IMS MATHS BOOK-16

$|z_1+z_2| \le |z_1|+|z_2|$

$|z_1-z_2| \ge |z_1|-|z_2|$

$\overline{z_1+z_2} = \bar{z_1}+\bar{z_2}$

Set – I

* Complex Analysis *

Introduction:–

In the field of real numbers, the equation $x^2+1=0$ has no solution. To permit the solution of this and similar equations (i.e. $x^2-2x+3=0$ etc), the real number system was extended to the set of complex numbers. Euler introduced the symbol 'i' with the property that $i^2=-1$. He also called i as the imaginary unit.

– A number of the form $a+ib$ where a,b are real numbers, was called complex number.

If we write $z=x+iy$ then z is called a complex variable.

Also x is called real part of z and is denoted by $R(z)$ i.e. $R(z)=x$ and y is called imaginary part of z and is denoted by $I(z)$ i.e. $I(z)=y$.

– Some times we express z as $z=(x,y)$

– If $x=0$, i.e. $z=iy$ then z is called pure imaginary number.

– The conjugate of $z=x+iy$ is $\bar{z}=x-iy$.

– $Re(z) = x = \dfrac{z+\bar{z}}{2}$

– $I(z) = y = \dfrac{z-\bar{z}}{2i}$

* Fundamental operations with Complex Numbers:–

Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$

Subtraction: $(a+ib)-(c+id) = (a-c)+i(b-d)$

Multiplication: $(a+ib)(c+id) = (ac-bd)+i(bc+ad)$

Division: $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)}$

$= \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$= \dfrac{ac+bd}{c^2+d^2} + i\left(\dfrac{bc-ad}{c^2+d^2}\right)$

if $c^2+d^2 \neq 0$.

* Absolute value:

– The absolute value (or) modulus of a complex number $z=a+ib$ is denoted by $|z|$ and is defined as

$|z| = |a+ib|$

$= \sqrt{a^2+b^2}$

Evidently $|z|^2 = a^2+b^2$

$= (a+ib)(a-ib)$

$= z\bar{z}$

$\therefore |z|^2 = z\bar{z}$

Also $\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$

* Geometrical Representation of Complex Numbers:–

Consider the complex number $z=x+iy$.

A ~~complex number~~ can be regarded as an ordered pair of reals. i.e. $z=(x,y)$.

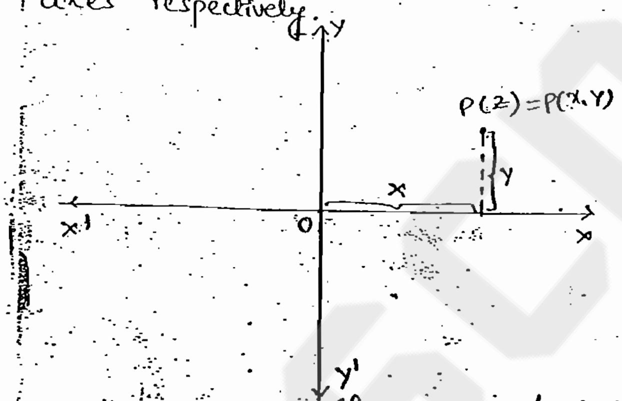
This form of z suggests that z can be represented by a point P whose coordinates are x & y relative to the rectangular axes x & y.

To each complex number there corresponds one and only one point in the xy-plane and conversely to each point in the plane there exists one and only one complex number. Due to this fact, the complex number z is referred to the point z in this plane.

This plane is called complex plane or Gaussian plane or Argand plane.

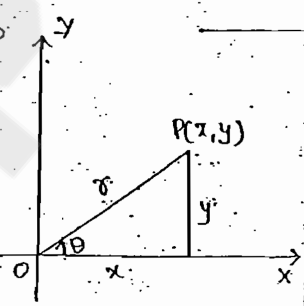
The representation of complex numbers is called Argand diagram.

The complex number x+iy is called complex coordinate and x, y axes are called real and imaginary axes respectively.



* Polar form of Complex Numbers:

Consider the point P in the complex plane corresponding to a non-zero complex number.



From the figure,

$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\Rightarrow x = r\cos\theta$, $y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$

i.e. $= |x+iy|$

$= |z|$

$\boxed{\therefore r = |z|}$

and $\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

It follows that,

$z = x+iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ ——— (1)

It is called polar form of the complex number z.

r and θ are called polar coordinates of z.

→ r is called modulus (or) absolute value of z.

→ The angle θ which the line OP makes with the +ve x-axis, is called argument (or) amplitude of z.

and is denoted by $\theta = \arg(z)$ (or) $\theta = \mathrm{amp}(z)$

→ The argument of z is not unique, since the equation (1) does not alter, if we replace θ by 2π+θ. So θ can have infinite number of values which differ from each other by 2π.

→ If a value of θ satisfies (1) and lies b/w −π & π.

i.e. $-\pi < \theta \leq \pi$ then that value of θ is called principal value of the argument.

Note:— It is evident from the definition of difference and modulus that $|z_1 - z_2|$

is the distance b/w two points z_1 & z_2

i.e $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$\therefore |z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

It follows that for fixed complex number z_0 and a real number 'r'. The equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r.

*** Point Set :-** Any collection of points in the complex (two dimensional) plane is called a point set and each point is called a member (or) element of the point set.

— The set of complex numbers is denoted by 'C'.

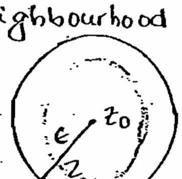
*** ϵ neighbourhood of a complex number z_0 :-**

The set of all points $z \in C$ satisfying the condition $|z - z_0| < \epsilon$ is defined as ϵ - neighbourhood of the z_0.

— A deleted neighbourhood of z_0 is neighbourhood of z_0 in which the point z_0 is omitted

i.e. $0 < |z - z_0| < \epsilon$.

— In general ϵ - neighbourhood of z_0 is denoted by $N(z_0, \epsilon)$.

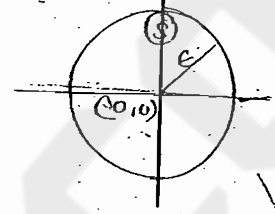


*** Bounded Set :-** A set 's' is said to be bounded if it is contained in some neighbourhood of the origin.

(or)

A set 's' is called bounded if we can find a constant ϵ such that $|z| < \epsilon \ \forall \ z \in s$.

— If a set is not bounded then it is said to be unbounded.



*** Interior Point :-**

A point z_1 of a set 's' is said to be an interior point of the set 's' if there exist a neighbourhood of z_1 which is contained completely in the set 's'.

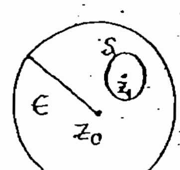
— If every neighbourhood of z_1 contains some points of 's' and some points that doesnot belong to 's' is called a boundary point.

— A point z_0 which is neither interior nor boundary point is called exterior point.

Example :-

Let $A = \{z \in C \,/\, |z - z_0| < \epsilon\}$

$B = \{z \in C \,/\, |z - z_0| \le \epsilon\}$



In this example every point of A is an interior point but not B.

*** Open Set :-** A set 's' is called an open set if every point in 's' is an interior Point.

Ex:- (i) the empty set

ii) the set of all complex numbers.

iii) $\{z: |z|>r\}, r\geq 0$

iv) $\{z: r_1<|z|<r_2\}, 0\leq r_1<r_2$

* **Limit Point** :- A point z_0 is said to be a limit point of 's' if every deleted neighbourhood of z_0 contains a point of 's'.

— Limit point is also known as Cluster point (or) point of accumulation.

— The limit Point of the set may (or) may not belong to the set.

Ex:- ① The limit points of open set $|z|<1$ are $|z|\leq 1$.

i.e. all the points of the set and all the points on the boundary $|z|=1$

2) The set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$ has '0' as a limit point.

3) The set $\left\{\frac{3+2ni}{1+n} \,\middle|\, n=1,2,3\ldots\right\}$

$$=\left\{\frac{3+2i}{2}, \frac{3+4i}{3}, \frac{3+6i}{4}, \ldots\right\}$$

has $2i$ as a limit point.

* **Closed Set** :- A set is said to be closed if it contains all its limit points.

Ex:- ① the empty set

(2). the set of all complex numbers.

(3). $\{z: |z|>r\}, r\geq 0$.

(4). $\{z: r_1\leq |z|\leq r_2\}, 0\leq r_1<r_2$

(5) the union of any two closed sets.

* **Closure of a set** :-

the union of a set and its limit points is called closure.

* **Domain (Region)** :-

— A set 's' of points in the complex plane is said to be connected set if any two of its points can be joined by a continuous curve, all of whose points belong to 's'.

— An open connected is called an open domain (or) open region

— If the boundary point of 's' are also added to an open domain then it is called closed domain.

* **Complex Variable** :-

If a symbol z takes any one of the values of a set of complex numbers, then z is called a complex variable. (or)

Let D be an arbitrary non-empty point set of XOY-plane. If z is allowed to denote any point of D, then z is called a complex variable and D is the domain of definition of z (or) simply domain.

* **Functions of a Complex Variable**

we say that 'w' is a function of the complex variable z with domain D and Range R, if D and R are two non-empty point sets of complex plane, if to each z in D there corresponds at least one w in R and to each w of R, there is at least one z of D to which w corresponds.

Then we symbolically write $w = f(z)$.

— The variable z is sometimes called independent variable and 'w' is called dependent variable.

The value of a function at $z=a$ is written as $f(a)$.

Thus if $f(z) = z^2$, $f(2i) = (2i)^2 = -4$

— If we have only one value 'w' of R to each value of z in D, then we say that 'w' is a single valued function of z (or) $f(z)$ is single valued.

— If more than one value of 'w' corresponds to each value of z, we say that 'w' is a multi valued (or) multiple valued function.

Ex:- ① Let $w = z^2$. Then corresponding to each value of z, we get only one value to 'w'.

So w is a single valued function.

This is because:

The function $w = z^2$ may be expressed as $w = f(z)$

$= f(x+iy)$

$= f(x,y)$

$= (x+iy)^2$

$= x^2-y^2+i(2xy)$

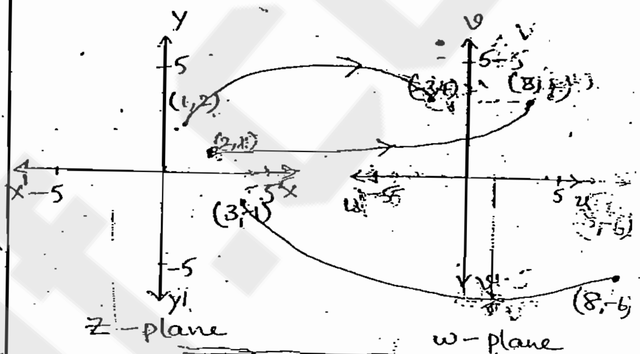
where $Re(w) = x^2-y^2$

$= u(x,y)$ say

and $Im(w) = 2xy$

$= v(x,y)$ say

$\boxed{\therefore w = u+iv}$



Example ②: Let $w = z^{1/2}$

Here to each value of z we get two values to 'w'. So we say multi-valued function.

This is because:

$$w = z^{1/2} = (x+iy)^{1/2}$$

$$= \sqrt{r}\, e^{i\theta/2} \quad \text{where} \quad x = r\cos\theta,\ y = r\sin\theta$$

Let $\theta = \theta_1$ then $w = \sqrt{r}\, e^{i\theta_1/2}$;

$\theta = \theta_1 + 2\pi$ then $w = \sqrt{r}\, e^{i(\theta_1+2\pi)/2}$

$$= \sqrt{r}\left[\cos(180+\theta_1/2) + i\sin(180+\theta_1/2)\right]$$

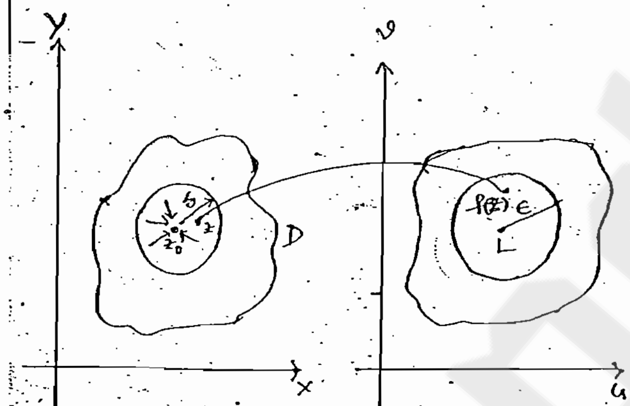
$$= \sqrt{r}\left[-\cos\theta_1/2 - i\sin\theta_1/2\right]$$

$$= -\sqrt{r}\, e^{i\theta_1/2}$$

we can verify that 'ω' gets the same values for θ_1 and $\theta_1+4\pi$.

* Limit of a Function:-

Let $f(z)$ be a function of a complex variable z. Then we say that $\lim\limits_{z\to z_0} f(z)=L$, if for any given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ (depending on ϵ) such that $|f(z)-L|<\epsilon$ whenever $0<|z-z_0|<\delta$.



The above results can also be written as, Let f be a function of two real variables x & y. we say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=L$ if for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-L|<\epsilon$ for every $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$.

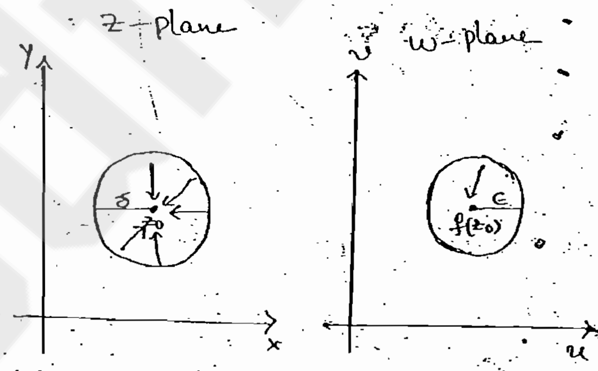
* Continuity of a Function:-

If $f(x_0,y_0)=L$, then we say that $f(x,y)$ is continuous at (x_0,y_0) (or) $f(z_0)=L$.

i.e. the value of the function at $z=z_0$ is equal to 'L', then we say that $f(z)$ is continuous at $z=z_0$.

(or)

$f(z)$ is continuous at $z=z_0$ if $\lim\limits_{z\to z_0} f(z)=f(z_0)$ i.e. if given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ depending on ϵ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$.



Note:- Here we are silent about how z approaches z_0, i.e. along which path it approaches z_0 is immaterial.

Note:- Let us consider $f(z)=z^2+3z+5$

Let $z=x+iy$ then $z^2=x^2-y^2+i2xy$

$$\therefore f(z)=z^2+3z+5$$
$$=(x^2-y^2+3x+5)+i(2xy+3y)$$
$$=f_1(x,y)+if_2(x,y).$$

$\therefore \lim_{x\to 0} f(x,0) = 0$

Similarly $\lim_{y\to 0} f(0,y) = 0$

$\therefore \lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} f(0,y)$

$\therefore$ The function is tending to zero along coordinate axes.

Now if we approach $(0,0)$ along the straight line path $y=mx$, we get

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

Since the limit depends upon the value of 'm'.

$\therefore f(x,y)$ approaches different values along different straight lines.

$\therefore$ The limit at the origin does not exist

$\therefore$ the function is not continuous at $(0,0)$

$\rightarrow$ Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{(x+y^2)^3} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

Sol'n: $\lim_{x\to 0} f(x,0) = \lim_{y\to 0} f(0,y) = 0$

$\therefore$ The function is tending to zero along the Coordinate axes.

Now if we approach $(0,0)$ along the straight line path $y=mx$, we get

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{m^2x^4}{(x+m^2x^2)^3}$$

$$= \lim_{x\to 0} \frac{m^2x}{(1+m^2x)^3}$$

$$= 0$$

$\therefore f(x,y)$ is tending to zero as $(x,y)\to(0,0)$ along any straight line.

But along the Parabola $x=y^2$

$$\lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{y^4 y^2}{(y^2+y^2)^3} = \frac{1}{8}$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$

$\rightarrow$ Let $f(x,y) = \begin{cases} \dfrac{x^3-2y^3}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

Show that $f(x,y)$ is Continuous at $(0,0)$.

Method (1):

Now we have

$$|f(x,y)-f(0,0)| = \left|\frac{x^3-2y^3}{x^2+y^2} - (0,0)\right|$$

$$= \left|\frac{x^3-2y^3}{x^2+y^2}\right|$$

Now

$$|x^3-2y^3| \leq |x^3| + |2y^3|$$

$$= |x|^3 + 2|y|^3$$

$$= |x||x|^2 + 2|y||y|^2$$

In general we can write

$$f(z) = u(x,y) + iv(x,y)$$

(or) $w = u + iv$

where u & v are functions of real variables x & y.

Theorem The function $f(z) = u(x,y) + iv(x,y)$ is continuous at a point $z_0 = x_0 + iy_0$ iff $u(x,y)$ & $v(x,y)$ are both continuous at the point (x_0, y_0).

Proof:- Suppose $f(z)$ is continuous at $z = z_0$ for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Now we have $|z - z_0| < \delta$.

$|u(x,y) - u(x_0,y_0)| \le |f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$

$(\because z = x + iy \Rightarrow |x| \le |z|$ & $|y| \le |z|.)$

$\therefore |u(x,y) - u(x_0,y_0)| < \epsilon$ whenever $|(x,y) - (x_0,y_0)| < \delta$.

Similarly $|v(x,y) - v(x_0,y_0)| < \epsilon$ whenever $|(x,y) - (x_0,y_0)| < \delta$

$\therefore u(x,y)$ & $v(x,y)$ are continuous at (x_0, y_0).

Conversely suppose that $u(x,y)$ & $v(x,y)$ are both continuous at (x_0,y_0)

$\therefore$ Given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$|u(x,y) - u(x_0,y_0)| < \epsilon/2$ &

$|v(x,y) - v(x_0,y_0)| < \epsilon/2$

whenever $|(x,y) - (x_0,y_0)| < \delta$.

Now we have

$$|f(z) - f(z_0)| = |u(x,y) + iv(x,y) - [u(x_0,y_0) + iv(x_0,y_0)]|$$

$$= |(u(x,y) - u(x_0,y_0)) + i(v(x,y) - v(x_0,y_0))|$$

$$\le |u(x,y) - u(x_0,y_0)| + |v(x,y) - v(x_0,y_0)|$$

$< \epsilon/2 + \epsilon/2$ whenever $|(x,y) - (x_0,y_0)| < \delta$

$= \epsilon$

$\therefore |f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

$\therefore f(z)$ is continuous at $z = z_0$.

Note:- In the case of a function of single real variable, there are only two directions to travel, a limit exists iff the RHL and LHL coincide.

— In the case of a function of two variables there are infinitely many directions are possible.

Problems:

Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \ne (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

whether this function is continuous (or) not!

Sol^n:- Along either of the coordinate axes.

Let us suppose that $(x,y) \to (0,0)$ along x-axes then $y = 0$.

$$\therefore f(x,y) = f(x,0) = \frac{x(0)}{x^2+0} = 0.$$

$\leq |z||x|^2 + 2|z||y|^2$

$\left(\because z = x+iy;\ |x| \leq |z| \ \& \ |y| \leq |z|\right)$

$= |z| \left(|x|^2 + 2|y|^2\right)$

$= |z| (x^2+2y^2)$

$< \sqrt{x^2+y^2}\ 2(x^2+y^2)$

$(\because x^2+2y^2 < 2(x^2+y^2))$

$\therefore \dfrac{|x^3-2y^3|}{|x^2+y^2|} < 2\sqrt{x^2+y^2}$

$\Rightarrow \left|\dfrac{x^3-2y^3}{x^2+y^2} - (0,0)\right| < 2|z-0|$
$< \epsilon$

whenever $|z-0| < \epsilon/2 = \delta$ (choosing)

$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$ whenever $|z-0| < \delta$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$\therefore f(x,y)$ is continuous at $(0,0)$

Method (2):

Switching to polar coordinates,

$x = r\cos\theta,\ y = r\sin\theta$

we show that $|f(r,\theta)| < \epsilon$ whenever $|r| < \delta$

Now we have

$|f(x,y) - f(0,0)| = \left|\dfrac{x^3-2y^3}{x^2+y^2} - (0,0)\right|$

$\Rightarrow |f(r,\theta)| = \left|\dfrac{r^3\cos^3\theta - 2r^3\sin^3\theta}{r^2}\right|$

$= |r|\ |\cos^3\theta - 2\sin^3\theta|$

$\leq |r| \left[|\cos^3\theta| + 2|\sin^3\theta|\right]$

$\leq |r| [1+2(1)]$

$= 3|r| < \epsilon$

whenever $|r| < \epsilon/3 = \delta$ (choosing)

$\therefore |f(r,\theta)| < \epsilon$ whenever $|r| < \delta$.

H.W. If $f(0,0) = 0$, which of the following functions are continuous at the origin?

(a) $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$ α (n.c)

(b) $f(x,y) = \dfrac{x^2y^2}{(x^2+y^2)^2}$ α (n.c)

(c) $f(x,y) = \dfrac{x^3y^2}{(x^2+y^2)^2}$ (continuous)

(d) $f(x,y) = \dfrac{x+ye^{-x^2}}{1+y^2}$ (continuous)

(e) $f(x,y) = \dfrac{(x+y^2)^2}{x^2+y^2}$

Sol'n: (c) $\lim\limits_{x\to 0} f(x,0) = 0 = \lim\limits_{y\to 0} f(0,y)$

$\therefore f(x,y)$ is tending to zero along the coordinate axes.

Choosing $y = mx$:

$\lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0} \dfrac{x^3m^2x^2}{(x^2+m^2x^2)^2}$

$= \lim\limits_{x\to 0} \dfrac{m^2x}{(1+m^2)}$

$= 0$

Choosing $x = my$

$$\lim_{y\to 0} f(my, y) = \lim_{y\to 0} \frac{m^3y^3\, y^2}{(m^2y^2+y^2)^2}$$

$$= \lim_{y\to 0} \frac{m^3 y}{(m^2+1)^2}$$

$$= 0$$

$\therefore f(x,y)$ is tending to zero as $(x,y) \longrightarrow (0,0)$ along any straight line.

$\therefore f(x,y)$ is continuous at $(0,0)$

(or)

$$f(x,y) = \frac{x^3y^2}{(x^2+y^2)^2};\ (x,y) \neq (0,0)$$

Now we have

$$|f(x,y) - f(0,0)| = |f(x,y) - (0,0)|$$

$$= \left|\frac{x^3y^2}{(x^2+y^2)^2}\right| \quad \text{--- (1)}$$

Now $|x^3y^2| = |x|^3|y|^2$

$$= |x|\,(x^2 \cdot y^2)$$

$$\leq \sqrt{x^2+y^2}\,(x^2+y^2)(x^2+y^2)$$

$(\because |x| \leq |z|\ \&\ |y| \leq |z|$

$\Rightarrow x^2 \leq |z|^2\ \&$

$y^2 \leq |z|^2)$

$$= (x^2+y^2)^2\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{|x^3y^2|}{|(x^2+y^2)^2|} \leq \sqrt{x^2+y^2}$$

$$\left|\frac{x^3y^2}{(x^2+y^2)^2}\right| \leq |z| < \epsilon$$

where $|z| < \frac{\epsilon}{1} = \delta$

$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$ where $|z - 0| < \delta$.

$\therefore f(x,y)$ is continuous at $(0,0)$

* Differentiability :—

If a function $f(z)$ is single valued in a domain D, then the derivative of $f(z)$ at $z=z_0$ is denoted by $f'(z_0)$ and is defined as $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$.

Provided that the limit exists. In this case we say that $f(z)$ is differentiable at $z=z_0$.

(or)

A function 'f' is said to be differentiable at z, if $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists and is denoted by $f'(z)$.

i.e. $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

Note :— 'h' approaches '0' through points in the plane, not just along the real axis.

Every differentiable function $f(z)$ is continuous But the converse is not true.

Proof : Let $f(z)$ be differentiable at $z=z_0$.

$\therefore f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

Now we have

$f(z)-f(z_0) = \frac{f(z)-f(z_0)}{z-z_0}(z-z_0)$

$\Rightarrow \lim_{z\to z_0}[f(z)-f(z_0)] = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\times(z-z_0)\right]$

$= \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right]\lim_{z\to z_0}(z-z_0)$

$= f'(z_0).(0)$

$= 0.$

$\Rightarrow \lim_{z\to z_0}(f(z)-f(z_0)) = 0$

$\Rightarrow \lim_{z\to z_0} f(z) = \lim_{z\to z_0} f(z_0)$

$= f(z_0)$

$\therefore f(z)$ is continuous at $z=z_0$

Converse: For example

$f(z) = |z|$

$= \sqrt{x^2+y^2}$ is continuous at $(0,0)$

but not differentiable at $(0,0)$.

Since:

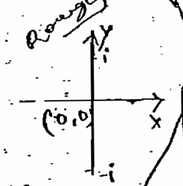
$\lim_{z\to 0} f(z) = 0 = f(0)$

$\therefore f(z)$ is continuous at $(0,0)$.

But $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to 0}\frac{|z|}{z}$

$= \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{x+iy}$

$= \lim_{(x,y)\to(0,0)} \frac{|x+iy|}{x+iy}$



$\frac{|iy|}{iy} = \frac{\sqrt{y^2}}{iy} = \frac{1}{i} = -i$

$= \begin{cases} 1 & \text{for +ve real values of } z \\ -1 & \text{for -ve real values of } z \\ -i & \text{for values of } z \text{ on the +ve imaginary axis.} \\ i & \text{for values of } z \text{ on the -ve imaginary axis} \end{cases}$

$\therefore f'(0)$ does not exist.

(or)

Since $\lim_{h\to 0} f(z+h) = \lim_{h\to 0} |z+h|$

the value of the limit at $z=0$ is $\lim_{h\to 0} |0+h| = \lim_{h\to 0} |h| = 0$

Hence $|z|$ is continuous at the origin.

But $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{|z+h|-|z|}{h}$

$= \lim_{h\to 0} \frac{|0+h|-0}{h}$

$= \lim_{h\to 0} \frac{|h|}{h} \longleftarrow (1)$

Let $h = h_1+ih_2$ and let $h\to 0$ along the real axis then $h_2=0$ & $h_1 \to 0$.

$\therefore \lim_{h\to 0} \frac{|h|}{h} = \lim_{h_1\to 0} \frac{|h_1|}{h_1} = \pm 1$.

If $h\to 0$ along the imaginary axis, $h_1=0$ and $h_2\to 0$ then

$\lim_{h\to 0} \frac{|h|}{h} = \lim_{h_2\to 0} \frac{|ih_2|}{h_2} = \pm i$

$\therefore$ the function is behaving different along the different paths.

$\Longrightarrow$ the function is not differentiable at the origin.

Note:- The rules of differentiation of real functions are also valid for complex function.

$\rightarrow$ Show that $f(z) = z^2$ is differentiable every where.

Sol'n: $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h}$

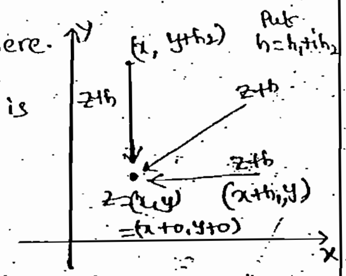
$= \lim_{h\to 0} \frac{\cancel{z^2}+2zh+h^2-\cancel{z^2}}{h}$

$= \lim_{h\to 0} (2z+h)$

$= 2z$

whatever be the path along which $h\to 0$, the limit exist and z^2 is defined everywhere.

$\therefore$ The function is differentiable Every where.



$\rightarrow$ where is $|z|$ differentiable?

Sol'n: Let $f(z) = |z|$

Now $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$= \lim_{h\to 0} \frac{|z+h|-|z|}{h}$

$$= \lim \frac{(|z+h|-|z|)(|z+h|+|z|)}{h(|z+h|+|z|)}$$

$$= \lim_{h\to 0} \frac{1}{|z+h|+|z|} \lim_{h\to 0} \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h}$$

$$= \frac{1}{|z+0|+|z|} \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \frac{\cancel{z\bar{z}} + z\bar{h} + h\bar{z} + h\bar{h} - \cancel{z\bar{z}}}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \left(\bar{z} + \bar{h} + z\frac{\bar{h}}{h}\right) \quad \text{—— (1)}$$

Let us approach z along a line parallel to x-axis.

$\therefore h$ is a real number.

$\therefore \bar{h} = h$. Then from (1),

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \bar{z} + h + \frac{z\cancel{h}}{\cancel{h}}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} (\bar{z} + h + z)$$

$$= \frac{1}{2|z|} \lim_{h\to 0} (2x+h) \quad (\because z+\bar{z} = 2x)$$

$$= \frac{\cancel{2}x}{\cancel{2}|z|} = \frac{x}{|z|} \quad \text{—— (2)}$$

Again let us approach z along a line parallel to y-axis.

$\therefore h$ is purely imaginary.

$\therefore h = ih_1$

Then from (1),

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} \left(\bar{z} - ih_1 + z\left(\frac{-ih_1}{ih_1}\right)\right)$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} (\bar{z} - ih_1 - z)$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} (-2iy - ih_1)$$

$$= \frac{1}{2|z|}(-2iy) = \frac{-iy}{|z|} \quad \text{—— (3)}$$

(2) & (3) are unequal if $x \neq 0$ and $y \neq 0$.

$\therefore |z|$ doesnot have a derivative if $z \neq 0$.

Now if $z=0$, $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{|h|}{h}$

$$\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \begin{cases} \pm 1 & \text{if approached parallel to } x\text{-axis} \\ \pm i & \text{if approached parallel to } y\text{-axis.} \end{cases}$$

$\therefore$ the limit would not exist even now

$\therefore f(z) = |z|$ is no where differentiable.

* Cauchy - Riemann Equations:

Let $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at $z = x+iy$. Then the first order partial derivatives of u and v exist at (x,y) and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also $f'(z) = u_x + iv_x$, where these partial derivatives are evaluated at (x,y).

Proof:- It is given that $f'(z)$ exists at z.

$\therefore f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$ exists.

Since $f(z) = u(x,y) + iv(x,y)$

$\Rightarrow f(x+iy) = u(x,y) + iv(x,y)$

Now we can write

$$\frac{f(z+h) - f(z)}{h} = \frac{f((x+a)+i(y+b)) - f(x+iy)}{a+ib}$$

where $h = a+ib$

$$= \frac{u(x+a, y+b) + iv(x+a, y+b) - u(x,y) - iv(x,y)}{a+ib}$$

$$= \frac{u(x+a, y+b) - u(x,y)}{a+ib} + i\frac{v(x+a, y+b) - v(x,y)}{a+ib} \quad \text{—— (1)}$$

Let us suppose that $h\to 0$ along the real axis then $b = 0$ and $a \to 0$.

(along a line ∥ lel to x-axis)

$$\therefore \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{a\to 0} \frac{u(x+a,y) - u(x,y)}{a} + i\lim_{a\to 0} \frac{v(x+a,y) - v(x,y)}{a}$$

Since the limit on the L.H.S exists, the limits on the R.H.S must also exist.

In addition to this we can observe that the limits on R.H.S are nothing but partial derivatives of u & v with respect to x.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= u_x + iv_x \quad \text{—— (2)}$$

Now let us suppose that $h\to 0$ along a line parallel to the imaginary axis through the point z.

Then $a = 0$ and $b \to 0$.

$\therefore$ from (1),

$$f'(z) = \lim_{b\to 0} \frac{u(x,y+b) - u(x,y)}{ib} + \lim_{b\to 0} i\left(\frac{v(x,y+b) - v(x,y)}{ib}\right)$$

Since the limit on L.H.S. exists, the limits on R.H.S also exist.

the limits on R.H.S are partial derivatives of u & v with respect to y.

$$\therefore f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

Now comparing (2) & (3), we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating the real & imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. $u_x = v_y$, $v_x = -u_y$ or $u_y = -v_x$

These equations are known as Cauchy-Riemann Equations.

Note: from ②, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial}{\partial x}(u+iv)$$
$$= \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}$$

Similarly from ③,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$
$$= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$
$$= -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial f}{\partial y}$$

From these we get,

$$\boxed{f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

This equation provides a method for calculating the derivative, if the derivative is known to exist.

Ex:- $f(z) = z^2$

$$= (x+iy)^2 = x^2-y^2+i(2xy)$$

is everywhere differentiable.

So that $f'(z) = \frac{\partial f}{\partial x}$

$$= 2x + i(2y)$$
$$= 2(x+iy) = 2z$$

→ Problems:

(1) Consider $f(z) = \bar{z}^2$

$$= (x-iy)^2$$
$$= x^2-y^2-2ixy$$

$\therefore u(x,y) = x^2-y^2$, $v(x,y) = -2xy$

$$\therefore \frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = -2y$$

$$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial y} = -2x$$

In general $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

But at (0,0) these conditions are satisfied.

∴ The Cauchy-Riemann (CR) conditions are not satisfied except at (0,0).

∴ ~~the function is not differentiable except at (0,0).~~

(2) Consider $f(z) = z^2$

$$= (x+iy)^2$$
$$= x^2-y^2+2ixy$$

$\therefore u(x,y) = x^2-y^2$ & $v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

∴ Cauchy-Riemann conditions are satisfied every where.

Now let us check whether the function is differentiable (or) not.

For that consider

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{(z+h)^2 - z^2}{h}$$

$$= \underset{(a+ib)\to 0}{\text{Lt}} \frac{(x+a)^2-(y+b)^2+2i(x+a)(y+b)-x^2+y^2-2ixy}{a+ib}$$

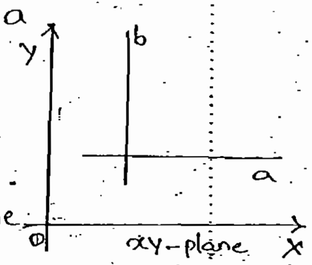
where $h = a+ib$

Let $h \to 0$ along the real axis of ab-plane, then $b=0$ and $a \to 0$

$$f'(z) = \underset{a\to 0}{\text{Lt}} \frac{2ax+a^2+2i(ay)}{a}$$

$$= 2(x+iy)$$

$$= 2z$$



Let $h \to 0$ along the Imaginary axis of the ab-plane

$\therefore a = 0$ and $b \to 0$

$$\therefore f'(z) = \underset{b\to 0}{\text{Lt}} \frac{-2by - b^2 + 2i(xb)}{ib}$$

$$= \underset{b\to 0}{\text{Lt}} \frac{-2y + 2ix}{i} = 2z$$

Since the limit is not depending on the value 'h' (or) a, b;

we say that the function is differentiable for all values of z.

Note:- The Cauchy-Riemann conditions are necessary condition only.

That is when a function $f(z)$ is differentiable then Cauchy-Riemann conditions are satisfied.

But they are not sufficient conditions

i.e. even though Cauchy-Riemann conditions are satisfied, the function may not be differentiable at that point.

This can be verified in the following example.

Let $f(z) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$

Let us observe that the behaviour of the function at the origin.

Let us verify Cauchy-Riemann Conditions at origin.

From the given function we get

$u = \dfrac{xy^2}{x^2+y^2}$, $v = 0$ when $z \neq 0$

$u = 0$, $v = 0$ when $z = 0$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Now $$\left.\frac{\partial u}{\partial x}\right|_{z=0} = \underset{h\to 0}{\text{Lt}} \frac{u(h,0)-u(0,0)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{0-0}{h} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{z=0} = \underset{h\to 0}{\text{Lt}} \frac{u(0,k)-u(0,0)}{h}$$

$$= \underset{k\to 0}{\text{Lt}} \frac{0-0}{k} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

$\therefore$ Cauchy Riemann Conditions are satisfied at the origin.

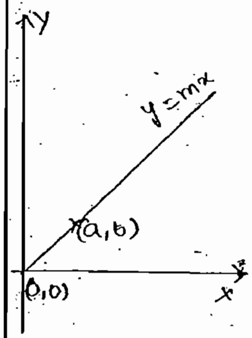
Now let us check the differenbility at the origin.

We say that the function is differentiable at the origin if .

$\lt_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ exists.

Now let $h = a+ib$ and $h\to 0$ along a line $y = mx$ of xy-plane then

$$\lt_{h\to 0} \frac{f(h)-f(0)}{h} = \lt_{h\to 0} \frac{f(h)}{h}$$



$$= \lt_{h\to 0} \frac{f(a+ib)}{a+ib} \quad \text{ie } (a+ib)\to 0$$

$$= \lt_{(a+ib)\to 0} \frac{ab^2}{a^2+b^2} \cdot \frac{1}{a+ib}$$

$$= \lt_{a\to 0} \frac{a(ma)^2}{a^2+b^2} \cdot \frac{1}{a+ima}$$

$$= \lt_{a\to 0} \frac{m^2a^3}{a^2(1+m^2)} \cdot \frac{1}{a(1+im)}$$

$$= \lt_{a\to 0} \frac{m^2}{(1+im)(1+m^2)}$$

$$= \frac{m^2}{(1+im)(1+m^2)}$$

∴ the value of the limit dependens upon the value of m

∴ the derivative of $f(z)$ at $z=0$ doesnot exist.

∴ The function is not differentiable at the origin.

Note:- If the Cauchy-Riemann equations are not satisfied then the function is nowhere differentiable.

problems:

→ Determine where the following functions satisfy the Cauchy-Riemann equations and where the functions are differentiable.

(a) $f(z) = x^2 - y^2$ (b) $f(z) = 2xyi$ $(0,0)$ differentiable at $(0,0)$

(c) $f(z) = z \cdot Re(z)$ $(0,0)$ diff at $(0,0)$ (d) $f(z) = z|z|$

(e) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$ CR-satis $(0,0)$ but not diff at $(0,0)$

Sol^n (a) $f(z) = x^2 - y^2$

Comparing with $f(z) = u(x,y) + iv(x,y)$

$u(x,y) = x^2 - y^2$; $v(x,y) = 0$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 0$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$$

But at $(0,0)$ these conditions are satisfied.

∴ Cauchy-Riemann equations are not satisfied except at $(0,0)$.

Now let us check the differentibility at $(0,0)$.

We say that the function is differentiable at the origin if $\lt_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ exists.

Let $h = a+ib$ and $h\to 0$ along a line $y = mx$ of xy-plane,

then $$\lt_{h\to 0} \frac{f(h)-f(0)}{h} = \lt_{(a+ib)\to 0} \frac{f(a+ib)-0}{a+ib}$$

$$= \lt_{(a+ib)\to 0} \frac{a^2-b^2}{a+ib}$$

$$= \lim_{a\to 0} \frac{a^2-(ma)^2}{a+i(ma)}$$

$$= \lim_{a\to 0} \frac{a^2(1-m^2)}{a(1+im)}$$

$$= 0$$

$\therefore$ At $(0,0)$ the given function $f(z) = x^2-y^2$ is differentiable.

(d) $f(z) = z|z|$,

Sol^n: $f(z) = (x+iy)\sqrt{x^2+y^2}$

$$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$$

$u(x,y) = x\sqrt{x^2+y^2}$; $v(x,y) = y\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$ at $(x,y) \neq (0,0)$.

Now let us check at the origin:

At $z=0$:

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x\sqrt{x^2}}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{0-0}{y} = 0$$

and $$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= 0$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$\therefore f(z)$ satisfies the Cauchy-Riemann equations at the origin.

Let us check the differentiability at $(0,0)$

Now $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0} \frac{z|z|}{z} = \lim_{z\to 0} |z|$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$

Along any path $f'(0) = 0$

$\therefore f(z)$ is differentiable at $(0,0)$.

* Analytic Function :-

— Consider a single valued function $f(z)$ in a domain D.

The function $f(z)$ is said to be analytic at a point $z = z_0$ if it is differentiable every where in the neighbourhood of z_0 (i.e. if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists).

— Thus analyticity is a region based property.

— A function $f(z)$ is analytic in a domain D if it is analytic at every point in the domain.

— A function $f(z)$ is analytic at every point in the complex plane is called an entire function.

— If $f'(z)$ exists at every point of a domain D except at a finite number of exceptional points, then $f(z)$ is said to be analytic in D and is reffered to as analytic function in D. These exceptional points are called singular points (or) singularities of the function.

— If $f'(z)$ exists at every point of D, then we say that $f(z)$ is regular in D.

— The terms regular and holomorphic are also sometimes used as synonyms for analytic.

* Singular Point:-

A Point $z = z_0$ is said to be a singular point of a function $f(z)$ if $f'(z_0)$ does not exist.

Examples:

→ The function $f(z) = |z|^2 = x^2+y^2$ is differentiable only at origin but not differentiable at any other point. So it is not analytic at any other point.

Sol^n: Consider $\dfrac{f(z+h)-f(z)}{h} = \dfrac{|z+h|^2-|z|^2}{h}$

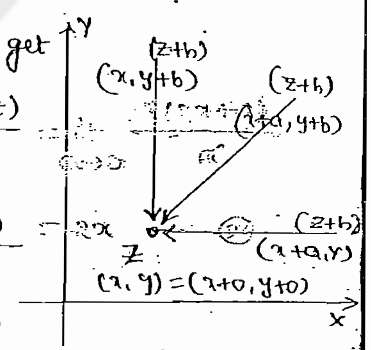
Let $h = a+ib$ then $\dfrac{|(x+a)+i(y+b)|^2 - |x+iy|^2}{a+ib}$

$$\therefore \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{(x+a)^2+(y+b)^2-(x^2+y^2)}{a+ib}$$

$$= \lim_{(a+ib)\to 0} \frac{2ax+a^2+2by+b^2}{a+ib} \quad \text{—— (1)}$$

Let the point $z+h$ tends to z along a line parallel to real axis then as $h\to 0$, $b=0$ & $a \to 0$.

$\therefore$ from (1), we get



$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{a\to 0} \frac{a(2x+a)}{a} = 2x \quad \text{—— (2)}$$

Similarly let us suppose that $z+h$ tends to z along a line parallel to imaginary axis, then as $h\to 0$, $a=0$ & $b\to 0$.

$\therefore$ from (1), we get

$$f'(z) = \lim_{b\to 0} \frac{b(2y+b)}{ib} = -i2y \quad \text{—— (3)}$$

Comparing (2) & (3), we say that the given function $|z|^2$ is not differentiable when $x\neq 0$, $y \neq 0$, i.e. other than origin. At the origin, we have.

$$\underset{h\to 0}{Lt}\frac{f(z+h)-f(z)}{h} = \underset{h\to 0}{Lt}\frac{|0+h|^2-|0|^2}{h}$$

$$= \underset{h\to 0}{Lt}\frac{|h|^2}{h}$$

Now as $h\to 0$ along real axis, h is any real number say h_1.

Then $\underset{h_1\to 0}{Lt}\frac{|h_1|^2}{h_1} = \underset{h_1\to 0}{Lt}\frac{h_1^2}{h_1} = 0$

As $h\to 0$ along imaginary axis, h is an imaginary number say $h = ih_2$

then $\underset{h\to 0}{Lt}\frac{|h|^2}{h} = \underset{h_2\to 0}{Lt}\frac{|ih_2|^2}{h_2}$

$$= \underset{h_2\to 0}{Lt}\frac{h_2^2}{h_2}$$

$$= 0$$

$\therefore$ At origin the given function $|z|^2$ is differentiable.

Ex(2): The function $f(z) = x^2y^2$ is differentiable at all points on each of the coordinate axis, but is still no where analytic.

Ex(3): All polynomials are entire functions and $f(z) = \frac{1}{1-z}$ is analytic anywhere except at $z=1$.

* Now we can state the necessary and sufficient condition for a function to be analytic in a domain D as below:

Necessary and Sufficient Conditions:

Let $f(z) = u(x,y) + iv(x,y)$ be defined in a domain D with $u(x,y)$ and $v(x,y)$ having continuous partial derivatives throughout D. Then the necessary and sufficient condition for a function $f(z)$ to be analytic in D is the satisfaction of Cauchy-Riemann Conditions.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

→ Show that an analytic $g(z)$ is independent of $\bar{z}$.

Sol'n:- Let $z = x+iy$ then

$\therefore x = \frac{z+\bar{z}}{2}$ & $y = \frac{z-\bar{z}}{2i}$

$$\therefore \frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(g(z))$$

$$= \frac{\partial}{\partial \bar{z}}(g(x+iy))$$

$$= \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + i\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial g}{\partial x}(1/2) + \frac{\partial g}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left[\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right]$$

Since $g(z)$ is analytic

$\therefore$ by Cauchy-Riemann Conditions, we have

$$\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(-i\frac{\partial g}{\partial y} + i\frac{\partial g}{\partial y}\right)$$

$$= 0. \qquad \left(\because f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}\right)$$

$\therefore$ then analytic function $g(z)$ is independent of $\bar{z}$.

2007 Prove that the function f defined by $f(z)=\begin{cases}\frac{z^5}{|z|^4}, & z\neq 0\\ 0, & z=0\end{cases}$

is not differentiable at $z=0$.

Sol^n: We say that $f(z)$ is not differentiable at the origin if

$\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Let $h=a+ib$

then $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}=\lim\limits_{(a+ib)\to 0}\frac{f(a+ib)}{a+ib}$

$$=\lim_{(a+ib)\to 0}\frac{(a+ib)^5}{|a+ib|^4}\times\frac{1}{(a+ib)}$$

$$=\lim_{(a+ib)\to 0}\frac{(a+ib)^4}{|a+ib|^4}$$

$$=\left[\lim_{(a+ib)\to 0}\frac{a+ib}{|a+ib|}\right]^4 \quad \text{--- (1)}$$

Let $h\to 0$ along the real axis of ab-plane then $b=0$ and $a\to 0$.

$$\therefore f'(0)=\left(\lim_{a\to 0}\frac{a}{|a|}\right)^4$$

$$=1$$

Let $h\to 0$ along the imaginary axis of ab-plane then $a=0$ and $b\to 0$.

$$\therefore f'(0)=\left[\lim_{b\to 0}\frac{ib}{|ib|}\right]^4$$

$$=\left[\lim_{b\to 0}\frac{ib}{b}\right]^4=1.$$

Let $h\to 0$ along a line $y=mx$ of xy-plane then from (1).

$$f'(0)=\left[\lim_{a\to 0}\frac{a+ima}{\sqrt{a^2+m^2a^2}}\right]^4$$

$$=\left[\lim_{a\to 0}\frac{1+im}{\sqrt{1+m^2}}\right]^4=\frac{1+im}{\sqrt{1+m^2}}$$

$\therefore$ The value of the limit depends upon the value of 'm'.

$\therefore$ The derivative of $f(z)$ at $z=0$ does not exist.

$\therefore$ The function is not differentiable at the origin.

(or)

Now $f'(0)=\lim\limits_{z\to 0}\frac{f(z)-f(0)}{z-0}$

$$=\lim_{z\to 0}\frac{z^5}{|z|^4}\times\frac{1}{z}$$

$$=\lim_{z\to 0}\frac{z^4}{|z|^4}=\left[\lim_{z\to 0}\frac{z}{|z|}\right]^4$$

$$=\left[\lim_{(x+iy)\to 0}\frac{x+iy}{\sqrt{x^2+y^2}}\right]^4 \quad \text{--- (1)}$$

$z\to 0$ along a line $y=mx$ of xy-plane then from (1)

$$f'(0)=\left[\lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}}\right]^4=\left[\lim_{x\to 0}\frac{1+im}{\sqrt{1+m^2}}\right]^4$$

$$=\left[\frac{1+im}{1+m^2}\right]^4$$

$\therefore$ The value of the limit depends upon the value of m

$\therefore f'(z)$ does not exist at $z=0$.

Problem: $f(z)=e^z$. Is entire function?

Sol^n: $f(z)=e^z$

$$=e^{x+iy}$$

$= e^x . e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x \cos y + ie^x \sin y$

Comparing with $\cancel{f(z) = u(x,y) + iv(x,y)}$

$u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$; $\frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Cauchy-Riemann Conditions are satisfied.

Since $e^x \cos y$ & $e^x \sin y$ are always continuous everywhere.

$\therefore$ The partial derivatives are continuous everywhere.

$\therefore f(z)$ is an analytic everywhere.

→ Place restriction on the constants a, b & c so that the following functions are entire.

(a) $f(z) = x + ay - i(bx + cy)$

(b) $f(z) = ax^2 - by^2 + icxy$

(c) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

(d) $f(z) = a(x^2+y^2) + ibxy + c$

Sol^n:- (a) $f(z) = x + ay - i(bx + cy)$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic

$\therefore f(z)$ has continuous partial derivatives and satisfy Cauchy-Riemann equations.

Now $u(x,y) = x + ay$; $v(x,y) = -(bx+cy)$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a$; $\frac{\partial v}{\partial y} = -c$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = -c$

$\Rightarrow \boxed{c = -1}$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow a = b$

(c) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic.

$\therefore f(z)$ has continuous partial derivatives & satisfies Cauchy-Riemann Conditions

Now $u(x,y) = e^x \cos ay + c$;

$v(x,y) = e^x \sin(y+b)$

$\frac{\partial u}{\partial x} = e^x \cos ay + 0$; $\frac{\partial v}{\partial x} = e^x \sin(y+b)$

$\frac{\partial u}{\partial y} = -ae^x \sin ay$; $\frac{\partial v}{\partial y} = e^x \cos(y+b)$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow e^x \cos ay = e^x \cos(y+b)$

$\Rightarrow \cos ay = \cos(y+b)$

$\Rightarrow a = 1, b = 2k\pi, k = 1, 2 - -$

and c is any Complex number.

* Problems Related to the test of Analyticity of a function:

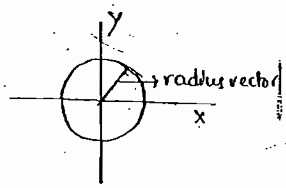
If $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

show that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not $z \to 0$ in any manner.

i.e. f is not differentiable at $z=0$.

Sol'n: $\dfrac{f(z)-f(0)}{z} = \dfrac{f(z)-0}{z}$

$$= \frac{f(z)}{z}$$



$$= \frac{x^3y(y-ix)}{(x^6+y^2)z}$$

$$= \frac{-ix^3y(\cancel{x+iy})}{(x^6+y^2)\cancel{z}}$$

$$= \frac{-ix^3y}{x^6+y^2}$$

Along the path $y = mx$ (radius vector)

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3(mx)}{x^6+(mx)^2}$$

$$= \lim_{x\to 0} \frac{-ix^4 m}{x^2(x^4+m^2)}$$

$$= \lim_{x\to 0} \frac{-ix^2 m}{x^4+m^2}$$

$$= 0$$

$\therefore \dfrac{f(z)-f(0)}{z} \longrightarrow 0$ as $z \longrightarrow 0$.

Now along the path $y = x^3$.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3y^3}{x^6+(x^3)^2}$$

$$= \lim_{x\to 0} \frac{-i}{2} = -\frac{i}{2} \neq 0.$$

$\therefore \lim_{z\to 0} \dfrac{f(z)-f(0)}{z} \neq 0$.

along any path except radius vector.

→ Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at $(0,0)$, although Cauchy-Riemann are satisfied at the point.

Sol'n: Given that $f(z) = \sqrt{|xy|}$

Here $u(x,y) = \sqrt{|xy|}$; $v(x,y) = 0$.

At the point $(0,0)$

$$\left.\frac{\partial u}{\partial x}\right| = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = 0.$$

$\therefore$ Cauchy-Riemann equations are satisfied at the point $(0,0)$.

Again $f'(0) = \lim_{z\to 0} \dfrac{f(z)-f(0)}{z-0}$

$$= \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}-0}{(x+iy)}$$

Let $(x,y) \longrightarrow (0,0)$ along $y = mx$

$$\therefore f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x+imx}$$

$$= \lim_{x\to 0} \left(\frac{\sqrt{|m|}}{1+im} \right)$$

$$= \frac{\sqrt{|m|}}{1+im}$$

which depends on m.

$\therefore f'(0)$ does not exist.

$\therefore f(z)$ is not analytic at $(0,0)$

[1998] Prove that the function $f(z) = u+iv$ where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$$

is continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist at $z=0$.

Sol^n:- $f(z) = u+iv$

$$\Rightarrow u+iv = f(z)$$

$$= \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} ; z \neq 0.$$

$$= \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} ; z \neq 0.$$

$$\Rightarrow u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

where $x \neq 0, y \neq 0$.

(I) To prove that $f(z)$ is continuous everywhere:

when $z \neq 0$, ~~u & v~~ both are rational functions of x & y with non-zero denominators.

do it by ε-δ method.

$\therefore u$ & v are continuous at all those points for which $z \neq 0$.

$\therefore f(z)$ is continuous at $z \neq 0$.

At the origin:

$u=0, v=0$ ($\because f(0)=0$).

$\therefore u$ & v are both continuous at the origin.

$\therefore f(z)$ is continuous at $(0,0)$

$\therefore f(z)$ is continuous everywhere

(II) To show that Cauchy-Riemann Equations are satisfied at $z=0$;

Since $f(0)=0$

$$\Rightarrow u(0,0) + iv(0,0) = 0.$$

$$\Rightarrow u(0,0) = 0 = v(0,0)$$

Now
$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$= \lim_{x\to 0} (1) = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y-0}{y} = 1.$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z=0$.

$\therefore$ Cauchy - Riemann equations are satisfied.

(ii) To Prove that $f'(0)$ doesnot exist:

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Let $(x,y) \longrightarrow (0,0)$ along the coordinate axes:

$$f'(0) = \lim_{x\to 0} \frac{x^3(1+i)}{x^2(x)} \text{ (along } x\text{-axis)}$$

$$= \lim_{x\to 0} (1+i) = 1+i$$

and $f'(0) = \lim_{y\to 0} \frac{-y^3+(iy^3)}{y^2(iy)} = \frac{-1+i}{i}$

$= 1+i$

Along $y=x$, then

$$f'(0) = \lim_{x\to 0} \frac{2x^3 i}{2x^2(x+ix)} = \frac{i}{1+i}$$

$= \frac{1}{2}(1+i)$

Since the values of $f'(0)$ are not unique along different paths.

$\therefore f'(0)$ does not exist.

$\therefore f(z)$ is not analytic at $z=0$.

[1991] Show that the function

$$f(z) = \begin{cases} e^{-z^{-4}} & z\neq 0 \\ 0 & z=0 \end{cases}$$ is not analytic at $z=0$, although Cauchy-Riemann equations are satisfied at the point. How would you explain this?

Sol^n:- To show that Cauchy-Riemann equations are satisfied at $z=0$.

$w = f(z) = u+iv$

Since $f(z)=0$ for $z=0$.

$0 = f(0) = u(0,0) + iv(0,0)$

$\Rightarrow u(0,0) = 0$

$v(0,0) = 0$

Since $f(z) = e^{-z^{-4}}$ for $z\neq 0$.

$$u+iv = e^{-\left[\frac{(x+iy)}{(x-iy)}(x-iy)\right]^{-4}}$$

$$= e^{-\left[\frac{x^2+y^2}{x-iy}\right]^{-4}}$$

$$= e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^2-y^2-2xyi\right]^2}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^4+y^4+4x^2y^2-2x^2y^2 + 4y^3xi-4x^3yi\right]}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[(x^4+y^4-6x^2y^2) - 4ixy(x^2-y^2)\right]}$$

$$\therefore u(x,y) = e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \cos\left[\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

$$v(x,y) = +e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \sin\left[\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

At $z=0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0}\left(\frac{e^{-x^{-4}}}{x}\right) = \lim_{x\to 0}\left(\frac{e^{-1/x^4}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x e^{1/x^4}}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{1+\frac{1}{x^4}+\left(\frac{1}{x^4}\right)^2+\cdots}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x+\frac{1}{x^3}+\frac{1}{x^7}+\cdots}\right]$$

$$=0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0}\frac{e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = 0$$

$\therefore$ Cauchy–Riemann equations are satisfied at $z=0$.

(II) To show that $f(z)$ is not analytic at $z=0$.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{-z^{-4}}$$

Let $z\to 0$ along the path

$$z = re^{i\pi/4}$$

$\therefore r\to 0$ as $z\to 0$.

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0} e^{-[re^{i\pi/4}]^{-4}}$$

$$= \lim_{r\to 0} e^{-\left[r^{-4}\left(e^{i\pi/4}\right)^{-4}\right]}$$

$$= \lim_{r\to 0} e^{r^{-4}}$$

$$= \lim_{r\to 0} e^{1/r^4} = \infty$$

$\therefore \lim_{z\to 0} f(z)$ doesnot exist.

$\therefore f(z)$ is not continuous at $z=0$.

$\therefore f(z)$ is not differentiable at $z=0$

$\therefore f(z)$ is not analytic at $z=0$.

(III) Explanation

The function $f(z)$ is analytic at $z=0$ if (i) Cauchy–Riemann equations satisfied at $z=0$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous at $z=0$.

Here the first one is satisfied and the second one is not satisfied.

$\therefore$ The function is not analytic at $z=0$

* Polar form of Cauchy-Riemann Equations :-

Let $f = u + iv$ be differentiable function with continuous partial derivatives ~~(i.e. analytic)~~ at a point $z = re^{i\theta}$, where u, v, r, θ are all real and $r \neq 0$.

Then the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof:- Since the partial derivatives are continuous, the chain rule may be applied.

Let us take into consideration of the relations $x = r\cos\theta$, $y = r\sin\theta$.

then $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{--- (1)}$$

Similarly $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$$= -\frac{\partial v}{\partial x}(r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \quad \text{--- (2)}$$

Since Cauchy-Riemann Conditions are satisfied.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

from (2), we have

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}(r\sin\theta) + \frac{\partial u}{\partial x}(r\cos\theta)$$

$$= r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$= r\frac{\partial u}{\partial r} \quad (\text{from (1)})$$

$\therefore \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{--- (I)}$$

Similarly Consider

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\cdot\frac{\partial u}{\partial y} \quad \text{--- (3)}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\cdot\frac{\partial v}{\partial x} + \sin\theta\cdot\frac{\partial v}{\partial y}$$

$$= -\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial u}{\partial x} \quad (\because \text{by Cauchy-Riemann equations}) \quad \text{--- (4)}$$

Comparing (3) & (4), we have

$$\boxed{-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}} \quad \text{--- (II)}$$

$\therefore$ (I) and (II) are known as the polar form of Cauchy-Riemann Conditions.

→ Why is not the polar form of Cauchy-Riemann Conditions valid at the origin.

Sol'n: At origin $r = 0$,

the Cauchy-Riemann Conditions of the polar form give

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} = 0.$$

$\Rightarrow$ u, v are independent of θ which is not true.

[generally u & v are depending on r & θ.
$x = r\cos\theta, y = r\sin\theta$
if $r=0$
$x=0, y=0$
i.e. if $r=0$
$(x,y)=(0,0)$]

$\therefore$ Cauchy – Riemann Conditions are not valid at origin.

$\rightarrow$ If $f(z)$ is differentiable, Show that $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$$= \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

This last expression is the Jacobian of u & v with respect to x & y.

Sol'n:- Let $f(z) = u(x,y) + iv(x,y)$ ——(1)

where $u(x,y)$ and $v(x,y)$ are functions of x & y.

Since $f(z)$ is differentiable

$\therefore$ the first order partial derivatives of u & v exist and they must satisfy the Cauchy - Riemann equations;

i.e. $u_x = v_y$ and $u_y = -v_x$ ——(2)

and $f'(z) = u_x + iv_x$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad (\text{from } 2)$$

Since $z = x+iy$

$\Rightarrow |z| = \sqrt{x^2+y^2}$

$$\therefore |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$= \sqrt{\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Now $|f'(z)|^2 = \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

(from 2)

$\rightarrow$ Let $f(z)$ and $g(z)$ be differentiable at z_0 with $f(z_0) = g(z_0) = 0$.

If $g'(z_0) \neq 0$ then $\lim\limits_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

Sol'n:

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim\limits_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim\limits_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}}$$

$$= \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{g(z)-g(z_0)}\right]$$

$$= \lim_{z\to z_0} \frac{f(z)}{g(z)} \quad (\because f(z_0) = g(z_0) = 0)$$

$\rightarrow$ Problems:

(1) Let $f(z) = |z|^2$ & $g(z) = z$

then $\lim\limits_{z\to 0} \frac{f(z)}{g(z)} = \frac{f'(0)}{g'(0)}$ $[\because f(0) = g(0) = 0]$

$$= \frac{\lim\limits_{z\to 0} \frac{f(z)-f(0)}{z-0}}{\lim\limits_{z\to 0} \frac{g(z)-g(0)}{z-0}}$$

$$= \frac{\lim\limits_{z\to 0} \frac{|z|^2}{z}}{\lim\limits_{z\to 0} \cancel{\frac{z}{z}}}$$

$$= \frac{\lim\limits_{z\to 0} \frac{z\bar{z}}{z}}{\lim\limits_{z\to 0} (1)}$$

$$= \lim_{z\to 0} \bar{z} = 0$$

→ (2) for $f(z) = \sin z$

we have $\lim\limits_{z\to 0} \frac{f(az)}{f(z)} = \lim\limits_{z\to 0}\left(\frac{\sin az}{\sin z}\right)$

$$= \frac{a\cos(0)}{\cos(0)}$$

$$= a(1)$$

$$= a$$

where 'a' is any complex number.

→ Evaluate the following limits exist. (a) $\lim\limits_{z\to 0} \frac{e^z-1}{3z}$ (b) $\lim\limits_{z\to 0} \frac{z^2}{|z|}$

(c) $\lim\limits_{z\to 0} \frac{2\sin z}{e^z-1}$ (d) $\lim\limits_{z\to 0} z\sin 1/z$

Sol^n:- (b) $\lim\limits_{z\to 0} \frac{z^2}{|z|}$

Let $f(z) = z^2$; $g(z) = |z|$

then $f(0) = g(0) = 0$.

$$\therefore \lim_{z\to 0} \frac{z^2}{|z|} = \frac{f'(0)}{g'(0)} = \frac{\lim\limits_{z\to 0} \frac{f(z)-f(0)}{z-0}}{\lim\limits_{z\to 0} \frac{g(z)-g(0)}{z-0}}$$

$$= \frac{\lim\limits_{z\to 0} \frac{z^2}{z}}{\lim\limits_{z\to 0} \frac{|z|}{z}}$$

$$= \frac{\lim\limits_{z\to 0} z}{\lim\limits_{z\to 0} \frac{|z|}{z}} \quad \text{—— (I)}$$

Now $\lim\limits_{z\to 0} \frac{|z|}{z} = \begin{cases} 1 \text{ for +ve real values of } z. \\ -1 \text{ for −ve real values of } z. \\ -i \text{ for values of } z \text{ on the +ve imaginary axis.} \\ i \text{ for values of } z \text{ on the −ve imaginary axis.} \end{cases}$

∴ from (I),

$$\lim_{z\to 0} \frac{z^2}{|z|} = 0$$

(c) $\lim\limits_{z\to 0} \frac{2\sin z}{e^z-1}$

Let $f(z) = 2\sin z$

$g(z) = e^z - 1$ then $f(0) = g(0) = 0$

Now $\lim\limits_{z\to 0} \frac{2\sin z}{e^z-1} = \frac{f'(0)}{g'(0)}$

$$= \frac{2\cos(0)}{e^0} = \frac{2}{1}$$

$$= 2$$

(d) $\lim\limits_{z\to 0} z.\sin 1/z = \lim\limits_{z\to 0} \frac{\sin 1/z}{1/z}$

$$= \lim_{z\to 0} \frac{\cos(1/z)(-1/z^2)}{(-1/z^2)}$$

$$= \lim_{z\to 0} \cos(1/z)$$

$$= \cos\infty$$

Limit does not exist.

→ Let $f(z)$ be analytic with continuous partials in a domain D that excludes the origin then show that $f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{iz}\frac{\partial f}{\partial \theta}$ at all points in D.

Sol'n: Let $f(z)$ be analytic with continuous partials in a domain D.

Let $f(z) = u+iv$

where $u = u(x,y)$ &

$v = v(x,y)$.

We know that

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad \text{--- (1)}$$

Putting $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \boxed{r^2 = x^2+y^2}$$

Consider $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$

$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$

$= -i\frac{\partial f}{\partial y}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ (from (1))

$= -i\frac{\partial f}{\partial y}(\cos\theta + i\sin\theta)$

$\therefore \frac{\partial f}{\partial r} = -i\frac{\partial f}{\partial y}e^{i\theta}$

$\Rightarrow -i\frac{\partial f}{\partial y} = e^{-i\theta}\frac{\partial f}{\partial r}$

$\Rightarrow \boxed{f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}}$ from (1).

Now consider

$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$= \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$

$= -i\frac{\partial f}{\partial y}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$ (from (1))

$= \frac{\partial f}{\partial y}(r\cos\theta + ir\sin\theta)$

$\Rightarrow \frac{\partial f}{\partial \theta} = \frac{-1}{i}f'(z)\,(x+iy)$

$= if'(z).z$ $\quad \left\{\because x = r\cos\theta,\ y = r\sin\theta\right.$

$\Rightarrow \boxed{f'(z) = \frac{1}{iz}\frac{\partial f}{\partial \theta}}$

Note: If $f(z)$ is analytic at a point then $f(z)$ has derivatives of all orders at that point.

That means the real and imaginary parts have continuous partial derivatives of all orders at that point.

In particular, the existence of $f'(z)$ tells us that $f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ is continuous (In view of 4th page theorem.), the derivatives of its real and imaginary components are also continuous.

* Harmonic functions:-

Definition:- A continuous real-valued function $u(x,y)$ defined in a domain D is said to be harmonic in D if it has continuous first and second order partials that satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This equation is known as Laplace's equation.

→ If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its real and imaginary parts u, v are harmonic in D.

Sol^n:– Given that the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain.

$$\therefore f'(z) = \frac{\partial f}{\partial x} \quad \text{--- (1)}$$

(or)

$$f'(z) = -i\frac{\partial f}{\partial y} \quad \text{--- (2)}$$

Since the analytic function has derivatives of all orders.

$$\therefore (1) \equiv f''(z) = \frac{\partial}{\partial x}(f'(z))$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} \quad \text{--- (3)}$$

$$(2) \equiv f''(z) = -i\frac{\partial}{\partial y}(f'(z))$$

$$= -i\frac{\partial}{\partial y}\left(-i\frac{\partial f}{\partial y}\right) = (-1)\frac{\partial^2 f}{\partial y^2} \quad \text{--- (4)}$$

From (3) & (4) we have,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 f = 0 \quad \text{--- (5)}$$

which is valid for any analytic function $f(z)$. i.e. If $f(z) = u(x,y) + iv(x,y)$ is an analytic in a domain D then from (5),

$$\nabla^2 f = 0$$

$$\Rightarrow \nabla^2(u + iv) = 0$$

$$\Rightarrow \nabla^2 u + i\nabla^2 v = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad \text{--- (6)}$$

∴ Both u and v are harmonic functions. Such functions u & v are called conjugate harmonic functions (or) conjugate functions simply. i.e. If $f(z) = u + iv$ is analytic then u, v both are harmonic functions since they satisfy Laplace's equation.

In such a case, u & v called conjugate harmonic functions (or) v is called a harmonic conjugate of u.

→ v is a harmonic conjugate of u iff u is a harmonic conjugate of $-v$.

Sol^n: Let $f(z) = u + iv$ be an analytic. Then v is harmonic conjugate of u.

Since $f(z)$ is analytic.

$\Rightarrow if(z)$ is also analytic.

$$\therefore if(z) = i(u + iv)$$

$$= -v + iu$$

From this we say that u harmonic conjugate of $-v$.

Let $if(z)$ be an analytic then u is haromic conjugate of $-v$.

$\therefore if(z) = -v + iu$

$\therefore i[if(z)] = i[-v+iu]$

$\Rightarrow -f(z) = -iv - u$

$= -(u+iv)$

$\Rightarrow f(z) = u+iv.$

$\therefore v$ is harmonic conjugate of u.

Note :- Laplace's equation furnishes us with a necessary condition for a function to be the real (or imaginary) Part of analytic function.

Ex: Verify whether $u(x,y) = x^2+y$ can be real part of an analytic function.

Sol'n :- If the given function $u(x,y)$ is to be real part of an analytic function, it ~~has to~~ satisfy the laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

But $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+0 = 2$

$\neq 0$

$\therefore$ Hence it cannot be.

* Calculation of Harmonic Conjugate :-

By using Cauchy - Riemann conditions, we can calculate the harmonic conjugate when real part of an analytic function is given.

→ show that $u = x^3 - 3y^2x$ is harmonic and determine ~~its~~ harmonic conjugate.

Sol^n: It is given that $u = x^3 - 3xy^2$ —— ①

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$

$\frac{\partial u}{\partial y} = -6xy \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

$\therefore u$ is harmonic.

Now let us try to find its harmonic conjugate.

Consider $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 3y^2$ (by Cauchy-Riemann equation)

Integrating with respect to y, we get,

$v = 3x^2y - y^3 + \phi(x)$ —— ②

where $\phi(x)$ is a constant function of ~~the integration~~.

Partially differentiating ② partially with respect to x the above relation we get,

$$\frac{\partial v}{\partial x} = 6xy + \phi'(x) \quad \text{--- (3)}$$

By Cauchy-Riemann equations,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow -6xy = -(6xy + \phi'(x))$$

$$\Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c \text{ (Constant)}$$

$$\therefore (2) \equiv v = 3x^2y - y^3 + c$$

$\therefore$ which is the required harmonic conjugate.

Hence the corresponding analytic function is $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

→ Show that the following functions are harmonic and determine their harmonic conjugates.

(a) $u = ax + by$; a & b are real constants.

(b) $u = \frac{y}{x^2+y^2}$; $x^2 + y^2 \neq 0$.

(c) $u = x^3 - 3xy^2$

(d) $u = \text{Arg } z$; $-\pi < \text{Arg } z < \pi$
$\hookrightarrow [\text{Arg } z = \theta = \tan^{-1}(y/x)]$

(e) $u = e^{x^2-y^2}\cos(2xy)$ (f) $u = (x-1)^3 - 3xy^2 + 3y^2$

→ Show that the function $u(x,y) = x + e^{-x}\cos y$ is harmonic and find its harmonic conjugate, determine $f(z)$ interms of z.

Sol^n:- It is given that

$$u(x,y) = x + e^{-x}\cos y \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial x} = 1 - e^{-x}\cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^{-x}\cos y,$$

$$\frac{\partial u}{\partial y} = -e^{-x}\sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u(x,y)$ is harmonic function.

Now let us try to find harmonic conjugate.

Consider $\frac{\partial u}{\partial x} = 1 - e^{-x}\cos y = \frac{\partial v}{\partial y}$

[by Cauchy-Riemann Conditions]

$$\Rightarrow \frac{\partial v}{\partial y} = 1 - e^{-x}\cos y$$

Integrating partially with respect to y, we get,

$$v = y - e^{-x}\sin y + \phi(x) \quad \text{--- (2)}$$

where $\phi(x)$ is a constant function of the integration.

Differentiating (2) partially with respect to x, we get

$$\frac{\partial v}{\partial x} = e^{-x}\sin y + \phi'(x) \quad \text{--- (3)}$$

by Cauchy-Riemann Conditions.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow -e^{-x}\sin y = -(e^{-x}\sin y + \phi'(x))$$

$$\Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c \text{ (Constant)}$$

$$\therefore (2) \equiv v = y - e^{-x}\sin y + c$$

which is required harmonic conjugate of u.

Now the analytic function

$$f(z) = u + iv$$

$$= x + e^{-x}\cos y + i(y - e^{-x}\sin y + c)$$

$= (x+iy) + e^{-x}(\cos y - i\sin y) + ic$

$= z + e^{-x}e^{-iy} + ic$

$= z + e^{-(x+iy)} + ic$

$= z + e^{-z} + ic$

2005 If $f(z) = u+iv$ is analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, determine $f(z)$ in terms of z.

Sol'n: It is given that $u-v = e^x(\cos y - \sin y)$ —— (I)

$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$ —— ①

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y)$ —— ②

② $\equiv -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^x(\sin y + \cos y)$

(by Cauchy-Riemann Conditions)

$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^x(\sin y + \cos y)$ —— ③

solving ① & ③

$\frac{\partial u}{\partial x} = e^x\cos y$ —— ④ & $\frac{\partial v}{\partial x} = e^x \sin y$ —— ⑤

Integrating ⑤ with respect to x, we get

$v = e^x \sin y + \phi(y)$ —— ⑥

Differentiating with respect to y, we get

$\frac{\partial v}{\partial y} = e^x\cos y + \phi'(y)$

by Cauchy-Riemann Conditions,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow e^x\cos y = e^x\cos y + \phi'(y)$

$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c$ (Constant)

$\therefore$ ⑥ $\equiv \boxed{v = e^x \sin y + c}$ —— ⑦

Now from (I),

$u = v + e^x(\cos y - \sin y)$

$\Rightarrow u = e^x \sin y + c + e^x(\cos y - \sin y)$ (from ⑦)

$\therefore \boxed{u = e^x(\cos y) + c}$ —— ⑧

$\therefore f(z) = u + iv$

$= e^x\cos y + c + i(e^x \sin y + c)$

$= e^x[\cos y + i\sin y] + c + ic$

$= e^x e^{iy} + \lambda$ (where $\lambda = c+ic$)

$= e^{x+iy} + \lambda$

$\boxed{f(z) = e^z + \lambda}$

* Some Simple Methods to Construct an analytic function f(z):

① Milne-thomson's Method:—

Since $f(z) = u(x,y) + iv(x,y)$

and $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ —— ①

This relation can be regarded as a formal identity in two independent Variables z & $\bar{z}$.

Putting $\bar{z} = z$; we get

$x = z$, and $y = 0$

and $f(z) = u(z,0) + iv(z,0)$

we have $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ (By Cauchy-Riemann conditions)

If we write

$$\frac{\partial u}{\partial x} = \phi_1(x,y) \;\&\; \frac{\partial u}{\partial y} = \phi_2(x,y)$$

$$\therefore f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$

$$= \phi_1(z,0) - i\phi_2(z,0)$$

Integrating it, we get

$$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

where C is an arbitrary Constant.

$\therefore f(z)$ is constructed when $u(x,y)$ is given.

Similarly,

if $v(x,y)$ is given, it can be shown that

$$f(z) = \int[\varphi_1(z,0) + i\varphi_2(z,0)]dz + C$$

where $\varphi_1(x,y) = \frac{\partial v}{\partial y}$ &

$$\varphi_2(x,y) = \frac{\partial v}{\partial x}$$

Problem

[1997] → Find the analytic function of which the real part is $e^x(x\cos y - y\sin y)$

Sol^n: Here $u(x,y) = e^x(x\cos y - y\sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$= \phi_1(x,y) \text{ (say)}$$

and $$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$= \phi_2(x,y) \text{ (say)}$$

By Milne's method,

we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$= e^z(z\cos(0) - 0) + e^z\cos(0) - i[0]$$

$$= e^z z + e^z$$

$$= e^z(z+1)$$

integrating, we get

$$f(z) = \int e^z(z+1)\,dz + C$$

$$= e^z(z-1) + e^z + C$$

$$f(z) = e^z \cdot z + C$$

Method 2

If the real part of an analytic function $f(z)$ is a given harmonic function $u(x,y)$ then $f(z) = 2u(z/2, z/2i) - u(0,0) + Ci$

where C is real.

Problem → Construct the analytic function $f(z) = u + iv$, where $u = \sin x\cosh y$.

Sol^n: Here $u = \sin x\cosh y$

$$u(z/2, z/2i) = \sin(z/2)\cdot\cosh(z/2i)$$

$$= \sin(z/2)\left[\frac{e^{z/2i} + e^{-z/2i}}{2}\right]$$

$$= \sin(z/2)\left[\frac{e^{-zi/2} + e^{zi/2}}{2}\right]$$

$$= \sin(z/2)\left[\frac{e^{zi/2} + e^{-zi/2}}{2}\right]$$

$$= \sin(z/2)\cos(z/2)$$ $\left[\because \cos x = \frac{e^{xi} + e^{-xi}}{2},\ \sin x = \frac{e^{xi} - e^{-xi}}{2}\right]$

and $u(0,0) = 0$

$$\therefore f(z) = 2u(z/2, z/2i) - u(0,0) + Ci$$

where C is real constant.

$$= 2\sin(z/2)\cos(z/2) - 0 + Ci$$

$$= \sin z + Ci$$

H.W. → $u(x,y) = x^3 - 3xy^2 + 3x + 1$

H.W. → $u(x,y) = y^3 - 3x^2y$

If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ interms of z.

Sol^n: Since ~~$f(z)=u+iv$~~

$\Rightarrow if(z)=iu-v$

Adding, we get

$f(z)+if(z)=(u-v)+i(u+v)$

$\Rightarrow (1+i)f(z)=(u-v)+i(u+v)$

$\Rightarrow F(z)=U+iV$

where $F(z)=(1+i)f(z)$

$U=u-v$ and $V=u+v$

$\frac{\partial U}{\partial x}=(x-y)(2x+4y)+(x^2+4xy+y^2)$
$=\phi_1(x,y)$ say

and $\frac{\partial U}{\partial y}=(x-y)(4x+2y)-(x^2+4xy+y^2)$
$=\phi_2(x,y)$ say

By Milne's method

$F'(z)=\phi_1(z,0)-i\phi_2(z,0)$

$=z(2z)+z^2-i[z(4z)-z^2]$

$=2z^2+z^2-i(4z^2-z^2)$

$=3z^2(1-i)$

Integrating, we get

$F(z)=z^3(1-i)+c$

$\Rightarrow (1+i)f(z)=-i\left(1-\frac{1}{i}\right)z^3+c$

$=-i(1+i)z^3+c$

$\Rightarrow f(z)=-iz^3+\frac{c}{1+i}$

$\Rightarrow f(z)=-iz^3+c_1$, where $c_1=\frac{c}{1+i}$

If $u+v=\frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ and $f(z)=u+iv$ is an analytic function of $f(z)$ then find $f(z)$ interms of z.

Sol^n: Since $f(z)=u+iv$

$\Rightarrow if(z)=-v+iu$

Adding, we get

$f+if=(u-v)+i(u+v)$

$(1+i)f=(u-v)+i(u+v)$

$\Rightarrow F(z)=U+iV$

where $F(z)=(1+i)f$

$U=u-v$ & $V=u+v$

$F'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x}$

Since $f(z)$ is analytic
$\Rightarrow F(z)$ is analytic
$\Rightarrow$ C-R equations are satisfied.
$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}$ & $\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$

Since $V=u+v$

$=\frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$

$\frac{\partial V}{\partial x}=\frac{(e^{2y}+e^{-2y}-2\cos 2x)(4\cos 2x)-(2\sin 2x)(4\sin 2x)}{(e^{2y}+e^{-2y}-2\cos 2x)^2}$
$=\varphi_2(x,y)$ (say)

$\frac{\partial V}{\partial y}=\frac{-(2e^{2y}-2e^{-2y})2\sin 2x}{(e^{2y}+e^{-2y}-2\cos 2x)^2}=\varphi_1(x,y)$ say

By Milne's method

$F'(z)=\varphi_1(z,0)+i\varphi_2(z,0)$ where $\varphi_1=\frac{\partial V}{\partial y}$
$\varphi_2=\frac{\partial V}{\partial x}$

$\therefore F'(z)=0+i\frac{(2-2\cos 2z)(4\cos 2z)-8\sin^2 2z}{(2-2\cos 2z)^2}$

$=\frac{2i(\cos 2z-1)}{(1-\cos 2z)^2}=\frac{-2i}{1-\cos 2z}=\frac{-2i}{2\sin^2 z}$
$=-i\operatorname{cosec}^2 z$

$\therefore F(z)=-i\int \operatorname{cosec}^2 z\,dz+c$

$=i\cot z+c$

$\therefore (1+i)f(z)=i\cot z+c$

$f(z)=\left(\frac{i}{1+i}\right)\cot z+\frac{c}{1+i}$

$f(z)=\frac{\cot z}{1-i}+c_1$ where $c_1=\frac{c}{1+i}$

H.W. 2005 If $f(z)=u+iv$ and $u-v=e^x(\cos y-\sin y)$ find $f(z)$ interms of z.

* Applications of Cauchy's Theorem *

In this lesson we consider two most important applications of Cauchy's theorem. They are ——

(1) Cauchy's Integral formula

(2) Taylors Theorem

In Cauchy's Integral formula we establish that when $f(z)$ is analytic in a domain D, then $f(z)$ will have derivatives of all orders in D. This formula also helps in evaluating integrals in certain cases. Let us suppose that $\int_C f(z)dz$ is to be evaluated. Then this formula can be applied if $F(z)$ is of the form $\frac{f(z)}{(z-z_0)^n}$, $n=0,1,2,\ldots$ where z_0 is a point inside C and $f(z)$ is analytic inside and on a simple closed contour C.

In Taylor's theorem we prove that if $f(z)$ is analytic in a domain D, then $f(z)$ can be expressed as infinite convergent series at every point of a circle of convergence.

Here we also prove a theorem called Morera's theorem, which is a partial converse of Cauchy's theorem.

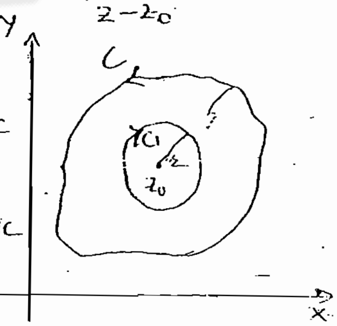
* Cauchy's First Integral Formula :-

(I) Let $f(z)$ be analytic in a simply-connected domain containing a simply closed curve 'C'. If z_0 is inside 'C' then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$

Proof: Given function $\frac{f(z)}{z-z_0}$ is not defined at $z=z_0$.

It is not analytic at $z=z_0$.

Since $f(z)$ is analytic at $z=z_0$.



∴ given $\epsilon>0$, ∃ a $\delta>0$ such that

$|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$

Now choose a $\gamma>0$ and less than δ.

Let it be so small that the positively oriented circle $|z-z_0|=\gamma$ and is denoted by C_1 is interior to C. Then

$|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|=\gamma$.

Observe that the contour C & C_1 is a boundary of a multiply connected domain in which $\frac{f(z)}{z-z_0}$ is analytic.

∴ By Cauchy's theorem for multiply-connected domain, we get

$$\int_{C-C_1} \frac{f(z)}{z-z_0}\,dz = 0.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_{C_1} \frac{f(z_0)+f(z)-f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_1} \frac{f(z_0)}{z-z_0}\,dz + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\int_{C_1} \frac{1}{z-z_0}\,dz + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)(2\pi i) + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - (2\pi i)f(z_0) = \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{--- (1)}$$

Consider

$$\left|\int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \int_{C_1} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$

$$\le \frac{\epsilon}{r}\int_{C_1} |dz|$$

$$= \frac{\epsilon}{r}(2\pi r) \quad (\because \text{length of the circle} = 2\pi r)$$

$$= 2\pi\epsilon$$

∴ from (1)

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - (2\pi i)f(z_0)\right| = \left|\int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le 2\pi\epsilon.$$

Since this relation must be satisfied for every $\epsilon > 0$, we get

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = 0 \quad (\because \epsilon \to 0)$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\boxed{\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$

Note: If $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$ for all points z_0 inside 'C', is $f(z)$ an analytic inside and on C.?

Sol^n: Yes.

If $f(z)$ is not analytic inside and on C, we cannot obtain integral formula.

Theorem (II) * **Cauchy's General Integral Formula**:-

Let $f(z)$ be analytic in a simply connected domain containing the simple contour 'c'. Then $f(z)$ has derivatives of all orders at each point z_0 inside 'c' with $f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$.

Proof: Since $f(z)$ is analytic in a simply connected domain D containing the simple closed contour 'c' and z_0 is any point inside 'c'

$$\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \text{--- (1)}$$

Now choose h such that z_0+h lies inside C.

Then $f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}dz$ ——②

Now $\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz$

(from ① & ②) ——③

on taking the limit as $h\to 0$, the LHS reduces to $f'(z_0)$.

The integrand in RHS reduces to

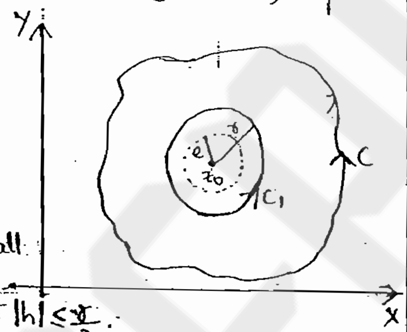
$$\frac{f(z)}{(z-z_0)^2}$$

For Proving the above, consider

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}dz - \frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}dz\right|$$

$$= \left|\frac{h}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2(z-z_0-h)}dz\right| \quad ④$$

Given a circle $C_1 : |z-z_0| = r$ contained in C.



Choose 'h' small enough so that $|h| \le \frac{r}{2}$.

Now by Cauchy's theorem for multiply-connected region we have.

$$\frac{h}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2(z-z_0-h)}dz = \frac{h}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^2(z-z_0-h)}dz$$

Since $f(z)$ is continuous on C_1,

$\therefore$ it is bounded on C_1

$\therefore |f(z)| \le M$ (say)

Then we have

$$\frac{|h|}{2\pi}\int_{C_1}\frac{|f(z)|}{|z-z_0|^2|z-z_0-h|}|dz| \le \frac{|h|M}{2\pi r^2}\int_{C_1}\frac{1}{|z-z_0-h|}|dz|$$

$$\le \frac{|h|M}{2\pi r^2}\int_{C_1}\frac{|dz|}{|z-z_0|-|h|} \le \frac{|h|M}{2\pi r^2\left(\frac{r}{2}\right)}\int_{C_1}|dz|$$

$$\le \frac{|h|M}{\pi r^3}\cdot 2\pi r = |h|\left(\frac{2M}{r^2}\right).$$

$\therefore$ As $h\to 0$

④ tends to zero.

on using ③ we get

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

$$= \lim_{h\to 0}\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

By Continuing the above process, we get

$$\frac{f'(z_0+h)-f'(z_0)}{h} = \frac{1}{2\pi i}\int_C\frac{1}{h}\left[\frac{f(z)}{(z-z_0-h)^2}-\frac{f(z)}{(z-z_0)^2}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(z-z_0)^2-(z-z_0-h)^2}{h(z-z_0)^2(z-z_0-h)^2}f(z)dz$$

$$= \frac{1}{2\pi i}\int_C\frac{2(z-z_0)-h}{(z-z_0)^2(z-z_0-h)^2}f(z)dz$$

As $\lim_{h\to 0}$, we get

$$f''(z_0) = \frac{1\cdot 2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$$

Hence by using induction, we can Prove that $f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

Hence the theorem.

Note :- The above two theorems tells us that the values of $f(z), f'(z), \ldots$ at any point z_0 inside c, can be expressed interms of the values of the function $f(z)$ on the boundary 'c'.

Note :- Here it is Proved that, if $f(z)$ is analytic in a simply connected domain D, then all order derivatives of $f(z)$ exist in D.

Note :- Cauchy's integral formula helps us to evaluate certain Complex integrals along a contour.

Ex-① : Evaluate $\int_C \frac{e^z \sin z}{(z-2)^2} dz$ where $C: |z| = 3$.

Sol'n : Comparing the given integral with

$$\int_C \frac{f(z)}{(z-z_0)^2} dz$$

we get $f(z) = e^z \sin z$; $z_0 = 2$

Since $e^z \sin z$ is analytic in $|z| = 3$.

and $z_0 = 2$ is a point inside $|z| = 3$,

$\therefore$ we can apply Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0) \quad \text{——— ①}$$

Now $f(z) = e^z \sin z$

$\Rightarrow f'(z) = e^z (\sin z + \cos z)$

$\Rightarrow f'(2) = e^2 (\sin 2 + \cos 2)$ $(\because z_0 = 2)$

From ①, we have

$$\int_C \frac{f(z)}{(z-2)^2} dz = 2\pi i e^2 (\sin 2 + \cos 2)$$

Ex-② Evaluate $\int_{|z|=2} \frac{z^3+3z-1}{(z-1)(z+3)} dz$.

Sol'n : Comparing the given integral with $\int_C \frac{f(z)}{z-z_0} dz$, we get

$$f(z) = \frac{z^3+3z-1}{z+3}, \quad z_0 = 1$$

Since $f(z)$ is analytic in $|z| = 2$ and $z_0 = 1$ is a point inside $|z| = 2$.

we apply Cauchy's integral formula

$$\int \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0) \quad \text{——— ①}$$

Since $f(z_0) = f(1)$

$$= \frac{(1)^3 + 3(1) - 1}{1+3} = 3/4$$

$$\therefore \int_{|z|=2} \frac{(z^3+3z-1)/z+3}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i (3/4)$$

$$= 3/2 \pi i$$

Note: we cannot apply cauchy's integral formula by taking $f(z) = \frac{z^3+3z-1}{z-1}$; $z_0 = -3$

because $z_0 = -3$ is not inside $|z| = 2$.

H.W. Evaluate $\int_{|z|=3} \frac{z^3+3z-1}{(z-1)(z+2)} dz$

$= 12\pi i$

(by using partial fractions)

HW Evaluate (a) $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ $=4\pi i$

(b) $\int_C \frac{e^{2z}}{(z+1)^4}dz$ where c is the circle $|z|=3$. $=\frac{8}{3}\pi i e^{-2}$

HW Evaluate the following integrals, where C is the circle $|z|=3$.

(a) $\int_C \frac{e^z}{z-2}dz$ Ans: $2\pi i e^2$

(b) $\int_C \frac{e^{z^2}}{(z-2)^2}dz$ Ans: $8\pi i e^4$

(c) $\int_C \frac{3z^4+2z-6}{(z-2)^3}dz$ Ans: $144\pi i$

→ Evaluate

(a) $\int_{|z|=2} \frac{1}{z^4-1}dz$ Ans: 0 $\left[\frac{1}{z^4-1}=\frac{1}{(z^2-1)(z^2+1)}=\frac{1}{(z-1)(z+1)(z-i)(z+i)}\right]$

(b) $\int_{|z|=2} \frac{1}{z^2+1}dz$ Ans: 0

→ Evaluate the integral $\int_C \frac{z}{(16-z^2)(z+i)}dz$ where C is circle.

(a) $|z|=2$ Ans: $2\pi/17$

(b) $|z+4|=2$ Ans: $\pi/17 - \frac{4\pi i}{17}$

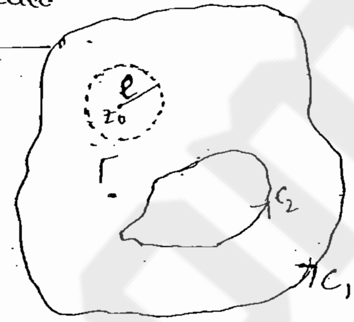
(c) $|z|=5$ Ans: $\frac{2\pi}{17} - \frac{8\pi i}{17}$

Theorem (III) * **Cauchy's Integral Formula for multiply Connected Regions:-**

Let $f(z)$ be analytic in a multiply connected domain $C = C_1 \cup C_2$. If z_0 is inside c then

$$f(z_0)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}dz-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}dz$$

Proof: Let $f(z)$ be analytic in a multiply (doubly) connected region R whose boundary is $C=C_1\cup C_2$. Let z_0 be inside c.



Construct a circle Γ contained in c and whose centre is z_0. Then in the region $C-\Gamma$, the function $\frac{f(z)}{z-z_0}$ is analytic then by Cauchy's theorem

$$\int_{C-\Gamma}\frac{f(z)}{z-z_0}dz=0$$

$$\Rightarrow \int_{C_1}\frac{f(z)}{z-z_0}dz-\int_{C_2}\frac{f(z)}{z-z_0}dz-\int_{\Gamma}\frac{f(z)}{z-z_0}dz=0 \quad \text{---(I)}$$

But $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz=f(z_0)$

(by Cauchy's integral formula)

∴ from (I), we have

$$f(z_0)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}dz-\oint_{C_2}\frac{f(z)}{z-z_0}dz.$$

Hence the result.

Note:- similarly we can show that the conclusion of the theorem (II) remains valid for the multiply-connected region R.

Taylor's Theorem :-

Let $f(z)$ be analytic in a domain D whose boundary is 'c'. z_0 is any point in C. Then $f(z)$ can be expressed as

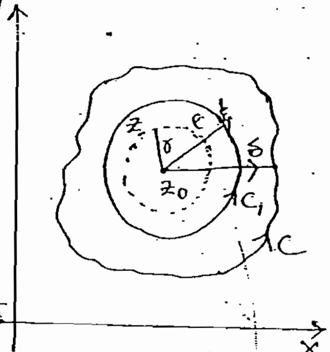
$$f(z) = f(z_0) + \frac{(z-z_0)f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + \frac{f^n(z_0)}{n!}(z-z_0)^n + \cdots$$

The series converges for $|z-z_0|<\delta$, where δ is the distance of z_0 to the nearest point on 'c'.

Proof: construct a circle c_1 with radius ρ and centre z_0.

Let $\rho<\delta$

Let z be any point in c_1. Then by Cauchy's integral formula



$$f(z) = \frac{1}{2\pi i}\int_{c_1}\frac{f(\xi)}{\xi - z}\,d\xi \quad \text{——} ①$$

Let $|z-z_0| = r$. Then $r = |z-z_0| < |\xi - z_0| = \rho$

Now $\dfrac{1}{\xi - z} = \dfrac{1}{(\xi-z_0)-(z-z_0)}$

$$= \frac{1}{(\xi-z_0)\left(1-\frac{z-z_0}{\xi-z_0}\right)}$$

$$= \frac{1}{(\xi-z_0)}\left(1-\frac{z-z_0}{\xi-z_0}\right)^{-1}$$

$$= \frac{1}{(\xi-z_0)}\left[1+\frac{z-z_0}{\xi-z_0}+\left(\frac{z-z_0}{\xi-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{\xi-z_0}\right)^{n-1}+\sum_{k=n}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^{k}\right]$$

$$\sum_{k=n}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^{k} = \left(\frac{z-z_0}{\xi-z_0}\right)^{n}+\left(\frac{z-z_0}{\xi-z_0}\right)^{n+1}+\cdots$$

$$= \left(\frac{z-z_0}{\xi-z_0}\right)^{n}\left[1+\left(\frac{z-z_0}{\xi-z_0}\right)+\cdots\right]$$

$$= \left(\frac{z-z_0}{\xi-z_0}\right)^{n}\left(1-\frac{z-z_0}{\xi-z_0}\right)^{-1}$$

$$= \frac{1}{(\xi-z_0)}\left[1+\left(\frac{z-z_0}{\xi-z_0}\right)+\left(\frac{z-z_0}{\xi-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{\xi-z_0}\right)^{n-1}+\frac{\left[(z-z_0)/(\xi-z_0)\right]^n}{1-\frac{z-z_0}{\xi-z_0}}\right]$$

Substituting this in ①, we get

$$f(z) = \frac{1}{2\pi i}\int_{c_1}\frac{f(\xi)}{(\xi-z_0)}\,d\xi + \frac{z-z_0}{2\pi i}\int_{c_1}\frac{f(\xi)}{(\xi-z_0)^2}\,d\xi + \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{c_1}\frac{f(\xi)}{(\xi-z_0)^n}\,d\xi + R_n \quad \text{——} ②$$

where $R_n = \dfrac{1}{2\pi i}\displaystyle\int_{c_1}\left(\frac{z-z_0}{\xi-z_0}\right)^n\frac{f(\xi)}{(\xi-z)}\,d\xi$

By using the general form of Cauchy's integral formula in ②, we get

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \frac{(z-z_0)^{n-1}}{(n-1)!}f^{n-1}(z_0) + R_n \quad \text{——} (I)$$

The result follows if we can show that $\lim_{n\to\infty} R_n = 0$.

Since $f(z)$ is continuous on c_1, $\exists$ a constant M such that $|f(z)| \le M$ on c_1.

$$\therefore |R_n| = \left|\frac{1}{2\pi i}\int_{c_1}\frac{(z-z_0)^n}{(\xi-z_0)^n}\frac{f(\xi)}{\xi-z}\,d\xi\right|$$

$$\le \frac{1}{2\pi}\int_{c_1}\left|\frac{z-z_0}{\xi-z_0}\right|^n\left|\frac{f(\xi)}{\xi-z}\right||d\xi|$$

$$\le \frac{M}{2\pi}\frac{r^n}{\rho^n}\int_{c_1}\frac{1}{|\xi-z|}|d\xi| \quad \text{——} ③$$

Now consider

$$\frac{1}{|\xi-z|} = \frac{1}{|\xi-z_0-(z-z_0)|} \le \frac{1}{|\xi-z_0|-|z-z_0|} = \frac{1}{\rho - r}$$

Substituting this in ③, we get

$$|R_n| \le \frac{M}{2\pi}\frac{r^n}{\rho^n}\frac{1}{\rho-r}\int_{C_1}|d\xi|$$

$$= \frac{M}{2\pi}\left(\frac{r}{\rho}\right)^n \frac{2\pi\rho}{\rho-r}$$

Since $\frac{r}{\rho}<1$, $\left(\frac{r}{\rho}\right)^n \to 0$ as $n\to\infty$

$\therefore \lim_{n\to\infty}|R_n| \le \lim \frac{M\rho}{\rho-r}\left(\frac{r}{\rho}\right)^n = 0$

$\therefore$ from ①, we have

$$f(z) = \sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n \quad ④$$

Ex: Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$.

Sol^n: $f(z) = \sin(z) \Rightarrow f(\pi/4) = \sin\pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \Rightarrow f(\pi/4) = \cos\pi/4 = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$f'''(z) = -\cos z \Rightarrow f'''(\pi/4) = -1/\sqrt{2}$

$f^{iv}(z) = \sin z \Rightarrow f^{iv}(\pi/4) = 1/\sqrt{2}$

⋮

Substituting these values in ④, we get

$$\sin(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(z-\pi/4) - \frac{1}{\sqrt{2}.2!}(z-\pi/4)^2 - \frac{1}{\sqrt{2}.3!}(z-\pi/4)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]$$

HW: Expand $f(z) = \log(1+z)$ in a Taylor's series about $z=0$.

* Note:- we know that a power series represents an analytic function inside its circle of convergence.

The Taylor's theorem is converse to the above. Thus a function $f(z)$ is analytic at a point z_0 iff

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in some disk $|z-z_0| \le r$.

where $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{n+1}}dz$

Note: we know that if $f(z)$ is analytic inside and on a simple closed contour C and z_0 is any point inside C, then the Cauchy's theorem says theorem $\int_C f(z)dz = 0$. But the converse of this theorem is not true.

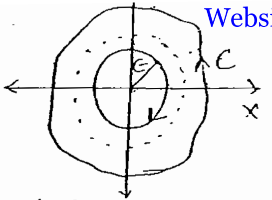
That is even iff $\int_C f(z)dz = 0$, where C is closed contour containing a point z_0 inside it, $f(z)$ need not be analytic inside and on C.

For example $\int_C \frac{1}{z^2}dz = 0$ along every simple closed curve 'C' having the origin as the interior point. This is because $f(z)$ is analytic in the region between C and some circle $|z| = \epsilon$ contained in C.

$\therefore$ By Cauchy's theorem for multiply-connected regions

$$\int_C \frac{1}{z^2}dz = \int_{|z|=\epsilon}\frac{1}{z^2}dz = \int_0^{2\pi}\frac{i\epsilon e^{i\theta}}{\epsilon^2 e^{i2\theta}}d\theta = 0.$$

But $\frac{1}{z^2}$ is not analytic at $z=0$.



Therefore in C, now we are going to state the theorem called Morera's theorem which is a partial converse to Cauchy's theorem.

→ **Morera's Theorem:**

Let $f(z)$ be continuous in a domain D and $\int_C f(z)dz=0$ along every simple closed contour 'c' contained in D; then $f(z)$ is analytic in D.

Note:- In view of Morera's theorem we can say that a necessary and sufficient condition for a continuous function to be analytic in a simply connected domain is that the integral of it is independent of the path.

LAURENT SERIES:

Earlier we have observed that $f(z)$ can be expressed in Taylor's series given by $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, if $f(z)$ is analytic in a disc $|z-z_0|<R$.

Now we consider that $f(z)$ is analytic in annulus given by $R_1 < |z-z_0| < R_2$, $R_1<R_2$ instead of a disc.

In this lesson we show that such a function $f(z)$ can be expressed in a power series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, which is known as Laurent series. This Laurent series will have two parts named as (i) Principal part (ii) Analytic part.

The analytic part is equalent to Taylor's series. The principal part helps us in determining the nature of singularities.

*** Laurent Series:**

we know that when $f(z)$ is analytic at a point z_0 then $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

that is valid in some neighbourhood of z_0. Now let us consider a function $f_1(z)$ defined as $f_1(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

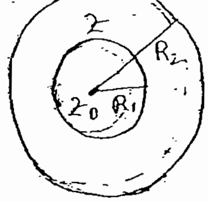
this can be viewed as a series in the variable $\frac{1}{z-z_0}$. Then let this series be convergent in $\frac{1}{|z-z_0|}<R$.

$\therefore |z-z_0| > \frac{1}{R} = R_1$ (say)

i.e. $|z-z_0| > R_1$

Now suppose that the series

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R_2.



Then $f_2(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic for $|z-z_0|<R_2$. If $R_2>R_1$, then $f_1(z)$ and $f_2(z)$ are both analytic in the annulus $R_1<|z-z_0|<R_2$.

Hence the function

$$f(z)=f_1(z)+f_2(z)=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$=\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

is analytic in the annulus $R_1<|z-z_0|<R_2$

Let us take $a_{-n}=b_n$

then $f(z)=\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$

$$=\sum_{n=-1}^{-\infty}a_n(z-z_0)^n+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n \quad \text{——— (1)}$$

A series of the above form is known as Laurent's series.

* LAURENT'S THEOREM :-

Suppose $f(z)$ is analytic in the annulus $R_1<|z-z_0|<R_2$. Then the representation $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ is valid throughout the annulus.

Further more the coefficients are given by $a_n=\frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi$

where C is any simple closed contour contained in the annulus that makes a clockwise revolution about the point z_0

→ Principal Part :- the series of negative powers of $(z-z_0)$ in $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ is called the principal part of the Laurent series of $f(z)$.

This part is convergent every where outside the circle $|z-z_0|=R_1$.

→ Analytic Part :- the series of positive powers of $(z-z_0)$ in $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ is called the analytic part of Laurent series of $f(z)$.

This part is convergent every where inside the circle $|z-z_0|=R_2$.

Problems 2005 → Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a laurent series valid for (a) $1<|z|<3$ (b) $|z|>3$ (c) $0<|z+1|<2$ (d) $|z|<1$

Sol'n: The given function resolving into partial fractions,

$$f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\frac{1}{z+1}\right)-\frac{1}{2}\left(\frac{1}{z+3}\right) \quad \text{——— (1)}$$

(a) $1<|z|<3$

Now consider $\frac{1}{z+1}$

$$\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

Here we have taken z common because the required range is $|z|>1$.

$$\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$= \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots \quad \text{(By using binomial expansion)}$$

This expansion is possible only when

$\left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$

which is required range.

Now consider $\frac{1}{z+3}$.

$$\frac{1}{z+3} = \frac{1}{3(1+\frac{z}{3})} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

Here we have taken 3 common because the required range is $|z|<3$.

$$\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right)$$

$$= \frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\cdots$$

This expansion is possible only when $\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$.

which is the required range.

Substituting these in ①, we get

$$f(z) = \frac{1}{2}\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right) - \frac{1}{2}\left(\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right)+\left(-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\cdots\right)$$

$$= \cdots + \frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}+\frac{z^2}{54}+\cdots$$

which is required Laurent expansion valid for $1<|z|<3$.

Principal part: $\frac{1}{2z}-\frac{1}{2z^2}+\cdots$

Analytic part: $-\frac{1}{6}+\frac{z}{18}+\frac{z^2}{54}+\cdots$

(b) $|z|>3$:

Now consider $\frac{1}{z+1}$

$$\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots$$

This expansion is valid for $\left|\frac{1}{z}\right|<1$

$\Rightarrow |z|>1$

which is valid for $|z|>3$. ($\because |z|>3>1$)

which is required range.

Now consider $\frac{1}{z+3}$

$$\frac{1}{z+3} = \frac{1}{z(1+\frac{3}{z})} = \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\cdots\right)$$

$$= \frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\cdots$$

This expansion is valid for $\frac{3}{z}<1$

$\left|\frac{3}{z}\right|<1 \Rightarrow |z|>3$

which is the required range.

$\therefore$ from ①,

$$f(z) = \frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right]-\frac{1}{2}\left[\frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\cdots\right]$$

$$= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\cdots$$

which is the required Laurent expansion valid for $|z|>3$.

(c) $0<|z+1|<2$:

Let $z+1=u$ then $0<|u|<2$

Now from ①,

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)}$$

$$= \frac{1}{2u(1+\frac{u}{2})}$$

$$= \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1-\frac{u}{2}+\frac{u^2}{4}-\cdots\right]$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \cdots$$

This expansion is possible only when

$$0<\left|\frac{u}{2}\right|<1$$

$$\Rightarrow 0<|u|<2$$

which is the required range.

∴ The required laurent series valid for

$0<|u|<2$

i.e. $0<|z+1|<2$ is

$$\frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \cdots$$

(d) $|z|<1$:

Now consider $\frac{1}{z+1}$

$$\therefore \frac{1}{z+1} = (1+z)^{-1} = (1-z+z^2-\cdots)$$

This expansion is valid only when $|z|<1$ which is required range.

Now Consider $\frac{1}{z+3}$

$$\therefore \frac{1}{z+3} = \frac{1}{3(1+\frac{z}{3})} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right]$$

This expansion is valid for

$$\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$$

which is also valid for $|z|<1$.

∴ from ①,

$$f(z) = \frac{1}{2}\left[1-z+z^2-\cdots\right] - \frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right]$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \cdots$$

which is the required laurent expansion valid for $|z|<1$.

This is a Taylor's Series.

[2002] show that when $0<|z-1|<2$, the function $f(z) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of $(z-1)$ as

$$\frac{-1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}.$$

Sol'n: Let $z-1=u$ then $0<|z-1|<2$

$$\Rightarrow 0<|u|<2$$

$$\frac{z}{(z-1)(z-3)} = \frac{u+1}{u(u-2)} = \frac{-1}{2u} + \frac{3}{2(u-2)}$$

Hence for $|u|<2$,

we have $f(z) = \frac{-1}{2u} - \frac{1}{4(1-\frac{u}{2})}$

$$= \frac{-1}{2u} - \frac{3}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$= -\frac{1}{2u} - \frac{3}{4}\left[1+\frac{u}{2}+\frac{u^2}{2^2}+\cdots\right]$$

$$= \frac{-1}{2(z-1)} - \frac{3}{4}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]$$

$$= \frac{-1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

H.W. → Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

H.W. → Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

i) $0<|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$.

→ Find the Laurent's expansion of $\frac{z^2}{z^4-1}$ is valid for $0<|z-i|<\sqrt{2}$.

Sol^n: Here $f(z) = \frac{z^2}{z^4-1}$

$$= \frac{z^2}{(z^2+1)(z^2-1)}$$

$$= \frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$$

Consider $\frac{1}{z+i} = \frac{1}{(z-i+2i)}$

$$= \frac{1}{2i\left[1+\frac{z-i}{2i}\right]}$$

$$= \frac{1}{2i}\left[1+\frac{z-i}{2i}\right]^{-1}$$

This expansion is possible if $\left|\frac{z-i}{2i}\right|<1$

$\therefore \left|\frac{z-i}{2i}\right|<1 \Rightarrow \frac{|z-i|}{2}<1$

$$\Rightarrow |z-i|<2$$

$$\therefore \frac{1}{z+i} = \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n \quad \text{--- (1)}$$

Now $\frac{1}{z+1} = \frac{1}{z-i+1+i}$

$$= \frac{1}{(1+i)\left[1+\frac{z-i}{1+i}\right]}$$

$$= \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{1+i}\right)^n$$

This expansion is possible only when $\left|\frac{z-i}{1+i}\right|<1$.

$\Rightarrow |z-i| < |1+i| = \sqrt{1+1} = \sqrt{2}$

$\Rightarrow |z-i|<\sqrt{2}$

$$\therefore \frac{1}{z+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-i)^n \quad \text{--- (2)}$$

Now Consider

$$\frac{1}{z-1} = \frac{1}{z-i-(1-i)} = \frac{1}{(1-i)\left[1-\frac{z-i}{i-1}\right]}$$

$$= \frac{1}{1-i}\left[1-\frac{z-i}{i-1}\right]^{-1}$$

The expansion is possible if $\left|\frac{z-i}{i-1}\right|<1$

i.e. $|z-i| < |i-1| = \sqrt{2}$

$\Rightarrow |z-i|<\sqrt{2}$

$$\therefore \frac{1}{z-1} = \frac{1}{i-1}\sum_{n=0}^{\infty}\left(\frac{z-i}{i-1}\right)^n = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(i-1)^{n+1}}$$

$\therefore$ The Laurent expansion of the given function is $\frac{z^2}{z^4-1} = \frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$

$$= \frac{1}{4}\left[\frac{i}{z+i} - \frac{i}{z-i} - \frac{1}{z+1} + \frac{1}{z-1}\right]$$

$$= \frac{i}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n - \frac{i}{4(z-i)}$$

$$-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(i-1)^{n+1}}(z-i)^n + \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-i)^n$$

Principal Part: $\frac{-i}{4(z-i)}$

→ Express $\sin z \cdot \sin\left(\frac{1}{z}\right)$ in a Laurent series valid for $|z|>0$.

Sol'n: we know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

and $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \cdots$

$$\therefore \sin z \cdot \sin\left(\frac{1}{z}\right) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots\right)$$

$$= \cdots + \left(\frac{1}{5!} + \frac{1}{7!}\cdot\frac{1}{3!} + \frac{1}{9!}\cdot\frac{1}{5!} + \cdots\right)\frac{1}{z^4} + \left(-\frac{1}{3!} - \frac{1}{5!}\cdot\frac{1}{3!} - \frac{1}{7!}\cdot\frac{1}{5!} + \cdots\right)\frac{1}{z^2} + \left(1 + \left(\frac{1}{3!}\right)^2 + \left(\frac{1}{7!}\right)^2 + \cdots\right) - \left(\frac{1}{3!} + \frac{1}{3!}\cdot\frac{1}{5!} + \frac{1}{5!}\cdot\frac{1}{7!} + \cdots\right)z^2 + \cdots$$

H.W. Find the Laurent series for the following functions valid for the given region (a) $e^{z^2} + e^{1/z^2}$; $|z|>0$.

(b) $\frac{1}{(z-a)(z-b)}$; $0<|z-a|<|a+b|$ Here $0<|a|<|b|$.

(c) $\frac{\sin z}{z^2}$; $|z|>0$.

H.W. Find the principal part for the following Laurent series:

$\frac{\sin z}{z^4}$; $|z|>0$ Ans: $\frac{1}{z^3} - \frac{1}{6z}$

→ For the function $f(z) = \frac{2z^3+1}{z^2+z}$ find (i) a Taylor's series valid in the neighbourhood of the point $z=i$.

(ii) a Laurent series valid within the annulus of which centre is the origin.

Sol'n: (i) we have $f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}$ — (1)

Let $f_1(z) = 2(z-1)$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{1}{z+1}$

Taylor's expansion for $f_1(z)$ about $z=i$ is given by $f_1(z) = \sum_{n=0}^{\infty} \frac{f_1^n(i)}{n!}(z-i)^n$ — (2)

$f_1(z) = 2(z-1) \Rightarrow f_1(i) = 2(i-1)$

$f_1'(z) = 2 \Rightarrow f_1'(i) = 2$

$f_1''(z) = 0 \Rightarrow f_1''(i) = 0$

$\therefore f_1^n(i) = 0$ for $n \geq 2$.

$\therefore$ from (2),

$$\boxed{f_1(z) = 2(i-1) + 2(z-i)}$$

Similarly we can find

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}}(z-i)^n \;\&$$

$$f_3(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$$

$\therefore$ from (1), we have

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n$$

which is the required Taylor's expansion.

(ii) For $|z|<1$, Laurent series for $f(z)$ is given by

$$f(z) = 2(z-1) + \frac{2}{z} + (1+z)^{-1}$$

$$= 2(z-1) + \frac{2}{z} + \left(1 - z + z^2 - z^3 + \cdots\right)$$

Principal part: $\frac{2}{z}$.

→ obtain the Taylor's (or) Laurent's series which represents the function $f(z) = \frac{1}{(1+z^2)(z+2)}$ when (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

* Classification of Singularities:

Upto now we have considered the functions which are analytic in and on a closed contour 'c'. Now we consider the domains that contain the points, where $f(z)$ is not analytic.

A point is said to be singular point of a function $f(z)$, if the function is not analytic at that point. These singularities are classified into three types, called (i) Removable singularity (ii) Pole (iii) Essential singularity. The behaviour of the function at these points is shown to be as:

i) At removable singularity the function can be redefined such that the function is analytic at that point.

ii) At a pole $f(z)$ tends to ∞ as z approaches the singular point.

iii) At an essential singularity the function comes arbitrarily close to every complex number in the deleted neighbourhood of that point.

The method of finding singularities for a given function is considered.

* Singularities:-

Definition: A single valued function $f(z)$ is said to have a singularity at a point if the function is not analytic at that point.

* Isolated Singular point:

If a function is analytic in some deleted neighbourhood of a singular point then that point is said to be an isolated singular point.

Ex①: Consider $f(z) = \frac{\sin z}{z}$, $z=0$ is an isolated singular point of $f(z)$, since $f(z)$ is analytic in any deleted neighbourhood of $z=0$.

Ex②: $f(z) = \frac{z+3}{z^2(z^2+1)}$ possesses three isolated singular points $z=0$, $z=i$ & $z=-i$

Ex③: $f(z) = \frac{1}{\sin(\pi/z)}$ has an infinite number of isolated singularities all of which lie on the real axis from $z=-1$ to $z=1$. These isolated singularities are at $z = \pm\frac{1}{n}$, $n=1,2,3,\ldots$

$z=1$
$0<|z-1|<\delta$
$z\in(1-\delta, 1+\delta)$

The origin $z=0$ is also a singularity but it is not isolated. Since every neighbourhood of '0' contains other singularities of the function.

Ex (4): The function $\log z$ has a singularity at the origin which is not isolated.

Since every neighbourhood of '0' contains points on the negative real axis, where $\text{Log}\, z$ ceases to be analytic.

Def: Let $z = z_0$ be an isolated singularity of a function $f(z)$.

Since singularity isolated, there exists a deleted neighbourhood $0 < |z - z_0| < r$ in which $f(z)$ is analytic. Then $f(z)$ has a laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

The part $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ of the laurent series is called the principal part of $f(z)$ at $z = z_0$.

→ **Riemann's Theorem**:-

If a function $f(z)$ has an isolated singularity at $z = z_0$ and is bounded in some deleted neighbourhood of z_0, then $f(z)$ can be defined at z_0 in such a way as to be analytic at z_0.

→ If $f(z)$ has an isolated singularity at $z = z_0$ and is bounded in some deleted neighbourhood of z_0, then $\lim f(z)$ exists.

→ **Removable Singularity**:

Let $f(z)$ has an isolated singularity at z_0. then z_0 is said to be removable singularity if $\lim_{z \to z_0} f(z)$ exists.

Example: $f(z) = \frac{\sin z}{z}$;

$z = 0$ is an isolated singular point

Also $\lim_{z \to 0} \frac{\sin z}{z} = 1$ (exists)

$\therefore z = 0$ is a removable singular point.

Theorem (1) If $f(z)$ has an isolated singularity at z_0 and $f(z) \to \infty$ as $z \to z_0$. then $f(z)$ has a pole at z_0.

Definition: Let $f(z)$ be analytic in a deleted neighbourhood of z_0 and 'n' be a +ve integer such that $\lim_{z \to z_0} (z - z_0)^n f(z)$

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0, \infty. \quad \text{---} (1)$$

then $f(z)$ is said to have a pole of order 'n' at z_0.

when $n = 1$, the pole is said to be a simple pole.

Note(1): From the nature of the condition that is satisfied by $f(z)$ when it has a pole of order 'n' at z_0. we conclude that it must be of the form $\frac{F(z)}{(z - z_0)^n} = f(z)$.

where $F(z)$ is analytic at z_0 and $F(z_0) \neq 0$.

$\cdot f(z)$ is not defined at z_0.

$f(z) = \frac{\quad}{(z-2)^3}$ has a pole of order 3 at $z=2$.

(2) $f(z) = \frac{3z-2}{(z-1)^2(z+1)(z-4)}$ has a pole of order '2' at $z=1$ and simple poles at $z=-1$ & $z=4$.

Note (2): The necessary and sufficient condition for an isolated singularity to be a pole is:

Necessary Condition:

If $f(z)$ has an isolated singularity at $z=z_0$ and $f(z) \to \infty$ as $z \to z_0$ then $f(z)$ has a pole at $z=z_0$.

Sufficient Condition:

Given that at $z=z_0$ is a pole of order K, then we have to prove that

$\lim_{z\to z_0} f(z) = \infty$.

By the definition of a pole

$\lim_{z\to z_0} (z-z_0)^K f(z) \neq 0$
$= A$ (Say)

Now, we consider

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \left[(z-z_0)^K \cdot f(z)\right] \frac{1}{(z-z_0)^K}$$

$= A \times \infty$

$= \infty$

$\therefore \lim_{z\to z_0} f(z) = \infty$

i.e. $f(z) \to \infty$ as $z \to z_0$.

→ If $f(z)$ has a pole at $z=z_0$, then $f(z)$ may be expressed as

$f(z) = \sum_{n=-K}^{\infty} b_n (z-z_0)^n$ where K is the order of the pole.

→ **Isolated Essential Singularity:**

An isolated singularity that is neither a removable singularity nor a pole is said to an isolated essential singularity.

Theorem → Cosorati – Weierstrass Theorem:

If $f(z)$ has an isolated essential singularity at $z=z_0$ then $f(z)$ comes arbitrarily close to every complex value in each deleted neighbourhood of z_0.

For example: $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z=2$.

— If a function is single valued and has singularity, then the singularity is either a pole or an essential singularity.

For this reason a pole is sometimes called non-essential singularity.

Equivalently, $z=z_0$ is an essential singularity if we cannot find any +ve integer 'n' such that

$\lim_{z\to z_0} (z-z_0)^n f(z) = A \neq 0$.

Example ②: $f(z) = e^{1/z}$

$z=0$ is an essential singularity of $f(z)$.

Note: If $z = z_0$ is an essential singularity of $f(z)$ then principal part of the laurent expansion has an infinitely many terms.

From the above example

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots,$$

$z=0$ is an essential singularity.

* Singularity at $z=\infty$:

A singularity of $f(z)$ at $z=\infty$ is a removable, a pole (or) essential according as the singularity of $f(1/z)$ at $z=0$ is removable, a pole or essential.

Ex: ① $f(z) = z^2+1$ has a pole of order '2' at $z=\infty$

Since $f(1/z) = \frac{1}{z^2}+1$

$$\Rightarrow z^2 f(1/z) = 1+z^2$$

$$\lim_{z\to 0} z^2 f(1/z) = \lim_{z\to 0}(1+z^2)$$

$$= 1$$

$$\neq 0.$$

$\therefore f(1/z)$ has a pole of second order at $z=0$.

Ex ②: $f(z) = e^z$ has an isolated essential singularity at $z=\infty$ because $f(1/z) = e^{1/z}$ has an isolated essential singularity at $z=0$.

→ Method of determining the nature of isolated singularities of a given function:

Step 1: observe at what points the given function is not analytic i.e., not defined.

Step 2: At those points determine the value of the limit of $f(z)$ as z tends to those points.

Step 3: (i) If the limit exists then that point is a removable singularity.

(ii) If the value of the limit is ∞, then that point is a pole.

(iii) If the point is neither removable nor a pole, then say that it is a essential singularity.

1999 → Find all the finite isolated singularities of $\frac{1}{\sin z - \cos z}$

Sol^n: If the denominator is equal to zero, the given function is not defined.

$\therefore \sin z - \cos z = 0 \Rightarrow z = \pi/4$

$$\therefore \lim_{z\to\pi/4} f(z) = \lim_{z\to \pi/4}\frac{1}{\sin z-\cos z} = \infty$$

Hence $z = \pi/4$ is a simple pole.

$$= \lim_{z \to \pi/4} \frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}}\sin z - \frac{1}{\sqrt{2}}\cos z}$$

$$= \lim_{z \to \pi/4} \frac{1/\sqrt{2}}{\cos\frac{\pi}{4}\sin z - \sin\frac{\pi}{4}\cos z}$$

$$= \lim_{z \to \pi/4} \frac{1/\sqrt{2}}{\sin(z-\pi/4)}$$

$$\Rightarrow \lim_{z \to \pi/4} \frac{\cancel{1/\sqrt{2}\,(z-\pi/4)}}{\sin(z-\pi/4)}$$

$$= 1/\sqrt{2} \neq 0$$

$$\lim_{z \to \pi/4} (z-\pi/4) \cdot f(z) = 1/\sqrt{2} \neq 0$$

where $f(z) = \dfrac{1/\sqrt{2}}{\sin(z-\pi/4)}$

→ Describe the singularity of $f(z) = \dfrac{z^2}{z+1}$ at $z = \infty$.

Sol'n: $f(z) = \dfrac{z^2}{z+1}$

$$\therefore f(1/z) = \frac{1/z^2}{\frac{1}{z}+1}$$

$$= \frac{1}{z(z+1)}$$

$\therefore z = 0,\ z = -1$ are simple poles for $f(1/z)$.

$\therefore f(z)$ has pole at $z = \infty$.

Method of determining the nature of singularities using the principal part:

With the help of the no. of terms that are present in the principal part of Laurent series for a function $f(z)$, the nature of the singularity can be determined. For that consider the following cases.

Let us consider the special case when $R_1=0$ $0<|z-z_0|<R_2$

Then $f(z)$ is analytic in the deleted nbd of z_0.

So z_0 is an isolated singularity.

Case(i): This isolated singularity will be removable singularity iff the principal part is zero. For proving this, first let us suppose that the principal part is zero and then prove that the singularity is removable. Now the Laurent series becomes $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. This is equivalent to Taylor's series of $f(z)$ at z_0, which is cgt in $|z-z_0|<\delta$, $\delta>0$. Thus $f(z)$ can be defined such that it is analytic at z_0. Hence z_0 is removable singularity.

In vice versa let us suppose that z_0 is removable singularity and then prove that the principal part is zero.

Since z_0 is removable singularity, $f(z)$ can be defined at z_0 such that it is analytic at z_0 also. Hence $f(z)$ is analytic in $|z-z_0|\leq\delta$. Hence by Taylor's theorem $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Hence

Case(ii): In this case the principal part has finitely many terms. This is the situation iff the singularity at z_0 is a pole. To prove this, first let us suppose that the principal part has finite no. of terms i.e, say K terms, then we prove that z_0 is a pole of order K. For that let us consider the Laurent series for $f(z)$ at z_0.

$$f(z) = \sum_{n=-K}^{\infty} a_n (z-z_0)^n = \frac{a_{-K}}{(z-z_0)^K} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0$$

$$+ a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\therefore (z-z_0)^K f(z) = a_{-K} + a_{-K+1}(z-z_0) + \cdots + a_{-2}(z-z_0)^{K-2}$$

$$+ a_{-1}(z-z_0)^{K-1} + a_0(z-z_0)^K + a_1(z-z_0)^{K+1} +$$

$$a_2(z-z_0)^{K+2} + \cdots$$

$$\therefore \lim_{z\to z_0} (z-z_0)^K f(z) = a_{-K} \neq 0, \infty$$

$\therefore z_0$ is a pole of $f(z)$ of order K.

In converse let z_0 be a pole of order K for $f(z)$, then we have to prove that the Laurent series will have finitely K terms.

Now we have $\lim_{z\to z_0} (z-z_0)^K f(z) \neq 0, \infty$.

$\therefore$ Let $(z-z_0)^K f(z) = F(z)$.

Then $F(z)$ is analytic in $|z-z_0| < R$

$\therefore$ By Taylor's theorem $F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

$$\therefore F(z) = (z-z_0)^K f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\therefore f(z) = \frac{a_0}{(z-z_0)^K} + \frac{a_1}{(z-z_0)^{K-1}} + \cdots + a_K + a_{K+1}(z-z_0) + \cdots$$

$$\therefore f(z) = \frac{b_{-k}}{(z-z_0)^k} + \frac{b_{-k+1}}{(z-z_0)^{k-1}} + \cdots + b_0 + b_1(z-z_0) + \cdots$$

$$= \sum_{n=-k}^{\infty} b_n (z-z_0)^n$$

Hence the principal part of $f(z)$ is having finite no. of terms, i.e, k terms.

Case(iii) In this let the principal part has infinitely many terms. Then z_0 cannot be removable or a pole. Hence by a process of elimination z_0 can be an essential singularity.

Note 1: The coefficients of a Laurent series are usually not found by evaluating the integrals in terms of which they are defined. In fact determining 'a_n' by other means will enable us to evaluate the integral by which they are defined.

In general if the given function $f(z)$ consists trignometric, (or) exponential or logarithmic function, we make the known standard expression and determine the Laurent series.

But in the case of algebraic functions we make use of binomial expansion and find the Laurent series.

(1) Example: Find the nature and location of the singularities of the function $f(z) = \frac{1}{z(e^z-1)}$

Prove that it can be expanded in the form $\frac{1}{z} - \frac{1}{z} + a_0 + a_1 z^2 + a_2 z^4 + \cdots$ where $0<|z|<2\pi$.

The $f(z) = \frac{1}{z(e^z-1)}$

$\therefore$ The of $f(z)$ are given by

$\Rightarrow \; 1 = 0 \Rightarrow e^z = 1 = e^{2n\pi i} \Rightarrow z = 2n\pi i$
$(n = 0, \pm 1, \pm 2, \ldots)$

Hence of f(z) at $z = 0$ and
$z = 2n\pi i \quad (n = \pm 1, \pm 2, \ldots)$

$z = 0$ al a factor of $e^z - 1$.

Hence uble pole of $f(z)$.

The others, namely $\pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots$ poles.

It follows can be expanded as a Laurent region $0 < |z| < 2\pi$ in powers of z. Since a double pole, the principal part of f consist of two terms only.

Now consider

$$\frac{(e^z-1)}{}$$

$$\frac{1}{\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1\right]}$$

$$\frac{1}{\left[1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right]}$$

$$= \left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)\right]^{-1}$$

$$1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right) + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)^2$$
$$- \left(\frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right)^3 + \left(\frac{z}{2} + \frac{z^2}{3!} + \cdots\right)^4 + \cdots$$

$$1 - \frac{z}{2!} + z^2\left(-\frac{1}{3!} + \frac{1}{4}\right) + z^3\left(-\frac{1}{4!} + \frac{1}{6} - \frac{1}{8}\right) +$$
$$\left(-\frac{1}{120} + \frac{1}{3} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16}\right)z^4 + \cdots$$

$$-\frac{z}{2} + \frac{1}{12}z^2 + 0 + \frac{1}{360}z^4 + \cdots$$

$$\frac{1}{2z} + \frac{1}{12} + \frac{1}{360}z^2 + \cdots$$

for each of the following conditions. Name the singularity in each case and give the region of convergence of each series.

a) $\frac{e^{2z}}{(z-1)^3}$; $z=1$ (b) $(z-3)\sin\frac{1}{z+2}$; $z=-2$ (c) $\frac{z-\sin z}{z^3}$; $z=0$

(d) $\frac{z}{(z+1)(z+2)}$; $z=-2$

(e) $\frac{1}{z^2(z-3)^2}$; $z=3$

Sol: (a) $\frac{e^{2z}}{(z-1)^3}$; $z=1$

Let $z-1=u$, then $z=1+u$.

and $\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2\cdot e^{2u}}{u^3}$

$$= \frac{e^2}{u^3}\left[1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\frac{(2u)^4}{4!}\cdots\right]$$

$$= \frac{e^2}{u^3}+\frac{2e^2}{u^2}+\frac{2e^2}{u}+\frac{4e^2}{3}+\frac{2e^2}{3}u+\cdots$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\cdots$$

$z=1$ is a pole of order 3 (or) triple pole.

The series cgs for all values of $z \neq 1$

(b) $(z-3)\sin\frac{1}{z+2}$; $z=-2$.

let $z+2=u$

then $(z-3)\sin\frac{1}{z+2} = (u-5)\sin\left(\frac{1}{u}\right)$

$$= (u-5)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}+\cdots\right\}$$

$$= 1-\frac{5}{u}-\frac{1}{3!u^2}+\frac{5}{3!u^3}+\frac{1}{5!u^4}+\cdots$$

$$= 1-\frac{5}{z+2}-\frac{1}{3!(z+2)^2}+\frac{5}{6(z+2)^3}\cdots$$

Since the principal part of $f(z)$ consists of an infinite no. of terms,

$\therefore$ $z=-2$ is an essential singularity

The series cgs for all values of $z\neq -2$

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots\right\}$$

$$= \frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\cdots$$

Since this expansion contains no negative powers of z;

$\therefore$ $z=0$ is a removable singularity.

The series cgs for all values of z.

(d) $\frac{z}{(z+1)(z+2)}$; $z=-2$.

Let $z+2=u$

then $\frac{z}{(z+1)(z+2)} = \frac{u-2}{u(u-1)} = \frac{2-u}{u(1-u)}$

$$= \frac{2-u}{u}(1-u)^{-1}$$

$$= \frac{2-u}{u}\left[1+u+u^2+u^3+\cdots\right]$$ ($\because$ By the binomial expansion it is possible when $|u|<1$ i.e. $|z+2|<1$)

$$= \left(\frac{2}{u}+2+2u+2u^2+\cdots\right)-\left(1+u+u^2+u^3+\cdots\right)$$

$$= \frac{2}{u}+1+u+u^2+u^3+\cdots$$

$$= \frac{2}{(z+2)}+1+(z+2)+(z+2)^2+\cdots$$

$z=-2$ is a pole of order 1.

(or) Simple pole.

The series cgs for all values of z s.t $0<|z+2|<1$

$\frac{1}{z^2(z-3)^2}$; $z=3$

Let $z-3=u$.

then $\frac{1}{z^2(z-3)^2} = \frac{1}{(u+3)^2u^2} = \frac{1}{9u^2\left[1+\frac{u}{3}\right]^2}$

$$= \frac{1}{9u^2}\left[1+\frac{u}{3}\right]^{-2}$$ by the binomial theorem when $\left|\frac{u}{3}\right|<1$

$$= \frac{1}{9u^2}\left[1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3+\cdots\right]$$

$$= \frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4u}{243}+\cdots = \frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4(z-3)}{243}+\cdots$$

$z=3$ is a pole of order 2 (or) double pole.

The series cgs for all values of z s.t $0<|z-3|<3$

In the previous discussion, we considered the evaluation of complex integrals by using parametrization, Cauchy's theorem, Cauchy's integral formula etc.

At present, we consider another important method of evaluating certain integrals by using residues. We define the residue of $f(z)$ as the coefficient of $\frac{1}{z-z_0}$ in the Laurent expansion of $f(z)$. We also consider the different methods of evaluating residues, which help us in evaluating integrals.

Residue: If $f(z)$ is analytic in a deleted nbd of z_0 then by Laurent theorem we may write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$
$$\cdots + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \cdots$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ —— (1)

$n = 0, \pm 1, \pm 2, \cdots$

In the special case $n=-1$,

we have from (1)

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz \Rightarrow \int_C f(z)\,dz = 2\pi i a_{-1}$$

where C is a simple closed contour enclosing z_0 and contained in the nbd of z_0. Then the coefficient a_{-1} is called the Residue of $f(z)$ at z_0.

The above relationship helps in evaluating the integral $\int_C f(z)\,dz$. Therefore evaluation of an integral is akin to (similar to) evaluating a coefficient in the Laurent expansion of the function.

Example: ① Evaluate $\int_C e^{1/z}\,dz$.

Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

The residue at $z=0$ is $1 = a_{-1}$.

$\therefore \int_C e^{1/z}\,dz = 2\pi i \cdot 1$
$= 2\pi i$

Example ② Evaluate $\int_C \sin\left(\frac{1}{z^2}\right) dz$.

Sol for $z \neq 0$, $\sin\left(\frac{1}{z^2}\right) = \frac{1}{z^2} - \frac{1}{3!}\left(\frac{1}{z^2}\right)^3 + \frac{1}{5!}\left(\frac{1}{z^2}\right)^5 + \cdots$

Since there is no term of z^{-1}, $a_{-1} = 0$.

Hence $\int_C \sin\left(\frac{1}{z^2}\right) dz = 0$.

→ Since $\int_C \sin\left(\frac{1}{z^2}\right) dz = 0$ along any simple closed contour contains the origin, why is not $\sin\left(\frac{1}{z^2}\right)$ analytic?

Sol At origin $\sin(1/z^2)$ is not defined.
so it is not continuous.
so we cannot apply Morera's theorem to conclude that $\sin(1/z^2)$ is analytic.

Residue Theorem

→ Let $f(z)$ be single-valued and analytic inside and on a simple closed curve 'c' except at the singularities $a, b, c, \ldots$ inside which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$

Then $\int_C f(z)\,dz = 2\pi i(a_{-1}+b_{-1}+c_{-1}+\cdots)$

i.e. the integral of $f(z)$ around 'C' is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C.

OR

→ suppose $f(z)$ is analytic inside and on a simple closed contour 'c' except for isolated singularities at $z_1, z_2, \ldots z_n$ inside 'c'. Let the residues at $z_1, z_2, \ldots z_n$ respectively be $\alpha_1, \alpha_2, \ldots \alpha_n$. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \alpha_k$$

proof About each singularity z_k, construct a circle C_k contained inside 'c' s.t. $C_k \cap C_L = 0$ when $K \neq L$. Then by Cauchy's theorem,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

where the integration along each interior is counter clock wise.

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz + \frac{1}{2\pi i}\int_{C_2} f(z)\,dz + \cdots + \frac{1}{2\pi i}\int_{C_n} f(z)\,dz.$$

$$= \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$= \sum_{k=1}^{n} \alpha_k.$$

$$\therefore \int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \alpha_k.$$

→ Evaluate $\int_C \frac{1}{(z-1)(z-2)}\,dz$ along different simple closed contours C.

Sol^n The given function $f(z) = \frac{1}{(z-1)(z-2)}$ has simple poles at $z=1$ and $z=2$.

$\therefore$ The residue of $f(z)$ at $z=1$ is

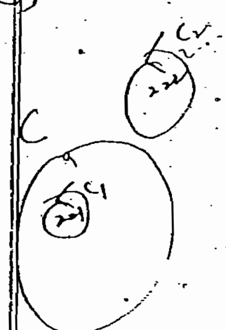
$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-2)} = -1$$

and the residue of $f(z)$ at $z=2$ is

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{1}{(z-1)(z-2)} = 1$$

Hence if $z=1$ is inside C and $z=2$ is outside C, then

$$\int_C \frac{1}{(z-1)(z-2)}\,dz = 2\pi i(-1) = -2\pi i$$



If $z=1$ is outside C and $z=2$ is inside C

then $\int_C \frac{1}{(z-1)(z-2)}\,dz = 2\pi i(1) = 2\pi i$

if both $z=1$ and $z=2$ are inside C.

then $\int_C \frac{1}{(z-1)(z-2)}\,dz = 2\pi i(-1+1) = 0$

C because the shape of C won't effect the value of the integral.

Note(2): If a function $f(z)$ has a simple pole at $z=z_0$ then the residue by using the formula

$$\lim_{z\to z_0} (z-z_0) f(z).$$

H.W → Evaluate $\int_C \frac{dz}{z(z-3)}$ along different simple closed contours C.

→ Formula for Calculating Residues:

Let a function $f(z)$ has a pole of order k at $z=z_0$. Then the residue is calculated by using the formula $a_{-1} = \frac{1}{(k-1)!} \cdot \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]$

This formula can be obtained as below:

Let $f(z)$ has a pole of order k at $z=z_0$, then $f(z)$ can be expanded in Laurent series as

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$$

$$= a_{-k}\frac{1}{(z-z_0)^k} + a_{-k+1}\frac{1}{(z-z_0)^{k-1}} + \cdots + a_{-1}\frac{1}{(z-z_0)} +$$

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$= \frac{1}{(z-z_0)^k}\left[a_{-k} + a_{-k+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + a_1(z-z_0)^{k+1} + \cdots\right]$$

$$\Rightarrow f(z)\cdot(z-z_0)^k = \left[a_{-k} + a_{-k+1}(z-z_0) + a_{-k+2}(z-z_0)^2 + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots\right]$$

Differentiating w.r.t z we get

$$\frac{d}{dz}(z-z_0)^k f(z) = a_{-k+1} + 2a_{-k+2}(z-z_0) + \cdots + a_{-1}(k-1)(z-z_0)^{k-2} + a_0 k(z-z_0)^{k-1} + \cdots$$

Repeating this process $(k-1)$ times, we get

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = a_{-k}(k-1)! + k(k-1)(k-2)\cdots 3\cdot 2\cdot a_0(z-z_0) + (k+1)k(k-1)\cdots 4\cdot 3\cdot a_1(z-z_0)^2 + \cdots$$

$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = a_{-1}(k-1)! + 0 + 0 + \cdots$$

$$\therefore a_{-1} = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]$$

Example: Evaluate $\int_C \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\,dz$, where C is a simple closed contour containing $z=1$

Sol: Let $f(z) = \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}$ $\quad z_0 = 1$

$f(z)$ has a pole of order 3 at $z=1$

$$\therefore a_{-1} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3 f(z)\right)$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\right]$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[z^4 - z^3 - 17z + 3\right]$$

$$= \frac{1}{2!}\lim_{z\to 1}\left(12z^2 - 6z - 0\right)$$

$$= \frac{1}{2!}\left[(12)(1)^2 - 6(1)\right]$$

$$= \frac{1}{2!}(12-6)$$

$$= \frac{1}{2}(6) = 3.$$

$$\therefore \int_C \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\,dz = 2\pi i\, a_{-1} = 2\pi i(3) = 6\pi i$$

$(z-1)(z+1)^2$

(b) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

(c) $f(z) = e^z \operatorname{cosec}^2 z$

at all poles in the finite plane.

Sol^n → (a) $f(z) = \dfrac{z}{(z-1)(z+1)^2}$

here $z=1$ and $z=-1$ are poles of orders one and two respectively

∴ Residue at $z=1$ is

$$a_{-1} = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+1)^2}$$

$$= \lim_{z\to 1}\frac{z}{(z+1)^2} = \frac{1}{4}$$

Residue at $z=-1$ is

$$a_1 = \lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z}{(z-1)(z+1)^2}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \lim_{z\to -1}\frac{-1}{(z-1)^2} = -\frac{1}{4}$$

Sol → (c) It is given that $f(z) = e^z \operatorname{cosec}^2 z$

$$= \frac{e^z}{\sin^2 z} = \frac{e^z}{(\sin z)^2}$$

∴ The function has double poles at $z=0, \pm\pi, \pm 2\pi, \ldots$

i.e., $z = m\pi$, where $m = 0, \pm 1, \pm 2, \ldots$

Using the general formula for calculating the residues we get

$$a_{-1} = \frac{1}{(1)!}\lim_{z\to m\pi}\frac{d}{dz}\left[(z-m\pi)^2\frac{e^z}{(\sin z)^2}\right].$$

$$\lim_{z\to m\pi} \frac{\cdots - e^z (z-m\pi)^2\, 2\sin z \cos z}{(\sin z)^4}$$

$$= \lim_{z\to m\pi} \frac{e^z (z-m\pi)^2[\sin z - 2\cos z] + 2e^z (z-m\pi)\sin z}{(\sin z)^3}$$

Let $z - m\pi = t \Rightarrow z = m\pi + t$.

when $z \to m\pi$, $t \to 0$.

then this limit can be written as

$$\lim_{t\to 0} e^{m\pi+t}\left\{\frac{t^2[\sin(m\pi+t) - 2\cos(m\pi+t)] + 2t\sin(m\pi+t)}{[\sin(m\pi+t)]^3}\right\}$$

either m is even or odd, the limit becomes

$$\lim_{t\to 0} e^{m\pi+t}\left\{\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{(\sin t)^3}\right\}$$

$$= \lim_{t\to 0} e^{m\pi+t}\left\{\lim_{t\to 0}\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{(\sin t)^3}\right\}$$

$$= e^{m\pi}\left\{\lim_{t\to 0}\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{\sin^3 t}\right\}$$

This limit can be easily evaluated by applying L'Hospital's rule many times after making the following adjustment:

$$= e^{m\pi}\cdot\lim_{t\to 0}\left\{\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{t^3}\cdot\frac{t^3}{\sin^3 t}\right\}$$

$$= e^{m\pi}\lim_{t\to 0}\left(\frac{t^2 \sin t - 2\cos t + 2t\sin t}{t^3}\right)\cdot\lim_{t\to 0}\frac{t^3}{\sin^3 t}$$

$$= e^{m\pi}\lim_{t\to 0}\left(\frac{t^2 \sin t - 2\cos t + 2t\sin t}{t^3}\right)\cdot 1$$

$= e^{m\pi}$ (by using L'Hospital's Rule many times)

$\therefore$ The residues are $e^{m\pi}$, $m = 0, \pm 1, \pm 2, \ldots$

Evaluation of real definite integrals:

Some types of real definite integrals can be evaluated by using the Residue theorem:

Let us first recollect what an improper integral is:

Let $f(x)$ be continuous over a semi-infinite interval $x \geq 0$ i.e. $[0, \infty)$. If $\lim_{R\to\infty} \int_0^R f(x)\,dx$ exists, the improper integral $\int_0^\infty f(x)\,dx$ is said to converge and its value is the value of the limit.

$$\therefore \int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx.$$

Next if $f(x)$ is continuous for all x then the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as (1)

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx. \quad \text{—— (2)}$$

When both the individual limits exist. The improper integral is said to converge also. Its value is defined as the sum of those two limits. To the same integral another type of value called Cauchy's principal value is also defined as:

$$\text{Principal value of } \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \quad \text{—— (3)}$$

provided this single limit exists.

If integral (2) cgs, the value obtained is the same as the cauchy principal value.

On the other hand, when $f(x) = x$ (for example), the cauchy principal value of integral (1) is zero, whereas that integral doesnot converge according

... in most of the cases the two values of the improper integral will be the same.

In problems, we consider in this lesson the Cauchy's principal value is only obtained.

Some real integrals are evaluated using the Residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour C.

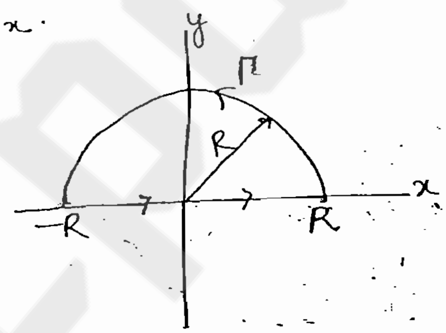
The following type of functions are considered.

Type ①

(a) $\int_{-\infty}^{\infty} f(x)\,dx$; $f(x)$ is a rational function.

For evaluating this type of integrals we consider $\int_C f(z)\,dz$ along a contour C consisting of a line along the x-axis from $-R$ to R and a semi-circle Γ having this line as diameter as shown in the figure. Then let $R \to \infty$.

If $f(x)$ is even function this can be used to evaluate $\int_0^{\infty} f(x)\,dx$.

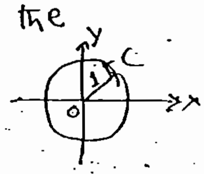


Type (2)

(b) $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$;

$F(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta, \cos\theta$.

In this case we take $z = e^{i\theta}$. Then $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Then the given integral will be equivalent to $\int_C F(z)\,dz$, where C is the unit circle $|z| = 1$ (as shown in the figure) with centre at the origin



$z = x + iy = e^{i\theta}$

$\Rightarrow z = \frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2i} i\left(z - \frac{1}{z}\right)$

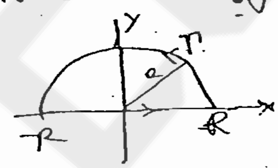
$= \cos\theta + i\sin\theta$

$\int_{-\infty} \quad \sin mz$

Here we consider $\int_C f(z) e^{imz} dz$, where C is a contour as that of type (1).

4) Miscellaneous integrals involving particular contours will be discussed.

→ If $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that $\underset{R\to\infty}{Lt} \int_\Gamma F(z) dz = 0$ where Γ is the semi-circular ~~arc~~ of radius R shown in figure.



Sol: W.K.T if $f(z)$ is continuous on a contour C having length L, with $|f(z)| \le M$ on C then $\left|\int_C f(z) dz\right| \le ML$.

$\therefore$ we have

$$\left|\int_\Gamma F(z) dz\right| \le \frac{M}{R^k} \cdot \pi R \quad (\because \text{length of arc } L = \pi R)$$

$$= \frac{\pi M}{R^{k-1}}$$

$2\pi R \quad \frac{2\pi R}{2} = \pi R$

Letting $R \to \infty$

$$\underset{R\to\infty}{Lt} \left|\int_\Gamma f(z) dz\right| = 0$$

$$\therefore \underset{R\to\infty}{Lt} \int_\Gamma F(z) dz = 0$$

→ S.T for $z = Re^{i\theta}$, $|f(z)| \le \frac{M}{R^k}$, $k > 1$; if $f(z) = \frac{1}{z^6+1}$

Sol: If $z = Re^{i\theta}$.

$$|f(z)| = |f(Re^{i\theta})|$$

$$= \left|\frac{1}{R^6 e^{6i\theta} + 1}\right| \le \frac{1}{|R^6 e^{6i\theta}| - 1} \quad \left[\because |z_1 + z_2| \ge |z_1| - |z_2| \Rightarrow \frac{1}{|z_1+z_2|} \le \frac{1}{|z_1| - |z_2|}\right]$$

$R^6 - 1 \quad = \quad \dfrac{}{R^6}$ if R is large enough (say $R > 2$).

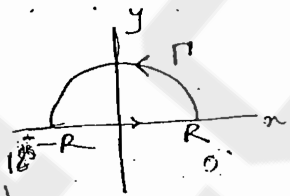
$\therefore |f(z)| \leq \dfrac{2}{R^6}$

Here $M = 2$, $K = 6$

problems on Type (1)

→ Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

Sol^n: For evaluating the given integral first let us consider $\displaystyle\int_C \frac{dz}{1+z^6}$, where C is the closed contour as shown in the figure consisting of the line from $-R$ to R and the semicircle Γ, traversed in the +ve (counter clockwise) sense.



The function $\dfrac{1}{z^6+1}$ will have poles at

$z^6 = e^{i(2n+1)\pi}$, where $n = 0, 1, 2, 3, 4, 5$.

i.e., $z = e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$

are the simple poles.

Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lie within C $[e^{5\pi i/6}]$

$\therefore \displaystyle\int_C \frac{1}{z^6+1}\,dz = 2\pi i\,[\text{Residue at } e^{\pi i/6} + \text{Residue at } e^{3\pi i/6} + \text{Residue at } e^{5\pi i/6}]$

i.e., $\displaystyle\int_C \frac{1}{z^6+1}\,dz = 2\pi i\left[\lim_{z\to e^{i\pi/6}} (z - e^{i\pi/6}) f(z) + \lim_{z\to e^{3\pi i/6}} (z - e^{3\pi i/6}) f(z) + \lim_{z\to e^{5\pi i/6}} (z - e^{5\pi i/6}) f(z)\right]$ —— (1)

Now consider

$\displaystyle\lim_{z\to e^{i\pi/6}} (z - e^{i\pi/6}) f(z) = \lim_{z\to e^{\frac{i\pi}{6}}} (z - e^{i\pi/6}) \frac{1}{z^6+1}$

$= \dfrac{0}{0}$ form.

$\displaystyle = \lim_{z\to e^{i\pi/6}} \frac{1}{6z^5}$ (on using L'Hospital's Rule)

$$\lim_{z \to e^{3\pi i/6}} \left\{ (z - e^{3\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{3\pi i/6}} \frac{1}{6z^5} \quad \left(\text{by using L-Hospital's Rule}\right) \quad \frac{0}{0}\text{ form}$$

$$= \frac{1}{6} e^{-5\pi i/2} \quad \text{——— (3)}$$

$$\lim_{z \to e^{5\pi i/6}} \left\{ (z - e^{5\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{5\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6} e^{-25\pi i/6} \quad \text{——— (4)}$$

Substituting (2), (3) & (4) in (1)

$$\int_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right\}$$

$$= \frac{2\pi i}{6} \left\{ \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} \right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2} \right) + \left(\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6} \right) \right\}$$

$$= \frac{\pi i}{3} \left\{ \cos\left(\pi - \frac{\pi}{6}\right) - i\sin\left(\pi - \frac{\pi}{6}\right) + \cos\left(2\pi + \frac{\pi}{2}\right) - i\sin\left(2\pi + \frac{\pi}{2}\right) + \cos\left(4\pi + \frac{\pi}{6}\right) - i\sin\left(4\pi + \frac{\pi}{6}\right) \right\}$$

$$= \frac{\pi i}{3} \left\{ -\cancel{\cos\frac{\pi}{6}} - i\sin\frac{\pi}{6} + \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} + \cancel{\cos\frac{\pi}{6}} - i\sin\frac{\pi}{6} \right\}$$

$$= \frac{\pi i}{3} \left\{ -2i\sin\frac{\pi}{6} + 0 - i \right\} = \frac{\pi i}{3}(-i)\left\{ 2\left(\frac{1}{2}\right) + 1 \right\}$$

$$= \frac{\pi}{3}(2) = \frac{2\pi}{3}$$

$$\Rightarrow \boxed{\int_C \frac{dz}{z^6+1}\, dz = \frac{2\pi}{3}}$$

$$\text{i.e., } \int_C \frac{1}{z^6+1}\, dz = \int_{-R}^{R} \frac{1}{x^6+1}\, dx + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

$$\Rightarrow \int_{-R}^{R} \frac{1}{x^6+1}\, dx + \int_0^{\pi} \frac{dz}{z^6+1} = \frac{2\pi}{3} \quad \text{—— (I)}$$

x-axis, $y=0$

first let us consider

$$\left|\int_0^{\pi} \frac{dz}{1+z^6}\right| \le \int_0^{\pi} \frac{|dz|}{|z^6+1|}$$

let $z = Re^{i\theta}$ then $|dz| = R\,d\theta$

$$|z^6+1| = |z^6-(-1)| \geq |z^6|-|1| = |z|^6 - 1 = R^6-1$$

$$\therefore \frac{1}{|z^6+1|} \le \frac{1}{R^6-1}$$

$$\therefore \left|\int_0^{\pi} \frac{dz}{1+z^6}\right| \le \int_0^{\pi} \frac{R\,d\theta}{R^6-1} = \frac{R}{R^6-1}\int_0^{\pi} d\theta$$

$$= \frac{R\pi}{R^6-1}$$

As $R \to \infty$, $\int_0^{\pi} \frac{dz}{1+z^6} = 0$ —— (II)

Taking the limit of both sides of (I) as $R \to \infty$ and using (II)

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

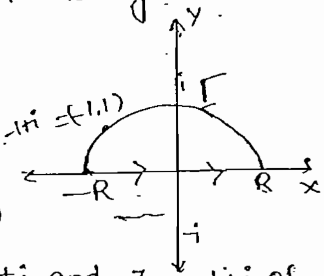
$$\therefore \int_0^{\infty} \frac{1}{x^6+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} \qquad \left(\because \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx \text{ if } f(x) \text{ is even function}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{3}\right)$$

$$\int_0^{\infty} \frac{1}{x^6+1}dx = \pi/3.$$

→ Show that $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

Sol^n Consider $\int_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)}$, where 'C' is the closed contour consisting of the line from $-R$ to R (along x-axis) and semicircle Γ having this line as diameter.



The function

$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$$

has poles at $z = \pm i$ and $z = -1 \pm i$ of orders 2 and 1 respectively.

But only the poles are $z = i$ and $z = -1+i$ lie with in 'C'.

$$\therefore \int \frac{z^2}{(z^2+1)^2(z^2+2z+2)} dz = 2\pi i \left[\begin{array}{l} (\text{Residue at } z = i) + \\ \text{Residue at } z = (-1+i) \end{array} \right] \quad (1)$$

Now residue at $z = i$ of order 2 is

$$\frac{1}{(2-1)!} \lim_{z \to i} \frac{d^{2-1}}{dz^{2-1}} \left[(z-i)^2 f(z) \right]$$

$$= \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \cdot \frac{z^2}{(z^2+1)^2(z^2+2z+2)} \right]$$

$$= \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \frac{z^2}{(z-i)^2(z+i)^2(z^2+2z+2)} \right]$$

$$= \frac{9i-12}{100}$$

Now residue at $z = -1+i$ of order 1 is

$$= \lim_{z \to (-1+i)} \left[(z-(-1+i)) \cdot \frac{z^2}{(z^2+1)^2(z^2+2z+2)} \right]$$

$$= \lim_{z \to (-1+i)} \frac{z^2}{(z-(-1-i)) \cdot (z^2+1)^2}$$

$$= \frac{3-4i}{25}$$

$$\therefore (1) \equiv \int_C f(z) dz = 2\pi i \left[\frac{9i-12}{100} + \frac{3-4i}{25} \right]$$

$$= \frac{7\pi}{50} \quad (2)$$

Now $\int_C \frac{z^2}{(z^2+1)^2(z^2+2z+2)} dz =$

$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

(from (2))

$$\therefore \int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_0^{\pi} \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50} \quad (3)$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero.

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$$

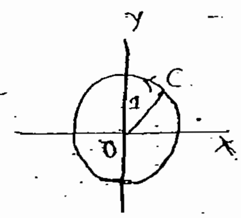
(2001) show that $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$.

Problems on Type(2):

→ show that $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta = \frac{-\pi}{3}$

Sol^n: The given function is of the type(2).

Let the contour 'c' be a unit circle $|z| = 1$ as shown in the figure.



Let $z = e^{i\theta}$ then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{z + \frac{1}{z}}{2}$$

$$= \frac{z^2+1}{2z}$$

$$\therefore \frac{\cos\theta}{5+4\cos\theta} = \frac{z^2+1}{2z} \cdot \frac{1}{5+4\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{z^2+1}{2(2z^2+5z+2)}$$

$$= \frac{z^2+1}{2(2z+1)(z+2)}$$

since $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$$

$$\therefore \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \int_C \frac{(z^2+1)}{z(z+2)(2z+1)}\frac{dz}{iz} \quad \text{--- (I)}$$

where c is a circle of unit radius with centre at origin.

$\therefore f(z) = \dfrac{z^2+1}{(2i)(z)(z+2)(2z+1)}$ has

Simple poles at $z=0, z=-2, z=-\frac{1}{2}$

But only the poles are $z=0, z=-\frac{1}{2}$ of order 1 lie with in 'c'.

$$\therefore \int_C f(z)dz = 2\pi i \begin{bmatrix}(\text{Residue at } z=0) + \\ (\text{Residue at } z=-\tfrac{1}{2})\end{bmatrix} \quad \text{--- (1)}$$

Now residue at $z=0$ is

$$\lim_{z\to 0} \frac{z.(z^2+1)}{(2i)z(z+2)(2z+1)} = \frac{1}{4i}$$

and residue at $z=-\frac{1}{2}$ is

$$\lim_{z\to -\frac{1}{2}} \frac{(z+\frac{1}{2})(z^2+1)}{(2i)(z)(z+2)(2z+1)} =$$

$$\lim_{z\to -\frac{1}{2}} \frac{(z+\frac{1}{2})(z^2+1)}{(2i)(z)(z+2)2(z+\frac{1}{2})}$$

$$= \frac{-5}{12i}$$

$$\therefore (1) \equiv \int_C f(z)dz = 2\pi i\left[\frac{1}{4i} - \frac{5}{12i}\right]$$

$$= 2\pi i\left(\frac{3-5}{12i}\right)$$

$$= -\frac{\pi}{3}$$

$\therefore$ from (I)

$$\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \frac{-\pi}{3}$$

[2003] → Prove that $\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ $(a>0)$

Sol'n: Let $I = \displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+(1-\cos 2\theta)}$$

$$= \int_{\phi=0}^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi} \quad \text{Putting } 2\theta=\phi \quad \text{--- (i)}$$

which is in the form of the type (2)

Let the contour c be the unit circle $|z|=1$ with centre at origin.

Let $z=e^{i\phi}$ then $\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$= \frac{1}{2z}(z^2+1)$$

$$\therefore \frac{a}{2a^2+1-\cos\phi} = \frac{a}{2a^2+1-\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{2az}{4a^2z+2z-z^2-1}$$

$$= \frac{-2az}{z^2-(2a^2+1)2z+1}$$

Since $z=e^{i\phi} \Rightarrow dz = ie^{i\phi}d\phi$

$$\Rightarrow dz = iz\,d\phi$$

$$\Rightarrow \boxed{d\phi = \frac{dz}{iz}}$$

$$\therefore (1) \equiv \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{2a}{i}\int_C \frac{dz}{[2z(2a^2+1)-z^2-1]}$$

where 'c' is unit circle of radius with centre at the origin.

$$= 2ai\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$= 2ai\int_C f(z)\,dz \quad \text{--- (ii)}$$

where $f(z) = \dfrac{1}{z^2 - 2(2a^2+1)z+1}$

Now the poles of $f(z)$ are given by

$z^2 - 2(2a^2+1)z + 1 = 0$

$\Rightarrow z = \dfrac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2 - 4}}{2}$

$= \dfrac{2(2a^2+1) \pm \sqrt{16a^4 + 16a^2}}{2}$

$$\boxed{z = (2a^2+1) \pm 2a\sqrt{a^2+1}}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$

$\beta = (2a^2+1) - 2a\sqrt{(a^2+1)}$

Clearly $|\alpha| > 1$ $(\because a > 0)$

Since $|\alpha\beta| = 1$

we have $|\beta| < 1$

Hence the only pole inside 'c' is at $z = \beta$.

$\therefore$ Residue at $z = \beta$ is

$\lim_{z \to \beta} (z-\beta) \dfrac{1}{(z-\alpha)(z-\beta)} = \dfrac{1}{\beta - \alpha}$

$= \dfrac{1}{-4a\sqrt{a^2+1}}$

Now from (ii),

$\displaystyle\int_0^\pi \frac{a^2 d\theta}{a^2 + \sin^2\theta} = 2ai \int_C f(z)\,dz$

$= (2ai)\, 2\pi i\, [\text{residue at } z = \beta]$

$= (2ai)(2\pi i)\left[\dfrac{1}{-4a\sqrt{a^2+1}}\right]$

$= \dfrac{\pi}{\sqrt{a^2+1}}$

Note: (1) $\displaystyle\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

HW Prove that $\displaystyle\int_0^{2\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{1}{2}\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$

$= \dfrac{\pi}{2\sqrt{1+a^2}}$

2001 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + 8\cos^2\theta}$

HW Prove that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\,d\theta$

$= \dfrac{2\pi}{b^2}\left[a - \sqrt{(a^2 - b^2)}\right]$

2006 Prove that $\displaystyle\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} = \frac{\pi a^2}{1-a^2}$; $-1 < a < 1$ i.e. $|a| < 1$.

Sol^n: Let $I = \displaystyle\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$

$\left(\because \displaystyle\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(a-x) = f(x)\right)$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}$ —— (i)

which is in the form of the type (2)

Let the contour 'c' be the unit circle $|z| = 1$ with centre at the origin.

Let $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$= \dfrac{z^2+1}{2z}$

and $\cos 2\theta = \dfrac{e^{2i\theta} + e^{-2i\theta}}{2}$

$= \dfrac{z^2 + \frac{1}{z^2}}{2}$

$= \dfrac{z^4 + 1}{2z^2}$

$$\therefore \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{\frac{z^4+1}{2z^2}}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2}$$

$$= \frac{z^4+1}{z\left(2z-2a(z^2+1)+2a^2z\right)}$$

$$= \frac{z^4+1}{z\left[2z(1-az)+2a^2z-2a\right]}$$

Since $z=e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$

$\Rightarrow dz = iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore (1) \equiv \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{2}\int_C \frac{(z^4+1)\frac{dz}{iz}}{2z\left[z(1-az)+a^2z-a\right]}$$

where C is unit circle of radius with centre at the origin.

$$= \frac{1}{4i}\int_C \frac{(z^4+1)\,dz}{z^2\left[z(1-az)+a^2z-a\right]}$$

$$= \frac{1}{4i}\int_C \frac{(z^4+1)dz}{z^2\left[z(1-az)-a(1-az)\right]}$$

$$= \frac{1}{4i}\int_C \frac{z^4+1}{z^2\left[(z-a)(1-az)\right]}dz$$

$$= \frac{1}{4i}\int_C f(z)\,dz \qquad \text{(ii)}$$

where $f(z) = \frac{z^4+1}{\left[z^2(z-a)(1-az)\right]}$

$\therefore f(z)$ has poles at $z=0,\ z=a,\ z=1/a$ of orders $2, 1$ and 1 respectively.

But only the poles are $z=0$ and $z=a$ inside C.

$$\therefore \int_C f(z)\,dz = 2\pi i\left[(\text{Residue at } z=0) + (\text{Residue at } z=a)\right] \qquad \text{(iii)}$$

Now residue at $z=0$ of order 2 is

$$\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-az^2-a+a^2z)}\right]$$

$$= -\left(\frac{1+a^2}{a^2}\right)$$

and Residue at $z=a$ is

$$\lim_{z\to a}\left[(z-a)f(z)\right] = \frac{a^4+1}{a^2(1-a^2)}$$

$\therefore$ from (iii)

$$\int_C f(z)\,dz = 2\pi i\left[\frac{-(1+a^2)}{a^2} + \frac{a^4+1}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left[\frac{-(1-a^4)+a^4+1}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left[\frac{2a^4}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left(\frac{2a^2}{1-a^2}\right) = \frac{4\pi i a^2}{1-a^2}$$

$\therefore$ from (ii), we have

$$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{4i}\left(\frac{4\pi i a^2}{1-a^2}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

HW → Apply Calculus of residues to prove that $\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \pi/6$.

$\boxed{2004}$ → Prove that $\int_0^{2\pi}\frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p}$, $0<p<1$.

Sol^n: Clearly the given function is of the type (2).

Let the contour 'c' be a unit circle $|z|=1$ with centre at origin.

Let $z=e^{i\theta}$ then $\cos 3\theta = \dfrac{e^{i3\theta}+e^{-i3\theta}}{2}$

$$= \frac{z^3+\frac{1}{z^3}}{2}$$

$$= \frac{z^6+1}{2z^3}$$

$$\therefore \cos^2 3\theta = \left(\frac{z^6+1}{2z^3}\right)^2$$

$$\boxed{\cos^2 3\theta = \frac{(z^6+1)^2}{4z^6}}$$

and $\cos 2\theta = \dfrac{e^{2i\theta}+e^{-2i\theta}}{2}$

$$= \frac{\left(z^2+\frac{1}{z^2}\right)}{2}$$

$$\boxed{\cos 2\theta = \frac{1}{2z^2}(z^4+1)}$$

Since $z=e^{i\theta} \Rightarrow \boxed{d\theta = \dfrac{dz}{iz}}$

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\,d\theta = \int_C \frac{\frac{(z^6+1)^2}{4z^6}\left(\frac{dz}{iz}\right)}{1-2p\left(\frac{z^4+1}{2z^2}\right)+p^2}$$

$$= \frac{1}{4i}\int_C \frac{(z^6+1)^2\,dz}{z^7\left[\frac{z^2-p(z^4+1)+p^2z^2}{z^2}\right]}$$

$$= \frac{1}{4i}\int_C \frac{(z^6+1)^2\,dz}{z^5\left[z^2-p(z^4+1)+p^2z^2\right]}$$

$$= \frac{1}{4i}\int_C f(z)\,dz \quad \text{--------- (i)}$$

where $f(z) = \dfrac{(z^6+1)^2}{z^5\left[z^2-p(z^4+1)+p^2z^2\right]}$

$$= \frac{(z^6+1)^2}{z^5\left[p(z^2)^2-(1+p^2)z^2+p\right]}$$

$$= \frac{-(z^6+1)^2}{z^5(pz^2-1)(z^2-p)}$$

Now $f(z)$ poles at $z=0,\ z=\pm\frac{1}{\sqrt{p}}$, $z=\pm\sqrt{p}$ of orders 5, 1 and 1 respectively.

Now residue at $z=0$ of order 5 is

$$\frac{1}{(5-1)!}\lim_{z\to 0}\frac{d^4}{dz^4}\left[z^5\cdot\frac{(z^6+1)^2}{z^5(pz^2-1)(z^2-p)}\right]$$

but this method is much more laborious to get the solution. so that we better to stop this method in particular problems.

<u>Now we follow the following Procedure:</u>

Let $I = \displaystyle\int_0^{2\pi}\frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \frac{1}{2}\int_0^{2\pi}\frac{1+\cos 6\theta}{1-2p\cos 2\theta+p^2}\,d\theta$

$$= \text{real part of } \frac{1}{2}\int_0^{2\pi}\frac{1+e^{i6\theta}}{1-2p\cos 2\theta+p^2}\,d\theta \quad (1)$$

Let the contour 'c' be a unit circle $|z|=1$ with centre at the origin.

Let $z=e^{i\theta} \Rightarrow \cos 2\theta = \dfrac{e^{i2\theta}+e^{-i2\theta}}{2}$

$$= \frac{z^2+z^{-2}}{2} = \frac{z^4+1}{2z^2}$$

$$\therefore \frac{1+e^{6i\theta}}{1-2p\cos 2\theta+p^2} = \frac{1+z^6}{1-2p\left(\frac{z^4+1}{2z^2}\right)+p^2}$$

$$= \frac{(1+z^6)z^2}{z^2-p(z^4+1)+p^2z^2}$$

$$= \frac{z^2(1+z^6)}{z^2(1-pz^2)-p(1-pz^2)}$$

$$= \frac{z^2(1+z^6)}{(z^2-p)(1-pz^2)}$$

Since $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Now consider

$$\int_0^{2\pi} \frac{1+e^{6i\theta}}{1-2p\cos 2\theta + p^2} d\theta = \int_C \frac{z^3(1+z^6)\frac{dz}{iz}}{(z^2-p)(1-pz^2)}$$

$$= \frac{1}{i}\int_C \frac{z(1+z^6)dz}{(z^2-p)(1-pz^2)}$$

$$= \frac{1}{i}\int_C f(z)dz \quad \text{——② where } C \text{ is unit circle of radius with Centre at origin.}$$

where $f(z) = \dfrac{z(1+z^6)}{(z^2-p)(1-pz^2)}$

The function $f(z)$ has the poles at

$z = \pm\frac{1}{\sqrt{p}}$, $z = \pm\sqrt{p}$ of orders one.

But only the poles are $z = \pm\sqrt{p}$ are lie within C.

$$\therefore \int_C f(z)\,dz = \int_C \frac{z(1+z^6)}{(z^2-p)(1-pz^2)} dz$$

$$= 2\pi i\left[(\text{Residue at } z=\sqrt{p}) + (\text{Residue at } z=-\sqrt{p})\right] \text{——③}$$

Now residue at $z=\sqrt{p}$ is of order 1 is

$$\lim_{z\to\sqrt{p}} \frac{(z-\sqrt{p})z(1+z^6)}{(z-\sqrt{p})(z+\sqrt{p})(1-pz^2)}$$

$$= \frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)} = \frac{1+p^3}{2(1-p^2)}$$

Residue at $z = -\sqrt{p}$ is of order 1 is

$$\lim_{z\to-\sqrt{p}} \frac{(z+\sqrt{p})z(1+z^6)}{(z+\sqrt{p})(z-\sqrt{p})(1-pz^2)}$$

$$= \frac{-\sqrt{p}(1+p^3)}{-2\sqrt{p}(1-p^2)} = \frac{1+p^3}{2(1-p^2)}$$

$$\therefore ③ \equiv \int_C f(z)dz = 2\pi i\left[\frac{1+p^3}{2(1-p^2)} + \frac{1+p^3}{2(1-p^2)}\right]$$

$$= \frac{2\pi i}{2(1-p^2)}(2)(1+p^3)$$

$$= \frac{2\pi i(1+p^3)}{1-p^2}$$

$$= \frac{2\pi i(1+p)(1-p+p^2)}{(1-p)(1+p)} \quad \left[\because a^3+b^3 = (a+b)(a^2-ab+b^2)\right]$$

$$= \frac{2\pi i(1-p+p^2)}{1-p}$$

$$\therefore ② \equiv \int_0^{2\pi} \frac{1+e^{6i\theta}}{1-2p\cos\theta+p^2} d\theta = \frac{1}{i}\int_C f(z)dz$$

$$= \frac{1}{i}(2\pi i)\frac{(1-p+p^2)}{1-p}$$

$$= \frac{2\pi(1-p+p^2)}{(1-p)} \text{——④}$$

$\therefore$ from ① and ④

$$I = \int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{1-2p\cos\theta+p^2} = \text{real part of } \frac{1}{2}\left[\frac{2\pi(1-p+p^2)}{1-p}\right]$$

where $0<p<1$.

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{1-2p\cos\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p} \quad \text{where } 0<p<1.$$

→ Prove that $\displaystyle\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n \text{ (n being +ve integer)}$$

Sol^n: Let $I = \displaystyle\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{(3+2\cos\theta)} d\theta$

$$= \text{real part of} \int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{(3+2\cos\theta)} d\theta \text{——①}$$

Let the contour 'C' be a unit circle $|z|=1$ with centre at the origin.

Let $z = e^{i\theta} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $e^{in\theta} = z^n$

$$\frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} = \frac{\left(1+z+\frac{1}{z}\right)^n z^n}{\left(3+z+\frac{1}{z}\right)}$$

$$= \frac{(z^2+z+1)^n z^n}{z^n(z^2+3z+1)/z}$$

$$= \frac{z(z^2+z+1)^n}{(z^2+3z+1)}$$

Since $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

Consider $\int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} d\theta = \int_C \frac{z(z^2+z+1)^n}{(z^2+3z+1)} \frac{dz}{iz}$

$$= \frac{1}{i}\int_C \frac{(z^2+z+1)^n}{z^2+3z+1} dz$$

where c is unit circle of radius with centre at origin

$$= \frac{1}{i}\int_C f(z)\,dz \quad \text{--- (2)}$$

where $f(z) = \frac{(z^2+z+1)^n}{(z^2+3z+1)}$

Now the poles of $f(z)$ are given by

$z^2+3z+1 = 0$

$\Rightarrow z = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$

Let $\alpha = \frac{-3+\sqrt{5}}{2}$; $\beta = \frac{-3-\sqrt{5}}{2}$

clearly $|\beta| > 1$

Since $|\alpha\beta| = 1 \Rightarrow |\alpha| < 1$ ($\because |\beta| > 1$)

Hence the only pole inside C is $z = \alpha$ of order 1.

$$\therefore \int_C f(z)\,dz = \int_C \frac{(z^2+z+1)^n}{z^2+3z+1} dz$$

$$= 2\pi i\,(\text{Residue at } z=\alpha) \quad \text{--- (3)}$$

Now the residue at $z=\alpha$ is

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha} \frac{\cancel{(z-\alpha)}(z^2+z+1)^n}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{(\alpha^2+\alpha+1)^n}{\alpha-\beta}$$

$$= \frac{\left(1-\frac{3}{2}+\frac{\sqrt{5}}{2}+\frac{7-3\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$= \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$\therefore$ from (3)

$$\int_C f(z)\,dz = 2\pi i\,\frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$\therefore$ from (2)

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} d\theta = \frac{1}{i}\,2\pi i\,\frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n \quad \text{--- (4)}$$

from (1) & (4)

$$I = \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta = \text{real part of } \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

$$\therefore \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} = \frac{2\pi(3-\sqrt{5})^n}{\sqrt{5}}$$

→ Prove that $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \frac{2\pi}{n!}$

Sol'n: Let $I = \int_0^{2\pi} e^{\cos\theta} e^{-(n\theta-\sin\theta)i}\,d\theta$

$$= \int_0^{2\pi} e^{\cos\theta} e^{-in\theta} e^{i\sin\theta}\,d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta+i\sin\theta} e^{-in\theta}\,d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\,d\theta$$

Putting $z=e^{i\theta}$ and let the contour 'C' be a unit circle $|z|=1$ with centre at the origin.

$$I = \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\,d\theta = \int_C e^z z^{-n}\frac{dz}{iz}$$

$$\left(\because z=e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}\right)$$

$$= \frac{1}{i}\int_C \frac{e^z}{z^{n+1}}\,dz$$

$$= \frac{1}{i}\int_C f(z)\,dz \quad \text{——— (1)}$$ where C is unit circle of radius with centre at origin.

where $f(z) = \frac{e^z}{z^{n+1}}$

Clearly $f(z)$ has a pole at $z=0$ of order $n+1$.

$\therefore \int_C f(z)\,dz = 2\pi i$ (Residue at $z=0$)

Now the residue at $z=0$ of order $(n+1)$ is

$$\lim_{z\to 0}(z-0)^{n+1}f(z) = \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left(z^{n+1}\frac{e^z}{z^{n+1}}\right)$$

$$= \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}(e^z)$$

$$= \frac{1}{n!}\lim_{z\to 0} e^z = \frac{1}{n!}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\,\frac{1}{n!}$$

$\therefore$ from (1)

$$I = \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\,d\theta = \frac{1}{i}(2\pi i)\frac{1}{n!} = \frac{2\pi}{n!}$$

$$\therefore \int_0^{2\pi} e^{\cos\theta+i\sin\theta} e^{-in\theta}\,d\theta = \frac{2\pi}{n!}$$

$$\Rightarrow \int_0^{2\pi} e^{\cos\theta} e^{-i(n\theta-\sin\theta)}\,d\theta = \frac{2\pi}{n!}$$

Equating real & imaginary parts

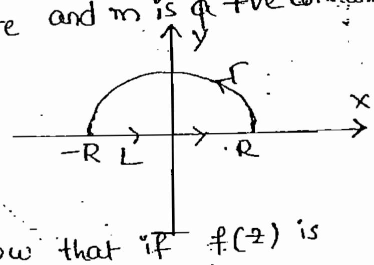
$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \frac{2\pi}{n!}$$

and $\int_0^{2\pi} e^{\cos\theta}\sin(n\theta-\sin\theta)\,d\theta = 0$

$$\therefore \int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta = \frac{2\pi}{n!}$$

Problems on Type (3): $\int_{-\infty}^{\infty} F(x)\begin{bmatrix}\cos mx\\ \sin mx\end{bmatrix}dx$

→ If $|F(z)| \le \frac{M}{R^K}$ for $z=Re^{i\theta}$, where $K>0$ and M are constants, Prove that $\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$, where Γ is the semicircular arc of radius R, shown in figure and m is a +ve constant.



Sol'n: We know that if $f(z)$ is continuous on a contour 'C' having length L, with $|f(z)| \le M$. Then $\left|\int_C f(z)\,dz\right| \le ML$.

$\therefore$ we have

$$\left|\int_\Gamma f(z)\,dz\right| \le \frac{M}{R^k}\cdot \pi R \quad (\because \text{length of arc } L = \pi R)$$

$$= \frac{\pi M}{R^{k-1}}$$

Now since $z = Re^{i\theta}$,

$$\int_\Gamma e^{imz} F(z)\,dz = \int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}\,d\theta$$

then

$$\left|\int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta}) iRe^{i\theta}\,d\theta\right| \le \int_0^\pi \left|e^{imRe^{i\theta}} F(Re^{i\theta}) iRe^{i\theta}\right| d\theta$$

$$= \int_0^\pi \left|e^{imR\cos\theta - mR\sin\theta} F(Re^{i\theta}) iRe^{i\theta}\right| d\theta$$

$$= \int_0^\pi e^{-mR\sin\theta} |F(Re^{i\theta})|\, R\,d\theta \quad (\because |e^{imR\cos\theta}| = 1)$$

$$\le \frac{M}{R^{k-1}} \int_0^{\pi} e^{-mR\sin\theta}\,d\theta$$

$$= \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \quad \text{——— (1)}$$

For $0 \le \theta \le \pi/2$, $\sin\theta \ge \frac{2\theta}{\pi}$

[Since, consider $f(\theta) = \frac{\sin\theta}{\theta}$. Then

$$f(\pi/2) = \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

Similarly $f(0) = \frac{\sin 0}{0} \Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

But $1 > \frac{2}{\pi}$.

The value of $f(\theta)$ varies from 1 to $\frac{2}{\pi}$ as θ varies from 0 to $\frac{\pi}{2}$.

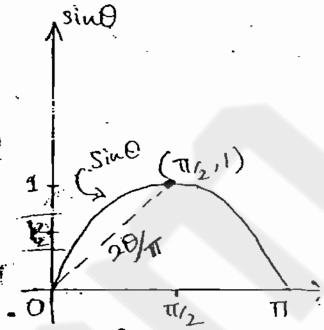
Now let us show that $f(\theta)$ is decreasing function in the given interval.

For that

Consider

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$$

$$= \frac{\cos\theta - \frac{\sin\theta}{\theta}}{\theta}$$



$\because$ cosine is a decreasing function in the interval $(0, \pi/2)$.

$\therefore f'(\theta) < 0$.

$\therefore f(\theta) = \frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ for all values of θ in $(0, \pi/2)$]

from (1), $\frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \le \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{\frac{-2mR\theta}{\pi}} d\theta$

$$= \frac{\pi M}{mR^k}\left(1 - e^{-mR}\right)$$

As $R\to\infty$, this approaches zero.

Since m & k are +ve.

i.e. $\lim_{R\to\infty} \int_\Gamma e^{imz} F(z)\,dz = 0$

$\boxed{2002}$ → Show that $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}, a > 0$

Sol'n: clearly which is in the form of type 1.

Now consider $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \int_C \frac{1}{z^2+1} e^{iaz}\,dz$

where C is the contour consisting of the semi-circle Γ of radius R. Together with the part of the real axis from $-R$ to R.

$$= \int_C f(z)\,dz \quad \text{——— (1)}$$

where $f(z) = \frac{e^{iaz}}{1+z^2}$

The function $f(z)$ has simple poles at $z=+i$ and $z=-i$, of which $z=i$ only inside C

$$\therefore \int_C f(z)dz = 2\pi i\{\text{Residue at } z=i\} \quad \text{--- (2)}$$

Now Residue at $z=i$ is,

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

$$(2) \equiv \int_C f(z)\,dz = 2\pi i\left(\frac{e^{-a}}{2i}\right)$$

$$= \pi e^{-a}$$

i.e. $$\int_C \frac{e^{iaz}}{1+z^2}dz = \pi e^{-a} \quad \text{--- (3)}$$

But $$\int_C \frac{e^{iaz}}{1+z^2}dz = \int_{-R}^{R}\frac{e^{iax}}{1+x^2}dx + \int_\Gamma \frac{e^{iaz}}{1+z^2}dz$$

$$\Rightarrow \int_{-R}^{R}\frac{\cos ax}{1+x^2}dx + i\int_{-R}^{R}\frac{\sin ax}{1+x^2}dx + \int_\Gamma \frac{e^{iaz}}{1+z^2}dz = \pi e^{-a} \quad (\text{from (3)}) \quad \text{--- (4)}$$

Now consider

$$\left|\int_\Gamma \frac{e^{iaz}}{1+z^2}dz\right| = \left|\int_0^\pi e^{iaz}\frac{1}{1+z^2}dz\right|$$

Let $z=Re^{i\theta} \Rightarrow dz = iRe^{i\theta}d\theta$

$$\therefore \left|\int_0^\pi e^{iaz}\frac{1}{1+z^2}dz\right| \le \int_0^\pi \left|\frac{e^{iaRe^{i\theta}}}{1+R^2e^{i2\theta}}iRe^{i\theta}\right|d\theta$$

$$\le \int_0^\pi \frac{|e^{iaR\cos\theta}||e^{-aR\sin\theta}|}{|1+R^2e^{i2\theta}|}R\,d\theta$$

$$\le \int_0^\pi \frac{|e^{-aR\sin\theta}|}{R^2-1}R\,d\theta \quad (\because |a+b| = |a-(-b)| \ge |a|-|b|)$$

$$= \frac{R}{R^2-1}\int_0^\pi e^{-aR\sin\theta}d\theta$$

$$= \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-aR\sin\theta}d\theta$$

But we know that $\sin\theta \ge \frac{2\theta}{\pi}$ when $0<\theta\le\pi/2$

$$\therefore aR\sin\theta \ge \frac{2aR\theta}{\pi}, a>0.$$

$$\Rightarrow e^{aR\sin\theta} \ge e^{2aR\theta/\pi}$$

$$\Rightarrow e^{-aR\sin\theta} \le e^{-2aR\theta/\pi}$$

$$\therefore \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-aR\sin\theta}d\theta \le \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta$$

$$= \frac{2R}{R^2-1}\left[\frac{e^{-2aR\theta/\pi}}{-2aR/\pi}\right]_0^{\pi/2}$$

$$= \frac{-\pi}{a(R^2-1)}\left[e^{-aR}-1\right]$$

$$= \frac{\pi}{a(R^2-1)}\left[1-e^{-aR}\right]$$

This tends to '0' as $R\to\infty$. Now as $R\to\infty$ from (4), we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}dx + i\int_{-\infty}^{\infty}\frac{\sin x}{1+x^2}dx = \pi e^{-a}.$$

Comparing the real part, we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}dx = \pi/e^a$$

$$\Rightarrow 2\int_0^\infty \frac{\cos x}{1+x^2}dx = \pi e^{-a} \quad \left(\because f(x)=\frac{\cos x}{1+x^2} \text{ is even function}\right)$$

$$= 2\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2}e^{-a}$$

Hence the result.

HW → Apply the Calculus of residues to evaluate $\int_0^\infty \frac{x\sin x}{x^2+a^2}dx; (a>0)$ $\left[\text{Hint } \int_C \frac{ze^{iz}}{z^2+a^2}dz\right]$

HW → Find $\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx$

* Complex Integration And Cauchy's theorem *

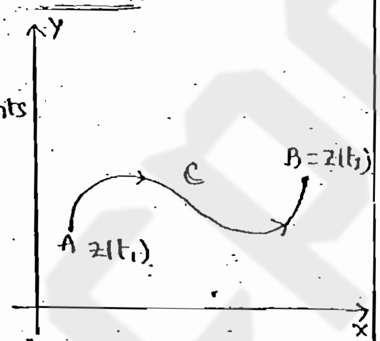
*** Continuous Curve or arc :-**

Let $\phi(t)$ and $\psi(t)$ be two real valued continuous functions of a real variable 't' in $t_1 \le t \le t_2$ then $z = x+iy = \phi(t)+i\psi(t) = z(t)$, $t_1 \le t \le t_2$ defines a continuous curve in the z-plane joining the points $z(t_1)$ and $z(t_2)$. Then the point $A: z(t_1)$ is called the initial point of the curve and $B: z(t_2)$ is called the terminal point of the curve.

– If B does not coincide with A, it is also called an arc.

→ **Simple Open Curve :-**

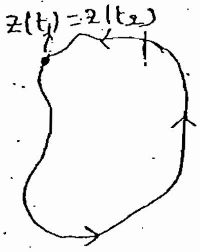
If the initial and terminal points of curve doesnot coincide & does not intersect itself, the curve is said to be simple open curve.



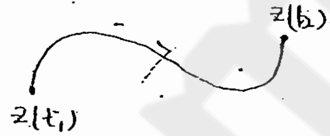
→ **closed curve :-** If the initial and terminal points of a curve 'C' coincide (i.e. $z(t_1) = z(t_2)$) then the given curve is called a closed curve.

→ **Simple closed Curve :-**

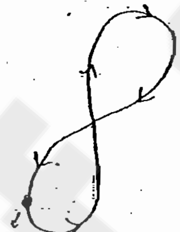
A closed curve which doesnot intersect itself is called simple closed curve (Jordan curve).



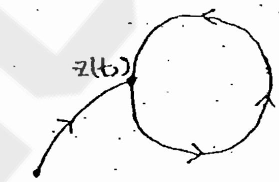
Simple closed (or) Jordan Curve.



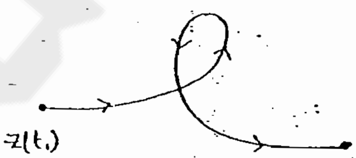
Simple open curve



closed, but not simple



not closed, not simple



not closed, not simple.

*** Smooth Curve :**

If $\phi(t)$ and $\psi(t)$ (i.e. $z(t)$) have continuous derivatives in $t_1 \le t \le t_2$ the curve is called a smooth curve (or) arc.

*** Piece-wise (or) Sectionally Smooth Curve (or) Contour :-**

A curve which is composed of a finite number of smooth arcs is called piecewise smooth (C1)

sectionally smooth curve (C) contour.

Ex:- Boundary of a square.

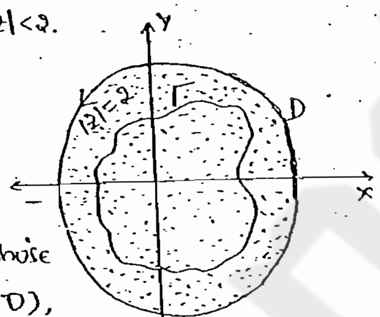
* Simply - Connected domain (Region):

A domain D is said to be simply-connected if every simple closed curve contained in D contains only points of R inside. (or)

A domain D is called simply-connected if any simple closed curve which lies in D can be shrunk to a point without leaving D.

For example: Suppose D is the domain defined by $|z|<2$.

If Γ is any simple closed curve lying in D (i.e. whose points are in D), we see that it can be shrunk to a point which lies in D, and does not leave D, so that D is simply-connected.



Multiply - Connected :-

A domain which is not simply connected is called multiply-connected.

For example: If D is the domain defined by $1<|z|<2$ (shown shaded in figure (i))

then there is a simple closed curve Γ lying in D which can not possibly be shrunk to a point without leaving D, so that D is multiply-connected.

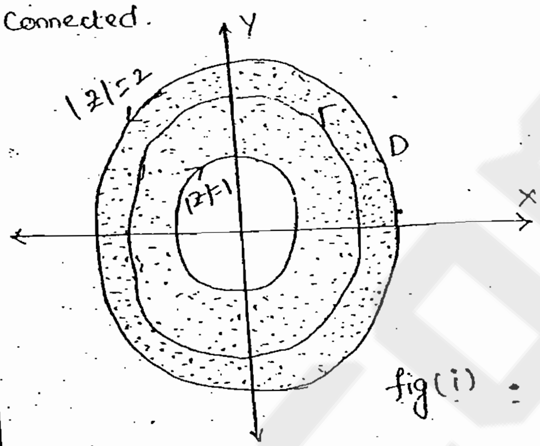


fig (i)

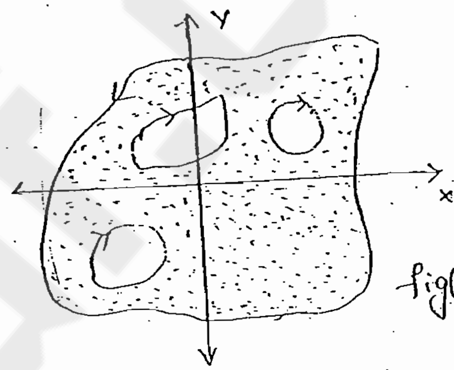


fig (ii)

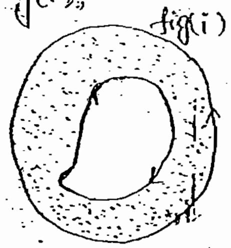
Geometrically, a simply connected domain is one which does not have any holes in it, while a multiply connected domain is one which does. Thus the multiply-connected domains of figures (i) & (ii) have respectively one and three holes in them.

* Positive orientation :-

The boundary 'C' of a domain is said to have positive orientation or to be traversed in the +ve sense, if a person walking on 'C' always has the domain to his left.

Ex: Let us observe the fig(i),

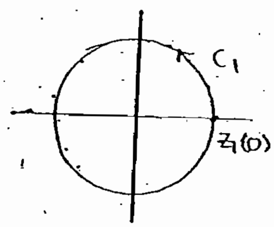
fig(i)



In the multiply-connected domain the outer boundary has +ve orientation, if traversed in counter clockwise, where as the inner boundary has +ve orientation if traversed clockwise as indicated in the figure (1).

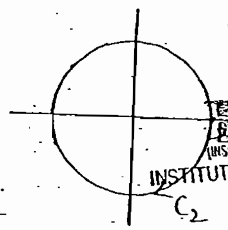
Ex (2):

$C_1 : z_1(t) = e^{it}$
$= \cos t + i \sin t;$
$0 \le t \le 2\pi$



$C_2 : z_2(t) = e^{-it}$
$= \cos t - i \sin t;$
$0 \le t \le 2\pi$

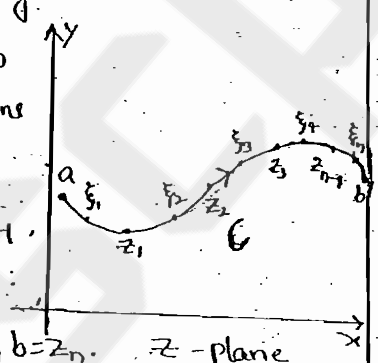
$z(t) = \phi(t) + i\psi(t)$
$x = \phi(t), y = \psi(t)$
$t = 0 : x = 1, y = 0$



* Complex Line Integrals:

Let $f(z)$ be continuous at all points of a curve 'c' which we shall assume has a finite length.

Subdivide 'c' into n parts by means of points $z_1, z_2, \ldots z_{n-1}$, chosen arbitrarily and call $a = z_0, b = z_n$.



On each arc joining z_{k-1} to z_k (where $k : 1 \to n$) choose a point ξ_k.

Form the sum

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1}).$$

On writing $z_k - z_{k-1} = \Delta z_k$

this becomes
$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$
$$= \sum_{k=1}^{n} f(\xi_k)\, \Delta z_k.$$

Let the number of subdivisions 'n' increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum 'S_n' approaches a limit which does not depend on the mode of subdivision and we denote this limit by $\int_a^b f(z)dz$ (or) $\int_C f(z)dz$

This is called the complex line integral or line integral of $f(z)$ along the curve 'c'.

In such a case $f(z)$ is said be integrable along 'c'.

*Note (1): If $f(z)$ is analytic at all points of a region R and if 'c' is a curve lying in R then $f(z)$ is certainly integrable along 'c'.

Note(2): Suppose that the smooth curve 'c' is parametrised by $z(t) = x(t) + iy(t)$, $a \le t \le b$.

then we can write

$$\int_C f(z)dz = \int_a^b f(z)dz = \int_a^b f(z(t))\, z'(t)\,dt$$

Note(3): In the case of a real function $f(x)$, the integral of $f(x)$, i.e. $\int_a^b f(x)\,dx$ depends upon the nature of the function and end points of the interval $[a,b]$.

But in the case of complex functions

The value of $\int_a^b f(z)\,dz$ depends upon the nature of the function $f(z)$ and on the path 'C', joining the two points, not just the end points of 'C'.

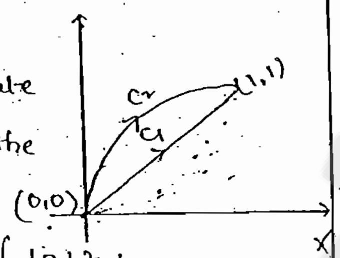
This can be observed from the following examples.

Examples: Find $\int_C |z|^2 dz$ along the curves (a) $C = C_1 : z_1(t) = t+it$ $(0 \le t \le 1)$

(b) $C = C_2 : z_2(t) = t^2+it$ $(0 \le t \le 1)$

Sol^n: For these two curves the initial point $(0,0)$ and the terminal point is $(1,1)$.

Now let us evaluate the integral along the curves



(a) $\int_C f(z)\,dz = \int_C |z|^2 dz$

$$= \int_{C_1} |z|^2 dz = \int_0^1 |t+it|^2 (1+i)\,dt$$

$z = t+it$, $z(t) = x+iy$, $x = t$, $y = t$, $\varphi(t)$, $y = x$

given

$$\frac{dz}{dt} = 1+i$$
$$dz = (1+i)dt$$

$$= (1+i)\int_0^1 (t^2+t^2)\,dt$$

$$= (1+i)\left(\frac{2t^3}{3}\right)_0^1$$

$$= \frac{2}{3} + \frac{2}{3}i$$

∴ In this case the path 'C_1' is the straight line joining $(0,0)$ and $(1,1)$.

(b) $$\int_C f(z)\,dz = \int_{C_2} |z|^2 dz = \int_0^1 |t^2+it|^2 (2t+i)\,dt$$

$$= \int_0^1 2t\,(t^4+t^2)\,dt + i\int_0^1 (t^4+t^2)\,dt$$

In this case, the path C_2 is a parabola joining $(0,0)$ & $(1,1)$.

Now comparing these two, we get

$$\int_{C_1} |z|^2 dz \ne \int_{C_2} |z|^2 dz.$$

But this need not be the case in all cases.

Example (H.10)

verify that $\int_C z^2 dz$ along both the paths

(a) $C_1 : z(t) = t+it$, $(0 \le t \le 1)$

(b) $C_2 : z(t) = t^2+it$, $(0 \le t \le 1)$

is equal.

Parametrization :-

In general the equations of curves in a plane will be in terms of variables x and y. Then x, y are expressed in terms of a parameter.

Example (1)

Find the parametrized curve tracing of the straight line segment from $z = i$ to $z = 1-i$.

Sol: In general the equation of the straight line in terms of a parameter is given by $z(t) = (a+bt) + i(c+dt)$, where a, b, c, d are determined by depending upon the initial and terminal points of the line segment and the interval of 't' which accounts the points on the given curve.

The first point $t = t_1$ is given by $z = i$.

$\therefore i = (a+bt_1) + i(c+dt_1)$.

Comparing the real and imaginary parts, we get

$a+bt_1 = 0$, $c+dt_1 = 1$

Similarly the second point $t = t_2$ is given as $z = 1-i$

$\therefore 1-i = (a+bt_2) + i(c+dt_2)$

From this we get $a+bt_2 = 1$; $c+dt_2 = -1$

$\therefore$ There are four equations in six unknowns a, b, c, d, t_1, t_2.

Now choose t_1, t_2 are arbitrarily

$\therefore$ let $t_1 = 0$, $t_2 = 1$.

Thus solving the above equations we get $a = 0$, $c = 1$, $b = 1$, $d = -2$

$\therefore$ The parametric equation of the given line segment is

$$z(t) = t + i(1-2t); \quad 0 \le t \le 1.$$

Example (2): Find the parametric equation of the parabola $y = 2x^2 - 3$ that initial point $z = -1-i$ and the terminal point $z = 2+5i$

In this case the x is of second degree and y is of first degree. So we take the general parametric equation as

$$z(t) = (a+bt) + i(c+dt^2) \quad \text{——} (1)$$

Let $t \in [0,1]$. Then determine the values of a, b, c & d. In this case $(-1,-1)$ and $(2,5)$ are the initial & terminal points.

When $t = 0$, $a = -1$, $c = -1$

$t = 1$; $b = 3$, $d = 6$.

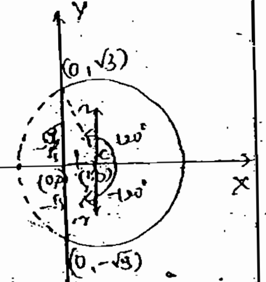
$\therefore$ The parametric equation of the given parabola is

$$z(t) = (3t-1) + i(6t^2 - 1); \quad 0 \le t \le 1.$$

Example (3): The part of the circle $|z-1| = 2$ in the right half of the z-plane.

The general parametric equation of a circle is $z(t) = z_0 + Re^{it}$, where z_0 is the centre of the circle, R the radius of the circle.

In the present case, centre of the circle is at $(1,0)$ and $R=2$



$\therefore$ The parametric equation of the circle is

$$z(t) = 1 + 2e^{it}$$
$$= (1+2\cos t) + i2\sin t \quad \text{—— (1)}$$

But in this case we require the circle of the right half of the z-plane.

For the first point $(0, -\sqrt{3})$ let us find

on using (1), we get,

$$0 + (-\sqrt{3})i = (1+2\cos t) + i2\sin t$$

$$\cos t = -\tfrac{1}{2}, \quad \sin t = -\sqrt{3}/2$$

$\therefore$ t satisfying the above two is $t = -\frac{2\pi}{3}$

Similarly let us consider the end point $(0, \sqrt{3})$.

Then $\cos t = -1/2$, $\sin t = \sqrt{3}/2$

Solving these two we get

$$t = 2\pi/3$$

$\therefore$ The required parametric equation is

$$z(t) = (1 + 2\cos t) + 2i\sin t, \quad -\frac{2\pi}{3} \le t \le \frac{2\pi}{3}$$

Example (?) — Evaluate $\int_C x\,dz$ along the line segment 'C' from origin to $(1+i)$.

Soln: The parametric equation of the curve in this example is $z(t) = t + it$, $0 \le t \le 1$.

$\therefore x(t) = t$, $y(t) = t$ & $dz = (1+i)dt$

$$\int_C x\,dz = \int_0^1 t(1+i)\,dt = (1+i)\left(\frac{t^2}{2}\right)_0^1$$

$$= \frac{1+i}{2}$$

$\rightarrow$ Evaluate $\int_C y\,dz$, $\int_C \bar{z}\,dz$ along the line segment 'C' from

(i) origin to $1+i$

(ii) origin to $1-i$.

Note: Suppose $C: z = z(t)$ is a smooth curve defined on the interval $[a,b]$. Breaking this interval into two subintervals $[a,c]$ and $[c,b]$,

we obtain two curves C_1 & C_2 from $z(t)$ by restricting the parametre 't' to the intervals $[a,c]$ and $[c,b]$ respectively. For any function $f(z)$ continuous on 'C'

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

$$= \int_a^c f(z(t))z'(t)\,dt + \int_c^b f(z(t))z'(t)\,dt$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

By the sum of two curves, we

antiderivative ?

Sol^n: $\int_{-i}^{i} |z|\,dz = \begin{cases} i & \text{along a straight line } z(t) = i(2t-1);\ 0 \le t \le 1 \\ 2i & \text{along a circle } z = e^{it};\ -\pi/2 \le t \le \pi/2 \end{cases}$

$F'(z) = f(z)$

$F(z) = \int_{-i}^{i} f(z)$

The integral (solution) depends on the path of integration. Hence the integrand is not analytic in a domain containing the path.

So antiderivative of $|z|$ doesnot exist.

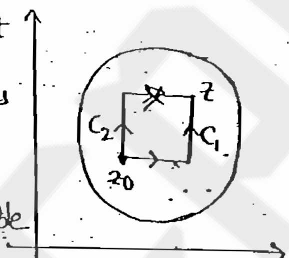
Theorem-(II)

* Now we Prove that Cauchy-Goursat theorem in a circular disc:

→ Let $f(z)$ be analytic in a domain containing the closed circle $|z-z_0| \le r$.

Then $\int_{|z-z_0|=r} f(z)\,dz = 0$.

Proof: we know that when $f(z)$ is Continuous in a domain D and there is a differentiable function $F(z)$ such that $F'(z) = f(z)$ in D, then $\int_C f(z)\,dz = 0$.



where 'C' is a simple closed Contour in D.

In this theorem we want to use of the above result and Prove the theorem, because the given Contour $|z-z_0| = r$ is closed and $f(z)$ is analytic in D implies $f(z)$ is continuous in D. Therefore now we have to establish that the antiderivative $F(z)$ to $f(z)$ exists in D.

For this consider any Point z in $|z-z_0| \le r$. Let us denote the curve consisting of horizontal line segment from $z_0 = x_0 + iy_0$ to $x + iy_0$ follow by vertical line segment from $x+iy_0$ to $z = x+iy$ by C_1.

Similarly let us denote the curve consisting of the vertical line segment from $z_0 = x_0 + iy_0$ to $x_0 + iy$ and the horizontal line segment from $x_0 + iy$ to $x+iy$ by C_2.

Now let us define,

$$F(z) = \int_{C_1} f(z)\,dz$$

$$= \int_{x_0}^{x} f(t+iy_0)\,dt + \int_{y_0}^{y} f(x+it)\,i\,dt \quad \text{--- (1)}$$

because, the first line is $z(t) = t + iy_0$; $x_0 \le t \le x$

the second line is $z(t) = x + it$; $y_0 \le t \le y$

Since $C_1 - C_2$ is a rectangle in D.

by Cauchy's - Goursat theorem, we get,

$$\int_{C_1 C_2} f(z)\,dz = 0.$$

$$\Rightarrow \int_{C_1-C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

from ①, $F(z) = \int_{C_2} f(z)\,dz$

$$\Rightarrow F(z) = \int_{y_0}^{y} f(x_0+it)\,dt + \int_{x_0}^{x} f(t+iy)\,dt \quad \text{②}$$

because : the first line is

$z(t) = x_0 + it;\ y_0 \le t \le y$

the second line is $z(t) = t+iy;\ x_0 \le t \le x$.

There is a result in differential calculus, which states that

$$\frac{\partial}{\partial \alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}\,dx + f(b(\alpha),\alpha)\frac{db}{d\alpha} - f(a(\alpha),\alpha)\frac{da}{d\alpha}$$

Keeping this in view differentiating ①, with respect to y and observe the following:

In the present case two variables are y and t. But the other term 'x' present in ① can be treated a parameter and one variable y is absent in $f(x+it)$. Treating that way we get,

$$\frac{\partial}{\partial y}\int_{y_0}^{y} f(x+it)\,i\,dt = i\left[\int_{y_0}^{y} \frac{\partial f(x+it)}{\partial y}\,i\,dt + f(x+iy)\frac{d(y)}{dy} - f(x+iy_0)\frac{d(y_0)}{dy}\right]$$

$$= i\left[0 + f(x+iy) - 0\right]$$

$$= i f(z)$$

$\therefore$ from ①,

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int_{x_0}^{x} f(t+iy_0)\,dt + \int_{y_0}^{y} f(x+it)\,i\,dt\right]$$

$$= \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)\,dt + \frac{\partial}{\partial y_0}\int_{y_0}^{y} f(x+it)\,dt$$

$$= 0 + if(z) \quad \left(\because \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)\,dt = \int_{x_0}^{x} \frac{\partial f(t+iy_0)}{\partial y}\,dt + f(x+iy_0)\frac{dx}{dy} - f(x_0+iy_0)\frac{dx_0}{dy} = 0\right)$$

$$\left.\begin{array}{l} \boxed{\dfrac{\partial F}{\partial y} = i f(z)} \\ \Rightarrow \boxed{-i\dfrac{\partial F}{\partial y} = f(z)} \end{array}\right\} \longrightarrow ③$$

Similarly let us take the partial derivative of ② with respect to 'x'. But before that let us observe that the following.

In this case 'x' and 't' are two variables and y is treated as a parameter. then

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} f(t+iy)\,dt$$

$$= \int_{x_0}^{x} \frac{\partial f(t+iy)}{\partial x}\,dt + f(x+iy)\frac{d(x)}{dx} - f(x_0+iy)\frac{d(x_0)}{dx}$$

$$= 0 + f(x+iy)\cdot 1 - 0$$

$$= f(x+iy)$$

$$= f(z) \text{ ——— } (4)$$

Comparing (3) & (4), we get,

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

Since $f(z)$ is analytic in the domain $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are continuous in the domain.

$\therefore$ Now $F(z)$ is such that its real and imaginary parts have continuous partial derivatives which satisfy Cauchy-Riemann conditions

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}.$$

So we conclude that $F(z)$ is analytic in the domain as z is arbitrary, so we get

$$F'(z) = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

$$\Longrightarrow F'(z) = f(z).$$

So antiderivative of $f(z)$ exists in D.

Hence we get $\int_C f(z)\,dz = 0.$

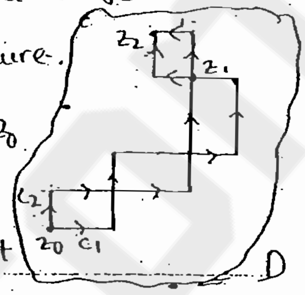
Theorem (III): (Cauchy – Goursat):

If $f(z)$ is analytic in a simply connected domain D and 'C' is closed contour lying in D then

$$\int_C f(z)\,dz = 0.$$

Proof: For proving the theorem first we try to establish the existence of an antiderivative $F(z)$ to the given function $f(z)$.

For this let us follow the following procedure.



Let us fix point z_0 in D and choose any arbitrary point z in D.

Then let us construct two curves C_1 & C_2 joining the points z_0 & z as shown in figure.

Now define $F(z) = \int_{C_1} f(z)\,dz$

Since $C_1 - C_2$ is a closed curve and integral around each rectangle is zero, we get

$$\int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_{C} f(z)\,dz = \int_{C_2} f(z)\,dz \text{ ——— } (1)$$

Suppose that $z_1 = x_1 + iy_1$ is the last point of intersection of C_1 & C_2 b/w z_0 and $z = x+iy$.

Suppose that in this rectangle, C_1 consists of horizontal followed by verticle line where as C_2 consists of verticle line followed by horizonta as shown in figure.

the value of the integral $\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} = f(z)$

$f(z)$ from z_0 to z_1 is the same along both contours C_1 & C_2 and we denote by a constant 'K', since z_1 is a constant then consider

$$F(z) = \int_{C_1} f(z)\,dz = K + \int_{x_1}^{x} f(t+iy_1)\,dt + \int_{y_1}^{y} f(x+it)\,i\,dt \quad \text{---} (2)$$

from (1),

$$F(z) = \int_{C_2} f(z)\,dz = \int_{y_1}^{y} f(x+it)\,i\,dt + \int_{x_1}^{x} f(t+iy)\,dt \quad \text{---} (3)$$

Now Differentiating (2) with respect to 'y' we get,

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int_{y_1}^{y} f(x+it)\,i\,dt$$

$$= i f(x+iy) = i f(z)$$

Similarly differentiating (3) with respect to 'x' we get

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int_{x_1}^{x} f(t+iy)\,dt\right)$$

$$= f(x+iy)$$

$$= f(z)$$

Comparing the above two results, we get,

$\therefore F(z)$ satisfies the Cauchy's-Riemann equations.

since $f(z)$ continuous in D, the partials $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ of $F(z)$ is such that its real and imaginary parts satisfy Cauchy's-Riemann equations and have continuous partials.

So $F(z)$ is analytic in D.

$$\therefore F'(z) = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

$\therefore$ antiderivative $f(z)$ exists in D.

Hence $\int_C f(z)\,dz = 0$.

Problem

→ what can be said about $\int_C \frac{1}{z}\,dz$ if the closed contour C pass through the origin.

Sol'n: At $z=0$, the integrand $\frac{1}{z}$ is undefined.

so it is not analytic on C

$$\therefore \int_C \frac{1}{z}\,dz \neq 0.$$

→ Suppose $f(z)$ is analytic on a closed contour 'C'. Does $\int_C f(z)\,dz = 0$?

Sol'n: Need not be.

Because, it is given that $f(z)$ is analytic only on 'C', but the nature

SOLVED PROBLEMS

LINE INTEGRALS

4.1 Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t$, $y = t^2 + 3$; (b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4); (c) a straight line from (0, 3) to (2, 4).

Solution

(a) The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Then, the given integral equals

$$\int_{t=0}^{1} [2(t^2+3) + (2t)^2]\,2\,dt + [3(2t) - (t^2+3)]\,2t\,dt = \int_0^1 (24t^2 + 12 - 2t^3 - 6t)\,dt = \frac{33}{2}$$

(b) Along the straight line from (0, 3) to (2, 3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = \frac{44}{3}$$

Along the straight line from (2, 3) to (2, 4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = \frac{5}{2}$$

Then, the required value $= 44/3 + 5/2 = 103/6$.

(c) An equation for the line joining (0, 3) and (2, 4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then, the line integral equals

$$\int_{y=3}^{4} [2y + (2y-6)^2]\,2\,dy + [3(2y-6) - y]\,dy = \int_3^4 (8y^2 - 39y + 54)\,dy = \frac{97}{6}$$

The result can also be obtained by using $y = \frac{1}{2}(x + 6)$.

4.2 Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by: (a) $z = t^2 + it$, (b) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$, respectively. Then the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\,d(t^2 + it) = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}$$

Another method. The given integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then, the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) + i\int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$

$$= \int_0^2 (2t^3 + t)\,dt + i\int_0^2 (-t^2)\,dt = 10 - \frac{8i}{3}$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from (0, 0) to (0, 2) for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy + i\int_{y=0}^{2} (0)(dy) - y(0) = \int_{y=0}^{2} y\,dy = 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0, 2) to (4, 2) for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2\cdot 0 + i\int_{x=0}^{2} x\cdot 0 - 2\,dx = \int_0^4 x\,dx + i\int_0^4 -2\,dx = 8 - 8i$$

Then, the required value $= 2 + (8 - 8i) = 10 - 8i$.

4.3 Suppose $f(z)$ is integrable along a curve C having finite length L and suppose there exists a positive number M such that $|f(z)| \le M$ on C. Prove that

$$\left|\int_C f(z)\,dz\right| \le ML$$

Solution By definition, we have on using the notation of page 3.17,

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k \tag{1}$$

Now

$$\left|\sum_{k=1}^{n} f(\xi_k)\,\Delta z_k\right| \le \sum_{k=1}^{n} |f(\xi_k)|\,|\Delta z_k| \le M\sum_{k=1}^{n} |\Delta z_k| \le ML \tag{2}$$

where we have used the facts that $|f(z)| \le M$ for all points z on C and that $\sum_{k=1}^{n} |\Delta z_k|$ represents the sum of all the chord lengths joining points z_{k-1} and z_k, where $k = 1, 2, \ldots, n$, and that this sum is not greater than the length of C.

Taking the limit of both sides of (2), using (1), the required result follows. It is possible to show, more generally, that

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz|$$

4.22 Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z = a$ is inside the simple closed curve C.

Solution As in Problem 4.21,

$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n}$$

$$= \int_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon^n e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta} d\theta$$

$$= \frac{i}{\epsilon^{n-1}} \frac{e^{(1-n)i\theta}}{(1-n)i}\Bigg|_0^{2\pi} = \frac{1}{(1-n)\epsilon^{n-1}}[e^{2(1-n)\pi i} - 1] = 0$$

where $n \neq 1$.

4.23 If $F(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi} dz$, where C is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

Find the value of $F(3.5)$.

Solution

$$F(3.5) = \int_C \frac{4z^2 + z + 5}{z - 3.5} dz$$

Since $\xi = 3.5$ is the only singular point of $\frac{4z^2 + z + 5}{z - 3.5}$ and it lies outside the ellipse C, therefore, $\frac{4z^2 + z + 5}{z - 3.5}$ is analytic every where within C.

Hence, by Cauchy's theory,

$$\int_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0$$

4.24 Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices $-1, 1, 1 + i, -1 + i$.

Solution Let $f(z) = z^3$, since $f(z)$ is analytic within and on the boundary of the rectangle (say, C) and also $f'(z)$ is continuous at each point within and on C. Hence, applying Cauchy's theorem, we get

$$\oint_C z^3 dz = 0$$

Consider $\oint_C z^3 dz$

$= \oint_C (x + iy)^3 (dx + idy)$

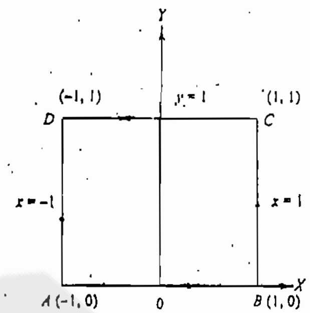
$$= \oint_C [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$= \int_{AB} (x+iy)^3 dx + i\,(x+iy)^3 dy$$

$$+ \int_{BC} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$+ \int_{CD} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$+ \int_{DA} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$



Along AB, $y = 0 \Rightarrow dy = 0$ $(-1 \le x \le 1)$
Along BC, $x = 1 \Rightarrow dx = 0$ $(0 \le y \le 1)$
Along CD, $y = 1 \Rightarrow dy = 0$ $(-1 \le x \le 1)$
Along DA, $x = -1 \Rightarrow dx = 0$ $(0 \le y \le 1)$

Fig. 4.21

$$= \int_{-1}^{1} x^3 dx + \int_0^1 i(1+iy)^3 dy + \int_1^{-1} (x+i)^3 dx + \int_1^0 i(-1+iy)^3 dy$$

$$= 0 + i\int_0^1 (1+iy)^3 dy + \int_1^{-1} (x+i)^3 dx + i\int_1^0 (iy-1)^3 dy$$

$$= i\left[\frac{(1+iy)^4}{4 \times i}\right]_0^1 + \left[\frac{(x+i)^4}{4}\right]_1^{-1} + i\left[\frac{(iy-1)^4}{4 \times i}\right]_1^0$$

$$= \frac{1}{4}\left[(1+iy)^4\right]_0^1 + \frac{1}{4}\left[(x+i)^4\right]_1^{-1} + \frac{1}{4}\left[(iy-1)^4\right]_1^0$$

$$= \frac{1}{4}[\{(1+i)^4 - 1\} + \{(i-1)^4 - (i+1)^4\} + \{1 - (i-1)^4\}] = 0$$

This verifies the Cauchy's theorem.

4.25 Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1| = 1$.

Solution If $f(z) = \frac{z+4}{z^2+2z+5}$, then poles of $f(z)$ are given by

$$z^2 + 2z + 5 = 0, \quad \therefore z = -1 + 2i, -1 - 2i$$

when $z = -1 + 2i$, then $|z+1| = |-1+2i+1| = 2 > 1$

$\therefore$ The pole $z = -1 + 2i$ lies outside the circle $|z+1| = 1$

Since both the poles lie outside the circle C, hence $f(z)$ is analytic everywhere within C. Also, $f'(z)$ is continuous within and on C.

By applying Cauchy's theorem, we get

$$\int_C f(z)dz = 0 \quad \text{i.e.} \quad \int_C \frac{z+4}{z^2+2z+5} dz = 0.$$

4.26 Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution Let $f(z) = \frac{z^2 - z + 1}{z - 1}$

Poles of $f(z)$ are given by

$$z - 1 = 0$$

i.e., $z = 1$

Since the pole $z = 1$ lie outside the circle C, hence $f(z)$ is analytic within and on C. Also, $f'(z)$ is continuous at each point within and on C.

So, by applying Cauchy's theorem, we get $\int_C f(z)dz = 0$.

4.27 Let C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1, 1) and (2, 3), find the value of

$$\int_C (12z^2 - 4iz)\, dz$$

Method 1. By Problem 4.17, the integral is independent of the path joining (1, 1) and (2, 3). Hence, any path can be chosen. In particular, let us choose the straight line paths from (1, 1) to (2, 1) and then from (2, 1) to (2, 3).

Solution

Case 1. Along the path from (1, 1) to (2, 1), $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$.

Then, the integral equals

$$\int_{x=1}^{2} \{12(x+i)^2 - 4i(x+i)\}\, dx = \{4(x+i)^3 - 2i(x+i)^2\}\Big|_1^2 = 20 + 30i$$

Case 2. Along the path from (2, 1) to (2, 3), $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy$, $dz = i\,dy$.

Then, the integral equals

$$\int_{y=1}^{3} \{12(2+iy)^2 - 4i(2+iy)\}i\, dy = \{4(2+iy)^3 - 2i(2+iy)^2\}\Big|_1^3 = -176 + 8i$$

Then, adding the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\, dz = (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} = -156 + 38i$$

It is clear that Method 2 is easier.

INTEGRALS OF SPECIAL FUNCTIONS

4.28 Determine (a) $\int \sin 3z \cos 3z\, dz$, (b) $\int \cot(2z+5)\, dz$.

Solution

(a) *Method 1.* Let $\sin 3z = u$. Then, $du = 3\cos 3z\,dz$ or $\cos 3z\,dz = du/3$. Then

$$\int \sin 3z \cos 3z\,dz = \int u\frac{du}{3} = \frac{1}{3}\int u\,du = \frac{1}{3}\frac{u^2}{2} + c = \frac{1}{6}u^2 + c = \frac{1}{6}\sin^2 3z + c$$

Method 2. $\int \sin 3z \cos 3z\,dz = \frac{1}{3}\int \sin 3z\,d(\sin 3z) = \frac{1}{6}\sin^2 3z + c$

Method 3. Let $\cos 3z = u$. Then $du = -3\sin 3z\,dz$ or $\sin 3z\,dz = -du/3$. Then

$$\int \sin 3z \cos 3z\,dz = -\frac{1}{3}\int u\,du = -\frac{1}{6}u^2 + c_1 = -\frac{1}{6}\cos^2 3z + c_1$$

Note that the results of Methods 1 and 3 differ by a constant.

(b) *Method 1.* $\int \cot(2z+5)\,dz = \int \frac{\cos(2z+5)}{\sin(2z+5)}\,dz$

Let $u = \sin(2z+5)$. Then, $du = 2\cos(2z+5)\,dz$ and $\cos(2z+5)\,dz = du/2$. Thus

$$\int \frac{\cos(2z+5)\,dz}{\sin(2z+5)} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln\sin(2z+5) + c$$

Method 2.
$$\int \cot(2z+5)\,dz = \int\frac{\cos(2z+5)}{\sin(2z+5)}\,dz = \frac{1}{2}\int\frac{d\{\sin(2z+5)\}}{\sin(2z+5)} = \frac{1}{2}\ln\sin(2z+5) + c$$

4.29 (a) Prove that $\int F(z)\,G'(z)\,dz = F(z)\,G(z) - \int F'(z)\,G(z)\,dz$.

(b) Find $\int z\,e^{2z}dz$ and $\int_0^1 z\,e^{2z}dz$.

(c) Find $\int z^2 \sin 4z\,dz$ and $\int_0^{2\pi} z^2 \sin 4z\,dz$.

(d) Evaluate $\int_C (z+2)e^{iz}dz$ along the parabola C defined by $\pi^2 y = x^2$ from $(0, 0)$ to $(\pi, 1)$.

Solution

(a) We have

$$d\{F(z)\,G(z)\} = F(z)\,G'(z)\,dz + F'(z)\,G(z)\,dz$$

Integrating both sides yields

$$\int d\{F(z)\,G(z)\} = F(z)\,G(z) = \int F(z)\,G'(z)\,dz + \int F'(z)\,G(z)\,dz$$

Then

$$\int F(z)\,G'(z)\,dz = F(z)\,G(z) - \int F'(z)\,G(z)\,dz$$

The method is often called *integration by parts.*

(b) Let $F(z) = z$, $G'(z) = e^{2z}$. Then $F'(z) = 1$ and $G(z) = \frac{1}{2}e^{2z}$, omitting the constant of integration. Thus, by part (a),

$$\int ze^{2z}dz = \int F(z)\,G'(z)\,dz = F(z)\,G(z) - \int F'(z)\,G(z)\,dz$$
$$= (z)\left(\tfrac{1}{2}e^{2z}\right) - \int 1\cdot\tfrac{1}{2}e^{2z}dz = \tfrac{1}{2}ze^{2z} - \tfrac{1}{4}e^{2z} + c$$

Hence $\int_0^1 ze^{2z}dz = \left(\tfrac{1}{2}ze^{2z} - \tfrac{1}{4}e^{2z} + c\right)\Big|_0^1 = \tfrac{1}{2}e^2 - \tfrac{1}{4}e^2 + \tfrac{1}{4} = \tfrac{1}{4}(e^2+1)$

(c) Integrating by parts choosing $F(z) = z^2$, $G'(z) = \sin 4z$ we have

$$\int z^2 \sin 4z\,dz = (z^2)(-\tfrac{1}{4}\cos 4z) - \int (2z)(-\tfrac{1}{4}\cos 4z)\,dz$$
$$= -\tfrac{1}{4}z^2\cos 4z + \tfrac{1}{2}\int z\cos 4z\,dz$$

Integrating this last integral by parts, this time choosing $F(z) = z$ and $G'(z) = \cos 4z$, we find

$$\int z\cos 4z\,dz = (z)\left(\frac{1}{4}\sin 4z\right) - \int (1)\left(\frac{1}{4}\sin 4z\right)dz = \frac{1}{4}z\sin 4z + \frac{1}{16}\cos 4z$$

Hence $\int z^2\sin 4z\,dz = -\frac{1}{4}z^2\cos 4z + \frac{1}{8}z\sin 4z + \frac{1}{32}\cos 4z + c$

and $\int_0^{2\pi} z^2 \sin 4z\,dz = -\pi^2 + \frac{1}{32} - \frac{1}{32} = -\pi^2$

The double integration by parts can be indicated in a suggestive manner by writing

$$\int z^2\sin 4z\,dz = (z^2)\left(-\frac{1}{4}\cos 4z\right) - (2z)\left(-\frac{1}{16}\sin 4z\right) + (2)\left(\frac{1}{64}\cos 4z\right) + c$$
$$= -\frac{1}{4}z^2\cos 4z + \frac{1}{8}z\sin 4z + \frac{1}{32}\cos 4z$$

where the first parentheses in each term (after the first) is obtained by differentiating z^2 successively, the second parentheses is obtained by integrating $\sin 4z$ successively, and the terms alternate in sign.

(d) The points $(0, 0)$ and $(\pi, 1)$ correspond to $z = 0$ and $z = \pi + i$. Since $(z+2)e^{iz}$ is analytic, we see by Problem 4.17 that the integral is independent of the path and is equal to

$$\int_0^{\pi+i}(z+2)\,e^{iz}dz = \left\{(z+2)\left(\frac{e^{iz}}{i}\right) - (1)(-e^{iz})\right\}\Big|_0^{\pi+i}$$
$$= (\pi + i + 2)\left(\frac{e^{i(\pi+i)}}{i}\right) + e^{i(\pi+i)} - \frac{2}{i} - 1$$
$$= -2e^{-1} - 1 + i(2 + \pi e^{-1} + 2e^{-1}).$$

4.30 Show that $\int \frac{dz}{z^2+a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2$.

Solution Let $z = a\tan u$. Then

$$\int \frac{dz}{z^2+a^2} = \int \frac{a\sec^2 u\, du}{a^2(\tan^2 u+1)} = \frac{1}{a}\int du = \frac{1}{a}\tan^{-1}\frac{z}{a}+c_1$$

Also,

$$\frac{1}{z^2+a^2} = \frac{1}{(z-ai)(z+ai)} = \frac{1}{2ai}\left(\frac{1}{z-ai}-\frac{1}{z+ai}\right)$$

and so

$$\int \frac{dz}{z^2+a^2} = \frac{1}{2ai}\int \frac{dz}{z-ai} - \frac{1}{2ai}\int \frac{dz}{z+ai}$$

$$= \frac{1}{2ai}\ln(z-ai) - \frac{1}{2ai}\ln(z+ai)+c_2 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right)+c_2$$

MISCELLANEOUS PROBLEMS

4.31 Prove Morera's theorem [page 4.5] under the assumption that $f(z)$ has a continuous derivative in $\mathcal{R}$.

Solution If $f(z)$ has a continuous derivative in $\mathcal{R}$, then we can apply Green's theorem to obtain

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$

$$= \iint_{\mathcal{R}} \left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy + i\iint_{\mathcal{R}}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dx\,dy$$

Then, if $\oint_C f(z)\,dz = 0$ around every closed path C in $\mathcal{R}$, we must have

$$\oint_C u\,dx - v\,dy = 0,\quad \oint_C v\,dx + u\,dy = 0$$

around every closed path C in $\mathcal{R}$. Hence from problem 4.8, the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied and thus (since these partial derivatives are continuous) it follows [Problem 3.5,) that $u + iv = f(z)$ is analytic.

4.32 A force field is given by $F = 3z + 5$. Find the work done in moing an object in this force field along the parabola $z = t^2 + it$ from $z = 0$ to $z = 4 + 2i$.

Solution Total work done $= \int_C F \cdot dz = \text{Re}\int_C \bar{F}\cdot dz = \text{Re}\left\{\int_C (3\bar{z}+5)dz\right\}$

$$= \text{Re}\left\{3\int_C \bar{z}\,dz + 5\int_C dz\right\} = \text{Re}\,\{3(10-\tfrac{8}{3}i)+5(4+2i)\} = 50$$

using the result of Problem 4.2.

4.33 Find: (a) $\int e^{ax}\sin bx\,dx$, (b) $\int e^{ax}\cos bx\,dx$.

Solution Omitting the constant of integration, we have

$$\int e^{(a+ib)x}dx = \frac{e^{(a+ib)x}}{a+ib}$$

which can be written

$$\int e^{ax}(\cos bx + i\sin bx)dx = \frac{e^{ax}(\cos bx + i\sin bx)}{a+ib} = \frac{e^{ax}(\cos bx + i\sin bx)(a-ib)}{a^2+b^2}$$

Then equating real and imaginary parts,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

4.34 Give an example of a continuous, closed, non-intersecting curve which lies in a bounded region $\mathcal{R}$ but which has an infinite length.

Solution Consider equilateral triangle ABC [Fig. 4.22] with sides of unit length. By trisecting each side, construct equilateral triangles DEF, GHJ and KLM. Then omitting sides DF, GJ and KM, we obtain the closed non-intersecting curve $ADEFBGHJCKLMA$ of Fig. 4.23.

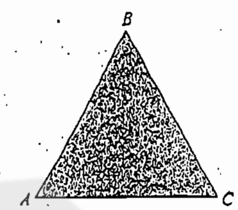


Fig. 4.22

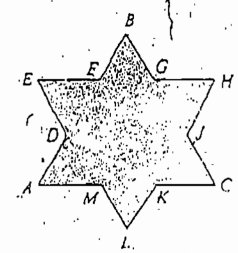


Fig. 4.23

The process can now be continued by trisecting sides DE, EF, FB, BG, GH, etc., and constructing equilateral triangles as before. By repeating the process indefinitely [see Fig. 4.24] we obtain a continuous closed non-intersecting curve which is the boundary of a region with finite area equal to

$$\frac{1}{4}\sqrt{3} + (3)\left(\frac{1}{3}\right)^2\frac{\sqrt{3}}{4} + (9)\left(\frac{1}{9}\right)^2\frac{\sqrt{3}}{4} + (27)\left(\frac{1}{27}\right)^2\frac{\sqrt{3}}{4}+\ldots$$

$$= \frac{\sqrt{3}}{4}\left(1+\frac{1}{3}+\frac{1}{9}+\ldots\right) = \frac{\sqrt{3}}{4}\,\frac{1}{1-1/3} = \frac{3\sqrt{3}}{8}$$

or 1.5 times the area of triangle ABC, and which has infinite length (see Problem 4.95).

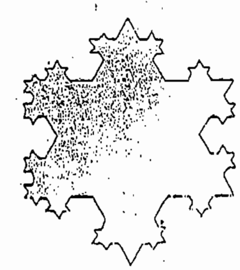


Fig. 4.24

4.35 Let $F(x, y)$ and $G(x, y)$ be continuous and have continuous first and second partial derivatives in a simply-connected region $\mathcal{R}$ bounded by a simple closed curve C. Prove that

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = -\iint_{\mathcal{R}}\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right]dx\,dy$$

Let $P = F\dfrac{\partial G}{\partial y}, Q = -F\dfrac{\partial G}{\partial x}$ in Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Then as required

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = \iint_{\mathcal{R}}\left(\frac{\partial}{\partial x}\left\{-F\frac{\partial G}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{F\frac{\partial G}{\partial y}\right\}\right)dx\,dy$$

$$= -\iint_{\mathcal{R}}\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right]dx\,dy$$

SUPPLEMENTARY PROBLEMS

LINE INTEGRALS

4.36 Evaluate $\int_{(0,1)}^{(2,5)} (3x + y)\,dx + (2y - x)\,dy$ along (a) the curve $y = x^2 + 1$, (b) the straight line joining (0, 1) and (2, 5), (c) the straight lines from (0,1) to (0, 5) and then from (0,5) to (2,5), (d) the straight lines from (0, 1) to (2,1) and then from (2, 1) to (2, 5)

4.37 (a) Evaluate $\oint_C (x + 2y)\,dx + (y - 2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta < 2\pi$ if C is described in a counterclockwise direction.
(b) What is the answer to (a) if C is described in a clockwise direction?

4.38 Evaluate $\int_C (x^2 - iy^2)\,dz$ along (a) the parabola $y = 2x^2$ from (1, 1) to (2, 8), (b) the straight lines from (1, 1) to (1, 8) and then from (1, 8) to (2, 8), (c) the straight line from (1, 1) to (2, 8).

4.39 Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1).

4.40 Evaluate $\int_C (z^2 + 3z)\,dz$ along (a) the circle $|z| = 2$ from (2, 0) to (0, 2) in a counterclockwise direction, (b) the straight line from (2, 0) to (0, 2), (c) the straight lines from (2, 0) to (2, 2) and then from (2, 2) to (0, 2).

4.41 If $f(z)$ and $g(z)$ are integrable, prove that

(a) $\int_a^b f(z)\,dz = -\int_b^a f(z)dz$

(b) $\int_C \{2f(z) - 3i\,g(z)\}dz = 2\int_C f(z)dz - 3i\int_C g(z)dz.$

4.42 Evaluate $\int_i^{2-i} (3xy + iy^2)\,dz$ (a) along the straight line joining $z = i$ and $z = 2 - i$, (b) along the curve $x = 2t - 2$, $y = 1 + t - t^2$.

4.43 Evaluate $\oint_C \bar{z}^2\,dz$ around the circles (a) $|z| = 1$, (b) $|z - 1| = 1$.

4.44 Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at (0, 0), (1, 0), (1, 1) and (1, 0), (c) the curve consisting of the parabolas $y = x^2$ from (0, 0) to (1, 1) and $y^2 = x$ from (1, 1) to (0, 0).

4.45 Evaluate $\int_C (z^2 + 1)^2\,dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$.

4.46 Evaluate $\int_C \bar{z}^2 dz + z^2 d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2 - 2i)\,z + (2 + 2i)\,\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$.

4.47 Evaluate $\oint_C \dfrac{dz}{z - 2}$ around (a) the circle $|z - 2| = 4$, (b) the circle $|z - 1| = 5$, (c) the square with vertices at $2 \pm 2i$, $-2 \pm 2i$.

4.48 Evaluate $\oint_C (x^2 + iy^2)\,ds$ around the circle $|z| = 2$ where s is the arc length.

GREEN'S THEOREM IN THE PLANE

4.49 Verify Green's theorem in the plane for $\oint_C (x^2 - 2xy)\,dx + (y^2 - x^3y)\,dy$ where C is a square with vertices at (0, 0), (2, 0), (2, 2), (0, 2).

4.50 Evaluate $\oint_C (5x + 6y - 3)\,dx + (3x - 4y + 2)\,dy$ around a triangle in the xy plane with vertices at (0, 0), (4, 0) and (4, 3).

4.51 Let C be any simple closed curve bounding a region having area A. Prove that

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

4.52 Use the result of Problem 4.51 to find the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta < 2\pi$.

4.53 Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ shown shaded in Fig. 4.25. [*Hint.* Parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le \theta < 2\pi$]

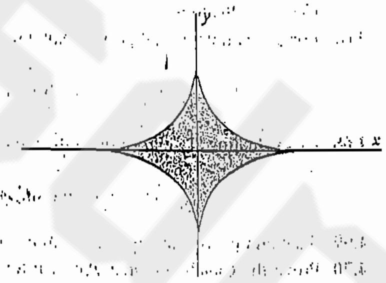


Fig. 4.25

4.54 Verify Green's theorem in the plane for $\oint_C x^2y\,dx + (y^3 - xy^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 16$.

4.55 (a) Prove that $\oint_C (y^2\cos x - 2e^y)\,dx + (2y\sin x - 2xe^y)\,dy = 0$ around any simple closed curve C.
(b) Evaluate the integral in (a) along the parabola $y = x^2$ from (0,0) to (π, π^2).

4.56 (a) Show that $\int_{(2,1)}^{(3,2)} (2xy^3 - 2y^2 - 6y)\,dx + (3x^2y^2 - 4xy - 6x)\,dy$ is independent of the path joining points (2, 1) and (3, 2). (b) Evaluate the integral in (a).

Theorem: Suppose $B(z, \bar{z})$ is continuous, and has continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x+iy$, and $\bar{z} = x-iy$, then Green's theorem can be written in complex form as

$$\oint_C B(z,\bar{z})\,dz = 2i \iint_{\mathcal{R}} \frac{\partial B}{\partial \bar{z}}\,dx\,dy$$

COMPLEX FORM OF GREEN'S THEOREM

4.57 If C is a simple closed curve enclosing a region of area A, prove that $A = \frac{1}{2i}\oint_C \bar{z}\,dz$.

4.58 Evaluate $\oint_C \bar{z}\,dz$ around (a) the circle $|z-2| = 3$, (b) the square with vertices at $z = 0, 2, 2i$ and $2+2i$, (c) the ellipse $|z-3| + |z+3| = 10$.

4.59 Evaluate $\oint_C (8\bar{z} + 3z)dz$ around the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.

4.60 Let $P(z, \bar{z})$ and $Q(z, \bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. Prove that

$$\oint_C P(z,\bar{z})dz + Q(z,\bar{z})d\bar{z} = 2i\iint_{\mathcal{R}}\left(\frac{\partial P}{\partial \bar{z}} - \frac{\partial Q}{\partial z}\right)dA$$

4.61 Show that the area in Problem 4.57 can be written in the form $A = \frac{1}{4i}\oint_C \bar{z}\,dz - z\,d\bar{z}$.

4.62 Show that the centroid of the region of Problem 4.57 is given in conjugate coordinates by $(\hat{z}, \hat{\bar{z}})$ where

$$\hat{z} = -\frac{1}{4Ai}\oint_C z^2 d\bar{z}, \qquad \hat{\bar{z}} = \frac{1}{4Ai}\oint_C \bar{z}^2 dz$$

4.63 Find the centroid of the region bounded above by $|z| = a > 0$ and below by $\operatorname{Im} z = 0$.

CAUCHY'S THEOREM AND THE CAUCHY–GOURSAT THEOREM

4.64 Verify Cauchy's theorem for the functions (a) $3z^2 + iz - 4$, (b) $5\sin 2z$, (c) $3\cosh(z+2)$ where C is the square with vertices at $1 \pm i, -1 \pm i$.

4.65 Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is (a) the circle $|z| = 1$, (b) the circle $|z-1| = 2$, (c) the ellipse $|z-3i| + |z+3i| = 20$.

4.66 Let C is the circle $|z-2| = 5$. Determine whether $\oint_C \frac{dz}{z-3} = 0$. (b) Does your answer to (a) contradict Cauchy's theorem?

4.67 For any simple closed curves, explain clearly the relationship between the observations

$$\oint_C (x^2 - y^2 + 2y)\,dx + (2x - 2xy)\,dy = 0 \text{ and } \oint_C (z^2 - 2iz)\,dz = 0$$

4.68 By evaluating $\oint_C e^z\,dz$ around the circle $|z| = 1$, show that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\theta + \sin\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta}\sin(\theta + \sin\theta)\,d\theta = 0$$

4.69 State and prove Cauchy's theorem for multiply-connected regions.

4.70 Prove the Cauchy-Goursat theorem for a polygon, such as $ABCDEFGA$ shown in Fig. 4.26, which may intersect itself.

4.71 Prove the Cauchy–Goursat theorem for the multiply-connected region $\mathcal{R}$ shown shaded in Fig. 4.27.

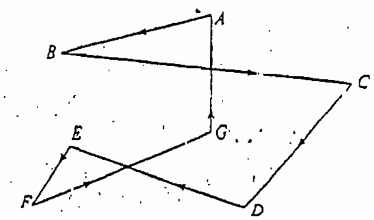


Fig. 4.26

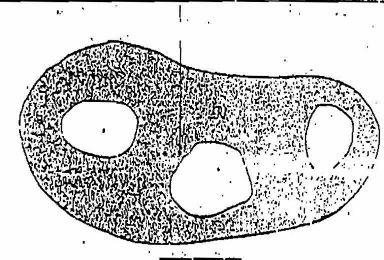


Fig. 4.27

4.72 (a) Prove the Cauchy–Goursat theorem for a rectangle and (b) show how the result of (a) can be used to prove the theorem for any simple closed curve C.

4.73 Let P and Q be continuous and have continuous first partial derivatives in a region $\mathcal{R}$. Let C be any simple closed curve in $\mathcal{R}$ and suppose that for any such curve.

$$\oint_C P\,dx + Q\,dy = 0$$

(a) Prove that there exists an analytic function $f(z)$ such that $\operatorname{Re}\{f(z)\,dz\} = P\,dx + Q\,dy$ is an exact differential.

(b) Determine p and q in terms of P and Q such that $\operatorname{Im}\{f(z)\,dz\} = p\,dx + q\,dy$ and verify that $\oint_C p\,dx + q\,dy = 0$.

(c) Discuss the connection between (a) and (b) and Cauchy's theorem.

4.74 Illustrate the results of Problem 4.73 if $P = 2x + y - 2xy$, $Q = x - 2y - x^2 + y^2$ by finding p, q and $f(z)$

4.75 Let P and Q be continuous and have continuous partial derivatives in a region $\mathcal{R}$. Suppose that for any simple closed curve C in $\mathcal{R}$, we have $\oint_C P\,dx + Q\,dy = 0$.

(a) Prove that $\oint_C Q\,dx - P\,dy = 0$. (b) Discuss the relationship of (a) with Cauchy's theorem.

CONSEQUENCES OF CAUCHY'S THEOREM

4.76 Show directly that $\int_{3+4i}^{4-3i} (6z^2 + 8iz)\,dz$ has the same value along the following paths C joining the points $3+4i$ and $4-3i$: (a) a straight line, (b) the straight lines from $3+4i$ to $4+4i$ and then from $4+4i$ to $4-3i$, (c) the circle $|z| = 5$. Determine this value.

4.77 Show that $\int_C e^{-2z}\,dz$ is independent of the path C joining the points $1-\pi i$ and $2+3\pi i$ and determine its value.

4.78 Given $G(z) = \int_{\pi-\pi i}^{z} \cos 3\zeta$ (a) Prove that $G(z)$ is independent of the path joining $\pi - \pi i$ and the arbitrary point z. (b) Determine $G(\pi i)$. (c) Prove that $G'(z) = \cos 3z$.

4.79 Given $G(z) = \int_{1+i}^{z} \sin\zeta^2 d\zeta$. (a) Prove that $G(z)$ is an analytic function of z. (b) Prove that $G'(z) \sin z^2$.

4.80 For the real line integral $\int_C P\,dx + Q\,dy$, state and prove a theorem corresponding to (a) Problem 4.17, (b) Problem 4.18, (c) Problem 4.20.

4.81 Prove Theorem 4.5, Page 4.7 for the region of Fig. 4.27.

4.82 (a) If C is the circle $|z| = \mathcal{R}$, show that

$$\lim_{R\to\infty} \oint_C \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz = 0$$

(b) Use the result of (a) to deduce that if C_1 is the circle $|z-2| = 5$, then

$$\oint_{C_1} \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz = 0$$

(c) Is the result in (b) true if C_1 is the circle $|z+1| = 2$? Explain.

INTEGRALS OF SPECIAL FUNCTIONS

4.83 Find each of the following integrals:

(a) $\int e^{-2z}\,dz$, (b) $\int z\sin z^2\,dz$, (c) $\int \frac{z^2+1}{z^3+3z+2}\,dz$, (d) $\int \sin^4 2z\cos 2z\,dz$ (e) $\int z^2\tanh(4z^3)\,dz$

4.84 Find each of the following integrals:

(a) $\int z\cos 2z\,dz$, (b) $\int z^2 e^{-z}\,dz$, (c) $\int z\ln z\,dz$, (d) $\int z^3\sinh z\,dz$.

4.85 Evaluate each of the following:

(a) $\int_{\pi i}^{2\pi i} e^{3z}\,dz$, (b) $\int_0^{\pi i}\sinh 5z\,dz$, (c) $\int_0^{\pi+i} z\cos 2z\,dz$.

4.86 Show that $\int_0^{\pi/2}\sin^2 z\,dz = \int_0^{\pi/2}\cos^2 z\,dz = \pi/4$.

4.87 Show that $\int \frac{dz}{z^2-a^2} = \frac{1}{2a}\ln\left(\frac{z-a}{z+a}\right) + c_1 = \frac{1}{a}\coth^{-1}\frac{z}{a} + c_2$.

4.88 Show that if we restrict ourselves to the same branch of the square root,

$$\int z\sqrt{2z+5}\,dz = \frac{1}{20}(2z+5)^{5/2} - \frac{5}{6}(2z+5)^{3/2} + c$$

4.89 Evaluate $\int \sqrt{1+\sqrt{z+1}}\,dz$, stating conditions under which your result is valid.

MISCELLANEOUS PROBLEMS

4.90 Use the definition of an integral to prove that along any arbitrary path joining points a and b,

(a) $\int_a^b dz = b-a$, (b) $\int_a^b z\,dz = \frac{1}{2}(b^2-a^2)$.

4.91 Prove the theorem concerning change of variables on page 4.2.
[*Hint*. Express each side as two real line integrals and use the Cauchy–Riemann equations.]

4.92 Let $u(x, y)$ be harmonic and have continuous derivatives, of order two at least, in a region $\mathcal{R}$.

(a) Show that the following integral is independent of the path in $\mathcal{R}$ joining (a, b) to (x, y).

$$v(x, y) = \int_{(a,b)}^{(x,y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

(b) Prove that $u + iv$ is an analytic function of $z = x + iy$ in $\mathcal{R}$.

(c) Prove that v is harmonic in $\mathcal{R}$.

4.93 Work Problem 4.92 for the special cases (a) $u = 3x^2y + 2x^2 - y^3 - 2y^2$, (b) $u = xe^x\cos y - ye^x\sin y$.
[See Problems 4.58(a) and (c), page 4.30.]

4.94 Using the definition of an integral, verify directly that where C is a simple closed curve and z_0 is any constant.

(a) $\oint_C dz = 0$, (b) $\oint_C z\,dz = 0$, (c) $\oint_C (z-z_0)\,dz = 0$

4.95 Find the length of the closed curve of Problem 4.34 after n steps and verify that as $n \to \infty$, the length of the curve becomes infinite.

4.96 Evaluate $\int_C \frac{dz}{z^2+4}$ along the line $x + y = 1$ in the direction of increasing x.

4.97 Show that $\int_0^\infty xe^{-x}\sin x\,dx = \frac{1}{2}$.

4.98 Evaluate $\int_{-2-2\sqrt{3}i}^{-2+2\sqrt{3}i} z^{1/2}\,dz$ along a straight line path if we choose that branch of $z^{1/2}$ such that $z^{1/2} = 1$ for $z = 1$.

4.99 Does Cauchy's theorem hold for the function $f(z) = z^{1/2}$ where C is the circle $|z| = 1$? Explain.

4.100 Does Cauchy's theorem hold for a curve, such as *EFGHFJE* in Fig. 4.28, which intersects itself? Justify your answer.

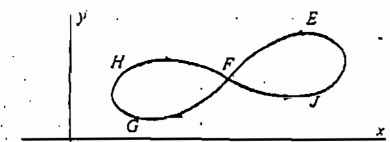


Fig. 4.28

4.101 If n is the direction of the outward drawn normal to a simple closed curve C, s is the arc length parameter and U is any continuously differentiable function, prove that

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds}$$

4.102 Prove *Green's first identity*,

$$\iint_{\mathcal{R}} (U\nabla^2 V)\,dx\,dy + \iint_{\mathcal{R}}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\right)dx\,dy = \oint_C U\frac{\partial V}{\partial n}\,ds$$

where $\mathcal{R}$ is the region bounded by the simple closed curve C, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, while n and s are as in Problem 4.101.

4.103 Use Problem 4.102 to prove *Green's second identity*

$$\iint_{\mathcal{R}} (U\nabla^2 V - V\nabla^2 U)\,dA = \oint_C \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)ds$$

where dA is an element of area of $\mathcal{R}$.

4.104 Write the result of Problem 4.35 in terms of the operator ∇.

4.105 Evaluate $\oint_C \frac{dz}{\sqrt{z^2+2z+2}}$ around the unit circle $|z| = 1$ starting with $z = 1$, assuming the integrand positive for this value.

4.106 *If n is a positive integer, show that*

$$\int_0^{2\pi} e^{\sin n\theta}\cos(\theta - \cos n\theta)\,d\theta = \int_0^{2\pi} e^{\sin n\theta}\sin(\theta - \cos n\theta)\,d\theta = 0$$

ANSWERS TO SUPPLEMENTARY PROBLEMS

4.36 (a) 88/3, (b) 32, (c) 40, (d) 24
4.37 (a) -48π, (b) 48π
4.38 (a) $\frac{511}{3} - \frac{49}{5}i$, (b) $\frac{518}{3} - 57i$, (c) $\frac{518}{3} - 8i$,
4.39 $-1 + i$
4.40 $-\frac{44}{3} - \frac{8}{3}i$ for all cases
4.42 (a) $-\frac{4}{3} + \frac{8}{3}i$, (b) $-\frac{1}{3} + \frac{79}{30}i$
4.43 (a) 0, (b) $4\pi i$
4.44 0 in all cases
4.45 $(96\pi^5 a^5 + 80\pi^3 a^3 + 30\pi a)/15$
4.46 248/15
4.47 $2\pi i$ in all cases
4.48 $8\pi(1 + i)$
4.49 Common value = -8
4.50 -18
4.52 πab
4.53 $3\pi a^2/8$
4.54 common value = 120π
4.55 (b) $-2^{\pi}e^{\pi/2}$
4.56 (b) 24
4.58 (a) $18\pi i$, (b) $8i$, (c) $40\pi i$
4.59 $6\pi i a^2$
4.63 $\bar{z} = 2ai/\pi$, $\hat{z} = -2ai/\pi$
4.74 One possibility is $p = x^2 - y^2 + 2y - x$, $q = 2x + y - 2xy$, $f(z) = iz^2 + (2 - i)z$.
4.76 $238 - 266i$
4.77 $\frac{1}{2}e^{-2}(1 - e^{-2})$
4.78 (b) 0
4.83 (a) $-\frac{1}{2}e^{-2z} + c$ (b) $-\frac{1}{2}\cos z^2 + c$ (c) $\frac{1}{3}\ln(z^3 + 3z + 2) + c$ (d) $\frac{1}{10}\sin^5 2z + c$

(e) $\frac{1}{12}\ln\cosh(4z^3) + c$
4.84 (a) $\frac{1}{2}z\sin 2z + \frac{1}{4}\cos 2z + c$ (b) $-e^{-z}(z^2 + 2z + 2) + c$ (c) $\frac{1}{2}z^2\ln z - \frac{1}{4} + c$
(d) $(z^3 + 6z)\cosh z - 3(z^2 + 2)\sinh z + c$
4.85 (a) 2/3, (b) $-2/5$, (c) $\frac{1}{4}\cosh 2 - \frac{1}{2}\sinh 2 + \frac{1}{2}\pi i\sinh 2$
4.89 $\frac{4}{5}(1 + \sqrt{z+1})^{5/2} - \frac{4}{3}(1 + \sqrt{z+1})^{3/2} + c$
4.96 $\pi/2$
4.98 32/3

SOLVED PROBLEMS

CAUCHY'S INTEGRAL FORMULAE

5.1 Let $f(z)$ is analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$, prove *Cauchy's integral formula*

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Solution

Method 1. The function $f(z)/(z-a)$ is analytic inside and on C except at the point $z = a$ (see Fig. 5.2). By Theorem 4.4, Page 4.7, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz \tag{1}$$

where we can choose Γ as a circle of radius ϵ with centre at a. Then an equation for Γ is $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$ where $0 \leqq \theta < 2\pi$. Substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, the integral on the right of (1) becomes

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})\, i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Thus we have from (1),

$$\oint_C \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \qquad (2)$$

Taking the limit of both sides of (2) and making use of the continuity of $f(z)$, we have

$$\oint_C \frac{f(z)}{z-a}dz = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$
$$= i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta$$
$$= i\int_0^{2\pi} f(a)\,d\theta = 2\pi i f(a) \qquad (3)$$

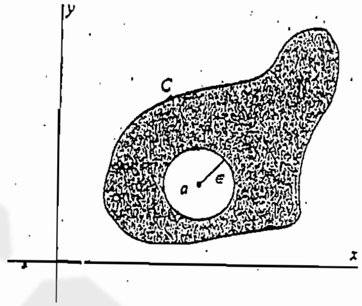


Fig. 5.2

so that we have, as required,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Method 2. The right side of equation (1) of Method 1 can be written as

$$\oint_\Gamma \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)-f(a)}{z-a}dz + \oint_\Gamma \frac{f(a)}{z-a}dz$$
$$= \oint_\Gamma \frac{f(z)-f(a)}{z-a}dz + 2\pi i f(a)$$

using Problem 4.21, Chapter 4. The required result will follow if we can show that

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}dz = 0$$

But by Problem 3.36, Chapter 3,

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}dz = \oint_\Gamma f'(a)\,dz + \oint_\Gamma \eta\,dz = \oint_\Gamma \eta\,dz$$

Then choosing Γ so small that for all points on Γ we have $|\eta| < \delta/2\pi$, we find

$$\left|\oint_\Gamma \eta\,dz\right| < \left(\frac{\delta}{2\pi}\right)(2\pi\epsilon) = \epsilon$$

Thus $\oint_\Gamma \eta\,dz = 0$ and the proof is complete.

5.2 If $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\Re$, prove that

Solution

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz$$

From Problem 5.1, if a and $a+h$ lie in $\Re$, we have

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$
$$= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero.

To show this we use the fact that if Γ is a circle of radius ϵ and centre a which lies entirely in $\Re$ (see Fig. 5.3), then

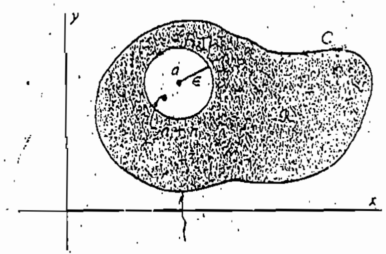


Fig. 5.3

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$
$$= \frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Choosing h so small in absolute value that $a+h$ lies in Γ and $|h| < \epsilon/2$, we have by Problem 5.7(c), and the fact that Γ has equation $|z-a| = \epsilon$,

$$|z-a-h| \geq |z-a| - |h| > \epsilon - \epsilon/2 = \epsilon/2$$

Also since $f(z)$ is analytic in $\Re$, we can find a positive number M such that $|f(z)| < M$.

Then since the length of Γ is $2\pi\epsilon$, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}\right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)(\epsilon^2)} = \frac{2|h|M}{\epsilon^2}$$

and it follows that the left side approaches zero as $h \to 0$, thus completing the proof.

It is of interest to observe that the result is equivalent to

$$\frac{d}{da}f(a) = \frac{d}{da}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{z-a}\right\}dz$$

which is an extension to contour integrals of *Leibnitz's rule* for differentiating under the integral sign

5.3 Prove that under the conditions of Problem 5.2

Solution

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \quad n = 0, 1, 2, 3, \ldots$$

The cases where $n = 0$ and 1 follow from Problems 5.1 and 5.2, respectively provided we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

To establish the case where $n = 2$, we use Problem 5.2 where a and $a+h$ lie in $\Re$ to obtain

$$\frac{f'(a+h)-f'(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right\} f(z)\,dz$$

$$= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz + \frac{h}{2\pi i}\oint_C \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3} f(z)\,dz$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero. The proof is similar to that of Problem 5.2, for using the fact that the integral around C equals the integral around Γ, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3} f(z)\,dz\right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)^2(\epsilon^3)} = \frac{4|h|M}{\epsilon^4}$$

Since M exists such that $|\{3(z-a)-2h\} f(z)| < M$.

In a similar manner, we can establish the result for $n = 3, 4, \ldots$ (see Problems 5.45 and 5.46).

The result is equivalent to (see last paragraph of Problem 5.2)

$$\frac{d^n}{da^n} f(a) = \frac{d^n}{da^n}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial a^n}\left[\frac{f(z)}{z-a}\right]dz$$

5.4 If $f(z)$ is analytic in a region $\mathcal{R}$, prove that $f'(z), f''(z), \ldots$ are analytic in $\mathcal{R}$.

Solution This follows from Problems 5.2 and 5.3.

5.5 Evaluate (a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$,

(b) $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is the circle $|z| = 3$.

Solution

(a) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$ respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad (1)$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

5.6 Evaluate $\int_C \frac{e^{-z}}{z+1}\,dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution Here, $f(z) = e^{-z}$ is an analytic function.

The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's theorem, we have $\int_C \frac{e^{-z}}{z+1}\,dz = 0$.

5.7 Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1| = 1$.

Solution The integrand has singularities, ~~where $z^2-1=0$ i.e.~~ at $z = 1$ and $z = -1$. The circle $|z-1| = 1$ has center at $z = 1$, $f(z) = 3z^2 + z$, is an analytic function.

Also,
$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\therefore \quad \int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz \qquad (1)$$

By Cauchy's integral formula,

$$\int_C \frac{3z^2+z}{z-1}\,dz = 2\pi i f(1) = 8\pi i \qquad \text{where } f(z) = 3z^2 + z$$

By Cauchy's theorem, $\int_C \frac{3z^2+z}{z+1}\,dz = 0$

$\therefore$ From (1), we have $\int_C \frac{3z^2+z}{z^2-1}\,dz = 4\pi i$.

5.8 Prove Cauchy's integral formula for multiply-connected regions.

Solution

We present a proof for the multiply-connected region $\mathcal{R}$ bounded by the simple closed curves C_1 and C_2 as indicated in Fig. 5.4. Extensions to other multiply-connected regions are easily made (see Problem 5.49).

Construct a circle Γ having centre at any point a in $\mathcal{R}$ so that Γ lies entirely in $\mathcal{R}$. Let $\mathcal{R}'$ consist of the set of points in $\mathcal{R}$ which are exterior to Γ. Then, the function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of $\mathcal{R}'$. Hence, by Cauchy's theorem for multiply-connected regions (Problem 4.15, Chapter 4),

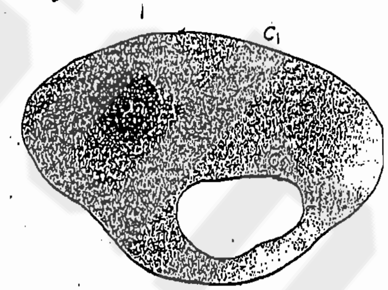


Fig. 5.4

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{\Gamma_2}\frac{f(z)}{z-a}dz = 0 \qquad (1)$$

But, by Cauchy's integral formula for simply-connected regions, we have

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-a}dz \qquad (2)$$

so that from (1),

$$f(a) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-a}dz \qquad (3)$$

Then, if C represents the entire boundary of $\mathcal{R}$ (suitably traversed so that an observer moving around C always has $\mathcal{R}$ lying to his left), we can write (3) as

$$f(a) = \frac{1}{2\pi i}\oint_C\frac{f(z)}{z-a}dz$$

In a similar manner, we can show that the other Cauchy integral formulae

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-a)^{n+1}}dz \qquad n = 1, 2, 3, \ldots$$

hold for multiply-connected regions (see Problem 5.49).

MORERA'S THEOREM

5.9 Prove *Morera's theorem* (the converse of Cauchy's theorem): If $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and if

$$\oint_C f(z)\,dz = 0$$

around every simple closed curve C in $\mathcal{R}$. Then $f(z)$ is analytic in $\mathcal{R}$.

Solution

If $\oint_C f(z)\,dz = 0$ independent of C, it follows by Problem 4.17, that $F(z) = \int_a^z f(z)\,dz$ is independent of the path joining a and z, so long as this path is in $\mathcal{R}$.

Then, by reasoning identical with that used in Problem 5.18, Chapter 4.18, it follows that $F(z)$ is analytic in $\mathcal{R}$ and $F'(z) = f(z)$. However, by Problem 5.2, it follows that $F'(z)$ is also analytic if $F(z)$ is. Hence, $f(z)$ is analytic in $\mathcal{R}$.

CAUCHY'S INEQUALITY

5.10 Let $f(z)$ be analytic inside and on a circle C of radius r and centre at $z = a$, prove *Cauchy's inequality*

$$|f^{(n)}(a)| \leqq \frac{M \cdot n!}{r^n} \quad n = 0, 1, 2, 3, \ldots$$

where M is a constant such that $|f(z)| < M$.

Solution

We have by Cauchy's integral formulae,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-a)^{n+1}}dz \qquad n = 0, 1, 2, 3, \ldots$$

Then, by Problem 4.3, Chapter 4, since $|z-a| = r$ on C and the length of C is $2\pi r$,

$$|f^{(n)}(a)| = \frac{n!}{2\pi}\left|\oint_C\frac{f(z)}{(z-a)^{n+1}}dz\right| \leqq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M\cdot n!}{r^n}$$

LIOUVILLE'S THEOREM

5.11 Prove *Liouville's theorem*: If for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded [i.e., we can find a constant M such that $|f(z)| < M$], then $f(z)$ must be a constant.

Solution

Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having centre at a and enclosing point b (see Fig. 5.5).

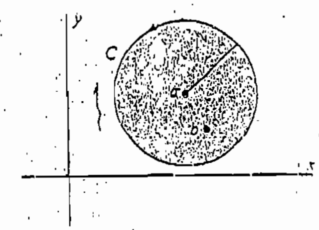


Fig. 5.5

From Cauchy's integral formula, we have

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C\frac{f(z)}{z-b}dz - \frac{1}{2\pi i}\oint_C\frac{f(z)}{z-a}dz$$

$$= \frac{b-a}{2\pi i}\oint_C\frac{f(z)\,dz}{(z-b)(z-a)}$$

Now we have

$$|z-a| = r, \; |z-b| = |z-a+a-b| \geqq |z-a| - |a-b| = r - |a-b| \geqq r/2$$

if we choose r so large that $|a-b| < r/2$. Then, since $|f(z)| < M$ and the length of C is $2\pi r$, we have by Problem 4.3, Chapter 4,

$$|f(b) - f(a)| = \frac{|b-a|}{2\pi}\left|\oint_C\frac{f(z)\,dz}{(z-b)(z-a)}\right| \leqq \frac{|b-a|\,M(2\pi r)}{2\pi(r/2)r} = \frac{2|b-a|\,M}{r}$$

Letting $r \to \infty$, we see that $|f(b) - f(a)| = 0$ or $f(b) = f(a)$, which shows that $f(z)$ must be a constant.

Another method. Letting $n = 1$ in Problem 5.10 and replacing a by z we have,

$$|f'(z)| \leqq M/r$$

Letting $r \to \infty$, we deduce that $|f'(z)| = 0$ and so $f'(z) = 0$. Hence $f(z) = $ constant, as required.

FUNDAMENTAL THEOREM OF ALGEBRA

5.12 Prove the *fundamental theorem of algebra*: Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$, where the degree $n \geqq 1$ and $a_n \neq 0$, has at least one root.

Solution

If $P(z) = 0$ has no root, then $f(z) = \dfrac{1}{P(z)}$ is analytic for all z. Also $|f(z)| = \dfrac{1}{|P(z)|}$ is bounded (and in fact approaches zero) as $|z| \to \infty$.

Then by Liouville's theorem (Problem 5.11), it follows that $f(z)$ and thus $P(z)$ must be a constant. Thus, we are led to a contradiction and conclude that $P(z) = 0$ must have at least one root or, as is sometimes said, $P(z)$ has at least one *zero*.

5.13 Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$, where the degree $n \geqq 1$ and $a_n \neq 0$, has exactly n roots.

Solution By the fundamental theorem of algebra (Problem 5.12), $P(z)$ has at least one root. Denote this root by α. Then $P(\alpha) = 0$. Hence

$$\begin{aligned} P(z) - P(\alpha) &= a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n - (a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n) \\ &= a_1(z-\alpha) + a_2(z^2 - \alpha^2) + \ldots + a_n(z^n - \alpha^n) \\ &= (z-\alpha)\, Q(z) \end{aligned}$$

where $Q(z)$ is a polynomial of degree $(n-1)$.

Applying the fundamental theorem of algebra again, we see that $Q(z)$ has at least one zero, which we can denote by β [which may equal α], and so $P(z) = (z-\alpha)(z-\beta)\, R(z)$. Continuing in this manner we see that $P(z)$ has exactly n zeros.

GAUSS' MEAN VALUE THEOREM

5.14 Let $f(z)$ be analytic inside and on a circle C with center at a. Prove Gauss' *mean value theorem* that the mean of the values of $f(z)$ on C is $f(a)$.

Solution By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz \tag{1}$$

If C has radius r, the equation of C is $|z-a| = r$ or $z = a + re^{i\theta}$. Thus, (1) becomes

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})\, ire^{i\theta}}{re^{i\theta}}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

which is the required result.

MAXIMUM MODULUS THEOREM

5.15 Prove the *maximum modulus theorem*: If $f(z)$ is analytic inside and on a simple closed curve C, then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.

Solution

Method 1. Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ does have a maximum value M for at least one value of z inside or on C. Suppose this maximum value is not attained on the boundary of C but is attained at an interior point a, i.e., $|f(a)| = M$. Let C_1 be a circle inside C with centre at a (see Fig. 5.6). If we exclude $f(z)$ from being a constant inside C_1, then there must be a point inside C_1, say b, such that $|f(b)| < M$ or, what is the same thing, $|f(b)| = M - \epsilon$ where $\epsilon > 0$.

Now, by the continuity of $|f(z)|$ at b, we see that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\big||f(z)| - |f(b)|\big| < \tfrac{1}{2}\epsilon \text{ whenever } |z-b| < \delta \tag{1}$$

$$|f(z)| < |f(b)| + \tfrac{1}{2}\epsilon = M - \epsilon + \tfrac{1}{2}\epsilon = M - \frac{1}{2}\epsilon \tag{2}$$

for all points interior to a circle C_2 with centre at b and radius δ, as shown shaded in the figure.

Construct a circle C_3 with centre at a which passes through b (dashed in Fig. 5.6). On part of this circle [namely that part PQ included in C_2], we have from (2), $|f(z)| < M - \frac{1}{2}\epsilon$. On the remaining part of the circle we have $|f(z)| \leqq M$.

If we measure θ counterclockwise from OP and let $\angle POQ = \alpha$, it follows from Problem 5.14 that if $r = |b-a|$,

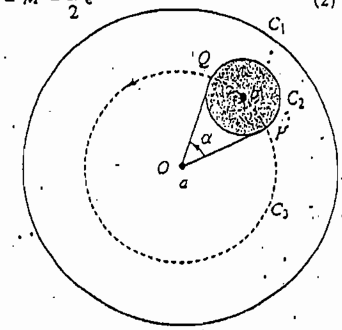


Fig. 5.6

$$f(a) = \frac{1}{2\pi}\int_0^{\alpha} f(a+re^{i\theta})\, d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} f(a+re^{i\theta})\, d\theta$$

Then

$$\begin{aligned} |f(a)| &\leqq \frac{1}{2\pi}\int_0^{\alpha} |f(a+re^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} |f(a+re^{i\theta})|\, d\theta \\ &\leqq \frac{1}{2\pi}\int_0^{\alpha} (M - \tfrac{1}{2}\epsilon)\, d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} M\, d\theta \\ &= \frac{\alpha}{2\pi}(M - \tfrac{1}{2}\epsilon) + \frac{M}{2\pi}(2\pi - \alpha) \\ &= M - \frac{\alpha\epsilon}{4\pi} \end{aligned}$$

i.e., $|f(a)| = M \leqq M - \dfrac{\alpha\epsilon}{4\pi}$, an impossible situation. By virtue of this contradiction, we conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Method 2. From Problem 5.14, we have

$$|f(a)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\, d\theta \tag{3}$$

Let us suppose that $|f(a)|$ is a maximum so that $|f(a+re^{i\theta})| \leqq |f(a)|$. If $|f(a+re^{i\theta})| < |f(a)|$ for one value of θ then, by continuity of f, it would hold for a finite arc, say $\theta_1 < \theta < \theta_2$. But, in such case, the mean value of $|f(a+re^{i\theta})|$ is less than $|f(a)|$, which would contradict (3). It follows, therefore, that in any neighbourhood of a, i.e., for $|z-a| < \delta$, $f(z)$ must be a constant. If $f(z)$ is not a constant, the maximum value of $|f(z)|$ must occur on C.

For another method, see Problem 5.66.

MINIMUM MODULUS THEOREM

5.16 Prove the *minimum modulus theorem*: Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.

Solution Since $f(z)$ is analytic inside and on C and since $f(z) \neq 0$ inside C, it follows that $1/f(z)$ is analytic inside C. By the maximum modulus theorem, it follows that $1/|f(z)|$ cannot assume its maximum value inside C and so $|f(z)|$ cannot assume its minimum value inside C. Then, since $|f(z)|$ has a minimum, this minimum must be attained on C.

5.17 Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Solution Let $f(z) = z$ for $|z| \leq 1$, so that C is a circle with centre at the origin and radius one. We have $f(z) = 0$ at $z = 0$. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r = 0$, i.e at $z = 0$.

THE ARGUMENT THEOREM

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5.18 Let $f(z)$ be analytic inside and on a simple closed curve C except for a pole $z = \alpha$ of order (multiplicity) p inside C. Suppose also that inside C $f(z)$ has only one zero $z = \beta$ of order (multiplicity) n and no zeros on C. Prove that

root

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = n - p$$

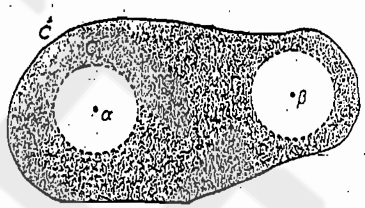

Fig. 5.7

Solution Let C_1 and Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ and $z = \beta$, respectively. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz \qquad (1)$$

Since $f(z)$ has a pole of order p at $z = \alpha$, we have

$$f(z) = \frac{F(z)}{(z-\alpha)^p} \qquad (2)$$

where $F(z)$ is analytic and different from zero inside and on C_1. Then taking logarithms in (2) and differentiating, we find

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha} \qquad (3)$$

so that

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}dz - \frac{p}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha} = 0 - p = -p \qquad (4)$$

Since $f(z)$ has a zero of order n at $z = \beta$, we have

$$f(z) = (z-\beta)^n G(z) \qquad (5)$$

where $G(z)$ is analytic and different from zero inside and on Γ_1.

Then by logarithmic differentiation, we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\beta} + \frac{G'(z)}{G(z)} \qquad (6)$$

so that

$$\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz = \frac{n}{2\pi i}\oint_{\Gamma_1} \frac{dz}{z-\beta} + \frac{1}{2\pi i}\oint \frac{G'(z)}{G(z)}dz = n \qquad (7)$$

Hence, from (1), (4), and (7), we have the required result

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz = n - p$$

5.19 Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Suppose that $f(z) \neq 0$ on C. If N and P are, respectively the number of zeros and poles of $f(z)$ inside C, counting multiplicities, prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P$$

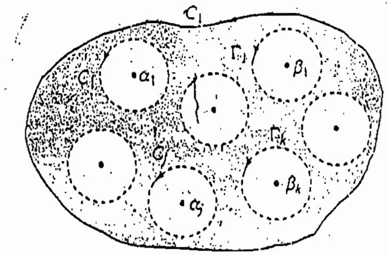

Fig. 5.8

Solution Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_k$ be the respective poles and zeros of $f(z)$ lying inside C [Fig. 5.8] and suppose their multiplicities are $p_1, p_2, \ldots, p_j$ and $n_1, n_2, \ldots, n_k$.

Enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_j$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. This can always be done since the poles and zeros are isolated.

Then, we have, using the results of Problem 5.18,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \sum_{r=1}^{j}\frac{1}{2\pi i}\oint_{\Gamma_r} \frac{f'(z)}{f(z)}dz + \sum_{r=1}^{k}\frac{1}{2\pi i}\oint_{C_r} \frac{f'(z)}{f(z)}dz$$

$$= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r$$

$$= N - P$$

5.20 Using argument principle, evaluate the integral $\int_C \frac{f'(z)}{f(z)}dz$, when $f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3}$ and C is the circle $|z| = 3$, taken in a positive sense.

The function has double zero at $z = i$, $z = -i$, poles of order three at $z = -1$, $z = -2$. Hence, N (no. of zeroes of $f(z)$ within C) $= 2 + 2 = 4$ and P (no. of poles of $f(z)$ within C) $= 3 + 3 = 6$. The zeroes and poles are inside the circle $|z| = 3$. Now, using argument principle, we get $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}dz = N - P = -2$.

Therefore, $\int_C \frac{f'(z)}{f(z)}\,dz = -4\pi i$.

5.21 Using argument principle, prove that $N - P = \frac{1}{2\pi}\Delta C \arg f(z)$, where ΔC denotes the variation in arg $f(z)$ as z moves once round C, N and P are number of zeroes and poles of $f(z)$ respectively.

Solution By argument principle, we have $N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$ (1)

Consider $f(z) = Re^{i\phi}$, therefore $R = |f(z)|$ and $\phi = \arg f(z)$.

Now, $f'(z)dz = df(z) = d(Re^{i\phi}) = e^{i\phi}(dR + iRd\phi) = \frac{f(z)}{R}(dR + iRd\phi)$.

Therefore from (1), $N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$

$$= \frac{1}{2\pi i}\int_C \frac{1}{R}(dR + iRd\phi) = \frac{1}{2\pi i}\int_C \frac{dR}{R} + \frac{1}{2\pi}\int_C d\phi \quad (2)$$

Now, $\int_C \frac{dR}{R} = [\log R]_C = 0$, since log R returns to its original value as z moves once round C.

and $\frac{1}{2\pi}\int_C d\phi = \frac{1}{2\pi}[\phi]_C = \frac{1}{2\pi}\Delta C \arg f(z)$, since we know that arg $f(z)$ does not return to its original value as z moves once round C and hence ΔC are $f(z)$ is not necessarily zero.

Hence from (2), $N - P = 0 + \frac{1}{2\pi}\Delta C \arg f(z) = \frac{1}{2\pi}\Delta C \arg f(z)$.

5.22 If $f(z) = z^5 - 3iz + 2z - 1 + i$, evaluate $\int_C \frac{f'(z)}{f(z)}\,dz$, where C encloses all the zeroes of $f(z)$.

Here, $f(z)$ has 5 zeroes, as we know that every polynomial of degree n has exactly n zeroes. Also, $f(z)$ has no poles. Thus N (no. of zeroes of $f(z)$ within C) = 5 and P (no. of poles of $f(z)$ within C) = 0. Now using argument principle, we get

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P = 5. \text{ Therefore, } \int_C \frac{f'(z)}{f(z)}\,dz = 10\pi i$$

5.23 Evaluate $\int_C \frac{f'(z)}{f(z)}\,dz$, where C is the circle $|z| = \pi$ and (a) $f(z) = \sin \pi z$ (b) $f(z) = \cos \pi z$.

(a) For $f(z) = \sin \pi z$. Zeroes of $f(z)$ are given by $\sin \pi z = 0 \Rightarrow \pi z = n\pi \Rightarrow z = n$ (where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$).

Now, zeroes of $f(z)$ within C are given by $z = 0, \pm 1, \pm 2, \pm 3$.

Therefore, N (no. of zeroes $f(z)$ of within C) = 7

and P (no. of poles of $f(z)$ within C) = 0. Now using argument principle, we get $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$

$= N - P = 7$. Therefore, $\int_C \frac{f'(z)}{f(z)}\,dz = 14\pi i$.

(b) For $f(z) = \cos \pi z$. Zeroes of $f(z)$ are given by

$\cos \pi z = 0 \Rightarrow \pi z = (2n+1)\frac{\pi}{2} \Rightarrow z = (2n+1).\frac{1}{2}$ (where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$).

Now zeroes of $f(z)$ within C are given by $z = \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{5}{2}, -\frac{3}{2}, -\frac{5}{2}$.

Therefore, N (no. of zeroes of $f(z)$ within C) = 6

and (no. of poles of $f(z)$ within C) = 0. Now using argument principle, we get

$$\frac{1}{-2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P = 6.$$

Therefore, $\int_C \frac{f'(z)}{f(z)}\,dz = 12\pi i$.

ROUCHE'S THEOREM

5.24 Prove *Rouché's theorem*: If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Solution Let $F(z) = g(z)/f(z)$ so that $g(z) = f(z)\,F(z)$ or briefly $g = fF$. Then, if N_1 and N_2 are the number of zeros inside C of $f + g$ and f, respectively, we have by Problem 5.18, using the fact that these functions have no poles inside C,

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz, \quad N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

Then

$$N_1 - N_2 = \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f'(1+F) + fF'}{f(1+F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \left\{\frac{f'}{f} + \frac{F'}{1+F}\right\}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz = \frac{1}{2\pi i}\oint_C F'(1 - F + F^2 - F^3 + \cdots)\,dz$$

$$= 0$$

using the given fact that $|F| < 1$ on C so that the series is uniformly convergent on C and term by term integration yields the value zero. Thus, $N_1 = N_2$ as required.

5.25 Use Rouché's theorem (Problem 5.24) to prove that every polynomial of degree n has exactly n zeros (fundamental theorem of algebra).

Solution Suppose the polynomial to be $a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$, where $a_n \neq 0$. Choose $f(z) = a_nz^n$ and $g(z) = a_0 + a_1z + a_2z^2 + \ldots + a_{n-1}z^{n-1}$.

If C is a circle having centre at the origin and radius $r > 1$, then on C we have

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}|}{|a_nz^n|}$$

$$\leq \frac{|a_0| + |a_1|\,r + |a_2|\,r^2 + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$\leq \frac{|a_0|\,r^{n-1} + |a_1|\,r^{n-1} + |a_2|\,r^{n-1} + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$\leq \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|\,r}$$

Then, by choosing r large enough, we can make $\left|\frac{g(z)}{f(z)}\right| \leq 1$, i.e. $|g(z)| < |f(z)|$. Hence, by Rouché's theorem the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z) = a_n z^n$. But since this last function has n zeros all located at $z = 0$, $f(z) + g(z)$ also has n zeros and the proof is complete.

5.26 Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution Consider the circle $C_1 : |z| = 1$. Let $f(z) = 12$, $g(z) = z^7 - 5z^3$. On C_1 we have

$$|g(z)| = |z^7 - 5z^3| \leq |z^7| + |5z^3| \leq 6 < 12 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$, i.e. there are no zeros inside C_1.

Consider the circle $C_2 : |z| = 2$. Let $f(z) = z^7$, $g(z) = 12 - 5z^3$. On C_2 we have

$$|g(z)| = |12 - 5z^3| \leq |12| + |5z^3| \leq 60 < 2^7 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$, i.e. all the zeros are inside C_2.

Hence all the roots lie inside $|z| = 2$ but outside $|z| = 1$, as required.

5.27 Prove that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

Solution Consider the circle $C_1 : |z| = 2$. Let $f(z) = z^5$, $g(z) = 15z + 1$. Both $f(z)$ and $g(z)$ being polynomial are analytic within and on C_1. On C_1, we have $\frac{|g(z)|}{|f(z)|} = \frac{|15z + 1|}{|z^5|} \leq \frac{15|z| + 1}{|z^5|} = \frac{31}{32} < 1$. Thus, $|g(z)| < |f(z)|$ on C_1. Hence, by Rouche's theorem, $f(z) + g(z) = z^5 + 15z + 1$ has same number of zeroes inside $|z| = 2$ as $f(z) = z^5$. But, $f(z) = z^5$ has five zeroes all located inside $|z| = 2$ so that $z^5 + 15z + 1$ has all the five zeroes inside $|z| = 2$.

Consider the circle $C_2 : |z| = \frac{3}{2}$. Let $f(z) = 15z$, $g(z) = z^5 + 1$. Both $f(z)$ and $g(z)$ being polynomial are analytic within and on C_2. On C_2, we have $\frac{|g(z)|}{|f(z)|} = \frac{|z^5 + 1|}{|15z|} \leq \frac{|z^5| + 1}{15|z|} = \frac{55}{144} < 1$. Thus $|g(z)| < |f(z)|$ on C_2. Hence, by Rouche's theorem, $f(z) + g(z) = z^5 + 15z + 1$ has same number of zeroes inside $|z| = \frac{3}{2}$ as $f(z) = 15z$. But, $f(z) = 15z$ has only one zero inside $|z| = \frac{3}{2}$ and hence the remaining four zeroes must lie in the annulus $\frac{3}{2} < |z| < 2$.

5.28 Using the Rouche's theorem, find the number of roots of $z^7 - 4z^3 + z + 1 = 0$, which lie interior to the $|z|$ unit circle $|z| = 1$.

Solution Consider the circle $C : |z| = 1$. Let $f(z) = -4z^3$, $g(z) = z^7 + z + 1$. Both $f(z)$ and $g(z)$ being polynomial, are analytic within and on C. On C, we have $\frac{|g(z)|}{|f(z)|} = \frac{|z^5 + z + 1|}{|-4z^3|} \leq \frac{|z|^7 + |z| + 1}{4|z^3|} = \frac{3}{4} < 1$.

Thus $|g(z)| < |f(z)|$ on C.

Hence, by Rouche's theorem $f(z) + g(z) = z^7 - 4z^3 + z + 1$ has same number of zeroes inside as $|z| = 1$ as $f(z) = -4z^3$. But $f(z)$ has zeroes of order three at origin interior to the unit circle $|z| = 1$. Thus, we conclude that $z^7 - 4z^3 + z + 1$ has three zeroes inside $|z| = 1$.

5.29 If $a > e$, then prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$, using the Rouche's theorem.

Consider the circle $C : |z| = 1$, i.e., $z = e^{i\theta}$. Given equation is $az^n - e^z = 0$. Let $f(z) = az^n$, $g(z) = -e^z$. Both $f(z)$ and $g(z)$ and analytic within and on C. On C, we have

$$\frac{|g(z)|}{|f(z)|} = \frac{|-e^z|}{|az^n|} \leq \frac{1 + |z| + \frac{|z|^2}{\lfloor 2} + \cdots}{a|z|^n} = \frac{e}{a} < 1, \text{ (as } |z| = 1 \text{ and } a > e).$$ Thus $|g(z)| < |f(z)|$ on C.

Hence, by Rouche's theorem, $f(z) + g(z) = az^n - e^z$ has same number of zeroes inside $|z| = 1$ as $f(z) = az^n$. Clearly $f(z)$ has n zeroes located $z = 0$ interior to the unit circle $|z| = 1$. Hence $az^n - e^z$ has all zeroes n inside $|z| = 1$.

POISSON'S INTEGRAL FORMULAE FOR A CIRCLE

5.30 (a) Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$, and let $z = re^{i\theta}$ be any point inside C. Prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(Re^{i\phi})\,d\phi$$

(b) If $u(r, \theta)$ and $v(r, \theta)$ are the real and imaginary parts of $f(re^{i\theta})$, prove that

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R, \phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R, \phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

The results are called *Poisson's integral formulae for the circle.*

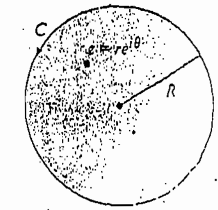

Fig. 5.9

Solution

(a) Since $z = re^{i\theta}$ is any point inside C, we have by Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\,dw \qquad (1)$$

The *inverse of the point* z with respect to C lies outside C and is given by $R^2/\bar{z}$. Hence, by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - R^2/\bar{z}}\,dw \qquad (2)$$

If we subtract (2) from (1), we find

$$f(z) = \frac{1}{2\pi i}\oint_C \left\{\frac{1}{w-z} - \frac{1}{w - R^2/\bar{z}}\right\} f(w)\,dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{z - R^2/\bar{z}}{(w-z)(w-R^2/\bar{z})} f(w)\,dw \tag{3}$$

Now, let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then, since $\bar{z} = re^{-i\theta}$, (3) yields

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\{re^{i\theta} - (R^2/r)e^{i\theta}\} f(Re^{i\phi})\, iRe^{i\phi}\,d\phi}{\{Re^{i\phi} - re^{i\theta}\}\{Re^{i\phi} - (R^2/r)e^{i\theta}\}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - R^2)\, e^{i(\theta+\phi)} f(Re^{i\phi})\,d\phi}{(Re^{i\phi} - re^{i\theta})(re^{i\phi} - Re^{i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\,d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{-i\phi} - re^{-i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

(b) Since $f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ and $f(Re^{i\phi}) = u(R,\phi) + i\,v(R,\phi)$, we have from part (a),

$$u(r,\theta) + iv(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\{u(R,\phi) + i\,v(R,\phi)\}\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2} + \frac{i}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

Then the required result follows on equating real and imaginary parts.

POISSON'S INTEGRAL FORMULAE FOR A HALF PLANE

5.31 Derive Poisson's formulae for the half plane [see Page 5.3].

Solution Let C be the boundary of a semicircle of radius R [see Fig. 5.10] containing ζ as an interior point. Since C encloses ζ but does not enclose $\bar{\zeta}$, we have by Cauchy's integral formula,

$$f(\zeta) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\zeta}\,dz, \quad 0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\bar{\zeta}}\,dz$$

Then by subtraction,

$$f(\zeta) = \frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}}\right\}dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{(\zeta - \bar{\zeta})\,f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

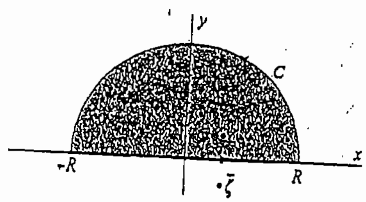

Fig. 5.10

Letting $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$, this can be written

$$F(\zeta) = \frac{1}{\pi}\int_{-R}^{R} \frac{\eta f(x)\,dx}{(x-\xi)^2 + \eta^2} + \frac{1}{\pi}\int_\Gamma \frac{\eta f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

where Γ is the semicircular arc of C. As $R \to \infty$, this last integral approaches zero [see Problem 5.85] and we have

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta f(x)\,dx}{(x-\xi)^2 + \eta^2}$$

Writing $f(\zeta) = f(\xi + i\eta) = u(\xi,\eta) + i\,v(\xi,\eta)$, $f(x) = u(x,0) + iv(x,0)$, we obtain as required,

$$u(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta u(x,0)\,dx}{(x-\xi)^2 + \eta^2}, \quad v(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta v(x,0)\,dx}{(x-\xi)^2 + \eta^2}$$

MISCELLANEOUS PROBLEMS

5.32 Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 and on the boundary [Fig. 5.11]. Prove that if z_0 is any point in $\mathcal{R}$, then

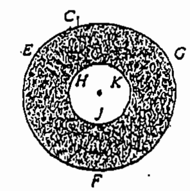

Fig. 5.11

Solution
$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

Method 1. Construct cross-cut EH connecting circles C_1 and C_2. Then $f(z)$ is analytic in the region bounded by $EFGEHKJHE$. Hence by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\oint_{EFGEHKJHE} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\oint_{EFGE} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{EH} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{HKJH} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{HE} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

since the integrals along EH and HE cancel.

Similar results can be established for the derivatives of $f(z)$.

Method 2. The result also follows from equation (3) of Problem 5.8 if we replace the simple closed curves C_1 and C_2 by the circles of Fig. 5.11.

5.33 Prove that $\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}2\pi$ where $n = 1, 2, 3, \ldots$.

Solution Let $z = e^{i\theta}$. Then, $dz = ie^{i\theta}d\theta = iz\,d\theta$ or $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$. Hence, if C is the unit circle $|z| = 1$, we have

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \oint_C \left\{\frac{1}{2}\left(z + \frac{1}{z}\right)\right\}^{2n} \frac{dz}{iz}$$

$$= \frac{1}{2^{2n} i}\oint_C \frac{1}{z}\left\{z^{2n} + \binom{2n}{1}(z^{2n-1})\left(\frac{1}{z}\right) + \cdots + \binom{2n}{k}(z^{2n-k})\left(\frac{1}{z}\right)^k + \cdots + \left(\frac{1}{z}\right)^{2n}\right\} dz$$

$$= \frac{1}{2^{2n} i}\oint_C \left\{z^{2n-1} + \binom{2n}{1} z^{2n-3} + \cdots + \binom{2n}{k} z^{2n-2k-1} + \cdots + z^{-2n}\right\} dz$$

$$= \frac{1}{2^{2n} i} \cdot 2\pi i \binom{2n}{n} = \frac{1}{2^{2n}}\binom{2n}{n} 2\pi$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{n!\,n!} 2\pi = \frac{(2n)(2n-1)(2n-2)\cdots(n)(n-1)\cdots 1}{2^{2n}\, n!\, n!} 2\pi$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} 2\pi$$

5.34 If $f(z) = u(x, y) + i\,v(x, y)$ is analytic in a region $\Re$, prove that u and v are harmonic in $\Re$.

Solution In Problem 3.6, we proved that u and v are harmonic in $\Re$, i.e. satisfy the equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$, under the *assumption* of existence of the second partial derivatives of u and v, i.e. the existence of $f''(z)$.

This assumption is no longer necessary since we have in fact proved in Problem 5.4 that if $f(z)$ is analytic in $\Re$ then *all* the derivatives of $f(z)$ exist.

5.35 Prove *Schwarz's theorem*: Let $f(z)$ be analytic for $|z| \leqq R$, $f(0) = 0$ and $|f(z)| \leqq M$. Then

$$|f(z)| \leqq \frac{M|z|}{R}$$

Solution

The function $f(z)/z$ is analytic in $|z| \leqq R$. Hence on $|z| = R$ we have by the maximum modulus theorem,

$$\left|\frac{f(z)}{z}\right| \leqq \frac{M}{R}$$

However, since this inequality must also hold for points inside $|z| = R$, we have for $|z| \leqq R$, $|f(z)| \leqq M|z|/R$ as required.

5.36 Let $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ where x is real. Show that the function $f(x)$ (a) has a first derivative at all values of x for which $0 \leqq x \leqq 1$ but (b) does not have a second derivative in $0 \leqq x \leqq 1$. (c) Reconcile these conclusions with the result of Problem 5.4.

Solution

(a) The only place where there is any question as to existence of the first derivative is at $x = 0$. But at $x = 0$ the derivative is

$$\lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin(1/\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x \sin(1/\Delta x) = 0$$

and so exists.

At all other values of x in $0 \leqq x \leqq 1$, the derivative is given (using elementary differentiation rules) by

$$x^2 \cos(1/x)\{-1/x^2\} + (2x)\sin(1/x) = 2x \sin(1/x) - \cos(1/x)$$

(b) From part (a), we have

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The second derivative exists for all x such that $0 < x \leqq 1$. At $x = 0$ the second derivative is given by

$$\lim_{\Delta x \to 0} \frac{f'(0 + \Delta x) - f'(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x \sin(1/\Delta x) - \cos(1/\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \{2 \sin(1/\Delta x) - (1/\Delta x)\cos(1/\Delta x)\}$$

which does not exist.

It follows that the second derivative of $f(x)$ does not exist in $0 \leqq x \leqq 1$.

(c) According to Problem 5.4, if $f(z)$ is analytic in a region $\Re$ then all higher derivatives exist and are analytic in $\Re$. The above results do not conflict with this, since the function $f(z) = z^2 \sin(1/z)$ is not analytic in any region which includes $z = 0$.

5.37 (a) If $F(z)$ is analytic inside and on a simple closed curve C except for a pole of order m at $z = a$ inside C, prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) How would you modify the result in (a) if more than one pole were inside C?

(a) If $F(z)$ has a pole of order m at $z = a$, then $F(z) = f(z)/(z-a)^m$ where $f(z)$ is analytic inside and on C, and $f(a) \neq 0$. Then, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^m}\,dz = \frac{f^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) Suppose there are two poles at $z = a_1$ and $z = a_2$ inside C, of orders m_1 and m_2 respectively. Let Γ_1 and Γ_2 be circles inside C having radii ϵ_1 and ϵ_2 and centres at a_1 and a_2 respectively (see Fig. 5.12). Then

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} F(z)\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} F(z)\,dz \quad (1)$$

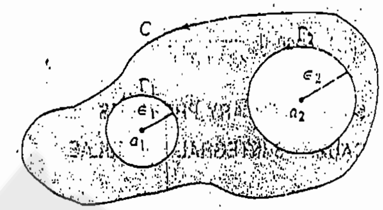

Fig. 5.12

If $F(z)$ has a pole of order m_1 at $z = a_1$, then

$$F(z) = \frac{f_1(z)}{(z-a_1)^{m_1}} \quad \text{where } f_1(z) \text{ is analytic and } f_1(a_1) \neq 0$$

If $F(z)$ has a pole of order m_2 at $z = a_2$, then

$$F(z) = \frac{f_2(z)}{(z-a_2)^{m_2}} \quad \text{where } f_2(z) \text{ is analytic and } f_2(a_2) \neq 0$$

Then by (1) and part (a),

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f_1(z)}{(z-a_1)^{m_1}}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f_2(z)}{(z-a_2)^{m_2}}\,dz$$

$$= \lim_{z\to a_1} \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\{(z-a_1)^{m_1}F(z)\}$$

$$+ \lim_{z\to a_2} \frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\{(z-a_2)^{m_2}F(z)\}$$

If the limits on the right are denoted by R_1 and R_2, we can write

$$\oint_C F(z)dz = 2\pi i(R_1 + R_2)$$

where R_1 and R_2 are called the *residues* of $F(z)$ at the poles $z = a_1$ and $z = a_2$.

In general if $F(z)$ has a number of poles inside C with residues $R_1, R_2, \ldots$, then $\oint_C F(z)\,dz = 2\pi i$ times the sum of the residues. This result is called the *residue theorem*. Applications of this theorem together with generalization to singularities other than poles, are treated in Chapter 7.

5.38 Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is the circle $|z| = 4$.

Solution The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ are at $z = \pm\pi i$ inside C and are both of order two.

Residue at $z = \pi i$ is $\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}$.

Residue at $z = -\pi i$ is $\lim_{z\to-\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2\frac{e^z}{(z+\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}$.

Then $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i$ (sum of residues) $= 2\pi i\left(\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$.

SUPPLEMENTARY PROBLEMS

CAUCHY'S INTEGRAL FORMULAE

5.39 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e_z}{z-2}\,dz$ if C is: (a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

5.40 Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z| = 5$.

5.41 Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is: (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2| + |z+2| = 6$.

5.42 Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices at: (a) $2 \pm i, -2 \pm i$; (b) $-i, 2-i, 2+i, i$.

5.43 Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.

5.44 Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z| = 2$.

5.45 If C is a simple closed curve enclosing $z = a$ and $f(z)$ is analytic inside and on C.

Prove that $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$.

5.46 Prove Cauchy's integral formulae for all positive integral values of n. [*Hint:* Use mathematical induction.]

5.47 Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$?(b) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$. If C is the circle $|z| = 1$.

5.48 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t > 0$ and C is the circle $|z| = 3$.

5.49 Prove Cauchy's integral formulae for the multiply-connected region of Fig. 4.26, Page 4.31.

MORERA'S THEOREM

5.50 (a) Determine whether $G(z) = \int_1^z \frac{d\zeta}{\zeta}$ is independent of the path joining 1 and z.

(b) Discuss the relationship of your answer to part (a) with Morera's theorem.

5.51 Does Morera's theorem apply in a multiply-connected region? Justify your answer.

5.52 (a) Suppose $P(x, y)$ and $Q(x, y)$ are conjugate harmonic functions and C is any simple closed curve, prove that $\oint_C P\,dx + Q\,dy = 0$.

(b) If for all simple closed curves C in a region $\mathcal{R}$, $\oint_C P\,dx + Q\,dy = 0$, is it true that P and Q are conjugate harmonic functions, i.e. is the converse of (a) true? Justify your conclusion.

CAUCHY'S INEQUALITY

5.53 (a) Use Cauchy's inequality to obtain estimates for the derivatives of $\sin z$ at $z = 0$ and (b) determine how good these estimates are.

5.54 (a) Show that if $f(z) = 1/(1-z)$, then $f^{(n)}(z) = n!/(1-z)^{n+1}$.

(b) Use (a) to show that the Cauchy inequality is "best possible", i.e., the estimate of growth of the nth derivative cannot be improved for all functions.

5.55 Prove that the equality in Cauchy's inequality (5.3), Page 5.2, holds if and only if $f(z) = kMz^n/r^n$ where $|k| = 1$.

5.56 Discuss Cauchy's inequality for the function $f(z) = e^{-1/z^2}$ in the neighbourhood of $z = 0$.

LIOUVILLE'S THEOREM

5.57 The function of a real variable defined by $f(x) = \sin x$ is (a) analytic everywhere and (b) bounded, i.e. $|\sin x| \leqq 1$ for all x but it is certainly not a constant. Does this contradict Liouville's theorem? Explain.

5.58 Where $A > 0$ and $B > 0$ are given constants a non-constant function $F(z)$ is such that $F(z + a) = F(z)$, and $F(z + bi) = F(z)$. Prove that $F(z)$ cannot be analytic in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

FUNDAMENTAL THEOREM OF ALGEBRA

5.59 (a) Carry out the details of proof of the fundamental theorem of algebra to show that the particular function $f(z) = z^4 - z^2 - 2z + 2$ has exactly four zeros. (b) Determine the zeros of $f(z)$.

5.60 Determine all the roots of the equatios (a) $z^3 - 3z + 4i = 0$, (b) $z^4 + z^2 + 1 = 0$.

GAUSS' MEAN VALUE THEOREM

5.61 Evaluate $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2(\pi/6 + 2e^{i\theta})\, d\theta$.

5.62 Show that the mean value of any harmonic function over a circle is equal to the value of the function at the centre.

5.63 Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$.

5.64 Prove that $\displaystyle\int_0^{\pi} \ln \sin \theta\, d\theta = -\pi \ln 2$. [*Hint.* Consider $f(z) = \ln(1 + z)$.]

MAXIMUM MODULUS THEOREM

5.65 Find the maximum of $|f(z)|$ in $|z| \leqq 1$ for the functions $f(z)$ given by (a) $z^2 - 3z + 2$, (b) $z^4 + z^2 + 1$, (c) $\cos 3z$, (d) $(2z + 1)/(2z - 1)$.

5.66 (a) If $f(z)$ be analytic inside and on the simple closed curve C enclosing $z = a$, prove that

$$\{f(a)\}^n = \frac{1}{2\pi i}\oint_C \frac{\{f(z)^n\}}{z - a}\, dz \qquad n = 0, 1, 2, \ldots$$

(b) Use (a) to prove that $|f(a)|^n \leqq M^n/2\pi D$ where D is the minimum distance from a to the curve C and M is the maximum value of $|f(z)|$ on C.

(c) By taking the nth root of both sides of the inequality in (b) and letting $n \to \infty$, prove the maximum modulus theorem.

5.67 Let $U(x, y)$ be harmonic inside and on a simple closed curve C. Prove that the (a) maximum and (b) minimum values of $U(x, y)$ are attained on C. Are there other restrictions on $U(x, y)$?

5.68 Verify Problem 5.67 for the functions (a) $x^2 - y^2$ and (b) $x^3 - 3xy^2$ if C is the circle $|z| = 1$.

5.69 Is the maximum modulus theorem valid for multiply-connected regions? Justify your answer.

THE ARGUMENT THEOREM

5.70 If $f(z) = z^5 - 3iz^2 + 2z - 1 + i$, evaluate $\displaystyle\oint_C \frac{f'(z)}{f(z)}\, dz$ where C encloses all the zeros of $f(z)$.

5.71 Let $f(z) = \dfrac{(z^2 + 1)^2}{(z^2 + 2z + 2)^3}$. Evaluate $\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{f'(z)}{f(z)}\, dz$ where C is the circle $|z| = 4$.

5.72 Evaluate $\displaystyle\oint_C \frac{f'(z)}{f(z)}\, dz$ if C is the circle $|z| = \pi$ and (a) $f(z) = \sin \pi z$, (b) $f(z) = \cos \pi z$, (c) $f(z) = \tan \pi z$.

5.73 If $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$, evaluate $\displaystyle\oint_C \frac{z f'(z)}{f(z)}\, dz$.

ROUCHE'S THEOREM

5.74 If $a > e$, prove that the equation $az^n = e^z$ has n roots inside $|z| = 1$.

5.75 Prove that $ze^z = a$ where $a \neq 0$ is real has infinitely many roots.

5.76 Prove that $\tan z = az$, $a > 0$ has (a) infinitely many real roots, (b) only two pure imaginary roots if $0 < a < 1$, (c) all real roots if $a \geqq 1$.

5.77 Prove that $z \tan z = a$, $a > 0$ has infinitely many real roots but no imaginary roots.

POISSON'S INTEGRAL FORMULAE FOR A CIRCLE

5.78 Show that $\displaystyle\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi = 2\pi$

(a) with, (b) without Poisson's integral formula for a circle.

5.79 Show that (a) $\displaystyle\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta - \phi)}\, d\phi = \frac{2\pi}{3} e^{\cos\theta}\cos(\sin\theta)$

(b) $\displaystyle\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5 - 4\cos(\theta - \phi)}\, d\phi = \frac{2\pi}{3} e^{\cos\theta}\sin(\sin\theta)$

5.80 (a) Prove that the function

$$U(r, \theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1 - r^2}\right), \quad 0 < r < 1, 0 \leqq \theta < 2\pi$$

is harmonic inside the circle $|z| = 1$.

(b) Show that $\displaystyle\lim_{r \to 1-} U(r, \theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

(c) Can you derive the expression for $U(r, \theta)$ from Poisson's integral formula for a circle?

5.81 If $f(z)$ is analytic inside and on the circle C defined by $|z| = R$ and if $z = re^{i\theta}$ is any point inside C, show that

$$f'(re^{i\theta}) = \frac{i}{2\pi}\int_0^{2\pi} \frac{R(R^2 - r^2)\, f(Re^{i\phi})\sin(\theta - \phi)}{[R^2 - 2Rr\cos(\theta - \phi) + r^2]^2}\, d\phi$$

5.82 Verify that the functions u and v of equations (5.7) and (5.8), Page 5.3, satisfy Laplace's equation.

POISSON'S INTEGRAL FORMULAE FOR A HALF PLANE

5.83 Find a function which is harmonic in the upper half plane $y > 0$ and which on the x axis takes the values -1 if $x < 0$ and 1 if $x > 0$.

5.84 Work Problem 5.83 if the function takes the values -1 if $x < -1$, 0 if $-1 < x < 1$, and 1 if $x > 1$.

5.85 Prove the statement made in Problem 5.31 that the integral over Γ approaches zero as $R \to \infty$.

5.86 Prove that under suitable restrictions on $f(x)$,

$$\lim_{\eta \to 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta f(x)}{(x-\xi)^2 + \eta^2}\, dx = f(\xi)$$

and state these restrictions.

5.87 Verify that the functions u and v of equations (5.10) and (5.11), Page 5.3, satisfy Laplace's equation.

MISCELLANEOUS PROBLEMS

5.88 Evaluate $\dfrac{1}{2\pi i} \oint_C \dfrac{z^2 dz}{z^2+4}$ where C is the square with vertices at ± 2, $\pm 2 + 4i$.

5.89 Evaluate $\oint_C \dfrac{\cos^2 tz}{z^3} dz$ where C is the circle $|z| = 1$ and $t > 0$.

5.90 (a) Show that $\oint_C \dfrac{dz}{z+1} = 2\pi i$ if C is the circle $|z| = 2$.
(b) Use (a) to show that

$$\oint_C \frac{(x+1)dx + y\,dy}{(x+1)^2 + y^2} = 0, \qquad \oint_C \frac{(x+1)dy - y\,dx}{(x+1)^2 + y^2} = 2\pi$$

and verify these results directly.

5.91 Find all functions $f(z)$ which are analytic everywhere in the entire complex plane and which satisfy the conditions (a) $f(2-i) = 4i$ and (b) $|f(z)| < e^2$ for all z.

5.92 If $f(z)$ is analytic inside inside and on a simple closed curve C, prove that

(a) $f'(a) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{-i\theta} f(a + e^{i\theta})\, d\theta$

(b) $\dfrac{f^{(n)}(a)}{n!} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{-in\theta} f(a + e^{i\theta})\, d\theta$

5.93 Prove that $8z^4 - 6z + 5 = 0$ has one root in each quadrant.

5.94 Show that (a) $\displaystyle\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 0$, (b) $\displaystyle\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 2\pi$.

5.95 Extend the result of Problem 5.32 so as to obtain formulae for the derivatives of $f(z)$ at any point in $\mathcal{R}$.

5.96 Prove that $z^3 e^{1-z} = 1$ has exactly two roots inside the circle $|z| = 1$.

5.97 If $t > 0$ and C is any simple closed curve enclosing $z = -1$, prove that

$$\frac{1}{2\pi i} \oint_C \frac{z e^{zt}}{(z+1)^3} dz = \left(t - \frac{t^2}{2}\right) e^{-t}$$

5.98 Find all functions $f(z)$ which are analytic in $|z| < 1$ and satisfy the conditions (a) $f(0) = 1$ and (b) $|f(z)| \geqq 1$ for $|z| < 1$.

5.99 Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C except that $f(z)$ has zeros at $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ of orders (multiplicites) $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$ respectively. Prove that

$$\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} p_k g(a_k) - \sum_{k=1}^{n} q_k g(b_k)$$

5.100 If $f(z) = a_0 z^n + a_1 z^{n-1} + a^2 z^{n-2} + \cdots + a_n$ where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and C encloses all the zeros of $f(z)$, evaluate (a) $\dfrac{1}{2\pi i} \oint_C \dfrac{z f'(z)}{f(z)} dz$, (b) $\dfrac{1}{2\pi i} \oint_C \dfrac{z^2 f'(z)}{f(z)} dz$ and interpret the results.

5.101 Find all functions $f(z)$ which are analytic in the region $|z| \leqq 1$ and are such that (a) $f(0) = 3$ and (b) $|f(z)| \leqq 3$ for all z such that $|z| < 1$.

5.102 Prove that $z^6 + 192z + 640 = 0$ has one root in the first and fourth quadrants and two roots in the second and thir quadrants.

5.103 Prove that the function $xy(x^2 - y^2)$ cannot have an absolute maximum or minimum inside the circle $|z| = 1$.

5.104 (a) If a function is analytic in a region $\mathcal{R}$, is it bounded in $\mathcal{R}$? (b) In view of your answer to (a), is it necessary to state that $f(z)$ is bounded in Liouville's theorem?

5.105 Find all functions $f(z)$ that are analytic everywhere, have a zero of order two at $z = 0$, satisfy the condition $|f'(z)| \leqq 6|z|$ for all z, and are such that $f(i) = -2$.

5.106 Prove that all the roots of $z^5 + z - 16i = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

5.107 If U is harmonic inside and on a simple closed curve C, prove that

$$\oint_C \frac{\partial U}{\partial n} ds = 0$$

where n is a unit normal to C in the z plane and s is the arc length parameter.

5.108 A theorem of Cauchy states that all the roots of the equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0$, where $a_1, a_2, \ldots, a_n$ are real, lie inside the circle $|z| = 1 + \max\{a_1, a_2, \ldots, a_n\}$, i.e. $|z| = 1$ plus the maximum of the values $a_1, a_2, \ldots, a_n$. Verify this theorem for the special cases (a) $z^3 - z^2 + z - 1 = 0$, (b) $z^4 + z^2 + 1 = 0$, (c) $z^4 - z^2 - 2z + 2 = 0$, (d) $z^4 + 3z^2 - 6z + 10 = 0$.

5.109 Prove the theorem of Cauchy stated in Problem 5.108.

5.110 Let $P(z)$ be any polynomial. If m is any positive integer and $\omega = e^{2\pi i/m}$, prove that

$$\frac{P(1) + P(\omega) + P(\omega^2) + \ldots + P(\omega^{m-1})}{m} = P(0)$$

and give a geometric interpretation.

5.111 Is the result of Problem 5.110 valid for any function $f(z)$? Justify your answer.

5.112 Prove *Jensen's theorem:* If $f(z)$ is analytic inside and on the circle $|z| = R$ except for zeros at $a_1, a_2, \ldots, a_m$ of multiplicities $p_1, p_2, \ldots, p_m$ and poles at $b_1, b_2, \ldots, b_n$ of multiplicities $q_1, q_2, \ldots, q_n$ respectively, and if $f(0)$ is finite and different from zero, then

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})|\, d\theta = \ln |f(0)| + \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) - \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right)$$

[*Hint.* Consider $\oint_C \ln z\, \{f'(z)/f(z)\}\, dz$ where C is the circle $|z| = R$.]

ANSWERS TO SUPPLEMENTARY PROBLEMS

5.39 (a) e^2, (b) 0
5.40 $2\pi i$
5.41 (a) $-2\pi i$, (b) 0
5.42 (a) 0, (b) $-\frac{1}{2}$
5.44 $-\pi i$
5.47 (a) $\pi i/32$, (b) $21\pi i/16$
5.48 $\frac{1}{2}(\sin t - t\cos t)$
5.59 (b) 1, 1, $-1 \pm i$
5.60 (a) i, $\frac{1}{2}(-i \pm \sqrt{15})$, (b) $\frac{1}{2}(-1 \pm \sqrt{3}\,i)$, $\frac{1}{2}(1 \pm \sqrt{3}\,i)$
5.61 1/4
5.63 17
5.70 $10\pi i$
5.71 -2
5.72 (a) $14\pi i$, (b) $12\pi i$, (c) $2\pi i$
5.73 $4\pi i$
5.83 $1 - (2/\pi)\tan^{-1}(y/x)$
5.84 $1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x+1}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x-1}\right)$
5.88 i
5.89 $-2\pi i t^2$
5.100 (a) $-a_1/a_0$, (b) $(a_1^2 - 2a_0a_2)/a_0^2$

Infinite Series Taylor's and Laurent's Series

SEQUENCES OF FUNCTIONS

The ideas of Chapter 2, Pages 2.8 and 2.9, for sequences and series of constants are easily extended to sequences and series of functions.

Let $u_1(z), u_2(z), \ldots, u_n(z)$ denoted briefly by $\{u_n(z)\}$, be a sequence of functions of z defined and single-valued in some region of the z plane. We call $U(z)$ the limit of $u_n(z)$ as $n \to \infty$, and write $\lim_{n\to\infty} u_n(z) = U(z)$, if given any positive number ϵ we can find a number N [depending in general on both ϵ and z] such that

$$|u_n(z) - U(z)| < \epsilon \qquad \text{for all } n > N$$

In such case we say that the *sequence converges* or is *convergent* to $U(z)$.

If a sequence converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the sequence. A sequence which is not convergent at some value (point) z is called *divergent* at z.

The theorems on limits given on Page 2.7 can be extended to sequences of functions.

SERIES OF FUNCTIONS

From the sequence of functions $\{u_n(z)\}$ let us form a new sequence $\{S_n(z)\}$ defined by

$$S_1(z) = u_1(z)$$
$$S_2(z) = u_1(z) + u_2(z)$$
$$\vdots$$
$$S_n(z) = u_1(z) + u_2(z) + \ldots + u_n(z)$$

where $S_n(z)$, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n(z)\}$.

The sequence $S_1(z), S_2(z), \ldots$ or $\{S_n(z)\}$ is symbolized by

$$u_1(z) + u_2(z) + \ldots = \sum_{n=1}^{\infty} u_n(z) \tag{6.1}$$

called an *infinite series*. If $\lim_{n\to\infty} S_n(z) = S(z)$, the series is called *convergent* and $S(z)$ is its *sum*; otherwise the series is called *divergent*. We sometimes write $\sum_{n=1}^{\infty} u_n(z)$ as $\Sigma u_n(z)$ or Σu_n for brevity.

As we have already seen, a necessary condition that the series (1) converge is $\lim_{n\to\infty} u_n(z) = 0$, but this is not sufficient. See, for example, Problem 2.150, and also Problems 6.68(c), 6.68(d) and 6.112(a).

If a series converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the series.

ABSOLUTE CONVERGENCE

A series $\sum_{n=1}^{\infty} u_n(z)$ is called *absolutely convergent* if the series of absolute values, i.e. $\sum_{n=1}^{\infty} |u_n(z)|$, converges.

If $\sum_{n=1}^{\infty} u_n(z)$ converges but $\sum_{n=1}^{\infty} |u_n(z)|$ does not converge, we call $\sum_{n=1}^{\infty} u_n(z)$ *conditionally convergent*.

UNIFORM CONVERGENCE OF SEQUENCES AND SERIES

In the definition of limit of a sequence of functions it was pointed out that the number N depends in general on ϵ and the particular value of z. It may happen, however, that we can find a number N such that $|u_n(z) - U(z)| < \epsilon$ for all $n > N$, where the same number N holds for all z in a region $\mathcal{R}$ [i.e. N depends only on ϵ and not on the particular value of z (point) in the region]. In such case we say that $u_n(z)$ *converges uniformly*, or is *uniformly convergent*, to $U(z)$ for all z in $\mathcal{R}$.

Similarly if the sequence of partial sums $\{S_n(z)\}$ converges uniformly to $S(z)$ in a region, we say that the infinite series (1) *converges uniformly*, or is *uniformly convergent*, to $S(z)$ in the region.

If we call $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \dots = S(z) - S_n(z)$ the *remainder* of the infinite series (1) after n terms, we can equivalently say that the series in uniformly convergent to $S(z)$ in $\mathcal{R}$ if given any $\epsilon > 0$ we can find a number N such that for all z in $\mathcal{R}$,

$$|R_n(z)| = |S(z) - S_n(z)| < \epsilon \qquad \text{for all } n > N$$

POWER SERIES

A series having the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \dots = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{6.2}$$

is called a *power series* in $z - a$. We shall sometimes indicate (6.2) briefly by $\Sigma a_n(z-a)^n$.

Clearly the power series (2) converges for $z = a$, and this may indeed be the only point for which it converges [see Problem 6.13(b)]. In general, however, the series converges for other points as well. In such case we can show that there exists a positive number R such that (6.2) converges for $|z - a| < R$ and diverges for $|z - a| > R$, while for $|z - a| = R$ it may or may not converge.

Geometrically if Γ is a circle of radius R with centre at $z = a$, then the series (2) converges at all points inside Γ and diverges at all points outside Γ, while it may or may not converge on the circle Γ. We can consider the special cases $R = 0$ and $R = \infty$ respectively to be the cases where (2) converges only at $z = a$ or converges for all (finite) values of z. Because of this geometrical interpretation, R is often called the *radius of convergence* of (2) and the corresponding circle is called the *circle of convergence*.

SOME IMPORTANT THEOREMS

For reference purposes we list here some important theorems involving sequences and series. Many of these will be familiar from their analogs for real variables.

A. General Theorems

Theorem 6.1 If a sequence has a limit, the limit is unique [i.e. it is the only one].

Theorem 6.2 Let $u_n = a_n + ib_n$, $n = 1, 2, 3, \dots$, where a_n and b_n are real. Then a necessary and sufficient condition that $\{u_n\}$ converge is that $\{a_n\}$ and $\{b_n\}$ converge.

Theorem 6.3 Let $\{a_n\}$ be a real sequence with the property that

$$\text{(i) } a_{n+1} \ge a_n \text{ or } a_{n+1} \le a_n, \qquad \text{(ii) } |a_n| < M \text{ (a constant)}$$

Then $\{a_n\}$ converges.

If the first condition in Property (i) holds the sequence is called *monotonic increasing*, while if the second condition holds it is called *monotonic decreasing*. If Property (ii) holds, the sequence is said to be *bounded*. Thus, the theorem states that every bounded monotonic (increasing or decreasing) sequence has a limit.

Theorem 6.4 A necessary and sufficient condition that $\{u_n\}$ converges is that given any $\epsilon > 0$, we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p > N$, $q > N$.

This result, which has the advantage that the limit itself is not present, is called *Cauchy's convergence criterion*.

Theorem 6.5 A necessary condition that Σu_n converge is that $\lim_{n\to\infty} u_n = 0$. However, the condition is not sufficients.

Theorem 6.6 Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence. Removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence or divergence.

Theorem 6.7 A necessary and sufficient condition that $\sum_{n=1}^{\infty}(a_n + ib_n)$ converge, where a_n and b_n are real, is that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

B. Theorems on Absolute Convergence

Theorem 6.8 If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges. In words, an absolutely convergent series is convergent.

Theorem 6.9 The terms of an absolutely convergent series can be rearranged in any order and all such rearranged series converge to the same sum. Also, the sum, difference, and product of absolutely convergent series is absolutely convergent.

These are not so for conditionally convergent series (see Problem 6.128).

C. Special Tests for Convergence

Theorem 6.10 (*Comparison tests*)

(a) If $\Sigma|v_n|$ converges and $|u_n| \le |v_n|$, then Σu_n converges absolutely.

(b) If $\Sigma|v_n|$ diverges and $|u_n| \ge |v_n|$, then $\Sigma|u_n|$ diverges but Σu_n may or may not converge.

Theorem 6.11 (*Ratio test*)

If $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = L$, then Σu_n converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

Theorem 6.12 (*nth Root test*)

If $\lim_{n\to\infty}\sqrt[n]{|u_n|} = L$, then Σu_n converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

Theorem 6.13 (*Integral test.*) If $f(x) \ge 0$ for $x \ge a$, then $\Sigma f(n)$ converges or diverges according as $\lim_{M\to\infty}\int_a^M f(x)dx$ converges or diverges.

Theorem 6.14 (*Raabe's test*)

If $\lim_{n\to\infty} n\left(1-\left|\dfrac{u_{n+1}}{u_n}\right|\right) = L$, then Σu_n converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L < 1$. If $L = 1$, the test fails.

Theorem 6.15 (*Gauss' test*)

Suppose $\left|\dfrac{u_{n+1}}{u_n}\right| = 1 - \dfrac{L}{n} + \dfrac{c_n}{n^2}$ where $|c_n| < M$ for all $n > N$, then Σu_n converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L \le 1$.

Theorem 6.16 (*Alternating series test*)

If $a_n \ge 0$, $a_{n+1} \le a_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \ldots = \Sigma(-1)^{n-1} a_n$ converges.

D. Theorems on Uniform Convergence

Theorem 6.17 (*Weierstrass M test*)

If $|u_n(z)| \le M_n$, where M_n is independent of z in a region $\mathcal{R}$ and ΣM_n converges, then $\Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$.

Theorem 6.18 The sum of a uniformly convergent series of continuous functions is continuous, i.e. if $u_n(z)$ is continuous in $\mathcal{R}$ and $S(z) = \Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

Theorem 6.19 Suppose $\{u_n(z)\}$ are continuous in $\mathcal{R}$, $S(z) = \Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$, then

$$\int_C S(z)dz = \int_C u_1(z)dz + \int_C u_2(z)dz + \ldots$$

or

$$\int_C \{\Sigma u_n(z)\}dz = \Sigma\int_C u_n(z)dz$$

In words, a uniformly convergent series of continuous function can be integrated term by term.

Theorem 6.20 If $u_n'(z) = \dfrac{d}{dz}u_n(z)$ exists in $\mathcal{R}$, $\Sigma u_n'(z)$ converges uniformly in $\mathcal{R}$ and $\Sigma u_n(z)$ converges in $\mathcal{R}$, then $\dfrac{d}{dz}\Sigma u_n(z) = \Sigma u_n'(z)$.

Theorem 6.21 Suppose $\{u_n(z)\}$ are analytic and $\Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$, then $S(z) = \Sigma u_n(z)$ is analytic in $\mathcal{R}$.

E. Theorems on Power Series

Theorem 6.22 A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.

Theorem 6.23

(a) A power series can be differentiated term by term in any region which lies entirely inside its circle of convergence.

(b) A power series can be integrated term by term along any curve C which lies entirely inside its circle of convergence.

(c) The sum of a power series is continuous in any region which lies entirely inside its circle of convergence.

These follows from Theorems 6.17-6.19 and 6.21.

Theorem 6.24 (*Abels theorem*)

Let $\Sigma a_n z^n$ have radius of convergence R and suppose that z_0 is a point on the circle of convergence such that $\Sigma a_n z_0^n$ converges. Then, $\lim_{z\to z_0} \Sigma a_n z^n = \Sigma a_n z_0^n$ where $z \to z_0$ from within the circle of convergence. Extensions to other power series are easily made.

Theorem 6.25 If $\Sigma a_n z^n$ converges to zero for all z such that $|z| < R$ where $R > 0$, then $a_n = 0$. Equivalently, if $\Sigma a_n z^n = \Sigma b_n z^n$ for all z such that $|z| < R$, then $a_n = b_n$.

TAYLOR'S THEOREM

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a + h$ be two points inside C. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \tag{6.3}$$

or writing $z = a + h$, $h = z - a$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{6.4}$$

This is called *Taylor's theorem* and the series (6.3) or (6.4) is called a *Taylor series* or *expansion* for $f(a+h)$ or $f(z)$.

The region of convergence of the series (6.4) is given by $|z-a|<R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z-a|=R$, the series may or may not converge. For $|z-a|>R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e. the series converges for all z.

If $a=0$ in (6.3) or (6.4), the resulting series is often called a *Maclaurin series*.

SOME SPECIAL SERIES

The following list show some special series together with their regions of convergence. In the case of multiple-valued functions, the principal branch is used.

1. $e^z = 1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dots+\dfrac{z^n}{n!}+\dots \qquad |z|<\infty$
2. $\sin z = z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dots(-1)^{n-1}\dfrac{z^{2n-1}}{(2n-1)!}+\dots \qquad |z|<\infty$
3. $\cos z = 1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dots(-1)^{n-1}\dfrac{z^{2n-1}}{(2n-2)!}+\dots \qquad |z|<\infty$
4. $\ln(1+z) = z-\dfrac{z^2}{2}+\dfrac{z^3}{3}-\dots(-1)^{n-1}\dfrac{z^n}{n}+\dots \qquad |z|<1$
5. $\tan^{-1} z = z-\dfrac{z^3}{3}+\dfrac{z^5}{5}-\dots(-1)^{n-1}\dfrac{z^{2n-1}}{2n-1}+\dots \qquad |z|<1$
6. $(1+z)^p = 1+pz+\dfrac{p(p-1)}{2!}z^2+\dots+\dfrac{p(p-1)\cdots(p-n+1)}{n!}z^n+\dots \qquad |z|<1$

This is the *binomial theorem or formula*. If $(1+z)^p$ is multiple-valued, the result is valid for that branch of the function which has the value 1 when $z=0$.

LAURENT'S THEOREM

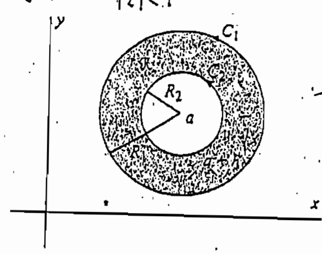

Fig. 6.1

Let C_1 and C_2 be concentric circles of radii R_1 and R_2 respectively and center at a [Fig. 6.1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and in the ring-shaped region $\mathfrak{R}$ [also called *annulus* or *annular region*] between C_1 and C_2, shown shaded in figure Let $a+h$ be any point in $\mathfrak{R}$. Then we have

$$f(a+h) = a_0+a_1h+a_2h^2+\dots+\frac{a_{-1}}{h}+\frac{a_{-2}}{h^2}+\frac{a_{-3}}{h^3}+\dots \qquad (6.5)$$

where

$$\left.\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz \qquad n=0,1,2,\dots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_2}(z-a)^{n-1}f(z)dz \qquad n=1,2,3,\dots \end{aligned}\right\} \qquad (6.6)$$

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

In the above integrations, we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 6.101]. Then, the coefficients (6.6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)^{n+1}}dz \qquad n=0,\pm1,\pm2,\dots \qquad (6.7)$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0+a_1(z-a)+a_2(z-a)^2+\dots+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\dots \qquad (6.8)$$

where

$$a_n = \frac{1}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta \quad n=0,\pm1,\pm2,\dots \qquad (6.9)$$

This is called *Laurent's theorem* and (6.5) or (6.8) with coefficients (6.6), (6.7) or (6.9) is called a *Laurent series* or *expansion*.

The part $a_0+a_1(z-a)+a_2(z-a)^2+\dots$ is called the *analytic part* of the Laurent series, while the remainder of the series, which consists of inverse powers of $z-a$ is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

CLASSIFICATION OF SINGULARITIES

It is possible to classify the singularities of a function $f(z)$ by examination of its Laurent series. For this purpose, we assume that in Fig. 6.1, $R_2=0$, so that $f(z)$ is analytic inside and on C_1 except at $z=a$, which is an isolated singularity [see Page 3.5]. In the following, all singularities are assumed isolated unless otherwise indicated.

1. **Poles.** If $f(z)$ has the form (6.8) in which the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\dots+\frac{a_{-n}}{(z-a)^n}$$

where $a_{-n}\neq 0$, then $z=a$ is called a *pole of order n*. If $n=1$, it is called a *simple pole*.

If $f(z)$ has a pole at $z=a$, then $\lim_{z\to a} f(z)=\infty$ [see Problem 6.32].

2. **Removable singularities.** If a single-valued function $f(z)$ is not defined at $z=a$ but $\lim_{z\to a} f(z)$ exists, then $z=a$ is called a *removable singularity*. In such case we define $f(z)$ at $z=a$ as equal to $\lim_{z\to a} f(z)$, and $f(z)$ will then be analytic at a.

Example 6.1 If $f(z)=\sin z/z$, then $z=0$ is a removable singularity since $f(0)$ is not defined but $\lim_{z\to a}\sin z/z=1$. We define $f(0)=\lim_{z\to a}\sin z/z=1$. Note that in this case

$$\frac{\sin z}{z}=\frac{1}{z}\left\{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\dots\right\}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\dots$$

3. **Essential singularities.** If $f(z)$ is single-valued, then any singularity which is not a pole or removable singularity is called an *essential singularity*. If $z=a$ is an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms.

Example 6.2 Since $e^{1/z}=1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+\dots$, $z=0$ is an essential singularity.

The following two related theorems are of interest (see Problems 6.154–6.156):

Casorati–Weierstrass theorem. In any neighbourhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ comes arbitrarily close to any complex number. In symbols, given any positive numbers δ and ϵ and any complex number A, there exists a value of z inside the circle $|z - a| = \delta$ for which $|f(z) - A| < \epsilon$.

Picard's theorem. In the neighbourhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ can take on any value whatsoever with perhaps one exception.

4. **Branch points.** A point $z = z_0$ is called a *branch point* of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0 [see Page 2.5]. A branch point is a non-isolated singularity. Since each of the branches of a multiple-valued function is analytic, all of the theorems for analytic functions, in particular Taylor's theorem, apply.

Example 6.3 The branch of $f(z) = z^{1/2}$, which has the value 1 for $z = 1$, has a Taylor series of the form $a_0 + a_1(z-1) + a_2(z-1)^2 + \ldots$ with radius of convergence $R = 1$ [the distance from $z = 1$ to the nearest singularity, namely the branch point $z = 0$].

5. **Singularities at infinity.** By letting $z = 1/w$ in $f(z)$, we obtain the function $f(1/w) = F(w)$. Then the nature of the singularity for $f(z)$ at $z = \infty$ [the point at infinity] is defined to be the same as that of $F(w)$ at $w = 0$.

Example 6.4 $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $F(w) = f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$. Similarly $f(z) = e^z$ has an essential singularity at $z = \infty$, since $F(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

ENTIRE FUNCTIONS

A function which is analytic everywhere in the finite plane [i.e. everywhere except at ∞] is called an *entire function* or *integral function*. The functions e^z, $\sin z$, $\cos z$ are entire functions.

An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely if a power series has an infinite radius of convergence, it represents an entire function.

Note that by Liouville's theorem [Chapter 5, Page 5.2] a function which is analytic *everywhere including* ∞ must be a constant.

MEROMORPHIC FUNCTIONS

A function that is analytic everywhere in the finite plane except at a finite number of poles is called a *meromorphic function*.

Example 6.5 $\dfrac{z}{(z-1)(z+3)^2}$ which is ~~analytic~~ everywhere in the finite plane except at the poles $z = 1$ (simple pole) and $z = -3$ (pole of order two) is a meromorphic function.

LAGRANGE'S EXPANSION

Let z be that root of $z = a + \zeta\phi(z)$ which has the value $z = a$ when $\zeta = 0$ Then, if $\phi(z)$ is analytic inside and on a circle C containing $z = a$, we have

$$z = a + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{[\phi(a)]^n\} \tag{6.10}$$

More generally, if $F(z)$ is analytic inside and on C, then

$$F(z) = F(a) + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{F'(a)\,[\phi(a)]^n\} \tag{6.11}$$

The expansion (6.11) and the special case (6.10) are often referred to as *Lagrange's expansions*.

ANALYTIC CONTINUATION

Suppose that we do not know the precise form of an analytic function $f(z)$ but only know that inside some circle of convergence C_1 with center at a [Fig. 6.2] $f(z)$ is represented by a Taylor series

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots \tag{6.12}$$

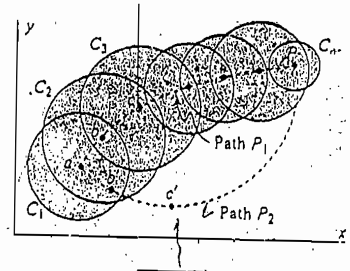

Fig. 6.2

Choosing a point b inside C_1, we can find the value of $f(z)$ and its derivatives at b from (6.12) and thus arrive at a new series

$$b_0 + b_1(z-b) + b_2(z-b)^2 + \ldots \tag{6.13}$$

having circle of convergence C_2. If C_2 extends beyond C_1, then the values of $f(z)$ and its derivatives can be obtained in this extended portion and so we have achieved ore information concerning $f(z)$.

We say, in this case, that $f(z)$ has been *extended analytically* beyond C_1 and call the process *analytic continuation* or *analytic extension*.

The process can of course be repeated indefinitely. Thus, choosing point c inside C_2, we arrive at a new series having circle of convergence C_3 which may extend beyond C_1 and C_2, etc.

The collection of all such power series representations, i.e. all possible analytic continuations, is defined as the analytic function $f(z)$ and each power series is sometimes called an *element* of $f(z)$.

In performing analytic continuations we must avoid singularities. For example, there cannot be any singularity in Fig. 6.2 which is both inside C_2 and on the boundary of C_1, since otherwise (6.13) would diverge at this point. In some cases the singularities on a circle of convergence are so numerous that analytic continuation is impossible. In these cases the boundary of the circle is called a *natural boundary* or barrier [see Problem 6.31]. The function represented by a series having a natural boundary is called a *lacunary* function.

In going from circle C_1 to circle C_n [Fig. 6.2], we have chosen the path of centres $a, b, c, \ldots, p$ which we represent breifly by *path* P_1. Many other paths are also possible, e.g. $a, b', c', \ldots, p$ represented briefly by *path* P_2. A question arises as to whether one obtains the same series representation valid inside C_n when one chooses different paths. The answer is *yes* so long as the region bounded by paths P_1 and P_2 has no singularity.

For a further discussion of analytic continuation, see Chapter 10.

SOLVED PROBLEMS

SEQUENCES AND SERIES OF FUNCTIONS

6.1. Using the definition, prove that $\lim_{n\to\infty}\left(1 + \frac{z}{n}\right) = 1$ for all z.

Solution Given any number $\epsilon > 0$, we must find N such that $|1 + z/n - 1| < \epsilon$ for $n > N$. Then $|z/n| < \epsilon$, i.e. $|z|/n < \epsilon$ if $n > |z|/\epsilon = N$.

6.2 (a) Prove that the series $z(1-z)+z^2(1-z)+z^3(1-z)+\dots$ converges for $|z|<1$, and (b) find its sum.

Solution The sum of the first n terms of the series is

$$S_n(z) = z(1-z)+z^2(1-z)+\dots+z^n(1-z)$$
$$= z-z^2+z^2-z^3+\dots+z^n-z^{n+1}$$
$$= z-z^{n+1}$$

Now $|S_n(z)-z| = |-z^{n+1}| = |z|^{n+1} < \epsilon$ for $(n+1)\ln|z| < \ln\epsilon$, i.e. $n+1 > \dfrac{\ln\epsilon}{\ln|z|}$ or $n > \dfrac{\ln\epsilon}{\ln|z|}-1$ if $z \neq 0$.

If $z=0$, $S_n(0)=0$ and $|S_n(0)-0|<\epsilon$ for all n.

Hence $\lim\limits_{n\to\infty} S_n(z) = z$, the required sum for all z such that $|z|<1$.

Another method.

Since $S_n(z) = z-z^{n+1}$, we have [by Problem 2.41, Chapter 2, in which we showed that $\lim\limits_{n\to\infty} z^n = 0$ if $|z|<1$]

$$\text{Required sum} = S(z) = \lim_{n\to\infty} S_n(z) = \lim_{n\to\infty}(z-z^{n+1}) = z$$

ABSOLUTE AND UNIFORM CONVERGENCE

6.3 (a) Prove that the series in Problem 6.2 converges uniformly to the sum z for $|z| \le \frac{1}{2}$.

(b) Does the series converge uniformly for $|z| \le 1$? Explain.

Solution

(a) In Problem 6.2, we have shown that $|S_n(z)-z|<\epsilon$ for all $n > \dfrac{\ln\epsilon}{\ln|z|}-1$, i.e. the series converges to the sum z for $|z|<1$ and thus for $|z|\le\frac{1}{2}$.

Now if $|z|\le\frac{1}{2}$, the largest value of $\dfrac{\ln\epsilon}{\ln|z|}-1$ occurs where $|z|=\frac{1}{2}$ and is given by $\dfrac{\ln\epsilon}{\ln(1/2)}-1=N$. It follows that $|S_n(z)-z|<\epsilon$ for all $n>N$ where N depends only on ϵ and not on the particular z in $|z|\le\frac{1}{2}$. Thus the series converges uniformly to z for $|z|\le\frac{1}{2}$.

(b) The same argument given in part (a) serves to show that the series converges uniformly to sum z for $|z|\le .9$ or $|z|\le .99$ by using $N=\dfrac{\ln\epsilon}{\ln(.9)}-1$ and $N=\dfrac{\ln\epsilon}{\ln(.99)}-1$ respectively.

However, it is clear that we cannot extend the argument to $|z|\le 1$ since this would require $N=\dfrac{\ln\epsilon}{\ln 1}-1$ which is infinite, i.e., there is no finite value of N which can be used in this case. Thus, the series does not converge uniformly for $|z|\le 1$.

6.4 (a) Prove that the sequence $\left\{\dfrac{1}{1+nz}\right\}$ is uniformly convergent to zero for all z such that $|z|\ge 2$.

(b) Can the region of uniform convergence in (a) be extended? Explain.

Solution

(a) We have $\left|\dfrac{1}{1+nz}-0\right|<\epsilon$ when $\dfrac{1}{|1+nz|}<\epsilon$ or $|1+nz|>1/\epsilon$. Now $|1+nz|\le|1|+|nz|=1+n|z|$ and $1+n|z|\ge|1+nz|>1/\epsilon$ for $n>\dfrac{1/\epsilon-1}{|z|}$. Thus, the sequence converges to zero for $|z|>2$.

To determine whether it converges uniformly to zero, note that the largest value of $\dfrac{1/\epsilon-1}{|z|}$ in $|z|\ge 2$ occurs for $|z|=2$ and is given by $\frac{1}{2}\{(1/\epsilon)-1\}=N$. It follows that $\left|\dfrac{1}{1+nz}-0\right|<\epsilon$ for all $n>N$ where N depends only on ϵ and not on the particular z in $|z|\ge 2$. Thus, the sequence is uniformly convergent to zero in this region.

(b) If δ is any positive number, the largest value of $\dfrac{(1/\epsilon)-1}{|z|}$ in $|z|\ge\delta$ occurs for $|z|=\delta$ and is given by $\dfrac{(1/\epsilon)-1}{\delta}$. As in part (a), it follows that the sequence converges uniformly to zero for all z such that $|z|\ge\delta$, i.e. in any region which excludes all points in a neighbourhood of $z=0$.

Since δ can be chosen arbitrarily close to zero, it follows that the region of (a) can be extended considerably.

6.5. Show that (a) the sum function in Problem 6.2 is discontinuous at $z=1$, (b) the limit in Problem 6.4 is discontinuous at $z=0$.

Solution

(a) From Problem 6.2, $S_n(z)=z-z^{n+1}$, $S(z)=\lim\limits_{n\to\infty}S_n(z)$. If $|z|<1$, $S(z)=\lim\limits_{n\to\infty}S_n(z)=z$. If $z=1$, $S_n(z)=S_n(1)=0$ and $\lim\limits_{n\to\infty}S_n(1)=0$. Hence $S(z)$ is discontinuous at $z=1$.

(b) From Problem 6.4, if we write $u_n(z)=\dfrac{1}{1+nz}$ and $U(z)=\lim\limits_{n\to\infty}u_n(z)$ we have $U(z)=0$ if $z\neq 0$ and 1 if $z=0$. Thus, $U(z)$ is discontinuous at $z=0$.

These are consequences of the fact [see Problem 6.16] that if a series of continuous functions is uniformly convergent in a region $\mathfrak{R}$, then the sum function must be continuous in $\mathfrak{R}$. Hence, if the sum function is not continuous, the series cannot be uniformly convergent. A similar result holds for sequences.

6.6. Prove that the series of Problem 6.2 is absolutely convergent for $|z|<1$.

Solution

Let
$$T_n(z)=|z(1-z)|+|z^2(1-z)|+\dots+|z^n(1-z)|$$
$$=|1-z|\{|z|+|z|^2+|z|^3+\dots+|z|^n\}$$
$$=|1-z||z|\left\{\frac{1-|z|^n}{1-|z|}\right\}$$

If $|z|<1$, then $\lim\limits_{n\to\infty}|z|^n=0$ and $\lim\limits_{n\to\infty}T_n(z)$ exists so that the series converges absolutely.

Note that the series of absolute values converges in this case to $\dfrac{|1-z||z|}{1-|z|}$.

SPECIAL CONVERGENCE TESTS

6.7 If $\Sigma|v_n|$ converges and $|u_n|\le|v_n|$, $n=1,2,3,\dots$ prove that $\Sigma|u_n|$ also converges (i.e., establish the comparison test for convergence).

Solution Let $S_n = |u_1| + |u_2| + \ldots + |u_n|$, $T_n = |v_1| + |v_2| + \ldots + |v_n|$.

Since $\Sigma |v_n|$ converges, $\lim_{n\to\infty} T_n$ exists and equals T, say. Also since $|v_n| \geq 0$, $T_n \leq T$.

Then $S_n = |u_1| + |u_2| + \ldots + |u_n| \leq |v_1| + |v_2| + \ldots + |v_n| \leq T$ or $0 \leq S_n \leq T$.

Thus, S_n is a bounded monotonic increasing sequence and must have a limit [Theorem 6.3, Page 6.3], i.e., $\Sigma |u_n|$ converges.

6.8 Prove that $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for any constant $p > 1$.

We have

$$\frac{1}{1^p} = \frac{1}{1^{p-1}}$$

$$\frac{1}{2^p} + \frac{1}{3^p} \leq \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leq \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}}$$

etc., where we consider 1, 2, 4, 8, ... terms of the series. It follows that the sum of any finite number of terms of the given series is less than the geometric series

$$\frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots = \frac{1}{1 - 1/2^{p-1}}$$

which converges for $p > 1$. Thus the given series, sometimes called the *p series*, converges.

By using a method analogous to that used here together with the comparison test for divergence [Theorem 6.10(b), Page 6.4], we can show that $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ diverges for $p \leq 1$.

6.9 Prove that an absolutely convergent series is convergent.

Given that $\Sigma |u_n|$ converges, we must show that Σu_n converges.

Solution Let $S_M = u_1 + u_2 + \ldots + u_M$ and $T_M = |u_1| + |u_2| + \ldots + |u_M|$.

Then
$$\begin{aligned} S_M + T_M &= (u_1 + |u_1|) + (u_2 + |u_2|) + \ldots + (u_M + |u_M|) \\ &\leq 2|u_1| + 2|u_2| + \ldots + 2|u_M| \end{aligned}$$

Since $\Sigma |u_n|$ converges and $u_n + |u_n| \geq 0$ for $n = 1, 2, 3, \ldots$, it follows that $S_M + T_M$ is a bounded monotonic increasing sequence and so $\lim_{M\to\infty} (S_M + T_M)$ exists.

Also since $\lim_{M\to\infty} T_M$ exists [because by hypothesis the series is absolutely convergent],

$$\lim_{M\to\infty} S_M = \lim_{M\to\infty} (S_M + T_M - T_M) = \lim_{M\to\infty} (S_M + T_M) - \lim_{M\to\infty} T_M$$

must also exist and the result is proved.

6.10 Prove that $\sum_{n=1}^{\infty} \dfrac{z^n}{n(n+1)}$ converges (absolutely) for $|z| \leq 1$.

Solution If $|z| \leq 1$, then $\left|\dfrac{z^n}{n(n+1)}\right| = \dfrac{|z|^n}{n(n+1)} \leq \dfrac{1}{n(n+1)} \leq \dfrac{1}{n^2}$.

Taking $u_n = \dfrac{z^n}{n(n+1)}$, $v_n = \dfrac{1}{n^2}$ in the comparison test and recognizing that $\sum \dfrac{1}{n^2}$ converges by Problem 6.8 with $p = 2$, we see that $\Sigma |u_n|$ converges, i.e. Σu_n converges absolutely.

6.11 Establish the ratio test for convergence.

Solution We must show that if $\lim_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = L < 1$, then $\Sigma |u_n|$ converges or, by Problem 6.9, Σu_n is (absolutely) convergent.

By hypothesis, we can choose an integer N so large that for all $n \geq N$, $\left|\dfrac{u_{n+1}}{u_n}\right| \leq r$ where r is some constant such that $L < r < 1$. Then

$$\begin{aligned} |u_{N+1}| &\leq r|u_N| \\ |u_{N+2}| &\leq r|u_{N+1}| < r^2 |u_N| \\ |u_{N+3}| &\leq r|u_{N+2}| < r^3 |u_N| \end{aligned}$$

etc. By addition,

$$|u_{N+1}| + |u_{N+2}| + \ldots \leq |u_N| (r + r^2 + r^3 + \ldots)$$

and so $\Sigma |u_n|$ converges by the comparison test since $0 < r < 1$.

6.12 Find the region of convergence of the series $\sum_{n=1}^{\infty} \dfrac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

Solution If $u_n = \dfrac{(z+2)^{n-1}}{(n+1)^3 4^n}$, then $u_{n+1} = \dfrac{(z+2)^n}{(n+2)^3 4^{n+1}}$. Hence, excluding $z = -2$ for which the given series converges, we have

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(z+2)}{4} \frac{(n+1)^3}{(n+2)^3}\right| = \frac{|z+2|}{4}$$

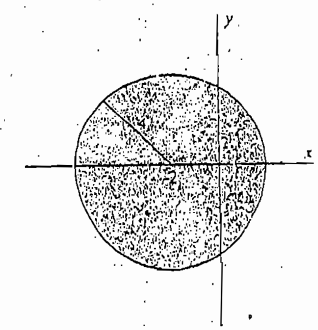

Fig. 6.3

Then the series converges (absolutely) for $\dfrac{|z+2|}{4} < 1$, i.e. $|z+2| < 4$. The point $z = -2$ is included in $|z+2| < 4$.

If $\dfrac{|z+2|}{4} = 1$, i.e. $|z+2| = 4$, the ratio test fails. However, it is seen that in this case

$$\left|\frac{(z+2)^{n-1}}{(n+1)^3 4^n}\right| = \frac{1}{4(n+1)^3} \leq \frac{1}{n^3}$$

and since $\sum \dfrac{1}{n^3}$ converges [p series with $p = 3$], the given series converges (absolutely).

It follows that the given series converges (absolutely) for $|z+2| \le 4$. Geometrically this is the set of all points inside and on the circle of radius 4 with centre at $z = -2$, called the *circle of convergence* [shown shaded in Fig. 6.3]. The *radius of convergence* is equal to 4.

6.13 Find the region of convergence of the series (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, (b) $\sum_{n=1}^{\infty} n!z^n$.

Solution

(a) If $u_n = \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, then $u_{n+1} = \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. Hence, excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|-\frac{z^2(2n-1)!}{(2n+1)!}\right| = \lim_{n\to\infty}\frac{(2n-1)!|z|^2}{(2n+1)(2n)(2n-1)!}$$

$$= \lim_{n\to\infty}\frac{|z|^2}{(2n+1)(2n)} = 0$$

for all finite z. Thus the series converges (absolutely) for all z, and we say that the series converges for $|z| < \infty$. We can equivalently say that the circle of convergence is infinite or that the radius of convergence is infinite.

(b) If $u_n = n!z^n$, $u_{n+1} = (n+1)!z^{n+1}$. Then excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = \lim_{n\to\infty}(n+1)|z| = \infty$$

Thus the series converges only for $z = 0$.

THEOREMS ON UNIFORM CONVERGENCE

6.14 Prove the Weierstrass M test, i.e. if in a region $\mathcal{R}$, $|u_n(z)| \le M_n$, $n = 1, 2, 3, \ldots$, where M_n are positive constants such that ΣM_n converges, then $\Sigma u_n(z)$ is uniformly (and absolutely) convergent in $\mathcal{R}$.

Solution The remainder of the series $\Sigma u_n(z)$ after n terms is $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \ldots$. Now

$$|R_n(z)| = |u_{n+1}(z) + u_{n+2}(z) + \ldots| \le |u_{n+1}(z)| + |u_{n+2}(z)| + \ldots$$
$$\le M_{n+1} + M_{n+2} + \ldots$$

But $M_{n+1} + M_{n+2} + \ldots$ can be made less than ϵ by choosing $n > N$, since ΣM_n converges. Since N is clearly independent of z, we have $|R_n(z)| < \epsilon$ for $n > N$, and the series is uniformly convergent. The absolute convergence follows at once from the comparison test.

6.15 Test for uniform convergence in the indicated region:

Solution

(a) $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}, |z| \le 1$; (b) $\sum_{n=1}^{\infty} \frac{z^1}{n^2+z^2}, 1 < |z| < 2$; (c) $\sum_{n=1}^{\infty} \frac{\cos nz}{n^3} |z| \le 1$.

(a) If $u_n(z) = \frac{z^n}{n\sqrt{n+1}}$, then $|u_n(z)| = \frac{|z|^n}{n\sqrt{n+1}} \le \frac{1}{n^{3/2}}$ if $|z| \le 1$. Calling $M_n = \frac{1}{n^{3/2}}$, we see that ΣM_n converges (p series with $p = 3/2$). Hence, by the Weierstrass M test, the given series converges uniformly (and absolutely) for $|z| \le 1$.

(b) The given series is $\frac{1}{1^2+z^2} + \frac{1}{2^2+z^2} + \frac{1}{3^2+z^2} + \cdots$. The first two terms can be omitted without affecting the uniform convergence of the series. For $n \ge 3$ and $1 < |z| < 2$, we have

$$|n^2+z^2| \ge |n^2| - |z^2| \ge n^2 - 4 \ge \tfrac{1}{2}n^2 \quad \text{or} \quad \left|\frac{1}{n^2+z^2}\right| \le \frac{2}{n^2}$$

Since $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges, it follows from the Weierstrass M test (with $M_n = 2/n^2$) that the given series converges uniformly (and absolutely) for $1 < |z| < 2$.

Note that the convergence, and thus uniform convergence, breaks down if $|z| = 1$ or $|z| = 2$ [namely at $z = \pm i$ and $z = \pm 2i$]. Hence, the series cannot converge uniformly for $1 \le |z| \le 2$.

(c) If $z = x + iy$, we have

$$\frac{\cos nz}{n^3} = \frac{e^{inz} + e^{-inz}}{2n^3} = \frac{e^{inx-ny} + e^{-inx+ny}}{2n^3} = \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3} + \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}.$$

The series

$$\sum_{n=1}^{\infty} \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3}$$

cannot converge for $y > 0$ and $y < 0$, respectively [since, in these cases, the nth term does not approach zero]. Hence the series does not converge for all z such that $|z| \le 1$, and so cannot possibly be uniformly convergent in this region.

The series does converge for $y = 0$, i.e. if z is real. In this case, $z = x$ and the series becomes $\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$. Then since $\left|\frac{\cos nx}{n^3}\right| \le \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, it follows from the Weierstrass M test (with $M_n = 1/n^3$) that the given series converges uniformly in any interval on the real axis.

6.16 Prove Theorem 6.18, Page 6.14, i.e. if $u_n(z)$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$ and $\sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent to $S(z)$ in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

Solution If $S_n(z) = u_1(z) + u_2(z) + \ldots + u_n(z)$, and $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \ldots$ is the remainder after n terms, it is clear that

$$S(z) = S_n(z) + R_n(z) \quad \text{and} \quad S(z+h) = S_n(z+h) + R_n(z+h)$$

and so
$$S(z+h) - S(z) = S_n(z+h) - S_n(z) + R_n(z+h) - R_n(z) \tag{1}$$

where z and $z + h$ are in $\mathcal{R}$.

Since $S_n(z)$ is the sum of a finite number of continuous functions, it must also be continuous. Then, given $\epsilon > 0$, we can find δ so that

$$|S_n(z+h) - S_n(z)| < \epsilon/3 \quad \text{whenever} \quad |h| < \delta \tag{2}$$

Since the series, by hypothesis, is uniformly convergent, we can choose N so that for all z in $\mathcal{R}$,

$$|R_n(z)| < \epsilon/3 \quad \text{and} \quad |R_n(z+h)| < \epsilon/3 \quad \text{for} \quad n > N \tag{3}$$

Then from (1), (2) and (3),

$|S(z+h) - S(z)| \le |S_n(z+h) - S_n(z)| + |R_n(z+h)| + |R_n(z)| < \epsilon$ for $|h| < \delta$ and all z in $\mathcal{R}$, and so the continuity is established.

6.17. Prove Theorem 6.19, Page 6.4, i.e. if $\{u_n(z)\}$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$, $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$, then

$$\int_C S(z)dz = \int_C \left(\sum_{n=1}^{\infty} u_n(z)\right)dz = \sum_{n=1}^{\infty}\int_C u_n(z)dz$$

Solution As in Problem 6.16, we have $S(z) = S_n(z) + R_n(z)$ and, so since these are continuous in $\mathcal{R}$ [by Problem 6.16], their integrals exist, i.e.,

$$\begin{aligned}\int_C S(z)dz &= \int_C S_n(z)dz + \int_C R_n(z)dz \\ &= \int_C u_1(z)dz + \int_C u_2(z)dz + \ldots + \int_C u_n(z)dz + \int_C R_n(z)dz\end{aligned}$$

By hypothesis, the series is uniformly convergent, so that given any $\epsilon > 0$ we can find a number N independent of z in $\mathcal{R}$ such that $|R_n(z)| < \epsilon$ when $n > N$. Denoting by L the length of C, we have [using Property e, Page 4.2]

$$\left|\int_C R_n(z)dz\right| < \epsilon L$$

Then $\left|\int_C S(z)dz - \int_C S_n(z)dz\right|$ can be made as small as we like by choosing n large enough, and the result is proved.

THEOREMS ON POWER SERIES

6.18 If a power series $\Sigma a_n z^n$ converges for $z = z_0 \ne 0$, prove that it converges (a) absolutely for $|z| < |z_0|$, (b) uniformly for $|z| \le |z_1|$ where $|z_1| < |z_0|$.

Solution

(a) Since $\Sigma a_n z_0^n$ converges, $\lim_{n\to\infty} a_n z_0^n = 0$ and so we can make $|a_n z_0^n| < 1$ by choosing n large enough, i.e., $|a_n| < \dfrac{1}{|z_0|^n}$ for $n > N$. Then

$$\sum_{N+1}^{\infty}|a_n z^n| = \sum_{N+1}^{\infty}|a_n||z|^n \le \sum_{N+1}^{\infty}\frac{|z|^n}{|z_0|^n} \tag{1}$$

But the last series in (1) converges for $|z| < |z_0|$ and so by the comparison test the first series converges, i.e. the given series is absolutely convergent.

(b) Let $M_n = \dfrac{|z_1|^n}{|z_0|^n}$. Then ΣM_n converges, since $|z_1| < |z_0|$. As in part (a), $|a_n z^n| < M_n$ for $|z| \le |z_1|$ so that, by the Weierstrass M test, $\Sigma a_n z^n$ is uniformly convergent.

It follows that a power series is uniformly convergent in any region which lies entirely inside its circle of convergence.

6.19 Prove that both the power series $\sum_{n=0}^{\infty} a_n z^n$ and the corresponding series of derivatives $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Solution Let $R > 0$ be the radius of convergence of $\Sigma a_n z^n$. Let $0 < |z_0| < R$. Then, as in Problem 6.18 we can choose N so that $|a_n| < \dfrac{1}{|z_0|^n}$ for $n > N$.

Thus the terms of the series $\Sigma|n a_n z^{n-1}| = \Sigma n|a_n||z|^{n-1}$ can for $n > N$ be made less than corresponding terms of the series $\Sigma n\dfrac{|z|^{n-1}}{|z_0|^n}$ which converges, by the ratio test, for $|z| < |z_0| < R$.

Hence $\Sigma n a_n z^{n-1}$ converges absolutely for all points such that $|z| < |z_0|$ (no matter how close $|z_0|$ is to R), i.e. for $|z| < R$.

If, however, $|z| > R$, $\lim_{n\to\infty} a_n z^n \ne 0$ and thus $\lim_{n\to\infty} n a_n z^{n-1} \ne 0$, so that $\Sigma n a_n z^{n-1}$ does not converge.

Thus R is the radius of convergence of $\Sigma n a_n z^{n-1}$. This is also true if $R = 0$.

Note that the series of derivatives may or may not converge for values of z such that $|z| = R$.

6.20 Prove that in any region which lies entirely within its circle of convergence, a power series (a) represents a continuous function, say $f(z)$, (b) can be integrated term by term to yield the integral of $f(z)$, (c) can be differentiated term by term to yield the derivative of $f(z)$.

Solution We consider the power series $\Sigma a_n z^n$, although analogous results hold for $\Sigma a_n(z-a)^n$.

(a) This follows from Problem 6.16 and the fact that each term $a_n z^n$ of the series is continuous.

(b) This follows from Problem 6.17 and the fact that each term $a_n z^n$ of the series is continuous and thus integrable.

(c) From Problem 6.19 the derivative of a power series convergence within the circle of convergence of the original power series and therefore is uniformly convergent in any region entirely within the circle of convergence. Thus the required result follows from Theorem 6.20, page 6.5.

6.21 Prove that the series $\sum_{n=1}^{\infty}\dfrac{z^n}{n^2}$ has a finite value at all points inside and on its circle of convergence but that this is not true for the series of derivatives.

Solution By the ratio test the series converges for $|z| < 1$ and diverges for $|z| > 1$. If $|z| = 1$, then $|z^n/n^2| = 1/n^2$ and the series is convergent (absolutely). Thus the series converges for $|z| \le 1$ and so has a finite value inside and on its circle of convergence.

The series of derivatives is $\sum_{n=1}^{\infty}\dfrac{z^{n-1}}{n}$. By the ratio test the series converges for $|z| < 1$. However, the series does not converge for all z such that $|z| = 1$, for example if $z = 1$, the series diverges.

TAYLOR'S THEOREM

6.22 Prove Taylor's theorem: If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Solution Let z be any point inside C. Construct a circle C_1 with centre at a and enclosing z (see Fig. 6.4). Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \tag{1}$$

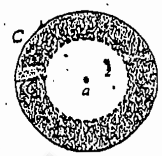

Fig. 6.4

We have

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left\{\frac{1}{1-(z-a)/(w-a)}\right\}$$

$$= \frac{1}{w-a}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{1-(z-a)/(w-a)}\right\}$$

or

$$\frac{1}{w-z} = \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots+\frac{(z-a)^{n-1}}{(w-a)^n}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \tag{2}$$

Multiplying both sides of (2) by $f(w)$ and using (1), we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw+\cdots+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n \tag{3}$$

where $U_n = \dfrac{1}{2\pi i}\oint_{C_1}\left(\dfrac{z-a}{w-a}\right)^n\dfrac{f(w)}{w-z}\,dw$

Using Cauchy's integral formulas

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw \qquad n = 0, 1, 2, 3, \ldots$$

(3) becomes

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + U_n$$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this we note that since w is on C_1,

$$\left|\frac{z-a}{w-a}\right| = \gamma < 1$$

where γ is a constant. Also we have $|f(w)| < M$ where M is a constant, and

$$|w-z| = |(w-a)-(z-a)| \geq r_1 - |z-a|$$

where r_1 is the radius of C_1. Hence from Property 5, Page 3.18, we have

$$|U_n| = \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}\,dw\right| \leq \frac{1}{2\pi}\frac{\gamma^n M}{r_1-|z-a|}\cdot 2\pi r_1 = \frac{\gamma^n M r_1}{r_1-|z-a|}$$

and we see that $\lim_{n\to\infty} U_n = 0$, completing the proof.

6.23 Let $f(z) = \ln(1+z)$, where we consider at branch which has the value zero when $z = 0$. (a) Expand $f(z)$ in a Taylor series about $z = 0$. (b) Determine the region of convergence for the series in (a). (c) Expand $\ln\left(\dfrac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$.

Solution

(a)

$$f(z) = \ln(1+z) \qquad f(0) = 0$$

$$f'(z) = \frac{1}{1+z} = (1+z)^{-1}, \qquad f'(0) = 1$$

$$f''(z) = -(1+z)^{-2} \qquad f''(0) = -1$$

$$f'''(z) = (-1)(-2)(1+z)^{-3} \qquad f'''(0) = 2!$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n+1)}(z) = (-1)^n n!(1+z)^{-(n+1)} \qquad f^{(n+1)}(0) = (-1)^n n!$$

Then

$$f(z) = \ln(1+z) = f(0) + f'(0)\,z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Another method. If $|z| < 1$,

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$$

Then integrating from 0 to z yields

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(b) The nth term is $u_n = \dfrac{(-1)^{n-1}z^n}{n}$. Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

and the series converges for $|z| < 1$. The series can be shown to converge for $|z| = 1$ except for $z = -1$.

This result also follows from the fact that the series converges in a circle which extends to the nearest singularity (i.e. $z = -1$) of $f(z)$.

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

which converges for $|z| < 1$. We can also show that this series converges for $|z| = 1$ except for $z = \pm 1$.

6.24 (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ (b) Determine the region of convergence of this series.

Solution

(a) $f(z) = \sin z, f'(z) = \cos z, f''(z) = -\sin z, f'''(z) = -\cos z, f^{IV}(z) = \sin z, \ldots$
$f(\pi/4) = \sqrt{2}/2, f'(\pi/4) = \sqrt{2}/2, f''(\pi/4) = -\sqrt{2}/2, f'''(\pi/4) = -\sqrt{2}/2, f^{IV}(\pi/4) = \sqrt{2}/2, \ldots$
Then, since $a = \pi/4$,

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \ldots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \ldots \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \ldots\right\} \end{aligned}$$

Another method. Let $u = z - \pi/4$ or $z = u + \pi/4$. Then we have,

$$\begin{aligned} \sin z &= \sin(u + \pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4) \\ &= \frac{\sqrt{2}}{2}(\sin u + \cos u) \\ &= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \ldots\right)\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \ldots\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \ldots\right\} \end{aligned}$$

(b) Since the singularity of $\sin z$ nearest to $\pi/4$ is at infinity, the series converges for all finite values of z, i.e. $|z| < \infty$. This can also be established by the ratio test.

AURENT'S THEOREM

6.25 Prove *Laurent's theorem*: If $f(z)$ is analytic inside and on the boundary of the ring-shaped region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 with centre at a and respective radii r_1 and $r_2 (r_1 > r_2)$ (see Fig. 6.5), then for all z in $\mathcal{R}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

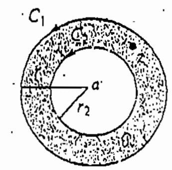

Fig. 6.5

where

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw \qquad n = 0, 1, 2, \ldots$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\, dw \qquad n = 1, 2, 3, \ldots$$

Solution By Cauchy's integral formula [see Problem 5.32, Page 5.19], we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z}\, dw \tag{1}$$

Consider the first integral in (1). As in Problem 6.22, equation (2), we have

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(w-a)\{1-(z-a)/(w-a)\}} \\ &= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{w-z} \end{aligned} \tag{2}$$

so that

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a}\, dw + \frac{z-a}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^2}\, dw \\ &\quad + \cdots + \frac{(z-a)^{n-1}}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^n}\, dw + U_n \\ &= a_0 + a_1(z-a) + \ldots + a_{n-1}(z-a)^{n-1} + U_n \end{aligned} \tag{3}$$

where

$$a_0 = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a}\, dw,\ a_1 = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^2}\, dw, \ldots, a_{n-1} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^n}\, dw$$

and

$$U_n = \frac{1}{2\pi i} \oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\, dw$$

Let us now consider the second integral in (1). We have on interchanging w and z in (2),

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(z-a)\{1-(w-a)/(z-a)\}} \\ &= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \left(\frac{w-a}{z-a}\right)^n \frac{1}{z-w} \end{aligned}$$

so that

$$\begin{aligned} -\frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{z-a}\, dw + \frac{1}{2\pi i} \oint_{C_2} \frac{w-a}{(z-a)^2} f(w)\, dw \\ &\quad + \cdots + \frac{1}{2\pi i} \oint_{C_2} \frac{(w-a)^{n-1}}{(z-a)^n} f(w)\, dw + V_n \end{aligned}$$

$$= \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + V_n \qquad (4)$$

where

$$a_{-1} = \frac{1}{2\pi i}\oint_{C_2} f(w)\,dw,\ a_{-2} = \frac{1}{2\pi i}\oint_{C_2} (w-a)\,f(w)\,dw, \cdots, a_{n-1} = \frac{1}{2\pi i}\oint_{C_2} (w-a)^{n-1} f(w)\,dw$$

and

$$V_n = \frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw$$

From (1), (3) and (4), we have

$$f(z) = \{a_0 + a_1(z-a) + \ldots + a_{n-1}(z-a)^{n-1}\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}\right\} + U_n + V_n \qquad (5)$$

The required result follows if we can show that (a) $\lim_{n\to\infty} U_n = 0$ and (b) $\lim_{n\to\infty} V_n = 0$. The proof of (a) follows from Problem 6.22. To prove (b), we first note that since w is on C_2,

$$\left|\frac{w-a}{z-a}\right| = \kappa < 1$$

where κ is a constant. Also we have $|f(w)| < M$ where M is a constant and

$$|z-w| = |(z-a)-(w-a)| \geq |z-a| - r_2$$

Hence, from Property e, Page 4.2, we have

$$|V_n| = \frac{1}{2\pi}\left|\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right|$$

$$\leq \frac{1}{2\pi}\frac{\kappa^n M}{|z-a|-r_2}2\pi r_2 = \frac{\kappa^n M r_2}{|z-a|-r_2}$$

Then $\lim_{n\to\infty} V_n = 0$ and the proof is complete.

6.26 Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$. (b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$. (c) $\dfrac{z-\sin z}{z^3}$; $z=0$.

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$. (e) $\dfrac{1}{z^2(z-3)^2}$; $z=3$.

Solution

(a) Let $z-1=u$. Then $z=1+u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z=1$ is a *pole of order* 3, or *triple pole*.
The series converges for all values of $z \neq 1$.

(b) Let $z+2=u$ or $z=u-2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} - \cdots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!u^3} + \frac{1}{5!u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

$z=-2$ is an *essential singularity*.
The series converges for all values of $z \neq -2$.

(c)

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

$z=0$ is a *removable singularity*.
The series converges for all values of z.

(d) Let $z+2=u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u}\cdot\frac{1}{1-u} = \left(\frac{2-u}{u}\right)(1+u+u^2+u^3+\cdots)$$

$$= \frac{2}{u} + 1 + u + u^2 + \cdots = \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

$z=-2$ is a *pole of order* 1, or *simple pole*.
The series converges for all values of z such that $0 < |z+2| < 1$.

(e) Let $z-3=u$. Then by the binomial theorem,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\}$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots$$

$z=3$ is a *pole of order* 2 or *double pole*.
The series converges for all values of z such that $0 < |z-3| < 3$.

6.27 Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for: (a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

Solution

(a) Resolving into partial fractions,

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e. $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series.*

LAGRANGE'S EXPANSION

6.28 Prove Lagrange's expansion on Page 190.

Solution Let us assume that C is taken so that there in only one simple zero of $z = a + \zeta\phi(z)$ inside C. Then, from Problem 6.99, Page 6.34, with $g(z) = z$ and $f(z) = z - a - \zeta\phi(z)$, we have

$$z = \frac{1}{2\pi i}\oint_C w\left\{\frac{1-\zeta\phi'(w)}{w - a - \zeta\phi(w)}\right\}dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1-\zeta\phi'(w)\}\left\{\frac{1}{1-\zeta\phi(w)/(w-a)}\right\}dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1-\zeta\phi'(w)\}\left\{\sum_{n=0}^{\infty}\zeta^n\phi^n(w)/(w-a)^n\right\}dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}dw + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C\left\{\frac{w\phi^n(w)}{(w-a)^{n+1}} - \frac{w\phi^{n-1}(w)\phi'(w)}{(w-a)^n}\right\}dw$$

$$= a - \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C \frac{w}{n}\frac{d}{dw}\left\{\frac{\phi^n(w)}{(w-a)^n}\right\}dw$$

$$= a + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i n}\oint_C \frac{\phi^n(w)}{(w-a)^n}dw$$

$$= a + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\frac{d^{n-1}}{da^{n-1}}[\phi^n(a)]$$

6.29 Show that the root of the equation $z = 1 + \xi z^s$, which is equal to 1 when $\xi = 0$ is given by

$$z = 1 + \xi + \frac{\xi^2}{2!}2s + \frac{\xi^3}{3!}3s(3s-1) + \frac{\xi^4}{4!}4s(4s-1)(4s-2) + \ldots$$

Solution Applying Lagrange's expansion, we have

$$z = a + \xi f(z) = a + \sum_{n=1}^{\infty}\frac{\xi^n}{n!}\frac{d^{n-1}}{da^{n-1}}[f(a)]^n$$

We obtain for $f(z) = z^s$

$$z = a + \sum_{n=1}^{\infty}\frac{\xi^n}{n!}\frac{d^{n-1}}{da^{n-1}}[a^{ns}] \quad (1)$$

For $n = 1, 2, 3, \ldots$, we have $\dfrac{d}{da}(a^{2s}) = 2s$, $\dfrac{d^2}{da^2}(a^{3s}) = 3s(3s-1)$,

$$\frac{d^3}{da^3}(a^{4s}) = 4s(4s-1)(4s-2)\dots \text{ etc.}$$

Hence, (1) becomes

$$z = 1 + \xi + \frac{\xi^2}{\underline{|2}}2s + \frac{\xi^3}{\underline{|3}}3s(3s-1) + \frac{\xi^4}{\underline{|4}}4s(4s-1)(4s-2) + \dots$$

ANALYTIC CONTINUATION

6.30 Show that the series (a) $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and (b) $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuatios of each other.

Solution

(a) By the ratio test, the series converges for $|z| < 2$ (shaded in Fig. 6.6). In this circle the series (which is a geometric series with first term $\frac{1}{2}$ and ratio $z/2$) can be summed and represents the function $\frac{1/2}{1-z/2} = \frac{1}{2-z}$.

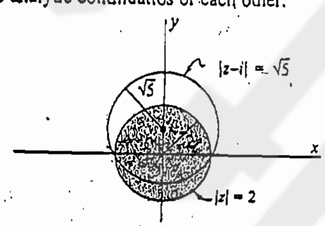

Fig. 6.6

(b) By the ratio test, the series converges for $\left|\frac{z-i}{2-i}\right| < 1$, i.e. $|z-i| < \sqrt{5}$, [see Fig. 6.6]. In this circle the series [which is a geometric series with first term $1/(2-i)$ and ratio $(z-i)/(2-i)$] can be summed and represents the function $\frac{1/(2-i)}{1-(z-i)/(2-i)} = \frac{1}{2-z}$.

Since the power series represent the same function in the regions common to the interiors of the circles $|z| = 2$ and $|z-i| = \sqrt{5}$, it follows that they are analytic continuations of each other.

6.31 Prove that the series $1 + z + z^2 + z^4 + z^8 + \dots = 1 + \sum_{n=0}^{\infty} z^{2^n}$ cannot be continued analytically beyond $|z| = 1$.

Solution Let $F(z) = 1 + z + z^2 + z^4 + z^8 + \dots$. Then $F(z) = z + F(z^2)$, $F(z) = z + z^2 + F(z^4)$, $F(z) = z + z^2 + z^4 + F(z^8)$, ...

From these it is clear that the values of z given by $z = 1$, $z^2 = 1$, $z^4 = 1$, $z^8 = 1$, ... are all singularities of $F(z)$. These singularities all lie on the circle $|z| = 1$. Given any small arc of this circle, there will be infinitely many such singularities. These represent an impassable barrier and analytic continuation beyond $|z| = 1$ is therefore impossible. The circle $|z| = 1$ constitutes a *natural boundary*.

MISCELLANEOUS PROBLEMS

6.32 Let $\{f_k(z)\}$, $k = 1, 2, 3, \dots$ be a sequence of functions analytic in a region $\mathcal{R}$. Suppose that

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

is uniformly convergent in $\mathcal{R}$. Prove that $F(z)$ is analytic in $\mathcal{R}$.

Solution Let $S_n(z) = \sum_{k=1}^{n} f_k(z)$. By definition of uniform convergence, given any $\epsilon > 0$ we can find a positive integer N depending on ϵ and not on z such that for all z in $\mathcal{R}$

$$|F(z) - S_n(z)| < \epsilon \quad \text{for all} \quad n > N \tag{1}$$

Now suppose that C is any simple-closed curve lying entirely in $\mathcal{R}$ and denote its length by L. Then, by Problem 6.16, since $f_k(z)$, $k = 1, 2, 3, \dots$ are continuous, $F(z)$ is also continuous so that $\oint_C F(z)\,dz$ exists. Also, using (1), we see that for $n > N$,

$$\left|\oint_C F(z)dz - \sum_{k=1}^{n}\oint_C f_k(z)dz\right| = \left|\oint_C \{F(z) - S_n(z)\}dz\right|$$

Because ϵ can be made as small as we please, we see that

$$\oint_C F(z)dz = \sum_{k=1}^{\infty}\oint_C f_k(z)dz$$

But, by Cauchy's theorem, $\oint_C f_k(z)\,dz = 0$. Hence

$$\oint_C F(z)\,dz = 0$$

and so by Morera's theorem (Page 5.1, Chapter 5), $F(z)$ must be analytic.

6.33 Prove that an analytic function cannot be bounded in the neighbourhood of an isolated singularity.

Solution Let $f(z)$ be analytic inside and on a circle C of radius r, except at the isolated singularity $z = a$ taken to be the centre of C. Then by Laurent's theorem $f(z)$ has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-a)^k \tag{1}$$

where the coefficients a_k are given by equation (6.7), page 6.7. In particular,

$$a_{-n} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{-n+1}}dz \quad n = 1, 2, 3, \dots \tag{2}$$

Now, if $|f(z)| < M$ for a constant M, i.e., if $f(z)$ is bounded, then from (2),

$$|a_{-n}| = \frac{1}{2\pi}\left|\oint_C (z-a)^{n-1} f(z)\,dz\right| \le \frac{1}{2\pi} r^{n-1} \cdot M \cdot 2\pi r = Mr^n$$

Hence since r can be made arbitrarily small, we have $a_{-n} = 0$, $n = 1, 2, 3, \dots$, i.e. $a_{-1} = a_{-2} = a_{-3} = \dots = 0$, and the Laurent series reduces to a Taylor series about $z = a$. This shows that $f(z)$ is analytic at $z = a$ so that $z = a$ is not a singularity, contrary to hypothesis. This contradiction shows that $f(z)$ cannot be bounded in the neighbourhood of an isolated singularity.

6.34 Prove that if $z \ne 0$, then

$$e^{\frac{1}{2}\alpha(z - 1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha)\, z_n$$

where

$$J_n(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \alpha\sin\theta)\,d\theta \qquad n = 0, 1, 2, \dots$$

Solution The point $z = 0$ is the only finite singularity of the function $e^{\frac{1}{2}\alpha(z-1/z)}$ and it follows that the function must have a Laurent series expansion of the form

$$e^{\frac{1}{2}\alpha(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha) z^n \tag{1}$$

which holds for $|z| > 0$. By equation (6.7), Page 6.7, the coefficients $J_n(\alpha)$ are given by

$$J_n(\alpha) = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{1}{2}\alpha(z-1/z)}}{z^{n+1}} dz \tag{2}$$

where C is any simple closed curve having $z = 0$ inside.

Let us in particular choose C to be a circle of radius 1 having centre at the origin; i.e., the equation of C is $|z| = 1$ or $z = e^{i\theta}$. Then (2) becomes

$$J_n(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{1}{2}\alpha(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} i e^{i\theta} d\theta \tag{3}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\alpha \sin\theta - in\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(\alpha \sin\theta - n\theta)\, d\theta + \frac{i}{2\pi} \int_0^{2\pi} \sin(\alpha \sin\theta - n\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - \alpha \sin\theta)\, d\theta$$

using the fact that $I = \int_0^{2\pi} \sin(\alpha \sin\theta - n\theta)\, d\theta = 0$. This last result follows since on letting $\theta = 2\pi - \phi$, we find

$$I = \int_0^{2\pi} \sin(-\alpha \sin\phi - 2\pi n + n\phi)\, d\phi = -\int_0^{2\pi} \sin(\alpha \sin\phi - n\phi)\, d\phi = -I$$

so that $I = -I$ and $I = 0$. The required result is thus established.

The function $J_n(\alpha)$ is called a *Bessel function* of the first kind of order n.

For further discussion of Bessel functions, see Chapter 10.

6.35 The *Legendre polynomials* $P_n(t)$, $n = 0, 1, 2, 3, \ldots$ are defined by *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

(a) Prove that if C is any simple closed curve enclosing the point $z = t$, then

$$P_n(t) = \frac{1}{2\pi i} \cdot \frac{1}{2^n} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz$$

This is called *Schlaefli's representation* for $P_n(t)$, or *Schlaefli's formula.*

(b) Prove that

$$P_n(t) = \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2 - 1} \cos\theta)^n d\theta$$

Solution

(a) By Cauchy's integral formulae, if C encloses point t,

$$f^{(n)}(t) = \frac{d^n}{dt^n} f(t) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-t)^{n+1}} dz$$

Then, taking $f(t) = (t^2 - 1)^n$ so that $f(z) = (z^2 - 1)^n$, we have the required result

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz$$

(b) Choose C as a circle with centre at t and radius $\sqrt{|t^2 - 1|}$ as shown in Fig. 6.7. Then an equation for C is $|z - t| = \sqrt{|t^2 - 1|}$ or $z = t + \sqrt{t^2 - 1}\, e^{i\theta}$, $0 \le \theta \le 2\pi$. Using this in part (a), we have

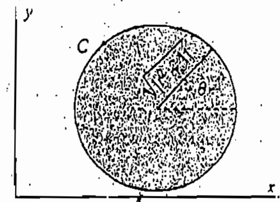

Fig. 6.7

$$P_n(t) = \frac{1}{2^n} \cdot \frac{1}{2\pi i} \int_0^{2\pi} \frac{\{(t + \sqrt{t^2 - 1}\, e^{i\theta})^2 - 1\}^n \sqrt{t^2 - 1}\, i e^{i\theta} d\theta}{(\sqrt{t^2 - 1}\, e^{i\theta})^{n+1}}$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{(t^2 - 1) + 2t\sqrt{t^2 - 1}\, e^{i\theta} + (t^2 - 1)e^{2i\theta}\}^n e^{-in\theta} d\theta}{(t^2 - 1)^{n/2}}$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{(t^2 - 1)e^{-i\theta} + 2t\sqrt{t^2 - 1} + (t^2 - 1)e^{i\theta}\}^n d\theta}{(t^2 - 1)^{n/2}}$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{2t\sqrt{t^2 - 1} + 2(t^2 - 1)\cos\theta\}^n d\theta}{(t^2 - 1)^{n/2}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2 - 1} \cos\theta)^n d\theta$$

For further discussion of Legendre polynomials, see Chapter 10.

SUPPLEMENTARY PROBLEMS

SEQUENCES AND SERIES OF FUNCTIONS

6.36 Using the definition, prove: (a) $\lim_{n\to\infty} \frac{3n - 2z}{n + z} = 3$, (b) $\lim_{n\to\infty} \frac{nz}{n^2 + z^2} = 0$.

6.37 If $\lim_{n\to\infty} u_n(z) = U(z)$ and $\lim_{n\to\infty} v_n(z) = v(z)$, prove that (a) $\lim_{n\to\infty} \{u_n(z) \pm v_n(z)\} = U(z) \pm V(z)$. (b) $\lim_{n\to\infty} \{u_n(z)\, v_n(z)\} = U(z)\, V(z)$, (c) $\lim_{n\to\infty} u_n(z)/v_n(z) = U(z)/V(z)$ if $V(z) \ne 0$.

6.38 (a) Prove that the series $\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots = \sum_{n=1}^{\infty} \frac{z^{n-1}}{2^n}$ converges for $|z| < 2$ and (b) find its sum.

6.39 (a) Determine the set of values of z for which the series $\sum_{n=0}^{\infty}(-1)^n(z^n + z^{n+1})$ converges and (b) find its sum.

6.40 (a) For what values of z does the series $\sum_{n=1}^{\infty}\frac{1}{(z^2+1)^n}$ converge and (b) what is its sum?

6.41 If $\lim_{n\to\infty}|u_n(z)| = 0$, prove that $\lim_{n\to\infty} u_n(z) = 0$. Is the converse true? Justify your answer.

6.42 Prove that for all finite z, $\lim_{n\to\infty} z^n/n! = 0$.

6.43 Let $\{a_n\}$, $n = 1, 2, 3, \ldots$ be a sequence of positive numbers having zero as a limit. Suppose that $|u_n(z)| \le a_n$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} u_n(z) = 0$.

6.44 Prove that the convergence or divergence of a series is not affected by adding (or removing) a finite number of terms.

6.45 Let $S_n = z + 2z^2 + 3z^3 + \ldots + nz^n$, $T_n = z + z^2 + z^3 + \ldots + z^n$. (a) Show that $S_n = (T_n - nz^{n+1})/(1 - z)$. (b) single Use (a) to find the sum of the series $\sum_{n=1}^{\infty} nz^n$ and determine the set of values for which the series converges.

6.46 Find the sum of the series $\sum_{n=0}^{\infty}\frac{n+1}{2^n}$.

ABSOLUTE AND UNIFORM CONVERGENCE

6.47 (a) Prove that $u_n(z) = 3z + 4z^2/n$, $n = 1, 2, 3, \ldots$, converges uniformly to $3z$ for all z inside or on the circle $|z| = 1$. (b) Can the circle of part (a) be enlarged? Explain.

6.48 (a) Determine whether the sequence $u_n(z) = nz/(n^2 + z^2)$ [Problem 6.36(b)] converges uniformly to zero for all z inside $|z| = 3$. (b) Does the result of (a) hold for all finite values of z?

6.49 Prove that the series $1 + az + a^2z^2 + \ldots$ converges uniformly to $1/(1 - az)$ inside or on the circle $|z| = R$ where $R < 1/|a|$.

6.50 Investigate the (a) absolute and (b) uniform convergence of the series

$$\frac{z}{3} + \frac{z(3-z)}{3^2} + \frac{z(3-z)^2}{3^3} + \frac{z(3-z)^3}{3^4} + \cdots$$

6.51 Investigate the (a) absolute and (b) uniform convergence of the series in Problem 6.39.

6.52 Investigate the (a) absolute and (b) uniform convergence of the series in Problem 6.40.

6.53 Let $\{a_n\}$ be a sequence of positive constants having limit zero; and suppose that for all z in a region $\mathcal{R}$, $|u_n(z)| \le a_n$, $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} u_n(z) = 0$ uniformly in $\mathcal{R}$.

6.54 (a) Prove that the sequence $u_n(z) = nze^{-nz^2}$ converges to zero for all finite z such that $\mathrm{Re}\{z^2\} > 0$, and represent this region geometrically. (b) Discuss the uniform convergence of the sequence in (a).

6.55 If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, prove that $\sum_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0$, converges absolutely.

6.56 Prove that if each of two series is absolutely and uniformly convergent in $\mathcal{R}$, their product is absolutely and uniformly convergent in $\mathcal{R}$.

SPECIAL CONVERGENCE TESTS

6.57 Test for convergence:

(a) $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$, (b) $\sum_{n=1}^{\infty}\frac{n}{3^n-1}$, (c) $\sum_{n=1}^{\infty}\frac{n+3}{3n^2-n+2}$,

(d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{4n+3}$, (e) $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n+2}}$.

6.58 Investigate the convergence of:

(a) $\sum_{n=1}^{\infty}\frac{1}{n+|z|}$, (b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+|z|}$, (c) $\sum_{n=1}^{\infty}\frac{1}{n^2+|z|}$, (d) $\sum_{n=1}^{\infty}\frac{1}{n^2+z}$.

6.59 Investigate the convergence of $\sum_{n=0}^{\infty}\frac{ne^{n\pi i/4}}{e^n-1}$.

6.60 Find the region of convergence of (a) $\sum_{n=0}^{\infty}\frac{(z+i)^n}{(n+1)(n+2)}$, (b) $\sum_{n=1}^{\infty}\frac{1}{n^2\cdot 3^n}\left(\frac{z+1}{z-1}\right)^n$, (c) $\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n}}{n!}$.

6.61 Investigate the region of absolute convergence of $\sum_{n=1}^{\infty}\frac{n(-1)^n(z-i)^n}{4^n(n^2+1)^{5/2}}$.

6.62 Find the region of convergence of $\sum_{n=0}^{\infty}\frac{e^{2\pi i n z}}{(n+1)^{3/2}}$.

6.63 Prove that the series $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$ diverges although the nth term approaches zero.

6.64 Let N be a positive integer and suppose that for all $n > N$, $|u_n| > 1/(n \ln n)$. Prove that $\sum_{n=1}^{\infty} u_n$ diverges.

6.65 Establish the validity of the (a) nth root test [Theorem 6.12], (b) integral test [Theorem 6.13], on Page 6.4.

6.66 Find the interval of convergence of $1 + 2z + z^2 + 2z^3 + z^4 + 2z^5 + \ldots$.

6.67 Prove Raabe's test (Theorem 6.14) on Page 6.4.

6.68 Test for convergence: (a) $\frac{1}{2\ln^2 2} + \frac{1}{3\ln^2 3} + \frac{1}{4\ln^2 4} + \cdots$, (b) $\frac{1}{5} + \frac{1\cdot 4}{5\cdot 8} + \frac{1\cdot 4\cdot 7}{5\cdot 8\cdot 11} + \cdots$, (c) $\frac{2}{5} + \frac{2\cdot 7}{5\cdot 10} + \frac{2\cdot 7\cdot 12}{5\cdot 10\cdot 15} + \cdots$, (d) $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \cdots$.

THEOREMS ON UNIFORM CONVERGENCE AND POWER SERIES

6.69 Determine the regions in which each of the following series is uniformly convergent:

(a) $\sum_{n=1}^{\infty} \frac{z^n}{3^n+1}$, (b) $\sum_{n=1}^{\infty} \frac{(z-i)^{2n}}{n^2}$, (c) $\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}$, (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+|z|^2}$

6.70 Prove Theorem 6.20, Page 6.5.

6.71 State and prove theorems for sequences analogous to Theorems 6.18, 6.19 and 6.20, Page 6.5, for series.

6.72 (a) By differentiating both sides of the identity

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \qquad |z| < 1$$

find the sum of the series $\sum_{n=1}^{\infty} nz^n$ for $|z| < 1$. Justify all steps.

(b) Find the sum of the series $\sum_{n=1}^{\infty} n^2 z^n$ for $|z| < 1$.

6.73 Let z be real and such that $0 \le z \le 1$, and let $u_n(z) = nze^{-nz^2}$. (a) Find $\lim_{n\to\infty} \int_0^1 u_n(z)\,dz$. (b) Find $\int_0^1 \left\{\lim_{n\to\infty} u_n(z)\right\} dz$. (c) Explain why the answers to (a) and (b) are not equal. [See Problem 6.54.]

6.74 Prove Abel's theorem [Theorem 6.24, Page 6.5].

6.75 (a) Prove that $\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots$ for $|z| < 1$.

(b) If we choose that branch of $f(z) = \tan^{-1} z$ such that $f(0) = 0$, use (a) to prove that

$$\tan^{-1} z = \int_0^z \frac{dz}{1+z^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

(c) Prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

6.76 Prove Theorem 6.25, Page 6.5.

6.77 (a) Determine $Y(z) = \sum_{n=0}^{\infty} a_n z^n$ such that for all z in $|z| \le 1$, $Y'(z) = Y(z)$, $Y(0) = 1$. State all theorems used and verify that the result obtained is a solution.

(b) Is the result obtained in (a) valid outside of $|z| \le 1$? Justify your answer.

(c) Show that $Y(z) = e^z$ satisfies the differential equation and condition in (a).

(d) Can we identify the series in (a) with e^z? Explain.

6.78 (a) Use series methods on the differential equation $Y''(z) + Y(z) = 0$, $Y(0) = 0$, $Y'(0) = 1$ to obtain the series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(b) How could you obtain a corresponding series for cos z?

TAYLOR'S THEOREM

6.79 Expand each of the following functions in a Taylor series about the indicated point and determine the region of convergence in each case.
(a) e^{-z}; $z = 0$ (b) $\cos z$; $z = \pi/2$ (c) $1/(1+z)$; $z = 1$ (d) $z^3 - 3z^2 + 4z - 2$; $z = 2$ (e) ze^{2z}; $z = -1$

6.80 If each of the following functions were expanded into a Taylor series about the indicated points, what would be the region of convergence? Do not perform the expansion.
(a) $\sin z/(z^2+4)$; $z = 0$ (b) $z/(e^z+1)$; $z = 0$ (c) $(z+3)/(z-1)(z-4)$; $z = 2$
(d) $e^{-z^2} \sinh(z+2)$; $z = 0$ (e) $e^z/z(z-1)$; $z = 4i$ (f) $\coth 2z$; $z = 0$
(g) $\sec \pi z$; $z = 1$

6.81 Verify the expansions 1, 2, 3, for e^z, $\sin z$, and $\cos z$ on Page 143.

6.82 Show that $\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots, \quad |z| < \infty$.

6.83 Prove that $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots, \quad |z| < 1$.

6.84 Show that: (a) $\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots, \quad |z| < \pi/2$

(b) $\sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \cdots, \quad |z| < \pi/2$

(c) $\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots, \quad 0 < |z| < \pi$

6.85 By replacing z by iz in the expansion of Problem 6.83, obtain the result in Problem 6.23(c) on Page 6.19.

6.86 How would you obtain series for (a) tanh z, (b) sech z, (c) csch z from the series in Problem 6.84?

6.87 Prove the uniqueness of the Taylor series expansion of $f(z)$ about $z = a$.

[*Hint.* Assume $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} d_n (z-a)^n$ and show that $c_n = d_n$, $n = 0, 1, 2, 3, \ldots$]

6.88 Prove the binomial Theoem 6.6 on Page 6.3.

6.89 If we choose that branch of $\sqrt{1+z^3}$ having the value 1 for $z = 0$, show that

$$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{2}z^3 + \frac{1\cdot 3}{2\cdot 4}z^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^9 + \cdots \qquad |z| < 1$$

6.90 (a) Choosing that branch of $\sin^{-1} z$ having the value zero for $z = 0$, show that

$$\sin^{-1} z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \cdots \qquad |z| < 1$$

(b) Prove that the result in (a) is valid for $z = i$.

6.91 (a) Expand $f(z) = \ln(3 - iz)$ in powers of $z - 2i$, choosing that branch of the logarithm for which $f(0) = \ln 3$, and (b) determine the region of convergence.

LAURENT'S THEOREM

6.92 Expand $f(z) = 1/(z-3)$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

6.93 Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for:
(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z - 1| > 1$, (c) $0 < |z - 2| < 1$.

6.94 Expand $f(z) = 1/z(z-2)$ in a Laurent series valid for (a) $0 < |z| < 2$, (b) $|z| > 2$.
6.95 Find an expansion of $f(z) = z/(z^2+1)$ valid for $|z-3| > 2$.
6.96 Expand $f(z) = 1/(z-2)^2$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$.
6.97 Expand each of the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case.

(a) $(1-\cos z)/z$, (b) e^{z^2}/z^3, (c) $z^{-1}\cosh z^{-1}$, (d) $z^2 e^{-z^4}$, (e) $z \sinh \sqrt{z}$.

6.98 Show that if $\tan z$ is expanded into a Laurent series about $z = \pi/2$, (a) the principal part is $-1/(z-\pi/2)$, (b) the series converges for $0 < |z-\pi/2| < \pi/2$, (c) $z = \pi/2$ is a simple pole.
6.99 Determine and classify all the singularities of the functions:
(a) $1/(2\sin z - 1)^2$, (b) $z/(e^{1/z}-1)$, (c) $\cos(z^2+z^{-2})$, (d) $\tan^{-1}(z^2+2z+2)$, (e) $z/(e^z-1)$.
6.100 (a) Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and (b) determine the region of convergence of this series. (c) Classify the singularities of $f(z)$.
6.101 Establish the result 6.7, page 6.7, for the coefficients in a Laurent series.
6.102 Prove that the only singularities of a rational function are poles.
6.103 Prove the converse of Problem 6.102, i.e., if the only singularities of a function are poles, the function must be rational.

LAGRANGE'S EXPANSION

6.104 Show that the root of the equation $z = 1 + \zeta z^p$, which is equal to 1 when $\zeta = 0$, is given by

$$z = 1 + \zeta + \frac{2p}{2!}\zeta^2 + \frac{(3p)(3p-1)}{3!}\zeta^3 + \frac{(4p)(4p-1)(4p-2)}{4!}\zeta^4 + \cdots$$

6.105 Calculate the root in Problem 6.104 if $p = 1/2$ and $\zeta = 1$, (a) by series and (b) exactly, and compare the two answers.

6.106 By considering the equation $z = \alpha + \frac{1}{2}\zeta(z^2-1)$, show that

$$\frac{1}{\sqrt{1-2a\zeta+\zeta^2}} = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{2^n n!} \frac{d^n}{da^n}(a^2-1)^n$$

6.107 Show how Lagrange's expansion can be used to solve Kepler's problem of determining that root of $z = a + \zeta \sin z$ for which $z = a$ when $\zeta = 0$
6.108 Prove the Lagrange expansion 6.11 on Page 6.9.

ANALYTIC CONTINUATION

6.109 (a) Prove that $F_2(z) = \dfrac{1}{1+i}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of $F_1(z) = \displaystyle\sum_{n=0}^{\infty} z^n$, showing graphically the regions of convergence of the series.
(b) Determine the function represented by all analytic continuations of $F_1(z)$.

6.110 Let $F_1(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$.
(a) Find an analytic continuation of $F_1(z)$, which converges for $z = 3-4i$.

(b) Determine the value of the analytic continuation in (a) for $z = 3-4i$.
6.111 Prove that the series $z^{1!} + z^{2!} + z^{3!} + \ldots$ has the natural boundary $|z| = 1$.

MISCELLANEOUS PROBLEMS

6.112 (a) Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges if the constant $p \le 1$.
(b) Prove that if p is complex the series in (a) converges if Re $\{p\} > 1$.
(c) Investigate the convergence or divergence of the series in (a) if Re $\{p\} \le 1$.
6.113 Test for convergence or divergence:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+i}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sin^2 n}{ie^n+(2-i)}n$ (c) $\displaystyle\sum_{n=1}^{\infty} n\sin^{-1}(1/n^3)$

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{(i)^n}{n\ln n}$ (e) $\displaystyle\sum_{n=1}^{\infty}\coth^{-1} n$ (f) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

6.114 Euler presented the following argument to show that $\displaystyle\sum_{-\infty}^{\infty} z^n = 0$:

$$\frac{z}{1-z} = z + z^2 + z^3 + \cdots = \sum_{1}^{\infty} z^n, \quad \frac{z}{z-1} = \frac{1}{1-1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{0}^{-\infty} z^n$$

Then adding, $\displaystyle\sum_{-\infty}^{\infty} z^n = 0$. Explain the fallacy.

6.115 Show that for $|z-1| < 1$, $z\ln z = (z-1) + \dfrac{(z-1)^2}{1\cdot 2} - \dfrac{(z-1)^3}{2\cdot 3} + \dfrac{(z-1)^4}{3\cdot 4} - \ldots$.
6.116 Expand $\sin^3 z$ in a Maclaurin series.
6.117 Given the series $z^2 + \dfrac{z^2}{1+z^2} + \dfrac{z^2}{(1+z^2)^2} + \dfrac{z^2}{(1+z^2)^3} + \cdots$;

(a) Show that the sum of the first n terms is $S_n(z) = 1 + z^2 - 1/(1+z^2)^{n-1}$.
(b) Show that the sum of the series is $1 + z^2$ for $z \ne 0$, and 0 for $z = 0$; and hence that $z = 0$ is a point of discontinuity.
(c) Show that the series is not uniformly convergent in the region $|z| \le \delta$ where $\delta > 0$.

6.118 If $F(z) = \dfrac{3z-3}{(2z-1)(z-2)}$, find a Laurent series of $F(z)$ about $z = 1$ convergent for $\frac{1}{2} < |z-1| < 1$.
6.119 Let $G(z) = (\tan^{-1} z)/z^4$. (a) Expand $G(z)$ in a Laurent series. (b) Determine the region of convergence of the series in (a). (c) Evaluate $\oint_C G(z)\,dz$ where C is a square with vertices at $2 \pm 2i$, $-2 \pm 2i$.
6.120 For each of the functions ze^{1/z^2}, $(\sin^2 z)/z$, $1/z(4-z)$ which have singularities at $z = 0$: (a) give a Laurent expansion about $z = 0$ and determine the region of convergence; (b) state in each case whether $z = 0$ is a removable singularity, essential singularity or a pole; (c) evaluate the integral of the function about the circle $|z| = 2$.

6.121 (a) Investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$. (b) Does your answer to (a) contradict Problem 6.8,

Page 6.12

6.122 (a) Show that the series $\frac{\sin z}{1^2+1} + \frac{\sin^2 z}{2^2+1} + \frac{\sin^3 z}{3^2+1} + \cdots$, where $z = x + iy$, converges absolutely in the region bounded by $\sin^2 x + \sinh^2 y = 1$. (b) Graph the region of (a).

6.123 If $|z| > 0$, prove that

$$\cosh(z + 1/z) = c_0 + c_1(z + 1/z) + c_2(z^2 + 1/z^2) + \ldots$$

where $c_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\phi \cosh(2\cos\phi)\, d\phi$

6.124 If $f(z)$ has simple zeros at $1 - i$ and $1 + i$, ~~double poles~~ at $-1 + i$ and $-1 - i$, but no other finite singularities, prove that the function must be given by

$$f(z) = \kappa \frac{z^2 - 2z + 2}{(z^2 + 2z + 2)^2}$$

where κ is an arbitrary constant.

6.125 Prove that for all z, $e^z \sin z = \sum_{n=1}^{\infty} \frac{2^{n/2}\sin(n\pi/4)}{n!} z^n$.

6.126 Show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, justifying all steps. [*Hint.* Use Problem 6.23.]

6.127 Investigate the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{z}{[1+(n-1)z][1+nz]}$.

[*Hint.* Resolve the nth term into partial fractions and show that the nth partial sum is $S_n(z) = 1 - \frac{1}{1+nz}$.]

6.128 Given $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to S, prove that the rearranged series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots = \tfrac{3}{2}S.$$

Explain.

[*Hint.* Take 1/2 of the first series and write it as $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$; then add term by term to the first series. Note that $S = \ln 2$, as shown in Problem 6.126.]

6.129 Prove that the *hypergeometric series*,

$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \cdots$$

(a) converges absolutely if $|z| < 1$, (b) diverges for $|z| > 1$, (c) converges absolutely for $z = 1$ if Re $(a + b - c) < 0$, (d) satisfies the differential equation $z(1 - z)Y'' + \{c - (a + b + 1)z\}Y' - abY = 0$.

6.130 Prove that for $|z| < 1$,

$$(\sin^{-1} z)^2 = z^2 + \frac{2}{3} \cdot \frac{z^4}{2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{z^6}{3} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{z^8}{4} + \cdots$$

6.131 Prove that $\sum_{n=1}^{\infty} \frac{1}{n^{1+i}}$ diverges.

6.132 Show that $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \cdots = 2\ln 2 - 1$.

6.133 Locate and name all the singularities of $\frac{z^6 + 1}{(z-1)^3(3z+2)^2} \sin\left(\frac{z^2}{z-3}\right)$.

6.134 By using only properties of infinite series, prove that

(a) $\left\{1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots\right\}\left\{1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots\right\} = \left\{1 + (a+b) + \frac{(a+b)^2}{2!} + \cdots\right\}$

(b) $\left\{1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots\right\}^2 + \left\{a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots\right\}^2 = 1$

6.135 If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ and $0 \le r < R$, prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

6.136 Use Problem 6.135 to prove Cauchy's inequality, namely

$$|f^{(n)}(0)| \le \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, \ldots$$

6.137 If a function has six zeros of order 4, and four poles of orders 3, 4, 7 and 8, but no other singularities in the finite plane, prove that it has a pole of order 2 at $z = \infty$.

6.138 State whether each of the following functions are entire, meromorphic or neither:

(a) $z^2 e^{-z}$, (b) $\cot 2z$, (c) $(1 - \cos z)/z$, (d) $\cosh z^2$, (e) $z \sin(1/z)$, (f) $z + 1/z$, (g) $\sin\sqrt{z}/\sqrt{z}$, (h) $\sqrt{\sin z}$.

6.139 If $-\pi < \theta < \pi$, prove that

$$\ln(2\cos\theta/2) = \cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \tfrac{1}{4}\cos 4\theta + \ldots$$

6.140 (a) Expand $1/\ln(1 + z)$ in a Laurent series about $z = 0$ and (b) determine the region of convergence.

6.141 If $S(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, prove that

$$\frac{S(z)}{1 - z} = a_0 + (a_0 + a_1)z + (a_0 + a_1 + a_2)z^2 + \ldots$$

giving restrictions if any.

6.142 Show that the following series

$$\frac{1}{1+|z|} - \frac{1}{2+|z|} + \frac{1}{3+|z|} - \frac{1}{4+|z|} + \cdots$$

(a) is not absolutely convergent but (b) is uniformly convergent for all values of z.

6.143 Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at all points of $|z| \le 1$ except $z = 1$.

6.144 Prove that the solution of $z = a + \zeta e^z$, which has the value a when $\zeta = 0$, is given by

$$z = a + \sum_{n=1}^{\infty} \frac{n^{n-1} e^{na} \zeta^n}{n!}$$

if $|\zeta| < |e^{-(a+1)}|$.

6.145 Find the sum of the series $1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots$.

6.146 Let $F(z)$ be analytic in the finite plane and suppose that $F(z)$ has period 2π, i.e. $F(z + 2\pi) = F(z)$. Prove that

$$F(z) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inz} \quad \text{where} \quad \alpha_n = \frac{1}{2\pi}\int_0^{2\pi} F(z)\, e^{-inz}\, dz$$

The series is called the *Fourier series* for $F(z)$.

6.147 Prove that the series

$$\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \ldots$$

is equal to $\pi/4$ if $0 < \theta < \pi$, and to $-\pi/4$ if $-\pi < \theta < 0$.

6.148 Prove that $|z| = 1$ is a natural boundary for the series $\sum_{n=0}^{\infty} 2^{-n} z^{3^n}$.

6.149 If $f(z)$ is analytic and not identically zero in the region $0 < |z - z_0| < R$, and suppose $\lim_{z \to z_0} f(z) = 0$. Prove that there exists a positive integer n such that $f(z) = (z - z_0)^n g(z)$ where $g(z)$ is analytic at z_0 and different from zero.

6.150 If $f(z)$ is analytic in a deleted neighbourhood of z_0 and $\lim_{z \to z_0} |f(z)| = \infty$, prove that $z = z_0$ is a pole of $f(z)$.

6.151 Explain why Problem 6.150 does not hold for $f(x) = e^{1/x^2}$ where x is real.

6.152 (a) Show that the function $f(x) = e^{1/z}$ can assume any value except zero.

(b) Discuss the relationship of the result of (a) to the Casorati-Weierstrass theorem and Picard's theorem.

6.153 (a) Determine whether the function $g(z) = z^2 - 3z + 2$ can assume any complex value.

(b) Is there any relationship of the result in (a) to the theorems of Casorati–Weierstrass and Picard? Explain.

6.154 Prove the Casorati-Weierstrass theorem stated on Page 6.8. [*Hint.* Use the fact that if $z = a$ is an essential singularity of $f(z)$, then it is also an essential singularity of $1/\{f(z) - A\}$.]

6.155 (a) Prove that along any ray through $z = 0$, $|z + e^z| \to \infty$.

(b) Does the result in (a) contradict the Casorati-Weierstrass theorem?

6.156 (a) Prove that an entire function $f(z)$ can assume any value whatsoever, with perhaps one exception.

(b) Illustrate the result of (a) by considering $f(z) = e^z$ and stating the exception in this case.

(c) What is the relationship of the result to the Casorati-Weierstrass and Picard theorems?

6.157 Prove that every entire function has a singularity at infinity. What type of singularity must this be? Justify your answer.

6.158 Prove that:

(a) $\dfrac{\ln(1+z)}{1+z} = z - \left(1 + \dfrac{1}{2}\right)z^2 + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)z^3 - \cdots, \quad |z| < 1$

(b) $\{\ln(1+z)\}^2 = z^2 - \left(1 + \dfrac{1}{2}\right)\dfrac{2z^3}{3} + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)\dfrac{2z^4}{4} - \cdots, \quad |z| < 1$

6.159 Find the sum of the following series if $|a| < 1$:

(a) $\sum_{n=1}^{\infty} n a^n \sin n\theta$, (b) $\sum_{n=1}^{\infty} n^2 a^n \sin n\theta$

6.160 Show that $e^{\sin z} = 1 + z + \dfrac{z^2}{2} - \dfrac{z^4}{8} - \dfrac{z^5}{15} + \cdots, \quad |z| < \infty$.

6.161 (a) Show that $e^{\sin z} = \sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ converges for $|z| \le 1$.

(b) Show that the function $F(z)$, defined as the collection of all possible analytic continuations of the series in (a), has a singular point at $z = 1$.

(c) Reconcile the results of (a) and (b).

6.162 Let $\sum_{n=1}^{\infty} a_n z^n$ converge inside a circle of convergence or radius R. There is a theorem which states that the function $F(z)$ defined by the collection of all possible continuations of this series, has at least one singular point on the circle of convergence. (a) Illustrate the theorem by several examples. (b) Can you prove the theorem?

6.163 Show that

$$u(r, \theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{U(\phi)\, d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\cos n\phi\, d\phi, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\sin n\phi\, d\phi$$

6.164 Let $\dfrac{z}{e^z - 1} = 1 + B_1 z + \dfrac{B_2 z^2}{2!} + \dfrac{B_3 z^3}{3!} + \cdots$,

(a) Show that the numbers B_n, called the *Bernoulli numbers*, satisfy the recursion formula $(B + 1)^n = B^n$ where B^k is formally replaced by B_k after expanding.

(b) Using (a) or otherwise, determine $B_1, \ldots, B_6$.

6.165 (a) Prove that $\dfrac{z}{e^z - 1} = \dfrac{z}{2}\left(\coth\dfrac{z}{2} - 1\right)$.

(b) Use Problem 6.164 and part (a) to show that $B_{2k+1} = 0$ if $k = 1, 2, 3, \ldots$.

6.166 Derive the series expansions:

(a) $\coth z = \dfrac{1}{z} + \dfrac{z}{3} - \dfrac{z^3}{45} + \cdots + \dfrac{B_{2n}(2z)^{2n}}{(2n)!\,z} + \cdots, \qquad |z| < \pi$

(b) $\cot z = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} + \cdots (-1)^n \dfrac{B_{2n}(2z)^{2n}}{(2n)!\,z} + \cdots, \qquad |z| < \pi$

(c) $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots (-1)^{n-1} \dfrac{2(2^{2n}-1)B_{2n}(2z)^{2n-1}}{(2n)!}, \qquad |z| < \pi/2.$

(d) $\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots (-1)^{n-1} \dfrac{2(2^{2n-1}-1)B_{2n} z^{2n-1}}{(2n)!} + \cdots, \qquad |z| < \pi$

[*Hint.* For (a), use Problem 6.165, for (b) replace z by iz in (a); for (c) use $\tan z = \cot z - 2\cot 2z$; for (d) use $\csc z = \cot z + \tan z/2$.]

ANSWERS TO SUPPLEMENTARY PROBLEMS

6.38 (a) $S_n(z) = \{1 - (z/2)^n\}/(2 - z)$ and $\lim_{n\to\infty} S_n(z)$ exists if $|z| < 2$, (b) $S(z) = 1/(2 - z)$.

6.39 (a) $|z| < 1$, (b) 1

6.40 (a) All z such that $|z^2 + 1| > 1$, (b) $1/z^2$

6.45 (b) $z/(1 - z)^2$, $|z| < 1$

6.46 4

6.50 (a) Converges absolutely if $|z - 3| < 3$ or $z = 0$. (b) Converges uniformly for $|z - 3| \le R$ where $0 < R < 3$; does not converge uniformly in any neighbourhood which inclues $z = 0$.

6.51 (a) Converges absolutely if $|z| < 1$. (b) Converges uniformly if $|z| \le R$ where $R < 1$.

6.52 (a) Converges absolutely if $|z^2 + 1| > 1$. (b) Converges uniformly if $|z^2 + 1| \ge R$ where $R > 1$.

6.54 (b) Not uniformly convergent in any region which includes $z = 0$.

6.57 (a) conv., (b) conv., (c) div., (d) conv., (e) div.

6.58 (a) Diverges for all finite z. (b) Converges for all z. (c) Converges for all z. (d) Converges for all z except $z = -n^2$, $n = 1, 2, 3, \ldots$.

6.59 Conv.

6.60 (a) $|z + i| \le 1$, (b) $|(z + 1)/(z - 1)| \le 3$, (c) $|z| < \infty$

6.61 Conv. abs. for $|z - i| \le 4$.

6.62 Converges if $\text{Im } z \ge 0$.

6.66 $|z| < 1$.

6.68 (a) conv., (b) conv., (c) div., (d) div.

6.69 (a) $|z| \le R$ where $R < 3$, (b) $|z - i| \le 1$, (c) $|z| \ge R$ where $R > 1$, (d) All z

6.72 (a) $z/(1 - z)^2$ [compare Problem 6.45], (b) $(1 + z)/(1 - z)^3$

6.73 (a) 1/2, (b) 0

6.77 (a) $Y(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

6.80 (a) $|z| < 2$, (b) $|z| < \pi$, (c) $|z - 2| < 1$, (d) $|z| < \infty$, (e) $|z - 4i| < 4$, (f) $|z| < \pi/2$, (g) $|z - 1| < 1/2$

6.91 (a) $\ln 5 - \dfrac{i(z - 2i)}{5} + \dfrac{(z - 2i)^2}{2\cdot 5^2} + \dfrac{i(z - 2i)^3}{3\cdot 5^3} - \dfrac{(z - 2i)^4}{4\cdot 5^4} - \cdots$ (b) $|z - 2i| < 5$

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6.92 (a) $-\dfrac{1}{3} - \dfrac{1}{9}z - \dfrac{1}{27}z^2 - \dfrac{1}{81}z^3 - \cdots$ (b) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots$

6.93 (a) $-\dfrac{1}{2}z - \dfrac{3}{4}z^2 - \dfrac{7}{8}z^3 - \dfrac{15}{16}z^4 - \cdots$ (b) $\cdots + \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + \dfrac{1}{2}z + \dfrac{1}{4}z^2 + \dfrac{1}{8}z^3 + \cdots$

(c) $-\dfrac{1}{2} - \dfrac{3}{z^2} - \dfrac{7}{z^3} - \dfrac{15}{z^4} - \cdots$ (d) $-(z - 1)^{-1} - 2(z - 1)^{-2} - 2(z - 1)^{-3} - \cdots$

(e) $1 - 2(z - 2)^{-1} - (z - 2) + (z - 2)^2 - (z - 2)^3 + (z - 2)^4 - \ldots$

6.97 (a) $\dfrac{z}{2!} - \dfrac{z^3}{4!} + \dfrac{z^5}{6!} - \cdots$; removable singularity

(b) $\dfrac{1}{z^3} + \dfrac{1}{z} + \dfrac{z}{2!} + \dfrac{z^3}{3!} + \dfrac{z^5}{4!} + \dfrac{z^7}{5!} + \ldots$; pole of order 3

(c) $\dfrac{1}{z} - \dfrac{1}{2!z^3} + \dfrac{1}{4!z^5} - \cdots$; essential singularity

(d) $z^2 - z^6 + \dfrac{z^{10}}{2!} - \dfrac{z^{14}}{3!} + \cdots$; ordinary point

(e) $z^{3/2} + \dfrac{z^{5/2}}{3!} + \dfrac{z^{7/2}}{5!} + \dfrac{z^{9/2}}{7!} + \cdots$; branch point

6.99 (a) $\pi/6 + 2m\pi$, $(2m + 1)\pi - \pi/6$, $m = 0, \pm 1, \pm 2, \ldots$; poles of order 2
(b) $i/2m\pi$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; essential singularity, $z = \infty$; pole of order 2
(c) $z = 0, \infty$; essential singularities
(d) $z = -1 \pm i$; branch points
(e) $z = 2m\pi i$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; removable singularity, $z = \infty$; essential singularity

6.100 (a) $e\left\{1 + 2(z - 2)^{-1} + \dfrac{2^2(z - 2)^{-2}}{2!} + \dfrac{2^3(z - 2)^{-3}}{3!} + \cdots\right\}$, (b) $|z - 2| > 0$, (c) $z = 2$: essential singularity, $z = \infty$, removable singularity

6.105 2.62 to two decimal accuracy

6.109 (b) $1/(1 - z)$

6.110 (b) $-3 - \frac{9}{4}i$.

6.113 (a) div., (b) conv., (c) conv., (d) conv., (e) div., (f) conv.

6.116 $\displaystyle\sum_{n=1}^{\infty} \frac{(3 - 3^{2n-1})z^{2n-1}}{4(2n-1)!}$

6.118 $\cdots - \dfrac{1}{8}(z - 1)^{-4} + \dfrac{1}{4}(z - 1)^{-3} - \dfrac{1}{2}(z - 1)^{-2} + (z - 1)^{-1} - 1(z - 1) - (z - 1)^2 - \cdots$

6.119 (a) $\dfrac{1}{z^3} - \dfrac{1}{3z} + \dfrac{z}{5} - \dfrac{z^3}{7} + \cdots$ (b) $|z| > 0$ (c) $-1/3$

6.120 (a) $z + z^{-1} + z^{-3}/2! + z^{-5}/3! + \ldots$; $|z| > 0$, $2z - 2z^3/3 + 4z^5/45 - \ldots$; $|z| \ge 0$, $z^{-1}/4 + 1/16 + z/64 + z^2/256 + \ldots$; $0 < |z| < 4$
(b) essential singularity, removable singularity, pole (c) $2\pi i$, 0, $\pi i/2$

6.121 (a) diverges

6.127 Not uniformly convergent in any region which includes $z = 0$; uniformly convergent in a region $|z| \geq \delta$, where δ is any positive number.

6.138 (a) entire, (b) meromorphic, (c) entire, (d) entire, (e) neither, (f) meromorphic, (g) entire, (h) neither

6.140 (a) $\frac{1}{z} + \frac{z}{2} - \frac{z}{12} + \frac{z^2}{24} + \frac{89z^3}{720} + \cdots$ (b) $0 < |z| < 1$

6.145 $e^{\cos\theta}\cos(\sin\theta)$

6.164 (b) $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$

The Residue Theorem and Applications

RESIDUES

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z = a$ chosen as the center of C. Then, as we have seen in Chapter 6, $f(z)$ has a Laurent series about $z = a$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{7.1}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm 1, \pm 2, \ldots \tag{7.2}$$

In the special case $n = -1$, we have from (7.2)

$$\oint_C f(z)\,dz = 2\pi i a_{-1} \tag{7.3}$$

Formally, we can obtain (7.3) from (7.1) by integrating term by term and using the results (Problems 4.21 and 4.22)

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \\ 0 & p = \text{integer} \neq 1 \end{cases} \tag{7.4}$$

Because of the fact that (7.3) involves only the coefficent a_{-1} in (7.1), we call a_{-1} the *residue* of $f(z)$ at $z = a$.

CALCULATION OF RESIDUES

To obtain the residue of a function $f(z)$ at $z = a$, it may appear from (7.1) that the Laurent expansion of $f(z)$ about $z = a$ must be obtained. However, in the case where $z = a$ is a pole of order k, there is a simple formula for a_{-1} given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\{(z-a)^k f(z)\} \tag{7.5}$$

If $k = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \tag{7.6}$$

which is a special case of (7.5) with $k = 1$ if we define $0! = 1$.

Example 7.1 If $f(z) = z/(z-1)(z+1)^2$, then $z = 1$ and $z = -1$ are poles of orders one and two, respectively. We have, using (7.6) and (7.5) with $k = 2$,

$$\text{Residue at } z = 1 \text{ is } \lim_{z \to 1} (z-1)\left\{\frac{z}{(z-1)(z+1)^2}\right\} = \frac{1}{4}$$

$$\text{Residue at } z = -1 \text{ is } \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz}\left\{(z+1)^2 \left(\frac{z}{(z-1)(z+1)^2}\right)\right\} = -\frac{1}{4}$$

If $z = a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

Example 7.2 If $f(z) = e^{-1/z}$. Then $z = 0$ is an essential singularity and from the known expansion for e^u with $u = -1/z$, we find

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \cdots$$

from which we see that the residue at $z = 0$ is the coefficient of $1/z$ and equals -1.

THE RESIDUE THEOREM

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}, b_{-1}, c_{-1} \ldots$ [see Fig. 7.1].

Then the *residue theorem* states that

$$\oint_C f(z)\, dz = 2\pi i (a_{-1} + b_{-1} + c_{-1} + \ldots) \tag{7.7}$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C. Note that (7.7) is a generalization of (7.3). Cauchy's theorem and integral formulae are special cases of this theorem (see Problem 7.91).

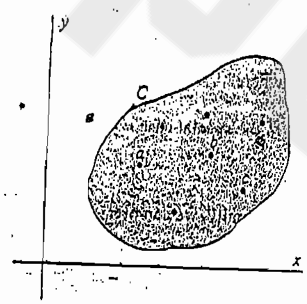


Fig. 7.1

EVALUATION OF DEFINITE INTEGRALS

The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour C, the choice of which may require great ingenuity. The following types are most common in practice.

1. $\int_{-\infty}^{\infty} F(x)dx$, where $F(x)$ is a rational function.

   Consider $\oint_C F(z)\, dz$ along a contour C consisting of the line along the x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as diameter [Fig. 7.2]. Then, let $R \to \infty$. If $F(x)$ is an even function, this can be used to evaluate $\int_0^{\infty} F(x)dx$. See Problems 7.7–7.10.

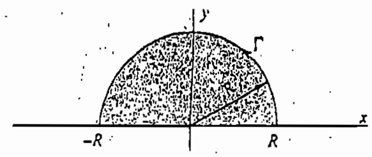


Fig. 7.2

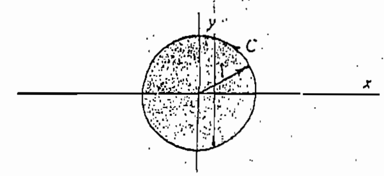


Fig. 7.3

2. $\int_0^{2\pi} G(\sin\theta, \cos\theta)d\theta$, where $G(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

   Let $z = e^{i\theta}$. Then $\sin\theta = (z - z^{-1})/2i$, $\cos\theta = (z + z^{-1})/2$ and $dz = ie^{i\theta}\, d\theta$ or $d\theta = dz/iz$. The given integral is equivalent to $\oint_C F(z)\, dz$ where C is the unit circle with center at the origin [Fig. 7.3]. See Problems 7.14–7.20.

3. $\int_{-\infty}^{\infty} F(x)\begin{Bmatrix}\cos mx \\ \sin mx\end{Bmatrix} dx$, where $F(x)$ is a rational function.

   Here, we consider $\oint_C F(z)e^{imz}\, dz$ where C is the same contour as that in Type 1. See Problems 7.21–7.26.

4. Miscellaneous integrals involving particular contours. See Problems 7.27–7.35.

SPECIAL THEOREMS USED IN EVALUATING INTEGRALS

In evaluating integrals such as those of Types 1 and 3 above, it is often necessary to show that $\int_\Gamma F(z)\, dz$ and $\int_\Gamma e^{imz} F(z)\, dz$ approach zero as $R \to \infty$. The following theorems are fundamental.

Theorem 7.1 Statement: Let AB be the arc $\alpha \le \theta \le \beta$ of the circle $|z| = R$. If $\lim_{z \to \infty} zf(z) = k$ where k is constant; then $\lim_{R \to \infty} \oint_{AB} f(z)\, dz = i(\beta - \alpha)k$, the integration along AB taken in anti-clock direction.

Proof: Since $\lim_{z \to \infty} zf(z) = k \therefore \lim_{z \to \infty} \{zf(z) - k\} = 0$ as k is constant.

$\therefore \quad |zf(z) - k| < \varepsilon$ for large values of z.

or $\quad zf(z) - k = \eta(z)$, where $|\eta(z)| < \varepsilon$

or $$f(z) = \frac{k + \eta(z)}{z} \qquad \int_{AB} f(z)dz = \int_{AB} \frac{k + \eta(z)}{z}\, dz = k\int_{AB} \frac{dz}{z} + \int_{AB} \frac{\eta(z)}{z}\, dz$$

But $$\int_{AB} \frac{dz}{z} = \int_\alpha^\beta \frac{i\,Re^{i\theta}}{Re^{i\theta}}\, d\theta, \text{ since } z = Re^{i\theta} \text{ on } AB,\ dz = i\,Re^{i\theta}\, d\theta$$

$$= i\int_\alpha^\beta d\theta = i(\beta - \alpha)$$

Hence,
$$\int_{AB} f(z)dz = i(\beta-\alpha)k + \int_{BA} \frac{\eta(z)}{z}dz$$

or
$$\int_{AB} f(z)dz - i(\beta-\alpha)k = \int_{AB} \frac{\eta(z)}{z}dz$$

$\therefore$
$$\left|\int_{AB} f(z)dz - i(\beta-\alpha)k\right| = \left|\int_{AB} \frac{\eta(z)}{z}dz\right| < \int_{AB} \frac{|\eta z|}{|z|}|dz|$$

$$< \varepsilon \int_{AB} \frac{1}{|z|}|dz|, \qquad \text{since } |\eta(z)| < \varepsilon$$

$$< \varepsilon \int_\alpha^\beta \frac{1}{R} R\,d\theta, \qquad \text{since } |z| = R, |dz| = R\,d\theta$$

$$= \varepsilon \int_\alpha^\beta d\theta = \varepsilon(\beta-\alpha) \to 0 \text{ as } \varepsilon \to 0 \text{ and } \varepsilon \to 0 \text{ as } z \to \infty$$

Thus, when $R \to \infty$, $\varepsilon \to 0$ and so we have $\lim_{R\to\infty}\left\{\int_{AB} f(z)dz - i(\beta-\alpha)k\right\} = 0$ or $\lim_{R\to\infty}\int_{AB} f(z)dz = i(\beta = \alpha)$.

Theorem 7.2 If $f(z) \to 0$ uniformly as $|z| \to \infty$, and $f(z)$ is meromorphic in upper half plane, then $\lim_{R\to\infty}\int_{C_R} e^{imz} f(z)dz = 0, (m > 0)$

where C_R denotes the semi-circle $|z| = R$, $I(z) > 0$.

Proof: Here R is large and can be made larger so as to include within it all the singularities and none on its boundary.

Since $\left.\right\}$ $\mathop{Lt}_{R\to\infty} f(z) = 0 \quad \therefore |f(z)| < \varepsilon$ for z on the large circle.

Now,
$$\left|\int_{C_R} e^{imz} f(z)dz\right| \le \int_{C_R} |e^{imz}||f(z)||dz| \le \varepsilon \int_{C_R} |e^{imz}||dz| \text{ since} |f(z)| < \varepsilon$$

$$= \varepsilon \int_0^\pi |e^{im(R\cos\theta + iR\sin\theta)}||Re^{i\theta} i\,d\theta| \quad \text{since } z = Re^{i\theta}$$

$$= \varepsilon \int_0^\pi e^{-mR\sin\theta} R\,d\theta$$

$$= 2\varepsilon R \int_0^{\pi/2} e^{-mR\sin\theta} d\theta \le 2\varepsilon R \int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta, \text{ by Jordan's inequality}$$

$$= \frac{\varepsilon\pi}{m}(1 - e^{-mR}) \text{ which} \to 0 \text{ when } R \to \infty \text{ and } \varepsilon \to 0$$

Hence,
$$\mathop{Lt}_{R\to\infty}\int_{C_R} e^{imz} f(z)dz = 0.$$

THE CAUCHY PRINCIPAL VALUE OF INTEGRALS

If $F(x)$ is continuous in $a \le x \le b$ except at a point x_0 such that $a < x_0 < b$, then if ϵ_1 and ϵ_2 are positive, we define
$$\int_a^b F(x)dx = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\int_a^{x_0-\epsilon_1} F(x)dx + \int_{x_0+\epsilon_2}^b F(x)dx\right\}.$$

In some cases, the above limit does not exist for $\epsilon_1 \ne \epsilon_2$ but does exist if we take $\epsilon_1 = \epsilon_2 = \epsilon$. In such case, we call
$$-\!\!\!\!\!\!\int_a^b F(x)dx = \lim_{\epsilon\to 0}\left\{\int_a^{x_0-\epsilon} F(x)dx + \int_{x_0+\epsilon}^b F(x)dx\right\}$$
the *Cauchy principal value* of the integral on the left.

Example 7.3 $\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\int_{-1}^{-\epsilon_1}\frac{dx}{x^3} + \int_{\epsilon_2}^{1}\frac{dx}{x^3}\right\} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\frac{1}{2\epsilon_2^2} - \frac{1}{2\epsilon_1^2}\right\}$ *does not exist. However, the Cauchy principal value with* $\epsilon_1 = \epsilon_2 = \epsilon$ *does exist and equals zero.*

DIFFERENTIATION UNDER THE INTEGRAL SING. LEIBNITZ'S RULE

A useful method for evaluating integrals employs *Leibnitz's rule* for differentiation under the integral sign. This rule states that
$$\frac{d}{d\alpha}\int_a^b F(x,\alpha)\,dx = \int_a^b \frac{\partial F}{\partial \alpha}\,dx$$

The rule is valid if a and b are constants, a is a real parameter such that $\alpha_1 \le \alpha \le \alpha_2$ where α_1 and α_2 are constants, and $F(x, \alpha)$ is continuous and has a continuous partial derivative with respect to a for $a \le x \le b$, $\alpha_1 \le \alpha \le \alpha_2$. It can be extended to cases where the limits a and b are infinite or dependent on α.

SOME SPECIAL EXPANSIONS

1. $\csc z = \dfrac{1}{z} + 2z\left(\dfrac{1}{z^2-\pi^2} - \dfrac{1}{z^2-4\pi^2} + \dfrac{1}{z^2-9\pi^2} - \cdots\right)$

2. $\sec z = \pi\left(\dfrac{1}{(\pi/2)^2-z^2} - \dfrac{3}{(3\pi/2)^2-z^2} + \dfrac{5}{(5\pi/2)^2-z^2} - \cdots\right)$

3. $\tan z = 2z\left(\dfrac{1}{(\pi/2)^2-z^2} + \dfrac{1}{(3\pi/2)^2-z^2} + \dfrac{1}{(5\pi/2)^2-z^2} + \cdots\right)$

4. $\cot z = \dfrac{1}{z} + 2z\left(\dfrac{1}{z^2-\pi^2} + \dfrac{1}{z^2-4\pi^2} + \dfrac{1}{z^2-9\pi^2} + \cdots\right)$

5. $\operatorname{csch} z = \dfrac{1}{z} - 2z\left(\dfrac{1}{z^2+\pi^2} - \dfrac{1}{z^2+4\pi^2} + \dfrac{1}{z^2+9\pi^2} - \cdots\right)$

6. $\operatorname{sech} z = \pi\left(\dfrac{1}{(\pi/2)^2+z^2} - \dfrac{3}{(3\pi/2)^2+z^2} + \dfrac{5}{(5\pi/2)^2+z^2} - \cdots\right)$

7. $\tanh z = 2z\left(\dfrac{1}{z^2+(\pi/2)^2} + \dfrac{1}{z^2+(3\pi/2)^2} + \dfrac{1}{z^2+(5\pi/2)^2} + \cdots\right)$

8. $\coth z = \dfrac{1}{z} + 2z\left(\dfrac{1}{z^2+\pi^2} + \dfrac{1}{z^2+4\pi^2} + \dfrac{1}{z^2+9\pi^2} + \cdots\right)$

SOLVED PROBLEMS

RESIDUES AND THE RESIDUE THEOREM

7.1 Let $f(z)$ be analytic inside and on a simple closed curve C except at point a inside C.

(a) Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \text{ where } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

i.e. $f(z)$ can be expanded into a converging Laurent series about $z = a$.

(b) Prove that

$$\oint_C f(z)\,dz = 2\pi i a_{-1}$$

Solution

(a) This follows from Laurent's theorem of Chapter 6.

(b) If we let $n = -1$ in the result of (a), we find

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz, \text{ i.e., } \oint_C f(z)\,dz = 2\pi i a_{-1}$$

We call a_{-1} the *residue* of $f(z)$ at $z = a$.

7.2 Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of points $a, b, c, \ldots$ inside C at which the residues are a_{-1}, b_{-1}, c_{-1}, ..., respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \ldots)$$

i.e., $2\pi i$ times the sum of the residues at all singularities enclosed by C.

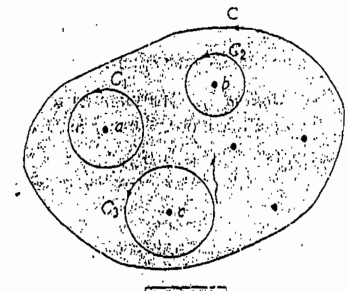


Fig. 7.4

Solution With centers at $a, b, c, \ldots$ respectively construct circles $C_1, C_2, C_3, \ldots$ that lie entirely inside C as shown in Fig. 7.4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 4.5, page 4.7, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \ldots \tag{1}$$

But by Problem 7.1,

$$\oint_{C_1} f(z)\,dz = 2\pi i\,a_{-1}, \quad \oint_{C_2} f(z)\,dz = 2\pi i\,b_{-1}, \quad \oint_{C_3} f(z)\,dz = 2\pi i c_{-1}, \ldots \tag{2}$$

Then, from (1) and (2), we have, as required.

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \ldots) = 2\pi i \text{ (sum of residues)}$$

The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of $f(z)$. It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 7.112 and 7.113).

7.3 Let $f(z)$ be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of $f(z)$ at a is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

Solution

Method 1. Suppose $f(z)$ has a pole a of order m, then the Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots \tag{1}$$

Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \dots \quad (2)$$

This represents the Taylor series about $z = a$ of the analytic function on the left. Differentiating both sides $m-1$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\,a_{-1} + m(m-1)\dots 2a_0(z-a) + \dots$$

Thus, on letting $z \to a$,

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\,a_{-1}$$

from which the required result follows.

Method 2. The required result also follows directly from Taylor's theorem on noting that the coefficient of $(z-a)^{m-1}$ in the expansion (2) is

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\Big|_{z=a}$$

Method 3. See Problem 5.37, Chapter 5, Page 5.21.

7.4 Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z\csc^2 z$ at all its poles in the finite plane.

Solution (a) $f(z)$ has a double pole at $z=-1$ and simple poles $z = \pm 2i$.

Method 1. Residue at $z=-1$ is

$$\lim_{z\to-1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} = \lim_{z\to-1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z\to2i}\left\{(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

Residue at $z=-2i$ is

$$\lim_{z\to-2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

Method 2. Residue at $z = 2i$ is

$$\lim_{z\to2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z^2+4)}\right\} = \left\{\lim_{z\to2i}\frac{z^2-2z}{(z+1)^2}\right\}\left\{\lim_{z\to2i}\frac{z-2i}{z^2+4}\right\}$$

$$= \frac{-4-4i}{(2i+1)^2}\cdot\lim_{z\to2i}\frac{1}{2z} = \frac{-4-4i}{(2i+1)^2}\cdot\frac{1}{4i} = \frac{7+i}{25}$$

using L'Hospital's rule. In a similar manner, or by replacing i by $-i$ in the result, we can obtain the residue at $z=-2i$.

(b) $f(z) = e^z\csc^2 z = e^z/\sin^2 z$ has double poles at $z = 0, \pm\pi, \pm2\pi, \dots$, i.e. $z = m\pi$ where $m = 0, +1, +2, \dots$

Method 1. Residue at $z = m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi}\frac{e^z[(z-m\pi)^2\sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2\cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to0}e^{u+m\pi}\left\{\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right\} = e^{m\pi}\left\{\lim_{u\to0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

$$\lim_{u\to0}\frac{u^3}{\sin^3 u} = \lim_{u\to0}\left(\frac{u}{\sin u}\right)^3 = 1$$

and thus write the limit as

$$e^{m\pi}\lim_{u\to0}\left(\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right) = e^{m\pi}\lim_{u\to0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit, we can instead use the series expansions $\sin u = u - u^3/3! + \dots$, $\cos u = 1 - u^2/2! + \dots$.

Method 2. (using Laurent's series).
In this method, we expand $f(z) = e^z\csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the coefficient of $1/(z-m\pi)$ as the required residue. To make the calculation easier, let $z = u + m\pi$. Then the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi+u}\csc^2(m\pi+u) = e^{m\pi}e^u\csc^2 u$. Using the Maclaurin expansions for e^u and $\sin u$, we find using long division

$$e^{m\pi}e^u\csc^2 u = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)}{\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right)^2} = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2}+\cdots\right)}{u^2\left(1-\frac{u^2}{6}+\frac{u^4}{120}-\cdots\right)^2}$$

$$= \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\cdots\right)}{u^2\left(1-\frac{u^2}{3}+\frac{2u^4}{45}+\cdots\right)} = e^{m\pi}\left(\frac{1}{u^2}+\frac{1}{u}+\frac{5}{6}+\frac{u}{3}+\cdots\right)$$

and so the residue is $e^{m\pi}$.

7.5 Find the residue of $F(z) = \dfrac{\cot z\coth z}{z^3}$ at $z = 0$.

Solution We have, as in Method 2 of Problem 7.4(b),

$$F(z) = \frac{\cos z\cosh z}{z^3\sin z\sinh z} = \frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)}$$

Taking $R \to \infty$ in (1), $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} + 0 = 2\pi i \sum R^+$ (2)

Pole of $f(z) = \dfrac{z^2}{(z^2+a^2)^3}$ are given by $(z^2+a^2)^3 = 0 \Rightarrow z = \pm ai$ (thrice)

Here, $z = ai$ is a pole of order 3 inside C, provided $R(a)$ is positive.

Therefore, residue at $ai = \dfrac{\varphi^2(ai)}{\lfloor 2}$

Here $\quad \phi(z) = (z-ai)^3 f(z) = (z-ai)^3 \cdot \dfrac{z^2}{(z^2+a^2)^3} = \dfrac{z^2(z-ai)^3}{(z-ai)^3(z+ai)^3} = \dfrac{z^2}{(z+ai)^3}$

$$\phi^1(z) = \frac{2z}{(z+ai)^3} - \frac{3z^2}{(z+ai)^4}, \varphi^2(z) = \frac{2}{(z+ai)^3} - \frac{6z}{(z+ai)^4} - \frac{6z}{(z+ai)^4} + \frac{12z^2}{(z+ai)^5}$$

$$\therefore \phi^2(ai) = \frac{2}{(2ai)^3} - \frac{6\times ai}{(2ai)^4} - \frac{6\times ai}{(2ai)^4} + \frac{12(ai)^2}{(2ai)^5} = \frac{1}{8a^3 i}$$

Therefore residue at $ai = \dfrac{1}{2}\cdot\dfrac{1}{8a^3 i} = \dfrac{1}{16a^3 i}$

Therefore from (2), $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = 2\pi i \times \frac{1}{16a^3 i} = \frac{\pi}{8a^3}$.

When $R(a)$ is negative, pole within C will be $z = -ai$

Hence replacing a by $-a$, we get $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = -\frac{\pi}{8a^3}$.

7.13 Evaluate $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^6+1)}$.

Solution Consider $\displaystyle\int_C \frac{z^2\,dz}{(z^6+1)} = \int_C f(z)\,dz$, where C is the contour consisting of a large semicircle Γ of radius R together with the part of real axis from $x = -R$ to $x = R$. By Cauchy's residue theorem,

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{x^2\,dx}{(x^6+1)} + \int_\Gamma \frac{z^2\,dz}{(z^6+1)} = 2\pi i \sum R^+ \quad (1)$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z.z^2}{(z^6+1)} = 0 \qquad \therefore \lim_{R\to\infty} \int_\Gamma \frac{z^2\,dz}{z^6+1} = 0$$

$$\text{Taking } R \to \infty \text{ (1)} \Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{x^6+1} = 2\pi i \sum R^+ \quad (2)$$

Poles of $f(z) = \dfrac{z^2}{z^6+1}$ are given by, $z^6 + 1 = 0 \Rightarrow z = (-1)^{1/6}$

$$\Rightarrow z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/6} = e^{\frac{i(2n+1)\pi}{6}};\ n = 0, 1, 2, 3, 4, 5.$$

Therefore $z = e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}, e^{\frac{7\pi i}{6}}, e^{\frac{9\pi i}{6}}, e^{\frac{11\pi i}{6}}$.

Out of these six poles, the amplitude of first three is less than π and as such they lie within C. If α denotes any of the above poles, then $\alpha^6 = -1$

Residue at $\alpha = \displaystyle\lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\cdot\frac{z^2}{z^6+1} = \lim_{z\to\infty}\frac{(z-\alpha)z^2}{z^6+1} = -\frac{\alpha^{-3}}{6}$

Residue at $e^{\frac{\pi i}{6}} = \dfrac{1}{6}\left(e^{\frac{\pi i}{6}}\right)^{-3} = -\dfrac{1}{6}i$

Residue at $e^{\frac{3\pi i}{6}} = \dfrac{1}{6}\left(e^{\frac{3\pi i}{6}}\right)^{-3} = \dfrac{i}{6}$

Residue at $e^{\frac{5\pi i}{6}} = \dfrac{1}{6}\left(e^{\frac{5\pi i}{6}}\right)^{-3} = -\dfrac{i}{6}$

$$\therefore \quad \sum R^+ = -\frac{i}{6} + \frac{i}{6} - \frac{i}{6} = -\frac{i}{6}$$

$$\therefore \quad (2) \Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{x^6+1} = 2\pi i \times \left(-\frac{i}{6}\right) = \frac{\pi}{3}.$$

$$\Rightarrow \quad 2\int_0^{\infty} \frac{x^2\,dx}{x^6+1} = \frac{\pi}{3} \Rightarrow \int_0^{\infty} \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$$

Definite Integrals of the Type $\int_0^{2\pi} G(\sin\theta, \cos\theta)\,d\theta$

7.14 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

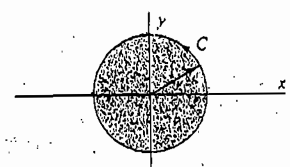


Fig. 7.6

Solution Let $z = e^{i\theta}$. Then $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{dz/iz}{3 - 2(z+z^{-1})/2 + (z - z^{-1})/2i}$$

$$= \oint_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i},$$

where C is the circle of unit radius with center at the origin (Fig. 7.6).

The poles of $2/\{(1-2i)z^2 + 6iz - 1 - 2i\}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)} = 2 - i, (2-i)/5$$

Only $(2-i)/5$ lies inside C.
Residue at

$$(2-i)/5 = \lim_{z\to(2-i)/5}\{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$$

$$= \lim_{z\to(2-i)/5}\frac{2}{2(1-2i)z+6i} = \frac{1}{2i}$$

by L'Hospital's rule.
Then

$$\oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right) = \pi,$$

the required value.

7.15 Given $a > |b|$, show that $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$.

Solution Let $z = e^{i\theta}$. Then, $\sin\theta = (e^{i\theta}-e^{-i\theta})/2i = (z-z^{-1})/2i$, $dz = ie^{i\theta}d\theta = iz\,d\theta$ so that

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \oint_C\frac{dz/iz}{a+b(z-z^{-1})/2i} = \oint_C\frac{2dz}{bz^2+2aiz-b}$$

where C is the circle of unit radius with center at the origin, as shown in Fig. 7.6.

The poles of $2/(bz^2+2aiz-b)$ are obtained by solving $bz^2+2aiz-b=0$ and are given by

$$z = \frac{-2ai\pm\sqrt{-4a^2+4b^2}}{2b} = \frac{-ai\pm\sqrt{a^2-b^2}\,i}{b}$$

$$= \left\{\frac{-a+\sqrt{a^2-b^2}}{b}\right\}i,\ \left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}i$$

Only $\left\{\left(-a+\sqrt{a^2-b^2}\right)/b\right\}i$ lies inside C, since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b}\cdot\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{(\sqrt{a^2-b^2}+a)}\right| < 1$$

when $a > |b|$.
Residue at

$$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}i = \lim_{z\to z_1}(z-z_1)\frac{2}{bz^2+2aiz-b}$$

$$= \lim_{z\to z_1}\frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}\,i}$$

by L'Hospital's rule
Then

$$\int_C\frac{2dz}{bz^2+2aiz-b} = 2\pi i\left(\frac{1}{\sqrt{a^2-b^2}\,i}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

the required value.

7.16 Show that $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = \frac{\pi}{12}$.

Solution Let $z = e^{i\theta}$. Then $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (e^{3i\theta}+e^{-3i\theta})/2 = (z^3+z^{-3})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = \oint_C\frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\frac{dz}{iz} = -\frac{1}{2i}\oint_C\frac{z^6+1}{z^3(2z-1)(z-2)}dz$$

where C is the contour of Fig. 7.6

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$ inside C.

Residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$$

Then

$$-\frac{1}{2i}\oint_C\frac{z^6+1}{z^3(2z-1)(z-2)}dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12}\text{ as required.}$$

7.17 Show that $\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

Solution Letting $z = e^{i\theta}$, we have $\sin\theta = (z-z^{-1})/2i$, $dz = ie^{i\theta}d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \oint_C\frac{dz/iz}{\{5-3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2}$$

where C is the contour of Fig. 7.6.

The integrand has poles of order 2 at $z = (10i\pm\sqrt{-100+36})/6 = (10i\pm 8i)/6 = 3i, i/3$. Only the pole $i/3$ lies inside C.
Residue at

$$z = i/3 \text{ is } \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$= \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z-i)^2(z-3i)^2}\right\} = -\frac{5}{256}$$

Then

$$-\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$

$$= \frac{\left(1-\frac{z^4}{6}+\cdots\right)}{z^5\left(1-\frac{z^4}{90}+\cdots\right)} = \frac{1}{z^5}\left(1 - \frac{7z^4}{45} \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z\to 0}\frac{1}{4!}\frac{d^4}{dz^4}\left\{z^5\frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

7.6 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ around the circle C with equation $|z| = 3$.

Solution The integrand $e^{zt}/\{z^2(z^2+2z+2)\}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2+2z+2=0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to 0}\frac{(z^2+2z+2)(te^{zt})-(e^{zt})(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\}\lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1-i$ is

$$\lim_{z\to -1-i}\left\{[z-(-1-i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\text{ (sum of residues)} = 2\pi i\left\{\frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t\right\}$$

that is,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t$$

Definite integrals of the type $\int_{-\infty}^{\infty} F(x)\,dx$

7.7 Let $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants. Prove that $\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$ where Γ is the semi-circular arc of radius R shown in Fig. 7.5.

Solution By Property (*e*), Page 4.2, we have

$$\left|\int_\Gamma F(z)dz\right| \le \frac{M}{R^k}.\pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma F(z)dz\right| = 0 \text{ and so } \lim_{R\to\infty}\int_\Gamma F(z)dz = 0$$

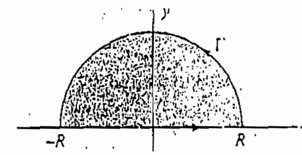


Fig. 7.5

7.8 Show that for $z = Re^{i\theta}$, $|f(z)| \le M/R^k$, $k > 1$ if $f(z) = 1/(z^6+1)$.

Solution Suppose $z = Re^{i\theta}$. Then

$$|f(z)| = \left|\frac{1}{R^6e^{6i\theta}+1}\right| \le \frac{1}{|R^6e^{6i\theta}|-1} = \frac{1}{R^6-1} \le \frac{2}{R^6}$$

where R is large enough (say $R > 2$, for example), so that $M = 2$, $k = 6$.

Note that we have made use of the inequality $|z_1+z_2| \ge |z_1| - |z_2|$ with $z_1 = R^6e^{6i\theta}$ and $z_2 = 1$.

7.9 Evaluate $\int_0^\infty \frac{dx}{x^6+1}$.

Solution Consider $\oint_C dz/(z^6+1)$, where C is the closed contour of Fig. 7.5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles at $1/(z^6+1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Then, using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}}\left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}}\left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}}\left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus

$$\oint_C\frac{dz}{z^6+1} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6}+\frac{1}{6}e^{-5\pi i/2}+\frac{1}{6}e^{-25\pi i/6}\right\} = \frac{2\pi}{3}$$

That is, $$\int_{-R}^{R}\frac{dx}{x^6+1}+\int_\Gamma\frac{dz}{z^6+1} = \frac{2\pi}{3} \quad (1)$$

Taking the limit of both sides of (1) as $R\to\infty$ and using Problems 7.7 and 7.8, we have

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^6+1} = \int_{-\infty}^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3} \quad (2)$$

Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^{\infty} \frac{dx}{x^6+1}$$

the required integral has the value $\pi/3$.

7.10 Show that $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

Solution The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour C of Fig. 7.5 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is

$$\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+1)^2(z-1)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Then

$$\oint_C \frac{z^2dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100}+\frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R}\frac{x^2dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 7.7, we obtain the required result.

7.11 Show that $\int_0^{\infty}\frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$.

Solution Consider $\int_C \frac{dz}{(1+z^2)^2} = \int_C f(z)dz$, where $f(z) = \frac{1}{(1+z^2)^2}$, C is the contour consisting of a large semi-circle Γ of radius R together with the part of real axis from $x = -R$ to $x = R$.

By Cauchy's residue theorem,

$$\int_C f(z)dz = \int_{-R}^{R}\frac{dx}{(1+x^2)^2} + \int_\Gamma \frac{dz}{(1+z^2)^2} = 2\pi i\sum R^+ \quad (1)$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z.\frac{1}{(1+z^2)^2} = \lim_{z\to\infty}\frac{z}{z^4\left(1+\frac{1}{z^2}\right)^2} = 0$$

$$\therefore \lim_{R\to\infty}\int_\Gamma \frac{dz}{(1+z^2)^2} = 0 \text{ and } \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(1+x^2)^2} = \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}$$

Taking $R \to \infty$ in (1), we get

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} + 0 = 2\pi i\sum R^+ \quad (2)$$

Poles of $f(z) = \frac{1}{(1+z^2)^2}$ are given by $(1+z^2)^2 = 0 \Rightarrow z = \pm i$ (twice) out of which only pole $z = i$ (order 2) lies inside C.

Therefore, residue at $i = \frac{\phi'(i)}{\lfloor 1} = \phi'(i)$

Here $\phi(z) = (z-i)^2 f(z) = (z-i)^2 \cdot \frac{1}{(1+z^2)^2} = \frac{(z-i)^2}{(z-i)^2(z+i)^2} = \frac{1}{(z+i)^2}$

$$\therefore \quad \phi'(z) = -\frac{2}{(z+i)^3}$$

Therefore, residue at $i = \phi'(i) = -\frac{2}{(2i)^3} = -\frac{2}{8i^3} = \frac{1}{4i}$

Therefore, from (2), $\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$

$$\Rightarrow 2\int_0^{\infty}\frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \Rightarrow \int_0^{\infty}\frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

7.12 Show that $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)^3} = \frac{\pi}{8a^3}$, provided $R(a)$ is positive. What is the value of this integral when $R(a)$ is negative.

Solution Consider $\int_C \frac{z^2dz}{(z^2+a^2)^3} = \int_C f(z)dz$, where $f(z) = \frac{z^2}{(z^2+a^2)^3}$, C is the contour consisting of a large semicircle Γ of radius R together with the part of real axis from $x = -R$ to $x = R$.

By residue's theorem,

$$\int_C f(z)dz = \int_{-R}^{R}\frac{x^2dx}{(x^2+a^2)^3} + \int_\Gamma \frac{z^2dz}{(z^2+a^2)^3} = 2\pi i\sum R^+ \quad (1)$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z\frac{z^2}{(z^2+a^2)^3} = \lim_{z\to\infty}\frac{z^3}{z^6\left(1+\frac{a^2}{z^2}\right)^3} = 0$$

$$\therefore \quad \lim_{R\to\infty}\int_\Gamma \frac{z^2dz}{(z^2+a^2)^3} = 0$$

Also $\quad \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2dx}{(x^2+a^2)^3} = \int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)^3}$

Another method. From Problem 7.15 we have for $a > |b|$,

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Then, by differentiating both sides with respect to a (considering b as constant) using Leibnitz's rule, we have

$$\frac{d}{da}\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \int_0^{2\pi} \frac{\partial}{\partial a}\left(\frac{1}{a + b\sin\theta}\right) d\theta = -\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2}$$

$$\frac{d}{da}\left(\frac{2\pi}{\sqrt{a^2 - b^2}}\right) = \frac{-2\pi a}{(a^2 - b^2)^{3/2}}$$

that is,

$$\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Letting $a = 5$ and $b = -3$, we have

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} = \frac{2\pi(5)}{(5^2 - 3^2)^{3/2}} = \frac{5\pi}{32}$$

7.18 Show that $\displaystyle\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{5}}(3 - \sqrt{5})^n$

Solution Let $I = \displaystyle\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta}\, d\theta =$ R.P. of $\displaystyle\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n e^{in\theta}}{3 + 2\cos\theta}\, d\theta$

$= $ R.P. of $\displaystyle\int_0^{2\pi} \frac{(1 + e^{i\theta} + e^{-i\theta})^n}{3 + e^{i\theta} + e^{-i\theta}} \cdot e^{in\theta} d\theta$, put $z = e^{i\theta}$, $\therefore dz = ie^{i\theta} d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$

$= $ R.P. of $\displaystyle\int_C \frac{\left(1 + z + \frac{1}{z}\right)^n}{\left(3 + z + \frac{1}{z}\right)} \cdot z^n \cdot \frac{dz}{iz}$, where C is a unit circle $|z| = 1$

$= $ R.P. of $\displaystyle\frac{1}{i}\int_C \frac{(1 + z + z^2)^n}{z^2 + 3z + 1}\, dz =$ R.P. of $\displaystyle\frac{1}{i}\int_C f(z)\, dz$ (1)

Poles of $f(z)$ are given by $z^2 + 3z + 1 = 0$, $\therefore z = \dfrac{-3 \pm \sqrt{9 - 4}}{2} = \dfrac{-3 \pm \sqrt{5}}{2}$

Let $\alpha = \dfrac{-3 + \sqrt{5}}{2}$, and $\beta = \dfrac{-3 - \sqrt{5}}{2}$. Clearly $z = \alpha$ is the only pole inside C.

Residue at $\alpha = \displaystyle\lim_{z \to \alpha} (z - \alpha) f(z) = \lim_{z \to \alpha} (z - \alpha) \cdot \frac{(1 + z + z^2)^n}{z^2 + 3z + 1}$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{(1 + z + z^2)^n}{1.(z - \alpha)(z - \beta)} = \frac{(1 + \alpha + \alpha^2)^n}{(\alpha - \beta)} = \frac{(3 - \sqrt{5})^n}{\sqrt{5}}$$

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i \sum R = 2\pi i \times \frac{(3 - \sqrt{5})^n}{\sqrt{5}}$

Therefore, $I = $ R.P. of $\dfrac{1}{i} \times \dfrac{2\pi i(3 - \sqrt{5})^n}{\sqrt{5}} = \dfrac{2\pi}{\sqrt{5}}(3 - \sqrt{5})^n$.

7.19 Apply Calculus of residues to prove that

$\displaystyle\int_0^{2\pi} e^{\cos\theta}.\cos(\sin\theta - n\theta)d\theta = \frac{2\pi}{\lfloor n}$, where n is a positive integer.

Solution Let $I = \displaystyle\int_0^{2\pi} e^{\cos\theta}.\cos(\sin\theta - n\theta)d\theta =$ R.P. of $\displaystyle\int_0^{2\pi} e^{\cos\theta}.e^{i(\sin\theta - n\theta)} d\theta$

$= $ R.P. of $\displaystyle\int_0^{2\pi} e^{\cos\theta + i\sin\theta}.e^{-in\theta} d\theta =$ R.P. of $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}.e^{-in\theta} d\theta$ Put $e^{i\theta} = z \Rightarrow d\theta = \dfrac{dz}{iz}$

$= $ R.P. of $\displaystyle\frac{1}{i}\int_C e^z z^{-n} \cdot \frac{dz}{z}$, where C is unit circle $|z| = 1$

$= $ R.P. of $\displaystyle\frac{1}{i}\int_C f(z)dz$, where $f(z) = \dfrac{e^z}{z^{n+1}}$ (1)

Poles of $f(z)$ are given by, $z^{n+1} = 0 \Rightarrow z = 0$. Clearly, $z = 0$ is a pole of order $(n + 1)$

Therefore, residue at $0 = \displaystyle\lim_{z \to 0} \frac{1}{\lfloor n + 1 - 1} \cdot \frac{d^n}{dz^n}\left[z^{n+1} \cdot \frac{e^z}{z^{n+1}}\right] = \lim_{z \to 0} \frac{1}{\lfloor n} e^z = \frac{1}{\lfloor n}$

By Cauchy's residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i \sum R = \frac{2\pi i}{\lfloor n}$

Therefore, $I = $ R.P. of $\dfrac{1}{i} \times \dfrac{2\pi i}{\lfloor n} = \dfrac{2\pi}{\lfloor n}$.

7.20 Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos n\theta d\theta}{1 + 2a\cos\theta + a^2}$ and $\displaystyle\int_0^{2\pi} \frac{\sin n\theta d\theta}{1 + 2a\cos\theta + a^2}$,

where $a^2 < 1$ and n is a positive integer.

Solution Both integrals are real and imaginary parts of $I = \displaystyle\int_0^{2\pi} \frac{e^{in\theta} d\theta}{1 + a(e^{i\theta} + e^{-i\theta}) + a^2}$

Put $e^{i\theta} = z \therefore d\theta = \dfrac{dz}{iz}$

$$= \int_C \frac{z^n}{1 + a(z + \frac{1}{z}) + a^2} \cdot \frac{dz}{iz} = \frac{1}{i}\int_C \frac{z^n dz}{az^2 + z + a^2 z + a}, \text{ where } C \text{ is unit circle } |z| = 1$$

$$= \frac{1}{i}\int_C f(z)dz, \text{ where } f(z) = \frac{z^n}{az^2 + z(1 + a^2) + a} \quad (1)$$

Poles of $f(z)$ are given by, $az^2 + z(1 + a^2) + a = 0 \Rightarrow z = -a, -\dfrac{1}{a}$

Since $a^2 < 1$ so $z = -\dfrac{1}{a}$ lies outside c and $z = -a$ lies within c.

Poles of $\psi(z) = \dfrac{e^{miz} z^3}{z^4 + a^4}$ are given by $z^4 + a^4 = 0 \Rightarrow z = -a^4 \Rightarrow z = (-1)^{1/4} a$

$$\Rightarrow z = a\left[\frac{\cos(2n+1)\pi}{4} + i\frac{\sin(2n+1)\pi}{4}\right] \Rightarrow z = ae^{i(2n+1)\pi/4}$$

Therefore, $z = ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}, ae^{\frac{5\pi i}{4}}, ae^{\frac{7\pi i}{4}}$, $n = 0, 1, 2, 3$.

Out of these poles, the amplitudes of $ae^{\frac{\pi i}{4}}$ and $ae^{\frac{3\pi i}{4}}$ are less than π and as such they lie within C. Now if α denotes any of the poles within C then $a^4 = -\alpha^4$

Residue at $z = \alpha$ is $\lim_{z\to\alpha}(z-\alpha)\psi(z) = \lim_{z\to\alpha}(z-\alpha)\dfrac{z^3 e^{miz}}{(z^4+a^4)} = \lim_{z\to\alpha}\dfrac{(z-\alpha)z^3 e^{miz}}{(z^2+\alpha^2)(z+\alpha)(z-\alpha)} = \dfrac{1}{4}e^{mi\alpha}$.

Therefore, residue (at $z = ae^{\frac{\pi i}{4}}$) $= \dfrac{1}{4}e^{ima\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)} = \dfrac{1}{4}e^{\frac{ima}{\sqrt{2}}}e^{-\frac{ma}{\sqrt{2}}}$

and residue (at $z = ae^{\frac{3\pi i}{4}}$) $= \dfrac{1}{4}e^{-\frac{ima}{\sqrt{2}}}e^{-\frac{ma}{\sqrt{2}}}$

Therefore, $\sum R^+ = \dfrac{1}{4}e^{-\frac{ma}{\sqrt{2}}}\left[e^{\frac{ima}{\sqrt{2}}} + e^{-\frac{ima}{\sqrt{2}}}\right] = \dfrac{1}{4}e^{-\frac{ma}{\sqrt{2}}}2\cos\dfrac{ma}{\sqrt{2}}$, $(\because e^{i\theta} + e^{-i\theta} = 2\cos\theta)$

So, (2) $\Rightarrow \int_{-\infty}^{\infty} e^{imx} f(x)dx = 2\pi i \times \dfrac{1}{2}e^{-\frac{ma}{\sqrt{2}}}\cos\dfrac{ma}{\sqrt{2}}$

$\Rightarrow \int_{-\infty}^{\infty}\dfrac{x^3}{a^4+x^4}(\cos mx + i\sin mx)dx = \pi i e^{-\frac{ma}{\sqrt{2}}}\cos\dfrac{ma}{\sqrt{2}}$, equating imaginary part, we have

$$\int_{-\infty}^{\infty}\frac{x^3}{a^4+x^4}\sin mx dx = \pi e^{-\frac{ma}{\sqrt{2}}}\cos\frac{ma}{\sqrt{2}} \Rightarrow \int_0^{\infty}\frac{x^3}{a^4+x^4}\sin mx dx = \frac{\pi}{2}e^{-\frac{ma}{\sqrt{2}}}\cos\frac{ma}{\sqrt{2}}.$$

Miscellaneous Definite Integrals

7.27 Show that $\int_0^{\infty}\dfrac{\sin x}{x}dx = \dfrac{\pi}{2}$.

Solution The method of Problem 7.22 leads us to consider the integral of e^{iz}/z around the contour of Fig. 7.5. However, since $z = 0$ lies on this path of integration and since we cannot integrate through a singularity, we modify that contour by indenting the path at $z = 0$, as shown in Fig. 7.8, which we call contour C' or $ABDEFGHJA$.

Since $z = 0$ is outside C', we have

$$\oint_{C'}\frac{e^{iz}}{z}dz = 0$$

or

$$\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}dx + \int_{HJA}\frac{e^{iz}}{z}dz + \int_{\epsilon}^{R}\frac{e^{ix}}{x}dx + \int_{BDEFG}\frac{e^{iz}}{z}dz = 0$$

Replacing x by $-x$ in the first integral and combining with the third integral, we find

$$\int_{\epsilon}^{R}\frac{e^{ix} - e^{-ix}}{x}dx + \int_{HJA}\frac{e^{iz}}{z}dz + \int_{BDEFG}\frac{e^{iz}}{z}dz = 0$$

or

$$2i\int_{\epsilon}^{R}\frac{\sin x}{x}dx = -\int_{HJA}\frac{e^{iz}}{z}dz - \int_{BDEFG}\frac{e^{iz}}{z}dz$$

Let $\epsilon \to 0$ and $R \to \infty$. By Problem 7.21, the second integral on the right approaches zero. Letting $z = \epsilon e^{i\theta}$ in the first integral on the right, we see that it approaches

$$\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta = -\lim_{\epsilon\to 0}\int_{\pi}^{0}ie^{i\epsilon e^{i\theta}}d\theta = \pi i$$

since the limit can be taken under the integral sign.

Then we have

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} 2i\int_{\epsilon}^{R}\frac{\sin x}{x}dx = \pi i \quad \text{or} \quad \int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$$

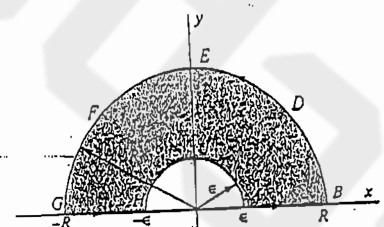


Fig. 7.8

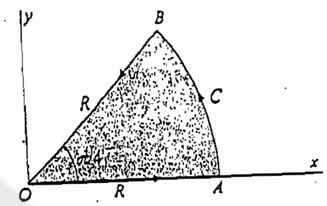


Fig. 7.9

7.28 Prove that

$$\int_0^{\infty}\sin x^2 dx = \int_0^{\infty}\cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Solution Let C be the contour indicated in Fig. 7.9, where AB is the arc of a circle with center at O and radius R. By Cauchy's theorem,

$$\oint_C e^{iz^2}dz = 0$$

or

$$\int_{OA} e^{iz^2}dz + \int_{AB} e^{iz^2}dz + \int_{BO} e^{iz^2}dz = 0 \qquad (1)$$

Now on OA, $z = x$ (from $x = 0$ to $x = R$); on AB, $z = Re^{i\theta}$ (from $\theta = 0$ to $\theta = \pi/4$); on BO, $z = re^{\pi i/4}$ (from $r = R$ to $r = 0$). Hence from (1)

$$\int_0^R e^{ix^2}dx + \int_0^{\pi/4} e^{iR^2e^{2i\theta}} iRe^{i\theta}d\theta + \int_R^0 e^{ir^2e^{\pi i/2}} e^{\pi i/4}dr = 0 \tag{2}$$

that is,

$$\int_0^R (\cos x^2 + i\sin x^2)dx = e^{\pi i/4}\int_0^R e^{-r^2}dr - \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta} iRe^{i\theta}d\theta \tag{3}$$

We consider the limit of (3) as $R \to \infty$. The first integral on the right becomes [see Problem 10.14]

$$e^{\pi i/4}\int_0^\infty e^{-r^2}dr = \frac{\sqrt{\pi}}{2}e^{\pi i/4} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}} \tag{4}$$

The absolute value of the second integral on the right of (3) is

$$\left|\int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta} iRe^{i\theta}d\theta\right| \le \int_0^{\pi/4} e^{-R^2\sin 2\theta}R\,d\theta = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}d\phi$$

$$\le \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\phi/\pi}d\phi = \frac{\pi}{4R}(1-e^{-R^2})$$

where we have used the transformation $2\theta = \phi$ and the inequality $\sin\phi \le 2\phi/\pi$, $0 \le \phi \le \pi/2$ (see Problem 7.21). This shows that as $R \to \infty$, the second integral on the right of (3) approaches zero. Then (3) becomes

$$\int_0^\infty (\cos x^2 + i\sin x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$$

and so, equating real and imaginary parts, we have as required,

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

7.29 Show that $\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Solution Consider $\oint_C (z^{p-1}/1+z)dz$. Since $z = 0$ is a branch point, choose C as the contour of Fig. 7.10 where the positive real axis is the branch line and where AB and GH are actually coincident with the x axis but are shown separated for visual purpose.

The integrand has the simple pole $z = -1$ inside C.

Residue at $z = -1 = e^{\pi i}$ is

$$\lim_{z\to -1}(z+1)\frac{z^{p-1}}{1+z} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}$$

Then

$$\oint_C \frac{z^{p-1}}{1+z}dz = 2\pi i e^{(p-1)\pi i}$$

or, omitting the integrand,

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$

We thus have

$$\int_\epsilon^R \frac{x^{p-1}}{1+x}dx + \int_0^{2\pi}\frac{(Re^{i\theta})^{p-1} iRe^{i\theta}d\theta}{1+Re^{i\theta}} + \int_R^\epsilon \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}dx + \int_{2\pi}^0 \frac{(\epsilon e^{i\theta})^{p-1} i\epsilon e^{i\theta}d\theta}{1+\epsilon e^{i\theta}} = 2\pi i e^{(p-1)\pi i}$$

where we have used $z = xe^{2\pi i}$ for the integral along GH, since the argument of z is increased by 2π in going around the circle $BDEFG$.

Taking the limit as $\epsilon \to 0$ and $R \to \infty$ and nothing that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx + \int_\infty^0 \frac{e^{2\pi i(p-1)}x^{p-1}}{1+x}dx = 2\pi e^{(p-1)\pi i}$$

or

$$(1 - e^{2\pi i(p-1)})\int_0^\infty \frac{x^{p-1}}{1+x}dx = 2\pi i e^{(p-1)\pi i}$$

so that

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{2\pi i e^{(p-1)\pi i}}{1-e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}$$

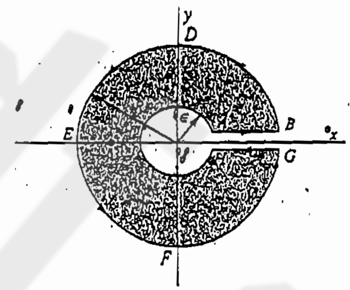


Fig. 7.10

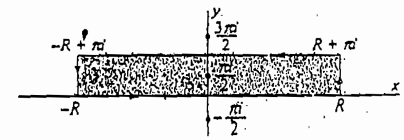


Fig. 7.11

7.30 Prove that $\int_0^\infty \frac{\cosh ax}{\cosh x}dx = \frac{\pi}{2\cos(\pi a/2)}$ where $|a| < 1$.

Solution Consider $\oint_C (e^{az}/\cosh z)\,dz$ where C is a rectangle having vertices at $-R$, R, $R + \pi i$, $-R + \pi i$ (see Fig. 7.11).

The poles of $e^{az}/\cosh z$ are simple and occur where $\cosh z = 0$, i.e., $z = \left(n + \frac{1}{2}\right)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$

The only pole enclosed by C is $\pi i/2$.

Residue of $e^{az}/\cosh z$ at $z = \pi i/2$ is

$$\lim_{z\to\pi i/2}(z - \pi i/2)\frac{e^{az}}{\cosh z} = \frac{e^{a\pi i/2}}{\sinh(\pi i/2)} = \frac{e^{a\pi i/2}}{i\sin(\pi/2)} = -ie^{a\pi i/2}$$

Then, by the residue theorem,

$$\oint_C \frac{e^{az}}{\cosh z}\,dz = 2\pi i(-ie^{a\pi i/2}) = 2\pi e^{a\pi i/2}$$

This can be written

$$\int_{-R}^{R}\frac{e^{ax}}{\cosh x}\,dx + \int_0^{\pi}\frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy + \int_R^{-R}\frac{e^{a(x+\pi i)}}{\cosh(x+\pi i)}\,dx + \int_{\pi}^{0}\frac{e^{a(-R+iy)}}{\cosh(-R+iy)}\,i\,dy = 2\pi e^{a\pi i/2} \quad (1)$$

As $R \to \infty$ the second and fourth integrals on the left approach zero. To show this, let us consider the second integral. Since

$$|\cosh(R+iy)| = \left|\frac{e^{R+iy}+e^{-R-iy}}{2}\right| \ge \frac{1}{2}\{|e^{R+iy}|-|e^{-R-iy}|\} = \frac{1}{2}(e^R - e^{-R}) \ge \frac{1}{4}e^R$$

we have

$$\left|\int_0^{\pi}\frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy\right| \le \int_0^{\pi}\frac{e^{aR}}{e^{R/4}}\,dy = 4\pi e^{(a-1)R}$$

and the result follows on noting that the right side approaches zero as $R \to \infty$ since $|a| < 1$. In a similar manner, we can show that the fourth integral on the left of (1) approaches zero as $R \to \infty$. Hence, (1) becomes

$$\lim_{R\to\infty}\left\{\int_{-R}^{R}\frac{e^{ax}}{\cosh x}\,dx + e^{a\pi i}\int_{-R}^{R}\frac{e^{ax}}{\cosh x}\,dx\right\} = 2\pi e^{a\pi i/2}$$

since $\cosh(x + \pi i) = -\cosh x$. Thus

$$\lim_{x\to\infty}\int_{-R}^{R}\frac{e^{ax}}{\cosh x}\,dx = \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx = \frac{2\pi e^{a\pi i/2}}{1+e^{a\pi i}} = \frac{2\pi}{e^{a\pi i/2}+e^{-a\pi i/2}} = \frac{\pi}{\cos(\pi a/2)}$$

Now

$$\int_{-\infty}^{0}\frac{e^{ax}}{\cosh x}\,dx + \int_0^{\infty}\frac{e^{ax}}{\cosh x}\,dx = \frac{\pi}{\cos(\pi a/2)}$$

Then, replacing x by $-x$ in the first integral, we have

$$\int_0^{\infty}\frac{e^{-ax}}{\cosh x}\,dx + \int_0^{\infty}\frac{e^{ax}}{\cosh x}\,dx = 2\int_0^{\infty}\frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{\cos(\pi a/2)}$$

from which the required result follows.

7.31 Prove that $\int_0^{\infty}\frac{\ln(x^2+1)}{x^2+1}\,dx = \pi\ln 2$.

Solution Consider $\oint_C \{\ln(z+i)/z^2+1\}\,dz$ around the contour C consisting of the real axis from $-R$ to R and the semicircle Γ of radius R (see Fig. 7.12).

The only pole of $\ln(z+i)/(z^2+1)$ inside C is the simple pole $z = i$, and the residue is

$$\lim_{z\to i}(z-i)\frac{\ln(z+i)}{(z-i)(z+i)} = \frac{\ln(2i)}{2i}$$

Hence, by the residue theorem,

$$\oint_C \frac{\ln(z+i)}{z^2+1}\,dz = 2\pi i\left\{\frac{\ln(2i)}{2i}\right\} = \pi\ln(2i) = \pi\ln 2 + \frac{1}{2}\pi^2 i \quad (1)$$

on writing $\ln(2i) = \ln 2 + \ln i = \ln 2 + \ln e^{\pi i/2} = \ln 2 + \pi i/2$ using principal values of the logarithm. The result can be written

$$\int_{-R}^{R}\frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma}\frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

or

$$\int_{-R}^{0}\frac{\ln(x+i)}{x^2+1}\,dx + \int_0^{R}\frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma}\frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

Replacing x by $-x$ in the first integral, this can be written

$$\int_0^{R}\frac{\ln(i-x)}{x^2+1}\,dx + \int_0^{R}\frac{\ln(i+x)}{x^2+1}\,dx + \int_{\Gamma}\frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

or, since $\ln(i-x) + \ln(i+x) = \ln(i^2 - x^2) = \ln(x^2+1) + \pi i$,

$$\int_0^{R}\frac{\ln(x^2+1)}{x^2+1}\,dx + \int_0^{R}\frac{\pi i}{x^2+1}\,dx + \int_{\Gamma}\frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i \quad (2)$$

As $R \to \infty$ we can show that the integral around Γ approaches zero (see Problem 7.117). Hence, on taking real parts, we find as required,

$$\lim_{R\to\infty}\int_0^{R}\frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^{\infty}\frac{\ln(x^2+1)}{x^2+1}\,dx = \pi\ln 2$$

7.32 Show that $\int_0^{\infty}\frac{\log(1+x^2)}{1+x^2}\,dx = \pi\log 2$.

Solution Consider $\int_C f(z)dz$, where $f(z) = \frac{\log(z+i)}{z^2+1}$ and C is the contour consisting of a large semicircle γ of radius R together with the part of the real axis from $x = -R$ to $x = R$.

Poles of $f(z)$ are given by $z^2 + 1 = 0 \Rightarrow z = i, -i$ are two simple poles. Out of these simple poles only $z = i$ lies within C.

$$\text{Residue (at } z = i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{\log(z+i)}{(z-i)(z+i)} = \frac{\log 2 + \frac{\pi i}{2}}{2i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\sum R^+ = 2\pi i\frac{\left(\log 2 + \frac{\pi i}{2}\right)}{2i} \Rightarrow \int_{-R}^{R}f(x) + \int_{\gamma}f(z)dz = \pi\left(\log 2 + \frac{\pi i}{2}\right)$$

$$\Rightarrow \int_{-R}^{R} \frac{\log(x+i)}{x^2+1}\,dx + \int_{\gamma} \frac{\log(z+i)}{z^2+1}\,dz = \pi\left(\log 2 + \frac{\pi i}{2}\right) \quad (1)$$

Now, $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z\log(z+i)}{(z-i)(z+i)} = \lim_{z\to\infty} \frac{z}{(z-i)} \cdot \lim_{z\to\infty} \frac{\log(z+i)}{(z+i)} = 0$

Therefore, $\lim_{R\to\infty} \int_{\gamma} \frac{\log(z+i)}{z^2+1}\,dz = 0$, Now, (1) $\Rightarrow \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{\log(x+i)}{(x^2+1)}\,dx = \pi\left(\log 2 + \frac{\pi i}{2}\right)$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(x^2+1)}{x^2+1}\,dx = \pi\log 2 \Rightarrow 2\int_0^{\infty} \frac{\frac{1}{2}\log(x^2+1)}{x^2+1}\,dx = \pi\log 2 \Rightarrow \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = \pi\log 2$$

7.33 Show that if $a \ge b \ge 0$, then $\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = \pi(b-a)$.

Solution Consider $\int_C f(z)\,dz$, where $f(z) = \frac{e^{i2az} - e^{i2bz}}{z^2}$

The given function has a pole at $z = 0$ on the real axis.

We choose the contour C to be a large semicircle $|z| = R$ indented at $z = 0$ and r be the radius of this small circle of indentation.

Hence by Cauchy's residue theorem, $\int_C f(z)dz = 2\pi i \sum R^+ = 0$

$$\int_{-R}^{r} f(x)dx + \int_{\gamma} f(z)dz + \int_r^R f(x)dx + \int_{\Gamma} f(z)dz = 0 \text{ or, } I_1 + I_2 + I_3 + I_4 = 0 \quad (2)$$

We have $I_4 = \int_{\Gamma} f(z)\,dz = \int_{\Gamma} \frac{e^{i2az} - e^{i2bz}}{z^2}\,dz$

Now $\lim_{R\to\infty} \int_{\Gamma} \frac{e^{i2az} - e^{i2bz}}{z^2}\,dz = 0$, $\left(\because \lim_{z\to\infty} \frac{1}{z^2} = 0\right)$

$I_2 = \int_{\gamma} f(z)\,dz$, γ is described in clock-wise direction and γ is the circle $|z| = r$

Now, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\frac{e^{i2az} - e^{i2bz}}{z^2} = \lim_{z\to 0} \frac{\left(1 + 2aiz + \frac{4a^2i^2z^2}{\underline{|2}} + \dots\right) - \left(1 + 2biz + \frac{4b^2i^2z^2}{\underline{|2}} + \dots\right)}{z}$

$= 2(a-b)i$, therefore, $\lim_{r\to 0} \int_{\gamma} f(z)dz = -i2(a-b)i(\pi - 0) = 2\pi(a-b)$, negative sign is taken because γ is described in clock-wise direction. Making $R \to \infty$ and $r \to 0$ in (1),

$$\int_{-\infty}^{0} f(x)dx + 2\pi(a-b) + \int_0^{\infty} f(x)dx + 0 = 0 \Rightarrow \int_{-\infty}^{\infty} f(x)dx = 0 - 2\pi(a-b)$$

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{i2ax} - e^{i2bx}}{x^2}\,dx = -2\pi(a-b)$, equating real part from both sides, we have

$$\int_{-\infty}^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = 2\pi(b-a) \Rightarrow \int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = \pi(b-a).$$

7.34 Prove that $\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4} - \frac{\pi}{4a^3} e^{-ma}\left(m + \frac{2}{a}\right)$, where $m > 0$, $a > 0$.

Solution Consider $\int_C f(z)dz$, where $f(z) = \frac{e^{imz}}{z(z^2+a^2)^2}$

Poles of $f(z)$ are given by $z(z^2+a^2)^2 = 0$ i.e. $z = 0$, $z = \pm ai$ (twice)

The given function $f(z)$ has pole at $z = 0$ on the real axis.

Other poles of $f(z)$ are $z = ai, -ai$ (each of order two). We choose only $z = ai$ (order two) which lies within C.

$$\text{Residue (at } z = ai) = \frac{\phi'(ai)}{\underline{|1}} = \phi'(ai) \quad (1)$$

Now, $\phi(z) = (z-ai)^2 f(z) = (z-ai)^2 \frac{e^{imz}}{z(z-ai)^2(z+ai)^2}$

$$\phi'(z) = \frac{e^{imz}[imz^2 - amz - 3z - ai]}{z^2(z+ai)^3}$$

$$\phi'(ai) = \frac{-e^{-am}.(am+2)}{4a^4}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \sum R^+ = 2\pi i.\frac{-e^{-am}.(am+2)}{4a^4}$$

or $\int_{-R}^{-r} f(x)dx + \int_{\gamma} f(z)dz + \int_r^R f(x)dx + \int_{\Gamma} f(z)dz = \frac{-\pi i e^{-am}}{2a^4}.(am+2)$

or $I_1 + I_2 + I_3 + I_4 = \frac{-\pi i e^{-am}}{2a^4}.(am+2) \quad (2)$

$I_4 = \int_{\Gamma} f(z)dz = \int_{\Gamma} \frac{e^{imz}}{z(z^2+a^2)^2}\,dz$, since $\lim_{z\to\infty} \frac{1}{z(z^2+a^2)^2} = 0 \therefore \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0$,

$I_2 = \int_{\Gamma} f(z)dz$, where γ is described in clockwise direction and γ is the circle $|z| = r$.

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z.e^{imz}}{z(z^2+a^2)^2} = \frac{1}{a^4}$$

$\therefore \lim_{r\to 0} \int_{\gamma} f(z)dz = -i\frac{1}{a^4}.(\pi - 0)$, where negative sign indicates clockwise direction of γ

Making $R \to \infty$, $r \to 0$ in (2)

$$\int_{-\infty}^{0} f(x)\,dx + \left(\frac{-\pi i}{a^4}\right) + \int_0^{\infty} f(x)dx + 0 = \frac{-\pi i e^{-am}}{2a^4}.(am+2)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi i}{a^4} - \frac{\pi i e^{-am}}{2a^4}\cdot(am+2) \Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2}\,dx = \frac{\pi i}{a^4}\left[1-\frac{e^{-am}}{2}(am+2)\right].$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi i}{a^4}\left[1-\frac{e^{-am}}{2}(am+2)\right]$$

Equating imaginary part, $\Rightarrow \displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{a^4}\left[1-\frac{e^{-am}}{2}(am+2)\right].$

$$\Rightarrow \int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4}\left[1-\frac{e^{-am}}{2}(am+2)\right] = \frac{\pi}{2a^4} - \frac{\pi}{4a^3}e^{-am}\left(m+\frac{2}{a}\right)$$

7.35 Prove that $\displaystyle\int_0^{\infty} \frac{x^4}{x^6-1}\,dx = \frac{\pi}{6}\sqrt{3}.$

Solution Consider $\int_C f(z)dz$, where $f(z) = \dfrac{z^4}{z^6-1}$

Poles of $f(z)$ are given by $z^6 - 1 = 0 \Rightarrow z = (1)^{\frac{1}{6}} \Rightarrow z = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{6}}$ $(n = 0, 1, 2, 3, 4, 5)$.

$$\Rightarrow z = \cos\frac{2n\pi}{6} + i\sin\frac{2n\pi}{6} \Rightarrow z = e^{\frac{in\pi}{3}}$$

$$\Rightarrow z = 1, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, -1, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}.$$

Out of these six poles, the last two having their amplitude more than π don't lie within C. The poles $\alpha = e^{\frac{\pi i}{3}}, \alpha^2 = e^{\frac{2\pi i}{3}}$ lie within C, where as the poles given by $z = 1, z = -1$ lie on the real axis.

We choose the contour C to be a large semicircle $|z| = R$ indented at $z = -1, z = 1$ by small semicircle γ_1, γ_2 of radius r_1 and r_2 respectively.

Points A, B, C, D are $-(1+r_1), -(1-r_1), 1-r_2, 1+r_2$ respectively. Let α denotes the pole within C.

$\therefore\ \alpha^6 - 1 = 0 \Rightarrow \alpha^6 = 1.$

Residue at $z = \alpha$ is $\lim\limits_{z\to\alpha}(z-\alpha)f(z)$

$$= \lim_{z\to\alpha}(z-\alpha)\frac{z^4}{z^6-1} = \lim_{z\to\alpha}\frac{z^4(z-\alpha)}{z^6-\alpha^6}\ [\because \alpha^6 = 1]\left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{z\to\alpha}\frac{5z^4 - \alpha.4z^3}{6z^5} = \frac{5\alpha^4 - \alpha.4\alpha^3}{6\alpha^5} = \frac{\alpha^4}{6\alpha^5} = \frac{1}{6\alpha}$$

Residue at $z = e^{\frac{\pi i}{3}}$ is $\dfrac{1}{6}e^{-\frac{\pi i}{3}} = \dfrac{e^{-\frac{\pi i}{3}}}{6}.$

Residue at $z = e^{\frac{2\pi i}{3}}$ is $\dfrac{1}{6}e^{-\frac{2\pi i}{3}} = -\dfrac{1}{6}e^{\pi i}.e^{-\frac{2\pi i}{3}} = -\dfrac{1}{6}e^{\frac{\pi i}{3}}.$

$$\sum R^+ = \frac{-1}{6}\left[e^{\frac{\pi i}{3}} - e^{-\frac{\pi i}{3}}\right] = -\frac{1}{6}.2i.\sin\left(\frac{\pi}{3}\right) = -\frac{i}{3}.\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}i}{6}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\sum R^+ = 2\pi i.\left(\frac{-\sqrt{3}i}{6}\right) = \frac{\sqrt{3}\pi}{3}$$

or $\displaystyle\int_{-R}^{-(1+r_1)} f(x)dx + \int_{\gamma_1} f(z)dz + \int_{-(1-r_1)}^{1-r_2} f(x)dx + \int_{\gamma_2} f(z)dz + \int_{1+r_2}^{R} f(x)dx + \int_{\Gamma} f(z)dz = \frac{\sqrt{3}\pi}{3}$

Making $r_1, r_2 \to 0$ and $R \to \infty$

$$\int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx + \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\Gamma} f(z)dz = \frac{\sqrt{3}\pi}{3}$$

or $$\int_{-\infty}^{\infty} f(x)dx + I_1 + I_2 + I_3 = \frac{\sqrt{3}\pi}{3} \qquad (1)$$

$I_1 = \int_{\gamma_1} f(z)dz$, γ_1 is described in clockwise direction.

$$\lim_{z\to-1}(z+1)f(z) = \lim_{z\to-1}(z+1).\frac{z^4}{z^6-1} = \lim_{z\to-1}\frac{z^5+z^4}{z^6-1} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{z\to-1}\frac{5z^4+4z^3}{6z^5} = \frac{5-4}{-6} = -\frac{1}{6}.$$

$$\lim_{r_1\to 0}\int_{\gamma_1} f(z)dz = -i\left(-\frac{1}{6}\right)(\pi - 0) = \frac{\pi i}{6}$$

$I_2 = \int_{\gamma_2} f(z)dz$, γ_2 is described in clockwise direction.

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1).\frac{z^4}{z^6-1} = \lim_{z\to 1}\frac{z^5-z^4}{z^6-1}. \quad \left[\frac{0}{0}\right]$$

$$= \lim_{z\to 1}\frac{5z^4-4z^3}{6z^5} = \frac{5-4}{6} = \frac{1}{6}$$

$$\lim_{r_2\to 0}\int_{\gamma_2} f(z)dz = -i\left(\frac{1}{6}\right)(\pi - 0) = -\frac{\pi i}{6}.$$

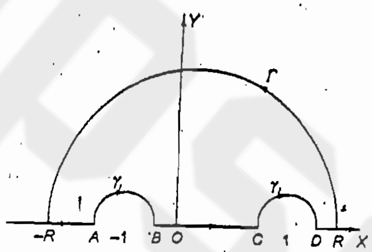
$$I_3 = \int_{\Gamma} f(z)dz = \int_{\Gamma}\frac{z^4}{z^6-1}dz$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z.\frac{z^4}{z^6-1} = \lim_{z\to\infty}\frac{z^5}{z^6\left(1-\frac{1}{z^6}\right)} = 0 \ \therefore \lim_{R\to\infty}\int_{\Gamma} f(z)dz = 0$$

Hence from (1),

$$\int_{-\infty}^{\infty} f(x)dx + \left[\frac{\pi i}{6} - \frac{\pi i}{6} + 0\right] = \frac{\sqrt{3}\pi}{3}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\sqrt{3}\pi}{3} \Rightarrow \int_{-\infty}^{\infty} \frac{x^4}{x^6-1}\,dx = \frac{\pi\sqrt{3}}{3} \Rightarrow \int_0^{\infty} \frac{x^4}{x^6-1}\,dx = \frac{\pi\sqrt{3}}{6}$$



7.36 Use the method of residue to show that

$$L^{-1}\left\{\frac{1}{(p+1)(p-2)^2}\right\} = \frac{1}{9}e^{-t} + \frac{t}{3}e^{2t} - \frac{1}{9}e^{2t}$$

Solution Using complex inversion formula, we have

$$L^{-1}\left\{\frac{1}{(p+1)(p-2)^2}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{e^{pt}}{(p+1)(p-2)^2}\,dp = \frac{1}{2\pi i}\int_C \frac{e^{pt}}{(p+1)(p-2)^2}\,dp$$

$$= \text{sum of the residues of } \frac{e^{pt}}{(p+1)(p-2)^2} \text{ at simple pole } p=-1 \text{ and double pole } p=2$$

$$= \lim_{p\to -1} \frac{e^{pt}}{(p-2)^2} + \lim_{p\to 2} \frac{d}{dp}\left(\frac{e^{pt}}{p+1}\right) = \frac{e^{-t}}{9} + \frac{(3t-1)e^{2t}}{9}$$

7.37 Find the Laplace Transform of the function $F(t) = \frac{e^{at}-1}{a}$. Prove, by the method of contour integration, that $F(t)$ is itself the inverse Laplace Transform of the function arrived at.

Solution Let $F(t) = \frac{e^{at}-1}{a}$ and $L\{F(t)\} = f(p)$ and

$$f(p) = L\left\{\frac{e^{at}-1}{a}\right\} = \frac{1}{a}\left[L(e^{at}) - L(1)\right]$$

$$= \frac{1}{a}\left[\frac{1}{p-a} - \frac{1}{p}\right] = \frac{1}{p(p-a)}$$

Also, by complex Inversion formula

$$L^{-1}\{f(p)\} = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} e^{pt} f(p)\,dp = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{e^{pt}}{p(p-a)}\,dp = \frac{1}{2\pi i}\int_C \frac{e^{pt}}{p(p-a)}\,dp$$

$$= \text{sum of the residues of } \frac{e^{pt}}{p(p-a)} \text{ at simple pole } p=0 \text{ and } p=a$$

$$= \lim_{p\to 0} \frac{e^{pt}}{p-a} + \lim_{p\to a} \frac{e^{pt}}{p} = -\frac{1}{a} + \frac{e^{at}}{a} = \frac{e^{at}-1}{a} = F(t).$$

7.38 Use the method of residue to find

$$L^{-1}\left\{\frac{1}{(p-1)(p+2)(p-3)}\right\}$$

Solution Using complex inversion formula, we have

$$L^{-1}\left\{\frac{1}{(p-1)(p+2)(p-3)}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{e^{pt}}{(p-1)(p+2)(p-3)}\,dp$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{pt}}{(p-1)(p+2)(p-3)}\,dp$$

$$= \text{sum of the residues of } \frac{e^{pt}}{(p-1)(p+2)(p-3)} \text{ at simple pole } p=1,\ p=-2 \text{ and } p=3$$

$$= \lim_{p\to 1} \frac{e^{pt}}{(p+2)(p-3)} + \lim_{p\to -2} \frac{e^{pt}}{(p-1)(p-3)} + \lim_{p\to 3} \frac{e^{pt}}{(p-1)(p+2)}$$

$$= -\frac{1}{6}e^{t} + \frac{1}{15}e^{-2t} + \frac{1}{10}e^{3t}.$$

7.39 Prove that $\int_0^{\pi/2} \ln \sin x\,dx = \int_0^{\pi/2} \ln \cos x\,dx = -\frac{1}{2}\pi \ln 2$.

Solution Letting $x = \tan\theta$ in the result of Problem 7.31, we find

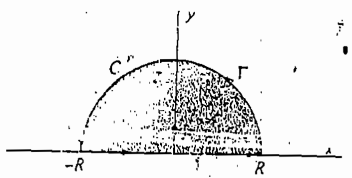
$$\int_0^{\pi/2} \frac{\ln(\tan^2\theta + 1)}{\tan^2\theta + 1}\sec^2\theta\,d\theta = -2\int_0^{\pi/2} \ln\cos\theta\,d\theta = \pi \ln 2$$

from which

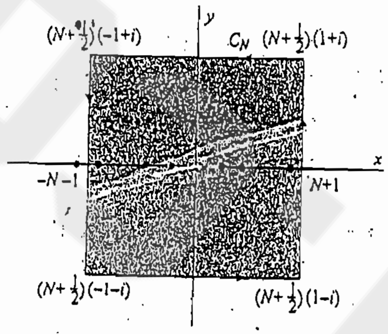
$$\int_0^{\pi/2} \ln\cos\theta\,d\theta = -\frac{1}{2}\pi\ln 2 \qquad (1)$$

which establishes part of the required result. Letting $\theta = \pi/2 - \phi$ in (1), we find

$$\int_0^{\pi/2} \ln\sin\phi\,d\phi = -\frac{1}{2}\pi\ln 2$$



y
C
Γ
−R
R



y
$(N+\frac{1}{2})(-1+i)$
C_N
$(N+\frac{1}{2})(1+i)$
x
$-N-1$
$N+1$
$(N+\frac{1}{2})(-1-i)$
$(N+\frac{1}{2})(1-i)$

IAS Previous Years Questions (1983–2012) Segment-wise

Complex Analysis

1983

- Obtain the Taylor and Laurent series expansions which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions (i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.
- Use the method of contour integration to evaluate $\int_0^\infty \frac{x^{a-1}}{1+x^2}dx, 0<a<2$.

1984

- Evaluate by contour integration method :

  (i) $\int_0^\infty \frac{x\sin mx}{x^4+a^4}dx$

  (ii) $\int_0^\infty \frac{x^{a-1}\log x}{1+x^2}dx$ *(1998, 1999)*
- Distinguish clearly between a pole and an essential singularity. If $z=a$ is an essential singularity of a function $f(z)$, then for an arbitrary positive integers $\eta, \in$ and ρ prove that $\exists$ a point z, such that $0<|z-a|<\rho$ for which $|f(z)-\eta|<\in$.

1985

- Prove that every power series represents an analytic function within its circle of convergence.
- Prove that the derivative of a function analytic in a domain is itself an analytic function.
- Evaluate, by the method of contour integration $\int_0^\infty \frac{x\sin ax}{x^2-b^2}dx$.

1986

- Let $f(z)$ be single valued and analytic with in and on a closed curve C. If z_0 is any point interior to C, then show that $f(z_0)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz$, where the integral is taken in the +ve sense around C.
- By contour integration method show that

  (i) $\int_0^\infty \frac{dx}{x^4+a^4}=\frac{\pi\sqrt{2}}{4a^3}$, where $a>0$.

  (ii) $\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$.

1987

- By considering the Laurent series for $f(z)=\frac{1}{(1-z)(z-2)}$ prove that if 'C' be a closed contour oriented in the contour clockwise direction, then $\int_C f(z)\,dx=2\pi i$.
- State and prove Cauchy's residue theorem
- By the method of contour integration, show that $\int_0^\infty \frac{\cos x}{x^2+a^2}dx=\frac{\pi e^{-a}}{2a}, a>0$.

1988

- By evaluating $\int \frac{dz}{z+2}$ over a suitable contour C, Prove that $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta}d\theta$ *(1997)*
- If f is analytic in $|Z|\le R$ and x, y lie inside the disc, evaluate the integral $\int_{|z|=R}\frac{f(z)dz}{(z-x)(z-y)}$ and deduce that a function analytic and bounded for all finite z is a constant.
- If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ has radius of convergence R and prove that $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^2 d\theta=\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$

- Evaluate $\int_C \frac{Ze^z}{(z-a)^3}$, if a lies inside the closed contour C.

- Prove that $\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2};(b>0)$ by the integrating e^{-z^2} along the boundary of the rectangle $|x| \le R, 0 \le y \le b$. *(1997)*

- Prove that the coefficients C_n of the expansion $\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} C_n z^n$ satisfy $C_n = C_{n-1} + C_{n-2}, n \ge 2$

  Determine C_n.

1989

- Find the singularities of $\sin\left(\frac{1}{1-z}\right)$ in the complex plane.

1990

- Let f be regular for |Z| < R, prove that, if $0<r<R$,

  $f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta)e^{-i\theta}\,d\theta$; where $u(\theta) = \text{Re } f(re^{i\theta})$

- Prove that the distance from the origin to the nearest zero of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is at least $\frac{r|a_0|}{M+|a_0|}$, where r is any number not exceeding the radius of the convergence of the series and $M = M(r) = \sup_{|z|=r}|f(z)|$

- Prove that $\int_{-\infty}^{\infty}\frac{x^4}{1+x^8}dx = \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{8}$ using residue calculus.

- Prove that if f=u+iv is regular through out the complex plane and au+bv-c ≥ 0 for suitable constants a,b,c then f is constant.

- Derive a series expansion of $\log(1+e^z)$ in powers of z.

- Determine the nature of singular points $\sin\left(\frac{1}{\cos \frac{1}{z}}\right)$ and investigate its behaviour at $z=\infty$.

1991

- A function f(z) is defined for finite values of z by f(0)=0 and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy Riemann equation are satisfied at the origin. Show also that f(z) is not analytic at the origin.

- If $|a| \ne R$ show that $\int_{|z|=R}\frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{\left|R^2-|a|^2\right|}$

- If $J_n(t) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)\,d\theta$, show that

  $e^{\frac{t}{2}\left(z-\frac{1}{z}\right)} = J_0(t) + zJ_1(t) + z^2J_2(t) + ---$

  $-\frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + ---$

- Examine the nature of the singularity of e^z at infinity.

- Evaluate the residues of the function $\frac{z^3}{(z-2)(z-3)(z-5)}$ at all singularities and show that their sum is zero.

  By integrating along a suitable contour show that

  $$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin a\pi} \quad \text{where } 0<a<1.$$

1992

- If $u = e^{-x}$ (x siny-ycosy), find 'v' such that f(z)= u+iv is analytic. Also find f(z) explicitly as a function of z. *(1997)*

- Let f(z) be analytic inside and on the circle C defined by $|z| = R$ and let $z = re^{i\theta}$ be any point inside C. prove that

  $$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(Re^{i\phi})}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi.$$

- Prove that all roots of $z^7-5z^3+12=0$ lies between the circles $|z|=1$ and $|z|=2$. *(1998,2006)*

- Find the region of convergence of the series whose n-th term is $\frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$

- Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for
  $(i)\ |z| > 3\ (ii)\ 1 < |z| < 3\ (iii)\ |z| < 1$ *(2005)*

- By integrating along a suitable contour evaluate $\int_0^\infty \frac{\cos mx}{x^2+1} dx$

1993

- In the finite Z- plane show that the function $f(z) = \sec\frac{1}{z}$. has infinitely many isolated singularities in a finite interval which includes '0'.

- Prove that (by applying Cauchy integral formula or otherwise)
  $$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} 2\pi,$$
  where n = 1,2,3...

- If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) and (2,3) find the value of $\int_C (12z^2 - 4iz)\,dz$

- Prove that $\sum_{n=1}^\infty \frac{z^n}{n(n+1)}$ converges absolutely for $|z| \le 1$.

- Evaluate $\int_0^\infty \frac{dx}{x^6+1}$ by choosing an appropriate contour.

1994

- How many zeros does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$ possess in (i) the first quadrant (ii) the fourth quadrant.

- Test for uniform convergence in the region $|Z| \le 1$ the series $\sum_{n=1}^\infty \frac{\cos nz}{n^3}$.

- Find Laurent series for (i) $\frac{e^{2z}}{(z-1)^3}$ about z=1 (ii) $\frac{1}{z^2(z-3)^2}$ about z = 3.

- Find the residues of $f(z) = e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

- By means of contour integration, evaluate $\int_0^\infty \frac{(\log_e u)^2}{u^2+1} du$

1995

- Let $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that 'u' is a harmonic function. Find a harmonic function v such that u+iv is an analytic function of z.

- Find the Taylor series expansion of the function $f(z) = \frac{z}{z^4+9}$ around $z = 0$. Find also the radius of convergence of the obtained series.

- Let C be the circle $|Z| = 2$ described contour clockwise. Evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2+1)} dz$

- Let $a \ge 0$. Evaluate the integral $\int_0^\infty \frac{\cos ax}{x^2+1} dx$ with the aid of residues. *(2006)*

- Let f be analytic in the entire complex plane. Suppose that there exists a constant A > 0 such that $|f(z)| \le A\,|z|$ for all z. Prove that there exists a complex number 'a' such that f(z)=az for all z.

- Suppose a power series $\sum_{n=0}^\infty a_n z^n$ converges at a point $z_0 \ne 0$.
  Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \ne 0$ $|z_1| < |z_0|$ and $z_1 \ne 0$ show that the series converges uniformly in the disc $\{z : |z| \le |z_1|\}$.

1996

- Evaluate $\lim_{z\to 0} \frac{1-\cos z}{\sin(z^2)}$

- Show that z=0 is not a branch point for the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity ?

- Prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n = 0, a_n \neq 0, n \geq 1$ has exactly 'n' roots.

- By using residue theorem, evaluate $\int_0^{\infty} \frac{\log_e(x^2+1)}{x^2+1}dx$

- About the singularity $z = -2$, find the Laurent expansion of $(z-3)\sin\frac{1}{z+2}$. Specify the region of convergence and nature of singularity at $z = -2$.

1997

- If $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_n}{(z-a)^n}$ find the residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2, \ldots A_n$, a & b are constant. What is the residue at infinity.

- Find the Laurent series for the function $e^{1/z}$ in $0 < |z| \leq \infty$.

Deduce that $\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)d\theta = \frac{1}{n!}$

$(n = 0, 1, 2, \ldots\ldots)$

(2001)

- Find the function f(z) analytic with in the unit circle which takes the values

$\frac{a-\cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$, $0 \leq \theta \leq 2\pi$ on the circle.

- Integrating e^{-z^2} along a suitable rectangular contour. Show that $\int_0^{\infty} e^{-x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$.

1998

- Show that the function

$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0$

$f(0) = 0$ is continuous and C-R conditions are satisfied at z=0, but $f'(z)$ does not exist at z=0.

- Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence and the nature of singularity at z = -2

- By using the integral representation of $f^n(0)$ prove that $\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{x^n e^{xz}}{n!\,z^{n+1}}dz$ where 'c' is any closed contour surrounding the origin. Hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}d\theta$

- Using residue theorem $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

1999

- Examine the nature of the function $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0$ $f(0) = 0$ in a region including the origin and hence show that Cauchy – Riemann equations are satisfied at the origin but f(z) is not analytic there.

- For the function $f(z) = \frac{-1}{z^2-3z+2}$, find Laurent series for the domain (i) 1 < |z| < 2 (ii) | z | >2 show further that $\oint_c f(z)dz = 0$ where 'c' is any closed contour enclosing the points z=1 and z=2.

- Using residue theorem show that $\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}dx = \frac{\pi}{2}e^{-a}\sin a, (a > 0)$ *(1984, 1998)*

- The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If f(2)=5 and f(-1)=2, find f(z).

❖ What kind of singularities the following functions have?

(i) $\frac{1}{1-e^z}$ at $z = 2\pi i$

(ii) $\frac{1}{\sin z - \cos z}$ at $z = \frac{\pi}{4}$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$

In case (iii) above what happens when 'a' is an integer (including a=0)?

2000

❖ Suppose $f(\xi)$ is continuous on a circle C. show that $\int_C \frac{f(\xi)}{(\xi - z)} d\xi$ as z varies inside of 'C', is differentiable under the integral sign. Find the derivative hence or otherwise derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C. (30)

2001

❖ Prove that the Riemann Zeta function ξ defined by $\xi(z) = \sum n^{-z}$ converges for $Re\, z > 1$ and converges uniformly for $Re\, z \geq 1+\in$ where $\in > 0$ is arbitrary small. (12)

❖ Show that $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$

2002

❖ Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for n=1,2,3,... then show that f(z)=g(z) for each z in $\mathbb{C}$ (12)

❖ Show that when 0 < | z-1 | < 2, the function $f(z) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of z-1 as

$$-\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$$ (15)

2003

❖ Use the method of contour integration to prove that $\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}; (a > 0)$. (15)

2004

❖ If all zeros of a polynomial p(z) lie in a half plane then show that zeros of the derivative p'(z) also lie in the same half plane. (15)

❖ Using contour integration, evaluate

$$\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{1 - 2p\cos 2\theta + p^2}, 0 < p < 1.$$ (15)

2005

❖ If f(z) = u + iv is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$ determine f(z) in terms of z. (12)

❖ Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for (i) 1<|z|<3 (ii) |z| >3 (iii) |z|< 1. (30)

2006

❖ With the aid of residues, evaluate

$$\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}; -1 < a < 1.$$ (15)

2007

❖ Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not differentiable at z=0 (12)

❖ Evaluate (by using residue theorem) $\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta}$ (15)

2008

- Find the residue of $\frac{\cot z \cot hz}{z^3}$ at $z = 0$. *(12)*
- Evaluate $\int_c \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2}\right] dz$ where c is the circle $|Z|=3$. State the theorem you use in evaluating above integral. *(15)*
- Let f(z) be entire function satisfying $f(z) \le k|z|^2$ for some +ve constant k and all z. show that $f(z) = az^2$ for some constant a. *(15)*

2009

- Let $f(z) = \frac{a_0 + a_1 z + \dots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \dots + b_n z^n}, b_n \ne 0.$, Assume that the zeroes of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\frac{a_{n-1}}{b_n}$. *(12)*
- If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$
  Show that: $\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$ *(30)*

2010

- Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of u(x, y). Hence find the analytic function f for which u(x, y) is the real part. *(12)*
- (i) Evaluate the line integral $\int_c f(z) dz$.
  Where $f(z) = z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.
  (ii) Find the image of the finite vertical strip R : x = 5 to x = 9, $-\pi \le y \le \pi$ of z – plane under the exponential function. *(15)*
- Find the Laurent series of the function $f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$ as $\sum_{n=-\infty}^{\infty} c_n z^n$ for $0 < |z| < \infty$
  Where $C_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \lambda\sin\phi) d\phi, n = 0, \pm1, \pm2, \dots,$ with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}(-\pi \le \phi \le \pi)$ as contour in this region. *(15)*

2011

- Evaluate by Contour integration, $\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}}$. *(15)*
- Find the Laurent Series for the function $f(z) = \frac{1}{1-z^2}$ with centre $z=1$. *(15)*
- Show that the series for which the sum of first n terms $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$ cannot be differentiated term-by-term at x=0. What happens at $x \ne 0$? *(15)*
- If f(z)=u+iv is an analytic function of z=x+iy and $u - v = \frac{\cos x + \sin x - e^{-y}}{\cos hy - \cos x}$, find f(z) subject to the condition, $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$ *(12)*

2012

- Show that the function defined by
  $$f(z) = \begin{cases} \frac{x^3 y^5 (x+iy)}{x^6 + y^{10}}, & z \ne 0 \\ 0, & z = 0 \end{cases}$$
  is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. *(12)*
- Use Cauchy integral formula to evaluate $\int_c \frac{e^{3z}}{(z+1)^4} dz$, where C is the circle $|z| = 2$. *(15)*
- Expand the function $f(z) = \frac{z}{(z+1)(z+3)}$ in Laurent series valid for
  (i) $1 < |z| < 3$ (ii) $|z| > 3$
  (iii) $0 < |z+1| < 2$ (iv) $|z| < 1$
  (15)
- Evaluate by Contour integration
  $$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}, a^2 < 1.$$ *(15)*

Previous Years Questions (2000–2012) Segment-wise

IFoS

Complex Anlysis

(According to the New Syllabus Pattern) Paper - II

2000

- Expand the function $f(z)=\log(z+2)$ in a power series and determine its radius of convergence.
- Prove that the function $f(z)=u+iv$ Where

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$f(0)=0$$

Satisfies Cauchy-Riemann equations at the origin, but $f'(0)$ does not exist.

2001

- Compute the Taylor series around $z=0$ and give the radius of convergence for $\frac{z}{z-1}$
- Show that the function $f(z)=\sqrt{|xy|}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied (13)
- By using the Residue Theorem evalutae the integral

$$\int_0^{2\pi}\frac{d\theta}{1-2a\sin\theta+a^2}\quad Where\ 0<a<1.$$ (14)

2002

- If $f(z)$ has a simple pole with residue K at the origin and is analytic on $0<|z|\leq|$ Show that

$$\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-a)(z-b)}dz=\frac{f(a)-f(b)}{a-b}+\frac{K}{ab}$$

Where $0<a<b<1$ and C is the circle $|z|=1$.

- If $f(a)=\oint_C\frac{3z^2+7z+1}{z-a}dz$ Where C is the circle $|z|=2$; Find

(i) $f(1-i)$; (ii) $f''(1-i)$; (iii) $f(1+i)$. (12)

- Under the bilinear transfonnation $w=\frac{3-z}{z-2}$ Find the images of

(1) $\left|z-\frac{5}{2}\right|=\frac{1}{2}$ and

(2) $\left|z-\frac{5}{2}\right|<\frac{1}{2}$ in the W-plane

2003

- If $w=f(z)=u(x,y)+iv(x,y)$, $z=x+iy$, is analytic in a domain, show that $\frac{\partial w}{\partial \bar{z}}=0$. Hence or otherwise show that $\sin(x+i3y)$ cannot be analytic
- Discuss the transformation $w=z+\frac{1}{z}$ and hence, show that
  (1) a circle in z-plane is mapped on an ellipse in the w-plane
  (2) a line in the z-plane is mapped into a hyperbola in the w-plane. (13)
- Find the Laurent series expansion of the function

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}$$ Valid in the region $2<|z|<3$. (13)

2004

- Investigate the continuity at (0,0) of the function

$$f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$ (10)

- Find the analytic function $f(z)=u(x,y)+iv(x,y)$ for which $u-v=e^x(\cos y-\sin y)$.