-1} \left\{ \frac{1}{P} \right\} + \frac{1}{5} L^{-1} \left\{ \frac{4(P+1)-3}{(P+1)^2+1} \right\}$$

$$= -\frac{4}{5} e^{-3t} + \frac{e^{-t}}{5} L^{-1} \left\{ \frac{4P-3}{P^2+1} \right\} \quad (\because L^{-1} \left\{ \frac{1}{P^2+1} \right\} = 1)$$

$$= -\frac{4}{5} e^{-3t} + \frac{e^{-5}}{5} \left[L^{-1} \left\{ \frac{4P}{P^2+1} \right\} + L^{-1} \left\{ \frac{-3}{P^2+1} \right\} \right]$$

$$= -\frac{4}{5} e^{-3t} + \frac{e^{-5}}{5} (4\cos t - 3\sin t)$$

$$L^{-1} \left\{ \frac{3P-2}{P^2-4P+20} \right\} = L^{-1} \left\{ \frac{3P-2}{(P-2)^2+16} \right\}$$

$$= L^{-1} \left\{ \frac{3(P-2)+4}{(P-2)^2+4^2} \right\}$$

$$= e^{2t} L^{-1} \left\{ \frac{3P+4}{P^2+4^2} \right\}$$

$$= e^{2t} \left[L^{-1} \left\{ \frac{3P}{P^2+4^2} \right\} + L^{-1} \left\{ \frac{4}{P^2+4^2} \right\} \right]$$

$$= e^{2t} [8\cos 2t + 4\sin 4t]$$

$$L^{-1} \left\{ \frac{3P+7}{P^2-2P-3} \right\} = L^{-1} \left\{ \frac{3P+7}{(P-1)^2-4} \right\}$$

$$= L^{-1} \left\{ \frac{3(P-1)+10}{(P-1)^2-4} \right\}$$

$$= e^t L^{-1} \left\{ \frac{3P+10}{P^2-2^2} \right\}$$

$$= e^t \left[3L^{-1} \left\{ \frac{P}{P^2-2^2} \right\} + 10L^{-1} \left\{ \frac{1}{P^2-2^2} \right\} \right]$$

$$= e^t [3\cosh 2t + \frac{10}{2} \sinh 2t]$$

$$= e^t [8\cosh 2t + 5\sinh 2t]$$

$$\therefore L^{-1} \left\{ \frac{1}{P^2-a^2} \right\} = \frac{1}{a} \sinh at$$

$$\rightarrow \text{Find (i) } L^{-1} \left\{ \frac{1}{(P+a)^n} \right\} \text{ (ii) } L^{-1} \left\{ \frac{P}{(P+1)^{5/2}} \right\}$$

$$\text{(iii) } L^{-1} \left\{ \frac{P}{(P+1)^5} \right\} \quad \text{(iv) } L^{-1} \left\{ \frac{3P+2}{4P^2+12P+9} \right\}$$

$$\rightarrow \text{Evaluate (i) } L^{-1} \left\{ \frac{1}{\sqrt{2P+3}} \right\} \text{ (ii) } L^{-1} \left\{ \frac{1}{(8P-27)^{1/3}} \right\}$$

$$\underline{\text{Sol'n (i)}}: L^{-1} \left\{ \frac{1}{(8P-27)^{1/3}} \right\} = L^{-1} \left\{ \frac{1}{8^{1/3} (P-27)^{1/3}} \right\}$$

$$\Rightarrow \frac{1}{2} L^{-1} \left\{ \frac{1}{(P-\frac{27}{8})^{1/3}} \right\}$$

$$= \frac{1}{2} e^{27/8 t} L^{-1} \left\{ \frac{1}{t^{1/3}} \right\}$$

$$= \frac{e^{27/8 t}}{2} \cdot \frac{t^{1/3-1}}{t^{1/3}}$$

$$= \frac{e^{27/8 t}}{2} \cdot \frac{t^{-2/3}}{t^{1/3}}$$

$$\rightarrow \text{If } L^{-1} \left\{ \frac{e^{-t/p}}{p^{1/2}} \right\} = \frac{\cos 2\sqrt{p}t}{\sqrt{\pi t}}, \text{ find}$$

$$L^{-1} \left\{ \frac{e^{-at/p}}{p^{1/2}} \right\} \text{ where } a > 0.$$

$$\underline{\text{Sol'n}}: \text{ Since } L^{-1} \{ f(ap) \} = \frac{1}{a} F(t/a)$$

$$\text{Hence } L^{-1} \left\{ \frac{e^{-t/p}}{\sqrt{p}} \right\} = \frac{\cos 2\sqrt{p}t}{\sqrt{\pi t}} \text{ gives}$$

$$L^{-1} \left\{ \frac{e^{-t/p_k}}{\sqrt{p_k}} \right\} = \frac{1}{k} \frac{\cos 2\sqrt{t/k}}{\sqrt{\pi k}}$$

$$\Rightarrow \frac{1}{\sqrt{k}} L^{-1} \left\{ \frac{e^{-t/p_k}}{\sqrt{p}} \right\} = \frac{1}{k} \frac{\cos(2\sqrt{t/k})}{\sqrt{\pi k}}$$

$$\Rightarrow L^{-1} \left\{ \frac{e^{-t/p_k}}{\sqrt{p}} \right\} = \frac{\sqrt{k}}{k} \frac{\cos 2(t/k)^{1/2}}{\sqrt{\pi k}}$$

Putting $k = \gamma_a$

$$\begin{aligned} L^{-1} \left\{ \frac{e^{-at/p}}{\sqrt{p}} \right\} &= \frac{\cos 2(at)^{1/2}}{\sqrt{\pi t}} \\ &= \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}} \end{aligned}$$

$$\boxed{\text{Imp: } L^{-1} \left\{ \frac{P}{(P^2+1)^2} \right\} = \frac{1}{2} t \sin t, \text{ find}}$$

$$L^{-1} \left\{ \frac{32P}{(16P^2+1)^2} \right\}$$

Sol'n: Given

$$L^{-1} \left\{ \frac{P}{(P^2+1)^2} \right\} = \frac{1}{2} t \sin t$$

Since $f(ap) = \frac{1}{a} F(t/a)$

$$L^{-1} \left\{ \frac{ap}{((ap)^2+1)^2} \right\} = \frac{1}{2} \left(\frac{1}{a} \right) \frac{t}{a} \sin \frac{t}{a}$$

Putting $a = 4$

$$L^{-1} \left\{ \frac{4P}{(16P^2+1)^2} \right\} = \frac{1}{2} \frac{1}{4} \frac{t}{4} \sin \frac{t}{4}$$

$$L^{-1} \left\{ \frac{4P}{(16P^2+1)^2} \right\} = \frac{1}{8} \frac{t}{4} \sin \frac{t}{4}$$

$$\Rightarrow 8L^{-1} \left\{ \frac{4P}{(16P^2+1)^2} \right\} = \frac{t}{4} \sin \frac{t}{4}.$$

$$\Rightarrow L^{-1} \left\{ \frac{32P}{(16P^2+1)^2} \right\} = t/4 \sin t/4$$

$$\Rightarrow \text{If } L^{-1} \left\{ \frac{P^2 - 1}{(P^2 + 1)^2} \right\} = t \cos t, \text{ Prove that}$$

$$L^{-1} \left\{ \frac{9P^2 - 1}{(9P^2 + 1)^2} \right\} = \frac{1}{9} t \cos \frac{t}{3}$$

$$\underline{\text{Sol'n}}: \text{ Given } L^{-1} \left\{ \frac{P^2 - 1}{(P^2 + 1)^2} \right\} = t \cos t$$

$$\text{then } L^{-1} \left\{ \frac{9P^2 - 1}{(9P^2 + 1)^2} \right\} = L^{-1} \left\{ \frac{(3P)^2 - 1}{((3P)^2 + 1)^2} \right\}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{P^2 - 1}{(P^2 + 1)^2} \right\} \quad \left[\because L^{-1}(f(\alpha P)) = \frac{1}{\alpha} F\left(\frac{t}{\alpha}\right) \right]$$

$$= \frac{1}{3} \frac{t}{3} \cos \frac{t}{3}$$

$$\rightarrow \text{If } L^{-1} \left\{ \frac{P}{P^2 - 16} \right\} = \cosh 4t, \text{ then Prove}$$

$$L^{-1} \left\{ \frac{P}{2P^2 - 8} \right\} = \frac{1}{2} \cosh 2t$$

$$\Rightarrow \text{Find } L^{-1} \left\{ \frac{-5P}{(P-2)^4} \right\} \quad \text{It is in the form of } L^{-1} \left\{ e^{-ap} f(p) \right\}$$

$$\underline{\text{Sol'n}}: \text{ Let } f(p) = \frac{1}{(P-2)^4}$$

$$\text{i.e. } L \{ F(t) \} = \frac{1}{(P-2)^4}$$

$$\Rightarrow F(t) = L^{-1} \left\{ \frac{1}{(P-2)^4} \right\}$$

$$= e^{2t} L^{-1} \left\{ \frac{1}{P^4} \right\}$$

$$= e^{2t} \frac{t^3}{3!} = \frac{1}{6} t^3 e^{2t}$$

Hence by second shifting theorem, we have

$$L^{-1} \left\{ e^{-ap} f(p) \right\} = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$\therefore L^{-1} \left\{ e^{-5P} f(p) \right\} = \begin{cases} F(t-5) & t > 5 \\ 0 & t < 5 \end{cases}$$

$$\therefore L^{-1} \left\{ \frac{e^{-5P}}{(P-2)^4} \right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)} & t > 5 \\ 0 & t < 5 \end{cases}$$

$$= \frac{1}{6}(t-5)^3 e^{2(t-5)}$$

in terms of Heaviside unit step function

$$\Rightarrow \text{find } L^{-1} \left\{ \frac{e^{4-3P}}{(P+4)^{5/2}} \right\}$$

$$\underline{\text{Sol'n}}: \text{ Let } f(p) = \frac{1}{(P+4)^{5/2}}$$

$$F(t) = L^{-1} \left\{ \frac{1}{(P+4)^{5/2}} \right\} = e^{-4t} L^{-1} \left\{ \frac{1}{P^{5/2}} \right\}$$

$$= e^{-4t} \frac{\frac{5}{2}-1}{t^{\frac{5}{2}}} \left| L^{-1} \left\{ \frac{1}{P^{n+1}} \right\} \right| = \frac{t^n}{T^{n+1}}, \quad n > -1$$

$$= e^{-4t} \frac{t^{3/2}}{3/2 \cdot 1/2 \sqrt{2}} = \frac{t^{3/2}}{3\sqrt{2}}$$

$$= \frac{1}{3} e^{-4t} \frac{t^{3/2}}{\sqrt{\pi}}$$

$$\therefore L^{-1} \left\{ \frac{e^{4-3P}}{(P+4)^{5/2}} \right\} = e^{4-4t} L^{-1} \left\{ \frac{e^{-3P}}{(P+4)^{5/2}} \right\}$$

∴ by second shifting theorem, we have

$$L^{-1} \left\{ e^{-ap} f(p) \right\} = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$\text{i.e. } e^{4-4t} L^{-1} \left\{ e^{-3P} f(p) \right\} = \begin{cases} e^{4-4(t-3)} F(t-3) & t > 3 \\ 0 & t < 3 \end{cases}$$

$$e^4 L^{-1} \left\{ \frac{e^{-3p}}{(p+4)^{3/2}} \right\}$$

$$= \begin{cases} e^{4 \frac{4}{3\sqrt{\pi}} e^{-4(t-3)} (t-3)^{3/2}} & t > 3 \\ 0 & t < 3 \end{cases}$$

$$= \begin{cases} \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2} & t > 3 \\ 0 & t < 3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2} H(t-3)$$

in terms of the heaviside unit step function.

→ For $a > 0$, Prove that $L^{-1}\{f(p)\} = F(t)$

$$\text{implies } L^{-1}\{f(ap+b)\} = \frac{1}{a} e^{-bt/a} F(t/a)$$

Sol'n: we have $L^{-1}\{f(p)\} = F(t)$

$$\Rightarrow f(p) = L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$$

$$\Rightarrow f(ap+b) = \int_0^\infty e^{-(ap+b)t} F(t) dt$$

$$= \int_0^\infty e^{-apt} e^{-bt} F(t) dt$$

$$= \int_0^\infty e^{-pt} e^{-bt/a} F(t/a) \frac{dt}{a}$$

Putting at=1
 $\Rightarrow dt = \frac{1}{a} dt$

$$= \int_0^\infty e^{-pt} \left\{ \frac{1}{a} e^{-bt/a} F(t/a) \right\} dt$$

by property of definite integral $\int_a^b f(x) dx = \int_a^b f(t) dt$.

$$= L \left\{ \frac{1}{a} e^{-bt/a} F(t/a) \right\}$$

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$$\therefore f(ap+b) = L \left\{ \frac{1}{a} e^{-bt/a} F(t/a) \right\}$$

$$\Rightarrow L^{-1}\{f(ap+b)\} = \frac{1}{a} e^{-bt/a} F(t/a)$$

→ Evaluate $L^{-1}\left\{ \frac{p+1}{p^2+6p+25} \right\}$

Sol'n: $L^{-1}\left\{ \frac{p+1}{p^2+6p+25} \right\} = L^{-1}\left\{ \frac{p+1}{(p+3)^2+16} \right\}$

$$= L^{-1}\left\{ \frac{(p+3)-2}{(p+3)^2+4^2} \right\}$$

$$= e^{-3t} L^{-1}\left\{ \frac{p-2}{p^2+4^2} \right\}$$

$$= e^{-3t} \left[L^{-1}\left\{ \frac{p}{p^2+4^2} \right\} - L^{-1}\left\{ \frac{2}{p^2+4^2} \right\} \right]$$

$$= e^{-3t} \left[\cos 2t - 2 \frac{1}{4} 8 \sin 4t \right]$$

$$= e^{-3t} \left[\cos 2t - \frac{1}{2} 8 \sin 4t \right]$$

→ Evaluate $L^{-1}\left\{ \frac{6p^2+22p+18}{p^3+6p^2+11p+6} \right\}$

Sol'n: $L^{-1}\left\{ \frac{6p^2+22p+18}{p^3+6p^2+11p+6} \right\} = L^{-1}\left\{ \frac{6p^2+22p+18}{(p+1)(p+2)(p+3)} \right\}$

$$= L^{-1}\left\{ \frac{1}{p+1} + \frac{2}{p+2} + \frac{3}{p+3} \right\}$$

$$= L^{-1}\left\{ \frac{1}{p+1} \right\} + L^{-1}\left\{ \frac{2}{p+2} \right\} + L^{-1}\left\{ \frac{3}{p+3} \right\}$$

$$= e^{-t} L^{-1}\left\{ \frac{1}{p} \right\} + 2e^{-2t} L^{-1}\left\{ \frac{1}{p} \right\} + 3e^{-3t} L^{-1}\left\{ \frac{1}{p} \right\}$$

$$= e^{-t} + 2e^{-2t} + 3e^{-3t}$$

\rightarrow Evaluate $\mathcal{L}^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

Solⁿ: $\mathcal{L}^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{p^2+2p+2} + \frac{2}{p^2+2p+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(p+1)^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$$

$$= e^{-t} \sin t + 2e^{-t} \sin 2t$$

\rightarrow Prove that $\mathcal{L}^{-1}\left\{\frac{4p+5}{(p-1)(p+2)}\right\}$

$$= 3te^t + 5e^t - \frac{5}{3}e^{-2t}$$

\rightarrow Evaluate $\mathcal{L}^{-1}\left\{\frac{4p+5}{(p-4)^2(p+3)}\right\}$

\rightarrow Find $\mathcal{L}^{-1}\left\{\frac{5p^2-15p-11}{(p+1)(p-2)^3}\right\}$

Solⁿ: $\frac{5p^2-15p-11}{(p+1)(p-2)^3}$

$$= \frac{-1}{3(p+1)} + \frac{1}{3(p-2)} + \frac{4}{(p-2)^2} - \frac{7}{(p-2)^3}$$

\rightarrow Prove that

$$\mathcal{L}^{-1}\left\{\frac{2p+1}{(p+2)^2(p-1)}\right\} = \frac{1}{3}t[e^t - e^{-2t}]$$

\rightarrow Find $\mathcal{L}^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

\rightarrow Prove that

$$\mathcal{L}^{-1}\left\{\frac{P}{(p^2-2p+2)(p^2+2p+2)}\right\} = \frac{1}{2} \sin t \cdot \sin ht$$

\rightarrow Prove that

$$\mathcal{L}^{-1}\left\{\frac{P}{p^4+p^2+1}\right\} = \frac{2}{\sqrt{3}} \sin b t / 2 \sin k / 2 \sqrt{3} t$$

$$\frac{P}{p^4+p^2+1} = \frac{P}{(p^2+1)^2-p^2} = \frac{P}{(p^2+1+p)(p^2+1-p)}$$

\rightarrow Find $\mathcal{L}^{-1}\left\{\frac{2p^3+2p^2+4p+1}{(p^2+1)(p^2+p+1)}\right\}$

\rightarrow show that

$$\mathcal{L}^{-1}\left\{\frac{P^2}{p^2+4a^4}\right\} = \frac{1}{2a} [\cosh at \sin ht + \sinh at \cosh ht]$$

Solⁿ: L.H.S = $\mathcal{L}^{-1}\left\{\frac{P^2}{(p^2+2a^2)^2-4a^2p^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{P^2}{(p^2+2a^2-2ap)(p^2+2a^2+2ap)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{P}{4a(p^2-2ap+2a^2)} - \frac{P}{4a(p^2+2ap+2a^2)}\right\}$$

$$= \frac{1}{4a} \mathcal{L}^{-1}\left\{\frac{(p-a)+a}{(p-a)^2+a^2} - \frac{(p+a)-a}{(p+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} \left[e^{at} \mathcal{L}^{-1}\left\{\frac{p+a}{p^2+a^2}\right\} - e^{-at} \mathcal{L}^{-1}\left\{\frac{p-a}{p^2+a^2}\right\} \right]$$

$$= \frac{1}{4a} \left[e^{at} \left(\mathcal{L}^{-1}\left\{\frac{p}{p^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{a}{p^2+a^2}\right\} \right) \right]$$

$$- e^{-at} \left(\mathcal{L}^{-1}\left\{\frac{-p}{p^2+a^2}\right\} - \mathcal{L}^{-1}\left\{\frac{a}{p^2+a^2}\right\} \right)$$

* Inverse Laplace Transform

of derivatives:

Theorem: If $\mathcal{L}\{f(p)\} = F(t)$, then

$$\mathcal{L}\{f^n(p)\} = (-1)^n t^n F(t), \text{ i.e.,}$$

$$\mathcal{L}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t), n=1,2,3.$$

Proof: we know that

$$\begin{aligned} \mathcal{L}\{t^n F(t)\} &= (-1)^n \frac{d^n}{dp^n} f(p) \\ &= (-1)^n f^n(p) \end{aligned}$$

$$\therefore \mathcal{L}\{f^n(p)\} = (-1)^n t^n F(t)$$

$$\text{(or)} \quad \mathcal{L}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t)$$

Note: The result of this theorem

can also be written as

$$\mathcal{L}\left\{\frac{d^n}{dp^n} f(p)\right\} = \mathcal{L}\{f^{(n)}(p)\} = (-1)^n t^n \mathcal{L}\{f(p)\}$$

Find $\mathcal{L}^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\}$

Sol:

$$\text{Let } f(p) = \frac{1}{p^2-a^2}$$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-2p}{(p^2-a^2)^2}$$

$$\Rightarrow \frac{p}{(p^2-a^2)^2} = -\frac{1}{2} \frac{d}{dp} f(p)$$

$$= -\frac{1}{2} \frac{d}{dp} \left(\frac{1}{p^2-a^2} \right) = \frac{1}{a} \sinhat$$

$$f(p) = \frac{1}{p^2-a^2}$$

$$\mathcal{L}\{F(t)\} = f(p)$$

$$F(t) = \mathcal{L}^{-1}\left\{\frac{1}{p^2-a^2}\right\}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2} \frac{d}{dp} \left(\frac{1}{p^2-a^2} \right)\right\}$$

$$= -\frac{1}{2} \mathcal{L}^{-1}\left\{\frac{d}{dp} \left(\frac{1}{p^2-a^2} \right)\right\}$$

$$= -\frac{1}{2} (-1)^1 + \mathcal{L}^{-1}\left\{\frac{1}{p^2-a^2}\right\} \text{ by formula}$$

$$= \frac{1}{2} t \left(\frac{1}{a} \right) \sinhat$$

$$= \frac{t}{2a} \sinhat \quad \mathcal{L}^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}$$

$$\overline{\rightarrow \text{Find (i)} \quad \mathcal{L}^{-1}\left\{\frac{p}{(p^2-16)^2}\right\}}$$

$$\text{(ii)} \quad \mathcal{L}^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} \quad \text{(iii)} \quad \mathcal{L}^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$$

Evaluate

$$\text{(i)} \quad \mathcal{L}^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\} \quad \text{(ii)} \quad \mathcal{L}^{-1}\left\{\frac{p+2}{(p^2+4p+5)^2}\right\}$$

$$\text{Sol: (i)} \quad \mathcal{L}^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$$

$$\text{Let } f(p) = \frac{1}{p^2+2p+2} = (p^2+2p+2)^{-1}$$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-(2p+2)}{(p^2+2p+2)^2} = \frac{-2(p+1)}{(p^2+2p+2)^2}$$

$$\Rightarrow \frac{p+1}{(p^2+2p+2)^2} = -\frac{1}{2} \frac{d}{dp} f(p)$$

$$= -\frac{1}{2} \frac{d}{dp} \left(\frac{1}{p^2+2p+2} \right)$$

$$\left(\because f(p) = \frac{1}{p^2+2p+2} \right)$$

$$\therefore \mathcal{L}^{-1}\left(\frac{p+1}{p^2+2p+2}\right) = -\frac{1}{2} \mathcal{L}^{-1}\left\{\frac{d}{dp} \left(\frac{1}{p^2+2p+2} \right)\right\}$$

$$= -\frac{1}{2} (-1)^1 + \mathcal{L}^{-1}\left\{\frac{1}{p^2+2p+2}\right\}$$

$$\begin{aligned}
 &= \int_0^\infty L(e^{-tx^2}) dx \\
 &= \int_0^\infty \frac{1}{P + x^2} dx \quad [\because L^{-1}\{e^{-at}\} = \frac{1}{P+a}] \\
 &= \int_0^\infty \frac{1}{(\sqrt{P})^2 + x^2} dx \\
 &= \left[\frac{1}{\sqrt{P}} \tan^{-1}\left(\frac{x}{\sqrt{P}}\right) \right]_0^\infty \\
 &= \left[\frac{1}{\sqrt{P}} \frac{\pi}{2} - 0 \right] = \frac{1}{\sqrt{P}} \frac{\pi}{2}.
 \end{aligned}$$

$\therefore L\{F(t)\} = \frac{1}{\sqrt{P}} \left(\frac{\pi}{2}\right)$
 $F(t) = L^{-1}\left\{\frac{1}{\sqrt{P}} \frac{\pi}{2}\right\}$
 $= \frac{\pi}{2} F\left(\frac{t}{\sqrt{P}}\right)$
 $F(t) = \frac{\pi}{2} \frac{1}{\sqrt{\pi t}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{t}}$
 $\therefore \int_0^\infty e^{-tx^2} dx = \frac{1}{2} \sqrt{\pi t} \quad \text{--- (2)}$

(vi) putting $t=1$ in eqn (2)

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

we know that $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$.

The complementary error function is defined

$$\begin{aligned}
 \operatorname{erfc}(t) &= 1 - \operatorname{erf}(t) \\
 &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx - \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \\
 &= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \quad [\because \operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx = 1] \\
 &= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx - \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \\
 \operatorname{erfc}(t) &= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx.
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\sqrt{P}} \binom{1}{2} \frac{\Gamma_{1/4} \Gamma_{3/4}}{\Gamma_{1/4 + 3/4}} \\
 &= \frac{1}{4\sqrt{P}} \frac{\Gamma_{1/4} \Gamma_{3/4}}{\Gamma_1} \quad (\because \Gamma_1 = 1) \\
 &= \frac{1}{4\sqrt{P}} \sqrt{\Gamma_{1/4}} \Gamma_{3/4} = \frac{1}{4\sqrt{P}} \sqrt{\frac{\pi}{4}} \Gamma_{1/4} \quad (\text{Here } P_i = \frac{1}{4}) \\
 &\quad (\because \Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin p\pi}, \text{ where } 0 < p < 1) \\
 &= \frac{1}{4\sqrt{P}} \frac{\pi}{\sin(\frac{\pi}{4})} \\
 &= \frac{\pi}{4 \times \frac{1}{\sqrt{2}} \sqrt{P}} = \frac{\pi}{2\sqrt{2}\sqrt{P}}
 \end{aligned}$$

$$\begin{aligned}
 2m-1 &= \frac{1}{2}; & 2n+1 &= \frac{1}{2} \\
 2m &= \frac{1}{2}; & 2n &= \frac{3}{2} \\
 m &= \frac{1}{4}; & n &= \frac{3}{4}
 \end{aligned}$$

(29)

$$\begin{aligned}
 \therefore L\{F(t)\} &= \frac{\pi}{2\sqrt{2}\sqrt{P}} \\
 \Rightarrow f(t) &= L^{-1}\left\{\frac{\pi}{2\sqrt{2}\sqrt{P}}\right\} \\
 &= \frac{\pi}{2\sqrt{2}} L^{-1}\left\{\frac{1}{\sqrt{P}}\right\} \\
 &= \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{\pi t}} = \frac{\sqrt{\pi}}{2\sqrt{2t}} = \frac{1}{2} \sqrt{\frac{\pi}{2t}} \\
 \therefore \int_0^\infty \cos t x^2 dx &= \frac{1}{2} \sqrt{\frac{\pi}{2t}}. \quad \text{--- (1)}
 \end{aligned}$$

(ii) Taking $t=1$ in eqn (1).

$$\int_0^\infty \cos x^2 dx = \underline{\underline{\frac{1}{2} \sqrt{\frac{\pi}{2}}}}.$$

$$\int_0^\infty e^{-tx^2} dx.$$

$$\text{Let } F(t) = \int_0^\infty e^{-tx^2} dx.$$

$$\therefore L\{F(t)\} = \int_0^\infty e^{pt} F(t) dt.$$

$$= \int_0^\infty e^{pt} \left\{ \int_0^\infty e^{-tx^2} dx \right\} dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{pt} e^{-tx^2} dt \right] dx, \text{ changing the order of integration}$$

$$= \Gamma(m) \Gamma(n) \frac{t^{m+n-1}}{\Gamma(m+n)}$$

putting $t=1$ in the above result, we get

$$B(m,n) = \int_0^1 u^{m-1} (1-u)^{n-1} du = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{where } m>0, n>0. \quad (2)$$

Deduction:

$$\text{Taking } dt = \sin^2 \theta.$$

$$dt = 2 \sin \theta \cos \theta d\theta.$$

$$u=0 \Rightarrow \theta=0.$$

$$u=1 \Rightarrow \theta=\pi/2.$$

: from (2), we have

$$\int_0^{\pi/2} \sin^{2m-2} \theta (\frac{1-\sin^2 \theta}{\sin \theta})^{n-1} 2 \sin \theta \cos \theta d\theta = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = B(m,n)$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{1}{2} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \\ = \frac{1}{2} B(m,n)$$

→ find $\int_0^{\infty} \left\{ \frac{1}{\sqrt{P(p-a)}} \right\}$ by the convolution theorem
and deduce the value of $\int_0^{\infty} \left\{ \frac{1}{P\sqrt{P+a}} \right\}$.

Sol: let $f(p) = \frac{1}{\sqrt{p}}$ and $g(p) = \frac{1}{p}$

$$\text{Then, } F(t) = \int_0^t f(p) dp = \int_0^t \frac{1}{\sqrt{p}} dp = \frac{t^{1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi t}}$$

$$\text{and, } G(t) = \int_0^t g(p) dp = e^{-at}$$

Now, using the convolution theorem, we have

$$\int_0^{\infty} \{f(p).g(p)\} = \int_0^{\infty} F(u) G(t-u) du$$

$$\int_0^{\infty} \left\{ \frac{1}{\sqrt{P(p-a)}} \right\} = \int_0^{\infty} \frac{1}{\sqrt{\pi t} \sqrt{u}} e^{at-u} du$$

$$= \frac{e^{at}}{\sqrt{\pi t}} \int_0^{\infty} \frac{1}{\sqrt{u}} e^{-au} du$$

$$\text{put } au=x^2 \Rightarrow u=\frac{x^2}{a} \quad \text{if } u=0 \text{ then } x=0 \\ u=t \text{ then } x=\sqrt{at} \quad \text{if } u=t \text{ then } x=\sqrt{at}$$

$$du=2x dx \Rightarrow du=\frac{2x dx}{a}$$

$$= \frac{e^{at}}{\sqrt{\pi t}} \int_0^{\sqrt{at}} e^{-x^2} \frac{2}{\sqrt{a}} dx$$

$$= \frac{e^{at}}{\sqrt{\pi t}} \frac{2}{\sqrt{a}} \int_0^{\sqrt{at}} e^{-x^2} dx = \frac{e^{at}}{\sqrt{\pi t}} \operatorname{erf}(\sqrt{at}) \quad (\text{by defn of erf})$$

$$\begin{aligned} \Rightarrow \frac{du}{dx} &= \frac{2}{a} \\ \Rightarrow \frac{du}{dx} &= \frac{2}{\sqrt{a}} \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty L(e^{-tx^2}) dx \\
 &= \int_0^\infty \frac{1}{P+x^2} dx \quad [\because L\{e^{at}\} = \frac{1}{P+a}] \\
 &= \int_0^\infty \frac{1}{(\sqrt{P})^2 + x^2} dx \\
 &= \left[\frac{1}{\sqrt{P}} \tan^{-1} \left(\frac{x}{\sqrt{P}} \right) \right]_0^\infty \\
 &= \left[\frac{1}{\sqrt{P}} \frac{\pi}{2} - 0 \right] = \frac{1}{\sqrt{P}} \frac{\pi}{2}.
 \end{aligned}$$

$$\therefore L\{f(t)\} = \frac{1}{\sqrt{P}} \left(\frac{\pi}{2} \right)$$

$$F(t) = L\left\{ \frac{1}{\sqrt{P}} \left(\frac{\pi}{2} \right) \right\}$$

$$= \frac{\pi}{2} L\left(\frac{1}{\sqrt{P}} \right)$$

$$F(t) = \frac{\pi}{2} \frac{1}{\sqrt{\pi t}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{t}}$$

$$\therefore \int e^{-tx^2} dx = \frac{1}{2} \sqrt{\pi t} \quad \rightarrow ②$$

(vi) putting $t=1$ in eqn ②.

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

→ we know that $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$.

The complementary error function is defined

$$\text{erfc}(t) = 1 - \text{erf}(t)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx - \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx \quad [\because \text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx]$$

$$= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx = 1 \quad (\because \int_0^\infty e^{-x^2} dx = 1)$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_t^\infty e^{-x^2} dx + \int_0^\infty e^{-x^2} dx \right] \quad \Rightarrow \text{erfc}(0) = 1$$

$$\boxed{\text{erfc}(t) = \frac{2}{\sqrt{\pi t}} \int_t^\infty e^{-x^2} dx.}$$

\rightarrow show that $L^{-1}\left\{\frac{1}{p\sqrt{p+4}}\right\} = \frac{1}{2} \operatorname{erf}(2\sqrt{t})$.

(27)

Soln: Let $f(p) = \frac{1}{p}$ and $g(p) = \frac{1}{\sqrt{p+4}}$

$$\text{Then } f(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p}\right\} = 1$$

$$\text{and } G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{(p+4)^{1/2}}\right\} = e^{-ut} L^{-1}\left\{\frac{1}{p}\right\}_2$$

$$\therefore G(t) = \frac{e^{-ut}}{\sqrt{\pi t}} = \frac{e^{-ut}}{\sqrt{\pi t}} \cdot \frac{\Gamma_2}{\Gamma_2} \quad (\Gamma_2 = \sqrt{\pi})$$

now, using the convolution theorem.

$$\begin{aligned} L^{-1}\{f(p) g(p)\} &= \int_0^t G(u) f(t-u) du \\ &= \int_0^t e^{-tu} \frac{1}{\sqrt{\pi u}} du \\ &= \frac{1}{\sqrt{\pi}} \int_0^t e^{-tu} \frac{1}{\sqrt{u}} du. \end{aligned}$$

$$\begin{aligned} \text{put } 4u = y^2 &\Rightarrow y = 2\sqrt{u} \\ \Rightarrow du &= \frac{1}{2} y dy. \end{aligned}$$

$$\begin{aligned} \Rightarrow du &= \frac{1}{2} \cdot \frac{2}{y} dy \\ \Rightarrow \frac{dy}{\sqrt{u}} &= dy. \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{\pi}} \int_0^{2\sqrt{t}} e^{-y^2} dy \\ &= \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_0^{2\sqrt{t}} e^{-y^2} dy \\ &= \frac{1}{2} \operatorname{erf}(2\sqrt{t}). \quad \left(\because \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \right) \end{aligned}$$

\rightarrow Apply convolution theorem to show that-

$$\int_0^t \sin u \cos(t-u) du = \frac{1}{2} \sin t.$$

Soln: By convolution theorem, we have

$$L\left\{\int_0^t f(u) G(t-u) du\right\} = L\{f(t)\} * L\{G(t)\} \quad (1)$$

Take $F(t) = \sin t$ and $G(t) = \cos t$.

Then eqn (1) reduces to

$$L\left\{\int_0^t \sin u \cos(t-u) du\right\} = L\{\sin t\} \cdot L\{\cos t\}$$

$$= \frac{1}{2} \cdot \frac{p}{p^2+1} \cdot \frac{p}{p^2+1} = \frac{p}{(p^2+1)^2}$$

$$\therefore \int_0^t \sin u \cos(t-u) du = L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}.$$

$$= \frac{1}{2} t \sin t.$$

$\boxed{\int_0^t f(u) \cos(t-u) du = \frac{1}{2} t \sin t}$

→ Evaluate $\int_0^t J_0(u) J_0(t-u) du$.

Solⁿ By convolution theorem:

$$L\left\{ \int_0^t f(u) G(t-u) du \right\} = L\{f(t)\} \cdot L\{G(t)\}. \quad \textcircled{1}$$

Take $F(t) = J_0(t)$ and $G(t) = J_0(t)$.

Then eqn ① reduces to

$$\begin{aligned} L\left\{ \int_0^t J_0(u) J_0(t-u) du \right\} &= L\{J_0(t)\} \cdot L\{J_0(t)\} \\ &= \left(\frac{1}{\sqrt{t^2+1}}\right)^2 \quad \left(\because L\{J_0(t)\} = \frac{1}{\sqrt{t^2+1}} \right) \\ &= \frac{1}{t^2+1} \\ \therefore \int_0^t J_0(u) J_0(t-u) du &= L^{-1}\left(\frac{1}{t^2+1}\right) \\ &= \sin t. \end{aligned}$$

→ Apply convolution theorem to prove that

$$B(m, n) = \int_0^1 u^{m-1} (1-u)^{n-1} du = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0, n > 0 \quad (\text{Beta function})$$

Hence deduce that

$$\int_0^{\pi/2} \sin^m \theta \cos^{n-1} \theta d\theta = \frac{1}{2} B(m, n) = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m+n)}$$

Solⁿ By the convolution theorem,

$$L\left\{ \int_0^t f(u) G(t-u) du \right\} = L\{f(t)\} \cdot G\{F(t)\} \quad \textcircled{1}$$

Take $F(t) = t^{m-1}$ and $G(t) = t^{n-1}$.

Then eqn ① reduces to

$$\begin{aligned} L\left\{ \int_0^t u^{m-1} (t-u)^{n-1} du \right\} &= L\{t^{m-1}\} \cdot L\{t^{n-1}\} \\ &= \frac{\Gamma(m) \Gamma(n)}{t^m t^n} = \frac{\Gamma(m) \Gamma(n)}{t^{m+n}}. \end{aligned}$$

$$\begin{aligned} \therefore \int_0^t u^{m-1} (t-u)^{n-1} du &= L^{-1}\left\{ \frac{\Gamma(m) \Gamma(n)}{t^{m+n}} \right\} \\ &= \Gamma(m) \Gamma(n) L^{-1}\left\{ \frac{1}{t^{m+n}} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a} \int_0^t \cos^2 au \sin at du - \frac{1}{a} \int_0^t \sin au \cos au \cdot \cos at du \\
 &= \frac{1}{2a} \int_0^t (1 + \cos 2au) du - \frac{1}{2a} \cos at \int_0^t \sin 2au du \\
 &= \frac{1}{2a} \sin at \left[u + \frac{\sin 2au}{2a} \right]_0^t - \frac{1}{2a} \cos at \left[-\frac{\cos 2au}{2a} \right]_0^t \\
 &= \frac{1}{2a} \sin at \left[t + \frac{1}{2a} \sin 2at - 0 \right] + \frac{1}{2a} \cos at \cdot \frac{1}{2a} (\cos 2at - 1) \\
 &= \frac{t}{2a} \sin at + \frac{1}{4a^2} \sin at \sin 2at + \frac{1}{2a^2} \cos at (-\sin^2 at) \\
 &= \frac{t}{2a} \sin at + \frac{1}{2a^2} 2 \sin at \cos at - \frac{1}{2a^2} \cos at \sin at \quad \left(\because \frac{1-\cos 2at}{2} = \sin^2 at \right) \\
 &= \underline{\underline{\frac{t \sin at}{2a}}} \quad \text{Ans.}
 \end{aligned}$$

(v)

$$\mathcal{L}^{-1} \left\{ \frac{1}{p(p^2+4)^2} \right\}$$

Now

$$\mathcal{L}^{-1} \left\{ \frac{1}{p(p^2+4)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{p^2} \cdot \frac{1}{(p^2+4)^2} \right\}$$

Let $f(p) = \frac{1}{p^2}$ and $g(p) = \frac{1}{(p^2+4)^2}$

Then $F(t) = \mathcal{L}^{-1}\{f(p)\} = \mathcal{L}^{-1}\left(\frac{1}{p^2}\right) = t$

and $G(t) = \mathcal{L}^{-1}\{g(p)\} = \mathcal{L}^{-1}\left\{ \frac{1}{(p^2+4)^2} \right\}$

$$\begin{aligned}
 &= \mathcal{L}^{-1} \left\{ \frac{1}{2} \frac{d}{dp} \left(\frac{1}{p^2+4} \right) \right\} \\
 &= -\frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{d}{dp} \left(\frac{1}{p^2+4} \right) \right\}
 \end{aligned}$$

$$= -\frac{1}{2} (p)^{-1} t \mathcal{L}^{-1} \left\{ \frac{1}{(p^2+4)} \right\}$$

$$= \frac{1}{2} t \frac{1}{2} \sin 2t$$

$$= \frac{t}{4} \sin 2t$$

Now, using the Convolution theorem, we have

$$\begin{aligned}
 \mathcal{L}^{-1}\{f(p)g(p)\} &= \int_0^t G(u) F(t-u) du \\
 &= \int_0^t \frac{u}{4} \sin 2u (t-u) du \\
 &= \frac{1}{4} \int_0^t (tu - u^2) \sin 2u du
 \end{aligned}$$

$$= \frac{1}{2} t \sin t$$

$$\therefore \int_0^t f(u) \cos(t-u) du = \frac{1}{2} t \sin t.$$

→ Evaluate $\int_0^t J_0(u) J_0(t-u) du$.

Solⁿ By convolution theorem,

$$L\left\{\int_0^t f(u) g(t-u) du\right\} = L\{f(t)\} \cdot L\{g(t)\}. \quad \textcircled{1}$$

Take $F(t) = J_0(t)$ and $G(t) = J_0(t)$.

Then eqn $\textcircled{1}$ reduces to

$$L\left\{\int_0^t J_0(u) J_0(t-u) du\right\} = L\{J_0(t)\} \cdot L\{J_0(t)\}.$$

$$= \left(\frac{1}{\sqrt{p^2+1}}\right)^2 \quad (\because L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}})$$

$$= \frac{1}{p^2+1}$$

$$\therefore \int_0^t J_0(u) J_0(t-u) du = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \sin t.$$

→ Apply convolution theorem to prove that

$$B(m, n) = \int_0^1 u^{m-1} (1-u)^{n-1} du = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0, n > 0 \quad (\text{Beta function})$$

Hence deduce that

$$\int_0^{\pi/2} \sin^n \theta \cos^{m-1} \theta d\theta = \frac{1}{2} B(m, n) = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m+n)}$$

Solⁿ By the convolution theorem,

$$L\left\{\int_0^t f(u) g(t-u) du\right\} = L\{f(t)\} \cdot G\{g(t)\} : \textcircled{1}$$

Take $F(t) = t^{m-1}$ and $G(t) = t^{n-1}$.

Then eqn $\textcircled{1}$ reduces to

$$L\left\{\int_0^t u^{m-1} (t-u)^{n-1} du\right\} = L\{t^{m-1}\} \cdot L\{t^{n-1}\}.$$

$$= \frac{\Gamma(m)}{p^m} \frac{\Gamma(n)}{p^n} = \frac{\Gamma(m) \Gamma(n)}{p^{m+n}}$$

$$\therefore \int_0^t u^{m-1} (t-u)^{n-1} du = L^{-1}\left\{\frac{\Gamma(m) \Gamma(n)}{p^{m+n}}\right\}$$

$$= \Gamma(m) \Gamma(n) L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

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on $x=0$ and the other end on $t=\infty$.

For this strip x varies from 0 to ∞ . Hence changing the order of integration $\textcircled{1}$ reduces to.

$$\begin{aligned}
 L\{H(t)\} &= \int_{x=0}^{\infty} F(x) \left\{ \int_{t=x}^{\infty} e^{-pt} G(t-x) dt \right\} dx \\
 &= \int_{x=0}^{\infty} F(x) \left\{ \int_{y=0}^{\infty} e^{-p(x+y)} G(y) dy \right\} dx \\
 &= \int_{x=0}^{\infty} F(x) \left\{ e^{-px} \int_{y=0}^{\infty} e^{-py} G(y) dy \right\} dx \\
 &= \int_{x=0}^{\infty} e^{-px} F(x) dx \left[\int_{y=0}^{\infty} e^{-py} G(y) dy \right] \\
 &= \left[\int_{x=0}^{\infty} e^{-px} F(x) dx \right] \left[\int_{t=0}^{\infty} e^{-pt} G(t) dt \right] \quad (\text{By the Property of definite integral}) \\
 &= L\{F(p)\} \cdot L\{G(p)\}
 \end{aligned}$$

$$\begin{aligned}
 H(t) &= L^{-1}\{f(p), G(p)\} \\
 L^{-1}\{f(p), g(p)\} &= H(t) \\
 L^{-1}\{f(p)g(p)\} &= \int_0^t F(x) G(t-x) dx \\
 &= \underline{F * G}
 \end{aligned}$$

Note: (1). The convolution theorem can be re-written as

$$\begin{aligned}
 L\left\{ \int_0^t F(x) G(t-x) dx \right\} &= L\{F(t)*G(t)\} \\
 &= L\{F(t)\} \cdot L\{G(t)\}
 \end{aligned}$$

(2). while using the convolution theorem, we use one of the following two forms.



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$$\mathcal{L}^{-1}\{f(p), g(p)\} = \int_0^t F(x) G(t-x) dx \quad (\text{or})$$

$$\mathcal{L}^{-1}\{f(p), g(p)\} = \int_0^t G(x) F(t-x) dx.$$

→ Use the convolution theorem to find ..

$$(i) \mathcal{L}^{-1}\left\{\frac{1}{(p+a)(p+b)}\right\} \quad (ii) \mathcal{L}^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\} \quad (iii) \mathcal{L}^{-1}\left\{\frac{1}{(p+1)(p-1)}\right\} \quad (iv) \mathcal{L}^{-1}\left\{\frac{1}{(p-1)(p+2)}\right\}$$

$$(v) \mathcal{L}^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} \quad (vi) \mathcal{L}^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} \quad (vii) \mathcal{L}^{-1}\left\{\frac{1}{(p-2)(p+1)}\right\}$$

Sol'n: (i) $\mathcal{L}^{-1}\left\{\frac{1}{(p+a)(p+b)}\right\}$

Let $f(p) = \frac{1}{p+a}$; $g(p) = \frac{1}{p+b}$

then, $F(t) = \mathcal{L}^{-1}\{f(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p+a}\right\} = e^{-at}$

and $G(t) = \mathcal{L}^{-1}\{g(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p+b}\right\} = e^{-bt}$

Now using the convolution theorem, we have

$$\mathcal{L}^{-1}\{f(p)g(p)\} = \int_0^t F(u)G(t-u)du$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(p+a)} \frac{1}{(p+b)}\right\} &= \int_0^t e^{-au} e^{-b(t-u)} du \\ &= e^{-bt} \int_0^t e^{(b-a)u} du \\ &= e^{-bt} \left[\frac{e^{(b-a)u}}{b-a} \right]_0^t = \frac{e^{-bt}}{b-a} \left[e^{(b-a)t} - 1 \right] = \frac{1}{(b-a)} [e^{-at} - e^{-bt}] \end{aligned}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{p+a} \frac{1}{p+b}\right\} = \frac{1}{(b-a)} \{e^{-at} - e^{-bt}\}$$

→ $\mathcal{L}^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

Sol'n: Let $f(p) = \frac{p}{p^2+a^2}$ and $g(p) = \frac{1}{p^2+a^2}$

then $F(t) = \mathcal{L}^{-1}\{f(p)\} = \mathcal{L}^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

and $G(t) = \mathcal{L}^{-1}\{g(p)\} = \mathcal{L}^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a} \sin at$

Now, using the convolution theorem, we have

$$\mathcal{L}^{-1}\{f(p)g(p)\} = \int_0^t F(u)G(t-u)du$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} &= \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u) du \\ &= \frac{1}{a} \int_0^t \cos au [\sin a(t-u) - \cos a(t-u)] du. \end{aligned}$$

Convolution

→ Another important general Property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms $f(p)$ and $g(p)$ whose inverses $F(t)$ and $G(t)$ (i.e. $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$).

we would like to calculate the inverse of the product

$h(p) = f(p) \cdot g(p)$ (i.e. $L^{-1}\{h(p)\} = L^{-1}\{f(p)g(p)\}$). From those known inverses $F(t)$ and $G(t)$. This inverse $H(t)$ is denoted by $(F * G)(t)$ and is called Convolution of $F(t)$ and $G(t)$.

→ Let $F(t)$ and $G(t)$ be two functions of class A then the convolution of the two functions $F(t)$ and $G(t)$ denoted by $F * G$ and is defined as

$$F * G = \int_0^t F(x) G(t-x) dx.$$

Properties of Convolution:

(i) $F * G = G * F$
(i.e, Commutative)

Sol'n: $F * G \Rightarrow \int_0^t F(x) G(t-x) dx$ Putting $t-x=y$
 $\Rightarrow x=t-y$
 $dx=-dy$

$$\begin{aligned} &= \int_0^t F(t-y) G(y) (-dy) \\ &= - \int_t^0 F(t-y) G(y) dy \\ &= \int_0^t G(y) F(t-y) dy \\ &= G * F \end{aligned}$$



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$$\text{H.W.} \rightarrow (ii) (F * G) * H = F * (G * H)$$

$$(iii) - F * (G + H) = (F * G) + (F * H)$$

$$\begin{aligned}\underline{\text{Sol'n:}} \quad F * (G + H) &= - \int_0^t F(x) (G(t-x) + H(t-x)) dx \\ &= \int_0^t F(x) [G(t-x) + H(t-x)] dx \\ &= \int_0^t F(x) G(t-x) dx + \int_0^t F(x) H(t-x) dx \\ &= (F * G) + (F * H)\end{aligned}$$

\rightarrow Convolution theorem (Convolution Property):—

Let $F(t)$ and $G(t)$ be two functions of class A and let

$$L^{-1}\{f(p)\} = f(t) \text{ and } L^{-1}\{g(p)\} = g(t) \text{ then}$$

$$L^{-1}\{f(p) \cdot g(p)\} = \int_0^t F(x) G(t-x) dx = F * G$$

Proof: We have to show that

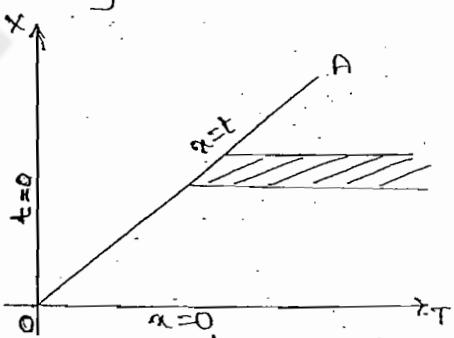
$$L\left\{ \int_0^t F(x) G(t-x) dx \right\} = f(p) \cdot g(p)$$

$$\text{Let } H(t) = \int_0^t F(x) G(t-x) dx = F * G.$$

$$\text{since } L\{H(t)\} = \int_{t=0}^{\infty} e^{-pt} H(t) dt \text{ (by definition of L.T.)}$$

$$\therefore L\{H(t)\} = \int_{t=0}^{\infty} e^{-pt} \left[\int_{x=0}^t F(x) G(t-x) dx \right] dt \quad \text{--- (1)}$$

In equation (1), the region of integration in the double integral is the infinite area below the line OA (with equation $x=t$) and above the line OT (with equation $x=0$). Here t & x are measured along OT and OX respectively.



To change the order of Integration:

Draw an elementary strip parallel to T-axis. one end lies

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$$\begin{aligned}
 &= \text{cost} + 1 = 1 - \text{cost} = f_1(t), \text{ say} \\
 \text{Let } L^{-1} \left\{ \frac{1}{P(P^r+1)} \right\}. &\rightarrow \text{Find } L^{-1} \left\{ \frac{1}{P(P^r+1)} \right\} \\
 &\rightarrow \cancel{\text{Find } L^{-1} \left\{ \frac{1}{P} \log \frac{P+2}{P+1} \right\}} \\
 &\text{So! } f(P) = \log \frac{P+2}{P+1} \\
 &= \log(P+2) - \log(P+1) \\
 &= f_1(P) = \frac{1}{P^r+1} \\
 &\therefore L^{-1} \left\{ \frac{1}{P^r+1} \right\} = L^{-1} \left\{ \frac{1}{P} - \frac{1}{P+1} \right\} \\
 &- L^{-1} \left\{ f_1(P) \right\} = L^{-1} \left(\frac{1}{P+2} \right) - L^{-1} \left(\frac{1}{P+1} \right) \\
 &= e^{-2t} - e^{-t} \quad [\because L^{-1} f_1(P) \\
 &= (-1)^r t L^{-1}(f_1(P))] \\
 &\rightarrow L^{-1} \left\{ f(P) \right\} = \frac{e^{-t} - e^{-2t}}{t} = F(t), \text{ say} \\
 &\therefore L^{-1} \left\{ \frac{1}{P} f(P) \right\} = \int_0^t F(x) dx \\
 &= \int_0^t \frac{e^{-x} - e^{-2x}}{x} dx \\
 &\therefore L^{-1} \left\{ \frac{1}{P} f(P) \right\} = \int_0^t \left[\frac{e^{-x} - e^{-2x}}{x} \right] dx \\
 &\rightarrow \text{Find } L^{-1} \left\{ \frac{1}{P} \left\{ \log \left(1 + \frac{1}{P^2} \right) \right\} \right\} \\
 &\rightarrow \cancel{\text{Find } L^{-1} \left\{ \frac{1}{P(P+1)^3} \right\}} \\
 &\text{So! Given } L^{-1} \left\{ \frac{1}{P(P+1)^3} \right\} \\
 &\text{Clearly it is in the form of } L^{-1} \left\{ \frac{f(P)}{P} \right\} \\
 &\text{Let } f(P) = \frac{1}{(P+1)^3}
 \end{aligned}$$



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$$\begin{aligned}
 \Rightarrow L^{-1}\{f(P)\} &= L^{-1}\left\{\frac{1}{(P+1)^3}\right\} \\
 &= e^{-t} L^{-1}\left\{\frac{1}{P^3}\right\} \\
 &= e^{-t} \frac{t^2}{2} = f(t), \text{ say} \\
 \text{Now } L^{-1}\left\{\frac{1}{P(P+1)^3}\right\} &= \int_0^t F(x) dx \\
 &= \int_0^t e^{-x} \frac{x^2}{2} dx \\
 &= \frac{1}{2} \int_0^t e^{-x} x^2 dx \\
 &= \frac{1}{2} \left[(-e^{-x} x^2)_0^t + 2 \int_0^t e^{-x} x dx \right] \\
 &= \frac{1}{2} \left[[-e^{-t} t^2 + 0] + 2 \left[-e^{-x} x \right]_0^t \right. \\
 &\quad \left. - 2 \left[e^{-x} \right]_0^t \right] \\
 &= \frac{1}{2} e^{-t} t^2 + (-e^{-t} t + 0) - (e^{-t} - 1) \\
 &= \frac{1}{2} e^{-t} t^2 - e^{-t} t - e^{-t} + 1 \\
 &= 1 - e^{-t} \left[\frac{t^2}{2} + t + 1 \right]
 \end{aligned}$$

\rightarrow If $L^{-1}\left\{\frac{P}{(P^2+1)^2}\right\} = \frac{t}{2} \sin t$,

find $L^{-1}\left\{\frac{1}{(P^2+1)^2}\right\}$.

Soln: $L^{-1}\left\{\frac{1}{(P^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{P} \cdot \frac{P}{(P^2+1)^2}\right\}$

which is in the form of $L^{-1}\left\{\frac{f(P)}{P}\right\}$

Here $f(P) = \frac{P}{(P^2+1)^2}$

and we know that
 $L^{-1}\left\{\frac{P}{(P^2+1)^2}\right\} = \frac{t}{2} \sin t = F(t)$, say

$$\begin{aligned}
 L^{-1}\left\{\frac{1}{P} \cdot \frac{P}{(P^2+1)^2}\right\} &= \int_0^t F(x) dx \\
 &= \int_0^t \frac{x}{2} \sin x dx
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow L^{-1}\left\{\frac{1}{(P^2+1)^2}\right\} &= \frac{1}{2} \left[-x \cos x + \sin x \right]_0^t \\
 &= \frac{1}{2} (\sin t - t \cos t)
 \end{aligned}$$

→ Find $\mathbb{E}^{-1}\left\{\frac{e^{-\sqrt{P}}}{P}\right\}$, and hence deduce that
 $\mathbb{E}^{-1}\left\{\frac{e^{-x\sqrt{P}}}{P}\right\} = \text{erfc } c\left(\frac{x}{2\sqrt{P}}\right)$.

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Soln: Let $f(c) = \frac{e^{-\sqrt{P}}}{P}$

$$\Rightarrow \mathbb{E}\{f(c)\} = e^{-\sqrt{P}}$$
$$\Rightarrow f(c) = \mathbb{E}\{e^{-\sqrt{P}c}\}$$

$$= \mathbb{E}\left\{1 - \sqrt{P} + \frac{(\sqrt{P})^2}{2!} - \frac{(\sqrt{P})^3}{3!} + \frac{(\sqrt{P})^4}{4!} - \dots\right\}$$
$$= \mathbb{E}\left[1 - \sqrt{P} + \frac{P}{2!} + \frac{P^2}{3!} + \frac{P^3}{4!} - \dots\right]$$
$$= \mathbb{E}\{1\} - \mathbb{E}\{P Y_2\} + \frac{1}{2!} \mathbb{E}\{P^2\} + \frac{1}{3!} \mathbb{E}\{P^3 Y_2\}$$
$$+ \frac{1}{4!} \mathbb{E}\{P^4\} + \dots$$
$$= \mathbb{E}\{1\} - \mathbb{E}\left\{\frac{1}{P Y_2}\right\} + \frac{1}{2!} \mathbb{E}\left\{\frac{1}{P^2}\right\} - \frac{1}{3!} \mathbb{E}\left\{\frac{1}{P^3 Y_2}\right\}$$
$$+ \frac{1}{4!} \mathbb{E}\left\{\frac{1}{P^4}\right\} - \dots$$

→ Heaviside's expansion theorem (or formula).

Let $F(p)$ and $G(p)$ be two polynomials in p where $F(p)$ has degree less than that of $G(p)$. If $G(p)$ has n distinct zeros α_r , ($r = 1, 2, \dots, n$).

i.e., $G(p) = (p-\alpha_1)(p-\alpha_2) \dots (p-\alpha_n)$. Then

$$\mathcal{L}^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^n \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}.$$

PROOF: Since $F(p)$ is a polynomial of degree less than that of $G(p)$ and $G(p)$ has n distinct zeros α_r , $r = 1, 2, \dots, n$.

$$\begin{aligned} \frac{F(p)}{G(p)} &= \frac{F(p)}{(p-\alpha_1)(p-\alpha_2) \dots (p-\alpha_n)} \\ &= \frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \dots + \frac{A_r}{p-\alpha_r} + \dots + \frac{A_n}{p-\alpha_n}. \end{aligned}$$

To compute A_r , multiplying both sides by $p-\alpha_r$.

$$\text{i.e., } \frac{F(p)}{G(p)} \cdot (p-\alpha_r) = (p-\alpha_r) \left[\frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \dots + \frac{A_r}{p-\alpha_r} + \dots + \frac{A_n}{p-\alpha_n} \right]$$

Taking limit as $p \rightarrow \alpha_r$, we get

$$\Rightarrow \lim_{p \rightarrow \alpha_r} \frac{F(p)}{G(p)} (p-\alpha_r) = A_r.$$

$$\begin{aligned} \Rightarrow A_r &= \lim_{p \rightarrow \alpha_r} \frac{F(p)}{G(p)} (p-\alpha_r) \\ &= F(\alpha_r) \lim_{p \rightarrow \alpha_r} \frac{(p-\alpha_r)}{G(p)} \quad (\text{Form } \frac{0}{0}) \\ &= F(\alpha_r) \lim_{p \rightarrow \alpha_r} \frac{1}{G'(p)} \quad \text{where } G(p) = \frac{(p-\alpha_1)}{(p-\alpha_2)} \dots \frac{(p-\alpha_r)}{(p-\alpha_{r+1})} \dots \frac{(p-\alpha_n)}{(p-\alpha_1)} \\ &= F(\alpha_r) \frac{1}{G'(\alpha_r)} \end{aligned}$$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)} \frac{1}{(p-\alpha_1)} + \frac{F(\alpha_2)}{G'(\alpha_2)} \frac{1}{(p-\alpha_2)} + \frac{F(\alpha_3)}{G'(\alpha_3)} \frac{1}{(p-\alpha_3)} + \dots + \frac{F(\alpha_r)}{G'(\alpha_r)} \frac{1}{(p-\alpha_r)} + \dots + \frac{F(\alpha_n)}{G'(\alpha_n)} \frac{1}{(p-\alpha_n)}$$

$$\text{Hence } \mathcal{L}^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} \mathcal{L}^{-1}\left(\frac{1}{p-\alpha_1}\right) + \frac{F(\alpha_2)}{G'(\alpha_2)} \mathcal{L}^{-1}\left(\frac{1}{p-\alpha_2}\right) + \dots + \frac{F(\alpha_r)}{G'(\alpha_r)} \mathcal{L}^{-1}\left(\frac{1}{p-\alpha_r}\right) + \dots + \frac{F(\alpha_n)}{G'(\alpha_n)} \mathcal{L}^{-1}\left(\frac{1}{p-\alpha_n}\right)$$

$$\begin{aligned}
 y(t) &= L^{-1} \left\{ \frac{P}{(P^2+4)(P^2+9)} \right\} + A L^{-1} \left\{ \frac{P}{P^2+9} \right\} + A L^{-1} \left\{ \frac{1}{P^2+9} \right\} \\
 &= \frac{1}{5} L^{-1} \left\{ \frac{P}{(P^2+4)} - \frac{P}{(P^2+9)} \right\} + L^{-1} \left\{ \frac{P}{P^2+9} \right\} + \frac{A}{3} L^{-1} \left\{ \frac{1}{P^2+9} \right\} \\
 &= \frac{1}{5} L^{-1} \left\{ \frac{P}{P^2+4} \right\} - \frac{1}{5} L^{-1} \left\{ \frac{P}{P^2+9} \right\} + \cos 3t + \frac{A}{3} \sin 3t
 \end{aligned}$$

$$y(t) = \frac{1}{5} \cos 2t - \frac{1}{5} \cos 3t + \cos 3t + \frac{A}{3} \sin 3t$$

$$\Rightarrow y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{A}{3} \sin 3t$$

$$y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{A}{3} \sin 3t \quad (\because y(\frac{\pi}{2}) = 1)$$

$$1 = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{A}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow -\frac{4}{5} = -\frac{A}{3}$$

$$\Rightarrow A = \frac{12}{5}$$

\therefore The required solution is

$$y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t$$

$$y(t) = \underline{\underline{\frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t}}$$

$$\rightarrow (D^2+3D+2)y = 0; \quad y=y_0 \text{ and } Dy=y_1 \text{ at } t=0$$

$$\rightarrow \text{solve } (D^2+3D+2)y = 20 \sin 2t; \quad y=-1, Dy=2 \text{ when } t=0$$

$$\rightarrow \text{solve } (D+1)^2 y = t; \quad y=-3 \text{ when } t=0$$

$$\text{and } y=-1 \text{ when } t=1$$

$$\rightarrow \text{solve } (D^2+2D+1)y = 3te^t, t>0$$

subject to the conditions $y=4, \frac{dy}{dt}=2$ when $t=0$

$$\rightarrow \text{solve } (D^2+1)y = t \cos 2t, \quad y=0, \quad \frac{dy}{dt}=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^2-3D+2)y = 1-e^{2t}, \quad y=1, \quad Dy=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^2+1)y = \sin t \sin 2t, \quad t>0$$

$$\text{if } y=1; \quad Dy=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^3-D)y = 2 \cos t, \quad y=3, \quad Dy=2, \quad D^2y=1 \text{ when } t=0$$

Solution of ordinary differential equations with Constant coefficients

(32)

The Laplace transform is very useful in solving ordinary linear differential eqns. with constant coefficients.

Suppose we wish to solve the n^{th} order ordinary differential eqn with constant coefficients

$$\frac{dy}{dt^n} + a_1 \frac{dy^{n-1}}{dt^{n-1}} + a_2 \frac{dy^{n-2}}{dt^{n-2}} + \dots + a_n y = f(t) \quad (1)$$

where $f(t)$ is a function of the independent variable t . and $a_0, a_1, a_2, \dots, a_n$ are constants, subject to the initial conditions

$$y(0) = k_0, \quad y'(0) = k_1, \quad y''(0) = k_2, \quad \dots \quad y^{(n)}(0) = k_{n+1} \quad (2)$$

where k_0, k_1, \dots, k_{n+1} are constants.

On taking the Laplace transform of both sides of eqn(1) and using conditions (2), we obtain an algebraic equation known as "subidaly eqn" for determination of $L\{y(t)\}$. The required solution is then obtained by finding the inverse Laplace transform of

$$L\{y(t)\} = f(t).$$

Notations

$$\rightarrow \frac{dy}{dt} = D y = y'(t) = y^{(1)}(t);$$

$$\frac{d^2y}{dt^2} = D^2 y = y''(t) = y^{(2)}(t); \quad \dots \quad \frac{d^n y}{dt^n} = D^n y = y^{(n)}(t) \text{ etc.}$$

\rightarrow At $t=0$, we have

$$y(0) = y_0, \quad y'(0) = y_1, \quad y''(0) = y_2, \quad \dots \quad y^{(n)}(0) = y_n.$$

→ Solve $\frac{d^2y}{dt^2} + y = 0$ under the conditions that $y=1$
 $\frac{dy}{dt}=0$ when $t=0$.

Sol: Given that $\frac{d^2y}{dt^2} + y = 0$, i.e. $y'' + y = 0 \quad \text{--- (1)}$

Taking Laplace transform of both sides of eqn(1),

wegner
 $L\{y''\} + L\{y\} = L(0)$

$$p^2 L\{y(t)\} - p y(0) - y'(0) + L\{y(t)\} = 0$$

$$p^2 L\{y(t)\} - p(1) - 0 + L\{y(t)\} = 0$$

$$(p^2 + 1) L\{y(t)\} = p$$

$$\Rightarrow L\{y(t)\} = \frac{p}{p^2 + 1}$$

Taking inverse Laplace transform, we get
 $\Rightarrow y(t) = L^{-1}\left\{\frac{p}{p^2 + 1}\right\}$
 $= \text{cost.}$

$y(t) = \text{cost.}$
which is the required solution.

→ Solve $(D^2 + m^2)x = a \cos nt$, $t > 0$.
 $x = x_0$ and $Dx = x_1$, when $t = 0$, $n \neq m$

Sol: Given eqn is

$$(D^2 + m^2)x = a \cos nt$$

$$x'' + m^2 x = a \cos nt \quad \text{--- (1)}$$

Taking Laplace transform of both sides of (1),

we get
 $L\{x''\} + L\{m^2 x\} = L\{a \cos nt\}$

$$p^2 L\{x(t)\} - p x(0) - x'(0) + m^2 L\{x(t)\} = \frac{ap}{p^2 + n^2} \quad \text{--- (2)}$$

Using the given conditions $x = x_0$ and $Dx = x_1$ when $t > 0$

eqn(2) reduces to

$$p^2 L\{x(t)\} - p x_0 - x_1 + m^2 L\{x(t)\} = \frac{ap}{p^2 + n^2}$$

$$\Rightarrow (p^2 + m^2) L\{x(t)\} = \frac{ap}{p^2 + n^2} + px_0 + x_1$$

$$\Rightarrow L\{x(t)\} = \frac{p}{p^2 + m^2} x_0 + \frac{1}{p^2 + m^2} x_1 + \frac{a \cdot p}{(p^2 + m^2)(p^2 + n^2)} \quad (33)$$

$$= x_0 \frac{p}{p^2 + m^2} + x_1 \frac{1}{p^2 + m^2} + \frac{a}{m^2 - n^2} \left[\frac{-p}{p^2 + m^2} + \frac{l}{p^2 + n^2} \right]$$

Taking the inverse Laplace transform, we get

$$x(t) = x_0 \mathcal{L}^{-1} \left\{ \frac{p}{p^2 + m^2} \right\} + x_1 \mathcal{L}^{-1} \left\{ \frac{1}{p^2 + m^2} \right\} + \frac{a}{m^2 - n^2} \left[\mathcal{L}^{-1} \left\{ \frac{-p}{p^2 + m^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{l}{p^2 + n^2} \right\} \right]$$

$$= x_0 \cos mt + x_1 \frac{1}{m} \sin mt + \frac{a}{m^2 - n^2} (-\cos nt + \cos nt)$$

$$x(t) = x_0 \cos mt + x_1 \frac{1}{m} \sin mt + \frac{a}{m^2 - n^2} [-\cos nt + \cos nt]$$

$$\rightarrow \text{Solve } (D+2)^2 y = 4e^{-2t}, y(0) = 1 \quad \& \quad y'(0) = 4.$$

$$\rightarrow \text{Solve } (D^2 - 2D + 2)y = 0, y = Dy = 1, \text{ when } t = 0$$

$$\rightarrow \text{Solve } (D^2 + 4D + 4)y = \sin nt, t > 0 \text{ with } x_0 \text{ and } x_1$$

for values of x_0 and Dx when $t = 0$ (i.e., $x_0 = x_1$, what = 0)

$$\checkmark \text{ Solve } (D^2 + 9)y = \cos 2t \quad \text{if } y(0) = 1, y'(0) = -1.$$

$$\text{Solve Given eqn } (D^2 + 9)y = \cos 2t \quad (1)$$

Taking Laplace transform of both sides

of eqn (1), we get

$$L\{y'' + 9y\} = L\{\cos 2t\}$$

$$\Rightarrow P^2 L\{y(t)\} - P y(0) - y'(0) = \frac{P}{P^2 + 4} \quad (2)$$

Using the given condition $y(0) = 1$ eqn (2) reduces to

$$P^2 L\{y(t)\} - P(1) = A + 9L\{y(t)\} = \frac{P}{P^2 + 4}$$

(Taking $y'(0) = A$ constant)

$$\Rightarrow (P^2 + 9)L\{y(t)\} = \frac{P}{P^2 + 4} + P + A.$$

$$\Rightarrow L\{y(t)\} = \frac{P}{(P^2 + 4)(P^2 + 9)} + \frac{P}{P^2 + 9} + \frac{A}{P^2 + 9}.$$

Taking inverse Laplace transform, we get

$$\begin{aligned}
 y(t) &= L^{-1} \left\{ \frac{P}{(P^2+4)(P^2+9)} \right\} + A L^{-1} \left\{ \frac{P}{P^2+9} \right\} + A L^{-1} \left\{ \frac{1}{P^2+9} \right\} \\
 &= \frac{1}{5} L^{-1} \left\{ \frac{P}{(P^2+4)} - \frac{P}{(P^2+9)} \right\} + L^{-1} \left\{ \frac{P}{P^2+9} \right\} + \frac{A}{3} L^{-1} \left\{ \frac{1}{P^2+9} \right\} \\
 &= \frac{1}{5} L^{-1} \left\{ \frac{P}{P^2+4} \right\} - \frac{1}{5} L^{-1} \left\{ \frac{P}{P^2+9} \right\} + \cos 3t + \frac{A}{3} \sin 3t
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= \frac{1}{5} \cos 2t - \frac{1}{5} \cos 3t + \cos 3t + \frac{A}{3} \sin 3t \\
 \Rightarrow y(t) &= \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{A}{3} \sin 3t
 \end{aligned}$$

$$y(0) = \frac{1}{5} \cos 0 + \frac{4}{5} \cos 0 + \frac{A}{3} \sin 0 \quad (\because y(0) = -1)$$

$$-1 = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{A}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow -\frac{4}{5} = -\frac{A}{3}$$

$$\Rightarrow A = \frac{12}{5}$$

∴ The required solution is

$$y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t$$

$$\underline{y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t}$$

$$\rightarrow (D^2+3D+2)y = 0 ; y=y_0 \text{ and } Dy=y_1 \text{ at } t=0$$

$$\rightarrow \text{solve } (D^2+3D+2)y = 20 \sin 2t ; y=-1, Dy=2 \text{ when } t=0$$

$$\rightarrow \text{solve } (D+1)^2 y = t ; y=-3 \text{ when } t=0$$

$$\text{and } y=-1 \text{ when } t=1$$

$$\rightarrow \text{solve } (D^2+2D+1)y = 3te^{-t}, t>0$$

subject to the conditions $y=4, \frac{dy}{dt}=2$ when $t=0$

$$\rightarrow \text{solve } (D^2+1)y = t \cos 2t, y=0, \frac{dy}{dt}=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^2-3D+2)y = 1-e^{-t}, y=1, Dy=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^2+1)y = \sin t \cos 2t, t>0$$

$$\text{if } y=1, Dy=0 \text{ when } t=0$$

$$\rightarrow \text{solve } (D^3-D)y = 2 \cos t, y=3, \frac{dy}{dt}=2, \frac{d^2y}{dt^2}=1 \text{ when } t=0$$

- Solve $(D^3 - D^2 - D + 1)y = 8te^t$
if $y = D^2y = 0$, $Dy = 1$ when $t=0$ (74)
- solve $(D^2 - 1)y = 1$, $y = Dy = D^2y = D^3y = 0$ at $t=0$
- solve $(D^2 + D)y = t^2 + 2t$. where $y(0) = 0$, $y'(0) = -2$
- solve $(D^4 + 2D^2 + 1)y = 0$ where $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$ and $y'''(0) = -3$.
- solve $(D^3 + 1)y = 1$, $t > 0$
 $y = Dy = D^2y = 0$ when $t=0$
- solve $(D^2 + n^2)y = a \sin(nt + \alpha)$, $y=0$, $Dy=0$ when $t=0$

Independence of Solution of Linear Differential Equations

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2 \log x. \quad \dots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $y' = 2Ax + B(2x \log x + x) \dots(2)$

Differentiating (2), $y'' = 2A + B(2 \log x + 2 + 1). \dots(3)$

We now eliminate A and B from (1), (2) and (3). To this end, we first solve (2) and (3) for A and B . Multiplying both sides of (3) by x , we get

$$xy'' = 2Ax + B(3x + 2x \log x). \quad \dots(4)$$

Subtracting (2) from (4), $xy'' - y' = 2Bx$ or $B = (xy'' - y')/2x$.

Substituting this value of B in (3), we have

$$2A = y'' - (1/2x)(xy'' - y')(3 + 2 \log x)$$

or $A = (1/4x)[2xy'' - (xy'' - y')(3 + 2 \log x)]$.

Substituting the above values A and B in (1), we have

$$y = (x/4)[2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \log x (xy'' - y')$$

or $4x = x(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x \log x (xy'' - y')$

or $x^2y'' - 3xy' + 4y = 0$, which is the required equation.

Ex. 8. Evaluate the Wronskian of the functions x and $x e^x$. Hence conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.

[Meerut 97]

Sol. Let $y_1 = x$ and $y_2 = x e^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x & x e^x \\ 1 & e^x + x e^x \end{vmatrix} = x e^x + x^2 e^x - x e^x = x^2 e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence y_1 and y_2 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax + Bx e^x, \quad \dots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $y' = A + B(e^x + xe^x) = A + B(1+x)e^x. \dots(2)$

Differentiating (2), $y'' = B[e^x + (1+x)e^x] = Be^x(2+x). \dots(3)$

We now eliminate A and B from (1), (2) and (3). To this we first solve (2) and (3) for A and B .

From (3), $B = y''/[e^x(2+x)].$

Substituting this value of B in (2), we have

$$A = y' - B(1+x)e^x = y' - \frac{1+x}{2+x}y'' = \frac{(2+x)y' - (1+x)y''}{2+x}$$

* Independence of Solutions of
Linear Differential Equations *
Wronskian and Its Properties

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SOLVED EXAMPLES

Ex. 1. If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of differential equation $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions. [Delhi B.Sc. (Hons) 1996]

Sol. The Wronskian of $y_1(x)$ and $y_2(x)$

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3\sin^2 3x - 3\cos^2 3x \\ &= -3(\sin^2 3x + \cos^2 3x) = -3 \neq 0. \end{aligned}$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

Ex. 2. Prove that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation $y'' + 4y = 0$ and these solutions are linearly independent. [Delhi (B.Sc.) (G) 1998]

Sol. Given equation is $y'' + 4y = 0$ (1)

Let $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ (2)

Now, $y_1' = 2\cos 2x$ and $y_1'' = -4\sin 2x$ (3)

$$y_1'' + 4y_1 = -4\sin 2x + 4\sin 2x = 0, \text{ by (2) and (3)}$$

Hence, $y_1(x) = \sin 2x$ is a solution of (1). Similarly we can prove that $y_2(x)$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2\sin^2 2x - 2\cos^2 2x \\ &= -2(\sin^2 2x + \cos^2 2x) = -2 \neq 0. \end{aligned}$$

Since $W(x) \neq 0$, $\sin 2x$ and $\cos 2x$ are linearly independent solutions of (1).

Ex. 3. Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$. [Delhi B.Sc. (P) 96, Delhi B.Sc. (H) 2002]

Sol. Given equation is $y'' - 2y' + 2y = 0$ (1)

Let $y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x$ (2)

From (2), $y_1' = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$ (3)

From (3), $y_1'' = e^x (\sin x + \cos x) + e^x (\cos x - \sin x) = 2e^x \cos x$ (4)

Independence of Solution of Linear Differential Equations

Now, $y_1''(x) - 2y_1'(x) + 2y_1(x) = 2e^x \cos x - 2e^x (\sin x + \cos x) + 2e^x \sin x = 0$,

showing that $y_1(x) = e^x \sin x$ is a solution of (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x (\sin x + \cos x) & e^x (\cos x - \sin x) \end{vmatrix}$$

$$= e^{2x} (\sin x \cos x - \sin^2 x) - e^{2x} (\sin x \cos x + \cos^2 x) = -e^{2x} \neq 0,$$

showing that $W(x) \neq 0$, and hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

The general solution of (1) is given by [Refer Art. 2.11]

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = e^x (c_1 \sin x + c_2 \cos x), \quad \dots(5)$$

where c_1 and c_2 are arbitrary constants.

$$\text{From (5), } y'(x) = e^x (c_1 \sin x + c_2 \cos x) + e^x (c_1 \cos x - c_2 \sin x) \dots(6)$$

Putting $x = 0$ in (5) and using the given result $y(0) = 2$, we get

$$y(0) = c_2 \text{ or } c_2 = 2.$$

Putting $x = 0$ in (6) and using the given result $y'(0) = -3$, we get

$$y'(0) = c_2 + c_1 \text{ or } -3 = 2 + c_1 \text{ or } c_1 = -5.$$

\therefore from (5), the solution of given equation satisfying the given properties is

$$y = e^x (2 \cos x - 5 \sin x).$$

Ex. 4. Show that e^{2x} and e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with the property that $y(0) = 0$ and $y'(0) = 1$. [Delhi B.Sc. (G) 1998]

Sol. Given equation is $y'' - 5y' + 6y = 0$. (1)

Let $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$. (2)

From (2) $y_1'(x) = 2e^{2x}$ and $y_1''(x) = 4e^{2x}$. (3)

$$\therefore y_1''(x) - 5y_1'(x) + 6y_1(x) = 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = 0,$$

showing that $y_1(x)$ is a solution of (1).

Similarly, we find that $y_2(x) = e^{3x}$ is a solution of (1),

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0,$$

showing that that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are linearly independent solutions of (1).

The general solution of (1) is given by

$$y(x) = c_1 e^{2x} + c_2 e^{3x}, c_1 \text{ and } c_2 \text{ being arbitrary constants.} \quad \dots(4)$$

$$\text{From (4), } y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}. \quad \dots(5)$$

Independence of Solution of Linear Differential Equations

Putting $x = 0$ in (4) and using $y(0) = 0$, we get $c_1 + c_2 = 0$ (6)

Putting $x = 0$ in (5) and using $y'(0) = 1$, we get $2c_1 + 3c_2 = 1$ (7)

Solving (6) and (7), $c_1 = -1$ and $c_2 = 1$ and so from (4), we have

$$y(x) = e^{3x} - e^{2x} \text{ as the required solution.}$$

Ex. 5. Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants c_1 and c_2 so that the solution $\sin x + 3 \cos x \equiv c_1 y_1(x) + c_2 y_2(x)$. [Delhi B.A. (P) 2002]

Sol. Given equation is $y'' + y = 0$ (1)

Here $y_1(x) = \sin x$ so that $y_1'(x) = \cos x$ and $y_1''(x) = -\sin x$ (2)

Hence $y_1''(x) + y_1(x) = -\sin x + \sin x = 0$, showing that $y_1(x)$ is a solution of (1). Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix} \\ &= \sin x (\cos x + \sin x) - \cos x (\sin x - \cos x) = 1 \neq 0, \end{aligned}$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $\sin x + 3 \cos x \equiv c_1 y_1(x) + c_2 y_2(x)$

or $\sin x + 3 \cos x \equiv c_1 \sin x + c_2 (\sin x - \cos x)$ (3)

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides of (3), we get

$$c_1 + c_2 = 1 \text{ and } -c_2 = 3 \text{ so that } c_1 = 4 \text{ and } c_2 = -3.$$

Ex. 6. Show that x and $x e^x$ are linearly independent on the x -axis.

Sol. The Wronskian $W(x)$ of x and $x e^x$ is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x e^x \\ \frac{dx}{dx} & \frac{d(x e^x)}{dx} \end{vmatrix} = \begin{vmatrix} x & x e^x \\ 1 & e^x + x e^x \end{vmatrix} \\ &= x(e^x + x e^x) - x e^x = x^2 e^x. \end{aligned}$$

We observe that $W(x) \neq 0$ for $x \neq 0$ on the x -axis. Hence x and $x e^x$ are linearly independent on the x -axis. [Refer corollary to theorem III of Art 2.6]

Ex. 7. Show that the Wronskian of the functions x^2 and $x^2 \log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation. If so, determine this differential equation. [Meerut 1988, 98]

Sol. Let $y_1(x) = x^2$ and $y_2(x) = x^2 \log x$.

The Wronskian $W(x)$ of y_1 and y_2 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \log x \\ 2x & 2x \log x + x \end{vmatrix} = x^2 (2x \log x + x) - 2x^3 \log x.$$

$\therefore W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$. Hence solution y_1 and y_2 can be linearly independent solutions of an ordinary differential equation.

Independence of Solution of Linear Differential Equations

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2 \log x, \quad \dots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $y' = 2Ax + B(2x \log x + x). \quad \dots(2)$

Differentiating (2), $y'' = 2A + B(2 \log x + 2 + 1). \quad \dots(3)$

We now eliminate A and B from (1), (2) and (3). To this end, we first solve (2) and (3) for A and B . Multiplying both sides of (3) by x , we get

$$xy'' = 2Ax + B(3x + 2x \log x). \quad \dots(4)$$

Subtracting (2) from (4), $xy'' - y' = 2Bx \quad \text{or} \quad B = (xy'' - y')/2x.$

Substituting this value of B in (3), we have

$$\begin{aligned} 2A &= y'' - (1/2x)(xy'' - y')(3 + 2 \log x) \\ \text{or} \quad A &= (1/4x)[2xy'' - (xy'' - y')(3 + 2 \log x)]. \end{aligned}$$

Substituting the above values A and B in (1), we have

$$y = (x/4)[2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \log x (xy'' - y')$$

$$\text{or } 4x = x(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x \log x (xy'' - y')$$

$$\text{or } x^2y'' - 3xy' + 4y = 0, \text{ which is the required equation.}$$

Ex. 8 Evaluate the Wronskian of the functions x and $x e^x$. Hence conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.

[Meerut 97]

Sol. Let $y_1 = x$ and $y_2 = x e^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x & x e^x \\ 1 & e^x + x e^x \end{vmatrix} = x e^x + x^2 e^x - x e^x = x^2 e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence y_1 and y_2 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax + Bx e^x, \quad \dots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $y' = A + B(e^x + xe^x) = A + B(1+x)e^x. \quad \dots(2)$

Differentiating (2), $y'' = B[e^x + (1+x)e^x] = Be^x(2+x). \quad \dots(3)$

We now eliminate A and B from (1), (2) and (3). To this we first solve (2) and (3) for A and B .

From (3), $B = y''/[e^x(2+x)].$

Substituting this value of B in (2), we have

$$A = y' - B(1+x)e^x = y' - \frac{1+x}{2+x}y'' = \frac{(2+x)y' - (1+x)y''}{2+x}$$

Independence of Solution of Linear Differential Equations

Substituting the above values of A and B in (1), we get

$$y = \left[\frac{(2+x)y' - (1+x)y''}{2+x} \right] x + \left[\frac{y''}{e^x(2+x)} \right] x e^x$$

$$\text{or } (2+x)y = x(2+x)y' - x(1+x)y'' + xy''$$

$$\text{or } x^2y'' - x(2+x)y' + (2+x)y = 0, \text{ which is required equation.}$$

Ex. 9.(a) Show that the solutions e^x, e^{-x}, e^{2x} of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ are linearly independent and hence or otherwise solve the given equation. [Delhi B.Sc. (G) 1993, 98 ; Meerut 87, 98]

Sol. Given equation is $y''' - 2y'' - y' + 2y = 0$ (1)

$$\text{Let } y_1 = e^x, y_2 = e^{-x} \text{ and } y_3 = e^{2x} \quad \dots (2)$$

$$\text{Here } y_1' = e^x, y_1'' = e^x \text{ and } y_1''' = e^x \quad \dots (3)$$

$$\therefore y_1''' - 2y_1'' - y_1' + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0, \text{ by (2) and (3)}$$

Hence $y_1 = e^x$ is a solution of (1). Similarly, we can show that e^{-x} and e^{2x} are also solutions of (1).

Now, the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} \\ &= (e^x \cdot e^{-x} \cdot e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \text{ by } C_2 \rightarrow C_2 - C_1 \\ &\quad C_3 \rightarrow C_3 - C_1 \\ &= -6e^{2x}, \text{ which is not identically zero on } (-\infty, \infty) \end{aligned}$$

Hence y_1, y_2, y_3 are linearly independent solutions of (1) [Refer corollary of theorem III of Art 2.6]. Since the order of the given equation (1) is three, it follows that the general solution of (1) will contain three arbitrary constants c_1, c_2, c_3 and is given by [Refer Art. 2.11]

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 \text{ i.e., } y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

Ex. 9.(b) Show that the solutions e^x, e^{-2x}, e^{2x} of $y''' - y'' - 4y' + 4 = 0$ are linearly independent and hence or otherwise solve the given equation.

Hint. Try yourself as in Ex. 9. (a) Ans. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x}$.

Ex. 10. Prove that the functions $1, x, x^2$ are linearly independent. Hence from the differential equation whose roots are $1, x, x^2$. [Meerut 1996, 97]

Sol. Let $y_1(x) = 1, y_2(x) = x$ and $y_3(x) = x^2$ (1)

Then the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}, \text{ using (1)}$$

$$\text{or } W(x) = 2 \neq 0 \text{ for any } x \in (-\infty, \infty).$$

Hence, y_1, y_2 , and y_3 are linearly independent.

Independence of Solution of Linear Differential Equations

2. Test the linear independence of the following sets of functions :

- | | |
|------------------------------------|-----------------------------|
| (i) $\sin x, \cos x.$ | [Ans. Linearly independent] |
| (ii) $1+x, 1+2x, x^2.$ | [Ans. Linearly independent] |
| (iii) $x^2-1, x^2-x+1, 3x^2-x-1.$ | [Ans. Linearly dependent] |
| (iv) $\sin x, \cos x, \sin 2x.$ | [Ans. Linearly independent] |
| (v) $e^x, e^{-x}, \sin ax.$ | [Ans. Linearly independent] |
| (vi) $e^x, x e^x, \sinh x.$ | [Ans. Linearly independent] |
| (vii) $\sin 3x, \sin x, \sin^3 x.$ | [Ans. Linearly dependent] |

3. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form the differential equation of second order having these two functions as independent solutions. [Ans. $y'' - 2y' + 2y = 0$]

4. Evaluate the Wronskian of the functions e^x and $x e^x$. Hence conclude whether or not they are linearly independent. If they are independent set up the differential equation having them as its independent solutions. [Ans. $y'' - 2y' + y = 0$]

5. Show that any two solutions of the equation $y'' + f(x)y' + g(x)y = 0$, $f(x)$ and $g(x)$ being continuous on an open interval I , are linearly independent, if and only if, their Wronskian is zero for some $x = x_0$ on I . [Meerut 1992]

[Hint. Proceed exactly as in theorem V of Art. 2.13 for $n = 2$.]

6. If the functions $p(x)$ and $q(x)$ are continuous on $\alpha < x < \beta$, and if the functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$, then prove that the Wronskian $W(y_1, y_2)$ is non-vanishing on $\alpha < x < \beta$.

[Hint. Proceed exactly as in theorem V of Art 2.13 for $n = 2$]

7. Show that linearly independent solutions of $y'' - 3y' + 2y = 0$ are e^x and e^{2x} . Find the solution $y(x)$ with the property that $y(0) = 0, y'(0) = 1$. [Ans. $y(x) = e^{2x} - e^x$]

(Delhi B.Sc. (G) 2000)

8. Show that the $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the real line, even though the Wronskian cannot be computed.

9. Show graphically that $y_1(x) = x^2$ and $y_2(x) = |x|$ are linearly independent on $-\infty < x < \infty$, however Wronskian vanishes for every real value of x .

10. Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ on any interval. [Nagpur 96]

11. Show that $y_1(x) = e^{-x/2} \sin(x\sqrt{3}/2)$ and $y_2(x) = e^{-x/2} \cos(x\sqrt{3}/2)$ are linearly independent solutions of the differential equation $y'' + y' + y = 0$. [Delhi B.Sc. (G) 1999, 2001]

[Hint: Proceed as in solved Ex. 2 on page 35]

(1)

IAS Previous Years Questions (1983–2012) Segment-wise

Ordinary Differential Equations and Laplace Transforms

1983

- ❖ Solve $x \frac{d^2y}{dx^2} + (x-1) \frac{dy}{dx} - y = x^2$.
- ❖ Solve $(y+yz) dx + (xz+z^2) dy + (y^2-xy) dz = 0$.
- ❖ Solve the equation $\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + y = t$ by the method of Laplace transform, given that $y = -3$ when $t = 0$, $y = -1$ when $t = 1$.

1984

- ❖ Solve $\frac{d^2y}{dx^2} + y = \sec x$.
- ❖ Using the transformation $y = \frac{u}{x}$ solve the equation $x y'' + (1+2k) y' + xy = 0$.
- ❖ Solve the equation $(D^2 + 1)x = t \cos 2t$, given that $x = 0$ by the method of Laplace transform.

1985

- ❖ Consider the equation $y'' + 5y = 2$. Find that solution ϕ of the equation which satisfies $\phi(1) = 3\phi'(0)$.
- ❖ Use Laplace transform to solve the differential equation $x^2 y'' + 2x y' + x y = e^x \left(\frac{d}{dt} \right)$ such that $x(0) = 2, x'(0) = -1$.
- ❖ For two functions f, g both absolutely integrable on $(-\infty, \infty)$, define the convolution $f * g$. If $L(f), L(g)$ are the Laplace transforms of f, g show that $L(f * g) = L(f)L(g)$.
- ❖ Find the Laplace transform of the function $f(t) = \int_{n=1}^{\infty} \frac{1}{(2n+1)\pi} \sin((2n+1)t) dt$ for $n = 1, 2, \dots$

1987

- ❖ Solve the equation $x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} = y + e^x$.
- ❖ If $f(t) = t^{p-1}, g(t) = t^{q-1}$ for $t > 0$ but $f(t) = g(t) = 0$ for $t \leq 0$, and $h(t) = f * g$ is the convolution of f, g show that $h(t) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} t^{p+q-1}; t \geq 0$ and p, q are positive constants. Hence deduce the formula

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

1988

- ❖ Solve the differential equation $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} = 2e^x \sin x$.
- ❖ Show that the equation $(12x+7y+1) dx + (7x+4y+1)$ $dy = 0$ represents a family of curves having as asymptotes the lines $3x+2y-1=0, 2x+y+1=0$.

Obtain the differential equation of all circles in a plane

$$\text{in the form } \frac{d^3y}{dx^3} \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} - 3 \frac{dy}{dx} \left(\frac{d^2y}{dx^2} \right)^2 = 0.$$

1989

- ❖ Find the value of y which satisfies the equation $(xy^3 - y^3 - x^2 e^x) + 3xy^2 \frac{dy}{dx} = 0$, given that $y=1$ when $x=1$.
- ❖ Prove that the differential equation of all parabolas lying in a plane $\frac{d}{dx} \left(\frac{dy}{dx} \right)^2 = 0$.
- ❖ Solve the differential equation

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} = 1 + x^2.$$

(2) Ordinary Differential Equations and Laplace Transforms

1990

- If the equation $\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$ (in unknown λ) has distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$. Show that the constant coefficients of differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n \frac{dy}{dx} + a_n = b$$

most general solution of the form

$$y = c_0(x) + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots + c_n e^{\lambda_n x}$$

where c_1, c_2, \dots, c_n are parameters. what is $c_0(x)$?Analyse the situation where the λ - equation in (a) has repeated roots.

Solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + y = 0$$

is explicit form. If your answer contains imaginary quantities, recast it in a form free of those.

- Show that if the function $\frac{1}{f(t)}$ can be integrated (w.r.t 't'), then one can solve $\frac{dy}{dx} = f(t)$ for any given f . Hence or otherwise.

$$\frac{dy}{dx} + \frac{x^2(1+3x^2)}{3x^2(1-x^2)} = 0$$

- Verify that $y = (\sin x)^2$ is a solution of $(1-x^2) \frac{dy}{dx} = -x \frac{dy}{dx} = 2$. Find also the most general solution.

1991

- If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)$. $ydx + f_2(xy) \cdot x dy = 0$, then show that $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny \neq 0$.
- Solve the differential equation $(x^2-2xy+y^2) dx + 2xy dy = 0$.
- Given that the differential equation $(2x^2y^2+y) dx - (x^3y-3x) dy = 0$ has an integrating factor of the form $x^h y^k$, find its general solution.

- Solve $\frac{d^4 y}{dx^4} - m^4 y = \sin mx$

- Solve the differential equation

$$\frac{d^4 y}{dx^4} - 2 \frac{d^3 y}{dx^3} + 8 \frac{dy}{dx} + 4y = 0$$

- Solve the differential equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} - 5y = xe^{-x}, \text{ given that } y=0 \text{ and } \frac{dy}{dx}=0, \text{ when } x=0.$$

1992

- By eliminating the constants a, b obtain the differential equation of which $xy = ac$ be $y = kx^2$ is a solution.

- Find the orthogonal trajectories of the family of semicubical parabolas $ay^3 = x^3$, where a is a variable parameter.

- Show that $(4x+3y+1) dx + (3x+2y+1) dy = 0$ represents hyperbolas having the following lines as asymptotes

$$4x+3y+1=0, 2x+y+1=0. \quad (1998)$$

- Solve the following differential equation $y (1+xy) dx + x (1-xy) dy = 0$

- Solve the following differential equation $(D^2+4) y = \sin 2x$ given that when $x=0$ then $y=0$ and $\frac{dy}{dx}=2$.

- Solve $(D^3-1)y = xe^x + \cos^2 x$.

- Solve $(x^2 D^2 + xD - 4) y = x^2$.

1993

- Show that the system of confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \text{ is self orthogonal.}$$

- Solve $\left\{ y \left(1 + \frac{1}{x} \right) + \cos y \right\} dx + \left[x + \log x - x \sin y \right] dy = 0$.

- Solve $\frac{d^2 y}{dx^2} + w_0^2 y = 0$ a coswt and discuss the nature of solutions w.r.t w_0 .

- Solve $(D^4 + D^2 + 1) y = e^{-x} \cos \left(\frac{\sqrt{3}x}{2} \right)$.

(3) Ordinary Differential Equations and Laplace Transforms

1994

- ❖ Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^3 y$.
- ❖ Show that if $\frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$ is a function of x only say, $f(x)$, then $e^{\int f(x) dx}$ is an integrating factor of $Pdx + Qdy = 0$.
- ❖ Find the family of curves whose tangent form angle $\frac{\pi}{4}$ with the hyperbola $xy = c$.
- ❖ Transform the differential equation

$\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^2 x = 2 \cos^3 x$ into one having z an independent variable where $z = \sin x$ and solve it.

- ❖ If $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$, (a, b and g being positive constants) and $x = a'$ and $\frac{dx}{dt} = 0$ when $t=0$; show that

$$x = a' + (a' - a) \cos \sqrt{\frac{g}{b}} t.$$

- ❖ Solve $(D^2 + D + 4)y = 8x^2 e^{2x} \sin 2x$ where, $D = \frac{dy}{dx}$.
- ❖ Solve $(2x^2 + 3y^2 - 7)x dx + (3x^2 + 2y^2 - 8)y dy = 0$.
- ❖ Test whether the equation $(x+y)^2 dx - (y^2 - 2xy - x^2) dy = 0$ is exact and hence solve it.

- ❖ Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10 \left(x^3 + \frac{1}{x} \right)$.

(1998)

- ❖ Determine all real valued solutions of the equation $y'' - y' + y = 0$.
- ❖ Find the solution of the equation $y'' + 4y = 8 \cos 2x$ given that $y=0$ and $y'=2$ when $x=0$.

1996

- ❖ Solve $x^2(y - px) = y p^2$.
- ❖ Solve $y \sin 2x dx + (1 + y^2 - \cos^2 x) dy = 0$.
- ❖ Solve $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 10y + 37 \sin 3x = 0$. Find the value of y when $x = \frac{\pi}{4}$, if it is given that $y=3$ and $\frac{dy}{dx} = 0$ when $x=0$.

- ❖ Solve $\frac{d^4y}{dx^4} + 2 \frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} = x^2 + 3e^{2x} + 4 \sin x$.
- ❖ Solve $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} = x + \log x$.

1997

- ❖ Solve the initial value problem $\frac{dy}{dx} = \frac{x^2 y}{x^3 y + 1}$, $y(0)=0$.
- ❖ Solve $(x-y^2+3x-y)dx + (x^2-y^2+x-3y)dy = 0$.
- ❖ Solve $\frac{d^4y}{dx^4} + 6 \frac{d^3y}{dx^3} + 11 \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} = 20e^{-2x} \sin x$.
- ❖ Make use of the transformation $y(x) = u(x) \sec x$ to obtain the solution of $y'' - 2y' \tan x + 5y = 0$; $y(0)=0$; $y'(0)=\sqrt{6}$.

- ❖ Solve $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2$; $y(0)=0$ and $y'(0)=2$.

1998

- ❖ Solve the differential equation $xy - \frac{dy}{dx} = y^2 e^{-x}$.
- ❖ Show that the equation $(4x+3y+1)dx + (3x+2y+1)dy = 0$ represents a family of hyperbolas having as asymptotes the lines $x+y=0$; $2x+y+1=0$. (1992)
- ❖ Solve the differential equation $y = 3px + 4p^2$.
- ❖ Solve $\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{4x}(x^2 + 9)$.



(4)

Ordinary Differential Equations and Laplace Transforms

- ❖ Solve the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = x \sin x.$$

- ❖ Solve the differential equation

$$\frac{xdx + ydy}{xdy - ydx} = \left(\frac{1-x^2-y^2}{x^2+y^2} \right)^k$$

- ❖ Solve $\frac{d^3y}{dx^3} - 3 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - 2y = e^x + \cos x$.

- ❖ By the method of variation of parameters solve the

differential equation $\frac{d^2y}{dx^2} + a^2y = \sec(ax)$.

2000

- ❖ Show that $3 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 8y = 0$ has an integral which is a polynomial in x. Deduce the general solution.

- ❖ Reduce $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$, where P, Q, R are functions of x, to the normal form.

Hence solve $\frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (4x^2 - 1)y = -3e^x \sin 2x$

- ❖ Solve the differential equation $y = x - 2a^2p + ap^2$. Find the singular solution and interpret it geometrically.

- ❖ Show that $(4x+3y+1)dx + (3x+2y+1)dy = 0$ represents a family of hyperbolas with a common axis and tangent at the vertex.

- ❖ Solve $x \frac{dy}{dx} - y = (x-1) \left(\frac{d^2y}{dx^2} - x^{-1}y \right)$ by the method of variation of parameters.

A continuous function $y(0)$ satisfies the differential equation

$$\frac{dy}{dt} = \begin{cases} 1+t & 0 \leq t < 1 \\ 2t+2t^2-3t^3, 1 \leq t \leq 5 \end{cases}$$

- ❖ If $y(0) = -e$, find $y(2)$.

12

- ❖ Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 3y = x \log x$.

12

- ❖ Solve $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)}{x^2}$.

15

- ❖ Find the general solution of $ayp^2 + (2x-b)p - y = 0$, $a > 0$.

15

- ❖ Solve $(D^2 + 1)^2 y = 24x \cos x$

given that $y=Dy=D^3y=0$ and $D^3y = 12$ when $x = 0$.

15

- ❖ Using the method of variation of parameters, solve

$$\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$$

15

2002

- ❖ Solve $x \frac{dy}{dx} + 3y = x^3 y^2$.

12

- ❖ Find the values of λ for which all solutions of $\frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - \lambda y = 0$ tend to zero as $x \rightarrow \infty$.

12

- Find the value of constant λ such that the following differential equation becomes exact.

$$(2xe^x + 3y^2) \frac{dy}{dx} + (3x^2 + \lambda e^x) = 0$$

Further, for this value of λ , solve the equation.

15

- ❖ Solve $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$.

15

- ❖ Using the method of variation of parameters, find

the solution of $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = xe^{\sin x}$ with

$y(0) = 0$ and $\left(\frac{dy}{dx} \right)_{x=0} = 0$.

15

- ❖ Solve $(D-1)(D-2 D+2)y = e^x$ where $D = \frac{d}{dx}$.

15

(5) Ordinary Differential Equations and Laplace Transforms

2003

- >Show that the orthogonal trajectory of a system of confocal ellipses is self-orthogonal. 12

Solve $x \frac{dy}{dx} + y \log y = xy^2$. 12

Solve $(D^2 - D)y = 4(c + \cos x + x^2)$, where $D = \frac{d}{dx}$. 15

- Solve the differential equation $(px^2 + y^2)(px + y) = (p + 1)^3$ where $p = \frac{dy}{dx}$, by reducing it to Clairaut's form using suitable substitutions. 15

Solve $(1+x)^2 y'' + (1+x)y' + y = \sin 2[\log(1+x)]$. 15

- Solve the differential equation $x^2 y'' - 4xy' + 6y = x^4 \sec^2 x$ by variation of parameters. 15

2004

- Find the solution of the following differential equation

$$\frac{dy}{dx} + y \cos x = \frac{-\sin 2x}{2} \quad 12$$

Solve $y(xy + x^2 y') dx + x(xy - x^2 y) dy = 0$. 12

Solve $(D^2 - 4D - 5)y = e^x(x + \cos x)$. 15

- Reduce the equation $(px-y)(py+x) = 2p$ where $p = \frac{dy}{dx}$ to Clairaut's equation and hence solve it. 15

Solve $(x+2)\frac{dy}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$. 15

- Solve the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1-x^2)y = x^2 \quad 15$$

- Find the orthogonal trajectory of a system of co-axial circles $x^2 + 2gx + c = 0$, where g is the parameter. 12

Solve $xy \frac{dy}{dx} = \sqrt{x^2 - y^2 - x^2 y^2 - 1}$. 12

Solve the differential equation $(x+1)^4 D^3 + 2(x+1)^3$

$$D^2 - (x+1)^2 D + (x+1)y = \frac{1}{x+1} \quad 15$$

Solve the differential equation $(x^2 + y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$,

where $p = \frac{dy}{dx}$, by reducing it to Clairaut's form by using suitable substitution. 15

Solve the differential equation $(\sin x - x \cos x)y'' - x \sin x y' + y \sin x = 0$. 15

given that $y = \sin x$ is a solution of this equation.

Solve the differential equation $y'' - 2xy' + 2y = x \log x, x > 0$ by variation of parameters. 15

2006

- Find the family of curves whose tangents form an angle $\frac{\pi}{4}$ with the hyperbolas $xy=c$, $c > 0$. 12

- Solve the differential equation

$$(xy^2 + e^{\frac{y}{x}})dx - x^2 y dy = 0 \quad 12$$

Solve $(1+y^2) + (x - e^{-\tan^{-1} y}) \frac{dy}{dx} = 0$. 15

- Solve the equation $x^2 p^2 + yp(2x+y) + y^2 = 0$ using the substitution $y = u$ and $xy=v$ and find its singular solution, where $p = \frac{dy}{dx}$. 15

- Solve the differential equation

$$x^2 \frac{d^3y}{dx^3} + 2x \frac{d^2y}{dx^2} - y = 10 \quad 15$$

- Solve the differential equation

$$(D^2 - 2D + 2)y = e^x \tan x, \text{ where } D = \frac{d}{dx},$$

by the method of variation of parameters. 15



2007

Solve the ordinary differential equation

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2} \sin 6x + \sin^3 3x, 0 < x < \frac{\pi}{2}. \quad 12$$

Find the solution of the equation

$$\frac{dy}{dx} + y^2 dx = -4x dx \quad 12$$

Determine the general and singular solutions of the equation $y = x \frac{dy}{dx} + a \frac{dy}{dx} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^k$, a being a constant. **15**Obtain the general solution of $[D^3 - 6D^2 + 12D - 8]$

$$y = 12 \left(e^{2x} + \frac{9}{4} e^{-2x} \right), \text{ where } D = \frac{d}{dx} \quad 15$$

Solve the equation $2x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} - 3y = x^3$.

Use the method of variation of parameters to find the general solution of the equation

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = 2e^{2x} \quad 15$$

2008

Solve the differential equation

$$y dx + (x + x^2 \ln x) dy = 0 \quad 12$$

Use the method of variation of parameters to find the general solution of $x^3 y'' - 4x y' - 6y = -x^4 \sin x$. **12**Using Laplace transform, solve the initial value problem $x^2 y'' + 2y' + 2y = 4t + e^{3x}$ with $y(0) = 1$, $y'(0) = -1$. **15**

Solve the differential equation

$$x^3 y'' - 3x^2 y' + xy = \sin(\ln x) + \dots \quad 15$$

Solve the equation $y - 2 \frac{dy}{dx} + y^2 = 0$ where $\frac{dy}{dx} = \frac{dy}{dx}$. **15****2009**

Find the Wronskian of the set of functions

$$\{3x^3, 3x^2\}.$$

on the interval $[-1, 1]$ and determine whether the set is linearly dependent on $[-1, 1]$. **12**Find the differential equation of the family of circles in the xy -plane passing through $(-1, 1)$ and $(1, 1)$. **20**

Find the inverse Laplace transform of

$$F(s) = 10 \left(\frac{s+1}{s+5} \right) \quad 20$$

$$\text{Solve: } \frac{dy}{dx} = \frac{y^2(x-y)}{3x^2 - x^2y - 4y^3}, y(0) = 1. \quad 20$$

2010

Consider the differential equation

$$y' = \alpha x, x > 0 \quad \text{where } \alpha \text{ is a constant. Show that}$$

- (i) if $\phi(x)$ is any solution and $\Psi(x) = \phi(x) e^{-\alpha x}$, then $\Psi(x)$ is a constant;
- (ii) if $\alpha < 0$, then every solution tends to zero as $x \rightarrow \infty$. **12**

Show that the differential equation

$$(3y - x) + 2y(y' - 3x)y' = 0 \quad \text{admits an integrating factor which is a function of } (x,y). \text{ Hence solve the equation.} \quad 12$$

Verify that

$$\frac{1}{2}(Mx + Ny)d(\log_e(xy)) + \frac{1}{2}(Mx - Ny)d(\log_e(\frac{x}{y})) = Mdx + Ndy$$

Hence show that-

- (i) if the differential equation $Mdx + Ndy = 0$ is homogeneous, then $(Mx + Ny)$ is an integrating factor unless $Mx + Ny \equiv 0$;

- (ii) if the differential equation

$Mdx + Ndy = 0$ is not exact but is of the form

$$f_1(xy)y dx + f_2(xy)x dy = 0 \quad \text{then } (Mx - Ny) \text{ is an integrating factor unless } Mx - Ny = 0. \quad 20$$

Show that the set of solutions of the homogeneous linear differential equation

$$y' + p(x)y = 0$$

(7) Ordinary Differential Equations and Laplace Transforms

<p>on an interval $I = [a, b]$ forms a vector subspace W of the real vector space of continuous functions on I. what is the dimension of W? 20</p> <p>Use the method of undetermined coefficients to find the particular solution of $y'' + y = \sin x + (1+x^2)e^x$ and hence find its general solution. 20</p> <p>2011</p> <p>Obtain the solution of the ordinary differential equation $\frac{dy}{dx} = (4x+y+1)^2$, if $y(0) = 1$. 10</p> <p>Determine the orthogonal trajectory of a family of curves represented by the polar equation $r = a(1 - \cos\theta)$, (r, θ) being the plane polar coordinates of any point. 10</p> <p>Obtain Clairaut's form of the differential equation $\left(x \frac{dy}{dx} - y\right)\left(y \frac{dy}{dx} + x\right) = x^2 \frac{dy}{dx}$. Also find its general solution. 15</p> <p>Obtain the general solution of the second order ordinary differential equation $y'' - 2y' + 2y = x + e^x \cos x$, where dashes denote derivatives w.r.t. to x. 15</p> <p>Using the method of variation of parameters, solve the second order differential equation $\frac{d^2y}{dx^2} + 4y = \tan 2x$. 15</p> <p>Use Laplace transform method to solve the following initial value problem:</p> $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t, x(0) = 2 \text{ and } \left.\frac{dx}{dt}\right _{t=0} = -1 \quad 15$ <p>2012</p> <p>Solve $\frac{dy}{dx} = \frac{2xye^{xy}}{x^2 + y^2 - ax}$. (12)</p> <p>Find the orthogonal trajectories of the family of curves $x^2 + y^2 = ax$. (12)</p> <p>Using Laplace transforms, solve the initial value problem $y'' + 2y' + y = e^{-t}$, $y(0) = -1$, $y'(0) = 1$. (12)</p>	<p>Show that the differential equation $(2xy \log y)dx + \left(x^2 + y^2 \sqrt{y^2 + 1}\right)dy = 0$ is not exact. Find an integrating factor and hence, the solution of the equation. (20)</p> <p>Find the general solution of the equation $y'' - y' = 12x^2 + 6x$. (20)</p> <p>Solve the ordinary differential equation $x(x-1)y'' - (2x-1)y' + 2y = x^2(2x-3)$. (20)</p>
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(19)

- ❖ Solve $\{x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3xD + 1\}y = (1 + \log x)^2$,

Where $D = \frac{d}{dx}$. (15)

- ❖ Solve $(D^4 + D^2 + 1)y = ax^2 + be^{-x} \sin 2x$, where

$D = \frac{d}{dx}$. (15)

- ❖ Solve $\frac{dy}{dx} = \tan(\frac{y}{x})$ (8)

- ❖ Solve and find the singular solution of (8)

$$x^3 p^2 + x^2 py + a^3 = 0$$

- ❖ Solve: $x^2 y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} - y \right)^2 = 0$ (10)

- ❖ Solve $\frac{d^4 y}{dx^4} + 2 \frac{d^3 y}{dx^3} + y = x^2 \cos x$. (10)

- ❖ Solve $x = y \frac{dy}{dx} - \left(\frac{dy}{dx} \right)^2$ (10)

- ❖ Solve $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = (1 + \log x)^2$ (10)



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(16)

Previous Years Questions (2000–2012) Segment-wise

Ordinary Differential Equations

(According to the New Syllabus Pattern) Paper I

2000

- ❖ Solve $(x+y)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$
 $P = \frac{dy}{dx}$ Interpret geometrically the factors in the P-and C-discriminants of the equation $8p^3x = y(12p^2 - 9)$

(20)

- ❖ Solve

$$(i) \frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \frac{a^2}{x^4} y = 0$$

$$(ii) \frac{d^2y}{dx^2} + (\tan x - 3\cos x) \frac{dy}{dx} + 2y\cos^2 x = \cos^4 x. \text{ by varying parameters.}$$

(20/200)

2001

- ❖ A constant coefficient differential equation has auxiliary equation expressible in factored form as $P(m) = m(m-1)^2(m^2+2m+5)^2$. What is the order of the differential equation and find its general solution.

$$P(m) = m(m-1)^2(m^2+2m+5)^2. \text{ What is the order of the differential equation and find its general solution.}$$

- ❖ Solve $x^2 \left(\frac{dy}{dx} \right)^2 + (2x-y) \frac{dy}{dx} + y^2 = 0$

(10)

- ❖ Using differential equations show that the system of confocal conics given by $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, is self orthogonal.

(10)

- ❖ Solve $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$ given that $y = e^{asinx^{-1}}$ is one solution of this equation.

(10)

- ❖ Find a general solution $y = (anx)^{\pi/2}$ for $\pi/2 < x < \pi/2$ by variation of parameters.

(10)

2002

- ❖ If $(B-a)^n e^{ax}$ is denoted by z , prove that

$z \frac{\partial z}{\partial n}, \frac{\partial^2 z}{\partial n^2}, \frac{\partial^3 z}{\partial n^3}$ all vanish when $n = a$. Hence show that $e^{ax}, xe^{ax}, x^2 e^{ax}, x^3 nx$ are all solutions of $(D-a)^4 y = 0$. Here D stands for $\frac{d}{dx}$.

- (10)
- ❖ Solve $4xp^2 - (3x+1)^2 = 0$ and examine for singular solutions and extraneous loci. Interpret the results geometrically.

(10)

- ❖ Form the differential equation whose primitive is

$$y = A \left(\sin x + \frac{\cos x}{x} \right) + B \left(\cos x + \frac{\sin x}{x} \right)$$

(10)

- ❖ Prove that the orthogonal trajectory of a system of parabolas belongs to the system itself.

- ❖ Using variation of parameters solve the differential equation

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x.$$

(10)

- ❖ Solve the equation by finding an integrating factor of $(x+2)\sin y dx + x \cos y dy = 0$.

(ii)

- ❖ Verify that $\phi(x) = x^2$ is a solution of

$$y'' - \frac{2}{x^2} y = 0 \text{ and find a second independent solution.}$$

(10)

- ❖ Show that the solution of $(D^{2n+1} - 1)y = 0$ consists of Ae^{ax} and n pairs of terms of the form

(10)

- $e^{ax}(b_r \cos \alpha x + c_r \sin \alpha x)$, where $a = \cos \frac{2\pi r}{2n+1}$ and $\alpha = \sin \frac{2\pi r}{2n+1}$, $r = 1, 2, \dots, n$ and b_r, c_r are arbitrary constants.

2003

- ❖ Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2gx + c = 0$ Where g is a parameter.

(10)

(17)

- Find the three solutions of $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$
Which are linearly independent on every real interval. (10)

- Solve and examine for singular solution:

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2.$$

(10)

- Solve $x^2 \frac{dy}{dx} - 2x^2 y + 10x^2 = 10\left(x + \frac{1}{x}\right)$ (10)

- Given $y = \frac{1}{x}$ is one solution of

$(x^3 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ find another linearly independent solution by reducing order and write the general solution. (10)

- Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax, \text{ a real.}$$

(10)

- Determine the family of orthogonal trajectories of the family $y = ce^{-x}$. (10)

- Show that the solution curve satisfying $(x^2 - xy) y' = 1$, where $y \rightarrow 1$ as $x \rightarrow \infty$ is a conic section. Identify the curve. (10)

- Solve $(1+x)^{-1} y' + (1+x)^{-2} y'' + y = 4 \cos(\ln(1+x))$, $y(0) = 1, y'(e-1) = \cos 1$. (10)

- Obtain the general solution $y'' + 2y' + 2y = 4e^{-x} \sin x$. (10)

- Find the general solution of $(xy^3 + y) dx + 2(x^2 y^2 + x + y^4) dy = 0$. (10)

- Obtain the general solution of $(D^4 + 2D^3 - D^2 - 2D) y = x^2 + e^{2x}$, Where $D = \frac{dy}{dx}$. (10)

- Form the differential equation that represents all parabolas each of which has latus rectum $4a$ and whose axes are parallel to the x -axis. (10)

- (i) The auxiliary polynomial of a certain homogenous linear differential equation with constant coefficients in factored form is

$$P(m) = m^4 (m-2)^6 (m^2 - 6m + 25)^3$$

What is the order of the differential equation and write a general solution?

- (ii) Find the equation of the one-parameter family of parabolas given by $y^2 = 2cx + c^2$, C real and show that this family is self-orthogonal. (10)

- Solve and examine for singular solution the following equation $P^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$ (10)

- Solve the differential equation $\frac{dy}{dx} + 9y = \sec 3x$ (10)

- Given $y = \frac{1}{x}$ is one solution solve the differential equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0$ reduction of order method. (10)

- Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 3y = 2e^{-x} - 10 \sin x$ by the method of undetermined coefficients. (10)

2006

- From $x^2 + y^2 + 2ax + 2by + c = 0$, derive differential equation not containing a, b or c . (10)

- Discuss the solution of the differential equation

$$y^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right] = a^2$$

- Solve $x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx}$ (10)

- Solve $\frac{d^4y}{dx^4} - y = x \sin x$ (10)

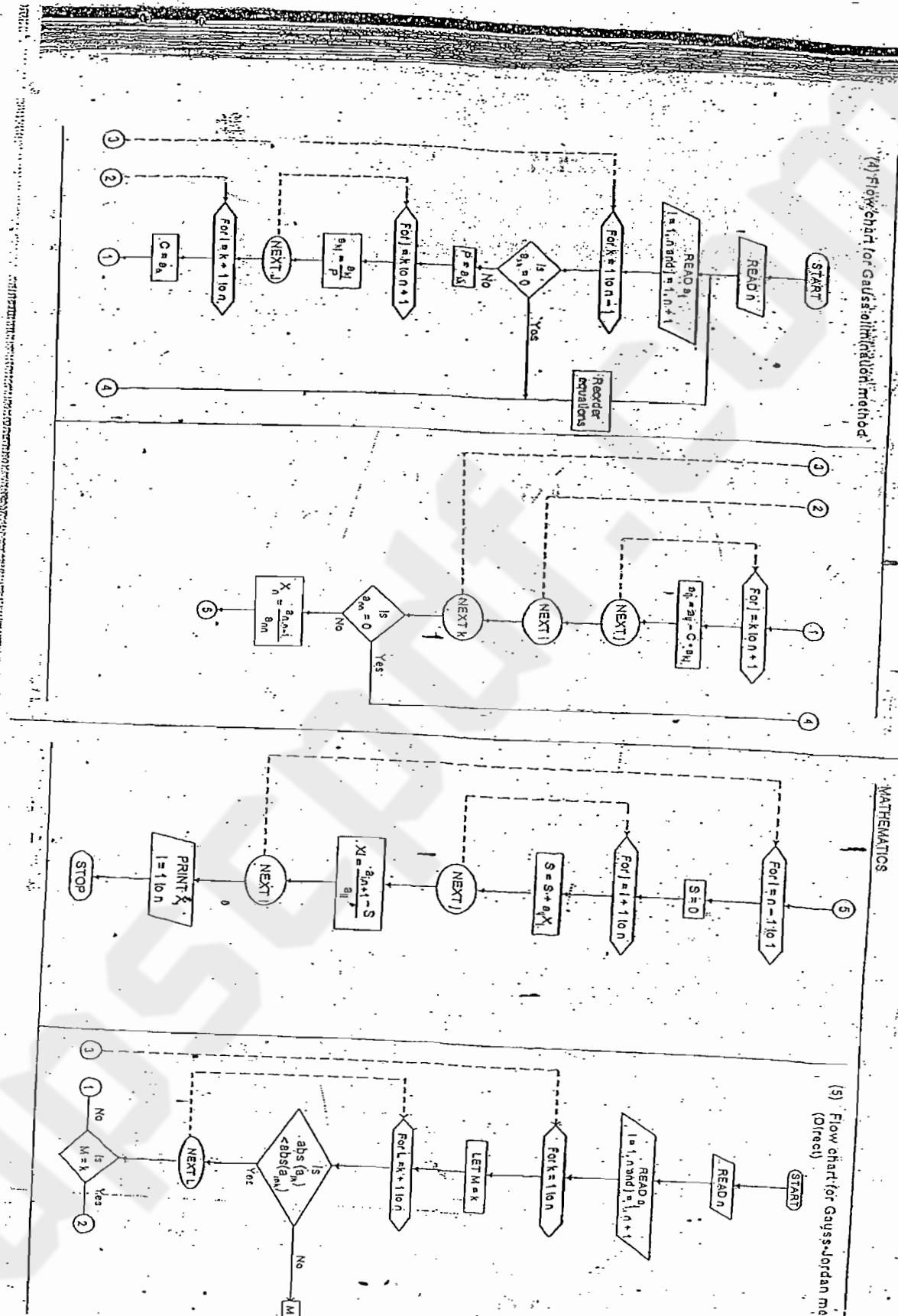
- Solve $x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 e^x$ (10/2008)

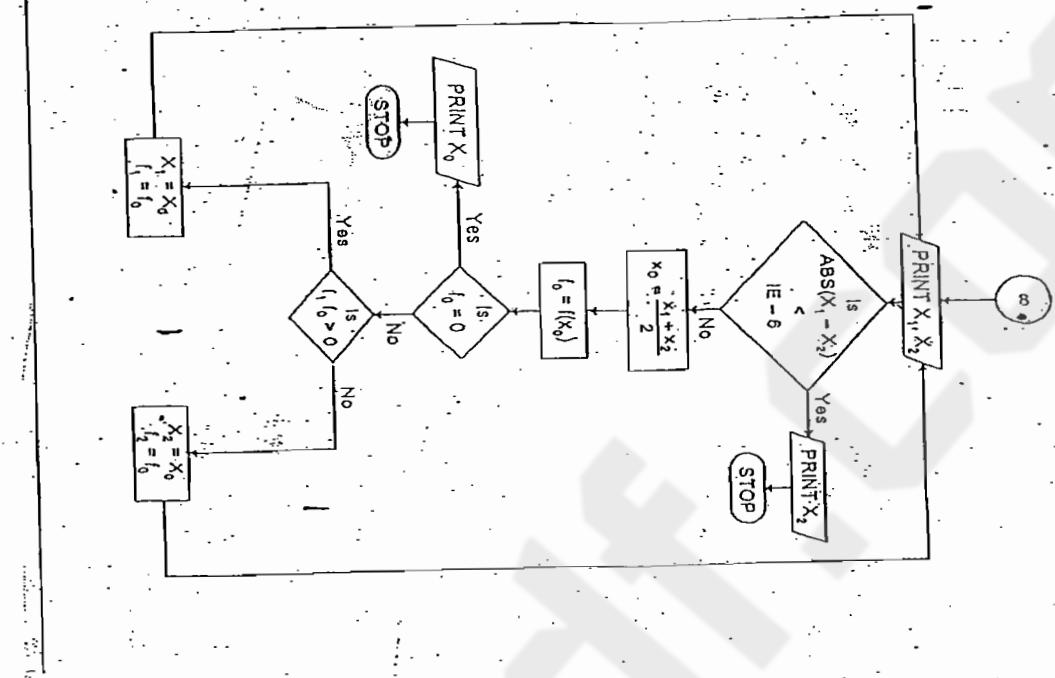
- Reduce $xy \left(\frac{dy}{dx} \right)^2 - (x^2 + y^2 + 1) \frac{dy}{dx} + xy = 0$ to Clairaut's form and find its singular solution. (10)



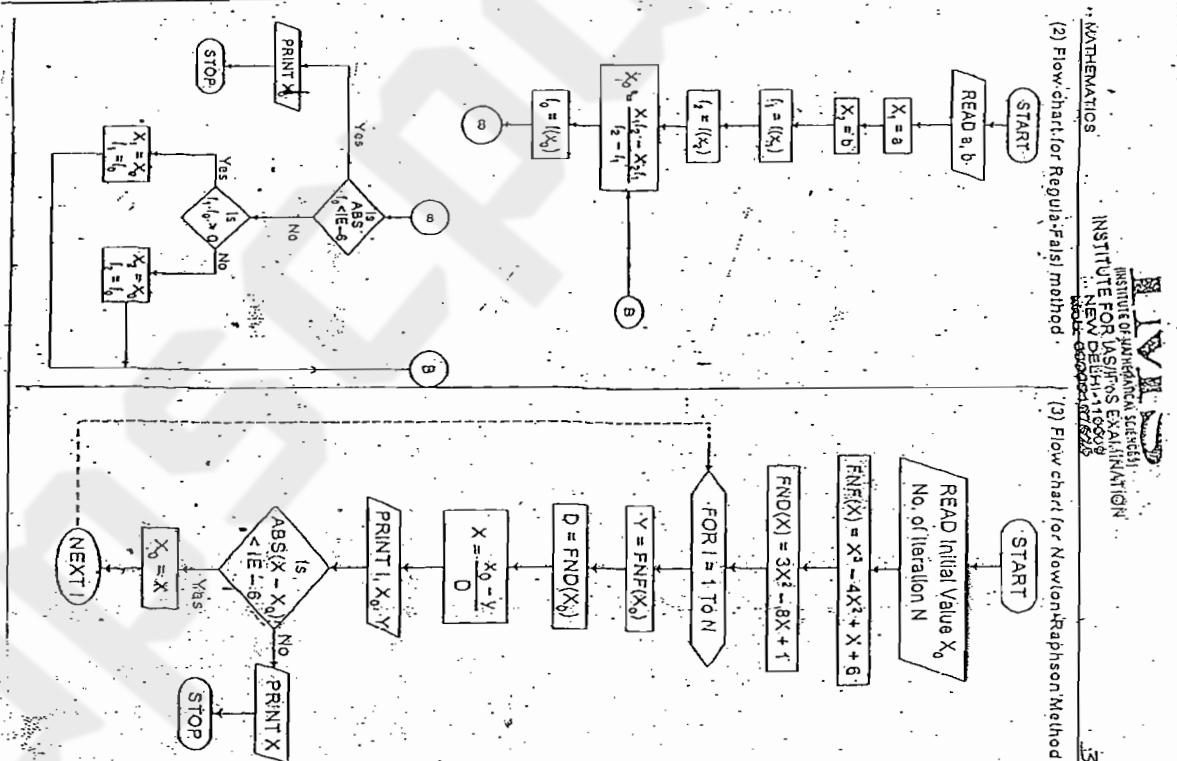
MATHEMATICS

(A) Flögelchart for Gauss-Jordan elimination method





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(13) Euler's methodStep 1: Read initial values (say) a, b, y_0 Step 2: Read number of points of x (say) n $h = \frac{b-a}{n-1}$ Step 4: Assign a to x Step 5: Assign $1 = 1$ Step 6: If $x >= (b+h)$, follow Step 9; otherwise follow Step 7 and 8.Step 7: Evaluate the Euler's formula $y_{i+1} = y_i + h(k_i, y_i)$ and print the results.Step 8: Increment i ($i = i+1$), following Step 6.

Step 9: Stop

(14) Runge-Kutta methodStep 1: Read initial values say a, b, y Step 2: Read number of points of x (say) n Step 3: Find the width of the interval (say) h Step 4: Assign the initial value a to x' Step 5: Compute the value of the parameters viz., k_1, k_2, k_3, k_4 and y' and substitute in the Runge-Kutta fourth order formula as follows:

$$y' = y + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

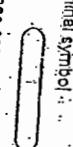
Step 6: Increment x' say $x' = x' + h$

Step 7: Continue steps 5 and 6 till we get

number of points equal to n .

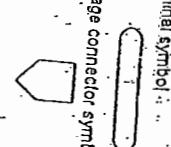
Step 8: Stop

- 5.FLOW CHARTING**
- 1. Flow chart is a pictorial representation of a sequence of instructions for solving a problem. System flow chart indicates the flow of data throughout a data processing system, as well as the flow into and out of the system.
 - 2. Flow chart symbols
 - 3. Guidelines to draw efficient flow charts
 - 4. Flow chart for Bisection method
 - 5. Flow chart for Solving Numerical Analysis Problems

MATHEMATICS**(1) Terminal symbol**

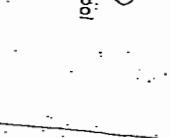
(6)

Draw flow chart symbols from top to bottom or left to right across the page to ensure their readability.

(2) Decision symbol

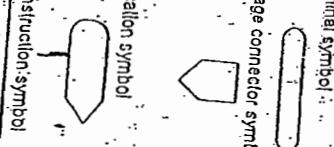
(7)

Avoid intersecting directed lines. Use connector symbol wherever required.

(3) Off-page connector symbol

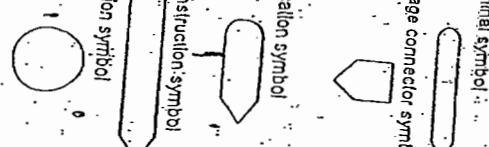
(8)

Always use pencil to draw flow chart so that erasing and redrawing are easier. To draw flow chart, use template. Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.

(4) Group instruction symbol

(9)

Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.

(5) Connection symbol

(10)

Flow chart for Solving Numerical Analysis Problems

- program. The main advantage of this algorithm is that while writing the program, writing is not concerned with the details of the statements or a programming language, they concentrate only on logic of the procedure. Further, the flow chart shows the logic pictorially, any problem can be identified and eliminated immediately.
- It is important to note that, for a beginner, it is recommended that a flow chart be drawn first in order to reduce the number of errors and omissions in the program. It is a good practice to have a flow chart which may help during the testing of the program as well as while debugging further modifications in the program.
- 5.1 Classification of flow charts**
- Flow charts can be divided into two broad categories:
- (1) Program flow charts and
 - (2) System flow charts
- Program flow chart is the pictorial representation of a sequence of instructions for solving a problem. System flow chart indicates the flow of data throughout a data processing system, as well as the flow into and out of the system.
- 5.2 Flow chart symbols**
- (1) Assignment symbol
 - (2) Decision symbol
 - (3) Module symbol
 - (4) Input/Output symbol
 - (5) General connector symbol
 - (6) Draw flow chart symbols from top to bottom or left to right across the page to ensure their readability.
 - (7) Avoid intersecting directed lines. Use connector symbol wherever required.
 - (8) Always use pencil to draw flow chart so that erasing and redrawing are easier. To draw flow chart, use template. Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.
 - (9) Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.
 - (10) Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.
- 5.3 Guidelines to draw efficient flow charts**
- (1) Analyse the nature of the problem completely and ensure that all the information relating to the problem is available.
 - (2) Decide the sequence of actions to be taken so that solution can be obtained in a finite number of steps.
 - (3) Prepare the general algorithm.
 - (4) Include refinements, wherever necessary.
 - (5) Draw flow chart on the basis of the algorithm.
- ```

graph TD
 START([START]) --> READ[Read Parameters]
 READ --> PRINT1[PRINT x1, y1]
 PRINT1 --> STOP1([STOP])
 STOP1 --> CALCULATE[Calculate Initial guess L, H]
 CALCULATE --> DECISION{Is L < H?}
 DECISION -- No --> STOP2([STOP])
 DECISION -- Yes --> X1_X2[X1 = X2]
 X1_X2 --> DECISION
 X1_X2 --> PRINT2[PRINT x1, y1]
 PRINT2 --> STOP3([STOP])
 STOP3 --> DECISION

```
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|                                                                                                                                         |                                                                                                                                          |
|-----------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Step 6:</b> Construct the difference table                                                                                           | <b>Step 7:</b> Print the results (i.e., $x$ and $y$ )                                                                                    |
| <b>Step 5:</b> Compute the value of $u = \frac{a-a_0}{h}$                                                                               | <b>Step 8:</b> If you want to try for another value follow Steps 4 to 8, otherwise follow Step 9                                         |
| <b>Step 6:</b> Expand the Newton-forward formula and substitute the values to get the function value for the given argument (say $a$ )  | <b>(10)</b> Simpson's One-third rule                                                                                                     |
| <b>Step 7:</b> Print the function value of the given argument (say $b$ )                                                                | <b>Step 1:</b> Read an initial value, say $a$                                                                                            |
| <b>Step 8:</b> If you want to continue, follow Steps 3 to 7, otherwise follow Step 9                                                    | <b>Step 2:</b> Read the final value, say $b$                                                                                             |
| <b>Step 9:</b> Stop                                                                                                                     | <b>Step 3:</b> Read number of divisions, say $n$                                                                                         |
| <b>(8)</b> Newton's - Backward formula                                                                                                  | <b>Step 4:</b> Find the width of the interval.<br>$l, a, h = \frac{b-a}{n}$                                                              |
| <b>Step 1:</b> Read $n$                                                                                                                 | <b>Step 5:</b> Assign $m=n/2$ (to get the area under the curve for $n$ intervals)                                                        |
| <b>Step 2:</b> Read $x_1$ and $y_{1,1}=10$                                                                                              | <b>Step 6:</b> Assign $S=0$ and $x=a$                                                                                                    |
| <b>Step 3:</b> Read the value of $x$ (say $a$ )                                                                                         | <b>Step 7:</b> Expand the Simpson's rule and it is as follows                                                                            |
| <b>Step 4:</b> Construct the difference table                                                                                           | $\frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \dots + 2y_{n-1} + 4y_n + y_{n+1})$ and compute the values of $h$ and $y_i$ and evaluate. |
| <b>Step 5:</b> Compute the value of $u = \frac{a-a_0}{h}$                                                                               | <b>Step 8:</b> Print the results.                                                                                                        |
| <b>Step 6:</b> Expand the Newton-forward formula and substitute the values, to get the function value for the given argument (say $a$ ) | <b>Step 9:</b> Stop                                                                                                                      |
| <b>Step 7:</b> Print the function value of the given argument (say $a$ )                                                                | <b>(11)</b> Trapezoidal rule                                                                                                             |
| <b>Step 8:</b> If you want to continue, follow Steps 3 to 7, otherwise follow Step 9                                                    | <b>Step 10:</b> Read two end points of the interval say $x_1, x_2$                                                                       |
| <b>Step 9:</b> Stop                                                                                                                     | <b>Step 11:</b> Read the allowed error in the integral say $\epsilon$                                                                    |
| <b>(9)</b> Lagrange's method                                                                                                            | <b>Step 12:</b> Read $y_1=10$                                                                                                            |
| <b>Step 11:</b> Read the number of values (say $n$ )                                                                                    | <b>Step 13:</b> $I \leftarrow 2x_1$                                                                                                      |
| <b>Step 2:</b> Read $x_{1,1}=10$                                                                                                        | <b>Parameters for Gaussian Integration</b>                                                                                               |
| <b>Step 3:</b> Read $y_1=10$                                                                                                            | <b>Step 14:</b> $h \leftarrow h/2$                                                                                                       |
| <b>Step 4:</b> Read the value of $x$ (say $a$ )                                                                                         | <b>Step 15:</b> $I_0 \leftarrow I_1$                                                                                                     |
| <b>Step 5:</b> Expand the lagrangian interpolation formula as per $n$                                                                   | <b>Step 16:</b> $I_1 \leftarrow h \times S_1$                                                                                            |
| <b>Step 6:</b> Substitute the values of $a$ , $x_1$ and $y_1$ and evaluate.                                                             | <b>Step 17:</b> Print the results $I_1, h$                                                                                               |
|                                                                                                                                         | <b>Step 18:</b> Stop                                                                                                                     |
|                                                                                                                                         | <b>(12)</b> Gaussian Quadrature formula                                                                                                  |
|                                                                                                                                         | <b>Step 19:</b> If $\epsilon$ transform the independent variable                                                                         |
|                                                                                                                                         | <b>Step 2:</b> Choose value of $n$                                                                                                       |
|                                                                                                                                         | <b>Step 3:</b> Expand the Gauss formula                                                                                                  |
|                                                                                                                                         | <b>Step 4:</b> Substitute the values for the parameters $A_i$ and $Z_i$ (use the following Table) and evaluate the formula               |

**4.2 Essential steps to be followed in developing a program**

From this, it can be stated that the following steps are very essential to develop a program.

- (1) Input step
- (2) Assignment step
- (3) Decision step
- (4) Repetitive step and
- (5) Output step.

**Remark:** The input step takes the three marks along with register number for each student. Register number is SUB1, SUB2, SUB3 and REGNO. The decision step checks whether all the students have been considered. The repetitive steps concentrate on calculation of output step provides the result.

**4.3 Illustrative examples of algorithms****1) Bisection method**

Assignment step assigns three marks and maximum number of iterations say n.

Step 1: Read initial values say  $x_0$  and  $x_1$ .

Step 2: Read the prescribed error say e.

Step 3: If  $f(x_0) \cdot f(x_1) > 0$ , then print the message "Solution is convergent and print  $x_2$ , otherwise follow steps 9 to 12.

Step 4: When  $|f(x_0)| < e$ , then print the message "Solution is convergent and print  $x_0$ , otherwise follow steps 7 to 12.

Step 5: If  $|f(x_1)| < e$ , then print the message "Solution is convergent and print  $x_1$ , otherwise follow steps 7 to 12.

Step 6: If  $|f(x_0)| > |f(x_1)|$ , then swap  $x_0$  and  $x_1$ .

Step 7: If  $|f(x_0)| > |f(x_1)|$ , then swap  $x_0$  and  $x_1$ .

Step 8: If  $|f(x_0)| > |f(x_1)|$ , then swap  $x_0$  and  $x_1$ .

Step 9: If  $|f(x_0)| > |f(x_1)|$ , then swap  $x_0$  and  $x_1$ .

Step 10: If  $|f(x_0)| > |f(x_1)|$ , then swap  $x_0$  and  $x_1$ .

Step 11: Print "Does not converge in n iterations".

Step 12: Print  $x_2$ .

Step 13: Stop.

**2) Newton-Raphson Method**

Assignment step assigns an initial value to x (say  $x_0$ )

Step 1: Assign an improved estimate of  $x_0$

Step 2: Evaluate  $f(x)$ ,  $f'(x_0)$  and  $f''(x_0)$

Step 3: Find the improved estimate of  $x_0$

$$x_1 = \frac{x_0 - f(x_0)}{f'(x_0)}$$

Step 4: If  $|x_1 - x_0| > e$ , follow steps 1 to 3.

Step 5: If  $|x_1 - x_0| < e$ , follow steps 6 to 10.

Step 6: If  $|x_1 - x_0| > e$ , follow steps 1 to 3.

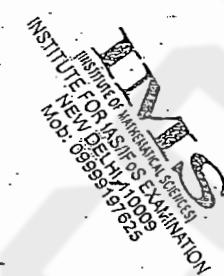
Step 7: If  $|x_1 - x_0| < e$ , follow steps 6 to 10.

Step 8: If  $|x_1 - x_0| > e$ , follow steps 1 to 3.

Step 9: If  $|x_1 - x_0| < e$ , follow steps 6 to 10.

Step 10: If  $|x_1 - x_0| < e$ , follow steps 6 to 10.

**MATHEMATICS****5.1 Numerical Methods****1) Direct methods****2) Iterative methods****3) Approximate methods****4) Numerical integration****5) Numerical differentiation****6) Solution of differential equations****7) Solution of partial differential equations****8) Solution of integral equations****9) Solution of boundary value problems****10) Solution of eigenvalue problems****11) Solution of systems of linear equations****12) Solution of non-linear equations****13) Solution of ordinary differential equations****14) Solution of partial differential equations****15) Solution of integral equations****16) Solution of boundary value problems****17) Solution of eigenvalue problems****18) Solution of systems of linear equations****19) Solution of non-linear equations****20) Solution of ordinary differential equations****21) Solution of partial differential equations****22) Solution of integral equations****23) Solution of boundary value problems****24) Solution of eigenvalue problems****25) 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## MATHEMATICS

### 4. ALGORITHMS

#### 4.0 Definition and origin of an algorithm

The term algorithm may be defined as sequence of instructions designed in such a way that if the instructions are executed in the specified sequence the desired results can be obtained. Further, the instructions should be unambiguous and the result should be obtained after a finite number of executed steps i.e., an algorithm must terminate and should not repeat one or more instructions infinitely. In other words, the algorithm represents the logic of the processing to be performed. The word algorithm originates from the word algorithm, which means the process of doing arithmetic with Arabic numerals. Later, the word algorithm combined with the word arithmetic to become algorithm.

#### 4.1 Development of an algorithm

We first construct an algorithm that gives a very general manner in which computer could produce the solution to a given problem. Such an algorithm is known as general algorithm. In addition, we add details in the general algorithm, in a step-by-step manner, so that the algorithm becomes more detailed, is called refinement of the general algorithm.

The following example shows how an algorithm is written for a given task. Consider the problem of calculating the average marks of a student in three different subjects.

An algorithm for this task involves the following steps:

- Step 1: Read a set of three marks.
- Step 2: Find the average by summing them and dividing by three.
- Step 3: Stop.

Successive refinements of this general algorithm add details until complete algorithm is achieved as per the user's specification.

- Step 1: SUB 1, SUB 2 and SUB 3 are placed in the variables TOTMARK, and place it in a variable TOTMARK by 3.
- Step 2: Compute the average mark dividing TOTMARK by 3.
- Step 3: It is important to note that hierarchical number system has been used to indicate that these are refinement of steps 1 and 2 of the general algorithm.

If you decide to calculate average marks for a class of 100 students, then the algorithm needs further refinement. The refined algorithm defines a new variable COUNTER which ensures that all the students are taken care for processing.

- Step 1: Initialise the counter, say COUNT = 1.
- Step 2: Read the student register number and his marks from an input device and place them in the variables REGNO, SUB 1, SUB 2 and SUB 3 respectively.
- Step 3: Compute the average marks for each student by summing of the corresponding marks and dividing by 3.
- Step 4: Display the register number and average marks of each student.
- Step 5: Start increment COUNT BY 1.
- Step 6: If COUNT <= 100, repeat steps 1 through 6 otherwise go to Step 7: Stop.

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This image is a collage of several mathematical documents from the Institute of Mathematical Sciences (IMSc). The documents are arranged in a grid-like pattern and feature large, stylized 'IMSc' and 'S'水印. The content includes various mathematical problems and their solutions, such as differential equations, trigonometric identities, and geometric proofs.

Some of the visible text and symbols include:

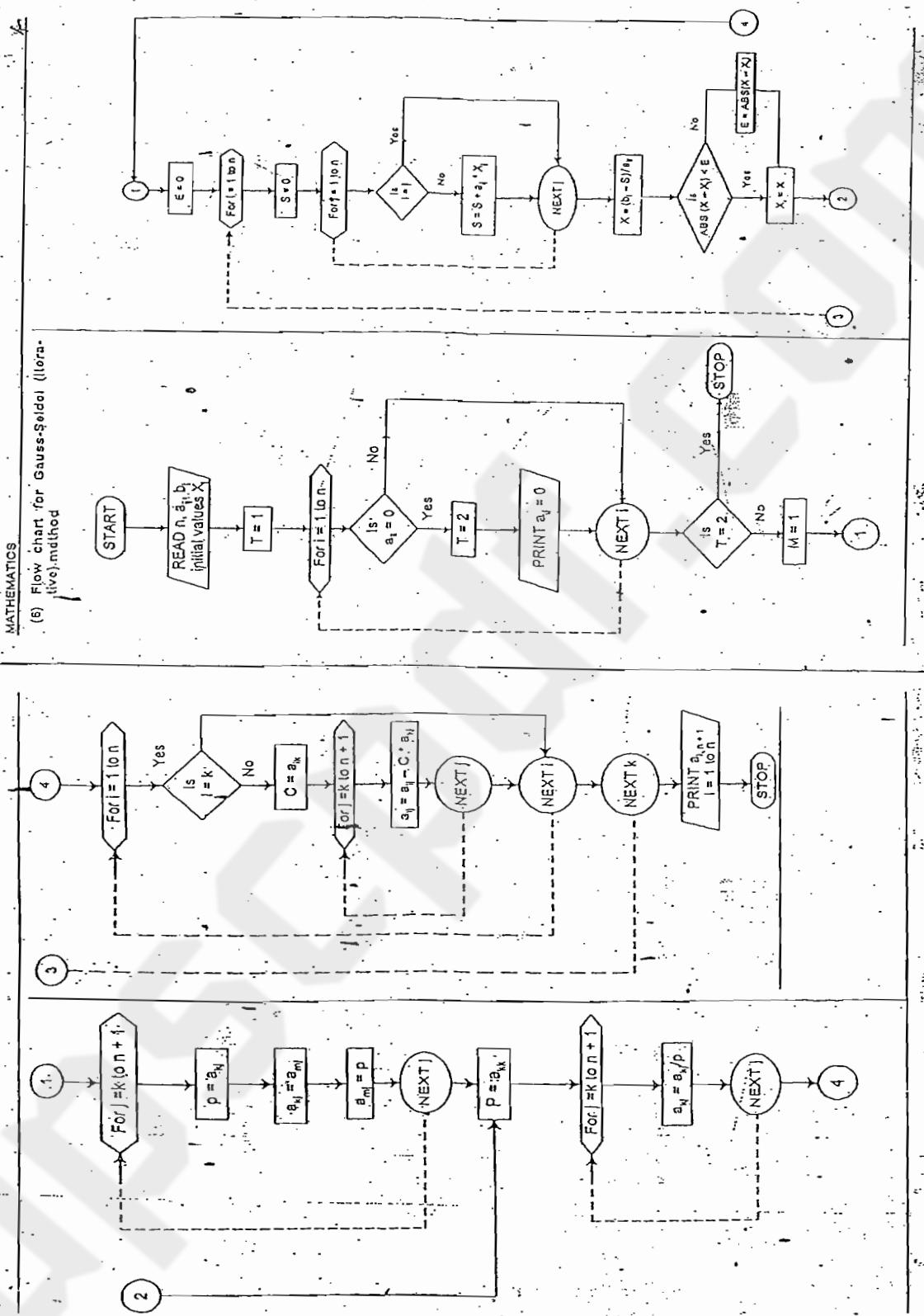
- INSTITUTE OF MATHEMATICAL SCIENCES
- IMSc
- S
- Differential Equations:
  - $\frac{dy}{dx} = \frac{x^2}{y^2}$
  - $\frac{dy}{dx} + 2\frac{dy}{dx}^2 + y = x^2 \cos x$
  - $x^2 \frac{dy}{dx} + 2x \frac{dy}{dx}^2 + y = 0$
  - $x^2 \frac{dy}{dx} + 2x \frac{dy}{dx}^2 + y = 1$
  - $x^2 \frac{dy}{dx} + 3x \frac{dy}{dx} + y = 1$
- Trigonometric Identities:
  - $\sec^2 y - \tan^2 y = 1$
- Geometric Proof:
  - A diagram showing a triangle with a point P inside it, connected to all three vertices. A line segment connects the midpoints of two sides.
- Calculus:
  - $D = \frac{d}{dx}$
  - $\int \sec^2 y dy = \tan y + C$
  - $\int \sec^2 y dy = \tan y + C$
- Algebra:
  - $(D^2 + D + 1)y = a\sin^2 x + b\sin x \cos x$



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| <p><b>2011</b></p> <p>Find the family of curves whose tangents form an angle of <math>\pi/4</math> with the x-axis.</p> <p>Solve <math>(x^2 - 1) \frac{dy}{dx} - 2x \frac{dy}{dx} = 0</math></p> <p>Solve by the method of variation of parameters the following equation:</p> <p><math>\frac{dy}{dx} + 2x \frac{dy}{dx} + (x^2 + 1)y = 0</math></p> <p>Find the general solution of <math>\frac{dy}{dx} + 2x \frac{dy}{dx} + (x^2 + 1)y = 0</math></p> <p>Show that <math>\cos(x+y)</math> is an integrating factor of <math>\frac{dy}{dx} + 2x \frac{dy}{dx} + (x^2 + 1)y = 0</math></p> <p>Find a second independent solution.</p> <p><b>2010</b></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p>Show that <math>e^{\frac{y}{2}}</math> is a solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p>Solve <math>\frac{dy}{dx} + 3y = 0</math> for <math>y</math>.</p> <p><b>2009</b></p> <p>Solve <math>\frac{dy}{dx} - 2 \tan x \frac{dy}{dx} + 5y = \sec x e^y</math></p> <p>Solve <math>p^2 + 2py \cot x = y^2</math> where <math>p = \frac{dy}{dx}</math></p> <p>Find the 2nd order ODE for which <math>x^2</math> and <math>x^2 e^x</math> are solutions.</p> | <p><b>2009</b></p> <p>Solve <math>\frac{dy}{dx} - \cot x \frac{dy}{dx} - (\cot x)y = \sin x</math></p> <p>Find the general solution of the equation <math>\frac{dy}{dx} + (1+x)\frac{dy}{dx} + y = 4\cos \log(1+x)</math></p> <p>Reduce the equation <math>x \frac{dy}{dx} = 2 \frac{dy}{dx}</math> to the following ordinary differential equation of the second degree:</p> <p><math>(2x^2 - 1) \frac{dy}{dx} - y = 0</math></p> <p>Solve the following ordinary differential equation of independent solutions:</p> <p><math>y^2(x) = x^2 \log x</math> are linearly independent solutions of the differential equation that has <math>y_1(x)</math> and <math>y_2(x)</math> as the following ordinary differential equation of the second degree:</p> <p><math>\frac{dy}{dx} + 3y = 0</math> solution of it.</p> <p><b>2008</b></p> <p>Show that the functions <math>y_1(x) = x^2</math> and <math>y_2(x) = x^2 \log x</math> are linearly independent solutions of the differential equation that has <math>y_1(x)</math> and <math>y_2(x)</math> as the following ordinary differential equation of the second degree:</p> <p><math>\frac{dy}{dx} + 3y = 0</math> solution of it.</p> <p><b>2007</b></p> <p>Find the orthogonal trajectories of the family of the curves <math>\frac{x^2}{a^2} + \frac{y^2}{b^2} + \lambda = 1</math>, <math>\lambda</math> being a parameter.</p> <p>Show that <math>\frac{dy}{dx}</math> and <math>\frac{d^2y}{dx^2}</math> are linearly independent</p> <p>Solutions of <math>\frac{d^2y}{dx^2} + 4y = 0</math> find the general solution which <math>y(0) = 0</math> and <math>\frac{dy}{dx}(0) = 1</math></p> <p>Find the family of curves whose tangents form an angle of <math>\pi/4</math> with the hypotenuse <math>xy = c</math>.</p> <p>Apply the method of variation of parameters to solve <math>\frac{dy}{dx} + 4x \frac{dy}{dx} + y = x^2 \sin x</math></p> <p>Find the family of curves whose tangents form an angle of <math>\pi/4</math> with the hypotenuse <math>xy = c</math>.</p> <p>Find the general solution of <math>(1-x^2) \frac{dy}{dx} - 2x</math></p> <p><b>2006</b></p> <p><math>(D^2 + a^2)y = \cos ax</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 2x</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p><b>2005</b></p> <p>Solve <math>\frac{dy}{dx} + 2 \frac{dy}{dx} + 3y = x^2</math></p> <p>Solve <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Show that <math>e^{\frac{y}{2}}</math> is a solution of <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p><b>2004</b></p> <p>Find the general solution of <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p><b>2003</b></p> <p>Solve <math>\frac{dy}{dx} + 2 \frac{dy}{dx} + 3y = x^2</math></p> <p>Solve <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Show that <math>e^{\frac{y}{2}}</math> is a solution of <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p><b>2002</b></p> <p>Solve <math>\frac{dy}{dx} + 2 \frac{dy}{dx} + 3y = x^2</math></p> <p>Solve <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Show that <math>e^{\frac{y}{2}}</math> is a solution of <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> <p><b>2001</b></p> <p>Solve <math>\frac{dy}{dx} + 2 \frac{dy}{dx} + 3y = x^2</math></p> <p>Solve <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Show that <math>e^{\frac{y}{2}}</math> is a solution of <math>\frac{dy}{dx} + 3y = x^2</math></p> <p>Find the general solution of <math>\frac{dy}{dx} - 4x \frac{dy}{dx} + y = 0</math></p> |
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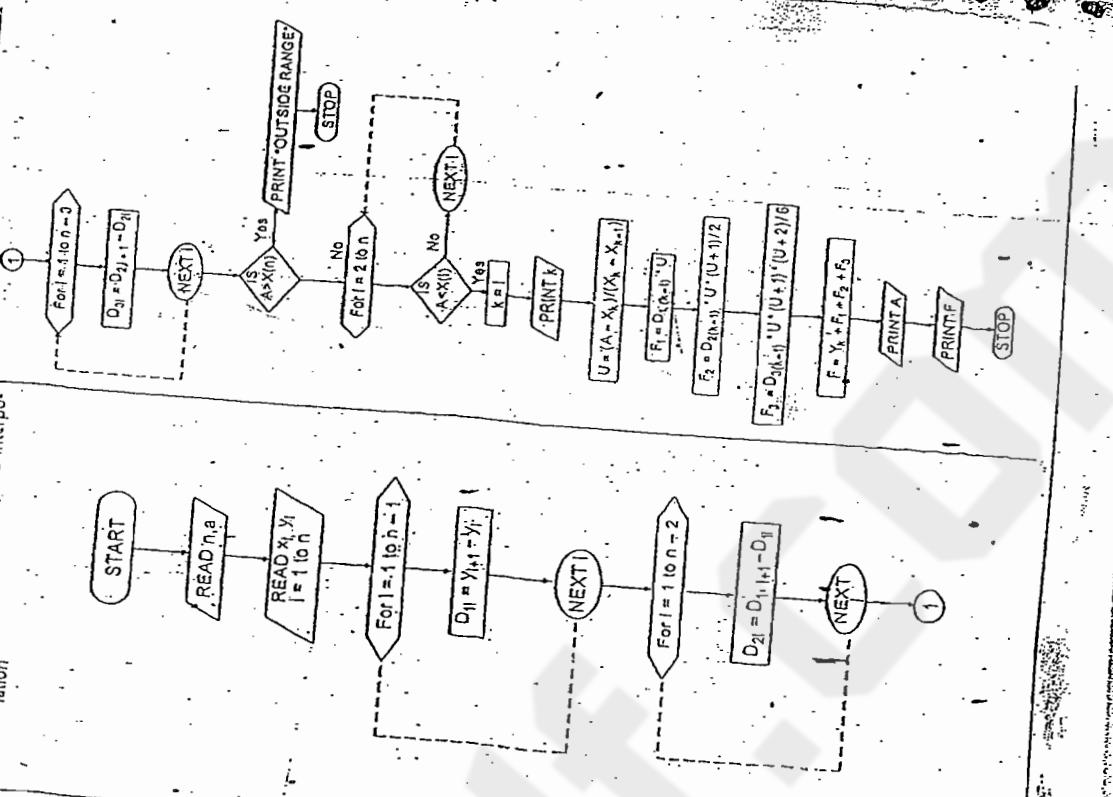
## MATHEMATICS

(6) Flow chart for Gauss-Seidel (iterative) method



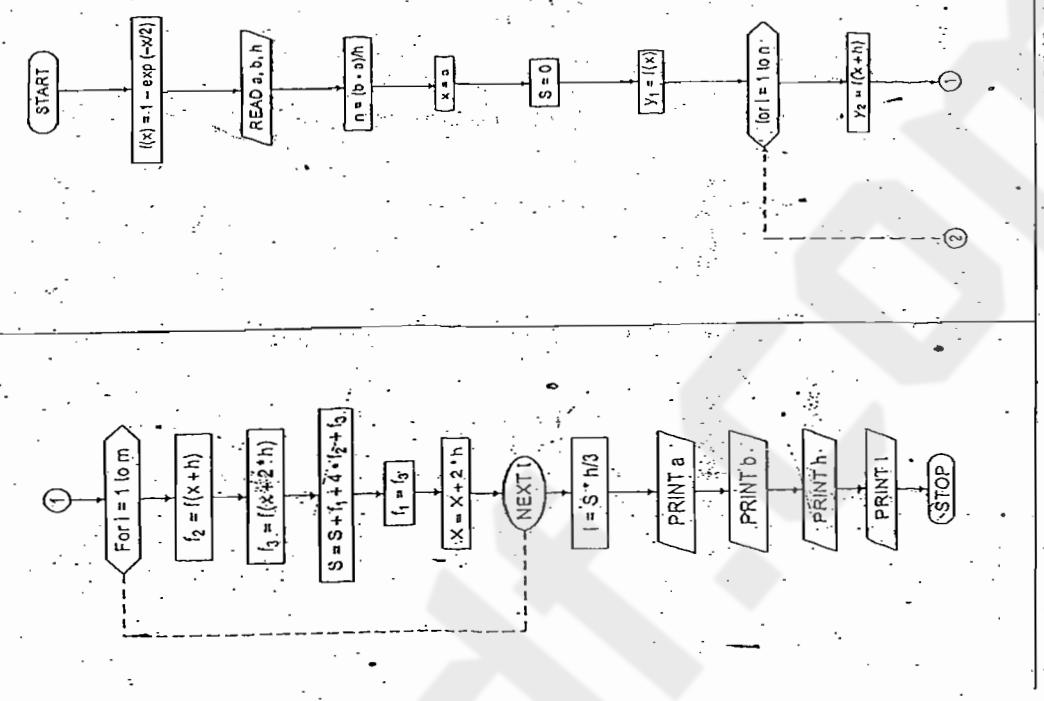
## MATHEMATICS

(8) Flow chart Newton-backward Interpolation



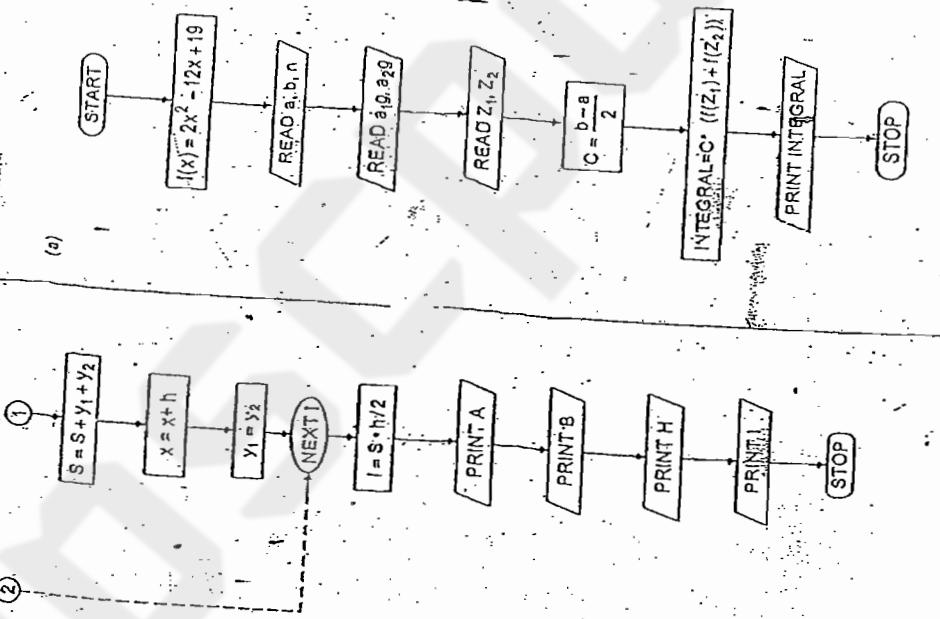
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 INSTITUTE OF MATHEMATICAL SCIENCES  
 INSTITUTE FOR IAS/IFS EXAMINATION  
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 Mobi: 09999191425  
 Flow chart for Trapezoidal rule



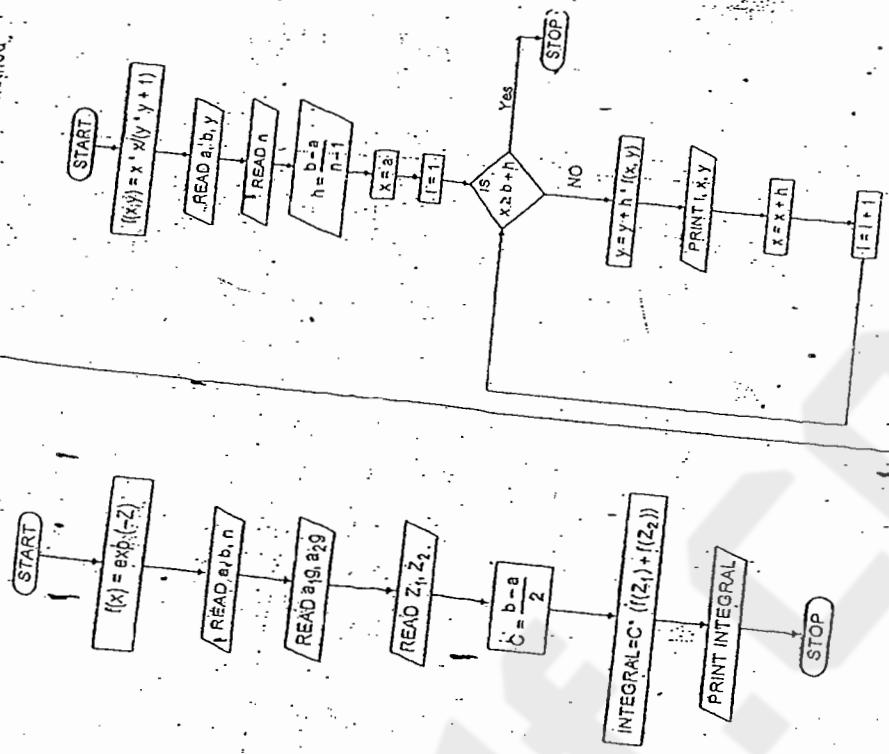
## MATHEMATICS

## (12) Flow chart for Gaussian Quadrature formula



(b)

## (13) Flow chart for Euler's method



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(1) Flow chart for 4th order Runge-Kutta  
method

