

IMS
MATHS
BOOK-05

Vector Space

Book - H.R. Vasanthi

VECTOR SPACESSet-I

INSTITUTE FOR ASPIRE EDUCATION
NEW DELHI 110003
Mob: 09999197625

1

Field: Let F be a non-empty set and $+^n$ and \cdot^n be binary operations on F . Then algebraic structure $(F, +, \cdot)$ is said to be field if the following properties are satisfied.

(I) $(F, +)$ is an abelian group.

====> a group in which the law of composition is commutative

i) Closure prop: $\forall a, b \in F \Rightarrow a+b \in F$ ii) Asso. prop: $\forall a, b, c \in F \Rightarrow (a+b)+c = a+(b+c)$.(iii) Existence of left identity: $\forall a \in F \exists 0 \in F$ s.t. $0+a=a$
Here '0' is the identity elt.(iv) Existence of left inverse: $\forall a \in F, \exists -a \in F$ s.t. $(-a)+a=0$ (left identity)Here $-a$ is the inverse of a in F .(v) comm. prop: $\forall a, b \in F; a+b=b+a$ (II) (F, \cdot) is an abelian groupi) Closure prop: $\forall a, b \in F \Rightarrow a \cdot b \in F$ ii) Asso. prop: $\forall a, b, c \in F \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$ iii) Existence of left identity: $\forall a \in F \exists 1 \in F$ s.t. $1 \cdot a = a$.Here 1 is the identity in F .(iv) Existence of left inverse: $\forall a \neq 0 \in F \exists \frac{1}{a} \in F$ s.t. $\frac{1}{a} \cdot a = 1$ ∴ $\frac{1}{a}$ is the inverse of a in F .(v) comm. prop: $\forall a, b \in F; ab = ba$ (vi) \cdot^n is distributive w.r.t $+^n$ i.e., $\forall a, b, c \in F \Rightarrow a \cdot (b+c) = ab+ac$.Ex: $(\mathbb{Q}, +, \cdot)$ is not a field. Indis. not factoring ($\frac{a}{b} \cdot \frac{c}{d} \neq \frac{a \cdot c}{b \cdot d}$) not integral $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are fields. $(\mathbb{Q}^*, +, \cdot)$, $(\mathbb{R}^*, +, \cdot)$, $(\mathbb{C}^*, +, \cdot)$ are not fields.Add \mathbb{Z}

Defn Subfield: Let F be a field and $K \subseteq F$.
If K is a field w.r.t same binary operations
in F then K is called subfield of F .

~~Ex~~ If R is a subfield of \mathbb{Q}
 \mathbb{Q} is a subfield of \mathbb{R}
 \mathbb{R} is " " \mathbb{C}

Defn External Composition:

Let A be any set. If $a, b \in A$ $\star a, b \in A$

then \star is said to be internal composition on A .

External Composition:

Let V and F be any two sets. If $a, b \in V$ $\forall a, b \in F$, $a \star b$
then \star is said to be an external composition in V over F .

vector Space or Linear Space

Let $(F, +, \cdot)$ be a field. The elements of F are called scalars.
Let V be a non-empty set whose elements are called vectors.

The following compositions are defined.

i) An internal composition in V called vector addition.
ii) An external composition in V over the field F called

scalar multiplication.

If these compositions satisfy the following axioms
then V is called vector space over the field F .

I. $(V, +)$ is an abelian group.

(i) closure prop : $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

(ii) Assoc. prop : $\forall \alpha, \beta, \gamma \in V \Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

(iii) existence of Identity:

$$\forall \alpha \in V, \exists 0 \in V, \text{ s.t. } \alpha + 0 = 0 + \alpha = \alpha$$

Here the identity elt $0 \in V$ is called zero vector.

(iv) existence of inverse:

$$\forall \alpha \in V, \exists -\alpha \in V \text{ s.t. } \alpha + (-\alpha) = -\alpha + \alpha = 0$$

(v) comm. prop:

$$\forall \alpha, \beta \in V \Rightarrow \alpha + \beta = \beta + \alpha$$

[II]. The two compositions i.e., scalar \times^n and vector \cdot .

$$\forall \alpha, b \in F; \alpha, \beta \in V \Rightarrow$$

$$(i) \alpha \cdot (\alpha + \beta) = \alpha \alpha + \alpha \beta$$

$$(ii) (\alpha + b)\alpha = \alpha \alpha + b\alpha$$

$$(iii) (ab)\alpha = a(b\alpha)$$

(iv) $1\alpha = \alpha$; 1 is the unity elt of the field F .

Note:

III. When V is a vector space over field F then we shall denote it by $V(F)$ and we say that $V(F)$ is a vector space.

IV. If F is the field R of real nos then V is called real vector space. Similarly $V(Q), V(C)$ are called rational, complex vector spaces respectively.

Problems:

$$(1) V = \mathbb{I}, F = \mathbb{Q}$$

Is $V(F)$ a vector space?

Solⁿ Internal Composition:

$$\forall \alpha, \beta \in \mathbb{I} \Rightarrow \alpha + \beta \in \mathbb{I}$$

\therefore vector \mathbb{I}^n is an internal composition on \mathbb{I} .

External Composition:

$\forall a \in \mathbb{Q}, a \in \mathbb{I} \Rightarrow a\alpha$ need not be an integer.

$$\text{Ex: } a = \frac{1}{2} \in \mathbb{Q}, \alpha = 3 \in \mathbb{I} \Rightarrow \frac{1}{2} \cdot 3 = \frac{3}{2} \notin \mathbb{I}.$$

\therefore scalar \mathbb{I}^n is not an external composition of \mathbb{I} over \mathbb{Q} .

$\therefore I(Q)$ is not a vector space

Note: If $V \subseteq F$ then $V(F)$ is not a vector space
(except $V = \{0\} \subseteq F$)

$$(2) \rightarrow V = \mathbb{R}; F = \emptyset$$

$$\mathbb{Q} \subseteq \mathbb{R}$$

$$\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$$

$$F \subseteq V$$

$$\text{and } \forall \alpha \in \mathbb{Q} \subseteq \mathbb{R}, \alpha \in \mathbb{R} \Rightarrow \alpha \in \mathbb{R}$$

\therefore External and internal compositions are satisfied.

I i) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$

\therefore Closure prop. is satisfied.

ii) $\forall \alpha, \beta, \gamma \in \mathbb{R}$

$$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

\therefore Asso. prop. is satisfied.

iii) $\forall \alpha \in \mathbb{R} \exists 0 \in \mathbb{R}$ s.t. $\alpha + 0 = 0 + \alpha = \alpha$

\therefore Identity prop. is satisfied.

$\therefore 0$ is identity elt.

iv) $\forall \alpha \in \mathbb{R} \exists -\alpha \in \mathbb{R}$ s.t. $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (Identity elt in \mathbb{R})

\therefore Inverse of α is $-\alpha$.

\therefore Inverse prop. is satisfied.

v) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta = \beta + \alpha$

\therefore Comm. prop. is satisfied

$\therefore (\mathbb{R}, +)$ is an abelian group.

II $\forall a, b \in Q \subseteq \mathbb{R}; \alpha, \beta \in \mathbb{R}$

(i) $\bar{\alpha}(\alpha + \beta) = \alpha\bar{\alpha} + \beta\bar{\alpha}$ (LHD in \mathbb{R})

(ii) $(\alpha + b)\bar{\alpha} = \alpha\bar{\alpha} + b\bar{\alpha}$ (RHD in \mathbb{R})

(iii) $(ab)\bar{\alpha} = a(b\bar{\alpha})$ (Asso. prop. in \mathbb{R})

(iv) $1\cdot\bar{\alpha} = \bar{\alpha} \quad \forall \alpha \in \mathbb{R}. \quad (1 \text{ is identity w.r.t } x^n \text{ in } \mathbb{R})$

$\therefore R(Q)$ is vector space.

Note: If $F \subseteq V$ then $V(F)$ is a vector space.

Similarly $C(Q)$, $C(R)$ are also vector spaces

→ A field k can be regarded as a vector space over any subfield F of k . (3)

Soln: Given that k is a field and F is a subfield of k .

∴ F is also field w.r.t some b.o.s defined in k .

Let us consider the elts of k as vectors.

$$\forall \alpha, \beta \in k \Rightarrow \alpha + \beta \in k.$$

and let us consider the elts of the subfield F as scalars.

$$\text{Note } \alpha \in F \subseteq k, \alpha \in k \Rightarrow \alpha \in k.$$

∴ Internal and external compositions are satisfied.

I. Since k is a field.

∴ $(k, +)$ is an abelian group

II. $\forall a, b \in F \subseteq k ; \alpha, \beta \in k$

$$(i) a(\alpha + \beta) = a\alpha + a\beta \quad (\text{LDL in } k)$$

$$(ii) (\alpha + b)\alpha = a\alpha + b\alpha \quad (\text{RDL in } k)$$

$$(iii) (ab)\alpha = a(b\alpha) \quad (\text{Res. prop. in } k)$$

$$(iv) 1\alpha = \alpha \quad \forall \alpha \in k. \text{ and } 1 \text{ is the identity elt of the subfield } F.$$

(∴ 1 is also Identity elt of the field k).

$$\therefore 1\alpha = \alpha \Rightarrow \alpha \in k.$$

∴ $k(F)$ is a vector space.

Note: If F is any field, then F itself is a vector space over the field F .

i.e., $F(F)$ is a vector space.

→ $V = \text{Set of all vectors and } F \text{ is a field of real nos.}$

Soln: $\forall \bar{z}, \bar{P} \in V \Rightarrow \bar{z} + \bar{P} \in V \text{ and}$

$\alpha \in F, \bar{\alpha} \in V \Rightarrow \alpha \bar{\alpha} \in V$

∴ Internal and external compositions are satisfied.

[I] (i) $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} \in V$

∴ closure prop. is satisfied

(ii) $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V \Rightarrow (\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$

∴ Assoc. prop. is satisfied.

(iii) $\forall \bar{\alpha} \in V \exists \bar{\alpha} \in V$ s.t. $\bar{\alpha} + \bar{\alpha} = \bar{\alpha}$

∴ $\bar{0}$ is the identity vector in V .

(iv) $\forall \bar{\alpha} \in V \exists -\bar{\alpha} \in V$ s.t. $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \bar{0}$ (Zero vector)

∴ inverse of $\bar{\alpha}$ is $-\bar{\alpha}$

(v) $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$

∴ comm. prop. is satisfied.

[II] $\forall a, b \in F \Rightarrow \bar{\alpha}, \bar{\beta} \in V$

(i) $a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$

(ii) $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$

(iii) $(ab)\bar{\alpha} = a(b\bar{\alpha})$

(iv) $1\bar{\alpha} = \bar{\alpha} \forall \bar{\alpha} \in V$

∴ $V(F)$ is a vector space.

→ V = set of all $m \times n$ matrices with their elts as real numbers
and $F = \mathbb{R}$.

Note: If V = the set of all $m \times n$ matrices with their elts as rational numbers and $F = \mathbb{R}$, then $V(F)$ is not a vector space.

Because there is no external composition.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \in V$; $\sqrt{7} \in F$ then $\sqrt{7}A = \begin{bmatrix} \sqrt{7} & 2\sqrt{7} & 3\sqrt{7} \\ 0 & \sqrt{7} & 2\sqrt{7} \end{bmatrix} \notin V$
(∴ the elts of resulting matrix are not rational numbers)

Similarly, if V = the set of all $m \times n$ matrices with their elts as real numbers.

(4)

and $F = \mathbb{C}$ (complex numbers)

then $V(F)$ is not vector space

→ If V = the set of all $m \times n$ matrices with their elts as integers, and $F = \mathbb{Q}$ (rational numbers) then $V(F)$ is not a vector space.

→ V = the set of all ordered n -tuples and F is any field.

Soln. Let $V = \{(a_1, a_2, \dots, a_n) / a_1, a_2, \dots, a_n \in F\}$

Let $\alpha, \beta \in V$

Choose $\alpha = (a_1, a_2, \dots, a_n)$

$\beta = (b_1, b_2, \dots, b_n)$

where $a_1, a_2, \dots, a_n \in F$

$b_1, b_2, \dots, b_n \in F$

$$\begin{aligned} \Rightarrow \alpha + \beta &= (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \\ &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in V \end{aligned}$$

$\because (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in F$

External composition is satisfied.

and $a \in F, \alpha \in V$

$$\Rightarrow a\alpha = a(a_1, a_2, \dots, a_n)$$

$$= (aa_1, aa_2, \dots, aa_n) \in V$$

$\because aa_1, aa_2, \dots, aa_n \in F$

External composition is satisfied.

[P]. (ii) $\forall \alpha, \beta \in V$

$$\Rightarrow \alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in V$$

(ii) $\forall \alpha, \beta, \gamma \in V$

$$(\alpha + \beta) + \gamma = [(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)] + (c_1, c_2, \dots, c_n)$$

$$\begin{aligned}
 &= (a_1+b_1, a_2+b_2, \dots, a_n+b_n) + (c_1, c_2, \dots, c_n) \\
 &= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \dots, (a_n+b_n)+c_n) \\
 &= (a_1+(b_1+c_1), a_2+(b_2+c_2), \dots, a_n+(b_n+c_n)) \\
 &\quad \text{(by assoc. prop. w.r.t +)} \\
 &= (a_1, a_2, \dots, a_n) + (b_1+c_1, b_2+c_2, \dots, b_n+c_n) \\
 &= (a_1, a_2, \dots, a_n) + [(b_1, b_2, \dots, b_n) + (c_1, c_2, \dots, c_n)] \\
 &= \alpha + (\beta + \gamma)
 \end{aligned}$$

\therefore Assoc. prop. is satisfied.

(iii) we have $0 = (0, 0, \dots, 0) \in V$ where $0 \in F$.

$$\begin{aligned}
 \text{if } \alpha &= (a_1, a_2, \dots, a_n) \in V \text{ where } a_1, a_2, \dots, a_n \in F \\
 \text{then } 0+\alpha &= (0, 0, 0, \dots, 0) + (a_1, a_2, \dots, a_n) \\
 &= (0+a_1, 0+a_2, \dots, 0+a_n) \\
 &= (a_1, a_2, \dots, a_n) \\
 &= \alpha
 \end{aligned}$$

Similarly $\alpha+0 = \alpha$.

$$\therefore 0+\alpha = 0+\alpha = \alpha$$

$\therefore 0 = (0, 0, 0, \dots, 0)$ is the identity elt in V .

(iv) If $\alpha = (a_1, a_2, \dots, a_n) \in V$ where $a_1, a_2, \dots, a_n \in F$.

$$\text{then } -\alpha = -(a_1, a_2, \dots, a_n)$$

$$= (-a_1, -a_2, \dots, -a_n) \in V$$

$$\begin{aligned}
 \text{Now } (-\alpha) + \alpha &= ((-a_1)+a_1, (-a_2)+a_2, \dots, (-a_n)+a_n) \\
 &= (0, 0, \dots, 0)
 \end{aligned}$$

Similarly $\alpha + (-\alpha) = 0$.

$\therefore (-\alpha) + \alpha = \alpha + (-\alpha) = 0$. $\therefore -\alpha$ is the inverse of α in V .

$$\begin{aligned}
 \text{(v)} \quad & \forall \alpha, \beta \in V \Rightarrow \alpha + \beta = (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \\
 & = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \\
 & = (b_1 + a_1, b_2 + a_2, \dots, b_n + a_n) \\
 & = (b_1, b_2, \dots, b_n) + (a_1, a_2, \dots, a_n) \\
 & = \beta + \alpha.
 \end{aligned}$$

$(V, +)$ is an abelian group.

ii for $\alpha, \beta \in V; a, b \in F$

$$\begin{aligned}
 \text{(i)} \quad \alpha(\alpha + \beta) &= \alpha[(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)] \\
 &= \alpha[a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] \\
 &= (\alpha(a_1 + b_1), \alpha(a_2 + b_2), \dots, \alpha(a_n + b_n)) \\
 &= (aa_1 + ab_1, aa_2 + ab_2, \dots, aa_n + ab_n) \\
 &= (aa_1, aa_2, \dots, aa_n) + (ab_1, ab_2, \dots, ab_n) \quad (\text{by LDL in } F) \\
 &= a(a_1, a_2, \dots, a_n) + a(b_1, b_2, \dots, b_n) \\
 &= a\alpha + a\beta.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \alpha((a+b)\alpha) = (a+b)(a_1, a_2, \dots, a_n) \\
 &= ((a+b)a_1, (a+b)a_2, \dots, (a+b)a_n) \\
 &= (aa_1 + ba_1, aa_2 + ba_2, \dots, aa_n + ba_n) \\
 &= (aa_1, aa_2, \dots, aa_n) + (ba_1, ba_2, \dots, ba_n) \quad (\text{by RDL in } F) \\
 &= a(a_1, a_2, \dots, a_n) + b(a_1, a_2, \dots, a_n) \\
 &= a\alpha + b\alpha.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & (ab)\alpha = (ab)(a_1, a_2, \dots, a_n) \\
 &= ((ab)a_1, (ab)a_2, \dots, (ab)a_n) \\
 &= (a(ba_1), a(ba_2), \dots, a(ba_n)) \quad (\text{by associativity}) \\
 &= a(ba_1, ba_2, \dots, ba_n) \\
 &= a[b(a_1, a_2, \dots, a_n)] \\
 &= a(b\alpha).
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & L = \{ (a_1, a_2, \dots, a_n) \\
 & = \{ (a_1, a_2, \dots, a_n) \\
 & = \{ (a_1, a_2, \dots, a_n) \mid a_i \in F \} \quad (\because a_i \in F) \\
 & = \{ a_1, a_2, \dots, a_n \mid a_i \in F \} \\
 & \therefore V(F) \text{ is a vector space.}
 \end{aligned}$$

Note:

1. → The vector space of all ordered n -tuples over F is denoted by $V_n(F)$.

→ Sometimes denote it by $F^{(n)}$ or F^n .

$$\therefore V_n(F) \text{ or } F^{(n)} = \{ (a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in F \}$$

2. $V_2(F) = \{ (a_1, a_2) \mid a_1, a_2 \in F \}$ is a vector space of all ordered pairs over F .

Similarly $V_3(F) = \{ (a_1, a_2, a_3) \mid a_1, a_2, a_3 \in F \}$ is the vector space of all ordered triple or triplets over F .

→ $F[x] =$ the set of all polynomials and F is any field.

$$\begin{aligned}
 \text{S1} \quad F[x] &= \{ f(x) \mid f(x) = \sum_{i=0}^{\infty} a_i x^i \\
 &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_i \in F \}
 \end{aligned}$$

Now $f(x), g(x) \in F[x]$

$$\therefore \text{Choose } f(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$g(x) = \sum_{i=0}^{\infty} b_i x^i; \text{ where } a_i, b_i \in F$$

$$\begin{aligned}
 \Rightarrow f(x) + g(x) &= (a_0 x^0 + a_1 x + a_2 x^2 + \dots) + (b_0 + b_1 x + b_2 x^2 + \dots) \\
 &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots \\
 &= \sum (a_i + b_i) x^i \in F[x].
 \end{aligned}$$

∴ External Composition is satisfied.

IMS

INSTITUTE OF APPLIED SCIENCES
INSTITUTE FOR IAS/IFS EXAMINATION
NEW DELHI-110009
Mob: 09999197625

(6)

Now $f(x) \in F[x]$; $a \in F$

$$\begin{aligned} af(x) &= a(a_0 + a_1 x + a_2 x^2 + \dots) \\ &= a a_0 + (a a_1)x + (a a_2)x^2 + \dots \\ &= \sum (a a_i)x^i \in F[x] \end{aligned}$$

External composition is satisfied. ($\because a, a_i \in F, i=0, 1, 2, \dots \Rightarrow a a_i \in F$)

(I) (i) $\forall f(x), g(x) \in F[x]$, where $f(x) = \sum a_i x^i$

$$\begin{aligned} \Rightarrow f(x) + g(x) &= \sum a_i x^i + \sum b_i x^i \quad a_i, b_i \in F \\ &= \sum (a_i + b_i) x^i \in F[x] \quad \{a_i + b_i \in F\} \end{aligned}$$

(ii) $\forall f(x), g(x), h(x) \in F[x]$

$$\begin{aligned} \Rightarrow [f(x) + g(x)] + h(x) &= [\sum a_i x^i + \sum b_i x^i] + \sum c_i x^i \\ &= \sum (a_i + b_i + c_i) x^i \\ &= \sum [(a_i + b_i) + c_i] x^i \\ &= \sum [a_i + (b_i + c_i)] x^i \quad (\text{by assoc prop}) \\ &= \sum a_i x^i + \sum (b_i + c_i) x^i \\ &= \sum a_i x^i + \left[\sum b_i x^i + \sum c_i x^i \right] \\ &= f(x) + [g(x) + h(x)] \end{aligned}$$

∴ Assoc. prop. is satisfied.

(iv).

we have $0 = 0 + 0x + 0x^2 + \dots$
(zero polynomial)
 $= \sum 0x^i \in F[x]$

if $f(x) = \sum a_i x^i \in F[x]$; $a_i \in F, i=0, 1, 2, \dots$

then $0 \cdot f(x) = \sum 0x^i + \sum a_i \cdot 0x^i$
 $= \sum (0 + a_i) x^i$
 $= \sum a_i x^i$
 $= f(x)$

Ans

Similarly $f(x) + 0 = f(x)$

$$\therefore 0 + f(x) = f(x) + 0 = f(x) \in F[x]$$

∴ Identity elt is the zero polynomial

(iv) If $f(x) \in F[x]$ then $-f(x) \in F[x]$.

$$\text{i.e., } f(x) = a_0 + a_1 x + \dots \in F[x]; a_0, a_1, a_2 \in F$$

$$\text{then } -f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \dots \in F[x]$$

$$= -a_0 - a_1 - a_2 - \dots \in F.$$

we have

$$\begin{aligned} (-f(x)) + f(x) &= (-a_0 + a_0) + (-a_1 + a_1)x + (-a_2 + a_2)x^2 \\ &= 0 + 0x + 0x^2 \\ &= 0 \text{ (zero polynomial)} \end{aligned}$$

Similarly $f(x) + (-f(x)) = 0$

$$(-f(x)) + f(x) = f(x) + (-f(x)) = 0$$

$\therefore -f(x)$ is the inverse polynomial of $f(x)$ in $F[x]$

(v) If $f(x), g(x) \in F[x]$

$$\begin{aligned} \Rightarrow f(x) + g(x) &= (a_0 + b_0)x^0 + (a_1 + b_1)x^1 + \dots \\ &= (b_0 + a_0)x^0 + (b_1 + a_1)x^1 + \dots \quad (\text{By abelian property of } F) \\ &= (b_0 + b_1)x^0 + \dots + (a_0 + a_1)x^1 + \dots \\ &= g(x) + f(x) \end{aligned}$$

∴ Commutative property is satisfied.

$(F[x], +)$ is an abelian group.

(vi) $\forall f(x), g(x) \in F[x]; a, b \in F$

$$\begin{aligned} (i) \quad a(f(x) + g(x)) &= a[(a_0 + b_0)x^0 + (a_1 + b_1)x^1 + \dots] \\ &= a[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots] \\ &= a(a_0 + b_0) + a(a_1 + b_1)x + a(a_2 + b_2)x^2 + \dots \\ &= (aa_0 + ab_0) + (aa_1 + ab_1)x + (aa_2 + ab_2)x^2 + \dots \quad (\text{By abelian property of } F) \\ &= (aa_0 + (ab)_0)x^0 + (aa_1 + (ab)_1)x^1 + (aa_2 + (ab)_2)x^2 + \dots \\ &\quad + (ab_0 + (ba)_0)x^0 + (ab_1 + (ba)_1)x^1 + (ab_2 + (ba)_2)x^2 + \dots \end{aligned}$$

$$= a(a_0 + a_1x + a_2x^2 + \dots) + b(b_0 + b_1x + b_2x^2 + \dots)$$

$$= a f(x) + b g(x)$$

(7)

$$(i) (af+b) f(x) = (af+b)(a_0 + a_1x + a_2x^2 + \dots)$$

$$= [(af+b)a_0] + [(af+b)a_1]x + [(af+b)a_2]x^2 + \dots$$

$$= (aa_0 + ba_0) + (aa_1 + ba_1)x + \dots$$

$$= (aa_0 + (a+b)a_1)x + \dots + (ba_0 + (b-a)a_1)x^2 + \dots$$

$$= a(a_0 + a_1x + \dots) + b(a_0 + a_1x + \dots)$$

$$= a f(x) + b f(x).$$

(ii) similarly

$$(iii) (ab) f(x) = ab f(x).$$

$$(iv) f(cx) = f(x). \quad \forall c \in F.$$

Let F be the field and let P_n be the set of all polynomials (of degree atmost n) over the field F .
S.T P_n is vector space over the field F .

Ex: Let $P_n = \{ f(x) / f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \}$ where $a_0, a_1, \dots, a_n \in F$

$\# f(x), g(x) \in P_n$

$$\text{Choose } f(x) = a_0 + a_1x + \dots + a_nx^n$$

$$g(x) = b_0 + b_1x + \dots + b_nx^n$$

$$a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n \in F.$$

$$\Rightarrow f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$$

$\in P_n$ $(\because a_0 + b_0, a_1 + b_1, \dots, a_n + b_n \in F)$
and polynomial of degree atmost n

$\# f(x) \in P_n; c \in F$

$$\Rightarrow cf(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \dots + (ca_n)x^n$$

$\in P_n$ $(\because ca_0, ca_1, \dots, ca_n \in F)$
and polynomial of degree atmost n

\therefore External and Internal Composition atmost n are satisfied.

(I) (i) $\forall f(x), g(x) \in P_n \Rightarrow f(x)+g(x) \in P_n$

\ Closure prop. is satisfied.

(ii) $\forall f(x), g(x), h(x) \in P_n$

$$\Rightarrow (f(x)+g(x))+h(x)$$

$$= f(x) + (g(x) + h(x))$$

\ Assoc. prop. is satisfied.

(iii) $\forall f(x) \in P_n \exists I(x) = a_0 + a_1x + \dots + a_nx^n \in P_n$

$$\text{ s.t } f(x) + I(x) = f(x).$$

$$\text{ Similarly } I(x) + f(x) = f(x)$$

$$\therefore f(x) + I(x) = I(x) + f(x) = f(x)$$

$\therefore I(x)$ is the identity polynomial in P_n . $\forall f(x) \in P_n$.

iv) Inverse prop.

(V) Commutative prop:

$\therefore (P_n, +)$ is an abelian group.

(II) $\forall f(x), g(x) \in P_n ; a, b \in F$

$$\text{ we have (i) } a(f(x) + g(x)) = a((a_0 + a_1x + a_2x^2 + \dots + a_nx^n) + (b_0 + b_1x + \dots + b_nx^n))$$

$$= a(a_0 + b_0) + a(a_1 + b_1)x + \dots + a(a_n + b_n)x^n$$

$$= a(a_0 + b_0) + a(a_1 + b_1)x + \dots + a(a_n + b_n)x^n$$

$$= (aa_0 + ab_0) + (aa_1 + ab_1)x + \dots + (aa_n + ab_n)x^n$$

$$= (aa_0 + aa_1x + \dots + aa_nx^n) + (ab_0 + ab_1x + \dots + ab_nx^n) \quad (\text{By LDL in } F)$$

$$= a(a_0 + a_1x + \dots + a_nx^n) + b(a_0 + a_1x + \dots + a_nx^n)$$

$$= a f(x) + b g(x)$$

$$(ii) \text{ S.t } (a+b)f(x) = af(x) + bf(x)$$

$$(iii) \text{ S.t } (ab)f(x) = a(bf(x))$$

$$(iv) \text{ S.t } 1f(x) = f(x) \quad \forall f(x) \in P_n \quad \therefore P_n(F) \text{ is a vector space}$$

→ Let F be any field and S be any non-empty set.

Let V be the set of all functions from S to F .

Ans: 13m

$$\text{i.e., } V = \{f | f: S \rightarrow F\}$$

(8)

Let us define sum of two vectors f and g in V as follows -

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$

and the product of the scalar ' c ' in F and the function f in V as follows:

$$(cf)(x) = c f(x) \quad \forall x \in S$$

then $V(F)$ is vector space.

Soln

$$\forall f, g \in V \Rightarrow (f+g)x = f(x) + g(x) \quad \forall x \in S$$

(By defn)
Since $f(x), g(x) \in F$ and F is a field.

$$\Rightarrow f(x) + g(x) \in F$$

$$\therefore (f+g)(x) = f(x) + g(x) \in F$$

$$\therefore (f+g) : S \rightarrow F$$

$$\therefore f+g \in V$$

Internal composition is satisfied.

$$\forall f \in V, c \in F \Rightarrow (cf)(x) = c f(x) \quad \forall x \in S$$

(By defn)
Since $f(x) \in F$, $c \in F$ and F is a field.

$$\therefore c f(x) \in F$$

$$\therefore cf : S \rightarrow F$$

$$\Rightarrow cf \in V; \forall c \in F, f \in V$$

External composition is satisfied.

i (i) $\forall f, g \in V \Rightarrow f+g \in V$ Closure prop. is satisfied

(ii) $\forall f, g, h \in V$

$$\Rightarrow [(f+g)+h](x) = (f+g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ = f(x) + [g(x) + h(x)]$$

$$= [f + (g + h)](x)$$

By ass. prop w.r.t F
i.e., f(x), g(x), h(x) EF
 $\Rightarrow [f(x) + g(x)] + h(x)$

$$\therefore (f+g)+h = f+(g+h)$$

$$= f(x) + [g(x) + h(x)]$$

(iii) If $f \in V$, $I(x) = 0$ EF then $I \in V$. (i.e. $I: S \rightarrow S$)

$$\text{Now } (I+f)(x) = I(x) + f(x)$$

$$= 0 + f(x)$$

$$= f(x) \in f(x) \in F$$

$$\therefore I+f = f \in f \in V$$

$$\text{Also } f+I = f \in f \in V$$

$$\therefore I+f = f+I = f \in f \in V$$

\therefore Identity elt $\equiv I \in V$

(iv) if $f \in V$, then $-f = (-1)f \in V$

$$\text{Now } [f + (-f)](x) = f(x) + (-f)(x)$$

$$= f(x) + [-1f(x)]$$

$$= f(x) - f(x)$$

$$= 0 = I(x)$$

$$\therefore f + (-f) = 0 = I$$

$$\text{Also } (-f) + f = 0 = I$$

$$\therefore f + (-f) = -f + f = 0 = I$$

\therefore Inverse of f is $-f$ in V .

(v) $\forall f, g \in V \Rightarrow (f+g)(x) = f(x) + g(x)$ (By defn)

$$= g(x) + f(x) \quad \text{By assoc. in } F$$

$$= (g+f)(x) \quad \text{i.e., } f(x), g(x) \in F$$

$$\therefore f+g = g+f \quad \Rightarrow f(x) + g(x) = g(x) + f(x)$$

\therefore Commutative prop is satisfied.

II $\forall a, b \in F$, $f, g \in V$

$$(i) [a(f+g)](x) = a(f+g)(x) \quad (\text{By defn})$$

$$= a(f(x) + g(x)) \quad (\text{By defn})$$

$$= af(x) + ag(x) \quad (\text{By LD L inf})$$

$$= (af)(x) + (ag)(x) = (af+ag)(x) \quad \therefore a(f+g) = af + ag$$

$$\begin{aligned}
 \text{(i)} [(af+bf)](x) &= (af+bf)x \quad (\text{by defn}) \\
 &= af(x) + bf(x) \\
 &= (af)(x) + (bf)(x) \\
 &= (af+bf)(x) \\
 \therefore (af+bf) &= af+bf.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} [(ab)f](x) &= (ab)f(x) \quad (\text{by defn}) \\
 &= a(bf(x)) \\
 &= a(bf)(x) \\
 \therefore (ab)f &= a(bf).
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} [1 \cdot f](x) &= 1 \cdot f(x) \quad (\text{by defn}) \\
 &= f(x) \quad \text{by identity in } f \\
 &\quad + f(x) \in f \\
 \therefore 1f &= f \quad \forall f \in V \\
 \therefore V(P) &\text{ is a vector space}
 \end{aligned}$$

Let V be the set of all pairs (x, y) of real numbers and let F be the field of real numbers. Examine in each of the following cases whether V is a vectorspace over the field of real numbers or not?

$$(1) (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\therefore c(x_1, y_1) = (cx_1, cy_1)$$

not (2)
Exhibit

$$(x_1, y_1) + (x_1, y_1) = (x_1 + x_1, y_1 + y_1)$$

$$c(x_1, y_1) = (cx_1, cy_1)$$

not (3)
Exhibit

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$c(x_1, y_1) = (1Cx_1, 1Cy_1).$$

not (4)
Exhibit

$$(x_1, y_1) + (x_1, y_1) = (x_1 + x_1, y_1 + y_1)$$

$$c(x_1, y_1) = (c^2x_1, c^2y_1)$$

part (5) $(x_1, y_1) + (x_1, y_1) = (x_1+x_1, y_1+y_1)$
~~by defn~~ $\therefore ((x_1, y_1)) = (0, 0)$.

Soln \Rightarrow I Let $\alpha = (x_1, y_1), \beta = (x_2, y_2) \in V$
 where $x_1, y_1, x_2, y_2 \in \mathbb{R}$

(i) $\alpha + \beta = (x_1, y_1) + (x_2, y_2)$
 $= (x_1+x_2, y_1+y_2) \in V \quad (\because x_1+x_2, y_1+y_2 \in \mathbb{R})$
 \therefore closure prop. is satisfied

(ii) $(\alpha + \beta) + r = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$
 $= (x_1+x_2, y_1+y_2) + (x_3, y_3)$
 $= ((x_1+x_2)+x_3, (y_1+y_2)+y_3) \quad (\text{by defn})$
 $= (x_1+(x_2+x_3), y_1+(y_2+y_3)) = (x_1, y_1) + (x_2, y_2) + (x_3, y_3) \quad (\text{by assoc. prop.})$
 $= (x_1, y_1) + (x_2+x_3, y_1+y_3)$
 $= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$
 $= \alpha + (\beta + r)$

\therefore Assoc. prop. is satisfied

(iii) $\forall \alpha = (x_1, y_1) \in V \exists (0, 0) \in V, \alpha \in \mathbb{R}$

Let $\alpha + 0 = (x_1, y_1) + (0, 0)$
 $= (x_1+0, y_1+0)$
 $= (x_1, y_1)$
 $= \alpha$

$\therefore 0 + \alpha = \alpha$

$\therefore 0 + \alpha = \alpha + 0 + \alpha$

$\therefore (0, 0)$ is the identity in V .

(iv) $\forall \alpha \in V \exists -\alpha = (-x_1, -y_1) \in V; \quad x_1, y_1 \in \mathbb{R}$
 $\therefore \alpha + (-\alpha) = (x_1, y_1) + (-x_1, -y_1)$

$$= (x-y, y-y) \quad (\text{By defn})$$

$$= (0, 0)$$

(10)

$$\therefore b(\alpha) + \alpha = (0, 0)$$

$$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = (0, 0)$$

$\therefore -\alpha$ is the inverse of α .

$$(V) \forall \alpha, \beta \in V \Rightarrow \alpha + \beta = \beta + \alpha \quad (\text{by defn})$$

\therefore comm. prop. is satisfied.

$\therefore (V, +)$ is an abelian group.

II $\forall \alpha, \beta \in V; a, b \in \mathbb{R}$

$$(i) a(\alpha + \beta) = a[(x_1, y_1) + (x_2, y_2)]$$

$$= a(x_1 + x_2, y_1 + y_2) \quad (\text{By defn})$$

$$= (ax_1 + ax_2, ay_1 + ay_2) \quad (\text{by defn})$$

$$\text{and } a\alpha + a\beta = a(x_1, y_1) + a(x_2, y_2)$$

$$= (ax_1, y_1) + (ax_2, y_2)$$

$$= (ax_1 + ax_2, y_1 + y_2) \quad (2)$$

\therefore from (1) & (2)

$$a(\alpha + \beta) \underset{(1)}{=} a\alpha + a\beta$$

$$(ii) (a+b)\alpha = (a+b)(x_1, y_1)$$

$$= ((a+b)x_1, y_1) \quad (1)$$

$$\text{and } a\alpha + b\alpha = a(x_1, y_1) + b(x_1, y_1)$$

$$= (ax_1, y_1) + (bx_1, y_1)$$

$$= ((a+b)x_1, y_1) \quad (2)$$

\therefore from (1) & (2) we have

$$(a+b)\alpha \underset{(1)}{\neq} a\alpha + b\alpha$$

$\therefore V(\mathbb{R})$ is not a vector space.

Sol(2)

Let $\alpha = (x, y) \in V$; $x, y \in R$

$$\text{Then } 1\alpha = 1(x, y) = (1x, 1y) \quad (\text{By defn}) \\ = (x, y)$$

But $(x, 0) \neq (x, y)$ (if $y \neq 0$)

$$\therefore 1\alpha \neq \alpha \quad \forall \alpha \in V$$

 $V(R)$ is not a vector space.

10/1

Let $V(F)$ be a vector space and $\vec{0}$ be the zero vector of V . Then

(i) $a\vec{0} = \vec{0}$ $\forall a \in F$

(ii) $0\alpha = \vec{0} \quad \forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha) \quad \forall a \in F, \forall \alpha \in V$

(iv) $(-a)\alpha = -(a\alpha) \quad \forall a \in F, \forall \alpha \in V$

(v) $a(\alpha + \beta) = a\alpha + a\beta \quad \forall a \in F, \forall \alpha, \beta \in V$.

(vi) $a\alpha = \vec{0} \Rightarrow a = 0 \text{ or } \alpha = \vec{0}$

proof:

(i) we have $\alpha\vec{0} = a(0+\vec{0}) \quad (\because 0 = 0+\vec{0})$
 $= a\vec{0} + a\vec{0} \quad (\because a(\alpha+\beta) = a\alpha+a\beta)$
 $\Rightarrow a\vec{0} + a\vec{0} = a\vec{0} + a\vec{0} \quad (\because a\vec{0} \in V \text{ and } a\vec{0} + a\vec{0} = a\vec{0})$

Given that $V(F)$ be a vector space
 V is an abelian group w.r.t addition.

Therefore by right cancellation law in V ,

we get $\vec{0} = a\vec{0}$

$\Rightarrow a\vec{0} = \vec{0} \quad \forall a \in F$

(ii). we have $0\alpha = (0+0)\alpha \quad (\because 0 = 0+0)$
 $= 0\alpha + 0\alpha$

$\Rightarrow 0\alpha + 0\alpha = 0\alpha + 0\alpha$

$(\because 0\alpha \in V \text{ and } 0+0\alpha = 0\alpha)$

Since V is an abelian group w.r.t addition.

Therefore by right cancellation law in V ,

we get $0 = \alpha\alpha$

$$\alpha\alpha = 0 \Leftrightarrow \alpha \in F.$$

(iii) we have $\alpha[\alpha + (-\alpha)] = \alpha\alpha + \alpha(-\alpha)$

$$\Rightarrow \alpha 0 = \alpha\alpha + \alpha(-\alpha)$$

$$\Rightarrow 0 = \alpha\alpha + \alpha(-\alpha)$$

$\Rightarrow \alpha(-\alpha)$ is the additive inverse of $\alpha\alpha$

$$\Rightarrow \alpha(-\alpha) = -\alpha\alpha$$

$$\therefore \alpha(-\alpha) = -\alpha\alpha \text{ for } \forall \alpha \in F.$$

(iv) we have $[\alpha + (-\alpha)]\alpha = \alpha\alpha + (-\alpha)\alpha$

$$\Rightarrow 0\alpha = \alpha\alpha + (-\alpha)\alpha$$

$$\Rightarrow 0 = \alpha\alpha + (-\alpha)\alpha$$

$\Rightarrow (-\alpha)\alpha$ is the additive inverse of $\alpha\alpha$.

$$\Rightarrow (-\alpha)\alpha = -\alpha\alpha$$

$$\therefore (-\alpha)\alpha = -\alpha\alpha \text{ for } \forall \alpha \in F.$$

(v) we have $\alpha(\alpha - \beta) = \alpha(\alpha + (-\beta))$

$$= \alpha\alpha + \alpha(-\beta)$$

$$= \alpha\alpha + [-(\alpha\beta)] \quad (\because \alpha(-\beta) = -\alpha\beta)$$

$$= \alpha\alpha - \alpha\beta$$

$$\therefore \alpha(\alpha - \beta) = \alpha\alpha - \alpha\beta \text{ for } \forall \alpha, \beta \in F.$$

(vi) Let $\alpha\alpha = 0$ and $\alpha \neq 0$.

Then α' exists because ' α ' is a non-zero element of the field F .

$$\therefore \alpha\alpha = 0 \Rightarrow \alpha'(\alpha\alpha) = \alpha'0$$

$$\Rightarrow (\alpha'\alpha)\alpha = 0$$

$$\Rightarrow 1\alpha = 0 \\ \Rightarrow \alpha = 0$$

Again let $a\alpha = 0$ and $\alpha \neq 0$.

Then to prove that $a = 0$.

If possible suppose that $a \neq 0$.

Then \bar{a} exists.

$$\therefore a\alpha = 0 \Rightarrow \bar{a}(a\alpha) = \bar{a}0 \\ \Rightarrow (\bar{a}a)\alpha = 0 \\ \Rightarrow 1\alpha = 0 \\ \Rightarrow \alpha = 0$$

Thus we get a contradiction.

∴ a must be a zero vector.

Therefore a must be equal to 0.

Hence $\alpha \neq 0$ and $a\alpha = 0$.

$$\Rightarrow a = 0$$

$$\therefore a\alpha = 0 \Rightarrow a = 0 \text{ or } \alpha = 0$$

$\alpha = 0$

Let $V(F)$ be a vector space. Then

(i) If $a, b \in F$ and α is a non-zero vector of V , we have $a\alpha = b\alpha \Rightarrow a = b$

(ii) If $\alpha, \beta \in V$ and a is a non-zero element of F , we have

$$a\alpha = a\beta \Rightarrow \alpha = \beta.$$

proof:

i)

we have $\alpha x = \beta x$

$$\Rightarrow \alpha x - \beta x = 0$$

$$\Rightarrow (\alpha - \beta)x = 0$$

$$\Rightarrow \alpha - \beta = 0$$

$$\Rightarrow \alpha = \beta$$

ii) we have $\alpha x = \beta x$

$$\Rightarrow \alpha x - \beta x = 0$$

$$\Rightarrow x(\alpha - \beta) = 0$$

$$\Rightarrow \alpha - \beta = 0, \text{ since } x \neq 0$$

$$\Rightarrow \alpha = \beta$$

→ On \mathbb{R}^n , define two operations

$$\alpha \oplus \beta = \alpha - \beta$$

$$c \cdot \alpha = -c\alpha$$

The operations on the right are the usual ones.
which of the axioms for a vector space are
satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

Let V be the set of all complex-valued functions
 f on the real line such that (for all $t \in \mathbb{R}$)
 $f(-t) = \bar{f}(t)$.

The bar denotes complex conjugation. Show that

V , with the operations

$$(f+g)(t) = f(t) + g(t)$$

$$(cf)(t) = c f(t)$$

is a vector space over the field of real numbers.
Give an example of a function in V which is not
real-valued.

Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$$u+v = u \cdot v \text{ for all } u, v \in R^+$$

$$\alpha u = u^\alpha \text{ for all } u \in R^+ \text{ and real scalar } \alpha.$$

Prove that R^+ is a vector space.

→ Which of the following subsets of V_4 are vector spaces for coordinate wise addition and scalar multiplication?

The set of all vectors $(x_1, x_2, x_3, x_4) \in V_4$ such that

(a) $x_4 = 0$ (b) $x_4 = 0$ (c) $x_2 > 0$ (d) $x_3^2 > 0$ (e) $x_3^2 < 0$

(f) $2x_1 + 3x_2 = 0$ (g) $x_1 + \frac{2}{3}x_2 - 3x_3 + x_4 = 1$. (h) $x_1 = 1$

Ans: (a), (b), (f) and (h) are vector spaces.

→ Which of the following subsets of P are vector spaces?

The set of all polynomials p such that

(a) degree of $p \leq n$ (b) degree of $p = 3$

(c) degree of $p \geq 4$ (d) $p(1) = 0$

(e) $p(1) = 1$ (f) $p'(1) = 0$

(g) p has integral coefficients.

Ans: (a), (d) and (f) are vector spaces.

Notations:

$C[a, b]$ = the set of all real-valued functions defined and continuous on the closed interval $[a, b]$.

$C^1[a, b]$ = the set of all real-valued functions defined on $[a, b]$ and whose first derivatives are continuous on $[a, b]$.

$C^n[a, b]$ = the set of all real-valued functions defined on $[a, b]$, differentiable n -times and whose n th derivatives are continuous on $[a, b]$. These functions are called n -times continuously differentiable functions.

→ Which of the following subsets $C[0, 1]$ are vector spaces?

The set of all functions $f \in C[0, 1]$ such that

(a) $f(y_1) = 0$ (b) $f(3/4) = 0$ (c) $f'(x) = x f(x)$ Yes

(d) $f(0) = f(1)$ (e) $f(x) < 0$ at a finite number of points in $[0, 1]$

(f) f has a local minima at $x = y_1$

(g) f has a local extrema at $x = y_2$

Ans: (a), (c), (d) & (f) are vector spaces.

→ Is \mathbb{Z}_5 a vector space over \mathbb{Z}_2 ?

Solⁿ: NO.

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is not subfield of

$$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

because:

$$2+3=0 \text{ in } \mathbb{Z}_5 \text{ but } 2+3 \neq 0 \text{ in } \mathbb{Z}_7$$

Hence \mathbb{Z}_5 is not a vector space over \mathbb{Z}_2 .

→ Let $K = \mathbb{Z}_3$, the integers modulo 3. How many elements are there in the vector space $V = K^4$?

Solⁿ: There are three choices 0, 1 or 2, for each of the four components of a vector in V .

$$\text{Hence } V \text{ has } 3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81 \text{ elements.}$$

→ Can \mathbb{C}^2 (pairs of complex numbers) be defined as a vector space: (a) over \mathbb{R} ? (b) over \mathbb{Q} ?

(c) over \mathbb{C} ? (d) over \mathbb{Z} ?

Solⁿ: (a), (b), (c), are vector spaces.

whereas (d) is not a vector space.

because \mathbb{Z} is not a field.

→ Can \mathbb{R}^n be defined as a vector space:

(a) over \mathbb{Q} (b) over \mathbb{R} (c) over \mathbb{C} ?

Solⁿ: (a), (b) are vector spaces.

whereas (c) is not a vector space

because \mathbb{C} is not a subfield of \mathbb{R} .

→ Let $V = \{ \langle a_n \rangle : a_n \in \mathbb{R} \text{ or } \mathbb{C} \}$ i.e., V is the set of all real sequences. prove that V is a vector space over \mathbb{R} , where addition and scalar-multiplication are defined component wise.

Miscellaneous results and notations

101

Let $f(I)$ be the set of all real-valued functions defined on the interval I .
With pointwise addition and scalar multiplication,
 $f(I)$ becomes a real vector space.

The zero of this space is the function

0 given by $0(x) = 0$ for all $x \in I$.

Note: If, instead of the real-valued functions, we use complex-valued functions defined on I and pointwise addition and scalar multiplication, then we get a complex vector space (using complex scalars).

We denote this complex vector space by $f_c(I)$.

Let $P(I)$ denote the set of all polynomials p with real coefficients defined on the interval I .

Where p is a function whose value at x is $p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$ for all $x \in I$, where α_i 's are real numbers and n is a non-negative integer.

Using pointwise addition and scalar multiplication as for functions, we find that $P(I)$ is a real vector space.

If we take complex coefficients for the

polynomials and use complex scalars; then we get the complex vector space $\mathcal{P}_c(\mathbb{I})$.

In both cases the vector space 0 of the space is the zero polynomial given by

$$0(x) = 0 \text{ for all } x \in \mathbb{I}$$

→ $\mathcal{C}[a,b]$, $\mathcal{C}^{(n)}[a,b]$, $\mathcal{C}^{(n)}_c[a,b]$ are real vector spaces under pointwise addition and scalar multiplication.

we have sum of two continuous (differentiable) functions is continuous (differentiable) and any scalar multiple of a continuous (differentiable) function is continuous (differentiable).

By changing the domain of definition of continuity and differentiability to the open interval (a, b) , we get, similarly, the real vector space $\mathcal{C}(a, b)$ and $\mathcal{C}^{(n)}(a, b)$ for each positive integer n .

Note: By changing real-valued functions to complex-valued functions and using complex scalars, we get the complex vector spaces $\mathcal{C}_c[a,b]$ and $\mathcal{C}_c(a,b)$.

Let $\mathcal{C}^\infty[a,b]$ stand for the set of all functions defined on $[a, b]$ and having derivatives of all orders on $[a, b]$. This is a real vector space for the usual operations. It is called the space of infinitely differentiable functions on $[a, b]$.

IIMS

INSTITUTE OF MANAGEMENT SCIENCES
INSTITUTE FOR IAS/IFS EXAMINATION
NEW DELHI-110009
Mob: 09999197625

(11)

SUBSPACE:

Let $V(F)$ be a vector space and $w \subseteq V$ if w is a vector space w.r.t the internal and external compositions in V then w is called a subspace of V .

Theorem:

- $V(F)$ is a vector space, w is a subset of V ($w \subseteq V$);
 w is a subspace of $V(F)$ iff the internal and external compositions are satisfied in w .
 - i.e. (i) $\forall \alpha, \beta \in w \Rightarrow \alpha + \beta \in w$.
 - (ii) $\forall a \in F, \alpha \in w \Rightarrow a\alpha \in w$.

proofNecessary part:

Let w be a subspace of $V(F)$:

∴ By defn w is a vector space w.r.t the internal and external Compositions in V :
Internal and external compositions are satisfied in w .

satisfied in w .

i.e., (i) $\forall \alpha, \beta \in w \Rightarrow \alpha + \beta \in w$ and
(ii) $\forall a \in F, \alpha \in w \Rightarrow a\alpha \in w$.

Sufficient condition:

Let $w \subseteq V$ and internal and external compositions be satisfied in w .

i.e. (i) $\forall \alpha, \beta \in w \Rightarrow \alpha + \beta \in w$
(ii) $\forall a \in F, \alpha \in w \Rightarrow a\alpha \in w$.

Proof:

[1]. (i) $\forall \alpha, \beta \in w \subseteq V \Rightarrow \alpha + \beta \in w$. (by hypothesis.)

∴ Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in w \subseteq V \Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ by ass. prop.
∴ Ass. prop. is satisfied

(iii) $\forall \alpha, \beta \in W \subseteq V$

$$\Rightarrow \alpha + \beta = \beta + \alpha. \quad (\text{By comm. prop. in } V)$$

∴ comm. prop. is satisfied in W .(iv) Take $a=0 \in F, \alpha \in W \Rightarrow a\alpha = 0 \in W \subseteq V$ (by hyp.)

$$\Rightarrow a \in W$$

$$\therefore 0 + \alpha = \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V. \quad (\text{By identity prop. in } V)$$

(v) $1 \in F \Rightarrow 1 \in W$.

$$\text{Take } a = 1 \in F; \alpha \in V$$

$$\Rightarrow a\alpha = 1\alpha = \alpha \quad (\text{By hyp.})$$

$$\Rightarrow 1\alpha = \alpha$$

$$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = 0 \quad (\text{By inverse prop. in } V)$$

∴ inverse of α is $-\alpha$.∴ $(W, +)$ is an abelian group.

[T]

 $\forall \alpha, \beta \in W \subseteq V, \alpha, b \in F$

$$(i) \alpha(\alpha + \beta) = \alpha\alpha + \alpha\beta$$

$$(ii) (\alpha + b)\alpha = \alpha\alpha + b\alpha$$

$$(iii) (ab)\alpha = a(b\alpha)$$

$$(iv) 1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$$

By axioms w.r.t.
external compositions
in V .

∴ $W(F)$ is a vector space.∴ $W(F)$ is a subspace of $V(F)$.

→ $V(F)$ is a vector space, W is a subset of $V(F)$ and
(i.e., $W \subseteq V$); W is a subspace of $V(F)$ iff $a, b \in F$ and
 $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.

Proof: N.C.:Let W be a subspace of $V(F)$.

By defn. W is a vector space w.r.t. internal
and external compositions in V .

∴ $a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha, b\beta \in W$ (By external composition in V)

$$\Rightarrow a\alpha, b\beta \in W$$

(By internal composition in W)

NP

$\Rightarrow \alpha\beta + b\beta \in W$ (by internal comp in W)

I (i) Take $a=b=1$ ϵF

$$\text{Iff, } \alpha, \beta \in W \subseteq V \Rightarrow 1\alpha + 1\beta \in W \text{ (by hyp)}$$

$$\Rightarrow \alpha + \beta \in W$$

Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V$

$$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

(iii) $\forall \alpha, \beta \in W \subseteq V$ (By assoc. prop. in V)

$$\Rightarrow \alpha + \beta = \beta + \alpha \text{ (By comm. prop. in } V)$$

\therefore Assoc. prop and Comm. prop are satisfied in W .

(iv) Take $a=b=0$ ϵF

0 ϵF , $\alpha, \beta \in W \subseteq V$

$$\Rightarrow 0\alpha + 0\beta \in W \text{ (by hyp)}$$

$$\Rightarrow 0 \in W$$

$\forall d \in W \subseteq V \Rightarrow 0 \in W$ s.t

$$\alpha + 0 = \alpha = 0 + \alpha \text{ (By identity in } V)$$

$\therefore 0$ is identity elt in W .

(v) $i \in F \Rightarrow -i \in F$

Take $a=-i \in F \Rightarrow b=i \in F$

$\alpha, \beta \in W \subseteq V$

$$\Rightarrow (-1)\alpha + 0\beta \in W \text{ (by hyp)}$$

$$\Rightarrow -\alpha \in W$$

\therefore If $\alpha \in W \subseteq V$ then $-\alpha \in W \subseteq V$

$$\therefore \alpha + (-\alpha) = \alpha + (-\alpha) = 0 \text{ (By inverse axiom in } V)$$

\therefore inverse of α is $-\alpha$

$(W, +)$ is an abelian group.

- II** $\forall \alpha, \beta \in W \subseteq V; a, b \in F$
- | | | |
|---|----|--|
| (i) $a(\alpha + \beta) = a\alpha + a\beta$ | {} | By axioms w.r.t external compositions in V . |
| (ii) $(\alpha + \beta)a = a\alpha + a\beta$ | {} | |
| (iii) $(ab)\alpha = a(b\alpha)$ | {} | |
| (iv) $1d = d \quad \forall d \in W \subseteq V$ | | |

$\therefore W(F)$ is a vector space

$\therefore W(F)$ is a subspace of $V(F)$.

$\rightarrow V(P)$ is a vectorspace; $W \subseteq V$; W is a subspace of $V(F)$ iff $(i) + \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$
 $(ii) \alpha \in F, \alpha \in W \Rightarrow \alpha \in W$.

Proof:

N.C. 2: Let w be a subspace of V .

\therefore By defn w is a vector space w.r.t the internal and external compositions in V .

By internal composition

$$\begin{aligned} & \forall \alpha, \beta \in W \Rightarrow \alpha \in W, -\beta \in W \quad (\text{By inverse axiom in } w) \\ & \Rightarrow \alpha + (-\beta) \in W \quad (\text{By closure prop in } w) \\ & \Rightarrow \alpha - \beta \in W \end{aligned}$$

By external composition

$$\alpha \in F, \alpha \in W \Rightarrow \alpha \in W.$$

S.C.: Let $W \subseteq V$; (i) $\forall \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$
(ii) $\forall \alpha \in F, \alpha \in W \Rightarrow \alpha \in W$

I. (i) Take $\alpha = 0 \in F$

$$\begin{aligned} & \alpha \in F, \alpha \in W \subseteq V \\ & \Rightarrow 0 \in W \quad (\text{by hyp}) \\ & \Rightarrow 0 \in W. \end{aligned}$$

$$\therefore 0 + \alpha = \alpha + 0 = \alpha. \quad \forall \alpha \in W \subseteq V$$

(by Identity axiom of V)

\therefore Identity prop. is satisfied in W .

and '0' is the identity in W .

(ii) Take $\alpha = 0 \in W, \beta = 1 \in W$

$$\begin{aligned} & \Rightarrow 0 - \alpha \in W \quad (\text{by hyp}) \\ & \Rightarrow -\alpha \in W. \end{aligned}$$

$$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = 0. \quad (\text{by inverse axiom of } V)$$

\therefore Inverse prop. is satisfied in W . and inverse of α is $-\alpha$.

$$(ii) \alpha, \beta \in W \subseteq V \Rightarrow \alpha, -\beta \in W \quad (\because \alpha \in W \Rightarrow -\alpha \in W)$$

$$\Rightarrow \alpha - (-\beta) \in W$$

$$\Rightarrow \alpha + \beta \in W$$

\therefore closure prop. is satisfied in W .

$$(iv) \forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

$$(v) \forall \alpha, \beta \in W \subseteq V \Rightarrow \alpha + \beta = \beta + \alpha$$

\therefore comm. prop. is satisfied.

$(W, +)$ is an abelian group.

II. $\forall \alpha, \beta \in W \subseteq V, \alpha, \beta \in F$

$$(i) \alpha(\lambda + \beta) = \lambda\alpha + \alpha\beta$$

$$(ii) (\alpha + b)\alpha = \alpha\alpha + b\alpha \quad \left\{ \begin{array}{l} \text{by axioms w.r.t. external} \\ \text{composition in } V \end{array} \right.$$

$$(iii) (ab)\alpha = a(b\alpha) \quad \left\{ \begin{array}{l} \text{by axioms w.r.t. internal} \\ \text{composition in } V \end{array} \right.$$

$$(iv) 1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$$

$\therefore W(F)$ is a vector space.

$\therefore W(F)$ is a subspace of $V(F)$.

$\rightarrow V(F)$ is a vector space and $W \subseteq V$; W is a subspace of $V(F)$.

$W(F)$ iff $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$.
 (\Leftarrow)

Proof

N.C.P

Let W be a subspace of $V(F)$.

\therefore by defn. W is a vector space w.r.t. the internal & external compositions in V .

By external composition in W -

$\forall \alpha \in F, \alpha \in W \Rightarrow \alpha \in W$

By internal composition

$\forall \alpha, \beta \in W, \alpha + \beta \in W$.

S.C. Let $W \subseteq V$, &

$\forall \alpha, \beta \in W, \alpha \in F \Rightarrow \alpha + \beta \in W$.

I (i) Take $a = 1 \in F$

$$1 \in F, \alpha, \beta \in W \Rightarrow 1 + \alpha + \beta \in W \text{ (by hyp)} \\ \Rightarrow \alpha + \beta \in W$$

\therefore Closure prop. is satisfied in W

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) $\alpha + \beta = \beta + \alpha$

\therefore Ass. & comm. prop. is satisfied in W .

(iv) $1 \in F \Rightarrow -1 \in F$

Take $a = -1 \in F, \beta \in W$

$$-1 \in F, \alpha, \beta \in W \Rightarrow (-1)\alpha + \beta \in W \text{ (by hyp)} \\ \Rightarrow -\alpha + \beta \in W$$

$$\therefore 0 + \alpha = \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V$$

(by identity of V)

\therefore Identity prop. is satisfied in W .

0 is the identity in W .

(v) $-1 \in F; \alpha, 0 \in W \Rightarrow -1 \cdot \alpha + 0 \in W \text{ (by hyp)}$

$$\Rightarrow -\alpha \in W$$

$\therefore \alpha + (-\alpha) = (-1)\alpha + \alpha = 0$ (By inverse axiom of V)

\therefore Inverse prop. is satisfied in W .

$\therefore -\alpha$ is inverse of α .

$\therefore (W, +)$ is an abelian group.

II $\forall \lambda, \mu \in W \subseteq V, a, b \in F$

(i) $a(\lambda + \mu) = a\lambda + a\mu$

(ii) $(a + b)\lambda = a\lambda + b\lambda$

(iii) $(ab)\lambda = a(b\lambda)$

(iv) $1\lambda = \lambda \quad \forall \lambda \in W \subseteq V$

$\therefore W(F)$ is a vector space

$\therefore W(F)$ is a subspace of $V(F)$

Algebra of subspaces

(14)

→ The intersection of any two subspaces of a vector space $V(F)$ is also a subspace of $V(F)$.

proof: Let w_1 & w_2 be any two subspaces of $V(F)$.

Let $w = w_1 \cap w_2$

$a, b \in F; \alpha, \beta \in w$

$\Rightarrow a, b \in F; \alpha, \beta \in w_1 \cap w_2$

$\Rightarrow a, b \in F; (\alpha, \beta \in w_1 \text{ and } \alpha, \beta \in w_2)$

$\Rightarrow ax + b\beta \in w_1 \text{ and } ax + b\beta \in w_2$

($\because w_1$ & w_2 are two

$w_1 \cap w_2$ is also subspace of $V(F)$ subspaces)

\therefore The intersection of two subspaces is also a subspace.

→ The arbitrary intersection of subspaces i.e., the intersection of any family of subspaces of a vector space is also a subspace.

proof: Let w_1, w_2, \dots be the given family of subspaces of the vector space $V(F)$.

Let $w = w_1 \cap w_2 \cap \dots$

$= \bigcap_{i \in N} w_i \quad (i=1, 2, \dots)$

$a, b \in F, \alpha, \beta \in w \Rightarrow a, b \in F; \alpha, \beta \in \bigcap_{i \in N} w_i$

$\Rightarrow a, b \in F, \alpha, \beta \in w_i \forall i \in N$

$\Rightarrow ax + b\beta \in w_i \quad \forall i \in N \quad (\because w_i \text{ is a}$

$\Rightarrow ax + b\beta \in \bigcap_{i \in N} w_i = w \quad \text{Subspace for all} \quad i \in N)$

$\therefore w = \bigcap_{i \in N} w_i$ is a subspace of $V(F)$

The intersection of any family of subspaces of a vector space is also a subspace.

→ the union of two subspaces, of a vector space
need not be a subspace.

$$\text{Ex}^5 \quad V_3(F) = \{ (a_1, a_2, a_3) / a_1, a_2, a_3 \in F \} \text{ is a vector space}$$

$$\text{Let } W_1 = \{ (a_1, a_2, b) / a_1, b \in F \} \subseteq V_3$$

$$\text{and } W_2 = \{ (a_1, 0, y) / a_1, y \in F \} \subseteq V_3$$

$$a_1, a_2 \in F;$$

$$\alpha = (0, a_1, b),$$

$$\beta = (0, c_1, d) \in W_1$$

$$\therefore a_1, b, c_1, d \in F$$

$$\Rightarrow a_1\alpha + a_2\beta = a_1(0, a_1, b) + a_2(0, c_1, d)$$

$$= (0, a_1a_1, a_1b) + (0, a_2c_1, a_2d)$$

$$= (0, a_1a_1 + a_2c_1, a_1b + a_2d) \in W_1$$

$\therefore W_1$ is a subspace. ($\because 0, a_1a_1 + a_2c_1, a_1b + a_2d \in W_1$)

$$\text{Now } a_1, a_2 \in F; \alpha = (a_1, 0, y_1), \beta = (a_2, 0, y_2) \in W_2$$

$$\Rightarrow a_1\alpha + a_2\beta = a_1(a_1, 0, y_1) + a_2(a_2, 0, y_2)$$

$$= (a_1a_1, 0, a_1y_1) + (a_2a_2, 0, a_2y_2)$$

$$= (a_1a_1 + a_2a_2, 0, a_1y_1 + a_2y_2) \in W_2$$

$\therefore W_2$ is a subspace of $V_3(F)$ ($\because a_1a_1 + a_2a_2, 0, a_1y_1 + a_2y_2 \in W_2$)

if $F \neq \emptyset$, then we have

$$(0, \frac{1}{2}, 3) \in W_1, (1, 0, 3) \in W_2$$

$$\Rightarrow (0, \frac{1}{2}, 3), (1, 0, 3) \in W_1 \cup W_2$$

$$\Rightarrow (0, \frac{1}{2}, 3) + (1, 0, 3) = (1, \frac{1}{2}, 6) \notin W_1 \cup W_2$$

(\because neither $(1, \frac{1}{2}, 6) \in W_1$

$\therefore W_1 \cup W_2$ is not closed under vector addition.

$\therefore W_1 \cup W_2$ is not a subspace of $V_3(F)$.

→ The union of two subspaces is a subspace iff one is contained in the other.

proof. Let w_1 and w_2 be two subspaces of the vector space $V(F)$.

N.C. Let $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$.

$$w_1 \subseteq w_2 \Rightarrow w_1 \cup w_2 = w_2 \text{ (subspace of } V(F))$$

$$w_2 \subseteq w_1 \Rightarrow w_1 \cup w_2 = w_1 \text{ (subspace of } V(F))$$

$\therefore w_1 \cup w_2$ is a subspace of $V(F)$.

S.C. Let $w_1 \cup w_2$ be a subspace of $V(F)$

then we prove that $w_1 \subseteq w_2$ or $w_2 \subseteq w_1$

By possible suppose that $w_1 \not\subseteq w_2$ or $w_2 \not\subseteq w_1$

if $w_1 \not\subseteq w_2$

let $\alpha \in w_1$ then $\alpha \notin w_2$

if $w_2 \not\subseteq w_1$
let $\beta \in w_2$ then $\beta \notin w_1$

Now $\alpha \in w_1$, $\beta \in w_2$

$$\Rightarrow \alpha + \beta \in w_1 \cup w_2$$

$$\Rightarrow \alpha + \beta \in w_1 \cup w_2 \quad (\because w_1 \cup w_2 \text{ is a subspace})$$

$$\Rightarrow \alpha + \beta \in w_1 \text{ or } \alpha + \beta \in w_2$$

Now $\alpha + \beta \in w_1$, $\alpha \in w_1$

$$\Rightarrow (\alpha + \beta) - \alpha \in w_1 \quad (\because w_1 \text{ is a subspace})$$

$$\Rightarrow \beta \in w_1$$

which is contradiction to $\beta \notin w_1$

and $\alpha + \beta \in w_2$, $\beta \in w_2$

$$\Rightarrow (\alpha + \beta) - \beta \in w_2 \quad (\because w_2 \text{ is a subspace})$$

$$\Rightarrow \alpha \in w_2$$

which is contradiction to $\alpha \notin w_2$

Our assumption that $w_1 \not\subseteq w_2$ or $w_2 \not\subseteq w_1$ is wrong

$$\therefore w_1 \subseteq w_2 \text{ or } w_2 \subseteq w_1$$

Note: ① Let $V(F)$ be any vector space.
Then V itself and the subset of V
consisting of the zero vector alone are
always subspaces of V .

These two subspaces are called improper
subspaces.

If V has any other subspace then it is
called a proper subspace.

② The subspace of V consisting of the
zero vector only is called the zero subspace
of V .

Problem:

\rightarrow Let $W = \{(a_1, a_2, 0) / a_1, a_2 \in F\} \subseteq V_3(F)$.
 Then S.T W is a subspace of $V_3(F)$.

Soln: Let $a, b \in F$; $\alpha, \beta \in W$.

$$\text{Choose } \alpha = (a_1, a_2, 0)$$

$$\beta = (b_1, b_2, 0)$$

where $a_1, a_2, b_1, b_2 \in F$

$$\Rightarrow a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$

$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$

$$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$$

$\therefore W$ is a subspace of $V_3(F)$. $a_1, a_2, b_1, b_2 \in F$

\rightarrow Let $W = \{(x_1, x_2, x_3) / a_1x_1 + a_2x_2 + a_3x_3 = 0$
 a_1, a_2, a_3 are fixed elts in F

$x_1, x_2, x_3 \in F\} \subseteq V_3(F)$.

Soln: $\forall a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = 0$

$\beta = (y_1, y_2, y_3) : a_1y_1 + a_2y_2 + a_3y_3 = 0$

$$\Rightarrow a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$

$$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$$

$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$$

$$\text{and } a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$$

$$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$$

$$= a(0) + b(0)$$

$$= 0$$

$$\therefore a\alpha + b\beta \in W$$

$\therefore W$ is a subspace of $V_3(F)$.

→ P.T. the set of all solutions (a_1, b_1, c_1) of the equation
 $a_1 + b_1 + 2c_1 = 0$ is a subspace of the vector space

Soln Let $W = \{(a_1, b_1, c_1) / a_1 + b_1 + 2c_1 = 0;
a_1, b_1, c_1 \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$.

Let $a_1, b_1 \in \mathbb{R}$, $\alpha, \beta \in W$

$$\text{Choose } \alpha = (a_1, b_1, c_1);$$

$$a_1 + b_1 + 2c_1 = 0$$

$$\beta = (a_2, b_2, c_2);$$

$$a_2 + b_2 + 2c_2 = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$

→ S.T. the set W of the elts of the vector space $V_3(\mathbb{R})$
of the form $(x+2y, y, -x+3y)$ - where $x, y \in \mathbb{R}$. Is a
subspace of $V_3(\mathbb{R})$.

Soln Let $W = \{(x+2y, y, -x+3y) / x, y \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$

let $a, b \in F : \alpha, \beta \in W$

$$\text{Choose } \alpha = (x_1+2y_1, y_1, -x_1+3y_1)$$

$$\beta = (x_2+2y_2, y_2, -x_2+3y_2)$$

$$\Rightarrow a\alpha + b\beta = a(x_1+2y_1, y_1, -x_1+3y_1) + b(x_2+2y_2, y_2, -x_2+3y_2)$$

$$= (ax_1+2ay_1, ay_1, -ax_1+3ay_1) + (bx_2+2by_2, by_2, -bx_2+3by_2)$$

$$= (ax_1+bx_2+2(ay_1+by_2), ay_1+by_2, -ax_1+bx_2+3(ay_1+by_2))$$

$\in W$.

$$\therefore a\alpha + b\beta \in W$$

$\therefore W$ is a Subspace of $V_3(\mathbb{R})$.

→ which of the following sets of vectors

$\alpha = (a_1, a_2, \dots, a_n)$ in \mathbb{R}^n are subspace of \mathbb{R}^n ? 17

- (i) all α s.t $a_1 \leq 0$
- (ii) all α s.t a_3 is an integer
- (iii) all α s.t $a_2 + 4a_3 = 0$
- (iv) all α s.t $a_1 + a_2 + \dots + a_n = k$ (constant)

Soln (i) Let $W = \{\alpha / \alpha = (a_1, a_2, \dots, a_n) \text{ and } a_1 \leq 0\} \subseteq \mathbb{R}^n$

If $a_1 = -3$ then $a_1 < 0$

Let $\alpha = (-3, a_2, a_3, \dots, a_n) \in W$

and if $a = -2 \in \mathbb{R}$

then $a\alpha = -2(-3, a_2, \dots, a_n)$

$= (6, 2a_2, \dots, 2a_n) \notin W$

$\therefore \alpha \notin W, a \in \mathbb{R} \Rightarrow a\alpha \notin W \quad (\because a \neq 0)$

$\therefore W$ is not a subspace of \mathbb{R}^n .

(ii) Let $W = \{\alpha / \alpha = (a_1, a_2, \dots, a_n) \text{ and } a_3 \text{ is an integer}\} \subseteq \mathbb{R}^n$

If $a_3 = -3$ is an integer.

Let $\alpha = (a_1, a_2, -3, \dots, a_n) \in W$

and $a = \frac{1}{2} \in \mathbb{R}$

then $a\alpha = \left(\frac{a_1}{2}, \frac{a_2}{2}, -\frac{3}{2}, \frac{a_4}{2}, \dots, a_n\right) \notin W$

$(-\because -\frac{3}{2} \text{ is not an integer}).$

$\therefore \alpha \notin W, a \in \mathbb{R} \Rightarrow a\alpha \notin W$

$\therefore W$ is not a subspace of \mathbb{R}^n .

(iii) Let $W = \{\alpha / \alpha = (a_1, a_2, \dots, a_n) \text{ and } a_2 + 4a_3 = 0\} \subseteq \mathbb{R}^n$

Note $a_1, b \in \mathbb{R}, \alpha, \beta \in W$

Choose $\alpha = (a_1, a_2, \dots, a_n)$ and $a_2 + 4a_3 = 0$

$\beta = (b_1, b_2, \dots, b_n)$ and $b_2 + 4b_3 = 0$

$$\begin{aligned} \Rightarrow ax+b\beta &= a(a_1, a_2, a_3, \dots, a_n) + b(b_1, b_2, b_3, \dots, b_n) \\ &= (aa_1+bb_1, aa_2+bb_2, aa_3+bb_3, \dots, aa_n+bb_n) \end{aligned} \quad (1)$$

Now we have

$$\begin{aligned} (aa_1+bb_1) + 4 \cdot (aa_2+bb_2) + (aa_3+bb_3) &= a(a_1+4a_2) + b(b_1+b_2) \\ &= a(0) + b(0) \\ &= 0. \end{aligned}$$

$$\therefore (1) \Rightarrow ax+b\beta \in W$$

$\therefore W$ is a subspace of \mathbb{R}^n .

(iv) Let $W = \{ \alpha / \alpha = (a_1, a_2, \dots, a_n) \text{ and } a_1+a_2+\dots+a_n = k \}$ and $a, b \in \mathbb{R}$.
 $\therefore W \subseteq \mathbb{R}^n$

Let $a, b \in \mathbb{R}$, $\alpha, \beta \in W$.

Choose $\alpha = (a_1, a_2, \dots, a_n)$ and $a_1+a_2+\dots+a_n = k$
 $\beta = (b_1, b_2, \dots, b_n)$ and $b_1+b_2+\dots+b_n = k$.

$$\begin{aligned} \Rightarrow ax+b\beta &= a(a_1, a_2, \dots, a_n) + b(b_1, b_2, \dots, b_n) \\ &= (aa_1+bb_1, aa_2+bb_2, \dots, aa_n+bb_n) \end{aligned} \quad (1)$$

Now we have

$$\begin{aligned} (aa_1+bb_1) + (aa_2+bb_2) + \dots + (aa_n+bb_n) \\ &= a(a_1+a_2+\dots+a_n) + b(b_1+b_2+\dots+b_n) \\ &= ak + bk \\ &= (a+b)k. \end{aligned}$$

If $k=0$ then (1) $\Rightarrow ax+b\beta \in W$.

$\therefore W$ is a subspace of \mathbb{R}^n .

If $k \neq 0$ then $ax+b\beta \notin W$.

$\therefore W$ is not a subspace of \mathbb{R}^n .

\rightarrow So W is not a subspace of $\mathbb{R}^3 = V$, where $W = \{(a, b, c) / a+b+c=1\}$

Sol: Let $\alpha = (0, 1, 0)$, $\beta = (1, 0, 0) \in W$.

then $\alpha+\beta = (1, 1, 0) \notin W$ ($\because 1+1+0=2 \neq 1$)

$\therefore W$ is not a subspace of $V = \mathbb{R}^3$.

\rightarrow S.T. w is not subspace of $V = \mathbb{R}^3$.

where $w = \{(a_1, b, c) / a_1, b, c \in \mathbb{Q}\} \subseteq \mathbb{R}^3$. (18)

Soln Let $a_1 = \sqrt{2} \notin \mathbb{Q}$, $a = (1, 2, 3) \in \mathbb{R}^3$

$$\Rightarrow a_1 a = \sqrt{2}(1, 2, 3)$$

$$= (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin w$$

$\therefore w$ is not a subspace of V . ($\because \sqrt{2}, 2\sqrt{2}, 3\sqrt{2} \notin \mathbb{Q}$)

\rightarrow S.T. w is not a subspace of $V = \mathbb{R}^n$.

where $w = \{(a_1, a_2, \dots, a_n) / a_1 \geq 0\}$.

Soln If $a_1 = 3$ then $a_1 > 0$.

$$a = (3, a_2, a_3, \dots, a_n)$$

If $a = -2 \in \mathbb{R}$

$$\text{then } ad = (-6, -2a_2, -2a_3, \dots, -2a_n) \notin w$$

$\therefore w$ is not a subspace of \mathbb{R}^n . ($\because a_1 = -6 < 0$)

\rightarrow S.T. w is not a subspace of \mathbb{R}^n .

where $w = \{(a_1, a_2, \dots, a_n) / a_2 = a_1^2\} \subseteq \mathbb{R}^n$.

Let $a \in \mathbb{R}$; $a = (a_1, a_2, \dots, a_n) \in w$ and

$\Rightarrow ad$ need not be an elt of w . $a_2 = a_1^2$

for example

let $a = \frac{1}{2} \in \mathbb{R}$, $a = (2, 4, a_3, \dots, a_n) \in w$

$$\Rightarrow ad = (1, 2, \frac{a_3}{2}, \dots, \frac{a_n}{2}) \notin w$$

($\because 2 \neq 1^2$)
i.e., $a_2 \neq a_1^2$

\rightarrow Let V be the real vector space of all functions f from \mathbb{R} into \mathbb{R} .

which of the following sets of functions are subspaces of V .

- (i) $w = \{f / f(3) = 0\}$ (iii) $w = \{f / f(-x) = -f(x)\}$
- (ii) $w = \{f / f(\pi) = f(1)\}$ (iv) $w = \{f / f(\pi) = 2 + f(1)\}$

$$(v) W = \{f \mid f(x) = [f(x)]^2\}$$

(vi) W consists of the continuous functions.

(vii) W consists of the differentiable functions.

Soln (i) Let $a, b \in \mathbb{R}$; $f, g \in W$ s.t $f(0) = 0$ & $g(0) = 0$

$$\Rightarrow (af + bg)(0) = (af)(0) + (bg)(0)$$

$$= a f(0) + b g(0)$$

$$= a(0) + b(0)$$

$$= 0$$

$$\therefore af + bg \in W.$$

$\therefore W$ is a subspace of V .

(ii) $a, b \in \mathbb{R}$; $f, g \in W$ i.e., $f(1) = f(0)$ and
 $g(1) = g(0)$

$$\Rightarrow (af + bg)(1) = (af)(1) + (bg)(1).$$

$$= af(1) + b g(1)$$

$$= af(0) + b g(0)$$

$$= (af)(0) + (bg)(0)$$

$$= (af + bg)(0)$$

$$\therefore af + bg \in W$$

$\therefore W$ is a subspace of V .

(iii) $a, b \in \mathbb{R}$; $f, g \in W$
i.e., $f(-x) = -f(x)$ &
 $g(-x) = -g(x)$

$$\Rightarrow (af + bg)(-x) = (af)(-x) + (bg)(-x)$$

$$= a f(-x) + b g(-x)$$

$$= a [-f(x)] + b [-g(x)]$$

$$= -[a f(x) + b g(x)]$$

$$= -[(af)(x) + (bg)(x)]$$

$$= -(af + bg)(x)$$

$$\therefore af + bg \in W.$$

$\therefore W$ is a subspace of V .

(iv) $a, b \in \mathbb{R}; f, g \in W$ i.e., $f(7) = 2f(f(1))$ &
 $g(7) = 2f(g(1))$

$$\begin{aligned}\Rightarrow (af + bg)(7) &= (af(7)) + (bg)(7) \\ &= af(2f(1)) + bg(2f(g(1))) \\ &= 2af(f(1)) + 2bg(g(1)) \\ &= (2a + 2b)f(f(1)) \\ &= (2a + 2b)f(f(1)) \quad \text{--- (1)}\end{aligned}$$

Let $a=1, b=1$ then

$$\begin{aligned}(f+g)(7) &= 4 + (f+g)(1) \\ &\neq 2 + (f+g)(1)\end{aligned}$$

$\therefore f+g \notin W$
 W is not a subspace

(v) Let $a, b \in \mathbb{R}; f, g \in W$ i.e., $f(x) = [f(x)]^2$ &
 $g(x) = [g(x)]^2$

$$\begin{aligned}\Rightarrow (af + bg)(x^2) &= a f(x^2) + bg(x^2) \\ &= a[f(x)]^2 + b[g(x)]^2 \quad \text{--- (1)}\end{aligned}$$

Now $(af + bg)(x^2) = [(af + bg)(x)]^2$

$$\begin{aligned}&= [a f(x) + b g(x)]^2 \\ &= a^2[f(x)]^2 + b^2[g(x)]^2 \\ &\quad + 2ab f(x) g(x) \quad \text{--- (2)}\end{aligned}$$

from (1) & (2)

$$a^2[f(x)]^2 + b^2[g(x)]^2 \neq a^2[f(x)]^2 + b^2[g(x)]^2 + 2ab f(x) g(x)$$

$\therefore af + bg \notin W$

$\therefore W$ is not a subspace

If f and g are continuous functions and f, g
 $a, b \in \mathbb{R}$ then $af + bg$ is also continuous function.

$$af + bg \in W$$

$\therefore W$ is a subspace of V .

(vi) If f and g are differentiable functions
and $a, b \in \mathbb{R}$ then

$$af + bg \text{ is also differentiable.}$$

$\therefore W$ is a subspace of V .

→ Let $W = \{(x_1, x_2, \dots, x_n) \in V_n / x_1 = 0\}$. Prove that (2)

W is a subspace of V_n

→ Prove that $W = \{(x_1, x_2, \dots, x_n) \in V_n^C / x_1 + x_2 + \dots + x_n = 0\}$,

where V_n^C → the set of all ordered n -tuples of complex numbers
 x_i 's are given constants

is a subspace of V_n^C .

→ Which of the following sets are subspaces of V_3 ?

- (a) $\{(x_1, x_2, x_3) / x_1 x_2 = 0\}$
- (b) $\{(x_1, x_2, x_3) / \frac{x_3}{x_1} = \sqrt{2}\}$
- (c) $\{(x_1, x_2, x_3) / \sqrt{2}x_1 = \sqrt{3}x_2\}$
- (d) $\{(x_1, x_2, x_3) / x_3 \text{ is an integer}\}$
- (e) $\{(x_1, x_2, x_3) / x_1^2 + x_2^2 + x_3^2 \leq 1\}$
- (f) $\{(x_1, x_2, x_3) / x_1 + x_2 + x_3 \geq 0\}$
- (g) $\{(x_1, x_2, x_3) / x_1 = \sqrt{2}x_2 \text{ and } x_3 = 3x_2\}$
- (h) $\{(x_1, x_2, x_3) / x_1 - 2x_2 = x_3 - 3x_2\}$
- (i) $\{(x_1, x_2, x_3) / x_1 = 2x_2 \text{ or } x_3 = 3x_2\}$

Ans: (c), (g) & (h) are subspaces of V_3 .

→ Which of the following sets are subspaces of P ?

- (a) $\{P \in P / \text{degree of } P = 4\}$
- (b) $\{P \in P / \text{degree of } P \leq 3\}$
- (c) $\{P \in P / \text{degree of } P \geq 5\}$
- (d) $\{P \in P / \text{degree of } P \leq 4 \text{ and } P'(0) = 0\}$
- (e) $\{P \in P / P(1) = 0\}$

Ans: (b), (d) & (e) are subspaces of P .

→ Which of the following sets are subspaces of $C([0, b])$?

- (a) $\{f \in C([a, b]) / f(x_0) = 0, x_0 \in [a, b]\}$
- (b) $\{f \in C([a, b]) / f'(x) = 0 \text{ for all } x \in [a, b]\}$
- (c) $\{f \in C([a, b]) / f(\frac{a+b}{2}) = 1\}$
- (d) $\{f \in C([a, b]) / f(x) = x^n \text{ for } n \in \mathbb{N}\}$
- (e) $\{f \in C([a, b]) / 2f''(x) + 3xf'''(x) - f'(x) + x^2 f(x) = 0\}$
- (f) $\{f \in C([a, b]) / \int_a^b f(x) dx = 0\}$

Ans: (a), (b), (d), (e) and (f) are subspaces of $C([a, b])$.

→ $C[a, b]$ is a subspace of $f[a, b]$.

because the sum of two continuous functions is continuous and any scalar multiple of a continuous function is again continuous, we find that addition and scalar multiplication are closed in $C[a, b]$.

This observation not only proves that $C[a, b]$ is a vector space, but also that it is a subspace of $f[a, b]$.

Note: The spaces $C[a, b]$, $C^{(0)}[a, b]$, $C^{(n)}[a, b]$ and $P[a, b]$ are subspaces of $f[a, b]$.

Further, note that

- $P[a, b]$ is a subspace of $C[a, b]$.
- $C^{(0)}[a, b]$ is a subspace of $C[a, b]$.
- $C^{(n)}[a, b]$ is a subspace of $C[a, b]$ for every positive integer n .
- $C^{(n)}[a, b]$ is a subspace of $C^{(m)}[a, b]$ for every $m < n$.
- $P[a, b]$ is a subspace of $C^{(n)}[a, b]$ for every positive integer n .
- Similar results are true for functions defined on (a, b) .

Let V be the vector space of all real sequences $\langle a_n \rangle$.

i) prove that $W = \{ \langle a_n \rangle \in V : \lim_{n \rightarrow \infty} a_n = 0 \}$ is a subspace of V .

ii) prove that $U = \{ \langle a_n \rangle \in V : \sum_{n=1}^{\infty} |a_n| \text{ is finite} \}$ is a subspace of V and is contained in W .

Sol: Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in W$, $\langle b_n \rangle \in W$.

$$\therefore \lim_{n \rightarrow \infty} a_n = 0 \text{ & } \lim_{n \rightarrow \infty} b_n = 0$$

$$\text{Now } \alpha \langle a_n \rangle + \beta \langle b_n \rangle = \langle \alpha a_n + \beta b_n \rangle.$$

$$\text{where } \lim_{n \rightarrow \infty} (\alpha a_n + \beta b_n) = \alpha \lim_{n \rightarrow \infty} a_n + \beta \lim_{n \rightarrow \infty} b_n \\ = \alpha \cdot 0 + \beta \cdot 0 \\ = 0.$$

$$\therefore \alpha \langle a_n \rangle + \beta \langle b_n \rangle \in W.$$

$\therefore W$ is a subspace of V .

iii) Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in U$, $\langle b_n \rangle \in U$.

$$\therefore \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ are finite.}$$

i.e., each one of them is a convergent series

It follows that $\alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ is finite

$$\text{i.e., } \alpha \langle a_n \rangle + \beta \langle b_n \rangle \in U.$$

Hence U is a subspace of V .

Let $\langle a_n \rangle \in U$ be arbitrary.

Then $\sum_{n=1}^{\infty} a_n$ is a convergent series and

$$\text{So } \lim_{n \rightarrow \infty} a_n = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} H(a_n) = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

$$\Rightarrow (a_n) \in W$$

$$\therefore \text{Hence } U \subseteq W.$$

Let V be the vector space of all 2×2 matrices over the field \mathbb{R} of real numbers.

$$\text{Let } (i) \quad W_1 = \{ A \in V \mid A^T = A \}$$

$$(ii) \quad W_2 = \{ A \in V \mid \det A = 0 \}$$

Show that W_1 and W_2 are not subspaces of V .

Show that W is a subspace of V where W consists of all matrices which commute with a given matrix T ; that is, $W = \{ A \in V \mid AT = TA \}$.

Solⁿ: Given that W consists of all matrices which commute with a given matrix

$$\text{i.e., } W = \{ A \in V \mid AT = TA \},$$

$$\text{since } OT = O = TO$$

$$\therefore O \in W.$$

$\therefore W$ is non-empty.

Now suppose $A, B \in W$

$$\text{i.e., } AT = TA \quad \text{and } BT = TB$$

for any scalars $a, b \in F$.

$$\begin{aligned} (aA + bB)T &= (aA)T + (bB)T \\ &= a(AT) + b(BT) \end{aligned}$$

$$\begin{aligned} &= a(TA) + b(TB) \\ &= T(aA) + T(bB) \\ &= T(aA + bB). \end{aligned}$$

(29)

Thus $aA + bB$ commutes with T .

$$\Rightarrow aA + bB \in W.$$

Hence W is a subspace of V .

→ Show that W is a subspace of V , where W consists of the bounded functions.

[A function $f \in V$ is bounded if there exists $M_{f,0} > 0$ such that $|f(x)| \leq M_{f,0}$ for every $x \in \mathbb{R}$.]

Sol: Since $0(x) = 0$ for every $x \in \mathbb{R}$.

Clearly 0 is bounded.

i.e., W is non-empty.

Now let $f, g \in W$ with M_f and M_g bounded

for f and g respectively. i.e., $|f(x)| \leq M_f$ & $|g(x)| \leq M_g$.

Then for any scalars a, b and $\forall x \in \mathbb{R}$

$$\begin{aligned} |(af + bg)(x)| &= |a f(x) + b g(x)| \\ &\leq |a f(x)| + |b g(x)| \\ &= |a| |f(x)| + |b| |g(x)| \\ &\leq |a| M_f + |b| M_g. \end{aligned}$$

$\Rightarrow |a| M_f + |b| M_g$ is a bound for the function $af + bg$.

Thus W is a subspace of V .

→ which of the following sets of vectors

$\alpha = (a_1, a_2, \dots, a_n)$ in \mathbb{R}^n are subspaces of \mathbb{R}^n

- (n > 3)
- (a) all α such that $a_1 \geq 0$.
 - (b) all α such that $a_1 + 3a_2 = a_3$.
 - (c) all α such that $a_2 = a_1$.
 - (d) all α such that $a_1 a_2 = 0$.
 - (e) all α such that a_2 is rational.

✓ Linear Combinations

Defn: Let $V(F)$ be a vector space.

$S = \{a_1, a_2, \dots, a_n\} \subseteq V$ then any vector

$\alpha = a_1 \overline{a_1} + a_2 \overline{a_2} + \dots + a_n \overline{a_n}$ where $a_1, a_2, \dots, a_n \in F$

is called a linear combination of the
vectors a_1, a_2, \dots, a_n .

✓ Linear Span: Let $V(F)$ be a vector space.

$S = \{a_1, a_2, \dots, a_n\} \subseteq V$. Then the
collection of all linear combinations of
a finite number of elements of 'S' is called
linear span of S and is denoted by $L(S)$.

i.e., $L(S) = \{a_1 \overline{a_1} + a_2 \overline{a_2} + \dots + a_n \overline{a_n} / a_1, a_2, \dots, a_n \in F\}$

Smallest subspace containing any subset of $V(F)$.

Def: Let $V(F)$ be a vector space and S be any subset of V (i.e., $S \subseteq V$). If U is a subspace of V containing S and U is contained in every subspace of V containing S then U is called the smallest subspace of V containing S .

→ The smallest subspace of V containing S is also called the subspace of V generated or spanned by S and is denoted by $\{S\}$. i.e., $\{S\} = U$

→ If $\{S\} = V$ then we say that V is spanned by S .

Theorem:

→ If $V(F)$ is a vector space, $S \subseteq V$, then the linear span of S is the smallest subspace of $V(F)$ containing S :

(i.e., $L(S)$ is a subspace of $V(F)$ generated by S i.e., $L(S) = \{S\}$.)

Proof: Given that $V(F)$ is a vector space

and $S \subseteq V$.

Let $S = \{d_1, d_2, \dots, d_n\} \subseteq V$

and $L(S) = \{a_1d_1 + a_2d_2 + \dots + a_nd_n \mid a_1, a_2, \dots, a_n \in F\}$

$\subseteq V$

NOW $\rightarrow a, b \in F ; d, \beta \in L(S)$

Choose $a = a_1d_1 + a_2d_2 + \dots + a_nd_n$

$\beta = b_1d_1 + b_2d_2 + \dots + b_nd_n$

where a 's, b 's $\in F$ and d 's $\in S$

$$\begin{aligned}
 & \Rightarrow a\alpha + b\beta = a(a_1x_1 + a_2x_2 + \dots + a_nx_n) \\
 & \quad + b(b_1x_1 + b_2x_2 + \dots + b_nx_n) \\
 & = (aa_1 + bb_1)x_1 + (aa_2 + bb_2)x_2 + \dots + (aa_n + bb_n)x_n \\
 & \in L(S). \quad (\because aa_1 + bb_1, aa_2 + bb_2, \dots, aa_n + bb_n \in F)
 \end{aligned}$$

$\therefore L(S)$ is a subspace of $V(F)$.

Let $\alpha_i \in S ; i=1, 2, \dots, n$

$$\text{then } \alpha_i = 1\alpha_i$$

= linear combination of α_i

$$\in L(S)$$

$$\therefore \alpha \in L(S)$$

$$\therefore S \subseteq L(S)$$

Now let W be any subspace of $V(F)$ containing S .
 $\therefore S \subseteq W$.

if $\alpha \in L(S)$, then α = the linear combination of
 a finite no. of elts of S .

$$\in W \quad (\because S \subseteq W)$$

\therefore If $\alpha \in L(S)$ then $\alpha \in W$

$$\therefore L(S) \subseteq W.$$

$$\therefore L(S) \subseteq W \subseteq V.$$

$\therefore L(S)$ is the smallest subspace of V containing S .

$$\therefore L(S) = \{S\}.$$

Note: If in any case, we are to prove that
 $L(S) = V$ then we are enough to prove that $V \subseteq L(S)$
 because w.r.t $L(S) \subseteq V$ ($\because L(S)$ is a subspace of V)

In order to prove that $V \subseteq L(S)$

for this each elt of V can be expressed
 as linear combination of a finite no. of elts of S .

\therefore Each elt of V will also be the elt. of $L(S)$.

i.e., let $\alpha \in V \Rightarrow \alpha = \text{the l.c. of finite no. of elts of } S$. (27)

$\therefore \alpha \in L(S)$

$\therefore V \subseteq L(S)$.

$\therefore V \subseteq L(S) \text{ and } L(S) \subseteq V$

$\Rightarrow L(S) = V$.

Ex

The subset $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $V_3(F)$.
(i.e., $S \subseteq V_3(F)$) generates or spans the entire vector space $V_3(F)$ i.e., $L(S) = V_3$.

Sol

N.K.T: $L(S) \subseteq V_3 \quad \text{--- (1)}$

Let $\alpha = (a, b, c) \in V_3$ then

$$\alpha = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

$\in L(S)$

$\therefore \alpha \in L(S)$

$\therefore V_3 \subseteq L(S) \quad \text{--- (2)}$

\therefore from (1) & (2) we have $L(S) = V_3$.

Defn

Linear sum of two subspaces

Let w_1 & w_2 be any two subspaces of $V(F)$

then the set $\{\alpha_i + \beta_j / \alpha_i \in w_1, \beta_j \in w_2\} \subseteq V$

is called linear sum of w_1 & w_2 and is denoted

by $w_1 + w_2$.

$$\text{i.e., } w_1 + w_2 = \{\alpha_i + \beta_j / \alpha_i \in w_1, \beta_j \in w_2\} \subseteq V$$

Theorem: Let w_1 and w_2 be two subspaces of $V(F)$,

then the linear sum $w_1 + w_2$ is a subspace of $V(F)$

and $w_1 + w_2 = L(w_1 \cup w_2)$.

i.e., $w_1 + w_2 = \{w_1 \cup w_2\}$.

Imp.

proof: Given that $V(F)$ is a vector space. w_1 & w_2 are two subspaces of $V(F)$.

$$w_1 + w_2 = \{d_i + d_j \mid d_i \in w_1, d_j \in w_2\} \subseteq V.$$

Let $a, b \in F$; $d, \beta \in w_1 + w_2$ Choose $\alpha = d_i + d_j$; $d_i \in w_1, d_j \in w_2$

$$\beta = d_k + d_l; \quad \begin{matrix} d_k \in w_1 \\ d_l \in w_2 \end{matrix}$$

$$\Rightarrow ad + b\beta = a(d_i + d_j) + b(d_k + d_l)$$

$$= (ad_i + bd_k) + (ad_j + bd_l)$$

$$\in w_1 + w_2$$

Since w_1 is a subspace

$$ad_i + bd_k \in w_1$$

and w_2 is a subspace

$$ad_j + bd_l \in w_2$$

∴ $w_1 + w_2$ is a subspace of $V(F)$.

$$\text{Now } 0 \in w_1, x \in w_2 \Rightarrow 0+x \in w_1 + w_2$$

$$\Rightarrow x \in w_1 + w_2$$

$$\therefore w_2 \subseteq w_1 + w_2 \quad \text{--- (1)}$$

$$y \in w_1, 0 \in w_2 \Rightarrow y+0 \in w_1 + w_2$$

$$\Rightarrow -y \in w_1 + w_2$$

$$\therefore w_1 \subseteq w_1 + w_2 \quad \text{--- (2)}$$

∴ from (1) & (2) we have

$$w_1 \cup w_2 \subseteq w_1 + w_2 \subseteq V$$

W.K.T Linear Span of $w_1 \cup w_2$ (i.e., $L(w_1 \cup w_2)$)is the smallest subspace of $V(F)$ containing $w_1 \cup w_2$

$$\therefore L(w_1 \cup w_2) \subseteq w_1 + w_2 \quad \text{--- (3)}$$

$$\text{Let } x \in w_1 + w_2 \Rightarrow x = d_i + d_j \in w_1 + w_2$$

$$\text{Now } d_i \in w_1, d_j \in w_2 \Rightarrow d_i, d_j \in w_1 \cup w_2$$

$$\text{Now } x \in w_1 + w_2$$

$$\begin{aligned} \Rightarrow x &= d_i + d_j \\ &= 1d_i + 1d_j \end{aligned}$$

= l.c. of finite no. of els of $w_1 \cup w_2$. 25
 $\in L(w_1 \cup w_2)$

$$\therefore w_1 + w_2 \subseteq L(w_1 \cup w_2) \quad \text{--- (4)}$$

from (3) & (4)

$$\text{we have } L(w_1 \cup w_2) = w_1 + w_2.$$

→ If S, T are subsets of $V(F)$ then

$$(i) S \subseteq T \Rightarrow L(S) \subseteq L(T).$$

$$(ii) L(S \cup T) = L(S) + L(T)$$

$$(iii) L(L(S)) = L(S).$$

Proof: Let $S = \{d_1, d_2, \dots, d_n\} \subseteq V$ then

any vector $\alpha = a_1 d_1 + a_2 d_2 + \dots + a_n d_n \in L(S)$

Since $S \subseteq T$

$$\Rightarrow S = \{d_1, d_2, \dots, d_n\} \subseteq T.$$

$$\therefore \alpha \in L(T).$$

∴ If $\alpha \in L(S)$ then $\alpha \in L(T)$.

$$\therefore L(S) \subseteq L(T).$$

(ii) let $S = \{d_1, d_2, \dots, d_n\} \subseteq V$

$$\text{and } T = \{\beta_1, \beta_2, \dots, \beta_p\} \subseteq V$$

$$\text{then } S \cup T = \{d_1, d_2, \dots, d_n, \beta_1, \beta_2, \dots, \beta_p\} \subseteq V.$$

let $\alpha \in L(S \cup T)$ then

$$\alpha = a_1 d_1 + a_2 d_2 + \dots + a_n d_n + b_1 \beta_1 + b_2 \beta_2 + \dots + b_p \beta_p$$

$$\text{since } a_1 d_1 + a_2 d_2 + \dots + a_n d_n \in L(S)$$

$$\text{and } b_1 \beta_1 + b_2 \beta_2 + \dots + b_p \beta_p \in L(T)$$

$$\therefore \alpha \in L(S) + L(T)$$

$$\therefore L(S \cup T) \subseteq L(S) + L(T) \quad \text{--- (i)}$$

Let $\gamma \in L(S) + L(T)$ then $\gamma = \beta + \delta$.

where $\beta \in L(S)$ & $\delta \in L(T)$.

NOW $\beta = L.C.$ of finite no. of elts of S and
 $\delta = L.C.$ of finite no. of elts of T .
 $\therefore \beta + \delta = L.C.$ of finite no. of elts of $S \cup T$.
 $\therefore \gamma = \beta + \delta \in L(S \cup T)$

\therefore if $\gamma \in L(S) + L(T)$ then

$\gamma \in L(S \cup T)$

$$\therefore L(S) + L(T) \subseteq L(S \cup T) \quad \text{--- (2)}$$

\therefore from (1) & (2) we have

$$L(S \cup T) = L(S) + L(T).$$

(iii) $L(L(S))$ is the smallest subspace of V containing $L(S)$.

But $L(S)$ is a subspace of V .

\therefore the smallest subspace of V containing $L(S)$ is $L(S)$ itself.

$$\therefore L(S) \subseteq L(L(S)) \subseteq L(S) \subseteq V$$

$$\therefore L(L(S)) = L(S)$$

$$\boxed{L(S) \subseteq L(L(S)) \subseteq L(S) \subseteq V}$$

Defn: Linear dependence of vectors:

(26)

$V(F)$ is a vector space and $S = \{d_1, d_2, \dots, d_n\} \subseteq V$

If there exists at least one non-zero scalar $a_1, a_2, \dots, a_n \in F$

such that $a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$

then S is called linear dependent.

Linear Independence of vectors:

$\rightarrow V(F)$ is a vector space and $S = \{d_1, d_2, \dots, d_n\} \subseteq V$

If $a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$, $a_i \in F$, $1 \leq i \leq n$.

$$\Rightarrow a_1, a_2, \dots, a_n = 0$$

i.e., $a_i = 0$ for each $1 \leq i \leq n$

$\therefore V_n(F) = \{(a_1, a_2, \dots, a_n) / a_1, a_2, \dots, a_n \in F\}$
is a vector space.

$$S = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\} \subseteq V_n(F).$$

Now $a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$

$$\Rightarrow a_1(1, 0, 0, \dots, 0) + a_2(0, 1, 0, \dots, 0)$$

$$+ \dots + a_n(0, 0, \dots, 0, 1) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, 0, 0, \dots, 0) + (0, a_2, 0, \dots, 0) + \dots +$$

$$(0, 0, \dots, a_n) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, a_2, \dots, a_n) = (0, 0, \dots, 0)$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0$$

$\therefore S$ is L.I.P.

Ex S.T $S = \{(1, 2, 1), (3, 1, 5), (3, -4, 7)\} \subseteq \mathbb{R}^3$

L.D.

Soln & $a, b, c \in \mathbb{R}$

$$a(1, 2, 1) + b(3, 1, 5) + c(3, -4, 7) = (0, 0, 0)$$

$$\Rightarrow (a, 2a, a) + (3b, b, 5b) + (3c, -4c, 7c) = (0, 0, 0)$$

$$\Rightarrow (a+3b+3c, 2a+b-4c, a+5b+7c) = (0, 0, 0)$$

$$\Rightarrow a+3b+3c = 0 \quad (1)$$

$$2a+b-4c = 0 \quad (2)$$

$$a+5b+7c = 0 \quad (3)$$

Solving these equations, we get

$$(1) - (3) \Rightarrow -2b-4c = 0$$

$$\Rightarrow b = -2c \quad (4)$$

$$(1) \times 2 \Rightarrow a-6b+3c = 0$$

$$\Rightarrow a-3c = 0$$

$$\Rightarrow a = 3c \quad (5)$$

Substituting (4) & (5) in (2) we get

$$6c-2c-4c = 0$$

$$\Rightarrow 0 = 0$$

If non-zero values for a, b, c to satisfy the equations (1) & (3)

The given set is L.D.

IMS

INSTITUTE FOR IAS/IFS EXAMINATION
NEW DELHI-110009
Mob: 09999197625

Theorem

→ If two vectors are linearly dependent, then one of them is a scalar multiple of the other.

Soln Let α, β be two linearly dependent vectors of the vector space $V(F)$.

∴ At least one of the scalar $a, b \in F$ is non-zero.

$$\text{Let } a\alpha + b\beta = 0$$

If $a \neq 0$ then $a\alpha = -b\beta$.

$$\Rightarrow \alpha = \left(-\frac{b}{a}\right)\beta$$

∴ α is scalar multiple of β .

If $b \neq 0$ then $b\beta = -a\alpha$

$$\Rightarrow \beta = \left(-\frac{a}{b}\right)\alpha$$

∴ β is the scalar multiple of α .

∴ One of the vectors α and β is scalar multiple of the other.

Theorem

A set consisting of single non-zero vector is always L.I.

Proof

Let $V(F)$ be a vector space.

$$S = \{\alpha\} \subseteq V ; \alpha \neq 0$$

If $a \in F$ then $a\alpha = 0$

$$\Rightarrow a = 0 \quad (\because \alpha \neq 0)$$

∴ S is L.I.

Theorem: If the set $S = \{d_1, d_2, \dots, d_n\}$ consisting of vectors of $V(F)$ is L.I. then none of the vectors d_1, d_2, \dots, d_n can be zero vector.

Proof: Given that

$$S = \{d_1, d_2, \dots, d_n\} \subseteq V \text{ is L.I.}$$

$$\therefore a_1 d_1 + a_2 d_2 + \dots + a_n d_n = 0, a_1, a_2, \dots, a_n \in F$$

$$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$$

If possible let $a_k = 0 ; 1 \leq k \leq n$.

$$\text{then } a_1 + a_2 + \dots + a_k a_k + a_{k+1} + \dots + a_n = 0$$

Since $a_k \neq 0$

for any $a_k \neq 0$ in F.

$\therefore S$ is LD.

which is contradiction to the hypothesis

that S is L.B.

Our assumption that $a_k = 0 ; 1 \leq k \leq n$ is wrong.

\therefore None of the vectors a_1, a_2, \dots, a_n can be zero vector

Theorem A set of vectors which containing the zero vector

is LD.

Proof Let $V(F)$ be the vector space.

$$S = \{a_1, a_2, \dots, a_n\} \subseteq V$$

and $a_k = 0 ; 1 \leq k \leq n$.

Consider linear combination

$$a_1 a_1 + a_2 a_2 + \dots + a_k a_k + a_{k+1} a_{k+1} + \dots + a_n a_n = 0$$

$$\text{Taking } a_1 = a_2 = \dots = a_{k+1} = \dots = a_n = 0 \\ \text{and } a_k \neq 0$$

$$a_1 a_1 + a_2 a_2 + \dots + a_k a_k + a_{k+1} a_{k+1} + \dots + a_n a_n = 0$$

$$\Rightarrow a_k a_k = 0$$

$$\Rightarrow a_k \neq 0 (\because a_k = 0)$$

$\therefore S$ is L.D.

Theorem A subset of a L.I set is L.I.

Proof $V(F)$ is a vector space

$$S = \{a_1, a_2, \dots, a_n\} \subseteq V \text{ is L.I}$$

Now let $S' = \{a_1, a_2, \dots, a_k\} \subseteq S \quad (1 \leq k \leq n)$

then $a_1 a_1 + a_2 a_2 + \dots + a_k a_k = 0 ; a_1, a_2, \dots, a_k \in F$

$$\Rightarrow a_1x_1 + a_2x_2 + \dots + a_kx_k + 0x_{k+1} + 0x_{k+2} + \dots + 0x_m = 0$$

$$\Rightarrow a_1 = a_2 = \dots = a_k = 0 \quad (\because S \text{ is LI})$$

$\therefore S^1 \text{ is LF.}$

(28)

Theorem: A superset of a linear dependent set of vectors is LD.

proof: Let $V(F)$ be a vector space.

and $S = \{d_1, d_2, \dots, d_n\} \subseteq V$ is LD.

Now let $S^1 = \{d_1, d_2, \dots, d_n, B_1, B_2, \dots, B_k\} \supseteq S$.

Since S is LD

$\therefore \exists$ at least one of the scalar $a_1, a_2, \dots, a_n \in F$
is not zero. s.t

$$a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$$

$$\Rightarrow a_1d_1 + a_2d_2 + \dots + a_nd_n + 0B_1 + 0B_2 + \dots + 0B_k = 0$$

Since in the above relation the scalar coefficients
not all zero.

$\therefore S^1$ is LD.

Theorem: Let $V(F)$ be vector space. and $S = \{d_1, d_2, \dots, d_n\} \subseteq V$
(contains non-zero vectors) if S is LD then one
of the vectors of S say d_i ($1 \leq i \leq n$) is a linear
combination of its preceding vectors.

proof: $V(F)$ is a vector space

$$S = \{d_1, d_2, \dots, d_n\} \subseteq V$$

and S contains non-zero vectors.

Since S is LD.

$\therefore \exists$ at least one scalar $a_1, a_2, \dots, a_n \in F$

$$\text{is non-zero s.t } a_1d_1 + a_2d_2 + \dots + a_nd_n = 0 \quad (1)$$

Suppose that the maximum value of k for which $a_k \neq 0$ is i .

i.e., $a_i \neq 0$ and $a_{i+1} = a_{i+2} = \dots = a_n = 0$

If this maximum value is one then $a_1 \neq 0$

$$\text{and } a_1 = a_2 = \dots = a_n = 0$$

$$① \equiv a_1 d_1 + 0 d_2 + \dots + 0 d_n = 0$$

$$\Rightarrow a_1 d_1 = 0$$

$$\Rightarrow a_1 = 0 \quad (\because a_1 \neq 0)$$

which is contradiction to the hypothesis that S contains non-zero vectors.

$$\therefore i \neq 1$$

$$\therefore 1 < i \leq n$$

$$② \equiv a_1 d_1 + a_2 d_2 + a_3 d_3 + \dots + a_{i-1} d_{i-1} + a_i d_i + \\ 0 d_{i+1} + 0 d_{i+2} + \dots + 0 d_n = 0$$

$$\Rightarrow a_i d_i = -a_1 d_1 - a_2 d_2 - \dots - a_{i-1} d_{i-1}$$

$$\Rightarrow a_i = \left(-\frac{a_1}{a_i} \right) d_1 + \left(-\frac{a_2}{a_i} \right) d_2 + \dots + \left(-\frac{a_{i-1}}{a_i} \right) d_{i-1}$$

$\therefore d_i \quad (1 < i \leq n)$ is a linear combination of its preceding vectors.

Let $V(F)$ be the vector space. $S = \{d_1, d_2, \dots, d_n\} \subseteq V$ (Contains non-zero vectors)

If one of the vectors of ' S ' say $d_i \quad (1 < i \leq n)$

is a linear combination of its preceding vectors

then S is LD.

Proof: Given that

$V(F)$ is a vector space.

$$S = \{d_1, d_2, \dots, d_n\} \subseteq V$$

and one of the vectors of S say α_i ($1 \leq i \leq n$)
is a linear combination of its preceding vectors. 20

$$\therefore d_i = a_1 d_1 + a_2 d_2 + \dots + a_{i-1} d_{i-1}$$

$$\Rightarrow a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_{i-1} \alpha_{i-1} + (-1) \alpha_i = 0$$

$$\Rightarrow a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_{i-1} \alpha_{i-1} + (-1) \alpha_i + 0 \alpha_{i+1} + 0 \alpha_{i+2} + \dots + 0 \alpha_n = 0$$

∴ Coefficient of $\alpha_i = -1 \neq 0$

∴ S is LD.

Theorem Let $V(F)$ be a vector space. $S = \{d_1, d_2, \dots, d_n\} \subseteq V$

If one of the vectors of S is a linear combination of all the remaining vectors then S is LD.

Proof: $V(F)$ is a vector space

$$S = \{d_1, d_2, \dots, d_n\} \subseteq V$$

and one of the vectors of S is a linear combination of all the remaining vectors

$$\therefore d_i = a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_{i-1} \alpha_{i-1} + a_i \alpha_i + a_{i+1} \alpha_{i+1} + \dots + a_n \alpha_n$$

$$\Rightarrow a_1 \alpha_1 + a_2 \alpha_2 + \dots + (-1) \alpha_i + a_{i+1} \alpha_{i+1} + a_{i+2} \alpha_{i+2} + \dots + a_n \alpha_n = 0$$

∴ The coefficient of $\alpha_i \neq 0$.

∴ S is LD.

Theorem If in a vector space $V(F)$, a vector β is a linear combination of the set of vectors d_1, d_2, \dots, d_n , then the set of vectors $\beta, d_1, d_2, \dots, d_n$ is LD.

sol: Since β is a linear combination of d_1, d_2, \dots, d_n

\therefore Scalars $a_1, a_2, \dots, a_n \in F$ s.t.

$$\beta = a_1d_1 + a_2d_2 + \dots + a_nd_n$$

$$\Rightarrow a_1d_1 + a_2d_2 + \dots + a_nd_n + (-1)\beta = 0$$

\therefore In the above relation the coefficient of $\beta = -1 \neq 0$

In the above relation not all the scalar coefficients are zero.

\therefore The set of vectors $d_1, d_2, \dots, d_n, \beta$ is LD.

\rightarrow write the vector $\alpha = (1, -2, 5)$ as a linear combination of the elements of the set $\{(1, 1, 1), (1, 2, 3), (2, -1, 1)\} \subseteq \mathbb{R}^3$.

$$\text{Soln: } \alpha = (1, -2, 5) = a(1, 1, 1) + b(1, 2, 3) + c(2, -1, 1)$$

$$= (a+b+2c, a+2b+c, a+3b+c)$$

$$\Rightarrow a+b+2c = 1 \quad \text{--- (1)}$$

$$a+2b+c = -2 \quad \text{--- (2)}$$

$$a+3b+c = 5 \quad \text{--- (3)}$$

$$(1) - (2) = -b+3c = 3 \quad \text{--- (4)}$$

$$(2) - (3) = -b-2c = -7 \quad \text{--- (5)}$$

$$(4) - (5) = 5c = 10$$

$$\Rightarrow [c = 2]$$

$$(4) \equiv -b = -3$$

$$\Rightarrow b = 3$$

$$(1) \equiv a+3+4 = 1$$

$$\Rightarrow [a = -6]$$

$$\therefore (1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) + 2(2, -1, 1)$$

\rightarrow Express $\alpha = (2, -5, 3)$ in \mathbb{R}^3 as a linear combination of the vectors $e_1 = (1, -3, 2)$, $e_2 = (2, -4, -1)$ and $e_3 = (1, -5, 1)$

\rightarrow Express the polynomial $\alpha = t^2 + 4t - 3$ as a linear combination of the polynomials $e_1 = t^2 - 2t + 5$, $e_2 = 2t^2 - 3t$ and $e_3 = t + 3$.

Solⁿ $\alpha = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$; where a, b, c are unknown scalars. (3D)

$$\Rightarrow t^2 + 4t - 3 = a(t^2 - 2t + 5) + b(2t^2 - 3t) + c(t + 3)$$

$$= (a+2b)t^2 + (-2a-3b+c)t + (5a+3c).$$

$$\Rightarrow a+2b=1 \quad \text{--- (1)}$$

$$-2a-3b+c=4 \quad \text{--- (2)}$$

$$5a+3c=-3 \quad \text{--- (3)}$$

$$2 \times (2) + 3 \times (1) \equiv -a+2c=11 \quad \text{--- (4)}$$

$$(3) + 5 \times (4) \equiv 13c=52 \quad (4) \equiv -a=3$$

$$\Rightarrow \boxed{c=4}; \quad \Rightarrow \boxed{a=-3}$$

$$(1) \equiv -3+2b=1$$

$$\Rightarrow \boxed{b=2}$$

(1) \equiv $\alpha = -3\mathbf{e}_1 + 2\mathbf{e}_2 + 4\mathbf{e}_3.$

\Rightarrow write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination

of the matrices -

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$

Solⁿ: $E = xA + yB + zC$ where x, y, z are unknown scalars. (1)

$$\begin{aligned} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} &= x \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & y \end{pmatrix} + \begin{pmatrix} 0 & 2x \\ 0 & -z \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} x & x+2x \\ x+y & y-z \end{pmatrix}$$

$$\boxed{x=3}$$

$$x+2z=1 \quad ; \quad x+y=1$$

$$\Rightarrow \boxed{z=-1}$$

$$\Rightarrow \boxed{y=-2}$$

(1) \equiv $E = 3A - 2B + (-1)C$

→ Determine whether α & β are L.D.
 where (a) $\alpha = (3, 4)$, $\beta = (1, -3)$
 (b) $\alpha = (2, -3)$, $\beta = (6, -9)$

Sol^b (a) Since no vector is a scalar multiple of the other.
 $\therefore \alpha$ & β are not L.D.

b). Since β is a scalar multiple of α .

$$\text{i.e., } (6, -9) = 3(2, -3)$$

$$\text{i.e., } \beta = 3\alpha$$

$\therefore \alpha$ & β are L.D. vectors

→ Determine whether α & β are L.D.

where (a) $\alpha = (4, 3, -2)$, $\beta = (2, -6, 7)$

(b) $\alpha = (-4, 6, -2)$, $\beta = (2, -3, 1)$

Sol^b a) neither is a scalar multiple of the other.
 $\therefore \alpha$ and β are not L.D.

- b) $\alpha = (-2)\beta$.

$\therefore \alpha$ and β are L.D.

→ S.T $S = \{(1, 2, 4), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

is a L.D subset of $V_3(\mathbb{R})$.

Since one of the vector of S is a linear combination
of all the remaining vectors.

$$\text{i.e., } (1, 2, 4) = 1(1, 0, 0) + 2(0, 1, 0) + 4(0, 0, 1)$$

$\therefore S$ is L.D.

Let \mathcal{W} be the set of all $(x_1, x_2, x_3, x_4, x_5)$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

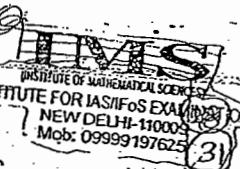
$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which

spans \mathcal{W} .

Ans

Echelon form of a matrix



A matrix 'A' is said to be in echelon form if the number of zeroes preceding the non-zero elt of a row increases row by row and the elts of last row or rows may be all zeros.

Ex: $\begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are all echelon matrices.

Note:

(1) The rank of matrix in echelon form is equal to the no. of non-zero rows of the matrix.

Ex: $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the matrix A in echelon form

\therefore The no. of non-zero rows in echelon form = 2

$$\therefore r(A) = 2$$

Note [2]: Let $\left. \begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array} \right\} \quad (1)$

given system of 3 non-homogeneous linear equations in 3 unknowns x, y, z.

Now write the single matrix equation

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3 \times 1}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3 \times 1}$

and the matrix $[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$

is called the augmented matrix of the given system of equations.

* working rule for finding the solutions of the equation $AX=B$:-

→ Now the augmented matrix $[A|B]$ reduce to an echelon form by applying only elementary row operations.

→ This echelon form will enable us to know the ranks of the augmented matrix $[A|B]$ and the coefficient matrix A .

Then the following cases arise:

(i) If $r(A) = r(A|B) = \text{no. of unknowns}$.

then the given system (I) is consistent and has unique solution.

(ii) If $r(A) = r(A|B) < \text{no. of unknowns}$.

then the given system (I) is consistent and has infinite solutions.

(iii) If $r(A) \neq r(A|B)$ then the given system is inconsistent and has no solution.

Note [3] Let $\left. \begin{array}{l} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{array} \right\}$ (1)

be the given system of 3 homogeneous linear equations in 3 unknowns x, y, z .

Now write the single matrix equation

$$AX=0$$

where coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3 \times 1}; \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3 \times 1}$$

Working rule for finding the solutions of the equation $Ax=0$.

(29)

→ Reduce the coefficient matrix A to echelon form by applying elementary row operations only.

This echelon form will help us to know the rank of the matrix A.

→ If $\rho(A) = \text{no. of unknowns}$,

then the system (ii) possesses a zero solution
(trivial solution),
i.e., $x=0, y=0, z=0$.

→ If $\rho(A) < \text{no. of unknowns}$,

then there will be a non-zero solution (non-trivial solution).

problem → Determine whether or not $\alpha = (3, 9, -4, -2)$ in \mathbb{R}^4 is a linear combination of $\alpha_1 = (1, -2, 0, 3)$, $\alpha_2 = (2, 3, 0, -1)$ and $\alpha_3 = (2, -1, 2, 1)$.

Soln:- Let $x, y, z \in \mathbb{R}$.

$$\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3;$$

$$\Rightarrow (3, 9, -4, -2) = x(1, -2, 0, 3) + y(2, 3, 0, -1) + z(2, -1, 2, 1)$$

$$\Rightarrow x + 2y + 2z = 3$$

$$-2x + 3y - z = 9$$

$$2x = -4$$

$$3x - 9 + z = -2$$

Now write the single matrix equation $Ax=B$

i.e.,
$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 3 & -1 \\ 0 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -4 \\ -2 \end{bmatrix}$$

Augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -2 & 3 & -1 & 9 \\ 0 & 0 & 2 & -4 \\ 3 & -1 & 1 & -2 \end{array} \right]$

$$R_2 \rightarrow R_2 + 2R_1$$

$$R_4 \rightarrow R_4 - 3R_1$$

$$\left[\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & -5 & -11 \end{array} \right]$$

$$R_4 \rightarrow R_4 + R_2$$

$$\left[\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -2 & 4 \end{array} \right]$$

$$R_4 \rightarrow R_4 + R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\therefore \text{L}(A) = \text{L}(A|B) = 3 = \text{no. of unknowns } x, y, z.$

\therefore The given system is consistent and has unique soln.

for solving the unknowns x, y, z .

we write the echelon matrix equation

$$\left[\begin{array}{ccc} 1 & 2 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 3 \\ 15 \\ -4 \\ 0 \end{array} \right]$$

$$\Rightarrow x + 2y + 2z = 3$$

$$7y + 3z = 15$$

$$2z = -4 \Rightarrow [z = -2]; [x = 1] \text{ and } [y = 3]$$

$$\therefore d = kx + 3y + 2z$$

$\therefore d$ is a linear combination of d_1, d_2, d_3 .

Note.

1. If the system of linear equations are consistent then it has one soln and the vector d is a linear combination of d_i ($1 \leq i \leq n$).

2. If the given system of linear equations are not consistent then it has no solution and the vector d is not a linear combination of d_i ($1 \leq i \leq n$)

→ P.T the set $\{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\} \subseteq \mathbb{R}^3$ is LD. (3B)

Sol: Let $a, b, c \in \mathbb{R}$ then

$$a(-1, 2, 1) + b(3, 0, -1) + c(-5, 4, 3) = (0, 0, 0).$$

$$\Rightarrow -a + 3b - 5c = 0 \quad \text{(1)}$$

$$2a + 4c = 0 \quad \text{(2)}$$

$$a - b + 3c = 0 \quad \text{(3)}.$$

Solving the above equations, we get

$$(1) + (3) \Rightarrow 2b - 2c = 0 \Rightarrow \boxed{b=c}$$

$$(2) \Rightarrow \boxed{a = -2c}$$

$$(3) \Rightarrow -2c - c + 3c = 0 \Rightarrow 0 = 0$$

∴ There are non-zero values for a, b, c to satisfy the equations (1), (2), (3).

∴ The given set is LD.

→ Determine whether or not the vectors

$(1, -2, 1), (2, 1, -1), (7, -4, 1)$ are LD.

Sol: If $a, b, c \in \mathbb{F}$, then

$$a(1, -2, 1) + b(2, 1, -1) + c(7, -4, 1) = (0, 0, 0).$$

$$\Rightarrow a + 2b + 7c = 0$$

$$2a + b - 4c = 0 \quad \left\{ \begin{array}{l} \\ \end{array} \right. \quad \text{(1)}$$

$$a - b + c = 0$$

$$\text{Coefficient matrix } A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{Now } |A| = 1(-3) - 2(2) + 7(1)$$

$$= -3 - 4 + 7 = 0$$

∴ $\text{r}(A) < \text{no. of unknowns } a, b, c$.

∴ The system of equations possess a non-zero solution.

∴ The given vectors are LD.

Note: (i) Consider the system of three linear equations

in three unknown variables.

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 0 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned} \quad \text{(1)}$$

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

→ If $|A| \neq 0$,

then the system ① possesses a trivial solution (zero soln)
i.e., $x=0, y=0, z=0$.

→ If $|A|=0$, the system ① possesses a non-trivial
solution (non-zero soln).

→ Determine whether $(2, -3, 7), (0, 0, 0), (3, -1, -4)$ are LD.

Method ①

Soln: Let $a, b, c \in \mathbb{R}$. Then

$$a(2, -3, 7) + b(0, 0, 0) + c(3, -1, -4) = (0, 0, 0)$$

$$\Rightarrow 2a + 0b + 3c = 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}$$

$$-3a + 0b - c = 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \text{--- ①}$$

$$7a + 0b - 4c = 0$$

The coefficient matrix $A = \begin{bmatrix} 2 & 0 & 3 \\ -3 & 0 & -1 \\ 7 & 0 & -4 \end{bmatrix}$

and $|A| = 0$

∴ The system of equations possess
a non-zero solution.

∴ The given vectors are LD.

Method ②

form the matrix 'A' whose rows are the given

vectors

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$$\Rightarrow |A| = 0$$

∴ The given vectors are LD.

Method ③

form the matrix 'A' whose rows are the given
vectors and reduce to echelon form

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc} 2 & -3 & 7 \\ 3 & -1 & -4 \\ 0 & 0 & 0 \end{array} \right]$$

$$R_2 \rightarrow 2R_2 - 3R_1 \quad \left[\begin{array}{ccc} 2 & -3 & 7 \\ 0 & 7 & -29 \\ 0 & 0 & 0 \end{array} \right]$$

which is an echelon form.

Since the echelon form has a zero row.

\therefore The given vectors are LD.

\rightarrow In $V_3(\mathbb{R})$, where \mathbb{R} is the field of real numbers, examine each of the following sets of vectors for linear dependence.

(i) $\{(2, 1, 2), (8, 4, 8)\}$ (ii) $\{(1, 2, 1), (0, 3, 1), (-1, 0, 1)\}$

(iii) $\{(2, 3, 5), (4, 9, 25)\}$.

\rightarrow P.T. the set $\{(1, 2, 1), (3, 1, 5), (2, -4, 7)\} \subseteq \mathbb{R}^3$ is LI.

\rightarrow Examine the vectors $(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0), (2, 1, 1, 1)$ are LI in \mathbb{R}^4 .

Sol: Now form the matrix 'A' whose rows are given vectors and reduce to echelon form.

$$A = \left[\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{array} \right]$$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 & \left[\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{array} \right] \\ R_3 &\rightarrow R_3 - R_1 & \\ R_4 &\rightarrow R_4 - 2R_1 & \left[\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{array} \right] \end{aligned}$$

$$\begin{aligned} R_2 &\rightarrow -\frac{1}{3}R_2 & \left[\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \end{array} \right] \\ R_3 &\rightarrow R_3 + R_2 & \end{aligned}$$

$$\begin{aligned} R_3 &\rightarrow R_3 + R_2 & \left[\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ R_4 &\rightarrow R_4 + R_2 & \end{aligned}$$

Clearly which is in echelon form.

Since this echelon form has two zero rows.

\therefore The given vectors are LD.

→ Determine, whether $(1, 2, -3), (1, -3, 2), (2, -1, 5)$ are LD.

Solⁿ: Now form the matrix A whose rows are given vectors and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 2 & -1 & 5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1 \quad \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & -5 & 11 \end{bmatrix}$$

Clearly which is in echelon form.

Since the echelon form has no zero rows.

∴ The given vectors are LI.

Q9: Let V be the vector space of functions from $\mathbb{R} \rightarrow \mathbb{R}$.

Show that $f, g, h \in V$ are LI.

where $f(t) = e^{2t}, g(t) = t^2, h(t) = t$.

Solⁿ: Let $a, b, c \in \mathbb{R}$ then $af + bg + ch = 0$

Now for every value of t,

we have

$$a f(t) + b g(t) + c h(t) = 0 \\ \Rightarrow a e^{2t} + b t^2 + c t = 0$$

if. $t=0$, then $a \cdot 1 + b(0) + c(0) = 0$

$$\Rightarrow a = 0 \quad \text{--- (1)}$$

if $t=1$ then $a e^{2(1)} + b(1)^2 + c(1) = 0$

$$\Rightarrow a e^2 + b + c = 0 \quad \text{--- (2)}$$

if $t=2$ then $a e^{2(2)} + b(2)^2 + c(2) = 0$

$$\Rightarrow a e^4 + 4b + 2c = 0 \quad \text{--- (3)}$$

$$(2) \equiv b + c = 0 \quad \text{--- (4)}$$

$$(3) \equiv 4b + 2c = 0 \quad (\because a = 0) \quad \text{--- (5)}$$

$$2 \times (4) - (5) \equiv -2b = 0$$

$$\Rightarrow b = 0 \quad \text{--- (6)}$$

$$\therefore (4) \equiv c = 0$$

∴ f, g, h are LI.

\Rightarrow S.T the functions $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ are LI.

Sol: Let $a, b, c \in \mathbb{R}$

$$\text{then } af + bg + ch = 0$$

for every value of t

$$\text{we have } af(t) + bg(t) + ch(t) = 0$$

$$\Rightarrow a \sin t + b \cos t + ct = 0 \quad \text{(1)}$$

$$\text{if } t=0 \text{ then } a(0) + b(1) + c(0) = 0$$

$$\Rightarrow [b=0] \quad (0)$$

$$\text{if } t=\pi \text{ then } a(\pi) + b(0) + c(\pi)b = 0$$

$$\Rightarrow a + c(\pi) = 0 \quad (2)$$

$$\text{if } t=\pi \text{ then } a(0) + b(-1) + c\pi = 0$$

$$\Rightarrow [-b + c\pi = 0] \quad (3)$$

$$\text{from (1) & (3)} \quad 0 + c\pi = 0$$

$$\Rightarrow [c=0] \quad (4)$$

$$\text{from (2) & (4)} \quad a + 0 = 0$$

$$\Rightarrow [a=0]$$

$\therefore f(t), g(t), h(t)$ are LI.

2005 find the values of k for which the vectors $(1, 1, 1, 1)$, $(1, 3, -2k)$, $(2, 2k-2, -k-2, 3k-1)$ and $(3, k+2, -3, 2k+1)$ are LI in \mathbb{R}^4 .

Sol: form the matrix A whose rows are given vectors.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{bmatrix}$$

Since the given vectors are LI \therefore

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{vmatrix} \neq 0$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$R_4 \rightarrow R_4 - 3R_1$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & k-1 \\ 0 & 2k-4 & -k-4 & 3k-3 \\ k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} 2 & -3 & k-1 \\ 2k-4 & -k-4 & 3k-3 \\ k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

proceeding in this way

second method:

diff (1) three times w.r.t t

(1)

(2)

(3)

→ If d_1, d_2 are vectors of $V(F)$ and $a, b \in F$.
 S.t. the set $\{d_1, d_2, ad_1 + bd_2\}$ is LD.

Soln Let $S = \{d_1, d_2, ad_1 + bd_2\} \subseteq V(F)$.

Since one of the vector of S is a l.c. of the remaining vectors.

$$\therefore ad_1 + bd_2 = ad_1 + bd_2$$

S is LD

→ Let d_1, d_2, d_3 be vectors of $V(F)$, $a, b \in F$.

S.t. the set $\{d_1, d_2, d_3\}$ is LD if the set $\{d_1 + ad_2 + bd_3, d_2, d_3\}$ is LD.

Soln Since the set $\{d_1 + ad_2 + bd_3, d_2, d_3\} \subseteq V$ is LD.

∴ Atleast one non-zero scalar $x, y, z \in F$ s.t.

$$x(d_1 + ad_2 + bd_3) + y(d_2) + z(d_3) = 0$$

$$\Rightarrow xd_1 + (ax + y)d_2 + (bx + z)d_3 = 0$$

If $x \neq 0$ then the set $\{d_1, d_2, d_3\}$ is LD.

If $x=0$ then atleast one of y & z is not zero

∴ Atleast one of $ay + bz$ & $bz + x$ is not zero

∴ the set $\{d_1, d_2, d_3\}$ is LD.

→ If α, β, γ are LF vectors of $V(F)$. Where F is field of complex numbers then $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also LI.

Soln Let $a, b, c \in F$ then

$$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$$

$$\Rightarrow (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$$

Since α, β, γ are LF

$$\therefore a+c=0 \quad \text{--- (1)}$$

$$a+b=0 \quad \text{--- (2)}$$

$$b+c=0 \quad \text{--- (3)}$$

$$(1)-(2) \Rightarrow a-b=0 \quad \text{--- (4)}$$

$$(2)+(4) \Rightarrow 2a=0 \Rightarrow [a=0]$$

$$(4) \Rightarrow [b=0] \text{ and } (3) \Rightarrow [c=0]$$

∴ $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are LI.

→ Let $C(C)$ be a vector space. Then show that $\{1, i\} \subseteq C(C)$ is LD.

Sol: Let $S = \{1, i\} \subseteq C(C)$. Since one of the vectors of S is scalar multiple of other. i.e., $i = i(1)$ ∴ S is LD.

→ Let $C(R)$ be a vector space then show that $\{1, i\} \subseteq C(R)$ is LI.

Sol: Let $a, b \in R$ then $a(1) + b(i) = 0 + 0(i)$

$$\Rightarrow a = 0, b = 0$$

∴ $\{1, i\}$ is LI.

→ S.T. the set $\{(1+i, 2i), (1, 1+i)\} \subseteq C^2(C)$ is LD over the field of complex numbers.

Sol: Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq C^2(C)$

Since one of the vectors of S is a scalar multiple of other.

i.e., $(1+i, 2i) = (1+i)(1, 1+i)$

∴ S is LD.

→ S.T. $\{(1+i, 2i), (1, 1+i)\} \subseteq C^2(R)$ is LI over the field of real numbers.

Sol: Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq C^2(R)$

Let $a, b \in R$ then

$$a(1+i, 2i) + b(1, 1+i) = (0, 0)$$

$$\Rightarrow (a+i\bar{a}, 2\bar{a}) + (b, b+i\bar{b}) = (0, 0)$$

$$\Rightarrow a(a+i\bar{a}+b, 2\bar{a}+b+i\bar{b}) = (0, 0)$$

$$\Rightarrow a(1+i) + b = 0 \quad (1)$$

$$b(1+i) + 2ia = 0 \quad (2)$$

$$(1) \Rightarrow (a+b) + ia = 0 + i0$$

$$\Rightarrow a+b=0 \text{ & } [a=0]$$

$$\Rightarrow [b=0]$$

∴ S is LI.

→ In the vector space $F[x]$ of all polynomials over the field F , the infinite set $S = \{1, x, x^2, \dots\}$ is LI.

Soln Let $S' = \{x^{m_1}, x^{m_2}, \dots, x^{m_n}\}$ be any finite subset of S having n vectors.

Hence m_1, m_2, \dots, m_n are non-negative integers.

Let $a_1, a_2, \dots, a_n \in F$. s.t.

$$a_1 x^{m_1} + a_2 x^{m_2} + \dots + a_n x^{m_n} = 0 x^{m_1} + 0 x^{m_2} + \dots + 0 x^{m_n}$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0$$

Every finite subset of S is LI.

$\therefore S$ is LI.

→ Let $S = \{(1, 0), (0, 1)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(3, 5) \in L(S)$.

$$\text{Soln } (3, 5) = 3(1, 0) + 5(0, 1)$$

$$\Rightarrow (3, 5) \in L(S)$$

→ Let $S = \{(1, 0, 0), (0, 1, 0)\} \subseteq \mathbb{R}^3(\mathbb{R})$. Find $L(S)$.

Do $(3, 2, 0)$, and $(2, 5, 1)$ belong to $L(S)$?

$$\text{Soln } L(S) = \{\alpha(1, 0, 0) + \beta(0, 1, 0) / \alpha, \beta \in \mathbb{R}\} \subseteq \mathbb{R}^3$$

$$= \{(x, \beta, 0) / x, \beta \in \mathbb{R}\}$$

$$\therefore (3, 2, 0) \in L(S)$$

$$\text{but } (2, 5, 1) \notin L(S). (\because 1 \neq 0)$$

→ Let $S = \{(2, 3), (1, 4)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(4, 1) \in L(S)$.

$$\text{Soln } (4, 1) = \alpha(2, 3) + \beta(1, 4) : \alpha, \beta \in \mathbb{R}$$

$$\Rightarrow 2\alpha + \beta = 4$$

$$3\alpha + 4\beta = 1$$

$$\Rightarrow \alpha = 3, \beta = -2$$

$$\therefore (4, 1) = 3(2, 3) - 2(1, 4)$$

$$\therefore (4, 1) \in L(S)$$

→ Est the vector $(2, -5, 3)$ in the subspace of \mathbb{R}^3 spanned by the vectors $(1, -3, 2), (2, -4, 1), (1, -5, 7)$?

Solⁿ: Let $\alpha = (2, -5, 3), \alpha_1 = (1, -3, 2), \alpha_2 = (2, -4, 1), \alpha_3 = (1, -5, 7)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{R}^3(\mathbb{R})$

Let $\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3; a, b, c \in \mathbb{R}$

then $(2, -5, 3) = a(1, -3, 2) + b(2, -4, 1) + c(1, -5, 7)$

$$\Rightarrow a + 2b + c = 2 \quad (1)$$

$$-3a - 4b - 5c = -5 \quad (2)$$

$$2a - b + 7c = 3 \quad (3)$$

$$3 \times (1) + (2) \Rightarrow 2b - 2c = 1 \Rightarrow b - c = Y_2 \quad (4)$$

$$2 \times (1) - (3) \Rightarrow 5b - 5c = 1 \Rightarrow b - c = Y_5 \quad (5)$$

The equations (4) & (5) are inconsistent.

$\therefore \alpha$ cannot be expressed as l.c. of S.

$\therefore \alpha$ is not in the subspace of \mathbb{R}^3 spanned by S.

→ In the vector space \mathbb{R}^3 .

let $\alpha = (1, 2, 1), \beta = (3, 1, 5), \gamma = (3, -4, 7)$.

s.t. the subspaces spanned by $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \gamma\}$ are the same.

Solⁿ: Let $S = \{\alpha, \beta\} \subseteq V_3(\mathbb{R})$

$T = \{\alpha, \beta, \gamma\} \subseteq V_3(\mathbb{R})$.

and $L(S) \& L(T)$ be two subspaces spanned by

S & T.

We have to show $L(S) = L(T)$.

Since $S \subseteq T \Rightarrow L(S) \subseteq L(T) = 0$

Let $x \in L(T)$ then

$$x = ax + b\beta + c\gamma; a, b, c \in \mathbb{R}$$

Let $v = a_1\alpha + a_2\beta; a_1, a_2 \in \mathbb{R}$.

$$\Rightarrow (3, -4, 7) = a_1(1, 2, 1) + a_2(3, 1, 5)$$

$$\Rightarrow a_1 + 3a_2 = 3 \quad \text{(i)}$$

$$2a_1 + a_2 = -4 \quad \text{(ii)}$$

$$a_1 + 5a_2 = 7 \quad \text{(iii)}$$

$$(i) - (iii) \Rightarrow -2a_2 = -4$$

$$a_2 = 2$$

$$\text{and } a_1 = -3$$

$$\therefore (3) \Rightarrow y = -3x + 2\beta.$$

$$(2) \Rightarrow x = ax + b\beta + c(-3x + 2\beta)$$

$$= (a-3c)x + (b+2c)\beta$$

= L.C. of x & β .

$$x \in L(S).$$

$$\therefore L(T) \subseteq L(S). \quad \text{(4)}$$

from (1) & (4)

$$\text{we have } L(S) = L(T)$$

→ Is the vector $(3, -4, 6)$ in the subspace of \mathbb{R}^3 spanned by the vectors $(1, 2, -1)$, $(2, 2, 1)$ and $(1, -2, 3)$?

→ Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$, $\alpha_3 = (-1, 2, 5, 2)$. Show that $(4, -5, 9, -7)$ is spanned by $\alpha_1, \alpha_2, \alpha_3$.

→ Is the vector $(3, -1, 0, -1)$ in the subspace of \mathbb{R}^4 spanned by the vectors $\alpha_1 = (2, -1, 3, 2)$, $\alpha_2 = (-1, 1, 1, -3)$ and $\alpha_3 = (1, 1, 9, -8)$?

→ Let $V = \mathbb{R}^3(\mathbb{R})$ and $S = \{ \alpha_1 = (1, 1, 0), \alpha_2 = (0, -1, 1), \alpha_3 = (1, 0, 1) \}$. Prove that $(a, b, c) \in L(S)$ iff $a = b + c$.

Soln: By definition of $L(S)$, $(a, b, c) \in$

$$(a, b, c) \in L(S)$$

$$\Leftrightarrow (a, b, c) = \alpha(1, 1, 0) + \beta(0, -1, 1) + \gamma(1, 0, 1); \alpha, \beta, \gamma \in \mathbb{R}$$

$$\Leftrightarrow (a, b, c) = (\alpha + \gamma, \beta - \gamma, \alpha + \gamma)$$

$$\Leftrightarrow a = \alpha + \gamma, b = \beta - \gamma, c = \alpha + \gamma$$

$$\Leftrightarrow a = b + c$$

→ If v_1, v_2, v_3 are three vectors in a vector space $V(F)$ such that $v_1 + v_2 + v_3 = 0$, then show that $\{v_1, v_2\}$ spans the same subspace as $\{v_1, v_3\}$. 34(iv)

Soln: Let $S = \{v_1, v_2\}$ and $T = \{v_2, v_3\}$.

→ we shall prove that $L(S) = L(T)$

Let $x \in L(S)$.

Then $x = \alpha v_1 + \beta v_2$; $\alpha, \beta \in F$

$$\Rightarrow x = \alpha(-v_2 - v_3) + \beta v_2$$

$$\Rightarrow x = (\beta - \alpha)v_2 - \alpha v_3 \in L(T)$$

$\therefore L(S) \subseteq L(T)$.

Conversely, let $y \in L(T)$.

Then $y = av_2 + bv_3$; $a, b \in F$

$$\Rightarrow y = av_2 + b(-v_1 - v_3)$$

$$\Rightarrow y = -bv_1 + (a-b)v_2 \in L(S)$$

$\therefore L(T) \subseteq L(S)$.

Hence $L(S) = L(T)$

Let us consider the set of all $(x_1, x_2, x_3, x_4, x_5)$ in \mathbb{R}^5 which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which

Span W .

Basis and Dimension:

Ex 1. 11.

Vector Space - Y

Basis: Let $V(F)$ be a vector space. and $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq V$.

if (i) S is LI.

(ii) $L(S) = V$ i.e., V spanned by S .

i.e., each vector in V is a l.c. of finite no. of elts of S .

Then S is called basis of $V(F)$.

Ex:- $S = \{e_1, e_2, \dots, e_n\} \subseteq V_n(F)$

where $e_1 = (1, 0, 0, \dots, 0)$; $e_2 = (0, 1, 0, \dots, 0)$... $e_n = (0, 0, \dots, 1)$ is basis of $V_n(F)$.

Soln (i) To prove S is LI.

$$S = \{e_1, e_2, \dots, e_n\} \subseteq V_n(F)$$

where $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_n = (0, 0, \dots, 1)$

Let $a_1, a_2, \dots, a_n \in F$ then

$$a_1e_1 + a_2e_2 + \dots + a_ne_n = 0$$

$$\Rightarrow a_1(1, 0, 0, \dots, 0) + a_2(0, 1, 0, \dots, 0) + \dots + a_n(0, 0, \dots, 1) = (0, 0, \dots, 0)$$

$$\Rightarrow (a_1, a_2, \dots, a_n) = (0, 0, 0, \dots, 0)$$

$$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$$

$\therefore S$ is LI.

(ii) To prove that, $L(S) = V_n(F)$.

We have always $L(S) \subseteq V_n(F) \rightarrow (1)$

$$\text{Let } \alpha = (a_1, a_2, \dots, a_n) = a_1(1, 0, 0, \dots, 0) + a_2(0, 1, 0, \dots, 0) + \dots + a_n(0, 0, \dots, 1) \in L(S)$$

$\therefore \alpha \in L(S)$

$$\Rightarrow V_n(F) \subseteq L(S) \rightarrow (2)$$

\therefore from (1) & (2) $L(S) = V_n(F)$.

$\therefore S$ is a basis of $V_n(F)$.

Note II. The set $\{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)\}$ is called the standard basis of $V_n(F)$.

IMS

INSTITUTE OF MATHEMATICAL SCIENCES

INSTITUTE FOR IAS/IFCS EXAMINATION

NEW DELHI-110009

Mob: 09999197625

(2) $\{(1, 0), (0, 1)\}$ is a basis of $V_2(F)$

(3) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $V_3(F)$.

Ex: $S = \{1, i\}$ is a basis of $C(\mathbb{R})$.

Sol: S is LI

We have always $L(S) \subseteq C(\mathbb{R})$ — (1)

Let $\alpha \in C(\mathbb{R})$ then $\alpha = a + bi$; $a, b \in \mathbb{R}$
 $= a \cdot 1 + b(i) \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore C(\mathbb{R}) \subseteq L(S)$ — (2)

from (1) & (2) $L(S) = C(\mathbb{R})$

$\therefore S$ is a basis of $C(\mathbb{R})$

Ex: Let $F_3[x] = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in F\}$

then $\{1, x, x^2\} \subseteq F_3[x]$ is basis of $F_3[x]$ over F .

Sol: Let $S = \{1, x, x^2\} \subseteq F[x]$

S is LI

We have always $L(S) \subseteq F[x]$ — (1)

Let $\alpha = a_0 + a_1x + a_2x^2 \in F[x]$

then $a_0 + a_1x + a_2x^2 = a_0(1) + a_1(x^2) + a_2(x^3) \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore F[x] \subseteq L(S)$ — (2)

from (1) & (2) $L(S) = F[x]$

$\therefore S$ is a basis of $F[x]$.

\rightarrow S.T the set $\{(1, 0, 0), (0, 1, 0), (1, 1, 0), (1, 2, 3)\} \subseteq V_3(\mathbb{R})$

is not a basis of $V_3(\mathbb{R})$

Sol: Let $S = \{(1, 0, 0), (0, 1, 0), (1, 1, 0), (1, 2, 3)\} \subseteq V_3(\mathbb{R})$

(i) To check whether the set S is LI or not:

Let $a, b, c, d \in \mathbb{R}$ then

$$a(1, 0, 0) + b(0, 1, 0) + c(1, 1, 0) + d(1, 2, 3) = (0, 0, 0)$$

$$\Rightarrow (a+b+c, b+2c+d, 3c) = (0, 0, 0)$$

$$\begin{aligned} a+b+c &= 0 \\ b+2c+d &= 0 \\ \hline a+2c+d &\stackrel{?}{=} 0 \end{aligned}$$

$$\begin{aligned} 3c &= 0 \\ \therefore c &= 0 \end{aligned}$$

(31)

$$\textcircled{1} \Leftrightarrow a+b=0 \Rightarrow [a=-b]$$

$$\textcircled{2} \Leftrightarrow b+d=0 \Rightarrow [b=-d]$$

If $d=k \neq 0$, then $b=-k$ and $a=k$

\therefore \exists non-zero values for a, b, d to satisfy the equations (1), (2)

\therefore The given set of vectors are LD.

$\therefore S$ is not a basis set of $V_3(\mathbb{R})$.

Note: Any subset of $V_n(F)$ (i.e., $S \subseteq V_n(F)$) having more than n elts will be LD and it cannot be a basis set of $V_n(F)$.

Defn: Finite Dimensional vector space (FDVS)

\rightarrow The vector space $V(F)$ is said to finite dimensional vector space or finitely generated if there exists a finite subset S of V s.t $V = L(S)$.

Note: If there exists no finite subset which spans V then V is called an infinite dimensional vector space.

Ex: Let $S = \{(1, 0), (0, 1)\} \subseteq V_2(\mathbb{R})$ then $V_2(\mathbb{R})$ is FDVS.

Sol: Let $(a, b) \in V_2(\mathbb{R})$; $a, b \in \mathbb{R}$

then $(a, b) = x(1, 0) + y(0, 1) \because x, y \in \mathbb{R}$

$$\begin{aligned} \Rightarrow (a, b) &= (x, 0) + (0, y) \\ &= (x, y) \end{aligned}$$

$$\Rightarrow [x=a] ; [y=b]$$

$$\therefore (a, b) = a(1, 0) + b(0, 1) \in L(S)$$

$$\therefore (a, b) \in L(S)$$

$$\therefore V_2(\mathbb{R}) \subseteq L(S) \quad \text{--- (1)}$$

$$\text{w.r.t } L(S) \subseteq V_2(\mathbb{R}) \quad \text{--- (2)}$$

\therefore from (1) & (2) we have $V_2(\mathbb{R}) = L(S)$.

$\therefore V_2(\mathbb{R})$ is a FDVS.

\rightarrow Similarly $V_3(\mathbb{R}) = \{(a, b, c) / a, b, c \in \mathbb{R}\}$ is a FDVS.

Since $V_3(\mathbb{R}) = L(S)$ where $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq V_3(\mathbb{R})$

is a FDVS.

Since $v_n(\mathbb{R}) = L(S)$

where $S = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0)\} \subseteq v_n(\mathbb{R})$

Note: A vector space may have more than one basis.

Ex (1) $S = \{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2(\mathbb{R})$

(2) $T = \{(1, 1), (1, 0)\}$ is also a basis of $\mathbb{R}^2(\mathbb{R})$

Sol Let $a, b \in \mathbb{R}$ then

$$a(1, 1) + b(1, 0) = (0, 0)$$

$$\Rightarrow (a+b, a) = (0, 0)$$

$$\Rightarrow a+b=0, \quad [a=0]$$

$$\Rightarrow [b=0]$$

$\therefore T$ is LI.

w.k.t $L(T) \subseteq \mathbb{R}^2(\mathbb{R}) \rightarrow \textcircled{1}$

Let $(a, b) \in \mathbb{R}^2(\mathbb{R})$ then

$$(a, b) = b(1, 1) + (a-b)(1, 0)$$

$$\therefore \mathbb{R}^2(\mathbb{R}) \subseteq L(T) \rightarrow \textcircled{2}$$

\therefore from (1) & (2) $\mathbb{R}^2(\mathbb{R}) = L(T)$.

$\therefore T$ is a basis of $\mathbb{R}^2(\mathbb{R})$.

→ Show that the set $S = \{1, x, x^2, \dots, x^n\}$ of n+1 polynomials is a basis for the vector space $F_n[x]$ of all polynomials of degree n over the field F.

Sol Given that $S = \{1, x, x^2, \dots, x^n\} \subseteq F_n[x]$

(i) To prove S is LI

Let $a_0, a_1, a_2, \dots, a_n \in F$ then

$$a_0(1) + a_1(x) + a_2(x^2) + \dots + a_n(x^n) = 0 \quad (\text{zero polynomial})$$

$$\Rightarrow a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0 + 0x + 0x^2 + \dots + 0x^n$$

$$\Rightarrow a_0 = a_1 = a_2 = \dots = a_n = 0$$

$\therefore S$ is LI

(ii) To prove $L(S) = F_n[x]$

w.k.t $L(S) \subseteq F_n[x]$

Let $f(x)$ be any polynomial of degree n over F.

i.e., $f(x) \in F_n[x]$.

then $f(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$; where $b_0, b_1, \dots, b_n \in F$

\therefore S. of elts of S
 $\in L(S)$.

$\therefore f(x) \in L(S)$.

\therefore If $f(x_1) \in f(x_2)$ then $f(x_1) \in L(S)$.

$$\therefore F_n[x] \subseteq L(S) \quad \text{--- (2)}$$

$$\text{from (1) \& (2)} \quad L(S) = F_n[x]$$

$\therefore S$ is a basis of $F_n[x]$.

Note: The above basis 'S' is the standard basis of the vector space of all polynomials of degree n over F .

Infinite dimensional vector space :-

Defn: The vector space $V(F)$ is said to be infinite dimensional vector space or infinitely generated if there exists an infinite subset S of V s.t $L(S) = V$.

Ex: Show that the set $S = \{1, x, x^2, \dots, x^n, \dots\}$ is a basis of the vector space $F[x]$ of all polynomials over the field F .

Sol: Given that $S = \{1, x, x^2, \dots, x^n\} \subseteq F[x]$.

(i) To prove S is LI.

$S' = \{x^{m_1}, x^{m_2}, \dots, x^{m_n}\}$ be a finite subset of S

- having n vectors.

Here m_1, m_2, \dots, m_n are non-negative integers.

Let $a_1, a_2, \dots, a_n \in F$ then $a_1 x^{m_1} + a_2 x^{m_2} + \dots + a_n x^{m_n} = 0$ (zero)

$$\Rightarrow a_1 x^{m_1} + a_2 x^{m_2} + \dots + a_n x^{m_n} = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0$$

$\therefore S'$ is LI

\therefore Every finite subset of S is LI.

$\therefore S$ is LI.

(ii) To prove $L(S) = F[x]$.

w.r.t $L(S) \subseteq F[x]$

Let $f(x) \in F[x]$

i.e. $f(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$ be any polynomial of the degree n in $F[x]$.

$$= b_0(1) + b_1(x) + b_2x^2 + \dots + b_m x^m + 0x^{m+1} + 0x^{m+2} + \dots$$

$\in L(S)$ if elts of S

$$\in L(S)$$

$$\therefore f(x) \in L(S)$$

$$\therefore \text{If } f(x) \in F[x] \text{ then } f(x) \in L(S)$$

$$\therefore F[x] \subseteq L(S) \quad \text{--- (2)}$$

$$\therefore \text{from (1) \& (2) we have } L(S) = F[x]$$

$\therefore S$ is a basis of $F[x]$.

Note: 1. The vector space $F[x]$ is an infinite dimensional vector space. Because there exists no finite subset of $F[x]$ which spans $F[x]$.

2. The vector space $F[x]$ has no finite basis.

Existence of basis of a finite dimensional vectorspace

Theorem Every finite dimensional vector space $V(F)$ has a basis (or)

If $S = \{d_1, d_2, \dots, d_m\}$ spans $V(F)$.

i.e., $L(S) = V$ then there exists a subset of S which forms a basis of V .

Proof: Let $V(F)$ be a finite dimensional vectorspace.

then \exists a finite subset S of V s.t $L(S) = V$.

i.e., let, $S = \{d_1, d_2, \dots, d_m\} \subseteq V$ s.t $L(S) = V$.

If S is LI then S itself is a basis of V .

If S is LD then there exists a vector $d_i \in S$ is a linear combination of its preceding vectors

$$d_1, d_2, \dots, d_{i-1} \quad \text{--- (1)}$$

$$\text{i.e., } d_i = a_1 d_1 + a_2 d_2 + \dots + a_{i-1} d_{i-1}$$

where $a_1, a_2, \dots, a_{i-1} \in F$

Now if we omit this vector d_i from the set 'S' then the remaining set 'S'' having $m-1$ vectors $d_1, d_2, \dots, d_{i-1}, d_{i+1}, \dots, d_m$

$$\text{i.e., } S' = \{d_1, d_2, \dots, d_{i-1}, d_{i+1}, \dots, d_m\} \subseteq S$$

Clearly $S' \subset S \Rightarrow L(S') \subset L(S)$

$$\Rightarrow L(S') \subset V \quad (1) \quad (\because L(S) \subset V)$$

Let $v \in V$ then v is l.c. of elements of S :

$$\therefore v = b_1x_1 + b_2x_2 + \dots + b_{i-1}x_{i-1} + b_i x_i + b_{i+1}x_{i+1} + \dots + b_mx_m$$

where $b_1, b_2, \dots, b_{i-1}, b_i, b_{i+1}, \dots$

$$(2) \quad \therefore v = b_1x_1 + b_2x_2 + \dots + b_{i-1}x_{i-1} + b_i(a_1x_1 + a_2x_2 + \dots + a_{i-1}x_{i-1})$$

$$+ b_{i+1}x_{i+1} + \dots + b_mx_m$$

$$= (b_1 + b_i a_1)x_1 + (b_2 + b_i a_2)x_2 + \dots + (b_{i-1} + b_i a_{i-1})x_{i-1}$$

$$+ b_{i+1}x_{i+1} + \dots + b_mx_m$$

$$= \text{l.c. of } x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_m$$

= l.c. of elements of the set S' .

$$\in L(S')$$

$$\therefore v \in L(S')$$

$$\therefore V \subseteq L(S') \quad (2)$$

\therefore from (1) & (2)

$$V = L(S')$$

If S' is LI then S' is a basis of V (P).

If S' is LD then proceeding as above we get new set S'' of $m-2$ vectors which generates V . i.e., $L(S'') = V$.

Continuing in this way, after finite no. of steps, obtain a LI subset of S which generates V and therefore it is a basis of V .

At the most repeating the procedure we left with a subset having a single non-zero vector which generates V and we know that a set containing a single non-zero vector is LI.

It forms a basis of V .

Theorem: If V is a finite dimensional vector space, then any two bases have same number of elements.

Proof: Let S be a n -dimensional vector space such that it has a basis.

Let $S_1 = \{d_1, d_2, \dots, d_m\}$ and $S_2 = \{\beta_1, \beta_2, \dots, \beta_n\}$ be two bases of V .

Now we shall prove that $m=n$.

If possible let $m \neq n$ then $m > n$ or $m < n$

Suppose $m > n$:

Since $d_i \in V$ and S_2 is a basis of V , $\exists a_{ij} \in F$ s.t.

$$d_i = a_{1i}\beta_1 + a_{2i}\beta_2 + \dots + a_{ni}\beta_n \quad i=1, 2, \dots, m \quad (1)$$

Now consider the relation

$$x_1d_1 + x_2d_2 + \dots + x_md_m = 0, x_i \in F \quad (2)$$

from (1) & (2) we have

$$x_1(a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 + \dots + a_{n1}\beta_n) + x_2(a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 + \dots + a_{n2}\beta_n) + \dots + x_m(a_{1m}\beta_1 + a_{2m}\beta_2 + \dots + a_{nm}\beta_n) = 0$$

$$\Rightarrow (x_1a_{11} + x_2a_{12} + \dots + x_ma_{1m})\beta_1 + (x_1a_{21} + x_2a_{22} + \dots + x_ma_{2m})\beta_2 + \dots + (x_1a_{n1} + x_2a_{n2} + \dots + x_ma_{nm})\beta_n = 0 \quad (3)$$

Since $\beta_1, \beta_2, \dots, \beta_n$ are L.I.

from (3) we have

$$\left. \begin{aligned} x_1a_{11} + x_2a_{12} + \dots + x_ma_{1m} &= 0 \\ x_1a_{21} + x_2a_{22} + \dots + x_ma_{2m} &= 0 \\ &\vdots \\ x_1a_{n1} + x_2a_{n2} + \dots + x_ma_{nm} &= 0 \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m &= 0 \end{aligned} \right\} \quad (5)$$

\therefore This is a system of n homogeneous linear eqns in m unknown variables.

As $m > n$ i.e., $n < m$

i.e., no. of eqns are less than no. of unknowns.

\therefore The above system (5) of eqns have a non-zero solution.

i.e., there exist $x_1, x_2, \dots, x_m \in F$ not all zero to satisfy the eqn (2).

$\therefore d_1, d_2, \dots, d_m$ are LD

which contradicts that S_1 is a basis of $V(F)$.

$m \neq n$.

Similarly $m \neq n$.

$\therefore m = n$.
i.e., Any two bases of a FDS $V(F)$ have
the same no. of elts.

Dimension of a vector Space:

Defn: The no. of elts in any basis of a finite dimensional vector space $V(F)$ is called the dimension of the vector space $V(F)$ and is denoted by $\dim V$ or $\dim_F V$.

Note: 1. If a vector space $V(F)$ has a finite basis having n vectors then $\dim V = n$.

2. If $\dim V = n$ then V has a basis containing n vectors say $S = \{x_1, x_2, \dots, x_n\}$

it means the vectors x_1, x_2, \dots, x_n are LI and each vector $v \in V$ is expressible as

$$v = a_1 x_1 + a_2 x_2 + \dots + a_n x_n \text{ where } a_1, a_2, \dots, a_n \in F$$

Ex: 1) $\dim \mathbb{R}^2 = 2$.

Since $\{(1, 0), (0, 1)\}$ is a basis of \mathbb{R}^2 .

2) $\dim \mathbb{R}^3 = 3$
Since $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of \mathbb{R}^3 .

3) $\dim \mathbb{R}^n = n$
Since $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)\}$ is a basis of \mathbb{R}^n .

4) $\dim \mathbb{C}$ over \mathbb{R}
 $\{1, i\}$ is a basis of \mathbb{C} over \mathbb{R} .

5) If F is any field then $\dim_F F = 1$

Since $\{1\}$, a set consisting of the unity elt of F is a basis of F over F .

Similarly $\dim_{\mathbb{R}} \mathbb{R} = 1$; $\dim_{\mathbb{C}} \mathbb{C} = 1$.

Note: Every non-zero elt of F will form a basis of F .

→ A finite dimensional vector space $V(F)$ has dimension n iff n is the maximum no. of linearly independent vectors in any subset of V .

Proof N.C: Let $\dim V = n$ and let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V . Then $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I.

Let $T = \{\beta_1, \beta_2, \dots, \beta_m\}$ be any subset of V where $m > n$.

If we prove that T is LD set then n is maximum no. of L.E. vectors in any subset of V .

Since $\beta_i \in V$ and S is a basis of V ,

$\exists a_{ij} \in F$ s.t.

$$\beta_i = a_{1i}\alpha_1 + a_{2i}\alpha_2 + \dots + a_{ni}\alpha_n; i=1, 2, \dots, m. \quad (1)$$

Consider the relation

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + \dots + x_m\beta_m = 0; x_i \in F \quad (2)$$

from (1) & (2) we have

$$x_1(a_{11}\alpha_1 + a_{21}\alpha_2 + a_{31}\alpha_3 + \dots + a_{n1}\alpha_n) + x_2(a_{21}\alpha_1 + a_{22}\alpha_2 + \dots + a_{2n}\alpha_n) + \dots + x_m(a_{1m}\alpha_1 + a_{2m}\alpha_2 + \dots + a_{nm}\alpha_n) = 0$$

$$+ \dots + x_m(a_{1m}\alpha_1 + a_{2m}\alpha_2 + \dots + a_{nm}\alpha_n) = 0$$

$$\Rightarrow (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m)x_1 + (a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m)x_2 + \dots + (a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mm}x_m)x_m = 0$$

Since $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I.

$$\therefore a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m = 0$$

This is a system of n homogeneous linear equations in m unknown variables.

As $m > n$ i.e., $n < m$
i.e., the no. of equations are less than no. of unknowns

∴ The above system (3) of equations have non-zero solution.

i.e., there exist non-zero values of $\alpha_1, \alpha_2, \dots, \alpha_m$ to satisfy the relation (2).

$\therefore \beta_1, \beta_2, \dots, \beta_m$ are LD. ($m > n$)

$\therefore n$ is the maximum no. of LI vectors in any subset of V .

S.C.

Let n be the maximum of LI vectors in any subset of V .

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a LI subset of V .

NOW we have to prove that S is a basis of V .
For this we are enough to prove that $V = L(S)$.

Since $S \subset V$

$$\therefore L(S) \subset V \quad \text{--- (1)}$$

Let $\alpha \in V$ and $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a maximal LI set.

$\therefore T = \{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha\}$ is LD. $\quad \text{--- (2)}$

$\Rightarrow \exists$ at least one non-zero scalar $a_1, a_2, \dots, a_n, a \in E$

$$\text{s.t. } a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + a\alpha = 0 \quad \text{--- (3)}$$

If $a=0$ then

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0 \quad (\because S \text{ is LI})$$

$$\therefore a = a_1 = a_2 = \dots = a_n = 0$$

which contradicts (2).

$$\therefore a \neq 0$$

From (3) we have

$$a\alpha = -a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

$$\Rightarrow \alpha = \left(\frac{-a_1}{a}\right)\alpha_1 + \left(\frac{-a_2}{a}\right)\alpha_2 + \dots + \left(\frac{-a_n}{a}\right)\alpha_n$$

$\Rightarrow \alpha$ is L.C. of elts of S .

$$\Rightarrow \alpha \in L(S)$$

$$\therefore V \subset L(S) \quad \text{--- (4)}$$

from (1) & (4) we have $V = L(S)$

$\therefore S$ is a basis containing n vectors.
 $\dim V = n$

Theorem:

If $\dim V = n$ then any $n+1$ vectors are LD.

Proof: Theorem (I) first part.

Extension theorem:

Every finite linearly independent subset of a finite dimensional vector space V over F can be extended to form a basis of V .

(Or).

If V is a finite dimensional vector space over F and if $S_1 = \{d_1, d_2, \dots, d_r\}$ is any LI set of vectors in V . Prove that, unless ' S_1 ' is a basis, we can find the vectors $d_{r+1}, d_{r+2}, \dots, d_n$ in V s.t.

$$\{d_1, d_2, \dots, d_r, d_{r+1}, d_{r+2}, \dots\}$$

Proof: Let $\dim V = n$, then ' n ' is the maximum no. of LI vectors in any subset of V .

Since $S_1 = \{d_1, d_2, \dots, d_r\}$ is any LI set of vectors in V .

If S_1 spans V i.e., $L(S_1) = V$ then it forms a basis of V (here $r=n$)

Let $S_2 = \{d_1, d_2, d_3, \dots, d_r, d_{r+1}, \dots, d_n\}$ be the maximal LI subset of V .

If we P.T $L(S_2) = V$ then S_2 is a basis of V .

If we P.T $L(S_2) \neq V$ then S_2 is a basis of V . Let $d \in V$ then $T = \{d_1, d_2, \dots, d_r, d_{r+1}, d_{r+2}, \dots, d_n, d\}$ which contains $n+1$ (i.e. $n > r$)

it must be LD.

∴ If at least one non-zero scalar $a_1, a_2, a_3, \dots, a_n, a \in F$ s.t $a_1d_1 + a_2d_2 + \dots + a_rd_r + a_{r+1}d_{r+1} + \dots + a_nd_n + ad = 0$ (1)

If possible let $a=0$, then $a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$

$$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0 \quad (\because S_2 \text{ is LI})$$

$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0 = 0$
which is contradiction to T is LD.

$\therefore a \neq 0$

$$\begin{aligned} \text{From (1)} \quad & \exists a \in - (a_1d_1 + a_2d_2 + \dots + a_nd_n) \\ & \Rightarrow a = \left(\frac{-a_1}{a}\right)d_1 + \left(\frac{-a_2}{a}\right)d_2 + \dots + \left(\frac{-a_n}{a}\right)d_n \\ & \in L(S_2). \end{aligned}$$

$V \subseteq L(S_2) \quad (3)$

w.k.t. $L(S_2) \subseteq V \quad (4)$

from (3) & (4)

$$V = L(S_2)$$

$\therefore S_2$ is a basis.

Q. If $\dim V = n$ and $\{\beta_1, \beta_2, \dots, \beta_m\}$ is LI subset of V , then $m \leq n$.

(or)

If $\dim V = n$ then a LI subset S_1 of V cannot have more than n elements.

Proof: Let $\dim V = n$ then n is the maximum no. of LI vectors in any subset V .

Let $S_1 = \{\beta_1, \beta_2, \dots, \beta_m\}$ be a LI subset of V .

If it contains more than n elts then S_1 is LD.

\therefore A LI subset S_1 of V cannot have more than n elts.

Theorem II \rightarrow If $\dim V = n$ and $S = \{a_1, a_2, \dots, a_n\}$ is a LI subset of V then S is a basis of V .

Proof: Since S is LI subset of V , it can be extended to form a basis of V .

Since $\dim V = n$ & S contains n LI vectors.

$\therefore S$ itself forms a basis of V .

\rightarrow Theorem: If $\dim V = n$ and $S = \{a_1, a_2, \dots, a_n\}$ spans V then S is a basis of V .

Proof Since $\dim V = n$
 \therefore Any basis of V has exactly n elts.

\therefore Since S spans V .

i.e., $L(S) = V$.

\therefore there exists any subset of S which forms a basis of V . (By existence of a basis of a FDS theorem)

Since no basis of V can have fewer than n elts.

$\therefore S$ itself forms a basis of V .

Note: If a vector space $V(P)$ is of dimension n then any set of n linearly independent vectors in V forms a basis of V .

(This result is Theorem II)

Theorem: Let $S = \{d_1, d_2, \dots, d_n\}$ be a basis of a finite dimensional vector space $V(F)$ of dimension n . Then every elt α of V can be uniquely expressed as $\alpha = a_1 d_1 + a_2 d_2 + \dots + a_n d_n$ where $a_1, a_2, \dots, a_n \in F$.

Proof: Since $S = \{d_1, d_2, \dots, d_n\}$ is a basis of V .

$$\therefore L(S) = V.$$

\therefore Any vector $\alpha \in V$ can be expressed as

$$\alpha = a_1 d_1 + a_2 d_2 + \dots + a_n d_n \quad \text{--- (1)}$$

To Show that (1) is unique representation:

Let us suppose that

$$\alpha = b_1 d_1 + b_2 d_2 + \dots + b_n d_n \quad \text{--- (2)}$$

from (1) & (2), we have $a_1 d_1 + a_2 d_2 + \dots + a_n d_n = b_1 d_1 + b_2 d_2 + \dots + b_n d_n$

$$\Rightarrow (a_1 - b_1) d_1 + (a_2 - b_2) d_2 + \dots + (a_n - b_n) d_n = 0$$

$$\Rightarrow a_1 - b_1 = 0, a_2 - b_2 = 0, \dots, a_n - b_n = 0 \quad (\because S \text{ is LI})$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, \dots, a_n = b_n.$$

\therefore (1) is a unique expression of V as a l.c. of d_1, d_2, \dots, d_n .

Row Reduced Echelon matrix:

An echelon matrix is called a row reduced echelon matrix or row canonical form iff

- (i) the distinguished elts are equal to 1.
- and (ii) these elements (distinguished) are the only non-zero elements in their respective columns.

Note: The first non-zero elt in the rows of an echelon matrix are called distinguished elts of A.

$$\text{Ex: } \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are all row reduced echelon matrices

Now non-zero rows of an echelon matrix are L.I.

Proof: Let $R_1, R_2, \dots, R_{n-1}, R_n$ be the non-zero rows of an echelon matrix A.

If possible let $R_n, R_{n-1}, \dots, R_2, R_1$ be the L.D. then one of the rows say R_m is a l.c. of its preceding rows.

$$\text{i.e., } R_m = a_{m+1}R_{m+1} + a_{m+2}R_{m+2} + \dots + a_nR_n$$

Let k^{th} elt of R_m be its non-zero entry.

Since A is an echelon form,

\therefore The k^{th} elt of each $R_{m+1}, R_{m+2}, \dots, R_n$ is zero.

$$\begin{aligned}\textcircled{1} &\equiv \text{the } k^{\text{th}} \text{ elt of } R_m = k^{\text{th}} \text{ elt of } \begin{bmatrix} a_{m+1}R_{m+1} + a_{m+2}R_{m+2} + \dots + a_nR_n \\ \vdots \\ \text{zero} \end{bmatrix} \\ &= a_{m+1}(0) + a_{m+2}(0) + \dots + a_n(0) \\ &= 0\end{aligned}$$

$$\therefore k^{\text{th}} \text{ elt of } R_m = 0$$

\therefore which contradicts the assumption that k^{th} elt of R_m is non-zero.

$\therefore R_1, R_2, \dots, R_{n-1}, R_n$ are L.I.

Problems

① Give examples of two different bases of $V_3(\mathbb{R})$ over \mathbb{R} .

Soln Let $V_3(\mathbb{R}) = \{(a, b, c) / a, b, c \in \mathbb{R}\}$

$$\text{Let } S_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq V_3(\mathbb{R})$$

$$\text{and } S_2 = \{(0, 1, 0), (0, 0, 1), (2, 3, 4)\} \subseteq V_3(\mathbb{R})$$

NOW we show that the sets S_1 & S_2 both form basis for $V_3(\mathbb{R})$.

(I) Let $S_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq V_3(\mathbb{R})$

(i) To show S_1 is L.I.

Let $a_1, a_2, a_3 \in F$, then

$$a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = (0, 0, 0)$$

$$\Rightarrow (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) = (0, 0, 0)$$

$$\Rightarrow (a_1, a_2, a_3) = (0, 0, 0)$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$

$\therefore S_1$ is LI

(ii) To show $L(S_1) \subseteq V_3(\mathbb{R})$

W.K.T. $L(S_1) \subseteq V_3(\mathbb{R}) \quad \text{(1)}$

Let $\alpha \in V_3(\mathbb{R})$

$$\text{i.e., } \alpha = (a, b, c) \in V_3(\mathbb{R})$$

$$\text{then } (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) \in L(S)$$

$\therefore \alpha \in L(S)$

$$\therefore V_3(\mathbb{R}) \subseteq L(S) \quad \text{(2)}$$

\therefore from (1) & (2) we have

$$V_3(\mathbb{R}) = L(S_1)$$

$\therefore S_1$ is a basis of $V_3(\mathbb{R})$

(iii) $S_2 = \{(0, 1, 0), (0, 0, 1), (2, 3, 4)\} \subseteq V_3(\mathbb{R})$

Similar.

Let V be the vector space of all 2×2 matrices over the field F . Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

Sol: Let $V(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in F \right\}$

$$\text{Let } \alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

be four elements of V .

$$\text{Let } S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subseteq V$$

(i) To show S is LI:

If $a_1, a_2, a_3, a_4 \in F$ then $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$

$$\Rightarrow a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dots$$

$$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0. \therefore S \text{ is LI}$$

$\text{W.K.T } L(S) \subseteq V \quad \text{--- (1)}$

Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= ax_1 + bx_2 + cx_3 + dx_4 \in L(S).$$

$\therefore \alpha \in L(S)$

$\therefore V \subseteq L(S) \quad \text{--- (2)}$

\therefore from (1) & (2) $V = L(S)$

$\therefore S$ is a basis of V .

Since the no. of elts in the basis 'S' is 4.

$\therefore \dim V = 4$

Let V be the vector space of 2×2 matrices over \mathbb{R}

Find a basis $\{A_1, A_2, A_3, A_4\}$ for V s.t. $A_i^2 = A_i$ for each i .

Soln: $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in \mathbb{R} \right\}$

Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

be any four vector elts of V s.t. $A_i^2 = A_i$ for each

Let $S = \{A_1, A_2, A_3, A_4\} \subseteq V$.

(i) To show S is LI

(ii) To show $L(S) = V$

S.T the real field \mathbb{R} is a vector space of infinite dimension over the rational field \mathbb{Q} .

Soln: we prove that the set $\{1, \pi, \pi^2, \dots, \pi^n\}$ is LI over \mathbb{Q} for any +ve integer 'n'

Suppose $a_0(1) + a_1(\pi) + a_2(\pi^2) + \dots + a_n(\pi^n) = 0$, where $a_i \in \mathbb{Q}$ and all a_i 's are not zero.

They π is a root of the non-zero polynomial

$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ over \mathbb{Q} .

This is impossible, since π is a transcendental number.

$\therefore \{1, \pi, \pi^2, \dots, \pi^n\}$ is LI over \mathbb{Q} for all +ve integer 'n'.

Hence \mathbb{R} is of an infinite dimension over \mathbb{Q} .

→ Determine whether or not the vectors $(1, -3, 2)$, $(2, 4, 1)$ and $(1, 1, 1)$ form a basis of \mathbb{R}^3 .

Solⁿ: W.K.T $\dim(\mathbb{R}^3) = 3$

if we show that the given three vectors are linearly independent they form a basis of \mathbb{R}^3 .

NOW form the matrix A,

whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(3) + 2(1) + 2(-2) \\ &= 3 + 2 - 4 \\ &= 2 \neq 0 \end{aligned}$$

∴ The given vectors are L.P.

∴ They form a basis of \mathbb{R}^3 .

→ Let V be vector space of ordered pairs of complex numbers over the field R. i.e., let V be the vector space $C(\mathbb{R})$

S.T. the set $S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$ is a basis for V.

→ S.T. the vectors $(1, 0, -1)$, $(0, -3, 2)$ and $(1, 2, 1)$ form a basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

Solⁿ: W.K.T $\dim \mathbb{R}^3 = 3$

NOW form the matrix A
whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{Now } |A| &= 1(-7) - 0(-1) - 1(3) \\ &= -7 - 3 = -10 \neq 0 \end{aligned}$$

∴ The given vectors are L.P.

∴ They form a basis for $V(\mathbb{P})$.

→ S.T. the set $\{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ is a basis for $V_3(\mathbb{C})$.

Solⁿ: W.K.T $\dim V_3(\mathbb{C}) = 3$.

NOW form the matrix whose rows are given vectors

$$\begin{bmatrix} 1 & i & 0 \\ 2i & 1 & 1 \\ 0 & 1+i & 1-i \end{bmatrix}$$

Sol: w.k.t. $\dim \mathbb{R}^3 = 3$

Form the matrix A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

$$\therefore |A| = 0$$

∴ The given set S is LD.

∴ S cannot be a basis of \mathbb{R}^3 .

→ S.T. the set $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$ is a spanning set of \mathbb{R}^3 but not a basis of \mathbb{R}^3 .

Sol: To show $L(S) \subseteq \mathbb{R}^3$

$$\text{w.k.t } L(S) \subseteq \mathbb{R}^3 \quad \textcircled{5}$$

Let $(a, b, c) \in \mathbb{R}^3$ then

$$(a, b, c) = x_1(1, 0, 0) + x_2(1, 1, 0) + x_3(1, 1, 1) + x_4(0, 1, 0)$$

$$\therefore (a, b, c) = (x_1 + x_2 + x_3, x_2 + x_3 + x_4, x_3)$$

$$\Rightarrow x_1 + x_2 + x_3 = a \quad \textcircled{1}$$

$$x_2 + x_3 + x_4 = b \quad \textcircled{2}$$

$$x_3 = c \quad \textcircled{3}$$

$$\textcircled{2} \equiv x_2 + x_4 = b - c \quad \textcircled{4}$$

$$\textcircled{1} \equiv x_1 + x_2 = a - c \quad \textcircled{5}$$

Take $x_2 = b$; $x_4 = -c$
in $\textcircled{4}$

$$\textcircled{5} \equiv x_1 = a - b - c$$

$$\therefore (a, b, c) = (a - b - c)(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1) - c(0, 1, 0)$$

$\in L(S)$

$$\therefore \mathbb{R}^3 \subseteq L(S) \quad \textcircled{6}$$

from $\textcircled{5}$ & $\textcircled{6}$ $\mathbb{R}^3 = L(S)$

Since $\dim \mathbb{R}^3 = 3$ and S contains $4 = (3+1)$ vectors.

∴ S is LD.

∴ S cannot be a basis of \mathbb{R}^3

→ Let $\{a, b, c\}$ be a basis for the vector space \mathbb{R}^3 .

P.T the sets

$\{a+b, b+c, c+a\}$, $\{a, a+b, a+b+c\}$ are also bases of \mathbb{R}^3 .

Sol Since $\{a, b, c\}$ is a basis of \mathbb{R}^3 .

$$\therefore \dim \mathbb{R}^3 = 3$$

(i) Now let $x, y, z \in \mathbb{R}$ then

$$x(a+b) + y(b+c) + z(c+a) = 0$$

$$\therefore x=y=z=0$$

$\therefore \{a+b, b+c, c+a\}$ is L.I.

\therefore It is a basis of \mathbb{R}^3 .

(ii) Now let $x, y, z \in \mathbb{R}$ then

$$x(a+b) + y(a+b+c) + z(a+b+c) = 0$$

Ques: find the dimension of the subspace of \mathbb{R}^4 spanned

by the set $\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)\}$

hence find a basis for the subspace.

theorems-

Let $V(F)$ be a vector space and a subset $S = \{d_1, d_2, \dots, d_n\}$ of $V(F)$ (i.e., $S \subseteq V$) be a linearly independent set. If $\alpha \in V(F)$ and $\alpha \notin L(S)$ then show that $S_1 = \{d_1, d_2, \dots, d_n, \alpha\}$ is a linearly independent set.

(or)

If $S = \{d_1, d_2, \dots, d_n\}$ is a LI set of vectors in V and $\alpha \in V$ is such that $\alpha \notin L(S)$, then $\{\alpha, d_1, d_2, \dots, d_n\}$ is LI set.

proof:

Let $a_1, a_2, a_3, \dots, a_n \in F$

$$a_1d_1 + a_2d_2 + \dots + a_nd_n = 0 \quad (1)$$

If $a \neq 0$ then

$$\alpha = \left(\frac{-a_1}{a}\right)d_1 + \left(\frac{-a_2}{a}\right)d_2 + \dots + \left(\frac{-a_n}{a}\right)d_n$$

$\in L(S)$. This is a contradiction to the hypothesis.

$\Rightarrow \alpha \in L(S)$ which is a contradiction to the hypothesis that $\alpha \notin L(S)$.

$$\therefore a = 0$$

$$(1) \Rightarrow a_1d_1 + a_2d_2 + \dots + a_nd_n = 0$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0 \quad (\because S \text{ is LI})$$

$$\Rightarrow a = a_1 = a_2 = \dots = a_n = 0$$

$\therefore S_1 = \{d_1, d_2, \dots, d_n, \alpha\}$ is LI set.

problem: Extend the set $\{(1,1,1), (1,0,0)\}$ to form a basis of \mathbb{R}^3 .

sol: Let $\mathbb{R}^3 = \{(x, y, z) / x, y, z \in \mathbb{R}\}$

Method (1) Let $S = \{(1,1,1), (1,0,0)\} \subseteq \mathbb{R}^3$

Let $a, b \in \mathbb{R}$ then

$$a(1,1,1) + b(1,0,0) = (0,0,0)$$

$$\Rightarrow a+b=0 \Rightarrow \boxed{b=0}$$

$$\boxed{a=0}$$

$$\therefore a=b=0$$

$\therefore S$ is LI.

$$\text{Now } L(S) = \{a(1,1,1) + b(1,0,0) / a, b \in \mathbb{R}\}$$

$$= \{(a+b, a, a) / a, b \in \mathbb{R}\}$$

Let $\alpha = (0,0,1) \in V$ then $\alpha \notin L(S)$.

\therefore the set $S' = \{(1,1,1), (1,0,0), (0,0,1)\}$ is LI.

$\therefore S^1$ is a basis of \mathbb{R}^3 .

Similarly $(0, 1, 0) \notin L(S)$

\therefore The set $\{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$ is LI set.

\therefore It is also basis of \mathbb{R}^3 .

Method II

$$\mathbb{R}^3 = \{(x, y, z) / x, y, z \in \mathbb{R}\}$$

$S = \{(1, 1, 1), (1, 0, 0)\}$ LI set

$$\text{Since } a(1, 1, 1) + b(1, 0, 0) = (0, 0, 0)$$

where $a, b \in \mathbb{R}$

$$\Rightarrow (a+b, a, a) = (0, 0, 0)$$

$$\Rightarrow a+b=0$$

w.r.t the vectors $\therefore S$ is LI

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$$

form a standard basis of \mathbb{R}^3 .

\therefore The vectors $\alpha = (1, 1, 1), \beta = (1, 0, 0)$

$$e_2 = (0, 1, 0), e_3 = (0, 0, 1)$$
 span \mathbb{R}^3

but any basis of \mathbb{R}^3 contains exactly 3 LI ele.

let us check whether α, β, e_2 are LI or not.

Now form the matrix A whose rows the vectors α, β, e_2

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow |A| = 1(-1) - 1(0) \\ = -1 \neq 0$$

$\therefore \alpha, \beta, e_2$ are LI vectors.

\therefore The vectors form a basis.

Similarly the set $\{\alpha, \beta, e_3\}$ is also LI.

\therefore It is a basis of \mathbb{R}^3 .

Method III

$$\text{Let } S = \{(1, 1, 1), (1, 0, 0)\} \subseteq \mathbb{R}^3$$

$$\text{since } a(1, 1, 1) + b(1, 0, 0) = (0, 0, 0) \text{ where } a, b \in \mathbb{R}$$

$$\Rightarrow a=0, b=0$$

$\therefore S$ is LI

w.r.t the vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ form a standard basis of \mathbb{R}^3 .

$\text{Span } \mathbb{R}^3 : \tau_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

but any basis of \mathbb{R}^3 contains exactly 3 L.I.

Let us check whether α, β, e_2 are L.I or not.

Now form the matrix A whose rows are the vectors α, β, e_2 .
reduce it to echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

∴ The echelon matrix of A has no zero rows.

∴ The vectors α, β, e_2 are L.I.

∴ They form a basis of \mathbb{R}^3 .

and also the vectors $(1, 1, 1), (0, -1, -1)$ and $(0, 0, -1)$ are L.I.

(∵ The non-zero rows of the matrix are L.I.)

∴ These are also form a basis of \mathbb{R}^3 .

NOTE: The extension of linearly independent vectors to a basis is not unique.

→ Extend the set $\{(0, 0, 0, 1), (1, 1, 0, 0), (0, 1, -1, 0)\}$ to form a basis of \mathbb{R}^4 .

→ Extend the set $S = \{(1, 1, 0)\}$ to form two different bases of \mathbb{R}^3 .

soln Since $(1, 1, 0) \neq (0, 0, 0)$

∴ S is L.I. set.

and $L(S) = \{a(1, 1, 0) / a \in \mathbb{R}\}$

$$= \{(a, a, 0) / a \in \mathbb{R}\}$$

Since $(0, 0, 1) \notin L(S)$

∴ $S_1 = \{(1, 1, 0), (0, 0, 1)\}$ is L.I.

$$\text{Now } L(S_1) = \{a(1, 1, 0) + b(0, 0, 1) / a, b \in \mathbb{R}\}$$

$$= \{(a+b, a, 0) / a, b \in \mathbb{R}\}$$

Since $(1,1,1) \notin L(S)$

$S_2 = \{(1,1,0), (0,0,1), (0,1,1)\}$ is L.I.

$\therefore S_2$ is a basis of \mathbb{R}^3 .

Similarly, $\{(1,1,0), (0,0,1), (0,1,0)\}$ is also basis of \mathbb{R}^3 .

→ Extend the set $\{(3, -1, 2)\}$ to two different bases for \mathbb{R}^3 .

→ Can the set $\{(1,0,0,0), (0,1,0,0), (1,-1,0,0)\}$ be extended to form a basis of \mathbb{R}^4 ?

Soln: The given set of vectors are not L.I. vectors.

$$\text{Since } 1(1,0,0,0) + (-1)(0,1,0,0) + (-1)(1,-1,0,0) = (0,0,0,0)$$

\therefore The given set of vectors cannot be extended to form a basis.

→ Determine whether or not the following vectors form a basis.

(i) $(1, -1, 0), (1, 3, -1), (5, 3, -2)$ of $\mathbb{R}^3(\mathbb{R})$

(ii) $(1, 0, 1), (1, 1, 0), (1, 1, -1)$ of $\mathbb{R}^3(\mathbb{R})$

(iii) $(6, 2, 3, 4), (0, 5, -3, 1), (0, 0, 7, -2), (0, 0, 0, 4)$ of $\mathbb{R}^4(\mathbb{R})$

(iv) $(1, -2, 4, 1), (2, -3, 3, 1), (1, 0, 6, -5), (2, -5, 7, 5)$ of $\mathbb{R}^4(\mathbb{R})$

P9/6 → Given two linearly independent vectors $(1, 0, 1, 0)$ and $(0, -1, 0, 0)$ of \mathbb{R}^4 , find a basis of \mathbb{R}^4 which includes these two vectors.

→ Let V be the vectorspace of all 2×2 symmetric matrices over \mathbb{R} . Find a basis and the dimension of V .

Soln: $V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ $(\because A^T = A \text{ is symmetric})$

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq V$

(i) To show S is L.I.

Let $x, y, z \in \mathbb{R}$. Then

$$x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = y = z = 0$$

w.k.t $L(S) \subseteq V = 0$.

$$\text{Let } \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\in L(S).$$

$$\therefore V \subseteq L(S) \quad \text{(ii)}$$

$$\therefore \text{from (i) \& (ii)} \quad L(S) = V.$$

$\therefore S$ is a basis of V .

$$\therefore \dim V = 3.$$

→ Let V be the vector space of 3×3 symmetric matrices over F . Show that $\dim V = 6$ by exhibiting a basis of V .

$$\text{Soln: Let } V = \left\{ \begin{bmatrix} a & b & g \\ b & c & e \\ g & e & c \end{bmatrix} \mid a, b, c, b, g, e \in F \right\}$$

$$\text{Let } S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq V$$

Note: Dimension of the vector space V of all 2×2 symmetric matrices is $3 = 2+1$

→ Dimension of the vector space V of all 3×3 symmetric matrices is $6 = 3+2+1$

→ Dimension of the vector space V of all $n \times n$ symmetric matrices is $n+(n-1)+(n-2)+\dots+3+2+1$.

$$= \frac{n(n+1)}{2}$$

→ V be the vector space of 2×2 anti-symmetric matrices over F . Show that $\dim V = 1$ by exhibiting a basis of V .

$$\text{Soln: Let } V = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \mid a \in F \right\}$$

$$\text{Let } S = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \subseteq V.$$

→ Let V be the vector space of 3×3 -anti-symmetric matrices over F . Show that $\dim V = 3$ by exhibiting a basis of V .

$$\text{Soln: Let } V = \left\{ \begin{bmatrix} 0 & h & g \\ h & 0 & e \\ g & e & 0 \end{bmatrix} \mid h, g, e \in F \right\}$$

$$\text{Let } S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \subseteq V.$$

- Note: → The dimension of the vector space of 2×2 skew symmetric matrices over F is $1 (= 2-1)$
- The dimension of the vector space of 3×3 skew symmetric matrices over F is $3 (= 3-1)+(3-2)$
- The dimension of the vector space of $n \times n$ skew symmetric matrices over F is $(n-1)+(n-2)+\dots+2+1 = \frac{n(n-1)}{2}$

Note: Let V be the vector space of $m \times n$ matrices over a field F .

Let $E_{ij} \in V$ be the matrix with 1 as ij -entry and elsewhere 0. Then the set $\{E_{ij}\}$ is a basis of V and $\dim V = mn$. (This basis is called the standard basis of V).

Let $V(\mathbb{R})$ be the real vector space of all 2×3 matrices with real entries. Find a basis for $V(\mathbb{R})$. What is the dimension of $V(\mathbb{R})$?

Sol: Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq V(\mathbb{R})$

are L.I. and $L(S) = V$.

$\therefore S$ is a basis of V .

$$\therefore \dim V = 6$$

Let V be the set of all real valued functions $y = f(x)$ satisfying $\frac{d^2y}{dx^2} + 4y = 0$. Prove that V is a 2-dimensional real vector space.

Sol: $\frac{d^2y}{dx^2} + 4y = 0 \Rightarrow (D^2 + 4)y = 0$, where $D = \frac{dy}{dx}$. — (1)

A.E of (1) is $m^2 + 4 = 0$

$$\Rightarrow m = \pm 2i$$

G.S. of (1) is $y = C_1 \cos 2x + C_2 \sin 2x$ — (2)

where C_1 and C_2 are any real constants.

Since V is the set of all real valued functions

$$y = f(x) \text{ satisfying } \frac{d^2y}{dx^2} + 4y = 0$$

$\therefore V = \{y = C_1 \cos 2x + C_2 \sin 2x / C_1, C_2 \in \mathbb{R}\}$ is a vector space.

Let $S = \{\cos 2x, \sin 2x\} \subseteq V$.

(Here we must show V is a vector space)

The Wronskian of $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$

$$\text{is } W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}$$

$$1 - 2\sin x \cos x$$

$\therefore S$ is a LI subset of V .

By (i) $L(S) = V$.

$\therefore S$ is a basis of V over \mathbb{R} .

$\therefore V$ is a two dimensional real vector space.

→ Let V be the set of all real-valued functions $y = f(x)$

$$\text{satisfying } \frac{d^3y}{dx^3} - 7 \frac{dy}{dx} - 6y = 0.$$

S.T. $V(\mathbb{R})$ is a 3-dimensional real vector space. write down a basis of this vector space.

$$\begin{aligned} \text{Soln: } m^3 - 7m - 6 &= 0 \Rightarrow (m+1)(m-2)^2 = 0 \\ &\Rightarrow (m+1)(m-2)^2 = 0 \\ &\Rightarrow m = -1, 2, 3 \end{aligned}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1' & y_2' & y_3' \end{bmatrix}$$

$$\begin{bmatrix} y_1'' & y_2'' & y_3'' \end{bmatrix}$$

→ S.T. the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation

$$\frac{d^3y}{dx^3} + 6 \frac{dy}{dx} + 11y = 0$$

is a vectorspace over \mathbb{R} .

→ Give a basis for the vectorspace.

→ S.T. the matrices $\begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$ form a basis of $V(\mathbb{R})$.

→ S.T. the matrices where V is the vectorspace of all 2×2 symmetric matrices over reals.

→ S.T. the dimension of the vectorspace $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} is 2.

Soln: Let $\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} / a, b \in \mathbb{Q} \}$.

$$\text{Let } S = \{ 1, \sqrt{2} \} \subseteq \mathbb{Q}\sqrt{2}$$

(i) To show S is LI.

Let $x, y \in \mathbb{Q}$ then

$$x(1) + y(\sqrt{2}) = 0 + 0\sqrt{2}$$

$$\Rightarrow x + y = 0$$

(ii) $L(S) = \mathbb{Q}(\sqrt{2})$.

w.k.t $L(S) \subseteq \mathbb{Q}\sqrt{2}$ — (i)

and let $a + b\sqrt{2} \in \mathbb{Q}\sqrt{2}$ then $a + b\sqrt{2} = a(1) + b\sqrt{2}$

$$G \subseteq L(S)$$

$$\therefore \mathbb{Q}(\sqrt{2}) \subseteq L(S). \quad \text{— (ii)}$$

\therefore from (i) & (ii) $L(S) = \mathbb{Q}(\sqrt{2})$.

S is a basis and $\dim(\mathbb{Q}\sqrt{2}) = 2$.

~~11~~

→ S.T the dimension of vector space $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over \mathbb{Q} is 4.

$$\text{Soln} \quad \text{Let } \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \left\{ a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \mid a, b, c, d \in \mathbb{Q} \right\}.$$

$$\text{Let } S = \{ 1, \sqrt{2}, \sqrt{3}, \sqrt{6} \}.$$

→ S.T $f_1(t) = 1, f_2(t) = t, f_3(t) = (t-2)^2$ form a basis of P_3 , the space of polynomial with degree ≤ 2 .

Express $3t^2 - 5t + 4$ as a l.c. of f_1, f_2, f_3 :

$$\text{Soln} \quad \text{Let } f(t) = 3t^2 - 5t + 4 \in P_3.$$

$$\text{then } f(t) = xf_1(t) + yf_2(t) + zf_3(t), \text{ where } x, y, z \in F.$$

$$\Rightarrow 3t^2 - 5t + 4 = x(1) + y(t-2) + z(t-2)^2. \quad \text{--- (1)}$$

$$= xt + yt - 2y + t^2 - 4t + 4z.$$

$$\Rightarrow 3t^2 - 5t + 4 = zt^2 + (y-4z)t + (x-2y+4z)$$

$$\Rightarrow \boxed{z = 3}$$

$$y-4z = -5$$

$$\Rightarrow \boxed{y = 7}$$

$$x-2y+4z = 4$$

$$\Rightarrow \boxed{x = 6}$$

$$\therefore \text{--- (1)} \quad 3t^2 - 5t + 4 = 6(1) + 7(t-2) + 3(t-2)^2$$

$$= \text{l.c. of } f_1, f_2, f_3.$$

Dimension of a subspace:-

Theorem: If w is a subspace of a finite dimensional vector space $V(F)$ then w is finite dimensional and $\dim w \leq \dim V$.

Further $V = w \Leftrightarrow \dim V = \dim w$.

Proof: Given that w is a subspace of finite dimensional vector space $V(F)$.

Let $\dim V = n$

(i) To prove w is finite dimensional.

If possible suppose that w is not finite dimensional.

then w has infinite basis.

Take S_1 is an infinite basis of w .

∴ S_1 is L.I in w .

But s_1 is the largest set.
 $\therefore s_1$ is a LI subset of V having more than ' n ' elts.
which is contradiction.
our supposition is wrong.
 $\therefore W$ is a finite dimensional.

Take $\dim W = m$.

Now we have to S.T. $m \leq n$.

Let $S = \{d_1, d_2, \dots, d_m\}$ be a basis of W .

$\Rightarrow S_1$ is LI set in W .

$\Rightarrow S_1$ is LI set in V .

Any LI subset of vector space $V(F)$ can be extended to form a basis of V .

\therefore there exists a basis S' of V s.t. $S_1 \subseteq S'$.

\Rightarrow No. of elts in $S_1 \leq$ No. of elts in S' .

$\Rightarrow m \leq n$.

i.e., $\dim W \leq \dim V$.

(ii) If $V=W$ then

W is a subspace of V and

V is $= W$.

$\therefore \dim W \leq \dim V \& \dim V \leq \dim W$.

$\Rightarrow \dim V = \dim W$.

Conversely suppose that $\dim V = \dim W$.

Let $\dim V = \dim W = n$ (say).

Let S be a basis of W .

then $L(S) = W$ and S has ' n ' LI vectors.

Also S is subset of V . ($\because S \subseteq W \subseteq V$)

and S has n LI vectors (i.e., S is LI in V)

$\Rightarrow S$ is a basis of V .

$\Rightarrow L(S) = V$.

$V = W$

IMS
(INSTITUTE OF MATHEMATICAL SCIENCES)
INSTITUTE FOR IAS/AIOS EXAMINATION
NEW DELHI-1100309
Mob: 09999197625

Note ③. If $w = \{0\}$ then the dimension $w = 0$.

②. If w is a proper subspace of a finite-dimensional vector space V then w is finite dimensional and $\dim w < \dim V$.

③. If V is finite dimensional and w is a subspace of V such that $\dim V = \dim w$. Then $V = w$.

Ques. If w_1, w_2 are two subspaces of a finite dimensional vector space $V(F)$, then $\dim(w_1 + w_2) = \dim w_1 + \dim w_2 - \dim(w_1 \cap w_2)$.

Ans. Given that w_1 & w_2 are two subspaces of $V(F)$.

$\therefore w_1 + w_2, w_1 \cap w_2$ are also subspaces of $V(F)$.

Since $w_1, w_2, w_1 + w_2$ & $w_1 \cap w_2$ are subspaces of finite dimensional vector space $V(F)$.

$\therefore w_1, w_2, w_1 + w_2$ & $w_1 \cap w_2$ are all finite dimensional.

Let $\dim(w_1 \cap w_2) = k$ and

let $S = \{v_1, v_2, \dots, v_k\} \subseteq w_1 \cap w_2$ be a basis of $w_1 \cap w_2$.

then $S \subseteq w_1$ and $S \subseteq w_2$

Since S is LI and $S \subseteq w_1$

$\therefore S$ can be extended to form a basis of w_1 .

Let $S_1 = \{v_1, v_2, \dots, v_k, d_1, d_2, \dots, d_m\}$ be a basis of w_1 .

$\therefore \dim(w_1) = k+m$

Since S is LI and $S \subseteq w_2$.

$\therefore S$ can be extended to form a basis of w_2 .

Let $S_2 = \{v_1, v_2, \dots, v_k, \beta_1, \beta_2, \dots, \beta_t\}$ be a basis of w_2

$\therefore \dim(w_2) = k+t$

$$\therefore \dim w_1 + \dim w_2 - \dim(w_1 \cap w_2) = (k+m) + (k+t) - k \\ = k+m+t.$$

NOW we have to show that $\dim(w_1 + w_2) \leq k+m+t$.

For this we have to show that the set

$S_3 = \{v_1, v_2, \dots, v_k, d_1, d_2, \dots, d_m, \beta_1, \beta_2, \dots, \beta_t\}$
is a basis of $w_1 + w_2$.

Let $c_1, c_2, \dots, c_k, a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_t$
then

$$(c_1 y_1 + c_2 y_2 + \dots + c_k y_k) + a_1 d_1 + a_2 d_2 + \dots + a_m d_m + b_1 \beta_1 + \dots + b_t \beta_t = 0 \quad (1)$$

$$\Rightarrow b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t = -(c_1 y_1 + c_2 y_2 + \dots + c_k y_k + a_1 d_1 + a_2 d_2 + \dots + a_m d_m) \quad (2)$$

Now $-(c_1 y_1 + c_2 y_2 + \dots + c_k y_k + a_1 d_1 + a_2 d_2 + \dots + a_m d_m) \in W_1$,
and $b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t \in W_2 \quad (3)$ (as it is l.c. of elts of S_1)
(as it is l.c. of elts of S_2)

$$\textcircled{3} \in b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t \in W_1 \quad (\text{by (3)}) \quad (4)$$

∴ from (3) & (4) we have

$$b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t \in W_1 \cap W_2$$

∴ It can be expressed as a l.c. of the basis of $W_1 \cap W_2$.

∴ we have $b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t = d_1 y_1 + d_2 y_2 + \dots + d_k y_k$

$$\Rightarrow b_1 \beta_1 + b_2 \beta_2 + \dots + b_t \beta_t - d_1 y_1 - d_2 y_2 - \dots - d_k y_k = 0$$

Since S_2 is L.I set.

$$\therefore b_1 = b_2 = \dots = b_t = d_1 = d_2 = \dots = d_k = 0$$

$$\therefore \textcircled{1} \in c_1 y_1 + c_2 y_2 + \dots + c_k y_k + a_1 d_1 + a_2 d_2 + \dots + a_m d_m = 0$$

Since S_1 is L.I.

$$\therefore c_1 = c_2 = c_3 = \dots = c_k = a_1 = a_2 = \dots = a_m = 0$$

$$\therefore c_1 = c_2 = c_3 = \dots = c_k = a_1 = a_2 = \dots = a_m = b_1 = b_2 = \dots = b_t = 0$$

∴ $\underline{S_3}$ is L.I set.

(ii) To show $L(S_3) = W_1 + W_2$

$$\text{w.r.t } L(S_3) \subseteq W_1 + W_2 \quad \textcircled{A}$$

Let $\alpha \in W_1 + W_2$ then

$$\alpha = \sum_i d_i y_i \text{ where } y_i \in W_1 \text{ and } d_i \in W_2$$

Since d_i is a l.c. of the basis of W_1 and y_i is a l.c.

of the basis of W_2

∴ α is a l.c. of the basis of $W_1 + W_2$

= L.C. of w_1 & w_3

$\subseteq L(S_3)$

$\therefore \text{if } \alpha \in w_1 + w_3 \text{ then } \alpha \in L(S_3)$

$\therefore w_1 + w_3 \subseteq L(S_3)$

from (A) & (B) we have

$$L(S_3) = w_1 + w_2$$

$\therefore S_3$ is a basis of $w_1 + w_2$

$\therefore \dim(w_1 + w_2) \in \mathbb{N}$

$$\therefore \dim(w_1 + w_2) = \dim(w_1) + \dim(w_2) - \dim(w_1 \cap w_2)$$

Note: (i) If w_1 and w_2 are two subspaces of a F.D.V.S. $V(F)$ such that $w_1 \cap w_2 = \{0\}$, then $\dim(w_1 + w_2) = \dim w_1 + \dim w_2$.

Defn: Row-equivalence of two matrices:

A matrix A is said to be row-equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary row operations.

Defn: Column-equivalence of two matrices:

A matrix A is said to be column equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary column operations.

Note: Elementary row operations are:

(i) Interchange of the i^{th} & j^{th} rows: $R_i \leftrightarrow R_j$

(ii) Multiplying the i^{th} row by a non-zero scalar K: $R_i \rightarrow kR_i$

(iii) Adding to the i^{th} row k times the j^{th} row: $R_i \rightarrow R_i + kR_j$

Defn: Row Space of a matrix:

Let $A = [a_{ij}]$ be an $m \times n$ matrix over a field F

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ with }$$

$R_1 = (a_{11}, a_{12}, \dots, a_{1n}), R_2 = (a_{21}, a_{22}, \dots, a_{2n})$,
 $\dots, R_m = (a_{m1}, a_{m2}, \dots, a_{mn})$ as vectors in F^m or
 $V_n(F)$.

The linear span of these vectors.

i.e., $\{R_1, R_2, \dots, R_m\}$ is a subspace of F^m and is
 called the row space of A.

i.e., $\text{row sp}(A) = \text{Span}(R_1, R_2, \dots, R_m)$

Similarly, the space spanned by the column vectors

i.e., $\{C_1, C_2, \dots, C_n\}$ is a subspace of F^m and is
 called the column space of A.

where $C_1 = (a_{11}, a_{21}, a_{31}, \dots, a_{m1})$

$C_2 = (a_{12}, a_{22}, a_{32}, \dots, a_{m2})$

\vdots
 $C_n = (a_{1n}, a_{2n}, \dots, a_{mn})$

i.e., $\text{col sp}(A) = \text{Span}(C_1, C_2, \dots, C_n)$.

Note: [1]. Column space of A is the same as the row space of A^T .

i.e., $\text{col sp}(A) = \text{row sp}(A^T)$.

Theorem: Row equivalent matrices have the same row space.

Proof: Let A and B be two row equivalent matrices.

Then by definition of row equivalence, each row of B is either a row of A or l.c. of rows of A.

\therefore The row space of B is contained in the row space of A.

By applying the inverse elementary row operations B and obtain A by applying the inverse elementary row operations A and obtain B.

\therefore The rowspace of A is contained in the rowspace of B.

\therefore The rowspaces of A & B are same.

Note [1]. Column equivalent matrices have the same columnspace.

[2]. Let A and B be two row-reduced echelon matrices. Then A and B have the same row space iff they have the same non-zero rows.

→ Determine whether the following matrices have the same row space.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & -1 \\ 4 & 3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

Ques The matrices have the same row space iff their row reduced echelon matrices have the same non-zero rows.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1} \sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 - R_2} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & -1 \\ 4 & 3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 4R_1} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 3 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 - 3R_1} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 - 2R_2} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 3R_1} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

∴ A and C have the same row space.

and B has different row space.

Column space.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{pmatrix}$$

Sol: A and B have the same column space iff A^T & B^T have same row space.

Now A^T & B^T reduce to row canonical form

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{pmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 5R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & 2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{pmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix}$$

$$R_1 \rightarrow R_1 + R_3$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore A^T$ & B^T have the same row space.

$\therefore A$ and B have the same column space.

Note: As the non-zero rows of an echelon matrix are LR and row equivalent matrices have same row space.

it follows that

Dimension of row space of A = Maximum no. of LR rows
(i.e. dimension of subspace)

= maximum no. of LR rows of
echelon matrix of A

= no. of non-zero rows of
echelon matrix of A

→ Reduce the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$ to row-reduced echelon form

Also find a basis for the row space and its dimension.

$$\text{SOLM} \quad A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 2 & 1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{pmatrix}$$

$R_2 \rightarrow R_2$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 4 & 1 & -1 & -3 \end{pmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -9 & -3 \end{pmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -3 \end{pmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the row reduced echelon form.

∴ A basis for the rowspace is

$$\{(1, 0, 1, -1), (0, 1, -3, -1)\} \text{ and the dimension of rowspace is } 2.$$

Let $V = \text{Span}(v_1, v_2, v_3)$ and $W = \text{Span}(u_1, u_2)$ be two subspaces of \mathbb{R}^4 where $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, +1, -2)$, $v_3 = (3, 6, 3, -7)$ and $v_1 = (1, 2, -4, 11)$, $v_2 = (2, 4, -5, 14)$; s.t. $V = W$.

Now: form the matrix A whose rows are v_i 's ($i=1, 2, 3$) and

reduce it to row reduced echelon form.

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & 24 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

NOW form the matrix whose rows are u_i 's ($i=1, 2$)

and reduce it to row reduced echelon form.

$$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + 4R_2} \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & -8 \end{pmatrix}$$

Since the non-zero rows of the row reduced matrices are same,

∴ The rowspaces of A & B are equal.

$$\therefore U = V.$$

generated by $(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)$.

Also extend the basis of W to a basis of the whole space \mathbb{R}^4 .

Sol: Now form the matrix A whose rows are the given vectors and reduce it in echelon form.

$$A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$ $R_3 \rightarrow R_3 - 4R_2$

The non-zero rows in the echelon matrix

$(1, -4, -2, 1)$ and $(0, 1, 1, 1)$ form a basis of W .

and $\dim W = 2$

In particular, the original three given vectors are LD.
Since \mathbb{R}^4 is 4-dimensional vector space.

\therefore we require for L2 vectors which include the above two vectors.

\therefore the vectors $(1, -4, -2, 1), (0, 1, 1, 1), (0, 0, 1, 2)$, and $(0, 0, 0, 1)$

are LI over \mathbb{R} . (Since they form an echelon matrix)

\therefore These vectors form a basis of \mathbb{R}^4 .

\therefore It is an extension of the basis of W .

2004) Let S be the space generated by vectors $\{(0, 2, 6), (3, 1, 6), (4, -3, -2)\}$. What is the dimension of the space S ? find basis for S .

1985) Consider the basis $S = \{v_1, v_2, v_3\}$ of \mathbb{R}^3 where $v_1 = (1, 1, 1)$,

$$v_2 = (1, 1, 0), v_3 = (1, 0, 0)$$

Express $(2, -3, 5)$ in terms of the basis v_1, v_2, v_3 .

\rightarrow Let W be the subspace of \mathbb{R}^5 spanned by $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$ and $u_5 = (2, 7, 3, 3, 9)$. Find a subset of the vectors which form a basis of W .

Sol¹ - Let $\{u_1, u_2, u_3, u_4, u_5\}$ which spans W .

Method 1 Since $u_2 = 2u_1$, u_1 & u_2 are LD.

i.e. eliminate the vector u_2 from S .

\therefore if $S_1 = \{u_1, u_3, u_4, u_5\}$ then Subspace W of \mathbb{R}^5 spanned by S_1 .

Now there exists no real number c s.t. $u_3 = cu_1$,
 $\therefore u_3, u_1$ are LI.

Also $u_4 \neq cu_1$ & $u_5 \neq cu_1$.

Now let us check whether the vector u_4 is a l.c. of
 u_1, u_3, u_5 or not.

Let $u_4 = au_1 + bu_3 + cu_5$ where $a, b, c \in \mathbb{R}$

$$u_4 = (1, 4, 5, 1, 8) = 1(1, 2, -1, 3, 4) + 2(1, 3, 2, 1, 6) - 0(2, 7, 3, 3, 9)$$

$\therefore u_4$ is l.c. of u_1, u_3 and u_5 .

$\therefore S_1$ is LD.

Eliminate the vector u_4 from S_1 .

If $S_2 = \{u_1, u_3, u_5\}$ then Subspace W of \mathbb{R}^5 spanned by S_2 .

No vector of S_2 is a t.c. of others.

$\therefore S_2$ is L.E subset of S .

$\therefore S_2$ is a basis of W

Method 2: Form the matrix A whose rows are given vectors
and reduce the matrix to an echelon form but

with interchanging any zero rows.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \\ R_5 \rightarrow R_5 - 2R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_3, R_5 \rightarrow R_5 - 3R_3 \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

\therefore The non-zero rows are the first, third and fifth rows.

$\therefore u_1, u_3, u_5$ form a basis of W

$v_1 = (2, -3, 4)$, $v_2 = (1, 9, 3)$, $v_3 = (-2, -4, 1)$ and $v_4 = (3, 7, -2)$

Determine a basis of the subspace spanned by the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 1, -1)$, $v_3 = (1, -1, 4)$, $v_4 = (4, 2, -2)$.

Let V_1 and V_2 be the subspaces of \mathbb{R}^4 generated by

$$\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 1, -1)\}$$
 and

$$\{(1, 2, 1, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$$
 respectively.

Find the dimension of

$$(i) V_1 \cap V_2 \quad (ii) V_1 + V_2 \quad (iii) V_1 \cup V_2$$

Sol Let $S_1 = \{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 1, -1)\}$ and

$$S_2 = \{(1, 2, 1, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$$
.

(i) Form the matrix A whose rows are the vectors of S_1 and reduce it to an echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 2R_1, R_3 \rightarrow R_3 - R_2$$

: The echelon matrix of A has two non-zero rows.

$\therefore \{(1, 1, 0, -1), (0, 1, 3, 1)\}$ form a basis of V_1 .

$$\therefore \dim V_1 = 2$$

(ii) Proceed as in (i) ... do

$$\dim V_2 = 2$$

(iii) Since V_1 and V_2 are two subspaces of \mathbb{R}^4 .

$\therefore V_1 + V_2$ is also a subspace of \mathbb{R}^4 .
 $\therefore V_1 + V_2$ is the space generated by all the six vectors (i.e., $S_1 \cup S_2$).

Now form the matrix A whose rows are the given six vectors.

and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \\ 1 & -2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1, R_4 \rightarrow R_4 - R_1 \\ R_5 \rightarrow R_5 - 2R_1, R_6 \rightarrow R_6 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$$\begin{array}{l}
 R_3 \rightarrow R_3 - R_2 \\
 R_5 \rightarrow R_5 - R_4 \\
 R_6 \rightarrow R_6 - 2R_4 \\
 \sim \left[\begin{array}{cccc} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{R_3 \leftrightarrow R_4 \\ R_3 \rightarrow R_3 - R_2}} \left[\begin{array}{cccc} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
 \end{array}$$

The echelon matrix of A has
three non-zero rows.
 $\therefore \dim(V_1 \cap V_2) = 3$.

$$\begin{aligned}
 \text{(iv)} \quad \dim(V_1 \cap V_2) &= \dim V_1 + \dim V_2 - \dim(V_1 + V_2) \\
 &= 2 + 2 - 3 \\
 &= 1
 \end{aligned}$$

Note: Let \mathbb{R}^4 , let w_1 be the space generated by $(1, 1, 0, -1)$
 $(2, 4, 1, 0)$ and $(-2, -3, -3, 1)$ and
let w_2 be the space generated by $(-1, -2, -2, 2)$, $(4, 6, 4, -6)$
 $(1, 3, 4, -3)$.
find a basis for the space $w_1 + w_2$.

Let $V = \mathbb{R}^4(\mathbb{R})$
 $w = \{(a, b, c, d) \in \mathbb{R}^4 / a = bt + c, c = bt + d\}$.
find a basis and the dimension of w .

Sol: Set $d_1 = (1, 1, 0, -1)$ and $d_2 = (0, 1, -1, -2)$
then $d_1, d_2 \in w$ and are L.I.

Since $x_1d_1 + x_2d_2 = 0$ where $x_1, x_2 \in \mathbb{R}$.

$$\begin{aligned}
 \Rightarrow x(1, 1, 0, -1) + y(0, 1, -1, -2) &= 0 \\
 \Rightarrow (x, x+y, -y, -x-y) &= (0, 0, 0, 0) \\
 \Rightarrow x = 0, y = 0
 \end{aligned}$$

To show w is spanned by d_1, d_2 .

$$\text{Let } (a, b, c, d) \in w \text{ then } a = bt + c, c = bt + d$$

$$\begin{aligned}
 \text{Since } a(1, 1, 0, -1) + c(0, 1, -1, -2) \\
 &= (a, a+b, c, -a+2c) \\
 &= (a, b, c, d) \text{ by (i)}
 \end{aligned}$$

$\therefore w$ is spanned by $\{d_1, d_2\}$.
 $\{d_1, d_2\}$ is a basis of w and $\dim w = 2$.

$$\begin{aligned}
 w &= \{(a, b, c, d) / a = bt + c, c = bt + d\} \\
 \text{Let } d = (a, b, c, d) \\
 \therefore d = (bt + c, bt + d, bt + d, bt + d) \\
 &= (2bt + d, b, bt + d, bt + d) \\
 &= b(2, 1, 1, 0) + d(1, 0, 1, 1)
 \end{aligned}$$

$$\begin{aligned}
 w &= L(\{d_1, d_2\}) \\
 \text{where } d_1 &= (2, 1, 1, 0) \\
 d_2 &= (1, 0, 1, 1) \\
 \text{are L.I.}
 \end{aligned}$$

$\{d_1, d_2\}$ is basis of w .

be two subspaces of $V = \mathbb{R}^3(\mathbb{R})$.

find the dimension of $A \cap B$.

Sol: Let $(x, y, z) \in A$ then

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0)$$

$$\therefore A = L\{(e_1, e_2)\}$$

where $e_1 = (1, 0, 0)$

$e_2 = (0, 1, 0)$ are L.S.

\therefore The set $\{e_1, e_2\}$ is basis of A .

$$\therefore \dim A = 2$$

Let $(0, y, z) \in B$ then

$$(0, y, z) = y(0, 1, 0) + z(0, 0, 1)$$

$$\Rightarrow B = L\{(d_1, d_2)\} \text{ where } d_1 = (0, 1, 0)$$

$d_2 = (0, 0, 1)$
are L.S.

\therefore The set $\{d_1, d_2\}$ is basis of B .

$$\therefore \dim B = 2$$

$$\text{Now } A \cap B = \{(0, y, 0) / y \in \mathbb{R}\}$$

$$\text{Let } (0, y, 0) = y(0, 1, 0)$$

$\therefore A \cap B = L\{\beta\}$, where $\beta = (0, 1, 0)$ is L.S.

$\therefore \{\beta\}$ is a basis of $A \cap B$.

$$\therefore \dim(A \cap B) = 1$$

$$\begin{aligned} \text{Since } \dim(A \cup B) &= \dim A + \dim B - \dim(A \cap B) \\ &= 2 + 2 - 1 \\ &= 3. \end{aligned}$$

→ find the two subspaces A and B of $V = \mathbb{R}^4(\mathbb{R})$ s.t.
 $\dim A = 2$, $\dim B = 3$ and $\dim(A \cap B) = 1$.

Sol: Let $A = \{(x, y, 0, 0) / x, y \in \mathbb{R}\}$ and

$B = \{(0, y, z, t) / y, z, t \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^4(\mathbb{R})$.

It is easy to verify that A and B are subspaces of $V = \mathbb{R}^4(\mathbb{R})$.

Let $(x_1, y_1, z_1, t) \in A$ then

$$(x_1, y_1, z_1, t) = x(1, 0, 0, 0) + y(0, 1, 0, 0).$$

$$\Rightarrow A = L(\{e_1, e_2\}).$$

where $e_1 = (1, 0, 0, 0)$ & $e_2 = (0, 1, 0, 0)$ are L.I.

\therefore The set $\{e_1, e_2\}$ is basis of A.

$$\therefore \dim A = 2$$

Let $(0, y_1, z_1, t) \in B$ then

$$(0, y_1, z_1, t) = y(0, 1, 0, 0) + z(0, 0, 1, 0) + t(0, 0, 0, 1).$$

$$\Rightarrow B = L(\{d_1, d_2, d_3\})$$

where $d_1 = (0, 1, 0, 0)$

$d_2 = (0, 0, 1, 0)$

$d_3 = (0, 0, 0, 1)$ are L.I.

\therefore The set $\{d_1, d_2, d_3\}$ is a basis of B.

$$\therefore \dim B = 3$$

$$A \cap B = \{(0, y_1, z_1, t) / y_1, z_1, t \in \mathbb{R}\}$$

Let $(0, y_1, z_1, t) \in A \cap B$ then

$$(0, y_1, z_1, t) = y(0, 1, 0, 0)$$

$$A \cap B = L(\{B\})$$

where $B = (0, 1, 0, 0)$ is L.I.

\therefore The set $\{B\}$ is a basis of \mathbb{R}^4 .

$$\therefore \dim(A \cap B) = 1$$

$$\therefore \dim(A + B) = \dim A + \dim B - \dim(A \cap B)$$

$$= 2 + 3 - 1$$

$$= 4.$$

Co-ordinates:

Let $B = \{d_1, d_2, \dots, d_n\}$ be a basis of $V(F)$.

Since $B = \{d_i / i=1, 2, \dots, n\}$ spans V , the vector $a \in V$ is a l.c. of the d_i 's.

$$\text{i.e., } a = a_1 d_1 + a_2 d_2 + \dots + a_n d_n ; a_i \in F.$$

Since the d_i 's are L.I.

The 'n' scalars a_1, a_2, \dots, a_n are completely determined by the vector a and the basis set $B = \{d_i / i=1, 2, \dots, n\}$.

and call the n -tuple (a_1, a_2, \dots, a_n) the coordinate vector of a relative to the basis $\{e_i\}$ and is denoted by $[a]_B$ or $[a]$.

$$\text{i.e., } [a] = (a_1, a_2, \dots, a_n);$$

problem → find the co-ordinate vector of $a = (3, 1, -4)$ in \mathbb{R}^3 relative to the basis $f_1 = (1, 1, 1)$, $f_2 = (0, 1, 1)$, $f_3 = (0, 0, 1)$.

Sol: a is d.c. of f_1, f_2, f_3
using unknowns x, y and z

$$\text{i.e. } a = x f_1 + y f_2 + z f_3$$

$$\begin{aligned} \Rightarrow (3, 1, -4) &= x(1, 1, 1) + y(0, 1, 1) + z(0, 0, 1) \\ &= (x, x, x) + (0, y, y) + (0, 0, z) \\ &= (x, xy, xz + y) \end{aligned}$$

$$\Rightarrow \boxed{x=3},$$

$$\Rightarrow xy=1 \Rightarrow \boxed{y=2}$$

$$xz+y=4 \Rightarrow \boxed{z=-5}$$

$$\therefore [a] = (3, -2, 5)$$

H.W. → find the co-ordinate vector of $a = (3, 1, -4)$ relative to the usual basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

H.W. Let V be the vector space of polynomials with degree

$$v = \{at^2 + bt + c \mid a, b, c \in \mathbb{R}\}$$

The polynomials $e_1 = 1$, $e_2 = t - 1$, and $e_3 = (t-1)^2 = t^2 - 2t + 1$

form a basis for V . Let $a = 2t^2 - 5t + 6$.

find $[a]$, the co-ordinate vector a relative to the basis $\{e_1, e_2, e_3\}$. (Ans: $[a] = (3, -1, 2)$)

→ find the coordinate vector $[u]$ relative to the basis $\{1, t, t^2, t^3\}$ of V , where $u = 2 - 3t + t^2 + 2t^3$.

Sol: u is a d.c. of $1, t, t^2, t^3$; using unknowns x, y, z, w .
i.e., $u = x + yt + t^2z + t^3w$.

$$\Rightarrow 2-3t+t^2+2t^3 = x+yt+zt^2+wt^3$$

$$\Rightarrow x=2, y=-3, z=1, w=2$$

$$[v] = (2, -3, 1, 2)$$

Let W be the space generated by the polynomials

$$v_1 = t^3 - 2t^2 + 4t + 1, v_2 = 2t^3 - 3t^2 + 9t - 1, v_3 = t^3 + 6t - 5 \text{ and}$$

$$v_4 = 2t^3 - 5t^2 + 7t + 5. \text{ Find a basis and dimension of } W.$$

Soln Since W is spanned by polynomials of degree 3.

$\therefore W$ is a subspace of the space $V_3(\mathbb{R})$.
(the space of all real polynomials
of degree ≤ 3)

W.R.T $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$.
and the zero polynomial

\therefore The co-ordinate vectors of v_1, v_2, v_3, v_4 w.r.t the above

basis are

$$(1, 4, -2, 1), (-1, 9, -3, 2), (1, 6, 0, 1) \text{ and } (5, 7, -5, 2)$$

Now form the matrix A . whose rows are these co-ordinate
vectors and reduce it to an echelon form

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ 1 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix}$$

$$\sim \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 + SR_1 \\ R_4 \rightarrow R_4 - 5R_1 \end{array} \left(\begin{array}{cccc} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{array} \right)$$

$$\sim \begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 + R_2 \end{array} \left(\begin{array}{cccc} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

which is in the echelon form.

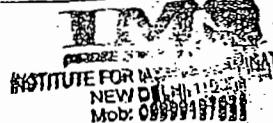
The non-zero rows of the echelon form of A form

a basis of the subspace W .

i.e., the vectors $(1, 4, -2, 1)$ and $(0, 13, -5, 3)$ form
a basis for W .

A basis for W consists of polynomials $t^3 - 2t^2 + 4t + 1$
and $3t^3 - 5t^2 + 13t$.

$$\therefore \dim W = 2$$

Set-IIILinear Transformation

60

* Defn.Vectorspace Homomorphism:

Let U and V be two vectorspaces over the same field F . Then the mapping $f: U \rightarrow V$ is called a homomorphism (or linear transformation), from U onto V , if (i)

$$f(x+\rho) = f(x) + f(\rho) \quad \forall x, \rho \in U$$

$$(ii) f(ax) = a.f(x) \quad \forall a \in F, x \in U$$

If f is onto function then V is called the homomorphic image of U .

If f is one-one onto function then f is called isomorphism. Then it is said that U isomorphic to V denoted by $U \cong V$.

Let $U(F)$ and $V(F)$ be two vectorspaces.

Then the function $T: U \rightarrow V$ is a linear transformation of U onto V iff

$$T(ax+b\rho) = aT(x) + bT(\rho) \quad \forall a, b \in F,$$

$$x, \rho \in U.$$

proof. Now suppose T is the function $T: U \rightarrow V$ is a linear transformation.

$$\therefore T(a, b \in F, x, \rho \in U)$$

$$T(ax+b\rho) = T(ax) + T(b\rho) \quad (\text{by definition})$$

$$= aT(x) + bT(\rho) \quad (\text{by definition})$$

conversely suppose T is

$$T(ax+b\rho) = aT(x) + bT(\rho) \quad \forall a, b \in F,$$

taking $a=1, b=1$ in F we get

$$T(x+\rho) = T(x) + T(\rho)$$

taking $b=0$ in F we get

$$T(ax) = aT(x)$$

$\therefore T$ is a linear transformation;

Note(3): The condition $T(cx+by) = aT(b) + bT(c)$

completely characterizes linear transformation.

Note(4): Suppose $T: V \rightarrow V$ is linear transformation.

Then for any $a_i \in F$ and any $x_i \in V$,

$$T(a_1x_1 + a_2x_2 + \dots + a_nx_n) = a_1T(x_1) + a_2T(x_2) + \dots + a_nT(x_n)$$

$$= a_1v_1 + a_2v_2 + \dots + a_nv_n$$

Note(5): If $T: V \rightarrow V$ (i.e. T transforms V into itself) then T is called a linear operator on V .

Note(6): If $T: V \rightarrow F$ (i.e. T transforms V into the field F) then T is called a linear function on V .

Zero Transformation:

Theorem: Let $V(F)$ and $V(F)$ be two vector spaces.

Let the mapping $T: V \rightarrow V$ be defined by

$$T(x) = \hat{0} \quad \forall x \in V$$

where $\hat{0}$ is the zero vector of V . Then T is a linear transformation.

Proof: If $a, b \in F$ and $x, y \in V$
 $\Rightarrow ax+by \in V$ ($\because V$ is a vector space).

By definition, we have

$$T(ax+by) = \hat{0}$$

$$= a\hat{0} + b\hat{0}$$

$$= aT(x) + bT(y)$$

$\therefore T$ is a linear transformation.

Such a L.T. is called the zero transformation
and is denoted by '0'.

61

Theorem: Let $V(F)$ be a vectorspace and the mapping $I: V \rightarrow V$ be defined by $I(x) = ax + bv$. Then, I is a linear operator from V into itself.

Proof: If $a, b \in F$; $x, y \in V$
 $\Rightarrow ax+by \in V$ ($\because V$ is a vectorspace).

By defn, we have

$$\begin{aligned} I(ax+by) &= ax+by \\ &= aI(x)+bI(y). \end{aligned}$$

I is a linear transformation.
from V into ~~itself~~ itself and is called
the identity operator.

* Negative of ~~Transformation~~

Theorem: Let $V(F)$ be two vectorspaces and $T: V \rightarrow V$ a linear transformation. Then the mapping $(-T)$ defined by $(-T)(x) = -T(x)$, $x \in V$.

is a linear transformation.

Proof: If $a, b \in F$ and $x, y \in V$
 $\Rightarrow ax+by \in V$ ($\because V$ is a vectorspace).

Now by definition,

$$\begin{aligned} (-T)(ax+by) &= -T(ax+by) \\ &= -[aT(x)+bT(y)] \quad (\because T \text{ is L.T.}) \\ &= -aT(x)-bT(y) \\ &= a(-T(x))+b(-T(y)) \\ &= a(-T(x))+b(-T(y)) \end{aligned}$$

$\Rightarrow -T$ is L.T.

* properties of Linear transformation:

Let $T: V \rightarrow V$ be a linear transformation from the vector space $V(F)$ to the vector space $V(F)$. Then

$$(i) T(\vec{0}) = \vec{0} \text{ where } \vec{0} \in V$$

$$(ii) T(-\lambda) = -T(\lambda) \quad \forall \lambda \in F$$

$$(iii) T(\lambda + \mu) = T(\lambda) + T(\mu) \quad \forall \lambda, \mu \in F.$$

Sol (i) $\lambda, \vec{0} \in V \Rightarrow T(\lambda), T(\vec{0}) \in V$.

$$\text{Now } T(\lambda) + T(\vec{0}) = T(\lambda + \vec{0}) \quad (\because T \text{ is L.T.}) \\ = T(\lambda) \\ = T(\lambda) + \vec{0} \quad (\because \vec{0} \in V)$$

$$\therefore T(\lambda) + T(\vec{0}) = T(\lambda) + \vec{0}$$

$$\text{by L.C.L., } T(\vec{0}) = \underline{\underline{\vec{0}}}.$$

(ii) $T(-\lambda) = T(-1 \cdot \lambda)$

$$= (-1) T(\lambda)$$

$$= -T(\lambda).$$

(iii) $T(\lambda + \mu) = T[\lambda + (\mu)]$

$$= T(\lambda) + T(\mu) \quad (\because T \text{ is L.T.})$$

$$= T(\lambda) + T(\mu) \quad (\text{by (ii)})$$

* Determination of Linear Transformation:

Let $V(F)$ and $W(F)$ be two vector spaces and $S = \{v_1, v_2, \dots, v_n\}$ be a basis

of V . Let $\{\delta_1, \delta_2, \dots, \delta_m\}$ be a set of m vectors in W . Then there exists a unique

linear transformation $T: V \rightarrow W$ s.t

$$T(v_i) = \delta_i \quad \text{for } i=1, 2, \dots, n.$$

Proof

Let $U(F)$ and $V(F)$ be two vector spaces.

Let $S = \{x_1, x_2, \dots, x_n\}$ be a basis of $U(F)$.

S is $L(T)$ and S' spans $V(F)$.

$$\text{ie } L(S) = V$$

Let $x \in U$, $\exists a_1, a_2, \dots, a_n \in F$ s.t.

$x = a_1 x_1 + a_2 x_2 + \dots + a_n x_n \quad (a_i \in F)$

(i) Existence of T :

Let $S_2 = \{S_1, S_2, \dots, S_n\} \subset V$

Then $S_1, S_2, \dots, S_n \in V \Rightarrow (a_1 S_1 + a_2 S_2 + \dots + a_n S_n) \in V$

We define $T: U \rightarrow V$ s.t.

$T(x) = a_1 S_1 + a_2 S_2 + \dots + a_n S_n$ where a_1, a_2, \dots, a_n are constants.

i.e. T is a linear map from U into V .

Now $x = 0 \cdot x_1 + 0 \cdot x_2 + \dots + 1 \cdot x_n + 0 \cdot x_{n+1} + \dots + 0 \cdot x_m$

By defn of T for

$$T(x) = T(0x_1 + 0x_2 + \dots + 1x_n + 0x_{n+1} + \dots + 0x_m)$$

$$= 0S_1 + 0S_2 + \dots + 1S_n + 0S_{n+1} + \dots + 0S_m \quad (\text{by (1)})$$

$$\sum S_i \neq \emptyset \quad (2)$$

(ii) To show that T is $L(T)$:

Let $a, b \in F$ and $x \in U$

$\therefore x = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

$$p = b_1 x_1 + b_2 x_2 + \dots + b_n x_n$$

$$\therefore T(x) = a_1 S_1 + a_2 S_2 + \dots + a_n S_n$$

$$\therefore T(p) = b_1 S_1 + b_2 S_2 + \dots + b_n S_n$$

$$\begin{aligned}
 a\alpha + b\beta &= a(a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n) + \\
 &\quad b(b_1\delta_1 + b_2\delta_2 + \dots + b_n\delta_n) \\
 &= (aa_1+bb_1)\delta_1 + \dots + (aa_n+bb_n)\delta_n \\
 T(a\alpha + b\beta) &= T[(aa_1+bb_1)\delta_1 + \dots + (aa_n+bb_n)\delta_n] \\
 &= (aa_1+bb_1)\delta_1 + \dots + (aa_n+bb_n)\delta_n \\
 &\quad (\text{by } (ii)) \\
 &= a(a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n) + \\
 &\quad b(b_1\delta_1 + b_2\delta_2 + \dots + b_n\delta_n) \\
 &= aT(\alpha) + bT(\beta) \\
 \therefore T \text{ is L.T.}
 \end{aligned}$$

(iii) To show that T is unique:

Let $T': U \rightarrow V$ be another L.T. s.t.

$$\begin{aligned}
 T(\alpha_i) &= \delta_{p_i} \quad ; \quad \text{for } i=1, 2, 3, \dots, n \\
 &\quad \text{and } p_1, p_2, \dots, p_n \in \{1, 2, 3, \dots, m\} \\
 T'(\alpha_i) &= T'(a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n) \\
 &= a_1 T'(\delta_1) + a_2 T'(\delta_2) + \dots + a_n T'(\delta_n) \\
 &= a_1 \delta_{p_1} + a_2 \delta_{p_2} + \dots + a_n \delta_{p_n} \\
 &= T(\alpha_i)
 \end{aligned}$$

$\therefore T' = T$ and hence T is unique.

Note:- In determining the L.T.

the assumption that

$\{\delta_1, \delta_2, \dots, \delta_n\}$ is basis of V is essential.

\rightarrow Let $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ be two ordered bases of n -dimensional vector space $V(\mathbb{R})$.
(62)

Let $\{a_1, a_2, \dots, a_n\}$ be an ordered set of n scalars such that

$$\alpha = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

$$\beta = a_1 f_1 + a_2 f_2 + \dots + a_n f_n. \text{ Then}$$

$T(\alpha) \in \mathbb{R}$ where T is the linear operator on V defined by $T(a_i) = \beta^i$, for $i = 1, 2, \dots, n$.

$$\underline{\text{Sol}}: T(\alpha) = T(a_1 e_1 + a_2 e_2 + \dots + a_n e_n)$$

$$= a_1 T(e_1) + a_2 T(e_2) + \dots + a_n T(e_n) \quad (\because T \text{ is L.O.})$$

$$= a_1 \beta_1 + a_2 \beta_2 + \dots + a_n \beta_n$$

$$= \beta \cdot \alpha$$

Problem: The mapping $V_1(\mathbb{R}^3) \rightarrow V_2(\mathbb{R}^2)$ is defined

by $T(x_1, y_1, z_1) = (x_1 - y_1, z_1)$. Show that T is a linear transformation.

Sol. Let $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$ be two vectors of $V_1(\mathbb{R}^3)$.

for $a, b \in \mathbb{R}$,

$$T(ax + by) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$

$$= T[(ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)]$$

$$= T[ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2]$$

$$= ((ax_1 + bx_2) - (ay_1 + by_2), (az_1 + bz_2) - (az_1 + bz_2))$$

$$= (a(x_1 - y_1) + b(x_2 - y_2), a(z_1 - z_2) + b(z_2 - z_1))$$

INSTAUTE FOR SCIENCE & TECHNOLOGY
NEW DELHI-110030
1962

$$\begin{aligned}
 &= (a(x_1-y_1), a(x_1-z_1)) + (b(x_2-y_2), b(x_2-z_2)) \\
 &= a(x_1-y_1, x_1-z_1) + b(x_2-y_2, x_2-z_2) \\
 &= aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2) \quad (\text{by defn}), \\
 &= aT(x) + bT(p) \\
 T(ax+bp) &= aT(x) + bT(p) \quad \forall a, b \in F, \\
 &\quad x, p \in V_2(\mathbb{R}) \\
 \therefore T \text{ is a linear transformation} \\
 &\text{from } V_2(\mathbb{R}) \text{ to } V_1(\mathbb{R})
 \end{aligned}$$

→ The mapping $T: V_2(\mathbb{R}) \rightarrow V_1(\mathbb{R})$ is
defined by $T(a, b, c) = ax + by + cz$.
Can T be a linear transformation?

Sol. Let $\alpha = (a, b, c)$ and $p = (x, y, z)$
be two vectors of $V_2(\mathbb{R})$.

for $p, q \in \mathbb{R}$,

$$\begin{aligned}
 T(p\alpha + q\beta) &= T[p(a, b, c) + q(x, y, z)] \\
 &= T[(pa, pb, pc) + (qx, qy, qz)] \\
 &= [pa + qx, pb + qy, pc + qz] \\
 &= (pa + qx)^v + (pb + qy)^v + (pc + qz)^v \\
 &= (pa + qx)v + (pb + qy)v + (pc + qz)v
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } pT(\alpha) + qT(\beta) &= \cancel{pT(\alpha) + qT(\beta)} \\
 &= pT(a, b, c) + qT(x, y, z) \\
 &= p(a^v + b^v + c^v) + q(x^v + y^v + z^v)
 \end{aligned}$$

$\therefore T(p\alpha + q\beta) \neq pT(\alpha) + qT(\beta)$.

$\therefore T$ is not a L.T from $V_2(\mathbb{R})$ to $V_1(\mathbb{R})$.

1 polynomials in the variable x over \mathbb{R}

(63)

Let $f(x) \in V(\mathbb{R})$; show that

(i) $D: V \rightarrow V$ defined by $Df(x) = \frac{d}{dx} f(x)$

(ii) $I: V \rightarrow V$ defined by $If(x) = \int_a^x f(t) dt$

are linear transformations.

Sol Let $f(x), g(x) \in V(\mathbb{R})$ and $a, b \in \mathbb{R}$

$$\begin{aligned} (i) D[a f(x) + b g(x)] &= \frac{d}{dx} [a f(x) + b g(x)] \\ &\quad (\text{by hypothesis}) \\ &= \frac{d}{dx} [a f(x)] + \frac{d}{dx} [b g(x)] \\ &= a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x) \\ &= a D f(x) + b D g(x) \end{aligned}$$

i.e. D is a linear transformation and
 D is called differential operator.

$$\begin{aligned} (ii) I[a f(x) + b g(x)] &= \int_a^x (a f(x) + b g(x)) dx \\ &= \int_0^x (a f(x)) dx + \int_0^x (b g(x)) dx \\ &= a \int_0^x f(x) dx + b \int_0^x g(x) dx \\ &= a I f(x) + b I g(x). \end{aligned}$$

i.e. I is L.T. and I is called integral operator.

Let $P_n(\mathbb{R})$ be the vector space of all polynomials of degree n over a field \mathbb{R} . If a linear operator T on $P_n(\mathbb{R})$ is such that

$$T f(x) = f(x+1), \quad f(x) \in P_n(\mathbb{R}).$$

$$\text{Show that } T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots + \frac{D^n}{n!}.$$

Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ $\forall x \in \mathbb{R}$.

$$\text{Now } \left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \dots + \frac{D^n}{n!} \right] f(x)$$

$$= \left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \dots + \frac{D^n}{n!} \right] (a_0 + a_1x + a_2x^2 + \dots + a_nx^n)$$

$$= (a_0 + a_1x + a_2x^2 + \dots + a_nx^n) + \frac{1}{1!}(0 + a_1 + a_2x + \dots + a_nx^{n-1})$$

$$+ \frac{1}{2!}(0 + 0 + 2a_2 + 6a_3 + \dots + n(n-1)a_{n-1}x^{n-2})$$

$$+ \dots + \frac{1}{n!}(0 + 0 + \dots + a_n)$$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + \dots + a_n(x+1)^n$$

$$= f(x+1).$$

$$= T \cdot f(x). \quad (\text{by hypothesis})$$

$$\therefore T = \left(1 + \frac{D}{1!} + \frac{D^2}{2!} + \dots + \frac{D^n}{n!} \right).$$

$\Rightarrow T$ is the mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by.

- $T(x_1, y_1, z_1) = (x_1, 0)$ a linear transformation.

Sol we have $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x_1, y_1, z_1) = (x_1, 0).$$

Let $\lambda, \rho \in \mathbb{R}^3$ where $\lambda = (x_1, y_1, z_1)$ &

$$\rho = (x_2, y_2, z_2)$$

for $a, b \in \mathbb{R}$,

$$a\lambda + b\rho = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$

$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$\therefore T(a\lambda + b\rho) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$= ((ax_1, 0)).$$

$$\text{and } aT(\lambda) + bT(\rho) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$$

$$= a(x_1, 0) + b(x_2, 0)$$

$$= ((ax_1 + bx_2, 0)).$$

clearly $T(c\alpha + b\beta) \neq cT(\alpha) + bT(\beta)$, (64)

Hence it is not a linear transformation.

Let T be a linear transformation from a vector space U into V (i.e. $T: U \rightarrow V$ is L.T.)
and the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ in U are such that $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I.

Let $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$ are L.I.

Sol. Given $T: U \rightarrow V$ is L.T.

and $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I. S.t.

Let there exist a_1, a_2, \dots, a_n such that
 $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \vec{0}$ (65)
 $(\because \vec{0} \in V)$

$$\Rightarrow T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) = T(\vec{0})$$

$$\Rightarrow a_1T(\alpha_1) + a_2T(\alpha_2) + \dots + a_nT(\alpha_n) = \vec{0} \quad (\text{As } T \text{ is L.T.})$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0 \quad (\vec{0} \in V) \quad (\because T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n) \text{ are L.I.})$$

from (65) a_1, a_2, \dots, a_n are L.O.

Let V be a vector space of w.v.s merely over the field F . M is a fixed matrix for V .
The mapping $T: V \rightarrow V$ is defined by

$T(A) = AM + MA$ where $A \in V$. Show that
 T is linear.

Sol. Let $a, b \in F$ and $A, B \in V$. Then

$$T(A) = AM + MA \text{ & } T(B) = BM + MB.$$

$$\begin{aligned} \therefore T(aA + bB) &= (aA + bB)M + M(aA + bB) \\ &= a(AM + MA) + b(BM + MB) \\ &= aT(A) + bT(B). \end{aligned}$$

$\therefore T$ is a linear transformation.

→ describe explicitly the linear transformation
 $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ s.t. $T(2,3) = (4,5)$ and $T(1,0) = (0,1)$

Sol: first of all we have to show $\{ \}$ the set

$S = \{(2,3), (1,0)\}$ is a basis of \mathbb{R}^2 .

for this we have to show S is L.I and
 $L(S) = \mathbb{R}^2$:

Let $a(2,3) + b(1,0) = \vec{0}$, $a, b \in \mathbb{R}$.

$$\Rightarrow (2a+b, 3a+0) = (0,0)$$

$$\Rightarrow 2a+b=0, 3a=0.$$

$$\Rightarrow a=0, b=0.$$

$\therefore S$ is L.S.

N.K.T $L(S) \subseteq \mathbb{R}^2$

Let $(x,y) \in \mathbb{R}^2$ then $(x,y) = c(2,3) + b(1,0)$

$$\Rightarrow (x,y) = (2a+b, 3a+0)$$

$$\Rightarrow 2a+b=x, 3a=y$$

$$\Rightarrow 2\left(\frac{y}{3}\right) + b = x, \boxed{a = \frac{y}{3}}$$

$$\Rightarrow \boxed{b = x - \frac{2y}{3}}$$

$$(x,y) = \frac{y}{3}(2,3) + \left(x - \frac{2y}{3}\right)(1,0)$$

$\in L(S)$.

$$(x,y) \in L(S)$$

$$\mathbb{R}^2 \subseteq L(S) \quad \text{②}$$

From ① & ② we have

$L(S) = \mathbb{R}^2$

$\therefore S$ is a basis of \mathbb{R}^2 .

Note: $\therefore S = \{(2,3), (1,0)\}$ is a basis of \mathbb{R}^2 and

$S' = \{(4,5), (0,1)\}$ is a set of two vectors in \mathbb{R}^2 .

$\therefore T$ is a unique linear transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ s.t}$$

(65)

$$T(x,y) = T\left[\frac{1}{3}(2,3) + \left(x - \frac{4y}{3}\right)(1,0)\right]$$

~~(from Ques.)~~

$$= \frac{1}{3}T(2,3) + \left(x - \frac{4y}{3}\right)T(1,0)$$

$$= \frac{1}{3}(4,5) + \left(x - \frac{4y}{3}\right)(0,1)$$

$$= \left(\frac{4x}{3}, \frac{5y}{3}\right) \text{ which is the reqd transformation.}$$

Show that $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of \mathbb{R}^3 .
Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a L.T. such that $T(1,0,0) = (1,1,0)$, $T(1,1,0) = (1,1,1)$, $T(1,1,1) = (1,1,2)$ and $T(1,1,2) = (1,1,3)$.

Ans → find $T(x_1, y_1, z)$ where $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$,

$$T(0,0,1) = -2$$

Ans → find a linear transformation

$$(i) T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ s.t } T(1,0) = (1,1) \text{ and } T(0,1) = (1,2)$$

$$(ii) T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ s.t } T(1,2) = (3,0) \text{ and } T(2,1) = (1,2)$$

$$(iii) T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ s.t } T(0,1,2) = (3,1,2) \text{ and } T(1,1,1) = (2,1,2)$$

$$(iv) T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ s.t } T(1,2) = (2, -1, 5) \text{ and } T(0,1) = (2, 1, -1)$$

* Sum of Linear Transformations

Defn Let T_1 and T_2 be two linear transformations from $V(F)$ into $V(F)$. Then their sum $T_1 + T_2$ is defined by $(T_1 + T_2)(x) = T_1(x) + T_2(x) \forall x \in V$.

Theorem Let $V(F)$ and $W(F)$ be two vector spaces. Let T_1 and T_2 be two linear transformations from V into W . Then the mapping $T_1 + T_2$ defined by $(T_1 + T_2)(x) = T_1(x) + T_2(x) \forall x \in V$ is a linear transformation.

proof: Given that $T_1: U \rightarrow V$ and $T_2: U \rightarrow V$ are linear transformation.
 and $(T_1 + T_2)(x) = T_1(x) + T_2(x) \forall x \in U$,
 $T_1(x), T_2(x) \in V$
 since $T_1(x), T_2(x) \in V$
 $\Rightarrow T_1(x) + T_2(x) \in V$.

Hence, $(T_1 + T_2): U \rightarrow V$
 Let $a, b \in F$ and $x, \rho \in U$. Then
 $(T_1 + T_2)(ax + b\rho) = T_1(ax + b\rho) + T_2(ax + b\rho)$ (by hyp)
 $= (aT_1(x) + bT_1(\rho))$
 $+ (aT_2(x) + bT_2(\rho))$ (since T_1 & T_2 are L.T.)
 $= a(T_1(x) + T_2(x)) + b(T_1(\rho) + T_2(\rho))$.
 $= a(T_1 + T_2)(x) + b(T_1 + T_2)(\rho)$. (by hyp)
 $\therefore T_1 + T_2$ is a L.T. from U onto V .

Scalar multiplication of a L.T.
 Let $T: U(F) \rightarrow V(F)$ be a linear transformation
 and $a \in F$. Then the function (aT) defined by
 $(aT)(x) = aT(x) \forall x \in U$. is a
 linear transformation.

proof
 Given that $T: U(F) \rightarrow V(F)$
 and $(aT)(x) = aT(x) \forall x \in U$, $x \in U$
 Now $T(x) \in V \Rightarrow a \cdot T(x) \in V$
 $\therefore (aT): U \rightarrow V$
 for $c, d \in F$ and $x, \rho \in U$
 $\Rightarrow (aT)(cx + d\rho) = aT(cx + d\rho)$ (by hyp).

$$= a [c T(x) + d T(p)] \quad (\because T \text{ is L.T.}) \quad (6)$$

$$= ac T(x) + ad T(p).$$

$$= c(aT)(x) + d(aT)(p).$$

Hence (aT) is a L.T. from $V_3(\mathbb{R})$ into V_1 .

problem:

Let $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ and $H: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ be two linear transformations defined by $T(x, y, z) = (x-y, y+z)$ and $H(x, y, z) = (2x, y-z)$.

Find (i) $H+T$. (ii) cH .

$$\begin{aligned} \text{sol}(i) (H+T)(x, y, z) &= H(x, y, z) + T(x, y, z) \\ &= (2x, y-z) + (x-y, y+z) \\ &= (3x-y, 2y) \end{aligned}$$

$$\begin{aligned} \text{(ii)} (cH)(x, y, z) &= cH(x, y, z) \\ &= c(2x, y-z) \\ &= (2cx, cy-cz) \end{aligned}$$

Let $G: V_3 \rightarrow V_2$ and $H: V_3 \rightarrow V_2$ be two linear operators defined by $G(e_1) = e_1 + e_2$, $G(e_2) = e_3$, $G(e_3) = e_2 - e_3$ and $H(e_1) = e_2$, $H(e_2) = 2e_1 - e_3$, $H(e_3) = 0$. Where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$.

Find (i). $G+H$ (ii) $2G$.

sol Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard basis of $V_3(\mathbb{R})$.

so it's

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0) \text{ and } e_3 = (0, 0, 1).$$

$$\text{Now, } G(e_1) = e_1 + e_2$$

$$\Rightarrow G(1, 0, 0) = (1, 0, 0) + (0, 1, 0)$$

$$= (1, 1, 0)$$

$$\therefore \boxed{G(1, 0, 0) = (1, 1, 0)}$$

$$G(e_2) = e_2$$

$$\Rightarrow \boxed{G(0, 1, 0) = (0, 0, 1)}$$

$$G(e_3) = e_2 - e_3$$

$$\Rightarrow G(0, 0, 1) = (0, 1, 0) - (0, 0, 1)$$

$$\therefore \boxed{G(0, 0, 1) = (0, 1, -1)}$$

$$\text{Again, } H(e_1) = e_3$$

$$\Rightarrow \boxed{H(1, 0, 0) = (0, 0, 1)}$$

$$H(e_2) = 2e_2 - e_3$$

$$\Rightarrow \boxed{H(0, 1, 0) = (0, 2, -1)}$$

$$\Rightarrow H(e_3) = 0$$

$$\Rightarrow \boxed{H(0, 0, 1) = (0, 0, 0)}$$

$$(i) (G+H)(e_1) = G(e_1) + H(e_1)$$

$$\Rightarrow (G+H)(1, 0, 0) = (1, 0, 0) + (0, 0, 1)$$

$$= (1, 1, 1)$$

$$(G+H)(e_2) = G(e_2) + H(e_2) \Rightarrow (G+H)(0, 1, 0) = (0, 1, 0)$$

$$(G+H)(e_3) = G(e_3) + H(e_3) \Rightarrow (G+H)(0, 0, 1) = (0, 1, -1)$$

$$(ii) (2G)(e_1) = 2G(e_1) = 2e_1 + 2e_2$$

$$(2G)(e_2) = 2G(e_2) = 2e_3$$

$$(2G)(e_3) = 2G(e_3) = 2e_2 - 2e_3 - \text{etc.}$$

* Product of Linear Transformations (67)

Let $U(F)$, $V(F)$ and $W(F)$ be three vector spaces and $T: V \rightarrow W$ and $H: U \rightarrow V$ are two linear transformations. Then the composite function TH (called the product of linear transformations) defined by

$$(TH)(x) = T[H(x)] \quad \forall x \in U.$$

is a linear transformation from U into W .

Note: The range of H is the domain of T .

Let H, H' be two linear transformations from $U(F)$ to $V(F)$. Let T, T' be the linear transformations from $V(F)$ to $W(F)$ and $a \in F$. Then

$$(i) T(H+H') = TH + TH'$$

$$(ii) (T+T')H = TH + T'H$$

$$(iii) a(TH) = (aT)H = T(aH)$$

* Algebra of Linear Operators

Let A, B, C be linear operators on a vector space $V(F)$. Also let O be the zero operator and I the identity operator on V . Then

$$(i) AO = OA = O$$

$$(ii) A\bar{I} = \bar{I}A = A$$

$$(iii) A(B+C) = AB + AC$$

$$(iv) (A+B)C = AC + BC$$

$$(v) A(BC) = (AB)C$$

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $H: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined

by $T(x, y, z) = (3x, y+z)$ and $H(x, y, z) = (2x-z, y)$

Compute (i) $T+H$ (ii) $4T-5H$ (iii) TH (iv) HT .

Sol Since T and H map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$,
the linear transformations $T+H$ and
 $4T-5H$ are defined.

$$(i) (T+H)(x_1, y, z) = T(x_1, y, z) + H(x_1, y, z)$$

$$= (3x, y+z) + (2x-z, y)$$

$$= (5x-z, y+z).$$

$$(ii) (4T-5H)(x_1, y, z) = 4T(x_1, y, z) - 5H(x_1, y, z)$$

$$= 4(3x_1, y+z) - 5(2x_1-z, y)$$

$$= (2x_1+5z, -y+4z).$$

(iii) and (iv) So $T+H$ and HT are not defined
because the range of T is not equal
to the domain of H and vice versa.

Let $T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are two
linear transformations defined by

$$T_1(x_1, y, z) = (3x_1, 4y-z)$$

$$T_2(x_1, y) = (-x_1, y). \text{ Compute } T_1 T_2 \text{ and } T_2 T_1.$$

Sol (i) Since the range of T_2 i.e. \mathbb{R}^2 is
not equal to the domain of T_1 (i.e. \mathbb{R}^3)

$\therefore T_1 T_2$ is not defined.

(ii) But the range T_1 i.e. \mathbb{R}^2 is equal
to the domain of T_2 .

$\therefore T_2 T_1$ is defined.

$$\therefore (T_2 T_1)(x_1, y, z) = T_2 [T_1(x_1, y, z)]$$

$$= T_2(3x_1, 4y-z)$$

$$= (-3x_1, 3y-z)$$

Let $P(\mathbb{R})$ be the vectorspace of all polynomials
 $\text{Let } x \in D, T$ be two linear operators on P
defined by $D[f(x)] = \frac{df}{dx}$ and
 $T[f(x)] = xf(x) \forall f(x) \in P(\mathbb{R})$
Show that (i) $TD \neq DT$ (ii) $(TD)^2 = T^2 D^2 + TD$

$$\underline{\text{Sol}} (i) (TD)f(x) = T[Df(x)] \\ = T\left[\frac{df}{dx}\right] \quad (\text{by hyp}) \\ = xf'(x)$$

$$\text{and } (DT)f(x) = D[Tf(x)] \\ = D[xf(x)] \quad (\text{by hyp}) \\ = \frac{d}{dx}(xf(x)) \\ = x \frac{d}{dx}f(x) + f(x)$$

clearly $TD \neq DT$
also $(DT)f(x) = (D(T)f(x)) = T(f(x)) = xf(x)$
 $\Rightarrow (DT - TD)f(x) = I f(x)$
 $\Rightarrow (DT - TD) = I$

$$(ii) (TD)^2 f(x) = [(TD)(TD)] f(x) \\ = (TD)[(TD)f(x)] \\ = (TD)[x f'(x)] \\ = T\left[D(x f'(x))\right] \\ = T\left[\frac{d}{dx}(x f'(x))\right] \\ = T\left[\left(x f''(x) + f'(x)\right)\right] \\ = T\left[x f''(x) + f'(x)\right] \\ = x [x f''(x) + f'(x)] \\ = x^2 f'''(x) + x f'(x)$$

now $(T^2 D^\sim) f(a)$

$$\begin{aligned}
 &= T^2 D [D f(a)] \\
 &= T^2 D \left[\frac{df}{da} \right] \\
 &= T^2 \left[\frac{d^2 f}{da^2} \right] \\
 &= T \left(T \left[\frac{df}{da} \right] \right) \\
 &= T \left(a \frac{df}{da} \right) \\
 &= a \left(a \frac{d^2 f}{da^2} \right) \\
 &= a^2 \frac{d^2 f}{da^2}.
 \end{aligned}$$

$$\therefore (T^2 D^\sim + TD)(f(a)) = (T^2 D^\sim)(f(a)) + (TD)f(a).$$

$$= a^2 \frac{d^2 f}{da^2} + a \frac{df}{da}$$

$$\therefore (TD)^\sim(f(a)) = (T^2 D^\sim + TD)f(a)$$

\checkmark f(a) CP

$$\underline{(TD)^\sim = T^2 D^\sim + TD.}$$

Ques. Let P be the polynomial space for one indeterminate x with real co-efficients.

Let $D: P \rightarrow P$ and $S: P \rightarrow P$ be two linear operators defined by

$$Df(x) = \frac{df}{dx} \quad \text{and} \quad Sf(x) = \int_0^x f(t) dt$$

Show that $DS = I$ and $SD \neq I$

where I is the identity transformation

~~Ans~~ Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $H: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two linear transformations defined by

$$T(x, y, z) = (x - 3y - 2z, y + 2z).$$

$$\therefore H(x, y) = (2x, 4x - y, 2x + 3y)$$

~~Ans~~ If $H \circ T$ and $T \circ H$: Is product commutative

~~Ans~~ Define on \mathbb{R}^2 linear operators H and T as follows $H(x, y) = (0, x)$ and $T(x, y) = (x, y)$, and show that

$$T \circ H = I, H \circ T \neq I \text{ and } T^2 = T.$$

* Transformations as vectors

Let $L(U, V)$ be the set of all linear transformations from a vector space $U(F)$ to a vector space $V(F)$. Then $L(U, V)$ be a vector space relative to the operations of vector addition and scalar multiplication defined as

$$(i) (T+H)(x) = T(x) + H(x)$$

$$(ii) (\alpha T)(x) = \alpha T(x) \quad \forall x \in U, \alpha \in F$$

$$T, H \in L(U, V).$$

The set $L(U, V)$ is also denoted by $\text{Hom}(U, V)$.

proof Let $L(U, V) = \{T: U \rightarrow V / T \text{ is a L.T.}\}$

Given $T: U \rightarrow V$ and

$H: U \rightarrow V$ are L.T. in $L(U, V)$.

$$\therefore (T+H)(x) = T(x) + H(x) \quad \forall x \in U.$$

$$T(x), H(x) \in V.$$

Since $T(x), H(x) \in V$

$$\Rightarrow T(x) + H(x) \in V.$$

$$\therefore (T+H): U \rightarrow V.$$

Let $a, b \in F$ and $\alpha, \beta \in V$ then

$$(T+H)(c\alpha + d\beta) = T(c\alpha + d\beta) + H(c\alpha + d\beta)$$

$$= (aT(\alpha) + bT(\beta)) + (cH(\alpha) + dH(\beta))$$

$(\because T \in H \text{ and } H \in L)$

$$= a(T+H)(\alpha) + b(T+H)(\beta).$$

$T+H$ is a L.T. from V into V ,
 $\therefore T+H \in L(V, V)$.

\therefore Internal composition is satisfied by $L(F)$

Given that $T: V \rightarrow V$ is a L.T. from V into V ,

$$\text{and } (aT)(\alpha) = aT(\alpha) + bT(\beta)$$

$$\text{Now } T(\alpha) \in V \Rightarrow aT(\alpha) \in V$$

$$\therefore (aT): V \rightarrow V.$$

for $c, d \in F$ and $\alpha, \beta \in V$

$$\begin{aligned} \Rightarrow (aT)(c\alpha + d\beta) &= aT(c\alpha + d\beta) \\ &= c(aT(\alpha) + dT(\beta)) \end{aligned}$$

$$= c(aT(\alpha)) + d(aT(\beta))$$

$$= c(aT)(\alpha) + d(aT)(\beta)$$

$\therefore (aT)$ is a L.T. from V into V .

$$\therefore aT \in L(V, V)$$

\therefore External composition is satisfied
by $L(V, V)$ over the field F .

Q. (i) $\forall T, H \in L(V, V) \Rightarrow T+H \in L(V, V)$
 \therefore closure prop is satisfied.

(ii) $\forall T, H, G \in L(V, V)$

$$\begin{aligned} [(T+H)+G](\alpha) &= (T+H)(\alpha) + G(\alpha) \\ &= T(\alpha) + H(\alpha) + G(\alpha) \end{aligned}$$

$$\begin{aligned}
 &= T(x) + [4(x) + 9(x)] \quad \because \text{is associative} \\
 &= T(x) + [4 + 9]x \\
 &= [T + (4 + 9)](x) \\
 (T+4)+4 &= T+(4+4) \\
 \therefore \text{assoc. prop. is satisfied for } L(v, v)
 \end{aligned}$$

(iii) Let '0' be the zero transformation from V into V .

$$\begin{aligned}
 \text{i.e. } 0(x) &= \hat{0} \quad \forall x \in V \\
 \text{now } (0+T)(x) &= 0(x) + T(x) \\
 &= \hat{0} + T(x) \\
 &= T(x) \quad (\because \hat{0} \text{ is additive identity})
 \end{aligned}$$

$$\begin{aligned}
 \therefore 0+T &= T \\
 \text{say } T+0 &= \hat{0} \\
 \therefore T \in L(v, v) \quad \forall & 0 \in L(v, v) \text{ &} \\
 \text{here } '0' \text{ is the additive identity in } & L(v, v).
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv) for } T \in L(v, v), \\
 \text{let us define } (-T) \text{ as } (-T)(x) = -T(x) \quad \forall x \in V. \\
 \text{then } (-T) \in L(v, v).
 \end{aligned}$$

$$\begin{aligned}
 \text{now } (-T+T)(x) &= (-T)(x) + T(x) \\
 &= -T(x) + T(x) \\
 &= \hat{0} \quad (\because \hat{0} \in V) \\
 &= 0(x) \\
 \therefore (-T)+T &= 0 \quad \forall T \in L(v, v). \\
 \text{say } T+(-T) &= 0 \quad \forall T \in L(v, v). \\
 \therefore \forall T \in L(v, v), \exists -T \in L(v, v) \text{ s.t.} & \\
 -T+T &= 0 = T+(-T)
 \end{aligned}$$

Here τ is additive inverse of τ
in $L(U, V)$.

$$(iv) (\tau + \text{it})(x) = \tau(x) + \text{it}(x) \\ = \text{it}(x) + \tau(x) \quad (\because \text{adding law is commutative}) \\ = (\text{it} + \tau)(x)$$

$\therefore \text{it} + \tau = \tau + \text{it}$.
commutative prop is satisfied by $L(U, V)$.

$(L(U, V), +)$ is an abelian grp.

$$(v) \forall a, b \in F, \tau, \eta, \theta \in L(U, V) : \\ \Rightarrow (i) [a(\tau + \eta)](x) = a(\tau + \eta)(x) \quad (\text{by } \text{hyp(i)}) \\ = a[\tau(x) + \eta(x)] \quad (\text{by hyp(ii)}) \\ = a\tau(x) + a\eta(x) \quad (\text{by hyp(iii)}) \\ = (\tau + \eta)(x) \quad (\text{by hyp(iv)}) \\ \therefore a(\tau + \eta) = \underline{a\tau + a\eta}$$

$$(vi) [(a+b)\tau](x) = (a+b)\tau(x) \quad (\text{by hyp(v)}) \\ = a\tau(x) + b\tau(x) \\ = (a\tau)(x) + (b\tau)(x) \quad (\text{by hyp(iv)}) \\ = (a\tau + b\tau)(x) \\ \therefore (a+b)\tau = \underline{a\tau + b\tau}$$

$$(vii) [(ab)\tau](x) = a[b\tau(x)] \quad (\text{by hyp(v)}) \\ = a[b\tau(x)] \\ = a[(b\tau)(x)] = [a(b\tau)](x)$$

$$\therefore (a \cdot b) T = a \cdot (b \cdot T)$$

$$(iv) (1 \cdot T)(x) = 1 \cdot T(x).$$

$\Rightarrow T(x)$, (multiplication by 1 is identity)
 \Leftarrow

$$\therefore 1 \cdot T = T.$$

$L(U, V)$ is a vectorspace over the field F .

$\rightarrow L(U, V)$ be the vectorspace of all linear transformations from $U(F)$ to $V(F)$.
So $\dim U = n \Rightarrow \dim V = m$.

$$\text{Then } \dim L(U, V) = mn.$$

proof. Given that $L(U, V)$ is the vectorspace of all linear transformations from $U(F)$ to $V(F)$
i.e. $L(U, V) = \{T: U \rightarrow V / T \text{ is a linear transformation}\}$.
since $\dim U = n$ and $\dim V = m$.

Let $B_1 = \{x_1, x_2, \dots, x_n\}$ and

$B_2 = \{e_1, e_2, \dots, e_m\}$ be the ordered bases of U and V respectively.

There exists uniquely a linear transformation T_{ij} from U to V such that

$$T_{ij}(x_i) = e_j, T_{ij}(x_1) = \hat{0}, \dots, T_{ij}(x_n) = \hat{0} \text{ where } e_1, \hat{0} \in V$$

i.e. $T_{ij}(x_i) = e_j, i=1, 2, \dots, n$
and $T_{ij}(x_k) = \hat{0} \quad k \neq i$

thus there are "mn" T_{ij} 's $\in L(U, V)$.

we shall show that $S = \{T_{ij}\}$ of mn elts is a basis for $L(U, V)$.

(i) TO prove S is $L(U, V)$:

Let $a_{ij}'s \in F$, let us suppose that
 $\sum_{i=1}^n \sum_{j=1}^m a_{ij} T_{ij} = \hat{0}$ $(\because \hat{0} \in L(U, V))$

for $\alpha_k \in V, k=1, 2, 3, \dots, n$ we get

$$\left[\sum_{i=1}^n \sum_{j=1}^m a_{ij} T_{ij} \right] (\alpha_k) = 0 (\alpha_k)$$

$$\Rightarrow \sum_{i=1}^n \sum_{j=1}^m a_{ij} T_{ij} (\alpha_k) = 0 \quad (\because \alpha \in V)$$

$$\Rightarrow \sum_{j=1}^m a_{kj} T_{kj} (\alpha_k) = 0 \quad \left(\because \sum_{j=1}^m a_{ij} T_{ij} (\alpha_k) + \sum_{j=1}^m a_{kj} T_{kj} (\alpha_k) + \dots + \sum_{j=1}^m a_{mj} T_{mj} (\alpha_k) = 0 \right)$$

where

$$1 \leq k \leq n.$$

$$\Rightarrow a_{k1} T_{k1} (\alpha_k) + a_{k2} T_{k2} (\alpha_k) + \dots + a_{kn} T_{kn} (\alpha_k) = 0$$

$$\Rightarrow a_{k1} p_1 + a_{k2} p_2 + \dots + a_{kn} p_n = 0. \quad \left(\text{by } \sum_{j=1}^m a_{kj} p_j = 0 \text{ if } k \neq n \text{ and so on.} \right)$$

$$\Rightarrow a_{k1} = a_{k2} = \dots = a_{kn} = 0. \quad (\because B_n \text{ is a basis of } V)$$

$\therefore S = \{T_{ij}\}$ is $L(V)$ set.

(ii). To show that $L(S) = L(U, V)$.

Let $T \in L(U, V)$ then the vector $T(\alpha_i) \in V$

It can be expressed as l.c. of $\{p_i\}$ of B_1 of B_2 .

$$\text{i.e. } T(\alpha_i) = b_{i1} p_1 + b_{i2} p_2 + \dots + b_{in} p_n.$$

In general for $i = 1, 2, \dots, n$

$$T(\alpha_i) = b_{i1} p_1 + b_{i2} p_2 + \dots + b_{in} p_n. \quad (1)$$

Consider the linear transformation:

$$H = \sum_{i=1}^n \sum_{j=1}^m b_{ij} T_{ij}$$

Clearly H is a linear combination of $S = \{T_{ij}\}$.

$$\therefore H \in L(U, V)$$

Let $\alpha_k \in V$ for $k = 1, \dots, n$.

$$\text{since } T_{ij} (\alpha_k) = 0 \text{ for } k \neq i \quad \& \quad T_{kj} (\alpha_k) = p_j$$

$$\begin{aligned} \text{we have } H(\alpha_k) &= \sum_{i=1}^n \sum_{j=1}^m b_{ij} T_{ij} (\alpha_k) \\ &= \sum_{j=1}^m b_{kj} T_{kj} (\alpha_k) \end{aligned}$$

$$= \sum_{j=1}^m b_{kj} p_j$$

i.e. $H(u) = b_{k1} p_1 + b_{k2} p_2 + \dots + b_{km} p_m$
 $= T(u)$ (by ①)

$\therefore H(u) = T(u)$ for each u
 $\Rightarrow H = T$

thus T is a linear combination of elements of S

i.e. $T(S) \subseteq L(U, V)$

$\therefore S$ is a basis set of $L(U, V)$.

$\therefore \dim L(U, V) = mn$

problems

→ find the dimension of $L(\mathbb{R}^3, \mathbb{R}^2)$

since $\dim \mathbb{R}^3 = 3$ and

$\dim \mathbb{R}^2 = 2$

$\therefore \dim(L(\mathbb{R}^3, \mathbb{R}^2)) = 6.$

→ find the dimension of $L(\mathbb{C}^3, \mathbb{R}^2)$.

Sol since \mathbb{C}^3 is a vector space over \mathbb{C}
and \mathbb{R}^2 is a vector space over \mathbb{R}
Hence $\dim(L(\mathbb{C}^3, \mathbb{R}^2))$ does not exist.

→ Let $V = \mathbb{C}^3$ be a vector space over \mathbb{R} .

→ find the dimension of $L(V, \mathbb{R}^2)$
(i.e. dim of $\text{Hom}(V, \mathbb{R}^2)$)

Sol since $V = \mathbb{C}^3$ is a vector space over \mathbb{R}

i.e. let $V = \{(a_1+ib_1, a_2+ib_2, a_3+ib_3) / a_i, b_i \in \mathbb{R}\}$

$\therefore \dim V = 6$

and obviously $\dim \mathbb{R}^2 = 2$

$\therefore \dim(\text{Hom}(V, \mathbb{R}^2)) = 6 \times 2 = 12.$

from this topic (2)

* Range and nullspace of a linear transformation.

Imp

Def Let $U(F)$ and $V(F)$ be two vectorspaces
and let $T: U \rightarrow V$ be a linear transformation.
The range of T is defined to be the set

$$\begin{aligned} \text{Range}(T) &= R(T) \\ &= \{T(u) | u \in U\}. \end{aligned}$$

Obviously the range of T is a subset
of V . i.e. $R(T) \subseteq V$.

Let $U(F)$ and $V(F)$ be two vectorspaces.
Let $T: U(F) \rightarrow V(F)$ be a linear transformation.
Then the range set $R(T)$ is a subspace of $V(F)$.

proof $\quad \text{for } \vec{0} \in V \Rightarrow T(\vec{0}) = \vec{0} \in R(T)$

$\therefore R(T)$ is non-empty set and
 $R(T) \subseteq V$.

Let $x_1, x_2 \in U$ and $p_1, p_2 \in R(T)$ be s.t.

$$T(x_1) = p_1 \text{ and } T(x_2) = p_2.$$

for $a, b \in F$, $T(ax_1 + bx_2) \in V$ ($\because V$ is vs).
 $\Rightarrow T(ax_1 + bx_2) \in R(T).$

$$\begin{aligned} \text{But } T(ax_1 + bx_2) &= aT(x_1) + bT(x_2) \\ &= ap_1 + bp_2 \quad (\because T \text{ is L.T.}) \\ &\in R(T). \end{aligned}$$

$\therefore a, b \in F$ and $p_1, p_2 \in R(T)$
 $\Rightarrow ap_1 + bp_2 \in R(T).$

$\therefore R(T)$ is subspace of $V(F)$.

$R(T)$ is called the range space.

Nullspace or Kernel:

Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \rightarrow V$ be a linear transformation.

The nullspace denoted by $N(T)$ is the set of all vectors $\alpha \in U$ s.t. $T(\alpha) = \vec{0}$ (zero vector of V).

The nullspace of $N(T)$ is also called the kernel of T .

$$\text{i.e. } N(T) = \{\alpha \in U / T(\alpha) = \vec{0} \in V\}$$

Obviously the nullspace $N(T) \subseteq U$.

Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \rightarrow V$ is a linear transformation, then nullspace $N(T)$ is a subspace of $U(F)$.

Proof. Let $N(T) = \{\alpha \in U / T(\alpha) = \vec{0} \in V\}$ -

$\because T(\vec{0}) = \vec{0} \Rightarrow \vec{0} \in N(T)$ ($\because \vec{0} \in U$, $\vec{0} \in V$).
 $\therefore N(T)$ is a non-empty subset of U .

$$\text{Now } \alpha, \beta \in N(T) \Rightarrow T(\alpha) = \vec{0}, T(\beta) = \vec{0}.$$

$$\begin{aligned} \text{For } a, b \in F, T(a\alpha + b\beta) &= aT(\alpha) + bT(\beta) \\ &= a\cdot\vec{0} + b\cdot\vec{0} \\ &= \vec{0}. \end{aligned} \quad (\because T \text{ is L.T.})$$

$$\therefore T(a\alpha + b\beta) = \vec{0}.$$

By definition $a\alpha + b\beta \in N(T)$.

$$\text{As } a, b \in F \text{ and } \alpha, \beta \in N(T) \Rightarrow a\alpha + b\beta \in N(T).$$

\therefore nullspace $N(T)$ is a subspace of $U(F)$.

Let $T: U(F) \rightarrow V(F)$ be a linear transformation.

If U is finite dimensional then the range space $R(T)$ is a finite dimensional subspace of $V(F)$.

PROOF

Let $S = \{x_1, x_2, \dots, x_n\}$ be the basis set
of $U(F)$.

Let $\beta \in R(T)$

then $\exists \alpha \in U$ such that $T(\alpha) = \beta$

$\Rightarrow \alpha = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ for $a_i \in F$

$$\Rightarrow T(\alpha) = T(a_1 x_1 + a_2 x_2 + \dots + a_n x_n)$$

$$\Rightarrow \beta = a_1 T(x_1) + a_2 T(x_2) + \dots + a_n T(x_n)$$

But $S' = \{T(x_1), T(x_2), \dots, T(x_n)\} \subset R(T) \Rightarrow L(S') \subseteq L(T)$

Now $\beta \in R(T)$ and $\beta = b_i$ of all's of S'

$$\Rightarrow \beta \in L(S') \Rightarrow L(T) \subseteq L(S') \quad (2)$$

from (1) & (2), we have $R(T) = L(S')$
Thus $R(T)$ is spanned by a finite set S .

$R(T)$ is finite dimensional subspace of $V(F)$.

* Dimension of Range and Kernel:

Let $T: U(F) \rightarrow V(F)$ be a linear transformation
where U is finite dimensional vector space.

Rank: Then the rank of T denoted by $r(T)$ is the

dimension of range space $R(T)$:

$$\text{i.e., } r(T) = \dim R(T).$$

nullity: The nullity of T denoted by $N(T)$ is the

dimension of null space $N(T)$.

$$N(T) = \dim N(T).$$

Theorem

Let $U(F)$ and $V(F)$ be two vector spaces and

$T: U \rightarrow V$ be a linear transformation. Let U be finite

dimensional then $r(T) + N(T) = \dim U$.

$$\text{i.e., } \text{rank}(T) + \text{nullity}(T) = \dim U$$

proof: The null space $N(T)$ is a subspace of finite dimensional space $U(F)$.

$\Rightarrow N(T)$ is finite dimensional.

Let $S = \{d_1, d_2, \dots, d_k\}$ be a basis of $N(T)$.

$$\therefore \dim N(T) = r(T) = k.$$

$$\therefore T(d_1) = \vec{0}, T(d_2) = \vec{0}, \dots, T(d_k) = \vec{0}. \quad (1)$$

As S is L.I. it can be extended to form a basis of $U(F)$.

Let $S_1 = \{d_1, d_2, \dots, d_k, \alpha_1, \alpha_2, \dots, \alpha_m\}$ be the extended basis of $U(F)$.

$$\dim U = k+m.$$

Now we show that the set of images of additional vectors $S_2 = \{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_m)\}$

is a basis of $R(T)$.

Clearly $S_2 \subseteq R(T)$.

To prove S_2 is L.I.

Let $a_1, a_2, \dots, a_m \in F$ such that

$$a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_m T(\alpha_m) = \vec{0}.$$

$$\Rightarrow T(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_m \alpha_m) = \vec{0}. \quad (\because T \text{ is L.T.})$$

$$\Rightarrow a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_m \alpha_m \in N(T).$$

But each vector in $N(T)$ is a l.c. of all the basis 'S'.

\therefore for some $b_1, b_2, \dots, b_k \in F$,

$$\text{let } a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_m \alpha_m = b_1 d_1 + b_2 d_2 + \dots + b_k d_k$$

$$\Rightarrow a_1 d_1 + a_2 d_2 + \dots + a_m d_m - b_1 d_1 - b_2 d_2 - \dots - b_k d_k = 0.$$

$$\Rightarrow a_1 = 0, a_2 = 0, \dots, a_m = 0, b_1 = 0, b_2 = 0, \dots, b_k = 0$$

($\because S_1$ is L.I.)

(74)

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in \text{range space } R(T)$, then $\exists \alpha \in U$ s.t.

$$T(\alpha) = \beta.$$

Now $\alpha \in U \Rightarrow \text{there exist } c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_m \text{ such that}$

$$\alpha = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k + d_1 \theta_1 + d_2 \theta_2 + \dots + d_m \theta_m$$

$$\begin{aligned} \Rightarrow T(\alpha) &= T(c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k + d_1 \theta_1 + d_2 \theta_2 + \dots + d_m \theta_m) \\ &= c_1 T(\alpha_1) + c_2 T(\alpha_2) + \dots + c_k T(\alpha_k) + d_1 T(\theta_1) + \\ &\quad d_2 T(\theta_2) + \dots + d_m T(\theta_m) \end{aligned}$$

$$\Rightarrow \beta = d_1 T(\theta_1) + d_2 T(\theta_2) + \dots + d_m T(\theta_m) \quad (\because \text{by (i)})$$

$$f(x+y) = f(x) + f(y) \Rightarrow \beta \in L(S_2).$$

$$\frac{d}{dx} f(x) = f'(x) \frac{d}{dx} \quad \therefore S_2 \text{ is a basis of } R(T).$$

and $\dim R(T) = m$

$$\therefore \dim R(T) + \dim R(T)^\perp = \dim U.$$

~~$$i.e., R(T) + R(T)^\perp = U.$$~~

problem:

$T: V_4(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is a linear transformation defined by $T(a, b, c, d) = (a-b+c-d, a+2c-d, a+b+2c-3d)$.
for $a, b, c, d \in \mathbb{R}$ then verify $R(T) + R(T)^\perp = \dim V_4(\mathbb{R})$.

Sol. Let $S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$
be the standard basis set of $V_4(\mathbb{R})$.

\therefore the transformation T on S will be

$$T(1, 0, 0, 0) = (1, 1, 1), \quad T(0, 1, 0, 0) = (-1, 0, 1)$$

$$T(0, 0, 1, 0) = (1, 2, 3), \quad T(0, 0, 0, 1) = (1, 4, -2)$$

$$\text{Let } S_1 = \{(1,1,1), (-1,0,1); (1,2,2) (1,-1, -3)\}$$

$$\therefore S_1 \subseteq R(T).$$

NOW we verify whether S_1 is L.I or not.

For, we find basis L.I set by forming the minors.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & -1 & -3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2, R_4 \rightarrow R_4 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

clearly which is in echelon form.

\therefore the non-zero rows of vectors

$\{(1,1,1), (0,1,2)\}$ constitute the L.I set

forming the basis of $R(T)$.

$$\Rightarrow \boxed{\dim R(T) = 2.}$$

Basis for nullspace of T :

$$N(T) = \{x \in V_4 / T(x) = \vec{0}\}$$

$$\text{Let } x \in N(T) \Rightarrow T(\vec{x}) = \vec{0}$$

$$\therefore T(a, b, c, d) = \vec{0} \text{ where } \vec{0} = (0, 0, 0) \in V_3$$

$$\Rightarrow (a-b+c+d, a+2c-d, a+b+3c-3d) = (0, 0, 0)$$

$$\Rightarrow a-b+c+d=0$$

$$a+2c-d=0$$

$$a+b+3c-3d=0$$

we have to solve
these for a, b, c, d .

coefficient matrix = $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

$$\sim \left[\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{array} \right] \quad R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad R_3 \rightarrow R_3 - 2R_2$$

clearly which is in echelon form.

\therefore The equivalent system of equations are

$$a - b + c + d = 0 \Rightarrow \boxed{b = 2d - c}$$

$$b + c - d = 0 \Rightarrow \boxed{c = d - 2b}$$

\therefore The number of free variables is 2 namely c, d and the values of a & b depend on these. and hence $\boxed{\text{Nullity}(T) = \dim(N(T)) = 2}$

choosing $c=1, d=0$, we get $a=-2, b=0$

$$\therefore (a, b, c, d) = (-2, 0, 1, 0)$$

choosing $c=0, d=1$, we get $a=1, b=2$

$$\therefore (a, b, c, d) = (1, 2, 0, 1)$$

$\therefore \{(-2, 0, 1, 0), (1, 2, 0, 1)\}$ constitute

a basis of $N(T)$.

$$\therefore \dim(R(T)) + \dim(N(T)) = 2 + 2$$

$$= 4 = \dim V_{\text{eff}}$$

→ Verify the Rank-Nullity theorem for the linear map

$$T: V_4 \rightarrow V_3 \text{ defined by } T(e_1) = f_1 + f_2 + f_3, T(e_2) = f_1 - f_2 + f_3$$

$T(e_3) = f_1$, $T(e_4) = f_1 + f_3$ where $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are standard basis V_4 and V_3 respectively.

Sol: Let $e_1 = (1, 0, 0, 0)$; $e_2 = (0, 1, 0, 0)$; $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$

$$\text{and } f_1 = (1, 0, 0), f_2 = (0, 1, 0), f_3 = (0, 0, 1)$$

$\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3\}$ are the standard basis of V_4 and V_3 respectively.

$$\text{we have } T(e_1) = f_1 + f_2 + f_3$$

$$\Rightarrow T(1, 0, 0, 0) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) \\ = (1, 1, 1)$$

$$T(e_2) = f_1 - f_2 + f_3$$

$$\Rightarrow T(0, 1, 0, 0) = (1, 0, 0) - (0, 1, 0) + (0, 0, 1) \\ = (1, -1, 1)$$

$$T(e_3) = f_1$$

$$\Rightarrow T(0, 0, 1, 0) = (1, 0, 0)$$

$$= f_1 + f_3$$

$$\Rightarrow T(0, 0, 0, 1) = (1, 0, 0) + (0, 0, 1) \\ = (1, 0, 1)$$

Let $d \in V_4$

Then d can be written as $d = a e_1 + b e_2 + c e_3 + d e_4$.

$$\text{Then } T(d) = T(ae_1 + be_2 + ce_3 + de_4)$$

$$= a T(e_1) + b T(e_2) + c T(e_3) + d T(e_4)$$

$$= a(1, 1, 1) + b(1, -1, 1) + c(1, 0, 0) + d(1, 0, 1)$$

$$= (a+b+c+d, a-b, a+b+d)$$

Consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_1 \\ R_2 \rightarrow R_2 - R_1 \end{array} \left| \begin{array}{ccc} 0 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{array} \right|$$

$$R_3 \rightarrow R_3 + 2R_2, R_4 \rightarrow R_4 - \frac{1}{2}R_2$$

$$\sim \left| \begin{array}{ccc} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right|$$

Clearly which is in echelon form.

∴ The non-zero rows of vectors

$\left\{ (1, -1, 1), (0, 2, 0), (0, 0, -1) \right\}$ constitute the L.I set forming the basis of $R(T)$.

$$\Rightarrow \boxed{\dim R(T) = 3}$$

Basis for null space of T :

$$N(T) = \left\{ \alpha \in V_4 / T(\alpha) = \vec{0} \right\}$$

$$\text{Let } \alpha \in N(T) \Rightarrow T(\alpha) = \vec{0}$$

$$\Rightarrow (a+b+c+d, a-b, a+b+d) = (0, 0, 0)$$

$$\Rightarrow a+b+c+d = 0 \quad (1)$$

$$a-b = 0 \quad (2)$$

$$a+b+d = 0 \quad (3)$$

We have to solve for a, b, c, d .

From (1) & (2) we get $c = -2a$

From (2) & (3) we get $d = -2b$

From (1), we get $b = 0$

The number of free variables is 2 namely 'a' and the values of d & b depends on 'a' and hence

$$\text{nullity}(T) = \dim N(T) = 1$$

76



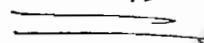
choosing $a=1$, we get $b=1, d=-2$
 $(c_1, c_2, d) = (1, 1, 0, -2)$.

$\{(1, 1, 0, -2)\}$ constitute a basis ~~for~~
 of $N(T)$.

$$\therefore \dim R(T) + \dim N(T) = 3 + 1$$

$$= 4$$

$$= \dim(V_4).$$



Q2 Let $T: V_4 \rightarrow V_3$ be a linear transformation defined by $T(x_1) = (1, 1, 1)$; $T(x_2) = (1, -1, 1)$; $T(x_3) = (1, 0, 0)$; $T(x_4) = (1, 0, 1)$.

Then verify that $\text{r}(T) + \text{r}(T) = \dim V_4$.

Q3 Find a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ whose range is spanned by $(1, 2, 0, -4), (2, 0, 1, -3)$.

Given $\text{r}(T)$ spanned by

$$\{(1, 2, 0, -4), (2, 0, 1, -3)\}.$$

Let us include a vector $(0, 0, 0, 0)$ in this set which will not effect the spanning property.

$$\text{So } T \text{ is } \{(1, 2, 0, -4), (2, 0, 1, -3), (0, 0, 0, 0)\}$$

Let $B = \{x_1, x_2, x_3\}$ be the basis of \mathbb{R}^3 .

W.K.T there exists a transformation

$$T \text{ s.t } T(x_1) = (1, 2, 0, -4)$$

$$T(x_2) = (2, 0, 1, -3)$$

$$T(x_3) = (0, 0, 0, 0)$$

Now if $\alpha \in \mathbb{R}^3 \Rightarrow \alpha = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$

$$\Rightarrow T(\alpha) = T(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3)$$

$$\Rightarrow T(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3) = aT(\mathbf{e}_1) + bT(\mathbf{e}_2) + cT(\mathbf{e}_3)$$

$$= a(1, 2, 0, -4) + b(2, 0, -1, -2)$$

$$+ c(0, 0, 0, 0)$$

$$\therefore T(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3) = (a+2b, 2a, -b, -4a-2b)$$

is the reqd transformation.

To find $f: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation.

whose range is spanned by

$$(1, -1, 2, 3) \text{ and } (2, 3, -1, 0)$$

Q1 consider the standard basis for \mathbb{R}^3

is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$.

Then $f(\mathbf{e}_1) = (1, -1, 2, 3)$

$$f(\mathbf{e}_2) = (2, 3, -1, 0) \text{ and}$$

$$f(\mathbf{e}_3) = (0, 0, 0, 0).$$

$$\text{N.K.T } (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$$

$$= f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$$

$$= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + x_3 f(\mathbf{e}_3)$$

$$= (x_1, -x_1, 2x_1, 3x_1) + (2x_2, 3x_2, -x_2, 0)$$

$$= (x_1 + 2x_2, -x_1 + 3x_2, 2x_1 - x_2, 0)$$

Q2 Let V be a vector space of all 2×2 matrices over reals. Let P be a fixed matrix of V , $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $T: V \rightarrow V$ be a linear operator defined by $T(A) = PA - A$, $A \in V$

→ find the nullity.

Sol: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

The nullspace $N(T)$ is the set of all 2×2 matrices whose T -image is $\vec{0}$.

$$\Rightarrow T(A) = PA = \vec{0}$$

$$\Rightarrow T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a-c=0, b-d=0$$

$$\Rightarrow a=c, b=d$$

the free variables are c, d
where $\dim N(T) = 2$.

H.W.

Describe explicitly the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose range space is spanned by $\{(1, 0, 1), (1, 2, 1)\}$.

→ find the null space, range, rank and nullity of the transformation -

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ defined by } T(x, y) = (x+y, x-y, y)$$

Sol:

$$(1) \text{ Let } x = (a, y) \in \mathbb{R}^2$$

$$\text{Then } N(T) = \{x \in \mathbb{R}^2 / T(x) = \vec{0}\}.$$

$$x \in N(T) \Rightarrow T(x) = \vec{0}$$

$$\Rightarrow T(a, y) = \vec{0} \text{ where } \vec{0} = (0, 0, 0)$$

$$\Rightarrow (a+y, a-y, y) = (0, 0, 0) \in \mathbb{R}^3$$

H.W.

$$\begin{cases} x-y=0 \\ y=0 \end{cases} \Rightarrow x=0, y=0 \\ \therefore x = (x_1 y) = (0, 0) \in \mathbb{R}^2$$

i. the nullspace of T consists of only zero vector of \mathbb{R}^2
 $\therefore \text{nullity } T = \dim N(T) = 0$

(ii) Range space of $T = \{ \mathbf{r} \in \mathbb{R}^3 / T(\mathbf{r}) = \mathbf{r} \text{ for all } \mathbf{r} \in \mathbb{R}^3 \}$

\therefore The range space consists of all vectors of the type $(x+y, x-y, y)$ for all $(x, y) \in \mathbb{R}^2$

(iii) $\dim R(T) + \dim N(T) = \dim \mathbb{R}^3$

$$\Rightarrow \dim R(T) + 0 = 3$$

$$\Rightarrow \dim R(T) = 3$$

\Rightarrow range of $T = \mathbb{R}^3$

→ show that $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given by

$T(x_1, y_1, z_1, t_1) = (2x_1, 2y_1, 0, 0)$ is a linear transformation. find its range and nullity.

Sol Let $\mathbf{x} = (x_1, y_1, z_1, t_1)$ and $\mathbf{r} = (x_2, y_2, z_2, t_2)$ be two vectors of \mathbb{R}^4 .

For $a, b \in \mathbb{R}$

$$\begin{aligned} T(ax + b\mathbf{r}) &= T[(a(x_1, y_1, z_1, t_1)) + b(x_2, y_2, z_2, t_2))] \\ &= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2, at_1 + bt_2) \\ &= (2(ax_1 + bx_2), 2(ay_1 + by_2), 0, 0)_{\mathbb{R}^4} \end{aligned}$$

$$= a(2x_1, 2y_1, 0, 0) + b(2x_2, 2y_2, 0, 0)$$

$$= a T(x_1) + b T(x_2)$$

$\therefore T$ is a linear transformation.

NOW we have $N(T) = \{(x_1, y_1, z_1, t) \in \mathbb{R}^4 \mid T(x_1, y_1, z_1, t) = (0, 0, 0, 0)\}$

$$\therefore (x_1, y_1, z_1, t) \in N(T)$$

$$\iff T(x_1, y_1, z_1, t) = (0, 0, 0, 0)$$

$$\iff (2x_1, 2y_1, 0, 0) = (0, 0, 0, 0)$$

$$\iff x_1 = 0, y_1 = 0$$

$$\therefore N(T) = \{(0, 0, z_1, t) \mid z_1, t \in \mathbb{R}\}$$

since $(0, 0, z_1, t) = z(0, 0, 1, 0) + t(0, 0, 0, 1)$

$\therefore N(T)$ is spanned by the set

$$S = \{e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$$

Clearly which is L.T

$\therefore S$ is basis of $N(T)$

$$\therefore \dim N(T) = 2$$

$$\boxed{\text{Nullity of } T = 2} \quad \text{i.e. } r(T) = 2$$

N.K.T

$$\therefore \dim R(T) + \dim N(T) = \dim \mathbb{R}^4$$

$$\Rightarrow \dim R(T) + 2 = 4$$

$$\Rightarrow \boxed{\dim R(T) = 2}$$

Show that $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by

$T(x_1, y_1, z_1, t) = (x_1y_1, x_1y_1, 0, 0)$ is a linear transformation. find rank and nullity

→ find the range, rank, kernel (Q)
and nullity of the linear transformation
 $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 + a_2, a_3 - a_1)$

sol Range space of $T = \{p \in \mathbb{R}^2 / T(K) = p\}$ for $K \in \mathbb{R}^3$.
The range space consists of all vectors of the
type $(a_1 + a_2, a_3 - a_1)$ for all $(a_1, a_2, a_3) \in \mathbb{R}^3$.
Let $p = (a, b) \in \mathbb{R}^2$ be arbitrary,
 $+ (a_1, a_2, a_3) = (a, b)$ for some
 $(a_1, a_2, a_3) \in \mathbb{R}^3$.

$$\Rightarrow (a_1 + a_2, a_3 - a_1) = (a, b)$$

$$\Rightarrow (a, b) = (a_1 + a_2, a_3 - a_1)$$

$$= (a_1 + a_2 + 0a_3, -a_1 + 0a_2 + 2a_3)$$

$$\equiv a_1(1, -1) + a_2(1, 0) + 2a_3(0, 1)$$

Here $s = \{(1, 0), (0, 1)\}$ is LI and

$\therefore R(T) \subseteq L(s) \text{ & } L(s) \subseteq R(T) \quad (1, -1) \in L(s)$.

$\Rightarrow L(s) = R(T)$. $\quad (\because (1, -1) \in (1, 0) + (0, 1))$

$\therefore s$ is a basis of $R(T)$.

$$\text{rank } T = \dim R(T) = 2.$$

NOW we have $\text{ker } T = \{(a_1, a_2, a_3) \in \mathbb{R}^3 / T(a_1, a_2, a_3) = (0, 0)\}$
Let $(a_1, a_2, a_3) \in \text{ker } T$ be arbitrary.

$$\text{Then } T(a_1, a_2, a_3) = (0, 0)$$

$$\text{i.e. } (a_1 + a_2, a_3 - a_1) = (0, 0) \quad (\text{by given})$$

$$\Rightarrow a_1 + a_2 = 0 \quad 2a_3 - a_1 = 0$$

$$\Rightarrow a_1 = a_2, \quad a_3 = \frac{a_1}{2}$$

$$\therefore \text{ker } T = \{(a_1, a_1, a_1/2) / a_1 \in \mathbb{R}\}.$$

$$\text{Since } (a_1, a_1, a_1/2) = a_1(1, 1, 1/2)$$

$\therefore \text{ker } T$ is spanned by the set $s = \{(1, 1, 1/2)\}$

$\therefore S$ is basis of ~~ker~~ T .

$\therefore \dim \text{ker } T = 1$

i.e. nullity $T = 1$.

→ Find the range, rank, Kernel and nullity of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$$

→ Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (x-y+2z, 2x+y-z, -x-2y)$$

Find the null space of T .

→ Let T be the linear transformation from \mathbb{R}^3 to \mathbb{R}^2 defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$$

for each $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Determine a basis for the null space of T .

What is the dimension of the Range space of T ?

→ Let $T: \mathbb{R}^5 \rightarrow \mathbb{R}^5$ be a linear mapping given by

$$T(a, b, c, d, e) = (b-d, d+e, b, 2d+e, b+e)$$

Obtain bases for its nullspace and range space.

→ Show that $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a linear transformation.

Where $f(x, y, z) = 3x+y-2z$. What is the dimension of the kernel? Find a basis for the kernel.

→ Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a linear transformation defined by $T(x, y, z) = (x+y, z+x)$.

Find a basis, dimension of each of the range

and null space of T .

→ Let $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ is defined by

$$T(a, b, c) = (a, b) + (a, b, c) \in \mathbb{R}^2$$

D.T. T is a linear transformation. Find the kernel of T .

Let $V(F)$ be a vectorspace and T be a linear operator on V . Prove that the following statements are true.

(i) The intersection of the range of T and

nullspace of T is the zero subspace.

$$\text{i.e. } R(T) \cap N(T) = \{\bar{0}\}.$$

(ii) If $T[T(w)] = \bar{0}$, then $T(w) = \bar{0}$.

Sol: (i) \Rightarrow (ii)

$$\text{Let } R(T) \cap N(T) = \{\bar{0}\}.$$

$$\text{Let } T(\rho) = \rho \quad \therefore \rho \in R(T).$$

$$\text{Now } T[T(w)] = \bar{0} \Rightarrow T(\rho) = \bar{0} \Rightarrow \rho \in N(T).$$

From (i) & (ii) $\rho \in R(T) \cap N(T)$

$$\text{But } R(T) \cap N(T) = \{\bar{0}\} \Rightarrow \rho = \bar{0}$$

$$\Rightarrow T(w) = \bar{0}.$$

$$\therefore T[T(w)] = \bar{0} \Rightarrow T(w) = \bar{0}.$$

(ii) \Rightarrow (i)

$$\text{Given } T[T(w)] = \bar{0} \Rightarrow T(w) = \bar{0}.$$

$$\text{Let } \rho \in R(T) \cap N(T)$$

$$\Rightarrow \rho \in R(T) \text{ and } \rho \in N(T)$$

$$\text{Now } \rho \in R(T) \Rightarrow T(x) = \rho \text{ for some } x \in V$$

$$\text{and } \rho \in N(T) \Rightarrow T(\rho) = \bar{0}$$

$$\Rightarrow T[T(w)] = \bar{0}$$

$$\Rightarrow T(w) = \bar{0}$$

$$\Rightarrow \rho = \bar{0} \quad (\because T(w) = \bar{0})$$

$$\therefore R(T) \cap N(T) = \{\bar{0}\}.$$

Note:- If $T: V \rightarrow W$ is a linear transformation
then $\rho(T) = \min(\dim V, \dim W)$.

→ Is there a linear transformation
 $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ for which $\text{rank } T = 3$ and
 $\text{nullity } T = 2$?

Sol If $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is a linear
transformation, then
 $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^4$
 $\Rightarrow 3 + 2 = 4$
this is impossible.

Hence T is not a linear
transformation.

Let T be a linear transformation
from \mathbb{R}^7 onto a 3-dimensional subspace of
 \mathbb{R}^5 . Find $\dim \ker T$.

Sol Let w be a 3-dimensional subspace
of \mathbb{R}^5 such that $T: \mathbb{R}^7 \rightarrow w$ is an
onto L.T.

We have
 $T(\mathbb{R}^7) = w \Rightarrow \dim T(\mathbb{R}^7) = \dim w = 3$

$$\therefore \dim \ker T = \dim(\mathbb{R}^7) - \dim T(\mathbb{R}^7) = 7 - 3 = 4$$

$$\therefore \text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^7$$

$$\Rightarrow 3 + \text{nullity}(T) = 7$$

$$\Rightarrow \text{nullity}(T) = 7 - 3 = 4$$

$$\Rightarrow \text{nullity}(T) = 4$$

$$\therefore \dim \ker T = 4$$

→ Let T be a linear transformation from \mathbb{R}^5 to \mathbb{R}^3 having a 2-dimensional Kernel. Find $\dim \text{range } T$. (Q)

Sol. Given that $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is L.T
and having a 2-dimensional Kernel.

$$\therefore \dim \text{Ker } T = 2 \Rightarrow \text{nullity}(T) = 2$$

$$\because \text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^5$$

$$\therefore \text{rank}(T) + 2 = 5$$

$$\Rightarrow \text{rank}(T) = 3$$

$$\Rightarrow \boxed{\dim \text{range } T = 3}$$

* Singular and Non-Singular Transformation:

Singular Transformation:

A linear transformation $T: U(\mathbb{R}) \rightarrow V(\mathbb{R})$ is said to be singular if the nullspace of T consists of at least one non-zero vector.

i.e. If there exists a vector $a \in U$

s.t. $T(a) = \vec{0}$ for $a \neq \vec{0}$ then T is singular.

Non-Singular Transformation:

A linear transformation $T: U(\mathbb{R}) \rightarrow V(\mathbb{R})$ is said to be non-singular if the nullspace consists of one zero vector alone.

i.e. $a \in U$ and $T(a) = \vec{0} \Rightarrow a = \vec{0}$

$$\Rightarrow N(T) = \{\vec{0}\}.$$

Theorem Let $V(F)$ and $V(\mathbb{R})$ be two vector spaces and $T: V \rightarrow V$ be a linear transformation. Then T is non-singular iff the set of images of a linearly independent set is linearly independent.

proof (i) Let T be non-singular and

let $S = \{x_1, x_2, \dots, x_n\}$ be a L.I. subset of V . Then its T -images set

$$S' = \{T(x_1), T(x_2), \dots, T(x_n)\}.$$

Now to prove S' is L.I.

for some $a_1, a_2, \dots, a_n \in F$,

$$a_1 T(x_1) + a_2 T(x_2) + \dots + a_n T(x_n) = \vec{0}$$

$$\Rightarrow T(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = \vec{0} \quad (\because \vec{0} \text{ ev})$$

$$\Rightarrow a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \vec{0} \quad (\because T \text{ is L.I.})$$

$$\Rightarrow a_1 = a_2 = \dots = a_n = 0 \quad (\because S \text{ is L.I.})$$

$\therefore S'$ is L.I.

(ii) Let the T -images of any L.I. set be L.I. then to prove T is non-singular.

Let $x \in V$ and $x \neq \vec{0}$. Then the set $A = \{x\}$ is L.I. set and image set

$$A' = \{T(x)\} \text{ is given to be L.I.}$$

$$\Rightarrow T(x) \neq \vec{0}$$

$$\therefore x \neq \vec{0} \Rightarrow T(x) \neq \vec{0}$$

$\therefore T$ is non-singular.

problem

A linear mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by
 $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$

Show that T is non-singular.

$$\Rightarrow (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z) = (0, 0, 0)$$

$$\Rightarrow x \cos \theta - y \sin \theta = 0 \quad (i)$$

$$x \sin \theta + y \cos \theta = 0 \quad (ii)$$

$$\therefore z = 0$$

Multiplying and adding eqns (i) & (ii).

$$\therefore x^2 + y^2 = 0$$

$$\Rightarrow x = 0, y = 0$$

$$\therefore x = 0, y = 0, z = 0$$

\therefore we have, $T(1, 1, 1) = \vec{0}$

$$\Rightarrow (1, 1, 1) = (0, 0, 0)$$

$\therefore T$ is non-singular

→ Show that a linear transformation $T: V \rightarrow V$ over the field F is non-singular iff T is one-one.

(i). Let T be non-singular

$$\text{i.e., } \alpha \in V, T(\alpha) = \vec{0} \Rightarrow \alpha = \vec{0}$$

Now for $\alpha_1, \alpha_2 \in V$,

$$T(\alpha_1) = T(\alpha_2)$$

$$\Rightarrow T(\alpha_1 - \alpha_2) = \vec{0} \quad (\because \vec{0} \in V)$$

$$\Rightarrow T(\alpha_1 - \alpha_2) = \vec{0} \quad (\because T \text{ is L.T})$$

$$\Rightarrow \alpha_1 - \alpha_2 = \vec{0} \quad (\because T \text{ is non-singular})$$

$$\Rightarrow \alpha_1 = \alpha_2$$

$\therefore T$ is one-one.

(ii). Let T be one-one.

\therefore zero at $\vec{0}$ of V is the T -image of only one element $\in V$.

\Rightarrow null space of T consists of only one element.

Since null space $N(T) \subset V$, it must consist of $\vec{0}$.

\Rightarrow null space $N(T)$ consists of only one $\vec{0}$ element.

$$\Rightarrow N(T) = \{\vec{0}\}$$

$\Rightarrow T$ is non-singular.

Let $T: U \rightarrow V$ be a linear transformation of $U(F)$ into $V(F)$ where $V(F)$ is finite dimensional. prove that U and the range space of T have the same dimension iff T is non-singular.

Sol: (i) Let $\dim U = \dim R(T)$

$$\text{By defn.} \quad \dim U = \dim R(T) + \dim N(T)$$

$$\Rightarrow \dim R(T) = \dim R(T) + \dim N(T)$$

$$\Rightarrow \dim N(T) = 0$$

\Rightarrow the null space of T is the zero space $\{\vec{0}\}$.

$\therefore T$ is non-singular.

(ii) Let T be non-singular. Then $N(T) = \{\vec{0}\}$ and nullity $T = 0$ i.e. $\dim(N(T)) = 0$.

$$\begin{aligned} \text{As } \dim U &= \dim R(T) + \dim N(T) \\ &= \dim R(T) + 0 \end{aligned}$$

$$\Rightarrow \dim U = \dim R(T).$$

Shashank
Maths

If U and V are finite dimensional vector spaces of the same dimension, then a linear mapping $T: U \rightarrow V$ is one-one iff it is onto.

Sol: T is one-one $\Leftrightarrow N(T) = \{\vec{0}\}$

$$\Leftrightarrow \dim N(T) = 0$$

$$\Leftrightarrow \dim R(T) + \dim N(T) = \dim U = \dim V$$

$$\Leftrightarrow R(T) = V$$

$$\Leftrightarrow T \text{ is onto.}$$

Inverse function:

Let $T: U \rightarrow V$ be a one-one onto mapping.

Then the mapping $T^{-1}: V \rightarrow U$ defined by

$$T^{-1}(v) = u \Leftrightarrow T(u) = v, \quad u \in U, v \in V, \text{ is called}$$

the inverse mapping of T .

Note: If $T: U \rightarrow V$ is one-one onto mapping

then the mapping $T^{-1}: V \rightarrow U$ is also one-one onto.

Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \rightarrow V$ be a one-one onto linear transformation. Then T^{-1} is a linear transformation and thus T is said to be invertible.

Sol. Let $\rho_1, \rho_2 \in V$ and $a, b \in F$

since T is one-one onto function,
there unique vectors $x_1, x_2 \in U$ s.t

$$T(x_1) = \rho_1 \quad \text{and} \quad T(x_2) = \rho_2$$

Hence by the definition of T^{-1}

$$x_1 = T^{-1}(\rho_1) \quad \text{and} \quad x_2 = T^{-1}(\rho_2)$$

also $x_1, x_2 \in U$ and $a, b \in F \Rightarrow ax_1 + bx_2 \in U$

$$\begin{aligned} \therefore T(ax_1 + bx_2) &= aT(x_1) + bT(x_2) \quad (\because T \text{ is } L) \\ &= a\rho_1 + b\rho_2 \end{aligned}$$

∴ by the defn of inverse

$$\begin{aligned} T^{-1}(a\rho_1 + b\rho_2) &= ax_1 + bx_2 \\ &= aT^{-1}(\rho_1) + bT^{-1}(\rho_2) \end{aligned}$$

$\therefore T^{-1}$ is a linear transformation
from V into U .

\rightarrow A linear transformation T on a finite dimensional vectorspace is invertible iff. T is non-singular.

Q. Let $U(\mathbb{F})$ and $V(\mathbb{F})$ be two vectorspaces and have the same dimension.

Let $T: U \rightarrow V$ be a linear transformation.

(i) Let T be non-singular.

$$\text{i.e. for } x \in U, T(x) = \vec{0} \Rightarrow x = \vec{0}$$

NOW TO PROVE T IS INVERTIBLE,

IT IS ENOUGH TO SHOW T IS ONE-ONE onto.

SINCE T IS NON-SINGULAR,

$$\text{i.e. for } x \in U, T(x) = \vec{0} \Rightarrow x = \vec{0}, N(T) = \{\vec{0}\}$$

$$\Rightarrow \dim N(T) = 0.$$

$$\text{FOR } x_1, x_2 \in U, T(x_1) = T(x_2)$$

$$\Rightarrow T(x_1) - T(x_2) = \vec{0}$$

$$\Rightarrow T(x_1 - x_2) = \vec{0} \quad (\because T \text{ is L.T.})$$

$$\Rightarrow x_1 - x_2 = \vec{0} \quad (\because T \text{ is non-singular})$$

$$\Rightarrow x_1 = x_2$$

$\therefore T$ IS ONE-ONE.

$$\text{N.W.T. } \dim U = \dim R(T) + \dim N(T)$$

$$= \dim R(T) \quad (\because \dim N(T) = 0).$$

ALSO $T: U \rightarrow V$ IS ONE-ONE

$$\Rightarrow V = R(T)$$

$\Rightarrow T$ IS INTO.

(ii) Let T BE INVERTIBLE SO THAT T IS

ONE-ONE onto.

T IS NON-SINGULAR. NOW TO PROVE T IS NON-SINGULAR.

$$\text{FOR } x \in U, T(x) = \vec{0} = T(\vec{0}) \quad (\because T \text{ IS LT.})$$

$$\Rightarrow T(x) = T(\vec{0}) \Rightarrow x = \vec{0} \quad (\because T \text{ IS non-singular})$$

Now Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces s.t. $\dim U = \dim V$.
 If $T: U \rightarrow V$ is a linear transformation
 then the following are equivalent.

- (1) T is invertible
- (2) T is non-singular
- (3) The range of T is V .
- (4) If $\{x_1, x_2, \dots, x_n\}$ is any basis of U , then $\{T(x_1), T(x_2), \dots, T(x_n)\}$ is a basis of V .
- (5) There is some basis $\{x_1, x_2, \dots, x_n\}$ of U s.t. $\{T(x_1), T(x_2), \dots, T(x_n)\}$ is a basis of V .

Here we shall have a series of implications $\textcircled{1} \Rightarrow \textcircled{2} \Rightarrow \textcircled{3} \Rightarrow \textcircled{4} \Rightarrow \textcircled{5} \Rightarrow \textcircled{1}$

- problems

\rightarrow If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear operator defined by $T(a, y, z) = (2a, 4a+y, 2a+3y-z)$.
 find T^{-1} .

Sol: since T is invertible -

$$T(x) = e \Rightarrow T^{-1}(e) = x, \quad x \in \mathbb{R}^3, \quad e \in \mathbb{R}^3.$$

$$\text{now } T(a, b, c) = (a, b, c) \Rightarrow T^{-1}(a, b, c) = (a, b, c)$$

$$\text{Now } (2a, 4a+y, 2a+3y-z) = (a, b, c)$$

$$\Rightarrow 2a = a, \quad 4a+y = b, \quad 2a+3y-z = c$$

$$\text{Solving } a = 0, \quad y = 2a - b, \quad z = 2c - 3b - c$$

$$\text{Hence } T^{-1}(a, b, c) = (0, 2a-b, 2c-3b-c).$$

\rightarrow The set $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$. $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is a linear operator defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$.

Show that T is non-singular and
find its inverse.

So! Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

Now $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(e_1) = e_1 + e_2 \Rightarrow T(1, 0, 0) = (1, 1, 0)$$

$$T(e_2) = e_2 + e_3 \Rightarrow T(0, 1, 0) = (0, 1, 1)$$

$$T(e_3) = e_1 + e_2 + e_3 \Rightarrow T(0, 0, 1) = (1, 1, 1)$$

Let $\alpha = (x, y, z) \in V_3(\mathbb{R})$

$$\therefore \alpha = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\Rightarrow T(\alpha) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$$

$$= x(1, 1, 0) + y(0, 1, 1) + z(1, 1, 1)$$

\therefore The transformation is given by

$$T(x, y, z) = (x+z, x+y+z, y+z).$$

Now if $T(\alpha, y, z) = \vec{0}$ then

$$(x+z, x+y+z, y+z) = (0, 0, 0)$$

$$\Rightarrow x+z=0, x+y+z=0, y+z=0.$$

$$\Rightarrow x=y=z=0$$

$$\therefore T(\alpha) = \vec{0} \Rightarrow \alpha = \vec{0}$$

Hence T is non-singular

∴ therefore T^{-1} exists.

$$\text{Let } T(x, y, z) = (a, b, c) \text{ then}$$

$$T^{-1}(a, b, c) = (x, y, z)$$

$$\text{Now } (x+z, x+y+z, y+z) = (a, b, c)$$

$$\Rightarrow x+z=a, x+y+z=b, y+z=c$$

$$\Rightarrow \boxed{x = b - c}, \boxed{y = a - c}, \boxed{z = a - b + c}$$

$$T(a, b, c) = (a+y, b)$$

$$= (b-c, b-a, a+b).$$

~~Ques.~~ Show that each of the following linear operators T on \mathbb{R}^3 is invertible and find T^{-1} .

$$(a) T(x, y, z) = (2x, 4x-y, 2x+3y-z)$$

$$(b) T(a, b, c) = (a-3b-2c, b-4c, c)$$

$$(c) T(a, b, c) = (3a, a+b, 2a+b+c)$$

$$(d) T(x, y, z) = (x+y+z, y+z, z)$$

$$(e) T(a, b, c) = (a-2b-5c, b-c, a).$$

~~Ques.~~ The set $\{e_1, e_2, e_3\}$ is the standard basis set of $V_3(\mathbb{R})$. The linear operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined below. Show that T is invertible and find T^{-1} .

$$(i) T(e_1) = e_1 + e_2, T(e_2) = e_1 - e_2 + e_3, T(e_3) = 3e_1 + 4e_3.$$

$$(ii) T(e_1) = e_1 - e_2, T(e_2) = e_2, T(e_3) = e_1 + e_2 - 7e_3.$$

$$(iii) T(e_1) = e_1 - e_2 + e_3, T(e_2) = 2e_1 - 5e_3, T(e_3) = 3e_1 - 2e_3.$$

~~2002~~ Show that the mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where $T(a, b, c) = (a-b, b-c, a+c)$ is linear and non-singular.

Matrix of Linear Transformation

Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces such that $\dim U = n$ and $\dim V = m$. Let $T: U \rightarrow V$ be a linear transformation.

Let $B_1 = \{x_1, x_2, \dots, x_n\}$ be the ordered basis of U and $B_2 = \{v_1, v_2, \dots, v_m\}$ be the ordered basis of V .

for every $x \in U \Rightarrow T(x) \in V$ and $T(x)$ can be expressed as a linear combination of elements of the basis B_2 :

If \exists : Here exists $a_i \in F$ s.t

$$T(x_1) = a_{11}v_1 + a_{12}v_2 + \dots + a_{1m}v_m$$

$$T(x_2) = a_{21}v_1 + a_{22}v_2 + \dots + a_{2m}v_m$$

$$T(x_j) = a_{j1}v_1 + a_{j2}v_2 + \dots + a_{jm}v_m \quad \text{④}$$

$$T(x_n) = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nm}v_m$$

Writing the co-ordinates $T(x_1), T(x_2), \dots, T(x_n)$ successively as columns of a matrix we get,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jj} & \dots & a_{jm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nm} \end{bmatrix} \text{ matrix}$$

This matrix represented as $[a_{ij}]_{m \times n}$ is called the matrix of the linear transformation T with respect to the bases B_1 and B_2 .

$$\text{Symbolically } [T : B_1, B_2] \text{ or } [T] = [a_{ij}]_{m \times n}$$

Hence the matrix $[a_{ij}]_{m \times n}$ completely determines the linear transformation through the relations given in (4).

Hence the matrix $[a_{ij}]_{m \times n}$ represents the transformation T .

Note: Let $T: V \rightarrow V$ be a linear operator such that $\dim V = n$.

If $B_1 = B_2 = B$ (say) then the above said matrix is called the matrix of T relative to the ordered basis B .

It is denoted by $[T : B] = [T]_B = [a_{ij}]_{n \times n}$.

problems

Let $T: V_2 \rightarrow V_2$ be defined by

$$T(x, y) = (x+y, xy, x^2)$$

Find $[T : B_1, B_2]$ where B_1 and B_2 are the standard bases of V_1 and V_2 .

Sol. B_1 is standard basis of V_2 and B_2 is standard basis of V_1 .

$$\therefore B_1 = \{(1, 0), (0, 1)\}$$

$$B_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

$$\text{Now } T(1, 0) = (1, 0, 1)$$

$$= 1(1, 0, 0) + 2(0, 1, 0) + 0(0, 0, 1)$$

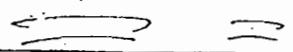
$$T(0, 1) = (1, -1, 1)$$

$$= 1(1, 0, 0) - 1(0, 1, 0) + 1(0, 0, 1)$$

NP

bases B_1 and B_2 is

$$[T; B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$



Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, y, z) = (3x+2y-4z, x-5y+3z)$$

Find the matrix of T relative to the bases $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$$B_2 = \{(1, 3), (2, 5)\}$$

sol

Let $(a, b) \in \mathbb{R}^2$ and

$$\text{let } (a, b) = p(1, 1) + q(2, 5)$$

$$= (p+2q, p+5q)$$

$$\Rightarrow p+2q = a, p+5q = b$$

$$\text{Solving } p = -5a+2b, q = 3a-b$$

$$\therefore (a, b) = (-5a+2b)(1, 3) + (3a-b)(2, 5) \quad (1)$$

$$\text{Now, } T(1, 1, 1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= -7(1, 3) + 4(2, 5) \quad (\text{from (1)})$$

$$T(1, 1, 0) = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

$$= -33(1, 3) + 19(2, 5) \quad (\text{from (1)})$$

$$T(1, 0, 0) = (3, 1)$$

$$= -13(1, 3) + 8(2, 5) \quad (\text{from (1)})$$

∴ the matrix of $L \cdot T$ relative to B_1 and B_2 is

$$[T: V_2(\mathbb{R})] = \begin{bmatrix} -7 & -2 & 2 \\ 4 & 19 & 8 \end{bmatrix}$$

\rightarrow If the matrix of a linear operator

T on $V_2(\mathbb{R})$ w.r.t the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Describe explicitly $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$

What is the matrix of T w.r.t the basis $\{(0,1,-1), (1,-1,1), (1,1,0)\}$.

Q) (i) Let the standard basis of $V_2(\mathbb{R})$ be

$$\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

Let $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (0,0,1)$

$$\text{Given } [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} T(x_1) &= 0x_1 + 1x_2 + (-1)x_3 \\ &= 0(1,0,0) + 1(0,1,0) + (-1)(0,0,1) \\ &= (0,1,-1) \end{aligned}$$

$$T(x_2) = 1x_1 + 0x_2 + (-1)x_3 = (1,0,-1)$$

$$T(x_3) = 1x_1 + (-1)x_2 + 0x_3 = (1,-1,0)$$

Let $(a,b,c) \in V_2(\mathbb{R})$ Then

$$\begin{aligned} (a,b,c) &= c(1,0,0) + b(0,1,0) + a(0,0,1) \\ &= a x_1 + b x_2 + c x_3. \end{aligned}$$

$$\therefore T(a,b,c) = a T(x_1) + b T(x_2) + c T(x_3)$$

$$= c(0,1,-1) + b(1,0,-1) + a(1,-1,0)$$

$$= (b+c, a-b, -a-c)$$

which is the reqd L.T.

(ii) Let $\beta_2 = \{e_1, e_2, e_3\}$, where

$$e_1 = (0, 1, -1), e_2 = (1, -1, 1), e_3 = (-1, 1, 0)$$

Using the transformation

$$\because T(a, b, c) = (a+b, a-b, -a),$$

we have

$$T(e_1) = T(0, 1, -1) = (0, 1, -1)$$

$$T(e_2) = T(1, -1, 1) = (0, -2, 0)$$

$$T(e_3) = T(-1, 1, 0) = (1, -1, 0)$$

Now let $(a, b, c) = x e_1 + y e_2 + z e_3$

$$= x(0, 1, -1) + y(1, -1, 1)$$

$$+ z(-1, 1, 0)$$

$$= (y-z, x-y+z, -x+y)$$

$$\begin{aligned} \Rightarrow y-z &= a \\ x-y+z &= b \\ -x+y &= c \end{aligned}$$

$x = a - b$

$y = a + b - c$

$z = b - c$

$$\therefore (a, b, c) = (a-b) e_1 + (a+b-c) e_2 + (b-c) e_3 \quad \text{①}$$

$$\therefore T(e_1) = (0, 1, -1)$$

$$= 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 \quad (\text{from ①})$$

$$T(e_2) = (0, -2, 0) = 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$$

$$+ (e_3) = (1, -1, 0) = 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3$$

$$\therefore [T : \beta_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $D: P_3 \rightarrow P_2$ be the polynomial differentiation transformation. $D(p) = \frac{dp}{dx}$.
Find the matrix of D relative to the standard bases.

$$B_1 = \{1, x, x^2, x^3\} \text{ and } B_2 = \{1, x, x^2\}.$$

$$\text{So } D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$D(x) = 1 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2$$

$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$$

∴ the matrix of D relative to B_1 ,

and B_2 is $[T: B_1, B_2]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let T be a linear transformation on \mathbb{R}^3 , whose matrix relative to the standard basis of \mathbb{R}^3 is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3x & 4 \end{bmatrix}.$$

Find the matrix of T relative to the basis $B = \{(1,1,1), (1,1,0), (0,1,1)\}$.

If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$T(x,y) = (2x - 3y, x+y).$$

Compute the matrix of T relative to the basis $B = \{(1,2), (2,3)\}$.

(89)

$$\text{Let } R_3[x] = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}.$$

Define $T: R_3[x] \rightarrow R_3[x]$ by $T(f(x)) = \frac{d}{dx} f(x)$ (1)

for all $f(x) \in R_3[x]$. Show that T is a

linear transformation. Also find the
matrix representation of T with reference

to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

Sol. Let $f(x), g(x) \in R_3[x]$ and $a, b \in \mathbb{R}$

By (1), we have

$$\begin{aligned} T(a f(x) + b g(x)) &= \frac{d}{dx} (a f(x) + b g(x)) \\ &= a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x) \end{aligned}$$

$$= a T(f(x)) + b T(g(x))$$

(T is a linear transformation.)

Now $T(1) = \frac{d}{dx}(1) = 0$

$$T(x) = \frac{d}{dx}(x) = 1$$

$$T(x^2) = \frac{d}{dx}(x^2) = 2x$$

Again $T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$$T(x) = 1 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

Hence the matrix representation of T
w.r.t. the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Now } T(1) = \frac{d}{dx}(1) = 0$$

$$T(1+\lambda) = \frac{d}{dx}(1+\lambda) = 1$$

$$T(1+\lambda+\lambda^2) = \frac{d}{dx}(1+\lambda+\lambda^2) = 1+2\lambda.$$

$$\text{Again } T(1) = 0 = 0 \cdot 1 + 0(1+\lambda) + 0(1+\lambda+\lambda^2).$$

$$T(1+\lambda) = 1 = 1 \cdot 1 + 0 \cdot (1+\lambda) + 0(1+\lambda+\lambda^2)$$

$$T(1+\lambda+\lambda^2) = 1+2\lambda = 1 \cdot 1 + 2 \cdot (1+\lambda) + 0(1+\lambda+\lambda^2)$$

hence the matrix representation

w.r.t the basis $\{1, 1+\lambda, 1+\lambda+\lambda^2\}$
of T .

$$\text{is: } \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Let $R_4[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 / a_3 \neq 0\}$.

Define $T: R_4[x] \rightarrow R_4[x]$ as

$$T(f(x)) = \frac{d}{dx}(f(x)) \text{ for all } f(x) \in R_4[x].$$

Let $\ell = \{1, x, x^2, x^3\}$ be an ordered

basis $R_4[x]$. Find $[T]_{\ell}$.

Let V be the vector space of polynomials of degree ≤ 3 over \mathbb{F} . Let T be a linear transformation defined on V

$$\text{by } T(a_0 + a_1x + a_2x^2 + a_3x^3) =$$

$$a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3. \quad (1)$$

compute the matrix of T relative to the bases (a) $\{1, x, x^2, x^3\}$ (b) $\{1+x, 1+x^2, 1+x^3\}$.

Sd

$$\textcircled{a} \quad T(1) = 1 = 1 + 0x + 0x^2 + 0x^3$$

$$T(x) = x+1 = 1 + 1 \cdot x + 0x^2 + 0x^3$$

$$T(x^2) = (x+1)^2 = 1 + 2x + 1 \cdot x^2 + 0x^3$$

$$T(x^3) = (x+1)^3 = 1 + 3x + 3x^2 + 1 \cdot x^3$$

Hence the matrix representation
of T w.r.t the basis $\{1, x, x^2, x^3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$T(1) = 1$$

$$T(1+x) = 1 + x$$

$$T(1+x^2) = 1 + x^2 = 1 + (1 + x^2 + 2x)$$

$$T(1+x^3) = 1 + (x+1)^3 = 1 + (x^3 + 1 + 3x^2 + 3x)$$

$$\text{Again } T(1) = 1 = 1 + 0 \cdot (1+x) + 0 \cdot (1+x^2) + 0 \cdot (1+x^3)$$

$$T(1+x) = 1 + (1+x) = 1 + 1 \cdot (1+x) + 0 \cdot (1+x^2) + 0 \cdot (1+x^3)$$

$$T(1+x^2) = 1 + 2 \cdot (1+x) + 1 \cdot (1+x^2) + 0 \cdot (1+x^3)$$

$$T(1+x^3) = 1 + (1+x^3) + (3x^2 + 3x)$$

$$= -5(1) + 3(1+x) + 3(1+x^2) + 1 \cdot (1+x^3)$$

Hence the matrix representation of T
w.r.t the basis $\{1, 1+x, 1+x^2, 1+x^3\}$ is

(P.T.O)

$$\begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider the vector space.

$X := \{ p(x) \mid p(x) \text{ is a polynomial of degree less than or equal to } 3 \text{ with real coefficients} \}$ over the

real field \mathbb{R} . Define the map $D: X \rightarrow X$

$$\text{by } D(p(x)) := p_1 + 2p_2 x + 3p_3 x^2$$

$$\text{where } p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3.$$

Is D a linear transformation on X ? If it is, then construct the matrix representation for D with respect to the ordered basis $\{1, x, x^2, x^3\}$ for X .

Sol: Let $p(x), q(x) \in X, a, b \in \mathbb{R}$.

Given map D defined by

$$D(p(x)) = p_1 + 2p_2 x + 3p_3 x^2$$

$$\text{where } p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$$

$$\text{i.e. } D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2.$$

$$\begin{aligned} \text{Now } D[a p(x) + b q(x)] &= D[a(p_0 + p_1 x + p_2 x^2 + p_3 x^3) \\ &\quad + b(q_0 + q_1 x + q_2 x^2 + q_3 x^3)] \\ &= D[(a p_0 + b q_0) + (a p_1 + b q_1)x + \\ &\quad (a p_2 + b q_2)x^2 + (a p_3 + b q_3)x^3] \\ &= (a p_1 + b q_1) + 2(a p_2 + b q_2)x \\ &\quad + 3(a p_3 + b q_3)x^2 \end{aligned}$$

$$\begin{aligned} &= a(p_1 + 2p_2 x + 3p_3 x^2) + b(q_1 + 2q_2 x + 3q_3 x^2) \\ &= a D(p(x)) + b D(q(x)) \end{aligned}$$

$\therefore D: X \rightarrow X$ is a linear transformation.

NOW

From ①,

$$D(0) = 0 = 0 + 0x + 0x^2 + 0x^3$$

$$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$$

$$D(x^2) = 2x = 0 + 2x + 0x^2 + 0x^3$$

$$D(x^3) = 3x^2 = 0 + 0x + 0x^2 + 3x^3$$

Hence the matrix representation of
D w.r.t the ordered basis $\{1, x, x^2, x^3\}$

is
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

