

**IMS**  
**MATHS**  
**BOOK-12**



(2)

**Special Linear  
Programming Problems**

	I	II	III	IV
A	2	0	2	6
B	2	4	5	0
C	6	5	0	6
D	0	3	5	2

- iv) There is now a single zero in 1st row in the position (1, 2). Encircle it to get the optimal assignment as AI<sub>1</sub>, BIV, CIII, DII.

**Step 5**

Adding costs corresponding to these assignments from the original profit maximizing matrix we get the maximum profit as,

$$7 + 8 + 9 + 5 = 29.$$

**EXERCISE 3:** There are 5 jobs to be done on 5 available machines. The following matrix shows the return in rupees on assigning various jobs to different machines. Determine an assignment which maximizes the total return.

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>	M <sub>5</sub>
J <sub>1</sub>	5	11	10	12	4
J <sub>2</sub>	2	4	6	3	5
J <sub>3</sub>	3	12	5	14	6
J <sub>4</sub>	6	14	4	11	7
J <sub>5</sub>	7	9	8	12	5

#### 12.4 SUMMARY

Assignment problem has been studied in this unit. Given an equal number of jobs and machines the problem consists of determining an optimal assignment. If the objective is to minimize the total cost incurred, the problem is known as cost minimizing assignment problem or simply assignment problem. In section 12.1, an assignment problem has been introduced systematically taking real life situations. Mathematical formulation of the problem has been developed in section 12.2. It has been shown that an assignment problem is represented completely by its cost-matrix.

An assignment problem is a special type of transportation problem and so of a linear programming problem. Consequently, it can be solved by the

	I	II	III	IV
A	0	0	2	5
B	3	5	6	0
C	6	5	0	5
D	1	4	6	0

## The Assignment Problem

## Step 3

The minimum uncovered element is 1, so

- subtracting 1 from all uncovered elements
- adding 1 to elements at intersection of horizontal and vertical lines viz. elements at positions (1, 4) and (3, 4).
- leaving all other covered-elements unchanged, we get,

	I	II	III	IV
A	0	0	2	6
B	2	4	5	0
C	6	5	0	0
D	0	3	5	0

Observe that now we require exactly 4 lines to cover all the zeros i.e. now  $r = n$ . So, we can go to step 4, and make optimal assignment.

## Step 4

- There is a single zero in 2nd row in the position (2, 4). Encircle this zero and cross all other zeros in its column i.e. 4th column.
- There is a single zero in 3rd row in the position (3, 3). Encircle this zero and cross all other zeros (if any) in its columns i.e. 3rd column.
- Now, there is only one unmarked zero in 4th row in the position (4, 1). Encircle this zero and cross all other zeros in its column i.e. 1st column.

**Special Linear Programming  
Problems**

	I	II	III	IV
A	-6	-7	-5	-2
B	-4	-3	-2	-8
C	-2	-4	-9	-4
D	-5	-3	-1	-7

**Step 1**

- i) Subtract the minimum element -7 from all elements of 1st row. Similarly, subtract -8, -9 and -7 respectively from all elements of 2nd, 3rd and 4th rows. The reduced matrix is,

	I	II	III	IV
A	1	0	2	5
B	4	5	6	0
C	7	5	0	5
D	2	4	6	0

(Note that this step amounts to subtracting each element of the original matrix (of the profit maximizing assignment problem) from the corresponding maximum element of their rows respectively. In other words, subtract all elements of 1st row from the maximum element i.e. 7 of the 1st row. Similarly, for the other rows).

- ii) Subtract the minimum element of each column from all elements of that column. The reduced matrix so obtained is,

	I	II	III	IV
A	0	0	2	5
B	3	5	6	0
C	6	5	0	5
D	1	4	6	0

**Step 2**

Cover all the zeros by minimum number of horizontal and vertical lines. Observe that only 3 lines can cover all the zeros. So,  $r = 3$ . As  $3 = r < n = 4$ , so we go to step 3.

Note that any arbitrary choice in step 4 (iii), of the zero to be encircled, would yield the same minimum total-cost.

#### The Assignment Problem

**EXERCISE 2:** Solve the cost-minimizing assignment problem.

	I	II	III	IV	V	VI
A	7	8	3	7	6	2
B	3	7	9	3	1	6
C	5	3	7	5	6	3
D	8	4	8	7	2	2
E	6	7	8	6	9	4
F	5	7	7	5	5	7

#### EXAMPLE 3

The owner of a small machine shop has 4 mechanists available to do 4 jobs. Jobs are offered with expected profits for each mechanist as follows:

	Mechanists			
	I	II	III	IV
A	6	7	5	2
B	4	3	2	8
C	2	4	9	4
D	5	3	1	7

Find by using the assignment method, the assignment of mechanists to jobs that will result in a maximum profit.

#### SOLUTION

From linear programming we know that a maximization problem can be converted into a minimization problem by replacing the costs with their negatives. It is also known that an assignment problem is a linear programming problem. So, we can convert the above maximizing assignment problem into the usual minimizing assignment problem, by replacing costs with their negatives and proceed with the Hungarian Method.

The corresponding minimizing assignment problem has the cost-matrix given below:

**Special Linear  
Programming Problems**

	$M_1$	$M_2$	$M_3$	$M_4$
$J_1$	0	3	3	7
$J_2$		8	7	0
$J_3$		0	0	5
$J_4$	6	0	0	7

As  $r = 4 = n$ , we can straightway go to step 4, and make the optimal assignment.

**Step 4**

- i) There is only one zero in 1st row in position (1, 1), so encircle this zero, and cross other zeros (if any) in its column i.e. 1st column.
- ii) There is only one zero in 2nd row in position (2, 4), so encircle this zero and cross other zeros (if any) in its column i.e. 4th column.

	$M_1$	$M_2$	$M_3$	$M_4$
$J_1$	(0)	3	3	7
$J_2$	3	8	7	(0)
$J_3$	4	(0)	X	5
$J_4$	6	X	(0)	7

- iii) Now, observe that 3rd and 4th rows as well as 2nd and 3rd columns contain two zeros each. To break this, and make an assignment, we pick any zero arbitrarily. Say, we pick zero in position (3, 2) and encircle it. Now, cross all zeros in its row i.e. 3rd row as well as its column i.e. 2nd column.
- iv) There is only one zero left in position (4, 3). Encircle it to get the optimal assignment as  $J_1 M_1, J_2 M_4, J_3 M_2, J_4 M_3$ .
- v)  $J_1 M_1, J_2 M_4, J_3 M_3, J_4 M_2$  is an alternative optional assignment.

**Step 5**

For determining minimum total cost, refer to the original cost-matrix of this problem and add the costs corresponding to  $J_1 M_1, J_2 M_4, J_3 M_2, J_4 M_3$ . This gives the minimum assignment cost as.

$$2 + 1 + 3 + 4 = 10.$$

**EXAMPLE 2****The Assignment Problem**

Solve the cost-minimizing assignment problem whose cost matrix is given below,

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>
J <sub>1</sub>	2	5	7	9
J <sub>2</sub>	4	9	10	1
J <sub>3</sub>	7	3	5	8
J <sub>4</sub>	8	2	4	9

**SOLUTION: Step 1**

- i) Subtracting the minimum element of each row from all elements of that row, the reduced cost-matrix is,

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>
J <sub>1</sub>	0	3	5	7
J <sub>2</sub>	3	8	9	0
J <sub>3</sub>	4	0	2	5
J <sub>4</sub>	6	0	2	7

- ii) subtracting the minimum element of each column from all elements of that column, we get

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	M <sub>4</sub>
J <sub>1</sub>	0	3	3	7
J <sub>2</sub>	3	8	7	0
J <sub>3</sub>	4	0	0	5
J <sub>4</sub>	6	0	0	7

**Step 2**

Cover all the zeros by least number of horizontal and vertical lines. Exactly 4 lines are required to cover all the zeros. So, r = 4.

- (iii) 3rd column contains only one zero in position  $C_{33}$ ,  
so, encircle it and cross all other zeros in  
its row. (i.e., 3rd row)
- (iv) There is only one zero in 3rd row, so encircle it.

Now, since each row and each column  
has a single encircled zero.

i.e., each row and each column has  
one and only one assignment,  
so an optimal assignment is reached.

The optimal assignment is:

$$A \rightarrow \text{II}, B \rightarrow \text{I}, C \rightarrow \text{II}, D \rightarrow \text{IV}.$$

Steps: The minimum assignment cost is

$$\underline{C_{13} + C_{21} + C_{32} + C_{44}} = 9 + 5 + 14 + 9 \\ = 37.$$

→ Solve the cost-minimizing assignment problem  
with the cost-matrix machine

Jobs

	I	II	III	IV	V
A	11	10	18	5	9
B	14	13	12	18	6
C	3	4	2	9	
D	15	18	19	7	12
E	10	11	19	6	14

The reduced cost-matrix so obtained is,

	I	II	III	IV
A	2	0	0	2
B	0	1	1	2
C	2	0	2	0
D	0	3	2	0

Again, cover the zeros by minimum number of horizontal and vertical lines.

We require exactly 4 lines to cover all the zeros.

As  $r=c=n$ ; optimal assignment can be made at this stage;

so go to step (4).

	I	II	III	IV
A	2	0	0	1
B	0	1	1	2
C	2	0	2	0
D	0	3	2	0

Step (1) for making assignments, proceed as follows:-

	I	II	III	IV
A	2	X	(0)	2
B	(0)	1	1	2
C	2	(0)	2	X
D	X	3	2	(0)

(i) 2<sup>nd</sup> row has only one zero in position (2,1).

So encircle this zero and cross all other zeros in its column (i.e., 1<sup>st</sup> column).

(ii) Now, the 4<sup>th</sup> row has only one zero in position (4,4), so encircle it and cross all other zeros in its column (i.e., 4<sup>th</sup> column).

- (ii) Subtracting the minimum elements of each column from elements of that column.  
we get -

	I	II	III
A	1	0	0
B	0	2	2
C	1	0	2
D	0	4	3

Step 2)

Cover all the zeros by minimum number of horizontal and vertical lines.

A systematic approach for this is to look for a row or column containing the maximum number of zeros.

See that we can cover all the zeros by 3 lines only.

So,  $x = 3 < 4 = n$ .

So go to step (3)

	1	0	0	2
	0	2	2	3
	1	0	2	0
	0	4	3	1

Step 3)

# Is the least uncovered element.

- (i) Subtract 1 from all the uncovered elements.  
(ii) add 1 to elements at intersection of the covering lines namely 1 at position (1,1)

and 1 at position (3,1).

- (iii) leave other covered elements unchanged.

In such a case encircle any one of the zeros which is not encircled arbitrarily and cross all other zeros in its row and column, both.

Continuing in this way we shall have exactly one encircled zero in each row and each column.

Assignments are made corresponding to each encircled zero.

Step(2):

for obtaining the minimum cost, refer to the original cost-matrix of the given problem. Optimum cost is obtained by adding costs  $c_{ij}$ 's at all the encircled-zero positions.

Note: If the cost matrix is not square i.e.,  $m \neq n$ , make it square by adding suitable number of dummy rows (or columns) with zero costs.

→ Solve the cost-minimizing assignment problem.

	I	II	III	IV
A	10	12	9	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Step 1:

Subtracting the minimum element of each row from all elements of that row.

we get

	I	II	III	IV
A	1	3	0	2
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

Step(3):

Here, the least number of lines needed to cover all the zeros is less than the order of the assignment problem.

Pick the minimum element not covered by these covering-lines and,

- (i) Subtract it from all uncovered elements
- (ii) add to all elements at intersection of two covering lines, and
- (iii) leave all other covered elements unchanged

Thus we get a new reduced matrix. Go to step(2)

Step(4):

Here the minimum number of lines needed to cover all the zeros is exactly equal to the order of the assignment.

An optimal assignment shall be made now.

- (i) Examine the rows successively until a row with exactly one zero is found. Encircle this zero and cross all other zeros in its column

- (ii) Similarly, examine the columns successively until a column with exactly one zero is found. Encircle this zero and cross all other zeros in its row.

Repeating the above steps either of the following situations encountered.

- a) Each row and each column has an encircled zero. In this case an optimal assignment has been made and the process terminates.

- b) There lie more than one zero in some rows and columns which are not encircled.

the minimum number of lines required to cover all the zeros was two.

In the third problem of order 3, the optimal assignment could be made only when the minimum number of lines required to cover all zeros in the reduced matrix is equal to three.

All the above observations contribute to the following steps of the Hungarian Method for solving an  $n \times n$  assignment problem.

### Hungarian Method:

#### Step 1:

- (i) Subtract the minimum element of each row from all elements of that row.
  - (ii) subtract the minimum element of each column from all elements of that column.
- The reduced matrix thus obtained, contains atleast one zero in each row and each column.

#### Step 2:

Cover all the zeros in the reduced cost matrix by minimum number of horizontal and vertical lines. Let the least number of such lines needed to cover all the zeros be  $r$ .

If  $r = n$ , an optimal assignment can be made at this stage.

In this case go to step 4.

If  $r < n$ , an optimal assignment cannot be made at this stage.

In this case go to step 3.

Such a situation can be systematically identified by observing that all the zeros in the above reduced matrix can be covered by a minimum of two lines only (shown as dotted lines below).

	0	1	
3	0	1	
2	0	3	

This can be resolved by creating new zeros from amongst the elements uncovered by these two dotted lines, for this purpose of the uncovered elements i.e., 1 is subtracted from all uncovered elements; added to the elements at intersection of the dotted lines, leaving other covered. The cost-matrix further reduces to, ~~the elements unchanged~~

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>
J <sub>1</sub>	0	3	0
J <sub>2</sub>	2	0	0
J <sub>3</sub>	1	0	2

Now, optimal assignment yielding zero total cost can be made as J<sub>1</sub>M<sub>1</sub>, J<sub>2</sub>M<sub>3</sub>, J<sub>3</sub>M<sub>2</sub> which corresponds to the total cost 2+3+2=7.

Now, it can be observed that we can make optimal assignments yielding zero total cost in the reduced cost matrix, only when the minimum number of dotted horizontal and vertical lines needed to cover all the zeros is equal to the order of the given assignment problem. (3x3 matrix) -

In the first two problems, each of order 2, we could make optimal assignments only when

	M <sub>1</sub>	M <sub>2</sub>
J <sub>1</sub>	0	0
J <sub>2</sub>	2	0

Keeping in mind the one job-one machine basis, the optimal assignment yielding total cost zero is J<sub>1</sub>M<sub>1</sub>, J<sub>2</sub>M<sub>2</sub> from the original cost-matrix of this problem the optimal assignment J<sub>1</sub>M<sub>1</sub>, J<sub>2</sub>M<sub>2</sub> corresponds to total cost ~~5+2 = 7~~.

→ Sometimes, even after this, optimal assignments in the reduced cost-matrix yielding zero total-cost cannot be located.

→ for example:

Consider the problem

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>
J <sub>1</sub>	2	4	2
J <sub>2</sub>	5	2	3
J <sub>3</sub>	4	2	5

After subtracting the minimum of each row (column) from all elements of that row (column) the reduced matrix so obtained is

	M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>
J <sub>1</sub>	0	2	0
J <sub>2</sub>	3	0	1
J <sub>3</sub>	2	0	3

It can be seen that on one job-one machine basis, an optimal assignment yielding total-cost zero cannot be obtained from this reduced matrix.

	M <sub>1</sub>	M <sub>2</sub>
J <sub>1</sub>	5	3
J <sub>2</sub>	2	6

In order to ensure that no element of the cost-matrix becomes negative, subtract the minimum element of each row from all the elements of that row.

we get,

	M <sub>1</sub>	M <sub>2</sub>
J <sub>1</sub>	2	0
J <sub>2</sub>	0	4

In the above reduced cost-matrix, the optimal assignment yielding total cost zero is J<sub>1</sub>M<sub>2</sub>, J<sub>2</sub>M<sub>1</sub>. So for the original problem the optimal assignment is J<sub>1</sub>M<sub>2</sub>, J<sub>2</sub>M<sub>1</sub>, yielding optimal value 3+2 = 5.

Let us consider

	M <sub>1</sub>	M <sub>2</sub>
J <sub>1</sub>	5	3
J <sub>2</sub>	6	2

Subtracting the minimum element of each row from all elements of that row, the reduced matrix obtained is,

	M <sub>1</sub>	M <sub>2</sub>
J <sub>1</sub>	2	0
J <sub>2</sub>	4	0

Now, subtracting the minimum element of each column from all elements of that column, the cost-matrix is further reduced to

with  $x_{ij} = 0 \text{ or } 1 ; \quad i=1,2,\dots,n; \quad j=1,2,\dots,n.$

An assignment problem is known from its cost-matrix  
[ $c_{ij}$ ], which is given as

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \\ \vdots & & & c_{ij} & \dots & c_{in} \\ c_{n1} & c_{n2} & \dots & c_{nj} & \dots & c_{nn} \end{bmatrix}$$

If each row refers to a job and each column refers to a machine, then  $c_{ij}$  is the cost of processing  $i^{th}$  job on  $j^{th}$  machine.

Note: An Assignment problem could be solved by Simplex method. It also happens to be an  $n \times n$  transportation problem with each  $a_i = b_j = 1$ . However, as an assignment problem is highly degenerate it will be frustrating to attempt to solve it by simplex method or transportation methods.

In fact a very convenient iterative procedure is available for solving an Assignment problem. It is called the Hungarian Method. Before we discuss this method, let us take up the following results.

→ The optimal solution of an assignment problem remains the same, if constant is added or subtracted from any row or column of the cost matrix.

for example:

Let us take the assignment problem.

Let us now generalise it and formulate it for  $n$  jobs

There be ' $n$ ' jobs which are to be processed on ' $n$ ' machines on one-job-one-machine basis

Let  $J_1, J_2, \dots, J_n$  be the ' $n$ ' jobs and let  $M_1, M_2, \dots, M_n$  be the ' $n$ ' machines.

Let  $c_{ij}$  be the cost of processing  $i^{\text{th}}$  job  $J_i$  on the machine  $M_j$ .

Let us formulate the problem of assigning jobs to machines so as to minimize the overall cost.

Let us define variable  $x_{ij}$  as follows,

$$x_{ij} = \begin{cases} 0 & \text{if } i^{\text{th}} \text{ job is not assigned to } j^{\text{th}} \text{ machine} \\ 1 & \text{if } i^{\text{th}} \text{ job is assigned to } j^{\text{th}} \text{ machine} \end{cases}$$

No job remains unprocessed and no machine remains idle.

Note that the number of jobs is equal to the number of machines.

The hypothesis of one job - one machine implies.

$$\sum_{i=1}^n x_{ij} = 1 \quad (j=1, 2, \dots, n)$$

$$\sum_{j=1}^n x_{ij} = 1 \quad (i=1, 2, \dots, n)$$

In each of these summations, only one term on the left has variable  $x_{ij}$  equal to one and the rest are zeros.

The assignment problem is mathematically stated as:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{subject to } \sum_{i=1}^n x_{ij} = 1 \quad (j=1, 2, \dots, n)$$

$$\sum_{j=1}^n x_{ij} = 1 \quad (i=1, 2, \dots, n)$$

## The Assignment problem

An assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. The objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

### for example:

A factory manager may wish to assign three different jobs to three machines in such a way that the total cost is minimized.

There are  $3! = 6$  ways of assigning 3 jobs to 3 machines. However, a problem of assigning 10 jobs to 10 machines requires  $10!$  assignments to be examined, which clearly is not a simple task. Hence, the need to evolve efficient method to solve an assignment problem.

However, a much more efficient method of solving such problems is available. This is the Hungarian method.

### Formulation of an Assignment problem:

Let us consider the case of a factory which has 3 jobs to be done on the 3 available machines. Each machine is capable of doing any of the three jobs. For each job, the machining cost depends

$$x_{41} = 30, x_{21} = 20, x_{32} = 30, x_{33} = 50, x_{34} = 10.$$

and the minimum cost of transportation is

$$\begin{aligned} &= (14 \times 30) + (15 \times 20) + (10 \times 30) + (8 \times 50) + (13 \times 10) \\ &= 1550. \end{aligned}$$

→ solve the TP:

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	R <sub>0</sub> ↓
S <sub>1</sub>	4	3	2	10 (6)
S <sub>2</sub>	1	5	0	13
S <sub>3</sub>	3	8	6	12
b <sub>j</sub> →	8	5	4	(2)

Since the total demand is more than total availability.

∴ The given problem is unbalanced.

∴ we introduce an artificial source  $s_4$  with availability  $a_4 = 160 - 140 = 20$ .

costs  $c_{ij}$ 's for all cells corresponding to artificial source  $s_4$  are taken as zeros.

By this we get the balanced TP.

	$D_1$	$D_2$	$D_3$	$D_4$	$a_{ij}$
$S_1$	25	17	25	14	30
$S_2$	15	10	18	24	50
$S_3$	16	20	8	13	60
$S_4$	0	0	0	0	20
$\sum$	30	30	50	50	
$b_j \rightarrow$	30	30	50	50	

Solving this balanced transportation problem by the u-v method,

The optimal solution is given as

	$D_1$	$D_2$	$D_3$	$D_4$	$a_{ij} - u_i - v_j$
$S_1$	25 (-4)	17 (-8)	25 (-10)	14 (3)	30 14
$S_2$	15 (20)	10 (30)	18 (-20)	24 (-5)	50 15
$S_3$	16 (-3)	20 (-12)	8 (60)	13 (6)	60 13
$S_4$	0 (10)	0 (-5)	0 (-5)	0 (10)	20 0
$\sum$	30	30	50	50	
$b_j \rightarrow$	30	30	50	50	
$v_j \rightarrow$	0	-5	-5	0	

From this, the optimal solution of the original unbalanced TP is given by:

case 2: When  $\sum_{i=1}^m a_i > \sum_{j=1}^n b_j$

Here the total availability at all sources is more than the total demand of all the destinations.

In such cases we do the following-

(i) Create an artificial destination  $D_{n+1}$  with

$$\text{demand } b_{n+1} = \sum_{i=1}^m a_i - \sum_{j=1}^n b_j.$$

This gives us a new problem with  $m$  sources and  $n+1$  destinations, and of course it balanced.

(ii) Set  $c_{ij}$ 's for all cells corresponding to the artificial destination  $D_{n+1}$ , equal to zero.

i.e.,  $c_{i,n+1} = 0$  for  $i = 1, 2, \dots, m$ .

(iii) Solve the above balanced TP by u-v method  
optimal solution of this problem, with variables  
 $x_{i,n+1}; i=1, 2, \dots, m$ ; corresponding to the  
artificial destination  $D_{n+1}$  deleted give an  
optimal solution for the given unbalanced TP.

Solve the TP:-

	$D_1$	$D_2$	$D_3$	$D_4$	$D_{n+1}$
$S_1$	25	17	25	14	30
$S_2$	15	10	18	24	50
$S_3$	16	20	8	13	60
	30	30	50	50	

Ques: In the given TP,

$$\text{Total demand} = 160$$

$$\text{Total availability} = 140.$$

### Unbalanced transportation problems.

31

Till now we have dealt with the transportation problem, assuming that the total demand of all the destinations is equal to the total availability at all the sources i.e.,  $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ .

But a situation may arise when the total available supply is not equal to the total requirement. Such type of T.P.'s are called unbalanced transportation problem.

Case I when  $\sum_{i=1}^m a_i < \sum_{j=1}^n b_j$

Here the total availability at all sources is less than the total demand of all the destinations.

In such cases we do the following.

(i) Create an artificial source  $S_{m+1}$  with

$$\text{availability } a_{m+1} = \sum_{j=1}^n b_j - \sum_{i=1}^m a_i$$

This gives us a new problem with  $(m+1)$  source and  $n$  destinations, and of course it is balanced.

(ii) Set  $c_{ij}$ 's in cells corresponding to this artificial source  $S_{m+1}$ , equal to zero.

$$\text{i.e., } c_{m+1,j} = 0 \text{ for } j=1, 2, \dots, n.$$

(iii) Solve the balanced T.P. thus obtained by U-V method. Optimal solution of this balanced T.P.

with variables  $a_{m+1,j}$ ;  $j=1, 2, \dots, n$  corresponding to the artificial source  $S_{m+1}$  deleted give an optimal solution for the given unbalanced T.P.



the cell (2,3) leaves the basis.

The new basic feasible solution is shown in table - 5.

Table - 5

9	12	9	6	9	10
7	3	7	7	5	5
5	4	6	7	5	5
6	5	10	11	3	4
6	8	11	2	2	10
6	8	11	2	2	10

(Crossed)

The no. of allocations =  $m+n-1 = 9$ .

Now compute the net evaluations which are shown in table - 6.

						W <sub>i</sub> ↓
9	12	9	6	9	10	2
7	3	7	7	5	5	0
6	5	10	11	3	4	2
6	8	11	2	2	10	1
6	8	11	2	2	10	0

All the net evaluations  $\leq 0$ .

∴ The new basic feasible solution is

Optimal.

Hence, the optimum solution is  
 $x_{11} = 5, x_{12} = 4, x_{26} = 2, x_{31} = 1, x_{33} = 1, x_{41} = 3,$   
 $x_{42} = 6, x_{45} = 4, x_{23} = 0$

The minimum transportation cost is  
 $= (5 \times 4) + (4 \times 6) + [f(x=0)] + (2 \times 5) + (1 \times 6) + (1 \times 9) +$   
 $(3 \times 6) + (2 \times 2) + (4 \times 2) = 112.$

Table-3

9 (1)	12 (1)	9 (3)	6 (2)	9 (1)	10 (1)
7 (2)	3 (4)	7 (1)	7 (2)	5 (1)	5 (2)
6 (1)	5 (1)	9 (1)	11 (2)	3 (1)	11 (1)
6 (3)	8 (1)	11 (1)	2 (1)	2 (1)	10 (1)

u<sub>ij</sub>

-3

0

-2

-3

$$V_j \rightarrow 9 \ 3 \ 12 \ 5 \ 5 \ 5$$

Since the net evaluations in two cells are the same.

∴ the current basic feasible solution is not optimal.

Choose unoccupied cell with the maximum Δ<sub>ij</sub>.

Clearly Δ<sub>23</sub> = 5 is the most positive.

∴ the cell (2,3) enters the basis.

- we allocate an unknown quantity θ to this cell (2,3) and identify a loop involving basic cells around the entering cell.
- Add and subtract θ alternately to and from the transition cells of the loop subject to the item requirements as shown in the table - 4.

Table-4

9	12	9 (3)	6	9	10
7 (2)	3 (4)	7 (1)	7 (2)	5 (1)	5 (2)
6 (1)	5 (1)	9 (1)	11 (2)	3 (1)	11 (1)
6 (3)	8 (1)	11 (1)	2 (1)	2 (1)	10 (1)

Now, assign minimum value θ so that one basic variable becomes zero and the other basic variables ≥ 0.

Taking θ =  $\frac{1}{2}$  in table - 4

x<sub>23</sub> becomes zero.  
i.e., x<sub>23</sub> = 0.

Table - 2

9	12	9(5)	6	7	5	6	5
7	3(4)	7	7	5(6)	5(2)	6+6 =6	
6	5	9(1)	11	3	11		2
6(3)	8	11	2(2)	2(4)	10		9

9 4 6 2 4+6  
=4

Now find the values of  $u_i$  &  $v_j$ .  
 As the maximum no. of allocations (basic cell)  
 exist for the 2nd and 4th rows.  
 putting either  $u_2 = 0$  or  $u_4 = 0$ .

$$\text{Let } u_2 = 0$$

$$\text{we have } u_2 + v_2 = 3 \Rightarrow v_2 = 3$$

$$u_2 + v_5 = 5 \Rightarrow v_5 = 5$$

$$u_2 + v_6 = 5 \Rightarrow v_6 = 5$$

$$u_4 + v_2 = 2 \Rightarrow u_4 = -3$$

$$u_4 + v_1 = 6 \Rightarrow v_1 = 9$$

$$u_4 + v_4 = 2 \Rightarrow v_4 = 5$$

$$u_3 + v_1 = 6 \Rightarrow u_3 = -3$$

$$u_3 + v_3 = 9 \Rightarrow v_3 = 12$$

$$u_1 + v_3 = 9 \Rightarrow u_1 = -3$$

and also the net evaluations  $\Delta_{ij} = u_i + v_j - c_{ij}$   
 for all unoccupied cells are exhibited  
 on Table - 3.

Number of allocations (basic cells) which is less than  $m+n-1 = 4+6-1 = 9$ .

∴ The solution is feasible, but not basic feasible.

i.e., the solution is degenerate.

In order to complete the basis and thereby remove degeneracy, we require only one more non-negative basic variable.

To break degeneracy, we allocate a very small +ve quantity  $\epsilon (\approx 0)$  (or allocate 0-) to occupied cell with minimum cost. Minimum entry in unoccupied position is in cell (3,5). If we allocate small quantity 'e' to cell (3,5) then  $m+n-1$  allocations (basic cells) will not be independent. because allocation at cell (3,5) forms a closed loop.

		(5)		
	(4)			(2)
(4)		(1)	e	
(3)			(2)	(3)

So small allocation 'e' cannot be made at cell (3,5).

Next, higher cost in unoccupied cells is in cell (3,2) and (2,5).

Let us make small positive quantity ' $\epsilon'$ ' allocation in cell (2,5), say.

∴ we have  $m+n-1 (\epsilon \neq 0)$  allocations which are independent because no closed loop is formed.

→ solve the following transportation problem:

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$a_i \downarrow$
$S_1$	9	12	9	6	9	10	5
$S_2$	7	3	7	7	5	5	6
$S_3$	6	5	9	11	3	11	2
$S_4$	6	8	11	2	2	10	9
$b_j \rightarrow$	4	4	6	2	4	2	

Soln:

The total supply and total demand being equal  
 $\text{ie } \sum a_i = \sum b_j = 22$ .  
 The transportation problem is balanced.

Using the Vogel's approximation method, the initial basic feasible solution is as shown in Table-1.

Table-1. Table-1

9	12	9	6	9	10	5	(3)	(3)	(2)	(2)	(2)
7	3	7	7	5	5	6	(4)	(2)	(2)	(2)	(2)
6	5	9	11	3	11	2	(1)	(2)	(2)	(2)	(2)
6	8	11	2	2	10	9	(0)	(0)	(4)	(2)	(2)
	4	6	2	4	2	2					
(0)	(2)	(2)	(4)	(1)							
(0)	(2)	(2)	(3)	(1)							
(0)	(2)	(2)		(1)							
(0)	(2)	(2)									
(0)	(2)										
(0)	(2)										

2)

The cell (1, 3) enters the basis.

∴ we allocate '0' in the cell (1, 3) and draw a closed path beginning and ending at 0-cell (i.e., (1, 3) cell).

Add and subtract 0, alternately to and from the transition cells of the loop subject to the lim requirements, as shown in Table - 7.

2	3	11	7
0 (5)		9	
11	0	6 (1)	1
8 (6)	15 (2)	9 (2)	

Taking  $\theta = 1$ ;  $x_{11}$  becomes zero (i.e.,  $x_{11} = 0$ )

the cell (1, 1) leaves the basis.

The new basic feasible solution is shown in Table - 8.

Table - 8

2	3	11	7
0 (5)		9 (1)	
11	0	6 (1)	1
8 (4)	15 (2)	9 (2)	

The no. of allocations = min - 1 = 2 (in Table - 8)  
as shown  
Now compute the net evaluations which are shown in Table - 9.

2 (1)	3 (5)	11 (6)	7 (2)
1 (5)	0 (2)	6 (1)	1 (0)
8 (4)	15 (1)	9 (2)	0

Since all the net evaluations are 50.

The current basic feasible solution is optimal.

The optimal (minimum) transportation cost  
 $\rightarrow 1 \times 2 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9) \times 100$   
 $= ₹ 10,000$

34

$x_{24}$  becomes zero.

i.e.,  $x_{24} = 0$  (non-basic).

i.e., the cell (2,4) leaves the basis.

The new basic feasible solution  
is shown in Table - 5.

2	3	11	7	1
①	(5)			
1	0	6	1	①
5	⑥	15	9	②

$$\therefore x_1 = 1, x_{12} = 5, x_{23} = 1, x_{31} = 6, x_{33} = 2 \text{ & } x_{24} = 2$$

∴ the total transportation cost of this

revised solution

$$= \text{Rs. } (1x2 + 5x3 + 1x6 + 6x5 + 2x15 + 2x9) \times 10 \\ = \text{Rs. } 10,100$$

As the no. of allocations for table - 5

$$= m+n-1 = 8+4-1 = 6.$$

we compute the net evaluations which are shown in table - 6.

Table - 6

2	3	11	7	1
①	(5)	(1)	(-1)	
1	0	6	1	3
(-5)	(G3)	(1)	(-1)	-9
5	8	15	9	0
⑥	(-2)	(2)	(2)	

→ 5 6 15 9

Since the cell (1,3) has a negative value,  
the current basic feasible solution  
is not optimal.

Since the net evaluations in two cells

are +ve, a better solution can be

found (i.e., the current basic feasible solution  
is not optimal).

Choose the unoccupied cell with the minimum  $\Delta_{ij}$ .

In case of a tie, select the one with lower original cost.

In Table - 3, cells  $(1,3)$  &  $(2,3)$  each have

$\Delta_{ij} = 1$  and out of these, cell  $(2,3)$  has  
lower original cost 6.

$\therefore$  Cell  $(2,3)$  enters the basis.

We allocate an unknown quantity  $\theta$ , to  
this cell  $(2,3)$  and identify a loop involving  
basic cells around this entering cell.

Add and subtract  $\theta$ , alternately to and from  
the transition cells of the loop subject to  
the rim requirements as shown in the  
table - 4.

2	3	11	7
①	⑤		
0	6	0	① - 6
5	8	15	9
⑥		③ - ①	① + ⑥

Now, assign a maximum value to  $\theta$  so that  
one basic variable becomes zero and the other  
basic variables remain  $> 0$ .

If we put  $\theta = 1$  in Table - 4

Table - 2

2	3	11	7	6
(1)	(5)			
1	0	6	1	(1)
5	8	15	9	(1)

7 5 3 2

The no. of allocations  
 $= m+n-1 = 4+4-1 = 7$   
 best val.

i.e.,  $x_{11} = 1, x_{12} = 5$   
 $x_{24} = 1, x_{31} = 6$   
 $x_{33} = 3$  and  $x_{34} = 1$

The transportation cost according to this route is  
 given by  $(2 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9) \times 100 = \text{Rs } 10,20$

now finding the values of  $u_i$  &  $v_j$ :

At the maximum no. of allocations (basic cells)  
 exist in the 3rd row.

$\therefore$  let  $u_3 = 0$

where  $u_3 + v_1 = 5 \Rightarrow v_1 = 5$

$u_3 + v_3 = 15 \Rightarrow v_3 = 15$

$u_3 + v_4 = 9 \Rightarrow v_4 = 9$

$u_1 + v_1 = 2 \Rightarrow u_1 = -3$

$u_1 + v_2 = 3 \Rightarrow v_2 = 6$

$u_1 + v_4 = 1 \Rightarrow v_4 = -8$

Also the net evaluations,  $\Delta_{ij} = u_i + v_j - c_{ij}$

for all unoccupied cells are exhibited

in Table - 3.

		$u_i + v_j$
2	3	11
(1)	(5)	(1)
1	0	6
(-4)	(-2)	(1)
5	8	15
(6)	(-2)	(9)

$v_j \rightarrow 5 \quad 6 \quad 15 \quad 9$

project site are as follows:

		Project sites			
		1	2	3	4
Factories	1	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

Determine the optimal distribution for the Company so as to minimize the total transportation cost.

For:

Express the supply from the factories, demand at sites and the unit shipping cost in the form of the following transportation table.

		Project sites				Supply (ai) ↓
		1	2	3	4	
Factories	1	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10

Demand:

(bj) →

Here the supply being equal to the demand

$$\text{i.e. } \sum a_i = \sum b_j = 17$$

The problem is balanced.

Find the initial basic feasible solution

Using the VAM, the initial basic feasible solution is shown in Table 2.

The net evaluations  $\Delta_{ij} = u_i + v_j - c_{ij}$ , if for all unoccupied cells are.

$$\Delta_{11} = u_1 + v_1 - c_{11} = 17 - 10 - 21 = -14.$$

$$\Delta_{12} = -8, \Delta_{13} = -26, \Delta_{23} = -5, \Delta_{34} = -9.$$

The values of  $\Delta_{ij}$ 's are recorded in the right bottom of the each cell as shown in Table - 3

Since all the net evaluations are negative the current basic feasible solution is optimal.

Hence the optimal allocation is given by

$$x_{11} = 11, x_{21} = 6, x_{12} = 3, x_{22} = 4, x_{32} = 7 \\ \text{and } x_{33} = 12$$

$\therefore$  The optimal (minimum) transportation cost =  $11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 23 + 9 \times 27 + 12 \times 16 = \underline{\underline{Rs. 196.}}$

A company has three cement factories located in cities 1, 2, 3 which supply cement to four projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truck loads of cement daily respectively and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truck loads. The transportation costs per truck load of cement (in hundreds of rupees) from each plant to each

finally the initial basic feasible solution  
is as shown below.

21	16	25	13	(11)
17 (1)	18 (3)	14 (2)	23 (4)	
32 (7)	27 (8)	18 (12)	41 (9)	19

$a_i \downarrow$

$b_j \rightarrow$  6 10 12 15

$$\begin{aligned} \text{The number of allocations} &= m+n-1 \\ &= 3+4-1 \\ &= 6 \text{ (basic variables)} \end{aligned}$$

Now, finding the values of  $u_i$  and  $v_j$ :

As the maximum number of basic cells exist  
in the 2nd row.

$$\therefore \text{let } u_2 = 0.$$

$$\text{we have, } u_1 + v_1 = 17 \Rightarrow v_1 = 17$$

$$u_2 + v_2 = 18 \Rightarrow v_2 = 18$$

$$u_3 + v_4 = 23 \Rightarrow v_4 = 23$$

$$u_1 + v_4 = 13 \Rightarrow u_1 = -10$$

$$u_3 + v_2 = 27 \Rightarrow u_3 = 9$$

$$u_3 + v_3 = 18 \Rightarrow v_3 = 9$$

Table 3

21	16	25	13	(11)
E1	E2	E3		-10
17 (1)	18 (3)	14 (2)	23 (4)	0
32 (7)	27 (8)	18 (12)	41 (9)	9

$$v_j \rightarrow 17 \quad 18 \quad 9 \quad 23$$

since  $C_{44} = 13$ , is the minimum cost, we allocate  
 $x_{14} = \min(11, 15) = 11$ . in the cell  $(1, 4)$ .  
 This exhausts the availability of the first row.  
 and therefore we cross it.

35

Table - 2				
14	18	16	23	18 9 (3)
32	27	18	41	19 (9)
32	10	12	4	
(15)	(5)	(4)	(18)	

The row and column differences are now computed for reduced table - 2 and displayed with in brackets.

The largest of these is  $(18)$  which is against the fourth column.

Since  $C_{14} = 23$  is the minimum cost, we allocate  $x_{14} = \min(13, 4) = 4$ . in the cell  $(1, 4)$ .

This exhausts the availability of fourth column.

and therefore cross it.

Proceeding in this way, the subsequent reduced transportation table and differences for the remaining rows and columns are as shown below:

Table - 3				
14	18	16	23	18 9 (4)
32	27	18	19 (9)	19 (9)
32	10	12		
(15)	(5)	(4)	(9)	

Table - 4				
18	14	16	23	18 9 (4)
27	18	19	19 (9)	19 (9)
27	10	12		
(7)	(4)	(9)	(9)	

Table - 5				
27	18	16	23	18 9
7	12			

→ Solve the following transportation problem:

		Destination				Availability (a <sub>i</sub> )
		A	B	C	D	
Source	R	21	16	25	13	
	S	37	18	14	23	13
	T	32	27	18	41	19

Requirement  
(b<sub>j</sub>): 6 10 12 15 43.

Soln: by finding the initial solution by Vogel's Approximation

Since  $\sum a_i = \sum b_j = 43$ .

The problem is balanced.

Find the initial basic feasible solution:

Using Vogel's Approximation method, the initial basic feasible solution is:

The differences between the smallest and next to the smallest cost in each row and each column are first computed and displayed inside parenthesis against the respective rows and columns.

Table-1

21	16	25	13 (11)	11 (3)
17	18	14	23	13 (3)
32	27	18	41	19 (4)

6 10 12 15 4  
(4) (2) (4) - (10)

The largest of these differences is (10) which is associated with the fourth column.

31

Table - 6

$a_{ij} \leq u_i - v_j$	10	12	13	8	14	19
	(1)	(-)	(-)	(17)	(-)	(-)
	15	18	12	16	19	20
	(9)	(-)	(13)	(-)	(-)	(-)
	17	16	13	14	10	18
	(-)	(11)	(-)	(3)	(24)	(1)
	19	18	20	21	12	13
	(-)	(-)	(-)	(-)	(-)	(44)

 $b_j \rightarrow 10 \quad 11 \quad 13 \quad 20 \quad 24 \quad 15$  $v_j \rightarrow 16 \quad 16 \quad 13 \quad 14 \quad 10 \quad 18$ 

In Table - 6,

we have  $\Delta_{ij} \leq 0$  for all non-basic cells.

∴ The current basic feasible solution

is optimal.

The optimal solution is given by

$$x_{11} = 1, x_{14} = 17, x_{21} = 9, x_{23} = 13,$$

$$x_{32} = 11, x_{34} = 3, x_{35} = 24, x_{36} = 1$$

$$\text{and } x_{46} = 14.$$

and the minimum cost of transportation

is given by

$$(10 \times 1) + (8 \times 17) + (15 \times 9) + (12 \times 13) + (16 \times 11) + (14 \times 3) \\ + (10 \times 24) + (8 \times 1) + (14 \times 13) = 1095.$$

This completes one iteration and we get an improved basic feasible solution.

The next two iterations are shown in Table-4

Table-5 and Table-6  
Table-4

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$a_{ij}$	$u_{ij}$
$D_1$	10	12	13	8	14	19	18	$0 = u_1$
	(10)	(2)	0	(-7)	(6)+8	(-10)	(-7)	
$D_2$	15	18	12	16	19	20	22	6
	(1)	(1)	(13)	(-2)	(-9)	(-2)		
$D_3$	17	16	13	14	10	18	39	6
	(-1)	(8)	(2)	(-1)	(14)+8	(2)	(1)	
$D_4$	19	18	20	21	12	13	14	1
	(-8)	(-5)	(-13)	(-12)	(-7)	(-4)		

$b_j \rightarrow 10 \quad 11 \quad 13 \quad 20 \quad 24 \quad 15$

$v_j \rightarrow 10 \quad 12 \quad 6 \quad 8 \quad 4 \quad 12$

Table-5:

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$a_{ij}$	$u_{ij}$
$0 = 1$	10	12	13	8	14	19	18	-6
	(10)+0	(2)	(-1)	(8)+8	(-10)	(-7)		
$S_1$	15	18	12	16	19	20	12	2
	(3)	(1)	(13)	(0)	(-7)	(0)		
$S_2$	17	16	13	14	10	18	39	0
	(-1)	(2)+0	(-3)	(-12)+8	(2)	(1)		
$S_3$	19	18	20	21	12	13	14	-5
	(-8)	(-1)	(-15)	(-12)	(-9)	(-4)		

$b_j \rightarrow 10 \quad 11 \quad 13 \quad 20 \quad 24 \quad 15$

$v_j \rightarrow 16 \quad 16 \quad 10 \quad 14 \quad 10 \quad 18$

$$u_1 + v_2 = 18$$

$$u_2 = 18 - 12$$

$$= 6$$

$$u_3 + v_4 = 14$$

$$u_4 = 14 - 8$$

$$= 6$$

$$u_5 + v_6 = 10$$

$$v_5 = 4$$

$$u_3 + v_6 = 18$$

$$u_6 = 12$$

$$u_4 - u_3 = 0$$

$$u_3 = 11, u_4 = 14, v_5 = 4$$

$$v_6 = 12$$

$$u_2 + u_4 = 18$$

$$u_2 = 12$$

$$u_3 + v_4 = 12$$

$$u_3 = 10$$

$$u_1 + u_4 = 8$$

$$u_1 = -6$$

$$u_4 + v_5 = 13$$

$$u_4 = -5$$

$$u_1 + u_4 = 10$$

$$u_1 = 16$$

we allocate an unknown quantity  $\theta$ , to this cell  $(i,j)$

and identify a closed-chain involving basic cells around this entering cell.

(loop)  
— Add and subtract  $\theta$ , alternately to and from the transition cells of the loop subject to the LHS requirements.

— Assign a maximum value to  $\theta$  so that one basic variable becomes zero and the other basic variables remain  $> 0$ .

If we put  $\theta = 6$  in Table - 2,

$x_{24}$  becomes zero.

i.e., the cell  $(d, 4)$  leaves the basis.

— putting  $\theta = 6$  in Table - 2 and turning  $(2, 4)$  as non-basic cell.

— we get the new basic feasible solution as shown in Table - 3:

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	A/R
$S_1$	10 (16)	12 (2)	13	8 (6)	14	19	18
$S_2$	15	18	12 (9)	16 (13)	19	20	21
$S_3$	12	16	13	14 (14)	10 (24)	18 (1)	39
$S_4$	19	18	26 (26)	21 (14)	12 (24)	13 (14)	149
b/r	10	11	13	20	24	25	

Table-2

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$a_{1j}$	$b_j$
$S_1$	10 - (10)	12 - (3)	13 - (-7)	8 - (2)	14 - (-8)	19 - (-5)	18 - 6	
$S_2$	15 - (1)	18 - (3) + 0	16 - (13)	16 - (6) - 0	19 - (-7)	20 - (0)	21 - 0	
$S_3$	17 - (-3)	16 - (6)	13 - (-3)	14 - (14)	10 - (24)	18 - (0)	39 - 2	
$S_4$	19 - (-10)	18 - (-7)	20 - (-15)	21 - (-12)	12 - (-7)	13 - (14)	14 - 7	
	$b_j \rightarrow$	10	11	13	20	24	15	
	$v_j \rightarrow$	66	18	12	16	12	20	

The net evaluations for the unoccupied cells (non-basic) are now calculated

as below:

for the cell  $(1, 3)$

$$\Delta_{13} = u_1 + v_3 - c_{13} = -6 + 12 - 13 = -7$$

and for the cell  $(1, 4)$

$$\Delta_{14} = u_1 + v_4 - c_{14} = -6 + 16 - 8 = 2$$

In this way  $\Delta_{ij}$  for all non-basic cells are evaluated and recorded in right bottom of each cell, as shown

in Table-2:

2<sup>nd</sup> iteration: Determining the cell to enter the basis.

Calculate  $\text{Max } \{\Delta_{ij} / \Delta_{ij} > 0\}$ .

Clearly  $\Delta_{14} = 2$  is the most gre.

$\therefore$  cell  $(1, 4)$  enters the basis.

Table-1

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	
$S_1$	10 (10)	12 (8)	13	8	14	19	18.8
$S_2$	15	18 (3)	12 (13)	16 (6)	19	20	22 19.6
$S_3$	17	16	13	14 (14)	10 (24)	18 (1)	39 35 1
$S_4$	19	18	20	21	12	13 (14)	14
$S_j$	10	11	13	18 14	20 14	24 14	18 14

Now finding the values of  $u_i$  and  $v_j$

- At maximum number of basic cells can't
- In the 2nd and 3rd row, we can start by putting either  $u_2$  or  $u_3$  equal to zero.

Let us put  $u_2 = 0$

- As  $(2, 2), (2, 3), (2, 4)$  are the basic cells in this row and for basic cells we

$$\text{know } \Delta_{ij} = u_i + v_j - c_{ij} = 0 \\ \text{i.e., } u_i + v_j = c_{ij}$$

$$u_2 + v_2 = 18 \Rightarrow v_2 = 18$$

$$u_2 + v_3 = 12 \Rightarrow v_3 = 12$$

$$u_2 + v_4 = 16 \Rightarrow v_4 = 16$$

$$v_1 + v_2 = 12 \Rightarrow u_1 + 18 = 12 \Rightarrow u_1 = -6$$

$$u_3 + v_4 = 14 \Rightarrow u_3 + 16 = 14 \Rightarrow u_3 = -2$$

$$u_3 + v_5 = 10 \Rightarrow u_3 + v_5 = 10 \Rightarrow u_5 = 12$$

$$u_3 + v_6 = 18 \Rightarrow -2 + v_6 = 18 \Rightarrow v_6 = 20$$

$$u_4 + v_6 = 13 \Rightarrow u_4 + 20 = 13 \Rightarrow u_4 = -7$$

$$u_1 + v_1 = 10 \Rightarrow -6 + v_1 = 10 \Rightarrow v_1 = 16$$

Add and subtract alternately to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

- (H) Assign a maximum value to  $\theta$  in such a way that the value of one basic variable becomes zero and the other basic variables remains non-negative.

The basic cell whose allocation has been reduced to zero, leaves the basis.

- (I). Return to step (3) and repeat the process until an optimum basic feasible solution has been obtained.

→ Solve the cost minimizing transportation problem.

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	air
$S_1$	10	12	13	18	14	19	18
$S_2$	15	18	12	16	19	20	22
$S_3$	17	16	13	14	10	18	39
$S_4$	19	18	20	21	12	13	14
$b_j \rightarrow$	10	11	13	20	24	15	

Soln: Using North-West Corner Rule, an initial basic feasible solution is given as

number of occupied cells is exactly equal to  $m+n-1$  2

(3) for each occupied cell in the current solution, solve the system of equations

$$u_i + v_j = c_{ij},$$

Starting initially with some  $u_i^0 = 0$  or  $v_j^0 = 0$

corresponding to a row or a column with maximum number of basic cells (or allocations) and entering the successively the values of  $u_i$  and  $v_j$  in the transportation table margins.

(4) Compute the net evaluations  $\Delta_{ij} = u_i + v_j - c_{ij}$  for all unoccupied basic cells and enter them in the right bottom corners of the corresponding cells.

(5) Examine the signs of each  $\Delta_{ij}$ . If all  $\Delta_{ij} \leq 0$ , then the current basic feasible solution is optimal.

If at least one  $\Delta_{ij} \geq 0$ , select the unoccupied cell, having the largest positive net evaluation to enter the basis.

(6) let the unoccupied cell  $(x, s)$  enter the basis. Allocate an unknown quantity, say ' $\delta'$ , to the cell  $(x, s)$ .

Identify a loop (or closed chain) that starts and ends at the cell  $(x, s)$  and connects some of the basic cells.

for the given TP, the following basic feasible solution is optimal, as  $\Delta_{ij} \leq 0$  for all non-basic cells.

			$u_i \downarrow$
			0
			3
4	3	1	
(-5)	(3)	(1)	
2	6	2	
(2)	(0)	(4)	
$v_j \rightarrow$	-1	3	1

$$\begin{aligned}
 u_1 + v_2 &= 3 \\
 u_1 + v_3 &= 1 \\
 u_2 + v_1 &= 2 \\
 u_2 + v_3 &= 4 \\
 \text{let } u_1 = 0 \\
 \Rightarrow v_2 &= 3 \\
 v_3 &= 1 \\
 u_2 &= 3 \\
 v_1 &= -1
 \end{aligned}$$

Hence  $x_{12} = 3$ ,  $x_{13} = 1$ ,  $x_{21} = 2$ ,  $x_{23} = 4$ .

is the optimal feasible solution.

The minimum cost of transportation is given

$$\text{by } (3 \times 3) + (1 \times 1) + (2 \times 2) + (2 \times 4) = 9 + 1 + 4 + 8 = 22.$$

### Working procedure for transportation problems

Various steps involved in solving transportation problem may be summarized in the following iterative procedure:

- (1) Find the initial basic feasible solution by using any of the three methods NWCM or LCM or NAM.

- (2) Check the number of occupied cells.

If there are less than  $m+n-1$ , there exists degeneracy and we introduce a very small positive assignment of  $\epsilon (\approx 0)$  (or assign zero) in suitable independent positions, so that the

If  $(p, q)$  be the cell leaving the basis, then by putting  $\theta = x_{pq}$ , allocation in no cell becomes zero. The cell (or one of the cells) where the updated allocation becomes zero is treated as non-basic cell for the new basic feasible solution.

from the example in Table-4, &

put  $\theta = x_{22} = 1$ .  
Substituting this value of  $\theta$  in all the occupied cells as well as the entering cell  $(2, 2)$ .  
we get the updated basic feasible solution as

Table-5

4	3	1
①	③	-
2	6	2
①	-	⑤

#### (6) Identification of the optimal stage:

Continuing the process we move from one basis to another in every step.

At some stage if we find  $\Delta_{ij} \leq 0$  for all non-basic cells, we declare optimality.

i.e., the basic feasible solution in hand is the optimal solution yielding the minimum cost of transportation.

- (c) From all the cells from which  $\theta$  is subtracted, choose the cell with minimum allocation. Let this be the cell  $(P, q)$ , then cell  $(P, q)$  leaves the basis.

From the example in Table-3, the closed chain as desired in (a) above, is shown as below:

Table-4

4	3	1
$\theta - \theta$	$\theta + \theta$	$\theta_2$
2	6	2
$\theta$	$\theta - \theta$	$\theta_3$

Allocation in the entering cell  $(2,1)$  is put equal to  $\theta$ : i.e.,  $x_{21} = \theta$ . Then closed chain is followed, and  $\theta$  is subtracted and added alternately to the corner cells.

i.e.,  $\theta$  is subtracted from 2 in the cell  $(1,1)$  added to 2 in cell  $(1,2)$ , subtracted from 1 in cell  $(2,2)$ .

As  $\theta$  has been subtracted from the cells  $(1,1)$  and  $(2,2)$  and as out of these entry 1 in the cell  $(2,2)$  is least, so we make the cell  $(2,2)$  leave the basis.

- (5) Updating the basis and determining the new basic feasible solution.

The value of  $\theta$  is then substituted in all the occupied cells as well as the entering cell  $(2,1)$ .

In the above example, as there is only one non-basic cell i.e. (2,1) with the net evaluation

$$\Delta_{21} = 5$$

∴ the cell (2,1) enter the basis.

#### (a) Selecting the cell to leave the basis

In the above step we have selected the cell (2,1) to enter the basis. In order to select the cell to leave the basis, we proceed as follows.

(a) With (r,s) as the starting and ending cell, determine a closed chain with all its other corner on occupied cells.

In the closed chain, a vertical line should be followed by a horizontal line and a horizontal line should be followed by a vertical line. i.e., two consecutive vertical lines or horizontal lines are not allowed.

(b) put  $\alpha_{rs} < 0$ . Subtract and add 0 alternately from  $\alpha_{ij}$ 's in the corner cells of the closed chain determined in (a) above, starting by subtracting 0 from the corner cell adjacent to cell (r,s) in the closed chain.

Even after adding and subtracting 0, as explained above, the row-sums and column-sums of allocations in the table continue to be equal to the availabilities and demands of various sources and destinations respectively.

Using  $\Delta_{ij} = u_i + v_j - C_{ij}$ , we have

$$\begin{aligned}\Delta_{13} &= u_1 + v_3 - C_{13} \\ &= 0 + (-1) - 1 \\ &= -2\end{aligned}$$

$$\begin{aligned}\text{and } \Delta_{21} &= u_2 + v_1 - C_{12} \\ &= 3 + 4 - 2 = 5.\end{aligned}$$

we denote the  $\Delta_{ij}$  values in the right-bottom corner of cell  $(i,j)$  as shown below.

Table-3

4	3	1
(2)	(2)	(2)
2	6	2
(5)	(1)	(3)

u.v

0

3

v.j → 4 3 -1

Here we do not record  $\Delta_{ij}$ 's for the basic (occupied) cells as we know that  $\Delta_{ij} = 0$  for all  $(i,j)$  basic cells.

### 3) Selecting the cell to enter the basis:

For determining the non-basic cell to enter the basis,

determine  $\max \{ \Delta_{ij} / \Delta_{ij} > 0 \} = \Delta_{re} (say)$

Then the unoccupied cell  $(r,s)$  enters the basis.  
i.e.  $(r,s)$  is the non-basic cell with most  
negative  $\Delta_{ij}$  entry.

If maximum is achieved for more than one unoccupied cell, then we can arbitrarily pick one of them and enter it in the basis.

2

While selecting such zero-entry cells, utmost care has to be taken so that all the  $(m+n-1)$  occupied cells (zero-entry or otherwise) in the basic feasible solution are linearly independent.

for this one has to ensure non-existence of a closed-chain formed by the occupied cells.

### (2) Determination of the net-evaluations:

Corresponding to each cell  $(i, j)$  we associate a quantity  $\Delta_{ij} = u_i + v_j - c_{ij}$  and call it the net evaluation for the cell  $(i, j)$ .

Note while determining the dual variables we have already taken the net-evaluations for all the basic cells to be zero.

So our interest now is confined to the determination of  $\Delta_{ij}$ 's for the non-basic cells.

For the above example, from Table-1 we find net evaluations for all the non-basic cells as below:

Table-2

	4	3	1	0
4	②	②		
3	2	6	2	
1		①	③	3

$$v_j \quad 4 \quad 3 \quad -1$$

-The remaining  $(m+n-1)$  dual variables are:  
they easily determined from the  $(m+n-1)$  equations.

In the above problem there are 4 basic cells,  
so there will be 4 equations of the type (1).  
given above.

-As 5 dual variables are to be evaluated,  
we assign one if three, equal to zero.  
There being 2 basic cells in each of the two  
rows and also in the 2nd column, we can  
assign zero to any one of  $u_1, u_2$  or  $v_2$ .  
Let us put  $u_1=0$ ; using this in the 4 equations

we have

$$\begin{aligned} u_1 + v_1 &= 4 \Rightarrow v_1 = 4 & (\because u_1 + v_1 = c_{1j} \\ u_1 + v_2 &= 3 \Rightarrow v_2 = 3 & \text{At } (1, j) \\ u_2 + v_1 &= 6 \Rightarrow u_2 = 6 - 3 = 3 \\ u_2 + v_3 &= 2 \Rightarrow v_3 = 2 - 3 = -1 \end{aligned}$$

in basic cell

If there are less than  $(m+n-1)$  occupied cells  
in a basic feasible solution, then we treat  
as many unoccupied cells as zero entry occupied  
cells as are necessary to raise the total number  
of occupied cells to  $(m+n-1)$ .

For example, if there are 6 occupied cells in  
a basic feasible solution of a  $4 \times 5$  transpor-  
tation problem, then we have to treat 2 unoccupied  
cells as zero entry occupied cells.

rows respectively whereas dual variables  $v_1, v_2, v_3$  are associated with the three columns respectively.

Generalising this, we conclude that for a problem with  $m$  sources and  $n$  destination, we will have  $(m+n)$  dual variables.

An initial basic feasible solution of the given T.P. by North-West Corner Method is

Table-1

4	3	1
2	6	2
1	5	0

$$\text{Define } \Delta_{ij} = u_i + v_j - c_{ij}; \text{ for } i=1, 2, \dots, m; \text{ and } j=1, 2, \dots, n.$$

The dual variables are determined by the criteria that for each basic cell  $(i, j)$

$$\text{we must have, } \Delta_{ij} = 0 \quad (1)$$

$$\text{i.e., } u_i + v_j - c_{ij} = 0 \Rightarrow u_i + v_j = c_{ij}$$

In case of an  $m \times n$  transportation problem, there being  $(m+n-1)$  basic cells, we get  $(m+n-1)$  equations of the above type.

In order to determine  $(m+n)$  dual variables from these  $(m+n-1)$  equations, we give an arbitrary value to one of the dual variables.

A good convention is to assign value zero to the dual variable corresponding to a row or a column with maximum number of basic cells.

The main steps of the U-V method are:

- (1) determination of the dual variables.
- (2) determination of the net evaluations.
- (3) selecting the cell to enter the basis.
- (4) selecting the cell to leave the basis.
- (5) updating the basis and determining a new basic feasible solution, and
- (6) identifying the termination stage.

### (1) Determination of the dual variables

Let us consider a transportation problem with 2 sources and 3 destinations, whose tabular form is given as:

	$D_1$	$D_2$	$D_3$	$a_{ij}$
$S_1$	4	3	1	4
$S_2$	2	6	2	b
$b_j \rightarrow$	2	3	5	

The rows and columns of this table we associate one variable each and call them dual variables.

- with rows we associate dual variable  $u_i$ 's and with columns  $v_j$ 's.
- In the above example dual variables  $u_1$  &  $u_2$  are associated with the first and the second

## Computational procedure for the Transportation problem:

Now, we discuss how to proceed from an initial basic feasible solution of the transportation problem to its optimal solution.

Like in linear programming problem it will be ensured that at each step we obtain a basic feasible solution yielding objective function value lesser than or at the most equal to that in the previous step.

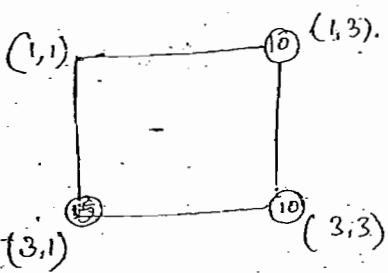
We are dealing with a minimization problem, so, we move from one basic feasible solution to a better basic feasible solution, ultimately, to reach one yielding the minimum cost of transportation. To do all this we have used a computational method both for a balanced as an unbalanced Transportation problem.

The method used for the balanced TP is known as U-V method.

## A Balanced transportation problem:

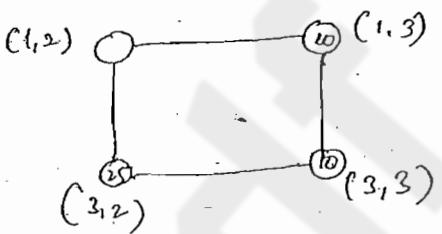
A Computational method to solve the balanced transportation problem is known as Modified Distribution Method or MODI Method or U-V method.



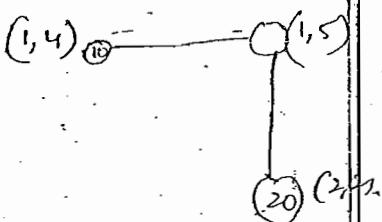
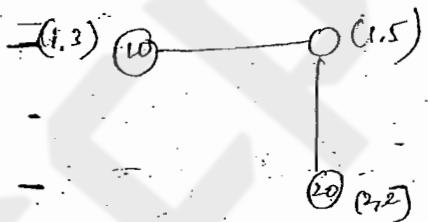


$\therefore (1,1)$  forms a closed chain with basic cells or variables at  $(1,3)$ ,  $(3,3)$ , and  $(3,1)$ .

$(1,2)$  also does not qualify as it forms a closed chain as shown below



$(1,5)$  qualifies as it does not form a closed chain.



Similarly,  $(2,1)$ ;  $(2,2)$ ;  $(2,3)$ ;  $(2,4)$  and  $(3,5)$  qualify while  $(3,4)$  does not.

we can put a zero or  $\textcircled{0}$  in any of these cells  $(1,5)$ ,  $(2,1)$ ,  $(2,2)$ ,  $(2,3)$  &  $(2,4)$

but not in  $(1,1)$ ,  $(1,2)$  and  $(3,4)$

→ But choose only one to make a total of  $(5+3)$  basic variables or cells

But only one cell is to be marked to make  $(3+2)-1 = 4$  variables.

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$a_i \geq 1$
$s_1$	4	5	3	1	7	20
$s_2$	10	8	8	6	2	20
$s_3$	3	6	4	5	4	50
	15	25	20	10	20	
$b_j$						

By using the Matrix Minima Method  
the basic feasible solution is :

4	8	3	1	7	20
x	x	x	(10)	(10)	x
10	8	8	6	2	20
x	x	x	x	x	(20)
3	6	4	5	4	50
(10)	(25)	(10)			
15	25	20	10	20	

Number of basic variables needed is  
 $(5+3)-1 = 7$ .

But we find only 6 of these. In the process  
there is an additional basic variable at zero  
level it to be identified.

Following the rule, we see that (1,1)  
does not qualify as we have the following  
closed chain.

The number of basic variables is 3.

$$\text{which is less than } (m+n)-1 = (3+2)-1 = 4$$

Such a solution in LPP we call a degenerate basic feasible solution.

Now we must identify the additional basic variable at zero level to make it a complete basic feasible solution having  $(m+n)-1$  variables.

Rule:

→ Select a variable as a candidate for basic variable at zero level.

Suppose it is  $x_{12}$  in the above example, i.e., cell  $(1,2)$ . It is called a candidate cell. Start joining it by horizontal lines and vertical lines alternately to basic variable cells with circled allocation.

In this process, if we can come back to the candidate cell, this is not qualified because they form linearly dependent set.

Otherwise, we can put a zero in the cell and circle it as  $\textcircled{0}$ .

In the above example,  $(1,2)$  can be joined only to  $(1,1)$  by a horizontal line and then  $(1,1)$  cannot be joined to any basic cell by a vertical line. So, we cannot come back to  $(1,2)$ .

$\therefore (1,2)$  qualifies and we can put a '0' in the cell  $(1,2)$  and circle it. i.e.,  $x_{12} = 0$

Note:  $(1,3)$  and  $(2,1)$  also qualify to become basic variables at zero level.

### Degenerate Basic feasible solution:

In a LPP, a basic feasible solution is said to be degenerate if certain basic variables (one or more) are at zero level i.e., the number of variables is less than 'n'. for the system of equation  $Ax=b$ ,  $x \geq 0$ ,  $A$  being  $m \times n$  matrix and rank of  $A$  is  $m$ .

However, in a LPP we were not faced with problem of identifying the basic variables at zero level. we got these variables at zero level in the process of solving the LPP.

But in case of Transportation problem while determining basic feasible solution by Northwest corner method, Matrix minima method or by Vogel's approximation method, we may discover that the number of basic variables are less than required  $(m+n-1)$  was number.

for example:

	$D_1$	$D_2$	$D_3$	$\text{Avl}$
$S_1$	5	15	10	50
$S_2$	10	8	20	55
	50	20	35	

By using the North-west Corner method the initial basic feasible solution is

5	15	10	x	x	50
10	8	20	35		55
x	x	20	35		
50	20	35			

Thus the cells are linearly independent positions. Hence the solution is a basic feasible solution.

20

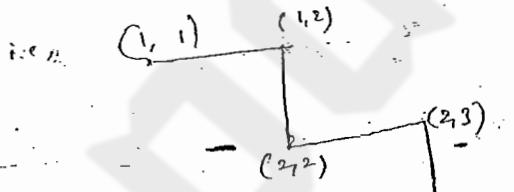
→ consider example (2)  
The solution of the TP by NWCM is:

5	7	6	4	act.
2	(5)	(20)	3	90
2	8	(20)	(30)	50
1	7	4	(20)	90

bj → 50 40 50 90

The basic cells are  $(1,1), (1,2), (2,2), (2,3), (3,3), (3,4)$

and these cells do not form a closed chain



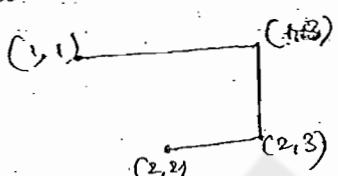
These cells are in linearly independent positions

→ Now we shall extend the closed chain rule to identify the cell at zero level in case the solution obtained by any of the methods or even at any stage is a degenerate solution.

	$P_1$	$P_2$	$P_3$	ast.
$S_1$	5 $\times 10$	15 $\times 10$	10 $\times 10$	50
$S_2$	10 $\times 8$	8 $\times 30$	20 $\times 35$	55
bij	40	30	35	

Basic cells are  $(1,1), (1,3), (2,2), (2,3)$

They do not form a closed chain as shown below:



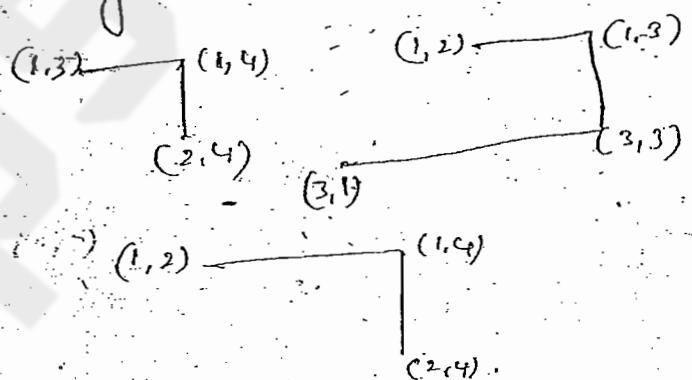
→ consider Example (3)

The solution of the TP by Matrix minima method is

5	7	6	4
2	x	40	10
1	70	30	x
0	50	40	5

The basic cells are  $(1,2), (1,3), (1,4), (2,4), (3,1), (3,3)$ .

In this even we try to form closed chains involving all the cells, we do not succeed.



- if we can form no circuit involving the basic cells above, the cells are in linearly independent positions. 2

- Applying this rule, we may observe that initial feasible solution obtained by North-West Corner Method, Matrix Minima Method or VAM is indeed a basic feasible solution satisfying the criterion of cells being in independent positions.

for example:

Consider the solution of the sewing machine transportation problem by North-West corner method P-8

5	15	10	50
10	(48)	(10)	20
10	8	(23)	(35)

∴ basic cells are  $(1,1), (1,2), (2,2), (2,3)$

The basic cells are  $(1,1), (1,2)$

$(1,1) \xrightarrow{\text{---}} (1,2)$

$\downarrow$   $(2,2) \xrightarrow{\text{---}}$

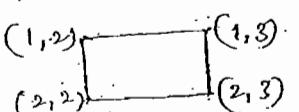
$(2,3)$

These cells do not form a closed chain.

These are in linearly independent positions.

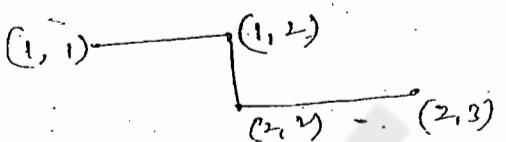
Consider the solution of the sewing machine transportation problem by Matrix Minima method, P-8

If we consider  $(1,2), (1,3), (2,3), (2,2)$  cells, which form a closed chain, as given below



Similarly,  $(1,1), (1,2), (2,2), (2,1)$  form a closed chain.

Now let us consider the cells  $(1,1), (1,2), (2,2), (2,3)$ .



In this case, drawing alternate horizontal and vertical lines, we do not come back to the starting cell  $(1,1)$ .

These cells do not form a closed chain and therefore, form a simply chain.

Similarly,  $(1,1), (1,3), (2,3)$  form a chain.  
 $(1,1), (1,3), (2,3)$  form a chain.

Now we give a method or rule - to test, if given any number of cells, these are in linearly independent positions or in linearly dependent positions.

### Closed Chain Rule:

Gives a feasible solution, if we can form a closed circuit involving all the basic cells or any subset or part of basic cells, then the cells are in linearly dependent positions.

19

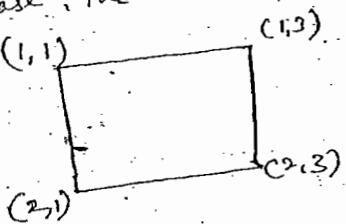
we have discussed the methods to find an initial basic feasible solution to given TP. Now we have to verify the fact that the solution so obtained is a basic feasible solution. So, we now introduce a rule or procedure for checking whether a given feasible solution is basic, i.e., cells are linearly independent. This is called - the chain rule.

Closed chain  
For example: Consider the sewing machine TP.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	art
S <sub>1</sub>	5 1	15 1	10 1	50
S <sub>2</sub>	10 1	8 1	20 1	55
	→ 40	50	35	

Consider the set of cells (1,1), (1,3), (2,3), (2,1). Starting with any one of the cells say (1,1) draw alternate horizontal and vertical lines to reach other cells. If doing so, we come back to the starting cell, we say that a closed chain is formed.

In this case, the circuit is as follows.



13	10	
95		125
275	= 125	400

Finally the initial basic feasible solution  
is as shown below.

$x_{11}$	$x_{12}$	$x_{13}$	$x_{14}$
200	50	-	-
16	18	$x_{23}$	10
21	24	13	10

$$\therefore x_{11} = 200, x_{12} = 50, x_{21} = 17.5, \\ x_{24} = 125, x_{33} = 27.5, x_{34} = 125.$$

These are basic variables

and other variables are non-basic.

Also the number of basic variables

$$= (m+n) - 1 \\ = (3+4) - 1 \\ = 6.$$

Since  $C_{11} = 11$ , is the minimum cost, we allocate  $x_{11} = \min(250, 200) = 200$  in the cell  $(1,1)$ . This exhausts the requirement of the first column and therefore cross off the first column.

The row and column differences are now computed for the resulting reduced transportation table as follows:

13	17	14	$250 - 200 = 50$ (1)
18	14	10	300 (4)
24	13	10	400 (3)
225	275	250	

- (5) (1) (0)

The largest of these penalties is (5) which is associated with the second column.

Since  $C_{12} = 13$ , is the minimum cost,

we allocate  $x_{12} = \min(50, 225) = 50$ .

This exhausts the availability of first row and therefore cross off the first row.

Proceeding in this way, the subsequent reduced transportation table and differences for the remaining rows and columns are

as shown below.

18	14	10	300 (4)
17	13	10	400 (3)
24	13	10	400 (3)
225-50 = 175	275	250	

(5) (1) (0)

14	10	125 (4)
13	10	400 (3)
275	250	

(1) (0)

Step 4: Repeat the procedure till all the rim requirements are satisfied.

→ Use Vogel's Approximation method to obtain an initial basic feasible solution of the TP.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	A <sub>j</sub>
S <sub>1</sub>	11	13	17	14	250
S <sub>2</sub>	16	18	14	10	300
S <sub>3</sub>	21	24	13	10	400

$$b_j \rightarrow 200 \quad 225 \quad 225 \quad 250$$

Since  $\sum a_{ij} = \sum b_j = 950$ .

∴ The problem is balanced.

The differences between the smallest and next to the smallest costs in each row and each column are first computed and displayed inside parenthesis against the respective rows and columns.

11 (200)	13	14	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)

$$- 200 \quad 225 \quad 225 \quad 250$$

(5) (6) (1) (0)

The largest of these penalties is (5) and is associated with the first column and the second row.

### Vogel's Approximation Method:

The Vogel's approximation method takes into account not only the least cost  $c_{ij}$  but also the costs that just exceed the least cost  $c_{ij}$  and therefore yields a better solution.

Vogel's Approximation Method consists of the following steps:

Step(1): Calculate penalties by taking differences between the minimum and next to minimum unit transportation costs in each row and each column.

Step(2): Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the row requirements to the lowest cost cell in that row or column.

In case of a tie allocate to the cell associated with the lower cost.

If the greatest difference corresponds to the  $i^{th}$  row and  $j^{th}$  column and  $c_{ij}$  is the lowest cost in the  $i^{th}$  row allocate as much as possible i.e.,  $\min(a_i, b_j)$  in the  $(i, j)^{th}$  cell and cross off the  $i^{th}$  row or  $j^{th}$  column.

Step(3): Recalculate the row and column differences for the reduced table and go to step (2).

	$D_1$	$D_2$	$D_3$	$D_4$	$c_{ij}$			
$s_1$	5	7	(4)	6	(10)	4	(20)	70
$s_2$	2	8	-	3	-	1	-	
$s_3$	x	x	-	x	-	(50)	-	50
bj	1	7	4	5	(40)	x	90	
	50	40	50	70				

All the demand requirements have been satisfied and hence an initial basic feasible solution is :

$$x_{12} = 40, x_{13} = 10, x_{14} = 20$$

$$x_{24} = 50, x_{31} = 50, x_{33} = 40$$

and Other variables are non-basic

Also the number of basic variables -

$$= (m+n) - 1$$

$$= (3+4) - 1$$

$$= 6.$$

and set  $x_{23} = x_{34} = 0$  and put a 'x'.

	$D_1$	$D_2$	$D_3$	$D_4$	
$S_1$	5	7	6	4	70
$S_2$	2	8	3	1	5
$S_3$	1	7	4	5	40
	(50)	x	(40)	x	40
	40	50	40	20	$= 10$

The next cheapest cost route is  $(1, 4)$  with  $C_{14} = 4$ .

availability  $a_1 = 70$ , demand  $D_4 = 20$

$$\min(70, 20) = 20$$

$\therefore$  set  $x_{14} = 20$  and encircle it as a basic variable.

5	7	6	4	(20)	$70 - 20 = 50$
2	8	3	1	(50)	
1	7	4	5		
(50)	x	(40)	x		
40	10	20			

Next cheapest cost route is  $(1, 3)$  with  $C_{13} = 6$ .

$$a_{13} = 50, b_3 = 10$$

$$\min(50, 10) = 10$$

$\therefore$  set  $x_{13} = 10$  and encircle it as a basic variable.

5	7	6	(10)	(60)	$50 - 10 = 40$
2	8	3	x	(50)	
1	7	4	5		
(50)	x	(40)	x		
40	10				

Last unused route is  $(1, 2)$  with  $C_{12} = 7$

$$a_1 = 40, b_2 = 40$$

$\therefore$  Set  $x_{12} = 40$  and encircle it.

	$D_1$	$D_2$	$D_3$	$D_4$	$a_i \downarrow$
$S_1$	5 x	7	6	4	70
$S_2$	2 x	8	3	1	50
$S_3$	1 <u>50</u>	7	4	5	$70 - 50 = 20$

$b_j \rightarrow 50, 40, 50, 70$

The next cheapest cost route is  $(2,4)$

with  $C_{24} = 1$

$$a_2 = 50, b_4 = 40$$

$$\min(50, 70) = 50$$

Set  $x_{24} = 50$  and encircle it as a basic variable.

and set  $x_{22} = x_{33} = 0$  as these are non-basic variables and put  $\alpha^* x$ .

	$D_1$	$D_2$	$D_3$	$D_4$	
$S_1$	5 x	7	6	4	70
$S_2$	2 x	8 x	3 x	1 <u>50</u>	70
$S_3$	1 <u>50</u>	7	4	5	40

$40 \quad 50 \quad 10-50 \\ = 20$

The next cheapest cost route among the unused routes  $C_{14} = C_{33} = 4$  in the cells  $(1,4)$  and  $(3,3)$ .

Choose any one of these cell say  $(3,3)$ .

Capacity availability  $a_3 = 40$ ,

Demand  $b_3 = 50$

$$\min(40, 50) = 40$$

Set  $x_{33} = 40$  and encircle it as a basic variable.

→ find the basic feasible solution by Matrix Minima method.

~~go to~~

Minima method:

	$D_1$	$D_2$	$D_3$	$D_4$	$a_i \downarrow$
$S_1$	5	7	6	4	40
$S_2$	2	8	3	1	50
$S_3$	1	7	4	5	90
$b_j \rightarrow$	50	40	50	70	

Minimum cost in the matrix is in the cell  $(3,1)$  linking third source  $S_3$  to destination  $D_1$ , and in the cell  $(2,4)$  linking source  $S_2$  to destination  $D_4$ . We can choose any one of the two cells. Let us choose  $(3,1)$ .

Availability  $a_3 = 90$ ; demand  $b_1 = 50$ .

$$\min(90, 50) = 50$$

Let  $x_{31} = 50$  and encircle it as a basic variable.

As demand of  $D_1$  is exactly met, put a cross 'x' in the cells  $(1,1)$  and  $(2,1)$ .

indicating that these are non-basic i.e.,  $x_{11} = x_{21} = 0$ .

This route is  $(2, 2)$ , connecting source  $S_2$  to market  $D_2$  with  $C_{22} = 8$ .

Availability at  $S_2 = 55$  and demand at  $D_2 = 30$

$$\min(55, 30) = 30$$

Set  $x_{22} = 30$  and encircle it.

and set  $x_{12} = 0$  and put a 'x'

	$D_1$	$D_2$	$D_3$	
$S_1$	5	15	10	$50 - 40 = 10$
$S_2$	10	8	20	$55 - 30 = 25$
	50	30	35	

The next cheapest route among uncrossed cells is  $C_{13} = 10$  in cell  $(1, 3)$ .

Availability at  $S_1 = 50 - 40 = 10$ ,

Demand at  $D_3 = 35$ .

$$\min(10, 35) = 10.$$

Set  $x_{13} = 10$  and encircle it.

	$D_1$	$D_2$	$D_3$	
$S_1$	5	15	10	
$S_2$	10	8	20	$25$
	50	30	35	$35 - 10 = 25$

The next cheapest and only uncrossed route is  $(2, 3)$  with  $C_{23} = 20$ .

Availability at  $S_2 = 0$  and demand at  $S_2 = 25$ .

Set  $x_{23} = 25$  and encircle it.

Now, all the item requirements have been satisfied and hence the initial basic feasible solution is

$$x_{11} = 40, x_{13} = 10, x_{22} = 30, x_{23} = 25$$

and other variables are non-basic.

$$\text{Also, the number of basic variables} = (m+n)-1 \\ = (2+3)-1 = 4$$

## MATRIX MINIMA METHOD (or) LEAST COST METHOD

Northwest Corner method considers only the constraints of availability and Supply of material, the cost of transportation is ignored, which is most important factor.

NOW, we discuss the method in which we transport starting with cheapest route and using the routes in ascending order of costs of transportation.

For example:

The Sewing Machine TP is

~~EP-3~~

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	a <sub>i,j</sub>
S <sub>1</sub>	5	15	10	50
S <sub>2</sub>	10	8	20	55
b <sub>j</sub>	40	30	35	

Soln: The cheapest route is the one connecting source S<sub>1</sub> to market D<sub>1</sub> where cost per unit

$$p_1 = 5 \text{, i.e., } C_{11} = 5.$$

Availability at S<sub>1</sub> is 50, demand at D<sub>1</sub> is 40.

$$\min [50, 40] = 40.$$

∴ S<sub>1</sub> can supply all the 40 units needed by D<sub>1</sub>.

$$\text{Set } x_{11} = 40 \text{ and encircle it.}$$

$$\text{Set } x_{21} = 0 \text{ and put 'x'}$$

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	
S <sub>1</sub>	5	15	10	50 - 40 = 10
S <sub>2</sub>	10	8	20	55
	40	30	35	

Next look for next cheapest cost among uncrossed cells.

$$\min(90, 20) = 20.$$

$$\therefore \text{Set } x_{33} = 20$$

Demand of  $D_3$  is exactly met.

Now consider the demand of  $D_4$ .

Demand of  $D_4 = 70$ ,

$$\text{availability of } S_4 = 90 - 20 = 70.$$

$$\therefore \text{Set } x_{34} = 70.$$

Hence an initial basic feasible solution to the given problem has been obtained and is displayed as follows:

	$D_1$	$D_2$	$D_3$	$D_4$	avl
$S_1$	5 (50)	7 (20)	6 x	4 x	70
$S_2$	2	8	3	1	50
$S_3$	x	(20)	(30)		
	50	40	50	70	
bj	50	40	50	70	

$$\therefore x_{11} = 50, x_{12} = 20, x_{22} = 20, x_{23} = 30.$$

$x_{33} = 20, x_{34} = 70$ . These are basic variables.

$$\text{and the number of basic variables} = (m+n)-1 \\ = (3+4)-1$$

Non basic variables are

$$x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$$

and set  $x_{21} = x_{31} = 0$  and put 'x'.  
 $\because$  demand of  $D_1$  is exactly met  
and it does not need any supply from  
 $S_2$  or  $S_3$ .

Availability at  $S_1 = 70 - 50 = 20$

and demand for  $D_2 = 40$

$$\min(20, 40) = 20$$

$$\therefore \text{set } x_{12} = 20.$$

and set  $x_{13} = x_{14} = 0$  and put 'x'  
 $\because$  Supply at  $S_1$  is exhausted. It cannot  
supply to markets  $D_3$  &  $D_4$ .

$D_2$  still needs  $40 - 20 = 20$  units which are  
to be supplied by next source  $S_2$ .

$$\min(20, 50) = 20$$

$S_2$  can supply all the 20 units need by  $D_2$ .

$$\therefore \text{set } x_{22} = 20.$$

and set  $x_{32} = 0$  and put 'x'.

$\because$  Demand of  $D_2$  is  
exactly met).

Next consider demand of  $D_3$  which is 50 units.

$S_2$  is left with  $50 - 20 = 30$  units.

$$\min(30, 50) = 30$$

$S_2$  can supply 30 units needed by  $D_3$ .

$$\therefore \text{set } x_{23} = 30.$$

and set  $x_{24} = 0$  and put 'x'.

Demand of  $D_3$  is  $50 - 30 = 20$  units more.

and  $S_3$  has 90 units.

- If  $b_1 < a_1$ , we move right horizontally to the 2<sup>nd</sup> column and make the 2<sup>nd</sup> allocation of magnitude  $x_{12} = \min(a_1 - a_{11}, b_1)$  in the cell (1,2).
- If  $b_1 = a_1$ , there is a tie for the 2<sup>nd</sup> allocation. One can make the 2<sup>nd</sup> allocation of magnitude  $x_{12} = \min(a_1 - a_{11}, b_1) = 0$  in the cell (1,2)  
or  $x_{21} = \min(a_2, b_1 - b_1) = 0$  in the cell (2,1)

Step 3: Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements ( $a_i$ 's availability &  $b_j$ 's demand)

Notes: This method does not take into account the costs of transports  $c_{ij}$ .

Example-II

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	a <sub>i</sub>
S <sub>1</sub>	5	7	6	4	-
S <sub>2</sub>	2	8	3	1	-
S <sub>3</sub>	1	7	4	5	-
b <sub>j</sub>	50	40	50	70	$\sum a_i = \sum b_j = 210$

Since  $\sum a_i = \sum b_j = 210$

∴ It is a balanced T.P.

Start with N-W. corner cell (1,1).

$$\min(a_1, b_1) = \min(50, 70) = 50.$$

Set  $x_{11} = 50$ .

$\therefore$  10 units can be supplied by  $S_1$  to  $D_2$ .  
 $\therefore$  set  $x_{12} = 10$  and encircle it.  
Now, availability at  $S_1$  exhausted.  
 $D_2$  still needs  $30 - 10 = 20$  units more. This can  
be supplied by  $S_2$  only (as availability of  
 $S_1$  is exhausted).  
 $\therefore$  let  $x_{22} = \min[55, 20] = 20$  and encircle it.  
Now,  $S_2$  has  $55 - 20 = 35$  units available  
and only 35 is the need of  $D_3$ .  
 $\therefore$  set  $x_{23} = 35$  and encircle it.  
 $\therefore$  we obtain a basic feasible solution  
of the TP with  $x_{11} = 40$ ,  $x_{12} = 10$ ,  $x_{21} = 20$ ,  
and  $x_{23} = 35$ . These are basic variables.  
All other variables are non-basic.  
and zero where we have put a 'x'.  
and the number of basic variables  $= (m+n)-1$   
 $= (2+3)-1$   
 $= 4$ .

North-West Corner Method consists of the following steps:

Step 1: Starting with the cell at the upper left  
(North-West) corner of the transportation matrix,  
we allocate as much as possible so that either the  
capacity of the first row is exhausted or the  
destination requirement of the first column is  
satisfied. i.e.,  $x_{11} = \min(a_1, b_1)$ .

Step 2: If  $b_1 > a_1$ , we move down vertically  
to the 2nd row and make the 2nd allocation  
of magnitude  $x_{21} = \min(a_2, b_1 - x_{11})$  in the  
cell (2,1).

directly and comfortably without adding artificial variable.

### North-West Corner Method

Let Sewing Machines TP, in a tabular form:

Example 1:

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	a <sub>i</sub>
S <sub>1</sub>	5 40	15 10	10 X	50
S <sub>2</sub>	10	8	20	
b <sub>j</sub>	X 40	30	35	55

In this method,

we start with extreme North-West corner cell (1,1) connecting source S<sub>1</sub> to destination D<sub>1</sub>. In this route, 50 units are available at S<sub>1</sub> and 40 are needed by D<sub>1</sub>.

The minimum of 50 and 40 i.e.,  $\min[50, 40] = 40$

can be transported from source S<sub>1</sub> to the market D<sub>1</sub>.

∴ we decide to set  $x_{11} = 40$ , write in the cell (1,1), and encircle it as shown.

Now, demand of D<sub>1</sub> is exactly met and therefore, it does not need any supply from S<sub>2</sub>.

∴  $x_{21} = 0$ .

we don't write it in the table as it is a non-basic variable. So we can put a 'x' in the cell (2,1).

S<sub>1</sub> has still  $50 - 40 = 10$  units available.

with it demand of D<sub>2</sub> is 30.

$$\min[10, 30] = 10$$

### An Initial Basic Feasible Solution of the Transportation problem:

Being a special type of LPP, TP could be solved by the simplex method. W.K.T the initial basic feasible solution for this problem can be determined by adding an artificial variable to each constraint and then applying phase-I procedure.

Now we will see that such an initial basic feasible solution can be obtained in a better way by exploiting the special structure of the problem. We shall now discuss the methods to obtain an initial basic feasible solution of a TP, viz.

- (1) North-West Corner Method (NWC Method)
- (2) Least cost or Matrix Minima Method
- (3) Vogel's Approximation method (VAM or Penalty Method).

Suppose we solve the TP by simplex method. Observe that in any TP with ' $m$ ' sources and ' $n$ ' destinations there are  $m+n$  constraints in  $m+n$  variables and each constraint is an equation. Thus, the first step is to add an artificial variable to each of the constraints resulting in  $m+n+m+n$  variables.

Next step is to apply Phase-II Method to get an IBFS to the problem. Here, even for a moderate size problem where  $m$  and  $n$  are not large, quite a large number of iterations are needed to obtain the IBFS.

Now we shall see that the structure of the problem allows us to write down the IBFS

then

$$D = \left[ \begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$= \begin{bmatrix} I_{4 \times 4} & B \\ 0 & I_{2 \times 2} \end{bmatrix}$$

where  $I_{4 \times 4}$  is a unit matrix of order 4,  
 $I_{2 \times 2}$  is a unit matrix of order 2, 0 is  
a zero matrix of order  $2 \times 4$  and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly  $|ID| = 1 \neq 0$ .Thus the rank of A is  $6 = 7 - 1 = (4+3) - 1$ .for a general given  $m \times n$  TP, A is $(m+n)$   $\times mn$  matrix and its rank is  $(m+n)-1$ .This leads us to a very important result that a basic feasible solution of TP consists of  $(m+n)-1$  variables.This means for a basic feasible solution of a TP, at the most  $(m+n)-1$  variables are +ve.

Also, we know for an LPP, the optimal solution is a basic feasible solution.

Thus the optimal solution to any TP with 'm' sources and 'n' destinations has at the most  $(m+n)-1$  +ve variables. This means, for optimal solution, out of 'mn' routes or links between sources and destinations, we need to use transportation on at the most  $(m+n)-1$  routes.For example: If there are 12 sources and 9 destinations, there are  $12 \times 9 = 108$  routes and variables. Out of these, only  $(12+9)-1 = 20$  variables are +ve (basic), and other 88 are non-basic. Thus out of possible 108 routes, we need to use only 20 routes.

for a general LP

$\mathbf{1}$  is a  $n$ -vector with element 1 each

$\mathbf{0}$  is a  $n$ -vector with element 0 each

and  $I_{3 \times 3}$  is a  $3 \times 3$  unit matrix.

(15)  $\rightarrow$  W.K.T in a general LPP, if the matrix of coefficient  $A$  is  $m \times n$  and rank of  $A$  is  $m$ , then the basic feasible solution of the system of equations  $A\mathbf{x} = \mathbf{b}$  has at the most  $m$  free variables.

Also, if rank is  $k < m$ ,  $m-k$  constraints are redundant.

This suggests that we must know the rank of matrix  $A$ .

If we denote rows of  $A$  by  $R_1, R_2, R_3, \dots, R_7$  and each  $R_i$  a row vector consisting of 12 elements.

Now multiply  $R_1, R_2, R_3, R_4$  by 1 and  $R_5, R_6, R_7$  by -1 and add, we get zero.

i.e.,  $R_1 + R_2 + R_3 + R_4 - (R_5 + R_6 + R_7) = 0$ .

This suggests that rows of  $A$  are linearly dependent.

i.e., the rank of  $A$  is  $< 7$ .

If we produce a  $6 \times 6$  non-singular submatrix of  $A$ , then the rank of  $A$  is 6.

Res To produce this submatrix  $D$ , we may omit the last row and take 3rd, 6th, 9th, 12th, 1st & 2nd column of  $A$ .

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are the activity vectors  
of  $x_{33}$  and  $x_{21}$  respectively.

We write activity vectors in the order  
 $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{41}, x_{42}, x_{43}$ .

the matrix of the coefficient A is  $7 \times 12$   
matrix where 7 is the number of  
constraints and 12 is the number of variables

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ I_{3 \times 3} & I_{3 \times 3} & I_{3 \times 3} & I_{3 \times 3} \end{bmatrix}$$

where  $I_{3 \times 3}$  is a row vector of 3 elements  
with element 1 each. i.e.,  $I = [1, 1, 1]$

$O = [0 \ 0 \ 0]$  and  $I_{3 \times 3}$  is a unit matrix  
of order 3.

For example:

take the variable  $x_{13}$ . It occurs in first source constraint and  $7^{th}$  constraint which  $(4+3)^{th} = 7^{th}$ .

first four constraints are source constraints as  $m=4$  and next three constraints i.e.,  $(4+1)^{st}, (4+2)^{nd}$  and  $(4+3)^{rd}$  are destination constraints.

$x_{13}$  occurs in  $1^{st}$  source constraint and  $3^{rd}$  destination constraint which is: The  $4+3=7^{th}$  constraint.

If a TP consists of ' $m$ ' sources and ' $n$ ' destinations it has a total of  $m+n$  constraints.

first  $m$  constraints are source constraints and next  $n$  i.e.,  $(m+1)^{th}, (m+2)^{th}, \dots, (m+n)^{th}$  are destination constraints.

A variable  $x_{ij}$  occurs in  $i^{th}$  source constraint and  $j^{th}$  destination constraint which is  $(m+j)^{th}$ .

(3) Observation (2) above lead us to write activity vector of any variable  $x_{ij}$ . For a general TP activity vector of any variable  $x_{ij}$  is a  $m+n$  vector for example:

$x_{13}$  occurs in  $1^{st}$  constraint with coefficient 1 and in the  $(4+3)^{th}=7^{th}$  constraint with coefficient 1 and with zero coefficient in remaining constraints.

its activity vector is given as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_{ij} \geq 0, i=1, 2, 3, 4; j=1, 2, 3.$$

The observations are as follows:

- (1)  $\rightarrow$  The total number of constraints is 7, namely  $A + t_3$  where 4 is number of sources and 3 is number of destinations.

The first 4 constraints - One arising from each availability restriction at the source, are called source constraints or row constraints and these are exhibited by rows in the table.

The last three constraints - one arising from each demand needed at the destination, are called column constraints or destination constraints and these are exhibited by columns of table.

It may be noted that the number of variables is  $4 \times 3 = 12$ .

There is a general TP with 'm' sources and 'n' destinations, the total number of constraints is  $m+n$  and table consists of  $m$  rows and  $n$  columns.

The total number of variables is  $m \times n (mn)$ .

- (2)  $\rightarrow$  From the above mathematical formulation, we may observe that:
- (i) each variable appears in exactly two constraints.
  - (ii) in each constraint, the coefficients of the variables are 1 each (i.e. coefficient of remaining variables are zero).

8

	$D_1$	$D_2$	$D_3$	$a_i \downarrow$
$S_1$	4	3	2	45
$S_2$	3	7	5	30
$S_3$	7	2	9	25
$S_4$	2	6	7	50
	40	20	90	
$b_j \rightarrow$				

\* Special Structure of the transportation problem:

If we examine closely mathematical model of any TP, we will observe some special characteristics of the problem.

Let us consider above example (2).

Its mathematical model - PS

$$\text{Minimize } Z = 4x_{11} + 3x_{12} + 2x_{13} + 3x_{21} + 7x_{22} + 5x_{23} \\ + 7x_{31} + 2x_{32} + 9x_{33} + 2x_{41} + 6x_{42} + 7x_{43}$$

Subject to

$$x_{11} + x_{12} + x_{13} = 45 \\ x_{21} + x_{22} + x_{23} = 30 \\ x_{31} + x_{32} + x_{33} = 25 \\ x_{41} + x_{42} + x_{43} = 50 \\ x_{11} + x_{21} + x_{31} + x_{41} = 40 \\ x_{12} + x_{22} + x_{32} + x_{42} = 20 \\ x_{13} + x_{23} + x_{33} + x_{43} = 10$$

Example (1)

Represent the following data of a TP in  
tabular form:

$$m=3 \quad n=4$$

$$a_1=15 \quad a_2=20 \quad a_3=30$$

$$b_1=10 \quad b_2=15 \quad b_3=22 \quad b_4=18$$

$$c_{11}=5 \quad c_{12}=7 \quad c_{13}=2 \quad c_{14}=1$$

$$c_{21}=2 \quad c_{22}=4 \quad c_{23}=3 \quad c_{24}=6$$

$$c_{31}=1 \quad c_{32}=3 \quad c_{33}=7 \quad c_{34}=4$$

SOL

	$D_1$	$D_2$	$D_3$	$D_4$	$a_i \downarrow$
$s_1$	5	7	2	1	15
$s_2$	2	4	3	6	20
$s_3$	1	3	7	4	30
	$b_j \rightarrow$	10	15	22	18

(2)

Time = 4 & no. of lectures = 8

$$a_1=45 \quad a_2=30 \quad a_3=90 \quad a_4=50$$

$$b_1=40 \quad b_2=20 \quad b_3=90$$

$$c_{11}=4 \quad c_{12}=3 \quad c_{13}=2$$

$$c_{21}=3 \quad c_{22}=7 \quad c_{23}=5$$

$$c_{31}=7 \quad c_{32}=2 \quad c_{33}=9$$

$$c_{41}=2 \quad c_{42}=6 \quad c_{43}=7$$

	Demi $D_1$	Bhopal $D_2$	Veluradu $D_3$	availability $a_i$
neerut $S_1$	$c_{11} = 5$ (1,1)	$c_{12} = 15$ (1,2)	$c_{13} = 10$ (1,3)	50
Kapur $S_2$	$c_{21} = 10$ (2,1)	$c_{22} = 8$ (2,2)	$c_{23} = 20$ (2,3)	55
Demand by	40	30	35	

— block at the intersection of  $S_1$  and  $D_1$  is called (1,1) cell.

— block at the intersection of  $S_2$  and  $D_3$  is called (2,3) cell and so on.

— Availability of the sources are shown at the end of horizontal rows and demand at the market is shown at the bottom of corresponding column.

— Horizontal strips are hereafter referred to as rows and vertical strips as columns.

— per unit transportation cost  $c_{ij}$  can be put in the north-west top of each cell.

— Thus the whole data of IP is put in the form of a table.

— The numbering of the cells is for our understanding only and thus the table (T-I) depicting the data is reproduced below in Table-I:

	$D_1$	$D_2$	$D_3$	all
$S_1$	5 15		10	50
$S_2$	10	8	20	55
bj →	40	30	35	Total 105

$$\text{Max Z} = 5x_{11} + 7x_{12} + 2x_{13} + 3x_{21} + \\ 6x_{22} + 5x_{23} + x_{31} + 12x_{32} + 4x_{33}$$

Subject to

$$x_{11} + x_{12} + x_{13} = 25$$

$$x_{21} + x_{22} + x_{23} = 35$$

$$x_{31} + x_{32} + x_{33} = 40$$

$$x_{11} + x_{21} + x_{31} = 30$$

$$x_{12} + x_{22} + x_{32} = 28$$

$$x_{13} + x_{23} + x_{33} = 42$$

$$x_{ij} \geq 0, i=1,2,3 \\ j=1,2,3$$

### \* Tabular Representation of a TP :-

In problem P-I, we had two sources  $S_1$  at Meerut and  $S_2$  at Kanpur. Also - we had three destinations  $D_1$  at Delhi,  $D_2$  at Bhopal and  $D_3$  at Dehradun.

Thus we have in all 6 links or routes connecting the sources to destinations from  $S_1$  to  $D_1, D_2, D_3$  respectively and 3 from  $S_2$  to  $D_1, D_2, D_3$  respectively. We represent these 6 links or routes by rectangular (or square) blocks formed by horizontal strips correspond to sources and vertical strips correspond to destinations or market.

This is shown in Table T-I below:

$$\sum_{i=1}^m a_i \neq \sum_{j=1}^n b_j$$

i.e.  $\sum_{i=1}^m a_i > \sum_{j=1}^n b_j$  or  $\sum_{i=1}^m a_i < \sum_{j=1}^n b_j$

6

The sewing machines company has 2 sources and 3 markets.  
such a problem is usually referred to as a  $2 \times 3$  TP; 2 being number of sources  
and 3 that of destinations.  
Now let us try to write mathematical model of TP.  
The required is (1) availabilities  $a_i$ ,  
(2) demands  $b_j$ ,  
(3) costs  $c_{ij}$ .

The variables that we introduce are  $x_{ij}$ .  
In the following, 'm' denotes number of sources and 'n' number of destinations.

Example:  
Write the mathematical model of the following TP.

$$m = 3, n = 3$$

$$a_1 = 25, a_2 = 35, a_3 = 40$$

$$b_1 = 30, b_2 = 28, b_3 = 42$$

$$c_{11} = 5, c_{12} = 7, c_{13} = 2$$

$$c_{21} = 3, c_{22} = 6, c_{23} = 5$$

$$c_{31} = 1, c_{32} = 12, c_{33} = 4$$

SOL

In a general TP we may have 'm' sources  $S_1, S_2, S_3, \dots, S_m$  with capacities  $a_1, a_2, \dots, a_m$ . Also 'n' may be number of destinations  $D_1, D_2, \dots, D_n$  with requirements of demand  $b_1, b_2, \dots, b_n$ .

Let  $x_{ij}$  be the units (to be determined) to be transported from source  $S_i$  to destination  $D_j$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$ .

We may name this problem P-II

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, 3, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, 3, \dots, n.$$

$$x_{ij} \geq 0, \quad i = 1, 2, 3, \dots, m \\ j = 1, 2, 3, \dots, n.$$

The problem is balanced if the total availability at all the sources is equal to the total demand at all the destinations.

Therefore for the problem to be balanced,

we must have

$$a_1 + a_2 + \dots + a_m = b_1 + b_2 + \dots + b_n$$

$$\text{i.e. } \sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

For the problem to be unbalanced we must we have

We can put the problem by a systematic manner and see that it is an L.P.P. [We may name this problem P-I].

$$\text{Max } Z = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}$$

Subject to:

$$x_{11} + x_{12} + x_{13} = 50 \quad (1)$$

$$x_{21} + x_{22} + x_{23} = 55 \quad (2)$$

$$x_{11} + x_{21} = 40 \quad (3)$$

$$x_{12} + x_{22} = 30 \quad (4)$$

$$x_{13} + x_{23} = 35 \quad (5)$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0 \quad (6)$$

$$\text{i.e. Max } Z = \sum_{i=1}^3 \sum_{j=1}^3 c_{ij} x_{ij} \quad (7)$$

$$\text{S.T.O.} \quad \sum_{j=1}^3 x_{1j} = 50 \quad (1)$$

$$\sum_{j=1}^3 x_{2j} = 55 \quad (2)$$

$$\sum_{i=1}^2 x_{ii} = 40 \quad (3)$$

$$\sum_{i=1}^2 x_{i2} = 30 \quad (4)$$

$$\sum_{i=1}^2 x_{i3} = 35 \quad (5)$$

$$x_{ij} \geq 0, i=1, 2, 1, 2, 3, j=1, 2, 3 \quad (6)$$

Now further (1) & (2) can be put in the form -  $\sum_{j=1}^3 x_{ij} = a_i, i=1, 2$

$$\sum_{j=1}^3 x_{ij} = a_i, i=1, 2$$

(3), (4), (5) can be put as

$$\sum_{i=1}^2 x_{ij} = b_j, j=1, 2, 3$$

$x_{ij} \geq 0, i=1, 2, j=1, 2, 3$

$$\text{i.e. } x_{ij} \geq 0, \quad i=1,2, \quad j=1,2,3.$$

(6)

.. is the restriction on variables.

The objective is to minimize the total transportation cost. Now  $c_{ij}$  is cost per unit for sending a machine from Meerut (S<sub>1</sub>) to Delhi.

we are transporting  $x_{11}$  units

$$\begin{array}{ccc} \therefore & 1 \text{ unit} & \rightarrow c_{11} \\ & x_{11} & \rightarrow ? \end{array}$$

$\therefore c_{11}x_{11}$  is the cost of transporting

$x_{11}$  units from Meerut (S<sub>1</sub>) to Delhi.

Similarly  $c_{12}x_{12}$  is the cost of transporting  $x_{12}$  units from Meerut (S<sub>1</sub>) to Bhopal.

$c_{13}x_{13}$  is the cost of transporting  $x_{13}$  units from Meerut (S<sub>1</sub>) to Deemedu.

Similarly we can find the cost of transportation from Kaspur (S<sub>2</sub>) to the three destinations.

Thus the total cost of transportation works out to be

$$Z = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + \text{Kapur} C_{22}x_{22} + \text{Kapur} C_{23}x_{23}$$

$$\therefore \sum_{i=1}^2 \sum_{j=1}^3 c_{ij}x_{ij} \quad \text{(7)}$$

Solving a TP means determining the values

of the variables  $x_{ij}$ ,  $i=1,2$ ;  $j=1,2,3$  such

that satisfy (1) - (7) and minimize Z as given in (7).

Also total units available at Kanpur ( $S_2$ ) are 55 and all the units are to be transported. we must have

$$x_{21} + x_{22} + x_{23} = 55 \quad \text{--- (2)}$$

NOW let us see the problem considering demands at the markets.

Delhi ( $D_1$ ) gets  $x_{11}$  from Heerut ( $S_1$ ) and  $x_{21}$  from Kanpur ( $S_2$ ).

Total demand of Delhi ( $D_1$ ) is 40 units which is to be exactly met. thus we have

$$x_{11} + x_{21} = 40 \quad \text{--- (3)}$$

Total demand of Bhopal ( $D_2$ ) is 30 units. Bhopal is getting  $x_{12}$  from Heerut ( $S_1$ ) and  $x_{22}$  from Kanpur ( $S_2$ ). Thus we have

$$x_{12} + x_{22} = 30 \quad \text{--- (4)}$$

Similarly demand of Dehradun ( $D_3$ ) is 35 units. Dehradun ( $D_3$ ) is getting  $x_{13}$  from Heerut ( $S_1$ ) and  $x_{23}$  from Kanpur ( $S_2$ ). Thus we have

$$x_{13} + x_{23} = 35 \quad \text{--- (5)}$$

thus (2) to (5) are five constraints on the variables and thus the constraints of the TP.

Also, each source either sends a pos number of machines to various centers or no machine. It can not send a -ve number of machines.

thus all the variables are non-negative i.e.  $x_{11} \geq 0, x_{12} \geq 0, x_{13} \geq 0,$

$$x_{21} \geq 0, x_{22} \geq 0, x_{23} \geq 0;$$

of transportation  $c_{ij}$  per unit from various sources to destinations, and solving a TP means determining the values of the variables  $x_{ij}$ , i.e. the amount of goods to be transported from various sources to the destinations such that the total transportation cost is the minimum. It is being assumed here that the TP is balanced.

However, if a TP is unbalanced, then we can formulate a balanced TP, the solution of which gives the solution of the given unbalanced TP.

### Mathematical formulation of a transportation problem:

As per notations introduced, let us consider the units of machine supplied from Meerut i.e. source  $S_1$  to destinations Delhi ( $D_1$ ), Bhopal ( $D_2$ ) and Dehradun ( $D_3$ ). The units supplied from  $S_1$  to  $D_1, D_2, D_3$  are  $x_{11}, x_{12}, x_{13}$  respectively.

Also, total units available at Meerut centre  $S_1$  are  $a_1 = 50$ .

Since all the units are to be exactly transported, we must have

$$x_{11} + x_{12} + x_{13} = 50 \quad \text{--- (1)}$$

Similarly, units supplied from Kapur  $S_2$  to markets at  $D_1, D_2$  and  $D_3$  are denoted by  $x_{21}, x_{22}, x_{23}$  respectively.

thus  $x_{12}$  means amount of goods to be sent from source 1 (Meerut in the example) to destination 2 (Bhopal in example).

$x_{23}$  means amount of goods to be sent from source 2 to destination 3.

(for example, it means from Kanpur to Dehradun)

Note that, in the ordered pair of suffices,

first indicates the source number and second the destination number.

The number of variables in the present example are six.

There are  $x_{11}, x_{12}, x_{13}$  from Meerut to Delhi, Bhopal and Dehradun and  $x_{21}, x_{22}, x_{23}$  from Kanpur to three markets.

Thus for a general TP having 'm' sources and 'n' destinations; therefore the number of variables are  $m \times n$  ( $2 \times 3$  in the example).

(iv) exactly on the same line, costs are denoted by  $c_{ij}$ .

thus  $c_{11}, c_{12}, c_{13}$  are the cost of transportation per unit from Meerut to the markets at Delhi, Bhopal and Dehradun and  $c_{21}, c_{22}, c_{23}$  are the costs

of transportation per unit from Kanpur to markets at Delhi, Bhopal and Dehradun.

Now we sum up: given the data -

availabilities are at various sources, demands by of the destinations and cost

the goods is demanded and is to be transported from the sources. These markets are called Destinations or markets and which denoted by  $D_1, D_2, \dots$  etc. Again there can be any number of destinations say 'n'.

Note: 'm' and 'n' can be equal or unequal.

- (ii) Availability at Meerut is 50 and at Kanpur is 55. In symbols availability at respective centres is denoted by  $a_1 = 50, a_2 = 55$ . In case, there are 'm' centres, we have 'm' values of availabilities as  $a_1, a_2, a_3, \dots, a_m$ . Demands at Delhi, Bhopal and Dehradun are respectively 40, 30 and 35. If it is denoted as  $b_1 = 40, b_2 = 30, b_3 = 35$ . If there are 'n' destinations, we have 'n' values as  $b_1, b_2, \dots, b_n$ .

- (iii) In a LPP, the variables are  $x_1, x_2, \dots, x_n$  and costs in objective functions are  $c_1, c_2, \dots, c_m$ . It is convenient in a TP to have two suffices with the variables as well as costs. It is because, we have a set of sources and another set of markets.

Let us see how we can do it. As pointed out, a TP is concerned with determining the amount of goods to be sent from each source to various destinations. Let  $a_{ij}$  be the amount of goods to be transported from  $i^{th}$  source to the  $j^{th}$  destination.

and also from Kapur to three markets such  
that demands of markets are exactly met and  
the total transportation cost is the minimum.

It may be noted that the problem  
being balanced, demands of the markets will  
be exactly met if and only if the manufacturing  
centres transport all their units to the  
various markets. What do we expect if the  
problem is unbalanced? Obviously,  
if the availability is more than demand, certain  
machines will be retained at the centres at  
Meerut and/or Kapur and  
if demand is more than availability, certain  
markets may not get the total machines  
required by them (i.e. there will be short  
supply).

In the above introduction, we are  
introduced to a specimen of a small TP  
we now gradually take to the world of  
mathematics, symbols and notations.

- (1) The centres like Meerut and Kapur where  
machines or goods is stored for transportation  
are called Sources, origin or warehouses.  
In the present example there are only  
two sources but there can be any number  
any number of sources say  $m$ . These can be  
denoted by  $S_1, S_2, \dots$  etc (or)  $O_1, O_2, \dots$  etc.  
In the example, there are three markets  
located at Delhi, Bhopal and Dehradoon where

Similarly, if demand at Dehradun was to stay other data is the same, total demand would have more than the total availability. Again, we say the problem is unbalanced.

Thus we see that if total availability is equal to total demand, problem is balanced and if total supply is different from total demand, problem is unbalanced.

Now let us come back to the problem. Our problem under discussion is a balanced one. The machines are to be transported from the centres at Meerut and Kanpur to the markets at Delhi, Bhopal and Dehradun so that demand of each market is exactly met. This is possible only when the centres transport their machines completely.

Now the transport charges: Rs. 5/- per piece from Meerut to Delhi, Rs. 15/- per piece from Meerut to Bhopal and Rs. 10/- per piece from Meerut to Dehradun.

Similarly, the transportation charges per piece from Kanpur to Delhi are Rs. 10/-, from Kanpur to Bhopal are Rs. 8/- and from Kanpur to Dehradun are Rs. 20/- So far, we have given the data of the whole problem.

In a transportation problem it concerned with determining the number of machines to be transported from Meerut to three markets

Set - IV

## The Transportation problem

This is a special type of linear programming problems (LPP) in which the objective is to transport a single commodity from various origins to different destinations at a minimum cost.

For example:

Suppose Usha sewing machine Co. has two manufacturing centres — one located at Meerut and the other at Kanpur. Manufacturing capacity of Meerut is 55 machines per day and that of Kanpur is 50 machines per day. At the end of each day, these centres have to supply the machines to their markets located at Delhi, Bhopal and Dehradun. The daily demands of these markets are fixed and are 40 machines at Delhi, 30 at Bhopal and 35 at Dehradun.

We may note that

(i) total supply at the two centres is  
 $50 + 55 = 105$  machines.

(ii) total demand at three markets is  
 $40 + 30 + 35 = 105$  machines.

Hence total availability at supply points is equal to total demand at the markets. When this is so, we say that the problem is balanced.

Suppose availability at Kanpur were 70 machines per day and other data remains unchanged, then availability was greater than total demand. In such a case, we say problem is unbalanced.



## LINEAR PROGRAMMING

## Problems 32 &amp; 33

Using dual simplex method, solve the following problems :

1. Maximize  $Z = -3x_1 - x_2$   
subject to  $x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2; x_1, x_2 \geq 0$ .
2. Minimize  $Z = 2x_1 + x_2$ ,  
subject to  $3x_1 + x_2 \geq 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 3, x_1, x_2 \geq 0$ .
3. Minimize  $Z = x_1 + 2x_2 + 3x_3$ ,  
subject to  $2x_1 - x_2 + x_3 \geq 4, x_1 + x_2 + 2x_3 \leq 8, x_2 - x_3 \geq 2; x_1, x_2, x_3 \geq 0$ .
4. Minimize  $Z = x_1 + 2x_2 + x_3 + 4x_4$   
subject to  $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10, 3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$   
 $5x_1 + 2x_2 + x_3 + 6x_4 \geq 15, x_1, x_2, x_3, x_4 \geq 0$ .

2000 15M Using duality or otherwise solve the linear programming problem

Maximize

$$Z = 18x_1 + 12x_2$$

subject to

$$2x_1 - 2x_2 \geq -3$$

$$3x_1 + 2x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

**Step 2. Find the initial basic solution.**

Setting the decision variables  $x_1, x_2, x_3$  each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0, s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

Initial solution is given by the table below :

$c_j$	-2	-2	-4	0	0	0		
$c_B$	Basis	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$
0	$s_1$	-2	(-3)	-5	1	0	0	-2
0	$s_2$	3	1	7	0	1	0	3
0	$s_3$	1	4	6	0	0	1	5
	$Z_j$	0	0	0	0	0	0	0
	$C_j$	-2	-2	-4	0	0	0	

**Step 3. Test nature of  $C_j$ .**

Since all  $C_j$  values are  $\leq 0$  and  $b_1 = -2$ , the initial solution is optimal but infeasible.

**Step 4. Mark the outgoing variable.**

Since  $b_1 < 0$ , the first row is the key row and  $s_1$  is the outgoing variable.

**Step 5. Calculate the ratio of elements of  $C_j$ -row to the corresponding negative elements of the key row.**

These ratios are  $-2/-2 = 1, -2/-3 = 0.67, -4/-5 = 0.8$ .

Since 0.67 is the smallest ratio,  $x_2$ -column is the key column and (-3) is the key element.

**Step 6. Iterate towards optimal feasible solution.**

Drop  $s_1$  and introduce  $x_2$  with its associated value -2 under  $c_B$  column. Then the revised dual simplex table is

$c_j$	-2	-2	-4	0	0	0		
$c_B$	Basis	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$
-2	$x_2$	2/3	1	5/3	1/3	0	0	2/3
0	$s_2$	7/3	0	16/3	1/3	1	0	7/3
0	$s_3$	-5/3	0	-2/3	4/3	0	1	7/3
	$Z_j$	-4/3	-2	-10/3	2/3	0	0	-4/3
	$C_j$	-2/3	0	-2/3	-2/3	0	0	

Since all  $C_j \leq 0$  and all  $b_i$ 's are  $> 0$ , this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 0 \text{ and } \max. Z' = -4/3 \text{ i.e. } \min. Z = 4/3.$$

## LINEAR PROGRAMMING

Since all  $C_j$  values are  $\leq 0$  and  $b_4 = -2$ , this solution is optimal but infeasible. We therefore proceed further.

(ii) *Mark the outgoing variable:*

Since  $b_4$  is negative, the fourth row is the key row and  $s_4$  is the outgoing variable.

(iii) *Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key row.*

This ratio is  $-2/-\frac{1}{2} = 4$  (neglecting other ratios corresponding to +ve or 0 elements of key row).

$\therefore x_1$ -column is the key column and  $\left(-\frac{1}{2}\right)$  is the key element.

(iv) *Drop  $s_4$  and introduce  $x_1$  with its associated value -3 under the  $c_B$  column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :*

	$c_j$	-3	-2	0	0	0	0	
$c_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$
0	$s_1$	0	0	1	0	-1	-1	6
0	$s_2$	0	0	0	1	-1	1	0
-2	$x_2$	0	1	0	0	0	1	3
-3	$x_1$	1	0	0	0	-10	-2	4
$Z_j$		-3	-2	0	0	3	4	-18
$C_j$		0	0	0	0	-3	-4	

Since all  $C_j$  values are  $\leq 0$  and all  $b$ 's are  $\geq 0$ , therefore this solution is optimal and feasible. Thus the optimal solution is  $x_1 = 4$ ,  $x_2 = 3$  and  $Z_{max} = -18$ .

Example 32.27. Using dual simplex method, solve the following problem:

$$\text{Minimize } Z = 2x_1 + 2x_2 + 4x_3$$

$$\text{subject to } 2x_1 + 3x_2 + 5x_3 \geq 2, 3x_1 + x_2 + 7x_3 \leq 3, x_1 + 4x_2 + 6x_3 \leq 5, x_1, x_2, x_3 \geq 0$$

Solution consists of the following steps :

Step 1. (i) Convert the given problem to maximization form by writing

$$\text{Maximize } Z' = -2x_1 - 2x_2 - 4x_3$$

(ii) Convert the first constraint into ( $\leq$ ) type. Thus it is equivalent to.

$$-2x_1 - 3x_2 - 4x_3 \leq -2$$

(iii) Express the problem in standard form.

Introducing slack variables,  $s_1, s_2, s_3$ , the given problem becomes

$$\text{Max. } Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } -2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$$

$$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$$

$$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Initial solution is given by the table below :

$c_j$	-3	-2	0	0	0	0	0	b
$c_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
0	$s_1$	-1	-1	1	0	0	0	-1
0	$s_2$	1	1	0	1	0	0	7
0	$s_3$	-1	(-2)	0	0	1	0	-10
0	$s_4$	0	1	0	0	0	1	3
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	0
$C_j = c_j - Z_j$		-3	-2	0	0	0	0	

*Step 3. Test nature of  $C_j$ .*

Since all  $C_j$  values are  $\leq 0$  and  $b_1 = -1$ ,  $b_3 = -10$ , the initial solution is optimal but infeasible. We therefore, proceed further.

*Step 4. Mark the outgoing variable.*

Since  $b_3$  is negative and numerically largest, the third row is the key row and  $s_3$  is the outgoing variable.

*Step 5. Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key row.*

These ratios are  $-3/-1=3$ ,  $-2/-2=1$  (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore,  $x_2$ -column is the key column and (-2) is the key element.

*Step 6. Iterate towards optimal feasible solution.*

(i) Drop  $s_3$  and introduce  $x_2$  alongwith its associated value -2 under  $c_B$  column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

$c_j$	-3	-2	0	0	0	0	0	b
$c_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	
0	$s_1$	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0	$s_2$	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
-2	$x_2$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0	$s_4$	$(-\frac{1}{2})$	0	0	0	$\frac{1}{2}$	1	-2
$Z_j = \sum c_B a_{ij}$		-1	-2	0	0	1	0	-10
$C_j = c_j - Z_j$		-2	0	0	0	-1	0	

### LIN EAR PROGRAMMING

the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

#### (2) Working procedure for dual simplex method :

*Step 1. (i) Convert the problem to maximization form, if it is not so.*

*(ii) Convert (≥) type constraints, if any to (≤) type by multiplying such constraints by -1.*

*(iii) Express the problem in standard form by introducing slack variables.*

*Step 2. Find the initial basic solution and express this information in the form of dual simplex table.*

*Step 3. Test the nature of  $C_j = c_j - Z_j$ :*

*(a) If all  $C_j \leq 0$ , and all  $b_i \geq 0$ , then optimal basic feasible solution has been attained.*

*(b) If all  $C_j \leq 0$  and at least one  $b_i < 0$ , then go to step 4.*

*(c) If any  $C_j \geq 0$ , the method fails.*

*Step 4. Mark the outgoing variable. Select the row that contains the most negative  $b_i$ . This will be the key row and the corresponding basic variable is the outgoing variable.*

*Step 5. Test the nature of key row elements :*

*(a) If all these elements are  $\geq 0$ , the problem does not have a feasible solution.*

*(b) If at least one element  $< 0$ , find the ratios of the corresponding elements of  $C_j$ -row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the incoming variable.*

*Step 6. Iterate towards optimal feasible solution. Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.*

**Example 32-26. Using dual simplex method :**

$$\text{maximize } -3x_1 - 2x_2,$$

$$\text{subject to } x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$$

**Solution** consists of the following steps :

*Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

*(ii) Express the problem in standard form*

Introducing slack variables  $s_1, s_2, s_3, s_4$  the given problem takes the form

$$\text{Max. } Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$\text{subject to } -x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$$

$$x_2 + s_4 = 3; x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$$

*Step 2. Find the initial basic solution*

Setting the decision variables  $x_1, x_2$  each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

As  $C_j$  is positive under some of the columns, this solution is not optimal.

(ii) Introduce  $y_4$  and drop  $s_1$ . Then the revised simplex table is

$c_B$	Basis	$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$	$b$
18	$y_4$	2/3	-1/3	0	1	2/3	-1/3	3/10
20	$y_3$	-1/3	2/3	1	0	-1/3	2/3	1/10
	$Z_f$	16/3	22/3	20	18	16/3	22/3	74/10
	$C_j$	-4/3	-4/3	0	0	-16/3	-22/3	

As all  $C_j \leq 0$ , this table gives the optimal solution.

Thus the optimal basic feasible solution is  $y_1 = 0, y_2 = 0, y_3 = 20, y_4 = 18$ ; max.  $W = 74$ .

Step 4. Derive optimal solution to the primal.

We note that the primal variables  $x_1, x_2$  correspond to the slack starting dual variables  $s_1, s_2$  respectively. In the final simplex table of the dual problem,  $C_j$  values corresponding to  $s_1$  and  $s_2$  are  $-16/3$  and  $-22/3$  respectively.

Thus, by rule I, we conclude that opt.  $x_1 = 16/3$  and opt.  $x_2 = 22/3$ .

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16, x_2 = 22/3; \text{ min. } Z = 74.$$

Obs. To check the validity of the duality theorem, the student is advised to solve the given L.P.P. directly by simplex method and see that min.  $Z = \text{max. } W = 74$ .

### Problems 32.7

Using duality solve the following problems (1 – 4) :

1. Minimize  $Z = 2x_1 + 9x_2 + x_3$ ,

subject to  $x_1 + 4x_2 + 2x_3 \geq 5, 3x_1 + x_2 + 2x_3 \geq 4$  and  $x_1, x_2 \geq 0$ .

2. Maximize  $Z = 2x_1 + x_2$ ,

subject to  $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1, x_1, x_2 \geq 0$ .

3. Maximize  $Z = 3x_1 + 2x_2$ ,

subject to  $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \leq 10, x_2 \leq 3, x_1, x_2 \geq 0$ . (Madras, 1996)

4. Maximize  $Z = 3x_1 + 2x_2 + 5x_3$ ,

subject to  $x_1 + 2x_2 + x_3 \leq 430, 3x_1 + 2x_3 \leq 460, x_1 + 4x_2 \leq 420, x_1, x_2, x_3 \geq 0$ .

### 32.8 (1) DUAL SIMPLEX METHOD

we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using  $M$ -method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in

### LINEAR PROGRAMMING

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables  $x_1, x_2$  correspond to the artificial starting dual variables  $A_1, A_2$  respectively. In the final simplex table of the dual problem,  $C_j$  corresponding to  $A_1$ , and  $A_2$  are 3 and 1 respectively after ignoring  $M$ . Thus by Rule II, we get opt.  $x_1 = 3$  and opt.  $x_2 = 1$ .

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1 ; \max. Z = 7.$$

Obs. The validity of the duality theorem is therefore checked since  $\max. Z = \min. W = 7$  from both the methods.

**Example 32.25.** Using duality solve the following problem :

$$\text{Minimize } Z = 0.7x_1 + 0.5x_2$$

$$\text{subject to } x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18, x_1, x_2 \geq 0.$$

(V.T.U., 2004)

$$\text{Sol. The dual of the given problem is Max. } W = 4y_1 + 6y_2 + 20y_3 + 18y_4,$$

$$\text{subject to } y_1 + y_3 + 2y_4 \leq 0.7, y_2 + 2y_3 + y_4 \leq 0.5, y_1, y_2, y_3, y_4 \geq 0.$$

**Step 1. Express the problem in the standard form.**

Introducing slack variables, the dual problem in the standard form becomes

$$\text{Max. } W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2,$$

$$\text{subject to } y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7,$$

$$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5, y_1, y_2, y_3, y_4 \geq 0.$$

**Step 2. Find an initial basic feasible solution.**

Setting non-basic variables  $y_1, y_2, y_3, y_4$  each equal to zero, the basic solution is

$$y_1 = y_2 = y_3 = y_4 = 0 \quad (\text{non-basic}); s_1 = 0.7, s_2 = 0.5 \quad (\text{basic}).$$

Since the basic variables  $s_1, s_2 > 0$ , the initial basic solution is feasible and non-degenerate.

Initial simplex table is

	$c_j$	4	6	20	18	0	0		
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$	$b$	$\theta$
0	$s_1$	1	0	1	2	1	0	0.7	0.7/1
0	$s_2$	0	1	(2)	1	0	1	0.5	0.5/2 ←
	$Z_j$	0	0	0	0	0	0	0	
	$C_j$	4	6	20	18	0	0		

As  $C_j$  is positive in some columns, the initial basic solution is not optimal.

**Step 3. Iterate towards an optimal solution.**

(i) Introduce  $y_3$  and drop  $s_2$ . Then the new simplex table is

	$c_j$	4	6	20	18	0	0		
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$	$b$	$\theta$
0	$s_1$	1	-1/2	0	(3/2)	1	-1/2	9/20	3/10 ←
20	$y_3$	0	1/2	1	1/2	0	1/2	1/4	1/2
	$Z_j$	0	10	20	10	0	10	5	
	$C_j$	4	-4	0	8	0	-10		

**Step 1. Express the problem in the standard form.**

Introducing the slack and the artificial variables, the dual problem in the standard form is

$$\text{Max. } W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2$$

$$\text{subject to } -y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$$

$$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$$

**Step 2. Find an initial basic feasible solution.**

Setting the non-basic variables  $y_1, y_2, y_3, s_1, s_2, A_1, A_2$ , each equal to zero, we get the initial basic feasible solution as

$$y_1 = y_2 = y_3 = s_1 = s_2 = 0 \text{ (non-basic)}; A_1 = 2, A_2 = 1. \text{ (basic)}$$

A Initial simplex table is

$c_j$	-2	-4	-3	0	0	-M	-M			
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	0
-M	$A_1$	-1	1	1	-1	0	1	0	2	2/1
-M	$A_2$	2	(1)	0	0	-1	0	1	1	1/1+
	$Z_j$	-M	-2M	-M	M	M	-M	-M	-3M	
	$C_j$	M-2	2M-4	M-3	-M	-M	0	0		

As  $C_j$  is positive under some columns, the initial solution is not optimal.

**Step 3. Iterate towards an optimal solution.**

(i) Introduce  $y_2$  and drop  $A_2$ . Then the new simplex table is

$c_j$	-2	-4	-3	0	0	-M	-M			
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	0
-M	$A_1$	-3	0	(1)	-1	1	1	-1	1	1/1-
-4	$y_2$	2	1	0	0	-1	0	-1	1	1/0
	$Z_j$	3M-8	-4	-M	M	4-M	-M	M-4	-M-4	
	$C_j$	6-3M	0	M-3	-M	M-4	0	4-2M		

As  $C_j$  is positive under some columns, this solution is not optimal.

(ii) Now introduce  $y_3$  and drop  $A_1$ . Then the revised simplex table is

$c_j$	-2	-4	-3	0	0	-M	-M			
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	
-3	$y_3$	-3	0	1	-1	1	-1	-1	1	
-4	$y_2$	2	1	0	0	-1	0	-1	1	1
	$Z_j$	1	-4	-3	3	1	-3	-1	-7	
	$C_j$	-3	0	0	-3	-1	3-M	1-M		

As all  $C_j \leq 0$ , the optimal solution is attained.

Thus an optimal solution to the dual problem is

$$y_1 = 0, y_2 = 1, y_3 = 1, \text{ Min. } W' = -\text{Max. } (W') = 7.$$

## LINEAR PROGRAMMING

## 32.10 DUALITY PRINCIPLE

If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and vice-versa.

(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal):

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

**Rule I.** If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the  $C_j$  row of the optimal dual simplex table and vice-versa.

**Rule II.** If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the  $C_j$  row of the optimal dual simplex table, after deleting the constant  $M$  and vice-versa.

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and vice-versa.

Now we shall workout two examples to demonstrate the primal dual relationships:

**Example 32.24.** Construct the dual of the following problem and solve both the primal and the dual:

$$\text{Maximize } Z = 2x_1 + x_2.$$

$$\text{subject to } -x_1 + 2x_2 \leq 2, x_1 + x_2 \leq 4, x_1 \leq 3, x_1, x_2 \geq 0.$$

**Solution using the primal problem.** Since only two variables are involved, it is convenient to solve the problem graphically.

In the  $x_1, x_2$ -plane, the five constraints show that the point  $(x_1, x_2)$  lies within the shaded region  $OABCD$  of Fig. 32.12. Values of the objective function  $Z = 2x_1 + x_2$  at these corners are  $Z(0) = 0$ ,  $Z(A) = 6$ ,  $Z(B) = 7$ ,  $Z(C) = 6$  and  $Z(D) = 1$ . Hence the optimal solution is  $x_1 = 3$ ,  $x_2 = 1$  and  $\text{max. } (Z) = 7$ .

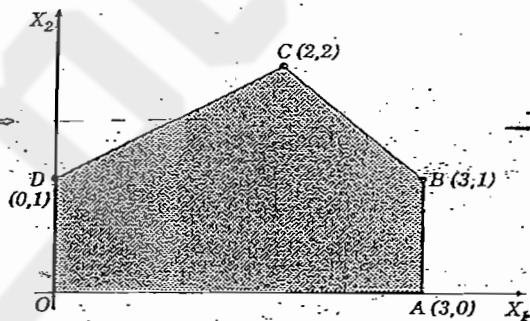


Fig. 32.12.

**Solution using the dual problem.** The dual problem of the given primal is :

$$\text{Minimize } W = 2y_1 + 4y_2 + 3y_3$$

$$\text{subject to } -y_1 + y_2 + y_3 \geq 2, 2y_1 + y_2 \geq 1, y_1, y_2 \geq 0.$$

Dual to this L.P.P.

$$\begin{aligned} \text{Maximize } & P = 6w_1 + 2w_2 + w_3 \\ \text{Sub. to } & 4w_1 + 2w_2 + 3w_3 \leq 7 \end{aligned}$$

$$2w_1 + w_2 + \frac{3}{2}w_3 \geq 5$$

$$-w_1 - w_2 - w_3 \geq -1$$

$$w_1 + w_2 + w_3 \geq 1$$

$$w_1, w_2, w_3 \geq 0$$

We can rewrite this L.P.P. as

$$\begin{aligned} \text{Maximize } & P = 6w_1 + 2w_2 + w_3 \\ \text{Sub. to } & 4w_1 + 2w_2 + 3w_3 \leq 7 \end{aligned}$$

$$2w_1 + w_2 + \frac{3}{2}w_3 \geq 5$$

$$w_1 + w_2 + w_3 = 1$$

$$w_1, w_2, w_3 \geq 0$$

which is precisely the given primal

verify that dual of the dual is the  
is the primal for the following L.P.P:

Maximize

$$Z = 4x_1 + 3x_2$$

Sub. to:

$$2x_1 + 6x_2 = 7$$

$$x_1 + 6x_2 \leq 3$$

$$2x_1 - x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

→ Dual of the dual is the primal.

for example:

$$\text{Max } Z = 6x_1 - 2x_2 + 2x_3$$

$$\text{Sub. to } 4x_1 - 2x_2 - 3x_3 \leq 7$$

$$2x_1 + x_2 - 4x_3 \geq 5$$

$$x_1 + x_2 + x_3 = 1 ; x_1, x_2, x_3 \geq 0$$

Sol: first, we shall write the given primal L.P.P. in canonical form.

$$\text{Max } Z = 6x_1 - 2x_2 + 2x_3$$

$$\text{Sub. to } 4x_1 - 2x_2 - 3x_3 \leq 7$$

$$-2x_1 - x_2 + 4x_3 \leq -5$$

$$x_1 + x_2 + x_3 \leq 1$$

$$-x_1 - x_2 - x_3 \leq -1 ; x_1, x_2, x_3 \geq 0$$

Dual to this L.P.P. is

$$\text{Min } W = 7y_1 - 5y_2 + 4y_3 - 4y_4$$

$$\text{Sub. to } 4y_1 - 2y_2 + 4y_3 - 4y_4 \geq 6$$

$$-2y_1 - y_2 + 4y_3 - 4y_4 \geq -2$$

$$-3y_1 + 4y_2 + 4y_3 - 4y_4 \geq 1$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Let us now write this dual in a canonical form, i.e. a minimization type with  $\leq$  type of constraints:

$$\text{we have } -\text{Max}$$

$$-W = -7y_1 + 5y_2 - y_3 + y_4$$

Sub. to

$$-4y_1 + 2y_2 - y_3 + y_4 \leq -6$$

$$2y_1 + y_2 - y_3 + y_4 \leq 2$$

$$3y_1 - 4y_2 - y_3 + y_4 \leq -1$$

$$4y_1 + 4y_2 + 4y_3 + 4y_4 \geq 0$$

$$x_1 + x_2 - 3x_3 \leq 8$$

$$4x_1 - x_2 + x_3 = 2$$

$$2x_1 + 3x_2 - x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

All coefficients in the present case  
are non-negative & the problem is  
equally unrestricted.  
Therefore, all dual variables should  
be unrestricted.

Now we may write the LPP as

minimize

$$Z = b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + b_m y_m$$

subject to

$$a_{11}y_1 + a_{12}y_2 + \dots + a_{1m}y_m \geq c_1$$

$$a_{21}y_1 + a_{22}y_2 + \dots + a_{2m}y_m \geq c_2$$

$$a_{31}y_1 + a_{32}y_2 + \dots + a_{3m}y_m \geq c_3$$

$y_1, y_2, \dots, y_m$  are all unrestricted.

2006 Given the programme

12 Max  $Z = 5x + 3y$

subject to

$$x + 3y \leq 12$$

$$3x + 4y \leq 9$$

$$x + 8y \leq 20$$

$$x, y \geq 0$$

Write its dual in the standard form

2008 Find the dual of the following LPP

12 Max  $Z = 2x_1 - x_2 + x_3$

such that

→ Write the dual of the following primal

Max

$$Z = 2x_1 + x_2 + 2x_3$$

Sub. to

$$4x_1 + 2x_2 + x_3 \leq 7$$

$$2x_1 + x_2 + 2x_3 = 5$$

$x_1, x_2, x_3 \geq 0$

→ Write the dual of the following problem in a form such that dual variables are all non-negative

Max

$$Z = 6x_1 + 4x_2 + x_3 + 7x_4 + 5x_5$$

Sub. to

$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$

$$2x_1 + x_2 + 3x_3 + 2x_4 + 9x_5 = 6$$

$x_1, x_2, x_3, x_4 \geq 0$

$x_5$  unrestricted

#### ⑥ Dual of a problem in a standard form:

Let us now consider a general L.P.P. in its standard form:

Max

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to.  $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Primal

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right.$$

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

$$\text{sol} \quad \text{put } z_1 = x_1^{1-x_1^n}$$

$$z_2 = x_2^{1-x_2^n}$$

we write the first inequality  
constraint in the form of two inequalities  
of ( $\leq$ ) type.  
we may rewrite the primal

$$\text{max } Z = 3x_1^{1-x_1^n} + 2x_2^{1-x_2^n} - 4x_3 + 6x_4$$

$$\begin{aligned} \text{sub. to: } & 6x_1 - 6x_1^n + 4x_2 - 4x_2^n - 3x_3 + 2x_4 \leq 2 \\ & 6x_1 - 6x_1^n + 4x_2 - 4x_2^n + 4x_3 + 2x_4 \leq 2 \\ & -6x_1 + 6x_1^n - 4x_2 + 4x_2^n + 2x_3 - 2x_4 \leq 3 \\ & 4x_1 - 4x_1^n - 3x_2 + 3x_2^n + 2x_3 + 2x_4 \leq 3 \\ & x_1^{1-x_1^n}, x_2^{1-x_2^n}, x_3, x_4 \geq 0 \end{aligned}$$

Therefore, dual is given by

$$\text{minimize } W = 2y_1 - 2y_2 + 3y_3$$

sub. to

$$\begin{aligned} 6y_1 - 6y_2 + 4y_3 &\geq 3 \\ -6y_1 + 6y_2 - 4y_3 &\geq -3 \\ 4y_1 - 4y_2 - 3y_3 &\geq 2 \\ -4y_1 + 4y_2 - 3y_3 &\geq -2 \\ -3y_1 + 2y_2 + 2y_3 &\geq -4 \\ 2y_1 - 2y_2 + y_3 &\geq 1 \end{aligned}$$

$$y_1, y_2, y_3 \geq 0$$

(OR)

$$\text{min } W = 2y_1 - 2y_2 + 3y_3$$

sub. to

$$\begin{aligned} 6y_1 - 6y_2 + 4y_3 &\geq 3, \quad 4y_1 - 4y_2 - 3y_3 \geq 2 \\ -3y_1 + 2y_2 + 2y_3 &\geq -4, \quad 2y_1 - 2y_2 + y_3 \geq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

H.M. Max -  
 $Z = 3x_1 + 6x_2 + 7x_3$

Subject. to -

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 \leq 11 \\ 7x_1 + 2x_2 + 4x_3 \leq 7 \\ x_1 + 5x_2 + 4x_3 \leq 5 \\ x_1, x_2 \text{ unrestricted}, x_3 \geq 0 \end{cases}$$

H.M. Min -  
 $Z = 6x_1 + 3x_2 + 2x_3$

Sub. to -

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &\leq 3 \\ 7x_1 + 2x_2 + 4x_3 &\leq 4 \\ 3x_1 + x_2 + 4x_3 &\leq 5 \\ x_1 &\geq 0, \\ x_2, x_3 &\text{ unrestricted} \end{aligned}$$

- ⑤ Dual of a problem with equality, Inequality constraints and Unrestricted variables:

for example:

$$\text{Max } Z = 8x_1 + 6x_2 + 7x_3 + 3x_4 - 2x_5$$

Sub. to:

$$4x_1 + 2x_2 + 5x_3 + 6x_4 + 2x_5 = 11$$

$$4x_1 + 3x_2 - 2x_3 - 5x_4 + 3x_5 \leq 13$$

$$x_1, x_2, x_3 \geq 0, \dots$$

$x_4$  and  $x_5$  unrestricted.

→ Write the dual of the following linear programming in 'A' form such that dual variables are all non-negative.

Maximize

$$Z = 2x_1 + 2x_2 - 4x_3 + x_4$$

Sub. to

$$6x_1 + 4x_2 - 3x_3 + 2x_4 \geq 2$$

$$4x_1 - 3x_2 + 2x_3 + x_4 \leq 3$$

$x_1, x_3$  unrestricted,  $x_2, x_4 \geq 0$ .

$$4y_1 + y_2 \geq 3$$

$$-4y_1 - y_2 \geq -3$$

$$2y_1 + 3y_2 \geq 4$$

$$-2y_1 - 3y_2 \geq -4$$

$$y_1, y_2 \geq 0$$

This can be written as

$$\text{min } w = 7y_1 + 5y_2$$

$$\text{sub. to } 4y_1 + y_2 \geq 3$$

$$4y_1 + y_2 \leq 3$$

$$2y_1 + 3y_2 \geq 4$$

$$2y_1 + 3y_2 \leq 4$$

$$y_1, y_2 \geq 0$$

$\Rightarrow$

$$\text{min } w = 7y_1 + 5y_2$$

$$\text{sub. to } 4y_1 + y_2 = 3$$

$$2y_1 + 3y_2 = 4$$

$$y_1, y_2 \geq 0$$

which is the reqd. dual form of

given primal L.P.P.

write the dual of the following:

How  $\rightarrow$  MAX

$$Z = 2x_1 + 3x_2 + 4x_3$$

sub. to

$$\begin{cases} x_1 - 5x_2 + 2x_3 \leq 7 \\ 2x_1 - 3x_2 + x_3 \leq 3 \end{cases}$$

$$x_1 + 2x_2 - 4x_3 \leq 2$$

$x_1, x_2 \geq 0, x_3$  unrestricted

(4) Dual of a problem with unrestricted variables:

Let us now consider a case in which there is no restriction on the variables i.e. when the variables involved in the problem may or may not be non-negative.

for example:

Max

$$Z = 3x_1 + 4x_2$$

Sub. to

$$\text{Primal} \quad \begin{cases} 4x_1 + 2x_2 \leq 7 \\ x_1 + 3x_2 \leq 5 \\ x_1, x_2 \text{ unrestricted} \end{cases}$$

Sol

$$x_1 = x_1^I - x_1^{II}$$

$$x_2 = x_2^I - x_2^{II}$$

$$\text{so that } x_1^I, x_1^{II}, x_2^I, x_2^{II} \geq 0.$$

and the primal can be written as

Maximize

$$Z = 2x_1^I - 2x_1^{II} + 4x_2^I - 4x_2^{II}$$

Sub. to

$$4x_1^I - 4x_1^{II} + 2x_2^I - 2x_2^{II} \leq 7$$

$$x_1^I - x_1^{II} + 3x_2^I - 3x_2^{II} \leq 5$$

$$x_1^I, x_1^{II}, x_2^I, x_2^{II} \geq 0$$

∴ Dual to this L.P.P. is

Maximize

$$D = 7y_1 + 5y_2$$

Sub. to

difference of two non-negative variables  
 $\therefore$  the above dual problem takes the form.

$$\text{Max } w = b_1 u_1 + b_2 y_2$$

Sub. to

$$a_1 u_1 + b_2 y_2 \geq c_1$$

$$a_2 u_1 + a_3 y_2 \geq c_2$$

$u_1$  unrestricted in sign,

$$y_2 \geq 0$$

construct the dual of the L.P.P.

$$\text{Max } Z = 4x_1 + 9x_2 + 2x_3$$

subject to

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 5$$

$$x_1, x_2, x_3 \geq 0$$

~~1989~~ Obtain the dual problem of the following L.P.P

$$\text{Max}$$

$$Z = x_1 - 2x_2 + 3x_3$$

Sub. to

$$-2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

$$\text{Max}$$

$$Z = 3x_1 - 2x_2 + 4x_3$$

Sub. to

$$-2x_1 + x_2 + 4x_3 + 2x_4 = 2$$

$$4x_1 - 2x_2 + 2x_3 - 2x_4 = 1$$

$$3x_1 + 2x_2 + 2x_3 + 2x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Sub. to

$$x_1 + x_2 \leq 5$$

$$2x_1 + 2x_2 + 2x_3 \leq 7$$

$$2x_2 + 2x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Max

$$Z = c_1 x_1 + c_2 x_2$$

Sub. to

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$

$$-a_{11}x_1 - a_{12}x_2 \leq -b_1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

NOW, this primal is in the proper form  
i.e. a maximization problem subject to  
all constraints ' $\leq$ ' type

∴ The dual problem is

Maximize

$$W = b_1 y_1 - b_1 y_2 + b_2 y_3$$

Sub. to

$$a_{11}y_1 + a_{12}y_2 + a_{21}y_3 \geq c_1$$

$$a_{12}y_1 - a_{11}y_2 + a_{22}y_3 \geq c_2$$

$$y_1, y_2, y_3 \geq 0$$

This can be written as

Maximize

$$W = b_1(y_1 - y_2) + b_2 y_3$$

Sub. to

$$a_{11}(y_1 - y_2) + a_{21}y_3 \geq c_1$$

$$a_{12}(y_1 - y_2) + a_{22}y_3 \geq c_2$$

$$y_1, y_2, y_3 \geq 0$$

The term  $(y_1 - y_2)$  appears in both objective function and all the constraints of the dual.

This will always happen whenever there is an equality constraint for the primal.  
Then the new variable  $(y_1 - y_2) \in U_1$   
becomes unrestricted in sign being the

then the dual problem is given by

Maximize

$$n = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

sub. to

$$3y_1 + 6y_2 - 7y_3 + 4y_4 + 4y_5 \leq 7$$

$$5y_1 + y_2 + 2y_3 + 2y_4 + 7y_5 \leq 12$$

$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

Write dual for the following L.P. problem:

Max

$$Z = 5x_1 + 3x_2$$

Sub. to

$$3x_1 + 5x_2 \leq 5$$

$$x_1 + 3x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Max

$$Z = 2x_1 + 3x_2 + 5x_3$$

Sub. to:

$$5x_1 + 6x_2 - 2x_3 \leq 8$$

$$-2x_1 + x_2 + 2x_3 \leq 2$$

$$x_1 + 5x_2 - 3x_3 \leq 1$$

$$-3x_1 + 3x_2 - 7x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

(3) formulation of dual problem when the primal has equality constraints:

Consider the problem

Max

$$Z = c_1x_1 + c_2x_2$$

Sub. to

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

so the equality constraint can be

written as

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \geq b_1$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \leq -b_1$$

NOW we restate the problem

Note:- all primal constraints are of  $\leq$  type  
non-negative RHS and all primal variables are non-negative.

### problems

→ write the dual of the following L.P.P.

$$\text{MAX } Z = x_1 + 2x_2$$

sub. to

$$2x_1 - 3x_2 \leq 3$$

$$4x_1 + x_2 \leq -4$$

$$x_1, x_2 \geq 0$$

Sol Dual to this L.P.P. is

$$\text{MIN } Z = 3y_1 - 4y_2$$

sub. to

$$2y_1 + 4y_2 \geq 1$$

$$-3y_1 + y_2 \geq 2$$

$$y_1, y_2 \geq 0$$

→ write the dual of the following L.P.P.

MAXIMIZE

$$Z = 3x_1 - 2x_2 + 4x_3$$

sub. to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$x_1, x_2, x_3 \geq 0$$

Since the problem is of minimization type,  
all constraints should be of ( $\geq$ ) type.  
we multiply 1st constraint throughout  
by -1, so that  $-7x_1 + 2x_2 + 2x_3 \leq -10$ .

Let  $y_1, y_2, y_3, y_4$  and  $y_5$  be the dual variables  
associated with the above five constraints.

the dual will have 'm' variables and 'n' constraints, i.e. the transpose of the body matrix of the primal problem gives the body matrix of the dual.

(v) The variables in both the primal and dual are non-negative.

Then the dual problem will be

Maximize

$$N = b_1 w_1 + b_2 w_2 + \dots + b_m w_m$$

Subject to constraints

$$a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m \geq c_1$$

$$a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m \geq c_2$$

$$\dots$$

$$\dots$$

$$\dots$$

$$\dots$$

$$a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m \geq c_n$$

$$w_1, w_2, \dots, w_m \geq 0$$

→ The primal-dual relationships can be conveniently displayed as below:

	Primal variables	
Dual variables	$w_1, a_{11}, a_{21}, \dots, a_{m1}$	$b_1$ R.H.S. of primal constraints.
	$w_2, a_{12}, a_{22}, \dots, a_{m2}$	$b_2$ R.H.S. of primal constraints.
	$\dots$	$\dots$
	$w_m, a_{1m}, a_{2m}, \dots, a_{mm}$	$b_m$ R.H.S. of the dual constraints.

→ The information regarding the primal-dual-objective type of constraints and the sign of dual variables can be summarized in the following table:

Standard primal objective	Dual objective function type	Constraints type	Variable sign
Maximization	Minimization	$\geq$	unrestricted
Minimization	Maximization	$\leq$	unrestricted

activities being programmed.

The notion of duality in linear programming was first introduced by von Neumann and was later explicitly given by Gale, Kuhn and Tucker.

## ② formulation of dual problem:

Consider the following L.P.P. :-

$$\text{Max } Z = c_1 z_1 + c_2 z_2 + \dots + c_n z_n$$

subject to the constraints

$$a_{11} z_1 + a_{12} z_2 + \dots + a_{1n} z_n \leq b_1$$

$$a_{21} z_1 + a_{22} z_2 + \dots + a_{2n} z_n \leq b_2$$

$$\dots$$

$$a_{m1} z_1 + a_{m2} z_2 + \dots + a_{mn} z_n \leq b_m$$

$$z_1, z_2, \dots, z_n \geq 0$$

To construct the dual problem, we adopt the following guidelines:

(i) The maximization problem for the primal becomes the minimization problem for the dual and vice versa.

(ii) ( $\leq$ ) type of constraints in the primal become ( $\geq$ ) type of constraints in the dual and vice versa.

(iii) The coefficients  $c_1, c_2, \dots, c_n$  in the objective function of the primal become

$b_1, b_2, \dots, b_m$  in the objective function of the dual and vice versa.

(iv) If the primal has 'n' variables and 'm' constraints,

These two problems have different mathematical formulations, one being a maximization problem whereas other being a minimization problem. Although they are expressed in terms of same basic data with different arrangement.

### (1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the duality theory.

Every linear programming problem has associated with it, another L.P.P. involving the same data and closely related optimal solutions. Such two problems are said to be duals of each other. While one of these is called the primal, the other is the dual.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it.

Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions for the

$$11w_1 + 12w_2 \geq 400$$

$$w_1, w_2 \geq 0$$

Let us consider the same problem from a different angle.

Consider a salesman, who sells nutrients by the form of Vitamin tablets and calories by the form of chocolate candy. Each milligram of vitamin A tablet costs Rs.  $w_1$ , Vitamin B tablet costs Rs.  $w_2$  and the amount of chocolate candy containing one calorie costs Rs.  $w_3$ .

To replace Krunchies, mother has to spend  $0.01w_1 + 1w_2 + 110w_3$  for which this amount should be less than 4 cents.

Similarly, to replace crispies, mother has to spend  $.25w_1 + .5w_2 + 120w_3$  for which amount is less than 5 cents.

On the other hand the salesman tries to minimize his revenue which is the total cost requirement of the mother and again the mathematical formulation of the problem is given by

Maximize

$$Z = w_1 + .5w_2 + 400w_3$$

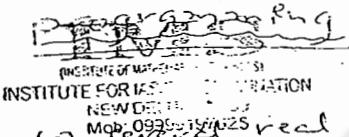
Sub. to

$$0.01w_1 + w_2 + 110w_3 \leq 4$$

$$0.25w_1 + 0.5w_2 + 120w_3 \leq 5$$

$$w_1, w_2, w_3 \geq 0$$

## Duality in Linear Programming



We were introduced to general real life problems that were formulated as L.P. models. Let us pick up one of those problems say the diet problem with different data namely the following.

A mother wishes her children to obtain certain amounts of nutrients from their breakfast cereals. The children have the choice of eating Krunchies or crispies or a mixture of the two; from their breakfast, they should obtain 1 mg of Vitamin A, 0.05 mg of Vitamin B and 400 calories. One ounce of Krunchies contains 0.01 mg of Vitamin A, 1 mg of Vitamin B and 110 calories.

1 ounce of crispies contains 0.25 mg of Vitamin A, 0.5 mg of Vitamin B and 120 calories. One ounce of Krunchies cost 4 rupees and 1 ounce of crispies cost 5 rupees.

If we formulate a linear programming model for the above problem,

assuming that any of the children eat  $x_1$  ounces of Krunchies and  $x_2$  ounces of crispies, then problem reduces to

minimize

$$Z = 4x_1 + 5x_2$$

subject to

$$0.01x_1 + 0.25x_2 \geq 1$$

$$1x_1 + 0.5x_2 \geq 0.5$$

from the above table,

$x_1$  is the incoming variable.

But the two rows have the same ratio under  $\sigma$ -column.

Now first column of unit matrix

has 1 and 0 for the tied rows.

Dividing these by the corresponding elements of the key column, we get 1 and  $\frac{9}{5}$

$\therefore x_2$ -row gives the smaller ratio

and therefore  $x_2$  is the first outgoing variable and (S) is the key element

thus the new simplex table:

		G	5	3	0	0	0	b	0	
ZB	BASIS	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$				
0	$S_1$	0	( $\frac{3}{5}$ )	1	$\frac{1}{5}$	0	0	0	$\rightarrow$	
5	$x_1$	1	$\frac{2}{5}$	0	$\frac{1}{5}$	0	2	$\rightarrow$	$S_1$	
0	$S_3$	0	$\frac{34}{5}$	0	$\frac{3}{5}$	1	6	$\frac{15}{17}$		
$Z = Z_B + C_S Z_S$		5	2	0	1	0	-10			
$Z = Z_B + C_S Z_S$		0	$\uparrow$	0	-1	0				

from the above table,

$x_2$  is the incoming variable

$S_1$  is the outgoing variable

Here ( $\frac{3}{5}$ ) is the key element and making it into unity and all other elements in its column to zero.

(ii) compare the resulting ratios (from left to right) first of unit matrix and then of the body matrix, column by column.

(iii) the outgoing variable lies in that row which first contains the smallest algebra ratio.

### problems

$$\rightarrow \text{Max } Z = 5x_1 + 3x_2$$

Sub. to

$$x_1 + x_2 \leq 2,$$

$$5x_1 + 2x_2 \leq 10,$$

$$3x_1 + 8x_2 \leq 12,$$

$$x_1, x_2 \geq 0.$$

Sol we write the given L.P.P. in standard form:

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

Sub. to

$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$

$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$

$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

where  $s_1, s_2$  &  $s_3$  are slack variables.

The I.P.F.S. is  $x_1 = x_2 = 0$  (non-basic)  
 $s_1 = 2, s_2 = 10, s_3 = 12$  (Basic)

$$\text{and } Z = 0.$$

Initial simplex table

$C_B$	$C_P$	$\bar{x}_1$	$\bar{x}_2$	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$	b	$Z_j$
0	$s_1$	1	1	1	0	0	2	$Z_1 = 0$
0	$s_2$	(5)	2	0	1	0	10	$Z_2 = 10$
0	$s_3$	3	8	0	0	1	12	$Z_3 = 12$
$Z_p = \Sigma C_B x_{Bj}$								
$Z_p = 0 + 0 + 0 + 0 + 0 + 0 = 0$								
$Z_p = C_p - Z_j$								
$Z_p = 5 - 10 = -5$								

### (3) Degeneracy:

We know that a basic feasible solution is said to be degenerate if any of the basic variables vanishes. This phenomenon of getting a degenerate basic feasible solution is called degeneracy which may arise

(i) at the initial stage, when at least one basic variable is zero in the initial basic feasible solution.

or (ii) at any subsequent stage, when the least five ratio's under O-column are equal for two or more rows.

In this case, an arbitrary choice of one of these basic variables may result in one or more basic variables becoming zero for the next iteration. At times, the same sequence of simplex iterations is repeated endlessly without improving the solution. These are termed as cycling type of problems. Cycling occurs very rarely. In fact, cycling has rarely occurred in practical problems.

To avoid cycling, we apply the following procedure:

(i) Divide each element in the tied rows by the five coefficients of the key column in that row.

$$\rightarrow MZ = z_1 + z_2$$

Sub. to

$$2z_1 + z_2 \geq 4$$

$$z_1 + 4z_2 \geq 7$$

$$z_1, z_2 \geq 0$$

$$\rightarrow MAX Z = z_1 + z_2$$

Subject to  $2z_1 + z_2 \leq 4$

$$z_1 + 4z_2 \geq 7$$

$$z_1, z_2 \geq 0$$

### \* Exceptional cases

#### (1) Tie for the incoming variable

When more than one variable has the same largest ratio value for  $Z_j$  row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the incoming variables arbitrarily. Such an arbitrary choice does not affect the optimal solution in any way.

#### (2) Tie for the outgoing variable

When more than one variable has the same least ratio under the  $\sigma$ -column, a tie for the choice of outgoing variable occurs. If the equal values of ratio are  $> 0$ , choose arbitrarily any one of them as leaving variable. Such an arbitrary choice does not affect the optimal solution.

If the equal values of ratios are zero, the simplex method fails and we make use the following degeneracy technique.

$C_B$	$C_{\text{R}} \text{ and } z_i$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$b$	
$\frac{15}{2}$	$x_1$	1	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{4}$	
0	$x_2$	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$	

$\bar{z}_j = \infty$  for  $x_1, x_2$   
 $\bar{z}_j = 0$  for  $x_3$

from the above table,

all  $C_j \leq 0$ ,

This gives optimal solution.

Hence an O.B.F.S to the given L.P.P is

$$x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{3}{4}$$

$$\text{and } \max Z = -\frac{75}{8}$$

$$\text{Hence } \min Z = -\max(-Z)$$

$$= -\max Z$$

$$= \frac{75}{8}$$

Use two phase method to solve the following L.P. problems.

$$\max Z = 5x_1 + 4x_2 + 3x_3$$

subject to

$$2x_1 + x_2 - 6x_3 = 20$$

$$5x_1 + 3x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0$$

$$\max Z = 3x_1 + 2x_2$$

Subj. to

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

$c_j$	0	0	0	0	0	-1	-1		
$C_B$	$B^{-1}B^T$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$a_1$	$a_2$	$b$
0	$x_1$	1	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
0	$x_3$	0	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$

$x_{ij} = \sum c_B a_{ij}$  0 0 0 0 0 0 0 0 0

$$C_j = g - z_j \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1$$

From the above table,

$$z_1 = C_j \leq 0$$

This table gives the optimal solution.

$$\text{Also } \max z_j = 0.$$

and no artificial variable appears in the basis. Therefore  $x_1 = \frac{5}{4}$ ,  $x_3 = \frac{3}{4}$  is an O.P.S. to the auxiliary L.P.P.

To find O.P.S. to the original problem

we proceed to phase-II.

### Phase-II:

Considering the actual costs

associated with the original variables, the objective function is

$$\text{max } Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0.5s_1 + 0.5s_2$$

Subject to

$$3x_1 + x_2 + s_1 + 0.5s_2 + A_1 + 0.5A_2 = 3$$

$$x_1 + x_2 + 0.5s_1 + 0.5s_2 + 0.5A_1 + A_2 = 2$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0.$$

Using final table of Phase-II, the optimal simplex table of Phase-II is as follows:

	$C_j$	0	0	0	0	0	-1	-1		
$C_B$	BASIS	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$A_1$	$A_2$	b	0
↑	$A_1$	(3)	-1	-1	-1	0	-1	0	3 - 1	→
↑	$A_2$	1	-1	1	0	-1	0	1	2 - 2	↑
		$\bar{C}_j^* = C_j - \bar{x}_{Bj} x_{Bj}$	-4	2	0	1	1	-1	-5	
		$C_j = \bar{C}_j^* + x_{Bj}^*$	4	-2	0	-1	1	0	0	

from the table above

$x_1$  is the entering variable,

$s_1$  is the outgoing variable.

Here (3) is the key element and make it into unity and make all other elements in its column to zero.

∴ The new simplex table is:

	$C_j$	0	0	0	0	0	-1	-1			
$C_B$	BASIS	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$A_1$	$A_2$	b	0	
0	$x_1$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	-3	
↑	$A_2$	0	$\frac{-1}{3}(4/3)$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	4 →	
		$\bar{C}_j^* = C_j - \bar{x}_{Bj} x_{Bj}$	0	$\frac{2}{3}$	$\frac{-4}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	-7	
		$C_j = \bar{C}_j^* + x_{Bj}^*$	0	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0		

from the above table,

$x_3$  is the entering variable,

$A_2$  is the outgoing variable,

Here  $(4/3)$  is the key element and

we convert it into unity and all other elements in its column equal to zero.

∴ The revised simplex table is:

We write the given LPP in the standard form:

$$\text{Max } Z_1 = -7x_1 + 3x_2 + 0x_3 + 0x_4 - A_1 - A_2$$

subject to

$$x_1 + x_2 + x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3 \quad \left\{ \begin{array}{l} \\ \end{array} \right.$$

$$x_1 + x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2 \quad \left\{ \begin{array}{l} \\ \end{array} \right.$$

$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

where  $s_1, s_2$  are surplus variables  
and  $A_1, A_2$  are artificial variables

Now we formulate an artificial objective function  $Z_1^*$  by assigning

(-1) cost to an artificial variable

(+1) cost to all other variables

$A_1$  &  $A_2$  and zero cost to all other variables

∴ we have

$$\text{Max } Z_1^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 - A_1 - A_2$$

subject to

$$x_1 + x_2 + x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3 \quad \left\{ \begin{array}{l} \\ \end{array} \right.$$

$$x_1 + x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2 \quad \left\{ \begin{array}{l} \\ \end{array} \right.$$

$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

Now the LPP is given by -

Setting  $x_1 = x_2 = s_1 = s_2 = 0$  (non-basic)

$A_1 = 3, A_2 = 2$  (basic)

and  $Z_1^* = -5 < 0$

Initial simplex table is:

all other elements in Pts. column equal to zero.

The new simplex table is:

$C_j$	$\bar{C}_B$	$x_1$	$x_2$	$S_1$	$S_2$	$\bar{Z}$	$b$
0	$x_2$	2	1	1	0	0	1
-7	A	-7	0	-4	-1	1	2
$Z^* = \bar{C}_B Z_B + C_j$							
$Z^* = 0 + (-7) \cdot 2 + 0 \cdot 1 + (-4) \cdot 0 + (-1) \cdot 1 + 1 \cdot 2$							
$Z^* = 9 - 7 = 2$							

From the above table all  $C_j$ 's  $\leq 0$ .

An optimum BPS to the auxiliary L.P.P. is obtained.

But  $\max Z^* = -2 < 0$  and artificial variable A is in the basis at the level.

∴ The original L.P.P. does not possess any feasible solution.

→ Use two-phase method to

$$\text{Max } Z = 7.5x_1 + 3x_2$$

subject to

$$3x_1 + x_2 - x_3 \geq 2$$

$$x_1 - x_2 + x_3 \geq 2$$

$$x_i \text{ being } \geq 0$$

Sol The objective of the function of the given L.P.P. is of minimization type!

So we convert it into maximization type

$$\text{we have } \text{Max } Z_1 = \text{Max } (-Z)$$

$$\therefore \text{Max } Z_1 = -7.5x_1 + 3x_2$$

where  $s_1$  is the slack variable  
 $s_2$  is the surplus variable and  
 $A$  is an artificial variable.

NOW we formulate an artificial objective function by assigning (-1) cost to an artificial variable  $A$  and zero cost to all other variables  $x_1, x_2, s_1, s_2$ .

$$\text{we have max } Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 - A$$

subject to

$$\begin{aligned} x_1 + x_2 + s_1 - 0s_2 + 0A &= 1 \\ x_1 + 4x_2 + 0s_1 - s_2 + A &= 6 \end{aligned} \quad \left. \right\} \quad (2)$$

$$x_1, x_2, s_1, s_2, A \geq 0$$

NOW the I BFS is given by

setting  $x_1 = x_2 = s_2 = 0$  (non-basic)

$$s_1 = 1, A = 6 \text{ (Basic).}$$

$$\text{and } Z^* = -6 (< 0).$$

### Simplex Table

	$C_j$	0	0	0	0	-1	
CB	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$A$	$b$
S	$s_1$	2	(1)	1	0	0	1
+ A		1	4	0	-1	1	6
							$\frac{6}{1} = 6$

$Z = 0 - 1 \cdot 1 + 4 \cdot 0 + 0 \cdot 1 - 1 \cdot 6 = -6$

from the above table:

$x_2$  is the entering variable,

$s_1$  is the outgoing variable,

and (1) is the key element and we make

To obtain a basic feasible solution, we prolong phase-I for pushing all the artificial variables out of the basis (without proceeding on to phase-II).

### Phase-II

The basic feasible solution found at the end of phase-I is used as the starting solution for the original problem in this phase i.e. the final simplex table of phase-I is taken as the initial simplex table of phase-II and the artificial objective function is replaced by the original objective function.

Note:- Before initiating phase-II, remove all artificial variables from the table, which were non-basic at the end of phase-I.

problem Use two-phase simplex method to

$$\text{MAX } Z = 5x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 1,$$

$$x_1 + 4x_2 \geq 6,$$

$$x_1, x_2 \geq 0.$$

### Phase-I

We write the given LPP in the standard form

$$\text{MAX } Z = 5x_1 + 2x_2 + 0s_1 + 0s_2 - 1A$$

subject to

$$2x_1 + x_2 + s_1 - 0s_2 - 1A = 1$$

$$x_1 + 4x_2 + 0s_1 - s_2 + A = 6$$

$$x_1, x_2, s_1, s_2, A \geq 0$$

①

Phase-II :

Step(1) : express the given LPP in the standard form by introducing slack, surplus and artificial variables.

Step(2) : formulate an artificial objective function  $Z^A = A_1x_1 + \dots + A_nx_n$  by assigning (-1) cost to each of the artificial variable  $A_i$  and zero cost to all other variables.

Step(3) : max  $Z^A$  subject to the constraints of the original problem using the simplex method. Then three cases arise:

(a) max  $Z^A < 0$  and at least one artificial variable appears in the optimal basis at a positive level.

In this case the original problem does not possess any feasible solution and the procedure comes to an end.

(b) max  $Z^A = 0$  and no artificial variable appears in the optimal basis.

In this case, a basic feasible solution is obtained and we proceed to phase-II for finding the optimal basic feasible solution to the original problem.

(c) max  $Z^A = 0$  and at least one artificial variable appears in the optimal basis at zero level.

Here a feasible solution to the auxiliary L.P.P. is also a feasible solution to the original problem with all artificial variables set = 0.

\* solve the following L.P. problems using M-method:

$\rightarrow$  Maximize  $Z = 3x_1 + 2x_2 + 3x_3$   
Subject to

$$2x_1 + x_2 + x_3 \leq 2$$

$$3x_1 + 4x_2 + 2x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

$\rightarrow$  Max  $Z = 2x_1 + x_2 + 3x_3$

Subject to

$$x_1 + x_2 + 2x_3 \leq 5$$

$$2x_1 + 3x_2 + 4x_3 = 12$$

$$x_1, x_2, x_3 \geq 0$$

$\rightarrow$  Max  $Z = 8x_2$

Subject to:

$$x_1 - x_2 \geq 0$$

$$2x_1 + 3x_2 = 6$$

$x_1, x_2$  unrestricted

$\rightarrow$  Max  $Z = 4x_1 + 3x_2 - x_3$

Subject to

$$x_1 + 2x_2 + 4x_3 \geq 12$$

$$3x_1 + 2x_2 + x_3 \geq 8$$

$$x_1, x_2 \geq 0$$

$\rightarrow$  Max  $Z = x_1 + 2x_2 + 2x_3 + 2x_4$

Subject to

$$x_1 + 2x_2 + 3x_3 = 15$$

$$2x_1 + x_2 + 5x_3 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\rightarrow$  Max  $Z = 4x_1 + 3x_2$

Subject to

$$2x_1 + 2x_2 \geq 10$$

$$-x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

\* Two-phase method:

The two-phase method is an alternative method to solve a given L.P.P. for which artificial variables are involved.

It is solved for two phases.

	$C_j$	3	2	0	0	-4		
$C_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$A_1$	$B$	0
0	$s_1$	2	(1)	1	0	0	2	2
-1	$A_1$	3	4	0	-1	1	12	3

$$Z_f = \sum C_B a_{Bj} - 3 + 4(-1) = 0, \quad 4 - 1 = 3$$

$$G = g - Z_f = 3 + 3(-1) = 0, \quad 2 + 4(-1) = 0$$

From the above table,

$x_2$  is the entering variable,

$s_1$  is the outgoing variable and (1) is the key elt. and all other elements in its column equal to zero.

Then the revised simplex table is

	$C_j$	3	2	0	0	-4		
$C_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$A_1$	$B$	
2	$x_2$	2	1	1	0	0	2	
-1	$A_1$	-5	0	4	-1	1	4	

$$Z_f = \sum C_B a_{Bj} = 4 + 5(-1) = 2 + 4(-1) = 4 - 4 = 0$$

$$G = x_2 - Z_f = -(1+5(-1)) = 0 = -(2+4(-1)) = 4 - 4 = 0$$

From the above table,

all  $C_j$ 's are  $\leq 0$  and artificial variable appears in the basis at non-zero level.

thus there exists a pseudo optimal solution to the problem.

Hence the optimal value of the objective function is  $\text{Min } Z = -4x_1 - 3x_2$

$$= \frac{12}{5}$$

$$\therefore \text{Max } Z = 3x_1 + 2x_2$$

Subject to the

$$2x_1 + x_2 \leq 2,$$

$$3x_1 + 4x_2 \geq 12, x_1, x_2 \geq 0.$$

Sol The objective function of the given LPP

is of maximization type.

Now we write the given LPP in standard form:

$$\text{Max } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - 1A_1$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 + 0A_1 \leq 2$$

$$3x_1 + 4x_2 + 0s_1 - s_2 + A_1 = 12,$$

$$x_1, x_2, s_1, s_2, A_1 \geq 0.$$

where  $s_1$  is slack variable,

$s_2$  is the surplus variable and

$A_1$  is the artificial variable.

NOW the IBS is

$$s_2 + x_1 - x_2 = 0 \text{ (non-basic)}$$

$$s_1 + 2x_1 + A_1 = 12 \text{ (basic)}$$

for which  $\boxed{Z = 0}$ .

NOW we put the above information by the simplex tableau:

$C_j$	-2	-1	0	0	-1		
$C_B$	Basis	$x_1$	$x_2$	$S_1$	$S_2$	$A_2$	$S$
-2	$x_1$	1	$\frac{1}{3}$	0	0	0	1
-1	$A_2$	0	$(S_1)$	-1	0	1	2
0	$S_2$	0	$\frac{1}{3}$	0	1	0	2
							$\frac{6}{5}$
$Z_j = C_j - C_B$		-2	$-\frac{2}{3} - \frac{S_1}{2}$	1	0	-1	$-\frac{2}{5} - \frac{S_1}{5}$
$C_j - C_B$		0	$(\frac{1}{3} + \frac{S_1}{2})$	0	0	0	

from the above table,

$x_2$  is the entering variable,

$A_2$  is the outgoing variable and  
empty for column for the next simplex

table  
here  $(S_1)$  is the key element and  
make it unity and all other elements

make it unity and all other elements

make it unity and all other elements

in the column equal to zero.

then the revised simplex table is:

$C_j$	-2	-1	0	0		
$C_B$	Basis	$x_1$	$x_2$	$S_1$	$S_2$	$S$
-2	$x_1$	1	0	$\frac{1}{3}$	0	$\frac{3}{5}$
-1	$x_2$	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
0	$S_2$	0	0	1	1	0
$Z_j = C_j - C_B$		-2	$-\frac{1}{3}$	$\frac{1}{5}$	0	$-\frac{12}{5}$
$C_j - C_B$		0	0	$-\frac{1}{5}$	0	

from the above table, all  $C_j \leq 0$

there remains no artificial variable  
as the basis.

the column  $B$  is an optimal BFS to the  
problem and is given by

$$x_1 = \frac{3}{5}, x_2 = \frac{6}{5} \text{ & } S_2 = 0.$$

$$\therefore \text{MAX } Z = -\frac{12}{5}$$

NOW the surplus variable

$s_1$  is not a basic variable since its value is -6. As negative quantities are not feasible,  $s_1$  must be prevented from appearing in the initial solution. This is done by taking  $\boxed{s_1 = 0}$ .

By setting other non-basic variables

$$x_1 = x_2 = 0,$$

we obtain the I.B.P.S as

$$x_1 = x_2 = 0, s_1 = 0, a_1 = 3, a_2 = 6, s_2 = 3.$$

Thus the initial simplex table is :

		C.R.						Z <sub>f</sub>			
		BASIS						-2	1	0	-4
CB	BASIS	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	s <sub>2</sub>	A <sub>1</sub>	A <sub>2</sub>	b			
-1	A <sub>1</sub>	(2)	1	0	0	1	0	3	$\frac{3}{1} = 3$	0	
-1	A <sub>2</sub>	4	3	-1	0	0	1	6	$\frac{6}{4} = \frac{3}{2}$		
0	s <sub>2</sub>	1	2	0	1	0	0	3	$\frac{3}{1} = 3$		
		$Z_f = \text{Excess } -4x_1 + x_2 + 0s_1 + 0s_2 + 1A_1 + (-4)A_2$						-94			
		$G = Z_f - Z_j + 4x_1 - 4x_2 + 4A_1 - 4A_2$						0 0 0			

From the above table,

the variable  $x_1$  is entering variable

$A_1$  is the outgoing variable and complete column for this variable in the next simplex table.

Here (2) is the key element and

convert it into unity and all other elements of its column to zero.

Then the new simplex table is:

Note :- The artificial variables are only a computational device for getting a starting solution. Once an artificial variable leaves the basis, it has served its purpose and we forget about it i.e. the column for this variable is omitted from the next simplex table.

problems

→ Use Charnes' penalty method to minimize  $Z = 2x_1 + 3x_2$

subject to

$$3x_1 + 2x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 3; x_1, x_2 \geq 0$$

Sol The objective function of the given LPP is of maximization type  
So, we convert it into maximization type

$$\text{Max } Z' = 4x_1 (-Z)$$

$$= -2x_1 - 3x_2$$

Now we write the given LPP in the standard form

$$\text{Max } Z' = -2x_1 - 3x_2 + 0S_1 + 0S_2 - M A_1$$

subject to

$$3x_1 + 2x_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 3$$

$$4x_1 + 3x_2 - S_1 + 0S_2 + 0A_1 + A_2 = 6$$

$$x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 3$$

$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

Where  $S_1$  is the surplus variable,

$S_2$  is the slack variable

$A_1, A_2$  are the artificial variables

Step(3): solve the modified L.P.P. by simplex method.

At any iteration of simplex method, one of the following three cases may arise:

(i) There remains no artificial variable

in the basis and the optimality condition is satisfied. Then the solution is an optimal basic feasible solution to the problem.

(ii) There is at least one artificial variable in the basis at zero level

(with zero value in b-column) and the optimality condition is satisfied. Then the solution is a degenerate optimal basic feasible solution.

(iii) There is at least one artificial variable

in the basis at non-zero level (with positive value in b-column) and the optimality condition is satisfied. Then the problem has no feasible solution. The final solution is <sup>not</sup> optimal,

since the objective function contains an unknown quantity  $A_{ij}$ . Such a solution satisfies the constraints but does not optimize the objective function and is therefore called pseudo optimal solution.

Step(4): Continue the simplex method until either an optimal basic feasible solution is obtained or an unbounded solution is indicated.

eliminate as soon as they become non-basic.

There are two similar method for solving such problems which we explain below:

- The "big M-method" or "method of penalties" due to A. Charnes and B. Dantzig.
- "two phase method" due to Dantzig, Orden and Wolfe.

The Big M-method or Method of Penalties

The big M-method or method of penalties consists of the following basic steps:

Step(1): Express the problem in standard form.

Step(2): Introduce non-negative variables to L.H.S. of all the constraints of ( $\geq$ ) or ( $\leq$ ) type. Such new variables are called

artificial variables. The purpose of introducing artificial variables is just to obtain an IFS. However, addition of

these variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the optimum simplex table. To achieve this, we assign very large penalty " $M$ " to

these artificial variables in the objective function, for maximization objective function.

$\text{Maximize } Z = z_1 + z_2 + z_3 + \dots + z_m$   
 $- z_1, z_2, \dots, z_m \geq 0$ , where  $z_1, z_2, \dots, z_m$  are surplus variables.

Now we may obtain ~~obtain~~ the initial basic feasible solution (IBFS) to this problem by letting  $x_1 = 0, x_2 = 0, \dots, x_n = 0$ .

Thus we obtain  $z_1 = -b_1, z_2 = -b_2, \dots, z_m = -b_m$ .

Now, the following relevant question:

arises: this basic solution is not feasible solution, it violates non-negativity restrictions, how can we apply simplex method to solve such a problem? for such a problem a slight modification is required.

for the problems with ( $\geq$ ) or ( $=$ ) constraints, the slack variables cannot provide a starting feasible solution.

To find a starting feasible solution for such cases, we use the methods of "artificial variables". The methods have acquired the name "artificial variables" because for these methods, we take the help of some variables which are fictitious and have no physical meaning. These variables are eliminated from the simplex.

$$\rightarrow \text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to

$$- 2x_1 + 3x_2 \leq 8$$

$$- 2x_2 + 5x_3 \leq 10$$

$$- 3x_1 + 2x_2 + 4x_3 \leq 15$$

$$- x_1, x_2, x_3 \geq 0$$

$$\rightarrow \text{Max } Z = x_1 - 2x_2 + 2x_3$$

subject to

$$- x_1 - x_2 + 2x_3 \leq 7$$

$$- 2x_1 + 4x_2 \leq 12$$

$$- 4x_1 + 3x_2 + 8x_3 \leq$$

$$x_1, x_2, x_3 \geq 0.$$

### Artificial variable techniques:

So far we have seen that the introduction of slack variables provided the initial basic feasible solution. But there are many problems wherein at least one of the constraints is of ( $\geq$ ) or (=) type and slack variables fail to give such a solution.

Suppose the given LPP is of the form:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Now we write in the standard form

$$\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_{n+2} = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m$$

$C_B$	$C_j$	107	1	2	0	0	0	
$C_B$	Basis	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$\sigma$
0	$x_4$	0	$\frac{1}{3}$	$-\frac{4}{3}$	1	0	$\frac{14}{3}$	$\frac{7}{3}$
0	$s_1$	0	$\frac{35}{6}$	$-\frac{2}{3}$	0	1	$\frac{16}{3}$	5
107	$x_1$	1	$-\frac{1}{3}$	$\frac{4}{3}$	0	0	$\frac{1}{3}$	0
$Z_p = \sum c_{Bj} a_{Bj}$								
$Z_p = 0 - \frac{1}{3} + \frac{4}{3} \cdot 107 = 107 - \frac{107}{3} = \frac{107}{3}$								
$G = 0 - \frac{1}{3} + \frac{110}{3} = \frac{113}{3}$								

As  $G_j$  is zero under some columns, the solution is not optimal. Here  $\frac{113}{3}$  being the largest the value of  $G_j$ ,  $x_3$  is the incoming variable. But all the values of  $\sigma$  being  $< 0$ ,  $x_3$  will not enter the basis.

This indicates that the solution to the problem is unbounded.

Q. Using the simplex method, solve the following L.P.P.

$$\rightarrow \text{Max } Z = x_1 + 3x_2 \rightarrow \text{Max } Z = 4x_1 + 5x_2$$

subject to

$$x_1 + 2x_2 \leq 10$$

$$x_1 \leq 5$$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

subject to

$$x_1 - 2x_2 \leq 2$$

$$2x_1 + 3x_2 \leq 6$$

$$x_1 + 2x_2 \leq 5$$

$$-x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

$$\rightarrow \text{Max } Z = 4x_1 + 10x_2 \rightarrow \text{Max } Z = 10x_1 + x_2 + 2x_3$$

i) subject to

$$2x_1 + x_2 \leq 50$$

$$2x_1 + 5x_2 \leq 100$$

$$2x_1 + 3x_2 \leq 90$$

$$x_1, x_2 \geq 0$$

subject to

$$x_1 + 3x_2 - x_3 \leq 10$$

$$4x_1 + 3x_2 + 2x_3 \leq 20$$

$$x_1, x_2, x_3 \geq 0$$

Now the IBSFS is given by  
 setting  $x_1 = x_2 = x_3 = 0$  (non-basic)  
 $x_4 = \frac{7}{3}, s_1 = 5, s_2 = 0$  (basic).

$\therefore$  The IBSFS is  $(0, 0, \frac{7}{3}, 5, 0)$   
 for which  $Z = 0$ .

Now we move from the current  
 BFS to the next better BFS.  
 Put the above information in  
 tableau form:

$C_j$	107	1	2	0	0	0	0	0
CB Basis.	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	b	$Z$
0	$x_4$	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{7}{3}/\frac{14}{3}$
0	$s_1$	16	$\frac{1}{2}$	-6	0	T	0	5
0	$s_2$	(3)	-1	-1	0	0	1	0
$Z = \sum c_j x_j$								
$G = g - z_j$								
107 1 2 0 0 0								

from the above table,  $x_4$  is the entering variable,  $s_2$  is outgoing variable and (3) is key element.

Convert the key element to unity and all other elements in its column to zero. Then we obtain the new proceed simple tableau:

$$x_1 = \frac{31}{5}, x_2 = \frac{58}{5}, x_3 = 0 \text{ (non-basic)}$$

$$\therefore Z_{\max}^1 = \frac{143}{5}$$

$$\text{i.e. } \max(Z^1) = \frac{143}{5}$$

$$\begin{aligned} \text{Hence } \min Z &= \max(-Z^1) \\ &= -\frac{143}{5} \end{aligned}$$

$\rightarrow$  solve the following LPP by  
Simplex method!

$$\text{Max } Z = 10x_1 + x_2 + 2x_3$$

Subject to the constraints:

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + 1/2x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Sol: The objective function of the given LPP is of maximization type.

Now we write the given LPP in the standard form.

$$\text{Max } Z = 10x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$$

Subject to

$$\frac{14}{3}x_1 + \frac{1}{2}x_2 - 2x_3 + x_4 + 0x_5 + 0x_6 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + x_5 + x_6 = 5$$

$$3x_1 - x_2 - x_3 + 0x_4 + x_5 + x_6 \leq 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Where  $x_4, x_5, x_6$  are the slack variables.

→ solve the following LPP by simplex method:

$$\text{Minimize } Z = x_1 - 3x_2 + 3x_3$$

subject to

$$+3x_1 - x_2 + 2x_3 \leq 7$$

$$2x_1 + 4x_2 \geq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

Sol: The objective function of the given is of minimization type.

converting it to the maximization

$$\text{we have } \max Z' = \min(-Z)$$

$$= -x_1 + 3x_2 - 3x_3$$

as the R.H.S of the second constraint is negative, we change it into the

$$\text{we have } -2x_1 - 4x_2 \leq 12$$

now we write the given LPP in standard form

$$\max Z' = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$$

$$-2x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

where  $s_1, s_2$  &  $s_3$  are slack

proceed this way, ~~and~~ we obtain the optimal solution



Note: (i) From the optimum tableau, we observe that the net evaluation corresponding to non-basic variable  $s_1$  is zero. This is an indication for the existence of an alternate basic feasible solution. Thus we can bring  $s_1$  into basis in place of  $s_3$  which satisfies the exist criterion. (It is giving minimum ratio). Therefore, by introducing  $s_1$  into the basis by place of  $s_3$ , the new optimum simplex tableau is as follows:

		1	2	0	0	0	
$C_B$	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$
2	$x_2$	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	3
1	$x_1$	1	0	0	$\frac{1}{3}$	$\frac{2}{3}$	6
0	$s_1$	0	0	1	$\frac{1}{3}$	$\frac{4}{3}$	8
$Z_f = \sum C_B Z_j$		1	2	0	1	0	12
$Z_g = q - Z_f$		0	0	0	-1	0	

From the above table, the alternate optimum soln is  $x_1 = 6$ ,  $x_2 = 3$ .

$$\boxed{\max Z = 12}$$

We observe that the basic feasible solution (BFS) has been changed but the optimum solution remains the same.

different basic feasible solution, we make the key element unity and make all other elements of the key column zero, subtracting proper multiples of key row from the other rows.

Here subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the third row. Also change the corresponding value with  $C_B$  column from 0 to 5, while replacing  $S_1$  by  $x_1$  under the basis.

$\therefore$  the second basic feasible solution is given by the following table:

$C_B$	Basis	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$	$b$
5	$x_1$	1	1	1	0	0	2
0	$S_2$	0	-3	-5	+ -	= 0	-
0	$S_3$	0	5	-3	0	1	6

$$Z_I = \sum C_B x_I = 5 \cdot 5 + 5 \cdot 5 + 0 \cdot 0 = 50$$

$$C_I = g - Z_I = 0 - 50 = -50$$

As  $C_j$  is either zero or negative i.e.  $C_j \leq 0$  under columns, the above table gives the optimal basic feasible solution.

$\therefore$  The optimal solution is  $x_1=2, x_2=0$

and maximum  $Z=50$ .

→ Use the simplex method to solve  
the following L.P.P.

$$\text{maximize } Z = x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 \leq 8,$$

$$x_1 + 2x_2 \leq 12,$$

$$x_1 - x_2 \leq 3; \quad x_1, x_2 \geq 0$$

Sol:

The objective function of the given L.P.P. is of maximization type and R.H.S. of all constraints are  $\geq 0$ .

Now we write the given L.P.P. for the standard form:

$$\text{max } Z = x_1 + 2x_2 + 0S_1 + 0S_2$$

subject to

$$x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 = 8$$

$$x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 12$$

$$x_1 - x_2 + 0S_1 + 0S_2 + S_3 = 3;$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

Where  $S_1, S_2, S_3$  are slack variables

NOW the initial basic feasible solution is given by

$$\text{setting } x_1 = x_2 = 0 \quad (\text{initial})$$

$$S_1 = 8, \quad S_2 = 12, \quad S_3 = 3 \quad (\text{basic})$$

∴ the I.B.F.S. is  $(0, 0, 8, 12, 3)$

for which  $Z = 0$ .

NOW we move from the current basic feasible solution to the next better basic feasible soln.

$C_B$	BASIS	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	$\theta$
2	$x_2$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	0	4	"
0	$s_2$	(2)	0	-1	1	0	4	$\frac{4-2}{-1} = 2 \rightarrow$
0	$s_3$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	7	$\frac{7-1}{\frac{1}{2}} = 14$
$Z_j = \sum C_B a_{ij}$								
$G = c_j - Z_j$								

from the above tableau,

$x_1$  is the incoming variable,  $s_2$  is the outgoing variable and (2) is the key elt.

Now convert the key element to unity and all other elements in its column to zero.

Then we get the new iterated simplex tableau as

$C_B$		1	2	0	0	0		
$C_B$	BASIS	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	$\theta$
2	$x_2$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	5	$\frac{5+1}{\frac{1}{2}} = 12$
1	$x_1$	1	0	- $\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{2-(-\frac{1}{2})}{\frac{1}{2}} = 3$
0	$s_3$	0	0	$\frac{3}{2}$	$\frac{1}{2}$	1	6	$\frac{6-1}{\frac{1}{2}} = 10$
$Z_j = \sum C_B a_{ij}$								
$G = c_j - Z_j$								

As  $C_j$  is either zero or negative (i.e  $C_j \leq 0$ )

under all columns, the above tableau

gives the optimal solution basic feasible solution.

∴ The optimal solution is  $x_1 = 2, x_2 = 5$   
and maximum  $Z = 12$ .

put the above information in tableau form:

		$C_j$	1	2	0	0	0		
$C_B$	BASIS		$x_1$	$x_2$	$S_1$	$S_2$	$S_3$	$b$	$\Theta$
0	$S_1$		-1	(2)	1	0	0	8	$\frac{8}{2} = 4$
0	$S_2$		1	2	0	1	0	12	$\frac{12}{2} = 6$
0	$S_3$		1	-1	0	0	1	3	-
		$Z_j = \sum c_{ij} x_i$	0	0	0	0	0	0	
		$C_j = c_j - Z_j$	1	2	0	0	0	0	

from the above table,  
 $x_2$  is incoming variable as  $C_j (=2)$   
 is maximum and the corresponding column  
 is known as key column.

The minimum ratio occurs  
 in the first row

$\therefore S_1$  is outgoing variable and the  
 common intersection element (2) is

the key element

Now convert the key element to  
 unity and all other elements in its  
 column to zero. Then we obtain  
 a new iterated simplex tableau

Step (4): Apply optimality test!

As  $g_i^j$ 's are +ve under some columns,

$\therefore$  the initial basic feasible solution is not optimal and we proceed to the next step.

Step (5): (i) Identify the incoming and outgoing variable

The above table shows that

$x_1$  is the incoming variable as  $C_j(z_s)$

$z_s$  maximum and the column in which it appears is the key column.

Dividing the elts under  $b$ -column by the corresponding elts of key-column, we find minimum ratio or  $z_s$  is 2 for two rows.

$\therefore$  Arbitrarily we choose the row containing  $s_1$  as the key row (shown marked by arrow on its right end)

The element at the intersection of key row and key column i.e. (1), is the key element

i.  $s_1$  is the outgoing variable

which will now become non-basic variable (i.e.  $s_1$ )

So, removing  $s_1$  and the left will contain  $x_1$ ,  $s_2$  and  $s_3$  as the basic variables.

(ii) Iterate towards the optimal solution:

To transform the initial set of eqns (constraint eqn (1)) with a basic feasible solution into an equivalent set of equations in it.

Step (3): Find an initial basic feasible solution.

There are three equations involving 5 unknowns and for obtaining a solution, we assign zero values any  $5-3=2$  of the variables.

Let us start from  $x_1 = 0, x_2 = 0$ .

From ①, we get the basic solution

$$s_1 = 2, s_2 = 10, s_3 = 12$$

Since all  $s_1, s_2, s_3$  are free  
the basic solution is feasible and  
non-degenerate.

The basic feasible solution is:

$$x_1 = x_2 = 0 \text{ (non-basic)}$$

$$\text{and } s_1 = 2, s_2 = 10, s_3 = 12 \text{ (basic).}$$

∴ Initial basic feasible solution is given by the following table.

		$C_j$	5	3	0	0	0	
CB	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	$\theta$
0	$s_1$	(I)	1	1	0	0	2	$\frac{2}{1} = 2$
0	$s_2$	5	2	0	1	0	10	$\frac{10}{2} = 5$
0	$s_3$	3	8	0	0	1	12	$\frac{12}{3} = 4$

$$B_j = \sum c_i a_{ij} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$C_j = c_j - Z_j \quad 5 \quad 3 \quad 0 \quad 0 \quad 0$$

[for  $x_1$  - column 1,  $Z_j = \sum c_i a_{ij} = 0(1) + 0(5) + 0(3) = 0$   
 $(j \neq 1)$   
 and for  $x_2$  - column 2,  $Z_j = \sum c_i a_{ij} = 0(0) + 0(2) + 0(8) = 0$   
 similarly  $Z_j(3) = \sum c_i a_{ij} = 0(2) + 0(10) + 0(4) = 0$

→ They make all other elements of the key column zero by subtracting proper multiples of key row from other rows.

Step(6): Go to step(4) and the computational procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

problems

2002 → Use simplex method to solve the following LPP:-

$$\text{Maximize } Z = 5x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 2$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

SOL

Step(1): The problem is of maximization type and all L's  $\geq 0$

Step(2): Express the problem in the standard form by introducing the slack variables  $s_1, s_2, s_3$

The problem in the standard form becomes:

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 2$$

$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$

$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

If more than one variable has the same maximum value, any of these variables may be selected arbitrarily as the entering variable.

Now divide the elements under b-column by the corresponding elements of key column and choose the row containing the minimum ratio & ( $\geq 0$ )

then replace the corresponding basic variable (by making its value zero). It is termed as the outgoing variable. The corresponding row is called the key row. (put an arrow on its right end).

The element at the intersection of the key row and key column is called the key element (which is shown bracketed).

It is also called pivotal element or leading element. If all the ratios are  $\leq 0$ , the problem is unbounded.

If all the ratios are  $\leq 0$ , the feasible solution can be made as large as we please without violating the feasibility condition. Hence the problem has an unbounded solution and no further iteration is required.

(ii) Iteration towards an optimal solution

Drop the outgoing variable and introduce the incoming variable along with its associated value under  $C_{\text{B}}$ -column.

Convert the key element to unity by dividing the key row by key element.

→ b-column denotes the values of the basic variables while remaining variables will always be zero.

→ The coefficients of the x's (decision variables) in the constant equation constitute the body matrix while the coefficients of the slack variables constitute the unit matrix.

Step(4): Apply optimality test.

compute  $C_j = c_j - Z_j$  where  $Z_j = \sum c_{kj} x_{kj}$

( $C_j$ -row is called net evaluation row and indicates the per unit increase in the objective function if the variable heading the column is brought into the solution).

- ✓ If all  $C_j$  are negative, then the initial basic feasible solution is optimal.
- If even one  $C_j$  is +ve, then the current feasible solution is not optimal (i.e. can be improved) and proceed to the next step.

Step(5): (i). Identify the incoming and outgoing variables.

If there are more than one the  $C_j$ , then the incoming variable is the one that heads the column containing maximum  $C_j$ . The column containing it is known as the key column (put one arrow at bottom).

for which,  $a_j^i$ ,  $j=1, 2, 3, \dots, (n-m)$ , are each zero, find all  $s_i$ .

If all  $s_i$ 's are  $\geq 0$ , the basic solution is feasible and non-degenerate.

If one or more of the  $s_i$  values are zero, then the solution is degenerate.

The above information can be expressed as:

	$C_B$	$c_1$	$c_2$	$c_3$	$\dots$	$c_m$	$b_1$	$b_2$	$b_3$	$\dots$	$b_n$
	BASIS	$x_1$	$x_2$	$x_3$	$\dots$	$x_m$	$s_1$	$s_2$	$s_3$	$\dots$	$s_n$
0	$s_1$	$a_{11}$	$a_{12}$	$a_{13}$	$\dots$	$a_{1m}$	1	0	0	$\dots$	$b_1$
0	$s_2$	$a_{21}$	$a_{22}$	$a_{23}$	$\dots$	$a_{2m}$	0	1	0	$\dots$	$b_2$
0	$s_3$	$a_{31}$	$a_{32}$	$a_{33}$	$\dots$	$a_{3m}$	0	0	1	$\dots$	$b_3$
		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

BODY MATRIX      UNIT MATRIX

Note:-  
 → The variables  $s_1, s_2, s_3$  etc are called basic variables and variables  $x_1, x_2, x_3$  etc are called non-basic variables.

→ Basis refers to the basic variables  $s_1, s_2, s_3, \dots$

→  $c_B$  - row denotes the coefficients of the variables in the objective function.

→  $C_B$  - column denotes coefficients of the basic variables for the objective function.

## \* Working procedure of the simplex method:

Assuming the existence of an initial basic feasible solution, an optimal solution to any L.P.P. by simplex method is found as follows:

Step(1): (i) check whether the objective function is to be minimized (or) maximized.

If  $Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$  is to be minimized, then convert it into a problem of maximization by writing

$$\text{maximize } Z^* = \text{minimize}(-Z)$$

(ii) check whether all b's are positive.

If any of the b's is negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step(2): Express the problem in the standard form.

Convert all inequalities of constraints into equalities by introducing slack/surplus variables by the constraints giving eqns of the form  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + s_1 + 0s_2 + 0s_3 + \dots$

Step(3): find an initial basic feasible solution.

If there are 'm' equations involving 'n' unknowns, then assign zero values to any ( $n-m$ ) of the variables for finding a solution. starting with a basic solution.

Now let us check whether our solut. is optimal. Once again, we need to write our objective function in terms of the current non-basic variables  $x_4$  and  $x_5$ . So, we need to write  $x_1$  in terms of  $x_4$  and  $x_5$  to eliminate  $x_1$ .

Also, to find the leaving variable we need to write all the current basic variable's  $x_1, x_2$  and  $x_3$  in terms of non-basic variables  $x_4$  and  $x_5$ .

From (16), (26) and (36), we have

$$x_1 + x_3 = 4 \quad (i)$$

$$x_2 + x_4 = 6 \quad (ii)$$

$$3x_1 + x_4 + x_5 = 6 \quad (iii)$$

$$(iii) \Rightarrow x_1 = \frac{6 - x_4 - x_5}{3} \Rightarrow x_1 = 2 - \frac{x_4}{3} - \frac{x_5}{3}$$

(16)

$$(i) \Rightarrow x_3 = 4 - x_1 \\ \Rightarrow x_3 = 4 - \left(2 - \frac{x_4}{3} - \frac{x_5}{3}\right)$$

$$\Rightarrow x_3 = 2 + \frac{x_4}{3} + \frac{x_5}{3} \quad (26)$$

$$(ii) \Rightarrow x_2 = 6 - \frac{x_4}{2} \quad (36)$$

The objective function becomes,

$$Z = 3x_1 + 3x_2 + 30$$

$$= 3\left(2 - \frac{x_4}{3} - \frac{x_5}{3}\right) + \frac{5x_4}{2} + 30$$

$$Z = 36 - \frac{7x_4}{2} - \frac{3x_5}{2}$$

In the above eqs the coefficients of all non-basic variables are all negative. So, the solution can not be improved further and we have obtained the optimal solution.

The objective function becomes

$$Z = 3x_1 + 5x_2$$

$$= 3x_1 + 5\left(6 - \frac{x_4}{2}\right)$$

$$\boxed{Z = 3x_1 - \frac{5x_4}{2} + 30}$$

In the above eqn, the coefficient  $x_1$  is +ve.

so we can improve  $Z$  by increasing  $x_1$  (It is entering variable).

since the coefficient of  $x_4$  is -ve,  $Z$  will decrease if we increase  $x_4$ . so, we keep  $x_4$  at '0' level (i.e.  $x_4 = 0$ )

We now have the following situation:

Table-II

<u>Basic variables</u>	<u>Leaving value of <math>x_1</math></u>
$x_2$	$\infty$
	$x_2 = 6 - \frac{x_4}{2}$
	$= 6$ ( $\because x_4 \geq 0$ )
$x$	
$x_3$	$x_3 = 4 - x_1$
	$x_1 \leq 4$ ( $\because x_3 \geq 0$ )
$x_5$	$x_5 = 6 - 3x_1 + x_4$
	$= 6 - 3x_1$
	$x_1 \leq 2$ ( $\because x_5 \geq 0$ )

Since  $x_5$ , gives  $x_1 \leq 2$  and RHS satisfied.

both  $x_1 \leq 4$  and  $x_1 \leq 2$ , we let  $\boxed{x_1 = 2}$ ,

then  $\boxed{x_5 = 0}$ . Hence the  $x_5$  is the leaving variable.

from the first row of Table-II, we get  $\boxed{x_2 = 6}$

from the second row of Table-II, we get  $x_3 = 4 - x_1$

$$\therefore \boxed{x_3 = 2}$$

from the third row of Table-II

$$\boxed{x_5 = 0}$$

the new basic feasible solution is

$$x_1 = 2, x_2 = 6, x_3 = 2, x_4 = 0, x_5 = 0$$

i.e.  $(2, 6, 2, 0, 0)$  for which  $\boxed{Z = 36}$

Let us now calculate the value of each variable for output only one variable.

variable  $x_1 = 0$

From the first row in Table I, we get  $\eta_3 = 1$

From the second row  $\lambda_4 = 0$  (i.e.  $\lambda_4 = 12 - 2(6) = 0$ )

from the third row  $z_5 = 6$  ( i.e  $z_5 = 18 - 3(4) - 2(6) = 6$  )

new basic feasible solution is

$$x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 6,$$

$Z = 30$ . and  $a_2, a_3$  and  $a_5$  are new basic variables.

Next, we shall check whether our solution is

optimal

Optimal. — for this, we need to write our objective function in terms of the current non-basic variables  $x_1$  and  $x_4$ . So, we need to write  $x_2$  and  $x_3$  in terms of the current non-basic variables  $x_1$  and  $x_4$  and use these two to eliminate  $x_1$  and  $x_4$  from the objective function. We

$x_1$  and  $x_2$  and use these  
Also to find the leaving variable we  
need to write all the current basic variables  
in terms of non-basic variables  $x_1$  and  $x_2$

$$\textcircled{1} \quad \exists \quad 1 \cdot x_3 = 4 - x_1 \quad \textcircled{15}$$

$$2x^2 = 12 - 74$$

$$x_2 = 6 - \frac{x_4}{2}$$

$$\begin{aligned} \textcircled{3} \quad \exists x_5 &= 18 - 3x_1 - 2x_2 \\ &= 18 - 3x_1 - 2\left(6 - \frac{x_4}{2}\right) \quad (\text{from } \textcircled{2}) \\ &= 18 - 3x_1 - 12 + x_4 \\ \cancel{x_5} &= \cancel{6} - 3x_1 + x_4 \end{aligned}$$

So the equation for Z would be the one to increase Z and hence should be chosen as the entering basic variable.

Here the choice of the entering variable is  $x_2$ .

How to identify the Leaving Basic Variable?

The possibilities for the leaving basic variable is that it is one of the non-basic variables  $x_3, x_4, x_5$ .

Here one of these  $x_3, x_4, x_5$  for which the entering variable  $x_2$  achieves the maximum value and none of  $x_3, x_4, x_5$  becomes negative. This is done as follows:

Basic Variables	$f_{24}$	Maximum value of $x_2$
$x_3$ :	$x_1 + x_3 = 4 \Rightarrow x_3 = 4 - x_1$ $\Rightarrow x_3 = 4$ ( $\because x_1 \geq 0$ )	No limit
$x_4$ :	$x_2 + x_4 = 12 \Rightarrow x_4 = 12 - x_2$ $\therefore x_4 \leq 12 - x_2 = 6$ ( $\because x_2 \geq 0$ )	
$x_5$ :	$x_1 + x_2 + x_5 = 18 \Rightarrow x_5 = 18 - x_1 - x_2$ $\therefore x_5 \leq \frac{18}{2} = 9$ ( $\because x_1, x_2 \geq 0$ )	

Since  $x_4$  (slack variable for the constraint  $x_2 \leq 12$ ) gives  $x_2 = 6$ , which satisfies the two conditions viz. that it is the maximum value for which none of  $x_3, x_4, x_5$  becomes negative, therefore,  $x_4$  is the leaving basic variable.

Note that  $x_2 = 9$  is greater than  $x_2 = 6$  but then if we take  $x_2 = 9$  then  $x_4$  becomes negative which we do not want.

But after introducing the slack or surplus variables, we take original variables viz  $x_1, x_2$  as the non-basic variables and the slack variables viz  $x_3, x_4, x_5$  as basic variables for the initial (starting) basic feasible solution.

Therefore from ①, ② and ③,

we get the basic variables  $x_3 = 4, x_4 = 12$   
 $x_5 = 18$

with the non-basic variables

$x_1, x_2$  chosen as  $x_1 = 0, x_2 = 0$ .

Therefore, the initial basic feasible

solution is  $(0, 0, 4, 12, 18)$

for which  $Z = 0$ .

for step (ii):

We move from the current basic feasible solution to the next better basic feasible solution. This is done by replacing one non-basic variable (called the entering basic variable) by a basic variable (called the leaving basic variable) and identifying the new basic feasible solution.

Now the question arises: How to identify the entering basic variable?

Since the objective (aim) is to increase the value of the objective function  $Z$ , therefore the variable that has the largest coefficient

Maximize Z

subject to

$$Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \geq 0$$

$$Z = 3x_1 + 5x_2 + x_3 + 0x_4 + 0x_5 \leq 5$$

$$0Z + x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 \leq 12$$

$$0Z + 0x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 \leq 18$$

$$0Z + 3x_1 + 2x_2 + 0x_3 + 0x_4 + x_5 \leq 18$$

where  $x_i \geq 0 ; i=1,2,3,4,5$ .

Now to use the algorithm, we have to find answers to the following corresponding questions:

(I) Initial step: How is the basic feasible solution selected?

(II) Iterative step: while seeking a better basic feasible solution, how is the direction of the movement chosen? Where do we stop? How to identify the new solution?

(III) Optimality Test: How to determine whether the latest basic feasible solution is the optimal solution?

for the step I, we can start with any basic feasible solution which is convenient to us. when the problem is in equality form, the obvious choice is the origin. In all the variables are taken as equal to zero.

e.g.  $x_1 = 0, x_2 = 0$  is the starting basic feasible solution.

Example:

$$\text{Maximize } Z = 3x_1 + 5x_2$$

subject to  $x_1 \leq 4$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0, x_2 \geq 0.$$

Sol By introducing the slack variables,

the problem becomes:

$$\text{Maximize } Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\text{subject to } x_1 + x_3 = 4 \quad (1)$$

$$2x_2 + x_4 = 12 \quad (2)$$

$$3x_1 + 2x_2 + x_5 = 18 \quad (3)$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0. \quad (4)$$

where  $x_3, x_4, x_5$  are slack variables.

Note that this problem is identical to the original (given) problem and its form is much more convenient for algebraic manipulation and for identification of the feasible solutions. While dealing with this problem it is easier to manipulate the equations of the objective function along with the constraint equations simultaneously. Therefore, before we use the steps of the algorithm, we rewrite the problem once again in an equivalent way in the equation form as follows:

for the algorithm of the simplex method,  
we have the following similar structure:

(ii) Structure of the simplex Algorithm:

The simplex Algorithm is an algebraic procedure in which each iteration involves a system of equations to obtain a new trial solution for the optimality test.  
According to the procedure, we have to take the following three steps:

(I) Initial step: start with a basic feasible solution of a given L.P.P.

(II) Iterative step: move to a better basic feasible solution.

(III) Optimality Test Step: the current basic feasible solution is optimal.

If yes - STOP

If No ! Repeat the iterative step.

we give a complete description of the general algorithm through the following example of a L.P.P.

But before we discuss the method,  
let us first understand the meaning of  
an Algorithm.

### Meaning of an Algorithm:

An algorithm is an iterative solution procedure. It is simply a process in which the steps are repeated (iterated) over and over again until the desired result is achieved.

thus, an algorithm is a procedure starting with the first step known as the initial step, developing a criteria to know when and where to stop and ~~to~~ reach the last step where the desired result is obtained.

This can be summarized in the following way:

#### (i) Structure of a General Algorithm:

first step: Ready to start the iterations

subsequent steps: performing the iterations

concluding steps: Has the desired result been achieved?

If Yes : Stop

If No : Repeat the iterations.

It was given by a famous American mathematician G.B. Dantzig in 1947. The word "simplex" has nothing to do with the method as such. Its origin can be traced back to a special problem that was studied in the early development of its algorithm. During World War-II, a group worked on allocation problems for the U.S. Air Forces. A few models were developed by this group to allocate resources in such a way so as to maximize or minimize some linear objective function.

— However, it was Dantzig — a member of this group, who ultimately formulated the general linear programming problem and devised the simplex method for its solution. Problems of linear programming type were formulated and discussed even before the method was developed by Dantzig. However, the simplex method is the most efficient and reliable procedure that is generally used to solve L.P.P. The method is extensively applied with the help of modern computers when the L.P.P. involves a large number of constraints and variables.

## The Simplex Method

### Introduction

The graphical method of solving an L.P.P. was discussed and it was shown that this method is giving difficulty when we are dealing with a L.P.P. involving three or more variables.

Also we know the meanings of various types of a L.P.P. have been explained namely feasible solutions, basic solutions, basic feasible solutions and optimal solutions.

It was shown that the set of all feasible solutions for a L.P.P. forms a convex set and that the optimal solution, if it exists, occurs at one of the extreme points of the convex set. Further that every extreme point of a convex set of feasible solutions of a L.P.P., corresponds to a basic feasible solution of the problem.

How to find that extreme point which corresponds to the optimal solution? In other words, how to identify an optimal solution out of the basic feasible solutions of a L.P.P.?

To answer this question, we use an algebraic method popularly called Simplex method.

The simplex method is a computational process suitable for a numerical solution of a linear programming problem.

Sol<sup>n</sup>: The vector form of the given linear program becomes

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix}$$

(1)  $[1, 0, 1, 0, 0, 0]^T$  is not basic feasible solution, although all its components are non-negative.

because: the vectors  $A_1$  and  $A_3$  associated with the  $x$ -variables not set equal to zero are not linearly independent

(2). The coefficient matrix A, comprising the column vectors

order  $2 \times 6$

Therefore a feasible solution must have at least  $6 - 2 = 4$  zero components (variables), which is not the case here.

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Note

If  $A_1$  or  $A_2$  are eliminated instead of  $A_3$ , either  $x_1$  or  $x_2$  are driven to zero. In these cases we will find that new solutions are not feasible.

How Consider the system of equations

5.

$$2x_1 + x_2 + 4x_3 = 11$$

$$3x_1 + x_2 + 5x_3 = 14$$

A feasible solution is  $x_1 = 2, x_2 = 3, x_3 = 1$ .

Reduce this feasible solution to a basic feasible solution.

→ for the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$

$$2x_1 + x_2 + 5x_3 - x_5 = 4$$

Here  $(1, 1, 1, 0)$  is a feasible solution.

Find a ~~basic~~ feasible solution.

write the constraint equations of the following

linear program in the vector form

$$\text{minimize } z = 2x_1 + 3x_2 + x_3 + 10x_4 + 15x_5 + 10x_6$$

$$\text{subject to } x_1 + 2x_2 + x_3 + 10x_4 + 15x_5 \leq 3$$

$$2x_1 + 3x_2 + 4x_3 + 7x_6 \leq 6$$

with: all variables non-negative

(1) Determine whether  $[1, 0, 1, 0, 0, 0]^T$  is a basic feasible solution to the linear program.

(2) Determine whether  $[1, 0, 0, 0, 2, 4]^T$  is a basic feasible solution to the linear program.

$$\begin{aligned} & \left. \begin{aligned} 3x_2 - x_3 &= -4 \\ 6x_2 + x_3 &= 10 \end{aligned} \right\} \\ \text{Solving, we get } & \boxed{x_2 = 2/3}, \boxed{x_3 = 6} \end{aligned}$$

To reduce the number of plus variables, the variable to be driven to zero is found by choosing  $r$  for which

$$\begin{aligned} \frac{x_r}{x_r} &= \min \left\{ \frac{x_2}{x_3} / x_3 > 0 \right\} \\ &= \min \left\{ \frac{x_1}{x_1}, \frac{x_2}{x_2}, \frac{x_3}{x_3} \right\} \\ &= \min \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{6} \right\} = \min \left\{ \frac{1}{2}, \frac{3}{2}, \frac{1}{6} \right\} \\ &= \frac{1}{6} \end{aligned}$$

Thus, we can remove vector  $A_3$  for which  $\frac{x_3}{x_3} = \frac{1}{6}$  and obtain new solution with not more than two non-negative (non-zero) variables.

The values of new variables are given by

$$\hat{x}_1 = x_1 - \frac{x_3}{x_3} x_1 = 1 - \frac{1}{6} \cdot \frac{2}{3} = \frac{2}{3}$$

$$\hat{x}_2 = x_2 - \frac{x_3}{x_3} x_2 = 1 - \frac{1}{6} \cdot \frac{8}{3} = \frac{8}{9}$$

Obviously, column  $A_1$  or  $A_2$  of a corresponding system of eqs. is given by

Hence the basic feasible solution to given system of eqs. is given by

$$x_1 = \frac{2}{3}, x_2 = \frac{8}{9}, x_3 = 0$$

SOL

The above given system of eqns may be put in matrix notations as

$$\begin{bmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$\Rightarrow AX=B$$

where  $A = \begin{bmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$  and  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ .

Let the columns of  $A$  be denoted by

$$A_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, A_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ & } A_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Here  $\epsilon(A) = 2$   
 i.e. basic solution to the given system of  
 eqns exist with not two variables  
 different from zero.  
 Also, the columns  $A_1, A_2, A_3$  are  
 linearly dependent (L.D.) (we can easily verify)

∴ scalars  $\lambda_1, \lambda_2, \lambda_3$  not all zero s.t.

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0.$$

$$\Rightarrow \begin{bmatrix} 2 \\ -5 \end{bmatrix}\lambda_1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix}\lambda_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\lambda_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2\lambda_1 + 3\lambda_2 - \lambda_3 = 0$$

$$-5\lambda_1 + 6\lambda_2 + \lambda_3 = 0$$

∴ This is a system of two eqns in three unknowns  $\lambda_1, \lambda_2, \lambda_3$ .

Let us choose one of the  $\lambda$ 's arbitrarily  
 say  $\lambda_1 = 1$ .

→ Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables.

$$2x_1 + x_2 - x_3 = 2$$

$$3x_1 + 2x_2 + x_3 = 3.$$

Investigate whether the basic solutions are degenerate basic solutions (or) not. Hence find the basic feasible solution of the system.

(Or)

Show that the following system of linear equations has two degenerate basic feasible solutions and the non-degenerate basic solution is not feasible.

$$-x_1 + x_2 - x_3 = 2, \quad 3x_1 + 2x_2 + x_3 = 3.$$

→ Find all the basic feasible solutions of the equations

$$2x_1 + 6x_2 + 2x_3 + 2x_4 = 3,$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Note: If there is a feasible solution to the system of constraints

$Ax = 0$   
 $x \geq 0$ , then there also exists a basic feasible solution to the system.

For example

Consider the system of equations

$$2x_1 + 3x_2 - x_3 = 4$$

$$-5x_1 + 6x_2 + x_3 = 2$$

A feasible solution is  $x_1 = 1, x_2 = 1, x_3 = 1$ . Reduce this feasible solution to a basic feasible solution.

Hence the optimised basic feasible solution is

$$x_1 = 0, x_2 = 0, x_3 = \frac{44}{17}, x_4 = \frac{45}{17}$$

and the maximum value of  $Z = 28.9$  -

2001 → Compute all basic feasible solutions of the linear programming problem.

$$\text{Max } Z = 2x_1 + 3x_2 - x_3 = 8$$
$$x_1 - 2x_2 + 6x_3 = -3$$

$$x_1, x_2, x_3 \geq 0$$

and hence indicate the optimal solution.

2002 → For the following system of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 - x_2 + 3x_3 = 4$$

Determine i) all basic solutions

ii) all basic feasible solutions

iii) a degenerate solution which is not a basic feasible solution.

HW: → find all the basic solutions to the following problem:

(2) Maximize  $Z = x_1 + 3x_2 + 3x_3$

$$\text{subject to: } x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 3x_2 + 5x_3 = 7$$

$$\text{with } x_1, x_2, x_3 \geq 0$$

which of the basic solutions are

a) non-degenerate basic feasible      b) optimal basic feasible?

→ For the following system of equations

$$x_1 + 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + 5x_3 = 5$$

Determine i) all basic solutions.

ii) all basic feasible solutions.

objective function.

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

$$\text{subject to } 2x_1 + 3x_2 - x_3 + 4x_4 = 8$$

$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Sol: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions =  $A(C_2) = 6$

The characteristics of the various basic solutions

are given below.

No. of basic solutions	Basic variables	Non-basic variables	values of basic variables	Is the solution feasible? (are all $x_j \geq 0$ )	value of $Z$	Is the solution optimal?
1.	$x_1, x_2$	$x_3 = 0$ $x_4 = 0$	$2x_1 + 3x_2 = 8$ $x_1 - 2x_2 = -3$ $\therefore x_1 = 1, x_2 = 2$	Yes	8	NO
2.	$x_1, x_3$	$x_2 = 0$ $x_4 = 0$	$2x_1 - x_3 = 8$ $x_1 + 6x_3 = -3$ $\therefore x_1 = -\frac{14}{13}, x_3 = -\frac{17}{13}$	NO	-	-
3.	$x_1, x_4$	$x_2 = 0$ $x_3 = 0$	$2x_1 + 8x_4 = 8$ $x_1 - 7x_4 = -3$ $\therefore x_1 = \frac{22}{9}, x_4 = \frac{7}{9}$	Yes	10.3	NO
4.	$x_2, x_3$	$x_1 = 0$ $x_4 \geq 0$	$3x_2 - x_3 = 8$ $-2x_2 + 6x_3 = -3$ $\therefore x_2 = \frac{45}{16}, x_3 = \frac{7}{16}$	Yes	10.2	NO
5.	$x_2, x_4$	$x_1 = 0$ $x_3 \geq 0$	$3x_2 + 4x_4 = 8$ $-2x_2 - 7x_4 = -3$ $\therefore x_2 = \frac{132}{39}, x_4 = \frac{-7}{13}$	NO	-	-
6.	$x_3, x_4$	$x_1 \geq 0$ $x_2 \geq 0$	$-x_1 + 4x_4 = 8$ $6x_3 - 7x_4 = -3$ $\therefore x_3 = \frac{44}{17}, x_4 = \frac{45}{17}$	Yes	28.9	Yes.

Sol: Since there are 3 variables and two constraints, a basic solution can be obtained by setting any one variable equal to zero and then solving resulting equation.

Also the total number of basic solutions =  $3! = 6$

The characteristics of the various basic solutions are as given below.

No. of basic solutions	Basic variables	Non basic variables	values of basic variables	Is the solution feasible? (Are all $x_i \geq 0$ ?)	Is the solution degenerate?
1	$x_1, x_2$	$x_3 = 0$	$x_1 + 4x_2 = 11$ $x_1 + 5x_2 = 14$ $\therefore x_1 = 3, x_2 = 5$	Yes	No
2	$x_2, x_3$	$x_1 = 0$	$x_2 + 4x_3 = 11$ $x_2 + 5x_3 = 14$ $\therefore x_2 = -1, x_3 = 3$	No	No
3	$x_1, x_3$	$x_2 = 0$	$2x_1 + 4x_3 = 11$ $3x_1 + 5x_3 = 14$ $\therefore x_1 = 1, x_3 = 2$	Yes	No

The basic feasible solutions are:

- (i)  $x_1 = 3, x_2 = 5, x_3 = 0$  | The second solution is non-degenerate which is not feasible solution.
- (ii)  $x_1 = 1, x_2 = 0, x_3 = 2$ .

→ find an optimal solution to the following LPP by computing all basic solutions and then finding one that maximizes the

## INSIGHTS INTO THE SIMPLEX METHOD

while solving an LPP graphically, we noticed the following important characteristics:

(a) whenever feasible solutions existed, the region of feasible solutions was convex, bounded by lines or planes (more than two variable case). For each such convex region, there were corners (or vertices) on the boundary and edges joining these corners.

(b) for each value of  $Z$ , the objective function could be represented by a line or plane and whenever the maximum or minimum value of  $Z$  was finite, the optimal solution occurred at some corner (or vertex) of the convex region of feasible solutions.

If the optimal solutions were not unique, there were points (on an edge) that were optimal, but in every event at least one vertex was optimal.

In case of unbounded optimal solution; however no corner point was optimal.

Interestingly, these observations, derived from simple graphical examples, hold true for the general LPP also, if we think of a geometrical representation in n-dimensional space.

The region of the feasible solutions is a convex region or a convex set, has corners or the vertices.

Similarly, chocolate & sugar constraint and fat.

Constraints are  $3x_1 + 2x_2 \geq 6$

$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$

$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8$

The non-negativity restrictions are

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

The problem becomes

Minimize  $Z = 5x_1 + 2x_2 + 3x_3 + 8x_4$

Subject to the constraints:

$400x_1 + 200x_2 + 150x_3 + 500x_4 \leq 500$

$3x_1 + 2x_2 \leq 6$

$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$

$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8$

Note: we solved linear programming, having 3 variables with the help of graph. we have seen that the feasible region was a 3-dimensional figure.

If there are more than 3 variables, we cannot draw a figure with four dimensions or more.

draw a figure with four dimensions or more.

i.e., we cannot solve this problem graphically.

We need some other method to solve a general LPP having any number of variables.

∴ the basic (non-basic) solution to  
the given system is:

Basic:  $x_1 = 2, x_2 = 1$ ; Non-basic  $x_3 = 0$

and all  $y_j \geq 0$  ( $j=1, 2, \dots$ )

∴ the solution is feasible.

∴ it is a basic feasible solution.

Similarly, the other two basic and non-basic  
solutions are:

Basic  $x_1 = 5, x_2 = -1$ ; Nonbasic  $x_3 = 0$

clearly which is not  
a feasible solution.  
(constraint)

and Basic  $x_2 = \frac{5}{3}, x_3 = \frac{2}{3}$ ; non-basic  $x_1 = 0$   
clearly which is a basic  
feasible solution.

Note:- The above first and 3rd basic solutions  
are non-degenerate solutions.

Find all the basic solutions of  
the following system of equations identifying  
in each case of the basic and non-basic  
variables:  
 $2x_1 + x_2 + 4x_3 = 1$   
 $3x_1 + x_2 + 5x_3 = 4$ .

Investigate whether the basic  
solutions are degenerate basic solutions  
or not.  
Hence find the basic feasible solution  
of the system.

→ Obtain all the basic feasible solutions to the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4$$

$$2\alpha_1 + \alpha_2 + 5\alpha_3 = 5.$$

Sol The given system of equations can be written in the matrix form as

$$Ax = b$$

where  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$ ,  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  and

$$= \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Since the number of linearly independent columns of  $A \in S_2$  (Lemma 2),  
 the maximum number of linearly independent columns of  $A \in S_2$  is 2.

Then we consider any of the  $2 \times 2$  basis matrix B:

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

[21], [22] - The variable  $s$  is not associated with the columns of  $B$  are  $x_3, x_1$  and  $x_2$ , respectively for three different cases.

$$\text{Let } B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Let  $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ .  
A basic solution to the system  
is obtained by taking  $n_3 = 0$  and solving  
the system  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ .

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 4 \\ x_1 + x_2 = 5 \end{cases} \Rightarrow x_1 = 2, x_2 = 3$$

Denote the columns of the  $2 \times 6$  coefficient matrix  $A$  in system ① by  $a_1, a_2, a_3, a_4, a_5$  &  $a_6$  respectively.

Then the constraint equation  $Ax = B$  can be rewritten in the vector form

$$\begin{matrix} \text{form} \\ a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ + a_6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{matrix}$$

~~Now~~ → Write the constraint equations for the following linear programs in vector form

$$\text{Minimize } Z = a_1 + 2a_2 + 3a_3 + 4a_4 + 0a_5 + 0a_6 + 0a_7$$

$$\text{subject to } a_1 + a_2 + a_3 + 3a_4 + a_5 \leq 9$$

$$2a_1 + a_2 + 3a_4 + a_6 \leq 9$$

$$-a_1 + a_2 + a_3 + a_7 = 0$$

with all variables non-negative.

→ find all the basic solutions of the following system of equations identifying in each case of the basic and non-basic variables:

$$2a_1 + 2a_2 + 4a_3 = 1$$

$$3a_1 + a_2 + 5a_3 = 14$$

Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic feasible solutions of the system.

problems

(4)

→ write the constraint equations of the linear program in the vector form

$$\text{Minimize } Z = 2x_1 + 3x_2 + x_3 + 0x_4 + 1x_5 + 0x_6$$

$$\text{subject to } x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$$

$$x_1 + 3x_2 + 4x_3 + x_6 = 6$$

With all variables non-negative.

Sol. In matrix notation,

standard form is

$$\text{Minimize } Z = C^T X$$

$$\text{subject to } AX = B \quad \text{(1)}$$

$$X \geq 0 \quad \text{(2)}$$

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X is vector of unknowns,  
including surplus and arti-  
ficial variables.

$$\text{i.e. } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$C^T$  is the row vector of the corresponding costs, i.e.  $C^T = [2, 3, 1, 0, 1, 0]^T$ ,

$A$  is coefficient matrix of the constraint equations.

$$\text{i.e. } A = \begin{bmatrix} 1 & -2 & 2 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

and  $B$  is the column vector of the right hand sides of the constraint equations.

$$\therefore C^T B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\textcircled{1} \quad \begin{cases} x + y + x_1 = 4 \\ -x + y + x_2 = 2 \end{cases} \quad \textcircled{2}$$

\textcircled{2} There are 4 variables and two constraint equations, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations.

Also the total number of basic solutions  
=  $4C_2 = 6$

Let  $x=0, y=0$  in \textcircled{2}  
we get  $x_1=4$   
 $x_2=2$ .

$$A = (0, 0, 4, 2)$$

Let  $x=0, x_1=0$   
we get  $y=4$  and  $x_2=-2$   
B = (0, 4, 0, -2)

$$\begin{array}{l} y \neq 4 \\ y + x_1 = 4 \\ \Rightarrow x_1 = 4 - y \\ x_2 = -2 \end{array}$$

Let  $y=0, x_1=0$   
we get  $x=4$  and  $x_2=6$   
C = (4, 0, 0, 6)

$$\begin{array}{l} x = 4 \\ x + x_2 = 6 \\ \Rightarrow x_2 = 2 \\ x_1 = 6 \end{array}$$

Let  $x=0, x_2=0$   
we get  $x_1=2, y=2$   
D = (0, 2, 2, 0)

$$\begin{array}{l} y = 2 \\ y + x_1 = 4 \\ x_1 = 4 - y \\ x_1 = 4 - 2 \\ x_1 = 2 \end{array}$$

Let  $y=0, x_2=0$   
we get  $x_1=6$   
 $x=-2$   
E = (-2, 0, 6, 0)

$$\begin{array}{l} x = -2 \\ x + x_1 = 6 \\ \Rightarrow x_1 = 6 - x \\ x_1 = 6 - (-2) \\ x_1 = 8 \end{array}$$

Let  $x_1=0, x_2=0$   
we get  $y=3$  &  $x=1$   
F = (1, 3, 0, 0)

(Continue in this way)

42(1)

from the graphical solution of the eqn (1) that extreme points of the convex set  $K$  of feasible solutions are

$$(0,0), \left(\frac{8}{3}, 0\right), (0,4), \left(\frac{12}{7}, \frac{20}{7}\right).$$

Thus, clearly we can see the correspondence between basic feasible solutions and extreme points of the Convex set of feasible solution.

$x^1$  corresponds to  $(0,0)$

$x^2$  corresponds to  $\left(\frac{8}{3}, 0\right)$

$x^3$  corresponds to  $(0,4)$

$x^4$  corresponds to  $\left(\frac{12}{7}, \frac{20}{7}\right)$

Also, Conversely every extreme point of  $K$  corresponds to some basic feasible solution.

- Ques. → put the following in slack form and describe which of the variables are zero at each of the vertices of the constraint set and hence determine the vertices algebraically.

$$\text{Maximize } Z = 4x + 3y$$

subject to

$$\begin{cases} x+y \leq 4 \\ -x+y \leq 2 \end{cases} \quad \text{--- (1)}$$

$$x, y \geq 0$$

Sol: Adding the slack variables to the constraints of the given LPP:

→ A basic feasible solution to a LPP corresponds to an extreme point of the convex set  $K$  of feasible solutions and conversely, every extreme point of  $K$  corresponds to a basic feasible solution to a LPP.

For example:

$$\text{Maximize } Z = 4x_1 + 5x_2$$

subject to

$$2x_1 + 8x_2 \leq 12$$

$$3x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

{ } → ①

- Adding the slack variables,  
the constraints become.

$$2x_1 + 8x_2 + x_3 = 12 \quad \{ \rightarrow ②$$

$$3x_1 + x_2 + x_4 = 8$$

From ②: There are 4 variables and two constraint

- equations; a basic solution can be obtained by setting any two ( $4-2=2$ ) variables equal to zero and then solving the resulting equations.

Also the total number of basic solutions

$$= 4C_2 = 6$$

All basic feasible solutions are

given by

$$x^1 = (0, 0, 12, 8), \quad x^2 = \left(\frac{8}{3}, 0, \frac{20}{3}, 0\right)$$

$$x^3 = (0, 4, 0, 4) \text{ and } x^4 = \left(\frac{12}{7}, \frac{20}{7}, 0, 0\right)$$

and other two basic solutions

$$x^5 = (0, 8, -12, 0) \text{ and } x^6 = (6, 0, 0, -10)$$

are not feasible.

42(19)

The set of all feasible solutions to a linear programming problem is a convex set.

Soln: W.K.T the constraints of a L.P.P can be converted into equations by means of introduction of slack or surplus variables.

Let us consider the constraint system of any L.P.P of the form

$$Ax = B, \quad x \geq 0$$

where  $A$  is an  $m \times n$  matrix.  
 $x$  is  $n \times 1$  matrix and  $B$  is an  $m \times 1$  matrix

Let the set  $K$  be the set of all feasible solutions of  $L.P.P. Ax = B$ .

$$\therefore K \subset \{x \mid Ax = B, x \geq 0\}$$

Now to prove that  $K$  is a convex set.

Let  $x_1, x_2 \in K$ .

Then we have

$$Ax_1 = B, \quad x_1 \geq 0$$

$$Ax_2 = B, \quad x_2 \geq 0$$

Now consider  $\lambda x_1 + (1-\lambda)x_2$  for  $0 \leq \lambda \leq 1$ .

$$\text{and } A[\lambda x_1 + (1-\lambda)x_2] = \lambda Ax_1 + (1-\lambda)Ax_2 \\ = \lambda B + (1-\lambda)B \\ = B$$

Since  $x_1, x_2, \lambda$  and  $1-\lambda$  are all  $\geq 0$ .

$$\therefore \lambda x_1 + (1-\lambda)x_2 \geq 0$$

Thus  $\lambda x_1 + (1-\lambda)x_2 \in K$  for  $0 \leq \lambda \leq 1$   
which implies that the set  $K$  is a convex set

- If one or more of the basic variables equal to zero, the basic feasible solution is degenerate.
- If all the basic variables are positive, the basic feasible solution is non-degenerate.
- Optimal basic feasible solution is that basic feasible solution which minimizes (or maximizes) the objective function of the L.P.P.

Note: The total number of basic solutions =  $n^m$

where 'n' is the number of unknown variables and  
'm' is the number of constraints.

→ A linear program is in standard form  
optimize  $Z = C^T X$   
subject to  $A X = B$  — ①  
with  $X \geq 0$  — ②

Denote the columns of the  $m \times n$  coefficient matrix A in system ① by  $A_1, A_2, \dots, A_n$  respectively.  
Then the matrix constraint eqn  $A X = B$  can be rewritten in the vector form

$$x_1 A_1 + x_2 A_2 + \dots + x_n A_n = B$$

the A-vectors and B are known  $m$ -dimensional vectors.

### Basic solution:

Gives a system of ' $m$ ' simultaneous linear equations in ' $n$ ' unknowns ( $m < n$ ).

$$Ax = b, \quad x^T \in \mathbb{R}^n,$$

where  $A$  is an  $m \times n$  matrix of rank  $m$ .

Let  $B$  be any  $m \times m$  submatrix of  $A$ ,

formed by  $m$  linearly independent columns of  $A$ . Then a solution obtained by setting  $n-m$  variables not associated with the columns of  $B$ , equal to zero (known as nonbasic variable) and solving the remaining system, is called a basic solution to the given system of equations.

The  $m$  variables, which may be all different or some zero, are called basic variables.  
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 submatrix  $B$  is a basis matrix with the columns of  $B$  as basis vectors.

Note:- If  $B$  is the basis submatrix then the basic solution to the system is  $x_B = B^{-1}b$ .

But  $x_B^T \in \mathbb{R}^m$ , and as such cannot be called a solution of the given system. If  $x_B$  is a basic solution, then a solution to the given system is

$$[x_B^T, 0]$$

where  $x_B^T \in \mathbb{R}^m$  and  $0 \in \mathbb{R}^{n-m}$ .

Basic feasible solution is that basic solution which also satisfies the non-negative restrictions.

$$2x_1 - 2x_2 + x_3 \leq 8$$

$$x_i \geq 0$$

$$\max z = 10x_1 + 11x_2$$

$$\text{Subject to } x_1 + 2x_2 \leq 150$$

$$3x_1 + 4x_2 \leq 200$$

$$6x_1 + x_2 \leq 175$$

$$x_1, x_2 \geq 0$$

$$\min z = 3x_1 + 2x_2 + 6x_3 + 6x_4$$

$$\text{Subject to } x_1 + 2x_2 + x_3 + 2x_4 \leq 1000$$

$$2x_1 + x_2 + 3x_3 + 7x_4 \leq 1500$$

$$x_i \geq 0 \quad i=1,2,3,4$$

$$\min z = 6x_1 + 3x_2 + 4x_3$$

$$\text{Subject to } x_1 + 6x_2 + x_3 = 10$$

$$2x_1 + 3x_2 + x_3 = 15$$

$$x_i \geq 0 \quad i=1,2,3$$

$$\max z = 7x_1 + 2x_2 + 3x_3 + x_4$$

$$\text{Subject to } 2x_1 + x_2 = 7$$

$$5x_1 + 8x_2 + 2x_4 = 10$$

$$x_1 + x_3 = 11$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$\min z = 10x_1 + 2x_2 - x_3$$

$$\text{Subject to } x_1 + x_2 \leq 50$$

$$x_1 + x_2 \geq 10$$

$$x_2 + x_3 \leq 30$$

$$x_2 + x_3 \geq 7$$

$$x_1 + x_2 + x_3 = 60$$

$$x_1, x_2, x_3 \geq 0$$

→ Convert the following LPP to the standard form.

$$\text{min } Z = 2x_1 + x_2 + 4x_3$$

$$\text{Subject to } -2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq 2$$

$x_1, x_2 \geq 0$  and  $x_3$  unrestricted in sign.

→ Reduce the following LPP to its standard form.

$$\text{Max } Z = x_1 - 3x_2$$

$$\text{Subject to } -x_1 + 2x_2 \leq 5$$

$$x_1 + 3x_2 = 10$$

$x_1$  and  $x_2$  being unrestricted in sign.

→ Reduce the following LPP to the standard form:

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$$\text{max } Z = x_1 + x_2 + 4x_3$$

$$\text{Subject to } -2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq 2$$

→ Reduce the following LPP in the standard form:

$$\text{min } Z = -3x_1 + x_2 + x_3$$

$$\text{Subject to } x_1 - 2x_2 + x_3 \leq 11$$

$$-4x_1 + x_2 + 2x_3 \geq 3$$

$$2x_1 - x_3 = -1$$

$$x_1, x_2 \geq 0, x_3 \geq 0 \text{ or } < 0$$

(i.e.  $x_3$  is unrestricted in sign)

→ put each of the following programs in matrix standard form.

$$\text{min } Z = x_1 + x_2 + 4x_3$$

$$\text{subject to } 5x_1 + 2x_2 - 3x_3 \geq -7$$

$$\text{minimize: } Z = 2x_1^I - 2x_1^{II} + 3x_2^I - 3x_2^{II} + 0x_3 + 0x_4 + 0x_5 + Mx_6$$

$$\text{Subject to: } 4x_1^I - 4x_1^{II} + 7x_2^I - 7x_2^{II} - x_3 + x_6 = 1$$

$$8x_1^I - 8x_1^{II} + 5x_2^I - 5x_2^{II} - x_4 + x_7 = 3$$

$$-6x_1^I + 6x_1^{II} + 9x_2^I + 9x_2^{II} + x_5 = 2$$

With: all variables non-negative.

An initial feasible solution to the program in

standard form is

$$x_6 = 1, x_7 = 3, x_5 = 2, \underline{x_1^I = x_1^{II} = x_2^I = x_2^{II} = x_3 = x_4 = 0}$$

$\rightarrow$  Reduce the following problem to the standard form:

$$\text{maximize } Z = 3x_1 + 5x_2 + 8x_3$$

$$\text{subject to: } 2x_1 + 5x_2 \leq 6$$

$$3x_1 + 2x_2 + 7x_3 \geq 5$$

$$3x_1 + 4x_3 \leq 3$$

$\rightarrow$  Express the following L.P.P. in the standard form

$$\text{minimize: } Z = 3x_1 + 2x_2 + 5x_3$$

$$\text{subject to: } -5x_1 + 2x_2 \leq 5$$

$$2x_1 + 3x_2 + 4x_3 \geq 7$$

$$2x_1 + 5x_3 \leq 3$$

With:  $x_1, x_2, x_3 \geq 0$

$\rightarrow$  Convert the following L.P.P. to standard form

$$\text{maximize: } Z = 3x_1 - 2x_2 + 4x_3$$

$$\text{subject to: } x_1 + 2x_2 + x_3 \leq 8$$

$$2x_1 - x_2 + x_3 \leq 2$$

$$4x_1 - 2x_2 - 3x_3 = -6$$

With:  $x_1, x_2 \geq 0$

2005 put the following program in standard form.

$$\text{minimize } z = 25x_1 + 80x_2$$

$$\text{subject to: } 4x_1 + 7x_2 \geq 1$$

$$8x_1 + 5x_2 \geq 3$$

$$6x_1 + 9x_2 \leq 2$$

and hence obtain an initial feasible solution.

Sol:

Since both  $x_1$  and  $x_2$  are unrestricted,

$$\text{we write } x_1 = x_1^I - x_1^{II}$$

$$x_2 = x_2^I - x_2^{II}$$

where all four new variables are required to be non-negative.

Substituting these quantities into the given program and then multiply the last constraint by -1 to force a non-ve RHS,

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$$\text{minimize: } z = 25x_1^I + 30x_2^I - 30x_1^{II} - 80x_2^{II}$$

$$\text{subject to: } 4x_1^I + 4x_2^I + 7x_1^{II} + 7x_2^{II} \geq 1$$

$$8x_1^I + 5x_2^I + 5x_1^{II} + 5x_2^{II} \geq 3$$

$$-6x_1^I + 6x_2^I - 9x_1^{II} - 9x_2^{II} \leq 2$$

with: all variables non-ve.

This program is converted into standard form by

subtracting surplus variables  $x_3$  and  $x_4$  respectively, from the left-hand sides of the first two constraints; adding

a slack variable  $x_5$  to the LHS of the third constraint;

and then adding artificial variables  $x_6$  and  $x_7$  respectively, to the RHS of the first two constraints.

artificial variable  $x_6$  only to the LHS of the third constraint.

- we have

$$\text{min } z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 + Mx_6$$

$$\begin{aligned} \text{Subject to } & 3x_1 + 4x_2 + x_4 = 5 \\ & 5x_2 + x_3 + 6x_5 = 7 \\ & 8x_1 + 9x_3 - x_5 + x_6 = 2 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (1)$$

$$x_i \geq 0, i=1, 2, 3, 4, 5, 6.$$

This program is in standard form, with an initial feasible solution  $x_4 = 5, x_5 = 7,$

$$x_6 = 2, x_1 = x_2 = x_3 = 0.$$

System (1) has standard <sup>natural</sup> form, if we define

$$x = [x_1, x_2, x_3, x_4, x_5, x_6]^T \quad c = [1, 2, 3, 0, 0, M]^T$$

$$A = \begin{bmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 0 & 5 & 1 & 6 & 0 & 0 \\ 8 & 0 & 9 & 0 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} \quad x_0 = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Note: In this case,  $x_2$  can be used to generate the initial solution rather than adding an artificial variable to the second constraint to achieve the same result.

In general, whenever a variable appears in one and only one constraint equation, and there with a one coefficient, that variable can be used to generate part of the initial solution by first dividing the constraint equation by the one coefficient and then setting the variable equal to the RHS of the equation; an artificial variable need not be added to the equation.

we have

$$\max z = x_1 + x_2 + 0x_3 + 0x_4$$

$$\text{subject to } \begin{cases} x_1 + 5x_2 + x_3 = 5 \\ 2x_1 + x_2 + x_4 = 4 \end{cases} \quad (1)$$

$x_i \geq 0, i=1, 2, 3, 4$ , has  
since each constraint equation has a slack  
variable, no artificial variables are  
required.

An initial feasible solution is

$$x_3 = 5, x_4 = 4; x_1 = x_2 = 0$$

System (1) is in the standard matrix form

if we define

$$x = [x_1, x_2, x_3, x_4]^T; c = [1, 1, 0, 0]^T \Rightarrow c^T = [1, 0, 0]$$

$$A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}; b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

→ put the following program in standard  
matrix form

$$\min z = x_1 + 2x_2 + 3x_3$$

$$\text{subject to } 3x_1 + 4x_2 + 5x_3 \leq 5$$

$$5x_1 + x_2 + 6x_3 = 7$$

$$8x_1 + 9x_2 + 7x_3 \leq 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Sol: adding a slack variable  $x_4$  to the LHS  
of the first constraint, subtracting a  
surplus variable  $x_5$  from the LHS of the  
third constraint, and then adding an

This program is in standard form  
with an initial feasible solution.

$$x_1 = 0.25, x_2 = 1 \text{ and}$$

$$x_1 = x_2 = 0.$$

(2)  $\rightarrow$  put the following program in  
standard form

$$\text{Max } Z = 5x_1 + 2x_2$$

$$\text{subject to } 6x_1 + 2x_2 \geq 6$$

$$4x_1 + 3x_2 \geq 4$$

$$x_1 + 2x_2 \geq 4$$

$$\text{and } x_1 \geq 0, x_2 \geq 0.$$

and hence obtain an initial feasible solution.

(3)  $\rightarrow$  put the following program in standard  
matrix form.

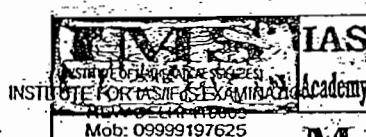
$$\text{Max } Z = x_1 + x_2$$

$$\text{subject to } x_1 + 5x_2 \leq 5$$

$$2x_1 + 3x_2 \leq 4$$

$$x_1 \geq 0, x_2 \geq 0.$$

Sol: Adding slack variables  $x_3$  and  $x_4$  respectively  
to the LHS of the constraints, and including  
these new variables with zero cost coefficient  
in the objective function.



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## MATHEMATICS

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problems:

- ① put the following program in standard form:  
Maximize  $Z = 8x_1 + 6x_2$   
subject to

$$0.20x_1 + 0.32x_2 \leq 0.25$$

$$x_1 + x_2 = 1$$

$$\text{and } x_1, x_2 \geq 0.$$

and hence obtain the initial feasible solution.  
S.O) To convert the first constraint  
into an equality,  
we have, add a slack variable  $x_3$  to the L.H.S.

L.H.S. ~~second constraint~~  
S.P. ~~second constraint~~  
Maximize  $Z = 8x_1 + 6x_2 + 0x_3$   
subject to  
the second constraint does not contain a slack  
equation, so we can add it to the L.H.S. of the second constraint  
variable, add an artificial variable  
variable, add an artificial variable  
to its L.H.S. Both new variables  
are included in the objective function,  
the slack variable with a zero cost  
coefficient and the artificial  
variable with a very large negative  
cost coefficient,  
yielding the program

$$\text{Maximize } Z = 8x_1 + 6x_2 + 0x_3 - Mx_4$$

Subject to

$$0.20x_1 + 0.32x_2 + x_3 = 0.25$$

$$x_1 + x_2 + x_4 = 1$$

$$\text{and } x_i \geq 0 \quad i=1, 2, 3, 4.$$

## \* Matrix form of L.P.P. in the standard form:

If linear programming is in standard form if the constraints are all modeled as equalities and if one feasible solution is known.

In matrix notation, standard form is

$$\text{Optimize (max/min)} Z = C^T X$$

$$\text{subject to: } AX = B$$

$$\text{and } X \geq 0$$

where  $X$  is the column vector of unknowns, including all slack, surplus and artificial variables;  $C^T$  is the row vector of the corresponding costs;  $A$  is the coefficients matrix of the constraints equations; and  $B$  is the column vector of the right-hand sides of the constraint equations.

If  $X_0$  denotes the vector of slack and artificial variables only, then an initial feasible solution is given by  $X_0 = B$ , where it is understood that all variables in  $X$  not included in  $X_0$  are assigned zero values.



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(37)

Accordingly, such variables are incorporated into the objective function with zero coefficients.

Artificial variables, however, do change the nature of the constraints, since they add to only one side of an inequality, the new system is equivalent to the old system of constraints if and only if the artificial variables are zero. To guarantee such assignments as the optimal solution, (in contrast to the initial ~~initial~~ INSTITUTE OF MATHEMATICAL EDUCATION NEW DELHI-110025, Mob. 09999197625), artificial variables are incorporated into the objective function with very large positive coefficients in a minimization program (or) very large negative coefficients in a maximization program. These coefficients, denoted by either  $M$  or  $-M$ , where  $M$  is understood to be a large positive number, represent the (severe) penalty incurred for making a unit assignment to the artificial variables.

Ex:-

slack variables ( $\leq$ )  $3x_1 + 8x_2 \leq 10$

$x_3$  slack variable converted into

$$3x_1 + 8x_2 + x_3 = 10$$

surplus variables ( $\geq$ )

$$2x_1 + 3x_2 \geq 11$$

$x_3$  surplus variable converted into

$$2x_1 + 3x_2 - x_3 = 11$$

Note:- (On restricted variables).

A variable which is unrestricted in sign (i.e. positive, negative or zero) is equivalent to the difference between two non-negative variables.

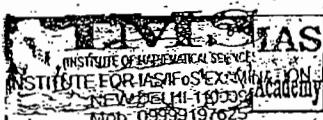
Thus if  $x_j$  is unrestricted in sign, it can be replaced by  $(x_j^1 - x_j^2)$ , where  $x_j^1$  and  $x_j^2$  are both non-negative i.e.  $x_j = x_j^1 - x_j^2$ , where  $x_j^1 \geq 0$  and  $x_j^2 \geq 0$ .

\* Generating an initial feasible solution:-

After all linear constraints (with non-negative right-hand sides)

have been transformed into

equations by introducing slack and surplus variables where necessary, add a new variable, called an artificial



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Eff

Let the constraints of a General L.P.P be  $\sum_{j=1}^n a_{ij}x_j \geq b_i$   $i = k+1, k+2, \dots$

Then, the non-negative variables anti which satisfy  $\sum_{j=1}^n a_{ij}x_j = b_{k+1} = b_k$   $i = k+1, k+2, \dots$

are called surplus variables.

\* Two forms of L.P.P.

Standard form



Maximize  $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to the constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$x_1, x_2, \dots, x_n \geq 0$  is known

as standard form.

The characteristics of this form are:

(i) The objective function is of the maximization type.

The minimization of a function (ex), is equivalent to the maximization

The new system is

$$\begin{aligned}x_1 + 2x_2 + x_3 &= 3 \\4x_1 + 5x_2 - x_4 &= 6 \\7x_1 + 8x_2 &= 15\end{aligned}\quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{--- (1)}$$

If now artificial variables  $x_5$  and  $x_6$  are respectively added to the RHS of the last two constraints for system (1), the constraint without a slack variable,

the result is

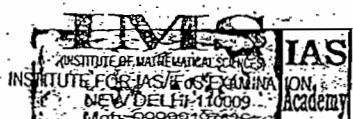
$$\begin{aligned}x_1 + 2x_2 + x_3 &= 3 \\4x_1 + 5x_2 - x_4 + x_5 &= 6 \\7x_1 + 8x_2 + x_6 &= 15\end{aligned}\quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{--- (2)}$$

A non-negative initial solution to

the system (2) is  $x_3 = 3$ ,  $x_5 = 6$ ,  $x_6 = 15$   
and  $x_1 = x_2 = x_4 = 0$ .

### \*Penalty costs:

The introduction of slack and surplus variables alters neither the nature of the constraints nor the objective.



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(36)

variable, to the left-hand side of each constraint equation that does not contain a slack variable. Each constraint equation will then contain either one slack variable or one artificial variable. A non-negative initial solution to this new set of constraints is obtained by setting each slack variable and each artificial variable equal to the right-hand side of equation in which it appears and setting all other variables, including the surplus = ~~variables~~, equal to zero.

Ex: The constraints

$$x_1 + 2x_2 \leq 3$$

$$4x_1 + 5x_2 \geq 6$$

$$x_1 + 8x_2 = 15$$
 is third

formed into a system of equations by adding a slack variable  $x_3$  to the left-hand side of the first constraint and subtracting a surplus variable  $x_4$  from the LHS of the second constraint.

if not, the procedure of jumping from one extreme point to another is repeated.

Since the number of vertices is finite, simplex method leads to an optimal vertex in a finite number of steps.

If at any stage, the procedure leads us to a vertex which has an edge leading to infinity and if the objective function value can be further improved by moving along that edge, the simplex method tells us that there is an unbounded solution.

However, to discuss simplex method, we need to know some basic concepts and results.

Now we shall introduce two important forms of a general linear programming problem, namely the standard form and canonical form. Also, we shall introduce some special types of variables namely the slack and surplus variables. We identify the nature of solutions such as feasible solutions, basic solutions, basic feasible solutions, optimal solutions etc.

#### Slack and Surplus variables

Let the constraints of General LPP be  $\sum_{j=1}^n a_{ij}x_j \leq b_i$ ,  $i=1, 2, \dots, k$

Then, the non-negative variables  $x_{n+i}$  which satisfy  $\sum a_{ij}x_i + x_{n+i} = b_i$ ,  $i=1, 2, \dots, k$ , are called slack variables.

The linear constraints of the problem are represented by half-spaces determined by the hyperplanes.

The optimal solution if it exists, corresponds to some vertex of the convex region.

The problem essentially is that of determining which particular vertex of the convex region corresponds to the optimal solution.

No method is so far known that will locate this optimal vertex in a single step and one has to adopt an iterative procedure for locating the same.

The best known and the most widely used procedure for locating the optimal vertex is the "Simplex method" that determines IIMSS vertex in a finite number of steps.

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The simplex method is a procedure which consists in moving step by step from a given vertex to an optimal vertex. At each step, it is possible to move only to an adjacent vertex.

The method consists of moving along an edge of the convex region of feasible solutions from one vertex to an adjacent one.

Of all the adjacent vertices the one yielding an improved value of the objective function over that of the preceding vertex, is chosen.

At each vertex point, the method tells us whether that extreme point is optimal, and

The following four foods are available for consumption: brownies, chocolate ice cream, Cola and Pineapple cheese cake. Each brownie costs 5Rs., each scoop of chocolate ice cream cost 2Rs., each bottle of Cola costs 3Rs. and each piece of Pineapple cheese cake costs 8Rs. Each day you require atleast 500 calories, 6 units of Chocolate, 10 units of sugar and 8 units of fat. The nutritional content per unit of each food is shown in the table below. formulate a linear programming model that can be used to satisfy your daily nutritional requirement at the minimum cost.

	<u>Calories</u>	<u>Chocolate</u>	<u>Sugar</u>	<u>Fat</u>
Brownie	400	3	2	2
Chocolate ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheese cake (1 piece)	500	0	4	5

Define the decision variables:

$x_1$  = number of brownies eaten daily

$x_2$  = number of scoops of chocolate ice cream eaten daily

$x_3$  = bottles of Cola drunk daily

$x_4$  = pieces of Pineapple cheese cake eaten daily.

Objective : To minimize the cost of diet.

$$\text{Total cost of diet} = 5x_1 + 2x_2 + 3x_3 + 8x_4$$

Thus the objective function is to

$$\text{minimize } Z = 5x_1 + 2x_2 + 3x_3 + 8x_4$$

Daily Calorie intake is  $400x_1 + 200x_2 + 150x_3 + 500x_4$

Minimum requirement is 500 units,

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$$

Solve Define the decision variables  
 $x_1$  = No. of units of product  $P_1$  to be produced

$x_2$  = No. of units of product  $P_2$  to be produced

$x_3$  = No. of units of product  $P_3$  to be produced

$x_4$  = No. of units of product  $P_4$  to be produced

Then the revenue  $Z$  that results from a given production program is

$$Z = 6x_1 + 5x_2 + 3x_3 + 7x_4$$

Since we cannot use more of each resource than is available and only 50 kg. of malt is available.

we have  $x_1 + x_2 + x_3 + x_4 \leq 50$

Similarly, for Hops

$$2x_1 + 2x_2 + x_3 + 2x_4 \leq 150$$

for yeast,

$$x_1 + x_2 + x_3 + 4x_4 \leq 80$$

finally, we impose the non-negativity restriction

on each product, i.e.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

The problem becomes

$$\text{Max } Z = 6x_1 + 5x_2 + 3x_3 + 7x_4$$

Subject to the constraints

$$x_1 + x_2 + x_3 + x_4 \leq 50$$

$$2x_1 + 2x_2 + x_3 + 2x_4 \leq 150$$

$$x_1 + x_2 + x_3 + 4x_4 \leq 80$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Consider a diet problem in which you want to minimize the cost of meeting a set of requirements. Suppose your diet requires that all the food you eat come from one of the four basic food groups say chocolate cake, ice cream, soda and cheese cake. At present,

The extreme points of the feasible region are

$$C_1(4,0,0), D_1\left(\frac{2}{3}, \frac{12}{7}, 0\right), A_2(0,4,0)$$

$$D_2\left(0, \frac{8}{3}, \frac{8}{3}\right), B_3(0,0,4), D_3\left(4, 0, \frac{8}{3}\right)$$

The objective function  $Z$  is maximum at the point  $D_2\left(0, \frac{8}{3}, \frac{8}{3}\right)$ .

and the maximum value of  $Z$  is  $\frac{88}{3}$ .

### Mathematical formulation in more than three variables:

- A company makes four products denoted by  $P_1, P_2, P_3$  and  $P_4$ . These products are made using the resources of water, malt, hops and yeast. Company has a free supply of water. Therefore, so it is the amount of other resources that restricts production capacity. The following table gives the amount of each resource required in the production of 1 unit of each product, the amount of each resource available, and the revenue received for one unit of each product. The problem faced by the company is to decide how much of each product should it make in order to maximize its revenue.

	$P_1$	$P_2$	$P_3$	$P_4$	Available.
Malt	1	1	0	3	- 50 kg
Hops	2	1	2	1	1.50 kg
Yeast	1	1	1	4	80 kg
Revenue	Rs. 6	Rs. 5	Rs. 3	Rs. 7	

→ Maximize  $Z = 0.5x_1 + 6x_2 + 5x_3$ .

Subject to the constraints

$$4x_1 + 6x_2 + 3x_3 \leq 24$$

$$x_1 + 1.5x_2 + 3x_3 \leq 12$$

$$3x_1 + x_2 \leq 12$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

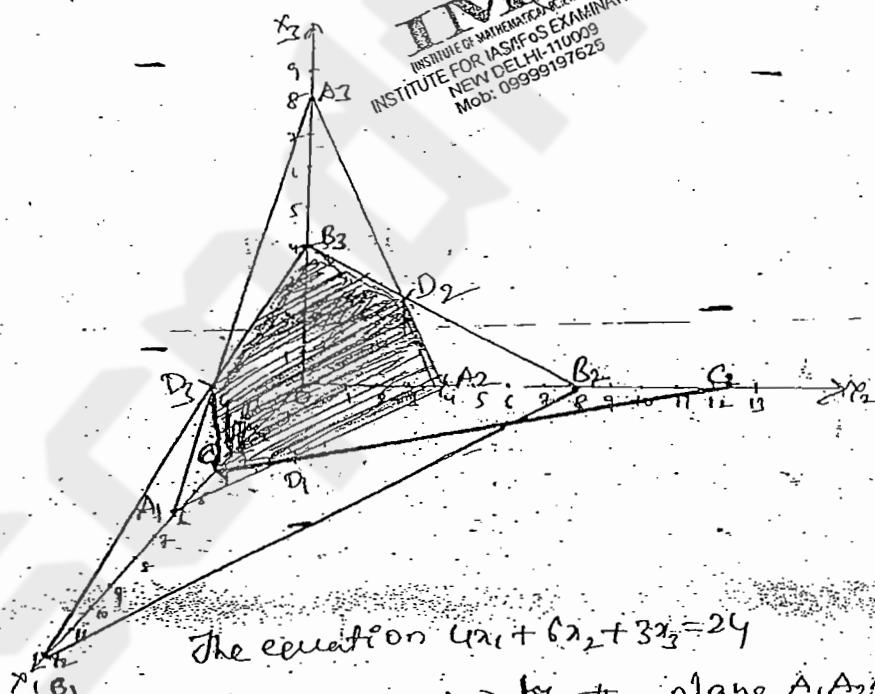
Soln: The equations

$$4x_1 + 6x_2 + 3x_3 = 24 \Rightarrow (8, 0, 0), (0, 4, 0), (0, 0, 8)$$

$$x_1 + 1.5x_2 + 3x_3 = 12 \Rightarrow (12, 0, 0), (0, 8, 0), (0, 0, 4)$$

represent planes -

and  $3x_1 + x_2 = 12$  is a line in  $x_1, x_2$  plane  
as shown in the figure



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The equation  $4x_1 + 6x_2 + 3x_3 = 24$

is represented by the plane  $A_1A_2A_3$

The ear  $x_1 + 1.5x_2 + 3x_3 = 12$  is represented

by  $B_1B_2B_3$

The ear  $3x_1 + x_2 = 12$  represented by  
the line  $C_1C_2$

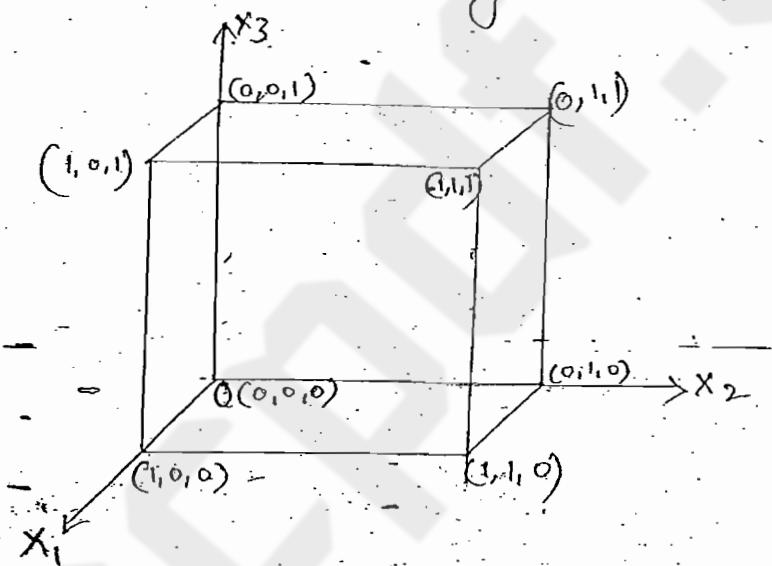
The objective function,  $Z$  is maximum at the point  $(1,0,0)$  and hence the maximum value of  $Z = 1$ .

→ Maximize  $Z = x_1 + x_2 + x_3$   
subject to the constraints:

$$x_1 \leq 1, x_2 \leq 1, x_3 \leq 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Sol<sup>n</sup>: If we draw its graph, it is the interior of a cube having its side of unit length as shown in the figure.



There are 8 vertices of the cube and these are the extreme points.

The objective function  $Z$  is maximum at the point  $(1,1,1)$ .

∴ the maximum value of  $Z = 3$ .

→ Solve the following LPP Graphically.

Maximize  $Z = x_1$

subject to the constraint

$$x_1 + x_2 + x_3 \leq 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Soln: Since there are 3 variables  $x_1, x_2, x_3$ , we are to draw the graph in a 3-dimensional space.

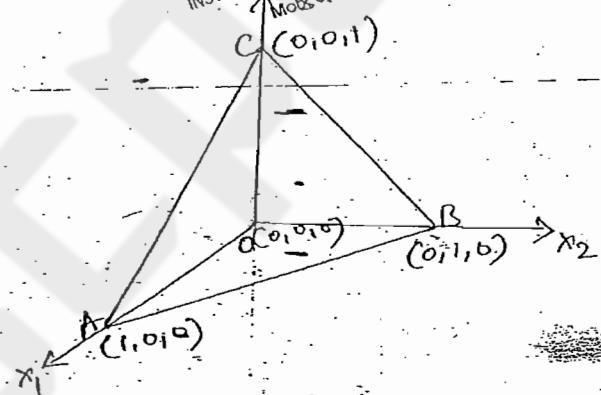
The equation  $x_1 + x_2 + x_3 = 1$  represents a plane.

If we take the intersection of this plane with the co-ordinate axes,

we get the points

A(1, 0, 0) ~~WVS~~  
as shown INSTITUTE OF MATHEMATICAL SCIENCES  
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and C(0, 0, 1). figure.



Since  $x_1 + x_2 + x_3 \leq 1$

∴ the feasible region is the portion containing the origin and satisfying the non-negative restriction.

The extreme points are given by  
(0, 0, 0), A(1, 0, 0), B(0, 1, 0) & C(0, 0, 1)

and Rs. 500. Formulate it as LPP to find the number of days should the company run each plant during the month so that the production cost is minimized while still meeting the market demand.

Soln: Suppose  $x_1, x_2, x_3$  be the number of days per month in which the company runs the plants I, II and III respectively. Amount of product A produced by three plants is

$$3000x_1 + 1000x_2 + 2000x_3$$

The demand of the product A is 24,000 bottles. We want to find  $x_1, x_2, x_3$  such that the market demand must be fulfilled.

$$\therefore \text{we must have } 3000x_1 + 1000x_2 + 2000x_3 \geq 24,000$$

Similarly for product B & C, we have

$$1000x_1 + 1000x_2 + 500x_3 \geq 16,000$$

$$\text{and } 2000x_1 + 4000x_2 + 3000x_3 \geq 48,000.$$

As the number of days cannot be negative.

$$x_1 \geq 0, x_2 \geq 0, \text{ and } x_3 \geq 0$$

Total cost of running the plants I, II and III

$$= 600x_1 + 400x_2 + 500x_3$$

We are interested in minimizing the total cost.

Hence the L.P.P. can be formulated as

$$\text{minimize } Z = 600x_1 + 400x_2 + 500x_3$$

subject to the constraints -

$$3000x_1 + 1000x_2 + 2000x_3 \geq 24,000$$

$$1000x_1 + 1000x_2 + 500x_3 \geq 16,000$$

$$2000x_1 + 4000x_2 + 3000x_3 \geq 48,000$$

If  $x_j$  units of component  $M_j$  are produced,  
the weekly profit  $Z$  is given by

$$Z = 6x_1 + 4x_2 + 7x_3$$

We wish to find the values of the variables which will satisfy all the constraints, the non-negativity restrictions and maximize  $Z$ . Thus, the mathematical formulation of the linear programming problem is:

$$\text{Maximize } Z = 6x_1 + 4x_2 + 7x_3$$

subject to the constraints

$$6x_1 + 5x_2 + 3x_3 \leq 100$$

$$3x_1 + 4x_2 + 9x_3 \leq 15$$


  
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→ A cold drinks company has three bottling plants, located at two different places. Each plant produces three different drinks A, B, and C. The capacities of three plants, in number of bottles per day are as follows:

	product A	product B	product C
Plant I	3000	1000	2000
Plant II	1000	1000	4000
Plant III	2000	500	3000

A market survey indicates that during any particular month there will be a demand of 24,000 bottles of A; 16,000 bottles of B and 48,000 bottles of C. The operating costs, per day, of running plants I, II & III are respectively Rs. 600, Rs 400.

an LPP such that the total profit is maximum.

Sol: Suppose  $x_j$  be the number of units of component  $M_j$  produced per week,  $j=1, 2, 3$ . We want to find the values of  $x_1, x_2, x_3$  which maximize the total profit.

Since the amount of steel, amount of brass and the labour are limited, we cannot arbitrarily increase the out put of any component.

Consider first the restriction imposed by the availability of steel.

Amount of steel used is

$$6x_1 + 5x_2 + 3x_3 \text{ per week.}$$

because, 6 kg. are required for each unit of component  $M_1$ , 5 kg are required for each unit of component  $M_2$  and 3 kg. for each unit of component  $M_3$ .

Since the total amount of steel available is restricted to 100 kg.

$$\therefore 6x_1 + 5x_2 + 3x_3 \leq 100.$$

Similarly for brass

$$3x_1 + 4x_2 + 9x_3 \leq 75.$$

As the labour is restricted to 20 man-weeks

$$\therefore x_1 + 2x_2 + x_3 \leq 20$$

Since we cannot produce negative quantities.

$$\therefore x_1 \geq 0, x_2 \geq 0 \text{ and } x_3 \geq 0$$

## Optimization in more than two variables

We now extend the method of a linear programming problem in more than two variables and further try the same methods for a L.P.P. in more than three variables. This will lead us to discover the limitations of the graphical method in solving the L.P.P. in more than three variables and hence the need for a more scientific algebraic method which is known as Simplex method to be discussed later.

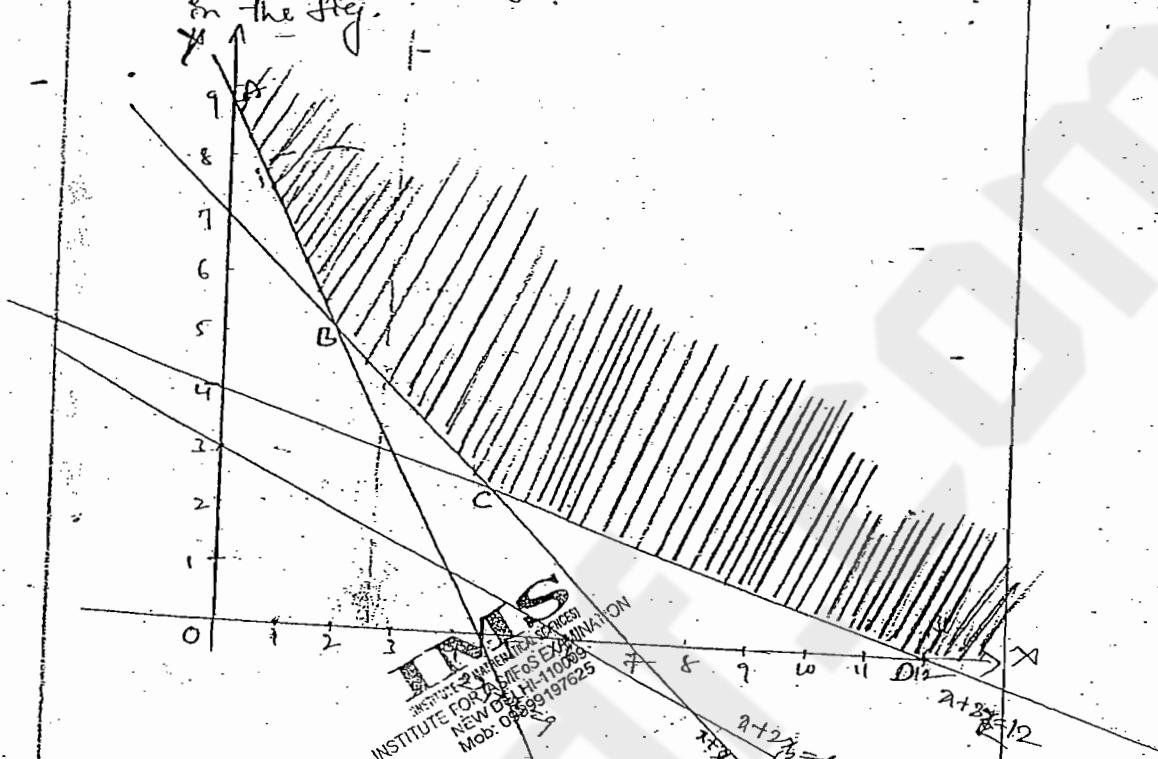
### Mathematical formulation in three variables:

→ A small scale ~~industry~~ produces three types of machine components  $M_1$ ,  $M_2$  and  $M_3$  made of steel and brass. The availability of steel, brass required for each component and the number of man-weeks of labour required to manufacture and assemble one unit of each component are as follows.

	$M_1$	$M_2$	$M_3$	Availability
Steel	6	5	2	100 kg
Brass	3	4	9	75 kg
Man-weeks	1	2	1	20 weeks

This labour is restricted to 20 man-weeks, steel is restricted to 100 kg per week and the brass to 75 kg per week. The industrialists' profit on each unit of  $M_1$ ,  $M_2$  and  $M_3$  is Rs/-6, Rs/-4 and Rs/-7 respectively. Give its mathematical formulation.

Q1 The graph of the given constraints as shown in the fig.



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The shaded region ABCD is the feasible region corresponding to the above constraints. The constraint  $x_1 + 2x_2 \geq 6$  does not intersect the feasible region.

If we remove the constraint  $x_1 + 2x_2 \geq 6$ , the feasible region remains unchanged. Such a constraint is called Redundant Constraint. Hence, the constraint  $x_1 + 2x_2 \geq 6$  is redundant.

The objective function  $Z = 5x_1 + 8x_2$   
consider the objective function lines  
 $5x_1 + 8x_2 = 60$ , it is given by P<sub>1</sub>Q<sub>1</sub>  
and  $5x_1 + 8x_2 = 40$ , it is given by P<sub>2</sub>Q<sub>2</sub>.

As  $40 < 60$ , we move from P<sub>2</sub>Q<sub>2</sub> to P<sub>1</sub>Q<sub>1</sub> parallel to itself until the point C of the feasible region is touched by this line.

The point so obtained gives the optimal value of Z. Minimum Z is obtained at the point C(4,2) and the minimum value of Z = 42.50.

The extreme points of CB are

$$C(0, 15) \text{ and } B\left(\frac{15}{2}, \frac{45}{4}\right)$$

and the value of  $Z$  at  $(0, 15)$  is.

$$15(0) + 30(15) = 450.$$

value of  $Z$  at  $\left(\frac{15}{2}, \frac{45}{4}\right)$  is

$$15\left(\frac{15}{2}\right) + 30\left(\frac{45}{4}\right) = 450.$$

i.e. maximum value of  $Z$  at both the extreme points is same and it is 450. If we take any point on the line segment joining C and B, then the value of  $Z$  at that point will also be the same.

with the help of the extreme points. also we can see that the ~~optimal~~ value of  $Z$  is 450.

$$\text{At } O(0, 0), Z = 0$$

$$\text{At } A(12, 0), Z = 360$$

~~$$\text{At } B\left(\frac{15}{2}, \frac{45}{4}\right) = 450$$~~

$$\text{At } C(0, 15), Z = 450.$$

$\therefore$  Maximum value of  $Z = 450$  at both the points  $(0, 15)$  and  $\left(\frac{15}{2}, \frac{45}{4}\right)$

$$\text{Max } Z = 2x_1 + 2x_2$$

subject to the constraints

$$x_1 + 3x_2 \leq 4$$

$$x_1 + 2x_2 \leq 6$$

$$x_1 \leq 3$$

$$x_1, x_2 \geq 0$$

$$\text{Minimize } Z = 5x_1 + 8x_2$$

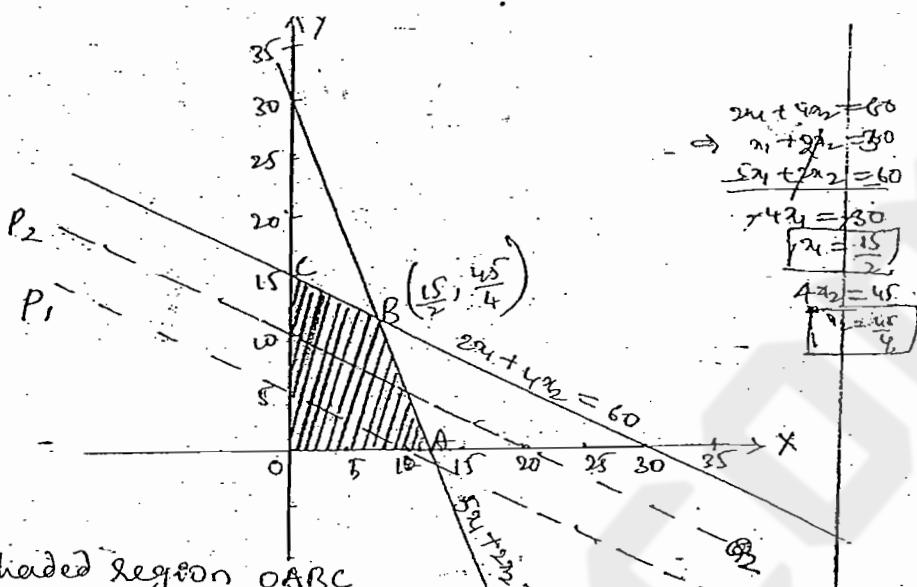
subject to the constraints

$$x_1 + 2x_2 \geq 6$$

$$x_1 + 2x_2 \geq 8$$

$$x_1 + 3x_2 \geq 12$$

$$2x_1 + x_2 \geq 9 ; x_1, x_2 \geq 0$$



The shaded region OABC represents the set of feasible solutions.

To locate a point on the feasible region which maximizes the objective function.

If we give a constant value, say 150 to  $Z$ , then the objective function may be written as

$$15x + 30y = 150$$

The line is given by  $P_1Q_1$ .

Now, we give another value 300 to  $Z$ ,

$$\text{i.e., } 15x + 30y = 300$$

The line is  $P_2Q_2$ .

The lines  $P_1Q_1$  and  $P_2Q_2$  both lie within the feasible region.

Increasing the value of  $Z$  and drawing different lines, we find that the objective function line coincides with the line CB.

Hence every point on the line segment CB of the feasible region provides the optimal value of  $Z$ .

In this case there is a common feasible region which satisfy the constraints but does not satisfy the non-negativity restriction  $x_1 \geq 0, x_2 \geq 0$ .

Hence there is no feasible region and the given problem is infeasible.

$$\text{Max } Z = 10x_1 + 15x_2$$

Subject to the constraints

$$x_1 \geq 5$$

$$x_2 \leq 10$$

$$x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

An Alternative optimum solution

A linear programming problem may have more than one optimal solution. This happens when the objective function line is parallel to a binding constraint (i.e. a constraint that is satisfied in the equality) ~~and~~ the optimal solution.

for example:

$$\text{Maximize } Z = 15x_1 + 30x_2$$

Subject to the constraints

$$2x_1 + 4x_2 \leq 60$$

$$5x_1 + 2x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

Sol<sup>n</sup>: Graph of the given constraints as shown in the figure.

$$2x_1 + 4x_2 = 60$$
$$\Rightarrow (0, 15) \text{ & } (30, 0)$$
$$5x_1 + 2x_2 = 60$$
$$\Rightarrow (0, 30) \text{ & } (12, 0)$$

In the graph, there is no point  $(x_1, x_2)$  which satisfy both the constraints. In this case the two constraints are inconsistent. There is no feasible region and hence the given problem is infeasible.

$\rightarrow$  Solve the following LPP:  $\rightarrow$  Maximize  $Z = 5x_1 + 2x_2$   
 Maximize  $Z = x_1 + x_2$  subject to the constraints  
 Subject to the constraints  $x_1 + x_2 \leq 2$   
 $x_1 + x_2 \leq 1$   $-3x_1 + 3x_2 \geq 12$   
 $-3x_1 + x_2 \geq 3$   $x_1, x_2 \geq 0$ .  
 $x_1 \geq 0, x_2 \geq 0$ .

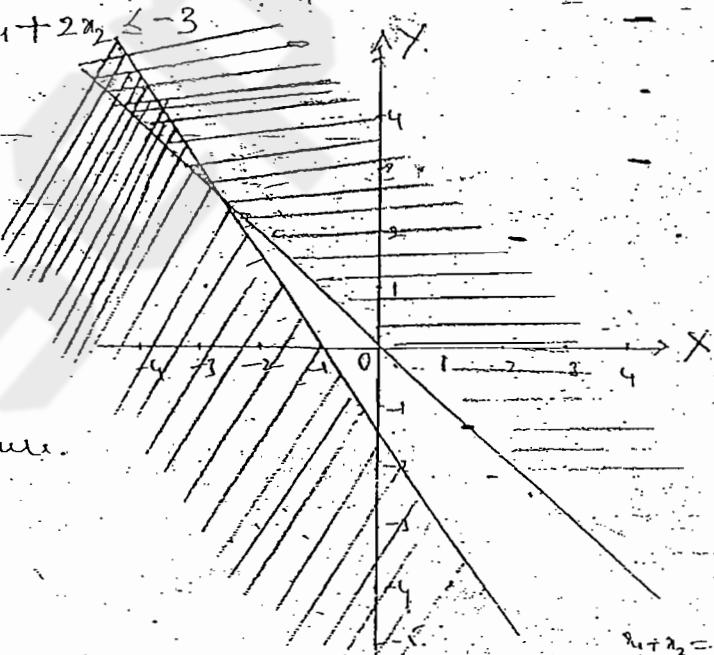
$\rightarrow$  Consider another example where the constraints are consistent but still the problem is infeasible.

$\rightarrow$  Maximize  $Z = 3x_1 - 2x_2$   
 Subject to the constraints

$$\begin{aligned} x_1 + x_2 &\geq 0 \\ -3x_1 + 2x_2 &\leq -3 \end{aligned}$$

Graph of the given constraints as shown:

in the figure.



$$\begin{aligned} x_1 + x_2 &= 0 \\ -3x_1 + 2x_2 &= -3 \\ (0, 1.5) &\text{ & } (-1, 0) \end{aligned}$$

Now consider linear programming problem which has no feasible region. In such a situation we say that the given problem is infeasible.

### No feasible solution:

When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the linear programming problem exists.

Maximize  $Z = 3x_1 - 2x_2$

Subject to the constraints

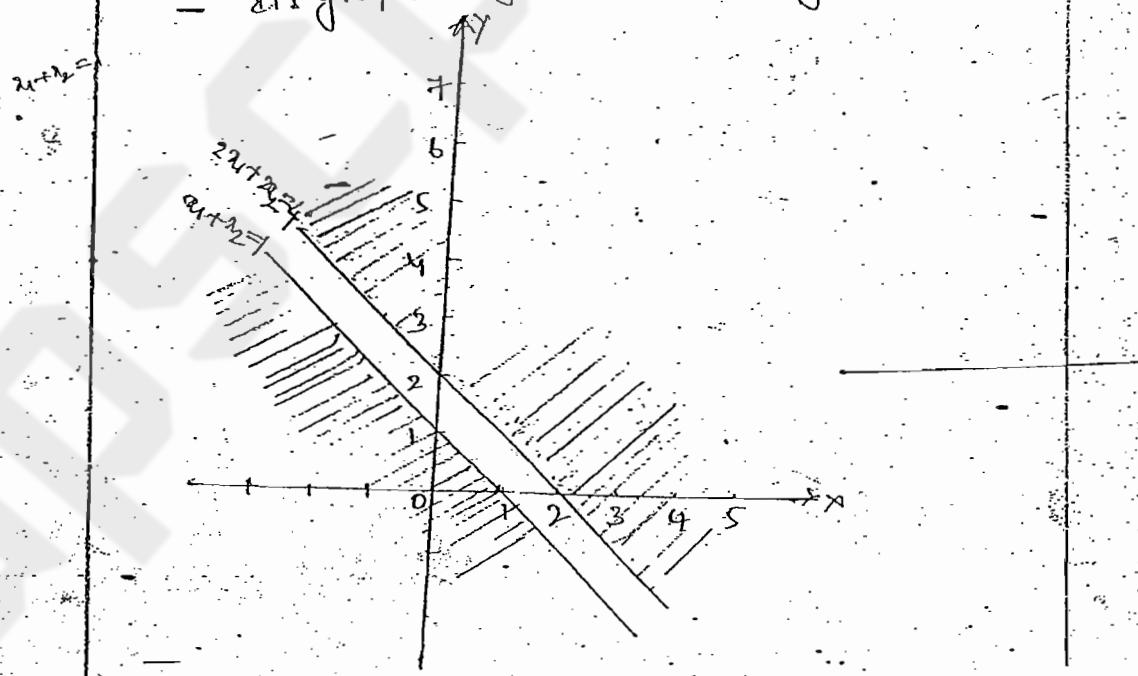
$$x_1 + x_2 \leq 4$$

$$2x_1 + 2x_2 \leq 7$$

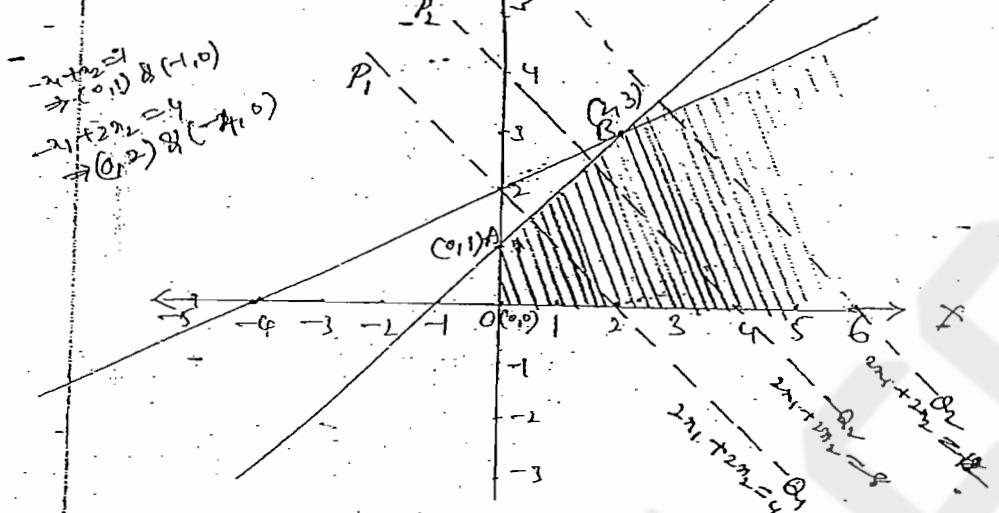
$$x_1, x_2 \geq 0$$

Sol: Clearly, the desired number pair  $(x_1, x_2)$  lies in the first quadrant only.

Its graph is given in the figure.



Graph of these constraints as shown in the figure.



The shaded portion is the feasible region and is unbounded. The extreme points are  $O(0,0)$ ,  $A(0,1)$  and  $B(2,3)$ .

At these points, the value of  $Z$  is finite.

As  $x_1$  and  $x_2$  both can approach to infinity, therefore the value of  $Z$  approaches to  $\infty$ .

$\therefore$  we say that the given problem is

- unbounded.

The objective function lines indicate that  $Z \rightarrow \infty$ .

H.W.

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to the constraints

$$x_1 - x_2 \leq 1$$

$$x_1 + 2x_2 \leq 3$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

$$\rightarrow \text{Max } Z = x_1 + 0.75x_2$$

Subject to the constraints

$$x_1 - x_2 \geq 0$$

$$-x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

In the previous problems, we have obtained a feasible region in each case. This feasible region in each case represents a convex set. Convex sets may be bounded or unbounded.

Refer to the graph of the problem discussed in (1) and (2). The convex polyhedron OABC is bounded.

In problem (3), the feasible region is not bounded. i.e., it is unbounded, because  $x_1$  and  $x_2$  both can go upto infinity. Since the problem is of minimization, the minimum value of  $Z$  exists, but  $\max Z = \infty$ .

In problem (4), the feasible region is the triangle ABC and it is bounded.

Now consider an example where the feasible region is unbounded and the maximum  $Z$  is infinite.

$$\rightarrow \text{Max } Z = 2x_1 + 2x_2$$

subject to the constraints

$$x_1 - x_2 \geq -1$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Soln: First constraint can be written as  
 $x_1 + x_2 \leq 1$ .

∴ we have  $\text{Max } Z = 2x_1 + 2x_2$

Subject to  $x_1 + x_2 \leq 1$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

### Bounded and unbounded sets

- Let  $S \subseteq \mathbb{R}^n$ . Then  $S$  is said to be bounded if there exists numbers  $k$  and  $K$  such that if  $x = (x_1, x_2, \dots, x_n) \in S$
- $$k \leq x_i \leq K$$
- The numbers  $k$  and  $K$  are known as a lower bound and upper bound respectively.
- We say that  $S$  is bounded below only if there exists a lower bound  $k$  of  $S$ . This is not a bounded set.
- Similarly,  $S$  is bounded above only if there exists an upper bound  $K$ .
- Also, there are sets which have neither a lower bound nor an upper bound.
- The sets which are bounded below only or bounded above only or neither bounded below nor bounded above are known as unbounded sets.
- In other words, the sets which are not bounded are called unbounded sets.
- Geometrically, a subset of  $\mathbb{R}^2$  is said to be bounded if its region can be enclosed by a closed curve say, a circle or a triangle.
- A set is said to be unbounded if its region cannot be enclosed in a closed curve.

we find that the line corresponding to the minimum value of  $Z$  passes through the point C.

Therefore, minimum  $Z$  is obtained at the point  $C(8,5)$  and minimum value of  $Z = 376$ .

We can verify minimum value of  $Z=376$  by calculating the value of  $Z$  at the extreme points.

$$\text{At } A(5,10) \quad Z = 400$$

$$\text{At } B(8,10) \quad Z = 496$$

$$\text{At } C(8,5) \quad Z = \underline{\underline{376}}$$

$$\therefore x=8 \text{ and } y=5$$

$$\text{and } \underline{\underline{Z=376}}$$

→ Solve the LPP Graphically:

$$\rightarrow \text{Minimize } Z = 3x_1 + 2.5x_2 \rightarrow \text{Minimize } Z = 4x_1 + 2x_2$$

subject to the constraints

subject to the constraints

$$2x_1 + 4x_2 \geq 40$$

$$2x_1 + 2x_2 \geq 2$$

$$3x_1 + 2x_2 \geq 50$$

$$3x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

$$\rightarrow \text{Minimize } Z = 20x_1 + 10x_2 \rightarrow \text{Minimize } Z = -2x_1 + 2x_2$$

subject to the constraints

subject to the constraints

$$x_1 + 2x_2 \leq 40$$

$$-2x_1 + 3x_2 \leq 10$$

$$3x_1 + 2x_2 \leq 30$$

$$x_1 + x_2 \leq 6$$

$$4x_1 + 3x_2 \leq 60$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

(4)  $\rightarrow$  Minimize  $Z = 32x_1 + 24x_2$   
subject to the constraints

$$x_1 \leq 8$$

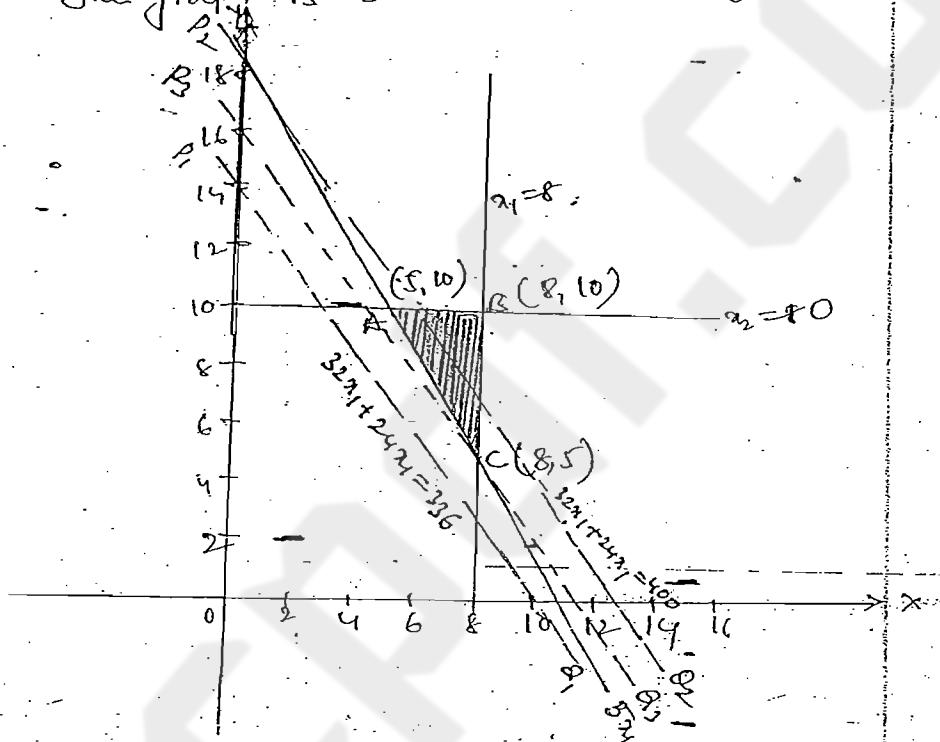
$$x_2 \leq 10$$

$$5x_1 + 3x_2 \leq 55$$

$$x_1, x_2 \geq 0$$

Sol<sup>n</sup>, let  $x_1 = 8$  and  $5x_1 + 3x_2 = 55 \Rightarrow (0, \frac{55}{3})$  and  $(11, 0)$

The graph is as shown in the figure



The feasible region is given by

$\triangle ABC$  with  $A(0, 0)$ ,  $B(8, 10)$

and  $C(8, 5)$  as the extreme points.

The objective function  $Z = 32x_1 + 24x_2$

Consider the objective function line

$$32x_1 + 24x_2 = 400 \text{ and } 32x_1 + 24x_2 = 336$$

and lines parallel to them.

point - C (2,4) : and the minimum value of  $Z$

$$Z = 52$$

$$\therefore x = 2 ; y = 4.$$

Isocost method:

we give a constant value, say 30 to  $Z$  and

draw the line  $P_1Q_1$  corresponding to  $6x + 10y = 30$

NOW, we give another value 60 to  $Z$  and draw

the line  $P_2Q_2$  corresponding to  $6x + 10y = 60$ .

At  $30 \leq 60$ , we move from  $P_2Q_2$  to  $P_1Q_1$  parallel

to itself until the farthest point C of the feasible region is touched by this line. The

point C so obtained gives the optimal value of  $Z$ .

The point C is the intersection of the equations

$$x + 2y = 10$$

$$6x + 10y = 60$$

Solving these equations, we find co-ordinates of

C as (2,4).

The optimal value of  $Z = 6x + 10y = 52$ .

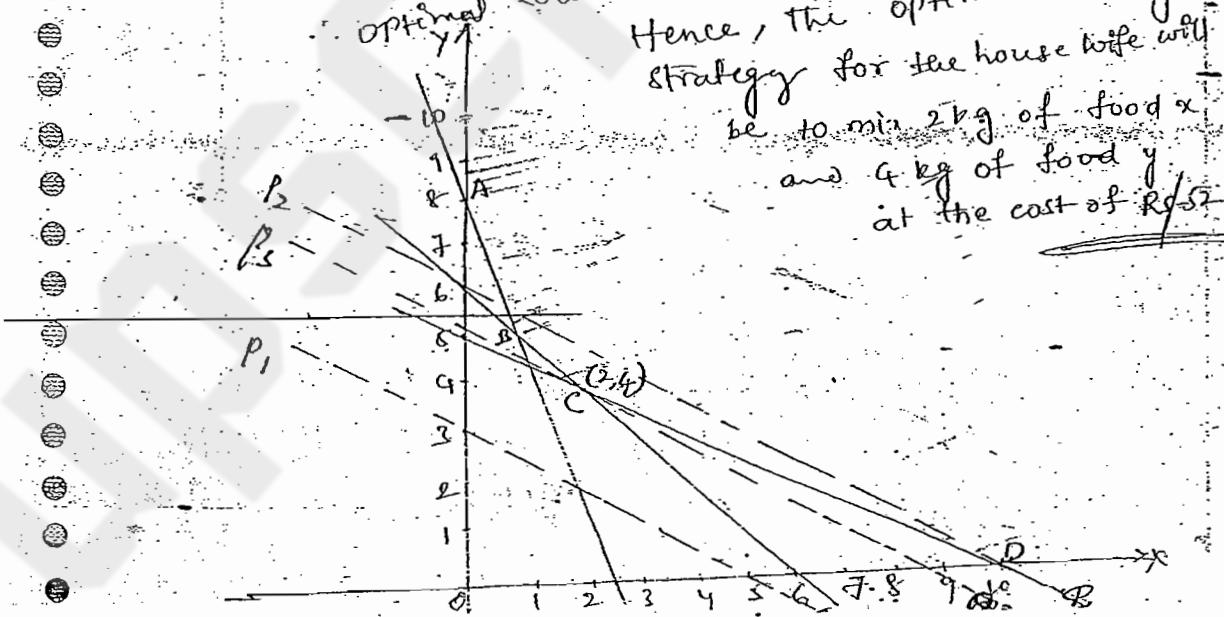
The optimal solution is  $x = 2 ; y = 4$ .

Hence, the optimal mixing strategy for the house wife will

be to mix 2 kg of food  $x$

and 4 kg of food  $y$

at the cost of Rs 52.



$$3x + y \geq 8$$

$$x, y \geq 0$$

We draw the lines  $x + 2y = 10$ ,  $2x + 2y = 12$  &  $3x + y = 8$ .

Since each of the constraints is greater than or equal to type, the points  $(x, y)$  satisfying all of them will form the region that falls towards the right of each of these straight lines.

Here the feasible region is open with vertices

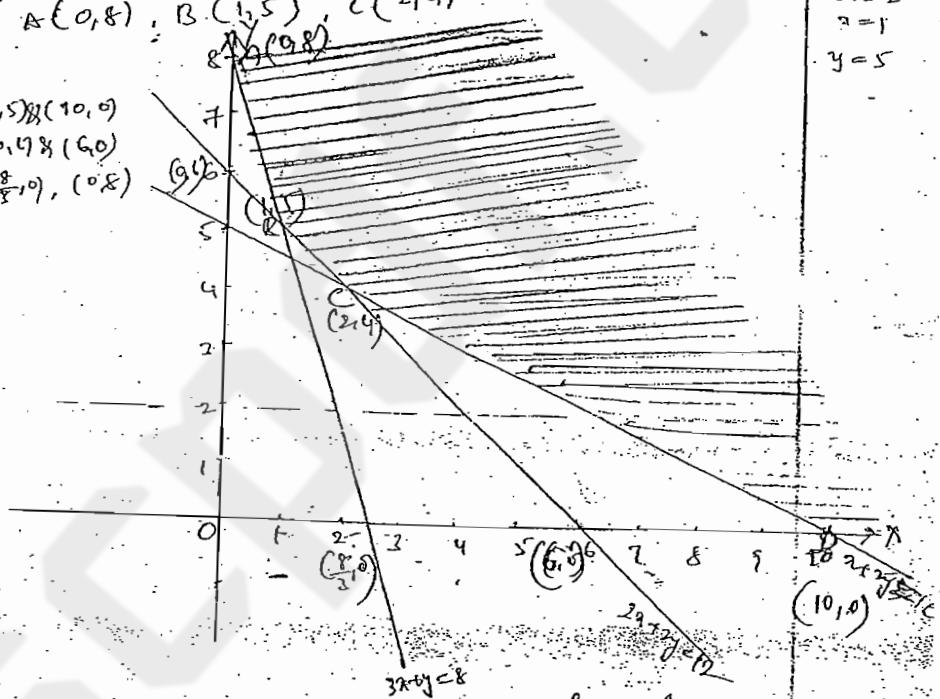
A, B, C and D.  
i.e., A(0, 8), B(1, 5), C(2, 4) and D(10, 0).

$$\begin{aligned}x + 2y &= 12 \\x + y &= 10 \\2x + 2y &= 12 \\x &= 2 \\y &= 1 \\y &= 5\end{aligned}$$

$$x + 2y = 10 \Rightarrow (0, 5) \text{ & } (10, 0)$$

$$x + y = 12 \Rightarrow (0, 12) \text{ & } (12, 0)$$

$$2x + y = 8 \Rightarrow \left(\frac{8}{3}, 0\right), (0, 8)$$



Note that (0, 0) is not a point of the feasible region.

The value of the objective function  $Z = 6x + 4y$

at the extreme points are

$$Z(0, 8) = 80$$

$$Z(1, 5) = 56$$

$$Z(2, 4) = 52$$

$$Z(10, 0) = 100$$

The minimum value of  $Z$  is at the

→ Solve the following LPP Graphically  
 $\rightarrow \text{Max } Z = 3x_1 + 4x_2 \quad \rightarrow \text{Max } Z = 5x_1 + 7x_2$   
 Subject to :  
 $4x_1 + 2x_2 \leq 80$        $x_1 + x_2 \leq 4$   
 $2x_1 + 5x_2 \leq 180$        $3x_1 + 8x_2 \leq 24$   
 $x_1, x_2 \geq 0$        $10x_1 + 7x_2 \leq 35$   
 $x_1, x_2 \geq 0$

→  $\text{Max } Z = 2x + y \quad \rightarrow \text{Max } Z = 22x_1 + 18x_2$   
 Subject to :  
 $5x + 10y \leq 50$        $260x_1 + 240x_2 \leq 5760$   
 $x + y \leq 1$        $x_1 + x_2 \leq 20$   
 $y \leq 4$        $x_1, x_2 \geq 0$   
 $x, y \geq 0$   
 $2x, y \geq 0$

③ → A house wife wishes to mix together two kinds of food, I & II, in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below.

	vitamin A	vitamin B	vitamin C
Food I	1	2	3
Food II	2	2	1

one kg of food I costs ₹6 and 1kg of food II costs ₹10. formulate the above problem as linear programming problem to find the least cost of the mixture which will produce the diet.

Let the mixture contain  $x$  kg of food I and  $y$  kg of food II. The formulation of the above problem is as follows.

Minimize  $Z = 6x + 10y$   
 Subject to the constraints  
 $x + y \geq 10$   
 $2x + 3y \geq 12$

The set of points satisfying

$$x_1 \geq 0, x_2 \geq 0 \text{ and } 2x_1 + 3x_2 \leq 12$$

is represented by the shaded area given by the  $\triangle AOB$ .

Similarly, the set of points satisfying

$$x_1 \geq 0, x_2 \geq 0 \text{ and } 3x_1 + 2x_2 \leq 8 \text{ is represented by the shaded area given by the } \triangle COE.$$

The feasible region or the solution space is the area of the graph which contains all pair of values that satisfy all the constraints.

The feasible region is bounded by the two axes and two lines  $2x_1 + 3x_2 = 12$  and  $3x_1 + 2x_2 = 8$ .

and it is the common shaded portion OABC

i.e. the four corners or extreme points of the polygon are

$$O(0,0), A(0,4), B\left(\frac{12}{7}, \frac{20}{7}\right), C\left(\frac{8}{3}, 0\right)$$

The values of the objective function  $Z = 4x_1 + 5x_2$  at these extreme points are

$$Z(O) = 0$$

$$Z(A) = 20$$

$$Z(B) = \frac{148}{7}$$

$$Z(C) = \frac{32}{3}$$

The maximum value of  $Z$  is at the point

$$B\left(\frac{12}{7}, \frac{20}{7}\right).$$

and the maximum value is  $Z = \frac{148}{7}$ .

$$x_1 = \frac{12}{7} \text{ and } x_2 = \frac{20}{7}$$

$\therefore$  The farthest point  $A = (30, 0)$  within the feasible region is touched by the line.

$\therefore$  At the point  $A = (30, 0)$  we obtain the maximum value of  $Z$ .

$\therefore$  Optimal solution if  $x = 30, y = 0$  and the optimal value of  $Z = 60x + 0$   
 $= 1800$

(2) Solve the following LPP Graphically.

$$\text{Maximize } Z = 4x_1 + 5x_2$$

Subject to the constraints:

$$2x_1 + 3x_2 \leq 12$$

$$3x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Soln:

Since every point which satisfies the constraint  $2x_1 + 3x_2 \leq 12$  and  $3x_1 + x_2 \leq 8$  lies in the first quadrant only.

$\therefore$  The desired pair  $(x_1, x_2)$  is restricted to the points of the first quadrant only.

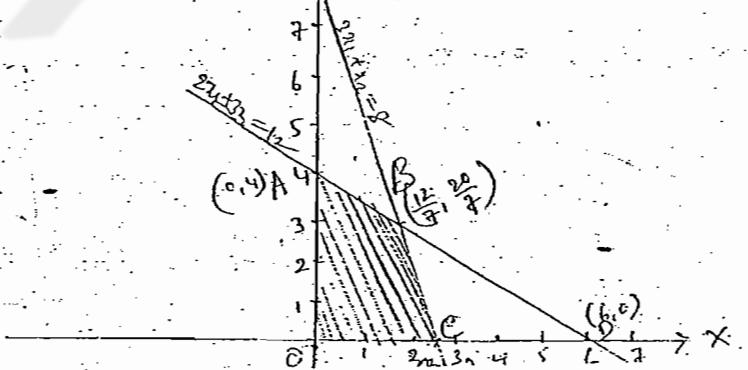
$$\text{Let } 2x_1 + 3x_2 = 12 \quad \text{(1)}$$

$$3x_1 + x_2 = 8 \quad \text{(2)}$$

$$(1) \Rightarrow 2x_1 + 3x_2 = 12 \Rightarrow (6, 0) \text{ and } (0, 4)$$

$$(2) \Rightarrow 3x_1 + x_2 = 8 \Rightarrow \left(\frac{8}{3}, 0\right) \text{ and } (0, 8)$$

$$8/3(0, 8)$$



The values of the objective function

$Z = 60x + 15y$  at these vertices are

$$Z(O) = 0$$

$$Z(A) = 60 \times 30 + 0 = 1800 \dots$$

$$Z(B) = 60 \times 20 + 15 \times 30 = 1650$$

$$Z(C) = 15 \times 50 = 750.$$

The maximum value of  $Z$  is 1800 at  $A = (30, 0)$ .

∴ The optimal solution to the problem is

$$x=30, y=0 \text{ and}$$

$$\text{Max } Z = 1800.$$

### (ii) Slack profit method:

Take a constant value say 600 (common multiple of 60 & 15) for  $Z$ .

$$\therefore 60x + 15y = 600$$

$$\Rightarrow 4x + y = 40$$

We draw this line which is represented by  $P_1Q_1$ .

Represented by  $P_1Q_1$ , we take another constant value 1200 for  $Z$ .

$$\therefore 60x + 15y = 1200$$

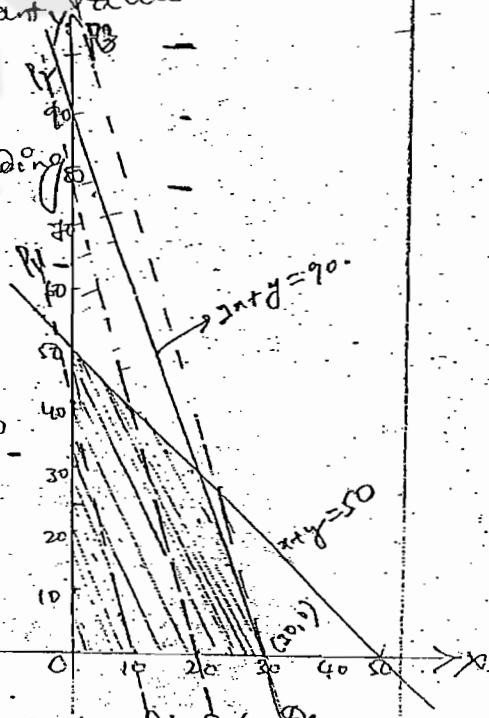
$$\Rightarrow 4x + y = 80$$

Draw the corresponding line  $P_2Q_2$ .

As  $600 < 1200$ ,

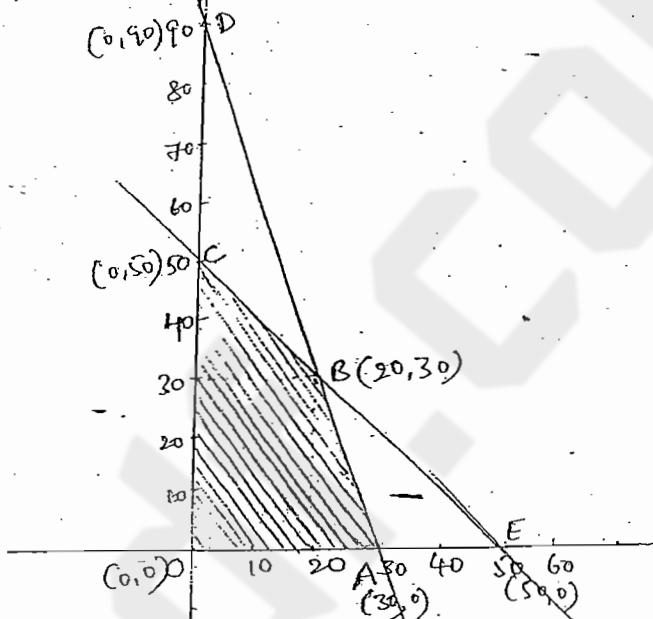
we move  $P_1Q_1$  from

$P_1Q_1$  to  $P_2Q_2$  parallel to itself as far as possible until the farthest point within the feasible region is touched by the line



line which passes through  $(0, 90)$ ,  $(30, 0)$  in  $xy$ -plane.

$\therefore$  All the points below and on this line satisfy the equation  $3x+y \leq 90$ .



The shaded region OABC which satisfies the equations (1) and (4).

The shaded region OABC is called as solution space or feasible region for the given problem.

Solving  $x+y=50$  and  
 $3x+y=90$ ,

we get  $x=20$ ,  $y=30$ .

$\therefore$  The required point  $B=(20, 30)$ .

### (i) Corner-point Method:

The vertices of the feasible region are  $O(0,0)$ ,  $A(30,0)$ ,  $B=(20,30)$  and  $C(0,50)$ .  
(The vertices  $O, A, B, C$  are also known as corner or extreme points)

the farthest point with in the feasible region be touched by the line. The co-ordinates of the point give maximum (minimum) value

of the objective function. The extremum may be either a point or a straight line. If it is a corner point, this yields the optimal solution. If it is a straight line, we have infinite number of optimal solutions.

Note: In Case of unbounded region, it either finds an optimal solution or declares an unbounded solution.

Unbounded solutions are not considered as optimal solutions. In real world problems, unlimited profit or loss is not possible.

### Problems

(1) Solve the following LPP graphically.

$$\text{Maximize } Z = 60x + 15y \quad \text{--- (1)}$$

Subject to the constraints

$$x+y \leq 50 \quad \text{--- (2)}$$

$$3x+y \leq 90 \quad \text{--- (3)}$$

$$x, y \geq 0 \quad \text{--- (4)}$$

Sol<sup>n</sup>: Let us consider the constraint

$x+y \leq 50$  as equality  $x+y=50$ ,

represents a line which passes through

$(0, 50)$  and  $(50, 0)$  in  $xy$ -plane.

The origin  $(0, 0)$  gives  $0+0=0 < 50$ .

$\therefore$  All the points below and on this line satisfy the inequality  $x+y \leq 50$ .

Let us consider the inequality  $3x+y \leq 90$ .

as equality  $3x+y=90$ , represents a

- If the problem is of maximization (minimization) the solution corresponding to largest (smallest) value of  $Z$  is the optimal solution of the LPP and the value of  $Z$  is the optimum value.

Note: finding all the corners is a long and tedious process for the problem having more constraints.

(2) Iso-profit or Iso-cost method  
- This is an alternative and more general method of finding the optimal solution of an LPP.

In this method, we first give any feasible constant value, say  $Z_1$ , to the objective function and draw the corresponding line of the objective function.

This line is called Iso-profit (or) Iso-cost line. since every point on the same line will give the same profit or cost  $Z_1$ .

Various steps of the method are as follows:

- i) Find the feasible region of LPP.
- ii) Assign a constant value  $Z_1$  to  $Z$  and draw the corresponding line of the objective function.
- iii) Assign another value  $Z_2$  to  $Z$  and draw the corresponding line of the objective function.
- iv) If  $Z_1 < Z_2$ , ( $Z_1 > Z_2$ ), then in case of maximization (minimization) move the line  $P_1Q_1$  corresponding to  $Z_1$  to the line  $P_2Q_2$  corresponding to  $Z_2$  parallel to itself as far as possible, until

## → The Graphical method of solving an LPP

The graphical method of solving an LPP is used when there are two variables.

If the problem has three or more variables, the graphical method is not suitable.

In that case, a very powerful method called Simplex method is used.

There are two techniques of solving an LPP by graphical method.

These are:

- (i) Corner point method
- (ii) Iso-profit method. (or) Iso-cost method.

### (i) Corner point Method:

The optimal solution to a LPP, if it exists, occurs at an extreme point (corner) of the feasible region.

The method comprises of the following steps:

- (i) find the feasible region of the LPP.
- (ii) find the co-ordinates of each vertex of the feasible region

The co-ordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at that point.

- (iii) Evaluate the value of the objective function  $Z$  at each corner point obtained in step (ii).

## → Some Important definitions Related with the general LPP:

Solution: A set of values of decision variables  $x_1, x_2, \dots, x_n$  which satisfy all the constraints of a General LPP is called a solution to General LPP.

feasible solution: Any solution to a General LPP which also satisfies the non-negative restrictions of the problem, is called a feasible solution to the General LPP.

Optimal feasible solution: Any feasible solution which optimizes (minimizes or maximizes) the objective function of General LPP is called an optimal feasible solution to the General LPP.

Note: The term optimum solution is also used for optimal solution.

Feasible regions: The common region determined by all the constraints and non-negativity restriction of an LPP is called a feasible region.

Convex region: A region is said to be convex if the line segment joining any two arbitrary points of the region lies entirely within the region. A feasible region of an LPP always a convex region.

while food F<sub>2</sub> contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. formulate the above problem as a linear programming problem to minimize the cost of mixture.

Number of units of vitamin A in  $x$  units of food

$F_1$  and  $y$  units of food  $F_2$  is  $2x + 4y$ .

The minimum daily requirement of vitamin A is 40 units.

$$2x + 4y \geq 40$$

Similarly, the number of units of vitamin B

in  $F_1$  and  $F_2$  is  $3x + 2y$ .

The daily minimum requirement of vitamin B is 50 units.

$$3x + 2y \geq 50$$

As the costs of one unit of  $F_1$  and  $F_2$  are

Rs. 3 and Rs. 2 respectively.

The total cost of purchasing  $x$  units of food  $F_1$  and  $y$  units of food  $F_2$  (in Rs) is

$$Z = 3x + 2.5y$$

which is the objective function.

The mathematical formulation of the problem is

$$\text{minimize } Z = 3x + 2.5y$$

Subject to the constraints

$$2x + 4y \geq 40$$

$$3x + 2y \geq 50$$

$$x, y \geq 0$$

A house wife wishes to mix two types of food  $F_1$  and  $F_2$  in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. food  $F_1$  costs Rs. 60/kg and food  $F_2$  costs Rs. 80/kg. food  $F_1$  contains 2 units/kg of vitamin A and 5 units/kg of vitamin B.

Hyp

→ A dealer wants to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items.

A fan costs him Rs. 260 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

→ Vitamin A and B are found in two different foods  $F_1$  and  $F_2$ . One unit of food  $F_1$  contains 2 units of Vitamin A and 3 units of Vitamin B. One unit of food  $F_2$  contains 4 units of Vitamin A and 2 units of Vitamin B. One unit of food  $F_1$  costs Rs. 3 and one unit of food  $F_2$  costs Rs. 2.50. The minimum daily requirement for a person of Vitamin A and B is 40 and 25 respectively. Assuming that anything in excess of daily minimum requirement of Vitamin A and B is not harmful, find out the mixture of food  $F_1$  and  $F_2$  at the minimum cost which meets the daily minimum requirement of vitamins A and B.

Let  $x$  = number of units of food  $F_1$   
 $y$  = number of units of food  $F_2$

$$\begin{aligned}x + 2y &\leq 12 \quad (\text{constraint on machine A}) \\2x + y &\leq 12 \quad (\text{constraint on machine B}) \\x + \frac{5}{4}y &\geq 5 \quad (\text{constraint on machine B}) \\x \geq 0, y \geq 0 & \quad (\text{non-negativity restriction})\end{aligned}$$

A retired person wants to invest an amount of upto Rs 20,000. His broker recommends investing in two types of bonds A and B, bond A yielding 10% return on the amount invested and bond B yielding 15% return on the amount invested.

After some consideration, he decides to invest atleast Rs 5000 in bond A and no more than Rs 8000 in bond B. He also wants to invest atleast as much in bond A as in bond B. What should his investment be?

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Mod: 09/09/1976  
Suggested if he wants to maximize his return investments. formulate LPP.

Ques Let  $x$  be the amount (in Rs) invested in bond A and  $y$  be the amount (in Rs) invested in bond B

The objective is to maximize his return on investment i.e,

$$\text{Maximize } Z = 0.10x + 0.15y$$

subject to the constraints

$$x + y \leq 20000 \quad (\text{sum of the investments})$$

$$x \geq 5000 \quad (\text{constraint on investment in bond A})$$

$$y \leq 8000 \quad (\text{constraint on investment in bond B})$$

$x \geq y$  (relation between investments)

$$x \geq 0, y \geq 0 \quad (\text{investment cannot be negative})$$

→ A manufacturer has 3 machines I, II and III installed in his factory. Machines I and II are capable of being operated for at the most 12 hours, whereas machine III must be operated atleast for 5 hours a day. He produces only two items, each requiring the use of the 3 machines.

The number of hours required for producing 1 unit of each of the items A and B on the 3 machines are given in the following table:

Items	Number of hours required on the machine		
	I	II	III
A	1	2	1
B	2	1	5/4

He makes a profit of Rs. 60 on item A and Rs. 40 on item B. Assuming that he can sell all that he produces, how many of each items should he produce so as to maximize his profit? Formulate the above problem as a linear programming problem.

Sol.: Let  $x$  be the number of items A and  $y$  be the number of items B produced.

The total profit on the production is Rs.  $60x + 40y$ .

The objective of manufacturer is to maximize the profit.

∴ The formulation of the problem is:

$$\text{Maximize } Z = 60x + 40y.$$

Subject to the constraints

Thus  $500x + 200y \leq 10000$  (constraint)

$$\text{Also, } x + y \leq 60 \quad (\text{constraint})$$

as the dealer has the space to store at the most 60 items.

If it is obvious that  $x \geq 0$ ,  $y \geq 0$  (non-negative restriction), as the number of chairs cannot be negative.

Profit on x tables = 50%

profit on x chairs =  $15x$   
profit on y chairs =  $15y$

Hence, the profit on  $x$  tables and  $y$  chairs  
 $= \text{Rs. } 50x + 15y$ . (Objective function)

obviously, dealer wishes to maximize the profit  $7\% + 15\%$ .

profit £ 50 & 15/-

*∴ The practical formulation of the*

L.P.  
Maximize  $Z = 50x + 15y$

subject to the constraints

$$5x + 2y \leq 100$$

$$x+y \leq 60$$

$$x \geq 0, y \geq 0.$$

Q. A firm manufactures two products A and B. One unit of product A needs 2 hours on machine I and 3 hours on machine II. One unit of the product B needs 3 hours on machine I and 1 hour on machine II. Daily capacity of machines I and II are 12 hours and 8 hours per day respectively. Profits obtained on selling one unit of A and one unit of B are Rs. 4/- and Rs. 5/- respectively. The problem is to determine the daily level of products A and B so as to maximize the profit.

Step 4: Add the non-negativity constraint on decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

The objective function, the set of constraints and the non-negative constraints together form a linear programming problem (LPP).

#### Example

A furniture dealer deals in only two items, viz, tables and chairs. He has Rs 10,000 to invest and a space to store at most 60 pieces. A table costs him Rs 500 and a chair Rs 200. He can sell a table at a profit of Rs 50 and a chair at a profit of Rs 15. Assume that he can sell all the items that he buys. Formulate this problem as an LPP, so that he can maximize the profit.

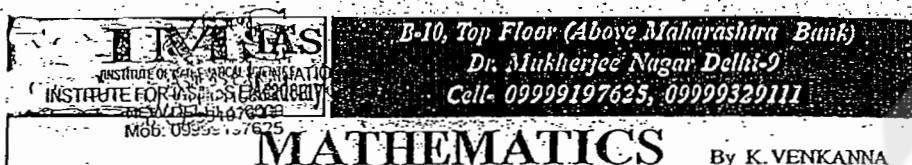
#### Solution

Let  $x$  and  $y$  denote the number of tables and chairs, respectively. ( $x$  and  $y$  are decision variables).

The cost of  $x$  tables = Rs  $500x$ .

The cost of  $y$  tables = Rs  $200y$ .

∴ The total cost of  $x$  tables and  $y$  chairs  
= Rs  $500x + 200y$   
which cannot be more than 10,000.



(vii)  $b_i$  ( $i=1, 2, \dots, m$ ) is the requirement  
(or) availability of the  $i^{\text{th}}$  constraint.

(viii)  $a_{ij}$  ( $i=1, 2, \dots, m$ ,  $j=1, 2, \dots, n$ ) is referred  
to as the technological coefficient.

### Mathematical formulation of a linear programming problem:

The procedure for mathematical  
formulation of LPP consists of the following  
major steps:

Step 1: Write down the decision variables  
and assign symbols  $x_1, x_2, \dots, x_n$  to them.

These decision variables are those quantities  
whose values we wish to determine.

Step 2: Formulate the objective function to be  
optimized (maximizing profit or minimizing cost)  
as a linear function of the decision variable.

Step 3: Formulate the other conditions of the form  
such as resource limitations, market,  
constraints, inter-relation between variables  
etc. as linear equations or inequalities involving  
of the decision variables.

### General formulation of LPP:

Maximize or minimize

$$x = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n (\leq, =, \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, =, \geq) b_2$$

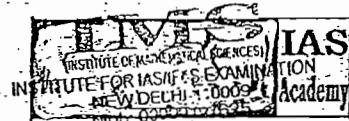
19. The following table gives the number of deaths from smallpox in the United States during the year 1800.

$$a_{m_1}x_1 + a_{m_2}x_2 + \dots + a_{m_n}x_n (\leq, =, \geq) b_m$$

$x_1, x_2, x_3, \dots, x_n$

-where -

- (i) the linear function  $Z$  which is to be maximized or minimized is the objective function of the LPP.
  - (ii)  $x_1, x_2, \dots, x_n$  are the decision variables.
  - (iii) the equations/inequalities ① are the constraints of LPP.
  - (iv) In the set of constraints ① the expression ( $\leq, =, \geq$ ) means that each constraint may take any one of the three signs.
  - (v)  $-C_j$  ( $j=1, 2, \dots, n$ ) represents per unit profit (or) cost to the  $j^{\text{th}}$  variable.
  - (vi) the set of inequalities ② is the set of non negative restrictions of the general linear programming problem.



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## MATHEMATICS

By K. VENKANNA  
*The person with 8 yrs of teaching exp.*

### (6) Advertising media selection problems:

In these problems, we find the optimum allocation of advertisements in different media in order to maximize the total effective audience / customers.

### Basic concepts of LPP:

The term programming means planning and refers to a process of determining a particular action from amongst several alternatives. The term 'linear' stands for indicating that all relationships involved in a particular problem are linear.

The general linear programming problem (LPP) calls for optimizing (maximizing or minimizing) a linear function for variables called the objective function subject to a set of linear equations and/or inequalities called the constraints or restrictions.

### General formulation of LPP:

Maximize or minimize

$$Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, =, \geq) b_1$$

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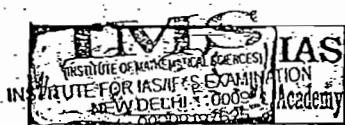
①

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, =, \geq) b_m$$

$$x_1, x_2, x_3, \dots, x_n \geq 0 \quad \text{②}$$

where

- (i) the linear function  $Z$  which is to be maximized or minimized is the objective function of the LPP.
- (ii)  $x_1, x_2, \dots, x_n$  are the decision variables.
- (iii) the equations/inequalities ① are the constraints of LPP.
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unit of the output etc in order to make maximum profit.

- (2). Diet problems:

In these problems, we determine the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimize the cost of desired diet such that it contains a certain minimum amount of each constituent/nutrient.

(3) Investment problems:

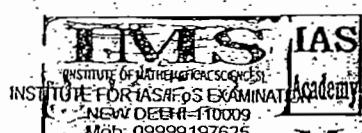
In these problems, we determine the amount which should be invested in a number of fixed income securities to maximize the return on investment.

(4). Transportation problems:

-- In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants or factories situated at different locations to different markets.

(5). Blending problems:

In these problems, we have to determine optimum amount of several constituents to be used in producing a set of products while determining the optimum quantity of each product to be produced.



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## MATHEMATICS

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### Optimization in two variables

We now initiate the study of Linear programming by taking a few examples involving only two variables and giving its mathematical formulation. Thereafter, we discuss its solution by Graphical Method and simultaneously give the intuitive idea of its feasible and optimal solutions.

Finally we discuss bounded and unbounded sets through the graphical method for solving a linear programming problem in two variables.

### Different Areas of Applications of LPP.

We shall now discuss some important areas of applications of LPP.

#### 1) Manufacturing problems:

In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of the product, warehouse, space per

as a point

$$x = \sum_{i=1}^m \mu_i x_i, \mu_i \geq 0, i=1, 2, 3, \dots, m$$

where  $\sum_{i=1}^m \mu_i = 1$

→ The set of all convex combinations of a finite number of points  $x_1, x_2, \dots, x_m$  in  $E^n$  is a convex set that is, the set

$$S = \left\{ x \mid x = \sum_{i=1}^m \mu_i x_i, \mu_i \geq 0, \sum_{i=1}^m \mu_i = 1 \right\}$$

is convex.

### Convex Hull:

Suppose A is a set which is not convex. Then the smallest convex set which contains A is called the Convex Hull of A. i.e., the convex hull of a set A is the intersection of all convex sets which contain A.

For example: the convex hull of the set-

$$A = \{(x, y) / x^2 + y^2 = 1\}$$

is the set

$$S = \{(x, y) / x^2 + y^2 \leq 1\}.$$

Here S is convex and it contains A.  
Observe that S is the smallest convex set containing A. In this way we can say that the convex hull of the point on the circumference of a circle is the circumference plus the interior of the circle. This is the smallest convex set containing the circumference.

→ The convex hull of a finite number of points  $x_1, x_2, \dots, x_m$  in  $E^n$  is the set of all convex combination of  $x_1, x_2, \dots, x_m$ .

i.e., the convex hull of  $x_1, x_2, \dots, x_m$  is the set

$$S = \{x \in E^n \mid x = \sum_{i=1}^m \mu_i x_i, \mu_i \geq 0, \sum_{i=1}^m \mu_i = 1\}.$$

→ The convex hull of the finite number of points is called Convex polyhedron spanned by these points.

Sol<sup>n</sup>: If  $x_1, x_2$  are any two points on the hyperplane  $Cx = d$ , then  $Cx_1 = d$  and  $Cx_2 = d$ .

The hyper plane will be convex if the point

$$x = \lambda x_2 + (1-\lambda)x_1 \text{ for } 0 \leq \lambda \leq 1.$$

lies on the hyperplane.

We have

$$Cx = C[\lambda x_2 + (1-\lambda)x_1]$$

$$= C\lambda x_2 + C(1-\lambda)x_1$$

$$= \lambda(Cx_2) + (1-\lambda)(Cx_1)$$

$$= \lambda d + (1-\lambda)d = d.$$

$$\therefore Cx = d.$$

Hence the hyperplane is a convex set.

→ A closed half plane is a convex set.

Ques: Consider ~~INST. FOR ADVANCED STUDY~~ Mod. No. 10005  
closed half space.

$$S_4 = \{x : Cx \leq d\}$$

Suppose  $x_1, x_2 \in S_4$  then  $Cx_1 \leq d, Cx_2 \leq d$ .

$$\text{Consider } x = \lambda x_2 + (1-\lambda)x_1, 0 \leq \lambda \leq 1.$$

$$\text{Now } Cx = \lambda Cx_2 + (1-\lambda)Cx_1, 0 \leq \lambda \leq 1.$$
  
$$\leq \lambda d + (1-\lambda)d \leq d.$$

$$\therefore Cx \leq d.$$

Hence  $S_4$  is a convex set.

→ Similarly, we can show that  $S_1, S_2$  &  $S_3$  are convex sets.

### Convex combination:

Let  $x_1, x_2, \dots, x_m$  be a finite number of points in a Euclidean space  $E^n$ . A convex combination of points  $x_1, x_2, \dots, x_m$  is defined

## Hyperplane and Half-Spaces

A hyper plane in  $E^n$  is defined to be a set S of points

$$S = \{x \in E^n / c_1 x_1 + c_2 x_2 + \dots + c_n x_n = d\}$$

$$\text{i.e., } S = \{x \in E^n / cx = d\}.$$

$$\text{where } c = [c_1, c_2, \dots, c_n] \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

W.R.T.  $c_1 x_1 + c_2 x_2 = d$  (called decision) is the eqn of the straight line in  $E^n$  and also this divides the plane into three parts.

Similarly, hyper plane  $cx = d$  in  $E^n$  divides  $E^n$  into three mutually exclusive and exhaustive regions. These are denoted by the sets

$$S_1 = \{x : cx < d\}$$

$$S_2 = \{x : cx = d\}$$

$$S_3 = \{x : cx > d\}$$

The sets  $S_1$  and  $S_3$  are called Open-half spaces.

The sets  $S_4 = \{x / cx \leq d\}$  and  $S_5 = \{x / cx \geq d\}$  are called closed half spaces.

Note:  $S_4 \cap S_5 = S_2$  which is the hyper plane  $cx = d$ .

→ A hyper plane is a convex set.

(Q) prove that the collection  $S = \{x \in E^n / cx = d\}$  is a convex set.

Note that the point  $x = \lambda x_1 + (1-\lambda)x_2$ ,  $0 < \lambda < 1$  is a point in between  $x_1$  and  $x_2$  ( $x_1 \neq x_2$ ).

According to the above definition an extreme point fails to satisfy this property.

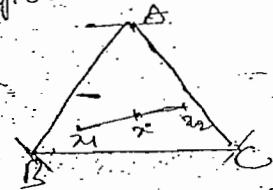
Consider a convex set 'S' formed by a triangle ABC and its interior. If  $x$  is a point inside the  $\triangle ABC$ , then it is possible to find points  $x_1$  and  $x_2$  in  $S$  such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0 < \lambda < 1,$$

If  $x$  is a boundary point in  $S$  different from points A, B and C, even then we can find points  $x_1$  and  $x_2$  in  $S$  such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0 < \lambda < 1.$$

But for the points A, B and C this is not possible. That is why the points A, B and C are called the extreme points.



Note: we should note the strict inequality imposed on  $\lambda$ .

An extreme point is a boundary point of the set; however, not all the boundary points of a convex set are necessarily extreme points.

If we consider circle, then every point on the circumference of the circle is an extreme point.

$$\Rightarrow \left(\frac{1}{2} + \frac{5}{2}\lambda\right)\left(2 - \frac{5}{3}\lambda\right) \leq 1 \text{ for } \lambda \in [0, 1]$$

$$\Rightarrow 1 + 5\lambda - \frac{5}{6}\lambda - \frac{25}{6}\lambda^2 \leq 1 \text{ for } \lambda \in [0, 1]$$

$$\Rightarrow \frac{25}{6}\lambda^2 - \frac{25}{6}\lambda \geq 0 \text{ for } \lambda \in [0, 1].$$

This inequality should hold for all values of  $\lambda$  such that  $0 \leq \lambda \leq 1$ .

But, if we take  $\lambda = \frac{1}{2}$ , then we get

$$\frac{25}{6}\lambda^2 - \frac{25}{6}\lambda = \frac{25}{12} - \frac{25}{24} = \frac{25}{24} > 0.$$

Hence the inequality is not satisfied for  $\lambda = \frac{1}{2}$ .

This contradiction shows that S is not convex.

Defn:

Extreme point or vertex of a convex set.

An extreme point (or vertex) of a convex set S is a point of the set which does not lie on any segment joining two other points of the set.

(B)

Let S be a convex set. A point  $x \in S$  is an extreme point of the convex set S if and only if there do not exist points  $x_1, x_2$  and  $(x_1 \neq x_2)$  in the set S such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0 < \lambda < 1.$$

$$\Rightarrow 3x_1x_2 + 2y_1y_2 \leq 6. \quad \text{--- (3)}$$

from (2) and (3), we get

$$\begin{aligned} 3(\lambda x_1 + (1-\lambda)x_2)^2 + 2(\lambda y_1 + (1-\lambda)y_2)^2 &\leq 6\lambda^2 + 6(1-\lambda)^2 \\ &\quad + 12\lambda(1-\lambda) \\ &= 6[\lambda + (1-\lambda)]^2 \\ &= 6. \end{aligned}$$

$$\text{Hence } 3(\lambda x_1 + (1-\lambda)x_2)^2 + 2(\lambda y_1 + (1-\lambda)y_2)^2 \leq 6$$

$$\therefore (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in S.$$

$\therefore S$  is a convex set.

→ Show that the set  $\{(x, y) / xy \leq 1, x \geq 0, y \geq 0\}$  is not convex.

For: In order to show that  $S$  is not convex we will take two points in  $S$  and show that their convex combination does not belong to  $S$ .

Clearly  $(3, \frac{1}{3})$  &  $(\frac{1}{2}, 2)$  belong to  $S$ .

Consider the combination of these points.

$$\text{i.e. } \lambda(3, \frac{1}{3}) + (1-\lambda)(\frac{1}{2}, 2), \quad 0 \leq \lambda \leq 1.$$

$$\Rightarrow \left(3\lambda + \frac{1}{2}(1-\lambda), \frac{1}{3} + 2(1-\lambda)\right), \quad 0 \leq \lambda \leq 1$$

$$\Rightarrow \left(\frac{1}{2} + \frac{5}{2}\lambda, 2 - \frac{5}{3}\lambda\right), \quad 0 \leq \lambda \leq 1.$$

$S$  will be convex if

$$\left(\frac{1}{2} + \frac{5}{2}\lambda, 2 - \frac{5}{3}\lambda\right) \in S \quad \forall \lambda \in [0, 1]$$

Hence  $\lambda x_2 + (1-\lambda)x_1 \in S_1, \forall \lambda \in [0, 1]$ .

$$\Rightarrow \lambda y_2 + (1-\lambda)y_1 \in S_2, \forall \lambda \in [0, 1].$$

$\therefore S_3$  is convex set.

$\rightarrow$  Show that the set  $S = \{(x, y) / 3x^2 + 2y^2 \leq 6\}$  is convex.

For:

Suppose  $p = (x_1, y_1) \in S, q = (x_2, y_2) \in S$  are any two points. Then

$$\begin{aligned} 3x_1^2 + 2y_1^2 &\leq 6 \\ 3x_2^2 + 2y_2^2 &\leq 6 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \rightarrow ①$$

$S$  will be convex if

$$\lambda(x_2, y_2) + (1-\lambda)(x_1, y_1) \in S; \quad 0 \leq \lambda \leq 1.$$

$$\text{i.e. } (\lambda x_2 + (1-\lambda)x_1, \lambda y_2 + (1-\lambda)y_1) \in S; \quad 0 \leq \lambda \leq 1.$$

NOW,

$$\begin{aligned} & 3(\lambda x_2 + (1-\lambda)x_1)^2 + 2(\lambda y_2 + (1-\lambda)y_1)^2 - \\ &= \lambda^2(3x_2^2 + 2y_2^2) + (1-\lambda)^2(3x_1^2 + 2y_1^2) \\ &+ 2\lambda(1-\lambda)[3x_2x_1 + 2y_2y_1] \\ &\leq 6\lambda^2 + 6(1-\lambda)^2 + 2\lambda(1-\lambda)[3x_2x_1 + 2y_2y_1] \quad ② \\ &\quad \text{(from ①)} \end{aligned}$$

NOW: Consider

$$3(x_1 - x_2)^2 + 2(y_1 - y_2)^2 \geq 0$$

$$\Rightarrow 3[x_1^2 + x_2^2 - 2x_1x_2] + 2[y_1^2 + y_2^2 - 2y_1y_2] \geq 0$$

$$\Rightarrow (3x_1^2 + 2y_1^2) + (3x_2^2 + 2y_2^2) - 2(3x_1x_2 + 2y_1y_2) \geq 0$$

$$\Rightarrow -2(3x_1x_2 + 2y_1y_2) \leq (3x_1^2 + 2y_1^2) + (3x_2^2 + 2y_2^2) \\ \leq 6 + 6 \quad (\because \text{from ①})$$

Both the points A and B belong to S.

$$\text{Take : } \lambda = \frac{2}{3} \text{ Then } \lambda A + (1-\lambda)B \\ = \frac{2}{3}(1,1) + \left(\frac{1}{3}\right)(-2,-2)$$

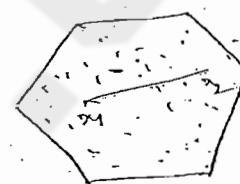
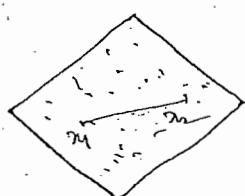
gives the point (0,0).

but (0,0) does not satisfy  $x_1 + x_2 \leq 1$

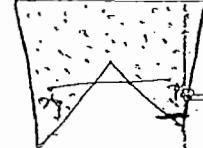
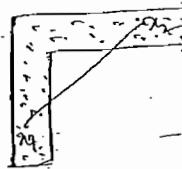
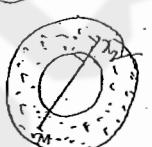
This shows that S is not a convex set.

Few more graphs of the sets which are convex and non-convex are given below.

convex sets



non-convex sets



If  $S_1$  and  $S_2$  are convex sets, then their intersection is ~~also~~ convex set.

Ques: Suppose  $S_3 = S_1 \cap S_2$

Let  $x_1, x_2$  be any two points in  $S_3$ .

Then  $x_1, x_2 \in S_1$  and  $x_1, x_2 \in S_2$ .

Since  $S_1$  and  $S_2$  are convex,

therefore  $\lambda x_2 + (1-\lambda)x_1 \in S_1$  for  $0 \leq \lambda \leq 1$ .

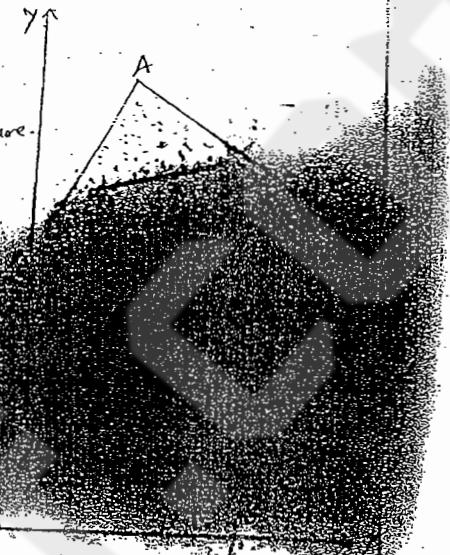
and  $\lambda x_2 + (1-\lambda)x_1 \in S_2$  for  $0 \leq \lambda \leq 1$

In other words, a set  $S$  is said to be convex if for any elements

$$x_1, x_2 \in S; \lambda x_1 + (1-\lambda)x_2 \in S \text{ for } 0 \leq \lambda \leq 1.$$

→ Consider a triangle ABC

with its interior is a convex set as shaded in figure.

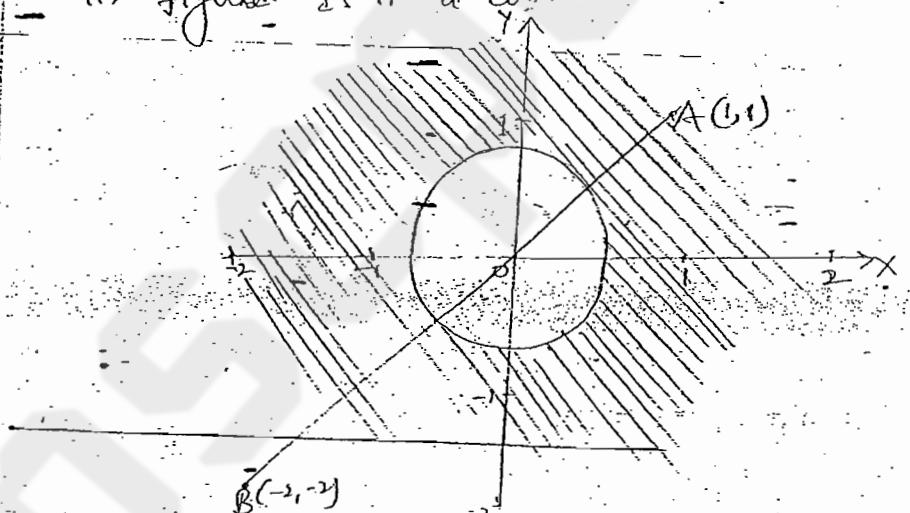


→ consider a set

$$S = \{(x, y) / x^2 + y^2 \leq 1\}$$

Clearly it is not convex

However, consider the set  $S = \{(x, y) / x^2 + y^2 \geq 1\}$   
which is the circle with its exterior as shown  
in figure. Is it a convex set?



Sol: Let the points  $A(1,1)$  and  $B(-2,-2)$ .

$$\text{Since } 1^2 + 1^2 > 1 \text{ and } (-2)^2 + (-2)^2 = 8 > 1.$$

## convex sets and their Geometry

### Notion of convex sets:

Let  $x_1$  and  $x_2$  be any two points in the Euclidean space  $E^n$ . Consider a line passing through the points  $x_1, x_2$  ( $x_1 \neq x_2$ ) in  $E^n$  defined as the set

$$S = \{x / x = \lambda x_1 + (1-\lambda)x_2, \text{ all real } \lambda\}$$

By giving different values to  $\lambda$  we will get the corresponding different points on the line. Suppose  $\lambda$  is chosen such that  $0 \leq \lambda \leq 1$ . Then

for  $\lambda = 0$ , we get  $x = x_1$  and

for  $\lambda = 1$ , we get  $x = x_2$ .

Thus we get ~~INSTITUTE OF MANAGEMENT & TECHNOLOGY  
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Mob: 09999197825~~ points between  $x_1$  and  $x_2$ . The line joining ~~INSTITUTE OF MANAGEMENT & TECHNOLOGY  
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Mob: 09999197825~~ points corresponding to the values of  $\lambda$  between 0 and 1, is often called a line segment.

In other words, the line segment joining the points  $x_1, x_2$  in  $E^n$  corresponding to the values of  $\lambda$  between 0 and 1 is a set of points denoted by  $S$ ,

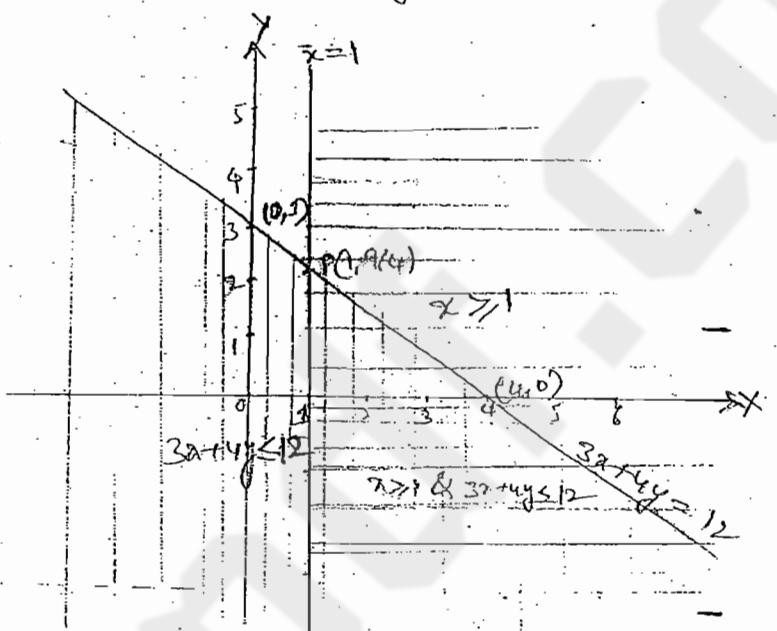
$$\text{where } S = \{x / x = \lambda x_1 + (1-\lambda)x_2, 0 \leq \lambda \leq 1\}$$

### Convex set

Defn: A set 'S' is said to be Convex if for any two points  $x_1, x_2$  in the set, the line segment joining these points is also in the set.

The set  $A = \{(x,y) | x \geq 1\}$  is shaded with horizontal lines and the set  $B = \{(x,y) | 3x+4y \leq 12\}$  is the set with vertical shading.

Now the set of points which satisfy both the inequalities i.e., the set  $A \cap B$  of points is the cross-hatched region shown in the fig.



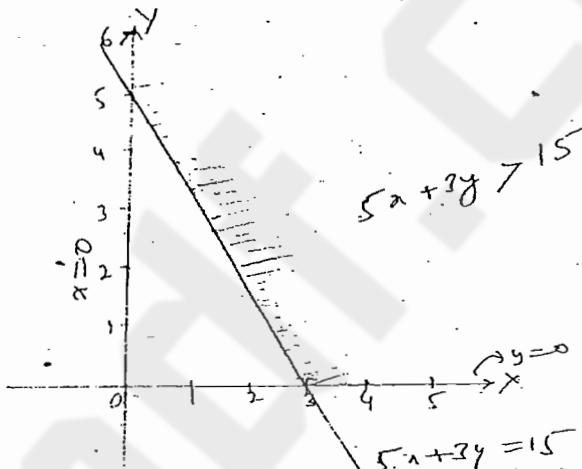
The corner point  $P(1,3)$  is the intersection of lines  
 $x=1$  and  $3x+4y=12$

→ Graph the set of points  $(x, y)$  satisfying the following three inequalities.

$$A = \{(x, y) \mid 5x + 3y \geq 15\}$$

$$B = \{(x, y) \mid x \geq 0\} \text{ and } C = \{(x, y) \mid y \geq 0\}$$

Soln: The desired set is the intersection of the three sets. The set ABC is the area which is shaded and bounded by the line  $5x + 3y = 15$ .



Note: The shaded region consists of the points lying only in the first quadrant due to the restrictions  $x \geq 0$  and  $y \geq 0$ .

→ Graph the set of points which satisfy the inequalities  $x \geq 1$  and  $3x + 4y \leq 12$ .

Soln: Draw the line  $x=1$ , which is a vertical line through the point  $(1, 0)$ .

The line  $3x + 4y = 12$  is the line joining the points  $(4, 0)$  and  $(0, 3)$ .

→ Draw the graph of the inequality  $15x + 8y \geq 60$ .  
first consider the line.

$$15x + 8y = 60$$

If we take  $y=0$ , then  $x=4$ .

If  $x=0$ , then  $y=15/2$ .

we can trace the line by joining the points  $(4, 0)$  and  $(0, 15/2)$ .

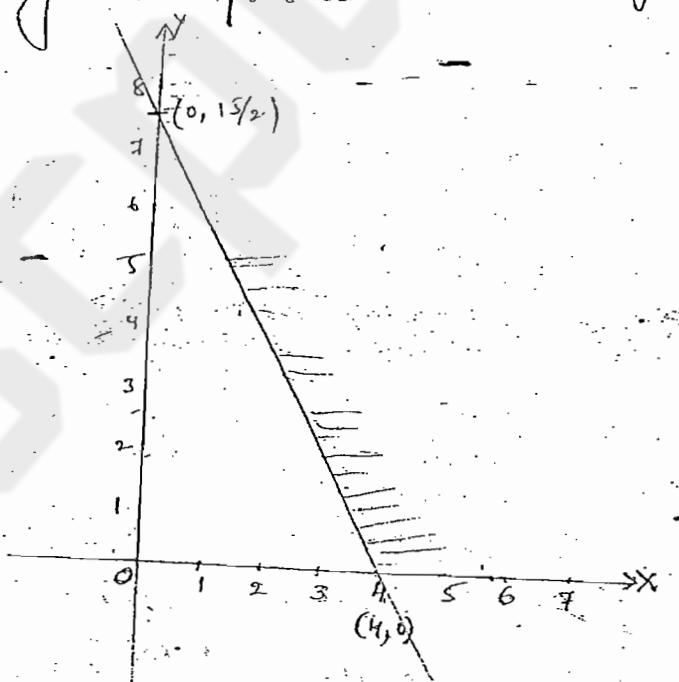
Let us now determine the location of the half plane.

for this, we put

$$x=0 \text{ and } y=0$$

$$15(0) + 8(0) = 0 \leq 60$$

This shows that  $15x + 8y \geq 60$  is that half plane in which origin does not lie. Hence the shaded region as shown in the figure, represents the  $15x + 8y \geq 60$ .



(iv) The set of points  $(x, y)$  such that  $3x+2y \geq 6$ .  
the other half plane bounded by the line  $3x+2y=6$ .

+ The inequality  $3x+2y \leq 6$  represents the set of points  $(x, y)$  which either lie on the line  $3x+2y=6$  or belong to the half-plane  $3x+2y < 6$ .

Similarly, the inequality  $3x+2y \geq 6$  represents the set of points  $(x, y)$  which either lie on the line  $3x+2y=6$  or belong to the half-plane  $3x+2y \geq 6$ .

Most of the inequalities that we study here will be of the form  $ax+by=c$ .

$$ax+by \leq c \quad \text{or} \quad ax+by \geq c.$$

In general we can say that a line  $ax+by=c$  divides the  $xy$ -plane into three regions.

viz,

- i) the set of points  $(x, y)$  such that  $ax+by=c$ , that is, the line itself.
- ii) the set of points  $(x, y)$  such that  $ax+by < c$ , i.e., one of the half-planes bounded by the line.
- iii) the set of points  $(x, y)$  such that  $ax+by > c$ , the other half plane bounded by the line.

If we put  $y=0$ , we get  $x=c/a$ , provided  $a \neq 0$ .  
 $x=c/a$  is the intercept of the line on  $x$ -axis.

Similarly on taking  $x=0$ , we get

$$y=c/b, b \neq 0.$$

as the intercept on  $y$ -axis.

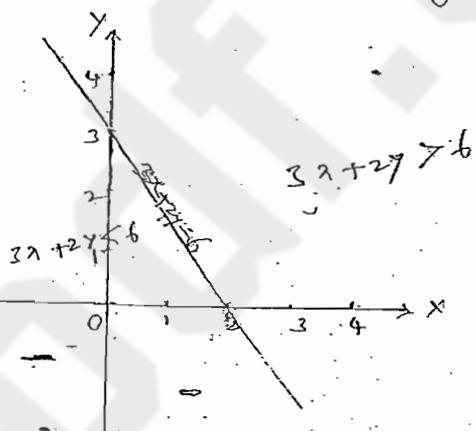
By joining the points  $(\frac{c}{a}, 0)$  &  $(0, \frac{c}{b})$ ,  $a \neq 0, b \neq 0$

we can trace the line.

For example,

Consider the line  $3x+2y=6$ .

Draw this line as shown in the figure



This line divides the plane into three sets or regions as shown in the figure.

These regions may be described as follows:

(i) The set of points  $(x, y)$  such that

$$3x+2y=6$$

i.e., those points which lie on the line.

(ii) The set of points  $(x, y)$  such that

$$3x+2y<6$$

The set of points  $(x, y)$  for which  $3x+2y<6$  is called the half-plane bounded by the line  $3x+2y=6$ .

In a linear programming problem, we have constraints expressed in the form of linear inequalities. Therefore, to study linear programming we must know the system of linear inequalities particularly their graphical solutions. Now we shall confine our discussion to the graphical solutions of inequalities.

Closely linked with the system of linear inequalities is the theory of convex sets. This theory has very important applications not only in linear programming but also in

Economics, Game theory etc.  
Due to these applications, a great deal of work has been done to develop the theory of convex sets.

Thus, now we discuss the inequalities and convex sets. In addition, we need the notion of extreme points, Hyper-plane and Half-space. These notions will be defined and explained with the help of some simple examples.

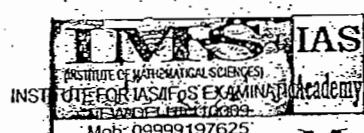
### Inequalities and their graphs:

We know that a general equation of a line is

$$ax + by = c,$$

where  $a, b, c$  are real constants.

It is also called a linear eqn. in two variables  $x$  and  $y$ .



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## MATHEMATICS

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### \* Linear programming \*

#### Introduction:

The linear programming originated during world war II (1939-1945), when the British and American military management called upon a group of scientists to study and plan the war activities, so that maximum damages could be inflicted on the enemy camps at minimum cost and loss. Because of the success in military operations, it quickly spread in all phases of industry and government organisations.

It was first coined in 1940 by McClosky and Trefthen (by using the term Operations Research) in a small town, Boudsey, of the United Kingdom.

In India, it came into existence in 1949, with opening of an operations research unit at the regional research laboratory at Hyderabad.

## Linear programming problems:

In the competitive world of business and industry, the decision maker wants to utilize his limited resources in a best possible manner. The limited resources may include material, money, time, man power, machine capacity etc. Linear programming can be viewed as a scientific approach that has evolved as an aid to a decision maker in business, industrial, agricultural, hospital, government and military organizations.

Now, suppose a vendor has a sum of Rs. 350 with which he wishes to purchase two types of tape, say, red and blue. Red tape costs Rs. 2 per metre and blue tape costs Rs. 3 per metre. He does not want to buy more than 40 metres of red tape. The question arises, "How many metres of red and blue tapes can he buy?" Assume that he buys  $x$  metres of red tape and  $y$  metres of blue tape.

The above problem can be stated mathematically

as follows:-

Find  $x$  and  $y$  such that

$$2x + 3y \leq 350 \quad (1)$$

$$x \leq 40 \quad (2)$$

$$x \geq 0, y \geq 0 \quad (3)$$

There can be a number of solution pairs  $(x, y)$ . Now, further suppose that the vendor sells red tape at a profit of Rs. 0.75 per metre while blue tape at a profit of Rs. 1 per metre. Obviously, vendor likes to pick up a pair  $(x, y)$  which gives him the maximum profit.

Now, the problem arises to find out the pair  $(x, y)$  which give maximum profit to the vendor, i.e., which will maximize  $0.75x + 1.y$ .

The above kind of problem is called a linear programming problem.







