

IMS

MATHS

BOOK-17

SET-1

(statics)

Basic Concepts

Mechanics is a science which deals with the behaviour of material bodies under the action of external forces.

Mechanics is a branch of science which deals with the conditions of rest or motion of bodies relative to their surroundings.

Mechanics is that branch of science which deals with the study of body in state of Motion (or) at rest under the effect of some forces.

Mechanics is the scientific bedrock of modern

Technology:

The boatman rowing his boat across a stream, the astronomer concerned with the motion of planets, the space technologist concerned with the spaceship; the engineer concerned with the erection of a dam etc, all have to apply mechanics.

Mechanics is a physical science:

Mechanics has its roots in the physical phenomena which occur in nature and it is sustained by human's desire to predict, to describe, to control and to understand such phenomena.

→ Mechanics is a formal science.

Mechanics has its own structure. The object of mechanics is the real world.

its study are not the real physical objects as they exist but abstract concepts - idealised models of the objects outside.

→ Mechanics is an applied science.

Mechanics is very much applicable to industry, defence and social sciences. It is an essential tool of the physicist, to the engineer to the space technologist and to the defence-chief. Hence the assertion that mechanics is an applied science and resembles mathematics.

* → Development, types and Branches of Mechanics

Development of Mechanics: The development of mechanics as a scientific discipline depends upon the following three

(i) Stage of Observation:

We can observe the physical world around us. for example: we can observe the motion of planets around the sun, motion of the moon around the earth, mutual attraction between physical bodies, earth quakes, tides etc.



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(ii) Postulation of laws: we seek the laws which govern the behaviour of objects in our physical world from our observations.

for example: Newton's laws of motion, Newtonian law of gravitation etc.

(iii) Validity and verification of laws: the results derived out of mathematical theory (developed out of the postulation of laws) are compared experimentally with the observations made. when the results of theory and experiment are identical, the theory can be used to build dams, bridges, to launch Appolo to moon etc. when the results of theory and experiment are not identical, new laws are postulated and new theories are developed.

In 1905 A. Einstein placed limitations on Newton law of motion in his theory of relativity and thus set the stage for the development of relativistic mechanics and quantum mechanics.

Types of Mechanics

(i) Newtonian Mechanics: This is based on the application of Newton's laws of motion directly; this is also known as classical mechanics or Analytical Mechanics.

Historically, Mechanics was the earliest branch of physics to be developed as an exact science.

It was left to Galileo (1564 - 1642) and Newton

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Forces in this system are also called forces in space; the force system may further be classified as follows.

→ Collinear Force System:

This system exists if all the forces act along the same line of action.

The system is always Coplanar.

→ Concurrent force system: This system exists if all lines of action of the forces intersect at a point. This system may be Coplanar or non-Coplanar.

Note: Concurrent force system of the two forces is always coplanar.

This is because a plane can be made through two concurrent straight lines.

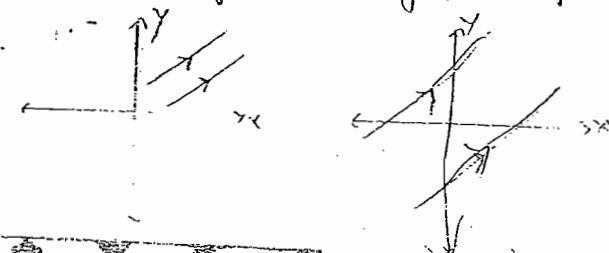
→ Non-Concurrent force system:

This system exists if the lines of action of the forces do not meet at one point.

This system may be Coplanar or non-Coplanar.

→ Parallel force system:

This system exists if the lines of action of all the forces are parallel. This system may be Coplanar (Or) non-Coplanar.



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and the sum of the components in the y -direction

$$y = \sum f_y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + \dots + F_n \sin \alpha_n \quad (1)$$

where $F_1 \sin \alpha_1, F_2 \sin \alpha_2, \dots, F_n \sin \alpha_n$ are resolved parts of F_1, F_2, \dots, F_n respectively along OY .

Let R be the resultant of the forces F_1, F_2, \dots, F_n and θ is the angle which R makes with OX .

$$\sum F_x = R \cos \theta \quad \text{and} \quad \sum F_y = R \sin \theta. \quad (\text{Algebraic sum of resolved parts of forces } F_1, F_2, \dots, F_n \text{ in any direction is equal to the resolved part of the resultant } R \text{ in that direction})$$

from (1),

$$x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \dots + F_n \cos \alpha_n = R \cos \theta$$

from (2),

$$y = F_2 \sin \alpha_1 + F_3 \sin \alpha_2 + \dots + F_n \sin \alpha_n = R \sin \theta$$

$$\text{i.e. } x = R \cos \theta \quad y = R \sin \theta$$

$$\Rightarrow R^2 (\cos^2 \theta + \sin^2 \theta) = x^2 + y^2$$

$$\Rightarrow R^2 = x^2 + y^2$$

$$\Rightarrow R = \sqrt{x^2 + y^2}$$

which gives the magnitude of the resultant

$$\frac{R \sin \theta}{R \cos \theta} = \frac{y}{x}$$

$$\Rightarrow \tan \theta = \frac{y}{x}$$

which gives the direction of the resultant provided $\cos \theta = \frac{x}{R}$ and $\sin \theta = \frac{y}{R}$ hold

$$\sin \theta = \frac{y}{R} \quad \text{hold}$$



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$$0^\circ, 180^\circ - C; (180^\circ - C) + (180^\circ - A)$$

$$0^\circ, 180^\circ - C, 360^\circ - (C+A)$$

$$0^\circ, 180^\circ - C, -(C+A).$$

The forces being proportional to $\cos A$, $\cos B$, $\cos C$
can be taken as:

$$P = k \cos A, Q = k \cos B, R = k \cos C \quad \text{where } k \text{ is a constant.}$$

Also, let the resultant be S , making an angle θ with the x -axis.

Then, resolving forces along the x -axis, we have

$$\begin{aligned} S \cos \theta &= P \cos 0^\circ + Q \cos (180^\circ - C) + R \cos (-C - A) \\ &= P - Q \cos C - R \cos B \end{aligned} \quad \text{--- (1)}$$

$$\begin{aligned} (\because \cos(-C-A) &= \cos(C+A) \\ &= \cos(180^\circ - B) \\ &= -\cos B) \end{aligned}$$

Also, resolving forces perpendicular to the x -axis, we have

$$\begin{aligned} S \sin \theta &= P \sin 0^\circ + Q \sin (180^\circ - C) + R \sin (-C - A) \\ &= Q \sin C - R \sin B \end{aligned} \quad (\because \sin(-C-A) = -\sin(C+A) = -\sin(180^\circ - B) = \sin B) \quad \text{--- (2)}$$

Multiplying and adding (1) & (2), we get

$$\begin{aligned} S^2 &= P^2 + Q^2 (\cos^2 C + \sin^2 C) + R^2 (\cos^2 B + \sin^2 B) \\ &\quad + 2QR (\cos B \cos C - \sin B \sin C) - 2RP \cos B - 2PQ \cos C \\ &= P^2 + Q^2 + R^2 - 2QR \cos A - 2RP \cos B - 2PQ \cos C. \\ (\because \cos B \cos C - \sin B \sin C &= \cos(B+C) = \cos(180^\circ - A) \\ &= -\cos A) \end{aligned}$$

Replacing P, Q, R by $k \cos A, k \cos B, k \cos C$; we get

$$S^2 = k^2 [\cos^2 A + \cos^2 B + \cos^2 C - 2 \cos A \cos B \cos C]$$

$$\text{But } \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C. \quad (\text{my trigonometry})$$

$$\therefore S^2 = k^2 [1 - 2 \cos A \cos B \cos C].$$

$$\Rightarrow S = k \sqrt{1 - 2 \cos A \cos B \cos C}. \quad \text{Ans.}$$

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If coplanar forces F_1, F_2, \dots, F_n acting at the point

O be in $\perp\!\!\!$ then,

$$\bar{R} = 0$$

$$\therefore |\bar{R}| = \sqrt{x^2 + y^2} = 0$$

which is true only if $x = 0$ and $y = 0$.

Hence the conditions are necessary.

Sufficient Conditions

If $x = 0$ and $y = 0$ then the forces are in $\perp\!\!\!$.

$$\therefore \bar{R} \cdot \hat{i} = x \text{ and } \bar{R} \cdot \hat{j} = y$$

$$\text{If } x = 0 \text{ and } y = 0$$

$$\text{then } \bar{R} \cdot \hat{i} = 0 \text{ and } \bar{R} \cdot \hat{j} = 0$$

Since \hat{i} and \hat{j} are not zero and \bar{R} cannot be $\perp\!\!\!$ to \hat{i} and \hat{j} as they are coplanar.

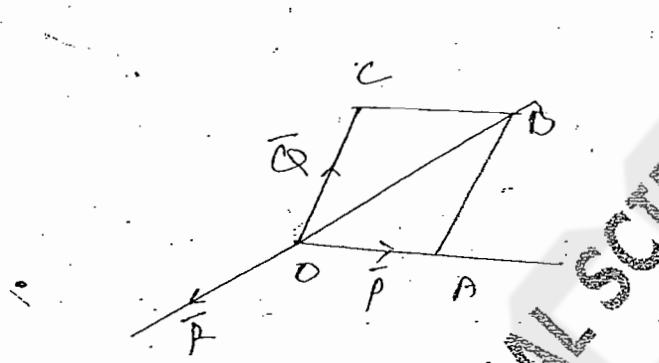
$$\therefore \text{we have } \bar{R} = 0$$

Hence the forces are in $\perp\!\!\!$.

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$$\text{or, } \bar{P} + \bar{Q} = -\bar{R} \quad \text{--- (1)}$$



Let $\bar{OA} = \bar{P}$, $\bar{OC} = \bar{Q}$
Completing the parallelogram $AA'BC$, we have,

$$\bar{AB} = \bar{A}'B = \bar{Q}$$

$$\text{and } \bar{OA} + \bar{AB} = \bar{OB} \quad \text{--- (2)}$$

From (2), we have,

$$\bar{OB} = -\bar{P}$$

$$\text{or, } \bar{BO} = \bar{P}$$

Thus the sides OA , AB and BO of $\triangle ABC$ represent the forces \bar{P} , \bar{Q} , \bar{R} in magnitude and direction, taken in order.



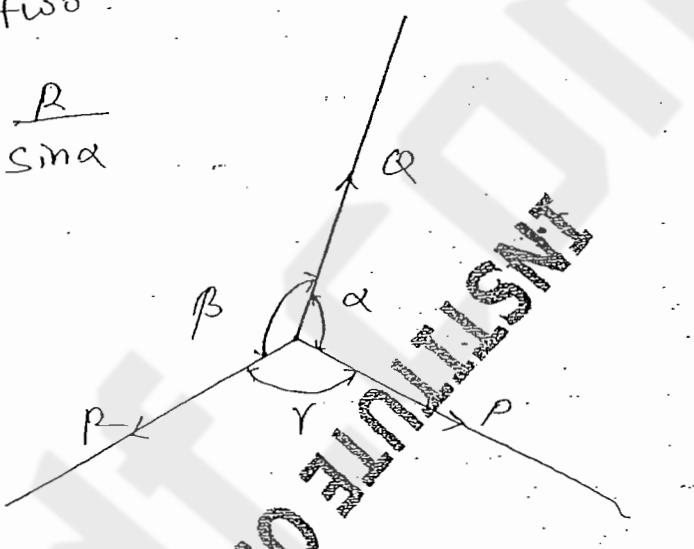
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Lami's Theorem

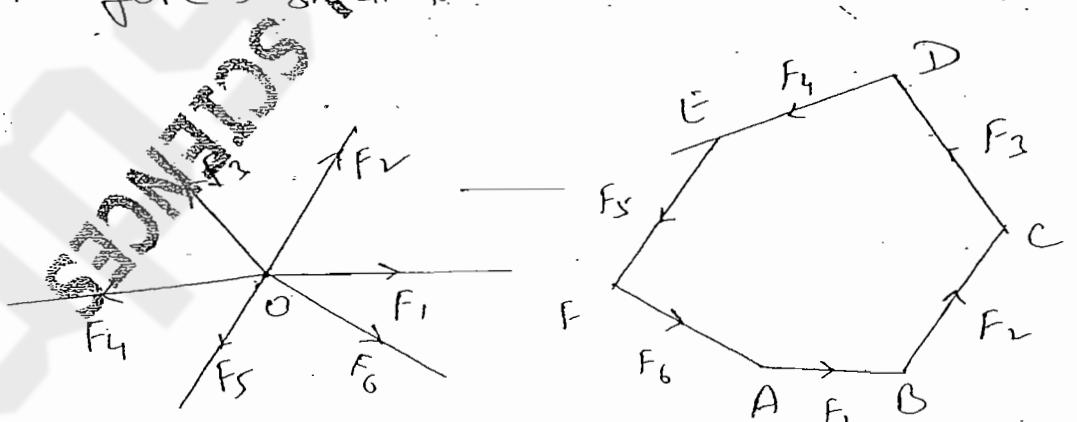
If three forces acting on a particle keep it in \equiv
 Each is proportional to the ~~sides~~ sine of the angle
 between the other two.

$$\text{i.e. } \frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha}$$



Polygon of forces

If any number of forces, acting on a particle
 be represented, in magnitude and direction, by
 the sides of a closed polygon, taken in order,
 the forces will be in \equiv .



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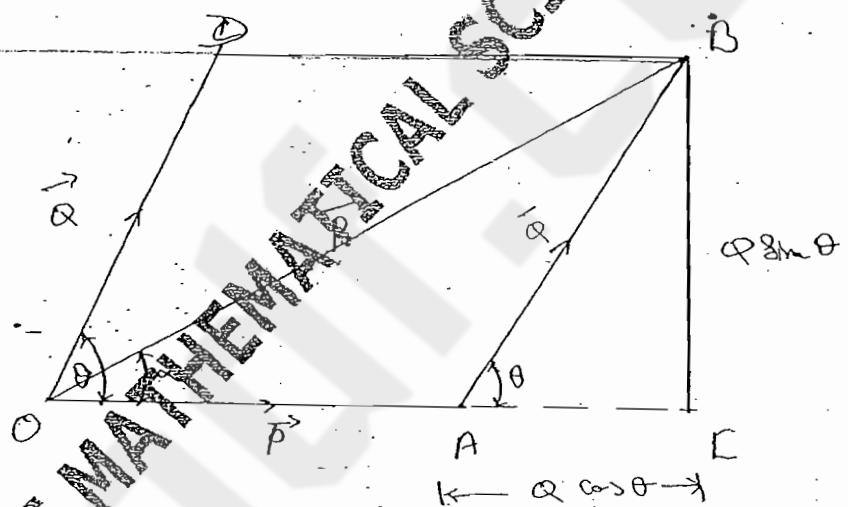
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- (Q.1) The resultant of forces P and Q is R . If Q is doubled in magnitude, R is doubled in magnitude. If Q is reversed, R is again doubled in magnitude. Show that

$$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$$

Soln:



Force representations are shown in figure.

$$\begin{aligned} R^2 &= \overline{P} + \overline{Q} \\ &= P^2 + Q^2 + 2PQ \cos \theta \quad (1) \end{aligned}$$

When Q is doubled, R is doubled.

$$\begin{aligned} \therefore (2R)^2 &= P^2 + (2Q)^2 + 2P(2Q) \cos \theta \\ \Rightarrow 4R^2 &= P^2 + 4Q^2 + 4PQ \cos \theta \quad (2) \end{aligned}$$

when Q is reversed, the resultant R is again doubled in magnitude.

$$\therefore \text{angle between } P \text{ and } Q = \pi - \theta$$

$$\therefore (2P)^2 = P^2 + (+Q)^2 + 2 \cdot P \cdot (+Q) \cos(\pi - \theta)$$

$$\Rightarrow 4P^2 = P^2 + Q^2 - 2PQ \cos\theta \quad (3).$$

Adding (1) & (3), we have

$$\begin{aligned} 5P^2 &= 2P^2 + 2Q^2 \\ \Rightarrow 2P^2 + 2Q^2 &= 5P^2 \quad (4) \end{aligned}$$

From, $2x(1) + (2)$, we have

$$= 3P^2 + 6Q^2$$

$$\therefore 3P^2 + 6Q^2 - 12P^2 = 0 \quad (5)$$

From (4) & (5), we have,

$$\frac{P^2}{6} = \frac{Q^2}{3} = \frac{P^2}{2}$$

$$\text{or, } \frac{P^2}{2} = \frac{Q^2}{3} = \frac{P^2}{6}$$

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$$\therefore P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$$

Proved

Q.2. The greatest resultant which two forces can have is P and least is Q . Show that if they act at an angle θ , the resultant is of magnitude

$$(P^2 \cos^2 \theta_{1/2} + Q^2 \sin^2 \theta_{1/2})^{1/2}$$

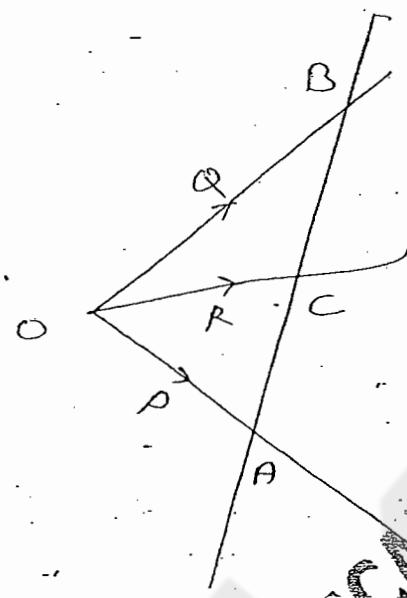
Hints. 14. F_1, F_2 be two forces. $F_1 = |F_1|$, $F_2 = |F_2|$

$$\therefore P = |P| = |F_1 + F_2|, Q = |F_1 - F_2|$$

$$\frac{P+Q}{2}, F_2 = \frac{P-Q}{2}$$

Forces P and Q act at O and have a resultant R . If any transversal cuts their line of action at A, B and C respectively then show that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$



in vector notation, \vec{P} can be written as

$$\vec{P} = P \frac{\vec{OA}}{|OA|}$$

$$= P \frac{\vec{OA}}{|OA|}$$

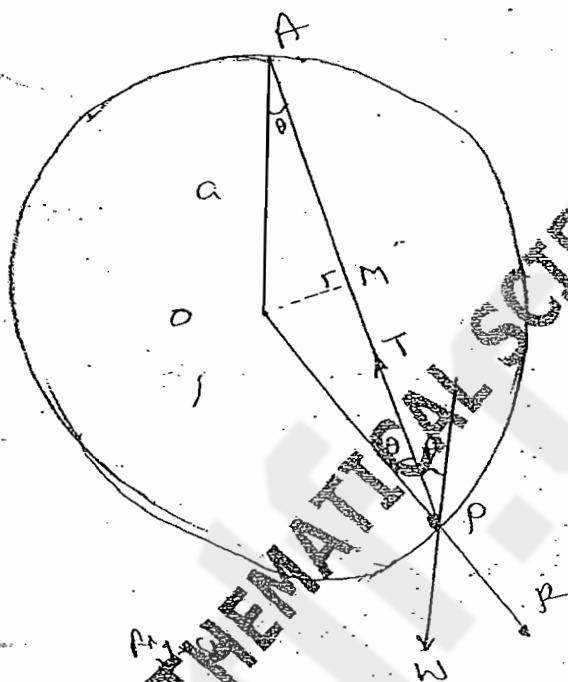
Similarly, $\vec{Q} = Q \frac{\vec{OB}}{|OB|}$

From L-H theorem, resultant \vec{R} of two forces \vec{P} and \vec{Q} is given by

$$\vec{R} = \left(\frac{P}{|OA|} + \frac{Q}{|OB|} \right) \vec{OC} \quad (1)$$



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Soln

Let 'O' be the centre of the circular wire of radius 'a'

One end of the string AP , of length 'l' is fixed at 'A' (at the highest point of the wire) and a weight 'W' is attached to the other end of the string, be it \Rightarrow when it is at the point P of the wire.

Free Body Diagram of the ring is shown below.



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But \vec{P} can be written as

$$\vec{P} = \frac{R}{OC} \vec{OC} \quad \text{--- (2)}$$

∴ from (1) & (2), we have,

$$\left(\frac{P}{OA} + \frac{Q}{OB} \right) \vec{OC} = \frac{R}{OC}$$

$$\Rightarrow \frac{P}{OA} + \frac{Q}{OB} \vec{OC}$$

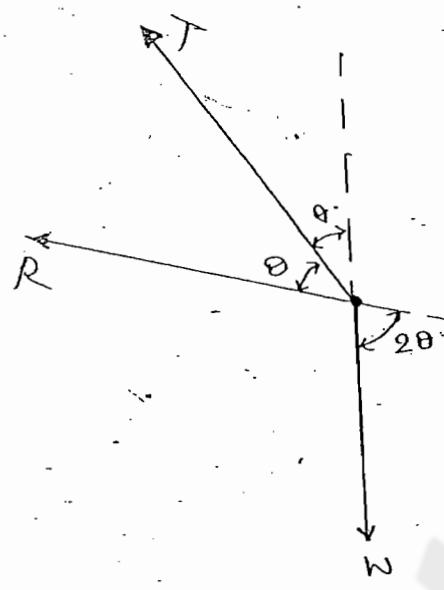
Proved

Q.4. One end of a light inextensible string of length 'l' is fastened to the highest point of a circular wire of a radius 'a', which is kept in a vertical plane. The other end of the string is attached to a small heavy ring of weight 'w' which slides on the wire. Find the tension of the string and the reaction of the wire.

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Where, W = weight of the ring T = Tension in the string R reaction force on the ring applied by circular wire along the normal to

Applying Lami's theorem, we have,

$$\frac{T}{\sin(\pi - 2\theta)} = \frac{W}{\sin\theta} = \frac{R}{\sin(\pi + \theta)}$$

$$\Rightarrow \frac{T}{\sin\theta} = \frac{-W}{\sin\theta} = \frac{R}{-\sin\theta}$$

 $\therefore R = -W$ (negative sign shows, it will act on the ring)

$$\text{Now } T = W \cdot \frac{\sin\theta}{\sin\theta}$$

Since,

$$\therefore l = AP \quad \text{and} \quad OA = a$$

 \therefore from figure (1)

$$l = 2 \cdot a \cdot \cos \theta$$

$$\therefore \cos \theta = \frac{l}{2a}$$

 \therefore Tension in the string

$$T = \frac{wl}{a}$$

Q) Reaction of the wire

$$R = w$$

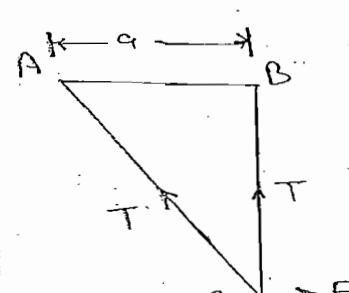
Ans

Q.S. A string of length 'l' is fastened to two points A and B at the same level at a distance 'a' apart. A ring of weight 'w' can slide on the string. If it is in \Rightarrow vertically below B, find that

$$F = w \frac{a}{l} \quad \text{and} \quad \text{that the tension in the string is } \frac{w(l^2 + a^2)}{2l}$$

$$\text{Hint: } AC + BC = l$$

apply Lami's theorem



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- (4) Three forces P, Q, R acting on a particle are in \perp . If the angle b/w P and Q is double the angle b/w P and R then prove that

$$P = \frac{Q^2 + R^2}{Q + R}$$

- (5) A and B are two fixed points at horizontal distance d apart. Two light strings of length a and b respectively support a mass m such that the tensions in the strings are in the ratio $a^2 : b^2$. Find the ratio $a : b$.

$$\frac{T_a}{T_b} = \frac{a^2}{b^2} \Rightarrow \frac{T_a}{m g} : \frac{T_b}{m g} = a^2 : b^2$$

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Sol: Let ABCD be the top of the table.

Let the legs be at L, M, N, K. The weight of the table acts at G, the centre of gravity of the table.

Let the weight P be placed at the corner C.

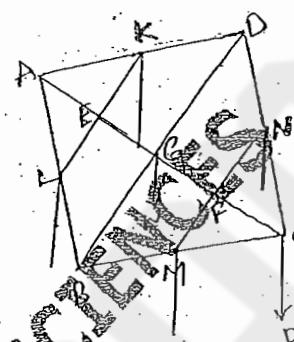
Taking moments of P and w about the line MN, we have

$$w \cdot |GFI| = P \cdot |FC|$$

$$\text{But, } |GFI| = |FC|$$

$$\therefore w \cdot |FC| = P$$

$$\Rightarrow P = w$$



Varrignon's theorem

The algebraic sum of the moments of two coplanar forces, performing a couple, about any point in their plane, is equal to the moment of their resultant about that point.

A force F acts at a point (3,4) of the xy-plane. The force is directed away from the origin and inclined at 60° to the x-axis. The horizontal component of F is 5 kg. wt.

(a) Determine the force F.

(b) Using Varrignon's theorem, calculate the moment of F about the origin.

(c) Hence find the perpendicular distance of the origin from the line of action of F.

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Draw AT \perp BC.From $\triangle DOL \sim \triangle DAT$,

we have,

$$\frac{AT}{OL} = \frac{AD}{OD} = \frac{3OD}{OD}$$

$$\Rightarrow OL = \frac{AT}{3}$$

Now if $S = \text{area } \triangle ABC$ then, $S = \frac{1}{2} \cdot AT \cdot BC$

$$\Rightarrow AT = \frac{2S}{BC}$$

$$OL = \frac{2}{3} \cdot \frac{S}{a}$$

$$\text{Similarly, } OM = \frac{2S}{3b} \quad \& \quad ON = \frac{2S}{3c}$$

Putting the value of OL , OM & ON in eqn (1),
we have,

$$P \cdot \frac{2S}{a} + Q \cdot \frac{2S}{3b} + R \cdot \frac{2S}{3c} = 0$$

$$\Rightarrow \frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0$$

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$$\begin{aligned} \vec{M} &= \vec{OA_r} \times \vec{F_r} \\ &= \vec{Oa_r} \times \vec{F_r} \end{aligned}$$

Q) Besides this couple or single force F_r if O is top is left at O .

Thus the single force F_r acting at O is equivalent to a force F_r acting at 'O' and a couple of moments $\vec{Oa_r} \times \vec{F_r}$.

Similarly other forces F_1, F_2, \dots, F_n acting at the point A_1, A_2, \dots, A_n whose position vectors referred to O are

$$\vec{a}_1, \vec{a}_2, \dots$$

respectively are equivalent to single forces F_1, F_2, \dots, F_n acting at 'O' and couples of moments $\vec{Oa}_1 \times \vec{F}_1, \vec{Oa}_2 \times \vec{F}_2, \dots, \vec{Oa}_n \times \vec{F}_n$.

Hence the given system of forces will be equivalent to forces $F_1, F_2, F_3, \dots, F_n$ acting at 'O' together with couples of moments

$$\vec{Oa}_1 \times \vec{F}_1, \vec{Oa}_2 \times \vec{F}_2, \dots, \vec{Oa}_n \times \vec{F}_n$$

If R is the resultant of the concurrent forces $F_1, F_2, F_3, \dots, F_n$ acting at 'O', then we have



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$$\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \dots + \bar{F}_n = \sum_{r=1}^n \bar{F}_r$$

and if M is the moment of the resultant couple of the above couples, then

$$M_r = \bar{a}_1 \times \bar{F}_1 + \bar{a}_2 \times \bar{F}_2 + \dots + \bar{a}_n \times \bar{F}_n = \sum_{r=1}^n \bar{a}_r \times \bar{F}_r$$

Hence the system of coplanar forces acting on a body is equivalent to a single force.

$$\bar{R} = \sum_{r=1}^n \bar{F}_r$$

Acting at an arbitrarily chosen point 'O' together with a couple of moment

$$M = \sum_{r=1}^n \bar{a}_r \times \bar{F}_r$$

* Necessary and sufficient conditions for \Rightarrow of rigid body.

The necessary and sufficient conditions for the \Rightarrow of a rigid body under the action of a system of coplanar forces acting at different points of it are that the sums of the resolved parts of the forces in any two mutually tv directions vanish separately and the sum of the moments of the forces about any point in the plane

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Cartesian form. From (1), we have,

$$x_i + y_j = \left(\frac{M}{Ry}\right)j + (R_n + Ryj)$$

Equating the coefficients of i and j on both sides, we have,

$$x = \frac{M}{Ry} + R_n, \quad y = + Ry$$

Eliminating ' t ' from them, we get

$$x = \frac{M}{Ry} + \frac{y}{Ry} \cdot R_n$$

$$\Rightarrow Ry \cdot x = R_n \cdot y$$

Which is the equation of the line of action of the resultant in Cartesian form.

Q. Three forces P, Q, R act along the sides of a triangle formed by the line $x+y=1$, $y-x=1$ and $y=2$. Find the equation of the line of action of the resultant.

Soln : Let the three forces P, Q, R act along the lines

$x+y=1$, $y-x=1$ and $y=2$ i.e. the lines of



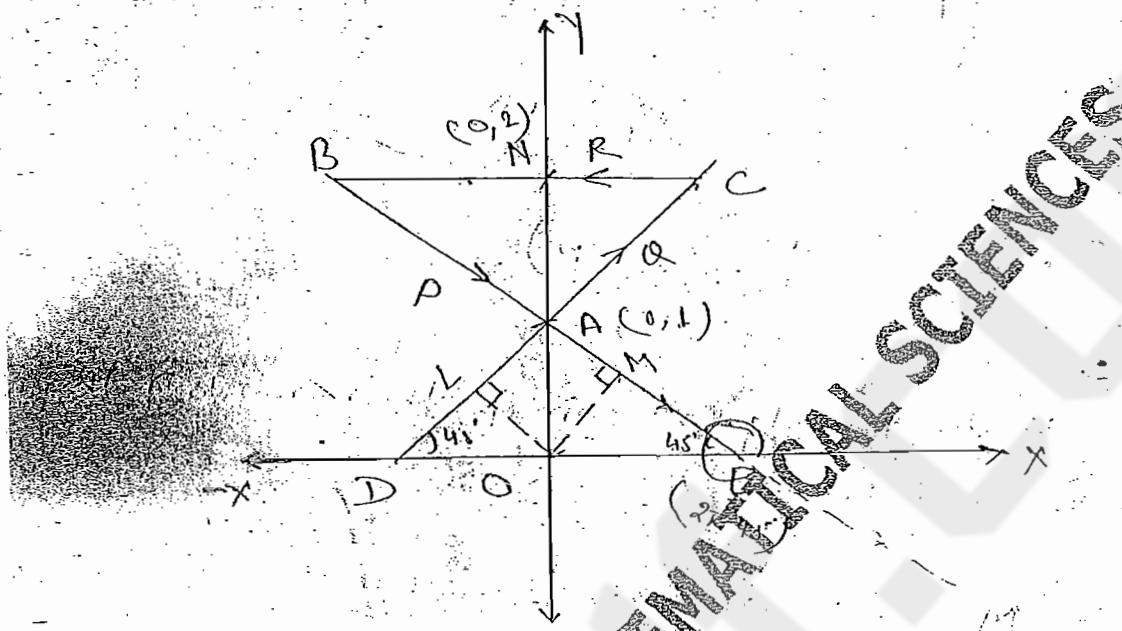
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(15)

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Let \vec{F} be the resultant force of \vec{P} , \vec{Q} & \vec{R}

$$\vec{F} = \vec{P} + \vec{Q} + \vec{R}$$

If F_x = resultant force in x -direction

$$F_x = P \cos(2\pi - 45^\circ) + Q \cos(45^\circ) + R \cos(\pi)$$

$$= \frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} - R$$

$$\therefore F_x = \frac{1}{\sqrt{2}} (P + Q - \sqrt{2}R)$$

if F_y = resultant force in y -direction

then,

$$F_y = P \sin(2\pi - 45^\circ) + Q \sin 45^\circ + R \sin 135^\circ$$

$$= -\frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} + R$$

$$\therefore F_y = \frac{1}{\sqrt{2}}(Q - P)$$

Let M be the algebraic sum of the moments of the forces about the origin 'O'.

$$M = -P \cdot OM - Q \cdot OL + R \cdot ON$$

$$= -P \cdot OA \sin 45^\circ - Q \cdot OA \sin 45^\circ + R \cdot 2$$

$$= \frac{1}{\sqrt{2}}(P + Q) + 2R$$

Projection of the fraction of resultant force

$$F_x \cdot x - F_y \cdot y = M$$

$$\Rightarrow \frac{1}{\sqrt{2}}(Q - P)x - \frac{1}{\sqrt{2}}(P + Q - \sqrt{2}R)y = -\frac{1}{\sqrt{2}}(P + Q - \sqrt{2}R)$$

$$\Rightarrow -(Q - P)x - (P + Q - \sqrt{2}R)y = 2\sqrt{2}R \quad (2)$$

\equiv

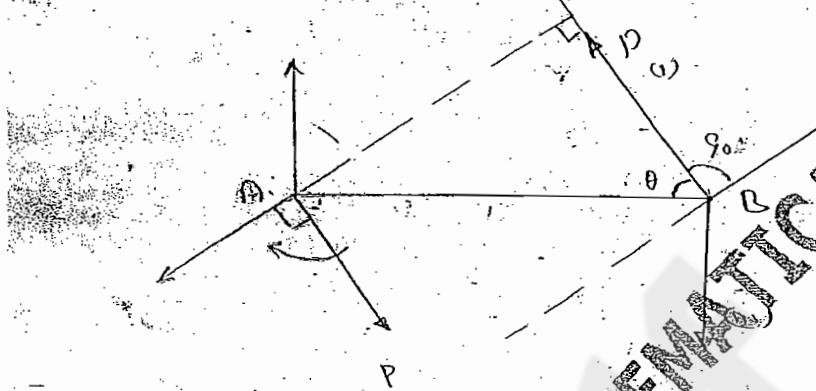


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- Q. Two effort utilize 114 forces acting at fixed points A and B form a couple of moment C if their lines of action are turned through one right angle they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment

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- Q. Weights w_1 , w_2 are fastened to a light inextensible string ABC at the point B, the end A being fixed. Prove that, If horizontal force 'P' is applied at 'C' and if the strings AB and BC are inclined at angle θ , then

$$P = (w_1 + w_2) \tan \theta = w_1 \tan \phi$$

Soln. Forces are shown in the figure.

14. T_1 = tension in the string AB

T_2 = tension in the string BC

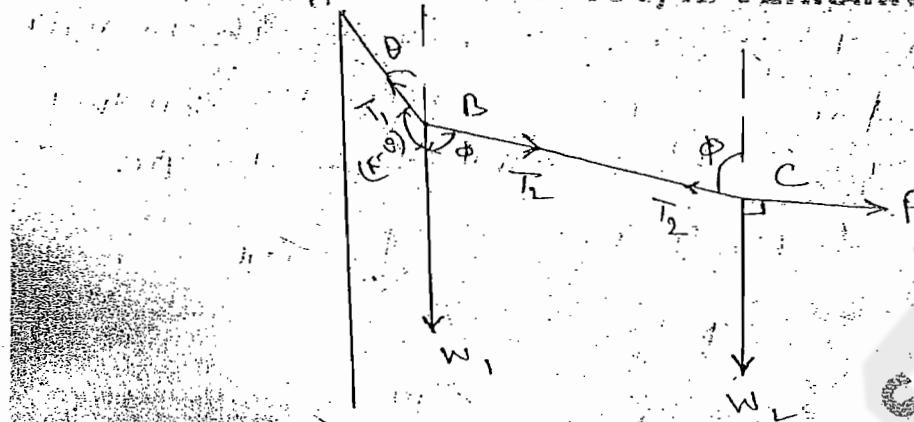


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Now, applying Lami's theorem at point P,

We have,

$$\frac{P}{\sin(\pi - \alpha)} = \frac{w_1}{\sin(\pi/2 + \theta)} \quad \text{Eqn 1}$$

$$\Rightarrow \frac{P}{\sin \phi} = \frac{T_2}{\sin \theta} = \frac{w_2}{\cos \theta}$$

$$P = w_2 \tan \phi \quad \& \quad T_2 = \frac{w_2}{\cos \theta}$$

Again, applying the Lami's theorem at B,

$$\frac{T_1}{\sin \phi} = \frac{T_2}{\sin(\pi - \theta)} = \frac{w_1}{\sin(\pi - \phi + \theta)}$$

$$\Rightarrow \frac{T_1}{\sin \phi} = \frac{T_2}{\sin \theta} = \frac{w_1}{\sin(\phi - \theta)}$$

$$\Rightarrow \frac{T_2}{\sin \theta} = \frac{w_1}{\sin(\phi - \theta)}$$

$$\frac{w_2}{\cos \phi \cdot \sin \theta} = \frac{w_1}{\sin(\phi - \theta)} \quad \text{Eqn 2}$$

$$\begin{aligned}
 & \text{W}_1 \xrightarrow{\cos\phi \cdot \sin\theta} \\
 & \text{W}_2 \xrightarrow{\sin\phi \cdot \cos\theta - \cos\phi \cdot \sin\theta} \\
 \Rightarrow & \text{W}_2 \xrightarrow{\tan\phi \cdot \cot\theta - 1} \\
 \Rightarrow & \text{W}_2 (\tan\phi \cdot \cot\theta - 1) = \text{W}_1 \\
 \Rightarrow & (\text{W}_2 \cdot \tan\phi) \cdot \cot\theta - \text{W}_2 = \text{W}_1 \\
 \Rightarrow & \text{P} = (\text{W}_1 + \text{W}_2) \quad (2) \\
 & \text{[From (1), } \text{P} = \text{W}_1 + \tan\phi]
 \end{aligned}$$

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\therefore from (1) & (2),

$$\text{P} = (\text{W}_1 + \text{W}_2) \tan\theta \stackrel{\text{Prove}}{=} \text{W}_1 + \tan\phi$$

Q4. A uniform circular disc of weight mw has a heavy particle of weight w attached to a point 'C' on its rim. If the disc is suspended from a point 'A' on its rim, 'B' being the lowest point, and if suspended from 'B', 'A' is the lowest point, show that the angle subtended by AB at the centre of the disc is $2 \sec^{-1}(e(n+1))$.



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Let O = centre of the circular disc.

Case 1:

Forces acting on the body is shown in figure.

$$\text{Let } \angle AOC = \theta$$

$$\angle BOC = \phi$$

In this case body is in equilibrium under the action of three forces nW , W and reaction at A .

Total moment about $A' \neq 0$

i.e. $MA = 0$

$$nW \cdot AL + W \cdot A'C = 0$$

$$n \cdot AL \cdot r = A'C = NC - NA$$

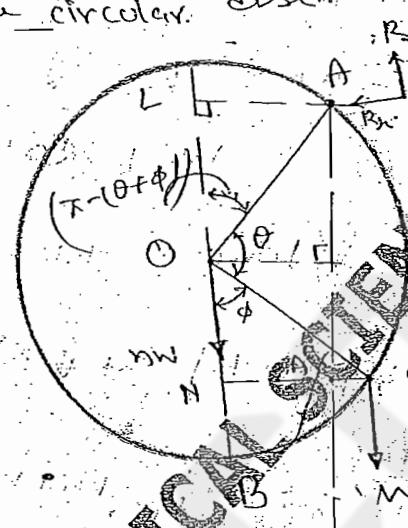
$$\Rightarrow n \cdot AL = NC - EA$$

$$\Rightarrow (n+1) AL = NC$$

$1. r = \text{radius of disc}$

$$\therefore (n+1) r \cdot \sin(\pi - (\theta + \phi)) = r \sin \phi$$

$$\Rightarrow (n+1) \sin(\theta + \phi) = \sin \phi \quad (1)$$



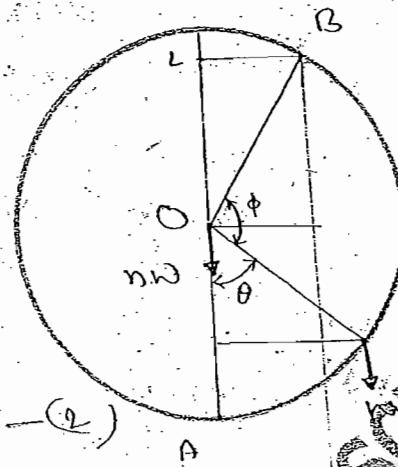
Case II:

Similarly,

$$\angle MCB = \theta$$

$$\Rightarrow (n+1) \sin(\theta + \phi)$$

$$= \sin\theta - (2)$$



(18)

From (1) & (2), we have

$$\theta = \phi$$

Putting $\theta = \phi$, in (1), we have

$$(n+1) \sin 2\theta = \sin\theta$$

$$\Rightarrow (n+1) 2 \cdot \sin\theta \cdot \cos\theta = \sin\theta$$

$$\Rightarrow 2(n+1) \cos\theta = 1$$

$$\cos\theta = \frac{1}{2(n+1)}$$

the required angle

$$\angle AOB = \theta + \phi = \theta + \theta = 2\theta$$

$$= 2 \cdot \sec^{-1} \left(\frac{1}{2(n+1)} \right)$$

Prove.

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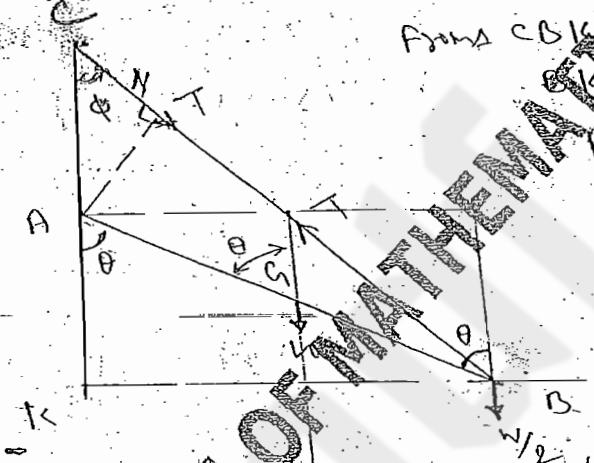
Q5. A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened at weight $\frac{W}{2}$. The weight of the rod being W , this end is fastened by a string of length $'l'$ to a point at a height 'C' vertically over the hinge. Show that the tension of the string is $\frac{W}{2}$.

$$\text{Hint: } \sum M_A = 0$$

$$\text{From } CBK \rightarrow ADK$$

$$BK = BK$$

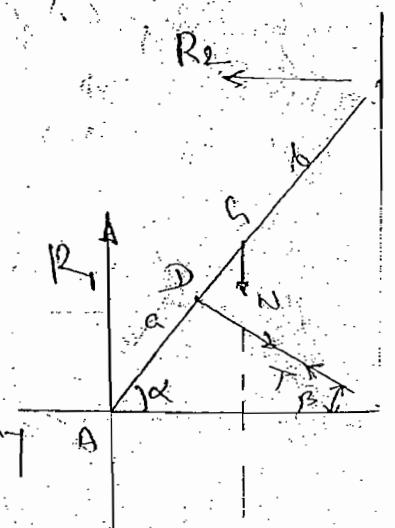
$$\text{Length of rod} = 2a (\text{say})$$



Q6. A beam of weight W , is divided by its centre of gravity into two portions AB and CB , whose lengths are a and b respectively. The beam

rests in a vertical plane on a smooth floor at A and against a smooth wall at C .

Q7. A string is attached to a hook at C and to the beam at point D . If T be the tension of the string and α & β be the inclinations of the beam and string respectively to the horizon, show that



(19)

F.Q: Two equal uniform rods AB, AC each of weight W are freely joined at A and rest with the extremities B and C on the inside of a smooth circular hoop whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being joined by a light string. Show that if the string is stretched, its tension is $W(\tan \alpha - 2 \tan \beta)$ where, α is the angle b/w the rods, and β the angle either rod subtends at the centre.

Soln:

Let AB, AC be the two rods, freely joined at A and each having weight W, be placed inside the smooth circular hoop.

If centre O is shown as figure

At the middle point D & E of respective rods, be joined by a string, and tension T is induced.

Let P = reaction at B & C by the circular hoop.

Given that

$$\angle BAK = \alpha = \angle CAK$$



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$$\text{and } \angle BOK = \angle COK = \beta$$

$$\angle ABO = \angle ACO = \alpha - \beta$$

For $\Rightarrow q$ -rod, AB (cm) AC

$$\sum F_y = 0$$

$$\Rightarrow 2P \cos \beta = w \Rightarrow P = \frac{w}{2 \cos \beta}$$

Now Consider the $\Rightarrow q$ -rod, along A'

$$\sum M_A = 0$$

$$\Rightarrow R \cdot AN - w \cdot AM - T = 0$$

$$\Rightarrow R \cdot AN = w \cdot AM + T \cdot AL$$

If H = length of each rod

then AY figure,

$$R \cdot \sin(\alpha - \beta) = w \cdot \frac{1}{2} \sin \alpha + T \cdot \frac{1}{2} \cos \alpha$$

$$\text{put } P = \frac{w}{2 \cos \beta} \text{ from (1)}$$

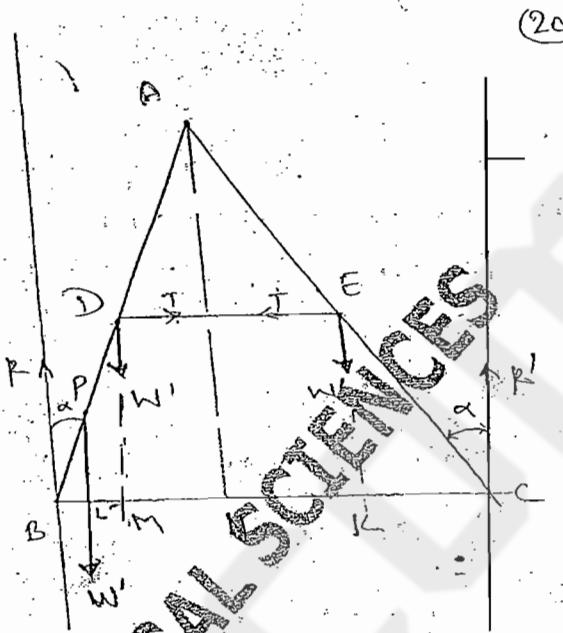
$$\frac{w}{2 \cos \beta} \cdot \sin(\alpha - \beta) = \frac{w}{2} \cdot \sin \alpha + \frac{T \cdot \cos \alpha}{2}$$

$$T \cdot \cos \alpha = \frac{2w}{\cos \beta} (\sin \alpha \cos \beta - \sin \beta \cos \alpha) = w \cdot$$

$$\Rightarrow T \cdot \cos \alpha = w(\sin \alpha - 2 \cdot \tan \beta \cdot \cos \alpha)$$

$$\therefore T = w(\tan \alpha - 2 \tan \beta) \quad \text{prove}$$

(Q. 2) A step ladder in the form of the letter A with each of its legs inclined at an angle α to the vertical is placed on a horizontal floor and is held up by a cord connecting the middle points of its legs, there being no friction anywhere. Show that when a weight W is placed on one of the steps at a height from the floor equal to n of the steps, if a height from the floor equal to h of the ladder, the increase in tension of the cord is $(h/n \tan \alpha)$.



Hints: $PL = \frac{1}{n} AN$

$$BM = MN = NK = \dots$$

$$\sum M_B = 0 \Rightarrow P = \frac{1}{2n} W + W'$$

$$\text{For leg } A, \sum M_A = 0$$

Since three forces acting upon a rigid body. If it is in equilibrium, they must either meet in a point or be parallel.

$$\therefore \bar{P} + \bar{Q} + \bar{R} = \bar{0} \text{ (for } \rightleftharpoons \text{ force)}$$



Also the sum of the moments of the forces about any point must be zero.

$$\therefore M_A = 0$$

$$\vec{AB} \times \bar{Q} + \vec{AC} \times \bar{R} = 0$$

$$\therefore \vec{AB} \times \bar{Q} = -\vec{AC} \times \bar{R}$$

$$\therefore \bar{n} \times \bar{Q} = \bar{n} \times \bar{R}$$



Since the forces \bar{Q} & \bar{R} are coplanar force, then their line of action must either intersect or be parallel.

If \bar{Q} , \bar{R} intersect, then so \bar{n} passes through point O.

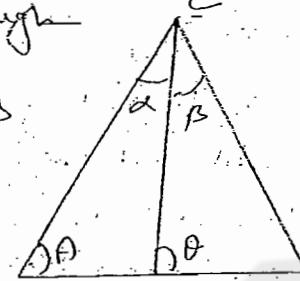
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- If a line CD be drawn through the vertex C of a $\triangle ABC$ meeting the opposite side AB in D and dividing it into two parts m and n as



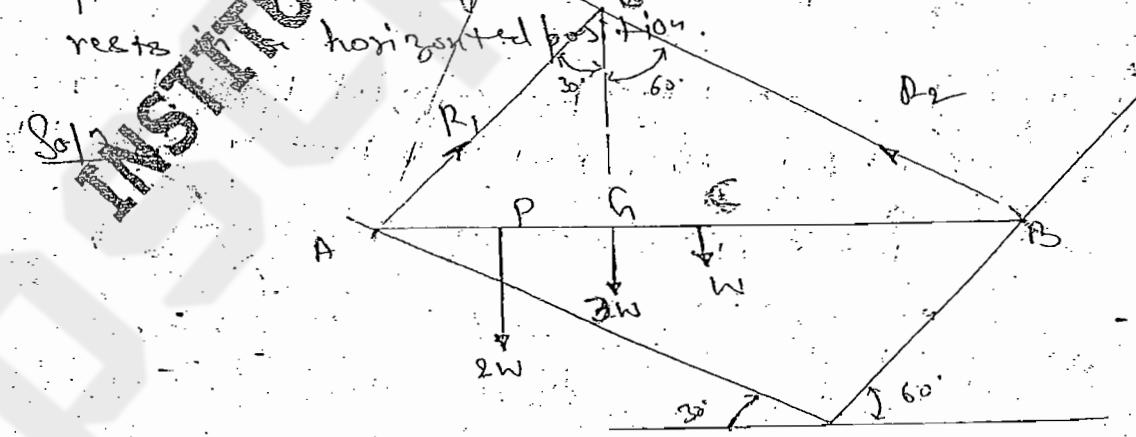
the angle 'C' into two parts α and β , and if

$$\angle CBD = \theta \text{, then}$$

$$(i) (m+n) \cot \theta = m \cot \alpha + n \cot \beta$$

$$(ii) (m+n) \cot \theta = n \cot \alpha - m \cot \beta$$

e.g. A uniform beam rests with its ends on two smooth inclined planes which makes angles of 30° and 60° with the horizontal respectively. A weight equal to twice the weight of the beam can slide along its length. Find the position of the sliding weight when the beam rests in a horizontal position.

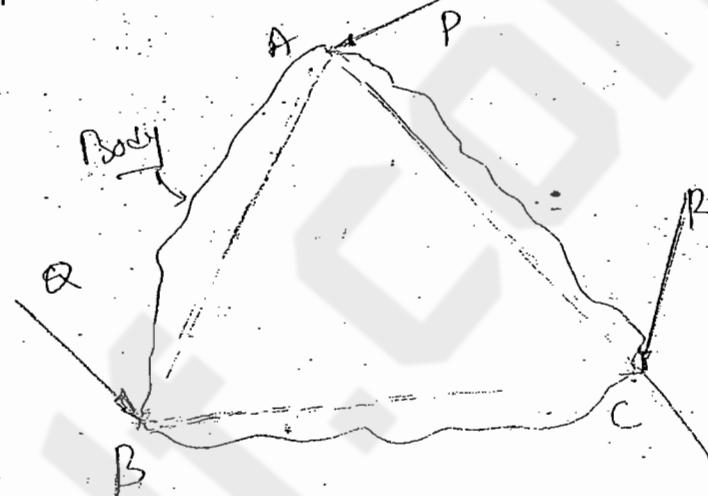


Let the beam AB rests in a horizontal position with its ends A, B on two smooth inclined planes. Find the ratio of the weights

20(i)

Equilibrium of a rigid body under the action of three forces only.

Theorem If three forces acting upon a rigid body keep it in \rightleftharpoons they must either meet in a point or be parallel.



Proof: Let the three forces \vec{P} , \vec{Q} , \vec{R} acting at the points A, B, C respectively of a rigid body, keep it in \rightleftharpoons

Since the forces are in \rightleftharpoons

$$\therefore \vec{P} + \vec{Q} + \vec{R} = 0 \quad \textcircled{1}$$

Also the sum of the moments of the forces about any point must be zero.

$$\therefore \sum M_A = 0$$

$$\Rightarrow \vec{AB} \times \vec{Q} + \vec{AC} \times \vec{R} = 0$$

$$\Rightarrow \vec{AB} \times \vec{Q} = \vec{CA} \times \vec{R} = \vec{n} \text{ (say)}$$

Thus \vec{n} is a vector fr to \vec{AB} , \vec{Q} , \vec{CA} and \vec{R} i.e.
vector \vec{n} is fr to the plane ABC and the \vec{Q} & \vec{R}

The forces \vec{Q} and \vec{P} must act in the plane ABC.

\vec{Q} & \vec{P} are coplanar, hence their lines of action must either intersect or be parallel.

I: If the \vec{Q} & \vec{P} intersect, then for $\vec{R} \rightleftharpoons$, their resultant must be equal and opposite to the third force \vec{P} .

Thus the third force \vec{P} must also act in the plane ABC.

As \vec{P} passes through the point of intersection of the forces \vec{Q} and \vec{P} , hence the three forces are coplanar and concurrent.

II: If the forces \vec{Q} and \vec{P} are parallel then their resultant is also parallel to them and for $\vec{R} \rightleftharpoons$ it must be equal and opposite to the third force \vec{P} .

Hence the three forces are coplanar and parallel.

Let the sliding weight ($2w$) act at P in \Rightarrow position.

(21)

R_1, R_2 = Reactions at A & B by the inclined plane on the beam AB, respectively.

For \Rightarrow the vertical line of action of the resultant force (weight) $3w$ at C will also pass through 'o' shown in fig.

in ΔAOb ,

$$Ah = Ob \cdot \tan 30^\circ = \frac{Ob}{\sqrt{3}}$$

and in ΔBOb , $Bh = Ob \cdot \tan 60^\circ = Ob\sqrt{3}$

$$\frac{Bh}{Ab} = 3$$

$$\Rightarrow \frac{Bh + Ah}{Ah} = 4$$

$$\Rightarrow \frac{AB}{Ah} = 4 \Rightarrow Ah = \frac{1}{4} AB$$

$$= 3+1$$

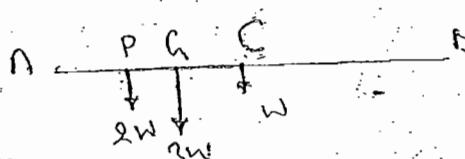
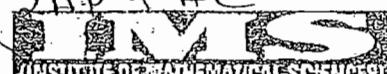
— (1)

But resultant of the weight w at C and $2w$ at P act at C. So for

$$\sum M_A = 0$$

$$\Rightarrow (3w) Ah = 2w \cdot AP + w \cdot AC$$

$$Ah = \frac{1}{3} (2AP + AC)$$



2w

3w

w

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$$AG = \frac{1}{3} \left(2AP + \frac{AB}{2} \right) \quad (1)$$

From (1) & (2), we have

$$\frac{1}{2}AB = \frac{1}{3} \left(2AP + \frac{AB}{2} \right)$$

$$\Rightarrow AP = \frac{AB}{8}$$

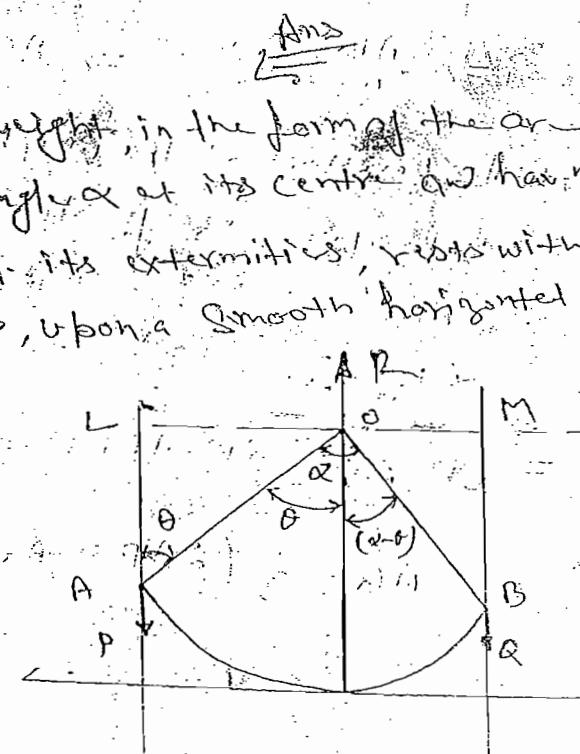
$$BP = AB - \frac{AB}{8} = \frac{7}{8}AB$$

$$\frac{AP}{BP} = \frac{1}{7}$$

Hence the beam will remain in the horizontal position on the inclined planes when the striking weight is at the point which divides the beam in the ratio

- (10) A wire, without weight, in the form of the arc of a circle subtending angle α at its centre and having weights P and Q at its extremities, rests with its concavity downwards, upon a smooth horizontal plane. Show that, if θ be the inclination to the vertical of the radius to the end of which P is suspended then

$$\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$



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Work done by a force

Let a force represented by the vector \vec{F} acts at the point A.

Let the point A be displaced to the point B,

Where, $\vec{AB} = \vec{d}$

Then the work done W by the force \vec{F} during the displacement \vec{d} if its point of application is defined as

$$W = \vec{F} \cdot \vec{d}$$

i.e. $W = \text{Scalar product of } \vec{F} \text{ and } \vec{d}$

Let θ be the angle between the vector \vec{F} & \vec{d} .

Qn) $F = |\vec{F}|$, $d = |\vec{d}| = AB$

$$W = F d \cos \theta$$

(2)

$= F \times (\text{displacement of the point of application of the } \vec{F} \text{ in the direction of the force})$

Hence, the work done by a force is equal to the magnitude of the force multiplied by the displacement



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of the point of application of the force in the direction of the force.

From Eq. (2), we have,

(i) if $\theta = \frac{\pi}{2}$, i.e. if the displacement of the point of application of the force is \vec{dr} to the direction of the force, then

$$W = 0$$

(ii) if $0 \leq \theta \leq \pi/2$ i.e. if the displacement of the point of application of the force is parallel to the line of action of the force in the direction of the force, then W is positive.

(iii) if $\pi/2 < \theta \leq \pi$, i.e. if the displacement of the point of application of the force, \vec{dr} to the line of action of the force is opposite to the direction of the force, then W is negative.

* The work done by force \vec{F} acting at the point P during small displacement \vec{dr} of its point of application is

$$\vec{F} \cdot \vec{dr}$$

Work done by a system of concurrent forces

(3)

- The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the separate forces.

Proof: Let there be n forces represented by the vectors $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ acting at a point P . Then along any displacement of P represented by the vector \vec{d} , the works done by the separate forces are respectively equal to:

$$\vec{F}_1 \cdot \vec{d}, \vec{F}_2 \cdot \vec{d}, \dots, \vec{F}_n \cdot \vec{d}$$

Total work done

$$W = (\vec{F}_1 \cdot \vec{d} + \vec{F}_2 \cdot \vec{d} + \dots + \vec{F}_n \cdot \vec{d})$$

$$= (\vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n) \cdot \vec{d}$$

$$\text{Hence, } \vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_n$$

But $\vec{R} \cdot \vec{d}$ is the work done by the resultant \vec{R} during the displacement \vec{d} of the point P .



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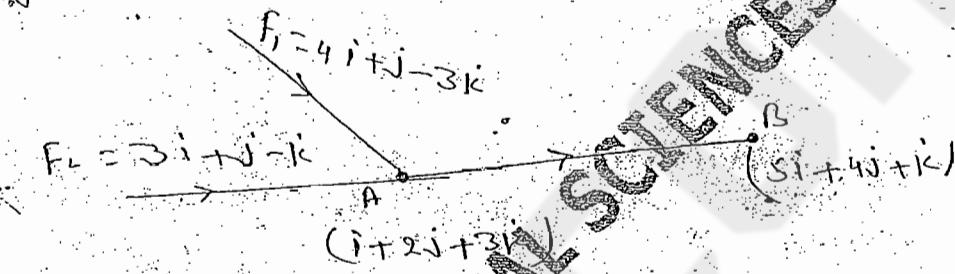
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Q. A Particle acted on by constant forces $4i + j - 3k$ and $3i + j - k$ is displaced from the point $i + 2j + 3k$ to the point $5i + 4j + k$. Find the total work done by the force.

Sol:

Let. R = resultant of two forces d = displacement of particle from $A \rightarrow B$.

$$\begin{aligned} R &= \vec{F}_1 + \vec{F}_2 \\ &= (4i + j - 3k) + (3i + j - k) \\ &= 7i + 2j - 4k \end{aligned}$$

$$\begin{aligned} d &= (5i + 4j + k) - (i + 2j + 3k) \\ &= 4i + 2j - 2k \end{aligned}$$

Total work done

$$\begin{aligned} W &= \vec{F} \cdot \vec{d} \\ &= (7i + 2j - 4k) \cdot (4i + 2j - 2k) \\ &= 28 + 4 + 8 \\ &= 40 \text{ units of work} \end{aligned}$$

Ans
↙

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(4)

Principle of Virtual Work

Virtual displacement (δs): of a point is any arbitrary infinitesimal change in the position of the point consistent with the constraints imposed on the motion of the point. This displacement can be imaginarily.

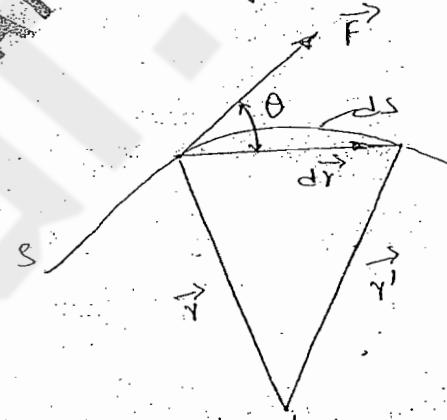
Work of a force and work of a couple moment

→ Work of a force on a particle is defined as

$$W = \vec{F} \cdot \vec{dr}$$

$$\text{or, } W = F ds \cos\theta$$

$$\text{or, } W = F r \cos\theta$$

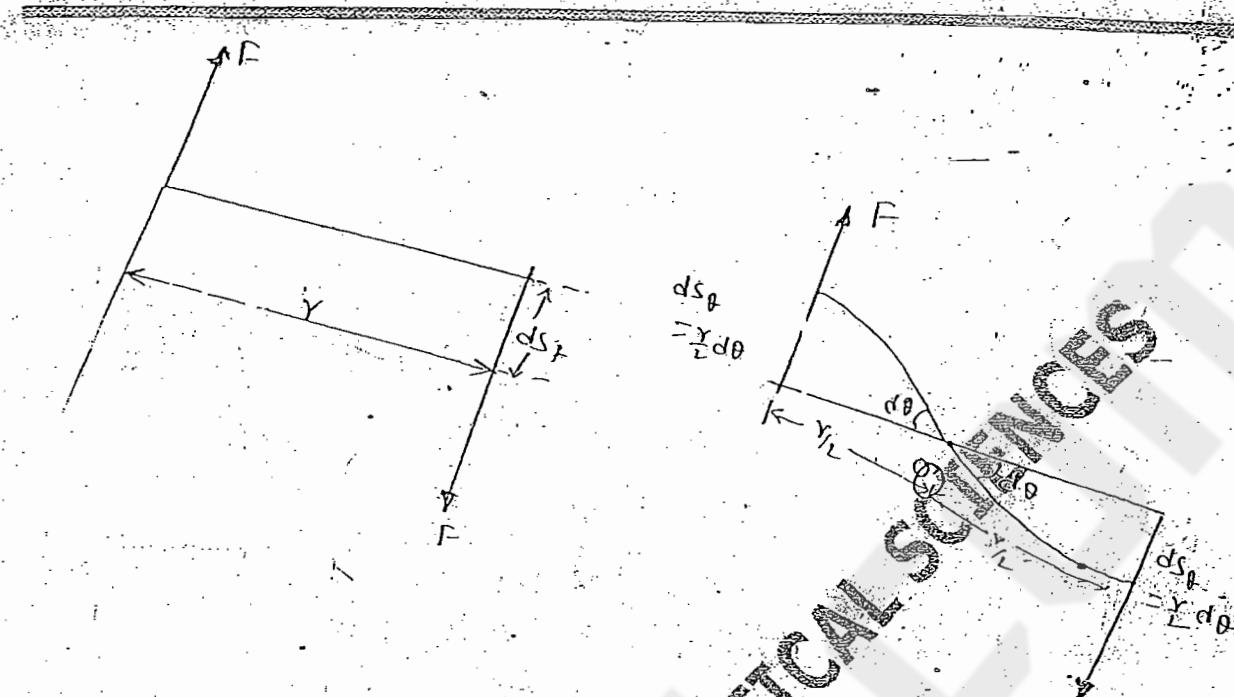


→ Any general differential displacement of a body can be considered as a combination of translation and rotation.

→ When a body is subjected to couple, the work corresponds to translation & zero. While the work corresponds to rotation is

$$dW = \vec{F} \cdot (\vec{r} d\theta) + \vec{F} \cdot (\vec{r} d\theta) = F r d\theta \\ = M d\theta$$

$$\text{or, } dW = \vec{M} \cdot \vec{d\theta}$$



→ For a body under static equilibrium, the virtual work is defined by external forces multiplying the 'virtual' movement along the direction of external forces, i.e.

$$\delta W = \vec{F} \cdot \vec{\delta r} \quad \text{or, } \delta W = M \cdot \vec{s} \theta$$

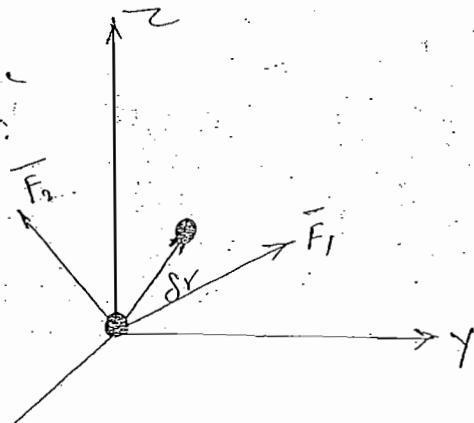
The virtual work for a particle under can be expressed as:

$$\delta W = \sum \vec{F} \cdot \vec{\delta r}$$

$$= (\sum F_{xi} i + \sum F_{yj} j + \sum F_{zk} k)$$

$$\therefore (\delta x_i + \delta y_j + \delta z_k)$$

$$= \sum F_i \delta x_i + \sum F_y \delta y + \sum F_z \delta z$$



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Principle of Virtual Work

(5)

the necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces is \Leftrightarrow is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation.

Proof Let a system of forces f_1, f_2, \dots, f_n act at the points of rigid body whose position vectors with respect to some origin O are $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$.

Suppose this system of forces is equivalent to a single force, $\vec{P} = \sum F_i$, acting at O , together with a couple of moment $\vec{M} = \sum r_i \times f_i$.

Then during any small displacement of the body consisting of uniform translation \vec{U} and a small rotation $\vec{\theta}$ about O ,

the sum of the works done by these forces

$$W = \vec{U} \cdot \vec{P} + \vec{\theta} \cdot \vec{M}. \quad (1)$$

The Conditions necessary : Let the given system be in \Leftrightarrow . Then $\vec{P} = 0$ and $\vec{M} = 0$.

From (1), the sum of the works done by the forces is zero. Hence the condition is necessary.

UT, IIT-JEE / IIT-JEE / CSIR EXAMINATIONS The Condition of Equilibrium by Superposition

The sum of the works done by the forces during any small displacement is zero. Then prove that the forces are in eq.

$$\vec{F} \cdot \vec{R} + \vec{G} \cdot \vec{G} = 0 \quad \text{--- (2)}$$

for any small displacement consisting of a uniform translation \vec{U} and a small rotation $\vec{\theta}$.

If $\vec{\theta} = 0$ and $\vec{U} \neq 0$

$$\text{then, } \vec{U} \cdot \vec{R} = 0$$

also if $\vec{U} \neq \vec{R}$

$$\vec{R} = 0$$

Now taking $\vec{\theta} \neq 0$ and $\vec{U} = 0$

$$\vec{\theta} \cdot \vec{G} = 0$$

if \vec{G} is not \vec{R} to $\vec{\theta}$ then

$$\vec{G} = 0$$

for any small displacement $\vec{\theta}$ and \vec{U} ,

we must have $\vec{R} = 0$ and $\vec{G} = 0$

* The equation (2) formed by the equating to zero the sum of the virtual works done by the forces is called the equation of virtual work.

* The above principle of virtual work and its proof equally holds whether the forces are coplanar or not and whether the forces act upon a particle or upon a rigid body.

(6)

- * Forces which are omitted in forming the equation of virtual work

The principle of virtual work gives us a very powerful method of solving the problems of mechanics. The mechanical advantage of this principle over other methods is that there are certain forces which are omitted in forming the equation of virtual work. Consequently the solution of the problem becomes easy by this method.

- (i) The work done by the tension of an inextensible string is zero during a small displacement.

Proof:

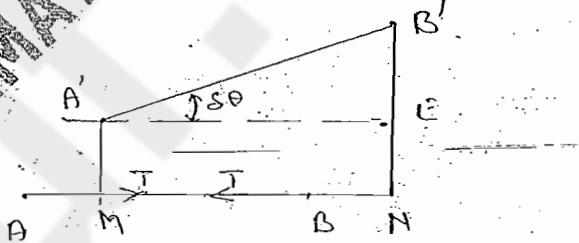
Let AB be a inextensible string of length 'l'.

T = tension in the string AB.

After a small displacement let the new position of AB is A'B'. $\therefore AB = A'B'$

∴ The work done by the tension of string AB, during the displacement

$$\begin{aligned}
 W &= T \cdot AM - T \cdot BN \\
 &= T(AB - MB) - T(MN - MB) \\
 &= T(AB - MN) \\
 &= T \cdot (AB - A'E) \\
 &= T \cdot (AB - A'B' \cos \theta) \quad [\because AB = A'B' = l] \\
 &= T \cdot l (1 - \cos \theta) \\
 &= T \cdot l \left\{ 1 - \left(1 - \frac{(8.8)^2}{21} + \frac{(8.8)^4}{41} - \dots \right) \right\}
 \end{aligned}$$



$$W = T \cdot l \cdot 0$$

$\therefore (80)^2 \approx 0$

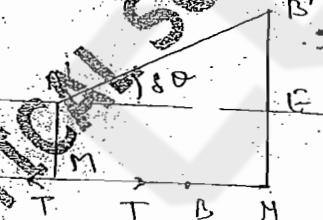
$$= 0$$

Hence the work done by the tension of an inextensible string is zero during a small displacement.

(ii) The work done by the thrust of an inextensible rod is zero during a small displacement. Similarly as above.

$$P = T \cdot A M + T \cdot B M$$

$$= 0$$



* from (i) and (ii) we can say that if the distance between two particles of a system is invariably, the work done by the reaction action and reaction between the two particles is zero.

(iii) The reaction R of any smooth surface with which the body is in contact does no work.

$$\Rightarrow R \perp AB \text{ for small displacement}$$

$$\therefore W = 0$$

But for rough surface

$$\text{Work done by the friction force} = -\int_{fr} AB$$

= virtual work

(iv) If a body rolls without sliding on any fixed surface, the work done by the reaction



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of the surface MATHEMATICS by K. VENKANNA the rolling body is -

七

Since the point Q - Contact of the body is for the moment of rest, (i) so the normal reaction and the force of friction at the point of contact have zero displacements.

(v) The work done by the mutual reaction b/w two bodies of a system is zero in any virtual displacement of the system.

Since the action & reaction are equal and opposite and so the work done by the action balances that done by the reaction.

(vi) If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.

Since, the displacement of the point of application of the force is zero.

Work done by the tension T of an extensible string of length l during a small displacement

$$= -T.SI. \quad [\text{Here, } A'B' = 1 + \delta I]$$

: increase

- * Similarly the work done by the thrust T of an extensible rod of length l during a small displacement

= T S1.

* Application of the principle of virtual work

While applying the principle of virtual work we can give any small displacement to the system, provided it is consistent with the geometrical condition of the system. This displacement should be such as to exclude the forces which are not required (and) to include those which are required in the final result.

After giving the displacement we must note the points and the lengths that change (and) that don't change during the displacement.

If any length or angle is + changing during the displacement, we should first find its value in terms of some variable symbol and then after solving the problem we should put its value in the position.

? //

In many cases we are required to find the tension of an inextensible string or the thrust/tension of an inextensible rod. In order to find such a tension/thrust we must give the system a displacement in which the length of the string or the rod changes, because otherwise Tension/thrust will not come in the equation of virtual work.

But according to the geometrical condition we cannot give such a displacement to the body. So to get over this difficulty we replace the



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(8)

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String or the rod by two equal and opposite forces T which are equivalent to the tension T throughout it. By doing so, evidently the equation of virtual work is not affected, while we become free to give the system a displacement in which the length of string/rod changes and consequently T will occur in the equation of virtual work and will thus be determined.

In any problem the virtual work done by the tension T of an extensible string of length l is $-T\delta l$ (n)

The virtual work done by the thrust T of an extensible rod of length l is $+T\delta l$

In order to find the virtual work done by a force other than a tension or a thrust we first mark (fixed point) fixed straight line. Then we measure the distance of the point of application of the force from this fixed point (line) while moving along the line of action of the force. If this displacement is π and the force is P , then the virtual work done by the force P during a small displacement is $P\pi n$ in magnitude. If this displacement is measured in the direction of the force P , the virtual work done by P is taken with +ve sign, and if the distance π is measured in the direction opposite to that of the force P , the virtual work done by P is taken with negative sign.

Equating to zero the total sum of the virtual work done by the forces, we get the equation of the Virtual work. Solving this equation we get the values of the required thing to be determined.

If $f(x)$ is a function of x , then during small displacement in which x changes to $x + \delta x$, we have,

$$\begin{aligned} Sf(x) &= f(x+\delta x) - f(x) \\ &= f(x) + \frac{\delta x}{\delta x} f'(x) + f(x) \\ &= f'(x) \cdot \delta x \end{aligned}$$

In many cases the only forces that remain in the equation of virtual work are those due to gravity. In such case if w is the total weight and \bar{z} the height or depth of its point of application / centre of gravity of the system, above / below a fixed horizontal level then by the Principle of V.W for centre of the body, we must have

$$w \cdot \delta \bar{z} = 0$$

$$\text{r.e. } \delta \bar{z} = 0$$



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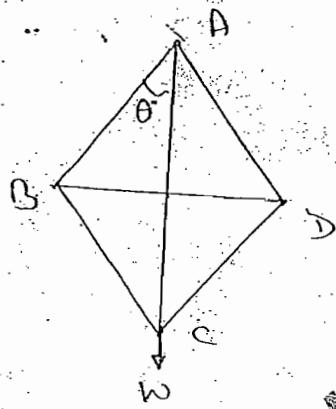
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(9)

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~~five weighted rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. if a weight 'W' is attached to C and the system be suspended from A, show that there is a thrust in BD equal to $\frac{10}{\sqrt{3}}$~~

Sol:



The system is suspended from A,
and a weight W is attached to C.

i.e. Reaction at A = weight at C.

AC must be vertical.

Let T = thrust in the rod BD

& $2a$ = length each rod

i.e. $AB = BC = CD = BD = 2a$ (in \rightleftharpoons)

Let $\angle BAC = \theta$

i.e. in $\triangle ABD$, $AB = AD = BD$

$\therefore \theta = 30^\circ$.

To find the thrust T in BD we shall have to give the system a displacement in which BD must change.

Replace BD by two equal & opposite force T as shown in fig. and then the distance BD can be changed.

Now give small displacement $\delta\theta$ clockwise as shown in fig.

Now,

$$BD = 2BO = 2AB \sin\theta = 2AB \sin\theta$$

$$AC = 2AO = 2AB \cos\theta = 2AB \cos\theta$$

By the principle of virtual work, we have,

$$T \delta(\text{gains}) - S(\text{losses}) = 0$$

$$\Rightarrow T \cdot 4\alpha \cdot \delta\theta - w \cdot 4a \cdot \sin\theta \cdot \delta\theta = 0$$

$$\Rightarrow (T \cos\theta - w \sin\theta) \delta\theta = 0$$

$$\delta\theta \neq 0$$

$$T \cos\theta = w \sin\theta$$

$$\Rightarrow T = w \tan\theta$$

$$\theta = 30^\circ$$

$$\therefore T = \frac{w}{\sqrt{3}} \quad \text{Proved}$$



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(10)

- (Q2) Four rods of equal weight m form a rhombus ABCD with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping them inclined at α to AC. Show that thrust in this stiffening rod is $(eW + 4m)$ tan α .

Soln :

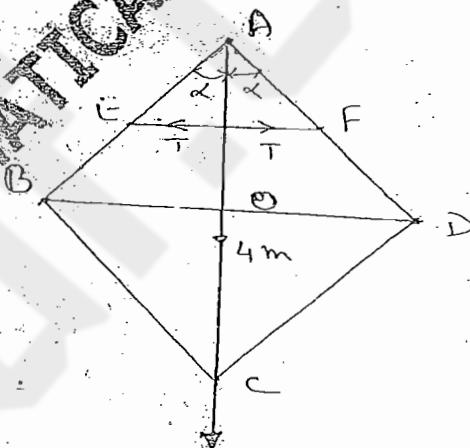
Let ABCD is a framework formed of four equal

rods each of weight m & length $2a$ (say). W = weight attached at C

Let EF = stiffening rod joining rods AB and AD.

Here, $AO = AC$ as \angle to BD as ABCD is rhombus.

$$\therefore \angle BAC = \angle DAC = \alpha$$

Let T = Thrust in rod EF.Total weight of all rods = $4m$ (will act at point O)Now replacing rod EF by two equal and opposite thrust T , as shown in figure.Give small δx displacement about A

$$EF = 2AE \sin\alpha \\ = 2a \sin\alpha$$

$$AO = AB \cos\alpha \\ = 2a \cos\alpha$$

$$\text{and } AC = 2AO \\ = 4a \cos\alpha$$

Now, by the principle of virtual work

$$T \cdot S \cdot (2a \sin\alpha) + 4m \delta (2a \cos\alpha) + W \delta (4a \cos\alpha) = 0 \\ \Rightarrow T \cdot 2a \cos\alpha \cdot S\delta + 4m (2a \cos\alpha \sin\alpha) \cdot S\delta + W (-4a \sin\alpha) \cdot S\delta = 0$$

$$\Rightarrow 2a \left\{ T \cdot \cos\alpha \cdot 4m \sin\alpha - 2W \sin\alpha \right\} \cdot S\delta = 0 \\ S\delta \neq 0$$

$$T \cdot \cos\alpha \cdot 4m \sin\alpha - 2W \sin\alpha = 0$$

$$T = (2W + 4m) \tan\alpha$$

Proved



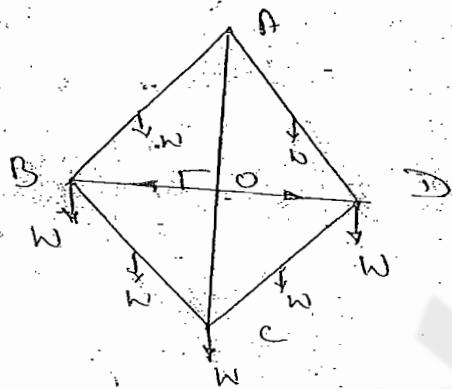
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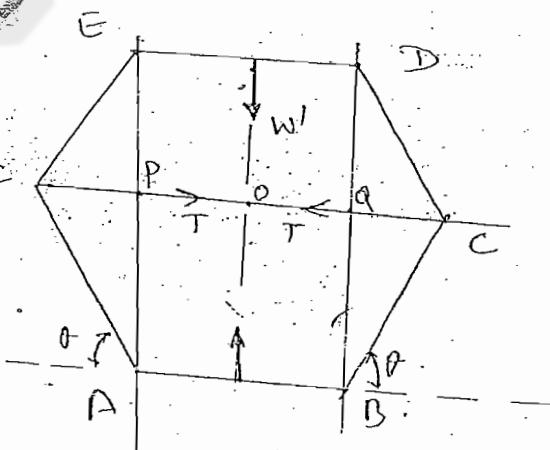
Q.3

(11)

A square framework, formed of uniform heavy rods of equal weights w , jointed together, is hung up by one corner. A weight w' is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.

Hint: $\theta = 45^\circ$

Q.4 A regular hexagon ABCDEF consists of six equal rods which are each of weight w and are freely joined together. The two opposite angles C and F are connected by a string, which is horizontal, AB being in contact with a horizontal plane. A weight w' is placed at the middle point of DE. Show that the tension of the string is $\frac{2(3w + w')}{\sqrt{3}}$

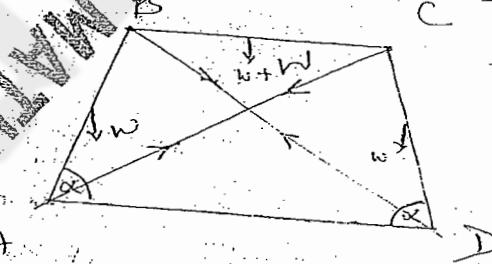


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Q.S. Three equal uniform rods AB, AC, CD each of weight w are freely jointed at B and C, and rest in a vertical plane A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass M of weight Mg be placed on BC at its middle point the tension of each string will be $\frac{w}{2}$ may not be $(w + Mg) \cos \alpha \cos \frac{\alpha}{2}$.

Hint:
fixed level = AD

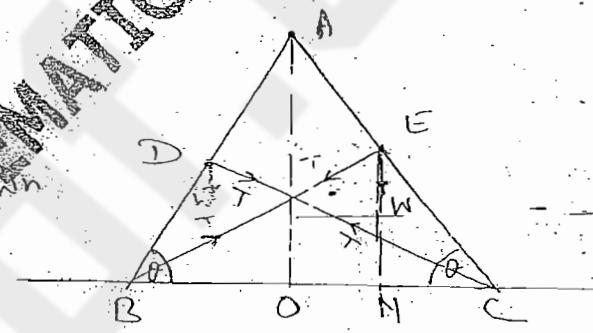


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(12)

- Q6. Two equal beams AC and AB, each of weight W , are connected by a hinge at A (0m) are placed in vertical plane with their extremities B and C resting on a smooth horizontal plane. They are prevented from falling by strings connecting B and C with the middle points D and E of the opposite beams. Show that the tension in each string is $T = \frac{1}{2} W \sqrt{1 + 9 \cot^2 \theta}$. Where ' θ ' is the inclination of each beam to the horizon.

Soln: Let AB and AC are equal beams, each of weight W is hinged at 'A' as shown in figure.



Let BE and DC are two strings where D and F are middle points of respective sides.

Let length of beam AB or AC = 1
Tension in each string = T

$$\therefore \angle ABC = \angle ACB = \theta$$

Here the fixed level is horizontal like BC.

\therefore Height of point E above BC

$$EN = \frac{1}{2} \sin \theta$$

and from $\triangle ABC$

$$BH = \frac{3}{4} BC \quad [\because \text{in } \triangle ACO, E \text{ is mid point of } AC \\ \text{and } EN \parallel AO \\ \therefore \text{mid point of } BC]$$

$$\Rightarrow BN = \frac{3}{4} \cdot 2l \cos\theta$$

$$= \frac{3}{2} l \cos\theta$$

Now in $\triangle BEN$.

$$BE^2 = EN^2 + BN^2$$

$$= (\frac{l}{2} \sin\theta)^2 + (\frac{3}{2} l \cos\theta)^2$$

$$\Rightarrow BE^2 = \frac{1}{4} l^2 (\sin^2\theta) + \frac{9}{4} l^2 \cos^2\theta$$

$$\Rightarrow BE = \frac{1}{2} l \sqrt{1 + 8 \cos^2\theta}$$

$$\therefore BE = \frac{1}{2} \sqrt{1 + 8 \cos^2\theta}$$

Let the system be given small displacement in which θ changes to $\theta + \delta\theta$. The level of the line BC laying on the horizontal plane remains fixed and the points B and C move on the plane. The point D and E are slightly displaced.

The equation of virtual work is

$$-2w \delta(EN) - 2w \delta(BN) = 0$$

$$-T \delta\left(\frac{1}{2} \sqrt{1 + 8 \cos^2\theta}\right) + w \delta\left(\frac{1}{2} \sin\theta\right) = 0$$

$$\Rightarrow T \cdot \frac{1}{2} \frac{1}{\sqrt{1 + 8 \cos^2\theta}} \cdot 16 \cos\theta \cdot (-\sin\theta) \delta\theta$$

$$+ \frac{w l}{2} \cos\theta \delta\theta = 0$$

$$\Rightarrow \left(T \cdot \frac{4 \cos\theta (-\sin\theta)}{\sqrt{1 + 8 \cos^2\theta}} + \frac{w l}{2} \cos\theta \right) \delta\theta = 0$$



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(13)

$$\Rightarrow T = \frac{w}{2} \cdot \sqrt{1+8\cos^2\theta}$$

$$= \frac{w}{8} \cdot \sqrt{\sin^2\theta + 9\cos^2\theta}$$

$$\Rightarrow T = \frac{w}{8} \sqrt{1+9\cot^2\theta}$$

Proved

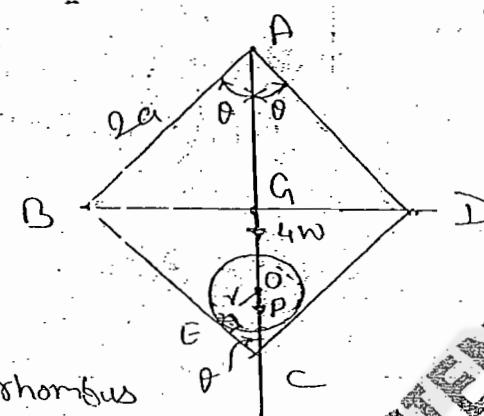
Q7. Four equal uniform bars each of weight w , are jointed together so as to form a rhombus. This is suspended vertically from one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing. Show that if θ be the angle at the fixed joint in the figure, then

$$\frac{\sin^2\theta}{\cos\theta} = \frac{P.r}{4(P+2w).a}$$

Where r is the radius of the sphere and a is the length of each bar

Soln: A rhombus ABCD formed of four rods each of weight w and length a is suspended from point A.

A sphere of weight P and radius r is placed inside the rhombus, as shown in figure.



The diagonal AC of rhombus must be vertical.

$$\text{we have, } \angle BAC = \angle DAC = \theta$$

Give the system a small geometrical displacement about AC in which θ changes to $\theta + \delta\theta$.

$$\text{Now, } AG = 2a \cos \theta$$

and depth of O,

$$AO = AC - OC$$

$$= 4a \cos \theta - 2a \cos (\theta + \delta\theta)$$

$$[\because \sin \theta = \frac{y}{OC}]$$

The equation of virtual work

$$4w \delta(AG) + P \delta(AO) = 0$$

$$\Rightarrow 4w \delta(2a \cos \theta) + P \delta(4a \cos \theta - 2a \cos(\theta + \delta\theta)) = 0$$

$$\Rightarrow 4w \cdot 2a(-\sin \theta) \cdot \delta\theta + P \left(-4a \sin \theta + 2a \cos \theta \cdot (-\cot \theta) \right) \delta\theta = 0$$



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(14)

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$$\Rightarrow \left\{ -8\omega a \sin\theta + p(-4a \sin\theta + r(\sec\theta \cdot \cot\theta)) \right\} 8\theta = 0$$

$$\Rightarrow -8\omega a \sin\theta + p(-4a \sin\theta + r \cdot \sec\theta \cdot \cot\theta) = 0$$

$$\Rightarrow 4a \sin\theta (2\omega + p) = pr \cdot \sec\theta \cdot \cot\theta$$

$$\Rightarrow 4a \sin\theta = \frac{pr}{2\omega + p} \cdot \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = \frac{pr}{4a(2\omega + p)} \quad \underline{\text{proved}}$$

(Q) A quadrilateral ABCD formed of four uniform rods freely joined to each other at their ends, the rods AB, AD being equal (n) also the rods BC, CD (n) freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is $(w + w') \sin\theta + w'$

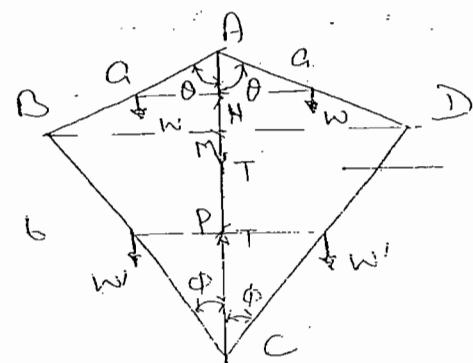
where w is the weight of upper rod and w' of lower rod and θ is equal to the angle BAD.

Soln:

The quadrilateral is suspended from the point A.

$$\text{Let } AB = AD = a$$

$$\text{and } BC = CD = b$$



The diagonal AC must be vertical and BD horizontal.

Let T = tension in the string AC.

$$\text{Let } LBAC = LDAC = \theta$$

$$\text{and } LBCA = LDCB = \phi$$

Now in \triangle position

$$\theta + \phi = 90^\circ \quad \text{--- (1)}$$

Now give the system a small symmetrical displacement about AC.

We have,

$$AC = AM + MC = a \cos \theta + b \cos \phi$$

The depth of middle point of AB or AD below A

$$AN = \frac{a}{2} \cos \theta$$

and the depth of the middle point of CB or CD below A

$$AP = a \cos \theta + \frac{b}{2} \cos \phi$$

Now The equation of Virtual Work

$$-T \delta(AC) + 2w \delta(AN) + 2w' \delta(AP) = 0$$

$$\Rightarrow -T \delta(a \cos \theta + b \cos \phi) + 2w \delta\left(\frac{a}{2} \cos \theta\right)$$

$$+ 2w' \delta\left(a \cos \theta + \frac{b}{2} \cos \phi\right) = 0$$

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$$\Rightarrow T(a \sin \theta + b \sin \phi) - w a \sin \theta \delta \theta$$

$$-2w' (a \sin \theta + \frac{1}{2} b \sin \phi \delta \phi) = 0$$

$$\Rightarrow \{ T a \sin \theta - w a \sin \theta - 2w' a \sin \theta \} \delta \theta$$

$$= \{ -T b \sin \phi + w' b \sin \phi \} \delta \phi = 0$$

From ΔABD and ΔBCD

$$a \sin \theta = b \sin \phi$$

$$\text{or, } a \cos \theta \delta \theta = b \cos \phi \delta \phi \quad (2)$$

From (1) and (2), we have,

$$T a \sin \theta - w a \sin \theta - 2w' a \sin \theta \cdot a$$

$$= \frac{-T b \sin \phi + w' b \sin \phi}{b \cos \phi}$$

$$\Rightarrow T \left(\frac{\sin \theta}{\cos \theta} + \frac{\sin \phi}{\cos \phi} \right) = (w + w') \tan \theta + w' \tan \phi$$

$$\Rightarrow T \frac{\sin \theta \cos \phi + \sin \phi \cos \theta}{\cos \theta \cos \phi} = (w + w') \frac{\sin \theta}{\cos \theta}$$

$$- w' \left(\frac{\sin \theta}{\cos \theta} + \frac{\sin \phi}{\cos \phi} \right)$$

$$\Rightarrow T \sin(\theta + \phi) = (w + w') \sin \theta \cos \phi$$

$$+ w' \sin(\theta + \phi)$$

$$\text{For } \Rightarrow, \theta + \phi = 90^\circ$$

$$\therefore T = (w + w') \sin^2 \theta + w' \quad \text{Proved}$$

Q.9.

Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C, the end A is fixed. Prove that if a horizontal force P is applied at C and AB, BC are inclined at angle θ, ϕ to the vertical, then

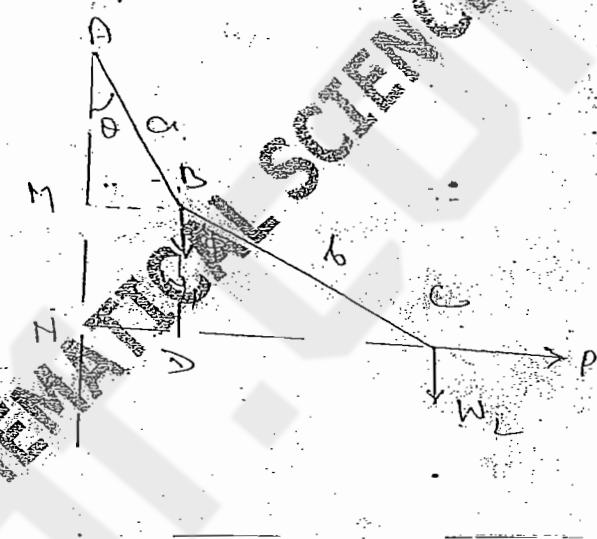
$$\begin{aligned} P &= (W_1 + W_2) \tan \theta \\ &= W_2 \tan \phi \end{aligned}$$

Hint:

$$\textcircled{1} \quad \theta \neq f(\phi)$$

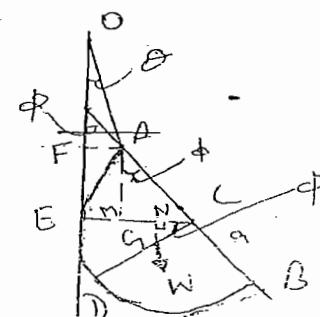
$$\textcircled{2} \quad \text{When } \phi \rightarrow \theta + \delta \theta \\ \delta \phi = 0$$

$$\textcircled{3} \quad \text{When } \phi \rightarrow \phi + \delta \phi \\ \delta \theta = 0$$



Q.9. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on the smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane back of the hemisphere to the vertical, prove that

$$\tan \phi = \frac{3}{8} + \tan \theta$$



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Soln:

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Let O be the fixed on the wall to which one end of the string is attached.

Let $l = \text{length of the string } AO$,

$a = \text{radius of the hemisphere}$.

G = point where centre of gravity of the sphere acts such that

$$OG = \frac{3}{8}a$$

$\theta = \text{angle made by the string on vertical wall}$.

$\phi = \text{angle made by the base BA of hemisphere}$

on vertical wall.

Now,

$$OG = OF + FM + MG \\ = l \cos \theta + a \cos \phi + \frac{3}{8}a \sin \phi$$

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$. the point O remains fixed and the length of the string AO doesn't change.

The equation of virtual equation is.

$$w \delta(l \cos \theta + a \cos \phi + \frac{3}{8}a \sin \phi) = 0$$

$$\text{or, } -l \sin \theta \delta\theta - a \sin \phi \delta\phi + \frac{3}{8}a \cos \phi \delta\phi = 0$$

$$\text{or, } l \sin \theta \cdot 2\theta = \left(\frac{3}{8}a \cos \phi - a \sin \phi \right) \delta\phi \quad \text{(1)}$$

From the figure,

$$\begin{aligned} F &= a \\ \Rightarrow EA + MC &= a \\ \Rightarrow RA + MC &= a \end{aligned}$$

$$\begin{aligned} \Rightarrow l \sin \theta + a \sin \phi &= a \\ \Rightarrow l \cos \theta \sin \phi + a \cos \phi \sin \theta &= 0 \end{aligned}$$

$$l \sin \theta \sin \phi = -l \cos \theta \sin \phi$$

Dividing ① by ②, we have

$$-\tan \theta = -\frac{38}{38 + l \cot \theta}$$

$$\Rightarrow -\tan \theta$$

Ans

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- 10). The middle point of the opposite sides of a jointed quadrilateral are connected by light rods of length l, l' . If T, T' be the tensions in these rods,

Prove that

$$\frac{T}{l} + \frac{T'}{l'} = 0$$

Soln:

Let ABCD be a quadrilateral, whose middle points of opposite sides are joined by two strings of length l and

Since, P, Q, R, S is the middle point

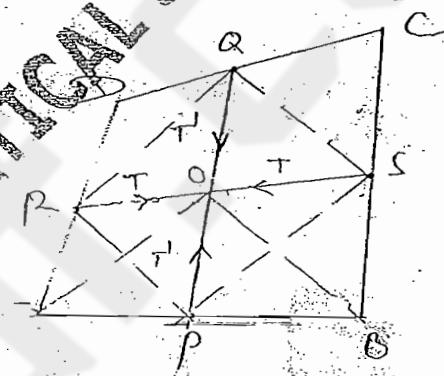
P, S, Q, R is a parallelogram

Replace the string PQ with two equal and opposite forces T .

and replace string PS with two equal and opposite forces $-T'$

Now give small displacement in which PQ and PS changes slightly.

The length of the rods AB, BC, CD, DA do not change.



The equation of virtual work

$$-TS(RS) - T'S(PQ) = 0 \quad (1)$$

$\Rightarrow \triangle OAB$

O is median

$$\begin{aligned} 1. \quad OA^L + OB^L &= 2PQ^2 + 2AP^2 \\ &= 2\left(\frac{1}{2}PA\right)^2 + 2\left(\frac{1}{2}PA\right)^2 \\ &= \frac{PA^2 + PB^2}{2} \end{aligned} \quad (2)$$

Similarly in $\triangle OCD$

$$OC^L + OD^L = \frac{1}{2}(PQ^2 + CD^2) = 0 \quad (3)$$

From (2) + (3),

$$OB^L + OC^L + OD^L = \frac{1}{2}(2PQ^2 + AB^2 + CD^2) \quad (4)$$

Similarly for $\triangle ACD$ and DOC , we have,

$$OA^L + OB^L + OC^L + OD^L = \frac{1}{2}(2RS^2 + AD^2 + CD^2) \quad (5)$$

from (4) and (5),

$$2PQ^2 + AB^2 + CD^2 = 2RS^2 + AD^2 + CD^2$$

$$\text{or, } 2(PQ^2 - RS^2) = \text{constant} \quad (6)$$

[Since, AB, BC, CD, AD are all of fixed length]

$$\text{or, } PQ \cdot S(PQ) = RS \cdot S(RS)$$



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$$\Rightarrow \frac{S(PS)}{S(PQ)} = \frac{PA}{PS}$$

But, $PS = 1$, $PA = ?$ given at $\frac{1}{1}$

$$\frac{S(PS)}{S(PQ)} = \frac{1}{1}$$

From

$$\frac{T_1}{T_1} = \frac{S(PA)}{S(PS)}$$

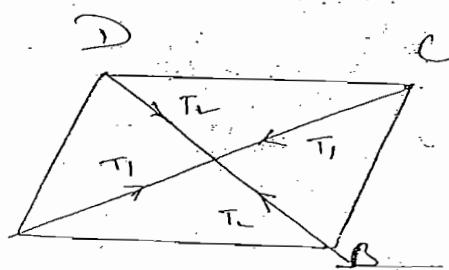
$$\Rightarrow \frac{1}{T_1} + \frac{1}{T_1}$$

- (Q. 11) Four rods are jointed together to form a parallelogram, the opposite sides are joined by the strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their length.

Sol 1: → Here gives only linear displacement

Hint: in parallelogram,

$$\begin{aligned} AC^2 + BD^2 &= AB^2 + BC^2 + CD^2 + DA^2 \\ &= \text{constant} \end{aligned}$$



- Q12. Four equal rods, each of length $2a$ and weight w , are freely jointed to form a square $ABCD$, which is kept in shape by a light rod BD and is supported in a vertical plane with CD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance $2b$ apart on the same level. Find the stress in the rod BD .

Soln: Let the rods AB and AC rest on two fixed smooth pegs E and F , which are at the same level and

$$EF = 2b$$

Let $2a$ = length of each rods AB, BC, CD, DA

T = effort in the rod BD

$$LBAe = LDAC$$

Now replace the rod BD by two equal and opposite forces T as shown in figure.

Now system is given small displacement in which θ changes to $\theta + \delta\theta$



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Here our reference line is EF.

forces contributing to the virtual work are

(i) The thrust in the rod BD

(ii) the weight $4w$ acting at G.

$$\begin{aligned} BD &= 2 \cdot 2a \sin \theta \\ &= 4a \sin \theta \end{aligned}$$

Ans

$$\begin{aligned} OG &= AG - AO \\ &= 2a \cos \theta \end{aligned}$$

$$\left[\tan \theta = \frac{b}{2a} \right]$$

The equation of virtual work is

$$T \delta(BD) + w \delta(OG) = 0$$

$$\Rightarrow T \delta(4a \sin \theta) + w \delta(2a \cos \theta - b \tan \theta) = 0$$

$$\Rightarrow T 4a \cos \theta \delta \theta + w (-2a \sin \theta + b \cos^2 \theta) \delta \theta = 0$$

$$\{ T 4a \cos \theta + w (b \cos^2 \theta - 2a \sin \theta) \} \delta \theta = 0$$

$$\therefore \delta \theta \neq 0$$

$$T 4a \cos \theta + w (b \cos^2 \theta - 2a \sin \theta) = 0$$

$$\Rightarrow T = \frac{w}{a} \left(2a \tan \theta - \frac{b \cos^2 \theta}{\cos \theta} \right)$$

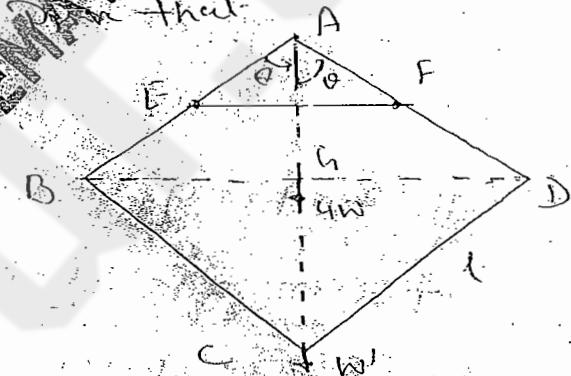
But in equilibrium position,
we have $\theta = 45^\circ$

$$T = \frac{w}{a} \cdot (2a - 2b\sqrt{2})$$

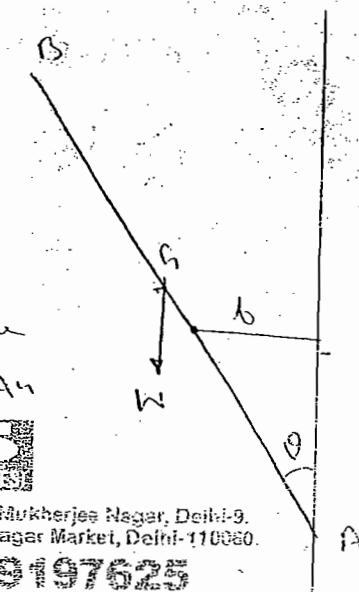
$$= \frac{2W}{a} (a - b\sqrt{2})$$

Q13. A rhombus is formed of rods each of weight W and length a with smooth joints. It rests symmetrically with its two upper sides in contact with two smooth pegs at the same level and at a distance $2a$ apart. A weight W is hung at the lowest point; if the sides of the rhombus make an angle θ with the vertical, then that

$$\sin^2 \theta = \frac{a(4W + W)}{a(4W + 2W)}$$



Q14. A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1} (\frac{b}{a})$.



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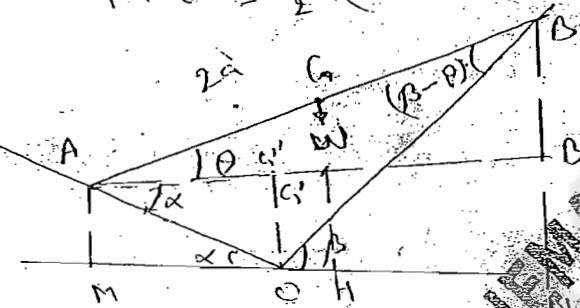
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Q. 15. A heavy uniform rod, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove by the principle of virtual work, that

$$\tan \theta = \frac{1}{2} (\cot \alpha - \cot \beta)$$



Hints. $AB = 2a \cos \theta$, $BB' = 2a \sin \theta$

$$OA = a \sin \theta \quad \text{--- (1)}$$

$$\frac{\text{or}}{\sin(\beta-\theta)} \quad \frac{2a}{\sin(\pi-(\alpha+\beta))} \Rightarrow OA = 2a \cdot \frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}$$

$$\therefore AB = OA + OB = a \sin \alpha = 2a \cdot \frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)} \cdot \sin \alpha$$

$$\begin{aligned} OH &= OA + AH \\ &= a \sin \theta + 2a \cdot \frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)} \cdot \sin \alpha. \end{aligned}$$

$$\therefore S(OH) = 0$$

$$\Rightarrow -a \cos \theta + 2a \cdot \frac{-\cos(\beta-\theta)}{\sin(\alpha+\beta)} \cdot \sin \alpha = 0$$

$$\Rightarrow a \cos \theta \cdot \sin(\alpha+\beta) = 2 \cdot \sin \alpha \cdot \cos(\beta-\theta)$$

$$\Rightarrow \sin(\alpha+\beta) = 2 \cdot \sin \alpha (\cos \beta + \tan \theta \cdot \sin \beta)$$

$$\therefore \tan \theta = \left\{ \frac{\sin(\alpha+\beta)}{2 \sin \alpha} - \cos \beta \right\} \frac{1}{\sin \beta}$$

$$\begin{aligned}\Rightarrow \tan \theta &= \left\{ \frac{\sin(\alpha+\beta)}{2 \cdot \sin \alpha \cdot \sin \beta} - \cot \beta \right\} \\ &= \left\{ \frac{\sin \alpha \cdot \cos \beta + \sin \beta \cdot \cos \alpha}{2 \sin \alpha \cdot \sin \beta} - \cot \beta \right\} \\ &= \left\{ \cot \beta + \cot \alpha - 2 \cot \beta \right\} \\ &= \left\{ \cot \alpha - \cot \beta \right\}\end{aligned}$$

Q. 16

- Two equal rods, AB and AC each of length 2a
 are freely jointed at A and rest on a smooth
 vertical circle of radius a. Show that if θ
 be the angl. between them, then

$$\sin^3 \theta = a \cdot \cos \theta$$

Sol. Let O be the centre of the

given fixed circle and M be the

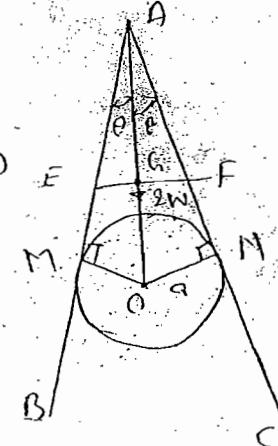
mid-point of each of the rods AB and

AC.

The rod AO is vertical.

Further, $\angle BAO = \theta = \angle CAO$

Give the rods a small displacement in
 which θ changes to $\theta + \delta\theta$. The point O remains fixed



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From figure,

$$C_{10} = OA - AB$$

$$\left[\sin \theta = \frac{OM}{OA} \Rightarrow OA = O \text{cosec } \theta \right]$$

$$AB = b \cos \theta$$

$$C_{10} = a \text{cosec } \theta - b \cos \theta$$

Equation of virtual work

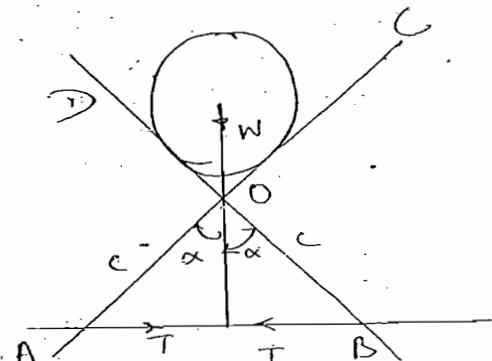
$$-2V \delta(C_1) = 0$$

$$\Rightarrow \delta(a \text{cosec } \theta - b \cos \theta) = 0$$

$$\Rightarrow -a \text{cosec } \theta \cdot \cot \theta + b \sin \theta = 0$$

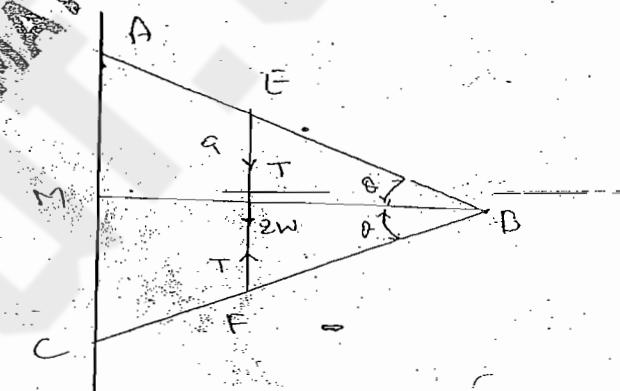
$$\Rightarrow -\lambda \text{cosec}^2 \theta = a \cos \theta \quad \text{Proved}$$

- Q.11:** The light rods AOC, BOD are smoothly hinged at O , & meet at a distance c from each of the ends A, B which are connected by a string of length $2c$ find. The rods rest in a vertical plane with the ends A, B on a smooth horizontal table. A smooth circular disc of radius a and weight W is placed on the rods above O with its plane vertical & that rods are tangential to the disc. Prove that the tension of the string is
- $$\frac{1}{2} W \left\{ \frac{a}{c} (\text{cosec } \theta + \tan \theta) \right\}$$



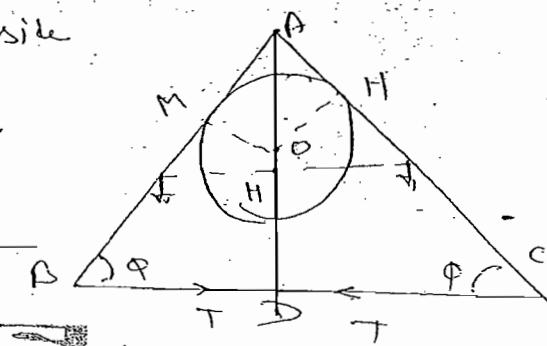
Q. 18 One end A of a uniform rod AB, of length $2a$ and weight w , is attached by a frictionless joint to a smooth vertical wall and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are jointed by an elastic string, natural length a and modulus of elasticity μ .

Prove that the system can not be in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2\sin^{-1}(\frac{1}{\sqrt{3}})$.



Q. 19. Two equal rods, each of weight w_1 and length l , are joined together and placed aside.

On a smooth horizontal cylindrical peg of radius 'r'. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal.

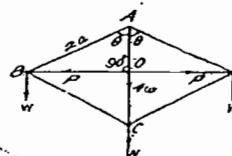


Tension in the string is $\frac{2w_1}{\cos \phi}$. If the string is slack, show that $\tan \phi = \frac{2w_1}{r}$.

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Ex.20 Four equal uniform rods, each of weight w , are freely jointed to form a rhombus $ABCD$. The frame work is suspended freely from A and a weight W is attached to each of the joints B , C , D . If two horizontal forces each of magnitude P acting at B and D keep the angle BAD equal to 120° , prove that $P = (W+w) 2\sqrt{3}$.

Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended from the point A and a weight W is attached to each of the points B , C and D . To save the system from collapsing two horizontal forces each of magnitude P act at B and D and in equilibrium $\angle BAD = 120^\circ$. Obviously the nature of the forces P is like that of thrust.



The total weight $4w$ of all the four rods AB , BC , CD and DA can be taken acting at the point of intersection O of the diagonals AC and BD . Obviously the line AC must be vertical and so BD is horizontal.

To find P we have to give the system a displacement in which the length BD must change and consequently the angle BAD will change so let us assume that $\angle BAC = \theta = \angle DAC$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta + 60^\circ$. The point A remains fixed and we shall measure the distances of the points of application of various forces from the point A . The points B , C , D and O change. The lengths of the rods AB , BC , CD and DA do not change while the length BD changes. The angle AOB will remain 90° .

We have

$$BD = 2BO = 2AB \sin \theta = 4a \sin \theta.$$

the depth of B or D or O below A

$$= AO = 2a \cos \theta.$$

and the depth of C below A

$$= AC = 2AO = 4a \cos \theta.$$

By the principle of virtual work, we have

$$P \delta (4a \sin \theta) + 4w \delta (2a \cos \theta) + 2W \delta (2a \cos \theta) = 0$$

$$+ W \delta (4a \cos \theta) = 0$$

$$\text{or } 4a P \cos \theta \delta \theta - 8aw \sin \theta \delta \theta - 4aW \sin \theta \delta \theta - 4aW \sin \theta \delta \theta = 0$$

$$\text{or } 4a [P \cos \theta - 2w \sin \theta - W \sin \theta] \delta \theta = 0$$

$$\text{or } P \cos \theta - 2(W+w) \sin \theta = 0$$

$$\text{or } P = 2(W+w) \tan \theta.$$

But in the position of equilibrium, $\theta = 60^\circ$.

$$\therefore P = 2(W+w) \tan 60^\circ = 2(W+w) \sqrt{3} = (W+w) 2\sqrt{3}.$$

Ex.21 Four equal uniform rods, each of weight W , are jointed to form a rhombus $ABCD$, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC = \theta$ by a light string joining B and D . Find the tension of the string.

Sol. $ABCD$ is a framework formed of four equal rods each of weight W and say of length $2a$. It is placed in a vertical plane with AC vertical and A resting on a horizontal plane. To keep the system in the form of a rhombus a light string joins B and D and prevents the points B and D from moving in the directions OB and OD respectively. Let T be the tension in the string BD . The total weight $4W$ of all the four rods may be taken acting at the point of intersection O of the diagonals AC and BD .

$$\text{Let } \angle DAC = \theta = \angle BAC.$$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta + 60^\circ$. The point A resting on the horizontal plane remains fixed. The points B , C , D and O will change. The lengths of the rods AB , BC , CD and DA will remain fixed while the length BD will change. The angle DOC will remain 90° .

$$\text{We have } BD = 2BO = 2AB \sin \theta = 4a \sin \theta,$$

$$\text{and the height of } O \text{ above the fixed point } A$$

$$= AO = 2a \cos \theta.$$

By the principle of virtual work, we have

$$-T \delta (4a \sin \theta) - 4W \delta (2a \cos \theta) = 0. \quad (1)$$

[Note that in the equation (1) the work done by the weight $4W$ has been taken with negative sign because the distance AO of its point of application O from the fixed point A is in a direction opposite to the direction of $4W$.]

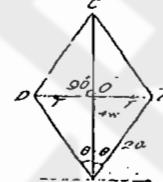
From the equation (1), we have

$$-4a T \cos \theta \delta \theta + 8aw \sin \theta \delta \theta = 0$$

$$\text{or } 4a [-T \cos \theta + 2W \sin \theta] \delta \theta = 0$$

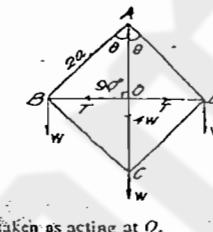
$$\text{or } -T \cos \theta + 2W \sin \theta = 0 \quad [\because \delta \theta \neq 0]$$

$$\text{or } T = 2W \tan \theta.$$



Ex.22 A square framework, formed of uniform heavy rods of equal weight W , jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.

Sol. $ABCD$ is a square framework formed of four rods each of weight W and say of length $2a$. It is suspended from the point A and a weight W is suspended from each of the three lower corners B , C and D . A light rod along the horizontal diagonal BD prevents the system from collapsing. Let T be the thrust in the rod BD . The total weight $4W$ of the rods AB , BC , CD and DA can be taken as acting at O .



To find T we shall have to give the system a displacement in which BD must change. So replace the rod BD by two equal and opposite forces T as shown in the figure and assume that $\angle BAC = \theta = \angle CAD$. [Note that the angle BAC will change during a displacement in which BD is to change.]

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta + 60^\circ$. The point A remains fixed and the points B , C , D and O change. The lengths of the rods AB , BC , CD and DA do not change while the length BD changes.

$$\text{We have } BD = 2BO = 2AB \sin \theta = 4a \sin \theta,$$

the depth of each of the points B , C and D below the fixed point A

$$= AO = 2a \cos \theta,$$

and the depth of C below A $= 2AO = 4a \cos \theta$.

By the principle of virtual work, we have

$$T \delta (4a \sin \theta) + 4W \delta (2a \cos \theta) + 2W \delta (2a \cos \theta) = 0$$

$$+ W \delta (4a \cos \theta) = 0$$

$$\text{or } 4a T \cos \theta \delta \theta - 8aw \sin \theta \delta \theta - 4aW \sin \theta \delta \theta - 4aW \sin \theta \delta \theta = 0$$

$$\text{or } 4a [T \cos \theta - 2W \sin \theta] \delta \theta = 0$$

$$\text{or } T \cos \theta - 2W \sin \theta = 0$$

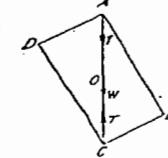
$$\text{or } T = 2W \tan \theta.$$

But in the position of equilibrium $\theta = 45^\circ$.

$\therefore T = 2W \tan 45^\circ = 2W$ = the total weight of the four rods.

Ex.23 Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$, which is suspended by the joint A , and is kept in shape by a string AC . Prove that the tension of the string is equal to half the weight of all the four rods.

Sol. $ABCD$ is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC . Let T be the tension in the string AC . The total weight W of all the four rods AB , BC , CD and DA can be taken as acting at O , the middle point of AC . Since the force of reaction at the point of suspension A balances the weight W at O , therefore the line AO must be vertical. Let $AC = 2x$.



Give the system a small displacement in which x changes to $x + \delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes. We have, $AO = x$.

By the principle of virtual work, we have

$$-T \delta (AC) + W \delta (AO) = 0$$

$$\text{or } -T \delta (2x) + W \delta (x) = 0$$

$$\text{or } 2T \delta x + W \delta x = 0$$

$$\text{or } [2T + W] \delta x = 0$$

$$\text{or } 2T + W = 0$$

$$[\because \delta x \neq 0]$$

$$\text{or } T = \frac{1}{2} W = \frac{1}{2} (\text{total weight of all the four rods}).$$

Ex.24 A string, of length a , forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W , which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is

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$$\frac{2W(2b^2-a^2)}{h\sqrt{(4b^2-a^2)}}$$

Sol. $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W . The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD . The total weight $4W$ of the rods AB , BC , CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD . We have $\angle AOB = 90^\circ$.

Let $\angle ABO = \theta$. Draw OM perpendicular to AB .

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The points O , C and D change. The lengths of the rods AB , AC , CD and DA do not change while the length BD changes. The $\angle AOB$ will remain 90° .

We have $BD = 2BO = 2AB \cos \theta = 2b \cos \theta$.

[Note that in the position of equilibrium $BD = a$. But during the displacement BD changes and so we have found BD in terms of θ .]

The depth of O below the fixed line $AB = MO$.

$\therefore BO \sin \theta = (AB \cos \theta) \sin \theta = b \sin \theta \cos \theta$.

By the principle of virtual work, we have:

$$-T\delta(2b \cos \theta) + 4W\delta(b \sin \theta \cos \theta) = 0$$

$$\text{or } 2bT \sin \theta \delta\theta + 4bW(\cos^2 \theta - \sin^2 \theta) \delta\theta = 0$$

$$\text{or } 2b[T \sin \theta - 2W(\sin^2 \theta - \cos^2 \theta)] \delta\theta = 0$$

$$\text{or } T \sin \theta - 2W(\sin^2 \theta - \cos^2 \theta) = 0 \quad [\because \delta\theta \neq 0]$$

$$\text{or } T = \frac{2W(\sin^2 \theta - \cos^2 \theta)}{\sin \theta} = \frac{2W(1 - 2 \cos^2 \theta)}{\sin \theta} = \frac{2W}{\sin \theta}(1 - 2 \cos^2 \theta).$$

In the position of equilibrium, $BD = a$ or $BO = \frac{a}{2}$. So in the position of equilibrium, $\cos \theta = \frac{BO}{AB} = \frac{a}{2b} = \frac{a}{2b}$.

$$\therefore T = \frac{2W(1 - 2(a^2/4b^2))}{\sin \theta} = \frac{2W(2b^2 - a^2)}{h\sqrt{(4b^2 - a^2)}}$$

Ex.25 Four equal uniform rods, each of weight W , are smoothly jointed so as to form a square $ABCD$; the side AB is fixed (clamped) in a vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC . Find the tension of the string.

Sol. $ABCD$ is a framework formed of four equal uniform rods each of weight W and say of length $2a$. The side AB is fixed in a vertical position with A uppermost. A string joins the middle points E and F of AD and DC respectively and in equilibrium $ABCD$ is a square.

Let T be the tension in the string EF . The total weight $4W$ of all the rods AB , BC , CD and DA acts at O , the point of intersection of the diagonals AC and BD . We have, $\angle AOD = 90^\circ$. Let $\angle BAC = \theta = \angle DAC$. Draw OM perpendicular to AB .

[Note that we have drawn $ABCD$ as a rhombus and not as a square because in a displacement in which EF is to change the figure will not remain a square. After finding the value of the tension T we shall use the fact that in the position of equilibrium the figure is a square].

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AB will remain fixed and so A is a fixed point. The points C , D and O will change. The lengths of the rods AB , BC , CD and DA do not change while the length EF changes. The $\angle AOD$ remains 90° .

We have $EF = \frac{1}{2}AC = AO = AD \cos \theta = 2a \cos \theta$. Also the depth of O below the fixed point A i.e., the distance of O from the fixed point A in the direction of the force $4W$

$$\therefore OM = AO \cos \theta = (2a \cos \theta) \cos \theta = 2a \cos^2 \theta.$$

By the principle of virtual work, we have:

$$-T\delta(2a \cos \theta) + 4W\delta(2a \cos^2 \theta) = 0$$

$$\text{or } 2aT \sin \theta \delta\theta - 16aW \cos \theta \sin \theta \delta\theta = 0$$

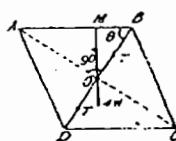
$$\text{or } 2aT \sin \theta (T - 8W \cos \theta) \delta\theta = 0$$

$$\text{or } T = 8W \cos \theta = 0 \quad [\because \delta\theta \neq 0]$$

$$\text{or } T = 8W \cos \theta.$$

But in the position of equilibrium, $\theta = 45^\circ$.

$$\therefore T = 8W \cos 45^\circ = 8W(1/\sqrt{2}) = 4W\sqrt{2}.$$



Ex.26 Six equal heavy beams are freely jointed at their ends to form a hexagon, and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two uppermost beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Find its tension in terms of W and θ , where W is the weight of each beam.

Sol. $ABCDEF$ is a hexagon formed of six equal heavy beams each of weight W and say of length $2a$. The frame is placed in a vertical plane with the beam AB resting on a horizontal plane. To save the system from collapsing, the middle points M and N of the beams FE and CD are connected by a light cord. Let T' be the tension in the cord MN .

The line FC is horizontal. We have $\angle EFC = \theta$, $\angle DCF = \theta$.

Draw EP and DQ perpendiculars to MN . The total weight $6W$ of all the six rods can be taken, acting at O , the middle point of FC . Draw OH and CK perpendicular to AB . We have $\angle CBK = \theta$.

Give the system a small symmetrical displacement about the vertical line OH in which θ changes to $\theta + \delta\theta$. The line AB on the horizontal plane remains fixed, and the distance of the point of application O of the weight $6W$ will be measured from AB . The lengths of the rods AB , BC etc. remain fixed while the length MN changes. The point O also changes.

$$\text{We have } MN = MP + PQ + QN = a \cos \theta + 2a + a \cos \theta = 2a + 2a \cos \theta$$

[Note that $PQ = ED = 2a$, because ED remains fixed].

Also the height of O above the fixed line AB

$$= HO = KC = 2a \sin \theta$$

$$\text{By the principle of virtual work, we have } -T'(2a + 2a \cos \theta) - 6W(2a \sin \theta) = 0$$

[The work done by $6W$ is taken with $-$ ive sign because the direction of HO is opposite to that of $6W$]

$$\text{or } -2aT' \sin \theta - 12aW \cos \theta \delta\theta = 0$$

$$\Rightarrow 2a(T \sin \theta - 6W \cos \theta) \delta\theta = 0$$

$$\Rightarrow T \sin \theta - 6W \cos \theta = 0 \quad (\because \delta\theta \neq 0)$$

$$\Rightarrow T = 6W \cot \theta$$

Ex.27 Six equal rods AB , BC , CD , DE , EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is $3W$.

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Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN . The total weight $6W$ of all the six rods AB , BC etc. can be taken acting at O , the middle point of MN . Let $\angle EFK = \theta = \angle CBA$.

Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The lengths of the rods AB , BC etc. remain fixed, the length MN changes and the point O also changes.

We have

$$MN = 2MO = 2KE = 2AF \sin \theta = 4a \sin \theta.$$

$$\text{Also the depth of } O \text{ below the fixed line } AB = MO = 2a \sin \theta.$$

By the principle of virtual work, we have:

$$-T\delta(4a \sin \theta) + 6W\delta(2a \sin \theta) = 0$$

$$\Rightarrow -4aT \sin \theta \delta\theta + 12aW \sin \theta \delta\theta = 0$$

$$\Rightarrow -4a(-T + 3W) \cos \theta \delta\theta = 0$$

$$\Rightarrow T = 3W \quad [\because \theta \neq 0 \text{ and } \cos \theta \neq 0].$$

Ex.28

Six equal bars are freely jointed at their extremities forming a regular hexagon $ABCDEF$ which is kept in shape by vertical strings joining the middle points of BC , CD and AE , EF respectively, the side AB being held horizontal and uppermost. Prove that the tension of each string is three times the weight of a bar.

Sol. $ABCDEF$ is a hexagon formed of six equal bars each of weight W and length $2a$. The rod AB is held horizontal and uppermost. The middle points M and N of BC and CD are joined by a string and the middle points P and Q of AE and EF are also joined by a string. Let T be the tension in each of the strings PQ and MN . The total weight $6W$ of all the six rods AB , BC etc. can be taken acting at G , the middle point of EF . Let $\angle HAF = \theta = \angle ABC$.

Give the system a small symmetrical displacement about the vertical line OG in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The lengths of the rods AB , BC etc. remain fixed, the lengths MN and PQ change and the point G also changes.

We have

$$PQ = MN = 2AJC \sin \theta = 2a \sin \theta.$$

$$\text{Also the depth of } G \text{ below } AB = OG = BC \sin \theta = 2a \sin \theta.$$



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By the principle of virtual work, we have

$$\begin{aligned} & -2T\delta(2a \sin \theta) + 6W\delta(2a \sin \theta) = 0 \\ \text{or } & -4aT \cos \theta \delta\theta + 12aW \cos \theta \delta\theta = 0 \\ \text{or } & 4a \cos \theta (-T + 3W) \delta\theta = 0 \\ \text{or } & -T + 3W = 0 \quad [\because \delta\theta \neq 0 \text{ and } \cos \theta \neq 0] \\ \text{or } & T = 3W \text{ i.e., the tension of each string is} \end{aligned}$$

three times the weight of a bar.

Ex.29 Six equal light rods are joined to form a hexagon ABCDEF which is suspended at A and F so that AF is horizontal. A rod BE, also light, keeps the figure from collapsing and is of such a length that the rods ending in the points B, E are inclined at an angle of 45° to the vertical. Equal weights W are suspended from B, C, D, E. Find the stress in BE.

Sol. ABCDEF is a hexagon formed of six equal light rods say each of length $2a$. It is suspended at A and F so that AF is horizontal. Equal weights W are suspended from each of the points B, C, D and E. A light rod joining B and E saves the system from collapsing. Let T be the stress in the rod BE. Since the rod BE prevents the points B and E from moving inwards, therefore the stress in the rod BE is a thrust.

$$\text{Let } ABE = \theta, FEB = \angle CBE = \angle DEB.$$

Replace the rod BE by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AF remains fixed. The points B, C, D and E change. The lengths of the rods AB, BC etc. do not change while the length BE changes.

We have

$$BE = AF + 2AM = 2a + 2.2a \cos \theta = 2a + 4a \cos \theta,$$

the depth of each of the points B and E below AF

$$= AM = 2a \sin \theta,$$

and the depth of each of the points C and D below AF

$$= 2AM = 4a \sin \theta.$$

By the principle of virtual work, we have

$$\begin{aligned} & T\delta(2a + 4a \cos \theta) + 2W\delta(2a \sin \theta) + 2W\delta(4a \sin \theta) = 0 \\ \text{or } & -4aT \sin \theta \delta\theta + 4aW \cos \theta \delta\theta + 8aW \cos \theta \delta\theta = 0 \\ \text{or } & 4a(-T \sin \theta + W \cos \theta + 2W \cos \theta) \delta\theta = 0 \\ \text{or } & -T \sin \theta + 3W \cos \theta = 0 \quad [\because \delta\theta \neq 0] \\ \text{or } & T = 3W \cot 45^\circ = 3W. \end{aligned}$$

But in the position of equilibrium each of the rods AB, BC, EF and ED makes an angle 45° with the vertical and so also with the horizontal BE. Therefore in the position of equilibrium, $\theta = 45^\circ$ and

$$T = 3W \cot 45^\circ = 3W.$$

Ex.30 Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are $(\frac{2}{3}\sqrt{3}/2)W$ and $(\sqrt{3}/2)W$, where W is the weight of each rod.

Sol. Let the length of each of the rods AB, BC etc. be $2a$ and let θ be the angle which each of the slant rods AB, AF, DC and DE makes with the vertical AD.

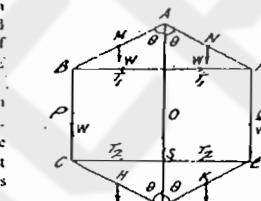
Let T_1 and T_2 be the thrusts in the rods BF and CE respectively. Here A is the fixed point. The weights of the rods AB, BC etc. act at their respective middle points as shown in the figure.

Let us first find the thrust T_1 .

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and keep the rod CF intact so that during any displacement the length CE does not change. Now give the system a small symmetrical displacement about the vertical line AD in which θ at the end A changes to $\theta + \delta\theta$ while θ at the end D does not change. The portion BCDEF moves as it is. The length BI changes while the length CE does not change so that during this small displacement the work done by the thrust T_2 of the rod CE is zero. The centres of gravity of all the six rods AB, BC etc. are slightly displaced.

$$\text{We have } BF = 4a \sin \theta.$$

In this case we cannot take the total weight of the rods AB, BC etc. act at the middle point O of AD. The depth of each of the points M and N below A is $a \cos \theta$, the depth of each of the points P and Q below A is $2a \cos \theta + a$, and the depth of each of the



points H and K below A is $2a \cos \theta + 2a - 1/2a$ where in this case SD is fixed.

By the principle of virtual work, we have

$$\begin{aligned} & T_1\delta(4a \sin \theta) + 2W\delta(2a \cos \theta) + 2W\delta(2a \cos \theta + a) - 0 \\ \text{or } & 4aT_1 \cos \theta \delta\theta - 10aW \sin \theta \delta\theta = 0 \\ \text{or } & 2a(2T_1 \cos \theta - 5W \sin \theta) \delta\theta = 0 \quad [\because \delta\theta \neq 0] \\ \text{or } & T_1 = \frac{5}{2}W \tan \theta. \end{aligned}$$

But in the position of equilibrium, the hexagon is a regular one and so $\theta = \pi/3$.

$$\text{Therefore } T_1 = \frac{5}{2}W \tan \frac{\pi}{3} = \frac{5}{2}W\sqrt{3}.$$

Now let us proceed to find the thrust T_2 .

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and so replace the rod CE by two equal and opposite forces T_2 as shown in the figure. Give the system a small symmetrical displacement about the vertical line AD in which θ at both the ends A and D changes to $\theta + \delta\theta$ so that both the lengths BF and CE change. In this case the total weight $6W$ of all the six rods AB, BC etc. can be taken acting at the middle point O of AD.

We have

$$BF = 4a \sin \theta, CE = 4a \sin \theta \text{ and } AO = 2a \cos \theta + a.$$

By the principle of virtual work, we have

$$\begin{aligned} & T_1\delta(4a \sin \theta) + T_2\delta(4a \sin \theta) + 6W\delta(2a \cos \theta + a) - 0 \\ \text{or } & 4aT_1 \cos \theta \delta\theta + 4aT_2 \cos \theta \delta\theta - 12aW \sin \theta \delta\theta = 0 \\ \text{or } & 4a(T_1 + T_2) \cos \theta - 3W \sin \theta \delta\theta = 0 \quad [\because \delta\theta \neq 0] \\ \text{or } & T_1 + T_2 = 3W \tan \theta. \end{aligned}$$

But in the position of equilibrium $\theta = \pi/3$.

$$\therefore T_1 + T_2 = 3W \tan \frac{\pi}{3} = 3W\sqrt{3}.$$

$$\therefore T_2 = 3W\sqrt{3} - T_1 = 3W\sqrt{3} - \frac{5W\sqrt{3}}{2} = \frac{W\sqrt{3}}{2}.$$

Ex.31 Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is

$$\tan B + \tan C$$

where W is the total weight of the rods AB and AC.

[Gorakhpur 79; Jivaji 78]

Sol. AB and AC are two

uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let

$AB = 2a$ and $AC = 2b$.

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore the total weight $W = W_1 + W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B + \delta B$ and C changes to $C + \delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE = DH + HE = a \cos B + b \cos C,$$

the height of any point of the line DE above BC
 $= DM - a \sin B.$

The equation of virtual work is -

$$\begin{aligned} & -T\delta(a \cos B + b \cos C) - W\delta(a \sin B) = 0 \\ \text{or } & aT \sin B \delta B + bT \sin C \delta C - aW \cos B \delta B = 0 \\ \text{or } & a(W \cos B - T \sin B) \delta B + bT \sin C \delta C = 0. \end{aligned} \quad \dots(1)$$

From the figure,

$$DM = a \sin B \text{ and } EN = b \sin C.$$

Since $DM = EN$, therefore $a \sin B = b \sin C$.

$$\therefore a(\sin B - b \sin C) = 0 \quad \dots(2)$$

$$\text{or } a \cos B \delta B + b \cos C \delta C = 0. \quad \dots(3)$$



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Dividing (1) by (2), we have

$$\begin{aligned} \frac{W \cos B - T \sin B}{\cos B} &= \frac{T \sin C}{\cos C} \\ \text{or } W - T \tan B &= T \tan C \\ \text{or } T' (\tan B + \tan C) &= W \\ \text{or } T' &= \frac{W}{\tan B + \tan C} \end{aligned}$$

Ex.32. Two uniform rods AB, BC of weights W and W' are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is

$$(W + W') \cos A \cos C / \sin B$$

Find the additional tension in the cord caused by suspending a weight W from B .

Sol. Draw figure and proceed as in Ex. 32.

In the first case we shall get

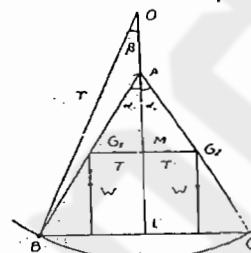
$$\begin{aligned} & (W + W') \cos A \cos C / \sin B \\ & = \frac{\tan A + \tan C}{\sin(A+C)} \cdot \frac{(W+W') \cos A \cos C}{\cos A \sin C} \\ & = \frac{(W+W') \cos A \cos C - (W+W') \cos A \cos C}{\sin(180^\circ - B)} \\ & = \frac{(W+W') \cos A \cos C}{\sin B} \end{aligned}$$

In the second case when a weight W' is also suspended from B , let T' be the tension in the cord. Write the new equation of virtual work and find T' .

$$\text{The required additional tension in the cord} \\ = T' - T = (2W' \cos A \cos C) / \sin B.$$

Ex.33. Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that if the string is stretched, its tension is $W(\tan \alpha - 2 \tan \beta)$, where 2α is the angle between the rods, and β the angle either rod subtends at the centre.

Sol. AB and AC are two uniform rods freely jointed at A and resting with their extremities B and C on the inside of a smooth circular hoop. The radius $OB=a$ of the circular hoop is greater than the length $2a$ of either rod. Let T be the tension in the string connecting the middle points G_1 and G_2 of the rods. The weights W and W of the two rods act at their middle points G_1 and G_2 and the total weight $W+W=2W$ will act at the middle point M of G_1G_2 . Given that $\angle BAL = \angle CAL = \alpha$ and $\angle BOL = \beta$.



Give the system a small displacement in which the angle α changes to $\alpha + \delta\alpha$ and β changes to $\beta + \delta\beta$. The smooth circular hoop remains fixed and hence its centre O can be taken as the fixed point. The lengths of the rods AB and AC do not change while the length of the string G_1G_2 changes.

The equation of virtual work is

$$\begin{aligned} & -T\delta(G_1G_2) + 2W\delta(MO) = 0 \\ \text{or } & -T\delta(2a \sin \alpha) + 2W\delta(r \cos \beta - a \cos \alpha) = 0 \\ \text{or } & a(-T \cos \beta + W \sin \alpha) \delta x + W r \sin \beta \delta \beta = 0 \quad \dots(1) \end{aligned}$$

In triangle OBL , $BL = r \sin \beta$,
and in triangle AHL , $BL = 2a \sin \alpha$.

$$\begin{aligned} & 2a \sin \alpha = r \sin \beta \\ \therefore & \delta(2a \sin \alpha) = \delta(r \sin \beta) \\ \text{or } & 2a \cos \alpha \delta \alpha = r \cos \beta \delta \beta. \quad \dots(2) \end{aligned}$$

Dividing (1) by (2), we get

$$\begin{aligned} & \frac{a(-T \cos \beta + W \sin \alpha)}{2a \cos \alpha} = \frac{W r \sin \beta}{r \cos \beta} \\ \text{or } & -T + W \tan \alpha = 2W \tan \beta \\ \text{or } & T = W(\tan \alpha - 2 \tan \beta). \end{aligned}$$

Ex.34. A frame, formed of four light rods, each of length a , freely jointed at A, B, C, D suspended from A . A mass m is suspended from B and D by two strings of length l ($l > a/\sqrt{2}$). The frame is kept in the form of a square by a string AC . Apply the method of virtual work to find the tension T in AC and show that when

$$l = a\sqrt{5}, T = 2mg/5.$$

Sol. The framework is suspended from A and so A is a fixed point from which the distances are to be measured. A mass m is

suspended from B and D by means of two strings BN and DN each of length l . Thus a weight mg acts at N . Let T be the tension in the string AC . In the position of equilibrium the figure is a square.

Let $\angle ABD = \theta$ and $\angle NBO = \phi$.

Give the system a small symmetrical displacement about the vertical AC in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA remain fixed and the length AC changes. The lengths of the strings BN and DN remain fixed so that the work done by their tensions is zero. The point N is slightly displaced.

We have

$$AC = 2AO = 2a \sin \theta,$$

and the depth of N below A

$$AN = AO + ON = a \sin \theta + l \sin \phi.$$

The equation of virtual work is

$$\begin{aligned} & l\delta(a \sin \theta) + mg \delta(l \sin \phi) = 0 \\ \text{or } & 2al \cos \theta \delta\theta + lmg \cos \phi \delta\phi = 0 \\ \text{or } & a \cos \theta (2T - mg) \delta\theta + lmg \cos \phi \delta\phi = 0. \quad \dots(1) \end{aligned}$$

Now from the ΔAOB , $BO = a \cos \theta$ and from the ΔBON , $BO = l \cos \phi$.

$$\therefore a \cos \theta = l \cos \phi.$$

$$\text{so that } -a \sin \theta \delta\theta = -l \sin \phi \delta\phi.$$

$$\text{or } a \sin \theta \delta\theta = l \sin \phi \delta\phi. \quad \dots(2)$$

Dividing (1) by (2), we have

$$\begin{aligned} & \frac{\cos \theta (2T - mg)}{\sin \theta} = \frac{mg \cos \phi}{\sin \phi} \\ \text{or } & \cos \theta (2T - mg) = mg \cot \phi \\ \text{or } & 2T - mg = mg \tan \theta \cot \phi \\ \text{or } & T = \frac{1}{2}mg(1 + \tan \theta \cot \phi). \end{aligned}$$

In the position of equilibrium $\theta = 45^\circ$,

$$\begin{aligned} BO &= a \cos 45^\circ = a/\sqrt{2}, \\ ON &= \sqrt{(BN^2 - BO^2)} = \sqrt{l^2 - (a^2/2)}, \\ &= (\sqrt{2l^2 - a^2})/\sqrt{2}, \end{aligned}$$

$$\begin{aligned} \text{so that } \cot \phi &= \frac{a}{\sqrt{l^2 - a^2}} = \frac{a}{\sqrt{(2l^2 - a^2)/2}} \\ &= \frac{a}{\sqrt{2l^2 - a^2}}. \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2}mg \left\{ 1 + \tan 45^\circ \cdot \frac{a}{\sqrt{2l^2 - a^2}} \right\} \\ &= \frac{1}{2}mg \left\{ 1 + \frac{a}{\sqrt{2l^2 - a^2}} \right\}. \end{aligned}$$

When $l = a\sqrt{5}$, the tension T

$$\begin{aligned} & = \frac{1}{2}mg \left\{ 1 + \frac{a}{\sqrt{2a^2(5-a^2)}} \right\} \\ & = \frac{1}{2}mg \left\{ 1 + \frac{a}{\sqrt{2a^2(5-a^2)}} \right\} \\ & = \frac{1}{2}mg(1+\frac{1}{2}) = \frac{3}{4}mg. \end{aligned}$$

Ex.35 A rod is movable about a point A , and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A . Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is

$$\frac{W \cos \alpha \cos \beta}{2 \sin(\alpha + \beta)},$$

where W is the weight of the rod, and α, β the inclinations of the rod and the string to the horizontal.

(Lucknow 76; Allahabad)

Sol. The rod AB is hinged at A . Let the length of the rod AB be a and the length of the string BC be l . At C there is a ring which can slide on a smooth horizontal wire AC .Let P be the horizontal force applied at the ring C to keep it at rest. The weight W of the rod AB acts at its middle point G .

$$\text{Let } \angle BAC = \alpha \text{ and } \angle BCA = \beta.$$

Give the system a small displacement in which α changes to $\alpha + \delta\alpha$ and β changes to $\beta + \delta\beta$. The point A remains fixed. The length of the rod AB remains fixed and the length of the string BC also remains fixed so that the work done by its tension is zero. The points G and C are slightly displaced. We have

$$\begin{aligned} & \text{the depth of } G \text{ below } A = MG \\ & = AG \sin \alpha = a \sin \alpha, \\ \text{and the horizontal distance of } C \text{ from } A = AC \\ & = AN + NC = a \cos \alpha + l \cos \beta. \end{aligned}$$

The equation of virtual work is

$$\begin{aligned} & l\delta(a \cos \alpha) + PA(a \cos \alpha + l \cos \beta) = 0 \\ \text{or } & la W \cos \alpha \delta\alpha + PA(a \cos \alpha + l \cos \beta) \delta\beta = 0 \\ \text{or } & a(lW \cos \alpha - PA \sin \alpha) \delta\alpha = lP \sin \beta \delta\beta. \quad \dots(1) \end{aligned}$$

From the figure, equating the values of BN found from the triangles ANB and CNB , we get

$$\begin{aligned} & a(lW \cos \alpha - PA \sin \alpha) \delta\alpha = lP \sin \beta \delta\beta. \end{aligned}$$

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$$\begin{aligned} & a \sin \alpha = l \sin \beta, \\ & \text{so that } a \cos \alpha \delta x = l \cos \beta \delta y. \quad \dots(2) \\ & \text{Dividing (1) by (2), we get} \\ & \frac{lW \cos \alpha - P \sin \alpha}{\cos \alpha} = \frac{P \sin \beta}{\cos \beta} \\ & \text{or } lW \cos \alpha \cos \beta - P \sin \alpha \cos \beta = P \cos \alpha \sin \beta \\ & \text{or } P(\sin \beta \cos \alpha + \cos \beta \sin \alpha) = \frac{1}{2} W \cos \alpha \cos \beta \\ & \text{or } P = \frac{W \cos \alpha \cos \beta}{2 \sin(\alpha + \beta)}. \end{aligned}$$

Ex.36 Weights W_1 , W_2 are fastened to a light inextensible string ABC at the points B , C the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB , BC are inclined at angles θ , ϕ to the vertical, then $P = (W_1 + W_2) \tan \theta = W_2 \tan \phi$.

Sol. Let the length of the portion AB of the string be a and that of BC be b . The point A is fixed and the vertical line AO through A is a fixed line.

From the fixed point A ,

$$\text{the depth of } B = AM = a \cos \theta.$$

and the depth of C

$$\begin{aligned} & = AN = AM + MN \\ & = AM + BD = a \cos \theta + b \cos \phi. \end{aligned}$$

Also the horizontal distance of the point C from the fixed line $AO = NC$

$$= ND = DC = MH = DC = a \sin \theta + b \sin \phi.$$

Now give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point A remains fixed, the length of the string remains unaltered and the points B and C are slightly displaced. The equation of virtual work is

$$\begin{aligned} & W_1 \delta(a \cos \theta) + W_2 \delta(a \cos \theta + b \cos \phi) \\ & + P \delta(a \sin \theta + b \sin \phi) = 0 \\ \text{or } & -aW_1 \sin \theta \delta\theta - aW_2 \sin \theta \delta\theta - bW_2 \sin \phi \delta\phi + aP \cos \theta \delta\theta \\ & + bP \cos \phi \delta\phi = 0 \\ \text{or } & a[P \cos \theta - (W_1 + W_2) \sin \theta] \delta\theta + b[W_2 \sin \phi - P \cos \phi] \delta\phi = 0. \quad \dots(1) \end{aligned}$$

where θ and ϕ are independent of each other.

Now consider a displacement when only θ changes and ϕ does not change so that $\delta\phi = 0$. Then putting $\delta\phi = 0$ in (1), we have

$$\begin{aligned} & a[P \cos \theta - (W_1 + W_2) \sin \theta] \delta\theta = 0 \\ \text{or } & P \cos \theta - (W_1 + W_2) \sin \theta = 0 \quad [\because \delta\theta \neq 0] \\ \text{or } & P = (W_1 + W_2) \tan \theta. \quad \dots(2) \end{aligned}$$

Again consider a displacement when only ϕ changes and θ does not change so that $\delta\theta = 0$. Thus putting $\delta\theta = 0$ in (1), we have

$$\begin{aligned} & b[W_2 \sin \phi - P \cos \phi] \delta\phi = 0 \\ \text{or } & W_2 \sin \phi - P \cos \phi = 0 \quad [\because \delta\phi \neq 0] \\ \text{or } & P = W_2 \tan \phi. \quad \dots(3) \end{aligned}$$

From (2) and (3), we have

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi.$$

Ex.37 Five equal uniform rods, freely jointed at their ends, form a regular pentagon $ABCDE$ and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is $W \cot(\pi/10)$, where W is the weight of the rod.

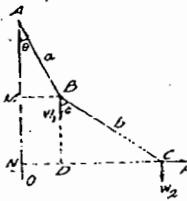
Sol. $ABCDE$ is a pentagon formed of five equal rods each of weight W and say of length $2a$. It is suspended from A and BE is jointed by a weightless bar. Let T be the thrust in the bar BE . The line AM joining A to the middle point M of CD is vertical and the line BE is horizontal. The weights of the rods AB , BC , CD , DE and EA act at their respective middle points. In the position of equilibrium the pentagon is a regular one so that each of its interior angles is $180^\circ - 72^\circ$ i.e., 108° or $\frac{3}{2}\pi$ radians.

Let θ be the angle which the two upper slant rods AB and AE make with the vertical and ϕ be the angle which the two lower slant rods CB and DE make with the vertical.

Replace the rod BE by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AM in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$.

The point A remains fixed. The lengths of the rods AB , BC etc. remain fixed, the length BE changes and the middle points of the rods AB , BC etc. are slightly displaced. The $\angle ANB$ remains 90° .



We have

$$BE = 2BN = 2 \cdot 2a \sin \theta = 4a \sin \theta,$$

the depth of the middle point of AB or AE below A = $a \cos \theta$,

the depth of the middle point of BC or ED below A

$$= 2a \cos \theta + a \cos \phi,$$

and the depth of the middle point M of CD below A

$$= 2a \cos \theta + 2a \cos \phi.$$

The equation of virtual work is

$$\begin{aligned} & T\delta(4a \sin \theta) + 2W\delta(a \cos \theta) + 2W\delta(2a \cos \theta + a \cos \phi) \\ & + W\delta(2a \cos \theta + 2a \cos \phi) = 0 \\ \text{or } & 4aT \cos \theta \delta\theta - 2aW \sin \theta \delta\theta - 4aW \sin \theta \delta\theta - 2aW \sin \phi \delta\phi - 2aW \sin \theta \delta\theta - 2aW \sin \phi \delta\phi = 0 \\ \text{or } & 4a(T \cos \theta - 2W \sin \theta) \delta\theta = 4aW \sin \phi \delta\phi. \quad \dots(1) \end{aligned}$$

From the figure, finding the length BE in two ways i.e., from the upper portion ABE and from the lower portion $BCDE$, we have

$$4a \sin \theta = 2a + a \sin \phi.$$

Differentiating, we get $4a \cos \theta \delta\theta = 4a \cos \phi \delta\phi$

$$\text{or } \cos \theta \delta\theta = \cos \phi \delta\phi. \quad \dots(2)$$

Dividing (1) by (2), we get

$$\frac{T \cos \theta - 2W \sin \theta}{\cos \theta} = \frac{W \sin \phi}{\cos \phi}$$

$$\text{or } T - 2W \tan \theta = 2W \tan \phi$$

$$\text{or } T = W(\tan \phi + 2 \tan \theta).$$

But in the position of equilibrium,

$$\theta = \frac{1}{2} \cdot \frac{3}{5} \pi = \frac{3}{10} \pi, \phi = \frac{3}{2} \pi - \frac{1}{2} \cdot \frac{1}{10} \pi.$$

$$\begin{aligned} \therefore T &= W \left(\tan \frac{1}{10} \pi + 2 \tan \frac{3}{10} \pi \right) = W \left[\tan \frac{1}{10} \pi + 2 \cot \frac{2}{10} \pi \right] \\ & \quad \left[\because \tan \frac{3}{10} \pi = \cot \left(\frac{1}{2} \pi - \frac{3}{10} \pi \right) = \cot \frac{2}{10} \pi \right] \\ &= W \left[\tan \frac{1}{10} \pi + 2 \cdot \frac{1 - \tan^2(\pi/10)}{2 \tan(\pi/10)} \right] \\ & \quad \left[\because \cot 2x = \frac{1 - \tan^2 x}{2 \tan x} \right] \\ &= W \cot(\pi/10). \end{aligned}$$

Ex.38 A regular pentagon $ABCDE$ formed of equal uniform rods each of weight W , is suspended from the point A and is maintained in shape by a light rod joining the middle points of BC and DE . Prove that the stress in the light rod is $2W \cot(\pi/10)$.

Sol. Proceed as in part (a).

Ex.39 A freely jointed framework is formed of five equal uniform rods each of weight W . The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles θ and ϕ respectively with the vertical, prove that the tension of the string is to the weight of the rod as $(4 \tan \theta + 2 \tan \phi) : (\tan \theta + \tan \phi)$.

Sol. Draw figure as in Ex. 30 (a). This question differs from the preceding one in having the string AM instead of the rod BE .

Let T be the tension in the string AM . The string AM is given to be inextensible, therefore before giving the displacement replace the string by two equal and opposite forces T so that the length AM may not be changed.

Here $AM = 2a \cos \theta + 2a \cos \phi$.

The equation of virtual work is

$$\begin{aligned} & -T\delta(2a \cos \theta + 2a \cos \phi) + 2W\delta(2a \cos \theta + a \cos \phi) \\ & + W\delta(2a \cos \theta + 2a \cos \phi) = 0 \end{aligned}$$

$$\text{or } 2aT \sin \theta \delta\theta + 2aT \sin \phi \delta\phi - 2aW \sin \theta \delta\theta - 4aW \sin \theta \delta\theta - 2aW \sin \phi \delta\phi = 0$$

$$\text{or } 2a \sin \theta (T - 4W) \delta\theta = 2a \sin \phi (2W - T) \delta\phi$$

$$\text{or } \sin \theta (T - 4W) \delta\theta = \sin \phi (2W - T) \delta\phi. \quad \dots(1)$$

Also from the figure, we have

$$4a \sin \theta = 2a + a \sin \phi,$$

$$\text{so that } 4a \cos \theta \delta\theta = 4a \cos \phi \delta\phi$$

$$\text{or } \cos \theta \delta\theta = \cos \phi \delta\phi. \quad \dots(2)$$

Dividing (1) by (2), we get

$$\tan \theta (T - 4W) = \tan \phi (2W - T)$$

$$\text{or } T(\tan \theta + \tan \phi) = W(2 \tan \phi + 4 \tan \theta)$$

$$\text{or } \frac{T}{W} = \frac{4 \tan \theta + 2 \tan \phi}{\tan \theta + \tan \phi}, \text{ which proves the required result.}$$

Ex.40 A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.

Sol. Let W_1 and W_2 be the weights of the beams AP and BQ .



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whose lengths are say $2l_1$ and $2l_2$ respectively. Let θ and ϕ be the angles which the radii OP and OQ make with the horizontal diameter COD of the board. Let a be the radius of the board.

Here COD is a fixed horizontal line. The weight W_1 of the beam AP acts at its centre of gravity G_1 whose height above $CD = MG_1 = l_1 + a \sin \theta$.

The weight W_2 of the beam BQ acts at G_2 whose height above CD is $l_2 + a \sin \phi$.

Let the beams be imagined to undergo a small displacement in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$. The equation of virtual work is $-W_1 \delta(l_1 + a \sin \theta) - W_2 \delta(l_2 + a \sin \phi) = 0$ or $-W_1 a \cos \theta \delta\theta - W_2 a \cos \phi \delta\phi = 0$ or $-W_1 \cos \theta \delta\theta - W_2 \cos \phi \delta\phi = 0 \quad \dots(1)$

If b be the distance between the tubes in which the beams slide, then from the figure

$$a \cos \theta = a \cos \phi = b = \text{constant}$$

so that, $-a \sin \theta \delta\theta - a \sin \phi \delta\phi = 0$

$$\text{or} \quad -\sin \theta \delta\theta - \sin \phi \delta\phi = 0 \quad \dots(2)$$

Dividing (1) by (2), we have

$$\frac{W_1}{W_2} \cot \phi = \frac{\tan \theta}{\tan \phi}, \text{ which gives the required ratio.}$$

Ex.41 A smoothly jointed framework of light rods forms a quadrilateral $ABCD$. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T , and the middle points R, S of the other pair by a light rod in a state of thrust X : show, by the method of virtual work, that $T/PQ = X/RS$.

Sol. $ABCD$ is a framework in the form of a quadrilateral formed of four light rods. The middle points P and Q of the rods AB and DC are joined by a string in a state of tension T and the middle points R and S of the rods AD and BC are joined by a light rod in a state of thrust X . The framework is to be taken as placed on some smooth horizontal plane.

Since P, S, Q, R are the middle points

of the sides of the quadrilateral $ABCD$, therefore $PSQR$ is a parallelogram. Consequently the diagonals PQ and RS of this parallelogram bisect each other at O .

Replace the string PQ by two equal and opposite forces T as shown in the figure and replace the rod RS by two equal and opposite forces X as shown in the figure. Now give the system a small displacement in which PQ changes to $PQ + \delta(PQ)$ and RS changes to $RS + \delta(RS)$. The lengths of the rods AB, BC, CD, DA do not change. The equation of virtual work is

$$-T\delta(PQ) + X\delta(RS) = 0$$

$$\text{or} \quad T\delta(PQ) = X\delta(RS)$$

$$\text{or} \quad \frac{\delta(PQ)}{\delta(RS)} = \frac{X}{T} \quad \dots(1)$$

Now let us find a relation between the parameters PQ and RS from the figure. Since OP is a median of the $\triangle OAB$, therefore

$$OA^2 + OB^2 = 2OP^2 + 2AP^2 = 2(\frac{1}{4}PQ^2) + 2(\frac{1}{4}AB)^2$$

$$= \frac{1}{2}(PQ^2 + AB^2). \quad \dots(2)$$

Similarly from $\triangle OCD$, we have

$$OC^2 + OD^2 = \frac{1}{2}(PQ^2 + CD^2). \quad \dots(3)$$

Adding (2) and (3), we get

$$OA^2 + OB^2 + OC^2 + OD^2 = \frac{1}{2}(2PQ^2 + AB^2 + CD^2) \quad \dots(4)$$

Doing the same thing with $\triangle OAD$ and $\triangle OBC$, we get

$$OA^2 + OB^2 + OC^2 + OD^2 = \frac{1}{2}(2RS^2 + BC^2 + DA^2). \quad \dots(5)$$

From (4) and (5), we get

$$\frac{1}{2}(2PQ^2 + AB^2 + CD^2) = \frac{1}{2}(2RS^2 + BC^2 + DA^2)$$

$$\text{or} \quad 2(PQ^2 - RS^2) = BC^2 + DA^2 - AB^2 - CD^2$$

$$\text{or} \quad PQ^2 - RS^2 = \text{constant}, \quad \dots(6)$$

since AB, BC, CD, DA are all of fixed lengths.

Differentiating (6), we get

$$2PQ \delta(PQ) - 2RS \delta(RS) = 0$$

$$\text{or} \quad \frac{\delta(PQ)}{\delta(RS)} = \frac{RS}{PQ} \quad \dots(7)$$

Equating the values of $\frac{\delta(PQ)}{\delta(RS)}$ from (1) and (7), we get

$$\frac{X}{T} = \frac{RS}{PQ} \text{ or } \frac{X}{RS} = \frac{T}{PQ}.$$



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Virtual Work

Ex.42 $ABCD$ is a rhombus with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C , and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD . B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is $2x$, show that the tension in the string is

$$W \tan x \left[\frac{c}{2l} \cosec^2 x - 1 \right]$$

Sol. The rods AB and AD of the frame rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2c$. The length of each rod of the rhombus is l and the rods forming the rhombus are weightless. A weight W is attached to the lowest point C . Let T be the tension in the string BD . We have

$$\angle BAC = x = \angle CAD.$$

The diagonal AC is vertical and BD is horizontal.

Give the system a small symmetrical displacement in which x changes to $x + \delta x$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains 90° . We have

$$BD = 2BO = 2AB \sin x = 2l \sin x.$$

Also the depth of the point C below EF

$$= MC = AC - AM = 2AO - AM$$

$$= 2AB \cos x - EM \cot x = 2l \cos x - c \cot x.$$

The equation of virtual work is

$$-T\delta(2l \sin x) + W\delta(2l \cos x - c \cot x) = 0$$

$$\text{or} \quad -2T \cos x \delta x - 2lW \sin x \delta x + Wc \cosec^2 x \delta x = 0$$

$$\text{or} \quad (-2T \cos x \delta x - 2lW \sin x \delta x + Wc \cosec^2 x \delta x) = 0 \quad [\because \delta x \neq 0]$$

$$\text{or} \quad 2lT \cos x = Wc \cosec^2 x - 2lW \sin x$$

$$\text{or} \quad T = \frac{1}{2l \cos x} [Wc \cosec^2 x - 2lW \sin x]$$

$$\text{or} \quad T = W \tan x \left[\frac{c}{2l} \cosec^2 x - 1 \right]$$

Ex.43 A rhombus $ABCD$ formed of four weightless rods each of length a freely jointed at the extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant $2c$ apart and in contact with AB and AD . Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If $2x$ be the angle of the rhombus at A , apply the principle of virtual work to find the tension of the string.

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and

$$EF = 2c.$$

The length of each rod of the rhombus is a and the rods forming the rhombus are light. Weights each equal to W are hung from the lowest corner C and from the middle points P and Q of the lower sides BC and CD . The diagonal AC is vertical and BD is horizontal. Let T be the tension in the inextensible string joining B and D . We have

$$\angle BAC = x = \angle DAC.$$

Replace the string BD by two equal and opposite forces T as shown in the figure so that the distance BD can be changed. Give the system a small symmetrical displacement in which x changes to $x + \delta x$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains 90° .

We have

$$BD = 2BO = 2AB \sin x = 2a \sin x.$$

The depth of C below EF is $MC = AC - AM$

$$= 2AO - AM = 2AB \cos x - EM \cot x$$

and the depth of P or Q below EF

$$= AN - AM = \frac{1}{2}AO - AM$$

$$= \frac{1}{2}a \cos x - c \cot x.$$

The equation of virtual work is

$$-T\delta(2a \sin x) + W\delta(2a \cos x - c \cot x) = 0$$

$$\text{or} \quad (-2aT \cos x + 2aW \sin x + Wc \cosec^2 x - 2aW \sin x) \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

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$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

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$$+ 2aW \sin x + 2cW \cosec^2 x \delta x = 0$$

$$+ 2aW \sin x + 2cW \cosec^2 x \$$

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Virtual Work

$$\text{or } \begin{aligned} & -2aT \cos x - 5aW \sin x + 3W \cosec^2 x = 0 \quad [\because \dot{\theta} = 0] \\ & +2a^2 \cos x = 3W \cosec^2 x - 5aW \sin x \\ & \therefore T = \frac{W(3\cosec^2 x - 5a \sin x)}{2a \cos x}. \end{aligned}$$

Ex.44 ABCD is a rhombus formed with four rods each of length l and of weight w joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is 2d and the angle at A is 2x, show that the tension in the string is

$$\tan x \left[\frac{d}{2l} (W+4w) \cosec^2 x - (W+2w) \right].$$

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and EF = 2d. The length of each rod forming the rhombus is l. The total weight 4w of the rods forming the rhombus can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W is attached to the lowest point C. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the string BD. We have

$$\angle BAC = x = \angle DAC.$$

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta x$. The line EF joining the pegs remain fixed and the distances will be measured from this line. The $\angle AGB$ remains 90°.

We have the length of the string BD

$$= 2BG = 2A \sin x - 2l \sin x$$

The depth of G below EF

$$= MG = AG - AM = l \cos x - d \cot x,$$

and the depth of C below EF

$$= MC = AC - AM = 2l \cos x - d \cot x.$$

The equation of virtual work is

$$\begin{aligned} & -T(2 \sin x) + 4w(l \cos x - d \cot x) \\ & \quad + W(2l \cos x - d \cot x) = 0 \\ \text{or } & -2l T \cos x - 4w \sin x - 8x : 4dw \cosec^2 x - 8x - 2l W \sin x \delta x \\ & \quad + dW \cosec^2 x \delta x = 0 \\ \text{or } & -2l T \cos x - 2l \sin x (2w + W) + d \cosec^2 x (4w + W) \delta x = 0 \\ \text{or } & -2l T \cos x - 2l \sin x (2w + W) + d \cosec^2 x (4w + W) = 0 \\ \text{or } & 2l T \cos x - d(W+4w) \cosec^2 x - 2l \sin x (W+2w) = 0 \end{aligned}$$

$$\begin{aligned} \text{or } & T = \frac{1}{2l \cos x} [d(W+4w) \cosec^2 x - 2l \sin x (W+2w)] \\ \text{or } & T = \tan x \left[\frac{d}{2l} (W+4w) \cosec^2 x - (W+2w) \right]. \end{aligned}$$

Ex.45. A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{1}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant $2b$ apart. A weight W is suspended from A, find the thrust in the rod BC.

Sol. ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and EF = 2b. Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{1}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let

$$\angle BAD = x = \angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are : (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

We have

$$BC = 2BD = 2AB \sin \theta = 2a \sin \theta.$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME \cot \theta = b \cot \theta.$$

The equation of virtual work is

$$\begin{aligned} & T\delta(2a \sin \theta) + W\delta(b \cot \theta) = 0 \\ \text{or } & 2a T \cos \theta \delta \theta - bW \cosec^2 \theta \delta \theta = 0 \\ \text{or } & (2a T \cos \theta - bW \cosec^2 \theta) \delta \theta = 0 \\ \text{or } & 2a T \cos \theta - bW \cosec^2 \theta = 0 \quad [\because \delta \theta \neq 0] \\ \text{or } & T = \frac{Wb \cosec^2 \theta}{2a} \end{aligned}$$

But in the position of equilibrium,

$$BC = \frac{1}{2}a \text{ and so } BD = \frac{1}{4}a.$$

$$\text{Therefore } \sin \theta = \frac{BD}{AB} = \frac{\frac{1}{4}a}{a} = \frac{1}{4}.$$



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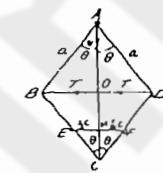
$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{1}{\sqrt{15}}.$$

$$\therefore T = \frac{Wb}{2a} \cdot \frac{16}{9} \cdot \frac{4}{\sqrt{15}} = \frac{32}{9\sqrt{15}} b W.$$

Ex.46 A rhomboidal framework ABCD is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude

$$W(2a^2c - b^2)/b^3 (4a^2 - b^2)^{1/2}.$$

Sol. The rods BC and CD of a rhomboidal framework ABCD are in contact with two smooth pegs E and F which are in the same horizontal line and EF = c. The rods forming the rhombus are light and the length of each rod forming the rhombus is a. Let T be the thrust in the light rod BD joining B and D. A weight W is placed at the highest joint A. In the position of equilibrium, BD = b. The diagonal AC is vertical and BD is horizontal. Let $\angle BAC = \theta = \angle CAD$.



Replace the rod BD by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The only forces contributing to the sum of virtual works are : (i) the weight W placed at A, and (ii) the thrust T in the rod BD. The reactions of the pegs E and F do not work. We have

$$BD = 2a \sin \theta = 2AB \sin \theta = 2a \sin \theta.$$

Also the height of A above the fixed line EF

$$= MA = CA - CM$$

$$= 2OA - CM = 2a \cos \theta - b \cot \theta.$$

The equation of virtual work is

$$\begin{aligned} & T\delta(2a \sin \theta) - W\delta(2a \cos \theta - b \cot \theta) = 0 \\ \text{or } & 2a T \cos \theta \delta \theta - 2a \cdot 2a \sin \theta \delta \theta - b \csc^2 \theta \delta \theta = 0 \\ \text{or } & (2a T \cos \theta + 2a W \sin \theta - b \csc^2 \theta) \delta \theta = 0 \quad [\because \delta \theta \neq 0] \\ \text{or } & 2a T \cos \theta + 2a W \sin \theta - b \csc^2 \theta = 0 \\ \text{or } & 2a T \cos \theta = b \csc^2 \theta - 2a W \sin \theta \\ \text{or } & T = W \frac{\frac{1}{2} \csc^2 \theta - 2a \sin \theta}{2a \cos \theta} \end{aligned}$$

But in the position of equilibrium, we have

$$BD = b \text{ so that } BO = \frac{1}{2}b.$$

from $\triangle AOB$, we have

$$\sin \theta = \frac{BO}{AB} = \frac{\frac{1}{2}b}{a} = \frac{b}{2a}.$$

$$\therefore \csc \theta = \frac{2a}{b} \text{ and } \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{b}{2a}\right)^2} = \frac{\sqrt{4a^2 - b^2}}{2a}.$$

Substituting in (1), we have

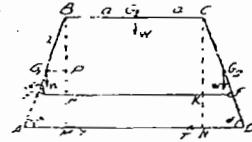
$$T = W \frac{\frac{1}{2} \left(\frac{4a^2 - b^2}{b^2} - 2a \right) / 2a}{2a \left(\sqrt{4a^2 - b^2} / 2a \right)} = W \frac{2a^2c - b^2}{b^3 (4a^2 - b^2)^{1/2}}.$$

Ex.47. Three rigid rods AB, BC, CD each of length $2a$, are smoothly jointed at B and C. The system is placed in a vertical plane so that rods AB, CD are in contact with two smooth pegs distant $2c$ apart in the same horizontal line, the rods AB, CD make equal angles α with the horizontal. Prove that the tension of the string AD which will maintain this configuration is

$$W \cosec \alpha \sec^2 \alpha ((3/c\alpha) - (3 + 2 \cos^3 \alpha)),$$

where W is the weight of either rod.

Sol. Three rigid rods AB, BC, CD, each of length $2a$ and weight W are smoothly jointed at B and C. The rods AB and CD are in contact with two smooth pegs E and F which are in the same horizontal line and EF = 2c. The rods BC and CD make equal angles α with the horizontal and the weight W acts at their midpoints.



Let T be the tension in the string AD joining A and D. The weights W of the rods AB, BC and CD act at their respective midpoints.

We have $\angle BAD = x = \angle CDA$.

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD do not change and the length AD changes.

We have,

$$\begin{aligned} AD &= AM = MN = ND \\ &= 2a \cos x + 2a + 2a \cos \alpha \\ &= 4a \cos x + 2a. \end{aligned}$$

The height of G or G' above the fixed line EF

$$\begin{aligned} & = HP = HB - PB = EH \tan \alpha - BG_1 \sin \alpha \\ & = \frac{1}{2}(2c - 2a) \tan \alpha - a \sin \alpha \end{aligned}$$

and the height of G' above EF

$$= HB = (c - a) \tan \alpha.$$

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The equation of virtual work is

$$\begin{aligned} -T\delta (4a \cos \alpha + 2a) - 2W\delta ((c-a) \tan \alpha - a \sin \alpha) \\ \text{or } 4a T \sin \alpha \delta \alpha - 2(c-a) W \sec^2 \alpha \delta \alpha \\ + 2a W \cos \alpha \delta \alpha - (c-a) W \sec^2 \alpha \delta \alpha = 0 \\ \text{or } [4a T \sin \alpha - 3(c-a) W \sec^2 \alpha] + 2a W \cos \alpha = 0 \\ \text{or } 4a T \sin \alpha - 3(c-a) W \sec^2 \alpha - 2a W \cos \alpha = 0 \\ \text{or } 4a T \sin \alpha = 3(c-a) W \sec^2 \alpha - 2a W \cos \alpha \\ \text{or } T = \frac{1}{4a \sin \alpha} [W(3c \sec^2 \alpha - 3a \sec^2 \alpha - 2a \cos \alpha)] \\ = \frac{1}{4a \sin \alpha} [W(3c/a - 3 + 2 \cos \alpha)]. \end{aligned}$$

Ex.48 Four light rods are joined together to form a quadrilateral $OABC$. The lengths are such that $OA=OC=a$, $AB=CB=b$.

The framework hangs in a vertical plane OA and OC resting in contact with two smooth pegs distant 1 apart and on the same horizontal level. A weight hangs at B . If θ, ϕ are the inclinations of OA , AB to the horizontal, prove that these values are given by the equations

$$a \cos \theta = b \cos \phi$$

and $\frac{1}{4} \sec^2 \theta \sin \phi = a \sin(\theta + \phi)$.

Sol. $OABC$ is a framework formed of four light rods such that $OA=OC=a$ and $AB=CB=b$.

The rods OA and OC are in contact with two smooth pegs P and Q which are in the same horizontal line and $PQ=1$. A weight W hangs at B . We have

$$\angle OAC = \theta \text{ and } \angle BAC = \phi.$$

Give the system a small displacement in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W acting at B .

$$\begin{aligned} \text{We have, the depth of } B \text{ below } P'Q &= LB = OB - OL \\ &= OD + DB - OL = a \sin \theta + b \sin \phi - \frac{1}{4} \tan \theta. \end{aligned}$$

The equation of virtual work is

$$\begin{aligned} W\delta(a \sin \theta + b \sin \phi - \frac{1}{4} \tan \theta) = 0 \\ \text{or } a \cos \theta \delta\theta + b \cos \phi \delta\phi - \frac{1}{4} \sec^2 \theta \delta\theta = 0 \\ \text{or } (\frac{1}{4} \sec^2 \theta - a \cos \theta) \delta\theta = b \cos \phi \delta\phi. \quad (1) \end{aligned}$$

Now let us find a relation between the parameters θ and ϕ from the figure. From the $\triangle OAD$, we have $AD = a \cos \theta$ and from the $\triangle BAD$, we have $AD = b \cos \phi$.

$$a \cos \theta = b \cos \phi. \quad (2)$$

Differentiating (2),

$$\begin{aligned} \text{or } -a \sin \theta \delta\theta - b \sin \phi \delta\phi \\ -a \sin \theta \delta\theta = b \sin \phi \delta\phi. \quad (3) \end{aligned}$$

Dividing (1) by (3), we get

$$\begin{aligned} \frac{\frac{1}{4} \sec^2 \theta - a \cos \theta}{a \sin \theta} = \frac{b \cos \phi}{b \sin \phi} \\ \text{or } \frac{\frac{1}{4} \sec^2 \theta \sin \phi}{a \sin \theta} = a (\sin \theta \cos \phi + \cos \theta \sin \phi) \\ \text{or } \frac{\frac{1}{4} \sec^2 \theta \sin \phi}{a \sin \theta} = a \sin(\theta + \phi). \quad (4) \end{aligned}$$

Thus θ and ϕ are given by the equations (2) and (4).

Ex.49 A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/2}$.

Sol. A uniform beam AB of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b . Suppose the rod makes an angle θ with the wall i.e., $\angle BAN = \theta$. The weight W of the rod acts at its middle point G .

Give the rod a small displacement in which θ changes to $\theta + \delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G . The reactions at A and C do not work.

We have, the height of G above the fixed point C

$$= NM = AM - AN = AG \cos \theta - CN \cot \theta$$

$$= a \cos \theta - b \cot \theta.$$

The equation of virtual work is

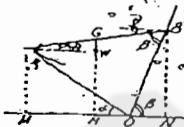
$$\begin{aligned} -W\delta(a \cos \theta - b \cot \theta) = 0 \\ \delta(a \cos \theta - b \cot \theta) = 0 \\ -a \sin \theta \delta\theta + b \cot^2 \theta \delta\theta = 0 \\ (-a \sin \theta + b \cot^2 \theta) \delta\theta = 0 \\ -a \sin \theta + b \cot^2 \theta = 0 \\ a \sin \theta = b \cot^2 \theta \text{ or } \sin^2 \theta = b/a \\ \theta = \sin^{-1}(b/a)^{1/2}, \end{aligned}$$

giving the inclination of the rod to the vertical in the position of equilibrium.

Ex.50 A heavy uniform rod, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclinations α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that

$$\tan \theta = \frac{1}{2} (\cot \alpha - \cot \beta).$$

Sol. Let AB be the rod of length $2a$ and G its middle point. Let AM , BN , and GH be the perpendiculars from A , B , and G on the horizontal line through O , the point of intersection of the inclined planes OA and OB . The weight W of the rod acts at G and in equilibrium the rod makes an angle θ with the horizontal.



Give the rod a small displacement in which θ changes to $\theta + \delta\theta$. The horizontal line MN through O is the fixed line from which the distances will be measured. The angles α and β remain fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G . The reactions at A and B do no work. We have

the height of G above the fixed line MN

$$= HG = \frac{1}{2} (AM + BN) = \frac{1}{2} (OA \sin \alpha + OB \sin \beta).$$

From the $\triangle AOB$ by the sine theorem of trigonometry, we have

$$\frac{OA}{\sin(\beta-\theta)} = \frac{OB}{\sin(\alpha+\theta)} = \frac{AB}{\sin(\pi-(\alpha+\beta))} = \frac{2a}{\sin(\pi-\alpha-\beta)}$$

$$\therefore OA = 2a \frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}, OB = \frac{2a \sin(\theta+\alpha)}{\sin(\alpha+\beta)}.$$

$$\therefore HG = \frac{1}{2} \cdot \frac{2a}{\sin(\alpha+\beta)} (\sin(\beta-\theta) \sin \alpha + \sin(\theta+\alpha) \sin \beta).$$

The equation of virtual work is

$$-W\delta(HG) = 0, \text{ or } \delta(HG) = 0$$

$$\text{or } \delta \left[\frac{a}{\sin(\alpha+\beta)} (\sin(\beta-\theta) \sin \alpha + \sin(\theta+\alpha) \sin \beta) \right] = 0$$

$$\text{or } \frac{a}{\sin(\alpha+\beta)} [-\cos(\beta-\theta) \sin \alpha + \cos(\theta+\alpha) \sin \beta] \delta\theta = 0$$

$$\text{or } -\cos(\beta-\theta) \sin \alpha + \cos(\theta+\alpha) \sin \beta = 0. \quad [\because \delta\theta \neq 0]$$

$$\text{or } (\cos \beta \cos \theta - \sin \beta \sin \theta) \sin \alpha + (\cos \theta \cos \alpha - \sin \theta \sin \alpha) \sin \beta = 0$$

$$\text{or } 2 \sin \alpha \sin \beta \sin \theta \cos \theta (\cos \alpha \sin \beta - \cos \beta \sin \alpha)$$

$$\text{or } \tan \theta = \frac{1}{2} (\cot \alpha - \cot \beta), \text{ giving the inclination of the rod to the horizontal in the position of equilibrium.}$$

Ex.51 An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2 \alpha)$ with the vertical, $2a$ being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.

Sol. ABC is an isosceles triangular lamina in which $AB=AC$.

The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.

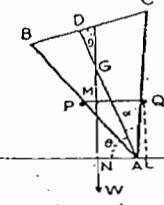
Let $PQ=a$ so that $BC=3a$.

If D is the middle point of BC , then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG = \frac{2}{3} AD.$$

The weight W of the lamina acts vertically downwards at G . We have

$$\angle BAD = \angle CAD = \alpha.$$



Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendiculars, therefore $\angle DAN=\theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line MN].

Now $\angle QPA = \angle PAN = \theta - \alpha$, and $\angle QAL = \pi - \theta + \alpha$.

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G . We have, the height of G above the fixed line PQ

$$\begin{aligned} MG &= NG - NM = NG - LQ \\ &= AG \sin \theta - AQ \sin(\pi - (\theta + \alpha)) \\ &= \frac{2}{3} AD \sin \theta - AQ \sin(\theta + \alpha). \end{aligned}$$

Now $AD = CD \cot \alpha = \frac{2}{3} a \cot \alpha$. Also from the $\triangle AQP$, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ} = \frac{PQ}{\sin PAQ} \text{ i.e., } \frac{AQ}{\sin(\theta-\alpha)} = \frac{a}{\sin 2\alpha}.$$

$$\therefore AQ = \frac{a}{\sin 2\alpha} \sin(\theta-\alpha).$$

$$\therefore MG = \frac{2}{3} a \cot \alpha \sin \theta - \frac{a}{\sin 2\alpha} \sin(2\alpha-\alpha) \sin(\theta+\alpha)$$

$$= a \cot \alpha \sin \theta - \frac{a}{2 \sin 2\alpha} 2 \sin(\theta-\alpha) \sin(\theta+\alpha)$$

$$= a \cot \alpha \sin \theta - \frac{a}{4 \sin^2 2\alpha} (\cos 2\alpha - \cos 0)$$

$$= a \cot \alpha \sin \theta - \frac{a \cos 2\alpha}{4 \sin \alpha \cos \alpha} + \frac{a \cos 0}{4 \sin \alpha \cos \alpha}$$

The equation of virtual work is

$$\begin{aligned} \therefore IV\delta(MG) &= 0, \text{ or } \delta(MG) = 0 \\ \text{or } \delta \left[a \cot \alpha \sin \theta - \frac{a \cos 2\theta}{4 \sin \alpha \cos \alpha} + \frac{a \cos 2\theta}{4 \sin \alpha \cos \alpha} \right] &= 0 \\ \text{or } \delta \left[a \cot \alpha \cos \theta - \frac{2a \sin 2\theta}{4 \sin \alpha \cos \alpha} \right] \delta\theta &= 0 \\ \text{or } a \cot \alpha \cos \theta - \frac{4a \sin \theta \cos \theta}{4 \sin \alpha \cos \alpha} &= 0 \quad [\because \delta\theta \neq 0] \\ \text{or } a \cos \theta &= 0 \text{ i.e., } \theta = \frac{\pi}{2}, \text{ giving one position of equilibrium in} \end{aligned}$$

which the lamina rests symmetrically on the pegs

$$\text{or } \cot \alpha = \frac{\sin \theta}{\sin \alpha \cos \alpha} = \infty \text{ i.e., } \sin \theta = \cos^2 \alpha \text{ i.e., } \theta = \sin^{-1}(\cos^2 \alpha), \text{ giving the other position of equilibrium.}$$

Ex.52 A square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either

$$\frac{\pi}{4} \text{ or } \frac{1}{2} \sin^{-1}\left(\frac{c^2 - a^2}{c^2}\right).$$

Sol. The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ = c$ and $AB = 2a$.

The weight W of the lamina acts at G , the middle point of the diagonal AC . Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

$$\angle PAM = \theta = \angle QPA.$$

We have $\angle BAC = \frac{1}{2}\pi$ constant.

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G . We have, the height of G above the fixed line PQ

$$\begin{aligned} &= LG = NG - NL = NG - MP \\ &= AG \sin(\frac{1}{2}\pi + \theta) - AP \sin \theta \\ &= a\sqrt{2} \sin(\frac{1}{2}\pi + \theta) - PQ \cos \theta \sin \theta \\ &= AG = \sqrt{2}a = \sqrt{2}a = a\sqrt{2}, \text{ and } AP = PQ \cos \theta \\ &= a\sqrt{2}(\sin(\frac{1}{2}\pi + \theta) + \cos(\frac{1}{2}\pi + \theta)) - c \cos \theta \sin \theta \\ &= a(\cos \theta - \sin \theta) - c \cos \theta \sin \theta. \end{aligned}$$

The equation of virtual work is

$$\begin{aligned} \therefore IV\delta(LG) &= 0, \text{ or } \delta(LG) = 0 \\ \text{or } \delta(a(\cos \theta + \sin \theta) - c \cos \theta \sin \theta) &= 0 \\ \text{or } [a(-\sin \theta + \cos \theta) - c(\cos^2 \theta - \sin^2 \theta)] \delta\theta &= 0 \\ \text{or } a(\cos \theta - \sin \theta) - c(\cos^2 \theta - \sin^2 \theta) &= 0 \quad [\because \delta\theta \neq 0] \\ \text{or } a(\cos \theta - \sin \theta)(a - c(\cos \theta + \sin \theta)) &= 0. \end{aligned}$$

\therefore either $\cos \theta - \sin \theta = 0$
i.e., $\sin \theta = \cos \theta$ i.e., $\tan \theta = 1$ i.e., $\theta = \frac{\pi}{4}$,

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

$$\begin{aligned} \text{or } &a - c(\cos \theta + \sin \theta) = 0 \\ \text{or } &c^2(\cos \theta + \sin \theta)^2 = a^2 \\ \text{or } &c^2(1 + 2\sin \theta \cos \theta) = a^2 \\ \text{or } &\sin 2\theta = \frac{a^2}{c^2} - 1 = \frac{a^2 - c^2}{c^2} \\ \text{or } &\theta = \frac{1}{2} \sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right). \end{aligned}$$

giving the other position of equilibrium.

Ex.53 A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove by the principle of virtual work that the side of length a makes with the vertical an angle θ given by $2c \cos 2\theta = b \cos \theta - a \sin \theta$.

Sol. Proceed as in part (a).

Ex.54 Two equal rods, AB and AC , each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a . Show that if 2θ be the angle between them, then $b \sin \theta = a \cos \theta$.

(BAS 2014 (OM))

Sol. Let O be the centre of the given circle and W be the weight of each of the rods AB and AC . If E and F are the middle points of AB and AC , then the total weight $2W$ of the two rods can be taken acting at G , the middle point of EF . The line AO is vertical. We have

$$\therefore \angle BAO = \angle CAO = \theta. \text{ Also } AB = 2b, AE = b. \text{ If the rod } AB \text{ touches}$$



the circle at M , then $\angle OMA = 90^\circ$ and OM = the radius of the circle = a .

Give the rods a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90° . We have, the height of G above the fixed point O

$$OG = OA - OM = OM \cos \theta - AE \cos \theta$$

$$= a \cos \theta - b \cos \theta.$$

The equation of virtual work is

$$\begin{aligned} \therefore IV\delta(OG) &= 0, \text{ or } \delta(OG) = 0 \\ \text{or } (-a \cos \theta \cot \theta - b \sin \theta) \delta\theta &= 0 \\ \text{or } -a \cos \theta \cot \theta + b \sin \theta &= 0 \\ \text{or } a \cos \theta \cot \theta &= b \sin \theta \quad [\because \delta\theta \neq 0] \\ \text{or } a \cos \theta &= b \sin \theta. \end{aligned}$$

Problems Involving elastic strings

Ex.55 Four equal jointed rods, each of length a are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is $a\sqrt{2}$.

Sol. $ABCD$ is a framework formed of four equal rods each of length a and say of weight W . It is suspended from the point A . A and C are connected by an elastic string and in equilibrium $ABCD$ is square. The diagonal AC is vertical and so BD is horizontal. Let T be the tension in the string AC . The total weight $4W$ of all the rods AB, BC, CD, DA can be taken acting at G , the point of intersection of the diagonals AC and BD . Let $\angle BAC = \angle DAC = \theta$.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the length AC changes, the point G is slightly displaced, the lengths of the rods AB, BC, CD, DA do not change, and the $\angle BGA$ remains 90° . We have $AC = 2AG = 2a \cos \theta$.

Also the depth of G below A is $AG = a \cos \theta$.

The equation of virtual work is

$$\begin{aligned} \therefore -T\delta(2a \cos \theta) + 4W\delta(a \cos \theta) &= 0 \\ 2aT \sin \theta \delta\theta - 4W \sin \theta \delta\theta &= 0 \\ 2a \sin \theta (T - 2W) \delta\theta &= 0 \\ T - 2W &= 0 \quad [\because \delta\theta \neq 0 \text{ and } \sin \theta \neq 0] \\ T &= 2W. \end{aligned}$$

Let l be the natural length of the elastic string AC . In the position of equilibrium, $\angle BAC = 45^\circ$ and so the extended length AC of the elastic string $= 2AG = 2a \cos 45^\circ = 2a/\sqrt{2} = a\sqrt{2}$.

By Hooke's law, the tension T in the elastic string AC is given by $T = \lambda \frac{AC - l}{l}$, where λ is the modulus of elasticity of the string.

$$= W \frac{a\sqrt{2} - l}{l} \quad [\because \lambda = W]$$

Equating the two values of T , we get

$$2W - W \frac{a\sqrt{2} - l}{l} = 0$$

$$\begin{aligned} \text{or } 2l &= a\sqrt{2} - l, \text{ or } 3l = a\sqrt{2} \\ \text{or } l &= a\sqrt{2}/3. \end{aligned}$$

Ex.56 One end of a uniform rod AB , of length $2a$ and weight W , is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC . The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A , and the angle between the rods is $2 \sin^{-1}(3/4)$.

Sol. AB and BC are two rods each of length $2a$ and weight W smoothly joined together at B . The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a .

Let T be the tension in the string EF .

The total weight $2W$ of the two rods can be taken acting at the middle point of EF . The line BG is horizontal and meets AC at its middle point M . Let $\angle BAC = \theta = \angle CAB$.

Give the system a small symmetrical displacement about B in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF = 2EG = 2EB \sin \theta = 2a \sin \theta$.

Also the depth of G below the fixed point A

$$\begin{aligned} \therefore AM &= AB \sin \theta = 2a \sin \theta \\ \text{The equation of virtual work is} & \\ \therefore -T\delta(2a \sin \theta) + 2W\delta(2a \sin \theta) &= 0 \\ (-2aT \cos \theta + 4W \cos \theta) \delta\theta &= 0 \\ 2a \cos \theta (-T + 2W) \delta\theta &= 0 \\ -T + 2W &= 0 \quad [\because \delta\theta \neq 0 \text{ and } \cos \theta \neq 0] \\ T &= 2W. \end{aligned}$$

Also by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda \frac{2a \sin \theta - a}{a}$$

where λ is the modulus of elasticity of the string
 $= 4W(2 \sin \theta - 1)$.

Statics

Vii

Equating the two values of T , we have

$$2W = 4W(2 \sin \theta - 1)$$

$$\text{or } 1 = 2(2 \sin \theta - 1), \text{ or } 1 = 4 \sin \theta - 2$$

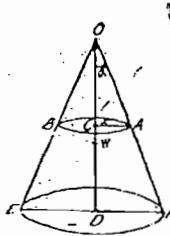
$$\text{or } 4 \sin \theta = 3, \text{ or } \sin \theta = 3/4, \text{ or } \theta = \sin^{-1}(3/4).$$

\therefore in equilibrium the whole angle between AB and BC
 $= 2\theta = 2 \sin^{-1}(3/4)$.

Ex.57 A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α . If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a \left(1 + \frac{W}{2\lambda\pi} \cot \alpha \right).$$

Sol. OEF is a smooth fixed cone of semi-vertical angle α , the axis OD of the cone being vertical. A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x . The weight W of the string acts at its centre of gravity C . Let T be the tension in this string.



Give the string a small displacement in which x changes to $x + \delta x$. The point O remains fixed, the point C is slightly displaced, $\angle \alpha$ is fixed and the length of the string slightly changes.

We have the length of the string AB in the form of a circle of radius $x + 2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.

Also the depth of the point of application C of the weight W below the fixed point O

$$= OC = AC \cot \alpha = x \cot \alpha$$

and so the work done by the weight W during this small displacement
 $= W\delta(x \cot \alpha)$.

Since the reactions at the various points of contact do no work, we have, by the principle of virtual work

$$-T\delta(2\pi x) + W\delta(x \cot \alpha) = 0$$

$$\text{or } -2\pi T \delta x + W \cot \alpha \delta x = 0 \quad \text{or} \quad (-2\pi T - W \cot \alpha) \delta x = 0$$

$$\text{or } -2\pi T + W \cot \alpha = 0 \quad [\because \delta x \neq 0]$$

$$\text{or } T = (W \cot \alpha)/2\pi.$$

By Hooke's law the tension T in the elastic string AB is given by

$$T = \lambda \frac{2\pi x - 2\pi a}{2\pi a} = \lambda \frac{x - a}{a}$$

Equating the two values of T , we get

$$\frac{W \cot \alpha}{2\pi} = \lambda \frac{x - a}{a}$$

$$\text{or } x - a = \frac{a}{2\pi \lambda} W \cot \alpha$$

$$\text{or } x - a = \left(1 + \frac{W}{2\pi \lambda} \cot \alpha \right).$$

which gives the radius of the string in equilibrium.

Ex.58 An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be 2α .

Sol. Proceed as in Ex. 54. Here in place of a heavy elastic string of weight W we have a heavy endless chain of weight W . If T is the tension in this chain, then proceeding as in Ex. 54, we get

$$T = (W \cot \alpha)/2\pi.$$



SET-4

STRINGS IN TWO DIMENSIONS

1

Flexible string: All those strings which offer no resistance on bending at any point are called flexible string. Here, the resultant action across any section of the string consists of a single force whose line of action is along the tangent to the curve formed by the string.

The normal section of the string is taken to be so small that it may be regarded as a curved line.

The Catenary

When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called catenary.

Uniform or Common Catenary

If the weight per unit length of the suspended string/chain is constant, then the catenary is called the uniform common catenary.

→ Here the word Catenary will always mean the common catenary.

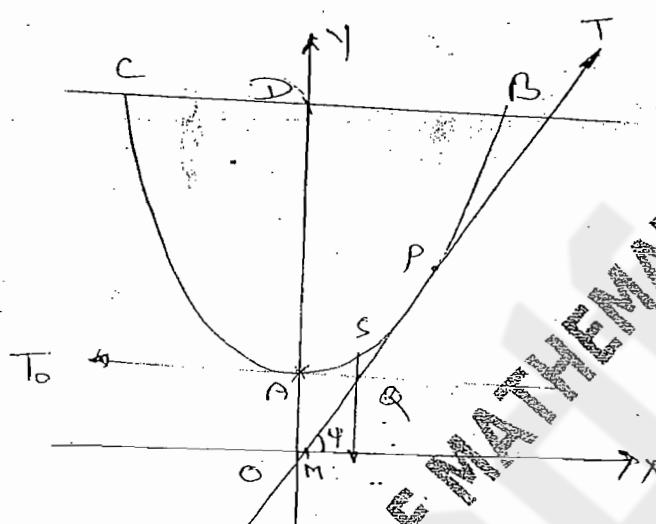


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*Intrinsic Equation of the Common Catenary

Let the uniform, flexible string BA hang in the form of a uniform catenary with A as the lowest point. Let P be any point on the portion AB of the



String and
S = Distance of P from A
measured along the arc
length of the string.

$w = \text{weight per unit length}$
of the string.

\therefore Weight of the portion AP = ws

The portion AP of the string is in \rightleftharpoons under the action of the following three forces

(i) the weight ws of the string AP acting vertically downward through it C.G.

(ii) Tension T_0 , at the lowest point A acting along the tangent to the curve at A, which is horizontal

(iii) Tension T , at P, acting along the tangent to the curve at P, inclined at an angle α to the horizontal

(2)

Since the string AP is in \Rightarrow under the action of three forces, acting in the same vertical plane.

\therefore the line of action of the weight 'ws' must pass through Q, which is the point of intersection of the lines of action of the tension T_0 and T.

Now, for \Rightarrow

$$\sum F_x = 0$$

$$T_0 = T \cos \psi \quad \text{--- (1)}$$

$$\text{and } \sum F_y = 0$$

$$T \sin \psi = ws \quad \text{--- (2)}$$

\therefore Dividing (2), we have,

$$\tan \psi = \frac{ws}{T_0} \quad \text{--- (3)}$$

Let $= nc = \text{weight of the length } 'c' \text{ of}$
the string.

$$\tan \psi = \frac{s}{c}$$

$$\Rightarrow s = c \tan \psi \quad \text{--- (4)}$$

Which is the intrinsic equation of the common catenary.



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- * $\therefore T_0 = T \cos \varphi$ i.e. the horizontal component of the tension at every point of the Catenary is the same and is equal to T_0 (the tension at the lowest point)
 - * $T \sin \varphi = w_s$ i.e. the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and that point.

Cartesian Equation of the Common Catenary

The intrinsic equation of the Common Catenary

$$s = c \tan \alpha \quad (i)$$

We know that $\frac{dy}{dx} = \tan \alpha$

$$s = c \frac{dy}{dx}$$

Differentiating both side w.r.t. x we have.

$$\frac{ds}{dx} = c \frac{d^2y}{dx^2}$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = c \frac{d^2y}{dx^2}$$

$$\text{put, } \frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}, \text{ we have}$$

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(3)

$$\Rightarrow \sqrt{1+p^2} = c \frac{dy}{dx} = c \frac{dp}{dx}$$

$$\text{or, } \frac{dx}{c} = \frac{dp}{\sqrt{1+p^2}}$$

Integrating, we have

$$\frac{x}{c} + A = \sinh^{-1}(p) \\ = \sinh^{-1}\left(\frac{dy}{dx}\right) \quad (2)$$

Where, A = Constant of integration.

If we choose the vertical line through the lowest point A' of the catenary as the axis of y, then at point

A, we have

$$x=0 \text{ and } \frac{dy}{dx}=0$$

$$\text{from (2), } A=0$$

$$\therefore \frac{x}{c} = \sinh^{-1}\left(\frac{dy}{dx}\right)$$

$$\text{or, } \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

Now, integrating both sides w.r.t. x, we have,

$$y = c \cosh\left(\frac{x}{c}\right) + \beta \quad (3)$$

where, β = Constant of Integration.

If we take the origin 'O' at a depth 'c' below the lowest point A of the catenary, then at A, we have,

$$x=0, y=c$$

∴ from (3), we have,

$$B=0$$

$$\therefore y = c \cosh\left(\frac{x}{c}\right) \quad (4)$$

which is the Cartesian equation of the common catenary.

(1) Axis of the catenary:

$\cosh\left(\frac{x}{c}\right)$ is an even function of x , therefore the curve is symmetrical about the axis of y , which is along the vertical through the lowest point of catenary. This vertical line of symmetry is called the axis of catenary.

(2) Vertex of the catenary: the lowest point 'A' of the common catenary at which the tangent is horizontal is called the vertex of the catenary.

(3) Diameter of the catenary: c of $y = c \cosh\left(\frac{x}{c}\right)$

(4) Directrix of the catenary: the horizontal line at a depth 'c' below the lowest point in y -axis is called the directrix of the catenary.



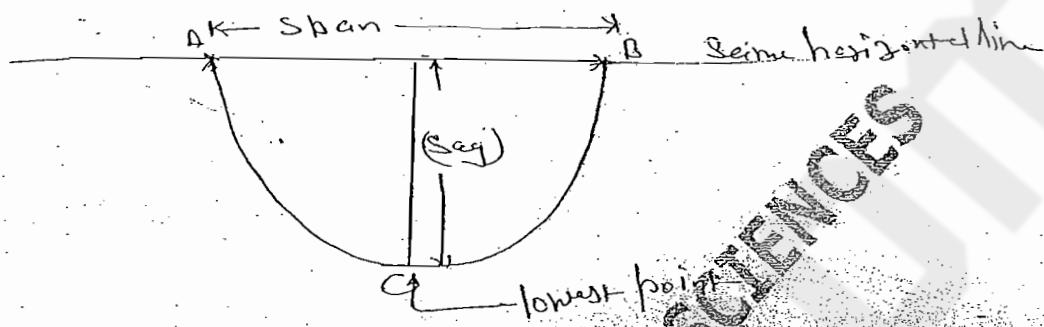
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(5) Span and Sag



* Relation between x and s

$$\therefore s = c \tan \varphi = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = s \quad \text{--- (1)}$$

Also, $y = c \cosh(x/c)$

$$\Rightarrow \frac{dy}{dx} = \sinh(x/c) \quad \text{--- (2)}$$

From (1) and (2), we have,

$$\frac{dy}{dx} = \sinh(x/c)$$

or, $s = c \sinh(x/c)$

which is the relation between x and s .

* Relation between y and s

$$\therefore y = c \cosh(x/c)$$

$$\text{and } s = c \sinh(x/c)$$

Squaring on both sides we have,

$$\therefore y^2 - s^2 = c^2 (\cos^2(\alpha_c) - \sin^2(\alpha_c))$$

$$\Rightarrow y^2 - s^2 = c^2$$

$$\text{or, } y^2 = c^2 + s^2$$

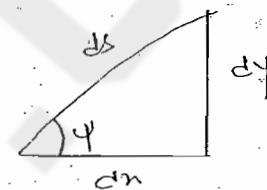
which is the relation between y and s .

* Relation between y and ψ

For any curve, we have,

$$\frac{dy}{ds} = \sin \psi$$

$$\begin{aligned} \therefore \frac{dy}{d\psi} &= \frac{dy}{ds} \cdot \frac{ds}{d\psi} \\ &= \sin \psi \cdot \frac{1}{d\psi} (c \cdot \tan \psi) \\ &= \frac{\sin \psi}{d\psi} \cdot c \sec^2 \psi \end{aligned}$$



$$= c \sec \psi \cdot \tan \psi$$

Integrating, we get,

$$y = c \sec \psi + A$$

where, A = Constant of integration.

When, $y = C$, $\psi = 0$, $\therefore A = 0$



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(5)

$$\therefore y = c \sec \varphi$$

which is the relation b/w y and φ .

* relation b/w x and φ

∴ for any curve,

$$\frac{dn}{ds} = \cos \varphi$$

$$\text{or, } \frac{dn}{d\varphi} = \frac{dn}{ds} \cdot \frac{ds}{d\varphi}$$
$$= \cos \varphi \cdot \frac{1}{c \sec \varphi + \tan \varphi} \quad [\because s = c \tan \varphi]$$

$$\text{or, } \frac{dn}{d\varphi} = \frac{1}{c \sec^2 \varphi}$$

$$\text{or, } \frac{dn}{d\varphi} = c \sec \varphi$$

$$\Rightarrow n = c \log (\sec \varphi + \tan \varphi) + B$$

Where, B is a constant of integration.

But when $x=0$, $\varphi=0 \quad \therefore B=0$

$$\therefore n = c \log (\sec \varphi + \tan \varphi)$$

* Relation between tension and ordinate

∴ We have,

$$T \cos \varphi = T_0 \quad \text{and} \quad T_0 = w c$$

$$\therefore T = T_0 \sec \varphi = w c \sec \varphi$$

$$\text{But, } y = c \sec \varphi$$

$$\boxed{T = w y}$$

This shows that the tension at any point of a catenary varies as the height of the point above the directrix.

* Radius of curvature at any point of a catenary

We have, $s = c \tan \varphi$.

$$r = \frac{ds}{d\varphi} = c \sec^2 \varphi$$

Q.T. If T be the tension at any point P of a catenary suspended from a straight string at the lowest point A , prove that

$$T^2 - T_0^2 = w^2$$

where, w being the weight of the arc AP of the catenary.



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Soln:

Let $AP = s$ and φ be the inclination
of the tangent at P to the
horizontal.if w = weight per unit length

$$\therefore w = ws$$

Let T = tension at point P of catenary T_0 = tension at the lowest point

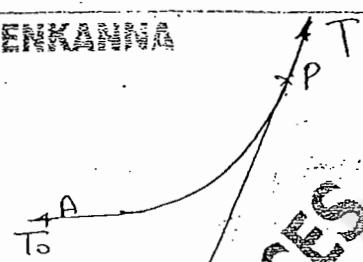
$$\therefore T \sin \varphi = T_0$$

$$\text{and } \sin \varphi = ws = w$$

squaring and adding, we have,

$$T^2 = T_0^2 + w^2$$

$$\Rightarrow T^2 - T_0^2 = w^2$$

Ans

Q2. Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their heights.

$$\text{Ans: } \therefore T \propto y$$

Q3. A rope of length 21 feet is suspended between two points at the same level and the lowest point of the rope is 6' feet below the point of suspension. Show that the horizontal component of the tension is $\frac{w(1^2 - b^2)}{2b}$ being the weight of the rope per foot of length.

Q4. A uniform chain of length l, is to be suspended from two points A and B, in the same horizontal line so that either terminated tension is n times that at the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{n^2 - 1}} \cdot \log \left(n + \sqrt{n^2 - 1} \right)$$



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MATHEMATICS by K. VENKADRA

Soln.

Let the uniform chain ACB of length l be suspended from two points A and B in the same horizontal level.

Let point $A = (x_1, y_1)$

and φ_1 = angle at which the

stangent at A ,

makes with the vertical

T = tension at point A

T_0 = tension at the lowest point C

Given that

$$T = n T_0 \quad \dots (1)$$

But

$$T = w l_1 \quad \text{and} \quad T_0 = w c$$

[where w = weight per unit length]

$$w l_1 = n w c$$

$$\Rightarrow w \cdot c \sec \varphi_1 = n w c$$

$$\left[\because \gamma = c \sec \varphi \right]$$

$$\Rightarrow \sec \varphi_1 = n \quad \dots (2)$$

Now, for the point A ,

$$s = \text{arc length } CA = \frac{l}{2}$$

From, $s = c \tan \varphi$, we have

$$\frac{t}{2} = c \cdot \tan \varphi_1$$

$$\Rightarrow c = \frac{1}{2 \tan \varphi_1} = \frac{1}{2 \sqrt{\sec^2 \varphi_1 - 1}}$$

$$\Rightarrow c = \frac{1}{2 \sqrt{n^2 - 1}}$$

QED,

$$\begin{aligned} x &= c \log (\sec \varphi_1 + \tan \varphi_1) \\ &= \frac{1}{2 \sqrt{n^2 - 1}} \log \left(n + \sqrt{n^2 - 1} \right) \end{aligned}$$

$$[\because \sec \varphi_1 = n]$$

Now the span AB = 2AD

$$= 2n \cdot \log \left(n + \sqrt{n^2 - 1} \right)$$

Prove



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MATHEMATICS by K. VENKANNA

Q.S. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \left(\frac{1 + \sqrt{1 + \mu^2}}{\mu} \right)$$

where, μ is the coefficient of friction.

Sol:

Let the end links A and B of a uniform chain slide along a fixed rough horizontal rod.

Let AB is the maximum span, then A and B are in the state of limiting equilibrium.

Let R_p = Normal reaction at nod A.

$F =$ resultant force of F_{fr} and R_p

Given $\mu = \tan \theta$ in case of limiting equilibrium.

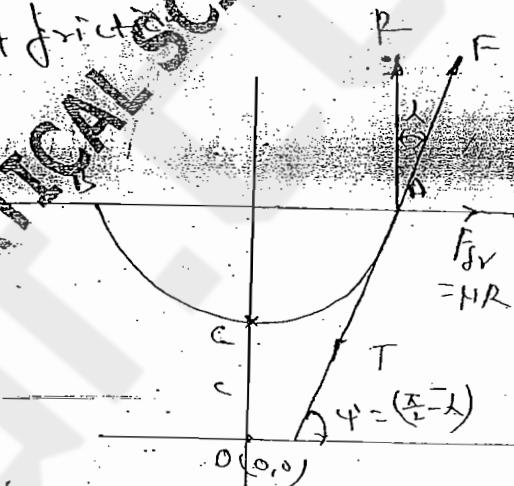
For θ , F and T will have the same magnitude but in opposite direction.

Length of the chain, $2s = 2c \tan \theta$

$$= 2c \tan(\frac{\pi}{2} - \lambda) \quad (\because \tan(\frac{\pi}{2} - \lambda) = \cot \lambda)$$

$$= 2c \cot \lambda$$

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i. length of the chain, $2s = \frac{2c}{H}$

Let point A = (x_1, y_1)

ii. maximum Span AB

$$= 2n = 2 \cdot c \log (\tan(\frac{\pi}{2} - H) + \sec H)$$

$$= 2c \cdot \log (\tan(\frac{\pi}{2} - H) + \sec(\frac{\pi}{2} - H))$$

$$= 2c \cdot \log (\cot H + \operatorname{cosec} H)$$

$$= 2c \cdot \log (\cot H + \sqrt{1 + \cot^2 H})$$

$$= 2c \cdot \log \left(\frac{1}{\tan H} + \sqrt{1 + \tan^2 H} \right)$$

iii. Required Span

$$\frac{\text{Span AB}}{\text{length of chain}} = \frac{2n}{2s}$$

$$= \frac{2c \cdot \log \left(\frac{1}{H} + \sqrt{1 + H^2} \right)}{\frac{2c}{H}}$$

$$= H \cdot \log \left(\frac{1 + \sqrt{1 + H^2}}{H} \right)$$

Pran



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MATHEMATICS by K. VENKANNA

- Q.6. The extremities of a heavy string of length a and weight w/m are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw . Show that the distance apart of the ring is

$$2l \lg(1 + \sqrt{2})$$

Soln: $T_x = wC = wL \Rightarrow C = 1$ $\therefore \theta = \lg(\sec \theta + \tan \theta)$
 $s = c \tan \theta$
 $\Rightarrow \tan \theta = \frac{s}{c} = \frac{1}{1} = 1$
 $\therefore \theta = 45^\circ$

- Q.7. A heavy uniform string of length l , is suspended from a fixed point A , and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distance between A and B are

$$a \sinh(l/a) \text{ and } \sqrt{(1 + a^2)} - a$$

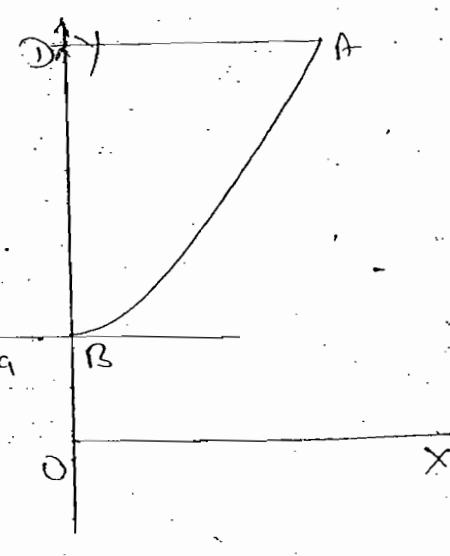
Solution \Rightarrow position the arc AB will represent half of the arc of the complete catenary with B as its lowest point.

$$\text{horizontal force, } F = wa \quad T_0 = F = wa$$

i.e. tension at the lowest point

$$T_0 = F = wa = wc$$

$$\Rightarrow a = c$$



Let point $A = (x_1, y_1)$

$$\text{arc } BA = 1 - s$$

From, $s = c \sinh(n_1)$, we have,

$$1 - s = a \cdot \sinh(n_1)$$

$$\Rightarrow x = a \cdot \sinh^{-1}(1/a)$$

\therefore Horizontal distance between A and B

$$= a \cdot \sinh(1/a)$$

Again from, $y^r = s^r + c$ we have,

$$y^r = t^r + c$$

$$C + BD$$

But

$$\begin{aligned} y^r &= y_1 - c = y_1 - a \\ &= \sqrt{t^r + a^2} - a \end{aligned}$$

\therefore Vertical distance between A and B is

$$(\sqrt{t^r + a^2} - a) \quad \text{Prove}$$



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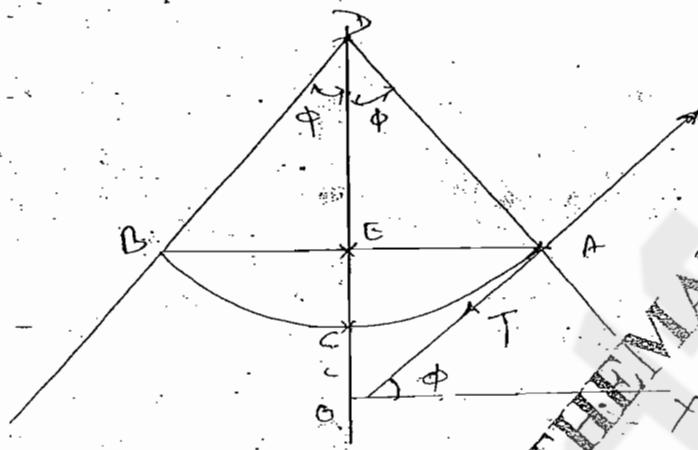
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MATHEMATICS BY K. MENAKANNA

- (8) The end links of a uniform chain of length 'l' can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles ϕ to the vertical. Prove that the sag in the middle point is $\frac{1}{2}l \cdot \tan\phi/2$.



Hints
for equilibrium at A.

$$T = P$$

$$\therefore \phi = \phi$$

$$\text{at } A, S = \frac{l}{2}$$

$$\therefore S = c \tan \phi$$

$$\Rightarrow \frac{l}{2} = c \tan \phi$$

$$\Rightarrow c = \frac{l}{2} \cdot \cot \phi \therefore$$

$$\text{sag} = EC = OE - OC$$

$$= Y_1 - c$$

$$= c \sin \phi - c$$

$$\therefore Y_1 = c \sin \phi$$

$$\therefore Y_1 = \frac{l}{2} \cdot \cot \phi \cdot \sin \phi$$

- (9) A uniform heavy chain is fastened at its extremities to two rings of equal weight, which glide on smooth rods intersecting in a vertical plane, and inclined at the same angle ϕ to the vertical. find the condition that the tension at the lowest point may be equal to half the weight of the chain and in that case, show that the vertical distance of the rings from the point of intersection of the rods is

- Sech by $(\sqrt{2} + 1)$

where l is the length of the chain.

Soln:

Let the rods inclined at the same angle α to the vertical y -axis intersect at the point O' .

Let A and B be the positions of the rings in \mathbb{R}^2 .

$$\text{L} \perp OC = c$$

Let w = weight per unit length of the chain.

\therefore total weight of the chain $= cw$.

$$T_0 = wc = wl$$

$$\Rightarrow c = l$$

Let point (x, y)

From $O\alpha = c \tan \varphi$, we have,

$$l = l \cdot \tan \varphi$$

$$\Rightarrow \varphi = \frac{\pi}{4}$$

Hence the condition that the tension at the lowest point may be equal to half the weight of the chain is that the tangents at the ends A and B of the chain will make an angle $\frac{\pi}{4}$ to the horizontal.



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Now, from $n = c \log(\tan \alpha + \sec \alpha)$ for the point A,
we have,

$$\begin{aligned} n_1 &= 1 \cdot \log(\tan \alpha_1 + \sec \alpha_1) \\ &= 1 \cdot \log(1 + \sqrt{2}) \end{aligned}$$

$$\text{i.e. } AD = 1 \cdot \log(1 + \sqrt{2})$$

From ΔAOD

$$OD = AD \cdot \cot \alpha$$

$$= 1 \cdot \log(1 + \sqrt{2}) \cdot \cot \alpha \quad \text{prove}$$

(10) A length T of a uniform chain has one end

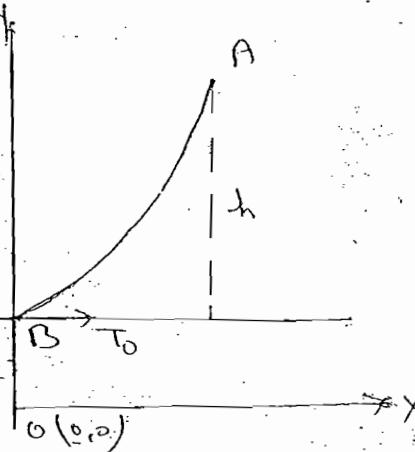
fixed at a height h above a rough table, and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length of the table

$$1 + \mu h = \sqrt{(\mu^2 + 1)h^2 + 2h^2}$$

where μ is the coefficient of friction.

Qn. Let one end of a uniform chain ABC of length T be fixed at A at a height h above the rough table and let the portion BC of the chain rest on the table.

$$f_r = HPC \quad wce$$



Let $BC = r$

The chain rests in limiting equilibrium with the portion AB in the form of an arc of a catenary with B as its vertex.

weight of the portion $BC = wr$

From figure

$$R = wr$$

$$\text{and } MR = T_0 = wc$$

$$\text{or, } Hwr^2 = wc$$

$$\text{or, } c = Hr \quad \text{(1)}$$

Now, the length of the arc $AB = l - r = s = l - r$

and ordinate of point $A = h + c$

$$\Rightarrow y_A = h + Hr$$

From $y_A = c + s$ for the point A , we have,

$$(h+Hr)^2 = (Hr)^2 + (l-r)^2$$

$$\text{or, } h^2 + H^2r^2 + 2Hr^2 = H^2r^2 + l^2 - 2lr$$

$$\text{or, } r^2 - 2(l+Hr)r + (l^2 - h^2) = 0$$



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MATHEMATICS BY K. VERMA

$$\frac{2(1+Hh) \pm \sqrt{4(1+Hh)^2 - 4(1^2 - h^2)}}{2}$$

$$r_1, r_2 = (1+Hh) \pm \sqrt{1^2 + H^2 h^2 + 2Hh - 1^2 + h^2}$$

$$= (1+Hh) \pm \sqrt{(H^2 + 1)h^2 + 2Hh}$$

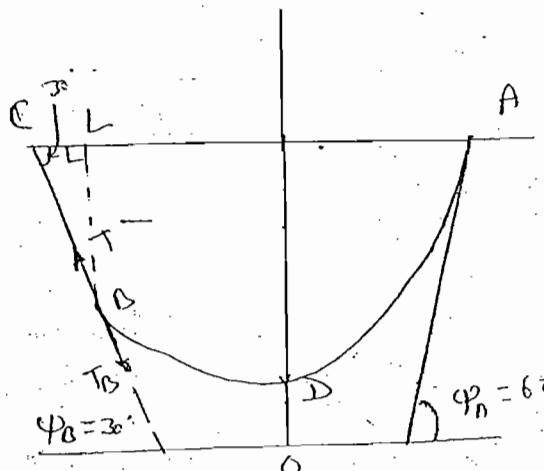
for + sign, $r_2 > r_1$

$$r_2 = (1+Hh) - \sqrt{(H^2 + 1)h^2 + 2Hh} \text{ prove}$$

- (*) A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The length of the string and chain are such that the ends of the chain at A and B make angles 60° and 30° respectively with the horizontal. Prove that the ratio of their lengths is

$$(\sqrt{3} - 1) : 1$$

Soln Let

 l = length of uniform chain AB a = length of light string BC

the chain AB being heavy will hang in the form of a catenary

while the string BC being light will hang in the form of a straight line.

$$\therefore T_B = T$$

Let D be the lowest point of the catenary

from eqn $y = c \sec \theta$, we get

$$y_A = c \sec \theta_A = c \sec 60^\circ$$

$$\therefore y_A = 2c$$

$$\text{and } y_B = c \sec \theta_B = c \sec 30^\circ$$

$$y_B = \frac{c}{\sqrt{3}}$$

where y_A and y_B are the ordinates of point A and B respectively.

ΔBLC

$$\begin{aligned} BL &= BC \sin 30^\circ \\ &= \frac{a}{2} \end{aligned}$$

$$[\because BC = a]$$

$$\therefore r = 2BL = 2(y_A - y_B) = 2\left(2c - \frac{c}{\sqrt{3}}\right)$$



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$$\therefore a = \frac{4c}{\sqrt{3}} (\sqrt{3} - 1)$$

If s_1 = length of arc BD(ii) s_2 = length of arc BA

$$\therefore s_1 + s_2 = l$$

$$\Rightarrow l = c \tan 60^\circ + c \tan 30^\circ \quad \therefore s = c \tan \theta$$

$$\text{or, } l = \beta_1 c + c$$

$$\frac{4c}{\sqrt{3}}$$

$$\frac{4c}{\sqrt{3}} (\sqrt{3} - 1)$$

$$\frac{a}{l}$$

Required

- (Q. 12) A uniform inextensible string of length l and weight W_l , carries at one end B, a particle of weight W which is placed on a smooth plane inclined at 30° to the horizontal. The other end of the string is attached to a point A, situated at a height h above the horizontal through B and in vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the

tangent at B to the catenary laying in the inclined plane if

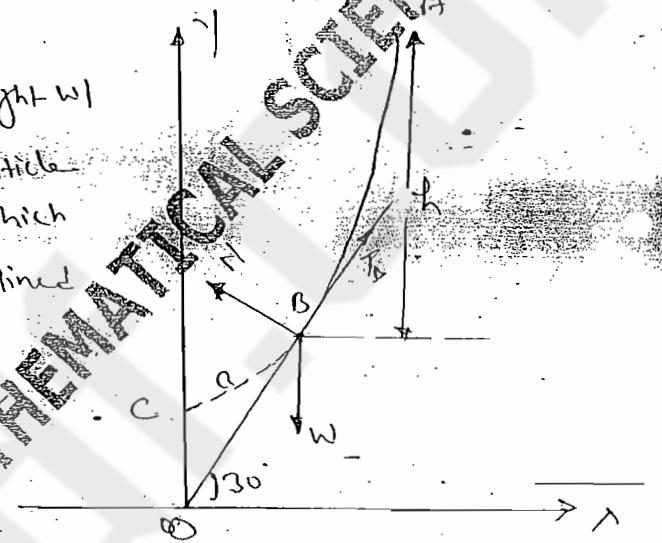
$$\frac{W}{w} = \frac{(l-h)(l+h)}{(h-\frac{l}{2}d)}$$

Sol:

Let AB be the string of weight w

of length l, carrying a particle of weight W at the end B, which is placed on the plane inclined at an angle 30° to the horizontal and the other end is fixed at A,

shown in figure.



Let C be the lowest point of catenary AB after extending, and OX be the directrix.

Let T_B = tension at B, inclined at angle θ to the horizontal.

Coordinate of B = (x_B, y_B)

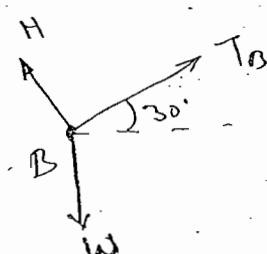
for particle B
along the inclined plane

$$T_B = W \cdot \cos 60^\circ$$

$$\Rightarrow T_B = \frac{W}{2}$$

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Now, $y_B = c \sin \varphi_B$ $\rightarrow c \sin 30^\circ$

$$\Rightarrow y_B = \frac{c}{\sqrt{3}}$$

and $T_B = w \cdot y_B = \frac{w c}{\sqrt{3}} \quad \text{(2)}$

∴ from (1) and (2), we have,

$$\frac{w}{2} = \frac{2w \cdot c}{\sqrt{3}}$$

$$\therefore c = \frac{\sqrt{3}}{4} \cdot \frac{w}{w}$$

Let $\text{arc } BC = c$ for point B, we have,

Now, from $s = c \tan \varphi$,

$$a = c \tan 30^\circ = \frac{\sqrt{3}}{4} \cdot \frac{w}{w} \cdot \frac{1}{\sqrt{3}}$$

$$\Rightarrow a = \frac{w}{4w}$$

for point A

$$s_A = l + a, \quad y_A = y_B + h$$

from $y^2 = c^2 + s^2$, we have,

$$(y_B + h)^2 = c^2 + (l + a)^2$$

and for point B, we have,

$$y_B^2 = c^2 + a^2$$

After subtracting we have,

$$(Y_0+h)^2 - Y_0^2 = (1+a)^2 - a^2$$

$$\text{or, } (2Y_0 + h) \cdot h = (1+2a) \cdot 1$$

$$\text{or, } h^2 + 2Y_0 \cdot h = 1^2 + 2a \cdot 1$$

$$\therefore Y_0 = \frac{2c}{\sqrt{3}} \text{ and } c = \frac{\sqrt{3}}{4} \cdot \frac{W}{n}$$

$$\Rightarrow h^2 + 2 \cdot \frac{2c}{\sqrt{3}} \cdot h = 1^2 + \frac{W}{2n} \quad | - h^2$$

$$\Rightarrow h^2 + \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{4} \cdot \frac{W}{n} = 1^2 + \frac{W}{2n} \quad | - h^2$$

$$\Rightarrow \frac{W}{n} \left(h - \frac{1}{2} \right) = 1^2 - \frac{h^2}{2n}$$

$$\Rightarrow \frac{W}{n} = \frac{1^2 - \frac{h^2}{2n}}{h - \frac{1}{2}}$$

$$\frac{(1+h)(1-h)}{(h-\frac{1}{2})}$$

Proved



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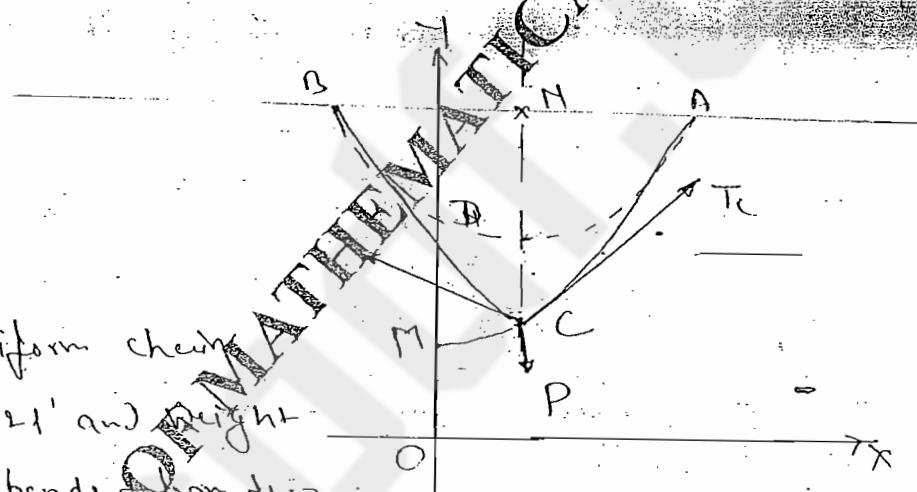
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MATHEMATICS BY K. VENKANNA

Q 13.

~~Dec 2010~~ A uniform chain of length $2l$ and weight w is suspended from two points A and B , in the same horizontal line. A load P is now suspended from the middle point D of the chain and the depth h of each point below AB is found to be h . Show that each terminal tension is $\frac{1}{2} \{ P + w \cdot \frac{2h}{l} \}$.

8.13



Let a uniform chain of length $2l$ and weight w is suspended from two points A and B , in the same horizontal line freely under gravity in the form of catenary ADB .

When a load P is attached at D of the chain, this will come down to C as shown in figure.

and then the two portions AC and BC of the chain each of length l will be the parts of two equal catenaries.

Let M be the lowest point and Ox be the direction of the catenary of which AC is an arc.

The weight per unit length of the chain

$$w = \frac{W}{2l}$$

For point C ..

$$2 T_c \cdot \sin \varphi_c = P \quad \text{--- (1)}$$

$$\text{But } T_c \sin \varphi_c = w a$$

where, $a = \text{arc } AC$

$$\text{from (1), } wa = P/2$$

$$\Rightarrow a = \frac{P}{2w} = \frac{P}{2W} \quad \text{--- (2)}$$

Let point A = (x_A, y_A)

and point C = (x_C, y_C)

$$s = s_A = \text{arc } MA = \text{arc } AC + \text{arc } CA \\ = a + l$$

$$\text{and } Y = Y_A$$

at point C



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PROBLEMS ON BREAKAWAY POINT (16)

MATHEMATICS BY K VENKANNA

$$S = S_c = \text{arc } NC = a$$

$$\text{and } T_c = T_A$$

$$\text{and Sag, } CN = h \text{ (given)}$$

$$\therefore T_c + h = T_A$$

$$\Rightarrow T_c = T_A - h$$

$$\text{from, } T^2 = C^2 + S^2, \text{ we have}$$

$$T_A^2 = C^2 + (a+h)^2$$

$$\text{and } T_c^2 = C^2 + a^2$$

After subtracting, we have,

$$T_A^2 - T_c^2 = (a+h)^2 - a^2$$

$$\text{or, } (T_A - h)^2 = (2ah + h^2)$$

$$\text{or, } (2T_A - h)h = 2ah + h^2$$

$$\text{or, } T_A = \frac{1}{2} \left\{ h + \frac{2ah + h^2}{h} \right\}$$

Hence, each of ferminal tension is given by

$$T = T_A = T_B = \omega T_n = \frac{\omega}{2I} \cdot \frac{1}{2} \left\{ h + \frac{2ah + h^2}{h} \right\}$$

$$\begin{aligned} \therefore T &= \frac{W}{4l} \left\{ h + \frac{2l \cdot \frac{2P}{2W} + l^2}{h} \right\} \\ &= \frac{W}{4l} \left\{ h + 2 \cdot P \cdot \frac{l^2}{Wh} + \frac{l^2}{h} \right\} \\ &= \frac{l}{2} \left\{ P \cdot \frac{l}{h} + W \frac{h^2 + l^2}{2lh} \right\} \end{aligned}$$

Proved

- Q14. A uniform chain of length $2l$ and weight $2W$, is suspended from two points in the same horizontal line. A load W is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h . Show that the tension is $\frac{1}{2}W \cdot \frac{h^2 + 2l^2}{h^2}$.

- Q15. A heavy string of uniform density and thickness is suspended from two given points in the same horizontal line. A weight, an n th part of the string, is attached at its lowest point. Show that if θ, ϕ be the inclination to the vertical of the tangent at the highest and lowest points of string $\tan \phi = (1+n) \tan \theta$.



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Q.16 Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a \left\{ \frac{3}{\log(2 + \sqrt{3})} + \frac{4\pi}{3} \right\}$$

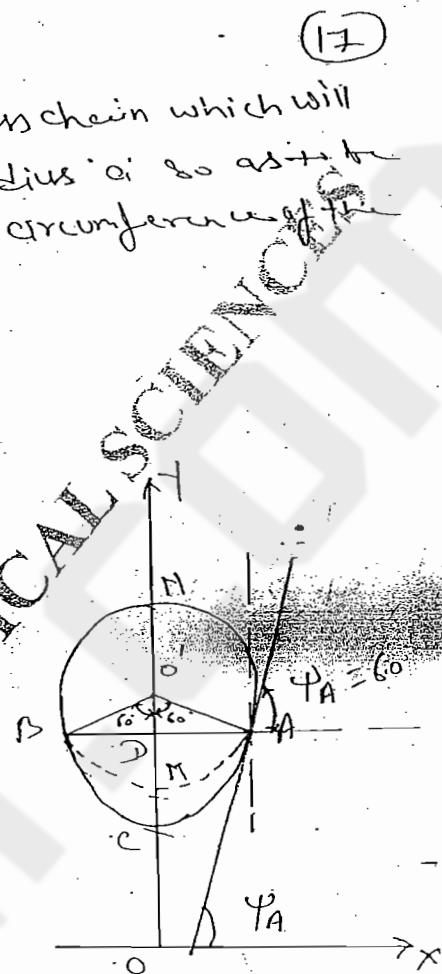
Sol:

Let $ANBMA$ be the circular pulley of radius a and $ANBCA$ the endless chain hanging over it.

Total length of the chain

$$\begin{aligned} &= \frac{2}{3}(\text{circumference of the pulley}) + \text{arc } BCA \\ &= \frac{2}{3}(2\pi a) + \text{arc } BCA \\ &= \frac{4}{3}\pi a + \text{arc } BCA \quad (1) \end{aligned}$$

Here, ACB forms an extenary, with C as a vertex.



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MATHEMATICS BY K. VENKATESH

Let $OC = c$ from $\triangle O'AD$, we have,

$$DA = OA \sin 60^\circ = a \cdot \frac{\sqrt{3}}{2}$$

$$\text{And from } x = c \log(\sec \psi + \tan \psi)$$

for the catenary ACB at point A, $\psi_A = 60^\circ$.

$$\therefore x = c \log(\sec 60^\circ + \tan 60^\circ) \\ = c \log(2 + \sqrt{3})$$

$$\text{But } x = DA$$

$$\therefore \frac{a \sqrt{3}}{2 \log(2 + \sqrt{3})}$$

$$\text{Now } s = c \tan \psi$$

at point A,

$$s_A = c \tan 60^\circ \\ = \frac{c \cdot \sqrt{3}}{2 \log(2 + \sqrt{3})} \sqrt{3}$$

Total length of arc $ACB = 2s_A$

$$= \frac{3a}{\log(2 + \sqrt{3})}$$

∴ from ①

Total length of the chain

$$l = \frac{4}{3}\pi a + \frac{3a}{2\log(2+\sqrt{3})}$$

$$= a \left\{ \frac{4}{3}\pi + \frac{3}{\log(2+\sqrt{3})} \right\}$$

Q.17 An endless uniform chain hung over two smooth pegs in the same horizontal line shown that,

when it is in a position of

equilibrium, the ratio of

the distance between

the vertices of the

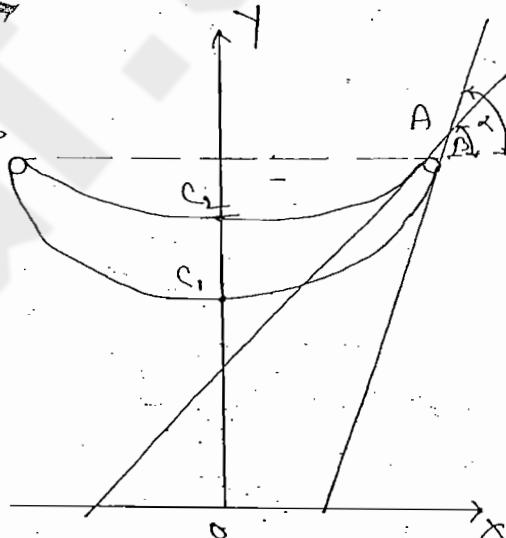
Catenary is to half the

length of the chain is the

tangent of half the angle

of inclination of the portions

near the pegs.

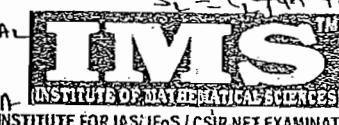


$$\frac{c_2 - c_1}{s_1 + s_2} = \tan(\alpha - \beta)$$

Hint: $T_{A1} = T_{A2}$, $s_1 = c_1 \tan \psi_1$, $s_2 = c_2 \tan \psi_2$

$$w\gamma_{1A} = w\gamma_{2A}$$

$$\therefore \gamma_{1A} = \gamma_{2A}$$



$$\therefore \gamma_{1A} = \tan \psi$$

$$c_1 \tan \psi_1 = c_2 \tan \psi_2$$

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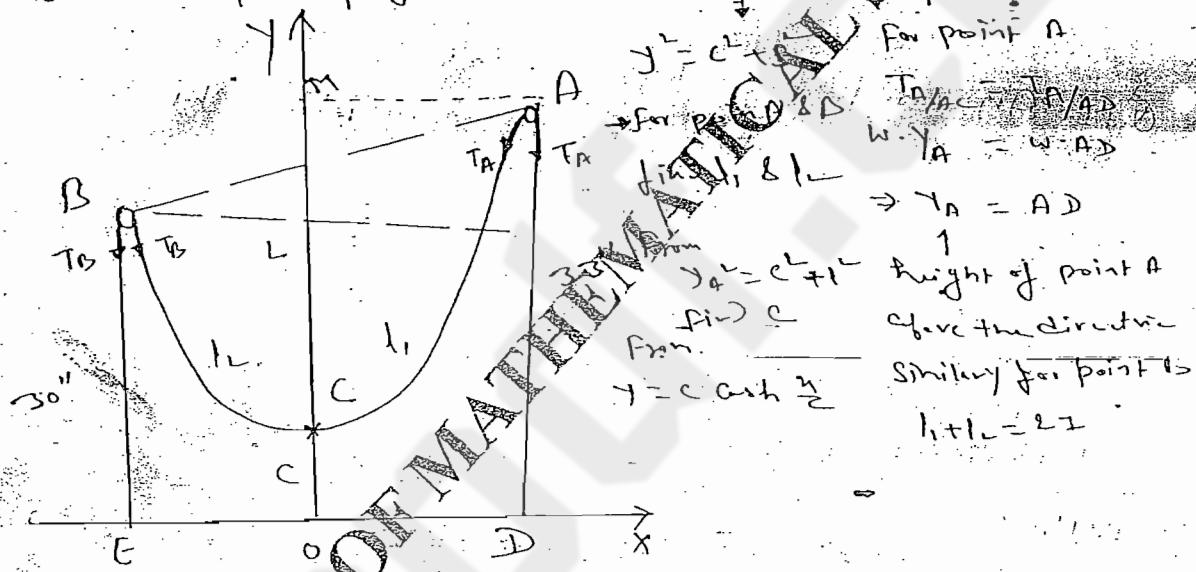
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(18)

A heavy uniform string, 90" long, hangs over two smooth pegs at different heights. The parts which hangs vertically are of length 30" and 33".

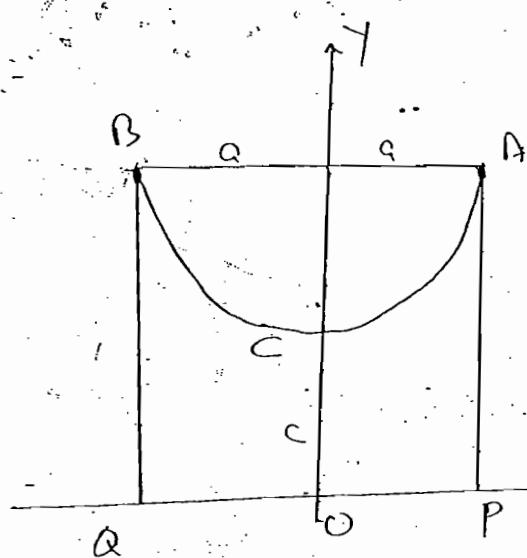
Prove that the vertex of catenary divides the whole string in the ratio of 4:5 and find the distance between the pegs.



(19) A string hangs over two smooth pegs which are at the same level: its free ends hanging vertically. Prove that when the string is of shortest possible length, the parameter of the catenary is equal to half the distance between the pegs, and find the whole length of the string.

Soln.

(19)



Suppose A and B are two smooth pegs at the same level and at a distance $2a$ apart

$$\therefore AB = 2a$$

A string hangs over the pegs A and B. The portion APQDQ of the string hangs vertically and the portion ACB is in the form of catenary. The vertex C and directrix is Ox .

The pegs are smooth.

Tension at point A due to string AP

= tension at the point A due to string ACB

$$\Rightarrow w_{AP} = w_{YA}$$

$$\Rightarrow AP = YA$$

[where
 $w = \frac{\text{weight}}{\text{length}}$]



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$$\therefore y_A = AP$$

shows that the point P lies on the directrix OX of the catenary ACB.

Similarly, the other free end Q of the string also lies on the directrix OX.

Lct. c = parameter of Catenary.

at. point A,

$$y = y_A = AP \quad \text{and } x = x_A = a$$

$$\therefore s = \text{arc } AC$$

$$= c \sinh\left(\frac{a}{c}\right)$$

anw

$$y_A = AP = c \cosh\left(\frac{a}{c}\right)$$

Total length of the string

$$l = 2x(\text{arc } AC + AP)$$

$$l = 2x \left(c \sinh\left(\frac{a}{c}\right) + c \cosh\left(\frac{a}{c}\right) \right)$$

$$l = 2c \left(\frac{e^{\frac{a}{c}} - e^{-\frac{a}{c}}}{2} + \frac{e^{\frac{a}{c}} + e^{-\frac{a}{c}}}{2} \right)$$

$$l = 2c \cdot e^{\frac{a}{c}} \quad \text{--- (1)}$$

Here, $l = f(c)$ only

(20)

For maximum or minimum value of 'I', we have,

$$\frac{dI}{dc} = 0$$

$$\Rightarrow 2 \cdot e^{\frac{a/c}{c}} + 2 \cdot c \cdot e^{\frac{a/c}{c}} \left(\frac{a}{c^2}\right) = 0$$

$$\Rightarrow 2 \cdot e^{\frac{a/c}{c}} \left(1 - \frac{a}{c}\right) = 0$$

$$e^{\frac{a/c}{c}} \neq 0$$

$$a = c$$

$$\text{Now, } \frac{d^2I}{dc^2} = -\frac{2a}{c^2} e^{\frac{a/c}{c}} + 2 \cdot c \cdot e^{\frac{a/c}{c}} \cdot \frac{a}{c^2}$$
$$= 2 \cdot c \cdot e^{\frac{a/c}{c}}$$

for $c=a$.

$\frac{d^2I}{dc^2}$ is minimum, when $c=a$

Thus when the string is of shortest possible

length, we have

$$c = a = \frac{1}{2}(2a)$$

= $\frac{1}{2} \times$ distance between the pegs

(Prove)



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Whole length of String from O,

$$l = 2\pi r \quad [\text{Computing } \frac{\pi}{2}]$$

Ans

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