IMS

MATHS

BOOK-14

Set - IV

The Sphere

Sphere:

Defn: A sphere is the locus of a point which moves so that its distance from a fixed point always remains constant.

The fixed point is called the centre and the constant distance is called the radius of the sphere.

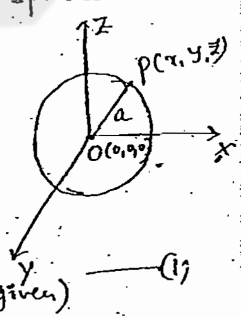
Equations of a sphere in different forms:

(a) Standard form:

To show that the eqn of the sphere whose centre is the origin and radius 'a' is $x^2+y^2+z^2=a^2$.

Proof: Let $P(x,y,z)$ be any point on the sphere.

Join OP.



Then

OP = radius of sphere $= a$ (given) ——(1)

By distance formula

$OP=\sqrt{x^2+y^2+z^2}$ ——(2).

From (1) & (2), we have

$\sqrt{x^2+y^2+z^2}=a$

$\Rightarrow x^2+y^2+z^2=a^2$

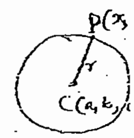
which is the required eqn of the sphere.

(b) Central form:

To find the eqn of a sphere whose centre is (a,b,c) and radius is r.

proof: Let $C(a,b,c)$ be the centre of the sphere.

Let $P(x,y,z)$ be the point on the sphere.



Join CP. Then

CP = radius of sphere $= r$ (given)

$\Rightarrow CP^2=r^2$

$\Rightarrow (x-a)^2+(y-b)^2+(z-c)^2=r^2$

which is the required eqn.

(C) General form:

To prove that the equation $x^2+y^2+z^2+2ux+2vy+2wz+d=$ represents a sphere and find its centre and radius.

proof: The given eqn is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

This can be written as

$$(x^2+2ux)+(y^2+2vy)+(z^2+2wz)+d=0$$

Adding $u^2+v^2+w^2$ to bothsides we get

$$(x^2+2ux+u^2)+(y^2+2vy+v^2)+(z^2+2wz+w^2) = -d+u^2+v^2+w^2$$

$$\Rightarrow (x+u)^2+(y+v)^2+(z+w)^2 = u^2+v^2+w^2-d.$$

$$\Rightarrow [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2 = \left[\sqrt{u^2+v^2+w^2-d}\right]^2 \quad \text{--- (2)}$$

which is clearly of the central form of the sphere.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad \text{--- (3)}$$

$\therefore$ (1) represents a sphere.

Now comparing (2) & (3), we have

$a=-u$, $b=-v$, $c=-w$ and

$$r=\sqrt{u^2+v^2+w^2-d}$$

$\therefore$ Centre of the sphere (1) is $(-u,-v,-w)$ and the radius is $\sqrt{u^2+v^2+w^2-d}$.

Note: If $u^2+v^2+w^2-d<0$ then the radius of the sphere imaginary and the centre $(-u,-v,-w)$ is real. In this case the sphere is called pseudo-sphere (or) a virtual sphere.

Working rule for finding the centre and radius of the sphere:

(1) First of all make the coefficients of $x^2, y^2, z^2 = 1$ if they are not so.

(2) Centre is

$$\left[-\tfrac{1}{2}\text{coeff of } x,\ -\tfrac{1}{2}\text{coeff of } y,\ -\tfrac{1}{2}\text{coeff of } z\right]$$

and radius is

$$\sqrt{\left(\tfrac{1}{2}\text{coeff of } x\right)^2+\left(\tfrac{1}{2}\text{coeff of } y\right)^2+\left(\tfrac{1}{2}\text{coeff of } z\right)^2 - \text{constant term}}$$

(1) Conditions for a sphere

The given eqn represents a sphere if

(i) it is a second degree in x, y, z

(ii) coeff of x^2 = coeff of y^2 = coeff of z^2,

and

(iii) it doesnot contain the terms involving the products xy, yz and zx.

(2) Since the general eqn of the sphere -

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Contains four unknown constants u, v, w, d.

So the sphere can be found to satisfy four conditions.

Four point form:

To find the eqn of a sphere passing through the given points.

Soln: Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the given four points.

Let the required eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

Since it passes through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

$$\therefore (x_1^2+y_1^2+z_1^2)+2ux_1+2vy_1+2wz_1+d \text{ ---}$$

$$(x_2^2+y_2^2+z_2^2)+2ux_2+2vy_2+2wz_2+d \text{ ---}$$

$$(x_3^2+y_3^2+z_3^2)+2ux_3+2vy_3+2wz_3+d \text{ ---}$$

$$(x_4^2+y_4^2+z_4^2)+2ux_4+2vy_4+2wz_4+d \text{ ---}$$

eliminating u, v, w, d from (1), (2), (3), (4) & (5) with the help of determinants, we have

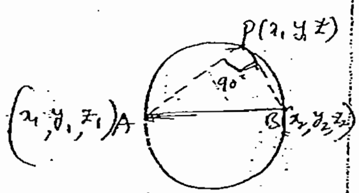
$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is the required equation.

Diameter form:-

To find the equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter.

Soln: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points.



Let $P(x, y, z)$ be any point on the sphere. Join AP and BP.

Since AB is diameter of the sphere then

$\angle APB$ = angle in the semi-circle $= 90°$

i.e, $AP \perp PB$ ——— ①

Now the d.r.'s of AP are $x-x_1,\ y-y_1,\ z-z_1$ and the d.r.'s of BP are $x-x_2,\ y-y_2,\ z-z_2$.

Since $AP \perp BP$

$$\therefore (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

which is the required eqn of the sphere.

Problems:

→ Find the radius and centre of the sphere $x^2+y^2+z^2-2x+4y-6z=2$.

Sol^n: This is comparing with the general eqn of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

we have

$2u=-2 \Rightarrow u=-1$ || $2v=4 \Rightarrow v=2$ || $2w=-6 \Rightarrow w=-3$

and $d=-2$

$\therefore$ centre is $(-u, -v, -w)$
$=(1, -2, 3)$

radius $= \sqrt{u^2+v^2+w^2-d}$
$= \sqrt{1+4+9+2}$
$= \sqrt{16} = 4$

H.W → Find the centres and radius of the following spheres.

(i) $x^2+y^2+z^2-6x+8y-10z+1=0$

(ii) $x^2+y^2+z^2+2x-4y-6z+5=0$

(iii) $2x^2+2y^2+2z^2-2x+4y+2z+3=0$

→ Obtain the eqn of the sphere described on the join of the points A $(2, -3, 4)$ B $(-5, 6, -7)$ as diameter.

Sol^n: Now let the required eqn of sphere on the join of two points (x_1, y_1, z_1) & (x_2, y_2, z_2) as diameter be

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$(x-2)(x+5)+(y+3)(y-6)+(z-4)(z+7)=0$$

$$\Rightarrow x^2+y^2+z^2+3x-3y+3z-56=0$$

→ Find the eqn of the sphere through the points $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Soln: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —(1)

Since it passes through $(0,0,0)$

$\therefore d=0$

(1) ≡ $x^2+y^2+z^2+2ux+2vy+2wz=0$ —(2)

Since it passes through $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$.

$\therefore$ we have

$a^2+2ua=0 \Rightarrow u=-\frac{a}{2}$

$b^2+2vb=0 \Rightarrow v=-\frac{b}{2}$

$c^2+2wc=0 \Rightarrow w=-\frac{c}{2}$

putting these values in (2) we get

$x^2+y^2+z^2-ax-by-cz=0$

which is the required equation.

Note: Eqn of the sphere OABC, where $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are three points on the axis is $x^2+y^2+z^2-ax-by-cz=0$

→ Find the eqn of the sphere circumscribing the tetrahedron $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=$

Soln: Three planes out of the given four planes taken at a time determine one vertex of the tetrahedron.

Hence the vertices of the tetrahedron are $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Remaining solution similar to previous problem.

→ Find the eqn of the sphere passing through the three points $(3,0,2)$, $(-1,1,1)$, $(2,-5,4)$ and having its centre on the plane $2x+3y+4z=6$ is

$x^2+y^2+z^2+4y-6z=1$.

Soln: Let the eqn of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —(1)

Since it passes through $(3,0,2)$, $(-1,1,1)$ and $(2,-5,4)$

$\therefore 9+4+6u+4w+d=0$

$\Rightarrow 6u+4w+13+d=0$ —(2)

$1+1+1-2u+2v+2w+d=0$

$\Rightarrow -2u+2v+2w+3+d=0$ ——— (3)

$4+25+16+4u-10v+8w+d=0$

$\Rightarrow 4u-10v+8w+45+d=0$ ——— (4)

Also centre $(-u,-v,-w)$ lies on the plane $2x+3y+4z-6=0$

$\therefore 2u+3v+4w+6=0$ ——— (5)

Solving the above eqns (2), (3), (4) & (5) we get u, v, w, d values.

putting these values in eqn (1)

which is the required sphere

→ Find the eqn to a sphere passing through the points $(1,-3,4)$, $(1,-5,2)$, $(1,-3,0)$ and having centre on the plane $x+y+z=0$.

→ Obtain the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the two points $(0,-2,-4)$, $(2,-1,-1)$.

Sol: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ——— (1)

Since its centre lies on the line $5y+2z=0=2x-3y$

$\therefore 5(-v)+2(-w)=0=2(-u)-3(-v)$

ie, $5v+2w=0$ ——— (2)

& $2u-3v=0$ ——— (3)

Since the sphere passes through the points $(0,-2,-4)$ and $(2,-1,-1)$

$\therefore 0+4+16+0-4v-8w+d=0$

$\Rightarrow -4v-8w+d+16=0$ ——— (4)

& $4+1+1+4u-2v-2w+d=0$

$\Rightarrow 4u-2v-2w+d+6=0$ ——— (5)

Solving the eqns (2), (3), (4) & (5)

we get

$u=-3$, $v=-2$, $w=5$

$d=12$.

$\therefore$ (1) ≡

$x^2+y^2+z^2-6x-4y+10z+12=0$

which is the required equation

→ P.T the eqn $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$ represents a sphere. Find its radius and centre.

[Ans: $\dfrac{\sqrt{\Sigma u^2-ad}}{a}$, $\left(-\dfrac{u}{a}, -\dfrac{v}{a}, -\dfrac{w}{a}\right)$]

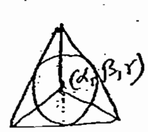
→ Find the eqn to the sphere through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$, $(1,2,3)$.

→ Find the eqn of the sphere through the four points $(4,-1,2)$, $(0,-2,3)$, $(1,-5,-1)$, $(2,0,1)$.

→ Find the eqn of the sphere through the four points $(0,0,0)$, $(-a,b,c)$, $(a,-b,c)$, $(a,b,-c)$ and determine its radius.

→ Find the eqn of the sphere inscribed in the tetrahedron whose faces are $x=0$, $y=0$, $z=0$ and $x+2y+2z=1$.

Solⁿ: The given faces are $x=0$, $y=0$, $z=0$ and $1-x-2y-2z=0$.



Let (α,β,γ) be the centre and 'r' the radius of the inscribed sphere. Then ⊥ distances of the centre from all the four faces are equal and each equal to radius.

$\therefore \dfrac{\alpha}{1}=\dfrac{\beta}{1}=\dfrac{\gamma}{1}=\dfrac{1-\alpha-2\beta-2\gamma}{\sqrt{1+4+4}}=$

$\therefore \alpha=\beta=\gamma=r$ and $1-\alpha-2\beta-2\gamma=r$

Eliminating α,β,γ, we get

$1-r-2r-2r=r$

$\Rightarrow 8r=1$

$\Rightarrow r=\dfrac{1}{8}$

$\therefore \alpha=\beta=\gamma=\dfrac{1}{8}$

$\therefore$ The centre is $(\alpha,\beta,\gamma)=\left(\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$ and the radius $(r)=\dfrac{1}{8}$

$\therefore$ The eqn of the sphere with centre $\left(\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$ and radius $\frac{1}{8}$ is

$$\left(x-\tfrac{1}{8}\right)^2+\left(y-\tfrac{1}{8}\right)^2+\left(z-\tfrac{1}{8}\right)^2=\left(\tfrac{1}{8}\right)^2$$

$$\Rightarrow x^2-\tfrac{1}{4}x+\tfrac{1}{64}+y^2-\tfrac{1}{4}y+\tfrac{1}{64}+z^2-\tfrac{1}{4}z+\tfrac{1}{64}=\tfrac{1}{64}$$

$$\Rightarrow x^2+y^2+z^2-\tfrac{1}{4}(x+y+z)+\tfrac{1}{32}=0$$

$$\Rightarrow 32(x^2+y^2+z^2)-8(x+y+z)+1=0$$

which is the required eqn.

→ A sphere is inscribed in the tetrahedron whose faces are $x=0, y=0, z=0$, $2x+6y+3z=14$. Find its centre, radius and write down its equation.

2002 → Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are $x=0, y=0, z=0$, $x+y+z=a$.

→ Find the eqn of the sphere inscribed in the tetrahedron whose faces are

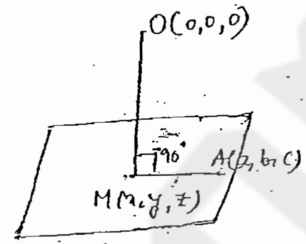
(i) $x=0, y=0, z=0, 2x-6y+3z+6=0$

2007 (ii) $x=0, y=0, z=0, 2x+3y+6z=6$

→ A plane passes through a fixed point (a, b, c), show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2+y^2+z^2-ax-by-cz=0$.

Sol^n: Let $A(a,b,c)$ be the fixed point on the variable plane α.

and let $M(x,y,z)$ be the foot of $\perp$ from the origin to the plane α.



$\therefore OM \perp MA$.

Now the d.r.'s of OM are x, y, z and the d.r.'s of MA are $x-a, y-b, z-c$.

Since $OM \perp MA$:

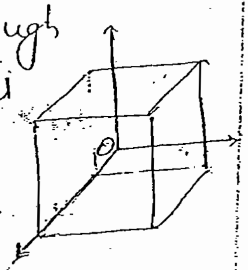
$\therefore x(x-a)+y(y-b)+z(z-c)=0$

$\Rightarrow x^2+y^2+z^2-ax-by-cz=0$

which is the required locus and it represents a sphere.

→ A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere.

Sol^n: Take the centre of the cube as the origin and the planes through the centre parallel to its faces as co-ordinate planes.



Let each of the eadge of the cube be equal to '2a'.

Then the eqns of the three pairs of parallel faces of the cube are

$x=a,\ x=-a,\ y=a,\ y=-a$

and $z=a,\ z=-a$.

Now let (α,β,γ) be any point in the locus.

Now the sum of squares of distances of P from the six faces is constant $=6k^2$ (say).

$$\left(\frac{\alpha-a}{1}\right)^2+\left(\frac{\alpha+a}{1}\right)^2+\left(\frac{\beta-a}{1}\right)^2+\left(\frac{\beta+a}{1}\right)^2+\left(\frac{\gamma-a}{1}\right)^2+\left(\frac{\gamma+a}{1}\right)^2=6k^2$$

$$\Rightarrow 2(\alpha^2+\beta^2+\gamma^2)+6a^2=6k^2$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2=3(k^2-a^2)$$

$\therefore$ Locus of $P(\alpha,\beta,\gamma)$ is

$$x^2+y^2+z^2=3(k^2-a^2).$$

which clearly represents a sphere.

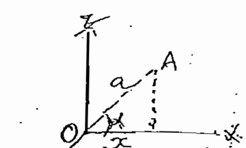
$\rightarrow$ OA, OB, OC are three mutually perpendicular lines through the origin whose direction cosines are $l_1,m_1,n_1;\ l_2,m_2,n_2;\ l_3,m_3,n_3$. If $OA=a,\ OB=b,\ OC=c$, show that the eqn of the sphere OABC is

$$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)-z(an_1+bn_2+cn_3)=0$$

Soln: Since l_1,m_1,n_1 are the actual d.c's of OA and $OA=a$.

$\therefore$ The co-ordinates of A are $(l_1a,\ m_1a,\ n_1a)$. | using (l_1,m_1,n_1)

Similarly, the co-ordinates of B & C are $(l_2a,\ m_2a,\ n_2a)$ and $(l_3a,\ m_3a,\ n_3a)$ respectively.



$\therefore \cos\alpha=l$

From fig. $\cos\alpha=\frac{x}{a}$

$\Rightarrow l=\frac{x}{a}$

$\Rightarrow x=la$

Also O is $(0,0,0)$.

Now let the required eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{---(1)}$$

Since it passes through $O(0,0,0)$

$\therefore$ ① $\equiv x^2+y^2+z^2+2ux+2vy+2wz=0$ — ② $(\because d=0)$

Since it passes through $A(l_1a, m_1a, n_1a)$

$\therefore l_1^2a^2+m_1^2a^2+n_1^2a^2+2ul_1a+2vm_1a+2wn_1a=0$

$\Rightarrow a+2ul_1+2vm_1+2wn_1=0$ — ③ $(\because l_1^2+m_1^2+n_1^2=0)$

Similarly for B and C

$b+2ul_2+2vm_2+2wn_2=0$ — ④

and $c+2ul_3+2vm_3+2wn_3=0$ — ⑤

Since the lines OA, OB, OC are mutually perpendicular.

$\therefore l_1, l_2, l_3;\ m_1, m_2, m_3;\ n_1, n_2, n_3$ are the d.c's of OX, OY, OZ referred to OA, OB, OC as axes.

So $l_1^2+l_2^2+l_3^2=1$, $m_1^2+m_2^2+m_3^2=1$ $n_1^2+n_2^2+n_3^2=1$ and

$l_1m_1+l_2m_2+l_3m_3=m_1n_1+m_2n_2+m_3n_3$
$=n_1l_1+n_2l_2+n_3l_3=0$

Now multiplying ③ by l_1; ④ by l_2; ⑤ by l_3 and adding, we get

$al_1+2ul_1^2+2vl_1m_1+2wn_1l_1$
$+bl_2+2ul_2^2+2vl_2m_2+2wn_2l_2$
$+cl_3+2ul_3^2+2vl_3m_3+2wn_3l_3=0$

$\Rightarrow al_1+bl_2+cl_3+2u(1)+2v(0)+2w(0)=0$

$\Rightarrow u=-\frac{1}{2}(al_1+bl_2+cl_3)$

Similarly $v=-\frac{1}{2}(am_1+bm_2+cm_3)$

$w=-\frac{1}{2}(an_1+bn_2+cn_3)$

Substituting in eqn ②, we get:

$x^2+y^2+z^2-x(al_1+bl_2+cl_3)$
$-y(am_1+bm_2+cm_3)$
$-z(an_1+bn_2+cn_3)=0$

which is the required equation.

1996 → Find the eqn of the sphere which passes through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and has its radius as small as possible.

Solⁿ: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — ①

Since it passes through $(1,0,0)$

$\therefore 1+2u+d=0$.

Since (1) passes through $(0,1,0)$

$1+2v+d=0$

Since (1) passes through $(0,0,1)$

$\therefore 1+2w+d=0$

from these, we have

$u=v=w=-\frac{(1+d)}{2}$

$\therefore (1) \equiv x^2+y^2+z^2-(1+d)x-(1+d)y-(1+d)z+d=0$ —— (2)

Centre is $(-u,-v,-w)$

$=\left(\frac{1+d}{2},\frac{1+d}{2},\frac{1+d}{2}\right)$

If 'R' is the radius of the sphere, then

$R^2=u^2+v^2+w^2-d$

$=3\left(\frac{1+d}{2}\right)^2-d$

$=\frac{3}{4}(1+d^2+2d)-d$

$=\frac{1}{4}(3+3d^2+6d-4d)$

$=\frac{1}{4}(3+3d^2+2d)$

$=\frac{3}{4}\left(d^2+\frac{2}{3}d+1\right)$

$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+\left(1-\frac{1}{9}\right)\right]$

$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+\frac{8}{9}\right]$

If $\left(d+\frac{1}{3}\right)=0$ then R^2 is least (i.e., R is least)

$\Rightarrow d=-\frac{1}{3}$

$\therefore (2) \equiv x^2+y^2+z^2-\left(1-\frac{1}{3}\right)x-\left(1-\frac{1}{3}\right)y-\left(1-\frac{1}{3}\right)z-\frac{1}{3}=0$

$\Rightarrow x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}=0$

$\Rightarrow 3(x^2+y^2+z^2)-2(x+y+z)-1=0$

which is the required eqn of the sphere.

1985 → A variable plane through a fixed point (a,b,c) cuts the co-ordinate axes in the points A, B, C. Show that the locus of the centres of the sphere OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Soln: Let the eqn of the plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ —— (1)

Since it passes through (a,b,c)

$\therefore \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ —— (2)

Since (1) cuts the axes in A, B, C.

$\therefore$ The co-ordinates of $A(\alpha,0,0)$, $B(0,\beta,0)$, $C(0,0,\gamma)$ and $(0,0,0)$.

Let the eqn of the sphere OABC be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (3)

Since it passes through $O(0,0,0)$.

$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$ —— (4)

and since it passes through $A(\alpha,0,0)$.

$\therefore \alpha^2+2u\alpha=0$

$\Rightarrow 2u=-\alpha$

Similarly $2v=-\beta$, $2w=-\gamma$

putting these values in (4), we get,

$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$ —— (5)

If (x_1,y_1,z_1) is the centre then $x_1=+\frac{\alpha}{2}$, $y_1=\frac{\beta}{2}$, $z_1=\frac{\gamma}{2}$

$\Rightarrow \alpha=2x_1$, $\beta=2y_1$, $\gamma=2z_1$

$\therefore (2) \equiv \frac{a}{2x_1}+\frac{b}{2y_1}+\frac{c}{2z_1}=1$

$\Rightarrow \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2$

$\therefore$ Locus of (x_1,y_1,z_1) is

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

→ A plane through a fixed point $(1,1,1)$ cuts the axes in A, B, C. Find the locus of the centre of the sphere OABC where O is origin.

→ A sphere of constant radius 2k passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron OABC.

Sol^n: Let co-ordinates of the points A, B, C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

The eqn of the sphere OABC is

$x^2+y^2+z^2-ax-by-cz=0$

Radius of this sphere

$=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2}=2k$ (given)

squaring on both sides, we get

$a^2+b^2+c^2=16k^2$ —— (1)

Let (x_1,y_1,z_1) be the co-ordinates of the tetrahedron OABC, then

$x_1=\frac{0+a+0+0}{4} \Rightarrow x_1=\frac{a}{4}$

$\Rightarrow a=4x_1$

Similarly $b=4y_1$ & $c=4z_1$

putting these values of a, b, c in (1), we get

$$16x_1^2 + 16y_1^2 + 16z_1^2 = 16k^2$$

$$\Rightarrow x_1^2 + y_1^2 + z_1^2 = k^2.$$

$\therefore$ Locus of the centroid (x_1, y_1, z_1) is

$$x^2 + y^2 + z^2 = k^2$$

1988, A sphere of constant radius k passes through the origin and meets the axes in A, B, C. prove that the centroid of the triangle ABC lies on the sphere

$$9(x^2 + y^2 + z^2) = 4k^2.$$

→ A variable sphere passes through the origin 'O' and meets the axes in A, B, C so that the volume of the tetrahedron OABC is constant. find the locus of the centre of the sphere.

2003 A sphere of constant radius 'r' passes through the origin 'O' and cuts the axes in A, B, C. find the locus of the foot of the perpendicular from 'O' to plane ABC.

Sol^n: Let the co-ordinates of the points A, B, C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Then the eq^n of the sphere OABC is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Its radius $= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}$
$= r$ (given)

$$\Rightarrow a^2 + b^2 + c^2 = 4r^2. \quad \text{--- (1)}$$

Now the eq^n of the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{--- (2)}$$

D.r's of the $\perp^r$ to this plane (2) are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

$\therefore$ Eq^{n}s of the line through $O(0,0,0)$ and $\perp^r$ to the plane (2) are

$$\frac{x-0}{1/a} = \frac{y-0}{1/b} = \frac{z-0}{1/c}$$

$\Rightarrow ax = by = cz$ ——— (3)

To find the locus ~~of foot~~ of $\perp^r$ from 'O' on the plane (2), i.e., the locus of the point of intersection of the plane (2) and line (3), we have to eliminate the unknown constants a, b, c from (1), (2) & (3).

Now from (3),

Let $ax = by = cz = \lambda$ (say)

$\Rightarrow a = \frac{\lambda}{x},\ b = \frac{\lambda}{y},\ c = \frac{\lambda}{z}$

putting these values in (1), we get

$$\lambda^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4r^2$$

$$\Rightarrow \lambda^2(x^{-2} + y^{-2} + z^{-2}) = 4r^2 \quad \text{——— (4)}$$

and putting the values of a, b, c in (2), we get

$$\frac{1}{\lambda}(x^2+y^2+z^2) = 1.$$

$$\Rightarrow \frac{1}{\lambda^2}(x^2+y^2+z^2)^2 = 1 \quad \text{——— (5)}$$

Multiplying (4) & (5), we get

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2}) = 4r^2.$$

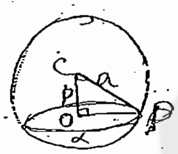
which is the required locus.

plane section of a sphere:

To prove that the section of a sphere by a plane is a circle.

Proof: Let 'C' be the centre of the sphere, 'a' its radius and α be the plane.

Draw CO $\perp^r$ from 'C' on the plane α and let $CO = p$.



'O' is the fixed point and p is a fixed length.

Let 'P' be any point on the section of the sphere by the plane 'α'. Join CP & OP

$\therefore CO \perp OP$

$\therefore$ In the right angled $\triangle COP$ we have

$OP^2 = CP^2 - CO^2$
$= a^2 - p^2$.

(or)

$OP = \sqrt{a^2-p^2}$ which is constant.

and O is fixed point.

$\therefore$ P lies on a circle whose centre is 'O' and radius is $\sqrt{a^2-p^2}$.

$\therefore$ The section of the sphere by a plane is a circle.

Equation of a circle:

Since the intersection of a sphere with a plane is a circle.

$\therefore$ in general a circle can be represented by the eqns of a sphere and a plane taken together.

i.e, the two eqns
$x^2+y^2+z^2+2ux+2vy+2wz+d=$
and $lx+my+nz=p$. taken together represent a circle.

Note:

I (1) The centre of the circle is the foot of $\perp^r$ from the centre of the sphere on the plane and

(2) Radius of the circle $= \sqrt{a^2-p^2}$, where 'a' is the radius of the sphere and p the length of $\perp$ from

the centre of the sphere on the plane.

[II]. The section of a sphere by a plane passing through the centre of the sphere is called a great circle. Its centre and radius is the same as that of the sphere.

problems :-

→ find the centre and ~~the~~ radius of the circle.

$x^2+y^2+z^2-2y-4z=11$, $x+2y+2z=15$.

Sol: The given sphere is

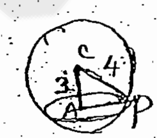
$$x^2+y^2+z^2-2y-4z-11=0 \quad \text{--- (1)}$$

Its centre is $(-u,-v,-w)=(0,1,2)$ and

radius $=\sqrt{1+4+11}=\sqrt{16}=4$

The given plane is

$$x+2y+2z=15 \quad \text{--- (2)}$$



Eqns (1) & (2) taken together represent a circle.

Now the centre of the circle is the foot of $\perp^r$ from the centre of the sphere (1) on the plane (2).

Now the dr's of the normal to the plane (2) are 1, 2, 2.

∴ Eqns of the line CA through C and $\perp^r$ to plane (2) are:

$$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{2}=r\ (\text{say}).$$

Any point on the line is

$$(r,\ 2r+1,\ 2r+2) \quad \text{--- (3)}$$

Let it be A

Since it lies on the plane (2):

$$(r)(1)+2(2r+1)+2(2r+2)=15$$

$$\Rightarrow 9r=9$$

$$\Rightarrow \boxed{r=1}$$

∴ (3) ≡ $A(1,\ 2(1)+1,\ 2(1)+2)$

$=(1,3,4)$

which is required centre of the circle.

Again $p=CA=$ distance from $C(0,1,2)$ to the plane $x+2y+2z=15$

$\therefore p = \frac{|0+2+4-15|}{\sqrt{1+4+4}}$

$= \frac{9}{3} = 3$

$\therefore$ The radius of the circle is

$AP = \sqrt{a^2 - p^2}$

$= \sqrt{(4)^2 - (3)^2}$

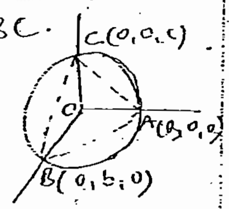
$= \sqrt{16-9}$

$= \sqrt{7}$

2001 → Find the eqns of the circle circumscribing the triangle formed by the three points $(a,0,0)$, $(0,b,c)$, $(0,0,c)$. Obtain also the co-ordinates of the centre of the circle.

Sol: Let the given points be $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

Then the circumcircle of ΔABC is the intersection of the plane ABC and the sphere OABC.



Now the plane ABC is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ —— (1) and the

eqn of the sphere OABC is $x^2+y^2+z^2 - ax - by - cz = 0$ —— (2)

$\therefore$ The eqns of the circle of ΔABC are

$x^2+y^2+z^2 - ax - by - cz = 0$,

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

remaining solution similar to previous problem.

→ Find the eqn of that plane which cuts the sphere $x^2+y^2+z^2 = a^2$ in a circle whose centre is (α, β, γ)

Sol: Since 'O' is the centre of the sphere and $A(\alpha,\beta,\gamma)$ is the centre of the circle.

$\therefore OA \perp^r$ to the required plane of the circle.

Now the d.r's of OA are $\alpha - 0, \beta - 0, \gamma - 0$

$\Rightarrow \alpha, \beta, \gamma$.

$\therefore$ The coeff of x, y, z in the eqn of the plane are α, β, γ ($\because$ OA is $\perp^r$ to the plane of circle).

$\therefore$ Eqn of the plane of the circle is

$$\alpha(x-\alpha)+\beta(x-\beta)+\gamma(x-\gamma)=0$$

1989 Show that the centres of all sections of the sphere $x^2+y^2+z^2=a^2$ by planes through a point (α, β, γ) lie on the sphere

$$x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0.$$

Soln: Let (x_1, y_1, z_1) be the centre of one of the sections, then the eqn of the plane is

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0 \quad (1)$$

Since it passes through the point (α, β, γ)

$$\therefore x_1(\alpha-x_1)+y_1(\beta-y_1)+z_1(\gamma-z_1)=0.$$

$$\Rightarrow x_1(x_1-\alpha)+y_1(y_1-\beta)+z_1(z_1-\gamma)=0.$$

$\therefore$ Locus of (x_1, y_1, z_1) is

$$x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0$$

which is the required eqn of the sphere.

$\rightarrow$ If 'r' be the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$
$$lx+my+nz=0.$$

prove that

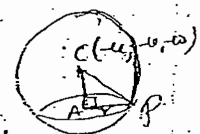
$$(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$$

Sol: The equations of the circle are

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad (1)$$

and $\quad lx+my+nz=0 \quad (2)$

The centre of the sphere (1) is $c(-u, -v, -w)$ and its radius $CP=\sqrt{u^2+v^2+w^2-d}$



Also $CA = \perp^r$ distance of $c(-u, -v, -w)$ from the plane (2)

$$= \frac{|l(-u)+m(-v)+n(-w)|}{\sqrt{l^2+m^2+n^2}}$$

$$= \frac{lu+mv+nw}{\sqrt{l^2+m^2+n^2}}$$

By the right angled ΔCAP

$AP^2 = CP^2 - CA^2$

$$\Rightarrow r^2 = (u^2+v^2+w^2-d) - \frac{(lu+mv+nw)^2}{l^2+m^2+n^2}$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (u^2+v^2+w^2)(l^2+m^2+n^2) - (lu+mv+nw)^2$$

By using the Lagrange's identity

$$= (mw-nv)^2 + (nu-lw)^2 + (lv-mu)^2$$

Hence the result.

$\boxed{\text{Lagrange's identity:}}$

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1l_2+m_1m_2+n_1n_2)^2 = (m_1n_2-n_1m_2)^2+(n_1l_2-l_2n_1)^2 + (l_1m_2-m_1l_2)^2$$

Note: The four points are said to be concyclic if the circle through any three points passes through the fourth point.

→ Show that the following sets of points are concyclic

(i) $(5,0,2)$, $(2,-6,0)$, $(7,-3,8)$, $(4,-9,6)$

(ii) $(-8,5,2)$, $(-5,2,2)$, $(-7,6,6)$, $(-4,3,6)$.

Sol: (i) Let the four given points be

$A(5,0,2)$, $B(2,-6,0)$, $C(7,-3,8)$, $D(4,-9,6)$.

Let us find the eqns of the circle ABC:

To find the eqn of the plane ABC:

Any plane through A is

$l(x-5)+m(y)+n(z-2)=0$ —— (1)

Since it passes through B & C we get

$3l+6m+2n=0$ —— (2)

and $2l-3m+6n=0$ —— (3)

Solving (2) & (3), we get

$$\frac{l}{6} = \frac{m}{-2} = \frac{n}{-3}.$$

$\therefore (1) \equiv \boxed{6x-2y-3z-24=0}$ — (4)

Now to find the eqn of the sphere OABC.

Let the eqn of the sphere through OABC be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — (5)

Since it passes through $O(0,0,0)$

$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$ — (6)

and it passes through the points A, B, C.

$29+10u+4w=0$ — (7)

$10+u-3v=0$ — (8)

$61+7u-3v+8w=0$ — (9)

Subtracting (8) from (9) we get

$51+6u+8w=0$ — (10)

Multiplying (7) by 2 and subtract (10) from it we get

$7+14u=0$

$\Rightarrow u=-\frac{1}{2}$

$(8) \equiv 3v=u+10$

$=-\frac{1}{2}+10$

$\Rightarrow v=\frac{19}{6}$

$(7) \equiv w=-6.$

putting u, v, w in (6), we get

$x^2+y^2+z^2-x+\frac{19}{3}y-12z=0$ — (11)

$\therefore$ The eqns of the circle through A, B, C are (i.e, intersection of sphere OABC and plane ABC)

$x^2+y^2+z^2-x+\frac{19}{3}y-12z=0$ — (12)

and $6x-2y-3z-24=0$ — (13)

The fourth point D(4, −9, 6) lies on circle ABC, if it lies both on the sphere (12) and plane (13).

Now D(4, −9, 6) lies on sphere (12) if

$16+81+36-4-57-72=0$

$\Rightarrow 0=0$ which is true

Similarly D(4, −9, 6) lies on plane (13) if

$24+18-18-24=0.$

$\Rightarrow 0=0$ which is true.

$\therefore$ D lies on the circle through A, B, C.

$\therefore$ The points A, B, C, D are concyclic.

Intersection of two Spheres:

We now consider two spheres and assume that the given spheres have points in common, i.e, intersect.

Assuming that two given spheres intersect, we show that the locus of the points of intersection of two spheres is a circle.

The co-ordinates of points, if any, common to the two spheres

$$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

$$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

satisfy both ~~these~~ eqns and, therefore also satisfy the eqn:

$$S_1-S_2 = 2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+d_1-d_2=0$$

which being a linear eqn in x, y, z represents a plane.

Now the points of intersection of the two spheres $S_1=0$, $S_2=0$ are the same as those of a one of these spheres and the plane $S_1-S_2=0$ and so it is a circle.

Note: The eqns of two spheres taken together also represents a circle.

→ Show that the sphere

$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ cuts

$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ in a great circle if

$$2(u_2^2+v_2^2+w_2^2)-d_2 = 2(u_1u_2+v_1v_2+w_1w_2)-d_1$$

$$\Rightarrow 2(u_1u_2+v_1v_2+w_1w_2) = 2r_2^2+d_1+d_2$$

where r_2 is the radius of the second sphere.

Sol: The plane of the circle, i.e, the plane in which their circle of intersection is $S_1 - S_2 = 0$

$\Rightarrow 2(u_1-u_2)x + 2(v_1-v_2)y + 2(w_1-w_2) + d_1-d_2 = 0$ — (1)

The circle of intersection is the great circle of the sphere S_2 only when the above plane passes through the centre of the sphere S_2,

i.e; the plane passes through $(-u_2, -v_2, -w_2)$.

$\therefore 2(u_1-u_2)(-u_2) + 2(v_1-v_2)(-v_2) + 2(w_1-w_2)(-w_2) + d_1-d_2 = 0$

$\Rightarrow 2(u_2^2+v_2^2+w_2^2) - d_2 = 2(u_1u_2+v_1v_2+w_1w_2) - d_1$

$\Rightarrow 2[(u_2^2+v_2^2+w_2^2) - d_2] + d_2 = 2(u_1u_2+v_1v_2+w_1w_2) - d_1$

$\Rightarrow 2r_2^2 + d_1 + d_2 = 2(u_1u_2+v_1v_2+w_1w_2)$

Spheres through a given circle:

Let the circle be given by the eqns

$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ —(1)

$P \equiv lx+my+nz-p=0$ —(2)

then the eqn $S+\lambda P=0$

i.e, $x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(lx+my+nz)=0$ —(3)

The equation (3) clearly represents a sphere

[∵ (i) it is a second degree equation.

(ii) coefficients of x^2, y^2, z^2 are equal.

and (iii) it doesnot contain the product terms xy, yz, zx.]

Also the co-ordinates of the points which satisfy (1) & (2) both, also satisfy (3).

Hence (3) represents a sphere through the curve of intersection of (1) & (2).

i.e, the given circle.

∴ The set of sphere through the circle $S=0$, $P=0$ is $\{S+\lambda P=0;\ \lambda \text{ is parameter}\}$

Similarly if the circle be given by the intersection of two spheres.

$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$.

then any sphere through this circle is $S+kS'=0$

∴ The set of spheres through the circle $S=0, S'=0$: $\{S+kS'=0;\ k \text{ is the parameter}\}$.

→ The eqn of the plane of the circle through the two spheres $S=0, S'=0$ is

$S-S' = 2(u-u')x+2(v-v')y+2(w-w')z+d-d'=0$

From this we see that the eqn of any sphere through the circle $S=0, S'=0$ is of the form $S+k(S-S')=0$

where k is parameter.

This form is sometimes more convenient.

Note: The general eqn of the sphere through the circle

$x^2+y^2+2gx+2fy+c=0,\ z=0$

is

$x^2+y^2+z^2+2gx+2fy+2kz+c=0.$

where k is the parameter

→ Find the equation of the sphere through the circle $x^2+y^2+z^2=9,\ 2x+3y+4z=5$ and the point $(1,2,3)$.

Sol^n: Given equations of the circle

$x^2+y^2+z^2=9$

$2x+3y+4z=5$ and the point $P(1,2,3)$.

Let the required eqn of sphere through a circle is

$(x^2+y^2+z^2-9)+\lambda(2x+3y+4z-5)=0$ — (3)

Since it passes through $P(1,2,3)$

$(1+4+9-9)+\lambda(2+6+12-5)=0$

$5+\lambda(15)=0$

$\boxed{\lambda=-1/3}$

(3) ≡

$x^2+y^2+z^2-9-\frac{1}{3}(2x+3y+4z-5)=0$

$\Rightarrow 3(x^2+y^2+z^2)-2x-3y-4z-22=0$

which is the required eqn of the sphere.

(2000) Find the eqn of the sphere through the circle $x^2+y^2+z^2=4,\ x+2y-z=2$ and the point $(1,-1,1)$.

→ Find the eqn to the sphere which passes through the point (α,β,γ) and the circle $x^2+y^2=a^2,\ z=0$

(Ans: $(x^2+y^2+z^2-a^2)\gamma+(a^2-\alpha^2-\beta^2-\gamma^2)z=0$)

Hint: $(x^2+y^2+z^2-a^2)+\lambda z=0$

→ Show that the eqn of the sphere having its centre on the plane $4x-5y-z=3$ and passing through the circle with equations

$x^2+y^2+z^2-2x-3y+4z+8=0,$

$x^2+y^2+z^2+4x+5y-6z+2=0$

is $x^2+y^2+z^2+7x+9y-11z-1=0$

(13)

Sol^n: The given circle is

$x^2+y^2+z^2-2x-3y+4z+8=0$ —— ①

$x^2+y^2+z^2+4x+5y-6z+2=0$ —— ②

①−② ≡ $3x+4y-5z-3=0$ —— ③

Now circle represented by ① & ② is same as the circle given by ① & ③.

Now any sphere through the circle given by ① & ③ is

$x^2+y^2+z^2-2x-3y+4z+8+\lambda(3x+4y-5z-3)=0$

$\Rightarrow x^2+y^2+z^2+(3\lambda-2)x+(4\lambda-3)y+(-5\lambda+4)z+(8-3\lambda)=0$

Its centre $(-u,-v,-w)$

$=\left(\frac{2-3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{5\lambda-4}{2}\right)$

Since it lies in the plane $4x-5y-z=3$ we get $\boxed{\lambda=3}$

④ ≡ $x^2+y^2+z^2+7x+9y-11z-1=0$.

→ Find the eqn of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0$, $x-2y+4z-9=0$ and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$

(Ans: $x^2+y^2+z^2+7y-8z+24=$

→ Show that the two circles $x^2+y^2+z^2-y+2z=0$, $x-y+z-2=$ $x^2+y^2+z^2+x-3y+z-5=0$, $2x-y+4z-1=0$; lie on the same sphere and find its equation.

Sol^n: The given circles are

$x^2+y^2+z^2-y+2z=0$, $x-y+z-2$ —— ①

$x^2+y^2+z^2+x-3y+z-5=0$, $2x-y+4z-1=0$ —— ②

Any sphere through ① is

$x^2+y^2+z^2-y+2z+\lambda_1(x-y+z-2)=$ —— ③

and any sphere through ② is

$x^2+y^2+z^2+x-3y+z-5+\lambda_2(2x-y+4z-1)=$ —— ④

The circles ① & ② will lie on same sphere if the eqns ③ & ④ represent the same sphere for same values of λ_1, λ_2:

∴ Equating the coefficients of like terms in ③ & ④, we get

$\lambda_1 = 2\lambda_2 + 1$ —⑤, $-1-\lambda_1 = -\lambda_2 - 3$ —⑥

$2+\lambda_1 = 4\lambda_2 + 1$ —⑦, $-2\lambda_1 = -\lambda_2 - 5$ —⑧

Solving ⑤ & ⑥, we get

$\lambda_1 = 3, \quad \lambda_2 = 1.$

and these values clearly satisfy remaining two eqns ⑦ & ⑧

∴ Two circles ① & ② lie on the same sphere whose eqn is (putting $\lambda_1 = 3, \lambda_2 = 1$ in ③ & ④)

we get

$$x^2+y^2+z^2+3x-4y+5z-6=0$$

→ Show that the two circles

$2(x^2+y^2+z^2)+8x-13y+17z-17=0,$
$2x+y-3z+1=0;$
$x^2+y^2+z^2+3x-4y+3z=0,$
$x-y+2z-4=0;$

lie on the same sphere and find its equation.

→ Prove that the circles

$x^2+y^2+z^2-2x+3y+4z-5=0,$
$5y+6z+1=0;$
$x^2+y^2+z^2-3x-4y+5z-6=0,$
$x+2y-7z=0$

lie on the same sphere and find its eqn.

→ Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity and find the eqn of sphere which has this circle for one of its great circle.

Sol: The given sphere

$x^2+y^2+z^2-x+z-2=0$ —①

and the plane

$x+2y-z-4=0$ —②

Centre of the sphere ① is

$C\left(\frac{1}{2}, 0, -\frac{1}{2}\right).$

and its radius

is $CP = \sqrt{\frac{1}{4}+0+\frac{1}{4}+2}$

$= \sqrt{5/2}.$

$CA = \perp^r$ distance from $C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ to the plane ②

$$= \frac{\left|\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4\right|}{\sqrt{1+4+1}}$$

$$= \frac{3}{\sqrt{6}} = \sqrt{3/2}$$

∴ Radius of circle

$$AP = \sqrt{CP^2 - CA^2}$$
$$= \sqrt{\frac{5}{2} - \frac{3}{2}}$$
$$= \sqrt{1} = 1.$$

∴ The plane ② meets the sphere ① in a circle of radius unity.

Now any sphere through the intersection of ① & ② is

$$x^2+y^2+z^2-x+z-2+k(x+2y-z-4)=0 \quad ③$$

If the circle of intersection of ① & ② is a great circle of sphere ③, then the centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ lies on the plane ②.

$$\therefore \frac{1-k}{2} + 2(-k) - \left(\frac{k-1}{2}\right) - 4 = 0$$
$$\Rightarrow \boxed{k=-1}$$

$$\therefore ③ \equiv x^2+y^2+z^2-2x-2y+2z+2=0$$

→ Obtain the eqn of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0$, $x+y+z=3$ as the great circle.

→ Find the eqn to the sphere which passes through the circle $x^2+y^2=4$, $z=0$ and is cut by the plane $x+2y+2z=$ in a circle of radius 3.

(14)

Sol^n: The given circle is

$$x^2+y^2-4=0, \quad z=0.$$

The eqns of this circle can be written as

$$x^2+y^2+z^2-4=0, \quad z=0$$

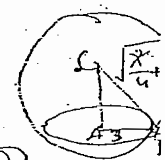
Any sphere through this circle is

$$(x^2+y^2+z^2-4)+\lambda z=0 \quad ①$$

Its centre $= \left(0, 0, -\frac{\lambda}{2}\right)$

and radius $= \sqrt{\frac{\lambda^2}{4}+4} = CP$

Now the sphere ① cut by the plane $x+2y+2z=0$ ② in a circle of the radius 3.



Draw $CA \perp$ to the plane ② from C.

∴ CA = $\perp^r$ distance from $\left(0,0,-\frac{\lambda}{2}\right)$ on the plane ②

$$= \frac{|0+0-\lambda|}{\sqrt{1+4+4}} = \frac{\lambda}{3}$$

Now from the right $\angle^d$ ΔCAP, $CA^2+AP^2=CP^2$

$$\Rightarrow \frac{\lambda^2}{9}+9 = \frac{\lambda^2}{4}+4$$
$$\Rightarrow \lambda = \pm 6$$

$$\therefore ① \equiv x^2+y^2+z^2\pm 6z-4=0$$

2008 → A sphere 'S' has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

2009 → Find the equation of the sphere having its centre on the plane $4x-5y-z=3$, and passing through the circle —

$$x^2+y^2+z^2-12x-3y+4z+8=0$$
$$3x+4y-5z+3=0$$

Tangent plane (Line) property:

If a plane (line) touch of the sphere, then $\perp^r$ distance from the centre of the sphere on the plane (line) must be equal to its radius of the sphere.

Intersection of a Sphere by a straightline:

To find the points where the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ meets the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Sol^n: The given line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)} \quad \text{--- (1)}$$

and Sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (2)}$$

Any point on the line (1) is

$$(lr+x_1, \; mr+y_1, \; nr+z_1) \quad \text{--- (3)}$$

If it lies on the sphere (2)

$$\therefore (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2+2u(lr+x_1)+2v(mr+y_1)+2w(nr+z_1)+d=0 \quad \text{(4)}$$

which is a quadratic in 'r' hence it gives two values of r.

$\therefore$ These values putting in (3) we get two points of intersection.

Note:

[1]. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2=a^2$ is

$$xx_1+yy_1+zz_1 = a^2.$$

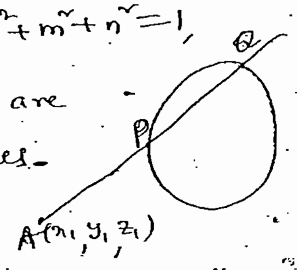
[2]. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

Power of a point w.r.t a Sphere:-

Let l, m, n be the actual d.c's of the line (1), so that $l^2+m^2+n^2=1$, and r_1, r_2 are the distances of the point $A(x_1, y_1, z_1)$ from the



points of intersections P and Q.

Now the eqn (4) reduces to

$r^2 + 2r[l(u+x_1) + m(v+y_1) + n(w+z_1)] + x_1^2+y_1^2+z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$ $[\because l^2+m^2+n^2=1]$

and $r_1 = AP$, $r_2 = AQ$ are its two roots.

$\therefore AP \cdot AQ = r_1 r_2$
$= x_1^2+y_1^2+z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$

$\left[\because ax^2+bx+c=0\right.$ product of two roots $\left.= \frac{c}{a}\right]$

which is independent of the d.c.'s. l, m, n and is thus constant.

i.e., if from a fixed point A, lines are drawn in any direction to intersect a given sphere in P and Q, then $AP \cdot AQ$ is constant. This constant $AP \cdot AQ$ is called the power of the point A w.r.t the sphere.

Note: The power of a point is obtained by substituting the co-ordinates of the point in the eqn of the sphere after making the R.H.S. zero.

→ Find the co-ordinates of the points where the line (i) $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5}$ intersects the sphere $x^2+y^2+z^2+2x-10y = 23$.

(ii) $\frac{x+2}{4} = \frac{y+9}{3} = \frac{z-8}{-5}$ meets the sphere $x^2+y^2+z^2 = 49$.

Sol: (i) The given line is

$\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = r$ (say) —— (1)

and the sphere is

$x^2+y^2+z^2+2x-10y-23=0$ —— (2)

Any point on the line (1) is

$(4r-3,\ 3r-4,\ -5r+8)$ —— (3)

It lies on the sphere (2).

$\therefore$ (2) ≡

$(4r-3)^2 + (3r-4)^2 + (-5r+8)^2 + 2(4r-3) - 10(3r-4) - 23 = 0$

$\Rightarrow 50r^2 - 150r + 100 = 0$

$\Rightarrow r^2 - 3r + 2 = 0$

$\Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1, 2$

putting $r=1, r=2$ in (3)

$\therefore$ (3) ≡ the required points of intersection are $(1,-1,3)$ & $(5,2,-2)$.

→ Find the locus of the middle point of the system of parallel chords of the sphere

(i) $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

(ii) $x^2+y^2+z^2=a^2$.

(or)

Show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chords.

Sol^n: (i)

Let all chords of the system be parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (1)

where l, m, n are actual d.c's.

Let (x_1, y_1, z_1) be the mid point of one of the chords.

Then its equations are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$

This lies on the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d$

$\Rightarrow (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2+2u(lr+x_1)+2v(mr+y_1)+2w(nr+z_1)+d=0$

$\Rightarrow r^2(l^2+m^2+n^2)+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]+x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$ — (2)

which is a quadratic in r.

Since (x_1, y_1, z_1) is the mid-point of the chord.

∴ the roots of (2) must be equal and opposite.

i.e, the sum of roots is zero $\left(-\frac{b}{a}\right)$

i.e, the co-efficient of $r=0$

∴ $l(u+x_1)+m(v+y_1)+n(w+z_1)=0$

∴ Locus of the mid point (x_1, y_1, z_1) is

$l(u+x)+m(v+y)+n(w+z)=0$

$\Rightarrow l(x+u)+m(y+v)+n(z+w)=0$

which is clearly a plane through the centre $(-u,-v,-w)$ and ⊥r to the line (1).

(ii) Ans: $lx+my+nz=0$

→ Show that the sum of the squares of the intercepts made by a given sphere on any three mutually $\perp^r$ straight lines through a fixed point is constant.

Sol: Let the given sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

and the three mutually $\perp^r$ lines through the fixed point $(0,0,0)$ (say), be

$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ —— (2) ; $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ —— (3)

and $\frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3}$ —— (4)

where l_1, m_1, n_1 etc. are the actual d.c's, so that

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1, \quad l_1^2+l_2^2+l_3^2=1 \\ l_1m_1+l_2m_2+l_3m_3=0, \\ l_1l_2+m_1m_2+n_1n_2=0 \end{array}\right\} \text{ etc.} \text{ —— (5)}$$

To find the intercept on line (2):

Any point on line (2) is (l_1r, m_1r, n_1r).

If it lies on the sphere (1), then

$r^2(l_1^2+m_1^2+n_1^2)+2r(l_1u+m_1v+n_1w)+d=0$

$\Rightarrow r^2+2r(l_1u+m_1v+n_1w)+d=0$ ($\because$ from (5))

It is a quadratic in r;

let the two roots be r_1, r_2, which are the distances from O of the two points of intersection say A_1, A_2 of the line and the sphere.

$\therefore$ If L_1 is the length of intercept on the first line, then

$L_1 = A_1A_2 = OA_2 - OA_1 = r_2 - r_1$

$\therefore L_1^2 = (r_2-r_1)^2$

$= (r_1+r_2)^2 - 4r_1r_2$

$= 4(ul_1+vm_1+wn_1)^2 - 4d$

$\left(\because r_1+r_2 = \frac{l_1u+m_1v+n_1w}{1}, \quad r_1r_2 = \frac{d}{1}\right)$

$= 4(u^2l_1^2+v^2m_1^2+w^2n_1^2+2uvl_1m_1 + 2vwm_1n_1+2wul_1n_1) - 4d$

Similarly,

$L_2 = 4(u^2l_2^2+v^2m_2^2+w^2n_2^2+ 2uvl_2m_2+2vwm_2n_2+ 2wul_1n_1)-4d.$

$L_3^2 = 4(u^2 l_3^2 + v^2 m_3^2 + w^2 n_3^2 + 2uv\, l_3 m_3 + 2vw\, m_3 n_3$
$+ 2wu\, n_3 l_3) - 4d.$

Adding, the sum of squares of the intercepts

$= L_1^2 + L_2^2 + L_3^2$

$= 4[u^2(l_1^2 + l_2^2 + l_3^2) + v^2(m_1^2 + m_2^2 + m_3^2)$
$+ w^2(n_1^2 + n_2^2 + n_3^2) + 2uv(l_1 m_1 + l_2 m_2 + l_3 m_3)$
$+ 2vw(m_1 n_1 + m_2 n_2 + m_3 n_3) +$
$2wu.(l_1 n_1 + l_2 n_2 + l_3 n_3)] - d$

$= 4[u^2(1) + v^2(1) + w^2(1) + 2uv(0) +$
$2wv(0) + 2wu(0)] - 12.d.$

$= 4(u^2 + v^2 + w^2) - 12d.$ (∵ from (5))

which is free from l_1, m_1, n_1 etc and is therefore constant for any set of lines

Hence the result

(OR)

Let the equation of the given sphere be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and
take the fixed point 'O' as the origin and any three mutually perpendicular lines through it as the co-ordinate axes.

The x-axis $(y = 0 = z)$ meets the sphere in points given by

$x^2 + 2ux + d = 0;$

so that if x_1, x_2 be its roots, the points of intersection are $(x_1, 0, 0), (x_2, 0, 0)$.

Also we have

$x_1 + x_2 = -2u$; $x_1 x_2 = d$.

$\therefore$ (intercept on x-axis)^2 = $(x_1 - x_2)^2$

$= (x_1+x_2)^2 - 4x_1x_2$

$= 4u^2 - 4d$

$= 4(u^2-d)$.

Similarly,

(intercept on y-axis)^2 = $4(v^2-d)$

(intercept on z-axis)^2 = $4(w^2-d)$.

The sum of the squares of the intercepts

$= 4(u^2+v^2+w^2-3d)$.

$= 4(\cancel{u^2}+v^2+w^2) - 12d$.

→ Show that the plane $lx+my+nz=p$ will touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ if

$(ul+mv+nw+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

2004 → Find the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$

Sol^n: equation of sphere is

$x^2+y^2+z^2-4x+2y-6z+5=0$

its centre $(2,-1,3)$.

and radius $= \sqrt{4+1+9-5}$
$= 3$

Any plane $\parallel$ to the plane $2x+2y-z=0$ is

$2x+2y-z=k$ —— (i)

If it touches the sphere, then length of $\perp^r$ from the centre of sphere must be equal to the radius of the sphere.

$\therefore \dfrac{|2(2)+2(-1)-(+3)-k|}{\sqrt{4+4+1}} = 3$

$\Rightarrow |4-2-3-k| = 3\sqrt{9}$

$\Rightarrow -1-k = \pm 9$

$\Rightarrow k = -1 \pm 9$

$\Rightarrow k = -10$ or 8

From (i), we have

$2x+2y-z=-10$ and

$2x+2y-z=8$

$\therefore$ The required tangent planes are

$2x+2y-z+10$ and

$2x+2y-z-8=0$

2004 → Find the equations of tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$; which are parallel to the plane $2x+y-z=4$.

→ Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$.

→ Find the value of 'a' for which the plane $x+y+z=a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

→ Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1,1,-1)$ and passes through the origin.

Sol: The given sphere is

$$x^2+y^2+z^2-x+3y+2z-3=0 \quad \text{--- (1)}$$

Equation of tangent plane at $(1,1,-1)$ to the sphere (1) is

$$x(1)+y(1)+z(-1)-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+(z-1)-3=0$$

$$\Rightarrow \frac{1}{2}x+\frac{5}{2}y-3=0$$

$$\Rightarrow x+5y-6=0 \quad \text{--- (2)}$$

The required sphere touching (1) at $(1,1,-1)$ is the sphere through the point circle of intersection of (1) and the tangent plane at $(1,1,-1)$ to the sphere i.e, the plane (2).

Now any sphere through the circle of intersection of (1) & (2) is

$$x^2+y^2+z^2-x+3y+2z-3+k(x+5y-6)=0 \quad \text{--- (3)}$$

If it passes through the origin $(0,0,0)$,

then $-3-6k=0$

$$\Rightarrow \boxed{k=-\frac{1}{2}}$$

$\therefore$ (3) ≡

$$x^2+y^2+z^2-x+3y+2z-3-\frac{1}{2}(x+5y-6)=0$$

$$2(x^2+y^2+z^2)-3x+y+4z=0$$

which is the required equation.

→ Show that the equation of the sphere which touches the sphere $4(x^2+y^2+z^2)+10x-25y-2z=0$ at the point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is

$$x^2+y^2+z^2+2x-6y+1=0$$

→ Find the equations of the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$.

→ Find the eqns of the sphere which pass through the circle $x^2+y^2+z^2-2x+2y+4z-3=0$, $2x+y+z=4$ and touches the plane $3x+4y=14$.

→ Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.

Sol^n: The given plane is

$$2x-2y+z+12=0 \quad \text{--(1)}$$

and the sphere is

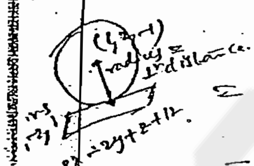
$$x^2+y^2+z^2-2x-4y+2z-3=0$$

The centre of the sphere is $(1,2,-1)$ and its radius is

$$\sqrt{1+4+1+3}=3.$$

Also the $\perp^r$ distance of the centre $(1,2,-1)$ from the plane (1)

$$= \frac{2(1)-2(2)+(-1)+12}{\sqrt{4+4+1}}$$



$$= \frac{2-4-1+12}{3}$$

$$= \frac{9}{3} = 3.$$

Since $\perp^r$ distance of the centre from the plane (1) = radius of the sphere

∴ The plane (1) touches the sphere (2).

The point of contact is the foot of perpendicular from the centre of the sphere on the plane.

Now the equations of the line through the centre $(1,2,-1)$ and $\perp^r$ to the plane (1) are

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}$$

Any point on this line is $(2r+1, -2r+2, r-1)$ --(3)

If it lies on the plane (1) then

$$2(2r+1)-2(-2r+2)+r-1+12=0$$

$\Rightarrow 9r+9=0$

$\Rightarrow \boxed{r=-1}$

(3) ≡ $(2(-1)+1, -2(-1)+2, -1-1)$

$= (-1,4,-2)$

which is the required point of contact.

→ Find the co-ordinates of the points on the sphere $x^2+y^2+z^2-4x+2y=4$

the tangent planes at which are parallel to the plane
$2x-y+2z=1$
(Ans: $(4,-2,2)$, $(0,0,2)$)

2008
→ find the equation of the tangent line to the circle
$x^2+y^2+z^2+5x-7y+2z-8=0$,
$3x-2y+4z+3=0$
at the point $(-3,5,4)$.

Sol^n:
Note: The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of circle.

The given sphere is
$x^2+y^2+z^2+5x-7y+2z-8=0$ ——— (1)
and the plane of the circle is $3x-2y+4z+3=0$ ——— (2)

Now equation of the tangent plane at $P(-3,5,4)$ to the sphere (1) is
$x(-3)+y(5)+z(4)+\frac{5}{2}(x-3)-\frac{7}{2}(y+5)+1(z+4)-8=0$
$\Rightarrow -x+3y+10z-58=0$.
$\Rightarrow x-3y-10z+58=0$ ——— (3)

The eqns (2) & (3) taken together represent the equation of the tangent line to the circle given by (1) & (2).

To find the d.r.'s of the tangent line:

Omitting the constant terms in (2) & (3), the equations are
$3x-2y+4z=0$
$x-3y-10z=0$
$\therefore \frac{x}{20+12}=\frac{y}{4+30}=\frac{z}{-9+2}$
$\Rightarrow \frac{x}{32}=\frac{y}{34}=\frac{z}{-7}$.

$\therefore$ The d.r.'s of the tangent line are $32, 34, -7$.
Also the tangent line passes through the given point $P(-3,5,4)$.
$\therefore$ The eqns of the tangent line to the circle at $P(-3,5,4)$ are $\frac{x+3}{32}=\frac{y-5}{34}=\frac{z-4}{-7}$.

→ Find the eqn of tangent line to the circle $3x^2+3y^2+3z^2-2x-3y-4z-22=0$, $3x+4y+5z-26=0$ at the point $(1,2,3)$.

→ Find the equations of the two tangent planes to the sphere $x^2+y^2+z^2=9$ which pass through the line $x+y=6,\ x-2z=3$.

Sol^n: The given line is

$x+y=6,\ x-2z=3.$

Any plane through this line is

$x+y-6+k(x-2z-3)=0$ ——— ①

If it touches the sphere $x^2+y^2+z^2-9=0$, then ⊥^{r} distance from the centre $(0,0,0)$ on the plane ① must be equal to the radius $(=3)$ of the sphere.

$\therefore \dfrac{-6-3k}{\sqrt{(1+k)^2+1+4k^2}}=3$

$\Rightarrow -2-k=\sqrt{5k^2+2k+2}$

$\Rightarrow (2-k)^2=5k^2+2k+2$

$\Rightarrow k^2+4k+4=5k^2+2k+2$

$\Rightarrow 4k^2-2k-2=0$

$\Rightarrow 2k^2-k-1=0$

$\Rightarrow 2k^2-2k+k-1=0$

$\Rightarrow 2k(k-1)+1(k-1)=0$

$(2k+1)(k-1)=0$ (15

$\Rightarrow k=-\frac{1}{2},\ 1.$

Putting these values in ①

the required planes are

$2x+y-2z=9$ and

$x+2y+2z=9$

→ Obtain the equations of the tangent planes to the sphere

(i) $x^2+y^2+z^2=9$ which can be drawn through the line $\frac{x-5}{2}=\frac{y-1}{-2}=\frac{z+1}{1}$

(ii) $x^2+y^2+z^2+6x-2z+1=0$ which pass through the line

$3(16-x)=3z=2y+30.$

(Hint: If the given line is symmetrical form then convert it into unsymmetrical form.)

→ Find the equations of spheres that pass through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touch the plane $2x+2y-z=11$.

Sol: Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

Since it passes through $(4,1,0)$

$\therefore 8u+2v+d+17=0$ —— (2)

Since (1) passes through $(2,-3,4)$ & $(1,0,0)$

$\therefore$ (1) ≡, we have

$4u-6v+8w+d+29=0$ —— (3)

$2u+d+1=0$ —— (4)

centre of the sphere (1) is $(-u,-v,-w)$

and radius $=\sqrt{u^2+v^2+w^2-d}$

Since the sphere touches the plane $2x+4y-z=11$

$\therefore$ length of the $\perp^r$ from the centre of the sphere to the plane $2x+4y-z-11=0$ must be equal to the radius of the sphere

$$\therefore \frac{2(-u)+4(-v)-(-w)-11}{\sqrt{4+4+1}}=\sqrt{u^2+v^2+w^2-d}$$

$\Rightarrow (-2u-4v+w-11)^2=9(u^2+v^2+w^2-d)$

$\Rightarrow 5u^2+5v^2+8w^2-8uv+4vw+4uw-44v+22w-9d-121=0$ —— (5)

from (4) $\boxed{u=-\frac{1}{2}(d+1)}$ —— (i)

from (2) $2v=-8u-d-17$ —— (ii)

$\therefore$ from (i) & (ii), we get

$\boxed{v=\frac{1}{2}(3d-13)}$

From (3), $w=\frac{5d-33}{4}$ —— (iii)

substituting these values of u, v, w in (5), we get

$72d^2-747d+1935=0$

$8d^2-83d+215=0$

$\Rightarrow d=5, \frac{43}{8}$.

$\therefore$ Substituting $d=5$ in (i) (ii) & (iii),

we get

$u=-3,\ v=1,\ w=-2$.

$\therefore$ (1) ≡ the required eqn of the sphere is

$x^2+y^2+z^2-6x+2y-4z+5=0$

Also, sub. $d=\frac{43}{8}$ in (i) (ii) & (iii)

we get $u=\ ;\ v=\ \ w=$ proceed like this.

and the required eqn of the sphere is

$16(x^2+y^2+z^2)-102x+50y-49z+86=0$

(20)

→ Find the locus of the centre of the sphere of constant radius which passes through a given point and touches the given line.

Soln: Take x-axis to be given line and perpendicular from the given point on the x-axis as the z-axis, then the co-ordinates of the given point on the z-axis are of the form $(0,0,c)$.

Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

It passes through $(0,0,c)$.

$\therefore c^2+2wc+d=0$ —— (2)

Given that the radius of the sphere is constant say 'λ'.

$\therefore u^2+v^2+w^2-d=\lambda^2$ —— (3)

The sphere meets the x-axis $(y=0, z=0)$.

where

$x^2+2ux+d=0$ —— (4)

Since the line i.e, x-axis touches the sphere, the two values of x given by (4) must be equal i.e, the discriminant of (4) is zero $(b^2-4ac=0)$

$\therefore 4u^2-4d=0$

$\Rightarrow \boxed{u^2=d}$ —— (5)

Eliminating d from (2), (3) & (4), we get

$u^2+2wc+c^2=0$ [by adding (2) & (5)]

and $v^2+w^2=\lambda^2$ [subtracting (5) from (3)]

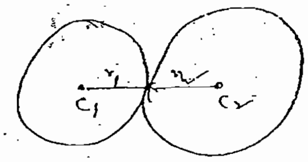
$\therefore$ The locus of the centre $(-u,-v,-w)$ of the sphere (1) is

$x^2-2cz+c^2=0$ and $y^2+z^2=\lambda^2$.

which represents a curve of intersection of two surfaces.

Touching Spheres:

(i) Two spheres touch externally if the distance between their centres is equal to the sum of their radii.



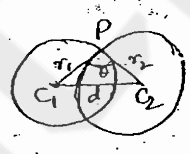
(ii) Two spheres touch internally if the distance between their centres is equal to the difference of their radii.

Angle of intersection of two spheres:

The angle of intersection of two spheres is the angle between their tangent planes at a common point of intersection. Since the radii of the spheres to a common point are $\perp^r$ to the tangent planes at that point, so the angle between the radii of spheres at the common point is equal to the angle between the tangent planes.

i.e, the angle of intersection of the spheres.

To find the angle:



Let C_1, C_2 be the centres of the spheres of radii r_1 & r_2.

Let P be their common point of intersection.

Let $C_1C_2 = d$.

The angle of intersection i.e, the angle between the tangent planes at P is the angle between the radii of the two spheres.

i.e, $\angle C_1PC_2 = \theta$, then

$$\cos\theta = \frac{(C_1P)^2 + (C_2P)^2 - (C_1C_2)^2}{2(C_1P)\cdot(C_2P)}$$

[Cosine formula]

$$= \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\therefore \theta = \cos^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

Orthogonal Spheres:

Two spheres are said to be orthogonal if the angle of intersection of two spheres is a right angle.

i.e, If the two spheres cut orthogonally then the square of the distance between the centres of two spheres = sum of squares of radii.

i.e $\boxed{d^2 = r_1^2 + r_2^2}$

Condition of orthogonality of two spheres:

To find the condition that the spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

to be orthogonal.

Sol^n: Two given spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \text{--- (1)}$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \text{--- (2)}$$

If the spheres cut orthogonally then square of distance between their centres = sum of the squares of their radii. --- (3)

Now the centres of the spheres (1) & (2) are $C_1(-u_1, -v_1, -w_1)$ and $C_2(-u_2, -v_2, -w_2)$ and their radii are

$\sqrt{u_1^2+v_1^2+w_1^2-d_1}$, $\sqrt{u_2^2+v_2^2+w_2^2-d_2}$

$\therefore$ (3) $\equiv (u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2 = (u_1^2+v_1^2+w_1^2-d_1) + (u_2^2+v_2^2+w_2^2-d_2)$

$\Rightarrow 2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2$

which is the required condition.

→ Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is

$$\frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}.$$

Sol^n: Let the common circle be $x^2+y^2=a^2;\ z=0$. --- (1)

where 'a' is radius of this circle.

Let the two given spheres through the circle (1) be

$$x^2+y^2+z^2-a^2+2\lambda_1 z=0 \quad \text{--- (2)}$$

and $$x^2+y^2+z^2-a^2+2\lambda_2 z=0 \quad \text{--- (3)}$$

Since r_1, r_2 are radii of the given spheres (2) & (3)

$$\therefore \left.\begin{aligned} r_1^2 &= \lambda_1^2+a^2 \\ r_2^2 &= \lambda_2^2+a^2 \end{aligned}\right\} \text{--- (4)}$$

(22)

Since the spheres (2) & (3) cut orthogonally.

$\therefore 2\lambda_1\lambda_2 = -a^2-a^2$

$\Rightarrow \lambda_1\lambda_2 = -a^2$

$\Rightarrow \lambda_1^2\lambda_2^2 = a^4$

$\therefore$ (4) $\equiv (r_1^2-a^2)(r_2^2-a^2) = a^4$

$\Rightarrow r_1^2 r_2^2 - a^2(r_1^2+r_2^2)+a^4 = a^4$

$\Rightarrow a^2(r_1^2+r_2^2) = r_1^2 r_2^2$

$\Rightarrow a^2 = \dfrac{r_1^2 r_2^2}{r_1^2+r_2^2}$

$\Rightarrow \boxed{a = \dfrac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}}$

Hence, the result.

1995 → Two spheres of radii r_1 and r_2 cut orthogonally prove that the area of the common circle is $\dfrac{\pi r_1^2 r_2^2}{r_1^2+r_2^2}$

Sol^n: From the above problem the radius of the common circle is $a = \dfrac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$

$\therefore$ The area of the common circle

$= \pi a^2 = \pi \dfrac{r_1^2 r_2^2}{r_1^2+r_2^2}$

→ Find the equation of the sphere that passes through the circle

$x^2+y^2+z^2-2x+3y-4z+6=0$

$3x-4y+5z-15=0$ and cuts the sphere

$x^2+y^2+z^2+2x+4y-6z+11=0$

orthogonally.

Sol^n: Given equations of circle

$x^2+y^2+z^2-2x+3y-4z+6=0$

$3x-4y+5z-15=0$ ——(1)

and given sphere

$x^2+y^2+z^2+2x+4y-6z+11=0$ ——(2)

Any sphere through the circle (1) is

$x^2+y^2+z^2-2x+3y+6+\lambda(3x-4y+5z-15)=0$ ——(3)

$\Rightarrow x^2+y^2+z^2+(-2+3\lambda)x+(3-4\lambda)y+(-4+5\lambda)z+6-15\lambda=0$

This will cut the sphere (2) orthogonally iff

$2\left(\dfrac{-2+3\lambda}{2}\right)(1)+2\left(\dfrac{3-4\lambda}{2}\right)(2)+2\left(\dfrac{-4+5\lambda}{2}\right) = (6-15\lambda)+11$

[$\because$ by using $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$]

$\Rightarrow \boxed{\lambda = -1/5}$

Putting this value of λ in (3), we get

$$5(x^2+y^2+z^2)-13x+19y-25z+45=0.$$

→ Find the equation of the sphere that passes through the two points $(0,3,0)$, $(-2,-1,-4)$ and cuts orthogonally the two spheres.

$$x^2+y^2+z^2+x-3z-2=0,$$
$$2(x^2+y^2+z^2)+x+3y+4=0.$$

2006 → Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and also cuts orthogonally the sphere.

$$x^2+y^2+z^2-4x+6y+4=0$$

Sol'n :- Given plane $3x+2y-z+2=0$ ——(1)

and the given sphere

$$x^2+y^2+z^2-4x+6y+4=0 \quad \text{——(2)}$$

Since the required sphere touches the plane (1) at $P(1,-2,1)$.

∴ Its centre lies on the normal to the plane at P.

Now the equations of normal to the plane (1) through $P(1,-2,1)$ are

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = r \text{ (say)} \quad \text{——(3)}$$

Any point on this line $C(3r+1,\ 2r-2,\ -r+1)$.

Let this point be the centre of the required sphere.

Now the radius of the required sphere

$$CP = \sqrt{(3r+1-1)^2+(2r-2+2)^2+(-r+1-1)^2}$$
$$= \sqrt{9r^2+4r^2+r^2}$$
$$= r\sqrt{14}$$

Since the required sphere cuts the sphere (2) orthogonally

∴ Square of distance between the centres = Sum of square of their radii. ——(4)

Now the centre of the sphere (2) $C'(2,-3,0)$

and radius $= \sqrt{4+9-4}$

$= 3$

∴ (4) ≡

$$(3r+1-2)^2+(2r-2+3)^2+(-r+1-0)^2 = 9+14r^2$$

$\Rightarrow r = -3/2$

$\therefore$ The Centre of the required sphere.

$C\left(-7/2, -5, 5/2\right)$

and the radius $CP = \frac{-3}{2}\sqrt{14}$

$= \frac{3}{2}\sqrt{14}$ (numerically)

$\therefore$ The required sphere is

$$\left(x+7/2\right)^2 + (y+5)^2 + \left(z-5/2\right)^2 = \left(\frac{3\sqrt{14}}{2}\right)^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12=0$$

→ Show that every sphere through the circle

$x^2+y^2-2ax+r^2=0,\ z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$.

Sol'n :- Any sphere through the first circle.

$x^2+y^2-2ax+\cancel{r^2}=0,\ z=0$

i.e. the circle

$x^2+y^2+z^2-2ax+r^2=0,\ z=0$

is $x^2+y^2+z^2-2ax+r^2+\lambda_1 z=0$ —— (1)

Again any sphere through the second circle.

$x^2+z^2=r^2,\ y=0.$

i.e. the circle $x^2+y^2+z^2=r^2,$ $y=0$ is

$x^2+y^2+z^2-r^2+\lambda_2 y=0$ —— (2)

(1) & (2) will cut orthogonall[y]

if $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

if $2(-a)(0)+2(0)\left(\frac{\lambda_2}{2}\right)+2\left(\frac{\lambda_1}{2}\right)(0)$

$= r^2-r^2$

if $0=0$

which is true.

Hence the result.

2007 → Show that the spheres

$x^2+y^2+z^2-x+z-2=0$ and

$3x^2+3y^2+3z^2-8x-10y+8z+14=0$

cut orthogonally. Find the centre and radius of their common circle.

Length of Tangent:

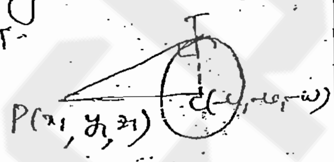
To find the length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Let $P(x_1, y_1, z_1)$ be a point out side the sphere

$S = x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Its centre is $C(-u,-v,-w)$ and radius $= \sqrt{u^2+v^2+w^2-d}$

Now let the tangent from $P(x_1, y_1, z_1)$ to the sphere meet at T, then radius CT at T must be at right angles to the tangent PT.



$\therefore \Delta PTC$ is right $\angle d$ triangle.

$\therefore PT^2 = PC^2 - CT^2$

$= (x_1+u)^2 + (y_1+v)^2 + (z_1+w)^2 - (u^2+v^2+w^2-d)$

$= x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$

$= S_{11}$

→ Find the length of the tangent drawn from the point $P(1,2,3)$ to the sphere $5(x^2+y^2+z^2) - x + 10y + 20z + 8$

Sol^n: Let $P(1,2,3)$ be the given point.

Let the tangent from $P(1,2,3)$ to the sphere $x^2+y^2+z^2-\frac{1}{5}x+2y+4z+\frac{8}{5}=$ meet at T.

$\therefore (PT)^2 = x_1^2+y_1^2+z_1^2-2ux_1+2vy_1+2wz_1+d$

$= 1+4+9-2\left(\frac{1}{10}\right)(1)+2(1)(2)+2(2)(3)+\frac{8}{5}$

$(PT)^2 = \frac{157}{5}$

$\Rightarrow PT = \sqrt{\frac{157}{5}}$ which is the required length of the tangent.

Radical plane of two Spheres:

The locus of a point whose powers w.r.t two spheres are equal i.e, the locus of a point where the square of the lengths of the tangents to the two spheres are equal, is a plane called the radical plane of the two spheres.

Equation of Radical plane of two spheres:

To find the equation of the radical plane of the spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

Sol^n: The given spheres are

$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ —— (1)

$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ —— (2)

Let $P(x,y,z)$ be any point on the radical plane

Then the power of P w.r.t sphere (1) = the power of P w.r.t sphere (2)

$\therefore x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2$

$\Rightarrow 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2$ —— (3)

which is the required eqn.

Note: The radical plane of two spheres $S_1=0$, $S_2=0$ (in both of which the coefficients of x^2, y^2, z^2 are equal to unity) is $S_1-S_2=0$

→ The radical plane of two spheres is perpendicular to the line joining their centres.

Sol^n: Let the spheres be

$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ —— (1)

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ —— (2)

The centres of (1) & (2) are $C_1(-u_1,-v_1,-w_1)$ & $C_2(-u_2,-v_2,-w_2)$.

$\therefore$ d.r.'s of line joining the centres C_1, C_2 are $u_1-u_2,\ v_1-v_2,\ w_1-w_2$.

Also the d.r.'s of the normal to the radical plane are proportional to

$2(u_1-u_2),\ 2(v_1-v_2),\ 2(w_1-w_2)$.

or $u_1-u_2,\ v_1-v_2,\ w_1-w_2$. ($\because$ Radical plane of two spheres is $2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$)

$\therefore$ The normal to the radical plane is parallel to the line C_1C_2 (or) the line C_1C_2 is $\perp^r$ to the radical plane.

Note: If the spheres intersect then the plane of their common circle is their radical plane.

Radical line of three spheres:

The three radical planes of three spheres intersect in a line.

i.e., If $S_1=0$, $S_2=0$, $S_3=0$ be the three spheres then their radical planes $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$ clearly meet in the line $S_1=S_2=S_3$ $\Leftrightarrow S_1-S_2=0, S_2-S_3=0$

This line is called the radical line or radical axis of three spheres.

Radical centre of four spheres:

The four radical lines of four spheres taken three at a time in a point which is called the radical centre of the four spheres.

Let the four spheres $S_1=0$, $S_2=0$, $S_3=0$, $S_4=0$.

Then the point common to the three planes $S_1=S_2=S_3=S_4$ is clearly common to the radical lines, taken three by three, of four spheres.

$S_1=S_2=S_3$
$S_1=S_2=S_4$
$S_2=S_3=S_4$
$S_1=S_3=S_4$

This point is the intersection of two lines $S_1-S_2=0$, $S_2-S_3=0$; $S_1-S_3=0$, $S_2-S_4=0$

This point is called radical centre.

Co-axial spheres:

A system of spheres any two members of which have the same radical plane is called a co-axial system of spheres.

Equation of co-axial system of spheres determined by two given spheres:

If $S_1=0$, $S_2=0$ be two spheres then $S_1+\lambda S_2=0$ represents a system of spheres, where λ is a parameter, such that any two members of the system have the same radical plane.

Let $S_1+\lambda_1 S_2=0$, $S_1+\lambda_2 S_2=0$ be any two members of the system $S_1+\lambda S_2=0$

Making the co-efficients x^2, y^2, z^2 unity in the two equations, we write them in the form

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}=0,\quad \frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0$$

The radical plane of these two spheres is

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0$$

$\Rightarrow (S_1+\lambda_1 S_2)(1+\lambda_2)-(S_1+\lambda_2 S_2)(1+\lambda_1)=0$

$\Rightarrow \lambda_2(S_1-S_2)-\lambda_1(S_1-S_2)=0$

$\Rightarrow (\lambda_2-\lambda_1)(S_1-S_2)=0$

$\Rightarrow S_1-S_2=0 \quad (\because \lambda_1\neq\lambda_2)$

Since the radical plane is independent of λ_1, λ_2, we see that every two members of the system have the same radical plane.

$\therefore S_1+\lambda S_2=0$ represents a system of co-axial spheres determined by two spheres $S_1=0$, $S_2=0$.

The co-axial system is also given by the eqn

$S_1+\lambda(S_1-S_2)=0$

Equation of co-axial system in the simplest form:

To prove that the equation of a co-axial system of spheres can be put in the form $x^2+y^2+z^2+2ux+d=0$ where 'u' is the parameter.

Soln: Let any two spheres of the system be

$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ — (1)

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ — (2)

(1) − (2)

Now take the line joining the centres as the x-axis.

$\therefore$ y & z co-ordinates of their centres become zero. i.e., $v_1=0, w_1=0, v_2=0, w_2=0$ and the equations of the spheres ① & ② become

$$x^2+y^2+z^2+2u_1x+d_1=0 \quad \text{③}$$
$$x^2+y^2+z^2+2u_2x+d_2=0 \quad \text{④}$$

Now the equation of their radical plane is

$$2(u_1-u_2)x+d_1-d_2=0$$

Let this be taken as the yz-plane i.e, $x=0$.

$\therefore d_1-d_2=0$

$\Rightarrow d_1=d_2=d$ (say)

$\therefore$ Eqns of spheres ③ & ④ become

$$x^2+y^2+z^2+2u_1x+d=0$$
$$x^2+y^2+z^2+2u_2x+d=0$$

$\therefore$ The equations of the co-axial system can be put in the form

$$x^2+y^2+z^2+2ux+d=0.$$

where 'u' is parameter.

Limiting points of a co-axial system:

The centres of two spheres of a co-axial system which have zero radius are called the limiting points of the system.

To find the limiting points of a system of co-axial spheres $x^2+y^2+z^2+2ux+d=0$

Sol: The given system of co-axial spheres is $x^2+y^2+z^2+2ux+d=0$.

Its centre is $(-u,0,0)$ and radius $\sqrt{u^2-d}$

Since for limiting points, radius $=0$

$\therefore \sqrt{u^2-d}=0 \Rightarrow u^2-d=0$
$\Rightarrow u=\pm\sqrt{d}$

$\therefore$ The centre $(-u,0,0)$ becomes $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$ which are the reqd limiting points.

→ Find the limiting points of the co-axial system defined by the spheres

$$x^2+y^2+z^2+3x-3y+6=0 \quad \text{①}$$
$$x^2+y^2+z^2-6y-6z+6=0 \quad \text{②}$$

Soln: The equation of any plane of circle two given spheres is ①-②

$\therefore\ 3x+3y+6z=0$

$\Rightarrow x+y+2z=0$

Now the eqn of co-axial system determined by the given spheres is

$x^2+y^2+z^2+3x-3y+\lambda(x+y+2z)=0$

$(\because S_1+\lambda(S_1-S_2)=0)$

$\Rightarrow x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0$ — (3)

where 'λ' is parameter.

Its centre $=\left(-\frac{(3+\lambda)}{2}, -\frac{(\lambda-3)}{2}, -\lambda\right)$ — (4)

and radius

$=\sqrt{\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6}$

For limiting point, equating this to zero, we get

$\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6=0$

$\Rightarrow \lambda^2=1 \Rightarrow \boxed{\lambda=\pm 1}$

$\therefore$ (4) $\equiv (-1,2,1)$ & $(-2,1,-1)$.

which are the required limiting points.

→ Show that the spheres which cut two given spheres along great circles all pass through two fixed points.

Sol^n: Let the two given spheres be

$x^2+y^2+z^2+2u_1x+d=0$ — (1)

$x^2+y^2+z^2+2u_2x+d=0$ — (2)

The eqn of another sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+c=0$ — (3)

where u, v, w, c are different for different spheres.

If (3) cuts (1) in a great circle then the centre $(-u_1,0,0)$ of (1) must be lie on the radical plane.

i.e., the plane of circle (1) & (3) is

$2(u-u_1)x+2vy+2wz+c-d=0$

$\therefore 2(u-u_1)(-u_1)+2v(0)+2w(0)+c-d=0$

$\Rightarrow 2uu_1-2u_1^2-c+d=0$ — (4)

Similarly (3) cuts (2) in a great circle if

$2uu_2-2u_2^2-c+d=0$ — (5)

(5)−(4) $\equiv$

$2u(u_1-u_2)-2(u_1^2-u_2^2)=0$

$\Rightarrow u-(u_1+u_2)=0$

$\Rightarrow \boxed{u=u_1+u_2}$

(4) $\equiv \boxed{c=2u_1u_2+d}$

u & c are constants dependent on only u_1, u_2, d, the given quantities.

∴ The sphere ③ cuts x-axis where putting $y=0, z=0$ in ③.

$③ \equiv x^2+2ux+c=0$

The roots of this equation are constant; depending upon the constants u & c only.

∴ Every sphere ③ cuts the x-axis at the same two points.

Hence the result.

→ Find the limiting points of the co-axial system of spheres

$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0$.

(Ans: $(2,-3,4)$, $(-2,3,-4)$)

→ Prove that the every sphere that passes through the limiting points of a co-axial system cuts every sphere of the system orthogonally.

Sol^n: Let the system of co-axial sphere be

$$x^2+y^2+z^2+2\lambda x+d=0 \quad \text{①}$$

The limiting points of system are $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$.

Let the eqn of the sphere through the limiting points be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad \text{②}$$

Since it passes through the limiting points $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$

∴ $d+2u\sqrt{d}+c=0$ &

$d-2u\sqrt{d}+c=0$

Solving these we get $u=0$ & $c=-d$.

$$② \equiv x^2+y^2+z^2+2vy+2wz-d=0 \quad \text{③}$$

Since ③ & ① cut orthogonally

iff $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

$\Rightarrow 0(2\lambda)+2v(0)+2w(0)=d-d$

$0=0$ which is true.

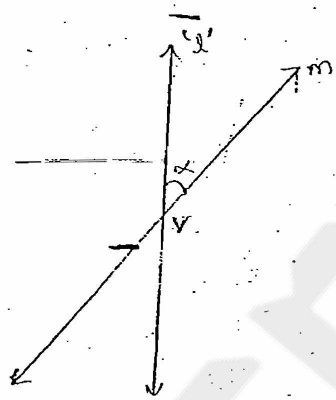
Hence the result.

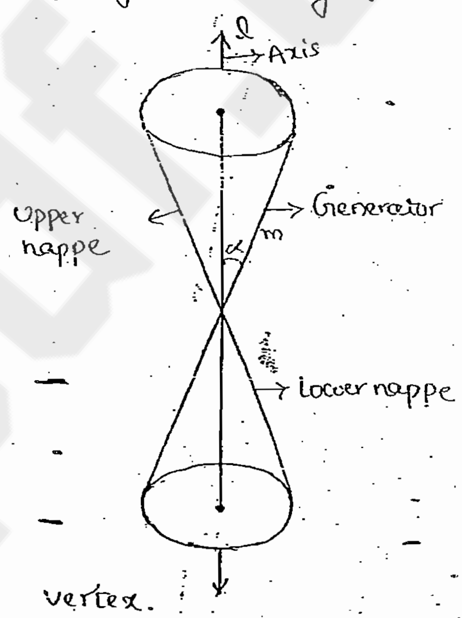
→ Show that the eqn $x^2+y^2+z^2+2\mu y+2\nu z+d=0$ where μ & ν are parameters represents a system of spheres passing through limiting points of the system $x^2+y^2+z^2+\lambda x+d=0$ and cutting every member of this system at right angles.

* Cone *

Def → Let 'l' be a fixed vertical line and 'm' be another line intersecting it at a fixed point V and inclined to it at an angle α.

Suppose we rotate the line 'm' around the line 'l' in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone here in after referred as cone and extending indefinitely far in both directions.





The point V is called vertex.
The line 'l' is the axis of the cone. The rotating line 'm' is called a generator of the cone. The vertex separates the cone into two parts called nappes.

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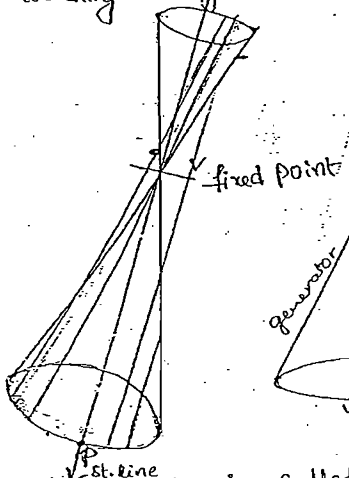
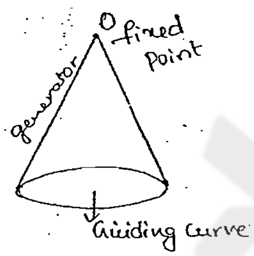
Cone

Set – V

Definition :-

A Cone is a surface generated by a straight line which passes through a fixed point and satisfying one more condition i.e. intersecting a given curve (or) touching a given surface.





→ A fixed point is called the vertex and the given curve (or) surface is called the guiding curve [or guiding surface] of the cone.

→ The straight line is known as the generator of the cone.

→ A cone whose equation is of second degree is known as quadric cone (or) quadratic cone.

* Equation of the Cone with Vertex at the origin :–

To show that the equation of a cone whose vertex is the origin is homogeneous in x, y, z.

Sol'n :- Let the equation of the cone with vertex as the origin be

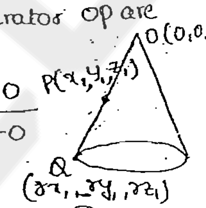
$$f(x, y, z) = 0 \quad \text{——— } (1)$$

Let $P(x_1, y_1, z_1)$ be any point on the cone.

$$\therefore f(x_1, y_1, z_1) = 0 \quad \text{—— } (2)$$

Equations of the generator OP are

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0}$$



$$\Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} \quad \text{——— } (3)$$

Any point on OP line is $Q(rx_1, ry_1, rz_1)$

Since the generator completely lies on the cone,

$$\therefore f(rx_1, ry_1, rz_1) = 0 \quad \text{for all values of } r \quad \text{——— } (4)$$

from (2) & (4) we have

$f(x_1, y_1, z_1)$ is homogeneous equation in x_1, y_1, z_1.

$\therefore f(x, y, z) = 0$ is homogeneous in x, y, z.

Conversely, any homogeneous equation in x, y, z represents a cone whose vertex is the origin.

Sol'n :- Let the homogeneous equation be

$$f(x, y, z) = 0 \quad \text{——— } (5)$$

If $P(x_1, y_1, z_1)$ is any point on the above surface then $f(x_1, y_1, z_1)=0$ ——⑥

Since the equation ⑤ is homogeneous, we have

$f(rx_1, ry_1, rz_1)=0$ ——⑦

for all values of r.

But the point $Q(rx_1, ry_1, rz_1)$ is any point on the line OP.

∴ Every point of the line OP lies on the surface ⑤

∴ The surface is generated by the line through 'O'.

∴ it represents a cone with vertex at the origin.

Note:— The second degree homogeneous

$ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$

represents a cone with vertex at the origin.

Note:— Method to make both equations homogeneous, when none of the two equations is a linear in x, y, z :—

(i) Make both equations homogeneous in x, y, z and t by introducing proper power of t, where t stands for 1.

(ii) Eliminate t from the two equations so obtained.

→ Find the equation to the cone with vertex at the origin and which pass through the curves given by the equations.

(i) $x^2+y^2+z^2-x-1=0$
$x^2+y^2+z^2-y-2=0$,

(ii) $x^2+y^2+z^2+x-2y+3z=4$,
$x^2+y^2+z^2+2x-3y+4z=5$.

Sol'n :— (i) The given equations

$x^2+y^2+z^2-x-1=0$ & $x^2+y^2+z^2+y-2=0$

$\Rightarrow x^2+y^2+z^2-xt-t^2=0$ ——① & $x^2+y^2+z^2+yt-2t^2=0$ ——② where $t=1$.

To eliminate t from ① & ②

Now ② − ① ≡ $-ty-tx+t^2=0$

$\Rightarrow t(t-x-y)=0$

$\Rightarrow t-x-y=0 \quad (\because t\neq 0)$

$\Rightarrow t=x+y$.

∴ ① ≡ $x^2+y^2+z^2-x(x+y)-(x+y)^2=0$

$\Rightarrow x^2+3xy-z^2=0$

which is the required equation of the cone.

→ Find the equations to the cone with vertex at the origin which pass through the curve.

$ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

Sol'n :— The given equations are

$ax^2+by^2+cz^2=1$ —— ①

and $lx+my+nz=p$ —— ②

② ≡ $\dfrac{lx+my+nz}{p}=1$

(2)

$\textcircled{1} \equiv ax^2+by^2+cz^2=(1)^2$

$\textcircled{1} \equiv ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{P}\right)^2$

$\Rightarrow P^2(ax^2+by^2+cz^2)=(lx+my+nz)^2$

which is the required equation of the cone.

→ find the equation to the cone at the origin which passes the curve $ax^2+by^2=2z$, $lx+my+nz=p$ — ① — ②

Sol'n :- $\textcircled{2} \equiv \frac{lx+my+nz}{P}=1$

$\textcircled{1} \equiv ax^2+by^2=2z\,(1)$.

$\Rightarrow ax^2+by^2=2z\left[\frac{lx+my+nz}{P}\right]$

$\Rightarrow P(ax^2+by^2)=2z(lx+my+nz)$

which is the required equation of the cone.

*** Equation of a cone with a given vertex and a given base Conic :-**

To find the equation to the cone whose vertex is the point (α,β,γ) and base the conic

$f(x,y)=ax^2+by^2+2hxy+2fy+2gx+c=0,\ z=0.$

Sol'n :- The equations of the conic are $ax^2+by^2+cz^2+2hxy+2fy+2gx+c=0,$
$z=0.$ — ①

The equations of any line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{— ②}$$

This line meets the plane $z=0$.

$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$

$\Rightarrow \frac{x-\alpha}{l}=\frac{-\gamma}{n}$ / $\frac{y-\beta}{m}=\frac{-\gamma}{n}$

$\Rightarrow x=\alpha-\frac{l\gamma}{n}$, $y=\beta-\frac{m}{n}\gamma$

If this point $(x,y,0)$ lies on the given conic then

$$a\left[\alpha-\frac{l\gamma}{n}\right]^2+b\left[\beta-\frac{m\gamma}{n}\right]^2+2h\left[\alpha-\frac{l\gamma}{n}\right]\left[\beta-\frac{m\gamma}{n}\right]+2g\left[\alpha-\frac{l\gamma}{n}\right]+c=0 \quad \text{— ③}$$

This is the condition for line ② to intersect the conic ①.

Now eliminating l,m,n from ② & ③

Now putting the values of l,m,n from ② in ③ we have

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2+2h\left[\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right]\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)+c=0$$

$$\Rightarrow a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2+2h(\alpha z-\gamma x)(\beta z-\gamma y)+2f(\beta z-\gamma y)(z-\gamma)+2g(\alpha z-\gamma x)(z-\gamma)+c(z-\gamma)^2=0 \quad \text{— ④}$$

which is the required equation of the cone.

Note:— (i) The equation of the cone is satisfied by the coordinates of the vertex (α,β,γ) i.e. putting α,β,γ for x,y,z in (4) we have

$$a(\alpha\gamma-\gamma\alpha)^2+b(\beta\gamma-\gamma\beta)+2h(\alpha\gamma-\gamma\alpha)(\beta\gamma-\gamma\beta)$$
$$+2f(\beta\gamma-\gamma\beta)(\gamma-\gamma)+2g(\alpha\gamma-\gamma\alpha)(\gamma-\gamma)+c(\gamma-\gamma)^2=0$$

$\Rightarrow 0=0$ which is true.

ii. The equation of the cone (4) also satisfied by the equation of the base cone.

putting $z=0$ in (4) we have

$$a\gamma^2x^2+b\gamma^2y^2+2h\gamma^2xy+2f\gamma^2y+2g\gamma^2x+c\gamma^2=0$$

through dividing with γ^2

$$ax^2+by^2+2hxy+2fy+2gx+c=0$$

→ Find the equation of the cone whose vertex is (α,β,γ) and whose base is

(i) $ax^2+by^2=1,\ z=0.$

(ii) $y^2=4ax,\ z=0.$

(iii) $x^2+y^2=4,\ z=2,$ vertex $(1,1,1)$

Sol^n (i) The given base conic is

$$ax^2+by^2=1,\ z=0 \quad \text{—(1)}$$

Now equation of any line through $w(\alpha,\beta,\gamma)$

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—(2)}$$

it meets the plane $z=0$ where.

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$\Rightarrow x-\alpha=-\frac{l}{n}\gamma,\quad y-\beta=-\frac{m}{n}\gamma$

$\Rightarrow x=\alpha-\frac{l}{n}\gamma,\quad y=\beta-\frac{m}{n}\gamma$

This point lies on the conic (1)

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+b\left(\beta-\frac{m}{n}\gamma\right)^2=1 \quad \text{—(3)}$$

Now eliminating l,m,n from (2)&(3) we have —

$$a\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]^2+b\left[\beta-\frac{y-\beta}{z-\gamma}\right]^2=1$$

$$\Rightarrow a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$$

which is the required equation of the cone.

H.W → Obtain the locus of the lines which pass through a point (α,β,γ) and through points of the conic.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

$$\left[\text{Ans: } \left(\frac{\alpha z-x\gamma}{a}\right)^2+\left(\frac{\beta z-y\gamma}{b}\right)^2=(z-\gamma)^2\right]$$

→ Find the equation of the cone whose vertex is the point $(1,1,0)$ and whose guiding curve is $y=0$, $x^2+z^2=4$.

$$\left[\text{Ans: } x^2-3y^2+z^2-2xy+8y-4=0\right]$$

IFS-2009 The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —(1), $z=0$ by the plane $x=0$ is a rectangular hyperbola. show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$

Sol^n:- Let the vertex P be (α,β,γ) and given guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ —— (1)

Now the equation of any line through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—— (2)}$$

it meets the plane $z=0$.

$$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\Rightarrow x-\alpha=-\frac{l}{n}\gamma,\quad y-\beta=-\frac{m}{n}\gamma$$

$$\Rightarrow x=\alpha-\frac{l\gamma}{n},\quad y=\beta-\frac{m}{n}\gamma,\quad z=0$$

This point lies on the ellipse (1)

$$\therefore \frac{1}{a^2}\left[\alpha-\frac{l}{n}\gamma\right]^2+\frac{1}{b^2}\left[\beta-\frac{m}{n}\gamma\right]^2=1 \quad \text{—— (3)}$$

Now eliminating l,m,n from (2)&(3) we have.

$$\frac{1}{a^2}\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+\frac{1}{b^2}\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2=1$$

$$\Rightarrow \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2 \quad \text{—— (4)}$$

which is required equation of cone.

This meets the plane $x=0$

$$\therefore (4) \equiv \frac{1}{a^2}(\alpha z-0)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$$

$$\Rightarrow \frac{\alpha^2z^2}{a^2}+\frac{\beta^2z^2+\gamma^2y^2-2\beta\gamma yz}{b^2}=z^2+\gamma^2-2z\gamma \quad (3)$$

This will be a rectangular hyperbola in yz-plane.

if Coefficient of y^2 + coefficient of z^2 = 0

if $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{b^2}-1=0$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

→ show that the equation of the cone whose vertex is the origin and whose base is the circle through the three points $(a,0,0),(0,b,0),(0,0,c)$ is $\sum a(b^2+c^2)yz=0$.

(or)

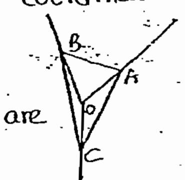
The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Sol^n:- The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{—— (1)}$$

Since it meets the coordinate axes in A, B, C the coordinates of A, B, C are $(a,0,0),(0,b,0),(0,0,c)$



Now the circle through A, B, C is the intersection of plane through A, B, C i.e.

i.e. Plane ① and any sphere through the points A, B, C say the sphere OABC.

Now the sphere OABC through the points O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c) is $x^2+y^2+z^2-ax-by-cz=0$ —②

∴ The guiding curve is the circle given by ① & ②

i.e. $x^2+y^2+z^2-ax-by-cz=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

② ≡ $x^2+y^2+z^2-(ax+by+cz)(1)=0$

$\Rightarrow x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(1)=0.$

$\Rightarrow x^2+y^2+z^2-x^2-\frac{b}{a}xy-\frac{c}{a}zx-\frac{a}{b}xy-y^2-\frac{c}{b}yz-\frac{a}{c}zx-\frac{b}{c}yz-z^2=0$

$\Rightarrow -y\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)-xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$

$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$

$\Rightarrow \sum a(b^2+c^2)yz=0$ —③

which is required equation of the cone.

→ find the equation of the cone whose vertex is (1,2,3) and guiding curve the circle $x^2+y^2+z^2=4$, $x+y+z=1$.

Sol'n :— Any generator through (1,2,3) is

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{x+y+z-6}{l+m+n}$$ —①

if it meets the plane $x+y+z=1$ then from ①, we have

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{1-6}{l+m+n}$$

$$\Rightarrow \frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{-5}{l+m+n}$$

$\Rightarrow x=1-\left[\frac{5l}{l+m+n}\right]$, $y=2-\left[\frac{5m}{l+m+n}\right]$ and $z=3-\left[\frac{5n}{l+m+n}\right]$.

i.e. the generator ① meets the plane $x+y+z=1$ in the point

$$\left[\frac{m+n-4l}{l+m+n}, \frac{2l-3m-2n}{l+m+n}, \frac{3l+3m-2n}{l+m+n}\right]$$

If this point lies on $x^2+y^2+z^2=4$ we get

$$(m+n-4l)^2+(2l-3m-2n)^2+(3l+3m-2n)^2=4(l+m+n)^2$$ —②

Eliminating l, m, n between ① & ② we get

$$[(y-2)+(z-3)-4(x-1)]^2+[2(x+1)-3(y-2)+2(z-3)]^2+[3(x+1)+3(y-2)-2(z-3)]^2=4[(x-1)+(y-2)+(z-3)]^2$$

$\Rightarrow (y+z-4x-1)^2+(2x-3y+2z-2)^2+(3x+3y-2z-3)^2=4(x+y+z-6)^2$

$\Rightarrow 5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z=$

which is the required equation.

→ Show that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ where $l^2+3m^2-3n^2=0$, is a generator of the cone $x^2+3y^2-3z^2=0$.

Sol^n :- The given line is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{--- (1)}$$

where $l^2+3m^2-3n^2=0$ —(2)

we can eliminate l,m,n from (1) & (2)

$l=x,\ m=y,\ n=z$

(2) $\equiv x^2+3y^2-3z^2=0$ —(3)

which is the required Cone.

∴ (1) lies on the Cone (3).

→ Show that the lines through the point (α,β,γ) whose d.c's satisfies $al^2+bm^2+cn^2=0$ generate the Cone.

$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$

Sol^n :- Any line through the point (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{--- (1)}$$

where $al^2+bm^2+cn^2=0$ —(2)

Eliminate l,m,n from (1) & (2) we have

$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$

which is the required cone.

Hence the result.

4

→ Show that the equation of the cone whose vertex is at the origin and the d.c's of whose generator satisfy the relation $3l^2-4m^2+5n^2=0$ is $3x^2-4y^2+5z^2=0$.

* Enveloping Cone of a Sphere:

Definition:— The locus of the tangent from a given point to sphere is a cone called the enveloping cone or tangent cone from the point to the sphere.

(or)

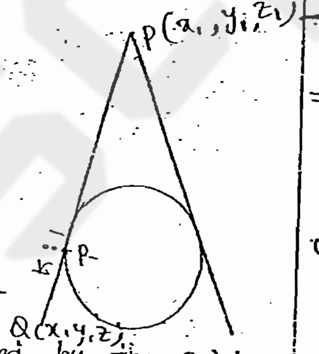
The cone formed by the tangent lines to a surface, drawn from a given point is called the enveloping cone of the surface with given point as its vertex.

→ To find Equation of the enveloping cone from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2=a^2$.

Sol'n:— The given equation of the sphere is $x^2+y^2+z^2=a^2$ —— ①

Let $P(x_1, y_1, z_1)$ be any given point.

Let $Q(x, y, z)$ be any point on a tangent from P to the given sphere.



Let PQ be divided by the point of contact R in the ratio $k:1$.

∴ The coordinates of R are

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right)$$

Since this point R lies in the sphere ①

$$\therefore \left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2+\left(\frac{kz+z_1}{k+1}\right)^2=a^2$$

$$\Rightarrow k^2x^2+x_1^2+2kxx_1+k^2y^2+y_1^2+2kyy_1+k^2z^2+z_1^2+2kzz_1=a^2(k^2+2k+1)$$

$$\Rightarrow k^2(x^2+y^2+z^2-a^2)+2k(xx_1+yy_1+zz_1-a^2)+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad \text{—— ②}$$

which is quadratic in k.

Since the line PQ touches the sphere, the two values of k must be equal

∴ Discriminant of ② $=0$

i.e. $b^2-4ac=0$

$$\therefore 4[xx_1+yy_1+zz_1-a^2]^2-4[x^2+y^2+z^2-a^2][x_1^2+y_1^2+z_1^2-a^2]=0$$

$$\Rightarrow [x^2+y^2+z^2-a^2][x_1^2+y_1^2+z_1^2-a^2]=[xx_1+yy_1+zz_1-a^2]^2$$

which is the required equation of the enveloping cone.

Note:— If $S=x^2+y^2+z^2-a^2$ so that $S=0$ is the equation of the sphere then $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$ i.e. S_1 is the result of substituting the point (x_1, y_1, z_1) in S.

and $T = xx_1 + yy_1 + zz_1 - a^2$ the expression of the tangent plane at (x_1, y_1, z_1) to the sphere.

then the enveloping cone is $SS_1 = T^2$.

→ Find the enveloping cone of the sphere $x^2+y^2+z^2-2x+4z=1$ with vertex at $(1,1,1)$.

Sol'n :- the equation of the sphere is $x^2+y^2+z^2-2x+4z=1$ —— ①

and the given vertex is $P(1,1,1)$

Let $S = x^2+y^2+z^2-2x+4z-1$

and $x_1 = 1,\ y_1 = 1,\ z_1 = 1$

$\therefore S_1 = (1)^2 + (1)^2 + (1)^2 - 2(1) + 4(1) - 1$

$= 4$

and $T = xx_1 + yy_1 + zz_1 - (x+x_1) + 2(z+z_1) - 1$

$= x(1) + y(1) + z(1) - (x+1) + 2(z+1) - 1$

$= \cancel{x} + y + z - \cancel{x} - \cancel{1} + 2z + \cancel{2} - \cancel{1}$

$= y + 3z$

$\therefore$ Equation of the enveloping cone is $SS_1 = T^2$.

$\Rightarrow (x^2+y^2+z^2-2x+4z-1)(4) = (y+3z)^2$

$\Rightarrow 4x^2+4y^2+4z^2-8x+16z-4 = y^2+9z^2+6yz$

$\Rightarrow 4x^2+3y^2-5z^2-8x+16z-6yz-4 = 0$

→ Show that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has its vertex at $(2,4,1)$ in a rectangular hyperbola.

Sol'n :- The given equation of the sphere is $x^2+y^2+z^2=11$ —— ①

and given vertex $(2,4,1)$.

Let $S = x^2+y^2+z^2-11$ and $x_1 = 2$, $y_1 = 4$, $z_1 = 1$.

$\therefore S_1 = 4+16+1-11 = 10$

$T = xx_1 + yy_1 + zz_1 - 11$

$= 2x + 4y + z - 11$

$\therefore$ the equation of the enveloping cone is $SS_1 = T^2$.

$\Rightarrow (x^2+y^2+z^2-11)(10) = (2x+4y+z-11)^2$ —— ②

This meets the plane $z=0$.

$\therefore (x^2+y^2+0-11)(10) = (2x+4y+0-11)^2$

$\Rightarrow (x^2+y^2-11)10 - (2x+4y-11)^2 = 0$.

This represents a rectangular hyperbola in the xy- plane.

if coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore (10-4) + (10-16) = 0$

$\Rightarrow 6-6 = 0$

$\Rightarrow 0 = 0$. which is true.

Hence the result.

* Quadratic Cone through the axes :-

→ Show that the general equation of a cone of second degree which pass through the axes is

$fyz + gzx + hxy = 0$.

where f, g, h are parameters.

Sol^n :- The general equation of the cone with its vertex at the origin is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ —— (1)

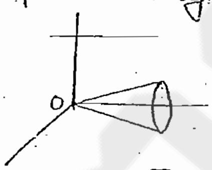
Since it passes through x-axis

∴ the d.c's of x-axis are 1, 0, 0 must satisfy (1)

$\therefore a(1)^2 + b(0)^2 + c(0)^2 + 2f(0) + 2g(0) + 2h(0) = 0$

$\Rightarrow \boxed{a=0}$

Similarly the cone passes through the axes of y & z.



we have $b=0, c=0$.

$\therefore (1) \equiv a(x^2) + 0(y^2) + 0(z^2) + 2fyz + 2gzx + 2hxy = 0$

$\Rightarrow 2fyz + 2gzx + 2hxy = 0$

$\Rightarrow fyz + gzx + hxy = 0$.

which is the required condition.

→ Show that a cone can be found so as to contain any two given sets of three mutually perpendicular concurrent lines as generators.

(or)

Show that a cone of second degree can be found to pass through any two sets of rectangular axes through the same origin.

Sol^n :- Take the three lines of one set as coordinate axes (i.e. OX, OY, OZ).

Let the lines OX', OY', OZ' of the second set be $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$, $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$, $\frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3}$.

Now general equation of the cone through axes (i.e. OX, OY, OZ) is $fyz + gzx + hxy = 0$ —— (1)

If it passes through OX' & OY' then the d.c's l_1, m_1, n_1 & l_2, m_2, n_2 of OX', OY', OZ' satisfy (1)

$\therefore fm_1n_1 + gn_1l_1 + hl_1m_1 = 0$ —— (2)

$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0$ —— (3)

Adding (2) & (3) we have

$f(m_1n_1 + m_2n_2) + g(n_1l_1 + n_2l_2) + h(l_1m_1 + l_2m_2) = 0$. —— (4)

But l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the d.c's of three mutually ⊥ lines.

$\therefore m_1n_1 + m_2n_2 + m_3n_3 = 0 \Rightarrow m_1n_1 + m_2n_2 = -m_3n_3$

$n_1l_1 + n_2l_2 + n_3l_3 = 0 \Rightarrow n_1l_1 + n_2l_2 = -n_3l_3$

& $l_1m_1 + l_2m_2 + l_3m_3 = 0 \Rightarrow l_1m_1 + l_2m_2 = -l_3m_3$

Putting these values in (4) we have

$-fm_3n_3 - gn_3l_3 - hl_3m_3 = 0$

$\Rightarrow fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$

i.e. ① is satisfied by the d.c's l_3, m_3, n_3 of OZ'

$\therefore$ The cone passes through the OZ' i.e. the cone passes through OX, OY, OZ and OX', OY', OZ' i.e. two sets of rectangular axes.

→ Find the equation of the cone which contains the three coordinate axes and the lines through the origin having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2.

Sol'n :- The equation of any cone through the three coordinate axes is $fyz + gzx + hxy = 0$ —①

Since it passes through lines with d.c's l_1, m_1, n_1 and l_2, m_2, n_2 and the d.c's of the generators satisfy the equation of the cone.

$\therefore fm_1n_1 + gn_1l_1 + hl_1m_1 = 0$ —②

$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0$ —③

$\therefore$ Eliminating f, g, h from ①,②,③ we have

$$\begin{vmatrix} yz & zx & xy \\ m_1n_1 & n_1l_1 & l_1m_1 \\ m_2n_2 & n_2l_2 & l_2m_2 \end{vmatrix} = 0$$

$\Rightarrow yz[n_1l_1l_2m_2 - n_2l_2l_1m_1]$
$- zx[m_1n_1l_2m_2 - m_2n_2l_1m_1]$
$+ xy[m_1n_1n_2l_2 - m_2n_2n_1l_1] = 0$

$\Rightarrow l_1l_2yz[n_1m_2 - n_2m_1] + m_1m_2xz[n_2l_1 - n_1l_2]$
$+ n_1n_2xy[m_1l_2 - m_2l_1]$

Which is required equation.

→ Find the equation to the cone which passes through the three coordinate axes as well as the two lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ —(I); $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$ —(II)

Sol'n :- the equation of any cone through the three coordinate axes

$fyz + gzx + hxy = 0$ —①

Since it passes through the lines (I) & (II) and d.c's of the generators satisfy the equation of the cone.

$\therefore f(-2)(3) + g(3)(1) + h(1)(-2) = 0$

& $f(-1)(1) + g(1)(3) + h(3)(-1) = 0$

$\Rightarrow f(-6) + g(3) + h(-2) = 0$ —②

& $f(-1) + g(3) + h(-3) = 0$ —③

Eliminating f, g, h from ①, ② & ③ we get

$$\begin{vmatrix} yz & zx & xy \\ -6 & 3 & -2 \\ -1 & 3 & -3 \end{vmatrix} = 0.$$

$yz(-9+6) - zx(18-2) + xy(-18+3) = 0$

$\Rightarrow 3yz + 16zx + 15xy = 0$

→ Find the equation of the quadric cone which passes through the 3 coordinate axes and three mutually perpendicular lines.

$\frac{1}{2}x = y = -z$, $x = \frac{1}{3}y = \frac{1}{5}z$, $\frac{1}{8}x = -\frac{1}{11}y = \frac{1}{5}z$

Sol'n :- Now the equation of any cone through the coordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{--- (1)}$$

Since it passes through the line

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$$

$\therefore f(1)(-1) + g(-1)(2) + h(2)(1) = 0$

$\Rightarrow -f - 2g + 2h = 0$

$\Rightarrow f + 2g - 2h = 0$ --- (2)

Similarly (1) passes through $\frac{x}{1} = \frac{y}{3} = \frac{z}{5}$

$\therefore f(3)(5) + g(5)(1) + h(1)(3) = 0$

$\Rightarrow 15f + 5g + 3h = 0$ --- (3)

Solving (2) & (3)

$$\frac{f}{6+10} = \frac{g}{-30-3} = \frac{h}{5-30}$$

$$\Rightarrow \frac{f}{16} = \frac{g}{-33} = \frac{h}{-25}$$

$\therefore$ putting these values of f, g, h in (1) we get

$$16(yz) + (-33)zx + (-25)xy = 0$$

$$\Rightarrow 16yz - 33zx - 25xy = 0$$

which is the required equation of the cone and the generator line $\frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$ also satisfy the equation of this cone.

→ Planes through OX & OY include an angle α, show that their line of intersection lies on the cone $z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha$.

Sol'n :- The equation of any plane through OX $(y=0, z=0)$ is $y + \lambda z = 0$ --- (1)

and the equation of any plane through OY $(x=0, z=0)$ is $x + \mu z = 0$ --- (2)

The angle between the two planes (1) & (2) is

$$\cos\alpha = \frac{0.1 + 1.0 + \mu\lambda}{\sqrt{1+\lambda^2}\,\sqrt{1+\mu^2}} = \frac{\mu\lambda}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}}$$

$$= \frac{\mu\lambda}{\sqrt{1+\mu^2+\lambda^2+\lambda^2\mu^2}}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{\sqrt{1+\mu^2+\lambda^2+\lambda^2\mu^2}}{\mu\lambda}$$

$$\tan^2\alpha = \sec^2\alpha - 1$$

$$= \frac{1+\mu^2+\lambda^2+\lambda^2\mu^2}{\mu^2\lambda^2} - 1$$

$$= \frac{1+\lambda^2+\mu^2}{\mu^2\lambda^2} \quad \text{--- (3)}$$

Eliminating λ, μ from (1), (2) & (3) we get

$$\tan^2\alpha = \frac{1 + \frac{y^2}{z^2} + \frac{x^2}{z^2}}{\left(\frac{y^2}{z^2}\right)\left(\frac{x^2}{z^2}\right)}$$

$$= \frac{z^2(x^2+y^2+z^2)}{x^2y^2}$$

$$\therefore z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha.$$

which is the required eq'n of the cone.

* Condition for general second degree equation to represent a cone :-

To find the condition that the equation

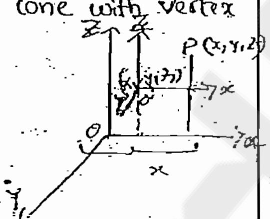
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

may represent a cone.

Sol^n :- The given equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{---}(1)$$

If it represents a cone with vertex at (x_1, y_1, z_1) say, then shifting the origin to the point (x_1, y_1, z_1) so that ~~we change~~ $x = X+x_1$, $Y = y+y_1$ and $z = Z+z_1$.



$\therefore$ The transformed equation is

$$a(X+x_1)^2+b(Y+y_1)^2+c(Z+z_1)^2+2f(Y+y_1)(Z+z_1)+2g(Z+z_1)(X+x_1)+2h(X+x_1)(Y+y_1)+2u(X+x_1)+2v(Y+y_1)+2w(Z+z_1)+d=0$$

$$\Rightarrow ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2x(ax_1+hy_1+gz_1+u)+2y(hx_1+by_1+fz_1+v)+2z(gx_1+fy_1+cz_1+w)+(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d)=0 \quad \text{---}(2)$$

Since (2) represents a cone with vertex at the origin, so it must be homogeneous in x, y, z.

$\therefore$ Coefficient of $x=0$, coefficient of y, coefficient of $z=0$ and constant term

i.e. $ax_1+hy_1+gz_1+u=0$ ---(3)

$hx_1+by_1+fz_1+v=0$ ---(4)

$gx_1+fy_1+cz_1+w=0$ ---(5)

and $ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0$ ---(6)

Now (6) can be written as

$$x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)+z_1(gx_1+fy_1+cz_1+w)+ux_1+vy_1+wz_1+d=0$$

$\Rightarrow ux_1+vy_1+wz_1+d=0$ ---(7) (using (3), (4) & (5))

Eliminating x_1, y_1, z_1 from (3), (4), (5) & (7) we get.

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

which is the required condition.

Note :- The vertex of the cone is obtained by solving any three of the four equations (3), (6), (5) and (7) for x_1, y_1, z_1.

Method for Numerical questions:

(i) Make the given equation homogeneous in x, y, z, t by introducing proper powers of t where $t=1$.

(ii) Let this be denoted by $F(x, y, z, t) = 0$.

(iii) Then the four equations ③, ④, ⑤ & ⑦ are obtained by equations $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ and $\frac{\partial F}{\partial t}=0$ where ultimately $t=1$.

(iv) Solve any three of the above four equations for x, y, z.

(v) Substitute these values of x, y, z in the fourth equation and if it is satisfied then the given equation represents a cone and values of x, y, z found in (iv) are the coordinates of the vertex.

→ Show that the equation $4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$ represents a cone with vertex $(-1,-2,-3)$.

Sol: Given equation is

$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$

Making given equation homogeneous, we get

$$F(x,y,z,t) = 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$$

Now $\frac{\partial F}{\partial x}=0$

$\Rightarrow 8x+2y+12t=0$

$\Rightarrow \boxed{4x+y+6t=0}$

$\frac{\partial F}{\partial y}=0$

$\Rightarrow -2y+2x-3z-11t=0$

$\Rightarrow \boxed{2x-2y-3z-11t=0}$

$\frac{\partial F}{\partial z}=0 \Rightarrow \boxed{4z-3y+6t=0}$

and $\frac{\partial F}{\partial t}=0 \Rightarrow \boxed{12x-11y+6z+8t=0}$

Putting $t=1$ in above equations, we get

$4x+y+6=0$ — ②

$2x-2y-3z-11=0$ — ③

$3y-4z-6=0$ — ④

$12x-11y+6z+8=0$ — ⑤

③ × 2 ≡ $4x-4y-6z-22=0$ — ⑥

Now ⑥ − ② $5y+6z+28=0$

$\Rightarrow 10y+12z+56=0$ — ⑦

④ × 3 $9y-12z-18=0$ — ⑧

⑦ + ⑧ ≡ $19y+38=0$

$\Rightarrow \boxed{y=-2}$

④ ≡ $3(-2)-4z-6=0$

$\Rightarrow -4z=12$

$\Rightarrow \boxed{z=-3}$

② ≡ $4x-2+6=0$

$\Rightarrow 4x=-4$

$\Rightarrow \boxed{x=-1}$

$\therefore$ These values of x, y & z as $x=-1$, $y=-2$ & $z=-3$ satisfy (5)

$\therefore$ The equation represents a cone and its vertex is $(-1,-2,-3)$.

→ Show that the equation $x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y-2z-20=0$ represents a cone with vertex $(1,-2,3)$.

→ Show that the equation $2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$ represents a cone whose vertex is $(-7/6, 1/3, 5/6)$.

Angle between two lines in which a plane through the vertex cuts a cone:–

→ Find the angle between the lines of intersection of the plane $x-3y+z=0$ and the cone $x^2-5y^2+z^2=0$

Sol'n :– The given plane is

$$x-3y+z=0 \quad \text{—(1)}$$

and given cone is $x^2-5y^2+z^2=0$ —(2)

Let the line of section be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{—(3)}$$

Since it lies on the plane (1)

$\therefore$ It is $\perp$lar to the normal to the plane.

$\therefore al+bm+cn=0$

$\Rightarrow l-3m+n=0$ —(4)

Also the line (3) lies on the cone (

$\therefore$ Its d.c's satisfies the equation of the cone.

$\therefore l^2-5m^2+n^2=0$ —(5)

(4) $\equiv \boxed{l=3m-n}$

(5) $\Rightarrow (3m-n)^2-5m^2+n^2=0$

$\Rightarrow 9m^2+n^2-6mn-5m^2+n^2=0$

$\Rightarrow 4m^2+2n^2-6mn=0$

$\Rightarrow 4m^2+2n^2-4mn-2mn=0$

$\Rightarrow 4m(m-n)-2n(m-n)=0$

$\Rightarrow (4m-2n)(m-n)=0$

$\Rightarrow m-n=0$ | $4m-2n=0$

$\Rightarrow m=n$ | $\cancel{4m=2n} \Rightarrow m=\frac{1}{2}n$

$\Rightarrow m-n=0$ | $0l+4m-2n=0$

$\Rightarrow 0l+m-n=0$ | also $l-3m+n=0$

from (1) $l-3m+n=0$

Solving

$\frac{l}{1-3}=\frac{m}{-1-0}=\frac{n}{0-1}$ | $\frac{l}{4-6}=\frac{m}{-2-0}=\frac{n}{0-4}$

$\frac{l}{-2}=\frac{m}{-1}=\frac{n}{-1}$ | $\Rightarrow \frac{l}{-2}=\frac{m}{-2}=\frac{n}{-4}$

$\Rightarrow \frac{l}{2}=\frac{m}{1}=\frac{n}{1}$ | $\Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{2}$

Putting these values of l,m,n in (3), the required lines of section are $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$ & $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$

If θ is angle between two lines of section then $\cos\theta = \frac{2(1)+1(1)+2(1)}{\sqrt{4+1+1}\sqrt{1+1+4}}$

$$\cos\theta = \frac{(5)}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\therefore \theta = \cos^{-1}(5/6)$$

Formulae :-

Let the plane be $ux+vy+wz=0$ — ①

and the cone be

$f(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — ②

then the angle between the lines cutting by plane ① in the cone ② is given by

$$\tan\theta = \frac{2P\sqrt{u^2+v^2+w^2}}{(a+b+c)(u^2+v^2+w^2)-F(u,v,w)}$$

where

$$P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} \text{ and}$$

$F(u,v,w) = au^2+bv^2+cw^2+2fvw+2gwu+2huv.$

Note :- ① The lines are $\perp$lar, if

$F(u,v,w) = (a+b+c)(u^2+v^2+w^2)$

② If the lines are coincident

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0.$$

→ Find the angles between the lines of section of the planes & cones.

(i) $10x+7y-6z=0$ and $20x^2+7y^2-108z^2=0$

(ii) $4x-y-5z=0$ and $8yz+3zx-5xy=0$

(iii) $x+y+z=0$ and $6xy+3yz-2zx=0$

(iv) $x+y+z=0$ and $x^2yz+xy-3z^2=0$

Ans (i) $\cos^{-1}(16/21)$ (ii) $\pi/2$

(iii) $\pi/3$ (iv) $\pi/6$

→ Find the equations to the lines in which plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$

Sol'n :- The given plane is

$2x+y-z=0$ — ①

and cone is $4x^2-y^2+3z^2=0$ — ②

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the equations of any one of the two lines in which the given plane meets the given cone.

$\therefore$ we have

$2l+m-n=0$ — ③, $4l^2-m^2+3n^2=0$ — ④

③ $\equiv n = 2l+m = 0$

④ $\equiv 4l^2-m^2+3(2l+m)^2=0$

$\Rightarrow 4l^2-m^2+3(4l^2+4lm+m^2)=0$

$\Rightarrow 4l^2-m^2+12l^2+12lm+3m^2=0$

$\Rightarrow 16l^2+2m^2+12lm=0$

$\Rightarrow 8l^2+m^2+6lm=0$

$\Rightarrow 8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0$

$\Rightarrow l/m = \frac{-6\pm\sqrt{36-32}}{16} = -\frac{1}{4}$ (or) $-\frac{1}{2}$

$\therefore l/m = -1/4$ & $l/m = -1/2$

$\Rightarrow l/m + 1/4 = 0$ & $l/m + 1/2 = 0$

$\Rightarrow 4l+m=0$ & $2l+m=0$

from ③ we have

$2l+m-n=0$ | $2l+m-n=0$

$\Rightarrow 4l+m+0n=0$ | $2l+m+0n=0$

& $2l+m-n=0$ | & $2l+m-n=0$

solving

$\frac{l}{-1}=\frac{m}{4}=\frac{n}{4-2}$ | $\frac{l}{-1}=\frac{m}{2}=\frac{n}{2-2}$

$\frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$ | $\Rightarrow \frac{l}{-1}=\frac{m}{2}=\frac{n}{0}$

$\therefore$ The required lines are

$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$; $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$

→ Find the equations of the lines of intersection of the following planes and cones.

(i) $x+3y-2z=0$ and $x^2+9y^2-4z^2=0$

(ii) $3x+4y+z=0$ and $15x^2-32y^2-7z^2=0$

2003 → (iii) $x+7y-5z=0$ and $3yz+14zx-30xy=0$

Ans:- (i) $x=2z$, $y=0$; $3y=2z$; $x=0$

(ii) $\frac{x}{-3}=\frac{y}{2}=\frac{z}{1}$; $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}$

(iii) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$; $\frac{x}{3}=\frac{y}{1}=\frac{z}{2}$

→ Show that the equation of the quadratic cone which contains the three coordinate axes and the lines in which the plane $x-5y-3z=0$ cuts the cone $7x^2+5y^2-3z^2=0$ is

$yz+10zx+18xy=0$.

* Mutually Perpendicular generators of a Cone :-

→ The necessary & sufficient condition for cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to have three mutually ⊥lar generators is that sum of coefficient of x^2, y^2, z^2 is zero. i.e. $a+b+c=0$.

→ If the general equation of the second degree

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone, then the condition that it may have three mutually perpendicular generators is $a+b+c=0$.

This result follows on shifting the origin to vertex. The coefficients of the second degree term remain unaffected.

→ Problems

(2006) → Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Sol'n :- The equation of the plane is

$$ax+by+cz=0 \quad \text{—— (1)}$$

and the cone is $yz+zx+xy=0$ —— (2)

Comparing (2) with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

$\therefore a=0,\ b=0,\ c=0$

$\Rightarrow a+b+c = 0+0+0$

$= 0$

∴ The cone (2) has three mutually ⊥lar generators.

The plane (1) will cut the cone (2) in ⊥ lines if the normal to the plane (1) through the vertex (0,0,0) [whose d.c's are proportional to a,b,c] lies on the cone (2)

if $bc+ca+ab=0$ (∵ d.c's of the generator satisfy the equation of the cone).

if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

(on dividing throughout by abc)

which is the required condition.

→ Prove that the plane $lx+my+nz=0$ cuts the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ in perpendicular lines if $(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0$.

Sol'n: The given plane is $lx+my+nz=0$ —— (1)

and cone is

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0 \quad \text{—— (2)}$$

Here the sum of the coefficients of $x^2, y^2, z^2 = (b-c)+(c-a)+(a-b)=0$.

(1)

$\therefore$ The cone (2) has three mutually $\perp$lar generators.

Now if the plane (1) cuts the cone (2) in perpendicular lines then normal to the plane (1) through vertex $(0,0,0)$ i.e. $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is the generator of the cone (2)

Since the d.c's of the generator satisfy the cone equation.

i.e. l, m, n must satisfy (2)

$\therefore (b-c)l^2+(c-a)m^2+(a-b)n^2$

$+2fmn+2gnl+2hlm=0$

which is the required condition.

$\rightarrow$ If $x=\frac{1}{2}y=z$ represents one of a set of three mutually perpendiculars of the cone $11yz+6zx-14xy=0$, find the equations of other two.

2008 $\rightarrow$ If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$ find the equations of the other two.

Sol'n:- The given cone is

$5yz-8zx-3xy=0$ —— (1)

and one of its three $\perp$lar generators is

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ —— (2)

The other two $\perp$lar generators are the lines which plane thr the vertex $(0,0,0)$ and $\perp$ to lin

i.e. the plane $x+2y+3z=0$ —

Let a line of section of (1) & (3)

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ —— (4)

Since (4) lies in the plane (3)

$\therefore$ It is $\perp$ to the normal to the pla

$\therefore l+2m+3n=0$ —— (5).

Also (4) lies on cone (1)

$\therefore$ i.e. the d.c's of (4) satisfies the equation of cone.

$\therefore 5mn-8nl-3lm=0$ —— (6)

(5) $\equiv l=-(2m+3n)$

$\therefore$ (6) $\equiv 5mn+8n(2m+3n)+3m(2m+3n)=0$

$\Rightarrow 6m^2+30mn+24n^2=0$

$\Rightarrow m^2+5mn+4n^2=0$

$\Rightarrow (m+n)(m+4n)=0$

$\Rightarrow m+n=0$	$m+4n=0$
$\Rightarrow 0l+m+n=0$	$\Rightarrow 0l+m+4n=0$
Also (5) $\equiv l+2m+3n=0$	Also $l+2m+3n=$
solving:	Solving:
$\frac{l}{3-2}=\frac{m}{1-0}=\frac{n}{0-1}$	$\frac{l}{3-8}=\frac{m}{4-0}=\frac{n}{0-1}$
$\frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$	$\frac{l}{-5}=\frac{m}{4}=\frac{n}{-1}$

$\therefore$ (4) $\equiv \frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ & $\frac{x}{-5}=\frac{y}{4}=\frac{z}{-1}$

which are the other two generators.

→ Show that the cone whose vertex is origin and which passes through the curve of intersection of the surface $3x^2-y^2+z^2=3a^2$ and any plane at a distance 'a' from the origin, has three mutually perpendicular generators.

→ Show that the cone whose vertex at the origin and which passes through the curve of intersection of the sphere $x^2+y^2+z^2=3a^2$ and any plane at a distance 'a' from the origin has three mutually ⊥lar generators.

Sol^n:- Given sphere is

$$x^2+y^2+z^2=3a^2 \quad \text{---} (1)$$

Any plane at a distance 'a' from the origin is $lx+my+nz=a$ — (2) (normal form)

where l,m,n are d.c's of normal to the plane.

Making (1) homogeneous with the help of (2),

the equation of the cone whose vertex is the origin and base, the curve of intersection of (1) & (2) is

$$x^2+y^2+z^2=3(lx+my+nz)^2$$

$$\Rightarrow x^2(1-3l^2)+y^2(1-3m^2)+z^2(1-3n^2)$$
$$-6mnyz-6nlzx-6lmxy=0 \quad \text{---} (3)$$

which is the required cone vertex at the origin.

Now in (3), we have

Coefficient of x^2 + Coefficient of y^2 + Coefficient of $z^2 = (1-3l^2)+(1-3m^2)+(1-3n^2)$

$$= 3-3(l^2+m^2+n^2)$$

$$= 3-3(1) \quad (\because l^2+m^2+n^2=1)$$

$$=0$$

∴ The cone (3) has three mutually ⊥lar generators.

→ Find the locus of the points from which three mutually perpendicular lines can be drawn to intersect the conic $z=0,\ ax^2+by^2=1$.

Sol^n:- The given conic is

$$z=0,\quad ax^2+by^2=1 \quad \text{---} (1)$$

Let (α,β,γ) be the point from which three mutually ⊥ lines can be drawn to intersect the conic (1).

Any line through (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{---} (2)$$

Since it meets the plane $z=0$

$$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\Rightarrow x=\alpha-\frac{l}{n}\gamma, \quad y=\beta-\frac{m}{n}\gamma$$

∴ this point lies on (1) if

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+b\left(\beta-\frac{m}{n}\gamma\right)^2=1 \quad \text{--- (3)}$$

Now eliminate l, m, n from (2) & (3)

we have

$$a\left[\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right]^2+b\left[\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right]^2=1$$

$$\Rightarrow a[\alpha z-\gamma x]^2+b[\beta z-\gamma y]^2=[z-\gamma]^2$$

$$\Rightarrow a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2-(z-\gamma)^2=0$$

This cone has three mutually $\perp$ar generators if

Coefficient of x^2 + coefficient of y^2 + coefficient of $z^2=0$

if $a\gamma^2+b\gamma^2+(a\alpha^2+b\beta^2-1)=0$

if $a\alpha^2+b\beta^2+(a+b)\gamma^2=1$

$\therefore$ Locus of the point (α,β,γ) is

$$ax^2+by^2+(a+b)z^2=1$$

Hence the result.

2007. Show that the plane $2x-y+2z=0$ cuts the cone $xy+yz+zx=0$ in perpendicular lines.

we find [illegible] line [illegible] when a plane cuts cone we get two lines these two lines $\perp$r.

* Tangent Plane :—

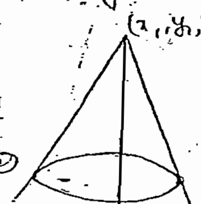
To find the equation of the tangent plane at the point (x_1,y_1,z_1) to the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Sol'n :- The given equation of the cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — ①

Equations of any line through (x_1,y_1,z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{—②}$$



Any point on this line is

$(x_1+lr,\ y_1+mr,\ z_1+nr)$

If it lies on the cone ① then

$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)+2h(lr+x_1)(mr+y_1)=0$

$\Rightarrow r^2[al^2+bm^2+cn^2+2fmn+2gnl+2hlm]+2r[l(ax_1+hy_1+gz_1)+m[hx_1+by_1+fz_1]+n(gx_1+fy_1+cz_1)]+ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0$ — ③

which is a quadratic equation in r.

Since (x_1,y_1,z_1) lies on the cone ①.

$\therefore ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0$ — ④

$\therefore$ ③ $\equiv r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hl\ldots$
$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)]+\ldots$ — Ⓐ

$\Rightarrow r[r(\ )+2(\ )]=0$

$\Rightarrow$ This equation has one root as zero.

If the line ② touches the cone, then the two values of r in Ⓐ must be equal.

But since one root is zero.

$\therefore$ Other root is also zero.

i.e. the coefficient of $r=0$.

i.e. $l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$ — ⑤

which is the condition for the line ②, to touch the cone ① at (x_1,y_1,z_1)

To find the locus of tangent line ②, we have to eliminate l,m,n from ② & ⑤

$\therefore$ Putting the values of l,m,n from ② in ⑤,

we have

$(x-x_1)(ax_1+hy_1+gz_1)+(y-y_1)(hx_1+by_1+fz_1)+(z-z_1)(gx_1+fy_1+cz_1)=0$

$\Rightarrow x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1$

$\Rightarrow x(ax_1+hy_1+gz_1)+$
$y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0$ —— ⑥

which is the required (∵ from ④) equation of the tangent plane.

* Working Rule to Tangent plane at (x_1, y_1, z_1) :-

In the equation of given cone (or) any surface change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, yz to $\frac{1}{2}(yz_1+y_1z)$, zx to $\frac{1}{2}(zx_1+z_1x)$, xy to $\frac{1}{2}(xy_1+x_1y)$, x to $\frac{1}{2}(x+x_1)$, y to $\frac{1}{2}(y+y_1)$, z to $\frac{1}{2}(z+z_1)$.

The equation obtained by this method will be same as equation ⑥.

Note:—

→ The tangent plane at any point of a cone passes through its vertex.

→ The vertex of the cone ① is $(0,0,0)$ and it clearly lies on the tangent plane ⑥.

→ The tangent plane at any point 'P' of a cone touches the cone (Intersection) along the generator through P.

Sol^n :- Let $P(x_1, y_1, z_1)$ be any point. The equation of the tangent plane at $P(x_1, y_1, z_1)$ is $x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)$
$+z(gx_1+fy_1+cz_1)=0$ —— ①

The equations of OP the generator through P are

$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0}$ | using $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}=r$ (say)

Any point on OP is $Q(rx_1, ry_1, rz_1)$

The equation of the tangent plane at Q is

$x(arx_1+hry_1+grz_1)+y(hrx_1+bry_1+frz_1)$
$+z(grx_1+fry_1+crz_1)=0$.

Dividing throughout by r.

$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)$
$+z(gx_1+fy_1+cz_1)=0$.

which is the same as the equation ① of the tangent plane at P.

∴ The tangent plane at P also touches the cone at any point of OP.

i.e. the generator through P.

∴ it touches the cone along OP.

This OP is called the generator of contact.

Note :- In the equation of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, We generally use the following notations:

(1). $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

(2). A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in D.

So that $A = bc - f^2$, $B = ca - g^2$,

$C = ab - h^2$, $F = gh - af$, $G = hf - bg$

$H = fg - ch$

(3). $BC - F^2 = Da$

Similarly $CA - G^2 = Db$, $AB - H^2 = Dc$

$GH - AF = fD$, $HF - BG = gD$,

$FG - CH = hD$.

where $D = abc + 2fgh - af^2 - bg^2 - ch^2$

(4) $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$

$= -(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)$.

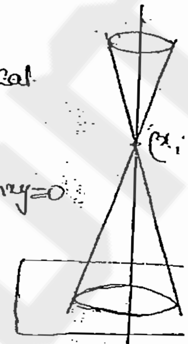
* Condition of tangency of a plane and Cone:—

The condition that the plane $lx + my + nz = 0$ may touch the Cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

is $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$

*Reciprocal Cone :-

The locus of the normals to the tangent planes through vertex of the cone is another cone called the reciprocal cone.



* The equation of reciprocal cone of the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

where A, B, C, D, F, G, H are cofactors of a, b, c, f, g, h in

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Problems :—

Show that the locus of the mid points of chords of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ drawn parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the plane

$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$.

sol'n :- Let $P(x_1, y_1, z_1)$ be the midpoint of one of the chords drawn ||el to the $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

Then equation of this chord is

$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ —— (1)

Any point on this line is

$(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2$
$+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)$
$+2h(lr+x_1)(mr+y_1)=0$

$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$
$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$
$+n(gx_1+fy_1+cz_1)]+$
$(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)=0$

which is a quadratic in r.

Since $P(x_1, y_1, z_1)$ is the midpoint of the chord.

$\therefore$ The two values of r should be equal in magnitude but opposite in sign.

$\therefore$ Sum of roots $=0$ (or)

the coefficient of $r=0$.

i.e. $l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$
$+n(gx_1+fy_1+cz_1)=0$.

$\Rightarrow x_1(al+hm+gn)+y_1(hl+bm+fn)$
$+z_1(gl+fm+cn)=0$.

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is

$x(al+hm+gn)+y(hl+bm+fn)+$
$z(gl+fm+cn)=0$ —— (2)

which is the required plane.

→ Find the locus of the chords of the cone which are bisected at a fixed point.

Sol^n :- Let $P(x_1, y_1, z_1)$ be the given fixed point and let any chord through P which is bisected at P be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{—— (2)}$$

Any point on this line is

$(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2$
$+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)$
$+2h(lr+x_1)(mr+y_1)=0$

$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$
$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$
$+n(gx_1+fy_1+cz_1)]$
$+ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1$
$=0$.

which is a quadratic in r.

Since $P(x_1, y_1, z_1)$ is the mid point of the chord (1)

$\therefore$ the two values of r should be equal in magnitude but opposite in sign.

$\therefore$ Coefficient of $r=0$.

$\therefore\ l(ax_1+hy_1+gz_1)+$
$m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$ — (2)

Eliminating l,m,n from (1) & (2) the locus of the chords which are bisected at P_1 is

$(x-x_1)(ax_1+hy_1+gz_1)+(y-y_1)$
$(hx_1+by_1+fz_1)+(z-z_1)(gx_1+fy_1+cz_1)=0$

which is the required equation.

→ Prove that the cones $ax^2+by^2+cz^2=0$ and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$ are reciprocal.

Sol^n:- The given first of the cone is $ax^2+by^2+cz^2=0$ — (1)

comparing with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

we have $a=a,\ b=b,\ c=c$

$f=0,\ g=0,\ h=0$.

$\therefore\ A=bc-f^2=bc-0=bc$

Similarly $B=ca-g^2=ca-0=ca$

$C=ab-h^2=ab-0=ab$

$F=gh-af=0-0=0,$

$G=hf-bg=0,\ H=fg-ch=0$

$\therefore$ The reciprocal cone of (1) is

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

$\Rightarrow\ bcx^2+cay^2+abz^2=0$

(on dividing through by abc)

$\Rightarrow\ \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

which is the second cone.

Note:- The condition for the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to have three mutually perpendicular tangent planes, if the reciprocal cone

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

has three mutually perpendicular generators for which $A+B+C=0$

i.e. $f^2+g^2+h^2=bc+ca+ab$

→ Prove that the perpendiculars drawn from the origin to the tangent plane to the cone $ax^2+by^2+cz^2=0$ lie on the cone $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$.

Sol^n:- The given cone is

$ax^2+by^2+cz^2=0$ — (1)

we required to find the reciprocal cone of (1)

comparing (1) with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

we have $a=a,\ b=b,\ c=c$

$f=0,\ g=0,\ h=0$

$\therefore\ A=bc-f^2=bc-0=bc$

$B=ca-g^2=ca-0=ca$

$C=ab-h^2=ab-0=ab$

$F = gh - af = 0 - 0$

$G = hf - bg = 0 - 0$

$H = fg - ch = 0 - 0$

$\therefore$ The reciprocal cone is

$-Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$\Rightarrow bcx^2 + cay^2 + abz^2 + 0 + 0 + 0 = 0$

$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

which is the required equation.

→ Show that the general equation of the cone which touches the three coordinate planes is

$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

Sol'n :- The general equation of a cone through the coordinate axes is

$fyz + gzx + hxy = 0$ —①

Its reciprocal cone is

$-Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ —②

where $A = bc - f^2 = 0 - f^2 = -f^2$

$B = ca - g^2 = -g^2$

$C = ab - h^2 = -h^2$

$F = gh - af = gh$

$G = hf - bg = hf$ & $H = fg - ch$
$= fg$

② $\equiv -f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$

$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx + 2fgxy = 4fgxy$

$\Rightarrow (fx + gy - hz)^2 = 4gfxy$

$\Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Rightarrow fx \pm 2\sqrt{fgxy} + gy = hz$

$\Rightarrow (\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

→ Find the equation of the cone which touches three coordinate planes and the planes $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$

Sol'n :- Required cone which touches the three coordinate planes and the planes $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$ is reciprocal line of a cone which passes through normals through the origin i.e. which passes through the three coordinate axes and two normals

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ —① & $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ —②

Now any cone equation through the coordinate axis is

$fyz + gzx + hxy = 0$ —③

If this cone passes through the lines ① & ②

$\therefore$ d.c's of these lines satisfy the equation of cone ③.

$\because 6f + 3g + 2h = 0$ and

$12f + 8g + 6h = 0$

$\therefore \frac{f}{2} = \frac{g}{-12} = \frac{h}{12} \Rightarrow \frac{f}{1} = \frac{g}{-6} = \frac{h}{6}$

$\therefore$ (3) $\equiv yz - 6zx + 6xy = 0$

$\Longrightarrow 2yz - 12zx + 12xy = 0$ — (4)

The required cone is the reciprocal cone of (4)

Comparing (4) with

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

we have

$a = b = c = 0$, $f = 1$, $g = -6$, $h = 6$

$A = bc - f^2 = -1$, $B = ca - g^2 = -36$

$C = ab - h^2 = -36$

$F = gh - af = -36$, $G = hf - bg = 6$

$H = fg - ch = -6$

$\therefore$ The reciprocal cone is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$-x^2 - 36y^2 - 36z^2 - 72yz + 12zx - 12xy = 0.$

$\Rightarrow x^2 + 36y^2 + 36z^2 + 72yz - 12zx + 12xy = 0$

which is the required equation of the cone which touches the three coordinate planes and the two given planes.

→ Prove that the cones

$ayz + bzx + cxy = 0$,

$(ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$ are reciprocal.

Sol^n :- The given cones are (16)

$ayz + bzx + cxy = 0$ — (1)

$(ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$ — (2)

we required to found the reciprocal cone of (2) is (1)

(2) $\equiv \sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0$

$\Rightarrow \sqrt{ax} + \sqrt{by} = -\sqrt{cz}$

$\Rightarrow (\sqrt{ax} + \sqrt{by})^2 = cz$

$\Rightarrow ax + by + 2\sqrt{ax}\sqrt{by} = cz$

$\Rightarrow ax + by - cz = -2\sqrt{abxy}$

$\Rightarrow (ax + by - cz)^2 = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 + 2abxy - 2bcyz - 2acxz = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 - 2abxy - 2bcyz - 2aczx =$ — (3)

For the reciprocal cone

This is comparing with

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$a = a^2$, $b = b^2$, $c = c^2$, $f = -bc$, $g = -ac$

$h = -ab$.

$\therefore A = bc - f^2 = b^2c^2 - b^2c^2 = 0$

$B = ca - g^2 = c^2a^2 - a^2c^2 = 0$

$C = ab - h^2 = a^2b^2 - a^2b^2 = 0$

$F = gh - af = (-ac)(-ab) + a^2bc$

$= a^2bc + a^2bc = 2a^2bc$

$G = hf - bg = 2ab^2c$, $H = 2abc^2$

The reciprocal cone is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$\Rightarrow 0 + 0 + 0 + 4a^2bc\,yz + 4ab^2c\,zx + 4abc^2\,xy = 0$

$\Rightarrow ayz + bzx + cxy = 0$

which is required equation.

→ Prove that the tangent planes to the cone $x^2-y^2+2z^2-3yz+4zx-5xy=0$ are perpendicular to the generator of the cone. $17x^2+8y^2+29z^2+2f\,yz+28yz-46zx-16xy=0$

Sol'n :- The given first cone is

$$x^2-y^2+2z^2-3yz+4zx-5xy=0 \quad \text{--- (1)}$$

We are required to find the reciprocal cone of (1)

∴ Comparing (1) with

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

We have $a=2,\ b=-2,\ c=4,$

$f=-3,\ g=4,\ h=-5$

Continue this we get the solution.

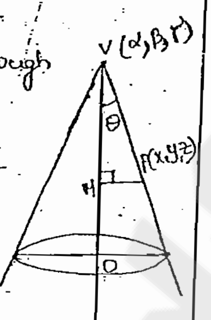
* The Right Circular Cone :-

Definition :- The surface generated by a straight line which passes through a fixed point and makes ~~moving~~ a constant angle with a fixed line through the fixed point is known as the right circular cone.

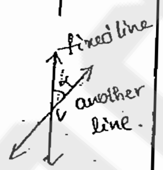
→ The fixed point is called the vertex.

→ The constant angle is called the semi-vertical angle.

→ The fixed line through the fixed point (i.e. vertex) is called the axis of the cone.



Note:- The section of a right circular cone by a plane perpendicular to its axis is a circle.

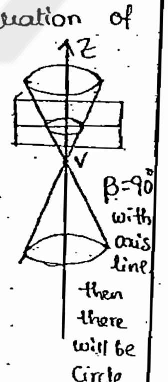


* Equations of a right circular Cone :-

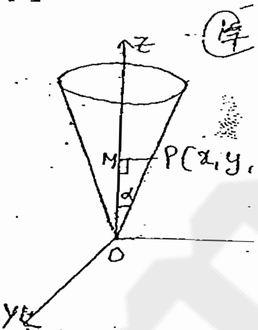
(a) Standard form

To show that the equation of the right circular cone whose vertex is the origin, axis OZ and semi-vertical angle α is:

$$x^2+y^2 = z^2\tan^2\alpha .$$



Let $P(x,y,z)$ be any point on the line.



Draw $PM \perp OZ$

$\therefore \angle MOP = \alpha$

Now, in the right angled ΔOMP,

$$\frac{OM}{OP} = \cos\alpha \quad \text{——— (1)}$$

Now OM = Projection of OP on OZ whose d.c's are $0,0,1$

$$= 0.(x-0)+0(y-0)+1(z-0)$$

$$= z \qquad [\text{Using } l(x_2-x_1)+m(y_2-y_1)+n(\ldots$$

Also $OP = \sqrt{x^2+y^2+z^2}$

$$(1) \equiv \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\alpha$$

$$\Rightarrow z^2 = (x^2+y^2+z^2)\cos^2\alpha$$

$$\Rightarrow z^2\sec^2\alpha = (x^2+y^2+z^2)$$

$$\Rightarrow z^2(1+\tan^2\alpha) = x^2+y^2+z^2$$

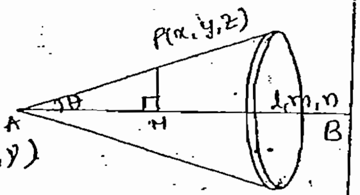
$$\Rightarrow x^2+y^2 = z^2\tan^2\alpha .$$

which is a required equation.

(b) General Form :-

To find the equation of a right circular cone whose vertex is (α,β,γ) semi vertical angle θ, and axis has d.c.'s l,m,n.

Sol^n :- Let $P(x,y,z)$ be any point on cone and AB, the axis of cone whose d.c's are l, m, n and passes through the vertex $A(\alpha, \beta, \gamma)$



Draw $PM \perp AB$

$\therefore \angle PAM = \theta$, the semi verticle angle

$\therefore$ right angle ΔAMP,

$$\frac{AM}{AP} = \cos\theta \quad \text{——— (1)}$$

AM = Projection of AP on the AB line whose d.c's are l, m, n:

$$= l(x-\alpha) + m(y-\beta) + n(z-\gamma)$$

and $AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$

$$(1) \equiv [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$
$$= [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]\cos^2\theta \quad \text{——— (2)}$$

which is the required equation of the cone.

Note :- (i) Put $\alpha = \beta = \gamma = 0$ in (2) then the equation of the right circular cone whose vertex is origin and axis with the d.c's l, m, n and semi vertical angle θ is.

$$(lx + my + nz)^2 = (x^2 + y^2 + z^2)\cos^2\theta$$

; If OZ is the axis of cone and $(0,0,0)$ as the vertex and θ, the semi vertical angle,

then putting $\alpha = \beta = \gamma = 0$, $l = 0, m = 0$, $n = 1$.

$$(2) \equiv z^2 = (x^2 + y^2 + z^2)\cos^2\theta$$
$$\Rightarrow z^2 \sec^2\theta = x^2 + y^2 + z^2$$
$$\Rightarrow z^2(1 + \tan^2\theta) = x^2 + y^2 + z^2$$
$$\Rightarrow z^2 \tan^2\theta = x^2 + y^2.$$

(iii) The semi vertical angle of a right circular cone admitting sets of three mutually perpendicular generators is $\tan^{-1}\sqrt{2}$.

For this, the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero and this means that

$$1 + 1 - \tan^2\theta = 0 \Rightarrow \theta = \tan^{-1}\sqrt{2}.$$

→ Find the equation to the right circular cone whose vertex is $P(2, -3, 5)$; axis PQ which makes equal angles with the axes and semi vertical angle is $30°$.

Sol^n:- Since the d.c's of the axis PQ which makes equal angles with the coordinate axes.

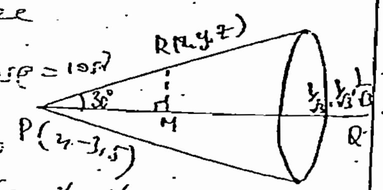
If a line PQ makes angles α, β, γ with axes,

we take

$\alpha = \beta = \gamma$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow l = m = n$



Since $l^2 + m^2 + n^2 = 1$

$\Rightarrow 3l^2 = 1$

$\Rightarrow l = \pm\frac{1}{\sqrt{3}}$

We take +ve sign.

$\therefore l = m = n = \frac{1}{\sqrt{3}}$.

Let $R(x,y,z)$ be any pt on the surface of the cone.

Draw $RM \perp PQ$

$\therefore \angle MPR = 30°$

In the rt. angled ΔRMP

$\cos 30° = \frac{MP}{PR}$.

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{MP}{PR}$ —(1)

Now MP = projection of PR on the axis PQ. —(2)

and d.r's of PR are $x-2, y+3, z-5$.

and d.c's of PQ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore (2) \equiv MP = \frac{1}{\sqrt{3}}(x-2) + \frac{1}{\sqrt{3}}(y+3) + \frac{1}{\sqrt{3}}(z-5)$

$= \frac{1}{\sqrt{3}}(x+y+z-4)$

and $PR = \sqrt{(x-2)^2+(y+3)^2+(z-5)^2}$

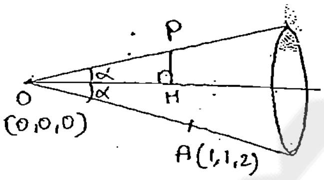
$\therefore (1) \equiv 5(x^2+y^2+z^2) - 8(yz+zx+xy) - 4x + 86y - 58z + 278 = 0$

→ Find the equation of the circular cone which passes through the point $(1,1,2)$ and has its vertex at the origin and the axis the line $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$.

Sol'n:- Let the d.c's of axis be l, m, n.

Given that the axis the line

$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$



$\therefore$ The d.c's of $(0,0,0)$ the OQ are proportional to $2,-4,3)$

$\therefore$ The actual d.c's of OQ are

$\frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}, \frac{3}{\sqrt{29}}$

Let α be the semi-vertical angle of the cone.

Since $A(1,1,2)$ lies on the cone.

$\therefore$ The d.c's of OA are proportional to $1-0, 1-0, 2-0$.

$\therefore$ The actual d.c's are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

The semi vertical angle α of a right circular cone is the angle between the axis & the generator of the cone.

$\therefore \alpha$ is the angle between OQ & OA.

$\therefore \cos\alpha = l_1l_2 + m_1m_2 + n_1n_2$

$= \left(\frac{2}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{-4}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{3}{\sqrt{29}}\right)\left(\frac{2}{\sqrt{6}}\right)$

$= \frac{1}{\sqrt{29}} \cdot \frac{1}{\sqrt{6}}[2-4+6]$

$\boxed{\cos\alpha = \frac{4}{\sqrt{29}\cdot\sqrt{6}}}$

Let $P(x,y,z)$ be any point on the cone.

Draw $PM \perp OQ$

$\therefore$ In the right angle ΔRMO

$\therefore \cos\alpha = \dfrac{MO}{PO}$

$\Rightarrow (MO)^2 = (PO)^2 \left(\dfrac{16}{29\times 6}\right)$ ——— (1)

Now MO = Projection of PO on OQ

$= l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$

$= \dfrac{2}{\sqrt{29}}(x-0) + \left(\dfrac{-4}{\sqrt{29}}\right)(y-0) + \dfrac{3}{\sqrt{29}}(z-0)$

$= \dfrac{1}{\sqrt{29}}\left[2x-4y+3z\right]$

and $PO = \sqrt{x^2+y^2+z^2}$

$\therefore$ (1) $\equiv \dfrac{1}{29}(2x-4y+3z)^2 = (x^2+y^2+z^2)\dfrac{16}{29\times 6}$

$\Rightarrow 3(4x^2+16y^2+9z^2-16xy-24yz+12xz)$

$= 8x^2+8y^2+8z^2$

$\Rightarrow 4x^2+40y^2+19z^2-48xy-72yz$

$+36xz = 0$

$\rightarrow$ Lines are drawn from the origin with the d.c's proportional $(1,2,2)$, $(2,3,6)$, $(3,4,12)$; find the direction cosines of the axis of right circular cone through them, and Prove that the semi vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Sol^n: Let l, m, n be the d.r's of the axis of the right circular cone.

Let O be the origin and P, Q, R be the given points.

Now the d.r's of OP, OQ, OR are $(1,2,2)$, $(2,3,6)$; $(3,4,12)$.

The d.c's of OP, OQ and OR are $1/3, 2/3, 2/3$; $2/7, 3/7, 6/7$ and $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.

Let α be the semi-vertical angle of the cone then

$\cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n$

$= \frac{3}{13}l + \frac{4}{13}m + \frac{12}{13}n$ ——— (A)

Now take first two members

$\frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n$

$\Rightarrow$ ~~$7l + 14m + 14n = 6l + 9m + 18n$~~

$\Rightarrow l + 5m - 4n = 0$ ——— (1)

From first & last we get

$2l + 7m - 5n = 0$ ——— (2)

Solving, we get

$\dfrac{l}{-1} = \dfrac{m}{1} = \dfrac{n}{1} = \pm\dfrac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}} = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ The d.c's of the axis are

$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$\therefore$ Putting these values in (A) we get

$\cos\alpha = \frac{1}{3}\left(-\frac{1}{\sqrt{3}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3}}\right)$

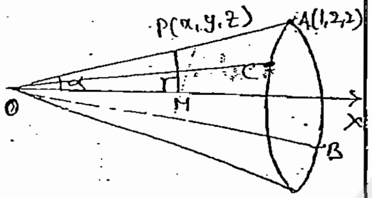
$= \frac{1}{3\sqrt{3}}(-1+2+2) = \frac{3}{3\sqrt{3}}$

$\cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

→ find the equation of the right circular cone generated by the straight lines drawn from the origin to cut the circle through the three points $(1,2,2)$, $(2,1,-2)$ and $(2,-2,1)$

Sol'n :- Let $A(1,2,2)$, $B(2,1,-2)$, $C(2,-2,1)$ be the given point.

Let l, m, n be the actual d.c's of the axis OX



Then OA, OB, OC make the same angle α with the axis OX, where α is the semi-vertical angle.

The direction ratios of OA, OB, OC are $(1,2,2)$, $(2,1,-2)$, $(2,-2,1)$

$\therefore$ The d.c's of OA, OB, OC are

$$\frac{1}{3}, \frac{2}{3}, \frac{2}{3};\ \frac{2}{3}, \frac{1}{3}, \frac{-2}{3};\ \frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$$

$$\therefore \cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m - \frac{2}{3}n = \frac{2}{3}l + \left(\frac{-2}{3}\right)m + \frac{1}{3}n \quad \text{---} (1)$$

from first two members we have

$$\frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m - \frac{2}{3}n$$

$$\Rightarrow l + 2m + 2n = 2l + m - 2n$$

$$\Rightarrow l - m - 4n = 0 \quad \text{---} (2)$$

From last two members we have

$$3m - 3n = 0 \Rightarrow 0l + m - n = 0 \quad \text{---} (3)$$

Solving (2) & (3) we have

$$\frac{l}{5} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{25+1+1}} = \pm\frac{1}{\sqrt{27}}$$

$$\therefore l = \frac{5}{\sqrt{27}},\ m = \frac{1}{\sqrt{27}},\ n = \frac{1}{\sqrt{27}}$$

$$\therefore (1) \equiv \cos\alpha = \frac{1}{3}\left(\frac{5}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right)$$

$$= \frac{1}{\sqrt{27}}\left(\frac{5+2+2}{3}\right)$$

$$= \frac{1}{\sqrt{27}} \times 9 = \frac{9}{9\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \boxed{\cos\alpha = \frac{1}{\sqrt{3}}}$$

Let $P(x,y,z)$ be any point on the cone.

Draw $PM \perp OX$

$\therefore \angle MOP = \alpha$

In the right angle ΔOMP,

$$\frac{OM}{OP} = \cos\alpha$$

$$(OM)^2 = (OP)^2 \frac{1}{3} \quad \text{---} (4)$$

OM = projection of OP on OX.

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

Continue this solution.

→ If α is the semi-vertical angle of the right circular cone which passes through the lines OX, OY, $x = y = z$, show that

$$\cos\alpha = (9 - 4\sqrt{3})^{-1/2}$$

Sol'n :- Let l, m, n be the d.c's of the axis of the cone. Since the axis makes the same angle α with each of the lines ox, oy and $x=y=z$.

Now the d.rs of ox, oy, $x=y=z$ are $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$

$\therefore$ the d.cs of ox, oy and $x=y=z$ are $(1,0,0)$, $(0,1,0)$ and $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

$$\cos\alpha = l(1)+m(0)+n(0) = l(0)+m(1)+n(0)$$
$$= l\left(\tfrac{1}{\sqrt{3}}\right)+m\left(\tfrac{1}{\sqrt{3}}\right)+n\left(\tfrac{1}{\sqrt{3}}\right) \quad \text{--- (1)}$$

from first two no's

we have $l=m \Rightarrow l-m+0n=0$ --- (2)

from last two no.s

we have $m = \frac{l}{\sqrt{3}}+\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}$

$$\Rightarrow \frac{l}{\sqrt{3}}+\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right)m+\frac{n}{\sqrt{3}}=0$$

$$\Rightarrow l+(1-\sqrt{3})m+n=0 \quad \text{--- (3)}$$

solving (2) and (3)

$$\frac{l}{-1+0}=\frac{m}{0-1}=\frac{n}{1-\sqrt{3}+1} \Rightarrow \frac{l}{-1}=\frac{m}{-1}=\frac{n}{2-\sqrt{3}}$$

$$\therefore \frac{l}{-1}=\frac{m}{-1}=\frac{n}{2-\sqrt{3}}=\pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+(2-\sqrt{3})^2}}$$

$$=\pm\frac{1}{\sqrt{2+4+3-4\sqrt{3}}}$$

$$=\pm\frac{1}{\sqrt{9-4\sqrt{3}}}$$

$$\frac{l}{-1}=\frac{-1}{\sqrt{9-4\sqrt{3}}} \Rightarrow l=\frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$m=\frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$n=\frac{-(2-\sqrt{3})}{\sqrt{9-4\sqrt{3}}}=\frac{\sqrt{3}-2}{\sqrt{9-4\sqrt{3}}}$$

from (1)

$$\cos\alpha=\frac{-1}{\sqrt{9-4\sqrt{3}}}$$

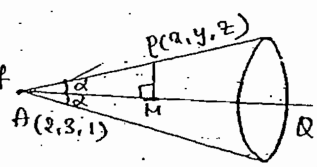
$$\alpha=\cos^{-1}\left(\frac{-1}{\sqrt{9-4\sqrt{3}}}\right)$$

$\rightarrow$ show that the equation of the right circular cone with vertex $(2,3,1)$, axis parallel to the line $-x=\frac{y}{2}=z$ and one of its generators having d.c's proportional to $(1,-1,1)$ is

$$x^2-8y^2+z^2+12xy-12yz+16zx-46x+36y+22z-19=0$$

Sol'n:- Let l,m,n be the d.c's of the axis of the right circular cone.

The given line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{1}$ is $\parallel$el to the axis.



$\therefore$ The d.c's of the axis are proportional to $-1,2,1$.

$\therefore$ The actual d.c's are $\frac{-1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}$

$\therefore l=\frac{-1}{\sqrt{6}}, m=\frac{2}{\sqrt{6}}, n=\frac{1}{\sqrt{6}}$.

Now, the d.c's of its generator are proportional to $1,-1,1$.

$\therefore$ The actual d.c's are

$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Now let α be the semi-vertical angle.

Then the semi vertical angle α of a right circular cone is the angle between the axis and the generator of the cone.

$\therefore \cos\alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{18}}[-1-2+1] = \frac{-2}{3\sqrt{2}}$

Let $P(x,y,z)$ be any point on the cone.

– Draw $PM \perp AQ$

$\therefore \angle MAP = \alpha$

In right angle ΔAMP,

$\cos\alpha = \frac{AM}{AP}$

$\Rightarrow (AM)^2 = (AP)^2 \cos^2\alpha$ —— (1)

Now AM = projection of AP on AQ

$= l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$

$= \left(\frac{-1}{\sqrt{6}}\right)(x-2) + \left(\frac{2}{\sqrt{6}}\right)(y-3) + \frac{1}{\sqrt{6}}(z-1)$

$= \frac{1}{\sqrt{6}}[-x+2y+z+2-6-1]$

$= \frac{1}{\sqrt{6}}[-x+2y+z-5]$

and $(AP)^2 = \sqrt{(x-2)^2+(y-3)^2+(z-1)^2}$

(1) $\equiv \frac{1}{6}\left[(-x+2y)+(z-2)\right]^2 = \left[(x-2)^2+(y-3)^2+(z-1)^2\right] \times \frac{8}{9}$

$\Rightarrow 3\left[x^2+4y^2-4xy+z^2+4-4z+2(-x+2y)(z-2)\right] = 4\left[x^2+y^2+z^2-4x-6y-2z+4+9+1\right]$

$\Rightarrow 3x^2+12y^2-12xy+3z^2+12-12z+6(-xz+2x+2yz-4y) = 4x^2+4y^2+4z^2-16x-24y-8z+56$

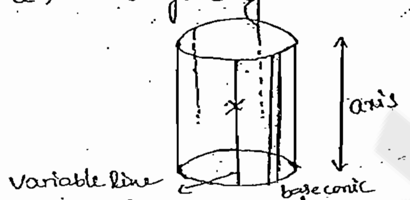
$x^2-8y^2+z^2+12xy-12yz+6zx-46x+36y+22z-19=0$

* The Cylinder * Set-VI

Definition

The surface generated by a variable line which is always parallel to a fixed line and intersects a given curve (or touches a given surface) is called the cylinder.

→ The variable line is called the generator, the fixed line the axis and the given curve (or surface) the guiding curve.



* Equation of a cylinder :-

To find the equation of the cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and the base conic is

$$f(x,y) = ax^2+by^2+2hxy+2gx+2fy+c=0,\quad z=0.$$

Sol^n :- The given line is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{---}(1)$$

and the base conic is

$$f(x,y) = ax^2+by^2+2hxy+2gx+2fy+c=0,\ z=0 \quad \text{---}(2)$$

Let (x_1,y_1,z_1) be any point on the generator of the cylinder and parallel to the line (1) then equations of generator line (i.e. a line through (x_1,y_1,z_1) and $\parallel$el to (1)) are:

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{---}(3)$$

It meets the plane $z=0$

∴ Putting $z=0$ in (3) we get

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n}$$

$$\therefore\ x = x_1-\frac{l}{n}z_1,\quad y=y_1-\frac{m}{n}z_1$$

∴ The point $\left(x_1-\frac{l}{n}z_1,\ y_1-\frac{m}{n}z_1\right)$

If this point lies on the conic (2) then $a\left[x_1-\frac{l}{n}z_1\right]^2+b\left[y_1-\frac{m}{n}z_1\right]^2+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+2g\left(x_1-\frac{l}{n}z_1\right)+2f\left(y_1-\frac{m}{n}z_1\right)+c=0$

∴ The locus of (x_1,y_1,z_1) is

$$a\left(x-\frac{l}{n}z\right)^2+b\left(y-\frac{m}{n}z\right)^2+2h\left(x-\frac{l}{n}z\right)\left(y-\frac{m}{n}z\right)+2g\left(x-\frac{l}{n}z\right)+2f\left(y-\frac{m}{n}z\right)+c=0.$$

$$\Rightarrow a(nx-lz)^2+b(ny-mz)^2+2h(nx-lz)(ny-mz)+2ng(nx-lz)+2nf(ny-mz)+cn^2=0$$

which is the required equation of the cylinder.

Note:- If the generators are $\parallel$el to z-axis, then $l=0$, $m=0$ and $n=1$.

∴ The equation of the cylinder becomes $ax^2+by^2+2hxy+2gx+2fy+c=0$, which is free from z.

→ If we required to find the equation of the cylinder whose generators are ∥lel to z-axis, and intersect a given conic then eliminate z from the equations of the conic.

∴ If given the equation of the cylinder:

→ If the generators are ∥lel to x-axis then eliminate 'x' and if the generators are ∥lel to y-axis then eliminate y from the equations of the conic to get equations of the cylinder.

Problems

→ Find the equation of a cylinder whose generating lines have the d.c's (l, m, n) and which passes through the circle $x^2+y^2=a^2, y=0$.

→ Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, and whose guiding curve is the ellipse $x^2+2y^2=1, z=3$.

→ Find the equation of the cylinder whose generators intersect the curve $ax^2+by^2=2z, lx+my+nz=p$ and are parallel to z-axis.

Sol^n: The given base conic is

$$ax^2+by^2=2z, \quad lx+my+nz=p \quad \text{—(1)}$$

Since the generators of the cylinder are ∥lel to the z-axis.

∴ The required equation of the cylinder free from the z-coordinate.

Now eliminate z from the equations (1), to get the required cylinder.

From first equation of (1) we have

$$z = \frac{ax^2+by^2}{2}$$

Putting in the second equation of (1),

$$lx+my+n\left(\frac{ax^2+by^2}{2}\right)=p$$

$$\Rightarrow 2lx+2my+n(ax^2+by^2)=2p$$

$$\Rightarrow n(ax^2+by^2)+2lx+2my-2p=0$$

which is the required cylinder.

→ Find the equation of the cylinder with generators parallel to x-axis and passing through the curve

$$ax^2+by^2+cz^2=1,$$
$$lx+my+nz=p.$$

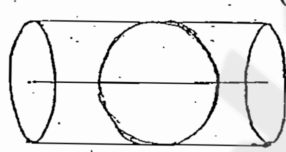
* Enveloping Cylinder of a Sphere:

To find the equation to the cylinder whose generators touch the sphere $x^2+y^2+z^2=a^2$ and are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ (or)

To find the locus of the tangent lines drawn to a sphere and parallel to a given line.

Sol^n :- The given sphere $x^2+y^2+z^2=a^2$ — ①

and the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — ②



Let (α,β,γ) be any point on the cylinder.

$\therefore$ Any line through (α,β,γ) ‖ let to ② is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (say) — ③

Any point on this line is $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$.

This point lies on the sphere ①, then

$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$

$\Rightarrow r^2(l^2+m^2+n^2)+2r(\alpha l+\beta m+\gamma n)+(\alpha^2+\beta^2+\gamma^2-a^2)=0$ — ④

Clearly which is a quadratic in r.

Since the generator ③ is a tangent line of the given sphere.

$\therefore$ the two values of r given ④ must be equal.

$\therefore$ The descriminant of ④ $=0$.

i.e. $b^2-4ac=0$.

$[2(l\alpha+m\beta+n\gamma)]^2=4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$

The locus of (α,β,γ) is

$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$

which is required equation of the cylinder and is known as the Enveloping cylinder of a sphere.

Problem

→ Find the enveloping cylinder of a sphere $x^2+y^2+z^2-2x+4y=1$ having its generators ~~parallel~~ parallel to the line $x=y=z$.

Ans: $x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$

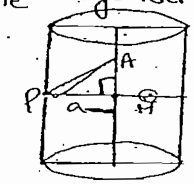
Sol^n:- Let (α,β,γ) be any point on the cylinder.

Continue in this way.

* Right Circular Cylinder :-

A surface generated by a line which intersects a fixed circle (is called guiding curve) and is ⊥lar to the plane of the circle is called right circular cylinder.

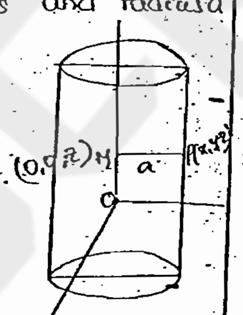
→ the normal to the plane of the circle through its centre is called the axis of the cylinder and the radius of the circle is the radius of the cylinder.



Equation of Right circular Cylinder :-

(a) Standard form :-

Show that the equation of the right circular cylinder whose axis is the z-axis and radius 'a' is $x^2+y^2=a^2$.



Let $P(x,y,z)$ be any point on the cylinder.

Draw $PM \perp$ z-axis

$\therefore OM = z$ and the coordinates of $M(0,0,z)$.

$\therefore MP$ = radius of the cylinder (given).

But $MP = \sqrt{(x-0)^2+(y-0)^2+(z-z)^2}$

$= \sqrt{x^2+y^2}$

$\therefore \sqrt{x^2+y^2} = a$

$\Rightarrow x^2+y^2=a^2$.

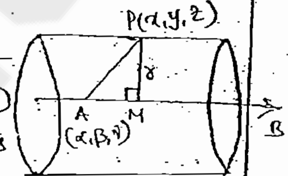
which is required equation.

(b) General Form :-

To find the equation to the right circular cylinder whose radius is 'r' and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

Sol :- Let AB be the axis of the cylinder whose equations are

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ —— (1)



where $A(\alpha,\beta,\gamma)$ is a point on it.

The d.r.s of AB are l,m,n.

$\therefore$ The actual d.c's are

$\frac{l}{\sqrt{\Sigma l^2}}, \frac{m}{\sqrt{\Sigma l^2}}, \frac{n}{\sqrt{\Sigma l^2}}$

Let $P(x,y,z)$ be any on the cylinder.

Draw $PM \perp AB$ axis.

and join PA.

PM = radius of the cylinder = r

In the right angled ΔPAM,

$AP^2 = AM^2 + PM^2$ —— (2)

$(AP)^2 = (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$

AM = Projection of AP on AB-axis.

$$= \frac{l}{\sqrt{\Sigma l^2}}(x-\alpha)+\frac{m}{\sqrt{\Sigma l^2}}(y-\beta)+\frac{n}{\sqrt{\Sigma l^2}}(z-\gamma)$$

$$= \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

$$\textcircled{1} \equiv (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$
$$= \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{l^2+m^2+n^2}+r^2$$

which is the required equation of the cylinder.

→ Find the equation of the right circular cylinder of radius 2 whose axis is the line

$$\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-2}{2}$$

→ The axis of the a right circular cylinder of radius 2 is

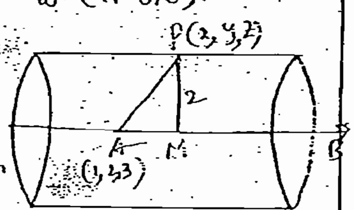
$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

show that its equation is

$$10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0.$$

→ Find the equation of the right circular cylinder of radius 2 whose axis passes through (1,2,3) and has d.c's proportional to (2,-3,6).

Sol: Let AB be axis of the cylinder which



passes through the point and has d.c's proportional 2

∴ Dividing each by

$$\sqrt{4+9+36} = \sqrt{49} = 7$$

∴ Actual d.c's are $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}$

Let $P(x,y,z)$ be any point on cylinder.

Draw $PM \perp AB$

∴ In right angled ΔAPM

$$AP^2 = AM^2+PM^2$$

Continue this solution.

→ Find equation to the right circular cylinder whose guiding circle is

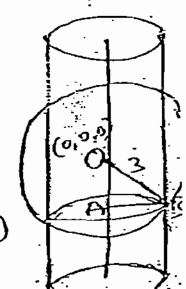
$$x^2+y^2+z^2=9, \quad x-y+z=3.$$

Note:- The axis of the cylinder is the line through centre of sph and ⊥ to the plane of the circle and radius of the cylinder is equ to radius of circle.

Sol:- The sphere is $x^2+y^2+z^2=9$ —(1)

and plane is $x-y+z=3$ —(2)

The centre of the sphere is O(0,0,0) and its radius is OB=3.



OA = ⊥ distance of O(0,0,0) from the plane (2)

$$= \frac{|0-0+0-3|}{\sqrt{1+1+1}} = \frac{|-3|}{\sqrt{3}} = \sqrt{3}$$

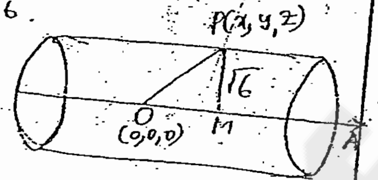
$\therefore AB$ = radius of the circle

$$= \sqrt{OB^2 - OA^2} = \sqrt{9-3} = \sqrt{6}$$

Again equation of the line through the centre $O(0,0,0)$ of the sphere and $\perp^r$ to plane ② are

$$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$$

which is the axis of the cylinder and radius $\sqrt{6}$.



The d.c's of the axis are proportional to $1,-1,1$.

The actual d.c's are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Let $P(x,y,z)$ be any point on the cylinder.

Join OP and draw $MP \perp OA$.

$$(OP)^2 = (OM)^2 + (MP)^2 \quad \text{--- ③}$$

Now $(OP)^2 = \sqrt{x^2+y^2+z^2}$

$(MP)^2 = 6$

& OM = Projection of OP on OA.

$$= \frac{1}{\sqrt{3}}(x) - \frac{1}{\sqrt{3}}(y) + \frac{1}{\sqrt{3}}(z)$$

$$= \frac{1}{\sqrt{3}}(x-y+z)$$

$$③ \equiv x^2+y^2+z^2 = \frac{(x-y+z)^2}{3} + 6$$

$$\Rightarrow x^2+y^2+z^2+xy+yz-zx-9=0$$

which is the required equation of the cylinder.

→ Find the equation of the right circular cylinder whose guiding curve is the circle through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Sol'n :- Let $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ be the given points.

Then the circle through A,B,C is the intersection of the plane ABC and the sphere OABC.

Now the equation of the plane ABC is $\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$ (intercept form)

$$x+y+z=1 \quad \text{--- ①}$$

and the sphere OABC is

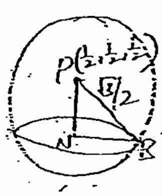
$$x^2+y^2+z^2-x-y-z=0 \quad \text{--- ②}$$

(Using $x^2+y^2+z^2-ax-by-cz=0$)

The centre of the sphere is $P(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

and radius $= \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$

$$= \frac{\sqrt{3}}{2}$$



from the right angle Δle. ΔPNB

$$NB^2 = \sqrt{PB^2 - NP^2} \quad \text{--- ③}$$

$NP = \perp$ distance from 'P' to the plane

$$= \frac{|\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1|}{\sqrt{1+1+1}} = \frac{1}{2\sqrt{3}}$$

(2) $\equiv NB = \sqrt{\frac{3}{4} - \frac{1}{4\times 3}}$

$= \sqrt{\frac{9-1}{4\times 3}} = \sqrt{\frac{8}{4\times 3}} = \sqrt{\frac{2}{3}}$

which is the radius of the circle.

∴ This is also radius of the cylinder

Now the equations of PN are
[i.e. through $P(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and
$\perp$ lar to the plane (1)]

$$\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z-\frac{1}{2}}{1}$$

which is the axis of the cylinder.

Now the d.c's of the axis are proportional to 1, 1, 1

The actual d.c's are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

and the radius is $\sqrt{2/3}$.

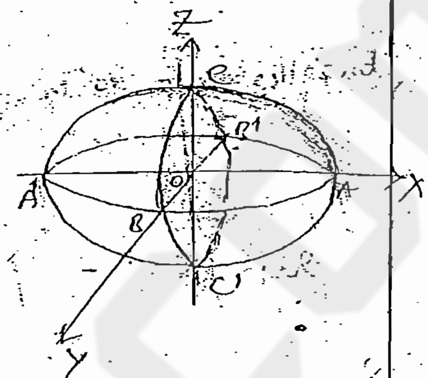
continue in this way we get the solution.

2005 find the right circular cylinder whose guiding curve is the circle through three points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$. find also the axis of the cylinder

(5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$; Hyperbolic paraboloid.

* Shapes of surfaces.

(1) Ellipsoid :- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



(i) centre: If (α, β, γ) is a point on the ellipsoid, then $(-\alpha, -\beta, -\gamma)$ is also a point on it.

The middle point of the join of these points is $(0,0,0)$, the origin.

Thus (α, β, γ), $(-\alpha, -\beta, -\gamma)$ are the points on a straight line through the origin and are equidistance from the origin.

Hence origin bisects every chord which passes through it and is therefore the centre of the surface.

(ii) Symmetry: Since there are only even powers of x, the surface is symmetrical about yz-plane.

Similarly, the surface is symmetrical about xz and xy planes.

If the point (α, β, γ) satisfies the eqn, then $(\alpha, \beta, -\gamma)$ also satisfies it. The line joining (α, β, γ) $(\alpha, \beta, -\gamma)$ is bisected at right angle by the xy-plane. It follows that the xy-plane bisects all chords perpendicular to it.

Similarly other co-ordinate planes also bisect chords $\perp^r$ to them.

Set – VII

The conicoid

(1)

A surface whose equation is of the second degree in x, y, z is called the conicoid.

i.e., the general equation of second degree in x, y, z

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a locus called a conicoid or Quadric.

– The above equation contains ten unknown constants which can be reduced to nine effective constants by dividing the equation throughout by 'a'.

Thus a conicoid can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

– By suitable transformation of axes, the above general equation can be reduced to one of the following standard forms.

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$: Ellipsoid

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; Hyperboloid of one sheet.

(3) $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; Hyperboloid of two sheets

(4) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$; Elliptic paraboloid.

These ~~three planes~~ are called principal planes.

The three lines of intersection of three principal planes taken in pairs are called principal axes.

In the present case co-ordinate axes are the principal axes.

(iii) Intersection with axes:

The surface meets x-axis $(y=0, z=0)$

$\therefore$ we have $\frac{x^2}{a^2}=1 \Rightarrow x=\pm a$

i.e the surface meets the x-axis in the points $A(a,0,0)$ and $A'(-a,0,0)$.

Similarly it meets y-axis $(x=0, z=0)$ at $B(0,b,0)$ and $B'(0,-b,0)$

and z-axis $(x=0, y=0)$ at $C(0,0,c)$ and $(0,0,-c)$

(iv) sections by co-ordinate planes: The surface meets the yz-plane i.e, $x=0$.

we have $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. which is an ellipse in that plane. (i.e y^2)

Similarly, it meets the zx-plane $(y=0)$ in the ellipse $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ in that plane

and it meets the xy-plane $(z=0)$ in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in that plane

(v) Generated by a variable curve:

The surface meets the plane $z=k$ in a curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2} \quad ; z=k, \quad -c\le k\le c \quad \text{(1)}$$

The surface is generated by the variable ellipse (1) in which k takes different values and whose plane is ∥ to the xy-plane (z=0) and centre (0,0,k) moves on the z-axis.

The ellipse (1) is real only if $1-\frac{k^2}{c^2}>0$

i.e. $k^2<c^2$

i.e. $|k|<c$

i.e. k lies between −c and c.

Similarly x and y cannot be numerically greater than a & b respectively.

So ~~that~~ we have for every point (x,y,z) on the surface

$-a\le x\le a$, $-b\le y\le b$, $-c\le z\le c$.

Hence, the surface lies between the planes $x=a$, $x=-a$; $y=b$, $y=-b$;

$z=c$, $z=-c$.

and therefore is a closed surface.

Hyperboloid of one sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(i) The origin bisects all chords which pass through it and is therefore the centre of the surface.

(ii) It is symmetrical about each of the co-ordinate planes. for only even powers x, y, z occur in its eqn; co-ordinate planes are the

... and co-ordinate axes are the principal axes of the surface

(iii) It meets the x-axis at $A(a,0,0)$, $A'(-a,0,0)$ the y-axis at $B(0,b,0)$, $B'(0,-b,0)$; and the z-axis in imaginary points. ($\because$ putting $x=0, y=0$, we get $-\frac{z^2}{c^2}=1$, $z^2=-c^2$)

(iv) Its section by the yz-plane $(x=0)$ is the hyperbola $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (i.e, DE, D'E')

– Its section by the zx-plane $(y=0)$ is the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ (i.e, FG, F'G')

– Its section by the xy-plane $(z=0)$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(v) The sections by the planes $z=k$ which are parallel to the xy-plane are the similar ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \quad z=k \text{ ——— (1)}$$

whose centre lie on z-axis and which increase in size as k increases.

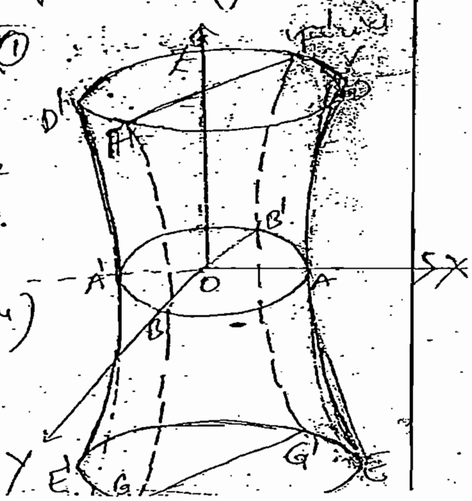
There is no limit to the increase of k.

The surface may, therefore, be generated by the variable ellipse (1) where k varies from $-\infty$ to $+\infty$.

The shape of the surface as shown in the figure.

(which is like Juggler's dabru) $\boxed{\text{i.e. मुगदर – Musha}}$

It is known as hyperboloid of one sheet



The hyperboloid of two sheets: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(i) origin is the centre; co-ordinate planes are the principal planes and co-ordinate axes are the principal axes of the surface.

(ii) It is symmetrical about each of the co-ordinate planes for only even powers of x, y, z occur in its equation.

(iii) It meets the x-axis at $A(a,0,0)$, $A'(-a,0,0)$ and the y and z-axes in imaginary points.

(iv) Its section by the xy-plane ($z=0$) is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (i.e. ACB, $A'C'B'$)

- Its section by the zx-plane ($y=0$) is the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ (i.e, DAE, $D'A'E'$)

- Its section by the yz-plane ($x=0$), is the imaginary ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$.

(v) The surface cuts the plane $x=k$ in an ellipse.

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1, \quad x=k.$$

which increases in size as k^2 increases, but is real when $\frac{k^2}{a^2}-1>0$.

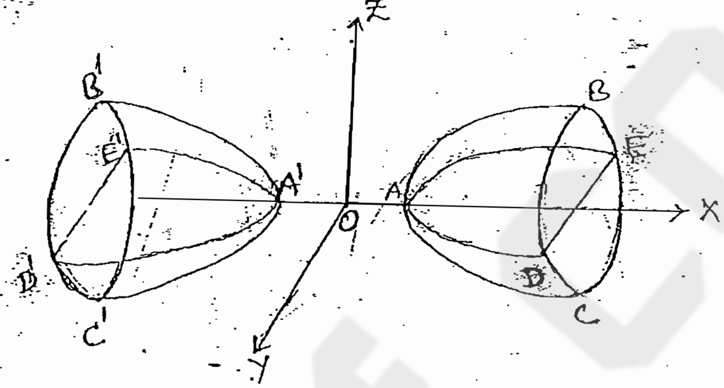
i.e, $k^2>a^2$.

i.e, $|k|>a$

i.e, when k does not lie between $-a$ and a.

Thus no portion of the surface lies between the planes $x = \pm a$.

This surface thus consists of two detached portions as shown in the figure.
It is known as hyperboloid of two sheets.



Its shape is like that of two tablas placed as shown in the figure.

* central conicoid :—

A conicoid whose all chords through the origin are bisected at the origin is called a central conicoid.

The equation $ax^2+by^2+cz^2=1$ —— ①

in general, represents a central conicoid.
All the above three equations

$$\left[\text{viz. } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\right]$$

are covered by this equation.

(i) when a, b, c are all +ve, ① represents an ellipsoid.

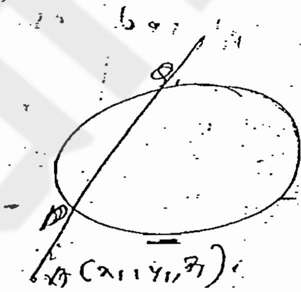
(ii) when two are +ve and one -ve, it represents a hyperboloid of one sheet, and (iii) when two are -ve and one +ve it represents a hyperboloid of two sheets.

The above equation for all values of a, b, c (-ve or +ve) represents a surface whose centre is origin and co-ordinate planes, the three principal planes.

The equation $ax^2+by^2+cz^2=1$ is called the standard form of central conicoid.

* **Intersection of a line and a conicoid:-**

To find the points of intersection of the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ with the conicoid $ax^2+by^2+cz^2=1$.



Sol: The given line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{---(1)}$$

and the conicoid is

$$ax^2+by^2+cz^2=1 \quad \text{---(2)}$$

Any point on the line (1) is

$$(lr+x_1, mr+y_1, nr+z_1). \quad \text{---(3)}$$

If it lies on the conicoid (2), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1.$$

$$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0. \quad \text{---(4)}$$

which is a quadratic in r, giving two values of r.

putting these values of r in (3),
we get the two points of intersection
P and Q.

Hence, every line meets a central
conicoid in two points.

The two values r_1 and r_2 of r obtained from the equation (4) are the measures of the distances of the points of intersection P and Q from the point (x_1, y_1, z_1) if (l, m, n) are the direction cosines of the line.

Note: The equation (4) will frequently be used in what follows.

Def → A chord of a central conicoid which passes through the centre is called a diameter.

→ Prove that the sum of the squares of the reciprocals of any three mutually perpendicular diameters of an ellipsoid is constant.

Sol. Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. — (1)

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the actual d.c's of three mutually $\perp$ diameters say PCP', QCQ', RCR' and let $2r_1, 2r_2, 2r_3$ be the lengths of the diameters.

Since the diameters of the ellipsoid are bisected at the centre

$C(0,0,0)$, $CP = CP' = r_1$; $CQ = CQ' = r_2$
$CR = CR' = r_3$.

Now as P is at a ~~distance r_1~~ from $C(0,0,0)$ ~~and~~ d.c's of CP are l_1, m_1, n_1

$\therefore$ The co-ordinates of P are $(l_1 r_1, m_1 r_1, n_1 r_1)$

Since P lies on ellipsoid (1)

$$\therefore \frac{l_1^2 r_1^2}{a^2} + \frac{m_1^2 r_1^2}{b^2} + \frac{n_1^2 r_1^2}{c^2} = 1.$$

$$\Rightarrow \frac{1}{r_1^2} = \frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}$$

now

$$\therefore \frac{1}{(PP')^2} = \frac{1}{(2r_1)^2} = \frac{1}{4r_1^2}$$

$$= \frac{1}{4}\left(\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}\right) \quad (2)$$

Similarly $$\frac{1}{(QQ')^2} = \frac{1}{4}\left(\frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}\right) \quad (3)$$

and $$\frac{1}{RR'^2} = \frac{1}{4}\left(\frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}\right) \quad (4)$$

Adding (2), (3) and (4), we have

$$\frac{1}{PP'^2} + \frac{1}{QQ'^2} + \frac{1}{RR'^2} = \frac{1}{4}\left[\frac{1}{a^2}(l_1^2 + l_2^2 + l_3^2) + \frac{1}{b^2}(m_1^2 + m_2^2 + m_3^2) + \frac{1}{c^2}(n_1^2 + n_2^2 + n_3^2)\right]$$

$$= \frac{1}{4}\left[\frac{1}{a^2}(1) + \frac{1}{b^2}(1) + \frac{1}{c^2}(1)\right]$$

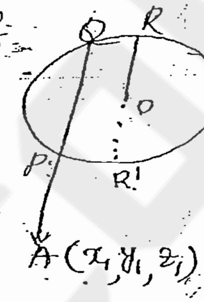
$$= \frac{1}{4}\left(1/a^2 + 1/b^2 + 1/c^2\right)$$

= constant. ($\because l_1, m_1, n_1, \ldots$ etc are the d.c's of three mutually $\perp$ lines).

→ A line through a given point ~~A meets~~ the central conicoid in P, Q. If R'OR is the diameter parallel to APQ, prove that $AP \cdot AQ : OR^2$ is constant.

Sol^n: Let $A(x_1, y_1, z_1)$ be the given point and let the conicoid be $ax^2+by^2+cz^2=1$. —— ①

Let l, m, n be the actual d.c.'s of the line through A which meets the conicoid in P and Q.



∴ Equations of the line APQ passing through $A(x_1, y_1, z_1)$ and with d.c.'s l, m, n are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{—— ②}$$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the conicoid ①, then

$$a(lr+x_1)^2 + b(mr+y_1)^2 + c(nr+z_1)^2 = 1$$

$$\Rightarrow r^2(al^2+bm^2+cn^2) + 2r(alx_1+bmy_1+cnz_1) + (ax_1^2+by_1^2+cz_1^2-1) = 0 \quad \text{—— ③}$$

which is a quadratic in 'r'.

Since l, m, n are the actual d.c.'s of the line ②.

∴ The two values of r in ② are the lengths AP and AQ.

∴ $AP \cdot AQ$ = product of roots

$$= \frac{ax_1^2+by_1^2+cz_1^2-1}{al^2+bm^2+cn^2} \quad \text{—— ④}$$

(r_1, r_2 are roots of $ax^2+bx+c=0$, $r_1r_2 = \frac{c}{a}$)

Now the equations of the diameter OR through $O(0,0,0)$ and $\parallel$ to line (2) are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}.$$

If $OR = r'$, then the co-ordinates of R are (lr', mr', nr').

Since R lies on the conicoid (1), then

$$al^2r'^2 + bm^2r'^2 + cn^2r'^2 = 1$$

$$\Rightarrow r'^2(al^2 + bm^2 + cn^2) = 1$$

$$\Rightarrow r'^2 = OR^2 = \frac{1}{al^2+bm^2+cn^2} \quad \text{--- (5)}$$

Dividing (4) & (5), we get

$$\frac{AP \cdot AQ}{OR^2} = \frac{ax_1^2+by_1^2+cz_1^2-1}{al^2+bm^2+cn^2} \times \frac{al^2+bm^2+cn^2}{1}$$

$$= ax_1^2 + by_1^2 + cz_1^2 - 1$$

$$= \text{constant}$$

Hence the result.

→ A is given point and POP' any diameter of a central conicoid. If OQ and OQ' are the diameters parallel to AP and AP', prove that $\frac{AP^2}{OQ^2} + \frac{AP'^2}{OQ'^2}$ is constant.

Sol: Let the central conicoid be $ax^2+by^2+cz^2 = 1$ --- (1)

Let A be the point (α, β, γ) and $P(x_1, y_1, z_1)$, $P'(-x_1, -y_1, -z_1)$ extremities of diameter POP'.

The d.c's of $\overline{AP}$ are proportional to

$x_1-\alpha,\ y_1-\beta,\ z_1-\gamma$ | using $x_2-x_1,\ y_2-y_1,\ z_2-z_1$

Dividing each by $\sqrt{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2}=AP$,

$\therefore$ the actual d.c's of AP are

$$l=\frac{x_1-\alpha}{AP};\ m=\frac{y_1-\beta}{AP}\ \&\ n=\frac{z_1-\gamma}{AP}. \quad (2)$$

$\therefore$ d.c's of OQ (∥ to AP) are also l, m, n.

Then if $OQ=r$, the co-ordinates of Q are (lr, mr, nr).

Since Q lies on the conicoid (1)

$$\therefore (1) \equiv a(lr)^2+b(mr)^2+c(nr)^2=1$$

$$\Rightarrow r^2(al^2+bm^2+cn^2)=1$$

$$\Rightarrow (OQ)^2\left[a\left(\frac{x_1-\alpha}{AP}\right)^2+b\left(\frac{y_1-\beta}{AP}\right)^2+c\left(\frac{z_1-\gamma}{AP}\right)^2\right]=1$$

($\because$ from (2) & OQ=r)

$$\Rightarrow \frac{OQ^2}{AP^2}\left[a(x_1-\alpha)^2+b(y_1-\beta)^2+c(z_1-\gamma)^2\right]=1$$

$$\Rightarrow \frac{AP^2}{OQ^2}=a(x_1-\alpha)^2+b(y_1-\beta)^2+c(z_1-\gamma)^2 \quad (3)$$

Now changing (x_1, y_1, z_1) to $(-x_1, -y_1, -z_1)$, i.e., P changes to P' and Q changes Q'.

$$\therefore \frac{AP'^2}{OQ'^2}=a(-x_1-\alpha)^2+b(-y_1-\beta)^2+c(-z_1-\gamma)^2$$
$$=a(x_1+\alpha)^2+b(y_1+\beta)^2+c(z_1+\gamma)^2 \quad (4)$$

Adding (3) and (4), we have

$$\frac{AP^2}{OQ^2}+\frac{AP'^2}{OQ'^2} = a\left[(x_1+\alpha)^2+(x_1-\alpha)^2\right]+b\left[(y_1+\beta)^2+(y_1-\beta)^2\right]$$
$$+c\left[(z_1+\gamma)^2+(z_1-\gamma)^2\right].$$
$$= 2a(x_1^2+\alpha^2)+2b(y_1^2+\beta^2)+2c(z_1^2+\gamma^2)$$
$$= 2(a\alpha^2+b\beta^2+c\gamma^2)+2(ax_1^2+by_1^2+cz_1^2).$$
$$= 2(a\alpha^2+b\beta^2+c\gamma^2)+2(1).$$

($\because$ $P(x_1,y_1,z_1)$ lies on (1) $\therefore ax_1^2+by_1^2+cz_1^2=1$)

$$= 2(a\alpha^2+b\beta^2+c\gamma^2+1)$$
$$= \text{constant}.$$

Hence the result.

Tangent planes

To find the equation of tangent plane at the point (x_1,y_1,z_1) of the central conicoid $ax^2+by^2+cz^2=1$.

Sol: The given conicoid is $ax^2+by^2+cz^2=1$ ——— (1)

Equation of a line through (x_1,y_1,z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{——— (2)}$$

Any point on (2) is $(lr+x_1, mr+y_1, nr+z_1)$

If it lies on (1), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1.$$
$$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+$$
$$(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \text{——— (3)}$$

But (x_1,y_1,z_1) lies on (1)

$$\therefore ax_1^2+by_1^2+cz_1^2=1. \quad \text{——— (4)}$$

$\therefore$ (3) becomes

$$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)=0$$

$$r[r(al^2+bm^2+cn^2)+2(alx_1+bmy_1+cnz_1)]=0 \quad \text{—(5)}$$

$\Rightarrow r=0.$

Since the line (2) touches the conicoid (1)

$\therefore$ it cuts (1) at two coincident points, which is so if the two values of 'r' in (5) are equal.

Since one root of (5) is zero,

$\therefore$ the other must also be zero.

$\therefore$ Coefficient of $r=0$

i.e, $alx_1+bmy_1+cnz_1=0$ —— (6)

eliminating l, m, n from (2) & (6),

the locus of line (2) is

$$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0$$

$\Rightarrow axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$

$\Rightarrow axx_1+byy_1+czz_1=1.$ ($\because$ by (4))

which is the required equation of tangent plane at (x_1, y_1, z_1)

Condition of Tangency:-

To find the condition that the plane $lx+my+nz=p$ should touch the conicoid $ax^2+by^2+cz^2=1$.

Sol: The given plane is $lx+my+nz=p$ —— (1)

and the conicoid is $ax^2+by^2+cz^2=1$ —— (2)

Let the plane (1) touch the conicoid (2) at the point (x_1, y_1, z_1).

Then ① should be identical with the tangent plane at (x_1, y_1, z_1) to ②.

Now the equation of tangent plane at (x_1, y_1, z_1) to ② is

$$axx_1 + byy_1 + czz_1 = 1. \longrightarrow ③$$

comparing ① and ③, we have

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p}.$$

$$\therefore x_1 = \frac{l}{ap},\ y_1 = \frac{m}{bp},\ z_1 = \frac{n}{cp} \longrightarrow ④$$

But since (x_1, y_1, z_1) being the point of contact lies on the conicoid ②.

$$\therefore a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1$$

$$\Rightarrow \boxed{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2}$$

which is the required condition of tangency

Note: From ④, the point of contact is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right).$$

→ Find the equations of two tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ which are parallel to the plane $lx + my + nz = 0$.

Sol: The given conicoid is $ax^2 + by^2 + cz^2 = 1 \longrightarrow ①$

and any plane $\parallel$ to $lx + my + nz = 0$ is

$$lx + my + nz = p \longrightarrow ②$$

If ② touches ①, then

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

$$p=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

$\therefore$ (2) $\Rightarrow$ the required tangent planes are

$$lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \text{—— (3)}$$

Note: The equation (3) represents tangent planes for all values of l, m, n.

Thus any tangent plane to conicoid (1) is

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

Director sphere:

To find the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid $ax^2+by^2+cz^2=1$

Sol: The given conicoid is

$$ax^2+by^2+cz^2=1 \quad \text{—— (1)}$$

Let $$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad \text{—— (2)}$$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad \text{—— (3)}$$

and $$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad \text{—— (4)}$$

be three mutually $\perp^r$ tangent planes

so that $l_1l_2+m_1m_2+n_1n_2=0$, etc. and

$l_1l_2+m_1m_2+l_3m_3=0$ etc. and

$l_1^2+m_1^2+n_1^2=1$ etc. and $l_2^2+m_2^2+n_2^2=1$ etc. $\Big\}$ —— (5)

The co-ordinates of the point of intersection satisfy the three equations ②, ③, ④ and its locus is therefore obtained by eliminating l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 from equations

squaring and adding ②, ③, ④.

we have

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$

$$=\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)+\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)+\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)$$

$$\Rightarrow x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)$$
$$+2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(m_1n_1+m_2n_2+m_3n_3)$$
$$+2zx(n_1l_1+n_2l_2+n_3l_3)=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+$$
$$\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

$$\Rightarrow x^2(1)+y^2(1)+z^2(1)+2xy(0)+2yz(0)+2zx(0)$$
$$=\frac{1}{a}(1)+\frac{1}{b}(1)+\frac{1}{c}(1) \quad (\because \text{from } ⑤)$$

$$\Rightarrow x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

which is the required locus and is a sphere concentric with the conicoid and is known as the director sphere.

→ Show that the length of the $\perp^r$ from the origin on the tangent plane at the point (x', y', z') of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given by $\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}$.

Sol^n: The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— ①

The equation of the tangent plane at (x', y', z') to ① is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1=0 \text{ —— } ②$$

Let $l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$ ...(2)

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad ...(3)$$

and $l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$ ...(4)

be three mutually $\perp$ tangent planes so that

$l_1l_2+m_1m_2+n_1n_2=0$ etc. and $l_1m_1+l_2m_2+l_3m_3=0$ etc.

and $l_1^2+m_1^2+n_1^2=1$ etc. and $l_1^2+l_2^2+l_3^2=1$ etc. ...(5)

The co-ordinates of the point of intersection satisfy the three equations (2), (3), (4) and its locus is therefore obtained by eliminating $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ from equations. Squaring and adding (2), (3), (4), we have

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$
$$=\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)+\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)+\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)$$

or
$$x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)$$
$$+2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(m_1n_1+m_2n_2+m_3n_3)$$
$$+2zx(n_1l_1+n_2l_2+n_3l_3)$$
$$=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

or
$$x^2(1)+y^2(1)+z^2(1)+2xy(0)+2yz(0)+2zx(0)$$
$$=\frac{1}{a}(1)+\frac{1}{b}(1)+\frac{1}{c}(1) \quad \textit{Using (5)}$$

or
$$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which is the required locus and is a sphere concentric with the conicoid and is known as the *director sphere.*

Example 1. *Show that the length of the perpendicular from the origin on the tangent plane at the point* (x', y', z') *of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is given by}$$

$$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}.$$

Sol. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

The equation of the tangent plane at (x', y', z') to (1) is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$$

or
$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1=0 \quad ...(2)$$

If p is the $\perp$ distance from the origin $(0, 0, 0)$ on (2), we have

$$p=\frac{0+0+0-1}{\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}}$$

or

$$-\frac{1}{p}=\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}$$

Squaring, $\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}$

which proves the required result.

Example 2. *If P, Q are any two points on the ellipsoid, the plane through the centre and the line of intersection of the tangent planes at P, Q bisects PQ.*

Sol. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

$$\therefore \quad \left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}&=1\\ \text{and} \quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}&=1\end{aligned}\right] \quad ...(2)$$

$[\because$ P, Q lie on (1)

Now equations of the tangents planes at P and Q to (1) are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad \text{or} \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1=0 \quad ...(3)$$

and $$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=1 \quad \text{or} \quad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1=0 \quad ...(4)$$

Now any plane through the line of intersection of (3) and (4) is

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)+k\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1\right)=0 \quad ...(5)$$

If it passes through the centre $(0, 0, 0)$ of the ellipsoid, then

$$0-1+k(0-1)=0 \quad \text{or} \quad k=-1.$$

$\therefore$ From (5), $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1-\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1\right)=0$$

or $$\frac{x(x_1-x_2)}{a^2}+\frac{y(y_1-y_2)}{b^2}+\frac{z(z_1-z_2)}{c^2}=0 \quad ...(6)$$

Now mid-point of PQ is

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

It lies on (6) if

$$\frac{(x_1+x_2)(x_1-x_2)}{2a^2}+\frac{(y_1+y_2)(y_1-y_2)}{2b^2}+\frac{(z_1+z_2)(z_1-z_2)}{2c^2}=0$$

or if $$\frac{x_1^2-x_2^2}{2a^2}+\frac{y_1^2-y_2^2}{2b^2}+\frac{z_1^2-z_2^2}{2c^2}=0$$

or if $$\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)-\frac{1}{2}\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}\right)=0$$ | Using (2)

or if $\frac{1}{2}(1)-\frac{1}{2}(1)=0$

or if $\frac{1}{2}-\frac{1}{2}=0$ which is true. Hence the result.

Example 3. (*a*) *A tangent plane to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meets the co-ordinate axes in A, B and C. Find the locus of the centroid of the (i) *triangle ABC,* (ii) *tetrahedron OABC.*

(Agra 1985, 87 ; Kanpur 1983)

(*b*) *If P be the point of contact of a tangent plane to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which meets the axes in A, B, C and PD, PE, PF are perpendiculars drawn from P to the axes, prove that

$$OD\,.\,OA=a^2,\ OE\,.\,OB=b^2,\ OF\,.\,OC=c^2.$$

Sol. (*a*) Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \quad ...(2)$$

Equation of tangent plane at $P(x_1, y_1, z_1)$ to (1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad ...(3)$$

This meets X-axis ($y=0$, $z=0$)

where $$\frac{xx_1}{a^2}=1 \quad \therefore \quad x=\frac{a^2}{x_1}.$$

Thus (3) meets X-axis in the point $A\left(\frac{a^2}{x_1}, 0, 0\right)$. Similarly it meets Y-axis in $B\left(0, \frac{b^2}{y_1}, 0\right)$, and Z-axis in $C\left(0, 0, \frac{c^2}{z_1}\right)$.

(*i*) Then if $G(\alpha, \beta, \gamma)$ be the centroid of $\triangle ABC$.

$$\alpha=\frac{\frac{a^2}{x_1}+0+0}{3}=\frac{a^2}{3x_1}, \text{ similarly } \beta=\frac{b^2}{3y_1},\ \gamma=\frac{c^2}{3z_1}$$

which give $$x_1=\frac{a^2}{3\alpha},\ y_1=\frac{b^2}{3\beta},\ z_1=\frac{c^2}{3\gamma}.$$

Putting these values of (x_1, y_1, z_1) in (2), we get

$$\frac{1}{a^2}\cdot\frac{a^4}{9\alpha^2}+\frac{1}{b^2}\cdot\frac{b^4}{9\beta^2}+\frac{1}{c^2}\cdot\frac{c^4}{9\gamma^2}=1,$$

or
$$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}=9.$$

$\therefore$ Locus of G (α, β, γ) is [changing (α, β, γ) to (x, y, z)]

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=9.$$

(*ii*) Please try yourself.

$$\left[\text{Ans. } \frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=16\right]$$

(*b*) Let $P(x_1, y_1, z_1)$ be the point of contact.

Then the equation of the tangent plane at $P(x_1, y_1, z_1)$ to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad \ldots(1)$$

If PD, PE, PF are $\perp$s drawn from P on the axes, then $OD=x_1$, $OE=y_1$, $OF=z_1$. (*Def. of co-ordinates*)

Now the plane (1) meets X-axis $(y=0, z=0)$ in the point A

where $\frac{xx_1}{a^2}=1$ or $xx_1=a^2$

or $OA \cdot OD=a^2$ $\quad [\because x=OA, x_1=OD]$

Similarly (1) meets Y-axis $(z=0, x=0)$ in the point B

where $\frac{yy_1}{b^2}=1$ or $yy_1=b^2$

or $OB \cdot OE=b^2$.

Similarly we can prove that $OC \cdot OF=c^2$.

Hence the result.

Example 4. *The tangent plane to the surface* $x^2+12y^2+4z^2=8$ *at the point* $(1, \frac{1}{2}, 1)$ *meets the co-ordinate axes at A, B, C. Find the centroid of* $\triangle ABC$. (Agra, 1986)

Sol. The tangent plane to the given surface at $(1, \frac{1}{2}, 1)$ is

$$x(1)+12y(\tfrac{1}{2})+4z(1)=8$$

or
$$x+6y+4z=8$$

which meets the co-ordinate axes at A, B and C.

$\Rightarrow$ $A(8, 0, 0)$; $B(0, 4/3, 0)$, $C(0, 0, 2)$

$\therefore$ Centroid of $\triangle ABC$ is,

$$\frac{8+0+0}{3},\ \frac{0+\frac{4}{3}+0}{3},\ \frac{0+0+2}{3}$$

i.e.,
$$\left(\frac{8}{3}, \frac{4}{9}, \frac{2}{3}\right).$$

Example 5. *A tangent plane to the conicoid $ax^2+by^2+cz^2=1$ meets the co-ordinate axes in P, Q and R. Find the locus of the centroid of the ΔPQR.*

Sol. Any tangent plane to the given conicoid is

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \ldots(i)$$

This plane meets the x-axis at P, so the co-ordinates of P, are

$$\left[\left(\frac{1}{l}\right)\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},\ 0,\ 0\right]$$

putting $y=0=z$ in (i).

Similarly Q and R are

$$\left[0,\ \frac{1}{m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},\ 0\right]$$

and

$$\left[0,\ 0,\ \frac{1}{n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}\right].$$

$\therefore$ If (x_1, y_1, z_1) be the centroid of ΔPQR, then

$$x_1=\frac{1}{3}\left[\frac{1}{l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}+0+0\right]$$

$$=\frac{1}{3l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$y_1=\frac{1}{3m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

and

$$z_1=\frac{1}{3n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$(3lx_1)^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$

or

$$\frac{9l^2}{a}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\cdot\frac{1}{ax_1^2}$$

Similarly,

$$\frac{9m^2}{b}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{by_1^2}$$

and

$$\frac{9n^2}{c}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{cz_1^2}$$

Adding,

$$9\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\left(\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}\right)$$

or

$$\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}=9$$

$\therefore$ The required locus is

$$\frac{1}{ax^2}+\frac{1}{by^2}+\frac{1}{cz^2}=9.$$

Example 6. *Show that the plane*

(a) $x+2y+3z=2$ *touches the conicoid* $x^2-2y^2+3z^2=2$.

(b) $3x+12y-6z-17=0$ *touches the conicoid*

(Bundelkhand 1985)

$$3x^2-6y^2+9z^2+17=0.$$

Find also the point of contact in each case.

Sol. (a) Let the plane $\quad x+2y+3z=2 \quad \ldots(1)$

touch the conicoid $\quad x^2-2y^2+3z^2=2 \quad \ldots(2)$

at the point (x_1, y_1, z_1).

The equation of tangent plane at (x_1, y_1, z_1) to (2) is

$$xx_1-2yy_1+3zz_1=2 \quad \ldots(3)$$

Since (3) and (1) are identical, $\therefore$ comparing (3) and (1), we get

$$\frac{x_1}{1}=\frac{-2y_1}{2}=\frac{3z_1}{3}=\frac{2}{2}$$

$$\therefore \quad x_1=1, \quad y_1=-1, \quad z_1=1 \quad \ldots(4)$$

Now the plane (1) will touch (2) if the point of contact (x_1, y_1, z_1) i.e., $(1, -1, 1)$ lies on the conicoid (2) i.e., if

$$(1)^2-2(-1)^2+3(1)^2=2 \quad \text{or} \quad 1-2+3=2 \text{ or } 2=2$$

which is true.

Hence the plane (1) touches the conicoid (2) and the point of contact is (x_1, y_1, z_1) i.e., $(1, -1, 1)$. | from (1)

(b) Please try yourself. [Ans. $(-1, 2, \frac{4}{3})$]

Example 7. *Find the equations to the tangent planes to the surfaces*

(a) $4x^2-5y^2+7z^2+13=0$, *parallel to the plane* $4x+20y-21z=0$.

(b) $x^2-2y^2+3z^2=2$, *parallel to the plane* $x-2y+3z=0$.

(M.D.U. 1987, 85)

Sol. (a) Any plane $\parallel$ to $4x+20y-21z=0$ is

$$4x+20y-21z=K \quad \ldots(1)$$

| Type $lx+my+nz=p$

The given conicoid is

$$4x^2-5y^2+7z^2+13=0$$

or

$$4x^2-5y^2+7z^2=-13$$

or

$$-\frac{4}{13}x^2+\frac{5}{13}y^2-\frac{7}{13}z^2=1 \quad \ldots(2)$$

[Form $ax^2+by^2+cz^2=1$]

(1) will touch the conicoid (1) if

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

or if $\dfrac{(4)^2}{\left(-\frac{4}{13}\right)}+\dfrac{(20)^2}{\left(\frac{5}{13}\right)}+\dfrac{(-21)^2}{\left(-\frac{7}{13}\right)}=K^2$

or if $-52+1040-819=K^2$

or if $K^2=1040-871=169$ $\therefore$ $K=\pm 13$

Putting these values of K in (1), the required tangent planes are

$$4x+20y-21z=\pm 13.$$

(b) Please try yourself. [Ans. $x-2y+3z=\pm 2$]

Example 8. *Find the co-ordinates of the point of contact of the plane $4x-6y+3z=5$ and the conicoid $2x^2-6y^2+3z^2=5$.* (Bundelkhand 1984)

Sol. Let the plane

$$4x-6y+3z=5 \quad \ldots(i)$$

touch the conicoid

$$2x^2-6y^2+3z^2=5 \quad \ldots(ii)$$

at (x_1, y_1, z_1).

The tangent plane to (ii) at (x_1, y_1, z_1) is

$$2xx_1-6yy_1+3zz_1=5 \quad \ldots(iii)$$

As (i) and (iii) represent the same plane, so comparing them, we have

$$\frac{2x_1}{4}=\frac{-6y_1}{-6}=\frac{3z_1}{3}=\frac{5}{5}$$

which gives $x_1=2, y_1=1, z_1=1$

$\therefore$ Required point is (2, 1, 1).

Example 9. (a) *Find the equations to the two tangent planes which contain the line given by*

$$7x-10y-30=0,\ 5y-3z=0$$

and touch the conicoid $7x^2+5y^2+3z^2=60.$

[Y. Imp.] (Agra 1986, 84; M.D.U. 1984)

(b) *Find the equations of the tangent planes to* $2x^2-6y^2+3z^2=5$ *which pass through the line* $x+9y-3z=0=3x-3y+6z-5.$

[Imp.] (M.D.U. 1986; K.U. 1983)

(c) *Find the equations of the tangent planes to* $7x^2-3y^2-z^2+21=0$ *which pass through the line* $7x-6y+9=0,\ z=3.$

Sol. (a) The given line is

$$7x+10y-30=0,\ 5y-3z=0$$

Any plane through this line is

$$7x+10y-30+k(5y-3z)=0$$

or $$7x+5(k+2)y-3kz=30 \quad \ldots(1)$$

| Form $lx+my+nz=p$

The given ellipsoid is

$$7x^2+5y^2+3z^2=60$$

or $$\frac{7}{60}x^2+\frac{5}{60}y^2+\frac{3}{60}z^2=1 \quad \ldots(2)$$

| Form $ax^2+by^2+cz^2=1$

The plane (1) touches the ellipsoid (2) if

$$\frac{(7)^2}{\left(\frac{7}{60}\right)}+\frac{25(k+2)^2}{\left(\frac{5}{60}\right)}+\frac{(-3k)^2}{\left(\frac{3}{60}\right)}=(30)^2$$

| Using $\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=p^2$

or $$\frac{49\times 60}{7}+\frac{25(k+2)^2\times 60}{5}+\frac{9k^2\times 60}{3}=900$$

or $$420+300(k^2+4k+4)+180k^2=900$$

or $$480k^2+1200k+720=0$$

or $$2k^2+5k+3=0 \quad \text{or} \quad (2k+3)(k+1)=0$$

$\therefore$ $k=-\dfrac{3}{2}$, or -1

Putting these values of k in (1), the required tangent planes are

$$7x+5\left(-\frac{3}{2}+2\right)y+\frac{9}{2}z=30 \text{ and } 7x+5(-1+2)y+3z=30$$

or $$7x+\frac{5}{2}y+\frac{9}{2}z=30 \text{ and } 7x+5y+3z=30$$

or $$14x+5y+9z=60 \text{ and } 7x+5y+3z=30.$$

(b) The given line is

$$x+9y-3z=0=3x-3y+6z-5$$

Any plane through this line is

$$x+9y-3z+k(3x-3y+6z-5)=0$$

or $$x(1+3k)+3(3-k)y-3(1-2k)z=5k \quad \ldots(1)$$

| Type $lx+my+nz=p$

and the given conicoid is

$$2x^2-6y^2+3z^2=5$$

or $$\frac{2}{5}x^2-\frac{6}{5}y^2+\frac{3}{5}z^2=1 \quad \ldots(2)$$

| Form $ax^2+by^2+cz^2=1$

The plane (1) touches (2) if

$$\frac{(1+3k)^2}{\left(\frac{2}{5}\right)}+\frac{9(3-k)^2}{\left(-\frac{6}{5}\right)}+\frac{9(1-2k)^2}{\left(\frac{3}{5}\right)}=(5k)^2$$

| Using $\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=p^2$

or $\quad \frac{5}{2}(1+3k)^2-\frac{45}{6}(3-k)^2+\frac{45}{3}(1-2k)^2=25k^2$

or $\quad (1+3k)^2-3(3-k)^2+6(1-2k)^2=10k^2$ | On dividing throughout by $\frac{5}{2}$

or $\quad 1+9k^2+6k-3(9+k^2-6k)+6(1+4k^2-4k)-10k^2=0$

or $\quad 1+9k^2+6k-27-3k^2+18k+6+24k^2-24k-10k^2=0$

or $\quad 20k^2-20=0$ or $k^2=1$ $\quad \therefore k=\pm 1$.

Putting these values of k in (1), the required tangent planes are

$(1+3)x+3(3-1)y-3(1-2)z=5$

and $\quad (1-3)x+3(3+1)y-3(1+2)z=-5$

or $\quad 4x+6y+3z-5=0$ and $-2x+12y-9z+5=0$

or $\quad 4x+6y+3z-5=0$ and $2x-12y+9z-5=0$.

(c) Please try yourself.
[Ans. $7x-6y-4z+21=0$; $14x-12y-z+21=0$]

Example 10. *Find the equations of the tangent plane to the surface $3x^2-6y^2+9z^2+17=0$ parallel to the plane $x+4y-2z=0$.* (Kanpur 1987)

Sol. Any plane parallel to the given plane is

$$x+4y-2z=p. \qquad \ldots(1)$$

If this plane touches the ellipsoid

$3x^2-6y^2+9z^2+17=0$

or $\quad 3x^2-6y^2+9z^2=-17$

or $\quad \left(-\frac{3}{17}\right)x^2+\left(\frac{6}{17}\right)y^2-\left(\frac{9}{17}\right)z^2=1$

then the condition of tangency is

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

or $\quad \left(\frac{1^2}{(-3/17)}\right)+\left(\frac{4^2}{(6/17)}\right)+\left(\frac{2^2}{(-9/17)}\right)=p^2$

or $\quad p^2=\left(-\frac{17}{3}\right)+\left(\frac{136}{3}\right)-\left(\frac{68}{9}\right)$

or $\quad 9p^2=-51+408-68=289$

or $\quad p=\pm\sqrt{\frac{289}{9}}=\pm\left(\frac{17}{3}\right)$

$\therefore$ From (1) the required tangent planes are

$x+4y-2z=\pm\left(\frac{17}{3}\right)$

or $\quad 3x+12y-6z=\pm 17$.

Example 11. *Find the equations to the tangent planes to $7x^2-3y^2-z^2+21=0$ which pass through the line $7x+6y+9=0$, $z=3$.* (M.D.U. 1983 ; K.U. 1985, 81)

Sol. Please try yourself. [Ans. $7x-6y-4z+21=0$, $14x-12y-z+21=0$]

Example 12. *Find the condition that the line $\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n}$ may touch the ellipsoid $3x^2+8y^2+z^2=c^2$.* (Agra 1983)

Sol. Given line is

$$\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n}$$

See page 8

Given ellipsoid is

$3x^2+8y^2+z^2=c^2$

$\Rightarrow \frac{3}{c^2}x^2+\frac{8}{c^2}y^2+\frac{1}{c^2}z^2=1$.

The condition for becoming a tangent line is

$a\alpha l+b\beta m+c\gamma n=0$

i.e. $\quad \frac{3}{c^2}\cdot(2)\cdot l+\frac{8}{c^2}\cdot(1)\,m+\frac{1}{c^2}\cdot 3\cdot n=0$

$\Rightarrow \quad 6l+8m+3n=0$.

Example 13. *If P is the point of contact of a tangent plane ABC to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and PD, PE, PF are perpendiculars from P on the axes, prove that $OD.OA=a^2$, $OE.OB=b^2$, $OF.OC=c^2$; A, B, C being the points where the tangent plane at P meets the co-ordinate axes.*

Sol. Let $P\to(\alpha,\beta,\gamma)$ so that the equation of tangent plane ABC is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1$$

It meets the co-ordinate axes in A, B, C.

$\therefore \quad OA=\frac{a^2}{\alpha}$, $OB=\frac{b^2}{\beta}$ and $OC=\frac{c^2}{\gamma}$

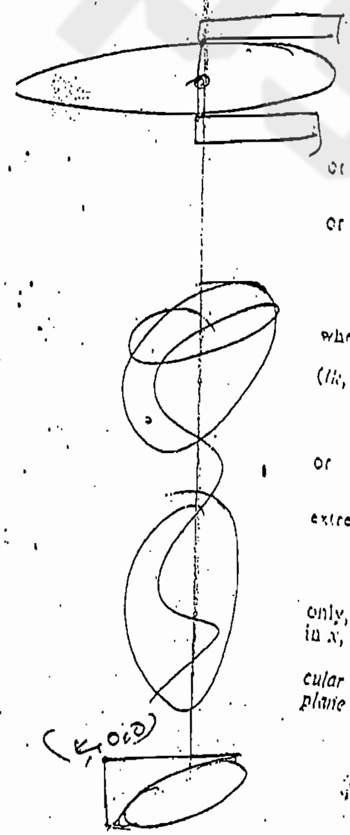
Also PD, PE, PF are perpendiculars from P on the axes.

$\therefore \quad OD=\alpha$, $OE=\beta$ and $OF=\gamma$

Hence $OD\cdot OA=\alpha\cdot\frac{a^2}{\alpha}=a^2$

$OE\cdot OB=\beta\cdot\frac{b^2}{\beta}=b^2$

and $\quad OF\cdot OC=\gamma\cdot\frac{c^2}{\gamma}=c^2$.



Example 14. *Show that the tangent planes at the extremities of any diameter of an ellipsoid are parallel.*

Sol. Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \dots(i)$$

As its centre is (0, 0, 0) so any diameter of this ellipsoid is a line through (0, 0, 0) and its equation is given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \dots(ii)$$

Any point on this diameter is (lr, mr, nr).

If this point lies on (*i*), then

$$\frac{l^2r^2}{a^2}+\frac{m^2r^2}{b^2}+\frac{n^2r^2}{c^2}=1$$

or $$r^2\left[\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right]=1$$

or $$r=\pm\frac{1}{\sqrt{\left(\frac{l}{a}\right)^2+\left(\frac{m}{b}\right)^2+\left(\frac{n}{c}\right)^2}}=\pm k, \qquad \dots(iii)$$

∴ The extremities of the diameter (*ii*) are (lk, mk, nk) and $(-lk, -mk, -nk)$ where k is given by (*iii*).

Now the equation of the tangent plane to the ellipsoid (*i*) at (lk, mk, nk) is given by

$$\frac{xlk}{a^2}+\frac{ymk}{b^2}+\frac{znk}{c^2}=1$$

or $$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=\frac{1}{k} \qquad \dots(iv)$$

Similarly the equation of the tangent plane to (*i*) at the other extremity $(-lk, -mk, -nk)$ of (*ii*) is

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=-\frac{1}{k} \qquad \dots(v)$$

Since the equations (*iv*) and (*v*) differ in the constant terms only, so these represent parallel planes (each being a linear equation in x, y, z).

Example 15. *Through a fixed point $(k, 0, 0)$ pairs of perpendicular lines are drawn to the conicoid $ax^2+by^2+cz^2=1$. Show that the plane through any pair touches the cone*

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Sol. Any line through the point $(k, 0, 0)$ is

$$\frac{x-k}{l}=\frac{y-0}{m}=\frac{z-0}{n} \qquad \dots(i)$$

Any point on (*i*) is $(k+lr, mr, nr)$, which is at a distance r from $(k, 0, 0)$.

∴ The distance of the points where the line (*i*) meets the given conicoid

$$ax^2+by^2+cz^2=1 \qquad \dots(ii)$$

are given by

$$a(k+lr)^2+b(mr)^2+c(nr)^2=1$$

or $$r^2(al^2+bm^2+cn^2)+2ak\,lr+(ak^2-1)=0 \qquad \dots(iii)$$

If the line (*i*) touches (*ii*) at $(k, 0, 0)$, then the two values of r given by (*iii*) must be coincident and the condition for the same is "$B^2=4AC$".

i.e. $$(2akl)^2=4(al^2+bm^2+cn^2)(ak^2-1)$$

or $$(al^2+bm^2+cn^2)(ak^2-1)=a^2k^2l^2 \qquad \dots(iv)$$

Now let the two perpendicular tangent lines through $(k, 0, 0)$ be

$$\frac{x-k}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$$

and $$\frac{x-k}{l_2}=\frac{y}{m_2}=\frac{z}{n_2} \qquad \dots(v)$$

Then from (*iv*), we get

$$(al_1^2+bm_1^2+cn_1^2)(ak^2-1)=a^2k^2l_1^2$$

and $$(al_2^2+bm_2^2+cn_2^2)(ak^2-1)=a^2k^2l_2^2$$

Adding these, we get

$$[a(l_1^2+l_2^2)+b(m_1^2+m_2^2)+c(n_1^2+n_2^2)](ak^2-1)=a^2k^2(l_1^2+l_2^2) \qquad \dots(vi)$$

If the line $\frac{(x-k)}{l_3}=\frac{(y-0)}{m_3}=\frac{(z-0)}{n_3}$ be the normal to the plane containing the tangent lines given by (*v*), then we obtain a set of three mutually perpendicular lines for which we have the relations $l_1^2+l_2^2+l_3^2=1=l_3^2+m_3^2+n_3^2$ etc.

i.e., $$l_1^2+l_2^2=m_3^2+n_3^2.$$

Similarly $m_1^2+m_2^2=n_3^2+l_3^2$, $n_1^2+n_2^2=l_3^2+m_3^2$.

Substituting these values in (*vi*), we get

$$[a(m_3^2+n_3^2)+b(n_3^2+l_3^2)+c(l_3^2+m_3^2)](ak^2-1)=a^2k^2(m_3^2+n_3^2)$$

or $$l_3^2[(b+c)(ak^2-1)]+m_3^2[(a+c)(ak^2-1)-a^2k^2]+n_3^2[(a+b)(ak^2-1)-a^2k^2]=0$$

or $$l_3^2(b+c)(ak^2-1)+m_3^2[c(ak^2-1)-a]+n_3^2[b(ak^2-1)-a]=0 \qquad \dots(vii)$$

Eliminating l_3, m_3, n_3 between (*vii*) and $\frac{x-k}{l_3}=\frac{y}{m_3}=\frac{z}{n_3}$ we find that the normal to the plane containing the lines given by (*v*) generates the cone

$$(x-k)^2(b+c)(ak^2-1)+y^2[c(ak^2-1)-a]+z^2[b(ak^2-1)-a]=0$$

and the plane itself touches the reciprocal cone

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Example 16. *Prove that the equation to the two tangent planes to the conicoid*

$$ax^2+by^2+cz^2=1$$

which pass through the line

$$u\equiv lx+my+nz-p=0,\quad u'\equiv l'x+m'y+n'z-p'=0 \text{ is}$$

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

[V. Imp.]

Sol. Any plane through the line

$u=0,\ u'=0$ is $u+ku'=0$

i.e., $(lx+my+nz-p)+k(l'x+m'y+n'z-p')=0$ ...(1)

or $(l+kl')x+(m+km')y+(n+kn')z=p+kp'$ ...(2)

| Form $lx+my+nz=p$

This will be tangent plane to the conicoid

$$ax^2+by^2+cz^2=1$$

if $\dfrac{(l+kl')^2}{a}+\dfrac{(m+km')^2}{b}+\dfrac{(n+kn')^2}{c}=(p+kp')^2$

| Using $\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=p^2$

or if $k^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)+2k\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$
$$+\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

Putting $k=\dfrac{-u}{u'}$, from (1), the required equation is

$$\frac{u^2}{u'^2}\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2\frac{u}{u'}\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

or $u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

Hence the result.

Example 17. (a) *Show that the plane*

$$lx+my+nz=p$$

will touch the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ if } a^2l^2+b^2m^2+c^2n^2=p^2.$$

[V. Imp.] (K.U. 1987)

(b) *Tangent planes are drawn to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

through the point (α, β, γ). Prove that the perpendiculars to them through the origin generate the cone

$$(\alpha x+\beta y+\gamma z)^2=a^2x^2+b^2y^2+c^2z^2.$$

[Imp.] (Allahabad 1984, 83, 80)

(c) *Tangent planes are drawn to the conicoid*

$$ax^2+by^2+cz^2=1$$

through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone

$$(\alpha x+\beta y+\gamma z)^2=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}.$$

[Imp.] (K.U. 1986 ; Kanpur 1981)

Sol. (a) The given plane is

$$lx+my+nz=p \quad ...(1)$$

and the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(2)$$

Let the plane (1) touch the ellipsoid (2) at (x_1, y_1, z_1). Then the plane (1) will be identical with the tangent plane at (x_1, y_1, z_1) to the surface (2) i.e.,

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad ...(3)$$

Comparing (1) and (3), we get

$$\frac{x_1/a^2}{l}=\frac{y_1/b^2}{m}=\frac{z_1/c^2}{n}=\frac{1}{p}$$

or $\dfrac{x_1}{a^2l}=\dfrac{y_1}{b^2m}=\dfrac{z_1}{c^2n}=\dfrac{1}{p}$,

$$x_1=\frac{a^2l}{p},\ y_1=\frac{b^2m}{p},\ z_1=\frac{c^2n}{p} \quad ...(4)$$

Since (x_1, y_1, z_1) being the point of contact lies on the ellipsoid (2).

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

or $$\frac{1}{a^2}\left(\frac{a^2 l}{p}\right)^2+\frac{1}{b^2}\left(\frac{b^2 m}{p}\right)^2+\frac{1}{c^2}\left(\frac{c^2 n}{p}\right)^2=1$$ | Using (4)

or $$a^2l^2+b^2m^2+c^2n^2=p^2$$ | C.T.M.

which is the required condition.

(b) Any plane through (α, β, γ) is
$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$
or $$lx+my+nz=l\alpha+m\beta+n\gamma \qquad \ldots(1)$$
| Form $lx+my+nz=p$

This will touch the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
if $$a^2l^2+b^2m^2+c^2n^2=(l\alpha+m\beta+n\gamma)^2 \qquad \ldots(2)$$
| Using $a^2l^2+b^2m^2+c^2n^2=p^2$ [Part (a)].

Now equations of the normal to (1) through (0, 0, 0) are
$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(3)$$
$\therefore$ Locus of the line (3) is [*eliminating l, m, n from* (2) *and* (3)]
$$a^2x^2+b^2y^2+c^2z^2=(\alpha x+\beta y+\gamma z)^2$$
which being a second degree homogeneous equation in x, y, z represents a cone.

(c) Any plane through (α, β, γ) is
$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$
$$lx+my+nz=l\alpha+m\beta+n\gamma \qquad \ldots(1)$$
| Form $lx+my+nz=p$

If it is the tangent plane to the conicoid
$$ax^2+by^2+cz^2=1,$$
then $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=(l\alpha+m\beta+n\gamma)^2 \qquad \ldots(2)$$
| Using $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

Now d.c.'s of the normal to plane (1) are proportional to l, m, n | Co-effts. of x, y, z

$\therefore$ Equations of $\perp$ to (1) through (0, 0, 0) *i.e.*, the normal to (1) through the origin are
$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(3)$$
To find the locus of line (3), we have to eliminate l, m, n, from (3) and (2). Putting the values of l, m, n from (3) in (2), we get
$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=(\alpha x+\beta y+\gamma z)^2$$
which is the required equation of cone.

Example 18. *Obtain the tangent planes for the ellipsoid*
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
which are parallel to $lx+my+nz=0$.

If $2r$ is the distance between the planes, show that a line through the origin and perpendicular to the planes lies on the cone
$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0.$$
[V. Imp.] (K.U. 1983 ; Rohilkhand 1982)

Sol. Any plane $\parallel$ to $lx+my+nz=0$ is
$$lx+my+nz=p \qquad \ldots(1)$$
This will touch the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
if $$a^2l^2+b^2m^2+c^2n^2=p^2$$
or if $$p=\pm\sqrt{a^2l^2+b^2m^2+c^2n^2}$$
Putting these values of p in (1), the required tangent planes are
$$lx+my+nz=\sqrt{a^2l^2+b^2m^2+c^2n^2} \qquad \ldots(2)$$
and $$lx+my+nz=-\sqrt{a^2l^2+b^2m^2+c^2n^2} \qquad \ldots(3)$$
Now one point on the plane (2) is (putting $x=0, y=0$)
$$P=\left[0, 0, \frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{n}\right]$$
Now distance between two $\parallel$ planes is the $\perp$ distance of a point on one plane from the other.

Since $2r$ is given to be the distance between two $\parallel$ planes (2) and (3).

$\therefore$ $2r=\perp$ distance of P from the plane (3)
$$=\frac{0+0+n.\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{n}+\sqrt{a^2l^2+b^2m^2+c^2n^2}}{\sqrt{l^2+m^2+n^2}}$$
$$=\frac{2\sqrt{a^2l^2+b^2m^2+c^2n^2}}{\sqrt{l^2+m^2+n^2}}$$
or $$r\sqrt{l^2+m^2+n^2}=\sqrt{a^2l^2+b^2m^2+c^2n^2}$$
Squaring $$r^2(l^2+m^2+n^2)=a^2l^2+b^2m^2+c^2n^2$$
or $$l^2(a^2-r^2)+m^2(b^2-r^2)+n^2(c^2-r^2)=0 \qquad \ldots(4)$$
Now equations of the line through (0, 0, 0) and $\perp$ to the tangent planes (2) or (3) are
$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(5)$$

Eliminating l, m, n from (5) and (4) [by putting the values of l, m, n from (5) in (4)], the required locus is

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0$$

which is a cone, being a homogenous equation in x, y, z.

Example 19. *If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation referred to rectangular axes is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

passes through the fixed point $(0, 0, k)$, show that it lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$ [Imp.]

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

Any line through $(0, 0, k)$ is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-k}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z-k}{n} \quad ...(2)$$

and any plane through this line (2) is

$$Lx+My+N(z-k)=0 \quad ...(3)$$

where

$$Ll+Mm+Nn=0 \quad ...(4)$$

If the plane (3) *i.e.*, $Lx+My+Nz=Nk$ touches the ellipsoid (1), then

$$L^2a^2+M^2b^2+N^2c^2=N^2k^2 \qquad | \text{ Using } l^2a^2+m^2b^2+n^2c^2=p^2$$

or

$$L^2a^2+M^2b^2+N^2(c^2-k^2)=0 \quad ...(5)$$

If the tangent planes are $\perp$, their normals are also $\perp$.

Now the lines whose d.c.'s L, M, N, are given by the equations (4) and (5), are normals to the planes (3).

From (4), $L=-\frac{Mm+Nn}{l}$. Putting this in (5), we get

$$\frac{(Mm+Nn)^2}{l^2}a^2+M^2b^2+N^2(c^2-k^2)=0$$

or

$$a^2(Mm+Nn)^2+b^2l^2M^2+l^2(c^2-k^2)N^2=0$$

or

$$M^2(a^2m^2+b^2l^2)+2MNmna^2+N^2(c^2l^2+a^2n^2-l^2k^2)=0.$$

Dividing throughout by N^2, we get

$$\frac{M^2}{N^2}(a^2m^2+b^2l^2)+2mna^2\cdot\frac{M}{N}+(c^2l^2+a^2n^2-l^2k^2)=0 \quad ...(6)$$

which is a quadratic in $\frac{M}{N}$. If L_1, M_1, N_1 and L_2, M_2, N_2 are the d.c.'s of the two lines, then, $\frac{M_1}{N_1}, \frac{M_2}{N_2}$ are the roots of (6), so that

$$\frac{M_1M_2}{N_1N_2}=\frac{c^2l^2+n^2a^2-l^2k^2}{a^2m^2+b^2l^2}=\frac{(c^2-k^2)l^2+n^2a^2}{m^2a^2+l^2b^2}$$

$$\therefore \frac{M_1M_2}{(c^2-k^2)l^2+n^2a^2}=\frac{N_1N_2}{m^2a^2+l^2b^2} \quad ...(7)$$

Similarly, eliminating M between (4) and (5), we have

$$\frac{N_1N_2}{l^2b^2+m^2a^2}=\frac{L_1L_2}{b^2n^2+(c^2-k^2)m^2} \quad ...(8)$$

$$\therefore \frac{L_1L_2}{(c^2-k^2)m^2+b^2n^2}=\frac{M_1M_2}{(c^2-k^2)l^2+n^2a^2}=\frac{N_1N_2}{a^2m^2+b^2l^2}$$ | From (7) and (8)

Since the two normals with d.c.'s L_1, M_1, N_1 and L_2, M_2, N_2 are $\perp$ $\therefore$

$$L_1L_2+M_1M_2+N_1N_2=0$$

or

$$(c^2-k^2)m^2+b^2n^2+(c^2-k^2)l^2+n^2a^2+a^2m^2+b^2l^2=0$$

or

$$l^2(b^2+c^2-k^2)+m^2(c^2+a^2-k^2)+n^2(a^2+b^2)=0 \quad ...(9)$$

Eliminating l, m, n from (2) and (9), the line (2) generates the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

Hence the result.

Example 20. *Find the locus of the feet of perpendiculars from the origin to the tangent planes to the surface* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *which cuts off from the axes, intercepts the sum of whose reciprocals is equal to constant* $\frac{1}{k}$. (K.U. 1983)

Sol. The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let $P(x_1, y_1, z_1)$ be the foot of $\perp$ from O(0, 0, 0) to any tangent plane to (1)

d.r.'s of OP are x_1-0, y_1-0, z_1-0 $\qquad | \; x_2-x_1, y_2-y_1, z_2-z_1$

or x_1, y_1, z_1

$\because$ OP is $\perp$ to the tangent plane,

$\therefore$ d.r.'s of OP are co-effs. of x, y, z in the equation of tangent plane.

$\therefore$ Equation of tangent plane (1) is

$$xx_1+yy_1+zz_1=p \quad ...(2)$$

$\because$ Plane (2) touches (1)

$$\therefore a^2x_1^2+b^2y_1^2+c^2z_1^2=p^2 \quad ...(3)$$

| Using $a^2l^2+b^2m^2+c^2n^2=p^2$

Plane (2) meets X-axis ($y=0, z=0$) where

$$xx_1=p \text{ or } x=\frac{p}{x_1}$$

Thus the plane cuts off intercept from the X-axis which is

The sum of reciprocals of intercepts $=\frac{1}{k}$ (given)

or $$\frac{x_1}{p}+\frac{y_1}{p}+\frac{z_1}{p}=\frac{1}{k}$$

or $$\frac{x_1+y_1+z_1}{p}=\frac{1}{k}$$

$\therefore$ $$p=k(x_1+y_1+z_1)$$

Eliminating p [By putting this value of p in (3)], we get

$$a^2x_1^2+b^2y_1^2+c^2z_1^2=k^2(x_1+y_1+z_1)^2$$

$\therefore$ Locus of foot of $\perp$ (x_1, y_1, z_1) is

$$a^2x^2+b^2y^2+c^2z^2=k^2(x+y+z)^2.$$

Example 21. *Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

to any of its tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$. [Imp.]

(*Kanpur 1981 ; K.U. 1986, 82 ; M.D.U. 1983 ; Allahabad 1986*)

Sol. The given conicoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let $P(x_1, y_1, z_1)$ be the foot of $\perp$ from centre $O(0, 0, 0)$ of (1) to any tangent plane to (1)

d.r.'s of OP are x_1, y_1, z_1

$\therefore$ Equation of tangent plane is

$xx_1+yy_1+zz_1=p$ ...(2) [$\because$ Tangent plane is $\perp$ to OP and hence co-effs. of x, y, z in tangent plane are d.r.'s of OP]

$\because$ Plane (2) touches ellipsoid (1)

$\therefore$ $a^2x_1^2+b^2y_1^2+c^2z_1^2=p^2$ ...(3) [Using $a^2l^2+b^2m^2+c^2n^2=p^2$]

Also $P(x_1, y_1, z_1)$ lies on (2) [$\because$ P is the foot of $\perp$ from (0, 0, 0) to the tangent plane (2)]

$\therefore$ $$x_1^2+y_1^2+z_1^2=p^2 \quad ...(4)$$

Eliminating p from (3) and (4), (by equating its values), we get

$$a^2x_1^2+b^2y_1^2+c^2z_1^2=(x_1^2+y_1^2+z_1^2)^2$$

$\therefore$ Locus of (x_1, y_1, z_1), the foot of $\perp$ is

$$a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2.$$

Example 22. *If P is the point on the ellipsoid* $x^2+2y^2+\frac{1}{3}z^2=1$, *such that the perpendicular from the origin on the tangent plane at P is of unit length, show that P lies on one or other of the planes* $3y=\pm z$.

Sol. Let P be the point (x_1, y_1, z_1).

Since it lies on the ellipsoid

$$x^2+2y^2+\tfrac{1}{3}z^2=1 \quad ...(1)$$

$\therefore$ $$x_1^2+2y_1^2+\tfrac{1}{3}z_1^2=1 \quad ...(2)$$

Now equation of tangent plane at $P(x_1, y_1, z_1)$ to (1) is

$$xx_1+2yy_1+\tfrac{1}{3}zz_1=1$$

or $$3xx_1+6yy_1+zz_1-3=0. \quad ...(3)$$

The $\perp$ distance from (0, 0, 0) to (3) $=1$ (given)

$$\pm\frac{0+0+0-3}{\sqrt{9x_1^2+36y_1^2+z_1^2}}=1$$

or $$9x_1^2+36y_1^2+z_1^2=9$$

or $$x_1^2+4y_1^2+\frac{1}{9}z_1^2=1 \quad ...(4)$$

Subtracting (2) from (4), we have

$$2y_1^2+\left(\frac{1}{9}-\frac{1}{3}\right)z_1^2=0$$

or $$2y_1^2-\frac{2}{9}z_1^2=0$$

or $$9y_1^2-z_1^2=0$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is $9y^2-z^2=0$

or $$(3y-z)(3y+z)=0$$

i.e., $$3y-z=0 \text{ or } 3y+z=0$$

Thus P lies either on $3y-z=0$ or $3y+z=0$

i.e., P lies on one of the planes $3y=\pm z$.

Hence the result.

Example 23. (*a*) *Prove that the locus of the point of intersection of three mutually perpendicular tangent planes to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $$x^2+y^2+z^2=a^2+b^2+c^2.$$

[V. Imp.] (*Rajputana 1983*)

(*b*) *Prove that the locus of points from which three mutually perpendicular tangent planes can be drawn to touch the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \; z=0$$

is the sphere $x^2+y^2+z^2=a^2+b^2$. [Imp.]

Sol. (*a*) Please try yourself as in Art. 9.

(*b*) Let $$lx+my+nz=p \quad ...(1)$$

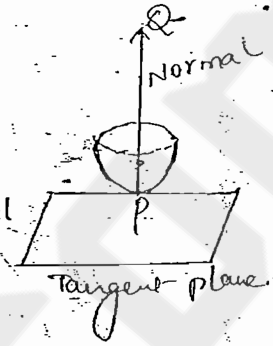
be a tangent plane to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \; z=0 \quad ...(2)$$

Set-VII

Normals

The normal at any point P of surface (quad.) is a line through the point of contact P and perpendicular to the tangent plane at P.



Equations of the normal:

To find the equations of the normal at the point (x_1, y_1, z_1) of the conicoid $ax^2+by^2+cz^2=1$.

Sol^n: The given conicoid is $ax^2+by^2+cz^2=1$ — ①

equation of tangent plane at (x_1, y_1, z_1) to ①

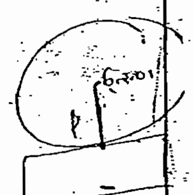
is $axx_1+byy_1+czz_1=1$ — ②

The d.c.'s of the normal to this plane are proportional to ax_1, by_1, cz_1.

∴ Equations ~~of the~~ normal at $P(x_1, y_1, z_1)$ to ①

[i.e., a line through (x_1, y_1, z_1) and $\perp^r$ to the tangent plane ②] are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{cz_1} \quad \text{— ③}$$



Actual d.c.'s form:

If p is the length of the perpendicular distance from the centre $(0,0,0)$ to the tangent plane ②

then $p = \dfrac{1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}$

The d.c's of the normal at (x_1, y_1, z_1) to (1) are proportional to ax_1, by_1, cz_1.

Dividing each by $\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}$

$\therefore$ The actual direction cosines are

$$\frac{ax_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}, \frac{by_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}, \frac{cz_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}$$

$\Rightarrow ax_1p, \; by_1p, \; cz_1p \quad \left(\because p = \frac{1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}\right)$

$\therefore$ The equations of normal at (x_1, y_1, z_1) in actual direction cosines form are

$$\frac{x-x_1}{ax_1p} = \frac{y-y_1}{by_1p} = \frac{z-z_1}{cz_1p}.$$

S.T. The equations of the normal at the point (x_1, y_1, z_1) of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2} = \frac{z-z_1}{z_1/c^2}$$

Note: The equations of the normal at (x_1, y_1, z_1) in the actual d.c's form are

$$\frac{x-x_1}{\frac{px_1}{a^2}} = \frac{y-y_1}{\frac{py_1}{b^2}} = \frac{z-z_1}{\frac{pz_1}{c^2}}$$

where p = length of $\perp^r$ distance from the centre $(0,0,0)$ to the tangent plane of the ellipsoid.

→ The normal at a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the principal planes in G_1, G_2, G_3

(i) Show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$

(ii) If $PG_1^2+PG_2^2+PG_3^2=K^2$, find the locus of P.

Sol^n: The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Let $P(x_1, y_1, z_1)$ be any point on the surface then the equations of the normal at $P(x_1,y_1,z_1)$ to (1) in actual d.r.s form are

$$\frac{x-x_1}{\frac{p x_1}{a^2}}=\frac{y-y_1}{\frac{p y_1}{b^2}}=\frac{z-z_1}{\frac{p z_1}{c^2}}=r\text{(say)} \quad \text{—— (2)}$$

where 'r' denotes the distance of any point on the normal from $P(x_1,y_1,z_1)$.

Any point on this normal is

$$\left(x_1+\frac{r p x_1}{a^2}, y_1+\frac{r p y_1}{b^2}, z+\frac{r p z_1}{c^2}\right)$$

If it lies on the YZ plane i.e, $x=0$,

then $x_1+\frac{r p x_1}{a^2}=0 \Rightarrow 1+\frac{pr}{a^2}=0$

$\therefore r=\frac{-a^2}{p}$

i.e $\boxed{PG_1=\frac{-a^2}{p}}$

Similarly $\boxed{PG_2=\frac{-b^2}{p}}$ & $\boxed{PG_3=\frac{-c^2}{p}}$

$\therefore$ (i) $PG_1 : PG_2 : PG_3 = \frac{-a^2}{p} : \frac{-b^2}{p} : \frac{-c^2}{p}$

$= a^2 : b^2 : c^2$.

(ii) We are given that, $PG_1^2+PG_2^2+PG_3^2=k^2$

$\Rightarrow \frac{a^4}{p^2}+\frac{b^4}{p^2}+\frac{c^4}{p^2}=k^2$

$\Rightarrow \boxed{\frac{1}{p^2}=\frac{k^2}{a^4+b^4+c^4}}$ —— (3)

But $p = \perp$. distance from $(0,0,0)$ on the tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \text{ at } (x_1,y_1,z_1)$$

$$= \frac{|-1|}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}} \quad \because \text{ to } (1).$$

$$\therefore \frac{1}{p^2} = \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4} = \frac{k^2}{a^4+b^4+c^4} \text{ (say)}$$

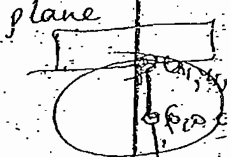
$\therefore$ Locus of $P(x_1,y_1,z_1)$ is [changing (x_1,y_1,z_1) to (x,y,z)]

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} = \frac{k^2}{a^4+b^4+c^4} \quad (4)$$

Also P lies on (1),

Thus P lies on the curve of intersection of two ellipsoides (1) and (4).

→ Find the length of the normal chord through $P(x_1,y_1,z_1)$ of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and prove that if it is equal to $4PG_3$, where G_3 is the point in which the normal chord meets the plane XOY, then P lies on the cone



$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0$$

Soln: The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Equations of the normal at $P(x_1,y_1,z_1)$ to (1) in the actual dic's form are

$$\frac{x-x_1}{\frac{px_1}{a^2}} = \frac{y-y_1}{\frac{py_1}{b^2}} = \frac{z-z_1}{\frac{pz_1}{c^2}} = r \text{ (say)}$$

where $p = \frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}}$ —— (2)

Any point on the normal at a distance 'r' from P is

$$A\left(x_1+\frac{px_1}{a^2}r,\ y_1+\frac{py_1}{b^2}r,\ z_1+\frac{pz_1}{c^2}r\right)$$

If 'r' is the length of the normal chord, then this point must lie on the ellipsoid (1)

$$\therefore \frac{1}{a^2}\left(x_1+\frac{px_1}{a^2}r\right)^2+\frac{1}{b^2}\left(y_1+\frac{py_1}{b^2}r\right)^2+\frac{1}{c^2}\left(z_1+\frac{pz_1}{c^2}r\right)^2$$

$$\Rightarrow r^2p^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2pr\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0 \quad (3)$$

But since $P(x_1,y_1,z_1)$ lies on (1)

$$\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$\therefore$ (3) becomes

$$r^2p^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2pr\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)=0$$

$$\Rightarrow rp\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2\left(\frac{1}{p^2}\right)=0 \ (\because \text{from}$$

$$\Rightarrow r=\frac{-2}{p^3\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)}$$

which is the length of the required normal chord.

Again the normal meets the xy-plane i.e., $z=0$ in

where $z_1+\frac{pz_1}{c^2}r=0$

$$\Rightarrow r=\frac{-c^2}{p}$$

$$\therefore PG_3=\frac{-c^2}{p}$$

Now if length of normal $= 4PG_3$; then

$$\frac{-2}{p^3\left[\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right]} = -4\cdot\frac{c^2}{p}$$

$$\Rightarrow \frac{1}{p^2} = 2c^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)$$

$$\Rightarrow \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4} = 2c^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)$$

($\because$ from ②)

$$\Rightarrow \frac{x_1^2}{a^6}(2c^2-a^2)+\frac{y_1^2}{b^6}(2c^2-b^2)+\frac{z_1^2}{c^4}=0$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0$$

which is the required surface

Example 3. *The normal at a variable point P of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meet the plane XOY in A and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

Find the locus of R if OR is drawn from the centre equal and parallel to AP.

Sol. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(1)}$$

Let $P(x_1, y_1, z_1)$ be the variable point on (1).

Then equations of the normal at $P(x_1, y_1, z_1)$ to (1) in the actual d.c.'s form are

$$\frac{x-x_1}{\dfrac{px_1}{a^2}}=\frac{y-y_1}{\dfrac{py_1}{b^2}}=\frac{z-z_1}{\dfrac{pz_1}{c^2}}=r \text{ (say)}$$

where $$p=\frac{1}{\sqrt{\left(\dfrac{x_1^2}{a^4}+\dfrac{y_1^2}{b^4}+\dfrac{z_1^2}{c^4}\right)}} \qquad \text{...(2)}$$

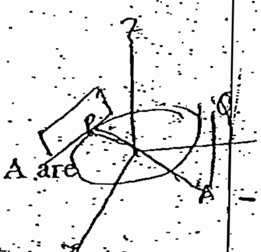
Any point on the normal at a distance r from P is

$$A\left(x_1+\frac{px_1}{a^2}r,\ y_1+\frac{py_1}{b^2}r,\ z_1+\frac{pz_1}{c^2}r\right) \qquad \text{...(3)}$$

The normal meets the XOY plane, *i.e.*, $z=0$, in A where

$$z_1+\frac{pz_1}{c^2}r=0 \quad \text{or} \quad r=\frac{-c^2}{p}$$

$$AP=r=\frac{-c^2}{p}$$



Putting this value of r in (3), the co-ordinates of A are

$$\left(x_1-\frac{c^2x_1}{a^2},\ y_1-\frac{c^2y_1}{b^2},\ 0\right)$$

$\therefore$ Equations of line AQ through A and $\parallel$ to OZ are

$$\frac{x-\left(x_1-\dfrac{c^2x_1}{a^2}\right)}{0}=\frac{y-\left(y_1-\dfrac{c^2y_1}{b^2}\right)}{0}=\frac{z-0}{1}=r(\text{say})$$

where $r=\frac{-c^2}{p}$

If each member $=AQ=AP=\frac{-c^2}{p}$, then the co-ordinates of Q are given by

$$x=x_1-\frac{c^2x_1}{a^2},\quad y=y_1-\frac{c^2y_1}{b^2},\quad z=\frac{-c^2}{p}$$

or $$x=\frac{(a^2-c^2)}{a^2}x_1,\quad y=\frac{b^2-c^2}{b^2}y_1,\quad z=\frac{-c^2}{p} \qquad \ldots(4)$$

The locus of Q is obtained by eliminating (x_1, y_1, z_1) from the equations (4). Now

$$\frac{z^2}{c^2}=\frac{c^2}{p^2}=c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right) \qquad \Big|\ \textit{Using } (2)$$

$$=\frac{c^2x_1^2}{a^4}+\frac{c^2y_1^2}{b^4}+\frac{z_1^2}{c^2}$$

$$=\frac{c^2x_1^2}{a^4}+\frac{c^2y_1^2}{b^4}+\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right) \qquad \Big|\ \because \text{P lies on (1)} \quad \therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$=\frac{(c^2-a^2)x_1^2}{a^4}+\frac{c^2-b^2}{b^4}y_1^2+1$$

$$=\frac{c^2-a^2}{a^4}\left(\frac{a^2x}{a^2-c^2}\right)^2+\frac{c^2-b^2}{b^4}\left(\frac{b^2y}{b^2-c^2}\right)^2+1 \qquad \Big|\ \textit{From } (4)$$

$$=-\frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}+1$$

or $$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1$$

which is the required locus of Q.

Second part. Equations of OR, a line through O(0, 0, 0) and parallel to normal at P, are

$$\frac{x-0}{\frac{px_1}{a^2}}=\frac{y-0}{\frac{py_1}{b^2}}=\frac{z-0}{\frac{pz_1}{c^2}}=AP=-\frac{c^2}{p} \text{ for R.}$$

Then if R be (x, y, z)

$$x=-x_1\frac{c^2}{a^2},\quad y=-y_1\frac{c^2}{b^2},\quad z=-z_1$$

so that $$x_1=\frac{-a^2x}{c^2},\quad y_1=\frac{-b^2y}{c^2},\quad z_1=-z.$$

But (x_1, y_1, z_1) lies on (1) $\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$

or $$\frac{1}{a^2}\cdot\frac{a^4x^2}{c^4}+\frac{1}{b^2}\cdot\frac{b^4y^2}{c^4}+\frac{z^2}{c^2}=1$$

or $$a^2x^2+b^2y^2+c^2z^2=c^4$$

which is the required locus of R.

Example 4. *The normals to an ellipsoid at the points P, P′ meet a principal plane in G, G′; show that the plane which bisects PP′ at right angles, bisects GG′.*

Sol. Let the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ...(1)$$

and let the principal plane be $\quad x=0 \qquad ...(2)$

Let the points P, P′ be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then since P, P′ lie on the ellipsoid (1),

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1$$

Subtracting, $\quad \frac{1}{a^2}(x_1^2-x_2^2)+\frac{1}{b^2}(y_1^2-y_2^2)+\frac{1}{c^2}(z_1^2-z_2^2)=0$

$$...(3)$$

The normal at $P(x_1, y_1, z_1)$ to (1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$

This meets the plane $x=0$, where

$$\frac{0-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$

or $$-a^2=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$

$$\therefore \quad y=y_1-y_1\frac{a^2}{b^2}, \qquad z=z_1-z_1\frac{a^2}{c^2}.$$

Thus the point G is $\left(0, y_1-\frac{a^2}{b^2}y_1, z_1-\frac{a^2}{c^2}z_1\right)$

Similarly G′ is $\left(0, y_2-\frac{a^2}{b^2}y_2, z_2-\frac{a^2}{c^2}z_2\right)$

The mid-point of GG′ is

$$G_1\left[0, \frac{y_1+y_2}{2}-\frac{a^2}{b^2}\left(\frac{y_1+y_2}{2}\right), \frac{z_1+z_2}{2}-\frac{a^2}{c^2}\left(\frac{z_1+z_2}{2}\right)\right]$$

i.e., $$G_1\left[0, \frac{y_1+y_2}{2}\left(1-\frac{a^2}{b^2}\right), \frac{z_1+z_2}{2}\left(1-\frac{a^2}{c^2}\right)\right]$$

Now mid-point of PP′ is $M\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right]$

and the d.c.'s of PP′ are proportional to $x_1-x_2, y_1-y_2, z_1-z_2$

$\therefore$ Equation of the plane through M, the mid-point of PP′ and $\perp$ to PP′ is

$$(x_1-x_2)\left(x-\frac{x_1+x_2}{2}\right)+(y_1-y_2)\left(y-\frac{y_1+y_2}{2}\right)+(z_1-z_2)\left(z-\frac{z_1+z_2}{2}\right)=0.$$

This passes through G_1 if

$$(x_1-x_2)\left(0-\frac{x_1+x_2}{2}\right)+(y_1-y_2)\left[\frac{y_1+y_2}{2}\left(1-\frac{a^2}{b^2}\right)-\frac{y_1+y_2}{2}\right]$$
$$+(z_1-z_2)\left[\frac{z_1+z_2}{2}\left(1-\frac{a^2}{c^2}\right)-\frac{z_1+z_2}{2}\right]=0$$

or if $$-\frac{1}{2}(x_1^2-x_2^2)-\frac{a^2}{2b^2}(y_1^2-y_2^2)-\frac{a^2}{2c^2}(z_1^2-z_2^2)=0$$

or if $$\frac{1}{a^2}(x_1^2-x_2^2)+\frac{1}{b^2}(y_1^2-y_2^2)+\frac{1}{c^2}(z_1^2-z_2^2)=0$$

which is true by (3). Hence the result.

Example 5. *The normals at P and P′, points of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meet the plane $z=0$ in G_3 and G_3' and make angles θ and θ' with PP′. Show that*

$$PG_3\cdot\cos\theta+P'G_3'\cos\theta'=0.$$

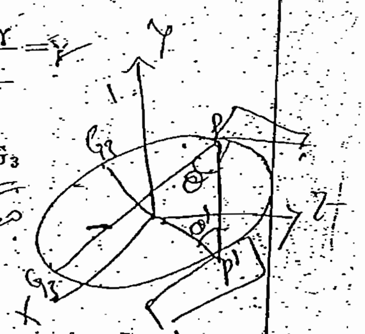
Sol. Let $P\to(\alpha,\beta,\gamma)$ and
$P'\to(\alpha',\beta',\gamma')$

Equations of normal at P are

$$\frac{x-\alpha}{\frac{p\alpha}{a^2}}=\frac{y-\beta}{\frac{p\beta}{b^2}}=\frac{z-\gamma}{\frac{p\gamma}{c^2}}=\Upsilon\ \text{(say)}$$

It meets the plane $z=0$ where

$$\frac{x-\alpha}{\frac{p\alpha}{a^2}}=\frac{y-\beta}{\frac{p\beta}{b^2}}=\frac{0-\gamma}{\frac{p\gamma}{c^2}}=\Upsilon$$



$$\Rightarrow\quad \Upsilon=-\frac{c^2}{p}=PG_3$$

$z=0$

Similarly

$$P'G_3'=-\frac{c^2}{p'}$$

D.C.'s of normal at P are

$$\frac{p\alpha}{a^2},\ \frac{p\beta}{b^2},\ \frac{p\gamma}{c^2}$$

D.C.'s of normal at P′ are

$$\frac{p'\alpha'}{a^2},\ \frac{p'\beta'}{b^2},\ \frac{p'\gamma'}{c^2}$$

D.R.'s of PP$'$ are $\alpha'-\alpha,\ \beta'-\beta,\ \gamma'-\gamma$

$\therefore$ D.C.'s of PP$'$ are

$$\frac{\alpha'-\alpha}{PP'},\ \frac{\beta'-\beta}{PP'},\ \frac{\gamma'-\gamma}{PP'}$$

Since θ is the angle between the normal at P and the line PP$'$, we have

$$\cos\theta=\frac{p\alpha}{a^2}\cdot\frac{\alpha'-\alpha}{PP'}+\frac{p\beta}{b^2}\cdot\frac{\beta'-\beta}{PP'}+\frac{p\gamma}{c^2}\cdot\frac{\gamma'-\gamma}{PP'}$$

$$\therefore\ PG_3\cos\theta=-\frac{c^2}{p}\cdot\frac{p}{PP'}\left[\frac{\alpha(\alpha'-\alpha)}{a^2}+\frac{\beta(\beta'-\beta)}{b^2}+\frac{\gamma(\gamma'-\gamma)}{c^2}\right]$$

$$=-\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)\right]$$

$$=-\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}-1\right]$$

$\because$ P(α, β, γ) lies on the given ellipsoid

$$\therefore\ \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1$$

Similarly $P'G_3'\cos\theta'$

$$=-\frac{c^2}{PP'}\left[1-\left(\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}\right)\right]$$

$$=\frac{c^2}{PP'}\left[\left(\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}\right)-1\right]$$

$$=-PG_3\cos\theta$$

$\Rightarrow\ P'G_3'\cos\theta'+PG_3\cos\theta=0.$

Example 6. *Prove that two normals to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie in the plane

$$lx+my+nz=0$$

and the line joining their feet has direction cosines proportional to

$$a^2(b^2-c^2)mn,\ b^2(c^2-a^2)nl,\ c^2(a^2-b^2)lm.$$

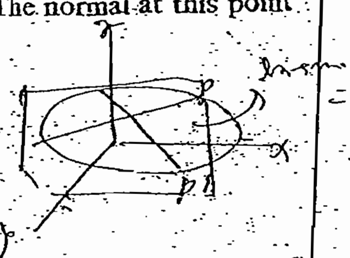
Also obtain the co-ordinates of these points.

Sol: The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

Let P (x_1, y_1, z_1) be any point on (1). The normal at this point P is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$



It lies on the plane $\quad lx+my+nz=0$

if $\quad lx_1+my_1+nz_1=0 \quad \ldots(2)$

and $\quad l\left(\frac{x_1}{a^2}\right)+m\left(\frac{y_1}{b^2}\right)+n\left(\frac{z_1}{c^2}\right)=0 \quad \ldots(3)$

Solving (2) and (3) by cross-multiplication,

$$\frac{x_1}{\frac{mn}{c^2}-\frac{mn}{b^2}}=\frac{y_1}{\frac{nl}{a^2}-\frac{nl}{c^2}}=\frac{z_1}{\frac{lm}{b^2}-\frac{lm}{a^2}}$$

or $$\frac{x_1}{mna^2(c^2-b^2)}=\frac{y_1}{nlb^2(a^2-c^2)}=\frac{z_1}{lmc^2(b^2-a^2)}$$

or $$\frac{\frac{x_1}{a}}{mna(b^2-c^2)}=\frac{\frac{y_1}{b}}{nlb(c^2-a^2)}=\frac{\frac{z_1}{c}}{lmc(a^2-b^2)}$$

$$=\frac{\sqrt{\Sigma\frac{x_1^2}{a^2}}}{\sqrt{\Sigma m^2n^2a^2(b^2-c^2)^2}}=\frac{\pm 1}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}$$

$=\pm\frac{1}{d}$ (say) $\quad \Big|\ \because\ P(x_1, y_1, z_1)$ lies on (1), $\therefore\ \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$

where $\quad d=\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}$

$\therefore$ The required two points are

$$\left[\frac{a^2mn(b^2-c^2)}{d}, \frac{b^2nl(c^2-a^2)}{d}, \frac{c^2lm(a^2-b^2)}{d}\right]$$

and $$\left[-\frac{a^2mn(b^2-c^2)}{d}, -\frac{b^2nl(c^2-a^2)}{d}, -\frac{c^2lm(a^2-b^2)}{d}\right]$$

| *On taking —ve sign*

The d.c.'s of the line joining these two points are proportional

to $\quad -\frac{a^2mn(b^2-c^2)}{d}-\frac{a^2mn(b^2-c^2)}{d}, \ldots\ldots, \ldots\ldots$

| *Using* $x_2-x_1, y_2-y_1, z_2-z_1$

i.e., $\quad a^2mn(b^2-c^2), b^2nl(c^2-a^2), c^2lm(a^2-b^2)$

Hence the result.

Number of normals from a given point

(I) To prove that there are six points on an ellipsoid the normals at which pass through a given point (α, β, γ).

Sol^n: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ — (1)

Equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}=\lambda$$

If it passes through (α, β, γ) then

$$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda \text{ (say)}$$

From first and last members, we have

$$\alpha-x_1=\frac{\lambda x_1}{a^2}$$

$$\Rightarrow \alpha = x_1\left(1+\frac{\lambda}{a^2}\right)$$

$$= x_1\left(\frac{a^2+\lambda}{a^2}\right)$$

$$\Rightarrow x_1=\frac{\alpha a^2}{a^2+\lambda}$$

Similarly, $y_1=\frac{\beta b^2}{b^2+\lambda}$; $z_1=\frac{\gamma c^2}{c^2+\lambda}$ — (2)

Since (x_1, y_1, z_1) lies on (1), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$\Rightarrow \frac{1}{a^2}\left[\frac{a^2\alpha}{a^2+\lambda}\right]^2+\frac{1}{b^2}\left[\frac{b^2\beta}{b^2+\lambda}\right]^2+\frac{1}{c^2}\left[\frac{c^2\gamma}{c^2+\lambda}\right]^2=1. \quad (\because \text{from (}$$

$$\Rightarrow \frac{a^2\alpha^2}{(a^2+\lambda)^2} + \frac{b^2\beta^2}{(b^2+\lambda)^2} + \frac{c^2\gamma^2}{(c^2+\lambda)^2} = 1$$

$$\Rightarrow a^2\alpha^2(a^2+\lambda)^2(c^2+\lambda)^2 + b^2\beta^2(a^2+\lambda)^2(c^2+\lambda)^2 + c^2\gamma^2(a^2+\lambda)^2(b^2+\lambda)^2 = (a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2$$

which, being an equation of the sixth degree, gives six values of λ, to each of which there corresponds a point (α, β, γ) as obtained from (2).

$\therefore$ There are six points on a central quadric (i.e., ellipsoid) the normals at which pass through a given point.

i.e., through a given point, six normals, in general can be drawn to a ~~central~~ quadric.

Note: Foot of normal:

From (2) $\left(\frac{a^2\alpha}{a^2+\lambda}, \frac{b^2\beta}{b^2+\lambda}, \frac{c^2\gamma}{c^2+\lambda}\right)$ are the co-ordinates of the foot of normal.

Cubic curve through the feet of six normals from a point:

$\rightarrow$ To show that the feet of the normals from (α, β, γ) to the ellipsoid are the six points of intersection of the ellipsoid and a certain cubic curve.

Sol: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——— (1)

If the normal at (x_1, y_1, z_1) to the ellipsoid

→ Show that in general six normals can be drawn from a given point (f, g, h) to the conicoid $ax^2+by^2+cz^2=1$. prove also that the six feet of the normals from (f, g, h) to the conicoid are the intersections of the conicoid with a cubic curve.

Quadric cone through six concurrent normals:

To show that, the six normals from (α, β, γ) to the ellipsoid lie on a cone of second degree

Soln: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. —— ①

Now since the normal at (x_1, y_1, z_1) passes through (α, β, γ) we have

$$x_1=\frac{a^2\alpha}{a^2+\lambda},\ y_1=\frac{b^2\beta}{b^2+\lambda},\ z_1=\frac{c^2\gamma}{c^2+\lambda}$$ —②

(from ① equation ②)

Let the equations of the normal from (α, β, γ) to the ellipsoid be —③

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

But w.k.t the equations of the normal form at (x_1, y_1, z_1) in the actual d.c's are

$$\frac{x-x_1}{\frac{px_1}{a^2}}=\frac{y-y_1}{\frac{py_1}{b^2}}=\frac{z-z_1}{\frac{pz_1}{c^2}}$$

(from pg-1 back) Note condn

where p = length of $\perp^r$ distance from the centre $(0,0,0)$ to the tangent plane of ① passes through the given point (α, β, γ),

then $x_1 = \frac{a^2\alpha}{a^2+\lambda}$, $y_1 = \frac{b^2\beta}{b^2+\lambda}$, $z_1 = \frac{c^2\gamma}{c^2+\lambda}$ (from above equ)

The feet of the normals (x_1, y_1, z_1) lie on the curve (changing (x_1, y_1, z_1) to (x, y, z)).

$$x = \frac{a^2\alpha}{a^2+\lambda}, \quad y = \frac{b^2\beta}{b^2+\lambda}, \quad z = \frac{c^2\gamma}{c^2+\lambda}$$

where λ is a parameter. —— (2)

To prove ~~that the~~ curve (2) is a cubic curve.

To test the degree of curve we see its intersection with any arbitrary plane.

The curve (2) meets an arbitrary plane

$$ux + vy + cz + d = 0 \quad \text{—— (3)}$$

$$\Rightarrow u\frac{a^2\alpha}{a^2+\lambda} + v\frac{b^2\beta}{b^2+\lambda} + w\frac{c^2\gamma}{c^2+\lambda} + d = 0$$

$$\Rightarrow ua^2\alpha(b^2+\lambda)(c^2+\lambda) + vb^2\beta(a^2+\lambda)(c^2+\lambda) + wc^2\gamma(a^2+\lambda)(b^2+\lambda) + d(a^2+\lambda)(b^2+\lambda)(c^2+\lambda) = 0$$

which is a cubic in λ, giving three values of λ.

Thus the curve (2) is cubic curve.

Since feet of the normals also lie on the ellipsoid (1), we can conclude that feet of the six normals from a given point are the six points of intersection of the ellipsoid and a cubic curve.

Then $l = \frac{p x_1}{a^2}$

$= \frac{p}{a^2} \cdot \frac{a^2 \alpha}{a^2+\lambda}$ (from ②)

$= \frac{p\alpha}{a^2+\lambda}$.

$\Rightarrow a^2+\lambda = \frac{p\alpha}{l}$ —— ④

Similarly $b^2+\lambda = \frac{p\beta}{m}$ —— ⑤ ; $c^2+\lambda = \frac{p\gamma}{n}$ —— ⑥

multiplying ④, ⑤ & ⑥ by b^2-c^2, c^2-a^2, a^2-b^2

and adding,

we get

$(b^2-c^2)(a^2+\lambda)+(c^2-a^2)(b^2+\lambda)+(a^2-b^2)(c^2+\lambda)$
$= (b^2-c^2)\frac{p\alpha}{l} + (c^2-a^2)\frac{p\beta}{m} + (a^2-b^2)\frac{p\gamma}{n}$

$\Rightarrow 0 + \lambda(0) = \frac{p\alpha}{l}(b^2-c^2) + (c^2-a^2)\frac{p\beta}{m} + (a^2-b^2)\frac{p\gamma}{n}$

$\Rightarrow \frac{\alpha}{l}(b^2-c^2) + \frac{\beta}{m}(c^2-a^2) + \frac{\gamma}{n}(a^2-b^2) = 0$ —— ⑦

Eliminating l, m, n from ③ and ⑦,

the locus of the normals ③ is

$$\frac{\alpha(b^2-c^2)}{x-\alpha} + \frac{\beta(c^2-a^2)}{y-\beta} + \frac{\gamma(a^2-b^2)}{z-\gamma} = 0.$$

$$\Rightarrow \alpha(b^2-c^2)(y-\beta)(z-\gamma) + \beta(c^2-a^2)(x-\alpha)(z-\gamma) + \gamma(a^2-b^2)(x-\alpha)(y-\beta) = 0$$

which is a cone of second degree.

Hence the result.

1983 Prove that the feet of the six normals from (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the curve of intersection of the ellipsoid and the cone $\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$

Solⁿ: The ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Equations ~~of~~ at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}$$

If it passes through (α, β, γ) then

$$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda \text{ (say)}$$

Then six feet of the normals from (α, β, γ) are given by

$$x_1=\frac{a^2\alpha}{a^2+\lambda}$$

$$y_1=\frac{b^2\beta}{b^2+\lambda} \text{ and } z_1=\frac{c^2\gamma}{c^2+\lambda}$$

These give

$$a^2+\lambda=\frac{a^2\alpha}{x_1},\quad b^2+\lambda=\frac{b^2\beta}{y_1}$$

$$\text{and } c^2+\lambda=\frac{c^2\gamma}{z_1}$$

Multiplying these equations by b^2-c^2, c^2-a^2, a^2-b^2 and adding, we get | Note this step

$$0+\lambda(0)=\frac{a^2\alpha(b^2-c^2)}{x_1}+\frac{b^2\beta(c^2-a^2)}{y_1}+\frac{c^2\gamma(a^2-b^2)}{z_1}$$

$\therefore$ (x_1, y_1, z_1) *i.e.*, the feet of the normals, lie on the cone

$$\frac{a^2\alpha(b^2-c^2)}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

Also the feet (x_1, y_1, z_1) of the normals lie on the ellipsoid (1). Thus the feet of the six normals lie on the curve of intersection of the ellipsoid and the above cone.

Note. *In the equation of the cone through the feet of six normals from a point to an ellipsoid.*

Co-eff. of $x^2=0$; co-eff. of $y^2=0$; co-eff. of $z^2=0$,

constant term $=0$. [Remember]

Example 2. *If P, Q, R; P', Q', R' are the feet of six normals from a point to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the plane PQR is given by

$$lx+my+nz=p;$$

then the plane $P'Q'R'$ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0. \quad \text{(K.U. 1984)}$$

[Imp.]

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \ldots(1)$$

and that the plane PQR is $lx+my+nz-p=0$ ...(2)

Let the required equation of plane $P'Q'R'$ be

$$l'x+m'y+n'z-p'=0 \quad \ldots(3)$$

The joint equation of the planes PQR and $P'Q'R'$ is

$$(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \quad \ldots(4)$$

The eqution of conicoid through the points of intersection of the ellipsoid (1) and pair of planes (4) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+k(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \quad \ldots(5)$$

If it is the same as the equation of the cone through the feet P, Q, R; P′, Q′, R′ of the six normals from the given point to the ellipsoid, then

$$Co\text{-}eff.\ of\ x^2=0 \quad i.e., \quad \frac{1}{a^2}+kll'=0 \quad \text{or} \quad l'=-\frac{1}{kla^2}$$

$$Co\text{-}eff.\ of\ y^2=0 \quad i.e., \quad \frac{1}{b^2}+kmm'=0 \quad \therefore \quad m'=-\frac{1}{kmb^2}$$

$$Co\text{-}eff.\ of\ z^2=0 \quad i.e., \quad \frac{1}{c^2}+knn'=0 \quad \therefore \quad n'=-\frac{1}{knc^2}$$

$$Constant\ term=0 \quad i.e., \quad -1+kpp'=0 \quad \therefore \quad p'=\frac{1}{kp}$$

Putting these values of l', m', n', p' in (3), the required plane P′Q′R′ is

$$-\frac{x}{kla^2}-\frac{y}{kmb^2}-\frac{z}{knc^2}-\frac{1}{kp}=0$$

or
$$\frac{x}{la^2}+\frac{y}{mb^2}+\frac{z}{nc^2}+\frac{1}{p}=0.$$

Hence the result.

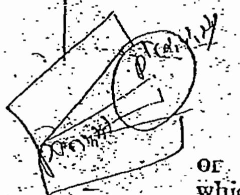
Article 14. Plane of Contact.

To find the equation of plane of contact of the point (x_1, y_1, z_1) with respect to conicoid $ax^2+by^2+cz^2=1$.

Let (x', y', z') be the point of contact any tangent plane to the conicoid $\qquad ax^2+by^2+cz^2=1 \qquad \ldots(1)$

$\therefore$ Tangent plane at (x', y', z') to (1) is

$$axx'+byy'+czz'=1$$



If it passes through the given point (x_1, y_1, z_1), then

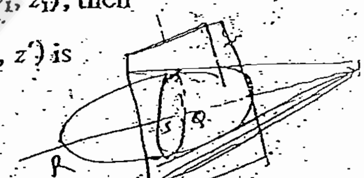
$$ax_1x'+by_1y'+cz_1z'=1$$

$\therefore$ Locus of the points of contact (x', y', z') is

$$ax_1x+by_1y+cz_1z=1$$

or
$$axx_1+byy_1+czz_1=1$$

which is the required plane of contact.



Article 15. Polar plane of a point.

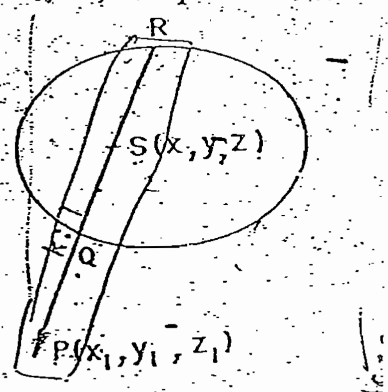
To find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the central conicoid $ax^2+by^2+cz^2=1$.

(M.D.U. 1985; K.U. 1986, 85; Manipur 1983)

The given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \ldots(1)$$

Let $P(x_1, y_1, z_1)$ be the given point and let PQR be any line through P which meets (1) in Q and R.



Let $S(x, y, z)$ be the harmonic conjugate of P w.r.t. Q and R.

Let Q divide PS in the ratio $k:1$.

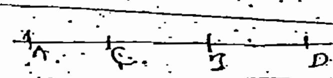
Then co-ordinates of Q are

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right)$$

Also the points C & D are called harmonic conjugates w.r.t. A and B. Thus C is harmonic conjugate of D and vice versa.

Harmonic Division:

If the line AB is divided internally & externally in the same ratio at C & D then C and D are said to divide harmonically.



Since Q lies on the conicoid (1),

$$\therefore \quad a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

or $$a(kx+x_1)^2+b(ky+y_1)^2+c(kz+z_1)^2-(k+1)^2=0$$

or $$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \ldots(2)$$

which is a quadratic equation in k.

Since PS is divided harmonically, *i.e.*, internally and externally in the same ratio at Q and R, $\therefore$ the quadratic (2) has equal and opposite roots.

$\therefore$ Sum of roots $=0$ *i.e.*, coeff. of $k=0$

or $$axx_1+byy_1+czz_1-1=0$$

or $$axx_1+byy_1+czz_1=1 \qquad | \text{ C.T.M.}$$

which is the equation of required polar plane of P.

Cor. *If P lies on the conicoid, the polar plane at P becomes the tangent plane at P.*

Article 16. Pole of a given plane.

To find the pole of the plane $lx+my+nz=p$, w.r.t. the conicoid $ax^2+by^2+cz^2=1$.

Let (x_1, y_1, z_1) be the required pole.

Then the polar plane of (x_1, y_1, z_1) w.r.t. conicoid

$$ax^2+by^2+cz^2=1$$

i.e., $$axx_1+byy_1+czz_1=1 \quad \ldots(1)$$

must be identical with the given plane

$$lx+my+nz=p \quad \ldots(2)$$

Comparing (1) and (2), we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{1}{p}$$

$$x_1=\frac{l}{ap},\ y_1=\frac{m}{bp},\ z_1=\frac{n}{cp}$$

Thus the pole is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

Example. *Prove that the locus of the poles of the tangent planes to $a^2x^2+b^2y^2-c^2z^2=1$ with respect to $\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1$ is the hyperboloid of one sheet. Find its equation.*

Sol. Let $lx+my+nz=p$ ...(1) be a tangent plane

to $a^2x^2+b^2y^2-c^2z^2=1$ ...(2)

$$\therefore \frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{-c^2}=p^2 \ldots(3) \quad \Big| \text{ Using } \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

Condition of tangency

Let (x_1, y_1, z_1) be the pole of plane (1) w.r.t.

$$\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1 \quad ...(4)$$

$\therefore$ Equation of polar plane of (x_1, y_1, z_1) w.r.t. (4) is

$$\alpha^2xx_1+\beta^2yy_1+\gamma^2zz_1=1 \quad ...(5)$$

Comparing (1) and (5),

$$\frac{\alpha^2x_1}{l}=\frac{\beta^2y_1}{m}=\frac{\gamma^2z_1}{n}=\frac{1}{p}$$

From first and fourth members

$l=\alpha^2px_1$, similarly $m=\beta^2py_1$ and $n=\gamma^2pz_1$

Putting these values of l, m, n in (3) [To eliminate l, m, n], we have

$$\frac{\alpha^4p^2x_1^2}{a^2}+\frac{\beta^4p^2y_1^2}{b^2}+\frac{\gamma^4p^2z_1^2}{-c^2}=p^2$$

Cancelling p^2, $\dfrac{\alpha^4x_1^2}{a^2}+\dfrac{\beta^4y_1^2}{b^2}-\dfrac{\gamma^4z_1^2}{c^2}=1$

$\therefore$ Locus of (x_1, y_1, z_1) [pole of (1) *w.r.t.* (4)] is

$$\frac{\alpha^4}{a^2}x^2+\frac{\beta^4}{b^2}y^2-\frac{\gamma^4}{c^2}z^2=1$$

which is a hyperboloid of one sheet. [$\because$ co-effs. of x^2 and y^2 are positive, but co-eff. of z^2 is negative]

Article 17. Conjugate points and conjugate planes.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let the conicoid be $ax^2+by^2+cz^2=1$. ...(1)

Then polar plane of (x_1, y_1, z_1) *w.r.t.* (1) is

$$axx_1+byy_1+czz_1=1$$

If it passes through $Q(x_2, y_2, z_2)$, then

$$ax_1x_2+by_1y_2+cz_1z_2=1$$

The symmetry of this result shows that the polar plane of Q also passes through P.

The two points such that the polar plane of each passes through the other are called the conjugate points.

Similarly it can be easily shown that if the pole of a plane S_1 lies on another plane S_2, then pole of S_2 must lie on S_1. *Two such planes (as S_2 and S_1 here) are called conjugate planes.*

Article 18. Polar lines

Two lines such that the polar plane of any point on one line passes through the other line are called conjugate lines *or* polar lines.

Polar of a line.

To find the equations of the polar of the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ w.r.t. the conicoid } ax^2+by^2+cz^2=1.$$

(*K.U.* 1986)

The given conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and the given line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(2)

Any point on line (2) is $(lr+x_1, mr+y_1, nr+z_1)$.

It's polar plane w.r.t. conicoid (1) is

$$ax(lr+x_1)+by(mr+y_1)+cz(nr+z_1)=1$$

or $$axx_1+byy_1+czz_1-1+r(alx+bmy+cnz)=0.$$

This passes through the line

$$\left.\begin{array}{r} axx_1+byy_1+czz_1-1=0 \\ alx+bmy+cnz=0 \end{array}\right\}$$

for all values of r.

Hence the equations of the polar line of (2) are

$$axx_1+byy_1+czz_1=1; \quad alx+bmy+cnz=0.$$

Method to write down the polar of

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

w.r.t. a central conicoid (equation in the standard form).

1. *Write down the polar plane of (x_1, y_1, z_1) w.r.t. conicoid thus getting* $axx_1+byy_1+czz_1=1$.

2. *Write down the polar plane of (l, m, n) and omit the constant term thus getting* $alx+bmy+cnz=0$.

3. *The above two equations are the required equations of the polar.*

Example 1. *Show that the equations of the polar of the line*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

w.r.t. quadric $x^2-2y^2+3z^2=4$ are $\frac{x+6}{3}=\frac{y-2}{3}=z-2$. (Kanpur 58)

Sol. The given line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ ...(1)

and the conicoid is $x^2-2y^2+3z^2=4$. ...(2)

$\therefore$ Any point on (1) is $(2r+1, 3r+2, 4r+3)$.

Polar plane of this point w.r.t. (2) is

$$x(2r+1)-2y(3r+2)+3z(4r+3)=4$$

or $$x+2rx-6yr-4y+12rz+9z-4=0$$

or $$(x-4y+9z-4)+2r(x-3y+6z)=0$$

which passes through the line

$$\left.\begin{array}{r} x-4y+9z-4=0 \\ x-3y+6z=0 \end{array}\right] \begin{array}{l} ...(3) \text{ for all values of } r \\ ...(4) \end{array}$$

$\therefore$ Equations (3) and (4) are the equations of the polar line of (1) w.r.t. conicoid (2).

To reduce the line given by (3) and (4) in symmetrical form.

To find d.r.'s of this line [omitting constant terms in (3) and (4)], we get

$$x-4y+9z=0$$
$$x-3y+6z=0$$

$$\therefore \quad \frac{x}{-24+27}=\frac{y}{9-6}=\frac{z}{-3+4} \text{ or } \frac{x}{3}=\frac{y}{3}=\frac{z}{1}.$$

Thus the d.r.'s of polar line are 3, 3, 1.

For any point put $z=2$ in (3) and (4).

(*As suggested by the question*)

$$\therefore \quad x-4y+14=0 \text{ and } x-3y+12=0$$

Solving, we have $x=-6$, $y=2$. Also $z=2$.

Hence one point on the polar is $(-6, 2, 2)$.

Thus the equations of the polar line in the symmetrical form are

$$\frac{x+6}{3}=\frac{y-2}{3}=\frac{z-2}{1}.$$

Hence the result.

Example 2. *Find the condition that the lines*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ and } \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

Sol. The polar of the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ w.r.t. the conicoid $ax^2+by^2+cz^2=1$ is given by

$$a\alpha x+b\beta y+c\gamma z-1=0, \; alx+bmy+cnz=0 \quad \ldots(1)$$

[*See Article 18 above*]

But $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \quad \ldots(2)$$

is given to be polar. Hence (2) should be identical with (1), *i.e.*, line (2) should lie on both the planes given by (1).

For this the point $(\alpha', \beta', \gamma')$ should lie on both the planes and the line (2) should be $\perp$ to the normal of each of the planes in (1).

$\therefore$ The required conditions are

$$a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'=1$$
$$a\alpha l'+b\beta m'+c\gamma n'=0$$
$$a\alpha' l+b\beta' m+c\gamma' n=0$$
$$all'+bmm'+cnn'=0.$$

Example 3. *Find the locus of straight lines drawn through a fixed point* (α, β, γ) *at right angles to their polars with respect to*

$$ax^2+by^2+cz^2=1.$$

(M.D.U. 1984; Kanpur 1982, 88; Lucknow 1980)

Sol. Any line through (α, β, γ) is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(1)

The polar of (1) w.r.t. the given conicoid is

$$\left.\begin{array}{l} a\alpha x+b\beta y+c\gamma z=1 \\ alx+bmy+cnz=0 \end{array}\right\} \qquad ...(2)$$

Omitting the constant terms in (2), the d.c.'s of line (2) are, given by

$$a\alpha x+b\beta y+c\gamma z=0$$
$$alx+bmy+cnz=0.$$

$$\therefore \quad \frac{x}{bc(n\beta-m\gamma)}=\frac{y}{ca(l\gamma-n\alpha)}=\frac{z}{ab(m\alpha-l\beta)}$$

Thus the d.c.'s of line (2) are proportional to

$$bc(n\beta-m\gamma),\ ca(l\gamma-n\alpha),\ ab(m\alpha-l\beta).$$

The lines (1) and (2) are $\perp$ | given

$$\therefore \quad lbc(n\beta-m\gamma)+mca(l\gamma-n\alpha)+nab(m\alpha-l\beta)=0$$

or
$$\alpha mna(b-c)+\beta nlb(c-a)+\gamma lmc(a-b)=0$$

or
$$\Sigma\frac{\alpha}{l}\left(\frac{1}{c}-\frac{1}{b}\right)=0 \qquad ...(3)$$

on dividing by $lmn\,abc$.

To find the locus of (1), eliminating l, m, n from (1) and (3), we have

$$\Sigma\left(\frac{\alpha}{x-\alpha}\right)\left(\frac{1}{c}-\frac{1}{b}\right)=0 \quad \text{or} \quad \Sigma\left(\frac{\alpha}{x-\alpha}\right)\left(\frac{1}{b}-\frac{1}{c}\right)=0.$$

Example 4. *If P, Q are the points, (x_1, y_1, z_1) (x_2, y_2, z_2), the polar of PQ w.r.t. $ax^2+by^2+cz^2=1$ is given by*

$$axx_1+byy_1+czz_1=1,\ axx_2+byy_2+czz_2=1.$$

Sol. Equations of line PQ are $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$

The polar of this line w.r.t. the given conicoid is

$$axx_1+byy_1+czz_1=1 \qquad ...(1)$$

and
$$ax(x_2-x_1)+by(y_2-y_1)+cz(z_2-z_1)=0 \qquad ...(2)$$

Adding (1) and (2), we have $axx_2+byy_2+czz_2=0$ ...(3)

Hence (1) and (3) are the required ~~equations~~ of of the polar.

Observations. *It is clear from the equations (1) and (3) that the polar of PQ is the line of intersection of polar planes of P and Q.*

Example 5. *Find the polar plane of the point $(2, -3, 4)$ with respect to the conicoid $x^2+2y^2+z_2=4$.* (Bundelkhand 1984)

Sol. Required polar plane is

$$x(2)+2y(-3)+z(4)=4$$

or
$$x-3y+2z=2.$$

Example 6. *Find the locus of straight line through a fixed point (α, β, γ) whose polar lines with respect to the quadratics $ax^2+by^2+cz^2=1$ and $a'x^2+b'y^2+c'z^2=1$ are coplanar.*

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=\gamma \text{ (say)} \quad \ldots(i)$$

The equations of the polar line of (i) w.r.t. $ax^2+by^2+cz^2=1$ are

$$a\alpha x+b\beta y+c\gamma z=1 \quad \ldots(ii)$$

$$alx+bmy+cnz=0 \quad \ldots(iii)$$

And the equations of polar line of (i) w.r.t. $a'x^2+b'y^2+c'z^2=1$ are

$$a'\alpha x+b'\beta y+c'\gamma z=1 \quad \ldots(iv)$$

$$a'lx+b'my+c'nz=0 \quad \ldots(v)$$

From (ii) and (iv), we have

$$(a-a')\alpha x+(b-b')\beta y+(c-c')\gamma z=0 \quad \ldots(vi)$$

From (iii) and (v), solving simultaneously, we have

$$\frac{lx}{(bc'-b'c)}=\frac{my}{(ca'-c'a)}=\frac{nz}{(ab'-a'b)} \quad \ldots(vii)$$

Eliminating x, y, z between (vi) and (vii), we get

$$\frac{(a-a')\alpha(bc-b'c)}{l}+\frac{(b-b')\beta(ca'-c'a)}{m}+\frac{(c-c')\gamma(ab'-a'b)}{n}=0 \quad \ldots(viii)$$

Eliminating l, m, n between (i) and (viii), we get the locus of the line as

$$\sum \frac{(a-a')\alpha(bc'-b'c)}{(x-\alpha)}=0.$$

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Example 7. *Prove that the locus of the poles of the tangent planes of $ax^2+by^2+cz^2=1$ with respect to $a'x^2+b'y^2+c'z^2=1$ is the conicoid* $\frac{(a'x)^2}{a}+\frac{(b'y)^2}{b}+\frac{(c'z)^2}{c}=1.$ (Allahabad 1982 ; Kanpur 1986)

Sol. Let $lx+my+nz=p$ ...(i)

be the tangent plane to the conicoid

$$ax^2+by^2+cz^2=1 \quad \ldots(ii)$$

Then $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2 \quad \ldots(iii)$$

Let (α, β, γ) be the pole of the plane (i) w.r. to

$$a'x^2+b'y^2+c'z^2=1. \quad \ldots(iv)$$

Then we have $a'\alpha x+b'\beta y+c'\gamma z=1$

Comparing (i) and (iv), we get

$$\frac{a'\alpha}{l}=\frac{b'\beta}{m}=\frac{c'\gamma}{n}=\frac{1}{p} \quad \ldots(v)$$

Eliminating l, m, n between (iii) and (v), we get

$$\frac{(a'\alpha p)^2}{a}+\frac{(b'\beta p)^2}{b}+\frac{(c'\gamma p)^2}{c}=p^2$$

$$\text{or} \quad \frac{(a'\alpha)^2}{a}+\frac{(b'\beta)^2}{b}+\frac{(c'\gamma)^2}{c}=1$$

$\therefore$ The required locus of (α, β, γ) is

$$\frac{(a'x)^2}{a}+\frac{(b'y)^2}{b}+\frac{(c'z)^2}{c}=1.$$

Example 8. *Show that the locus of the pole of the plane $lx+my+nz=p$ with respect to the system of conicoids $\Sigma[x^2/(a^2+k)]=1$ is a straight line perpendicular to the given plane, where k is a parameter.*

Sol. Let (α, β, γ) be the pole of the plane

$$lx+my+nz=p \qquad \ldots(i)$$

with respect to the conicoid

$$\frac{x^2}{(a^2+k^2)}+\frac{y^2}{(b^2+k)}+\frac{z^2}{(c^2+k)}=1 \qquad \ldots(ii)$$

The polar plane of (α, β, γ) w.r.t. this conicoid is

$$\frac{\alpha x}{(a^2+k)}+\frac{\beta y}{(b^2+k)}+\frac{\gamma z}{(c^2+k)}=1 \qquad \ldots(iii)$$

Since (*i*) and (*iii*) represents the same plane, therefore comparing them, we get

$$\frac{\alpha/(a^2+k)}{l}=\frac{\beta/(b^2+k)}{m}=\frac{\gamma/(c^2+k)}{n}=\frac{1}{p}$$

$$\text{where} \quad \alpha=(a^2+k)\frac{l}{p},\ \beta=(b^2+k)\frac{m}{p},\ \gamma=(c^2+k)\frac{n}{p}$$

$$\text{or} \quad \frac{\alpha-(a^2l/p)}{l}=\frac{k}{p}=\frac{\beta-(b^2m/p)}{m}=\frac{\gamma-(c^2n/p)}{n}$$

$\therefore$ The locus of (α, β, γ) is

$$\frac{x-(a^2l/p)}{l}=\frac{y-(b^2m/p)}{m}=\frac{z-(c^2n/p)}{n}$$

which is a straight line and its direction cosines being l, m, n is perpendicular to the plane (*i*).

Article 19. Enveloping Cone

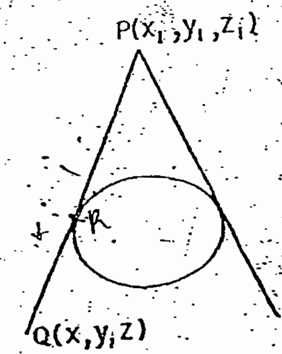
To find the equation of enveloping cone from the point (x_1, y_1, z_1) to the central conicoid $ax^2+by^2+cz^2=1$. *(M.D.U. 1984)*

The given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \ldots(1)$$

Let P be the point (x_1, y_1, z_1).

Let Q (x, y, z) be *any* point on a tangent from P to the conicoid.



The point which divides PQ in the ratio $k:1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

If it lies on (1), then

$$a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

which simplifies to

$$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad ...(2)$$

which is a quadratic in k.

Since the line PQ *touches* the conicoid (1), $\therefore$ (2) must have equal roots.

$$\therefore \quad 4(axx_1+byy_1+czz_1-1)^2$$
$$=4(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1) \quad | \text{ Using } b^2=4ac$$

or $\quad (axx_1+byy_1+czz_1-1)^2$

$$=(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)$$

which is the required equation.

Remember. If $S=0$ is the given surface, then with usual notations the enveloping cone is given by $SS_1=T^2$.

Example 1. *Find the locus of points from which three mutually perpendicular tangents can be drawn to the surface $ax^2+by^2+cz^2=1$.* **[Imp.]**

Sol. Let $P(x_1, y_1, z_1)$ be the point.

Then the three mutually $\perp$ tangents drawn from P will be three mutually $\perp$ generators of the enveloping cone with P as vertex. The equation of the enveloping cone is $SS_1=T^2$

or $\quad (ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)=(axx_1+byy_1+czz_1-1)^2$

Since this cone has three mutually $\perp$ generators,

$\therefore$ Co-eff. of x^2 + coeff. of y^2 + coeff. of $z^2=0$

i.e., $\quad a(by_1^2+cz_1^2-1)+b(ax_1^2+cz_1^2-1)+c(ax_1^2+by_1^2-1)=0$

or $\quad a(b+c)x_1^2+b(c+a)y_1^2+c(a+b)z_1^2=a+b+c$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is [changing (x_1, y_1, z_1) to (x, y, z)]

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c.$$

Example 2. *The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose vertex is P by the plane $z=0$ is (i) a parabola, (ii) a rectangular hyperbola. Find the locus of P.* **[Imp.]**

(M.D.U. 1985)

Sol. Let $P(x_1, y_1, z_1)$ be the vertex of enveloping cone of the ellipsoid

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad ...(1)$$

The enveloping cone of (1) is $SS_1=T^2$

i.e.,
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$$
$$=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or $\dfrac{x^2}{a^2}\left(\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)+\dfrac{y^2}{b^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{z_1^2}{c^2}-1\right)-\dfrac{2x_1y_1}{a^2b^2}xy+\ldots=0$ ...(2)

(1) and (2) represent a parabola in the XY plane if

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right).\ \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=\frac{x_1^2y_1^2}{a^4b^4}$$

[Using $ab=h^2$ if the equation is $ax^2+2hxy+by^2+\ldots=0$

or $\dfrac{1}{a^2b^2}\left(\dfrac{x_1^2y_1^2}{a^2b^2}+\dfrac{y_1^2z_1^2}{b^2c^2}-\dfrac{y_1^2}{b^2}+\dfrac{z_1^2x_1^2}{a^2c^2}+\dfrac{z_1^4}{c^4}-\dfrac{z_1^2}{c^2}-\dfrac{x_1^2}{a^2}-\dfrac{z_1^2}{c^2}+1\right)$
$=\dfrac{x_1^2y_1^2}{a^4b^4}$

or $\left(\dfrac{y_1^2z_1^2}{b^2c^2}+\dfrac{z_1^2x_1^2}{c^2a^2}+\dfrac{z_1^4}{c^4}-\dfrac{z_1^2}{c^2}\right)-\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)=0$

or $\dfrac{z_1^2}{c^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)-\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)=0$

or $\left(\dfrac{z_1^2}{c^2}-1\right)\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)=0$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is $\left(\dfrac{z^2}{c^2}-1\right)\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)=0$

$\therefore$ Either $\dfrac{z^2}{c^2}-1=0$ or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1=0.$

Rejecting the second equation [$\because$ it is the given ellipsoid and P does not lie on it], the locus is $\dfrac{z^2}{c^2}-1=0$ or $z=\pm c$.

(Kanpur 1988)

(*ii*) The equation (2) represents a rectangular hyperbola in the XY plane if co-eff. of x^2 + co-eff. $y^2=0$

i.e., if $\dfrac{1}{a^2}\left(\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)+\dfrac{1}{b^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{z_1^2}{c^2}-1\right)=0$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$\frac{1}{a^2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)=0$$

or $$\frac{x^2}{a^2b^2}+\frac{y^2}{a^2b^2}+\frac{z^2}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0$$

or $$\frac{x^2+y^2}{a^2b^2}+\frac{z^2(a^2+b^2)}{a^2b^2c^2}=\frac{a^2+b^2}{a^2b^2}$$

or $$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1.$$

Example 3. *Find the locus of luminous point if the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ *casts a circular shadow on the plane* $z=0$.

(Kanpur 1988)

Sol. Let $P(x_1, y_1, z_1)$ be the luminous point.

The enveloping cone of the given ellipsoid with vertex at P is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$$

$$=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2 \qquad \Big| \text{ Using } SS_1=T^2$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

This will be a circle if the *co-eff. of* $xy=0$

and *co-eff. of* x^2=*co-eff. of* y^2

i.e., if $\dfrac{x_1y_1}{a^2b^2}=0$ ...(1)

and $\dfrac{1}{a^2}\left(\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)=\dfrac{1}{b^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{z_1^2}{c^2}-1\right)$ ...(2)

From (1) either $x_1=0$ or $y_1=0$.

Case I. If $x_1=0$ from (2), we have

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{z_1^2}{c^2}-1\right)$$

$$\therefore \quad \frac{y_1^2}{a^2b^2}+\frac{z_1^2(b^2-a^2)}{a^2b^2c^2}=\frac{b^2-a^2}{a^2b^2}$$

or $$\frac{y_1^2}{b^2-a^2}+\frac{z_1^2}{c^2}=1.$$

Thus the locus of $P(x_1, y_1, z_1)$ is $x=0, \dfrac{y^2}{b^2-a^2}+\dfrac{z^2}{c^2}=1$ which is an ellipse in the YZ plane.

Case II. If $y_1=0$, (2) gives

$$\frac{1}{a^2}\left(\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$y=0, \frac{1}{a^2}\left(\frac{z^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)$$

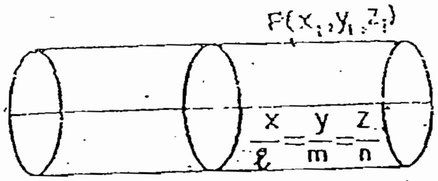
or $$y=0, \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1.$$

Article 20. Enveloping Cylinder

To find the equation of the enveloping cylinder of the central conicoid $ax^2+by^2+cz^2=1$ whose generators are parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$. *(Garhwal 1986)*

The given conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and the given line is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(2)



Let $P(x_1, y_1, z_1)$ be any point on a tangent ∥ to the line (2).

| Note this step

The equations of the tangent line through (x_1, y_1, z_1) and ∥ to (2) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)}.$$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$.

If it lies on (1), then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

Since the line (2) touches the conicoid (1), ∴ (3) has equal roots.

$$\therefore\ 4(alx_1+bmy_1+cnz_1)^2-4(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1)=0 \quad | \text{ Using } b^2-4ac=0$$

or $\quad (alx_1+bmy_1+cnz_1)^2=(al^2+bm^2+cn^2)[ax_1^2+by_1^2+cy_1^2-1]$

∴ Locus of (x_1, y_1, z_1) is

$$(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)$$

which is the required equation of enveloping cylinder.

Method to write down the enveloping cylinder.

If $S\equiv ax^2+by^2+cz^2-1$, so that $S=0$ is the equation of central conicoid then $s_1=al^2+bm^2+cn^2$, *i.e.*, s_1 is obtained by putting (l, m, n) in S and neglecting the constant term.

$t=alx+bmy+cnz$, where t is the expression for the tangent plane at (l, m, n) after omitting the constant term, then the enveloping cylinder is $Ss_1=t^2$. [Remember]

TPS-2003 **Example 1.** *Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the lines $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$, meet the plane $z=0$ in circles.*

Sol. The given ellipsoid is $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ ...(1)

and the given lines are $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ ...(2)

The equation of enveloping cylinder is $Ss_1=t^2$

$$\text{i.e., } \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left[\frac{1}{a^2}(0)^2+\frac{1}{b^2}(\pm\sqrt{a^2-b^2})^2+\frac{1}{c^2}(c)^2\right]$$
$$=\left[\frac{1}{a^2}(0)x+\frac{1}{b^2}(\pm\sqrt{(a^2-b^2)})y+\frac{1}{c^2}\cdot cz\right]^2$$

or $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{a^2-b^2}{b^2}+1\right)=\left(\pm\frac{\sqrt{a^2-b^2}}{b^2}y+\frac{z}{c}\right)^2$

This meets the plane $z=0$ where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2-b^2}{b^2}+1\right)=\left(\frac{\pm\sqrt{a^2-b^2}}{b^2}.y+0\right)^2$$

or
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).\frac{a^2}{b^2}=\frac{(a^2-b^2)y^2}{b^4}$$

or
$$x^2+\frac{a^2}{b^2}y^2-a^2=\frac{(a^2-b^2)y^2}{b^2}=\frac{a^2}{b^2}y^2-y^2$$

or $x^2+y^2=a^2$, which is a circle.

2008 **Example 2.** *Show that the enveloping cylinder of the ellipsoid $ax^2+by^2+cz^2=1$ with generators parallel to Z-axis meet the plane $z=0$ in ellipse.* or in parabolas.

Sol. Please try yourself as above.

[**Hint.** *Remember that an equation in x, y represents a parabola in XY plane if its second degree terms form a perfect square.*]

Article 21. Section with a given centre. [V. Imp.]

To find the locus of chords of the conicoid

$$ax^2+by^2+cz^2=1$$

which are bisected at the given point (x_1, y_1, z_1).

The given conicoid is $\quad ax^2+by^2+cz^2=1 \quad$ ...(1)

Any chord through (x_1, y_1, z_1) is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(2)

Any point on this chord is $(lr+x_1, mr+y_1, nr+z_1)$

If it lies on (1), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

If l, m, n are the actual d.c.'s of line (2), then here r is the distance of any point common to the conicoid (1) and the chord (2) from the given point (x_1, y_1, z_1).

If (x_1, y_1, z_1) is the middle point of chord (2), the points of intersection of (1) and (2) should be equidistant and on either side of (x_1, y_1, z_1), *i.e.*, the two values of r should be equal and opposite or the sum of roots in (3) is zero.

$\therefore$ Co-eff. of $r=0$ giving $alx_1+bmy_1+cnz_1=0$ ...(4)

Eliminating l, m, n from (2) and (4), the locus of chords (2) is

$$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0$$

or
$$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$$

or $\quad ax x_1+by y_1+cz z_1-1=ax_1^2+by_1^2+cz_1^2-1$ ...(5)

which is of the form $\quad T=S_1$. [Remember]

Note. The plane (5) meets the given conicoid in a conic whose centre is (x_1, y_1, z_1).

Article 22. To find the locus of middle points of a system of chords of the conicoid $ax^2+by^2+cz^2=1$ which are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. (M.D.U. 1983)

Any chord through (x_1, y_1, z_1) drawn ∥ to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(1)$$

Any point on this chord is $(lr+x_1, mr+y_1, nr+z_1)$

This lies on the given conicoid $ax^2+by^2+cz^2=1$ if

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(2)

If (x_1, y_1, z_1) is the mid point of (1), then the two values of r in (2) must be equal in magnitude but opposite in sign, *i.e.*, its sum of two roots is zero or the co-eff. of $r=0$

$$\therefore \quad alx_1+bmy_1+cnz_1=0$$

$\therefore$ Locus of (x_1, y_1, z_1) the mid-point is

$$alx+bmy+cnz=0$$

which is a plane through the centre of the conicoid.

Example 1. *Find the equation to the plane which cuts the surface (a) $2x^2+3y^2+5z^2=4$ in a conic whose centre is at the point (1, 2, 3).*

(b) $x^2+4y^2-5z^2=1$ in a conic whose centre is at the point (2, 3, 4).

Sol. (a) The given conicoid is $S\equiv 2x^2+3y^2+5z^2-4=0$. [Make R.H.S.=0]

Here $\quad S_1=2(1)^2+3(2)^2+5(3)^2-4$ | Putting (1, 2, 3) in S

$=2+12+45-4=55$

and $\quad T=2x(1)+3y(2)+5z(3)-4$

$=2x+6y+15z-4.$

The required plane is $T=S_1$, *i.e.*,

$$2x+6y+15z-4=55 \quad \text{or} \quad 2x+6y+15z=59.$$

(b) Please try yourself as above. [Ans. $x+6y-10z+20=0$]

Example 2. *Show that centre of the conic given by*

$$ax^2+by^2+cz^2=1, \; lx+my+nz=p$$

is the point $\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right)$

where $l^2+m^2+n^2=1$ *and* $p_0=\sqrt{\sum \frac{l^2}{a}}$.

Sol. Let (x_1, y_1, z_1) be the centre of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$.

Then the plane with (x_1, y_1, z_1) as the centre of section is $(T=S_1)$

i.e., $$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad ...(1)$$

The plane should be identical with $lx+my+nz=p$ ...(2)

Comparing the co-efficients in (1) and (2), we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{ax_1^2+by_1^2+cz_1^2}{p}=k \text{ (say)}.$$

From these we have, $x_1=\frac{lk}{a}, y_1=\frac{mk}{b}, z_1=\frac{nk}{c}$ ...(3)

and $$ax_1^2+by_1^2+cz_1^2=pk \quad ...(4)$$

Putting the values of x_1, y_1, z_1 from (3) in (4), we have

$$a\left(\frac{l^2k^2}{a^2}\right)+b\left(\frac{m^2k^2}{b^2}\right)+c\left(\frac{n^2k^2}{c^2}\right)=pk$$

or $$\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)k=p \text{ or } p_0^2k=p \quad \therefore \quad k=\frac{p}{p_0^2}.$$

Putting this value of k in (3), the centre of section (x_1, y_1, z_1) is $$\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right).$$ Hence the result.

Example 3. *Find the locus of centres of all plane sections of a conicoid*

(a) *which pass through a fixed point.*

(b) *which are at a constant distance from the centre.*

(c) *which are parallel to a given line.*

(d) *which pass through a given line.*

Sol. Let (x_1, y_1, z_1) be the centre of plane section of the conicoid $$ax^2+by^2+cz^2=1. \quad ...(1)$$

Then equation of the plane with (x_1, y_1, z_1) as centre is

$$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1 \quad | \text{ Using } T=S_1$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad ...(2)$$

(a) The plane (2) passes through a fixed point say (α, β, γ), then

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2$$

$\therefore$ Locus of (x_1, y_1, z_1) is $ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$ which is a conicoid.

(b) The plane (2) is at a constant distance k (say) from the centre (0, 0, 0).

$$\therefore \frac{ax_1^2+by_1^2+cz_1^2}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}=k$$

or $$(ax_1^2+by_1^2+cz_1^2)^2=k^2(a^2x_1^2+b^2y_1^2+c^2z_1^2)$$

$\therefore$ Locus of (x_1, y_1, z_1) is $(ax^2+by^2+cz^2)^2=k^2(a^2x^2+b^2y^2+c^2z^2)$.

(c) Let the given line be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. ...(3)

The plane (2) will be $\parallel$ to line (3) if its normal is $\perp$ to (3), *i.e.*,

if $$l(ax_1)+m(by_1)+n(cz_1)=0$$

or $$alx_1+bmy_1+cnz_1=0.$$

$\therefore$ Locus of (x_1, y_1, z_1) is $alx+bmy+cnz=0$ which is a plane.

(d) Let the given line be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$. ...(4)

The line (4) will lie on plane (2) if (*i*) the line (4) is $\perp$ to normal to the plane (2) *i.e.*,

$$lax_1+mby_1+ncz_1=0 \quad ...(5)$$

and (*ii*) one point (α, β, γ) on (4) lies on the plane (2) *i.e.*,

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2 \quad ...(6)$$

From (5) and (6), the locus of (x_1, y_1, z_1) is

$$alx+bmy+cnz=0,\ ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$$

which being the intersection of a plane and a conicoid represents a conic.

Example 4. *Find the centre of the conic given by the equations*

$$2x-2y-5z+5=0,\ 3x^2+2y^2-15z^2=4.$$

Sol. The conicoid is $\quad S\equiv 3x^2+2y^2-15z^2-4=0$ ...(1)

Let (x_1, y_1, z_1) be the centre of the given conic. Then equation of the plane which cuts (1) in a conic with centre (x_1, y_1, z_1) is $T=S_1$

i.e., $$3xx_1+2yy_1-15zz_1-4=3x_1^2+2y_1^2-15z_1^2-4$$

or $$3xx_1+2yy_1-15zz_1-(3x_1^2+2y_1^2-15z_1^2)=0 \quad ...(2)$$

Now this is the same as the given plane ...(3)

$$2x-2y-5z+5=0$$

Comparing (2) and (3), we get

$$\frac{3x_1}{2}=\frac{2y_1}{-2}=\frac{-15z_1}{-5}=\frac{-(3x_1^2+2y_1^2-15z_1^2)}{5}=k \text{ (say)} \quad (4)$$

$$x_1=\frac{2}{3}k,\ y_1=-k,\ z_1=\frac{k}{3}$$

and $$3x_1^2+2y_1^2-15z_1^2=-5k.$$

Putting the values of x_1, y_1, z_1 in the last equation, we get

$$3\left(\frac{4}{9}k^2\right)+12k^2-15\left(\frac{k^2}{9}\right)=-5k$$

or $$\frac{4}{3}k^2+2k^2-\frac{5}{3}k^2=-5k$$

or $$4k^2+6k^2-5k^2=-15k \text{ or } 5k^2=-15k \therefore k=-3.$$

$\therefore$ From (4), the centre is (x_1, y_1, z_1) *i.e.*, $(-2, 3, -1)$.

Example 5. *Prove that the centres of sections of*

$$ax^2+by^2+cz^2=1$$

by the planes which are at a constant distance p from the origin lie on the surface

$$(ax^2+by^2+cz^2)^2=p^2(a^2x^2+b^2y^2+c^2z^2).$$

Sol. If (α, β, γ) be the centre of the section of the given ellipsoid then equation of this section of the sphere is "$T=S_1$"

i.e. $\quad (a\alpha x+b\beta y+c\gamma z-1)=(a\alpha^2+b\beta^2+c\gamma^2-1)$

or $\quad -a\alpha x-b\beta y-c\gamma z+(a\alpha^2+b\beta^2+c\gamma^2)=0 \quad \ldots(i)$

The distance of this plane (*i*) from the origin (0, 0, 0) is given as *p*.

$$\therefore \quad p=\frac{a\alpha^2+b\beta^2+c\gamma^2}{\sqrt{[(a\alpha)^2+(b\beta)^2+(c\gamma)^2]}}$$

or $\quad p^2(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)=(a\alpha^2+b\beta^2+c\gamma^2)^2$

$\therefore$ The locus of the centre (α, β, γ) is

$$p^2(a^2x^2+b^2y^2+c^2z^2)=(ax^2+by^2+cz^2)^2.$$

Example 6. *Prove that the centre of the section of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *by the plane ABC whose equation is* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *is the centroid of the triangle ABC.*

Sol. The equation of the ellipsoid is

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \ldots(i)$$

and the equation of the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \ldots(ii)$$

Let (α, β, γ) be the centre of the section (*i*) by the plane (*ii*) then the equation of this section is "$T=S_1$"

i.e. $\quad \frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{\gamma z}{c^2}-1=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1$

or $\quad \frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad \ldots(iii)$

The equation (*ii*) and (*iii*) represent the same plane, so comparing them, we get

$$\frac{\left(\frac{\alpha}{a^2}\right)}{\left(\frac{1}{a}\right)}=\frac{\left(\frac{\beta}{b^2}\right)}{\left(\frac{1}{b}\right)}=\frac{\left(\frac{\gamma}{c^2}\right)}{\left(\frac{1}{c}\right)}=\frac{\left(\frac{\alpha^2}{a^2}\right)+\left(\frac{\beta^2}{b^2}\right)+\left(\frac{\gamma^2}{c^2}\right)}{1}=k \quad \text{(say)}$$

$\therefore \quad \alpha=ak, \beta=bk, \gamma=ck$ and

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=k$$

$$\therefore \quad \left(\frac{a^2k^2}{a^2}\right)+\left(\frac{b^2k^2}{b^2}\right)+\left(\frac{c^2k^2}{c^2}\right)=k$$

or $\quad 3k^2=k \quad$ or $\quad k=\frac{1}{3}$.

$\therefore \quad \alpha=ak=\frac{1}{3}a, \quad \beta=bk=\frac{1}{3}b, \quad \gamma=ck=\frac{1}{3}c.$

$\therefore$ The centre of the section of (*i*) by the plane (*ii*) is (α, β, γ) or $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

Also the co-ordinates of the vertices of $\triangle ABC$ are

$$A(a, 0, 0), B(0, b, 0), C(0, 0, c)$$

$\therefore$ The co-ordinates of the centroid of $\triangle ABC$ are $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

Hence the centre of the section of (*i*) by (*ii*) is the centre of $\triangle ABC$.

Example 7. *Find the locus of the mid-points of the chords of the conicoid $ax^2+by^2+cz^2=1$ which passes through (α, β, γ).*

(Allahabad 1981 ; Kanpur 1979 ; Lucknow 1982)

Sol. Let (x_1, y_1, z_1) be the mid-point of the chord of the given conicoid. Then the locus of the chords of the given conicoid with (x_1, y_1, z_1) as mid-point is "$T=S_1$"

where $\quad T=axx_1+byy_1+czz_1-1 \quad$ and

$$S_1=ax_1^2+by_1^2+cz_1^2-1$$

i.e. $\quad axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$

or $\quad axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$

If it posses through (α, β, γ), we have

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2$$

$\therefore$ The required locus of the mid-point (x_1, y_1, z_1) of the chords of the given conicoid is

$$ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$$

or $\quad ax(x-\alpha)+by(y-\beta)+cz(z-\gamma)=0.$

Example 8. *Show that the line joining a point P to the centre of a conicoid $ax^2+by^2+cz^2=1$ passes through the centre of the section of the conicoid by the polar plane of P.*

Sol. Let (x', y', z') be the co-ordinates of the point P. Then the polar plane of $P(x', y', z')$ with respect to the given conicoid is

$$axx'+byy'+czz'=1 \quad \ldots(i)$$

Let (α, β, γ) be the centre of the section of the given conicoid by the plane (*i*), then equation of this plane section can also be written as

"$T=S_1$" or

$$a\alpha x+b\beta y+c\gamma z-1=a\alpha^2+b\beta^2+c\gamma^2-1$$

or $\quad a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2 \quad \ldots(ii)$

Since the equations (*i*) and (*ii*) represent the same plane, so comparing them, we get

$$\frac{\alpha}{x'}=\frac{\beta}{y'}=\frac{\gamma}{z'} \quad \ldots(iii)$$

Also the equations of the line joining the point $P(x', y', z')$ to the centre $(0, 0, 0)$ of the given conicoid is

$$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}$$

If this line passes through the centre (α, β, γ) of the section of given conicoid be the plane (i), then

$$\frac{\alpha}{x'}=\frac{\beta}{y'}=\frac{\gamma}{z'}$$

which is true by virtue of (iii).

Hence the line joining $P(x', y', z')$ to the centre of the given conicoid passes through the centre (α, β, γ) of the section of the conicoid by the polar plane (i) of P.

Example 9. *Prove that the section of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

whose centre is at the point $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ *passes through the extremities of the axes.* (Rohilkhand 1985)

Sol. The ellipsoid is

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad ...(i)$$

The equation of the section of this ellipsoid with $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ as its centre is "$T=S_1$"

i.e.,
$$\frac{x\cdot\frac{1}{3a}}{a^2}+\frac{y\cdot\frac{1}{3b}}{b^2}+\frac{z\cdot\frac{1}{3c}}{c^2}-1$$

$$=\frac{\left(\frac{1}{3a^2}\right)^2}{a^2}+\frac{\left(\frac{1}{3b}\right)^2}{b^2}+\frac{\left(\frac{1}{3c}\right)^2}{c^2}-1$$

or
$$\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$$

or
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

which is the plane evidently passing through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, the three extremities of the axes of the ellipsoid given by (i).

Example 10. *Find the locus of centres of sections of* $ax^2+by^2+cz^2=1$ *which touch* $\alpha x^2+\beta y^2+\gamma z^2=1$. (Rohilkhand 1983)

Sol. Let (x_1, y_1, z_1) be the centre of the section of conicoid

$$ax^2+by^2+cz^2=1 \quad ...(i)$$

The equation of the section is $T=S_1$

or $axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$

or $ax_1x+by_1y+cz_1z=(ax_1^2+by_1^2+cz_1^2)$ ...(i)

If the plane (i) touches the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$, then we must have

$$\text{“}\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\text{”}$$

$$\therefore \frac{(ax_1)^2}{\alpha}+\frac{(by_1)^2}{\beta}+\frac{(cz_1)^2}{\gamma}=(ax_1^2+by_1^2+cz_1^2)^2$$

$\therefore$ The required locus of (x_1, y_1, z_1) is

$$\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}=(ax^2+by^2+cz^2)^2.$$

Example 11. *Prove that the middle point of the chords of $ax^2+by^2+cz^2=1$ which are parallel to $x=0$ and touch $x^2+y^2+z^2=\gamma^2$ lies on the surface*

$$by^2(bx^2+by^2+cz^2-bx^2)+cz^2(-cx^2+by^2+cz^2-c\gamma^2)=0.$$

(Kanpur 1982 ; Rohilkhand 1983)

Sol. The equation of any line having (α, β, γ) as mid-point and parallel to the plane $x=0$ is

$$\frac{x-\alpha}{0}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=\lambda \text{ (say)} \quad ...(i)$$

where m and n are variables.

Any point on this line is $(\alpha, \beta+m\gamma, \gamma+n\lambda)$. If this point lies on the conicoid $ax^2+by^2+cz^2=1$, then we have

$$a\alpha^2+b(\beta+m\lambda)^2+c(\gamma+n\lambda)^2=1$$

or $\lambda(bm^2+cn^2)+2\lambda(b\beta m+c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ ...(ii)

$\therefore$ (α, β, γ) is the mid-point of the chord (i) of the given conicoid, so that sum of the roots of equation (ii), which is a quadratic in λ, must be zero

i.e., $bm\beta+cn\gamma=0$ ...(iii)

Also the line (i) touches the sphere

$$x^2+y^2+z^2=r^2$$

$\therefore$ The length of perpendicular from the centre (0, 0, 0) of the sphere to (i) must be equal to the radius r of the sphere

i.e., $\left[\begin{vmatrix}-\alpha & -\beta \\ 0 & m\end{vmatrix}^2+\begin{vmatrix}-\beta & -\gamma \\ m & n\end{vmatrix}^2+\begin{vmatrix}-\gamma & -\alpha \\ n & 0\end{vmatrix}^2\right]\div(m^2+n^2)=r^2$

or $m^2\alpha^2+(n\beta-m\gamma)^2+\alpha^2n^2=r^2(m^2+n^2)$

or $(r^2-\alpha^2)(m^2+n^2)=(n\beta-m\gamma)^2$

or $(r^2-\alpha^2)\left[\left(\frac{m}{n}\right)^2+1\right]=\left[\beta-\left(\frac{m}{n}\right)\gamma\right]^2$ ...(iv)

Also from (iii), we have $\frac{m}{n}=\frac{(-c\gamma)}{(b\beta)}$

Substituting the value in (*iv*), we get

$$(r^2-\alpha^2)\left[\left(\frac{c^2\gamma^2}{b^2\beta^2}\right)+1\right]=\left[\beta+\left(\frac{c\gamma}{b\beta}\right)\right]^2$$

or
$$(r^2-\alpha^2)[c^2\gamma^2+b^2\beta^2]=[b\beta^2+\gamma^2c]^2$$

$\therefore$ The required locus of (α, β, γ) is

$$(r^2-x^2)(c^2z^2+b^2y^2)=(by^2+cz^2)^2$$

or $c^2r^2z^2+b^2r^2y^2-c^2x^2z^2-b^2y^2x^2=b^2y^4+c^2z^4-2bcy^2z^2$

or $by^2(bx^2+by^2+cz^2-bx^2)+cz^2(cz^2+by^2-cx^2-cr^2)=0.$

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Article 23. To trace the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$

The given surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ ...(1)

(*i*) **Symmetry.** Since the equation (1) contains even powers of x, y, z, so the surface is symmetrical about the YZ, ZX, and XY planes.

(*ii*) **Axes intersection.** The cone meets X-axis $(y=0, z=0)$ where $\frac{x^2}{a^2}=0$ or $x=0, 0$, *i.e.*, in two coincident points.

Thus cone meets X-axis at the origin. Similarly, it meets Y and Z-axis also at the origin.

(*iii*) **Sections by co-ordinate planes.** The cone (1) meets the YZ plane $(x=0)$, where $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ or $y=\pm\frac{b}{c}z$ which are two straight lines in that plane [on opposite sides of Z-axis and making equal angles with it].

Similarly, the cone (1) meets ZX plane $(y=0)$ in two lines $x=\pm\frac{a}{c}z$ which are equally inclined to Z-axis and on opposite sides of it.

Again it meets the XY plane $(z=0)$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ which is a point ellipse in *that plane*.

(*iv*) **Generated by a variable curve.** The surface (1) meets the plane $z=k$ [where putting $z=k$ in (1)].

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{k^2}{c^2}=0 \quad\text{or}\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$$

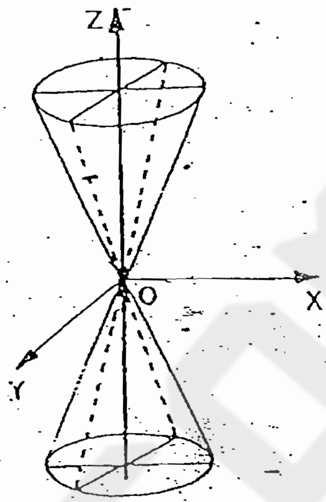
∴ Thus the cone (i) is generated by the variable ellipse

$$z=k, \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2} \quad ...(2) \ (k \text{ varies})$$

whose plane is ∥ to XY plane and centre $(0, 0, k)$ moves on Z-axis. The ellipse (2) is real for all values of k +ve or −ve and the semi-axes $\frac{ak}{c}$, $\frac{bk}{c}$ increase as k increases numerically and $\to\infty$ as $k\to\infty$.

∴ The cone extends to infinity both above and below the XY-plane.

Hence the shape is as shown in the adjoining figure.



Note 1. The standard equation of the cone is of the form

$$ax^2+by^2+cz^2=0.$$

Note 2. A cone can be regarded as a central conicoid whose centre is the vertex.

Article 24. Some important results about the cone

$$ax^2+by^2+cz^2=0.$$

(*i*) The tangent plane at (x_1, y_1, z_1) and the plane of contact of (x_1, y_1, z_1) and polar plane of (x_1, y_1, z_1) w.r.t. given cone is

$$axx_1+byy_1+czz_1=0.$$

(*ii*) The plane $lx+my+nz=0$ touches the cone if

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0.$$

(*iii*) The equation of plane which cuts the cone in a conic with centre (x_1, y_1, z_1) is given by $T=S_1$.

☞ *The student is advised to prove these results* as in Arts. 7, 8, 14, 15, 21.

Example 1. *Find the equation of the normal plane of the cone*

$$ax^2+by^2+cz^2=0,$$

through the generator

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Sol. [The normal plane *through the generator OP of a cone (vertex O) is the plane through OP and ⊥ to the tangent plane at any point of OP.*] [Remember]

The given cone is $\quad ax^2+by^2+cz^2=0 \quad ...(1)$

and the generator is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$

Any plane through the line (2) is

$$Ax+By+Cz=0 \quad \ldots(3)$$

where $$Al+Bm+Cn=0 \quad \ldots(4)$$

Any point on (2) is Q (lr, mr, nr). The tangent plane at Q to (1) is

$$ax(lr)+by(mr)+cz(nr)=0 \text{ or } alx+bmy+cnz=0 \quad \ldots(5)$$

If (3) is the normal plane through (2), then (3) is $\perp$ to (5).

$$\therefore \quad Aal+Bbm+Ccn=0 \quad \ldots(6)$$

Solving (4) and (6) by cross-multiplication, we have

$$\frac{A}{mn(c-b)}=\frac{B}{nl(a-c)}=\frac{C}{lm(b-a)}$$

Putting these values of A, B, C in (3) and taking out $-$ve sign common, we have

$$mn(b-c)x+nl(c-a)y+lm(a-b)z=0.$$

Dividing throughout by lmn, we get

$$\frac{(b-c)x}{l}+\frac{(c-a)y}{m}+\frac{(a-b)z}{n}=0,$$

which is the required normal plane.

Example 2. *Lines are drawn through the origin perpendicular to normal planes of the cone*

$$ax^2+by^2+cz^2=0.$$

Show that they generate the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

Sol. Let the line OP given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

be any generator of the cone

$$ax^2+by^2+cz^2=0 \quad \ldots(1)$$

Since the d.c.'s of the generator satisfy the equation of the cone,

$$\therefore \quad al^2+bm^2+cn^2=0 \quad \ldots(2)$$

Also equation of the normal plane through OP to (1) is

$$\left(\frac{b-c}{l}\right)x+\left(\frac{c-a}{m}\right)y+\left(\frac{a-b}{n}\right)z=0 \quad \ldots(3)$$

[*See Example* 1, above]

Equations of the line through (0, 0, 0) $\perp$ to (3) are

$$\frac{x}{\left(\frac{b-c}{l}\right)}=\frac{y}{\left(\frac{c-a}{m}\right)}=\frac{z}{\left(\frac{a-b}{n}\right)}$$

or $$\frac{l}{\frac{(b-c)}{x}}=\frac{m}{\frac{(c-a)}{y}}=\frac{n}{\frac{(a-b)}{z}} \quad \ldots(4)$$

To find the locus of line (4), we have to eliminate l, m, n from (4) and (2). Putting the values of l, m, n from (4) in (2), we get

$$a\left(\frac{b-c}{x}\right)^2+b\left(\frac{c-a}{y}\right)^2+c\left(\frac{a-b}{z}\right)^2=0$$

or
$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0$$

which is the required cone.

Example 3. *Prove that if a plane cuts the cone*

$$ax^2+by^2+cz^2=0$$

in perpendicular generators, it touches the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0.$$

Sol. Let the plane be $ux+vy+wz=0$ ...(1)

and the cone is $ax^2+by^2+cz^2=0$ ...(2)

Let a line of section of (1) and (2) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Since it lies on (1) and (2) both.

$\therefore$ $ul+vm+wn=0$ and $al^2+bm^2+cn^2=0$ ...(3)

The two lines given by (3) are $\perp$ if

$$u^2(b+c)+v^2(c+a)+w^2(a+b)=0 \quad ...(4)$$

Now the plane (1) will touch the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$$

if
$$\frac{u^2}{\left(\frac{1}{b+c}\right)}+\frac{v^2}{\left(\frac{1}{c+a}\right)}+\frac{w^2}{\left(\frac{1}{a+b}\right)}=0$$

$$\left| \text{Using } \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0 \right.$$

or if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ which is true by (4).

Hence the result.

Remember. *Two lines given by*

$$ul+vm+wn=0 \text{ and } al^2+bm^2+cn^2=0$$

are $\perp$ if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$.

Example 4. *Show that the perpendicular tangent planes to*

$$ax^2+by^2+cz^2=0$$

intersect in generators of the cone

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0 \quad \text{[Imp.]}$$

Sol. The given cone is $ax^2+by^2+cz^2=0$ ...(1)

Any tangent plane to (1) is $lx+my+nz=0$ ...(2)

where $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$$ ...(3)

Let the line of intersection of two tangent planes, (through origin) be

$$\frac{x}{L}=\frac{y}{M}=\frac{z}{N}$$ ...(4)

Since it lies on (2), $\therefore$ $Ll+Mm+Nn=0$ ...(5)

The two lines given by (5) and (3) are $\perp$,

$$\therefore \quad L^2\left(\frac{1}{b}+\frac{1}{c}\right)+M^2\left(\frac{1}{c}+\frac{1}{a}\right)+N^2\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

Eliminating L, M, N from this and (4), the required locus is

$$x^2\left(\frac{1}{b}+\frac{1}{c}\right)+y^2\left(\frac{1}{c}+\frac{1}{a}\right)+z^2\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

or $$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

Example 5. (*a*) *The locus of the asymptotes drawn from the origin to the conicoid*

$$ax^2+by^2+cz^2=1$$

is the asymptotic cone

$$ax^2+by^2+cz^2=0.$$

(*b*) *Prove that the hyperboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ and } -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

have the same asymptotic cone.

Sol. [Def. *An asymptote meets the given surface at two points an infinity*].

[Remember]

(*a*) The conicoid is $ax^2+by^2+cz^2=1$ ...(1)

Let the asymptote through (0, 0, 0) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$ ...(2)

Any point on (2) is (lr, mr, nr).

If it lies on (1), then

$$al^2r^2+bm^2r^2+cn^2r^2=1$$

or $$al^2+bm^2+cn^2=\frac{1}{r^2}$$

Since the asymptote (2) meets (1) an infinity $\therefore$ $r=\infty$.

$$\therefore \quad al^2+bm^2+cn^2=\frac{1}{\infty}=0$$ ...(3)

Eliminating l, m, n from (2) and (3), the locus of (2) is

$$ax^2+by^2+cz^2=0$$

which is a cone.

(*b*) Please try yourself as in part (*a*).

Example 6. *Any plane whose normal lies on the cone*

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0$$

cuts the surface

$$ax^2+by^2+cz^2=1$$

in a rectangular hyperbola. [Imp.]

Sol. Let the plane be $ux+vy+wz=0$ ...(1)

This cuts the surface $ax^2+by^2+cz^2=1$

in rectangular hyperbola.

Let the asymptote of this hyperbola be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(2)$$

The asymptote (2) lies on plane (1)

$$\therefore \quad ul+vm+wn=0 \qquad ...(3)$$

Also any point on (2) is (lr, mr, nr). This point will lie on the surface $ax^2+by^2+cz^2=1$, if

$$r^2(al^2+bm^2+cn^2)=1 \quad \text{or} \quad al^2+bm^2+cn^2=\frac{1}{r^2}$$

But $r\to\infty$, as the asymptote cuts the surface at ∞

$\therefore$ We have $al^2+bm^2+cn^2=0$ ...(4)

The asymptote of a rectangular hyperbola are $\perp$. Thus the two lines given by (3) and (4) are $\perp$.

$\therefore \quad u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ | *Refer Ex.* 15 (*i*), *page* 35

This shows that the normal $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$ to plane (1) lies on the cone

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0.$$

Set – VII

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PARABOLOID

Article 24. (*a*) To trace elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{c}.$$

The given paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$. ...(1)

(*i*) Symmetry: Since (1) contains even powers of x and y, so it is symmetrical about the YZ and ZX planes.

(*ii*) Intersection with axes. The surface (1) meets X-axis $(y=0, z=0)$ where $\frac{x^2}{a^2}=0$ or $x^2=0$ $\therefore$ $x=0, 0$.

$\therefore$ The surface (1) touches the X-axis at O(0, 0, 0).

Similarly it touches Y-axis at (0, 0, 0). The the surface (1) touches the XY plane. Also (1) meets Z-axis at (0, 0, 0).

(*iii*) Sections by co-ordinate planes. (1) meets the YZ plane $(x=0)$ where

$$\frac{y^2}{b^2}=\frac{2z}{c}, \text{ or } y^2=\frac{2b^2}{c}z$$

which is an upward parabola *in that plane*.

Similarly (1) meets the ZX plane $(y=0)$ in an upward parabola $x^2=\frac{2a^2}{c}z$ *in that plane*.

Again (1) meets the XY plane $(z=0)$ in

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$$

which is a point ellipse in that plane.

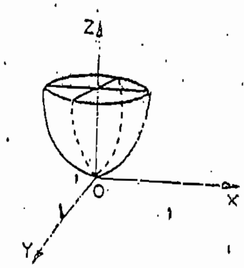
(*iv*) Generated by a variable curve. The surface (1) meets the plane $z=k$, where

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c} \quad\text{or}\quad \frac{x^2}{\left(\frac{2a^2k}{c}\right)}+\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1$$

Thus the surface is generated by a variable ellipse

$$z=k, \quad \frac{x^2}{\left(\frac{2a^2k}{c}\right)}+\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1 \qquad ...(2)$$

where k varies.



Its plane is ‖ to XY-plane and the centre $(0, 0, k)$ moves on Z-axis. Now the ellipse (2) is real if k is +ve. Thus the surface lies only above the XY-plane.

Also the semi-axis of the ellipse (2) are $a\sqrt{\frac{2k}{c}}$, $b\sqrt{\frac{2k}{c}}$ which increases as $k>0$ increases and $\to\infty$ as $k\to\infty$. Thus the surface extends to ∞ above the XY-plane.

The shape is as shown in the adjoining figure.

Article 24. (*b*) To trace the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$

The equation of the surface is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c} \qquad ...(1)$$

(*i*) Symmetry. Since the equation (1) contains even powers of x and y, so the surface is symmetrical about the YZ and ZX-planes.

(*ii*) Axes intersection. The surface (1) meets X-axis ($y=0$, $z=0$) where $\frac{x^2}{a^2}=0$ or $x^2=0$ or $x=0, 0$. Thus the surface (1) touches X-axis at the origin. Similarly it touches Y-axis at the origin.

Thus (1) touches XY-plane at O(0, 0, 0).

It meets Z-axis ($x=0$, $y=0$), where $\frac{2z}{c}=0$ or $z=0$ *i.e.*, at the origin.

(*iii*) Sections by co-ordinate planes. The surface (1) meets the YZ-planes ($x=0$) where

$$\frac{-y^2}{b^2}=\frac{2z}{c} \quad \text{or} \quad y^2=-2\frac{b^2}{c}z$$

which is a downward parabola *in that plane* (*assuming c to be* +ve).

Similarly (1) meets the ZX-plane in the upward parabola

$$x^2=\frac{2a^2}{c}z \textit{ in that plane.}$$

It meets the XY-plane ($z=0$) where

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{or} \quad y=\pm\frac{b}{a}x$$

which are two straight lines in that plane equally inclined to X-axis.

(*iv*) Generated by a variable curve. The surface (1) meets the plane $z=k$ where [putting $z=k$ in (1)],

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c} \quad \text{or} \quad \frac{x^2}{\left(\frac{2a^2k}{c}\right)}-\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1.$$

Thus the surface is generated by a variable hyperbola

$$\frac{x^2}{\left(\frac{2a^2k}{c}\right)}-\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1,\ z=k \quad \text{...(2) [as } k \text{ varies]}$$

whose plane is ∥ to the XY-plane, and centre (0, 0, k) moves on the Z-axis.

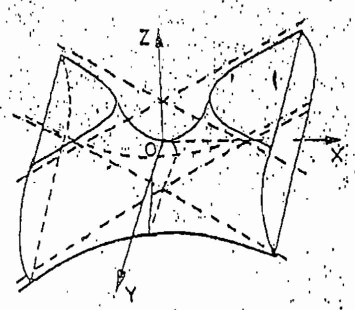
The hyperbola (2) has transverse axis ∥ to X-axis if k is +ve and ∥ to Y-axis if k is −ve. [∵ *c is assumed to be* +ve]

Also the transverse semi-axis is $a\sqrt{\frac{2k}{c}}$ which increases as k(+ve) increases and $\to\infty$ as $k\to\infty$.

Thus surface extends to infinity above the XY-plane.

Similarly the surface extends to infinity below the XY-plane.

∴ The surface extends to infinity both above and below the XY-plane. Hence the shape of the surface is as shown in the figure.



Note. The general equation of the paraboloid is of the form $ax^2+by^2=2z$, which is an elliptic or hyperbolic paraboloid according as a and b are of the same or opposite signs.

Article 25. Intersection of a line with the paraboloid.

Let the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{...(1)}$$

and the paraboloid be $\quad ax^2+by^2=2z \quad$ ...(2)

Any point on (1) is P($lr+x_1$, $mr+y_1$, $nr+z_1$). If it lies on (2) then $\quad a(lr+x_1)^2+b(mr+y_1)^2=2(nr+z_1)$

or $\quad r^2(al^2+bm^2)+2r(alx_1+bmy_1-n)+(ax_1^2+by_1^2-2z_1)=0 \quad$ ...(3)

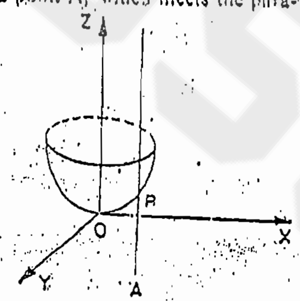
which gives two values of r.

This shows that every line meets the paraboloid in two points, *i.e. every plane section of a paraboloid is a conic.*

Again if $l=m=0$, $n=1$ from (3), one value of r is infinite showing that any line ∥ to the Z-axis meets the paraboloid in one point at an infinite distance from A(x_1, y_1, z_1) and in a finite point P whose distance from A is given by

$$r=\frac{ax_1^2+by_1^2-2z_1}{2}.$$

Such a line drawn through a point A, which meets the paraboloid in one point at an infinite distance from A and in a point P is called a diameter of the paraboloid and P is called the extremity of the diameter.



Thus a line ∥ to OZ is a diameter of the paraboloid $ax^2+by^2=2z$

Def. The diameter of a paraboloid which is ⊥ to the tangent plane at its extremity, is called the axis of the paraboloid, and its extremity is called the vertex of the paraboloid.

Thus OZ is the axis and O the vertex of the paraboloid

$$ax^2+by^2=2z.$$

Cor. A line ‖ to the axis of a paraboloid is diameter.

Article 26. Some standard results about the paraboloid

Following are some of the results about a paraboloid which can be easily proved. The student is advised to prove these results for himself.

Let the paraboloid be

$$ax^2+by^2=2z. \quad \ldots(1)$$

(*i*) The tangent plane to (1) at (x_1, y_1, z_1) is

$axx_1+byy_1=z+z_1$. (*K.U. 1970*) [*See Art. 1 (g) prove by General Methods*]

(*ii*) The condition of tangency for a given plane

$$lx+my+nz=p$$

and the paraboloid (1) is

$$\frac{l^2}{a}+\frac{m^2}{b}=-2np \quad \text{(K.U. 1973)}$$

[*For proof see page (xvii) of general methods.* *Art. 2(d)*]

and the *point of contact* is

$$\left(\frac{-l}{an}, \frac{-m}{bn}, \frac{-p}{n}\right)$$

and *any tangent plane* to (1) ‖ to $lx+my+nz=0$ is

$$2n(lx+my+nz)+\frac{l^2}{a}+\frac{m^2}{b}=0.$$

(*iii*) The plane of contact and the polar plane of (x_1, y_1, z_1) w.r.t. (1) is $axx_1+byy_1=z+z_1$.

(*iv*) The enveloping cone of (1) is given by $SS_1=T^2$

i.e., $(ax^2+by^2-2z)(ax_1^2+by_1^2-2z_1)=[axx_1+byy_1-(z+z_1)]^2$.

(*v*) The plane section of (1) with given centre (x_1, y_1, z_1) is given by $T=S_1$

i.e. $axx_1+byy_1-(z-y_1)=ax_1^2+by_1^2-2z_1$.

Example 1. *Find the condition that the plane $lx+my+nz=1$ may be a tangent plane to the paraboloid $x^2+y^2=2z$.*

Sol. Reproduce Art 2(*d*) page (xvii) General Methods replacing a, b, p by 1 each).

Example 2. (*a*) *Show that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ and find the co-ordinates of the point of contact.* (Agra 1988)

(*b*) *Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find the co-ordinates of the point of contact.* (Agra 1987 : Madurai 1983)

Sol. (*a*) Let the plane $8x-6x-z=5$ ...(1)

touch the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$, or $3x^2-2y^2=6z$ ...(2)

at the point (x_1, y_1, z_1).

Then the tangent plane to (2) at (x_1, y_1, z_1) is

$$3xx_1-2yy_1=3(z+z_1) \quad \text{or} \quad 3xx_1-2yy_1-3z=3z_1 \quad \ldots(3)$$

Now this plane is identical with (1).

∴ Comparing the co-effs. in (1) and (3), we have

$$\frac{3x_1}{8}=\frac{-2y_1}{-6}=\frac{-3}{-1}=\frac{3z_1}{5}$$

which gives $x_1=8, \; y_1=9, \; z_1=5.$

The plane (1) touches the paraboloid (2) if the point of contact (x_1, y_1, z_1), *i.e.* (8, 9, 5) lies on (2) *i.e.* if $3(64)-2(81)=6(5)$

or if $192-162=30$ or $30=30$ which is true.

Hence (1) touches (2) and the point of contact is (8, 9, 5).

(*b*) Please try yourself. [Ans. (3, 3, −3)]

Example 3. *Prove that the paraboloids*

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1}; \; \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2}; \; \frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$$

have a common tangent plane if

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

Sol. The given paraboloids are $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1}$ ...(1)

$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2}$ ...(2) and $\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$ ...(3)

Let the common tangent plane be $lx+my+nz=p$ ...(4)

Since it touches the paraboloid (1)

i.e., $\frac{c_1}{a_1^2}x^2+\frac{c_1}{b_1^2}y^2=2z.$

$$\therefore \frac{l^2}{\left(\frac{c_1}{a_1^2}\right)}+\frac{m^2}{\left(\frac{c_1}{b_1^2}\right)}=-2np \qquad \left| \; \text{Using } \frac{l^2}{a}+\frac{m^2}{b}=-2np \right.$$

or $l^2a_1^2+m^2b_1^2+2nc_1p=0$ ...(5)

Similarly (4) touches (2) and (3),

∴ $l^2a_2^2+m^2b_2^2+2nc_2p=0$ ...(6)

and $l^2a_3^2+m^2b_3^2+2nc_3p=0$ ...(7)

Eliminating l^2, m^2, $2np$ from (5), (6), (7) by determinants, we have

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix}=0 \text{ or } \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

Example 4. *Show that the equation to two tangent planes to the surface $ax^2+by^2=2z$ which pass through the line*

$$u \equiv lx+my+nz-p=0,\ u' \equiv l'x+m'y+n'z-p'=0$$

is

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2n'p'\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+np'+n'p\right)+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)=0.$$ [Imp.]

Sol. Any plane through the line $u=0,\ u'=0$ is $uk+u'=0$

$$lx+my+nz-p+k(l'x+m'y+n'z-p')=0 \quad \ldots(1)$$

or $\quad (l+kl')x+(m+km')y+(n+kn')z=p+kp'.$

If it touches the paraboloid $ax^2+by^2=2z$, then

$$\frac{(l+kl')^2}{a}+\frac{(m+km')^2}{b}=-2(n+kn')\cdot(p+kp').$$

$\Big|$ Using $\dfrac{l^2}{a}+\dfrac{m^2}{b}=-2np$

or $\quad k^2\left(\dfrac{l'^2}{a}+\dfrac{m'^2}{b}+2n'p'\right)+2k\left(\dfrac{ll'}{a}+\dfrac{mm'}{b}+n'p+np'\right)+\left(\dfrac{l^2}{a}+\dfrac{m^2}{b}+2np\right)=0.$

Putting $k=-\dfrac{u}{u'}$, from (1) in this, we get the required result.

Example 5. *Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid*

(i) $ax+by^2+2z=0$

(ii) $ax^2+by^2=2z.$

Sol. (i) Let $P(x_1, y_1, z_1)$ be the point. Then enveloping cone of $ax^2+by^2+2z=0$ of the tangents from $P(x_1, y_1, z_1)$ is $SS_1=T^2$

or $\quad (ax^2+by^2+2z)(ax_1^2+by_1^2+2z_1)=(axx_1+byy_1+z+z_1)^2 \quad \ldots(1)$

If the three lines drawn from P to touch the given paraboloid are mutually $\perp$, then the cone must have three mutually $\perp$ generators, *i.e.*, *sum of coeffs. of x^2, y^2, z^2 in (1) is zero.*

or $\quad a(by_1^2+2z_1)+b(ax_1^2+2z_1)-1=0$

or $\quad ab(x_1^2+y_1^2)+2z_1(a+b)-1=0.$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$ab(x^2+y^2)+2(a+b)z-1=0.$$

(ii) Please try yourself. [Ans: $ab(x^2+y^2)-2(a+b)z-1=0$]

Example 6. (a) *Find the equation of the plane which cuts the paraboloid $x^2-2y^2=z$ in a conic with its centre at $(2, \frac{3}{2}, 4)$.*

(b) *Find the centre of the conic $ax^2+by^2=2z,\ lx+my+nz=p.$*

Sol. (a) The paraboloid is $S \equiv x^2-2y^2-z=0$ and the centre (x_1, y_1, z) is $(2, \frac{3}{2}, 4)$.

Then the plane which has the centre $P(2, \frac{3}{2}, 4)$ is given by $T=S_1$, where T is the expression for tangent plane at P with R.H.S. zero, *i.e.* by

$$x(2)-2y\left(\frac{3}{2}\right)-\frac{1}{2}(z+4)=(2)^2-2\left(\frac{3}{2}\right)^2-4$$

or $\quad 2x-3y-\dfrac{1}{2}z-2=4-\dfrac{9}{2}-4$

or $\quad 4x-6y-z+5=0.$

(b) Let (x_1, y_1, z_1) be the centre of the conic given by

$$ax^2+by^2-2z=0 \quad \ldots(1)$$

and $\quad lx+my+nz-p=0 \quad \ldots(2)$

Then equation of the plane which cuts (1) in a conic with a centre at (x_1, y_1, z_1) is $\quad T=S_1$ *i.e.*

$$axx_1+byy_1-(z+z_1)=ax_1^2+by_1^2-2z_1$$

or $\quad axx_1+byy_1-z-(ax_1^2+by_1^2-z_1)=0 \quad \ldots(3)$

Now (2) and (3) are identical. $\therefore$ Comparing coeffs, we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{-1}{n}=\frac{ax_1^2+by_1^2-z_1}{p}.$$

Which give $x_1=\dfrac{-l}{an},\ y_1=\dfrac{-m}{bn}$ and $ax_1^2+by_1^2-z_1=\dfrac{-p}{n}$

Putting the values of x_1, y_1 from first two equations in the third equation, we get

$$a\cdot\frac{l^2}{a^2n^2}+b\cdot\frac{m^2}{b^2n^2}-z_1=-\frac{p}{n}$$

or $\quad z_1=\dfrac{l^2}{an^2}+\dfrac{m^2}{bn^2}+\dfrac{p}{n}.$

Thus the centre (x_1, y_1, z_1) of the conic is

$$\left(\frac{-l}{an}, \frac{-m}{bn}, \frac{l^2}{an^2}+\frac{m^2}{bn^2}+\frac{p}{n}\right).$$

Example 7. *Show that the locus of centres of a system of parallel plane sections of a paraboloid is a diameter.*

Prove also that the tangent plane at the extremity of the diameter is parallel to the plane sections. [Imp.]

Sol. Let the paraboloid be $\quad ax^2+by^2=2z \quad \ldots(1)$

and let (x_1, y_1, z_1) be the centre of one of the plane sections of (1) drawn parallel to a given plane

$$lx+my+nz=p \quad \ldots(2)$$

Then equation of the plane section of (1) whose centre is (x_1, y_1, z_1) is

$$axx_1+byy_1-(z+z_1)=ax_1^2+by_1^2-2z_1 \quad \ldots(3) \quad | \ T=S_1$$

Now (2) is $\parallel$ to plane (3),

$$\therefore \quad \frac{ax_1}{l}=\frac{by_1}{m}=\frac{-1}{n}$$

$\therefore$ Locus of (x_1, y_1, z_1) is $\frac{ax}{l}=\frac{by}{m}=\frac{-1}{n}$ ...(4)

which is the line of intersection of the planes

$nax+l=0$ [from first and third members of (4)]

and $nby+m=0$ [from second and third members of (4)]

which are respectively ∥ to planes $x=0$ and $y=0$.

Thus the line (4) is ∥ to the Z-axis ($x=0$, $y=0$) which is the axis of the paraboloid, and consequently (3) is a diameter of (1). Hence the first part.

Second part. To find the extremity of diameter (3), we have to solve (4) and (1).

From (4), $x=-\frac{l}{na}$, $y=-\frac{m}{nb}$.

Putting these values of x, y in (1), we get

$$a\left(-\frac{l}{na}\right)^2+b\left(-\frac{m}{nb}\right)^2=2z$$

or $$z=\frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}.$$

Hence the extremity of the diameter is

$$\left(-\frac{l}{na}, -\frac{m}{nb}, \frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}\right).$$

$\therefore$ Equation of the tangent plane to (1) at this extremity is

$$ax\left(-\frac{l}{na}\right)+by\left(-\frac{m}{nb}\right)=z+\left(\frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}\right)$$

| Using $axx_1+byy_1=z+z_1$

or $$lx+my+nz+\frac{l^2}{2na}+\frac{m^2}{2nb}=0.$$

which is clearly ∥ to the given plane (2). Hence the result.

Article 27. To find the locus of intersection of three mutually perpendicular tangent planes to the paraboloid

$ax^2+by^2=2z$ [V. Imp.]

The given paraboloid is $ax^2+by^2=2z$ ...(1)

Let $l_1x+m_1y+n_1z=p_1$ (l_1, m_1, n_1 being the actual d.c.'s) be one of the three mutually ⊥ tangent planes so that

$$\frac{l_1^2}{a}+\frac{m_1^2}{b}=-2n_1p_1 \qquad \textit{Condition of tangency}$$

or $$p_1=-\frac{1}{2n_1}\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}\right).$$

Putting the value of p_1 the equation of one of the three mutually ⊥ tangent planes is

$$l_1x+m_1y+n_1z=-\frac{1}{2n_1}\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}\right)$$

Multiplying both sides by n_1

or $$n_1l_1x+m_1n_1y+n_1^2z+\frac{l_1^2}{2a}+\frac{m_1^2}{2b}=0 \text{ (Note this step)} \quad ...(2)$$

Similarly the equations of other two tangent planes is

$$n_2l_2x+m_2n_2y+n_2^2z+\frac{l_2^2}{2a}+\frac{m_2^2}{2b}=0 \quad ...(3)$$

$$n_3l_3x+m_3n_3y+n_3^2z+\frac{l_3^2}{2a}+\frac{m_3^2}{2b}=0 \quad ...(4)$$

The locus of the point of intersection of (2), (3), (4) is given by eliminating l_1, m_1, n_1 etc. from these equations. Adding (2), (3), (4), we get

$$x\Sigma l_1n_1+y\Sigma m_1n_1+z\Sigma n_1^2+\frac{1}{2a}\Sigma l_1^2+\frac{1}{2b}\Sigma m_1^2=0$$

or $$x(0)+y(0)+z(1)+\frac{1}{2a}(1)+\frac{1}{2b}(1)=0$$

or $$z+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

[$\because$ l_1, m_1, n_1, etc. *are the d.c.'s of three mutually ⊥ lines,* $\therefore$ $\Sigma l_1^2=\Sigma m_1^2=\Sigma n_1^2=1$; $\Sigma m_1n_1=0$ etc.]

which is the required locus.

It is clearly a plane ∥ to XY plane, *i.e.* ⊥ to the Z-axis, the axis of the paraboloid.

Article 28. Normal to the paraboloid.

To find the equations of the normal at the point (x_1, y_1, z_1) of paraboloid

(*i*) $ax^2+by^2=2z$

(*ii*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z.$

(*i*) The given paraboloid is $ax^2+by^2=2z$ ...(1)

The tangent plane at (x_1, y_1, z_1) to (1) is

$axx_1+byy_1=z+z_1$ | *Using the rule of tangent plane*

or $axx_1+byy_1-z-z_1=0.$ ...(2)

The d.c.'s of normal to this tangent plane are proportional to ax_1, by_1, -1. | *Coeffs. of x, y, z*

$\therefore$ Equations of the normal at (x_1, y_1, z_1), [*i.e.* a line through (x_1, y_1, z_1) and ⊥ to (2)] are

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{-1}.$$

(*ii*) Please try yourself.

Ans. $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{-1}$

Article 29. Number of normals

To prove that there are five points on an ellipticparaboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$, the normals at which pass through a given point (α, β, γ)

[V. Imp.] *(Allahabad 1982; L.N.M. 1982)*

The given paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ ...(1)

The normal at (x_1, y_1, z_1) is $\dfrac{x-x_1}{\frac{x_1}{a^2}}=\dfrac{y-y_1}{\frac{y_1}{b^2}}=\dfrac{z-z_1}{-1}$

This passes through (α, β, γ), if

$$\frac{\alpha-x_1}{\frac{x_1}{a^2}}=\frac{\beta-y_1}{\frac{y_1}{b^2}}=\frac{\gamma-z_1}{-1}=\lambda \text{ (say)}$$

From first and last members,

$$\alpha-x_1=\frac{x_1\lambda}{a^2} \text{ or } \alpha=x_1\left(1+\frac{\lambda}{a^2}\right)=\frac{\lambda+a^2}{a^2}x_1$$

$$\therefore \quad x_1=\frac{a^2\alpha}{a^2+\lambda}. \quad ...(2)$$

Similarly $\quad y_1=\frac{b^2\beta}{b^2+\lambda} \quad$ and $z_1=\gamma+\lambda$.

But since (x_1, y_1, z_1) lies on (1), $\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2z_1$

or $$\frac{1}{a^2}\cdot\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{1}{b^2}\cdot\frac{b^4\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda)$$

or $$a^2\alpha^2(b^2+\lambda)^2+b^2\beta^2(a^2+\lambda)^2=2(\lambda+\gamma)(a^2+\lambda)^2(b^2+\lambda)^2 \quad ...(3)$$

This equation being of fifth degree in λ gives five values of λ. Putting these values of λ in (2), we get five points on the paraboloid the normals at which pass through (α, β, γ). Hence the result.

Cor. From (2), the foot of normal is

$$\left(\frac{a^2\alpha}{a^2+\lambda}, \frac{b^2\beta}{b^2+\lambda}, \gamma+\lambda\right).$$

Article 30. Prove that the feet of normals from a given point (α, β, γ) to an elliptic paraboloid are the five points of intersection of the elliptic paraboloid and a certain cubic curve.

Let as an exercise for the student.

Proceed exactly as in Article 12 in the case of an ellipsoid.

Example 1. *Prove that the normals from (α, β, γ) to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ lie on the cone*

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

[V. Imp.] *(M.D.U. 1986, 85)*

Sol. The given paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \quad ...(1)$$

Let any line through (α, β, γ) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad ...(2)$$

be the normal at (x_1, y_1, z_1) to (1).

The equation of the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-(z+z_1)=0 \quad ...(3)$$

Since (2) is normal to (3) $\therefore$ it is ∥ to the normal to (3)

$$\therefore \quad \frac{l}{\frac{x_1}{a^2}}=\frac{m}{\frac{y_1}{b^2}}=\frac{n}{-1}=k \text{ (say)} \quad ...(4)$$

Again if the normal at (x_1, y_1, z_1) to (1) passes through (α, β, γ),

then $x_1=\frac{a^2\alpha}{a^2+\lambda}$, $y_1=\frac{b^2\beta}{b^2+\lambda}$, $z_1=\gamma+\lambda$. [From Eqn. (2) of Art. 29] ...(5)

From (4), $l=k\frac{x_1}{a^2}=\frac{k}{a^2}\cdot\frac{a^2\alpha}{a^2+\lambda}=\frac{k\alpha}{a^2+\lambda}$ | Using (5)

or $$a^2+\lambda=\frac{k\alpha}{l} \quad ...(6)$$

$$m=k\frac{y_1}{b^2}=\frac{k}{b^2}\cdot\frac{b^2\beta}{b^2+\lambda}=\frac{k\beta}{b^2+\lambda} \text{ or } b^2+\lambda=\frac{k\beta}{m} \quad ...(7)$$

$$n=-k \quad ...(8)$$

Subtracting (7) from (6), we get

$$a^2-b^2=k\left(\frac{\alpha}{l}-\frac{\beta}{m}\right)$$

$$=-n\left(\frac{\alpha}{l}-\frac{\beta}{m}\right) \quad ...(9) \quad | \text{ Using (8)}$$

To find the locus, we have to eliminate l, m, n from (2) and (9). Putting the value of l, m, n from (2) in (9), we have

$$a^2-b^2=-(z-\gamma)\left(\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}\right)$$

or $$\frac{a^2-b^2}{z-\gamma}=-\frac{\alpha}{x-\alpha}+\frac{\beta}{y-\beta}$$

or $$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0$$

which is the required result.

Example 2. *Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2+y^2=2az$. Prove also that if the point lies on the surface*
$$27a(x^2+y^2)+8(a-z)^3=0,$$
then two of the three normal coincide. [V. Imp.]

Sol. The given paraboloid is $x^2-y^2=2az$

or $$\frac{x^2}{a}+\frac{y^2}{a}=2z \qquad \ldots(1)$$

The normal at (x_1, y_1, z_1) to (1) is
$$\frac{x-x_1}{\frac{x_1}{a}}=\frac{y-y_1}{\frac{y_1}{a}}=\frac{z-z_1}{-1} \qquad \ldots(2)$$

If it passes through the given point (α, β, γ), then
$$\frac{\alpha-x_1}{\frac{x_1}{a}}=\frac{\beta-y_1}{\frac{y_1}{a}}=\frac{\gamma-z_1}{-1}=\lambda \text{ (say)} \qquad \ldots(3)$$

which give $$x_1=\frac{a\alpha}{a+\lambda},\ y_1=\frac{a\beta}{a+\lambda},\ z_1=\gamma+\lambda \qquad \ldots(4)$$

or $$\frac{1}{a}\cdot\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{1}{a}\cdot\frac{a^2\beta^2}{(a+\lambda)^2}=2(\gamma+\lambda)$$

or $$a\alpha^2+a\beta^2=2(\gamma+\lambda)(a+\lambda)^2 \qquad \ldots(5)$$

which being a third degree equation in λ, gives three values of λ. Putting these values of λ in (4), we get three points (x_1, y_1, z_1) on the paraboloid (1) at which the normals pass thro' (α, β, γ). Thus from a given point three normals can be drawn to (1).

Rewriting (5) as
$$f(\lambda)=2(\gamma+\lambda)(a+\lambda)^2-a(\alpha^2+\beta^2)=0 \qquad \ldots(6)$$
$$\therefore\quad f'(\lambda)=2(a+\lambda)^2+4(\gamma+\lambda)(a+\lambda)=0 \qquad \ldots(7)$$

If two of the normals coincide, then (6) must have two equal roots showing that $f(\lambda)$ and $f'(\lambda)$ must have a common linear factor.

From (7), $$2(a+\lambda)[a+\lambda+2(\gamma+\lambda)]=0$$

or $$\lambda=-\frac{a+2\gamma}{3} \qquad \because a+\lambda\neq 0$$

Putting this value of λ in (6), we get
$$a(\alpha^2+\beta^2)=2\left[\left(\gamma-\frac{a+2\gamma}{3}\right)\left(a-\frac{a+2\gamma}{3}\right)^2\right]$$
$$=2\left(\frac{\gamma-a}{3}\right)\left(\frac{2a-2\gamma}{3}\right)^2=-\frac{8}{27}(a-\gamma)^3$$

or $$a(\alpha^2+\beta^2)+\frac{8}{27}(a-\gamma)^3=0$$

or $$27a(\alpha^2+\beta^2)+8(a-\gamma)^3=0$$

Hence (α, β, γ) lies on
$$27a(x^2+y^2)+8(a-z)^3=0.$$

Example 3. *Show that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ lie on the sphere*
$$x^2+y^2+z^2-z(a+\gamma)-\frac{\gamma}{2\beta}(\alpha^2+\beta^2)=0.$$ [Imp.]

Sol. If (x_1, y_1, z_1) be the foot of normal through (α, β, γ) to the paraboloid

$$x^2+y^2=2az \text{ or } \frac{x^2}{a}+\frac{y^2}{a}=2z, \text{ then}$$
$$x_1=\frac{a\alpha}{a+\lambda},\ y_1=\frac{a\beta}{a+\lambda},\ z_1=\gamma+\lambda \qquad \ldots(1)$$
[See Example 2]

But (x_1, y_1, z_1) lies on the paraboloid, so we have
$$\frac{1}{a}\cdot\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{1}{a}\cdot\frac{a^2\beta^2}{(a+\lambda)^2}=2(\gamma+\lambda)$$

or $$\frac{a(\alpha^2+\beta^2)}{(a+\lambda)^2}=2(\gamma+\lambda) \qquad \ldots(2)$$

Now (x_1, y_1, z_1) will lie on the given sphere
$$x^2+y^2+z^2-z(a+\gamma)-\frac{\gamma}{2\beta}(\alpha^2+\beta^2)=0$$

if $$\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{a^2\beta^2}{(a+\lambda)^2}+(\gamma+\lambda)^2-(\gamma+\lambda)(a+\gamma)-\frac{a}{2(a+\lambda)}(\alpha^2+\beta^2)=0$$

or if $$\frac{a^2(\alpha^2+\beta^2)}{(a+\lambda)^2}+(\gamma+\lambda)^2-(\gamma+\lambda)(a+\gamma)-\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)}=0$$

or if $$\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}[2a-(a+\lambda)]+(\gamma+\lambda)[\gamma+\lambda-a-\gamma]=0$$

or if $$\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}(a-\lambda)+(\gamma+\lambda)(\lambda-a)=0$$

or if $$(a-\lambda)\left[\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}-\gamma-\lambda\right]=0$$

or if $\dfrac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}=\gamma+\lambda$ $\because$ $a-\lambda\neq 0$

which is true by (2). Hence the result.

Example 4. *Prove that the perpendicular from* (α, β, γ) *to the polar plane w.r.t.* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ *lies on the cone*

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Sol. The polar plane of (α, β, γ) w.r.t. the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \text{ is } \frac{\alpha x}{a^2}+\frac{\beta y}{b^2}=z+\gamma \quad \ldots(1)$$

Any line through (α, β, γ) is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(2)

If (2) is $\perp$ to the polar plane (1), then it is $\parallel$ to the normal to (1).

$$\therefore \quad \frac{\frac{\alpha}{a^2}}{l}=\frac{\frac{\beta}{b^2}}{m}=\frac{-1}{n}=k \text{ (say)}$$

$$\therefore \quad \frac{\alpha}{l}=a^2k,\ \frac{\beta}{m}=b^2k \text{ and } k=\frac{-1}{n}$$

$$\therefore \quad \frac{\alpha}{l}-\frac{\beta}{m}=(a^2-b^2)k=\frac{-(a^2-b^2)}{n} \quad \ldots(3)$$

This shows that line (2) lies on the cone [putting l, m, n from (2) in (3)]

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}=-\frac{(a^2-b^2)}{z-\gamma}$$

or

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Hence the result.

CONJUGATE DIAMETERS

(*Vikram 1984*)

1. We know that if l, m, n be proportional to the d.c.s. of a given system of parallel chords of the conicoid $ax^2+by^2+cz^2=1$ and if (α, β, γ) be the mid. pt. of any one of them, then the locus of the mid. pts. (α, β, γ) of the parallel chords is the plane

$$alx+bmy+cnz=0$$

which passes through the centre (0, 0, 0) of the conicoid. ...(*i*)

This plane is called the *diametral plane conjugate to the direction* l, m, n. Conversely, any plane $Ax+By+Cz=0$ ...(*ii*) through the centre is the diametral plane conjugate to the direction l, m, n given by $\dfrac{al}{A}=\dfrac{bm}{B}=\dfrac{cn}{C}$ [identifying (*i*) and (*ii*)]

Thus every central plane is a diametral plane conjugate to some direction.

If P be any point on the conicoid, then the plane bisecting chords parallel to OP is called the *diametral plane of* OP.

Note. In what follows, we shall confine our attention to the ellipsoid only.

2. Definitions

Conjugate Semi-diameters. Any three semi-diameters are called *Conjugate Semi-diameters* if the plane containing any two of them is the diametral plane of the third.

Conjugate Planes. Any three diametral planes are called *Conjugate Planes* if each is the diametral plane of the line of intersection of the other two.

3. **Relations between the coordinates of the extremities of a system of conjugate diameters of an ellipsoid.**

Let P (x_1, y_1, z_1) be any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \ldots(i)$$

Then the diametral plane of OP (*i.e.* the plane bisecting chords parallel to OP) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \quad \ldots(ii)$$

Let Q (x_2, y_2, z_2) be any point on the section of (*i*) by the plane (*ii*)

$$\therefore \quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0 \quad \ldots(iii)$$

which shows that Q lies on the diametral plane of OP.

The equation (*iii*) is also the condition that the diametral plane $\dfrac{xx_2}{a^2}+\dfrac{yy_2}{b^2}+\dfrac{zz_2}{c^2}=0$ of OQ passes through P.

Thus if the diametral plane of OP passes through Q, then the diametral plane of OQ also passes through P.

Let the line of intersection of the diametral planes of OP and OQ meet the surface (*i*) in R (x_3, y_3, z_3).

Since R lies on the diametral planes of OP and OQ, the diametral plane of OQ *i.e.* the plane $\dfrac{xx_2}{a^2}+\dfrac{yy_2}{b^2}+\dfrac{zz_2}{c^2}=0$ should pass through P and Q.

(The three semi-diameters OP, OQ, OR are such that the plane containing any two of them is the diametral plane of third. Hence they are called conjugate semi-diameters).

Since the points P, Q, R lie on (*I*)

$$\therefore\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1;\qquad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1,$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1 \qquad \ldots(I)$$

Since the diametral plane of OP passes through Q and R, diametral plane of OQ passes through R and P, diametral plane of OR passes through P and Q

$$\therefore\quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0$$

$$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0,\quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0, \qquad \ldots(II)$$

By virtue of the relations (*I*), we observe that

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c};\ \frac{x_2}{a},\ \frac{y_2}{b};\ \frac{z_2}{c}\ \text{and}\ \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c}$$

can be regarded as the *direction-cosines* of any three lines.

($\because$ if $l^2+m^2+n^2=1$, then l, m, n are *d.cs.* of a line.)

Also, by virtue of the relation (*II*), these lines are mutually perpendicular. Hence, we have $\dfrac{x_1}{a},\ \dfrac{x_2}{a},\ \dfrac{x_3}{a},\ \dfrac{y_1}{b},\ \dfrac{y_2}{b},\ \dfrac{y_3}{b}$ and $\dfrac{z_1}{c},\ \dfrac{z_2}{c},\ \dfrac{z_3}{c}$ as the *d.cs.* of another set of three mutually perpendicular lines.

[$\because$ if l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the *d.cs.* of three mutually perpendicular lines, then l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 are also the *d.cs.* of three mutually perpendicular lines.]

Hence we have

$$\left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}&=1\\ \frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{b^2}&=1\\ \frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}&=1\end{aligned}\right\}\ \text{and}\ \left\{\begin{aligned}\frac{x_1y_1}{ab}+\frac{x_2y_2}{ab}+\frac{x_3y_3}{ab}&=0\\ \frac{y_1z_1}{bc}+\frac{y_2z_2}{bc}+\frac{y_3z_3}{bc}&=0\\ \frac{z_1x_1}{ca}+\frac{z_2x_2}{ca}+\frac{z_3x_3}{ca}&=0\end{aligned}\right.$$

i.e.
$$\left.\begin{aligned}x_1^2+x_2^2+x_3^2&=a^2\\ y_1^2+y_2^2+y_3^2&=b^2\\ z_1^2+z_2^2+z_3^2&=c^2\end{aligned}\right\}\ \ldots(III)\ \text{and}\ \left.\begin{aligned}x_1y_1+x_2y_2+x_3y_3&=0\\ y_1z_1+y_2z_2+y_3z_3&=0\\ z_1x_1+z_2x_2+z_3x_3&=0\end{aligned}\right\}\ \ldots(IV).$$

The coordinates of the conjugates semi-diameters are connected by the relations (I), (II), (III) and (IV) above.

4. The sum of the squares of three conjugate semi-diameters of an ellipsoid is constant. (*Agra 1986*)

Let OP, OQ, OR be three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let the coordinates of P, Q, R be (x_r, y_r, z_r) $r=1, 2, 3$. Then $x_1^2+x_2^2+x_3^2=a^2$, $y_1^2+y_2^2+y_3^2=b^2$, $z_1^2+z_2^2+z_3^2=c^2$

[See relations (III), Art. 3]

Now $OP^2+OQ^2+OR^2$

$$=(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)+(x_3^2+y_3^2+z_3^2)$$
$$=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$$
$$=a^2+b^2+c^2 \text{ which is constant.}$$

5. The volume of the parallelopiped formed by three conjugate semi-diameters of an ellipsoid as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Then

$$\left.\begin{aligned}x_1^2+x_2^2+x_3^2&=a^2\\ y_1^2+y_2^2+y_3^2&=b^2\\ z_1^2+z_2^2+z_3^2&=c^2\end{aligned}\right\}\ \text{and}\ \left.\begin{aligned}x_1y_1+x_2y_2+x_3y_3&=0\\ y_1z_1+y_2z_2+y_3z_3&=0\\ z_1x_1+z_2x_2+z_3x_3&=0\end{aligned}\right\}\ \ldots(A)$$

Volume of the parallelopiped with OP, OQ, OR as coterminous edge $=V=6\times$ volume of tetrahedron OPQR.

$$=6\times\tfrac{1}{6}\begin{vmatrix}0&0&0&1\\ x_1&y_1&z_1&1\\ x_2&y_2&z_2&1\\ x_3&y_3&z_3&1\end{vmatrix}=\begin{vmatrix}x_1&y_1&z_1\\ x_2&y_2&z_2\\ x_3&y_3&z_3\end{vmatrix}\ \text{(numerically)}$$

$$\therefore\ V^2=\begin{vmatrix}x_1&y_1&z_1\\ x_2&y_2&z_2\\ x_3&y_3&z_3\end{vmatrix}\times\begin{vmatrix}x_1&y_1&z_1\\ x_2&y_2&z_2\\ x_3&y_3&z_3\end{vmatrix}$$

$$=\begin{vmatrix}\Sigma x_1^2&\Sigma x_1y_1&\Sigma x_1z_1\\ \Sigma x_1y_1&\Sigma y_1^2&\Sigma y_1z_1\\ \Sigma x_1z_1&\Sigma y_1z_1&\Sigma z_1^2\end{vmatrix}=\begin{vmatrix}a^2&0&0\\ 0&b^2&0\\ 0&0&c^2\end{vmatrix}\ \text{Using (A)}$$

$$=a^2b^2c^2$$

$\therefore$ $T=abc$, which is constant.

6. The sum of the squares of the areas of the faces of the parallelopiped with any three conjugate semi-diameters as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities

Since diametral plane of OP passes through Q and R

$$\therefore \begin{cases} \dfrac{x_2x_1}{a^2}+\dfrac{y_2y_1}{b^2}+\dfrac{z_2z_1}{c^2}=0 \\ \dfrac{x_3x_1}{a^2}+\dfrac{y_3y_1}{b^2}+\dfrac{z_3z_1}{c^2}=0 \end{cases}$$

$$\text{or} \begin{cases} \left(\dfrac{x_2}{a}\right)\dfrac{x_1}{a}+\left(\dfrac{y_2}{b}\right)\dfrac{y_1}{b}+\left(\dfrac{z_2}{c}\right)\dfrac{z_1}{c}=0 \\ \left(\dfrac{x_3}{a}\right)\dfrac{x_1}{a}+\left(\dfrac{y_3}{b}\right)\dfrac{y_1}{b}+\left(\dfrac{z_3}{c}\right)\dfrac{z_1}{c}=0 \end{cases}$$

$$\therefore \frac{\dfrac{x_1}{a}}{\dfrac{y_2z_3-y_3z_2}{bc}}=\frac{\dfrac{y_1}{b}}{\dfrac{z_2x_3-z_3x_2}{ca}}=\frac{\dfrac{z_1}{c}}{\dfrac{x_2y_3-x_3y_2}{ab}}$$

$$=\frac{\sqrt{\Sigma\dfrac{x_1^2}{a^2}}}{\sqrt{\Sigma\left(\dfrac{y_2z_3-y_3z_2}{bc}\right)^2}}=\pm\frac{1}{\sin\theta}=\pm\frac{1}{\sin 90^\circ}=\pm 1$$

[Since P lies on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ $\therefore$ $\Sigma\dfrac{x_1^2}{a^2}=1$

and $\dfrac{x_2}{a}, \dfrac{y_2}{b}, \dfrac{z_2}{c}$; $\dfrac{x_3}{a}, \dfrac{y_3}{b}, \dfrac{z_3}{c}$ are the d.c.s. of two perpendicular straight lines.]

$$\therefore \frac{x_1}{a}=\pm\frac{y_2z_3-y_3z_2}{bc}, \quad \frac{y_1}{b}=\pm\frac{z_2x_3-z_3x_2}{ca}, \quad \frac{z_1}{c}=\pm\frac{x_2y_3-x_3y_2}{ab} \quad \text{...(A)}$$

Let A_1, A_2, A_3 be the areas of the triangles OQR, ORP and OPQ and let l_r, m_r, n_r ($r=1, 2, 3$) be the d.c.s. of normals to these planes.

Projecting the $\triangle$OQR on the plane $x=0$, we get a triangle with vertices (0, 0, 0), (0, y_2, z_2) and (0, y_3, z_3) having area $=\frac{1}{2}(y_2z_3-y_3z_2)$. The area of the projection is also $A_1 l_1$.

$$A_1 l_1=\tfrac{1}{2}(y_2z_3-y_3z_2)=\pm\frac{bc\,x_1}{2a} \qquad \text{using (A)}$$

Similarly $A_1m_1=\pm\dfrac{ca y_1}{2b}$, $\qquad A_1 n_1=\pm\dfrac{abz_1}{2c}$

Squaring and adding

$$A_1^2=\frac{b^2c^2x_1^2}{4a^2}+\frac{c^2a^2y_1^2}{4b^2}+\frac{a^2b^2z_1^2}{4c^2} \qquad (\because\ l_1^2+m_1^2+n_1^2=1)$$

Similarly, projecting the areas ORP and OPQ on the coordinate planes, we get

$$A_2^2=\frac{b^2c^2x_2^2}{4a^2}+\frac{c^2a^2y_2^2}{4b^2}+\frac{a^2b^2z_2^2}{4c^2}$$

$$A_3^2=\frac{b^2c^2x_3^2}{4a^2}+\frac{c^2a^2y_3^2}{4b^2}+\frac{a^2b^2z_3^2}{4c^2}$$

Adding, we get

$$A_1^2+A_2^2+A_3^2=\frac{b^2c^2}{4a^2}\,\Sigma x_1^2+\frac{c^2a^2}{4b^2}\,\Sigma y_1^2+\frac{a^2b^2}{4c^2}\,\Sigma z_1^2$$

$$=\frac{b^2c^2}{4a^2}(a^2)+\frac{c^2a^2}{4b^2}(b)^2+\frac{a^2b^2}{4c^2}(c^2)$$

$$=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2) \text{ which is constant.}$$

7. The sum of the squares of the projection of any three conjugate semi-diameters on any line is constant.

Let l, m, n be the d.cs. of any given line and OP, OQ, OR be the three conjugate semi-diameters.

Projection of OP on this line is $lx_1+my_1+nz_1$

" " OQ " " " " $lx_2+my_2+nz_2$

" " OR " " " " $lx_3+my_3+nz_3$

$\therefore$ the sum of the squares of the projections of OP, OQ and OR on this line.

$$=(lx_1+my_1+nz_1)^2+(lx_2+my_2+nz_2)^2+(lx_3+my_3+nz_3)^2$$

$$=l^2\Sigma x_1^2+m^2\Sigma y_1^2+n^2\Sigma z_1^2+2lm\Sigma x_1y_1+2mn\,\Sigma y_1z_1+2nl\Sigma z_1x_1$$

$$=l^2(a^2)+m^2(b^2)+n^2(c^2)+2ml(0)+2mn(0)+2nl(0)$$

$$=a^2l^2+b^2m^2+c^2n^2 \quad \text{... which is constant.}$$

8. The sum of the squares of the projections of any three conjugate semi-diameters on any plane is constant.

Let l, m, n be the d.c.s. of the normal to any given plane. Let OP, OQ, OR be the three conjugate semi-diameters.

Sum of the squares of the projections of OP, OQ and OR on this plane

$$=[OP^2-(lx_1+my_1+nz_1)^2]+[OQ^2-(lx_2+my_2+nz_2)^2]+[OR^2-(lx_3+my_3+nz_3)^2]$$

$$=(OP^2+OQ^2+OR^2)-[(lx_1+my_1+nz_1)^2+(lx_2+my_2+nz_2)^2+(lx_3+my_3+nz_3)^2]$$

$$=a^2+b^2+c^2-(a^2l^2+b^2m^2+c^2n^2) \quad [\text{using Arts. 4 and 7}$$

$$=a^2(1-l^2)+b^2(1-m^2)+c^2(1-n^2)$$

$$=a^2(m^2+n^2)+b^2(n^2+l^2)+c^2(l^2+m^2)$$

which is constant. $\qquad (\because\ l^2+m^2+n^2=1)$

Example 1. *If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the extremities of three conjugate semi-diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

find the equation of the plane through these points.

(Agra 1984, 81 ; Allahabad 1981 ; Rohilkhand 1980)

Sol. Let P, Q, R be the extremities (x_r, y_r, z_r), $r=1, 2, 3$ of the three conjugate semi-diameters OP, OQ, OR.

Let the equation of the plane PQR be

$$lx+my+nz=p \quad \ldots(i)$$

Since P, Q, R all lie on (*i*)

$$\therefore \quad lx_1+my_1+nz_1=p \quad \ldots(ii)$$

$$lx_2+my_2+nz_2=p \quad \ldots(iii)$$

$$lx_3+my_3+nz_3=p \quad \ldots(iv)$$

Multiplying (*ii*) by x_1, (*iii*) by x_2 and (*iv*) by x_3 and adding, we get

$$l\Sigma x_1^2+m\Sigma x_1y_1+n\Sigma z_1x_1=p(x_1+x_2+x_3)$$

or $$la^2+m(0)+n(0)=p(x_1+x_2+x_3)$$

$$\therefore \quad l=\frac{p}{a^2}(x_1+x_2+x_3)$$

Similarly multiplying (*ii*), (*iii*), (*iv*) by y_1, y_2, y_3 respectively and adding

$$m=\frac{p}{b^2}(y_1+y_2+y_3)$$

and multiplying (*ii*), (*iii*), (*iv*) by z_1, z_2, z_3 respectively and adding

$$n=\frac{p}{c^2}(z_1+z_2+z_3)$$

Substituting the values of l, m, n in (*i*), equation of plane PQR is

$$\frac{p}{a^2}(x_1+x_2+x_3)x+\frac{p}{b^2}(y_1+y_2+y_3)y+\frac{p}{c^2}(z_1+z_2+z_3)z=p$$

or $$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

Example 2. *Show that the plane PQR, where P, Q, R are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$ at the centroid of the triangle PQR.*

(Agra 1982 ; [illegible])

Sol. Proceeding as in Ex. 1, the equation of plane PQR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad \ldots(i)$$

The centroid of ΔPQR is

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

The tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3} \text{ at G is}$$

$$\frac{x\left(\frac{x_1+x_2+x_3}{3}\right)}{a^2}+\frac{y\left(\frac{y_1+y_2+y_3}{3}\right)}{b^2}+\frac{z\left(\frac{z_1+z_2+z_3}{3}\right)}{c^2}=\frac{1}{3}$$

or $$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1$$

which is the same as (*i*).

Hence the plane PQR touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3} \text{ at G.}$$

Example 3. *Prove that the pole of the plane PQR lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$, where OP, OQ, OR are three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.*

Sol. Equation of the plane PQR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad \ldots(i)$$

| See Ex. 1

Let (x', y', z') be the pole of the plane PQR.

Equation to the polar plane of (x', y', z') w.r.t. the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is}$$

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1 \quad \ldots(ii)$$

Comparing (*i*) and (*ii*), we have

$$\therefore \left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2+\left(\frac{z'}{c}\right)^2$$

$$=\frac{1}{a^2}(x_1+x_2+x_3)^2+\frac{1}{b^2}(y_1+y_2+y_3)^2+\frac{1}{c^2}(z_1+z_2+z_3)^2$$

$$=\frac{1}{a^2}\Sigma x_1^2+\frac{1}{b^2}\Sigma y_1^2+\frac{1}{c^2}\Sigma z_1^2+\frac{2}{a^2}\Sigma x_1x_2+\frac{2}{b^2}\Sigma y_1y_2+\frac{2}{c^2}\Sigma z_1z_2$$

$$=\frac{1}{a^2}(a^2)+\frac{1}{b^2}(b^2)+\frac{1}{c^2}(c^2)+\frac{2}{a^2}(0)+\frac{2}{b^2}(0)+\frac{2}{c^2}(0)$$

$$=3$$

Hence (x', y', z') lies on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3.$$

Example 4. *Prove that the locus of the foot of the perpendicular from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to the plane PQR through the extremities of three conjugate semi-diameters is*

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2.$$

Sol. Equation to the plane PQR through the extremities of three conjugate semi-diameters OP, OQ, OR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad ...(i)$$

| See Ex. 1

Let $D(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the centre $(0, 0, 0)$ to the plane (i).

Then the d.c.s. of OD, the normal to (i) are proportional to α, β, γ.

$$\therefore \frac{\frac{x_1+x_2+x_3}{a^2}}{\alpha}=\frac{\frac{y_1+y_2+y_3}{b^2}}{\beta}=\frac{\frac{z_1+z_2+z_3}{c^2}}{\gamma}=\frac{1}{\lambda}\text{ (say)} \quad ...(ii)$$

Since (α, β, γ) lies on (i)

$$\therefore \Sigma\frac{\alpha(x_1+x_2+x_3)}{a^2}=1 \quad ...(iii)$$

From (ii), we have

$$\frac{\alpha^2}{\lambda}+\frac{\beta^2}{\lambda}+\frac{\gamma^2}{\lambda}=\Sigma\frac{\alpha(x_1+x_2+x_3)}{a^2}=1 \quad \text{By } (iii)$$

$$\therefore \lambda=\alpha^2+\beta^2+\gamma^2 \quad ...(iv)$$

Also from (ii)

$$(x_1+x_2+x_3)^2=\left(\frac{a^2\alpha}{\lambda}\right)^2 \therefore \frac{(x_1+x_2+x_3)^2}{a^2}=\frac{a^2\alpha^2}{\lambda^2}$$

or $$\frac{\Sigma x_1^2+2\Sigma x_1x_2}{a^2}=\frac{a^2\alpha^2}{\lambda^2}$$

or $$\frac{a^2}{a^2}=\frac{a^2\alpha^2}{\lambda^2} \text{ since } \Sigma x_1^2=a^2, \Sigma x_1x_2=0$$

or $$\frac{a^2\alpha^2}{\lambda^2}=1$$

Similarly $\frac{b^2\beta^2}{\lambda^2}=1, \quad \frac{c^2\gamma^2}{\lambda^2}=1$

Adding $\frac{a^2\alpha^2}{\lambda^2}+\frac{b^2\beta^2}{\lambda^2}+\frac{c^2\gamma^2}{\lambda^2}=3$

or $a^2\alpha^2+b^2\beta^2+c^2\gamma^2=3\lambda^2=3(\alpha^2+\beta^2+\gamma^2)^2$. | using (iv)

Hence the locus of (α, β, γ) is

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2.$$

Example 5. *Find the locus of the equal conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \text{ (Vikram 1984 ; L.N.M. 1982)}$$

Sol. Let OP, OQ, OR be three equal conjugate semi-diameters. Then we have

$$OP^2+OQ^2+OR^2=a^2+b^2+c^2$$

$\because$ $OP^2=OQ^2=OR^2$

$\therefore$ each $=\frac{1}{3}(a^2+b^2+c^2)$ ...(i)

Let P be the point (x_1, y_1, z_1). Equations of OP are

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1} \quad ...(ii)$$

where $x_1^2+y_1^2+z_1^2=OP^2=\frac{1}{3}(a^2+b^2+c^2)$ ...(iii)

| using (i)

Also $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \quad ...(iv)$$

($\because$ P lies on the ellipsoid).

From (iii) and (iv), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=\frac{3(x_1^2+y_1^2+z_1^2)}{a^2+b^2+c^2} \quad ...(v)$$

| (each $=1$)

Eliminating x_1, y_1, z_1 from (*ii*) and (*v*), the required locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2}.$$

Example 6. *Prove that the plane through a pair of equal conjugate semi-diameters of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *touches the cone* $\frac{x^2}{a^2(2a^2-b^2-c^2)}+\frac{y^2}{b^2(2b^2-c^2-a^2)}+\frac{z^2}{c^2(2c^2-a^2-b^2)}=0.$

Sol. Let P, Q, R be the extremities of three equal conjugate semi-diameters OP, OQ, OR.

Let the coordinates of P be (x_1, y_1, z_1). Also let

The plane OQR through the conjugate semi-diameters OQ and OR is the diameter plane of OP.

$\therefore$ The equation of the plane OQR is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \quad ...(i)$$

The plane (*i*) will touch the cone $\Sigma \frac{x^2}{a^2(2a^2-b^2-c^2)}=0$

if $\Sigma \frac{x_1^2}{a^4}\cdot a^2(2a^2-b^2-c^2)=0$

$\left(\because\right.$ The plane $lx+my+nz=p$ touches $ax^2+by^2+cz^2=1$

if $\left.\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\right)$

or if $\Sigma \frac{x_1^2}{a^2}[3a^2-(a^2+b^2+c^2)]=0$

or if $\Sigma 3x_1^2-\Sigma \frac{x_1^2}{a^2}(a^2+b^2+c^2)=0$

or if $3\Sigma x_1^2=(a^2+b^2+c^2)\,\Sigma \frac{x_1^2}{a^2}$

or if $3(x_1^2+y_1^2+z_1^2)=(a^2+b^2+c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)$

or if $3\,.\,OP^2=(a^2+b^2+c^2)(1)$ $\quad |\because$ P lies on the ellipsoid

or if $3r^2=a^2+b^2+c^2$

which is true $\because OP^2+OQ^2+OR^2=a^2+b^2+c^2$

and $OP=OQ=OR=r$

Hence the result.

Example 7. *Prove that the locus of the section of the ellipsoid* $\sum\left(\frac{x^2}{a^2}\right)=1$ *by the plane PQR is the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}.$$

Sol. The equation of the plane PQR is

$$\left(\frac{x_1+x_2+x_3}{a^2}\right)x+\left(\frac{y_1+y_2+y_3}{b^2}\right)y+\left(\frac{z_1+z_2+z_3}{c^2}\right)z=1 \quad ...(i)$$

If (α, β, γ) be the centre of the section of the given ellipsoid by the plane PQR, then the equation of PQR can be written as "$T=S_1$"

i.e. $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad ...(ii)$$

$\therefore$ Equations (*i*) and (*ii*) represent the same plane, therefore, comparing them, we get

$$\frac{x_1+x_2+x_3}{\alpha}=\frac{y_1+y_2+y_3}{\beta}=\frac{z_1+z_2+z_3}{\gamma}=\frac{1}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}}$$

where $$\frac{\alpha}{a}=\left(\frac{x_1+x_2+x_3}{a}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Similarly,

$$\frac{\beta}{b}=\left(\frac{y_1+y_2+y_3}{b}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

and $$\frac{\gamma}{c}=\left(\frac{z_1+z_2+z_3}{c}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Squaring and adding, we get

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2\left[\left(\frac{x_1+x_2+x_3}{a^2}\right)^2+\left(\frac{y_1+y_2+y_3}{b^2}\right)^2+\left(\frac{z_1+z_2+z_3}{c^2}\right)^2\right]$$

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=3\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2$$

or $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{1}{3}$$

$\therefore$ The required locus of (α, β, γ) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}.$$

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Example 8. *Prove that the locus of the point of intersection of three tangent planes to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *which are parallel to conjugate diametral planes of* $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ *is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$

Sol. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \qquad \ldots(i)$$

Then the diametral plane of P w.r.t. to (*i*) is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=0 \qquad \ldots(ii)$$

Any plane parallel to (*ii*) is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=p_1 \qquad \ldots(iii)$$

If (*iii*) is a tangent plane to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, then

"$p^2=a^2l^2+b^2m^2+c^2n^2$",

or $$p_1^2=a^2\left(\frac{x_1}{\alpha^2}\right)^2+b^2\left(\frac{y_1}{\beta^2}\right)^2+c^2\left(\frac{z_1}{\gamma^2}\right)^2. \qquad \ldots(iv)$$

Similarly the equation of other planes parallel to OQ and OR are

$$\frac{xx_2}{\alpha^2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}=p_2 \qquad \ldots(v)$$

and $$\frac{xx_3}{\alpha^2}+\frac{yy_3}{\beta^2}+\frac{zz_3}{\gamma^2}=p_3 \qquad \ldots(vi)$$

where $$p_2^2=\sum\left[a^2\left(\frac{x_2}{\alpha^2}\right)^2\right]$$

and $$p_3^2=\sum\left[a^2\left(\frac{x_3}{\alpha^2}\right)^2\right]$$

Now for the locus of the point of intersection of (*iii*), (*iv*) and (*vi*), we square and add (*iii*), (*iv*), (*vi*) and get

$$\sum\frac{x^2}{\alpha^4}(x_1^2+x_2^2+x_3^2)+\sum\frac{2xy}{\alpha^2\beta^2}(x_1y_1+x_2y_2+x_3y_3)$$
$$=p_1^2+p_2^2+p_3^2$$

or $$\sum\left[\frac{x^2}{\alpha^4}(\alpha^2)\right]+\sum\frac{2xy}{\alpha^2\beta^2}(0)=\sum\frac{a^2}{\alpha^4}(x_1^2+x_2^2+x_3^2),$$

from (*iv*) and (*vii*) using $\Sigma x_1^2=\alpha^2$ and $x_1y_1=0$ etc.

or $$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^4}(\alpha^2)+\frac{b^2}{\beta^4}(\beta^2)+\frac{c^2}{\gamma^4}(\gamma^2),$$

Since $\Sigma x_1^2=\alpha^2$ etc.

or $$\frac{x^2}{a^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$$

Set – VII

Reduction of General Equation of Second degree

General equation of the second degree:-

The most general equation of second degree is written as $F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ——— (1)

(or)

$F(x,y,z) \equiv f(x,y,z)+2ux+2vy+2wz+d=0$ ——— (2)

where $f(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$ ——— (3)

The equation (1) contains ~~ten~~ unknowns constants which can be reduced to nine effective constants by dividing the equation throughout by 'd'.

Thus a surface can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

Note (1):- In all discussions in this chapter, we shall take $F(x,y,z)$ and $f(x,y,z)$ as defined in (1) and (3) above i.e $f(x,y,z)$ will be taken as the homogeneous part of $F(x,y,z)$.

Note:- (2) Here $\frac{\partial f}{\partial x} = 2(ax+hy+gz)$

$\frac{\partial f}{\partial y} = 2(hx+by+fz)$

$\frac{\partial f}{\partial z} = 2(gx+fy+cz)$; and

$\frac{\partial F}{\partial x} = 2(ax+hy+gz+u)$, $\frac{\partial F}{\partial y} = 2(hx+by+fz+v)$, $\frac{\partial F}{\partial z} = 2(gx+fy+cz+w)$

* Determination of the centre of surface $F(x,y,z)=0$:-

Let (x_1, y_1, z_1) be the centre of the surface $f(x,y,z)=0$.

Shifting the origin to the centre (x_1, y_1, z_1), the transformed equation of the surface

$$f(x+x_1, y+y_1, z+z_1)=0$$

i.e
$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1)+2h(x+x_1)(y+y_1)+2u(x+x_1)+2v(y+y_1)+2w(z+z_1)+d=0.$$

$$\Rightarrow f(x,y,z)+2x(ax_1+hy_1+gz_1+u)+2y(hx_1+by_1+fz_1+v)+2z(gx_1+fy_1+cz_1+w)+(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d)=0 \quad \text{--- (I)}$$

where
$$f(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

Now as the centre of (I) is origin, so it should be homogeneous in (x,y,z) (Since if (x',y',z') is a point on it, $(-x',-y',-z')$ must also lie on it as $(0,0,0)$, the mid-point of the chord joining (x',y',z') and $(-x',-y',-z')$, is the centre of the surface $f(x,y,z)=0$ and therefore only second degree terms must exist in (I).)

$\therefore$ From (I) we have $ax_1 + hy_1 + gz_1 + u = 0$ ...(II)

$hx_1 + by_1 + fz_1 + v = 0$ ...(III)

$gx_1 + fy_1 + cz_1 + w = 0$ ...(IV)

Also constant term in (I) can be rewritten as

$$x_1(ax_1 + hy_1 + gz_1 + u) + y_1(hx_1 + by_1 + fz_1 + v) + z_1(gx_1 + fy_1 + cz_1 + w) + (ux_1 + vy_1 + wz_1 + d)$$

$= ux_1 + vy_1 + wz_1 + d$, with the help of (II), (III), (IV).

$= d'$, (say)

Then (I) reduces to $f(x, y, z) + d' = 0$ ...(V)

where $d' = ux_1 + vy_1 + wz_1 + d.$ ...(VI)

And x_1, y_1, z_1 is obtained from (II), (III) and (IV), which can be obtained from

$\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$ replacing x, y, z by x_1, y_1, z_1.

Hence centre of the surface $F(x, y, z) = 0$, is given by solving

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0 \text{ for } x, y, z$$

and the equation of the surface referred to centre as origin is

$$f(x, y, z) + (ux_1 + vy_1 + wz_1 + d) = 0, \qquad \text{...(VII)}$$

where (x_1, y_1, z_1) is the centre of the surface. (Remember)

Note. The equations (II), (III), (IV) may or may not give a unique centre. There may be more than one centre, a line of centres or a plane of centres depending upon the nature of solutions of the above three equations.

From (II), (III), (IV) we find that the centre (x_1, y_1, z_1) lies on the planes

$$\left.\begin{aligned} ax + hy + gz + u &= 0 \\ hx + by + fz + v &= 0 \\ gx + fy + cz + w &= 0 \end{aligned}\right\} \qquad \text{...(VIII)}$$

and

These planes are known as central planes and any point common to these planes is a centre.

§ 12.03. Transformation of f (x, y, z).

To show that by the rotation of axes the expression $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transforms to $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(ab + bc + ca - f^2 - g^2 - h^2) - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(A + B + C) - D = 0,$

where A, B, C are the cofactors of a, b, c respectively in the determinant

$$D = \begin{vmatrix} c & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

We know that the expression $x^2 + y^2 + z^2$ is an invariant when the rectangular axes are rotated through the same origin.

(See chapter on Change of axes)

$\therefore$ If we put $x = l_1x + m_1y + n_1z$, $y = l_2x + m_2y + n_2z$

and $z = l_3x + m_3y + n_3z$ in $x^2 + y^2 + z^2$, then by the relations

$l_1^2 + m_1^2 + n_1^2 = 1,\ l_2^2 + m_2^2 + n_2^2 = 1,\ l_3^2 + m_3^2 + n_3^2 = 1;$

$l_1^2 + l_2^2 + l_3^2 = 1,\ m_1^2 + m_2^2 + m_3^2 = 1,\ n_1^2 + n_2^2 + n_3^2 = 1,$

$l_1l_2 + m_1m_2 + n_1n_2 = 0,\ l_2l_3 + m_2m_3 + n_2n_3 = 0,\ l_3l_1 + m_3m_1 + n_3n_1 = 0$

$l_1m_1 + l_2m_2 + l_3m_3 = 0,\ m_1n_1 + m_2n_2 + m_3n_3 = 0,\ n_1l_1 + n_2l_2 + n_3l_3 = 0,$

it remains unchanged.

Now if the axes are rotated in such a manner that $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ becomes $\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2$, then the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2) \qquad \text{...(i)}$$

should reduce to $\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2 - \lambda(x^2 + y^2 + z^2)$ ...(ii)

i.e. both the expressions (i) and (ii) will be the product of linear factors for the same value of λ.

Now if (i) *i.e.* if $(a-\lambda)x^2 + (b-\lambda) + (c-\lambda)z^2 + 2fyz + 2gzx + 2hxy$ is the product of two linear factors then we must have

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \qquad \text{...(iii)}$$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(ab + bc + ca - f^2 - g^2 - h^2) - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(A + B + C) - D = 0,$

where A, B, C are the cofactors of corresponding small letters in the determinant
$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

And the expression $(\lambda_1 - \lambda)x^2 + (\lambda_2 - \lambda)y^2 + (\lambda_3 - \lambda)z^2$ in (ii) will be the product of two linear factors, if

$$\begin{vmatrix} \lambda_1-\lambda & 0 & 0 \\ 0 & \lambda_2-\lambda & 0 \\ 0 & 0 & \lambda_3-\lambda \end{vmatrix} = 0 \text{ i.e., } (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = 0$$

i.e. when $\lambda = \lambda_1$ or λ_2 or λ_3.

The same values of λ should be obtained from (iii) also.

Hence $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic (iii) in λ, which is called discriminating cubic.

Let λ be any root (real or imaginary) of the discriminating cubic (iii) and that l, m, n be the principal direction cosines (which may also be real or imaginary) corresponding to this value of λ, then

$$al + hm + gn = \lambda l$$
$$hl + bm + fn = \lambda m$$
$$gl + fm + cn = \lambda n \qquad \text{[See chapter on Change of axes]}$$

or

$$(a - \lambda) l + hm + gn = 0$$
$$hl + (b - \lambda) m + fn = 0$$
$$gl + fm + (c - \lambda) n = 0$$

where λ is to be replaced by $\lambda_1, \lambda_2, \lambda_3$, to get the corresponding direction-cosines of the axes.

§ 12.04. Various Forms of General Equation of Second Degree.

The general equation of second degree viz. $F(x, y, z) = 0$, as given in § 12.01 Page 1 of this chapter, can be reduced to any one of the following forms :—

S. No.	Equation	Name of the surface
1.	$Ax^2 + By^2 + Cz^2 = 1$	Ellipsoid
2.	$Ax^2 + By^2 - Cz^2 = 1$	Hyperboloid of one sheet
3.	$Ax^2 - By^2 - Cz^2 = 1$	Hyperboloid of two sheets
4.	$Ax^2 + By^2 + Cz^2 = 0$	Cone
5.	$Ax^2 + By^2 + 2kz = 0$	Elliptic paraboloid
6.	$Ax^2 - By^2 + 2kz = 0$	Hyperbolic paraboloid
7.	$Ax^2 + By^2 + d = 0$	Elliptic cylinder
8.	$Ax^2 - By^2 + d = 0$	Hyperbolic cylinder
9.	$Ax^2 - By^2 = 0$	Pair of planes
If second degree terms form a perfect square, then		
10.	$Ax^2 + Bx + C = 0$	Pair of parallel planes
11.	$y^2 = Ax$	Parabolic cylinder
12.	$A(x^2 + y^2) + Bz = 0$	Paraboloid of revolution
13.	$A(x^2 + y^2) + Cz^2 = 1$	Ellipsoid of revolution
14.	$A(x^2 - y^2) + Cz^2 = 1$	Hyperboloid of revolution

Note. In discussion to follow we shall take

$$f(l, m, n) \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm,$$

whence

$$\frac{\partial f}{\partial l} = 2(al + bm + gn);$$

$$\frac{\partial f}{\partial m} = 2(hl + bm + fn), \quad \frac{\partial f}{\partial n} = 2(gl + fm + cn)$$

§ 12.05. Equation of surface referred to centre as origin.

From § 12.02, result (VIII) on Page 2 of this chapter we know that the centre (x_1, y_1, z_1) of $F(x, y, z) = 0$; lies on the planes given by

$$ax + hy + gz + u = 0 \qquad \text{...(i)}$$
$$hx + by + fz + v = 0 \qquad \text{...(ii)}$$
$$gx + fy + cz + w = 0 \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by A, H, G respectively and adding we get

$$Dx + (Au + Hv + Gw) = 0, \qquad \text{...(iv)}$$

where A, B, C, F, G, H are the cofactors of the corresponding small letters viz. a, b, c, f, g, h in the determinant $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Similarly multiplying (i), (ii), (iii) by H, B, F and G, F, C respectively and adding separately, we get

$$Dy + (Hu + Bv + Fw) = 0 \qquad \text{...(v)}$$
$$Dz + (Gu + Fv + Cw) = 0 \qquad \text{...(vi)}$$

$\therefore$ From (iv), (v) and (vi) the coordinates of the centre are given by

$$\frac{x}{Au + Hv + Gw} = \frac{y}{Hu + Bv + Fw} = \frac{z}{Gu + Fv + Cw} = -\frac{1}{D} \qquad \text{...(A)}$$

Cor. 1. The equation of a diametral plane of the surface (conicoid) $F(x, y, z) = 0$ is

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0 \qquad \text{...(B)}$$

and so any diametral plane passes through the centre or centres.

Cor. 2. From (II), (III), (IV) and (VI) of § 12.02 Page 2 of this chapter we have

$$ax_1 + hy_1 + gz_1 + u = 0$$
$$hx_1 + by_1 + fz_1 + v = 0$$
$$gx_1 + fy_1 + cz_1 + w = 0$$

and

$$ux_1 + vy_1 + wz_1 + (d - d') = 0$$

which on eliminating x_1, y_1, z_1 gives

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d - d' \end{vmatrix} = 0$$

or

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} - d' \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or $P - d'D = 0$ or $d' = P/D$, ...(C)

where

$$P = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \text{ and } D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Hence referred to centre as origin [See result (VII) of § 12.02, Page 2 of this chapter] the equation of the surface $F(x, y, z) = 0$ is

$$f(x, y, z) + (P/D) = 0 \quad \text{...(E)}$$

*§ 12.06. Some properties of determinant D.

We know $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and A, B, C, F, G, H denote the cofactors of the corresponding small letters in this determinant D.

$$\therefore \quad A = bc - f^2, \; B = ca - g^2, \; C = ab - h^2,$$
$$F = gh - af, \; G = hf - bg, \; H = fg - ch.$$

Also $BC - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$

$$= a^2bc - abg^2 - ach^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2afgh$$
$$= a(abc + 2fgh - af^2 - bg^2 - ch^2) = aD$$

Similarly $CA - G^2 = bD$, $AB - H^2 = cD$,

$$GH - AF = fD, \; HF - BG = gD, \; FG - CH = hD.$$

And from the properties of determinants (See Author's Algebra or Matrices) we know that

$$Aa + Hh + Gg = D, \; Ha + Bh + Fg = 0, \; Ga + Fh + Cg = 0$$

and similar other results.

(i) If $D = 0$, from above we have

$$BC = F^2, \; CA = G^2, \; AB = H^2, \; GH = AF, \; HF = BG, \; FG = CH$$

(ii) If $D = 0$ and $A = 0$, then we have $G = 0$, $H = 0$.

(iii) If $D = 0$ and $A = 0$, $B = 0$, then $F = 0$, $G = 0$, $H = 0$ but C may or may not be zero.

(iv) If $D = 0$ and $H = 0$, then either $A = 0$, $G = 0$ or $B = 0$, $F = 0$.

(v) If $D = 0$ and $A + B + C = 0$, then

$$A = B = C = F = G = H = 0. \quad \text{(Note)}$$

since, A, B, C have the same sign when $D = 0$ and so $A + B + C = 0$ gives $A = B = C = 0$, whence $F = 0 = G = H$.

§ 12.07. Some facts about planes (to be remembered).

Let there be two equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \quad \text{...(I)}$$

each representing a plane.

These two equations will represent the same plane, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} \quad \text{...(II)}$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = 0$$

The planes given by (I) will be parallel but not the same provided

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2} \quad \text{...(III)}$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0, \; \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} \neq 0$$

and will intersect in a line provided

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \quad \text{...(IV)}$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \; \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0.$$

§ 12.08. Various Cases.

Here we shall consider the various cases which depend on the solution of the equations (i), (ii), (iii) of § 12.05 on Page 5 of this chapter.

Case I. $D \neq 0$.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are finite and unique.

$\therefore$ The surface (conicoid) $F(x, y, z) = 0$ has a unique centre at a finite distance.

Case II. $D = 0$ and $Au + Hv + Gw \neq 0$.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are infinite, provided

$Au + Hv + Gw$, $Hu + Bv + Fw$ and $Gu + Fv + Cw$ are not zero.

Thus the surface $F(x, y, z) = 0$ has a single centre at infinity.

Case III. $D = 0$, $Au + Hv + Gw = 0$.

If we denote the equations (i), (ii) and (iii) of § 12.05 on Page 5 of this chapter by $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ respectively then we can see that

$$AS_1 + HS_2 + GS_3 = 0.$$

$\therefore$ The central planes (see definition on § 12.02 Page 2 of this chapter) have a common line of intersection.

Also if $A = bc - f^2 \neq 0$, then the planes $S_2 = 0$ and $S_3 = 0$ are neither identical nor parallel. So there is a definite line of intersection and the surface $F(x, y, z) = 0$ in this case possesses a line of centres at a finite distance.

We can easily see that when $D = 0$ and $Au + Hv + Gw = 0$ but $A \neq 0$, then $Hu + Bv + Fw = 0$ and $Gu + Fv - Cw$ are also zero, since in this case from § 12.06 (i) Page 6 we have $F = \sqrt{(BC)}$, $G = \sqrt{(CA)}$, $H = \sqrt{(AB)}$ ...(α)

$$\because Au + Hv + Gw = 0 \Rightarrow \sqrt{A}\,(\sqrt{A}u + \sqrt{B}v + \sqrt{C}w) = 0$$
$$\Rightarrow \sqrt{A}u + \sqrt{B}v + \sqrt{C}w = 0, \; \because A \neq 0, \quad \text{...(β)}$$

Now $Hu + Bv + Fw = \sqrt{(AB)}\,u + Bv + \sqrt{(BC)}\,w$

$$= \sqrt{B}\,[\sqrt{A}u + \sqrt{B}v + \sqrt{C}w] = 0, \text{ from (β)}$$

Similarly we can prove that $Gu + Fv + Cw = 0$

[Also we can see from above that if $D = 0$, $A = 0$, then from (α) we get
$$G = 0, \; H = 0.$$

Hence in this case $Au + Hv + Gw = 0$ but $Hu + Bv + Fw$ and $Gu + Fv + Cw$ may or may not be zero].

Case IV. A, B, C, F, G, H are all zero.

As in case III above, the central planes have a common line of intersection. But these planes are parallel as is evident from § 12.07 Page 6 of this chapter.

Also we assume that $fu - gv \neq 0$, because otherwise the two planes given by (i) and (ii) of § 12.05 Page 5 of this chapter would be identical and similarly $gu - hw \neq 0$, as otherwise the planes given by (i) and (iii) of § 12.05 Page 5 of this chapter would be identical.

Hence in this case central planes [given by (i), (ii) and (iii) of § 12.05 Page 5 of this chapter] are parallel but not coincident and so the surface $F(x, y, z) = 0$ has a line of centres at an infinite distance.

Case V. A, B, C, F, G, H are all zero and $fu = gv = hw$.

In this case if f, g, h are not zero, the central planes (as discussed above in Case IV) are identical and so the surface $F(x, y, z) = 0$ has a plane of centres.

In case all or two of f, g, h are zero, we can deal the case directly.

§ 12.09. Reduction of general equation.

In § 12.04 Page 4 of this chapter, we have seen the various forms of the surfaces represented by the general equation of second degree. Now we shall discuss in articles to follow the reduction to the standard forms depending upon the various cases as given in § 12.08 on Pages 7–8 Ch. XII.

§ 12.10. Case I. $D \neq 0$.

In this case there is a unique centre at a finite distance. Also none of the roots of the discriminating cubic (or λ-cubic) vanishes and so $D \neq 0$.

Here the forms to any one of which the given equation can reduce are :—

(i) $Ax^2 + By^2 + Cz^2 = 1$ (Ellipsoid)

(ii) $Ax^2 + By^2 - Cz^2 = 1$ (Hyperboloid of one sheet)

(iii) $Ax^2 - By^2 - Cz^2 = 1$ (Hyperboloid of two sheets)

(iv) $Ax^2 + By^2 + Cz^2 = 0$ (Cone)

Method of Procedure.

(i) Find the coordinates (x_1, y_1, z_1) of the centre of the given surface $F(x, y, z) = 0$ by solving the equations
$$\frac{\partial F}{\partial x} = 0, \; \frac{\partial F}{\partial y} = 0, \; \frac{\partial F}{\partial z} = 0.$$

(ii) Shift the origin to the centre (x_1, y_1, z_1) and then the equation of the surface referred to centre as origin is
$$f(x, y, z) + d' = 0, \text{ where } d' = ux_1 + vy_1 + wz_1 + d.$$

(iii) By rotation of axes, transform the given equation to the form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic, which can be reduced to any one of the forms given above.

(iv) The direction ratios of axes can be obtained by solving any two of the following three equations :—
$$(a - \lambda) l + hm + gn = 0$$
$$hl + (b - \lambda) m + fn = 0$$
$$gl + fm + (c - \lambda) n = 0.$$

Putting the three values of λ, the direction ratios of the three axes can be obtained and so their equations can be obtained *i.e.* we can find the equations of three lines through the centre and having above direction-ratios.

(v) The principal planes are given by
$$\lambda (lx + my + nz) + (ul + vm + wn) = 0.$$

(vi) If $d' = 0$, then the surface is a cone. (Remember)

Note : For the solution of a cubic equation, students should go through the section on solution of cubic equations from Author's Theory of Equations. It is not always possible to solve a cubic equation when all its roots are real, but with the help of Descarte's Rule of signs, we can find the number of its positive and negative roots.

Solved Examples on § 12.10.

*Ex. 1. Reduce the equation $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$ to the standard form. Find the nature of the conicoid, its centre and equations of its axes.

Sol. Let $F(x, y, z) \equiv 3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$

Then the coordinates of the centre are given by
$$\frac{\partial F}{\partial x} = 0, \; \frac{\partial F}{\partial y} = 0, \; \frac{\partial F}{\partial z} = 0,$$

i.e.
$$\left.\begin{array}{lll} 6x + 2z + 2y - 4 = 0 & \text{or} & 3x + y + z - 2 = 0 \\ 10y + 2z + 2x = 0 & \text{or} & x + 5y + z = 0 \\ 6z + 2y + 2x - 8 = 0 & \text{or} & x + y + 3z - 4 = 0 \end{array}\right\} \quad \text{...(I)}$$

Solving the equations of (I) we get the centre (x_1, y_1, z_1) as $(1/3, -1/3, 4/3)$ *i.e.* $x_1 = 1/3$, $y_1 = -1/3$, $z_1 = 4/3$.

Shifting the origin to the centre $(1/3, -1/3, 4/3)$, the equation of the surface reduces to
$$f(x, y, z) + d' = 0,$$
where $d' = ux_1 + vy_1 + wz_1 + d$ ...(II)
$$= (-2)(1/3) + (0)(-1/3) + (-4)(4/3) + 5 = -1$$

$\therefore$ From (II), the reduced equation of the surface is
$$(3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy) + (-1) = 0. \quad \text{...(III)}$$

as $f(x, y, z)$ is the homogeneous part of $F(x, y, z)$.

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or $(3-\lambda)[(5-\lambda)(3-\lambda)-1]-[(3-\lambda)-1]+[1-(5-\lambda)]=0$

or $$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0. \quad \text{...(IV)}$$

By trial we find that $\lambda = 2$ satisfies (IV), so we have $(\lambda - 2)$ as a factor of L.H.S. of (IV) and so we can rewrite (IV) as

$(\lambda - 2)(\lambda^2 - 9\lambda + 18) = 0$ or $(\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$.

$\therefore$ The roots of the discriminating cubic (IV) are 2, 3, 6.

$\therefore$ By rotation of axes, the given equation transforms to

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$ i.e. $2x^2 + 3y^2 + 6z^2 - 1 = 0$, ...(V)

substituting values of $\lambda_1, \lambda_2, \lambda_3$ and d'.

Then equation (V) can be rewritten as $2x^2 + 3y^2 + 6z^2 = 1$, which represents an ellipsoid.

The direction-ratios of axes can be obtained by solving two of the following three equations

$(a-\lambda)l + hm + gn = 0,\ hl + (b-\lambda)m + fn = 0,\ gl + fm + (c-\lambda)n = 0$

or $(3-\lambda)l + m + n = 0,\ l + (5-\lambda)m + n = 0,\ l + m + (3-\lambda)n = 0$ ...(VI)

Taking $\lambda = 2$, we have $l + m + n = 0,\ l + 3m + n = 0,\ l + m + n = 0.$

Solving $l + m + n = 0,\ l + 3m + n = 0$, we get

$$\frac{l}{1-3} = \frac{m}{1-1} = \frac{n}{3-1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$$

$\therefore$ The equations of the axis, corresponding to $\lambda = 2$, are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{0} = \frac{z-(4/3)}{-1} \quad \text{Ans.}$$

Similarly corresponding to $\lambda = 3$ and $\lambda = 6$ the direction ratios of the axes (*i.e.* the principal directions) are

$$\frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \quad \text{and} \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

As the equation of the corresponding axes are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-1} = \frac{z-(4/3)}{1};$$

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1};$$

Ex. 2. Reduce the equation $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$ to the standard form. Also find its centre and the equation referred to centre as origin. (Avadh 95)

Solution. Given $F(x, y, z) \equiv 3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0.$

$\therefore$ The coordinates of the centre are given by

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$$

i.e. $6x - 6 = 0$ or $x - 1 = 0$ or $x = 1$ ...(i)

$-2y + 6z + 6 = 0$ or $y - 3z - 3 = 0$ ...(ii)

and $-2z - 2 = 0$ or $z + 1 = 0$ or $z = -1$ ...(iii)

$\therefore$ Solving (i), (ii) and (iii) we get the centre (x_1, y_1, z_1) as $(1, 0, -1)$. **Ans.**

Shifting the origin to the centre $(1, 0, -1)$ the equation of the surface reduces to $f(x, y, z) + d' = 0,$

where $d' = ux_1 + vy_1 + wz_1 + d$

$= (-3)(1) + (3)(0) + (-1)(-1) - 2 = -4$

$\therefore$ From (iv), the equation of the surface referred to centre as origin is

$(3x^2 - y^2 - z^2 + 6yz) + (-4) = 0$ or $3x^2 - y^2 - z^2 + 6yz - 4 = 0$ **Ans.**

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -1-\lambda & 3 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0,$$

putting values of a, b, c, f, g, h

or $(3-\lambda)[(1+\lambda)^2 - 9] = 0$ or $(\lambda - 3)[\lambda^2 + 2\lambda - 8] = 0$

or $(\lambda - 3)(\lambda - 2)(\lambda + 4) = 0,$ or $\lambda = 2, 3, -4.$

$\therefore$ By rotation of axes, the given equation transforms to

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$ i.e. $2x^2 + 3y^2 - 4z^2 - 4 = 0$

or $2x^2 + 3y^2 - 4z^2 = 4$, which represents a hyperboloid of one sheet

[$\because$ it is of the form $Ax^2 + By^2 - Cz^2 = 1$]. **Ans.**

Ex. 3. Show that the equation $x^2 + y^2 + z^2 - 6yz - 2zx - 2xy - 6x - 2y - 2z + 2 = 0$ represents a hyperboloid of two sheets.

Solution. Comparing the given equation $F(x, y, z) = 0$ with the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$, we have $a = 1$, $b = 1, c = 1, f = -3, g = -1, h = -1, u = -3, v = -1, w = -1, d = 2.$ ...(i)

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$$

i.e. $2x_1 - 2y_1 - 2z_1 - 6 = 0$ or $x_1 - y_1 - z_1 = 3$ ...(ii)

$2y_1 - 6z_1 - 2x_1 - 2 = 0$ or $x_1 - y_1 + 3z_1 = -2$ ...(iii)

$2z_1 - 6y_1 - 2x_1 - 2 = 0$ or $x_1 + 3y_1 - z_1 = -2$ ...(iv)

Solving (ii), (iii) and (iv) we get $x_1 = 1/2,\ y_1 = -5/4,\ z_1 = -5/4$

$\therefore$ centre of the given surface is $(1/2, -5/4, -5/4)$.

Also $d' = ux_1 + vy_1 + wz_1 + d$

$= (-3)(1/2) + (-1)(-5/4) + (-1)(-5/4) + 2 = 3$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -3 \\ -1 & -3 & 1-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & \lambda-4 \\ -1 & -3 & 4-\lambda \end{vmatrix} = 0, \text{ applying } C_3 - C_2$$

or $\lambda^3 - 3\lambda^2 - 8\lambda + 16 = 0$ or $(\lambda - 4)(\lambda^2 + \lambda - 4) = 0$

$\therefore$ Either $\lambda = 4$ or $\lambda^2 + \lambda - 4 = 0$.

Now $\lambda^2 + \lambda - 4 = 0$ gives $\lambda = [-1 \pm \sqrt{(1+16)}]/2$.

Thus we find that two values of λ are +ve and one –ve.

$\therefore$ By rotation of axes, the given equation transforms to

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$ *i.e.* $-\dfrac{\lambda_1}{3}x^2 - \dfrac{\lambda_2}{3}y^2 - \dfrac{\lambda_3}{3}z^2 = 1$, $\because d' = 3$.

Now two values of λ being positive and one negative, from above the equation of the surface transforms to the form $Ax^2 + By^2 + Cz^2 = 1$, where two of A, B, C are negative and third positive; so that the given surface is a hyperboloid of two sheets.

Ex. 4. Reduce the equation $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$ to the standard form. What does it represent?

Sol. Comparing the given equation $F(x, y, z) = 0$ with the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$,

we have $a = 2, b = -7, c = 2, f = -5, g = -4, h = -5, u = 3, v = 6, w = -3, d = 5$.

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x} = 0, \; \frac{\partial F}{\partial y} = 0, \text{ and } \frac{\partial F}{\partial z} = 0.$$

i.e. $4x_1 - 8z_1 - 10y_1 + 6 = 0$ or $2x_1 - 5y_1 - 4z_1 + 3 = 0$; ...(ii)

$-14y_1 - 10z_1 - 10x_1 + 12 = 0$ or $5x_1 + 7y_1 + 5z_1 - 6 = 0$; ...(iii)

$4z_1 - 10y_1 - 8x_1 - 6 = 0$ or $4x_1 + 5y_1 - 2z_1 + 3 = 0$; ...(iv)

Solving (ii), (iii) and (iv) we get $x_1 = 1/3, y_1 = -1/3, z_1 = 4/3$.

$\therefore$ Centre of the given surface is $(1/3, -1/3, 4/3)$.

Also $d' = ux_1 + vy_1 + wz_1 + d$

$= 3(1/3) + 6(-1/3) + (-3)(4/3) + 5 = 1 - 2 - 4 + 5 = 0$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix} = 0$$

or $(2-\lambda)[-(7+\lambda)(2-\lambda) - 25] + 5[-5(2-\lambda) - 20] - 4[25 - 4(7+\lambda)] = 0$

or $\lambda^3 + 3\lambda^2 - 90\lambda + 216 = 0$, on simplifying

or $(\lambda - 3)(\lambda^2 + 6\lambda - 72) = 0$ or $(\lambda - 3)(\lambda - 6)(\lambda + 12) = 0$

or $\lambda = 3, 6, -12$

$\therefore$ Let $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = -12$

$\therefore$ By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

or $3x^2 + 6y^2 - 12z^2 + 0 = 0$, substituting values of $\lambda_1, \lambda_2, \lambda_3, d'$

or $x^2 + 2y^2 - 4z^2 = 0$, which is the required standard form and represents a cone. [See § 12.04 (4) Page 4 Ch. XII]

Also the vertex of the cone (and not centre) is $(1/3, -1/3, 4/3)$ as calculated above.

Exercises on § 12.10 (Case I).

Ex. 1. Reduce the equation $11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$ to the standard form and show that it represents an ellipsoid and find the equations of the axes. (*Avadh* 91; *Garhwal* 94, 92)

Ans. Centre $(-2, 2, -1)$; $3x^2 + 6y^2 + 18z^2 = 12$ (ellipsoid)

d.r's of the axes are 1, 1, 2; 2, 1, –2; –2, 2, –1.

Ex. 2. Reduce $3x^2 + 6yz - y^2 - z^2 - 6x + 6y - 2z - 2 = 0$ to the standard form. What surface does it represent?

Ans. $2x^2 + 3y^2 - 4z^2 = 4$; Hyperboloid of one sheet.

Ex. 3. Reduce $2x^2 - y^2 - 10z^2 + 20yz - 8zx - 28xy + 16x + 26y + 16z - 34 = 0$ to the standard form. What does it represent ?

Ans. $2x^2 - y^2 - 2z^2 = 1$; Hyperboloid of two sheets.

Ex. 4. For the conicoid $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$, show that

(i) all the roots of the discriminating cubic are real,

and (ii) principal directions are perpendicular to each other. (*Rohilkhand* 91)

Ex. 5. Reduce $x^2 + 3y^2 + 3z^2 - 2yz - 2x - 2y + 6z + 3 = 0$ to the standard form and show that the surface represented by it is an ellipsoid.

Ans. $x^2 + 2y^2 + 4z^2 = 1$.

§ 12.11. Case II. $D = 0$ and $Au + Hv + Gw \neq 0$.

$D = 0 \Rightarrow$ one root of the discriminating cubic is zero. (Note)

Here the forms to any one of which the given equation can reduce are

$Ax^2 + By^2 + Cz = 0$. (Elliptic Paraboloid)

and $Ax^2 - By^2 + Cz = 0$. (Hyperbolic Paraboloid)

Method of Procedure.

(i) Find the discriminating cubic viz $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$

One root of this cubic will be zero in this case. (Note)

(ii) Put $\lambda = 0$ in the above determinant and associate each row with l_3, m_3, n_3.

i.e. $al_3 + hm_3 + gn_3 = 0$, $hl_3 + bm_3 + fn_3 = 0$, $gl_3 + fm_3 + cn_3 = 0$.

Solve any two of these, which will give the direction ratios of the axis corresponding to $\lambda = 0$.

(iii) Evaluate $k = ul_3 + vm_3 + wn_3$, (Remember)

where l_3, m_3, n_3 are actual direction-cosines.

If $k \neq 0$, then reduced equation is

$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$, where λ_1, λ_2 are non-zero roots of the discriminating cubic.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1 and λ_2 have the same or opposite signs.

(iv) Vertex. The coordinates of the vertex of the paraboloid in this case are obtained by solving any two of the three equations

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l_3} = \frac{\left(\frac{\partial F}{\partial y}\right)}{m_3} = \frac{\left(\frac{\partial F}{\partial z}\right)}{n_3} = 2k,$$

with the equation $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ (Remember)

Solved Examples on § 12.11.

**Ex. 1. Determine completely what is represented by the equation

$$2x^2 + 2y^2 + z^2 + 2yz - 2zx - 4xy + x + y = 0.$$

Find the coordinates of its vertex and the equations to its axis. (*Garhwal* 93)

Solution. Here 'a' = 2, 'b' = 2, 'c' = 1, 'f' = 1, 'g' = − 1, 'h' = − 2, 'u' = 1/2, 'v' = 1/2, 'w' = 0 and 'd' = 0

∴ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 2-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0 \qquad \ldots(i)$$

or $(2-\lambda)\{(2-\lambda)(1-\lambda)-1\} + 2\{-2(1-\lambda)+1\} - \{-2+(2-\lambda)\} = 0$.

or $\lambda^3 - 5\lambda^2 + 2\lambda = 0$ or $\lambda(\lambda^2 - 5\lambda + 2) = 0$ or $\lambda = 0, \frac{1}{2}[5 \pm \sqrt{(17)}]$

∴ Let $\lambda_1 = \frac{1}{2}[5 + \sqrt{(17)}]$, $\lambda_2 = \frac{1}{2}[5 - \sqrt{(17)}]$, $\lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$2l_3 - 2m_3 - n_3 = 0$, $-2l_3 - 2m_3 + n_3 = 0$, $-l_3 + m_3 + n_3 = 0$

Solving last two equations simultaneously for l_3, m_3, n_3, we get

$$\frac{l_3}{2-1} = \frac{m_3}{-1+2} = \frac{n_3}{-2+2}$$

i.e.
$$\frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{0} = \frac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(1^2 + 1^2 + 0^2)}} = \frac{1}{\sqrt{2}}$$

∴ $l_3 = 1/\sqrt{2}$, $m_3 = 1/\sqrt{2}$, $n_3 = 0$.

These gives d.c.'s of the axis corresponding to $\lambda = 0$.

Now $k = ul_3 + vm_3 + wn_3 = (\frac{1}{2})(1/\sqrt{2}) + (\frac{1}{2})(1/\sqrt{2}) + 0 = 1/\sqrt{2}$

∴ The required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\frac{1}{2}[5 + \sqrt{(17)}]x^2 + \frac{1}{2}[5 - \sqrt{(17)}]y^2 + 2(1/\sqrt{2})z = 0$,

which represents an elliptic paraboloid as both λ_1, λ_2 are positive. (Note)

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the three equations

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l_3} = \frac{\left(\frac{\partial F}{\partial y}\right)}{m_3} = \frac{\left(\frac{\partial F}{\partial z}\right)}{n_3} = 2k.$$

along with $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

...See § 12.11 (iv) Page 14 Ch. XII

i.e. any two of the equations $4x - 2z - 4y + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$

or $2x - 2y - z = 0$;

$4y + 2z - 4x + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$ or $2x - 2y - z = 0$;

$2z + 2y + 2x = 2(0)(1/\sqrt{2})$ or $x - y - z = 0$

with $\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 0z\right] + \frac{1}{2}x + \frac{1}{2}y + 0 + 0 = 0$

i.e. $2x - 2y - z = 0$, $x - y - z = 0$, $x + y = 0$

Solving these we get $x = 0, y = 0, z = 0$ *i.e.* the coordinates of the vertex are (0, 0, 0). Ans.

∴ The equations to its axis are

$$\frac{x-0}{l_3} = \frac{y-0}{m_3} = \frac{z-0}{n_3} \quad \textit{i.e.} \quad \frac{x-0}{(1/\sqrt{2})} = \frac{y-0}{(1/\sqrt{2})} = \frac{z-0}{0}$$

i.e. $x = y, z = 0$. Ans.

*Ex. 2. Reduce the equation $3z^2 - 6yz - 6zx - 7x - 5y + 6z + 3 = 0$ to standard form and find its nature. (*Avadh* 94)

Sol. Here 'a' = 0, 'b' = 0, 'c' = 3, 'f' = − 3, 'g' = − 3, 'h' = 0, 'u' = − 7/2, 'v' = − 5/2, 'w' = 3 and 'd' = 3

∴ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & -3 \\ 0 & 0-\lambda & -3 \\ -3 & -3 & 3-\lambda \end{vmatrix} = 0 \qquad \ldots(i)$$

or $-\lambda[-\lambda(3-\lambda) - 9] + 0 - 3[-3\lambda] = 0$ or $\lambda^3 - 3\lambda^2 - 18\lambda = 0$

or $\lambda(\lambda^2 - 3\lambda - 18) = 0$ or $\lambda(\lambda - 6)(\lambda + 3) = 0$ or $\lambda = 0, 6, -3$

Let $\lambda_1 = 6$, $\lambda_2 = -3$, $\lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$0 \cdot l_3 + 0 \cdot m_3 - 3n_3 = 0$; $0 \cdot l_3 + 0 \cdot m_3 - 3n_3 = 0$, $-3l_3 - 3m_3 + 3n_3 = 0$

These gives $n_3 = 0$, $l_3 + m_3 = 0$

i.e. $\dfrac{l_3}{1} = \dfrac{m_3}{-1} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{\{1^2 + (-1)^2 + 0^2\}}} = \dfrac{1}{\sqrt{2}}$

$\therefore l_3 = 1/\sqrt{2}$, $m_3 = -1/\sqrt{2}$, $n_3 = 0$

These gives d.c's of the axis corresponding to $\lambda = 0$.

Now $k = u l_3 + v m_3 + w n_3 = -\dfrac{7}{2}\left(\dfrac{1}{\sqrt{2}}\right) - \dfrac{5}{2}\left(\dfrac{1}{\sqrt{2}}\right) + 3(0) = -\dfrac{1}{\sqrt{2}}$

$\therefore$ Required reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0 \quad \text{or} \quad 6x^2 - 3y^2 - \sqrt{2}z = 0,$$

which represents a hyperbolic paraboloid as λ_1 and λ_2 are of opposite signs. Ans.

Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$. (Rohilkhand 95)

Solution. Here $a' = 4$, $b' = -1$, $c' = -1$, $f' = 1$, $g' = 0$, $h' = 0$, $u' = -4$, $v' = -2$, $w' = 4$ and $d' = -2$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

or $(4-\lambda)[(1+\lambda)^2 - 1] = 0$ or $\lambda(\lambda+2)(\lambda-4) = 0$ or $\lambda = 0, -2, 4$.

$\therefore$ Let $\lambda_1 = -2$, $\lambda_2 = 4$, $\lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$4l_3 = 0, \ -m_3 + n_3 = 0, \ m_3 - n_3 = 0 \Rightarrow l_3 = 0, \ m_3 = n_3.$$

But $l_3^2 + m_3^2 + n_3^2 = 1$, so $0 + m_3^2 + m_3^2 = 1$ or $m_3 = 1/\sqrt{2} = n_3$

$\therefore$ We have $l_3 = 0$, $m_3 = 1/\sqrt{2}$, $n_3 = 1/\sqrt{2}$.

Now $k = u l_3 + v m_3 + w n_3 = -4(0) - 2(1/\sqrt{2}) + 4(1/\sqrt{2}) = \sqrt{2}$

$\therefore$ Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad -2x^2 + 4y^2 + 2\sqrt{2}z = 0 \quad$ or $\quad x^2 - 2y^2 - z\sqrt{2} = 0$,

which represents a hyperbolic paraboloid as λ_1 is $-$ve and λ_2 is $+$ ve.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations.

$$\frac{(\partial F/\partial x)}{l_3} = \frac{(\partial F/\partial y)}{m_3} = \frac{(\partial F/\partial z)}{n_3} = 2k$$

and $\quad k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ ...See § 12.11 (iv) Page 14.

i.e. any two of the equations

$$8x - 8 = 2(\sqrt{2})(0) \ \text{i.e.} \ x = 1;$$
$$-2y + 2z - 4 = 2(\sqrt{2})(1/\sqrt{2}) \ \text{i.e.} \ y - z + 3 = 0$$
$$-2z + 2y + 8 = 2(\sqrt{2})(1/\sqrt{2}) \ \text{i.e.} \ y - z + 3 = 0$$

with $\quad \sqrt{2}\left[0 \cdot x + \dfrac{1}{\sqrt{2}} y + \dfrac{1}{\sqrt{2}} z\right] - 4x - 2y + 4z - 2 = 0$

i.e. $x = 1$, $y - z + 3 = 0$, $4x + y - 5z + 2 = 0$

Solving these we get $x = 1$, $y = -9/4$, $z = 3/4$.

$\therefore$ Coordinates of the vertex are $(1, -9/4, 3/4)$ Ans.

And the equations of its axis are $\dfrac{x-1}{l_3} = \dfrac{y-(-9/4)}{m_3} = \dfrac{z-(3/4)}{n_3}$

or $\quad \dfrac{x-1}{0} = \dfrac{y+(9/4)}{1/\sqrt{2}} = \dfrac{z-(3/4)}{1/\sqrt{2}} \quad$ i.e. $\quad \dfrac{x-1}{0} = \dfrac{4y+9}{1} = \dfrac{4z-3}{1}$ Ans.

*Ex. 4. Show that the following equation represents a paraboloid. Find its vertex and equations to the axis.

$$4y^2 + 4z^2 + 4yz - 2x - 14y - 22z + 33 = 0.$$ (Rohilkhand 92, 90)

Solution. Here $a' = 0$, $b' = 4$, $c' = 4$, $f' = 2$, $g' = 0$, $h' = 0$, $u' = -1$, $v' = -7$, $w' = -11$ and $d' = 33$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0 \quad \text{...(i)}$$

or $\quad -\lambda[(4-\lambda)^2 - 4] = 0 \quad$ or $\quad \lambda[\lambda^2 - 8\lambda + 12] = 0$

or $\quad \lambda(\lambda-2)(\lambda-6) = 0 \quad$ or $\quad \lambda = 0, 2, 6$

$\therefore$ Let $\lambda_1 = 2$, $\lambda_2 = 6$ and $\lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3 we have $4m_3 + 2n_3 = 0$, $2m_3 + 4n_3 = 0 \Rightarrow m_3 = 0 = n_3$

But $l_3^2 + m_3^2 + n_3^2 = 1$, so $l_3^2 + 0 + 0 = 1 \Rightarrow l_3 = 1$

Now $k = u l_3 + v m_3 + w n_3 = -1(1) - 7(0) - 11(0) = -1$

$\therefore$ Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad 2x^2 + 6y^2 + 2(-1)z = 0 \quad$ or $\quad x^2 + 3y^2 - z = 0$,

which represents an elliptic paraboloid as both λ_1 and λ_2 are positive.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations

$$\frac{\partial F/\partial x}{l_3} = \frac{\partial F/\partial y}{m_3} = \frac{\partial F/\partial z}{n_3} = 2k$$

and $\quad k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$, ...See § 12.11 (iv) P. 14.

i.e. any two of the equations

$-2 = 2(-1)(0)$ which is absurd.

$8y + 4z - 14 = 2(-1).0 \Rightarrow 4y + 2z = 7$;

$8x + 4y - 22 = 2(-1)(0) \Rightarrow 2y + 4z = 11$;

with $(-1)[1 . x + 0 . y + 0 . z] - x - 7y - 11z + 33 = 0$ or $2x + 7y + 11z = 33$

i.e. $4y + 2z = 7,\ 2y + 4z = 11,\ 2x + 7y + 11z = 33$

Solving these we get $x = 1,\ y = 1/2,\ z = 5/2$

$\therefore$ Coordinates of the vertex are $(1, 1/2, 5/2)$ **Ans.**

And the equations of its axis are $\dfrac{x-1}{l_3} = \dfrac{y-(1/2)}{m_3} = \dfrac{z-(5/2)}{n_3}$

or $\dfrac{x-1}{1} = \dfrac{y-(1/2)}{0} = \dfrac{z-(5/2)}{0}$ or $\dfrac{x-1}{1} = \dfrac{2y-1}{0} = \dfrac{2z-5}{0}$

or $2y - 1 = 0,\ 2z - 5 = 0$ or $y = 1/2,\ z = 5/2.$

**Ex. 5. Prove that $z(ax + by + cz) + \alpha x + \beta y = 0$ represents a paraboloid and the equations to the axis are

$$ax + by + 2cz = 0,\ (a^2 + b^2) z + a\alpha + b\beta = 0. \quad (\textit{Rohilkhand } 93)$$

Sol. Given equation is $cz^2 + byz + azx + \alpha x + \beta y = 0$

$\therefore$ Here $'a' = 0,\ 'b' = 0,\ 'c' = c,\ 'b' = b/2,\ 'g' = a/2,\ 'h' = 0,\ 'u' = \alpha/2,$ $'v' = \beta/2\ 'w' = 0$ and $'d' = 0$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & a/2 \\ 0 & 0-\lambda & b/2 \\ a/2 & b/2 & c-\lambda \end{vmatrix} = 0 \quad \text{...(i)}$$

or $-\lambda[-\lambda(c-\lambda) - (b^2/4)] + (a/2)[a\lambda/2] = 0$

or $4\lambda^3 - 4a\lambda^2 - (a^2 + b^2)\lambda = 0$ or $\lambda[4\lambda^2 - 4a\lambda - a^2 - b^2] = 0$

or $\lambda = 0$ and $\lambda = \dfrac{4a \pm \sqrt{(16a^2 + 16a^2 + 16b^2)}}{8} = \dfrac{a \pm \sqrt{(2a^2 + b^2)}}{2}$

Let $\lambda_1 = \frac{1}{2}\left[a + \sqrt{(2a^2 + b^2)}\right],\ \lambda_2 = \frac{1}{2}\left[a - \sqrt{(2a^2 + b^2)}\right],\ \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$0 . l_3 + 0 . m_3 + (a/2) n_3 = 0,\ 0 . l_3 + 0 . m_3 + (b/2) n_3 = 0$$

and $$(a/2) l_3 + (b/2) m_3 + cn_3 = 0$$

These gives $n_3 = 0$ and $al_3 + bm_3 = 0$

i.e. $\dfrac{l_3}{b} = \dfrac{m_3}{-a} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(b^2 + a^2 + 0)}} = \dfrac{1}{\sqrt{(a^2 + b^2)}}$

$\therefore l_3 = b/\sqrt{(a^2 + b^2)},\ m_3 = -a/\sqrt{(a^2 + b^2)},\ n_3 = 0$

Now $k = ul_3 + vm_3 + wn_3$

$$= \frac{\alpha}{2} \cdot \frac{b}{\sqrt{(a^2 + b^2)}} + \frac{\beta}{2}\left[\frac{-a}{\sqrt{(a^2 + b^2)}}\right] + 0 = \frac{b\alpha - a\beta}{2\sqrt{(a^2 + b^2)}} \neq 0$$

$\therefore$ Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\frac{1}{2}\left[a + \sqrt{(2a^2 + b^2)}\right]x^2 + \frac{1}{2}\left[a - \sqrt{(2a^2 + b^2)}\right]y^2 + \dfrac{(b\alpha - a\beta)}{\sqrt{(a^2 + b^2)}} z = 0$ ...(ii)

Now as $a + \sqrt{(2a^2 + b^2)} > 0$ and $a - \sqrt{(2a^2 + b^2)} < 0$, so (ii) represents a hypebolic paraboloid.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations

$$\frac{\partial F/\partial x}{l_3} = \frac{\partial F/\partial y}{m_3} = \frac{\partial F/\partial z}{n_3} = 2k.$$

and $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ ...See § 12.11 (iv) P. 14 Ch. XII

i.e. any two of the equations

$$\frac{\alpha + az}{b/\sqrt{(a^2 + b^2)}} = \frac{\beta + bz}{-a/\sqrt{(a^2 + b^2)}} = \frac{ax + by + 2cz}{0} = \frac{2(b\alpha - a\beta)}{2\sqrt{(a^2 + b^2)}}$$

and $k\left[\dfrac{bx}{\sqrt{(a^2 + b^2)}} - \dfrac{ay}{\sqrt{(a^2 + b^2)}} + 0\right] + \dfrac{\alpha}{2}x + \dfrac{\beta}{2}y = 0$, on substituting the values

Thus we have $\dfrac{\alpha + az}{b} = \dfrac{\beta + bz}{-a}$, $ax + by + 2cz = 0$

and $$\frac{b\alpha - a\beta}{2\sqrt{(a^2 + b^2)}}\left[\frac{bx - ay}{\sqrt{(a^2 + b^2)}}\right] + \frac{1}{2}(\alpha x + \beta y) = 0$$

i.e. $(a^2 + b^2) z + a\alpha + b\beta = 0,\quad ax + by + 2cz = 0$

and $(b\alpha - a\beta)(bx - ay) + (\alpha x + \beta y)(a^2 + b^2) = 0$

Now if (x_1, y_1, z_1) be the vertex of the paraboloid then x, y, z satisfies above three equations

i.e. $$(a^2 + b^2) z_1 + a\alpha + b\beta = 0, \quad \text{...(iii)}$$

$$ax_1 + by_1 + 2cz_1 = 0 \quad \text{...(iv)}$$

and $$(b\alpha - a\beta)(bx - ay) + (\alpha x + \beta y)(a^2 + b^2) = 0 \quad \text{...(v)}$$

And the equations of the axis are $\dfrac{x - x_1}{l_3} = \dfrac{y - y_1}{m_3} = \dfrac{z - z_1}{n_3}$,

or $$\frac{x - x_1}{b/\sqrt{(a^2 + b^2)}} = \frac{y - y_1}{-a/\sqrt{(a^2 + b^2)}} = \frac{z - z_1}{0}, \quad \text{...(vi)}$$

substituting values of l_3, m_3, n_3

These give $z - z_1 = 0$ or $z = z_1 = -\dfrac{a\alpha + b\beta}{(a^2 + b^2)}$, from (iii)

or $$(a^2 + b^2) z + a\alpha + b\beta = 0 \quad \text{...(vii)}$$

Again from first two fractions of (vi), we get $a(x - x_1) + b(y - y_1) = 0$

or $ax + by = ax_1 + by_1 = -2cz_1$, from (iv)

$= -2c\left[-\dfrac{a\alpha + b\beta}{a^2 + b^2}\right]$, from (iii)

$= -2cz$, from (vii)

or $\quad ax + by + 2cz = 0$ ...(viii)

Hence from (vii) and (viii) the equations of the axis of the paraboloid are

$$(a^2 + b^2) z + a\alpha + b\beta = 0, \; ax + by + 2cz = 0$$

Exercises on § 12.11 (Case II)

Ex. 1. Prove that the surface represented by the equation $3x^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy + 4x + 6y + 2z + 1 = 0$ is an elliptic paraboloid.

Ans. Reduced form is $[8 + \sqrt{(38)}]x^2 + [8 - \sqrt{(38)}]y^2 - [18/\sqrt{(13)}]z = 0$

*Ex. 2. Find the coordinates of the vertex and equation to the axis of the elliptic paraboloid $4x^2 + y^2 + z^2 - 2zx - 2xy + x + y - 4z - 6 = 0$.

Ans. $(-1, 2, -1)$; $x + 1 = -\frac{1}{2}(y - 2) = \frac{1}{2}(z + 1)$.

*Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid

$$5x^2 - 16y^2 + 5z^2 + 8yz - 14zx + 8xy + 4x + 20y + 4z - 24 = 0.$$

Ans. $(1, 1, 1)$, $\frac{1}{2}(x - 1) = y - 1 = \frac{1}{2}(z - 1)$.

§ 12.12. Case III. $D = 0$, $Au + Hv + Gw = 0$, $A \neq 0$.

In this case the forms to any one of which the given equation can reduce are :—

(i) $Ax^2 + By^2 + C = 0$ (Elliptic cylinder)

(ii) $Ax^2 - By^2 + C = 0$ (Hyperbolic cylinder)

(iii) $Ax^2 - By^2 = 0$ (Pair of Planes)

In this case there is a line centres at a finite distance and the discriminating cubic has one root zero, say λ_3.

The line of centres is given by any two of $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$,

where $F(x, y, z) = 0$ is the equation of the given surface.

Let (α, β, γ) be the coordinates of any point lying on this line. Then shifting the origin to (α, β, γ) and rotating the axes in such a manner that these coincide with a set of mutually perpendicular principal directions, the given equation reduces to the form $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, where $k = u\alpha + v\beta + w\gamma + d$.

Nature : If $k = 0$, this represents a pair of planes.

If $k \neq 0$, this represents an elliptic or hyperbolic cylinder according as the non-zero values of λ (*i.e.* the non-zero roots of the discriminating cubic) are both of the same or opposite signs. (Remember)

The line of intersection of the principal planes corresponding to non-zero values of λ is the axis of the cylinder. It is parallel to the principal direction corresponding to λ_3 which is zero and is also the line of the centres.

Solved Examples on § 12.12.

**Ex. 1. Show that the surface $26x^2 + 20y^2 + 10z^2 - 4yz - 16zx - 36xy + 52x - 36y - 16z + 25 = 0$ represents an elliptic cylinder. Also find the equations to its axis.

Solution. Here the discriminating cubic is given by

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 26-\lambda & -18 & -8 \\ -18 & 20-\lambda & -2 \\ -8 & -2 & 10-\lambda \end{vmatrix} = 0$$

or $\quad (26 - \lambda)[(20 - \lambda)(10 - \lambda) - 4] + 18[-18(10 - \lambda) - 16] - 8[36 + 8(20 - \lambda)] = 0$

or $\quad \lambda^3 - 56\lambda^2 + 588\lambda = 0 \quad$ or $\quad \lambda(\lambda^2 - 56\lambda + 588) = 0$

or $\quad \lambda(\lambda - 14)(\lambda - 42) = 0 \quad$ or $\quad \lambda = 0, 14, 42.$

Let $\lambda_1 = 14$, $\lambda_2 = 42$ and $\lambda_3 = 0$. ...(i)

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $26l_3 - 18m_3 - 8n_3 = 0$, $-18l_3 + 20m_3 - 2n_3 = 0$, $-8l_3 - 2m_3 + 10n_3 = 0$.

Solving first and third of these simultaneously, we have

$$\frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{1} = \frac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(1^2 + 1^2 + 1^2)}} = \frac{1}{\sqrt{3}}$$

i.e. $\quad l_3 = 1/\sqrt{3} = m_3 = n_3.$

The line of centres is given by any two of $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 52x - 36y - 16z + 52 = 0$ *i.e.* $13x - 9y - 4z + 13 = 0$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 40y - 4z - 36x - 36 = 0$ *i.e.* $9x - 10y + z + 9 = 0$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 20z - 4y - 16x - 16 = 0$ *i.e.* $4x + y - 5z + 4 = 0.$

Let (α, β, γ) be any point on the line of centres.

Choosing $\alpha = -1$, $\beta = 0$, $\gamma = 0$ we find $(-1, 0, 0)$ is a point on the line of centres.

[Note. The method of choosing α, β, γ is not unique]

Now $k = u\alpha + v\beta + w\gamma + d = 26(-1) + (-18)(0) + (-8)(0) + 25 = -1 \neq 0.$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$

i.e. $\quad 14x^2 + 42y^2 - 1 = 0$, which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign. [See § 12.12 Page 20 Ch. XII]

Also the equations of the axis of cylinder are

$$\frac{x - (-1)}{l_3} = \frac{y - 0}{m_3} = \frac{z - 0}{n_3} \quad \text{or} \quad \frac{x + 1}{1} = \frac{y}{1} = \frac{z}{1}, \text{ from (iii)}$$

*Ex. 2. Prove that the surface represented by the equation

$$5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$ and find the equations to its axis. (Garhwal 95)

Solution. Here $a = 5$, $b = 5$, $c = 8$, $f = 4$, $g = 4$, $h = -1$, $u = 6$, $v = -6$, $w = 0$ and $d = 6$.

∴ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -1 & 4 \\ -1 & 5-\lambda & 4 \\ 4 & 4 & 8-\lambda \end{vmatrix} = 0$$

or $(5-\lambda)[(5-\lambda)(8-\lambda)-16]+[-(8-\lambda)-16]+4[-4-4(5-\lambda)]=0$

or $\lambda^3-18\lambda^2+72\lambda=0$ or $\lambda(\lambda^2-18\lambda+72)=0$

or $\lambda(\lambda-6)(\lambda-12)=0$ or $\lambda=0,6,12.$

$\therefore$ Let $\lambda_1=6,\ \lambda_2=12,\ \lambda_3=0.$ ...(ii)

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$5l_3-m_3+4n_3=0,\ -l_3+5m_3+4n_3=0,\ 4l_3+4m_3+8n_3=0.$

Solving last two equations simultaneously, we get

$$\frac{l_3}{4-10}=\frac{m_3}{-2-4}=\frac{n_3}{5+1} \quad \text{or} \quad \frac{l_3}{-1}=\frac{m_3}{-1}=\frac{n_3}{1}=\frac{1}{\sqrt{3}} \quad ...(iii)$$

Also the line of centres is given by any two of

$$\partial F/\partial x=0,\ \partial F/\partial y=0,\ \partial F/\partial z=0$$

$\frac{\partial F}{\partial x}=0 \Rightarrow 10x+8z-2y+12=0$ or $5x-y+4z+6=0$

$\frac{\partial F}{\partial y}=0 \Rightarrow 10y+8z-2x-12=0$ or $x-5y-4z+6=0$

$\frac{\partial F}{\partial z}=0 \Rightarrow 16z+8y+8x=0$ or $x+y+2z=0.$

Let (α,β,γ) be any point on the line of centres. Choosing $\gamma=0,\ \beta=1,\ \alpha=-1$ we find $(-1,1,0)$ is a point on the line of centres.

Now $k=u\alpha+v\beta+w\gamma+d=(6)(-1)+(-6)(1)+0+6=-6\neq 0$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0$

i.e. $6x^2+12y^2-6=0$ *i.e.* $x^2+2y^2-1=0$ ...(iv)

which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign.

Also (iv) can be rewritten as $\frac{x^2}{1}+\frac{y^2}{(1/2)}=1.$

And so if e be the required eccentricity, then

$b^2=a^2(1-e^2) \Rightarrow 1/2=(1-e^2)$ or $e=1/\sqrt{2}.$ **Ans.**

Also the equations of the axis of cylinder are

$$\frac{x-(-1)}{l_3}=\frac{y-1}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{-1}=\frac{y-1}{-1}=\frac{z}{1}$$ **Ans.**

**Ex. 3. Determine completely the surface represented by

$$2y^2-2yz+2zx-2xy-x-2y+3z-2=0.$$

Sol. Here 'a' $=0$, 'b' $=2$, 'c' $=0$, 'f' $=-1$, 'g' $=1$, 'h' $=-1$, 'u' $=-1/2$, 'v' $=-1$, 'w' $=3/2$ and 'd' $=-2$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0 \quad ...(i)$$

or $-\lambda[-(2-\lambda)\lambda-1]+[\lambda+1]+[1-(2-\lambda)]=0$

or $-\lambda^3+2\lambda^2+\lambda+\lambda+1+1-2+\lambda=0$ or $\lambda^3-2\lambda^2-3\lambda=0$

or $\lambda(\lambda^2-2\lambda-3)=0$ or $\lambda(\lambda+1)(\lambda-3)=0$ or $\lambda=0,-1,3$

or $\lambda_1=3,\ \lambda_2=-1,\ \lambda_3=0.$

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $-m_3+n_3=0,\ -l_3+2m_3-n_3=0,\ l_3-m_3=0$

From these on solving we get $l_3=m_3=n_3=1/\sqrt{3}$ (Note)

Further the line of centres is given by any two of

$$\partial F/\partial x=0,\ \partial F/\partial y=0,\ \partial F/\partial z=0$$

Now $\frac{\partial F}{\partial x}=0 \Rightarrow 2z-2y-1=0;$

$\frac{\partial F}{\partial y}=0 \Rightarrow 4y-2z-2x-2=0$ or $x-2y+z+1=0;$

$\frac{\partial F}{\partial z}=0 \Rightarrow -2y+2x+3=0$ or $2x-2y+3=0$

Let (α,β,γ) be any point on the line of centres. Choosing $\gamma=0,\ \beta=-1/2,\ \alpha=-2$ we find that $(-2,-1/2,0)$ is a point on the line of centres.

Now $k=u\alpha+v\beta+w\gamma+d=(-\frac{1}{2})(-2)+(-1)(-\frac{1}{2})+(\frac{3}{2})(0)-2=-\frac{1}{2}\neq 0$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0.$

i.e. $3x^2-y^2-(1/2)=0$

which represents a hyperbolic cylinder as λ_1, λ_2 are of different signs.

Also the equations of the axis of the cylinder are

$$\frac{x-(-2)}{l_3}=\frac{y-(-1/2)}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+2}{1}=\frac{y+(1/2)}{1}=\frac{z}{1}$$ **Ans.**

*Ex. 4 (a). Prove that the equation $5x^2-4y^2+5z^2+4yz-14zx+4xy+16x+16y+32z+8=0$ represents a pair of planes which pass through the line $x+2=y-1=z$ and are inclined at an angle $2\tan^{-1}(1/\sqrt{2})$.

Solution. Here 'a' $=5$, 'b' $=-4$, 'c' $=5$, 'f' $=2$, 'g' $=-7$, 'h' $=2$, 'u' $=8$, 'v' $=8$, 'w' $=-16$ and 'd' $=8$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & 2 & -7 \\ 2 & -4-\lambda & 2 \\ -7 & 2 & 5-\lambda \end{vmatrix} = 0 \quad ...(i)$$

or $(5-\lambda)[-(4+\lambda)(5-\lambda)-4]-2[2(5-\lambda)+14]-7[4-7(4+\lambda)]=0$

or $$\lambda^3 - 6\lambda^2 - 72\lambda = 0$$

or $\lambda(\lambda^2 - 6\lambda - 72) = 0$ or $\lambda(\lambda + 6)(\lambda - 12) = 0$ or $\lambda = 0, -6, 12$.

Let $\lambda_1 = 12,\ \lambda_2 = -6,\ \lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$5l_3 + 2m_3 - 7n_3 = 0,\ 2l_3 - 4m_3 + 2n_3 = 0,\ -7l_3 + 2m_3 + 5n_3 = 0.$$

Solving first two equations simultaneously, we get

$$\frac{l_3}{4-28} = \frac{m_3}{-14-10} = \frac{n_3}{-20-4} \text{ or } \frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \quad \text{(Note) ...(ii)}$$

Further the line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$$

Now $\dfrac{\partial F}{\partial x} = 0 \Rightarrow 10x - 14z + 4y + 16 = 0$ or $5x + 2y - 7z + 8 = 0$

$\dfrac{\partial F}{\partial y} = 0 \Rightarrow -8y + 4z + 4x + 16 = 0$ or $x - 2y + z + 4 = 0$

$\dfrac{\partial F}{\partial z} = 0 \Rightarrow 10z + 4y - 14x - 32 = 0$ or $7x - 2y - 5z + 16 = 0$.

Let (α, β, γ) be any point on the line of centres.

Choosing $\gamma = 0,\ \beta = 1,\ \alpha = -2$ we find that $(-2, 1, 0)$ is a point on the line of the centres.

Now $k = u\alpha + v\beta + w\gamma + d = 8(-2) + 8(1) - 16(0) + 8 = 0$.

Hence the reduced equation of the given surface is

$$\lambda_1 x^2 + \lambda_2 y^2 + k = 0 \text{ or } 12x^2 - 6y^2 + 0 = 0 \text{ or } 2x^2 - y^2 = 0 \quad \text{...(iii)}$$

which represents a pair of planes whose line of section is the line through $(-2, 1, 0)$ and direction ratios from (ii) are 1, 1, 1.

$\therefore$ The equations of this line through which the two planes given by (ii) pass are

$$\frac{x-(-2)}{1} = \frac{y-1}{1} = \frac{z-0}{1} \text{ or } x + 2 = y - 1 = z.$$

Again the planes represented by (iii) are $y^2 = 2x^2$,

i.e. $y = x\sqrt{2}$ and $y = -x\sqrt{2}$ *i.e.* $x\sqrt{2} - y = 0$ and $x\sqrt{2} + y = 0$

$\therefore$ The direction ratios of their normals are $\sqrt{2}, -1, 0$ and $\sqrt{2}, 1, 0$.

$\therefore$ If θ be the angle between these planes, then

$$\cos\theta = \frac{\sqrt{2}.\sqrt{2} + (-1).(1) + 0.0}{\sqrt{[(\sqrt{2})^2 + (-1)^2 + 0^2].[(\sqrt{2})^2 + (1)^2 + 0^2]}} = \frac{1}{3}$$

$$\therefore \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1}{3} \text{ or } 3 - 3\tan^2\frac{\theta}{2} = 1 + \tan^2\frac{\theta}{2} \text{ or } 2\tan^2\frac{\theta}{2} = 1$$

or $\tan(\theta/2) = 1/\sqrt{2}$ or $\theta = 2\tan^{-1}(1/\sqrt{2})$. Hence proved.

Ex. 4 (b). In Ex. 4 (a) above prove that the two planes pass through the line $x + 3 = y = z + 1$ and the angle between them is $\tan^{-1}(2\sqrt{2})$

Hint. Proceed exactly as in Ex. 4 (a) above.

Here prove that if (α, β, γ) be any point on the line of centres then choosing $\beta = 0, \gamma = -1,\ \alpha = -3$ we find that $(-3, 0, -1)$ is a point on the line of centres.

$\therefore$ The equations of the line through which the planes given by (iii) of Ex. 4 (a) above pass, are $\dfrac{x-(-3)}{1} = \dfrac{y-0}{1} = \dfrac{z-(-1)}{1}$ or $x + 3 = y = z + 1$

Also in Ex. 4 (a) above $\cos\theta = 1/3$ *i.e.* $\sec\theta = 3$

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta = 9 \Rightarrow \tan^2\theta = 8 \Rightarrow \tan\theta = 2\sqrt{2}$

$\Rightarrow \theta = \tan^{-1}(2\sqrt{2})$. Hence proved.

Exercises on § 12.12. (Case III)

Ex. 1. Reduce $2x^2 + 5y^2 + 2z^2 - 2yz + 4zx - 2xy + 14x - 16y + 14z + 26 = 0$ to the standard form. What does is represent ?

Ans. $2x^2 + y^2 = 1$, elliptic cylinder whose axis is $\dfrac{x+3}{-1} = \dfrac{y-1}{0} = \dfrac{z}{1}$

Ex. 2. Reduce $x^2 - y^2 + 4yz + 4zx - 6x - 2y - 8z + 5 = 0$ to the standard form. What does it represent ?

Ans. Hyperbolic cylinder $x^2 - y^2 = 1$, axis is $\dfrac{x-1}{-2} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$

Ex. 3. Find the condition that the homogeneous equation of second degree in x, y, z represent a pair of planes. (*Kanpur* 92)

§ 12.13. Case IV. A, B, C, F, G, H are all zero; $f\,u \neq g\,v$.

In this case there is a line of centres at infinity and the two roots of discriminating cubic are zero, say λ_2 and λ_3. Also third root $\lambda_1 \neq 0$.

If the axes through the same origin is so rotated that they are parallel to a set of three mutually perpendicular principal directions then the transformed equation is $\lambda_1 x^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2)$

$$+ 2z(ul_3 + vm_3 + wn_3) + d = 0 \quad \text{...(i)}$$

The direction cosines l_2, m_2, n_2 and l_3, m_3, n_3 corresponding to zero roots λ_2 and λ_3 satisfy the equation $\quad al + hm + gn = 0 \quad$ ...(ii)

Choose l_2, m_2, n_2 such that : $\quad ul_2 + vm_2 + wn_2 = 0 \quad$ ...(iii)

Then from (i), (iii) we have $\lambda_1 x^2 + 2px + 2rz + d = 0$, ...(iv)

where $p = ul_1 + vm_1 + wn_1,\quad r = ul_3 + vm_3 + wn_3$ ...(v)

From (iv), $\lambda_1\left(x^2 + \dfrac{2p}{\lambda_1}x + \dfrac{p^2}{\lambda_1^2}\right) + 2rz + \left(d - \dfrac{p^2}{\lambda_1}\right) = 0$ (Note)

or $$\lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + 2r\left[z + \frac{1}{2r}\left(d - \frac{p^2}{\lambda_1}\right)\right] = 0 \quad \text{...(vi)}$$

Shifting the origin to the point $\left[-\frac{p}{\lambda_1}, 0, -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}\right)\right]$

the equation (vi) transforms to $\lambda_1 x^2 + 2rz = 0$ or $x^2 + (2r/\lambda_1) z = 0$, ...(vii)

which is the required reduced form and represents a parabolic cylinder.

The latus rectum of a normal section is

$2r/\lambda_1$ i.e. $(2/\lambda_1)(ul_3 + vm_3 + wn_3)$, from (v).

Alternative method.

$\because A = 0 = B = C$, so we have $bc - f^2 = 0$, $ca - g^2 = 0$, $ab - h^2 = 0$.

These imply that a, b, c have the same sign, say positive

Also $F = 0$, $G = 0$, $H = 0$ give $gh - af = 0$, $hf - bg = 0$, $fg - ch = 0$

and so either f, g, h are all positive or two negative and one positive. (Note)

$\therefore f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

$= [\sqrt{a}x \pm \sqrt{b}y \pm \sqrt{c}z]^2$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu \neq gv$, then we proceed as follows :—

General equation $F(x, y, z) = 0$ can be rewritten as

$$[\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda]^2 = 2x[\lambda\sqrt{a} - u] + 2y[\lambda\sqrt{b} - v] + 2z[\lambda\sqrt{c} - w] + (\lambda^2 - d) \quad ...(I)$$

Now choose λ in such a way that the planes $\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda = 0$ and

$2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d) = 0$ are at right angles

so $\sqrt{a}(\lambda\sqrt{a} - u) + \sqrt{b}(\lambda\sqrt{b} - v) + \sqrt{c}(\lambda\sqrt{c} - w) = 0$

or $\lambda(a + b + c) = u\sqrt{a} + v\sqrt{b} + w\sqrt{c}$

or $\lambda = (u\sqrt{a} + v\sqrt{b} + w\sqrt{c})/(a + b + c)$ ...(II)

The equation (I) with the help of (II) can be rewritten as

$$\left[\frac{\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda}{\sqrt{(a+b+c)}}\right]^2 = k\left[\frac{2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}}\right],$$

where $k = 2[\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}]/(a + b + c)$.

i.e. The above equation takes the from $X^2 = kY$,

where $X = (\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda)/\sqrt{(a + b + c)}$

and $$Y = \frac{2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}}$$

This represents a parabolic cylinder.

§ 12.14. Case V. A, B, C, F, G, H are all zero and fu = gv = hw.

In this case there is a plane of centres and two roots λ_2, λ_3 (say) of the discriminating cubic are zero.

If l_1, m_1, n_1 be the principal direction cosines corresponding to the non-zero root λ_1 of the discriminating cubic, then

$$\frac{al_1 + hm_1 + gn_1}{l_1} = \frac{hl_1 + bm_1 + fn_1}{m_1} = \frac{gl_1 + fm_1 + cn_1}{n_1} \quad ...(i)$$

But $f^2 = bc$, $g^2 = ca$ and $h^2 = ab$, so

$al_1 + hm_1 + gn_1 = al_1 + \sqrt{(ab)}m_1 + \sqrt{(ca)}n_1 = \sqrt{a}[\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$

Similarly $hl_1 + bm_1 + fn_1 = \sqrt{b}[\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$,

and $gl_1 + fm_1 + cn_1 = \sqrt{c}[\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$

$\therefore$ From (i) we have $\frac{l_1}{\sqrt{a}} = \frac{m_1}{\sqrt{b}} = \frac{n_1}{\sqrt{c}}$ ...(ii)

Also here $fu = gv = hw$

$\Rightarrow \sqrt{(bc)}u = \sqrt{(ca)}v = \sqrt{(ab)}w$, $\because f^2 = bc$ etc.

$\Rightarrow u/\sqrt{(a)} = v/\sqrt{(b)} = w/\sqrt{(c)}$

From (ii), $u/l_1 = v/m_1 = w/n_1$ ...(iii)

Now if l_2, m_2, n_2 and l_3, m_3, n_3 be the principal direction cosines corresponding to zero roots λ_2 and λ_3, then

$ul_2 + vm_2 + wn_2 = l_1l_2 + m_1m_2 + n_1n_2 = 0$

$ul_3 + vm_3 + wn_3 = l_1l_3 + m_1m_3 + n_1n_3 = 0.$

Now as in § 12.13 Page 25 Ch. XII rotating the axes we find that the transformed equation is $\lambda_1 x^2 + 2x(ul_1 + vm_1 + wn_1) + d = 0$ (Note)

or $\lambda_1 x^2 + 2px + d = 0$, where $p = ul_1 + vm_1 + wn_1$

or $\lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + \left(d - \frac{p^2}{\lambda_1}\right) = 0$ or, $\lambda_1 x^2 + k = 0$,

changing the origin to $(-p/\lambda_1, 0, 0)$ and where $k = d - (p^2/\lambda_1)$

This equation represents a pair of planes which are identical or parallel according as $k = 0$ or $k \neq 0$

Alternative method.

As in the alternative method given in § 12.13 on Page 26, if A, B, C, F, G and H are zero, we can prove that $f(x, y, z) = [\sqrt{a}x \pm \sqrt{b}y \pm \sqrt{c}z]^2$.

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu = gv = hw$, then as above we can get

$u/\sqrt{a} = v/\sqrt{b} = w/\sqrt{c} = \mu$ (say) ...(iv)

Also the general equation $F(x, y, z) = 0$ in this case can be written as

$(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z)^2 - 2(ux + vy + wz) + d = 0$ (Note) or $(\sqrt{ax}+\sqrt{by}+\sqrt{cz})^2+2\mu(\sqrt{ax}+\sqrt{by}+\sqrt{cz})+d=0$, from (iv)

or $\sqrt{ax}+\sqrt{by}+\sqrt{cz}=-\mu\pm\sqrt{(\mu^2-d)}$, solving as a quadratic equation in $\sqrt{ax}+\sqrt{by}+\sqrt{cz}$

This represents a pair of parallel planes.

Solved Examples on § 12.13 — § 12.14 (Case IV and V).

*Ex. 1. Reduce the equation $x^2+y^2+z^2-2yz+2zx-2xy+x-4y-z+1=0$ to the standard form and find the latus rectum of the principal section. (*Avadh* 93)

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x-y+z)^2=-x+4y-z-1$

or $(x-y+z+\lambda)^2=(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)$ ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on the R.H.S. (Note)

Now choose λ in such a way that the planes $x-y+z+\lambda=0$ and $(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)=0$ are at right angles.

Then $1.(2\lambda-1)+(-1)\{-2(\lambda-2)\}+1.(2\lambda-1)=0\Rightarrow\lambda=1$

$\therefore$ From (i), the given equation of the surface can be rewritten as

$$(x-y+z+1)^2=x+2y+z$$

or $$3\left[\frac{x-y+z+1}{\sqrt{3}}\right]^2=\sqrt{6}\left[\frac{x+2y+z}{\sqrt{6}}\right]$$ (Note)

or $3X^2=\sqrt{6}Y$ or $X^2=(1/3)\sqrt{6}Y$, which represents a paraboloic cylinder and the latus rectum of the principal paraboloic section is $\sqrt{6}/3$. Ans.

**Ex. 2. Show that the equation $x^2+4y^2+9z^2+12yz+6zx+4xy-54x-52y+62z+113=0$ represents a parabolic cylinder, and that the foci of the normal parabolic section lie on the line

$$x+2y+3z+1=0=x+y-z-5.$$

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x+2y+3z)^2=54x+52y-62z-113$

or $(x+2y+3z+\lambda)^2=2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z+(\lambda^2-113)$, ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on R.H.S. (Note)

Now choose λ in such a way that the planes $x+2y+3z+\lambda=0$ and $2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z+\lambda^2-113=0$ are at right angles.

Then $1.\{2(\lambda+27)\}+2.\{4(\lambda+13)\}+3.\{2(3\lambda-31)\}=0$

or $2\lambda+54+8\lambda+104+18\lambda-186=0$ or $\lambda=1$.

$\therefore$ From (i), the given equation of the surface reduces to

$$(x+2y+3z+1)^2=56x+56y-56z-112$$

or $$(x+2y+3z+1)^2=56(x+y-z-2)$$

or $$14\left[\frac{x+2y+3z+1}{\sqrt{(1^2+2^2+3^2)}}\right]^2=56\sqrt{3}\left[\frac{x+y-z-2}{\sqrt{\{(1^2+1^2+(-1)^2\}}}\right]$$

or $$Y^2=4\sqrt{3}X,$$ ...(ii)

which represents a parabolic cylinder and the latus rectum of the normal parabolic section is $4\sqrt{3}$.

[Note : The vertex of the parabolic cylinder lie on the line of intersection of the planes $x+2y+3z+1=0$, $x+y-z-2=0$, the latter being a tangent plane which touches the cylinder along the vertices.]

The foci evidently lie on the line of intersection of the plane $x+2y+3z+1=0$ *i.e.* the plane through the axis and a plane parallel to the tangent plane $x+y-z-2=0$ but at a distance (1/4)th of latus rectum (*i.e.* $\sqrt{3}$) from it.

Now any plane parallel to the tangent plane $x+y-z-2=0$, ...(iii) $x+y-z+k=0$ and it should be at a distance $\sqrt{3}$ from the tangent plane.

Now any point on the tangent plane is (2, 0, 0), putting $y=0$, $z=0$ in $x+y-z-2=0$.

$\therefore$ distance of the plane $x+y-z+k=0$ from (2, 0, 0) must be $\sqrt{3}$.

i.e. $$\frac{2+0-0+k}{\sqrt{[1^2+1^2+(-1)^2]}}=\sqrt{3}\quad\text{or}\quad 2+k=3\quad\text{or}\quad k=1.$$

$\therefore$ Foci lie on the line of intersection of the planes

$x+2y+3z+1=0$ and $x+y-z+1=0$, from (iii) Hence proved.

**Ex. 4. Show that the equation $4x^2+9y^2+36z^2-36yz+24zx-12xy-10x+15y-30z+6=0$ represents a pair of parallel planes and find the reduced equation.

Solution. As the second degree terms of the given equation form a perfect square, so it can be rewritten as

$$(2x-3y+6z)^2=10x-15y+30z-6=5(2x-3y+6z)-6$$

or $(2x-3y+6z)^2-5(2x-3y+6z)+6=0$ ...(i)

or $X^2-5X+6=0$, where $X=2x-3y+6z$

or $(X-2)(X-3)=0$ or $X=2, X=3$

or $2x-3y+6z=2,\ 2x-3y+6z=3$ ...(ii)

Hence the given equation represents a pair of parallel planes given by (ii).

Also from (i) we have

$$49\left[\frac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}\right]^2=7\left[\frac{5(2x-3y+6z)}{\sqrt{(2^2+3^2+6^2)}}\right]-6$$ ...(iii)

Now choose $2x-3y+6z=0$ as $x=0$ *i.e.* if (x, y, z) be the coordinates of any point, then $x=\dfrac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}$

Then (iii) reduces to $49x^2 - 35x + 6 = 0$, which is the required reduced equation. Ans.

Exercises on § 12.13 — § 12.14 (Cases IV—V)

**Ex. 1. Reduce the equation $36x^2 + 4y^2 + z^2 - 4yz - 12zx + 24xy + 4x + 16y - 26z - 3 = 0$ to the standard form. Show that it represents a parabolic cylinder and find the latus rectum of a normal section. Also show that the foci of the normal parabolic sections lie on the line $6x + 2y - z + 1 = 0 = 2x - 3y + 6z + (90/41)$. (*Avadh* 91)

Ans. $41y^2 = 28x$, latus rectum $= 28/41$

Ex. 2. Reduce the equation $9x^2 + 4y^2 + 4z^2 + 8yz + 12zx + 12xy + 4x + y + 10z + 1 = 0$. Ans. $17y^2 = 7x$, a parabolic cylinder

Ex. 3. What surface is represented by the equation
$$x^2 + 4y^2 + z^2 + 2zx - 4yz - 4xy - 2x + 4y - 2z - 3 = 0\ ?$$
Reduce it to the standard form.

Ans. A pair of parallel planes, $6x^2 - 2\sqrt{6}x - 3 = 0$

Ex. 4. Show that $(3x - 4y + z)^2 + 9x - 12y + 3z - 10 = 0$ represents a pair of parallel planes. Also reduce it to the standard form $26x^2 - 3\sqrt{(26)}\,x - 10 = 0$.

§ 12.15. Conicoids of revolution.

Here two cases arise viz.

(i) Two roots of the discriminating cubic are equal and third root not equal to zero.

(ii) Two roots of the discriminating cubic are equal and third root equal to zero.

Under (i) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz^2 = 1 \qquad \text{(Ellipsoid of revolution)}$$

and
$$A(x^2 - y^2) + Bz^2 = 1 \qquad \text{(Hyperboloid of revolution)}$$

Under (ii) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz = 0 \qquad \text{(Paraboloid of revolution)}$$

and
$$A(x^2 + y^2) + D = 0 \qquad \text{(Right circular cylinder)}$$

$\therefore$ We conclude that if the two roots of the discriminating cubic are equal, then surface $F(x, y, z) = 0$ represents a surface (or conicoid) of revolution.

Here we proceed in the usual way and the direction ratios of the axis of rotation are obtained from the usual equations by taking that value of λ which is different from the equal values.

Solved Examples on § 12.15.

**Ex. 1. Show that the equation $x^2 + y^2 + z^2 + yz + zx + xy + 3x + y + 4z + 4 = 0$ represents a surface of revolution and determine the equations of its axis of rotation.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 1/2 & 1/2 \\ 1/2 & 1-\lambda & 1/2 \\ 1/2 & 1/2 & 1-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(1-\lambda)[(1-\lambda)^2 - (1/4)] - (1/2)[(1/2)(1-\lambda) - (1/4)] + (1/2)[(1/4) - (1/2)(1-\lambda)] = 0$

or $(1-\lambda)^3 - (3/4)(1-\lambda) + (1/4) = 0$

or $4(1-\lambda)^3 - 3(1-\lambda) + 1 = 0$ or $4\lambda^3 - 12\lambda^2 + 9\lambda - 2 = 0$

or $(\lambda - 2)(2\lambda - 1)^2 = 0$ or $\lambda = 2, 1/2, 1/2$.

$\therefore$ We observe that two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents a surface of revolution [either ellipsoid or hyperboloid of revolution]

...See § 12.15 (i) Page 30 Ch. XII.

The central planes are given by $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

i.e. $2x + y + z + 3 = 0,\ x + 2y + z + 1 = 0,\ x + y + 2z + 4 = 0$.

Solving these we get $x = -1, y = 1, z = -2$

$\therefore$ Centre of the given surface is $(-1, 1, -2)$.

$\therefore d' = u\alpha + v\beta + w\gamma + d = (3/2)(-1) + (1/2)(1) + (2)(-2) + 4 = -1$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$

or $(1/2)x^2 + (1/2)y^2 + 2z^2 - 1 = 0$ or $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{(1/2)} = 1$

which is an ellipsoid (of revolution), the squares of whose semiaxes are 2, 2, 1/2.

Now putting $\lambda = 2$ in the determinant given by (i) and associating each row with l, m, n, the direction cosines of the principal axis (or axis of revolution), we have

$$-l + (1/2)m + (1/2)n = 0,\ (1/2)l - m + (1/2)n = 0,\ (1/2)l + (1/2)m - n = 0$$

i.e. $-2l + m + n = 0,\ l - 2m + n = 0,\ l + m - 2n = 0$

and these gives $l = m = n = 1/\sqrt{3}$, $\because l^2 + m^2 + n^2 = 1$.

Now the required axis of rotation (or principal axis) is a line through the centre $(-1, 1, -2)$ of the surface of revolution and direction cosines $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ or direction ratios 1, 1, 1.

$\therefore$ The required equations of the axis of rotation are

$$\frac{x-(-1)}{1} = \frac{y-1}{1} = \frac{z-(-2)}{1} \quad \text{or} \quad x + 1 = y - 1 = z + 2 \qquad \text{Ans.}$$

*Ex. 2. Reduce to standard form the equation
$$7x^2 + y^2 + z^2 + 16yz + 8zx - 8xy + 2x + 4y - 40z - 14 = 0$$
and find the principal axis.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad\text{or}\quad \begin{vmatrix} 7-\lambda & -4 & 4 \\ -4 & 1-\lambda & 8 \\ 4 & 8 & 1-\lambda \end{vmatrix}=0 \quad ...(i)$$

or $(7-\lambda)[(1-\lambda)^2-64]+4[-4(1-\lambda)-32]+4[-32-4(1-\lambda)]=0$

or $\lambda^3-9\lambda^2-81\lambda+729=0$ or $(\lambda-9)(\lambda^2-81)=0$

or $(\lambda-9)(\lambda-9)(\lambda+9)=0$ or $\lambda=9, 9, -9.$

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0$

i.e. $14x-8y+8z+2=0,\ -8x+2y+16z+4=0,\ 8x+16y+2z-40=0$

i.e. $7x-4y+4z+1=0,\ -4x+y+8z+2=0,\ 4x+8y+z-20=0$

Solving these we get $x=1, y=2, z=0$.

$\therefore$ Centre of the given surface is $(1, 2, 0)$;

$\therefore d'=u\alpha+v\beta+w\gamma+d=(1)(1)+(2)(2)+(-20)(0)-14=-9$

$\therefore$ The reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad\text{or}\quad 9x^2+9y^2-9z^2-9=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution, the squares of whose semi-axes are 1, 1, 1.

Now putting $\lambda=-9$ in the determinant given by (i) and associating each row with l, m, n, the d.c.'s of the principal axis, we have.

$$16l-4m+4n=0,\ -4l+10m+8n=0,\ 4l+8m+10n=0$$

and these gives $\dfrac{l}{1}=\dfrac{m}{2}=\dfrac{n}{-2}=\dfrac{1}{3}$, $\because l^2+m^2+n^2=1$.

$\therefore$ The equations of the principal axis passing through the centre (1, 2, 0) and d.r.'s 1, 2, −2 are $\dfrac{x-1}{1}=\dfrac{y-2}{2}=\dfrac{z-0}{-2}$. **Ans.**

Ex. 3. Show that the equation $x^2+2yz=1$ represents a surface of revolution and find the axis of revolution.

Solution. Given $F(x, y, z)\equiv x^2+2yz-1=0$

$\therefore$ Here $a'=1, b=c=0, f=1, g=0=h=u=v=w, d'=-1$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad\text{or}\quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}=0 \quad ...(i)$$

or $(1-\lambda)[\lambda^2-1]=0$ or $\lambda=1, 1, -1.$

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0.$

i.e. $2x=0,\ 2z=0,\ 2y=0$ *i.e.* $x=0, y=0, z=0.$

$\therefore$ Centre of the given surface is (0, 0, 0).

$\therefore d'=u\alpha+v\beta+w\gamma+d=0+0+0-1=-1.$

$\therefore$ Reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad\text{or}\quad x^2+y^2-z^2-1=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution.

Now putting $\lambda=-1$ in the determinant given by (i) and associating each row with l, m, n, the d.c.'s of the axis of revolution (or principal axis) we have

$$2l=0,\ m+n=0,\ m+n=0 \Rightarrow \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$$

$\therefore$ The equations of required axis of revolution which passes through the centre (0, 0, 0) and whose d.r.'s are 0, 1, −1 are

$\dfrac{x-0}{0}=\dfrac{y-0}{1}=\dfrac{z-0}{-1}$ *i.e.* $x=0, y+z=0.$ **Ans.**

Ex. 4. Show that the surface represented by the equation

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$$

is a paraboloid of revolution, the coordinates of the focus being (1, 2, 3) and the equations to axis are $x=y-1=z-2$. (*Avadh* 95; *Rohilkhand* 97, 96, 94)

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad\text{or}\quad \begin{vmatrix} 1-\lambda & -(1/2) & -(1/2) \\ -(1/2) & 1-\lambda & -(1/2) \\ -(1/2) & -(1/2) & 1-\lambda \end{vmatrix}=0$$

or $(1-\lambda)[(1-\lambda)^2-(\frac{1}{4})]+(\frac{1}{2})[-(\frac{1}{2})(1-\lambda)-(\frac{1}{4})]-(\frac{1}{2})[(\frac{1}{4})+\frac{1}{2}(1-\lambda)]=0$

or $(1-\lambda)[8(1-\lambda)^2-2]-2[2(1-\lambda)+1]=0$ or $4\lambda^3-12\lambda^2+9\lambda=0$

or $\lambda(4\lambda^2-12\lambda+9)=0$ or $\lambda(2\lambda-3)^2=0$ or $\lambda=3/2, 3/2, 0$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The direction ratios of the axis are given by

$$al+hm+gn=0,\ hl+bm+fn=0,\ gl+fm+cn=0$$

i.e. $l-\frac{1}{2}m-\frac{1}{2}n=0,\ -\frac{1}{2}l+m-\frac{1}{2}n=0,\ -\frac{1}{2}l-\frac{1}{2}m+n=0$

i.e. $2l-m-n=0,\ -l+2m-n=0,\ -l-m+2n=0.$

These give $l=m=n=1/\sqrt{3}$

Now $k=ul+vm+wn$

or $k=(-3/2)(1/\sqrt{3})+(-3)(1/\sqrt{3})+(-9/2)(1/\sqrt{3})=-3\sqrt{3}\neq 0.$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$ (Note)

...See § 12.11 (iii) Page 14 Ch. XII

or $(3/2)x^2 + (3/2)y^2 + 2(-3\sqrt{3})z = 0$

or $x^2 + y^2 = 4\sqrt{3}z$, which represents a paraboloid of revolution.

Also the coordinates of the vertex of the paraboloid are obtained by solving any two of the three equations

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l} = \frac{\left(\frac{\partial F}{\partial y}\right)}{m} = \frac{\left(\frac{\partial F}{\partial z}\right)}{n} = 2k \quad \text{...See § 12.11 (iv) Page 14 Ch. XII}$$

or $\dfrac{2x - y - z - 3}{1/\sqrt{3}} = \dfrac{2y - z - x - 6}{1/\sqrt{3}} = \dfrac{2z - y - x - 9}{1/\sqrt{3}} = -6\sqrt{3}$.

or $2x - y - z - 3 = 2y - z - x - 6 = 2z - y - x - 9 = -6$

or $2x - y - z + 3 = 0,\ x - 2y + z = 0,\ x + y - 2z + 3 = 0$ ...(I)

with the equation $k(lx + my + nz) + ux + vy + wz + d = 0$

i.e. $-3\sqrt{3}\left(\dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y + \dfrac{1}{\sqrt{3}}z\right) + \left(-\dfrac{3}{2}\right)x + (-3)y + \dfrac{-9}{2}z + 21 = 0$

i.e. $3x + 4y + 5z - 14 = 0$ ...(II)

Solving $2x - y - z + 3 = 0,\ x - 2y + z = 0,\ 3x - 4y + 5z - 14 = 0$

we get $x = 0, y = 1, z = 2$. $\therefore$ The required vertex is $(0, 1, 2)$.

$\therefore$ Equations of the axis are $\dfrac{x-0}{1} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

or $x = y - 1 = z - 2$. Ans.

Also the focus will be a point on the axis whose actual direction cosines are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and will be at a distance $(1/4) \cdot 4\sqrt{3}$ i.e. $\sqrt{3}$, from the vertex $(0, 1, 2)$

$\therefore$ Coordinates of the focus are given by

$$\frac{x-0}{(1/\sqrt{3})} = \frac{y-1}{(1/\sqrt{3})} = \frac{z-2}{(1/\sqrt{3})} = \sqrt{3} \quad \text{(Note)}$$

or $x = 1, y = 2, z = 3$

$\therefore$ The required focus is $(1, 2, 3)$. Ans.

**Ex. 5. Show that $13x^2 + 45y^2 + 40z^2 + 12yz + 36zx - 24xy - 49 = 0$ represents a right circular cylinder whose axis is $x/6 = y/2 = z/-3$ and radius 1.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 13-\lambda & -12 & 18 \\ -12 & 45-\lambda & 6 \\ 18 & 6 & 40-\lambda \end{vmatrix} = 0$$

or $(13-\lambda)[(45-\lambda)(40-\lambda) - 36] + 12[-12(40-\lambda) - 108]$

$+ 18[-72 - 18(45-\lambda)] = 0$

or $\lambda(\lambda - 49)^2 = 0$ or $\lambda = 0, 49, 49$.

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The d. ratios of the axis are given by

$$al + hm + gn = 0,\ hl + bm + fn = 0,\ gl + fm + cn = 0$$

i.e. $13l - 12m + 18n = 0,\ -12l + 45m + 6n = 0,\ 18l + 6m + 40n = 0$.

Solving these we get $\dfrac{l}{6} = \dfrac{m}{2} = \dfrac{n}{-3} = \dfrac{1}{\sqrt{7}}$ (Note)

Now here $k = ul + vm + wn = 0$, $\because u = 0 = v = w$.

Also the line of centres is given by $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

or $26x - 24y + 36z = 0,\ -24x + 90y + 12z = 0,\ 36x + 12y + 80z = 0$

or $13x - 12y + 18z = 0,\ 8x - 30y - 4z = 0,\ 9x + 3y + 20z = 0$,

which gives $x = 0, y = 0, z = 0$.

$\therefore$ Any point on the line of centres is $(0, 0, 0)$

Also $d' = u\alpha + v\beta + w\gamma + d = 0 = 0 + 0 - 49 = -49$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + d' = 0$ (Note)

or $49x^2 + 49y^2 - 49 = 0$ or $x^2 + y^2 = 1$,

which is a right circular cylinder of radius 1, as any section of this surface by a plane $z = k$ is a circle $x^2 + y^2 = 1$, whose radius is 1.

And the equations of the axis are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad \text{i.e.} \quad \frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$$

Ex. 6. Prove that the equation $2y^2 + 4zx + 2x - 4y + 6z + 5 = 0$ represents a right circular cone. Show also that the semi-vertical angle of this cone is $\pi/4$ and its axis is given by $x + z + 2 = 0,\ y = 1$. (Garhwal 96)

Solution. The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0 \quad \text{...(i)}$$

or $-\lambda[-\lambda(2-\lambda)] + 2[-2(2-\lambda)] = 0$.

or $(2-\lambda)(\lambda^2 - 4) = 0$ or $\lambda = 2, 2, -2$

As two roots of this cubic are equal and third is not zero, so the given surface is a surface of revolution. (Note)

Also the line of centres is given by any two of $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

i.e. $4z + 2 = 0,\ 4y - 4 = 0,\ 4x + 6 = 0$.

$\therefore$ If (α, β, γ) be any point on the line of centres, then

$4\gamma + 2 = 0,\ 4\beta - 4 = 0,\ 4\alpha + 6 = 0 \Rightarrow \alpha = -3/2,\ \beta = 1,\ \gamma = -1/2$.

$\therefore$ Any point on the line of centres is $(-3/2, 1, -1/2)$.

$\therefore\ d' = u\alpha + v\beta + w\gamma + d = 1(-3/2) - 2(1) + 3(-1/2) + 5 = 0.$

$\therefore$ The reduced form of the equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad 2x^2 + 2y^2 - 2z^2 + 0 = 0$$

or
$$x^2 + y^2 - z^2 = 0 \quad \text{or} \quad x^2 + y^2 = z^2 \tan^2 45°$$

which represents a right circular cone of semi-vertical angle $\pi/4$.

[$\because$ $x^2 + y^2 = z^2 \tan^2 \alpha$ represents a cone whose semi-vertical angle is α].

Now putting the unequal value of λ viz. -2 in the determinant of (i) and associating each row with l, m, n we have $2l + 2n = 0$, $4m = 0$, $2l + 2n = 0$

These gives
$$\frac{l}{1} = \frac{m}{0} = \frac{n}{-1} = \frac{1}{\sqrt{2}}$$

$\therefore$ The equations of its axis are

$$\frac{x-(-3/2)}{l} = \frac{y-1}{m} = \frac{z-(-1/2)}{n} \quad \text{or} \quad \frac{x+(3/2)}{1} = \frac{y-1}{0} = \frac{z+(1/2)}{-1}$$

or $\quad -x - (3/2) = z + (1/2),\ y - 1 = 0 \quad$ or $\quad x + z + 2 = 0,\ y = 1$

Hence proved.

Exercises on § 12.15

Ex. 1. Prove that the equation $2x^2 + 5y^2 + z^2 - 4xy - 8x + 14y + 3 = 0$ is a surface of revolution. Also find the equations of its principal axis.

Ans. Reduced equation is $x^2 + y^2 + 6z^2 = 8$, axis $2x + y - 1 = 0 = z$.

Ex. 2. Find the reduced equation of the surface

$$x^2 - y^2 + 2yz - 2zx - x - y + z = 0. \text{ Also find its axis.}$$

Ans. $3(x^2 - y^2) = z$; $x - (1/3) = y + (1/3) = z$.

*__Ex. 3.__ Discuss the nature of the surface $yz + zx + xy = a^2$.

Ans. A hyperboloid of revolution ; reduced equation is
$$2z^2 - y^2 - z^2 = 2a^2; \text{ axis is } x = y = z.$$

Exercises on Chapter XII

Ex. 1. Reduce the surface $40x^2 + 50y^2 - 9z^2 - 8yz - 16zx + 26xy + 4x + 20y - 28z - 3 = 0$ into the standard form and find the latus rectum of a normal section. (Avadh 92)

Ex. 2. Reduce the equation $12x^2 + 10y^2 + 8z^2 - 9yz + zx - 13xy + 75x + 77y - 38z + 100 = 0$ into the standard form and also describe the nature of the surface and find the equations of its axes. (Avadh 92)