IMS

MATHS

BOOK-13

* The Riemann Integral *

Introduction :-

In elementary treatments, the process of integration is generally introduced as the inverse of differentiation.

If $F'(x) = f(x)$ for all x belonging to the domain of the function f; F is called an integral of the given function f.

Historically, however the subject of integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a sum. This notion of integral as summation is based on geometrical concepts.

A German mathematician G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical concepts.

Riemann's definition covered only bounded functions.

It was Cauchy who extended this definition to unbounded functions.

In the present chapter we shall study the Riemann integral of real valued, bounded functions defined on some closed interval.

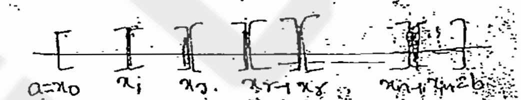
* Partition of a closed Interval

Let $I = [a,b]$ be a closed bounded interval. If $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ then the finite ordered set

$P = \{x_0, x_1, x_2, \dots, x_{r-1}, x_r, \dots x_n\}$

where $r = 1, 2, \dots n$ is called a partition of I.

The $(n+1)$ points $x_0, x_1, \dots x_{n-1}, x_n$ called partition points of P.



The n closed subintervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2], \dots I_r = [x_{r-1}, x_r] \dots I_n = [x_{n-1}, x_n]$ determined by P are called segment of the partition P.

Clearly $\bigcup_{r=1}^{n} I_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a,b]$

(or)

$P = \{[x_{r-1}, x_r]\}_{r=1}^{n}$

— The length of the r^{th} subinterval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r.

i.e. $\delta_r = x_r - x_{r-1}$; $r = 1, 2, \dots n$.

Note: (1) By changing the partition points, the partition can be changed and hence, there can be an infinite number of partitions of the interval I.

we shall denote it by $P[a,b]$ the set (or family) of all partitions of $[a,b]$.

② Partition is also known as dissection (or) net.

* Norm of a partition :-

The maximum of the lengths of the subintervals of a partition P is called norm (or) mesh of the partition P and is denoted by $\|P\|$ (or) $\mu(P)$.

i.e. $\|P\| = Max\{\delta_r / r = 1,2 \ldots n\}$

$= Max\{x_r - x_{r-1} / r = 1,2 \ldots n\}$

$= Max\{x_1 - x_0, x_2 - x_1, \ldots x_n - x_{n-1}\}$

Note (1): If $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a,b]$ then

$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \ldots + \delta_n$

$= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$

$= x_n - x_0$

$= b - a$

* Refinement of a partition:-

If P, P' be two partitions of $[a,b]$ and $P \subset P'$ then the partition P' is called a refinement of partition P on $[a,b]$. we also say 'P'' is finer than P.

i.e. If P' is finer than P, then every point of P is used in the construction of P' and P' has atleast one additional point.

→ If P_1, P_2 are two partitions of $[a,b]$ then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. Therefore $P_1 \cup P_2$ is called a common refinement of P_1 & P_2.

Note: If $P_1, P_2 \in P[a,b]$ and $P_1 \subset P_2$ then $\|P_2\| \leq \|P_1\|$.

* Upper and Lower Darboux Sums:

Let $f : [a,b] \rightarrow \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, \ldots x_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$, f is also bounded on each of the subintervals. (i.e. $I_r = [x_{r-1}, x_r], r = 1,2 \ldots n$)

Let M, m be the supremum and infimum of f in $[a,b]$ and M_r, m_r be the supremum and infimum of f in the rth subintervals.

$I_r = [x_{r-1}, x_r]$; $r = 1,2, \ldots n$.

The sum $M_1\delta_1 + M_2\delta_2 + \ldots + M_r\delta_r + \ldots + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$

is called the upper Darboux sum of f corresponding to the partition P, and is denoted by $U(P,f)$ or $U(f,P)$

→ The sum $m_1\delta_1 + m_2\delta_2 + \cdots + m_r\delta_r + \cdots + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$.

is called the lower Darboux sum of f corresponding to the partition P and is denoted by $L(P,f)$ or $L(f,P)$.

i.e. $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$

$L(P,f) = \sum_{r=1}^{n} m_r\delta_r$

* Oscillatory Sum:

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P=\{a=x_0, x_1, \cdots x_n = b\}$ be a partition of $[a,b]$.

Let m_r and M_r be the infimum and supremum of f on $I_r = [x_{r-1}, x_r]$ $r=1,2,\ldots n$. Then

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r$$

$$= \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$= \sum_{r=1}^{n} O_r\delta_r$$

where $O_r = M_r - m_r$ denotes the oscillation of f on I_r.

$U(P,f) - L(P,f) = \sum_{r=1}^{n} O_r\delta_r$ is called the oscillatory sum of f corresponding to the partition P and is denoted by $\omega(P,f)$.

i.e $\omega(P,f) = \sum_{r=1}^{n} O_r\delta_r$

→ If $f:[a,b]\to\mathbb{R}$ is bounded function and $P\in \mathcal{P}[a,b]$ then

$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

where m, M are the infimum and supremum of f on $[a,b]$.

Proof: Let $P=\{a=x_0, x_1, \cdots x_n=b\}$ be partition of $[a,b]$.

Since f is bounded on $[a,b]$

$\Rightarrow f$ is bounded on each subinterval of $[a,b]$.

i.e. f is bounded on $I_r = [x_{r-1}, x_r]$ $r=1,2,\cdots n$

Let m_r and M_r be the infimum & supremum of f on $I_r = [x_{r-1}, x_r]$

$\therefore m \le m_r \le M_r \le M$

$\Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$

$\Rightarrow \sum_{r=1}^{n} m\,\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$

$\Rightarrow m\sum_{r=1}^{n}\delta_r \le L(P,f) \le U(P,f) \le M\sum_{r=1}^{n}\delta_r$

$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

$\left[\because \sum_{r=1}^{n}\delta_r = b-a\right]$

Note: The above theorem implies that $L(P,f)$ & $U(P,f)$ are bounded if f is bounded.

* Upper and Lower Riemann Integrals:-

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P\in\mathcal{P}[a,b]$ then we have

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

where

m, M are infimum and supremum of 'f' on $[a,b]$.

For every $P \in P[a,b]$,

we have

$L(P,f) \leq M(b-a)$ and

$U(P,f) \geq m(b-a)$

$\Rightarrow$ the set $\{L(P,f)\}_{P\in P[a,b]}$ of lower sums is bounded above by $M(b-a)$.

$\therefore$ It has the least upper bound. (lub)

the set $\{U(P,f)\}_{P\in P[a,b]}$ of the upper sums is bounded below by $m(b-a)$.

It has the greatest lower bound (glb)

Now the $\sup\{L(P,f)\}_{P\in P[a,b]}$ is called lower Riemann Integral of f on $[a,b]$ and is denoted by

$$\underline{\int_a^b} f(x)dx$$

i.e. $\underline{\int_a^b} f(x)dx = \text{Lub}\{L(P,f)\}_{P\in P[a,b]}$

and the $\text{glb}\{U(P,f)\}_{P\in P[a,b]}$ is called Upper Riemann Integral of f on $[a,b]$ and is denoted by

$$\overline{\int_a^b} f(x)dx$$

i.e. $\overline{\int_a^b} f(x)dx = \text{glb}\{U(P,f)\}_{P\in P[a,b]}$

* Riemann Integral:

– A bounded f is said to be Riemann integrable (or R-integrable) on $[a,b]$ if its lower and upper Riemann integrals are equal.

i.e. if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$.

– The common value of these integrals is called the Riemann integral of f on $[a,b]$ and is denoted by $\int_a^b f(x)dx$.

i.e., $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$

Note: (1) The interval $[a,b]$ is called the range of the integration. The numbers a and b are called the lower and upper limits of integration respectively.

(2) The family of all bounded functions which are R-integrable on $[a,b]$ is denoted by $R[a,b]$. If f is R-integrable on $[a,b]$ then $f \in R[a,b]$.

(3) f is R-integrable on $[a,b]$

$\Rightarrow$ (i) f is bounded on $[a,b]$

(ii) $\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx = \int_{a}^{b} f(x)\,dx$

(4) A bounded function f on $[a,b]$ is such that

$$\int_{\underline{a}}^{b} f(x)\,dx \neq \int_{a}^{\bar{b}} f(x)\,dx$$

then f is not R-integrable on $[a,b]$

Problems :

→ Let $f(x) = x \ \forall x \in [0,1]$ and let $P = \{0, 1/3, 2/3, 1\}$ be a partition of $[0,1]$. Compute $U(P,f)$ and $L(P,f)$.

Sol^n: Partition set P divides the interval $[0,1]$ into subintervals.

$I_1 = [0, 1/3]$, $I_2 = [1/3, 2/3]$, $I_3 = [2/3, 1]$

Now $\delta_1 = 1/3 - 0 = 1/3$

$\delta_2 = 2/3 - 1/3 = 1/3$

$\delta_3 = 1 - 2/3 = 1/3$

Since $f(x) = x$ is an increasing function on $[0,1]$.

$\therefore$ $M_1 = 1/3$, $m_1 = 0$

$M_2 = 2/3$, $m_2 = 1/3$

$M_3 = 1$, $m_3 = 2/3$

$\therefore U(P,f) = \sum_{r=1}^{3} M_r \delta_r$

$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$

$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$

$= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = 2/3$

Now $L(P,f) = \sum_{r=1}^{3} m_r \delta_r$

$= 1/3$

→ Compute $L(P,f)$ and $U(P,f)$ of the function f defined by $f(x) = x^2$ on $[0,1]$ and $P = \{0, 1/4, 2/4, 3/4, 1\}$

Sol^n: The partition set P divides $[0,1]$ into subintervals $I_1 = [0, 1/4]$, $I_2 = [1/4, 2/4]$, $I_3 = [2/4, 3/4]$ and $I_4 = [3/4, 1]$

$\therefore$ $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1/4$

Since $f(x) = x^2$ is an increasing on $[0,1]$.

$\therefore$ $m_1 = 0$; $M_1 = 1/16$

$m_2 = 1/16$; $M_2 = 4/16$

$m_3 = 4/16$; $M_3 = 9/16$

$m_4 = 9/16$; $M_4 = 1$

$L(P,f) = \sum_{r=1}^{4} m_r \delta_r$

$= 7/32$

and $U(P,f) = \sum_{r=1}^{n} M_r \delta_r = 15/32$

→ If f is defined on $[a,b]$ by $f(x) = K \ \forall x \in [a,b]$ where K is constant then $f \in R[a,b]$ and $\int_a^b f(x)\,dx = K(b-a)$. (or) A constant function is R-integrable

Sol'n: Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ be any partition of $[a,b]$.

Let $I_r = [x_{r-1}, x_r]$, $r = 1, 2 \ldots n$ be the r^{th} subinterval of $[a,b]$.

Since $f(x) = k$ (Constant).

$\therefore M_r = m_r = k$.

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} k(x_r - x_{r-1})$$

$$= k \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$= k(b-a)$$

and $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$$= k(b-a)$$

Now $\underline{\int_a^b} f(x)dx = \underset{P \in P[a,b]}{\text{lub}} \{L(P,f)\}$

$$= k(b-a).$$

and $\overline{\int_a^b} f(x)dx = \underset{P \in P[a,b]}{\text{glb}} \{U(P,f)\}$

$$= k(b-a).$$

$$\therefore \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = k(b-a)$$

$\therefore f \in R[a,b]$

and $\int_a^b f(x)dx = k(b-a)$.

P-I 2000 Show that the function f defined by $f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$

is not Riemann integrable on any interval.

(or)

Show by an example that every bounded function need not be R-integrable.

Sol'n: Let f be ~~denoted~~ defined on $[a,b]$ by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

Clearly $f(x)$ is bounded on $[a,b]$ because $0 \le f(x) \le 1 \ \forall x \in [a,b]$.

Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ be a partition of $[a,b]$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2 \ldots n$ be r^{th} subinterval of $[a,b]$.

$\therefore M_r = 1$; $m_r = 0$.

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \cdot \delta_r$$

$$= b-a$$

and $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$$= \sum_{r=1}^{n} 0 \cdot \delta_r$$

$$= 0.$$

Now $\underline{\int_a^b} f(x)\,dx = \underset{P \in P[a,b]}{\text{Lub}} \{L(P,f)\}$

$$= 0$$

and $\overline{\int_a^b} f(x)\,dx = \underset{P \in P[a,b]}{\text{glb}} \{U(P,f)\}$

$$= b-a.$$

$$\therefore \underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx.$$

$\therefore f$ is not Riemann integrable on $[a,b]$.

$\therefore$ Every bounded function need not be a Riemann integrable.

$\rightarrow$ Let $f(x)$ be defined on $[0,1]$ as follows.

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

show that f is not Riemann integrable over $[0,1]$.

Sol'n:- $P = \{0 = x_0, x_1, \ldots, x_n = 1\}$ be a partition of $[0,1]$.

Let $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots$

$\rightarrow$ Evaluate $\int_0^1 dx$ by applying the definition of Riemann-integral.

Sol'n: Let $f(x) = 1 \; \forall x \in [0,1]$.

$\rightarrow$ f is defined on $[0,1]$ by $f(x) = x \;\; \forall x \in [0,1]$. then $f \in R[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Sol'n: Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\}$ be any partition of $[0,1]$.

$I_r = [x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]$, $r = 1, 2, \ldots n$

$\delta_r = x_r - x_{r-1}$; $r = 1, 2, \ldots n$

$= \frac{r}{n} - \left(\frac{r-1}{n}\right)$

$= \cancel{\frac{r}{n}} - \cancel{\frac{r}{n}} + \frac{1}{n}$

$\boxed{\delta_r = 1/n}$

Since $f(x) = x$ is an increasing on $[0,1]$.

$\therefore m_r = \frac{r-1}{n}$; $M_r = \frac{r}{n}$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n}$

$= \frac{1}{n^2} \sum_{r=1}^{n} (r)$

$= \frac{1}{n^2} (1 + 2 + \cdots + n)$

$= \frac{1}{n^2} \left[\frac{n(n+1)}{2}\right]$

$= \frac{(n+1)}{n(2)}$

$= \frac{1 + 1/n}{2} = \frac{1}{2}[1 + \frac{1}{n}]$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$

$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$

$= \frac{1}{n^2}[0 + 1 + 2 + \cdots + (n-1)]$

$= \frac{1}{n^2}[(1 + 2 + \cdots + n-1) + n - n]$

$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right]$

$= \left[\frac{1}{n^2} \cdot \frac{n^2(1 + 1/n)}{2} - \frac{1}{n}\right]$

$= \frac{1}{2}(1 + 1/n) - \frac{1}{n}$

Now $\underline{\int_0^1} f(x)\,dx = \underset{P \in P[0,1]}{\text{Lub}} \{L(P,f)\}$

$= \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n}\right]$

$= \frac{1}{2}(1+0) - 0$

$= \frac{1}{2}$

and $\overline{\int_0^1} f(x)\,dx = \underset{P \in P[0,1]}{\text{glb}} \{U(P,f)\}$

$= \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{2}\left(1+\frac{1}{n}\right)\right]$

$= \frac{1}{2}(1+0) - 0$

$= \frac{1}{2}$.

$\therefore \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \frac{1}{2}$.

$\therefore f \in R[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$

→ If f is defined on $[0,a]$; $a>0$ by $f(x) = x^2 \ \forall x \in [0,a]$ then $f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

Sol^n: Let $P_n = \left\{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \ldots \ldots \left(\frac{r-1}{n}\right)a, \left(\frac{r}{n}\right)a, \ldots \frac{na}{n} = a\right\}$

be any partition of $[0,a]$.

Let $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$; $r = 1,2,3 \ldots n$

$\delta_r = \frac{ra}{n} - \frac{ra}{n} + \frac{a}{n}$

$\boxed{\delta_r = \frac{a}{n}}$

Since $f(x) = x^2$ is an increasing function on $[0,a]$; $a>0$

$\therefore M_r = \left(\frac{ra}{n}\right)^2$; $m_r = \left[\frac{(r-1)a}{n}\right]^2$

$= \frac{r^2a^2}{n^2}$; $= \frac{(r-1)^2a^2}{n^2}$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \frac{r^2a^2}{n^2} \cdot \frac{a}{n}$

$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2$

$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)a^3$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \frac{(r-1)^2a^2}{n^2} \cdot \frac{a}{n}$

$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2$

$= \frac{a^3}{n^3}\left[0 + 1^2 + 2^2 + \cdots + (n-1)^2\right]$

$= \frac{a^3}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 - n^2\right]$

$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - n^2\right]$

$= \frac{a^3}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n}\right]$

Now $\underline{\int_0^a} f(x)\,dx = \underset{P \in P[0,a]}{\text{Lub}} \{L(P,f)\}$

$= \underset{n\to\infty}{\text{Lt}} \left[\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n}\right]$

$= \frac{a^3}{6}(1)(2) - 0$

$= \frac{a^3}{3}$

and $\overline{\int_0^a} f(x)\,dx = \underset{P \in P[0,a]}{\text{glb}} \{U(P,f)\}$

$$= \lim_{n\to\infty} \left[\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \frac{a^3}{6}(1+0)(2+0)$$

$$= \frac{a^3}{6}(2) = \frac{a^3}{3}$$

$$\therefore \underline{\int_0^a} f(x)\,dx = \overline{\int_0^a} f(x)\,dx = \frac{a^3}{3}$$

$\therefore f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

→ If f is defined on $[0,a]$, $a>0$ by $f(x) = x^3 \ \forall\ x \in [0,a]$ then $f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^4}{4}$.

→ Show that $f(x) = 3x+1$ is Riemann integrable on $[0,1]$ and $\int_0^1 (3x+1)\,dx = 5/2$.

Sol^n:- Let $f(x) = 3x+1 \ \forall\ x \in [0,1]$

Then f is bounded on $[0,1]$.

Let $P = \{0, 1/n, 2/n, 3/n, \ldots, \frac{(r-1)}{n}, \frac{r}{n}, \ldots, \frac{n}{n}=1\}$.

$$I_r = [x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]; \quad r = 1, 2, \ldots n$$

$$\therefore \delta_r = \frac{r}{n} - \left(\frac{r-1}{n}\right)$$

$$= \cancel{\frac{r}{n}} - \cancel{\frac{r}{n}} + \frac{1}{n} = 1/n$$

Since $f(x) = 3x+1$ is an increasing function on $[0,1]$.

$$\therefore M_r = 3\left(\frac{r}{n}\right)+1; \quad m_r = 3\left(\frac{r-1}{n}\right)+1$$

Now $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$$= \sum_{r=1}^{n} \left[3\left(\frac{r}{n}\right)+1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{r=1}^{n}\left[\frac{3r}{n}+1\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}\sum_{r=1}^{n} r + \sum_{r=1}^{n}(1)\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}(1+2+\cdots+n) + (1+1+\cdots\right.$$

$$= \frac{1}{n}\left[\frac{3}{n}\left(\frac{n(n+1)}{2}\right) + n(1)\right]$$

$$= \frac{1}{n}\left[\frac{3}{2}(n+1)+n\right]$$

$$= \frac{3}{2}\left(1+\frac{1}{n}\right)+1.$$

$$L(P;f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n}\left[3\left(\frac{r-1}{n}\right)+1\right]\frac{1}{n}$$

$$= \frac{1}{n}\left[3\sum_{r=1}^{n}\frac{(r-1)}{n} + \sum_{r=1}^{n}(1)\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}\sum_{r=1}^{n}(r-1) + \sum_{r=1}^{n}(1)\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}(0+1+2+\cdots+(n-1)+n-n) + (1+1+\cdots+n)\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}(1+2+\cdots+n) - 3 + n(1)\right]$$

$$= \frac{1}{n}\left[\frac{3}{n}\cdot\frac{n(n+1)}{2} - 3 + n\right]$$

$$= \frac{3}{2}\left(1+\frac{1}{n}\right) - \frac{3}{n} + 1$$

Now $\int_0^1 f(x)\,dx = \text{lub}\{L(P,f) : P \in P[0,1]\}$

$$= \lim_{n\to\infty}\left[\frac{3}{2}\left(1+\frac{1}{n}\right) - \frac{3}{n} + 1\right]$$

$$= 3/2 + 1 = 5/2$$

and $\overline{\int_0^1} f(x)\,dx = \text{glb}\{U(P,f)\}_{P\in P[0,1]}$

$= \lim_{n\to\infty}\left[\frac{3}{2}(1+\frac{1}{n})+1\right]$

$= \frac{3}{2}+1 = \frac{5}{2}$

$\therefore \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \frac{5}{2}$

$\therefore f\in R[0,1]$ and $\int_0^1 (3x+2)\,dx = \frac{5}{2}$

→ show that $f(x) = 2x+1$ is integrable on $[1,2]$ and $\int_1^2 (2x+1)dx = 4$

Sol'n: Let $f(x) = 2x+1\ \forall x\in[1,2]$ then $f(x)$ is bounded on $[1,2]$

Let $P = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, 1+\frac{3}{n}, \ldots, 1+\frac{r-1}{n}, 1+\frac{r}{n}, \ldots 1+\frac{n}{n}=2\}$

be any partition of $[1,2]$.

Let $I_r = \left[1+\frac{r-1}{n}, 1+\frac{r}{n}\right]$

$r = 1, 2, \ldots n$.

H.W. Prove that $\int_1^2 f(x)dx = 7$, where $f(x) = 2x+4$.

H.W. Prove that $f(x) = 3x+1$ is integrable on $[1,2]$ and $\int_1^2 (3x+1)dx = \frac{11}{2}$.

→ show that $f(x) = 2-3x$ is integrable on $[1,3]$ and $\int_1^3 (2-3x)dx = -8$.

→ show that $f(x) = x$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \frac{1}{2}(b^2-a^2)$

Sol'n: Let $f(x) = x\ \forall x\in[a,b]$ then $f(x)$ is bounded on $[a,b]$.

Let the partition $P = \{a=a, a+\frac{h}{n}, a+\frac{2h}{n}, \ldots a+\frac{nh}{n}\}$

where $h = b-a$ be dividing the interval $[a,b]$ into 'n' equal parts.

Let $I_r = \left[a+\frac{(r-1)h}{n}, a+\frac{rh}{n}\right]$, $r = 1, 2, 3, \ldots$

→ Let f be defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{2} & \text{when } x\in\mathbb{Q} \\ \frac{1}{3} & \text{when } x\in\mathbb{R}-\mathbb{Q}\end{cases}$$

Then show that f is bounded but not Riemann integrable on $[0,1]$.

→ A function f is bounded on $[a,b]$. show that

(i) when k is a +ve constant:

$\underline{\int_a^b} kf\,dx = k\underline{\int_a^b} f\,dx$ and $\overline{\int_a^b} kf\,dx = k\overline{\int_a^b} f\,dx$ and

(ii) when k is a −ve constant:

$\underline{\int_a^b} kf\,dx = k\overline{\int_a^b} f\,dx$ and $\overline{\int_a^b} kf\,dx = k\underline{\int_a^b} f\,dx$

Also deduce that if f is integrable on $[a,b]$, then kf is also Riemann integrable where k is a constant.

and $\int_a^b kf\,dx = k\int_a^b f\,dx$

Sol'n:- Let f be a bounded function on $[a,b]$ and Let $P = \{a=x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots x_n=b\}$ be any partition of $[a,b]$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1,2,\dots n$

Let m_r and M_r be the infimum and Supremum of f on $I_r = [x_{r-1}, x_r]$, $r=1,2,\dots n$

Let m_r' & M_r' be the infimum and supremum of kf on

$I_r = [x_{r-1}, x_r]$; $r = 1,2 \dots n$.

(i) when k is +ve constant:

$m_r' = km_r$ and $M_r' = kM_r$

$$U(P, kf) = \sum_{r=1}^{n} M_r' \delta_r$$

$$= \sum_{r=1}^{n} kM_r\delta_r$$

$$= k\sum_{r=1}^{n} M_r\delta_r = kU(P,f)$$

Similarly $L(P,kf) = kL(P,f)$.

$$\therefore \int_a^b kf\,dx = glb\{U(P,kf)\}_{P\in P[a,b]}$$

$$= glb\{kU(P,f)\}_{P\in P[a,b]}$$

$$= k\,glb\{U(P,f)\}_{P\in P[a,b]}$$

$$= k\int_a^b f(x)dx.$$

and $\int_a^b kf\,dx = lub\{L(P,kf)\}_{P\in P[a,b]}$

$$= lub\{kL(P,f)\}_{P\in P[a,b]}$$

$$= k\,lub\{L(P,f)\}_{P\in P[a,b]}$$

$$= k\int_a^b f(x)dx.$$

(ii) when k is −ve constant:

$m_r' = kM_r$ and $M_r' = km_r$

$$\therefore U(P,kf) = \sum_{r=1}^{n} M_r'\delta_r$$

$$= \sum_{r=1}^{n} km_r\delta_r$$

$$= k\sum_{r=1}^{n} m_r\delta_r$$

$$= k\,L(P,f)$$

Similarly $L(P,kf) = kU(P,f)$

$$\therefore \int_a^{\bar{b}} kf\,dx = glb\{U(P,kf)\}_{P\in P[a,b]}$$

$$= glb\{kL(P,f)\}_{P\in P[a,b]}$$

$$= k\,lub\{L(P,f)\}_{P\in P[a,b]}$$

$$= k\int_{\underline{a}}^{b} f\,dx.$$

Similarly $\int_{\underline{a}}^{b} kf\,dx = k\int_a^{\bar{b}} f\,dx$.

(iii) If f is integrable on $[a,b]$ then

$$\int_a^b f(x)dx = \int_a^{\bar{b}} f(x)dx = \int_{\underline{a}}^{b} f(x)dx.$$

$\therefore$ from parts (i) & (ii), we have

$$\int_a^b kf\,dx = \int_a^{\bar{b}} kf\,dx = k\int_a^b f\,dx.$$

$\Rightarrow kf$ is Riemann-Integrable on $[a,b]$

and $\int_a^b kf\,dx = k\int_a^b f\,dx$.

→ Let $f(x) = \sin x \ \forall x\in[0,\pi/2]$

Let $P = \{0, \pi/2n, 2\pi/2n, \dots \frac{n\pi}{2n}\}$ be a partition of $[0,\pi/2]$. Compute $U(P,f)$ and $L(P,f)$. Hence Prove that $f\in R[0,\pi/2]$.

Sol^n:- Let $f(x) = \sin x \ \forall x \in [0, \pi/2]$

then f is bounded on $[0, \pi/2]$.

Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \dots \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be a partition of $[0, \pi/2]$.

$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$; $r = 1, 2, 3 \dots$

Since f is increasing function on $[0, \pi/2]$

$\therefore m_r = \sin\left(\frac{(r-1)\pi}{2n}\right)$ and

$M_r = \sin\left(\frac{r\pi}{2n}\right)$

and $\delta_r = \frac{r\pi}{2n} - \frac{(r-1)\pi}{2n}$

$= \frac{\pi}{2n}$

Now $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$$= \sum_{r=1}^{n} \sin\left(\frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[\sin\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right) + \sin\left(\frac{3\pi}{2n}\right) + \dots + \sin\left(\frac{n\pi}{2n}\right)\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\left[\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right] \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\left[\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha + \overline{n-1}\beta) = \frac{\sin\left[\alpha + \frac{n-1}{2}\beta\right]\sin\frac{n\beta}{2}}{\sin \beta/2}\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{(n+1)\pi}{4n} \cdot \sin\frac{\pi}{4}}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sin\frac{(n+1)\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{\sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right)}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4n}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\left[\cos\left(\frac{\pi}{4n}\right) + \sin\left(\frac{\pi}{4n}\right)\right]}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{4n}\left[\cot\left(\frac{\pi}{4n}\right) + 1\right]$$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$$= \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{2n} \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[0 + \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \dots + \sin\frac{(n-1)\pi}{2n}\right]$$

$$= \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\left(\frac{\pi}{2n} + \frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n} + \frac{2\pi}{2n}\right) + \dots + \sin\left(\frac{\pi}{2n} + \frac{(n-2)\pi}{2n}\right)\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\left[\sin\frac{\pi}{2n} + \frac{(n-2)}{2}\frac{\pi}{2n}\right]\sin\left(\frac{n-1}{2}\frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \pi/2n\right)}$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\left[\frac{\pi}{2n} - \frac{\pi}{2n} + \frac{n\pi}{4n}\right]\sin\left[\frac{\pi}{4} - \frac{\pi}{4n}\right]}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{\pi}{4} \cdot \sin\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\left[\sin\frac{\pi}{4}\cos\frac{\pi}{4n} - \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{4n}\right]}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}}\left[\frac{\cos\frac{\pi}{4n} - \sin\frac{\pi}{4n}}{\sin\left(\frac{\pi}{4n}\right)}\right]\frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{2n}\cdot\frac{1}{2}\left[\cot\left(\frac{\pi}{4n}\right)-1\right]$$

Now $\int_0^{\pi/2} f(x)\,dx = \sup\{L(P,f)\}\ P\in P[0,\frac{\pi}{2}]$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left[\cot\left(\frac{\pi}{4n}\right)-1\right]$$

$$= \lim_{\frac{\pi}{4n}\to 0}\frac{\pi/4n}{\tan(\pi/4n)} - \lim_{n\to\infty}\frac{\pi}{4n}$$

$$= 1-0$$

$$= 1$$

and $\int_0^{\pi/2} f(x)\,dx = \inf\{U(P,f)\}\ P\in P[0,\pi/2]$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left[\cot\frac{\pi}{4n}-1\right]$$

$$= \lim_{\frac{\pi}{4n}\to 0}\frac{\pi/4n}{\tan(\pi/4n)} + \lim_{n\to\infty}\ldots$$

$$= 1+0$$

$$= 1$$

$\therefore \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx = 1.$

$\therefore f\in R[0,\pi/2]$ and $\int_0^{\pi/2} f(x)\,dx = 1$

→ If f be a function defined on $[0,\pi/4]$ by $f(x)=\begin{cases}\cos x & \text{if } x \text{ is rational}\\ \sin x & \text{if } x \text{ is irrational}\end{cases}$

then $f\notin R[0,\pi/4]$.

Sol^n:- Let $P = \{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \frac{3\pi}{4n}, \ldots \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots \frac{n\pi}{4n}=\frac{\pi}{4}\}$

be a Partition of $[0,\pi/4]$.

$$I_r = \left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right];\ r=1,2,\ldots$$

Since $\cos x \geq \sin x$

i.e. $\sin x \leq \cos x$ in $[0,\pi/4]$

$$\therefore m_r = \sin\left(\frac{(r-1)\pi}{4n}\right)$$

$$M_r = \cos\left(\frac{(r-1)\pi}{4n}\right)$$

and $\delta_r = \frac{\pi}{4n}$

Now $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$

$$= \sum_{r=1}^{n}\cos\frac{(r-1)\pi}{4n}\cdot\frac{\pi}{4n}$$

$$= \frac{\pi}{4n}\sum_{r=1}^{n}\cos\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos\frac{\pi}{4n} + \cos\frac{2\pi}{4n} + \cdots + \cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos\left(0+\frac{\pi}{4n}\right) + \cos\left(0+\frac{2\pi}{4n}\right) + \cdots + \cos\left(0+\frac{(n-1)\pi}{4n}\right)\right]$$

$$= \frac{\pi}{4n}\left[\frac{\cos\left(0+\frac{n-1}{2}\cdot\frac{\pi}{4n}\right)\sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\pi/4n\right)}\right]$$

$$= \frac{\pi}{4n}\left[\frac{\cos\frac{(n-1)\pi}{8n}\cdot\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8n}\right)}\right]$$

$$= \frac{\pi}{4n}\cdot\frac{1}{\sqrt{2}}\left[\frac{\cos\frac{(n-1)\pi}{8n}\cdot\sin\frac{\pi}{8}}{\sin(\pi/8n)}\right]$$

Similarly $L(P,f) =$

$$\frac{\pi}{4n}\cdot\frac{\sin\left[\frac{(n-1)\pi}{8n}\right]\cdot\sin\frac{\pi}{8}}{\sin(\pi/8n)}$$

Now

$$\int_0^{\pi/4} f(x)\,dx = \text{lub}\{L(P,f)\}\; P\in P[0,\pi/4]$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\sin\left[\frac{(n-1)\pi}{8n}\right]\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8n}\right)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(\pi/8n)}{\sin(\pi/8n)}\, 2\sin\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1\times 2\sin(\pi/8)\sin(\pi/8)$$

$$= 2\sin^2(\pi/8)$$

$$= 1-\cos\pi/4 = 1-\frac{1}{\sqrt{2}}$$

and $\int_0^{\pi/4} f(x)\,dx = \text{glb}\{U(P,f)\}\; P\in P[0,\frac{\pi}{4}]$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\cos\frac{(n-1)\pi}{8n}\cdot\sin\frac{\pi}{8}}{\sin(\pi/8n)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cdot 2\cos\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8}$$

$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \int_0^{\pi/4} f(x)\,dx \neq \int_0^{\pi/4} f(x)\,dx$$

$$\therefore f\notin R[0,\pi/4]$$

* Some Theorem Statements:-

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function and $P\in P[a,b]$ then

(i) $L(P,f)\le U(P,f)$

(ii) $L(P,-f) = -U(P,f)$

(iii) $U(P,-f) = -L(P,f)$

→ If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are bounded functions and $P\in P[a,b]$ then

(i) $U(P,f+g)\le U(P,f)+U(P,g)$

(ii) $L(P,f+g)\ge L(P,f)+L(P,g)$

(iii) $w(P,f+g)\le w(P,f)+w(P,g)$.

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

i.e. Lower Riemann Integral cannot exceed upper Riemann integral.

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function, then

$$m(b-a)\le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

where m & M are the infimum and supremum of f on $[a,b]$.

→ If $f\in R[a,b]$ then

(i) $m(b-a)\le \int_a^b f(x)\,dx \le M(b-a)$ if $b\ge a$

(ii) $m(b-a)\ge \int_a^b f(x)\,dx \ge M(b-a)$ if $b\le a$.

where m and M are the infimum and supremum of f on $[a,b]$.

Definition:

The meaning of $\int_a^b f(x)\,dx$ when $b \le a$

If f is bounded and integrable on [illegible] for $a>b$ i.e. $b<a$.

we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \text{ when } a>b$$

Also $\int_a^b f(x)\,dx = 0$ when $a=b$.

Darboux Theorem:-

If $f:[a,b]\to\mathbb{R}$ is a bounded function then for each $\epsilon>0$, there exists a $\delta>0$ such that

(i) $U(P,f) < \int_a^b f(x)\,dx + \epsilon$

(ii) $L(P,f) > \int_a^b f(x)\,dx - \epsilon$

for each $P \in P[a,b]$ with $\|P\| < \delta$.

→ A bounded function f is integrable on $[a,b]$ iff for each $\epsilon>0$, $\exists$ a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

Imp If $f:[a,b]\to\mathbb{R}$ is continuous function on $[a,b]$ then f is integrable on $[a,b]$.

Imp If $f:[a,b]\to\mathbb{R}$ is monotonic on $[a,b]$ then f is integrable on $[a,b]$.

Imp If the set of points of discontinuity of bounded function $f:[a,b]\to\mathbb{R}$ is finite, then f is integrable on $[a,b]$.

Imp If the set of points of discontinuity of a bounded function $f:[a,b]\to\mathbb{R}$ has a finite number of limit points then f is integrable on $[a,b]$.

*** Riemann Sum :-**

Let f be a real valued function defined on $[a,b]$.

Let $P=\{a=x_0, x_1, \dots, x_n=b\}$ be a partition of $[a,b]$.

Let $\xi_r \in [x_{r-1}, x_r]$, $r=1,2,\dots,n$

then sum $\sum_{r=1}^{n} f(\xi_r)\,\delta_r$ is called a Riemann sum of f on $[a,b]$ relative to P.

It is denoted by $S(f,P)$ or $S(P,f)$

i.e. $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\,\delta_r$ —— (1)

Note:(i)

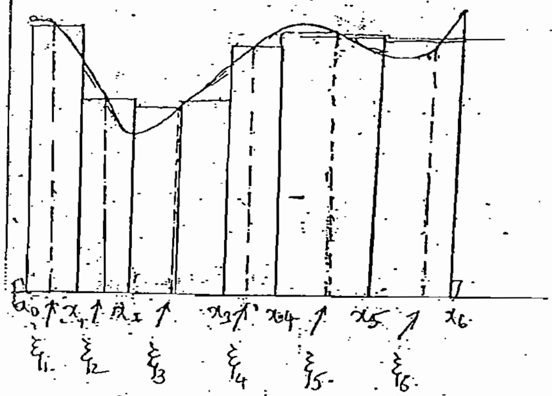
Since ξ_r is any arbitrary point of $[x_{r-1}, x_r]$, therefore corresponding to each partition P of $[a,b]$, there exist infinitely many Riemann sum.

Note(ii)

If the function f is +ve on $[a,b]$, then the Riemann sum (1) is the

Sum of the areas of 'n' rectangles whose bases are the subintervals $I_r = [x_{r-1}, x_r]$ and whose heights are $f(\xi_r)$.

A Riemann Sum



* Integral as the limit of a Sum (second definition of Riemann Integral):-

[Note: Earlier, we arrived at the integral of a function via the upper and the lower sums. The number M_r, m_r which appear in these sums are not necessarily the values of the function f (they are values of 'f' if 'f' is continuous)

we shall now show that $\int f dx$ can also be considered as the limit of a sequence of sums in which M_r and m_r are replaced by the values of 'f'.

Corresponding to a partition P of $[a,b]$, let us choose points $\xi_1, \xi_2, \xi_3 - - - \xi_n$ such that $x_{r-1} \le \xi_r \le x_r$ $(r = 1, 2 - - - n)$ and consider the sum $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\delta_r$

The sum $S(P,f)$ is called Riemann sum of 'f' on $[a,b]$ relative to P.]

Definition:

We say that $S(P,f)$ converges to L as $\|P\| \to 0$. i.e. $\lim_{\|P\| \to 0} S(P,f) = L$

i.e. if for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|S(P,f) - L| < \epsilon$.

For every partition

$P = \{a = x_0, x_1, - - - x_n = b\}$ with $\|P\| < \delta$ and for every choice of points $\xi_r \in [x_{r-1}, x_r]$

(or)

A function f is said to be integrable on $[a,b]$ if $\lim_{\|P\| \to 0} S(P,f)$ exists and

$$\lim_{\|P\| \to 0} S(P,f) = \int_a^b f(x)dx.$$

Note(i):

Since $\|P\| \to 0$ when $n \to \infty$ therefore $\lim_{\|P\| \to 0}$ can be replaced by $\lim_{n \to \infty}$ in the above definition.

Note(2):

$f \in R[a,b] \Rightarrow \lim S(P,f)$ exists.

Corollary(1): If 'f' is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h f(a+rh)$$

where $h = \frac{b-a}{n}$.

Let $P = \{a, a+h, a+2h, \ldots a+nh = b\}$ be a partition of $[a,b]$.

It divides the interval $[a,b]$ into n equal subintervals; each of length $h = \frac{b-a}{n}$

$\therefore \|P\| = \frac{b-a}{n}$

As $\|P\| \to 0$, $n \to \infty$

$I_r = [a+(r-1)h, a+rh]$

Let $\xi_r \in I_r$ such that

$a+(r-1)h \le \xi_r \le a+rh$; $r = 1,2 \ldots n$

Then $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(P,f)$

$$= \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(a+rh)h$$

(Taking $\xi_r = a+rh$)

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h f(a+rh)$$

Corollary-2:

If 'f' is integrable on $[0,1]$ then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

Sol^n: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{r-1}{n}, \frac{r}{n} \ldots \frac{n}{n} = 1\right\}$ be a partition of $[0,1]$.

It divides $[0,1]$ into 'n' equal subintervals, each of length $1/n$.

$\therefore \|P\| = 1/n$.

As $\|P\| \to 0$; $n \to \infty$

$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$; $r = 1,2 \ldots n$

$\therefore \delta_r = 1/n$.

Let $\xi_r \in I_r$ such that

$\frac{r-1}{n} \le \xi_r \le \frac{r}{n}$, $r = 1,2 \ldots n$.

then $\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$$

(Taking $\xi_r = \frac{r}{n}$)

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

Note: To evaluate the limit of a sum:

(i) write the limit of sum

$\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$

(ii) replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx

(iii) replace $\lim_{n\to\infty} \sum_{r=1}^{n}$ by $\int_0^1$

Note that the limits of integration are the values of $\frac{r}{n}$ for the first and last terms as $n \to \infty$).

$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left(\frac{1}{n}\right) = \int_0^1 f(x)\,dx.$

Corollary(3):

If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} (a h^{r} - a h^{r-1}) f(ah^{r})$

where $h = \left(\frac{b}{a}\right)^{1/n}$

Sol^n:- Let $P = \{a, ah, ah^2, \ldots ah^{r-1}, ah^r, \ldots ah^n = b\}$ be a partition of $[a, b]$

$I_r = [ah^{r-1}, ah^r]$; $r = 1, 2, \ldots n$

$\delta_r = ah^r - ah^{r-1}$

As $\|P\| \to 0$, $h \to 1$, $n \to \infty$.

Let $\xi_r \in I_r$ such that

$ah^{r-1} \le \xi_r \le ah^r$; $r = 1, 2, \ldots n$.

Then $$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(ah^r)(ah^r - ah^{r-1})$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r)$$

Problems:-

From definition, Prove that

$\int_1^2 f(x)dx = 6$ where $f(x) = 2x+3$.

Sol^n:- Let $f(x) = 2x+3 \;\forall\; x \in [1,2]$

Since f is bounded and continuous on $[1,2]$

$\therefore$ f is integrable on $[1,2]$

Let $P = \{1 = x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots x_n = 2\}$

$= \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots 1+\frac{r-1}{n}, 1+\frac{r}{n}, \ldots 1+\frac{n}{n} = 2\}$

be a partition of $[1,2]$ which divides $[1,2]$ into n equal subintervals

$\therefore$ Each of length $= \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$\therefore \|P\| = \frac{1}{n}$ and

$\therefore \|P\| \to 0$ as $n \to \infty$.

$I_r = [x_{r-1}, x_r] = \left[1+\frac{r-1}{n}, 1+\frac{r}{n}\right]$; $r = 1, 2, \ldots n$

Let $\xi_r \in I_r$ such that $1+\frac{r-1}{n} \le \xi_r \le 1+\frac{r}{n}$.

$r = 1, 2, \ldots n$.

i.e. $\xi_r \in I_r$ such that $x_{r-1} \le \xi_r \le x_r$

$$\therefore \int_1^2 f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\delta_r \quad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(1+\frac{r}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(2\left(1+\frac{r}{n}\right)+3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[5\sum_{r=1}^{n}(1) + \frac{2}{n}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty} \left[\frac{5}{n}\cdot n + \frac{2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[5 + \left(1+\frac{1}{n}\right)\right]$$

$= 6.$

$\rightarrow \int_1^2 x\,dx = 3/2$.

$\rightarrow \int_0^1 (2x^2 - 3x + 5)\,dx = 25/6$.

$\rightarrow$ Evaluate $\int_{-1}^{1} f(x)dx$ where $f(x) = |x|$.

Sol^n:- Let $f(x) = |x| \;\forall\; x \in [-1, 1]$.

$$\therefore f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\therefore f(x)$ is bounded and continuous on $[-1,1]$.

$\Rightarrow f(x)$ is integrable on $[-1,1]$.

Let, $P = \{-1 = x_0, x_1, x_2, \ldots x_n = 0, x_{n+1}, x_{n+2}, \ldots x_{n+n} = x_{2n} = 1\}$

$= \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \ldots -1+\frac{n+1}{n}, -1+\frac{n+2}{n} \ldots -1+\frac{n+n}{n} = 1\}$

be a partition of $[-1,1]$, which divides $[-1,1]$ into $2n$ equal subintervals, each of length $= \frac{b-a}{2n} = \frac{1-(-1)}{2n} = \frac{1}{n}$

$\therefore \|P\| = \frac{1}{n} \to 0$ as $n \to \infty$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots 2n$.

Let $\xi_r \in I_r$ such that $x_{r-1} \leq \xi_r \leq x_r$; $r = 1, 2, \ldots n$

and $\delta_r = 1/n$

$$\therefore \int_{-1}^{1} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{2n} f(\xi_r)\delta_r$$

$$\lim_{n\to\infty} \sum_{r=1}^{2n} f(x_r)\frac{1}{n} \quad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{2n} f\left(-1+\frac{r}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}\left[-\left(-1+\frac{r}{n}\right)\right] + \sum_{r=n+1}^{2n}\left(-1+\frac{r}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\cancel{n} - \frac{1}{n}\frac{n(n+1)}{2} - \cancel{n} + \frac{1}{n}\sum_{r=n+1}^{2n} r\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{1}{n}\left((n+1)+(n+2)+ \ldots + (n+n) = 2n\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{1}{n}\left\{\frac{n(n+1+2n)}{2}\right\}\right]$$

$\left[\because \text{in an AP } S_n = \frac{n}{2}(a+l)\right.$ a = first term, l = last term$\left.\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{(3n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[-\tfrac{1}{2}\left(1+\tfrac{1}{n}\right) + \left(3+\tfrac{1}{n}\right)\tfrac{1}{2}\right]$$

$$= -\tfrac{1}{2} + \tfrac{3}{2} = \frac{\cancel{2}}{\cancel{2}} = 1$$

$\rightarrow$ Evaluate $\int_{-1}^{2} f(x)\,dx$ where $f(x) =$

Sol^n: Since $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$

$\therefore f(x)$ bounded and continuous function on $[-1,2]$.

Let $P = \{-1 = x_0, x_1, x_2 \ldots x_n = 0, x_{n+1}, x_{n+2} \ldots x_{2n} = 1, x_{2n+1}, x_{2n+2} \ldots x_{2n+n} = x_{3n} = 2\}$

$= \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \ldots -1+\frac{n}{n},$

$-1+\frac{n+1}{n}, -1+\frac{n+2}{n}, \ldots -1+\frac{2n}{n} = 1,$

$-1+\frac{2n+1}{n}, -1+\frac{2n+2}{n}, \ldots -1+\frac{2n+n}{n} = 2\}$

be a partition of $[-1,2]$ which divides $[-1,2]$ into $3n$ equal sub intervals each sub interval length

$$\frac{b-a}{3n} = \frac{2-(-1)}{3n} = \frac{1}{n}$$

$\therefore \|P\| = \frac{1}{n} \to 0$ as $n \to \infty$.

$I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots 3n$

Let $\xi_r \in I_r$ such that $x_{r-1} \le \xi_r \le x_r$; $r = 1, 2, \dots 3n$.
and $\delta_r = \frac{1}{n}$

$$\therefore \int_{-1}^{2} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{3n} f(\xi_r)\,\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{3n} f(x_r)\frac{1}{n}$$
(Taking $\xi_r = x_r$)

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{3n} f\left(-1+\frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}\left[-\left(-1+\frac{r}{n}\right)\right] + \sum_{r=n+1}^{3n}\left(-1+\frac{r}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}(1) - \sum_{r=1}^{n}\frac{r}{n} - \sum_{r=n+1}^{3n}(1) + \sum_{r=n+1}^{3n}\left(\frac{r}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n - \frac{1}{n}\frac{n(n+1)}{2} - 2n + \frac{1}{n}\{(n+1) + (n+2) + (n+3) + \dots + 3n\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[-n - \frac{1}{2}(n+1) + \frac{1}{n}\cdot\frac{2n}{2}(n+1+3n)\right]$$

$\left(\because \text{in an A.P } S_{2n} = \frac{2n}{2}(a+l)\right)$

$$= \lim_{n\to\infty}\left[-1 - \frac{1}{2}\left(1+\frac{1}{n}\right) + \left(4+\frac{1}{n}\right)\right]$$

$$= -1 - \tfrac{1}{2} + 4$$

$$= -\tfrac{3}{2} + 4 = \frac{-3+8}{2}$$

$$= \tfrac{5}{2}$$

→ ~~Evaluate $\int_{-1}^{1} f(x)\,dx$~~
where $f(x) = |x|$

→ Show that $\int_0^a \sin x\,dx = 1 - \cos a$
where a is fixed Real number.

Sol^n:- Since $f(x) = \sin x$ is bounded and continuous on $[0, a]$.

$\therefore f$ is Riemann integrable on $[0, a]$.

Let $P = \{0 = x_0, x_1, x_2, \dots x_n = a\}$
$= \{0, \frac{a}{n}, \frac{2a}{n}, \dots \frac{na}{n} = a\}$ be a Partition of $[0, a]$ which divides $[0, a]$ into n equal subinterval each of length $= \frac{b-a}{n} = \frac{a-0}{n} = \frac{a}{n}$.

$\therefore \|P\| \to 0$ as $n \to \infty$

$I_r = [x_{r-1}, x_r] = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$; $r = 1, 2, \dots n$

$\delta_r = \frac{a}{n}$; $r = 1, 2 \dots n$.

$$\therefore \int_0^a f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\frac{a}{n}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{ra}{n}\right)\frac{a}{n}$$

$$= \lim_{n\to\infty} \frac{a}{n}\sum_{r=1}^{n} \sin\left(\frac{ra}{n}\right)$$

$$= \lim_{n\to\infty} \frac{a}{n}\left[\sin\frac{a}{n} + \sin\frac{2a}{n} + \dots + \sin\frac{na}{n}\right]$$

$$= \lim_{n\to\infty} \frac{\frac{a}{n}\left[\sin\left(\frac{a}{n} + \frac{n-1}{2}\cdot\frac{a}{n}\right)\sin\left(\frac{n}{2}\cdot\frac{a}{n}\right)\right]}{\sin\left(\frac{a}{2n}\right)}$$

$$\left[\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha+\overline{n-1}\,\beta) = \frac{\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\frac{n\beta}{2}}{\sin\beta/2}\right]$$

$$= \lim_{n\to\infty} \frac{\frac{a}{n}\left[\sin\left(\frac{a}{n}+\frac{a}{2}-\frac{a}{2n}\right)\sin\left(\frac{a}{2}\right)\right]}{\sin\left(\frac{a}{2n}\right)}$$

$$= \lim_{\frac{a}{2n}\to 0} 2\cdot\frac{\frac{a}{2n}}{\sin\left(\frac{a}{2n}\right)} \cdot \lim_{n\to\infty} \sin\left(\frac{a}{n}+\frac{a}{2}-\frac{a}{2n}\right)\sin\left(\frac{a}{2}\right)$$

$$= 2\cdot(1)\sin^2\left(\frac{a}{2}\right)$$

$$= 1-\cos a$$

H.W. show that $\int_0^a \cos x\,dx = \sin a$.

where a is a fixed number.

H.W. $\int_0^{\pi/2} \cos x\,dx = 1$

$\boxed{2004}$ P-II Show that the greatest integer function $f(x)=[x]$ is integrable on $[0,4]$ and $\int_0^4 [x]dx = 6$

Sol'n: $f(x)=[x]\ \forall x\in[0,4]$

$$\Rightarrow f(x) = \begin{cases} 0 & \text{when } 0\le x<1 \\ 1 & \text{when } 1\le x<2 \\ 2 & \text{when } 2\le x<3 \\ 3 & \text{when } 3\le x<4 \end{cases}$$

$\Rightarrow f$ is bounded and has only four points of finite discontinuity at $1,2,3,4$.

Since the points of discontinuity of f on $[0,4]$ are finite in number.

$\therefore f$ is integrable on $[0,4]$.

$$\therefore \int_0^4 [x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \int_3^4 [x]dx$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3$$

$$= 1(2-1)+2(3-2)+3(4-3)$$

$$= 1+2+3$$

$$= 6$$

H.W. Prove that $f(x)=x[x]$ is integrable on $[0,2]$ and $\int_0^2 x[x]dx =$

H.W. Prove that $f(x)=x-[x]$ is integrable on $[1,10]$ and $\int_1^{10} f(x)dx =$

$\boxed{2007}$ P-II Show that the function f defined by

$f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n=0,1,2,\ldots$

$f(0)=0$ is integrable on $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = 2/3$.

Sol'n:- $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n=0,1,2,\ldots$

$$= \frac{1}{2^0} = 1 \quad \text{when } \frac{1}{2^1} < x \le \frac{1}{2^0}$$

$$= \frac{1}{2} \quad \text{when } \frac{1}{2^2} < x \le \frac{1}{2^1}$$

$$= \frac{1}{2^2} \quad \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2}$$

$$\vdots$$

$$= \frac{1}{2^{n-1}} \quad \text{when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$$

$$\vdots$$

$$0 \quad \text{when } x=0$$

$\Rightarrow f$ is bounded and continuous on $[0,1]$ except at the points

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

The set of points discontinuity of f on $[0,1]$ is

$\{\frac{1}{2}, \frac{1}{2^2}, \ldots\}$ which has only one limit point '0'.

Since the set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore f$ is integrable on $[0,1]$.

Now $\int_{1/2^n}^{1} f(x)dx = \int_{1/2}^{1} f(x)dx + \int_{1/2^2}^{1/2} f(x)dx + \int_{1/2^3}^{1/2^2} f(x)dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} f(x)dx$

$= (1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{2^2}) + \frac{1}{2^2}(\frac{1}{2^2} - \frac{1}{2^3}) + \cdots + \frac{1}{2^{n-1}}(\frac{1}{2^{n-1}} - \frac{1}{2^n})$

$= (1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2^2}) + \frac{1}{2^2}(\frac{1}{2^3}) + \cdots + \frac{1}{2^{n-1}}(\frac{1}{2^n})$

$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots + \left(\frac{1}{2^2}\right)^{n-1}\right]$

$= \frac{1}{2}\left[\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}\right] = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$

Taking limit when $n \to \infty$, we get

$$\lim_{n\to\infty} \int_{1/2^n}^{1} f(x)dx = \lim_{n\to\infty} \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

$$\Rightarrow \int_0^1 f(x)dx = 2/3$$

→ show that f defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}; \ (n = 0, 1, 2, \ldots) \\ 0; & x = 0 \end{cases}$$

is integrable on $[0,1]$. Also show that $\int_0^1 f(x)dx = \frac{\pi^2}{6} - 1$

Sol'n: $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}; \ n = 1, 2, \ldots \\ 0, & x = 0 \end{cases}$

$= 1; \ \frac{1}{2} < x \le 1$

$= \frac{1}{2}; \ \frac{1}{3} < x \le \frac{1}{2}$

$= \frac{1}{3}; \ \frac{1}{4} < x \le \frac{1}{3}$

$\vdots$

$= \frac{1}{n-1}; \ \frac{1}{n} < x \le \frac{1}{n-1}$

$\vdots$

$= 0; \ x = 0$

$\Rightarrow f(x)$ is bounded and continuous on $[0,1]$ except at the points

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n+1}, \ldots$

The set of points of discontinuity

of f on $[0,1]$ is $\{1/2, 1/3, 1/4, ----\}$ which has one limit point '0'.

The set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore$ f is integrable on $[0,1]$.

Now $\int_{\frac{1}{n+1}}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + ---- + \int_{\frac{1}{n+1}}^{1/n} f(x)\,dx$

$= \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} \frac{1}{2}\,dx + \int_{1/4}^{1/3} \frac{1}{3}\,dx + --- + \int_{\frac{1}{n+1}}^{1/n} \frac{1}{n}\,dx$

$= (1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2}-\frac{1}{3}) + \frac{1}{3}(\frac{1}{3}-\frac{1}{4}) + ---- + \frac{1}{n}(\frac{1}{n}-\frac{1}{n+1})$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ ------ +\frac{1}{n^2}\right) - \left(\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+ --- \frac{1}{n(n+1)}\right)$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ ------ +\frac{1}{n^2}\right) - \left((1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+ ---- --- +(\frac{1}{n}-\frac{1}{n+1})\right)$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ ------ +\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)$

$\therefore \int_{\frac{1}{n+1}}^{1} f(x)\,dx = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ --- +\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)$

Now taking limit as $n\to\infty$, we get

$\lim_{n\to\infty} \int_{\frac{1}{n+1}}^{1} f(x)\,dx = \lim_{n\to\infty}\left[\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ --- +\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)\right]$

$\Rightarrow \int_0^1 f(x)\,dx = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+ ---- +\frac{1}{n^2}\right) - \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$

$= \frac{\pi^2}{6} - 1$

($\because$ The series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ --- +\frac{1}{n^2}+ ---$ converges to $\frac{\pi^2}{6}$)

HW ① show that the function f defined on $[0,1]$ as

$f(x) = 2rx, \quad \frac{1}{r+1} < x \le \frac{1}{r}, \quad r = 1, 2, 3, ----$

is integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$

* Properties of Riemann Integral

$\to$ If $f \in R[a,b]$ then $-f \in R[a,b]$.

$\to$ If $f \in R[a,b]$ then $|f| \in R[a,b]$

[Note: Converse is not true.
i.e. If $|f| \in R[a,b]$ then f need not be R-integrable on $[a,b]$.]

Ex:- $f:[a,b] \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$

$\to$ If $f, g \in R[a,b]$ then $f \pm g \in R[a,b]$ and $\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

→ If $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in R[a,b]$

→ If $f \in R[a,b]$ then $f^2 \in R[a,b]$

→ If $f, g \in R[a,b]$ and there exists $t > 0$ such that $|g(x)|$ $\forall x \in [a,b]$. then $\frac{f}{g} \in R[a,b]$.

→ If $f, g \in R[a,b]$ then $fg \in R[a,b]$.

Note: Even though f, g are not integrable on $[a,b]$, fg may be integrable on $[a,b]$

Ex:- Let $f : [a,b] \to \mathbb{R}$; $g : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 ; x \in \mathbb{Q} \\ 1 , x \in \mathbb{R} - \mathbb{Q} \end{cases} \quad \text{and } g(x) = \begin{cases} 1 , x \in \mathbb{Q} \\ 0 , x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then $(fg)x = f(x) \cdot g(x)$

$= 0 \ \forall x \in \mathbb{R}$

Since fg is constant function.

$\Rightarrow fg \in R[a,b]$

But f, g are not Riemann integrable on $[a,b]$.

→ If $f \in R[a,b]$ and $a < c < b$ then $f \in R[a,c]$, $f \in R[c,b]$.

and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

→ If $f \in R[a,c]$, $f \in R[c,b]$ and $a < c < b$ then $f \in R[a,b]$.

→ If $f \in R[a,b]$ and $f(x) \geq 0$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

Sol^n: Since $f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$.

Let m & M be the infimum & supremum of f on $[a,b]$.

Since $f(x) \geq 0$ $\forall x \in [a,b]$

Now for every $P \in P[a,b]$,

$L(P,f) \geq m(b-a)$
≥ 0

$\Rightarrow L(P,f) \geq 0$

Now $\int_a^b f(x)\,dx = \text{lub}\{L(P,f)\} \geq 0$
$P \in P[a,b]$

But $\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$ ($\because f \in R[a,b]$)

$\therefore \int_a^b f(x)\,dx \geq 0$.

2004 → If $f, g \in R[a,b]$ and $f(x) \geq g(x)$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Sol^n: $f, g \in R[a,b]$

$\Rightarrow f - g \in R[a,b]$

Since $f(x) \geq g(x)$ $\forall x \in [a,b]$

$\Rightarrow f(x) - g(x) \geq 0$ $\forall x \in [a,b]$

$\Rightarrow (f-g)(x) \geq 0$ $\forall x \in [a,b]$

We know that

If $f \in R[a,b]$ and $f(x) \geq 0$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$

$\therefore \int_a^b (f-g)(x)\,dx \geq 0$

$\Rightarrow \int_a^b (f(x)-g(x))\,dx \geq 0$

$\Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$

$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

Corollary: If $f, g, h \in R[a,b]$ and $f(x) \geq g(x) \geq h(x)\ \forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \geq \int_a^b h(x)\,dx.$

Note: $0 < x < 1$

$\Rightarrow 0 < x^2 < 1$

Since exponential function is an increasing function on $\mathbb{R}$.

$\therefore 0 < x^2 < 1$

$\Rightarrow e^0 < e^{x^2} < e^1$

$\Rightarrow 1 < e^{x^2} < e$

Take $f(x) = 1$, $g(x) = e^{x^2}$, $h(x) = e$

$\therefore$ we have $f(x) < g(x) < h(x)\ \forall x \in [0,1]$

$\Rightarrow \int_0^1 f(x)\,dx < \int_0^1 g(x)\,dx < \int_0^1 h(x)\,dx$

$\Rightarrow \int_0^1 1\,dx < \int_0^1 e^{x^2}\,dx < \int_0^1 e\,dx$

$\Rightarrow 1 < \int_0^1 e^{x^2}\,dx < e.$

→ If $f \in R[a,b]$ then $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

Sol^n:- Since $f \in R[a,b]$

$\Rightarrow |f| \in R[a,b]$

$\Rightarrow -|f| \in R[a,b].$

we have

$-|f(x)| \leq f(x) \leq |f(x)|\ \forall x \in [a,b]$

$\Rightarrow \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

$\therefore \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

→ If $f \in R[a,b]$ and m, M are infimum & supremum of f in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ and $\int_a^b f(x)\,dx = \mu(b-a)$ where $\mu \in [m, M]$.

Sol^n:- For every $P \in P[a,b]$ we have

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$ — ①

Now $\text{Lub}\{L(P,f)\}_{P \in P[a,b]} = \int_{\underline{a}}^{b} f(x)\,dx$

$= \int_a^b f(x)\,dx$

$(\because f \in R[a,b])$

$\Rightarrow L(P,f) \leq \int_a^b f(x)\,dx$ — ②

and $\text{glb}\{U(P,f)\}_{P \in P[a,b]} = \int_a^{\overline{b}} f(x)\,dx$

$= \int_a^b f(x)\,dx$

$(\because f \in R[a,b])$

$\Rightarrow U(P,f) \geq \int_a^b f(x)dx$ —— (3)

$\therefore$ from (1), (2) & (3) we have

$$m(b-a) \leq L(P,f) \leq \int_a^b f(x)dx \leq U(P,f) \leq M(b-a)$$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

$$\Rightarrow m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M \text{ for } a \neq b.$$

$\Rightarrow \frac{1}{b-a}\int_a^b f(x)dx$ is a number μ (say) lying between the bounds m & M.

$$\therefore \frac{1}{b-a}\int_a^b f(x)\,dx = \mu.$$

$$\Rightarrow \int_a^b f(x)\,dx = \mu(b-a)$$

where $m \leq \mu \leq M$.

For $a=b$, the result is trivially true.

Ex:- $f(x) = \sqrt{3+x^2}$ $\forall x \in [1,3]$

$\Rightarrow f(x)$ bounded & Riemann integrable on $[1,3]$.

Now $f'(x) = \frac{x}{\sqrt{3+x^2}} > 0 \;\forall x \in [1,3]$.

$\therefore f(x)$ is an increasing.

$\therefore m = f(1) = \sqrt{3+1^2} = 2$ $\qquad M = f(3) = \sqrt{3+3^2} = \sqrt{12}$

$$\therefore 2(3-1) \leq \int_1^3 \sqrt{3+x^2}\,dx \leq \sqrt{12}\,(3-1)$$

$$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^2}\,dx \leq 4\sqrt{3} < 7$$

$$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^2}\,dx < 7$$

→ If f is continuous on $[a,b]$, $\exists\, c \in [a,b]$ such that $\int_a^b f(x)dx = (b-a)f(c)$

Proof:- Since f is continuous on $[a,b]$

$\Rightarrow f \in R[a,b]$.

$\therefore \exists\, \mu \in [m,M]$ such that $\int_a^b f(x)dx = \mu(b-a)$ — (1)

Since f is continuous on $[a,b]$, it attains every value between its bounds m, M.

$\therefore \mu \in [m,M] \Rightarrow \exists$ a number $c \in [a,b]$ such that $f(c) = \mu$.

$$\therefore (1) \equiv \int_a^b f(x)\,dx = (b-a)f(c).$$

→ If $f \in R[a,b]$ and $|f(x)| \leq k$ $\forall x \in [a,b]$ and $k \in R^+$ then

$$\left|\int_a^b f(x)dx\right| \leq k(b-a).$$

Sol:- Since $|f(x)| \leq k$ $\forall x \in [a,b]$

$\Rightarrow -k \leq f(x) \leq k$ $\forall x \in [a,b]$

If m, M are the infimum and supremum of f on $[a,b]$ then we have

$-k \leq m \leq f(x) \leq M \leq k$ $\forall x \in [a,b]$

since $f \in R[a,b]$

$$\therefore m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$\therefore -k(b-a) \leq m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \leq k(b-a)$$

$$\Rightarrow -k(b-a) \leq \int_a^b f(x)dx \leq k(b-a)$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \leq k(b-a)$$

Ex: $f(x) = \frac{\sin x}{1+x^8}$ $\forall x \in [10,19]$

For $x \geq 10 \Rightarrow x^8 \geq 10^8$

$\Rightarrow 1+x^8 \geq 1+10^8$

$\Rightarrow |1+x^8| \geq |1+10^8|$

$\Rightarrow \frac{1}{|1+x^8|} \leq \frac{1}{1+10^8} < \frac{1}{10^8}$

Since $|\sin x| \leq 1$ $\forall x \in \mathbb{R}$

$\Rightarrow \left|\frac{\sin x}{1+x^8}\right| < \frac{1}{10^8} = 10^{-8}$ $\forall x \in [10,19]$

$\therefore$ By theorem,

$$\left|\int_a^b f(x)\,dx\right| \leq k(b-a)$$

where $|f(x)| \leq k$.

we have

$$\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| \leq 10^{-8}(19-10)$$

$$= 9 \times 10^{-8}$$

$$< 10^{-7}$$

*** Functions defined by integrals:**

Let $f \in R[a,b]$. Then for each $t \in [a,b]$, $[a,t] \subset [a,b]$ and hence $f \in R[a,t]$.

Therefore $\int_a^t f(x)\,dx$ is well defined.

The function $\phi(t) = \int_a^t f(x)\,dx$, $t \in [a,b]$

The function ϕ is called an integral function or Indefinite integral of the integrable function f.

Note: The integral function of 'f' may also be defined as

$$\phi(t) = \int_t^b f(x)\,dx, \quad t \in [a,b].$$

Continuity of the integral function:-

If $f \in R[a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$, $t \in [a,b]$ is continuous on $[a,b]$.

(or)

The integral function of an integrable is continuous.

Proof: Since $f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$.

$\Rightarrow \exists\, k \in \mathbb{R}^+$ such that $|f(x)| \leq k$ $\forall x \in [a,b]$

If $x_1, x_2 \in [a,b]$ such that $a \leq x_1 < x_2 \leq b$

then $|\phi(x_2) - \phi(x_1)| = \left|\int_a^{x_2} f(x)\,dx - \int_a^{x_1} f(x)\,dx\right|$

$$= \left|\int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx - \int_a^{x_1} f(x)\,dx\right|$$

$$= \left|\int_{x_1}^{x_2} f(x)\,dx\right|$$

$\leq k(x_2 - x_1)$ ($\because f \in R[a,b]$ and $|f(x)| \leq k$, $k \in$)

Now for each $\epsilon > 0$,

~~we have~~

$|\phi(x_2) - \phi(x_1)| < \epsilon$ whenever $|x_2 - x_1| < \frac{\epsilon}{k}$

choosing $\delta = \frac{\epsilon}{k}$

$\therefore |\phi(x_2) - \phi(x_1)| < \epsilon$ whenever $|x_2 - x_1| < \delta$.

$\therefore \phi(x)$ is uniformly continuous on $[a,b]$.

$\therefore \phi(x)$ is continuous on $[a,b]$

* **Derivability of the integral function:-**

If $f \in R[a,b]$ and f is continuous at $c \in [a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$. $\forall\, t \in [a,b]$

is derivable at 'c' and $\phi'(c) = f(c)$.

Proof: Since f is continuous at $c \in [a,b]$.

$\therefore$ For each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$; $x, c \in [a,b]$. — (1)

Take h, so that $|h| < \delta$.

Now $\phi(c+h) - \phi(c) = \int_a^{c+h} f(x)\,dx - \int_a^{c} f(x)\,dx$

$$= \int_a^c f(x)\,dx + \int_c^{c+h} f(x)\,dx - \int_a^c f(x)\,dx$$

$$= \int_c^{c+h} f(x)\,dx \quad \text{———} \quad (2)$$

Since f is continuous on $[a,b]$.

$\therefore \exists\, c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

$\therefore (2) \equiv \phi(c+h) - \phi(c) = h \cdot f(c)$

$$\therefore \left| \frac{\phi(c+h) - \phi(c)}{h} - f(c) \right| = \left| \frac{1}{h}\int_c^{c+h} f(x)\,dx - \frac{1}{h}\int_c^{c+h} f(c)\,dx \right|$$

$$= \left| \frac{1}{h} \int_c^{c+h} \{f(x) - f(c)\}\,dx \right|$$

$$\leq \frac{1}{h} \int_c^{c+h} |f(x) - f(c)|\,dx$$

$$< \frac{1}{h} \int_c^{c+h} \epsilon\,dx \quad (\text{from } (1))$$

$= \epsilon$ whenever $0 < |x - c| < \delta$.

$\therefore \left| \frac{\phi(c+h) - \phi(c)}{h} - f(c) \right| < \epsilon$ whenever $0 < |x - c| < \delta$ (i.e. $0 < |h| < \delta$)

i.e. $\lim_{h \to 0} \frac{\phi(c+h) - \phi(c)}{h} = f(c)$

i.e. $\phi'(c) = f(c)$

$\therefore \phi$ is derivable at c.

i.e. continuity of f at $c \in [a,b]$ is derivability of ϕ at c.

i.e. continuity of f on $[a,b]$ is derivability of ϕ on $[a,b]$.

Note:

(1) This theorem is some times referred to as the first fundamental theorem of integral calculus.

(2) The integral function ϕ defined by $\phi(t) = \int_t^b f(x)dx$ is continuous and derivable on $[a,b]$ under the conditions of the above two theorems.

(3) Since $\phi(t) = \int_t^b f(x)$ is Continuous on $t \in [a,b]$

$$\phi(0) = \int_0^b f(x)dx = \lim_{t \to 0} \phi(t).$$

(4) If f is continuous on $[a,b]$ then $\phi(t) = \int_a^t f(x)dx \ \forall t \in [a,b]$ is derivable at every $x \in [a,b]$ and $\phi'(x) = f(x)$.

Definition

If $f \in R[a,b]$ and if $\exists \phi : [a,b] \to$ such that $\phi'(x) = f(x) \ \forall x \in [a,b]$ then ϕ is called a primitive or antiderivative of f.

Note: (1) If f is continuous on $[a,b]$ then f possesses a primitive

$$\phi(t) = \int_a^t f(x)dx \ \forall t \in [a,b].$$

(2) Primitive of f is not unique.

If ϕ is a primitive of f then $\phi + c$, $c \in \mathbb{R}$ is also a primitive of f.

Ex: $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = \sin x$.

Since $f(x) = \sin x$ is Continuous on $[a,b]$.

$\therefore$ Primitive of $\sin x$ exists on $[a,b]$

$\therefore$ If $\phi : [a,b] \to \mathbb{R}$ is defined by $\phi(x) = -\cos x$ then we know that

$\phi'(x) = (-\cos x)' = \sin x \ \forall x \in [a,b]$

$\therefore -\cos x$ is the primitive of $\sin x$ on $[a,b]$.

(3) Continuity of a function is not a necessary condition for the existence of a primitive.

Ex: Consider the function ϕ defined on $[0,1]$ as

$$\phi(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then
$$\phi'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

we know that $\phi'(x)$ is not continuous at $x = 0$.

If $f(x) = \phi'(x)$ on $[0,1]$ then $f(x)$ is not continuous on $[0,1]$.

Even though $f(x)$ admits of a primitive $\phi(x)$ in $[0,1]$, it fails to be continuous in $[0,1]$.

* Fundamental theorem of integral Calculus:

If $f \in R[a,b]$ and ϕ is primitive of f on $[a,b]$ (i.e. $\phi'(x) = f(x)$ $\forall x \in [a,b]$)

then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Proof: ϕ is a primitive of f on $[a,b]$.

$\therefore \phi'(x) = f(x)\ \forall x \in [a,b]$ ——— (1)

Since $f \in R[a,b]$,

Consider a partition

$P = \{a = x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots x_n = b\}$ of $[a,b]$

and $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots n$.

let $\xi_r \in I_r$ such that $x_{r-1} \leq \xi_r \leq x_r$, $r = 1, 2, \ldots n$

we have

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r \quad \text{——— (2)}$$

Since ϕ is derivable on $[a,b]$

$\Rightarrow \phi$ is continuous and derivable on $[x_{r-1}, x_r]$; $r = 1, 2, \ldots n$.

$\therefore$ By Lagrange's Mean value theorem,

$$\phi'(\xi_r) = \frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}}, \quad r = 1, 2, \ldots n$$

$\Rightarrow \phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(\xi_r)$, $\xi_r \in I_r$, $r = 1, 2, \ldots n$

$$\Rightarrow \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} \phi'(\xi_r)\delta_r$$

$$= \sum_{r=1}^{n} f(\xi_r)\delta_r \quad \text{(from (1))}$$

$$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\delta_r = [(\phi(x_1) - \phi(x_0)) + (\phi(x_2) - \phi(x_1)) + \cdots + (\phi(x_n) - \phi(x_{n-1}))]$$

$$= \phi(x_n) - \phi(x_0)$$

$$= \phi(b) - \phi(a)$$

$$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\delta_r = \phi(b) - \phi(a)$$

$$\therefore \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{\|P\| \to 0} (\phi(b) - \phi(a))$$

$$\Rightarrow \int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Note: (1) The above theorem is sometimes called the second fundamental theorem of integral calculus.

(2) If ϕ' is continuous on $[a,b]$ then $\int_a^b \phi'(x)\,dx = \phi(b) - \phi(a)$.

(3) $\phi(b) - \phi(a)$ is denoted as $[\phi(x)]_a^b$.

Ex:- show that $\int_0^1 x^4\,dx = 1/5$

Sol'n: $f(x) = x^4$ is continuous on $\mathbb{R}$.

$\therefore$ It is continuous on $[0,1]$.

$\therefore \int_0^1 x^4\,dx$ exists.

Let $\phi(x) = \frac{x^5}{5}$ $\forall x \in [0,1]$

$\Rightarrow \phi'(x) = x^4$ exists on $[0,1]$

$\therefore$ $\phi(x)$ is derivable on $[0,1]$.

$\therefore$ $\phi'(x) = x^4 = f(x)\ \forall x \in [0,1]$.

$\therefore$ $\phi(x)$ is a primitive of 'f' on $[0,1]$.

$\therefore$ By fundamental theorem of Integral Calculus.

$$\int_0^1 x^4\,dx = \phi(1) - \phi(0)$$

$$= \tfrac{1}{5} - 0 = \tfrac{1}{5}$$

Ex 1(2) show that $\int_a^b \cos x\,dx = \sin b - \sin a$.

$f(x) = \cos x$ is continuous on $[a,b]$

$\therefore$ $\int_a^b \cos x$ exists.

Let $\phi(x) = \sin x\ \forall x \in [a,b]$

$\phi'(x) = \cos x\ \forall x \in [a,b]$

$\therefore$ $\phi(x)$ is derivable on $[a,b]$

$\therefore$ $\phi'(x) = \cos x = f(x)\ \forall x \in [a,b]$

$\phi(x)$ is a primitive of 'f' on $[a,b]$.

By fundamental theorem

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$= \sin b - \sin a$$

Show that $\int_a^b e^x\,dx = e^b - e^a$.

* Mean value theorems of Integral Calculus :-

If $f, g \in R[a,b]$ and $g(x) \geq 0$ (or) $g(x) \leq 0\ \forall x \in [a,b]$ then there exists a number μ with $m \leq \mu \leq M$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

Proof:- Let $g(x) \geq 0\ \forall x \in [a,b]$.

Since $f \in R[a,b]$

$\Rightarrow$ f is bounded on $[a,b]$.

$\therefore$ If m, M are the infimum and supremum of 'f' on $[a,b]$.

then $m \leq f(x) \leq M\ \forall x \in [a,b]$.

Since $g(x) \geq 0\ \forall x \in [a,b]$.

$\therefore$ $m\,g(x) \leq f(x)\,g(x) \leq M g(x)\ \forall x \in [a,b]$

$$\Rightarrow \int_a^b m\,g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b M g(x)$$

$$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

$\Rightarrow \exists\ \mu \in [m, M]$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx \quad \text{--- (1)}$$

Now suppose that $g(x) \leq 0\ \forall x \in [a,b]$

$\Rightarrow -g(x) \geq 0\ \forall x \in [a,b]$

$\Rightarrow \exists\ \mu \in [m,M]$, we have

$$\int_a^b f(x)(-g(x))\,dx = \mu \int_a^b (-g(x))\,dx$$

(from (1))

$$\Rightarrow \int_a^b f(x).g(x)\,dx = \mu \int_a^b g(x)\,dx$$

Note: If $g(x) = 1\ \forall x \in [a,b]$ then $g \in R[a,b]$ and $g(x) > 0\ \forall x \in [a,b]$.

$\therefore$ By Mean value Theorem, we have

$$\int_a^b f(x)\cdot(1)\,dx = \mu\int_a^b 1\,dx$$

$$= \mu(b-a) \text{ where } \mu\in[m,M]$$

$$\therefore \int_a^b f(x)\,dx = \mu(b-a)$$

Corollary: If f is continuous on $[a,b]$, $g\in R[a,b]$ and $g(x)\geq 0$ (or) $g(x)\leq 0$ $\forall x\in[a,b]$ then $\exists\, c\in[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Proof: Since f is continuous on $[a,b]$

$\Rightarrow f\in R[a,b]$

$\therefore \exists\, \mu\in[m,M]$ such that

$$\int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx$$

Since f is continuous on $[a,b]$, it attains every value between its bounds m, M.

$\therefore \mu\in[m,M] \Rightarrow \exists\, c\in[a,b]$ such that $f(c)=\mu$.

$$\therefore \int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Problems :-

→ show that $\exists\, \xi\in[0,\pi/2]$ such that $\int_0^{\pi/2} x\sin x\,dx = \xi$

sol: Let $f(x)=x$, $g(x)=\sin x$

then $f(x)$ is continuous and $g(x)$ is integrable on $[0,\pi/2]$

and $g(x)=\sin x\geq 0$ $\forall x\in[0,\pi/2]$

Applying first mean value theorem, we get,

$$\int_0^{\pi/2} x\sin x\,dx = \xi\int_0^{\pi/2}\sin x\,dx$$

$$= \xi\,(1)$$

$$= \xi$$

→ Prove that $\frac{1}{\pi}\leq\int_0^1 \frac{\sin\pi x}{1+x^2}\,dx\leq\frac{2}{\pi}$

Solⁿ: Let $f(x)=\frac{1}{1+x^2}$, $g(x)=\sin\pi x$ $\forall x\in[0,1]$

then f, g are continuous on $[0,1]$ and hence integrable on $[0,1]$.

and $g(x)\geq 0$ $\forall x\in[0,1]$.

Since f is decreasing on $[0,1]$.

$\therefore$ infimum $f = f(1) = \frac{1}{2}$

Supremum $f = f(0) = 1$.

$\therefore$ By first Mean value theorem.

$\exists\ \mu \in [1/2, 1]$

Such that $\int_0^1 f(x)g(x) = \mu \int_0^1 g(x)\,dx$

$$\Rightarrow \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \int_0^1 \sin \pi x\,dx$$

$$= \mu\left[\frac{-\cos \pi x}{\pi}\right]_0^1$$

$$= \mu\left[\frac{2}{\pi}\right] \quad \text{—— (1)}$$

Since f is continuous on $[0,1]$.

it attains every value between its bounds $1/2$ and 1.

$\therefore\ \mu \in [1/2, 1]$

$\Rightarrow$ a number $c \in [0,1]$ such that

$f(c) = \mu$

from (1)

$$f(c) = \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \quad \text{—— (2)}$$

But $0 \le c \le 1$ and f is decreasing on $[0,1]$

$\Rightarrow f(0) \ge f(c) \ge f(1)$

$\Rightarrow 1 \ge f(c) \ge 1/2$

$$\Rightarrow \frac{1}{2} \le \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le 1$$

$$\Rightarrow \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le \frac{2}{\pi}$$

H.W $\rightarrow$ Prove that $\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$

Sol'n: Let $f(x) = \frac{1}{\sin x}$, $g(x) = x$

H.W. $\rightarrow$ Prove that $\frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,dx \le \frac{1}{3}$.

Sol'n: Let $f(x) = \frac{1}{\sqrt{1+x^2}}$, $g(x)$:

H.W. $\rightarrow$ Prove that $\frac{\pi^3}{24} \le \int_0^{\pi} \frac{x^2}{5+3\cos x}\,dx$

Let $f(x) = \frac{1}{5+3\cos x}$; $g(x) = x^2$

H.W. $\rightarrow$ Prove that $\frac{\pi}{4} \le \int_0^{\pi/4} \sec x\,dx \le \frac{\pi}{2\sqrt{3}}$

Let $f(x) = \sec x$ $\quad g(x) = 1$.

$\rightarrow$ By applying Mean value theorem of integral calculus, show that $e^{1/4} < 4/3 < e^{1/3}$ by considering $\int_3^4 \frac{1}{x}\,dx$.

Sol'n: Let $f(x) = 1/x$, $g(x) = 1\ \forall x \in [3,4]$

then f, g are continuous on $[3,4]$

and hence integrable on $[3,4]$

and $g(x) = 1 > 0\ \forall x \in [3,4]$

Since $f(x)$ is decreasing function on $[3,4]$.

$\therefore$ infimum $f = f(4) = 1/4$

Supremum $f = f(3) = 1/3$

$\therefore$ By first Mean value theorem,

$$\int_3^4 f(x)\cdot g(x)\,dx = f(c)\int_3^4 g(x)\,dx$$

$$\Rightarrow \int_3^4 f(x)\,dx = f(c)\int_3^4 (1)\,dx$$

$$= f(c)(4-3)$$

$$= f(c) \text{ where } c \in [3,4].$$

Now $3 \le c \le 4$

$\Rightarrow f(3) \geq f(c) \geq f(4)$ ($\because f$ is decreasing)

$\Rightarrow \frac{1}{3} \geq f(c) \geq \frac{1}{4}$

$\Rightarrow \frac{1}{4} \le \int_3^4 f(x)\,dx \le \frac{1}{3}$

$\Rightarrow \frac{1}{4} \le [\log x]_3^4 \le \frac{1}{3}$

$\Rightarrow \frac{1}{4} \le \log\left(\frac{4}{3}\right) \le \frac{1}{3}$

$\Rightarrow \frac{1}{4} \le \log(4/3) \le \frac{1}{3}$

$\Rightarrow e^{1/4} \le 4/3 \le e^{1/3}$.

Bonnet's Mean Value Theorem:-

Let $g \in R[a,b]$ and let f be monotonically decreasing and non-negative on $[a,b]$. Then for some $\xi \in [a,b]$ such that $\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx$

(or)

Let $g \in R[a,b]$, and let f be monotonically increasing and non-negative on $[a,b]$. Then for some $\eta \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(b)\int_{\eta}^{b} g(x)\,dx$$

Problems :-

→ If $0<a<b$, show that $\left|\int_a^b \frac{\sin x}{x}\,dx\right| < \frac{2}{a}$.

Sol^n: Let $f(x) = \frac{1}{x}$; $g(x) = \sin x$ $\forall x \in [a,b], a>0$

Since $f(x) = \frac{1}{x}$ $\forall x \in [a,b], a>0$.

$\Rightarrow f(x)$ is monotonically decreasing on $[a,b]$.

and $f(x) > 0$ $\forall x \in [a,b], a>0$.

and $g(x) = \sin x$ $\forall x \in [a,b]$

$\Rightarrow g(x)$ is continuous on $[a,b]$

$\Rightarrow g \in R[a,b]$.

$\therefore$ The conditions of Bonnet's Mean Value theorem are satisfied.

$\therefore \exists\, \xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx$$

$$\Rightarrow \int_a^b \frac{\sin x}{x}\,dx = \frac{1}{a}\int_a^{\xi} \sin x\,dx$$

$$= \frac{1}{a}\left[-\cos x\right]_a^{\xi}$$

$$= \frac{1}{a}\left[-\cos\xi + \cos a\right]$$

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| = \frac{1}{a}\left|\cos a - \cos\xi\right|$$

$$\le \frac{1}{a}\left[|\cos a| + |\cos\xi|\right]$$

$$\le \frac{1}{a}(1+1) \quad (\because |\cos\theta| \le 1)$$

$$= \frac{2}{a}$$

$$\therefore \left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{2}{a}$$

(i) If $f(x) = x$, $g(x) = e^x$ verify the first mean value theorem in $[-1,1]$

(ii) verify Bonnet's mean value theorem in $[-1,1]$ for the functions $f(x) = e^x$ and $g(x) = x$.

Sol'n: (i) $f(x) = x$, $g(x) = e^x$ $\forall x \in [-1,1]$

$\Rightarrow f, g$ are continuous on $[-1,1]$

$\Rightarrow f, g \in R[-1,1]$

and $g(x) = e^x > 0$ $\forall x \in [-1,1]$.

$\therefore$ The conditions of first mean value theorem are satisfied.

Now $\int_{-1}^{1} f(x) g(x)\, dx = \int_{-1}^{1} x e^x dx$

$= [x e^x - e^x]_{-1}^{1}$

$= (e^1 - e^1) - (-e^{-1} - e^{-1})$

$= 0 + 2e^{-1}$

$= \frac{2}{e}$

Now $\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} e^x dx = [e^x]_{-1}^{1}$

$= e^1 - e^{-1}$

$= \frac{e^2 - 1}{e}$

Since f is continuous on $[-1,1]$; it attains every value between $f(-1) = -1$ and $f(1) = 1$.

If $\mu = \frac{2}{e^2 - 1}$ then $0 < \mu < 1$ as $e > 2$.

$\Rightarrow e^2 > 4$

$\Rightarrow e^2 - 1 > 3$.

Now $\exists\, c \in [-1,1]$ such that

$f(c) = \frac{2}{e^2 - 1}$

$\therefore$ we have $f(c) \int_{-1}^{1} g(x)\, dx = \frac{2}{e^2-1} \cdot \frac{e^2 - 1}{e}$ — (2)

$\therefore$ from (1) & (2) we have

$$\int_{-1}^{1} f(x) g(x)\, dx = f(c) \int_{-1}^{1} g(x)\, dx$$

$\therefore$ The first mean value theorem is verified.

(ii) Since $g(x) = x$ $\forall x \in [-1,1]$

$\Rightarrow$ It is continuous on $[-1,1]$

$\Rightarrow g \in R[-1,1]$

and $f(x) = e^x$ is monotonically increasing and +ve on $[-1,1]$.

$\therefore$ The conditions of Bonnet's mean value theorem are satisfied.

Now $\int_{-1}^{1} f(x) g(x)\, dx = \int_{-1}^{1} x e^x dx$

$= 2/e$

and $\int_{\eta}^{1} g(x)\, dx = \int_{\eta}^{1} x\, dx = \frac{1}{2}(1 - \eta^2)$ for $\eta \in [-1,1]$

$\therefore f(1) \int_{\eta}^{1} g(x)\, dx = e \cdot \frac{1}{2}(1 - \eta^2)$

Let us choose η such that

$\frac{2}{e} = \frac{e}{2}(1 - \eta^2)$

i.e. $\eta^2 = \frac{e^2 - 4}{e^2}$

$\Rightarrow \eta = \frac{\sqrt{e^2 - 4}}{e}$; $0 < \eta < 1$.

For this value of η, we have

$$\int_{-1}^{1} f(x) g(x)\, dx = f(1) \int_{\eta}^{1} g(x)\, dx.$$

$\therefore$ Bonnet's mean value theorem is verified.

→ Show that the Bonnet's Mean Value theorem doesnot hold on $[-1,1]$ for $f(x)=g(x)=x^2$.

Sol^n :- $f(x)=x^2$ is not monotonic on $[-1,1]$. Because for the interval $[-1,0]$, it is decreasing and for the interval $[0,1]$ it is increasing.

∴ The conditions of Bonnet's mean value theorem are not satisfied.

∴ Bonnet's mean value theorem doesnot hold on $[-1,1]$.

→ **Second Mean Value Theorem**:-

Let $g \in R[a,b]$ and let f be bounded and monotonic on $[a,b]$ then

$$\int_a^b fg = f(a)\int_a^\xi g + f(b)\int_\xi^b g.$$

Proof:— Let f be monotonically decreasing on $[a,b]$ then $f(x)-f(b)$ is monotonically decreasing and non-negative on $[a,b]$.

∴ By Bonnet's theorem $\exists\ \xi \in [a,b]$ such that

$$\int_a^b (f(x)-f(b))\,g(x)dx = (f(a)-f(b))\int_a^\xi g(x)dx$$

$$\Rightarrow \int_a^b f(x)\,g(x)\,dx - f(b)\int_a^b g(x)dx = f(a)\int_a^\xi g(x)dx - f(b)\int_a^\xi g(x)dx.$$

$$\therefore \int_a^b f(x)\,g(x)\,dx = f(a)\int_a^\xi g(x)dx + f(b)\left[\int_a^b g(x)dx - \int_a^\xi g(x)dx\right].$$

$$= f(a)\int_a^\xi g(x)dx + f(b)\left[\int_a^b g(x)dx + \int_\xi^a g(x)dx\right]$$

$$= f(a)\int_a^\xi g(x)\,dx + f(b)\int_\xi^b g(x)\,dx.$$

Now if f is monotonically increasing on $[a,b]$ then $-f$ is monotonically decreasing on $[a,b]$.

$$\therefore \int_a^b [-f(x)]\,g(x)dx = [-f(a)]\int_a^\xi g(x)dx + [-f(b)]\int_\xi^b g(x)dx.$$

$$\Rightarrow \int_a^b f(x)\,g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx$$

→ * **Integral as the limit of a Sum**:

(Formula given in Pg. No. 45)

→ Evaluate $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{n^2}{(n^2+r^2)^{3/2}}$

Sol^n: The given limit

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]^{3/2}} = \int_0^1 \frac{dx}{(1+x^2)^{3/2}}$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta$$

[Putting $x=\tan\theta$ $\Rightarrow dx=\sec^2\theta\,d\theta$; limits for θ are 0 to $\pi/4$]

$$= \int_0^{\pi/4} \cos\theta\, d\theta = [\sin\theta]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}} \quad \text{ans}$$

→ Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots \frac{1}{8n}\right]$$

Sol^n: The given limit

$$\lim_{n\to\infty}\left[\frac{n^2}{n^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{n^2}{8n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{n^2}{(n+n)^3}\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{n^3\left(1+\left(\frac{r}{n}\right)\right)^3} = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n}\cdot\frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} = -\frac{1}{2}\left[\frac{1}{(1+x)^2}\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{(1+1)^2} - \frac{1}{(1+0)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4} - 1\right]$$

$$= -\frac{1}{2}\left[\frac{1-4}{4}\right] = -\frac{1}{2}\cdot\frac{-3}{4} = \frac{3}{8}$$

→ Find the limit $\left\{\frac{n!}{n^n}\right\}^{1/n}$ when $n\to\infty$

Sol^n: Let $A = \lim_{n\to\infty}\left\{\frac{n!}{n^n}\right\}^{1/n}$

$$= \lim_{n\to\infty}\left[\frac{1.2.3\cdots n}{n.n.n.\cdots n}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right]^{1/n}$$

$$\log A = \lim_{n\to\infty}\frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log x\, dx$$

$$= \int_0^1 (\log x)\cdot 1\,dx$$

$$= [(\log x)x]_0^1 - \int_0^1 \frac{1}{x}x\,dx.$$

$$= 0 - [x]_0^1 = -1$$

$$A = e^{-1} = \frac{1}{e}$$

→ Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\cdots\right.\right.$ $\left.\cdots\left(1+\frac{n^4}{n^4}\right)^{1/n}\right]$

Sol^n:- Let $A = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\cdots\right.\right.$ $\left.\cdots + \left(1+\frac{n^4}{n^4}\right)^{1/n}\right]$

$$\log A = \lim_{n\to\infty}\log\left(1+\frac{1}{n^4}\right) + \frac{1}{2}\log\left(1+\frac{2^4}{n^4}\right) + \frac{1}{3}\log\left(1+\frac{3^4}{n^4}\right) + \cdots + \frac{1}{n}\log\left(1+\cdots\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}\log\left(1+\frac{r^4}{n^4}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{n}{r}\log\left(1+\left(\frac{r}{n}\right)^4\right)$$

$$= \int_0^1 \frac{1}{x}\cdot\log(1+x^4)\,dx.$$

$$= \int_0^1 \frac{1}{x}\left[x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \cdots\right]dx$$

$$= \int_0^1 \left[x^3 - \frac{x^7}{2} + \frac{x^{11}}{3} - \cdots\right]dx$$

$$= \left[\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \cdots\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{16} + \frac{1}{36} + \cdots$$

$$= \frac{1}{4}\left[1 - \frac{1}{4} + \frac{1}{9} + \cdots\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{1}{4}\cdot\frac{\pi^2}{12}$$

$$\Rightarrow \log A = \frac{\pi^2}{48}$$

$$\Rightarrow \boxed{A = e^{\pi^2/48}}$$

→ show that

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{3/n} + e^{6/n} + e^{9/n} + \cdots + e^{3n/n}\right) = \frac{e^3-1}{3}$$

Sol^n: $\lim_{n\to\infty}\frac{1}{n}\left(e^{3/n} + e^{6/n} + \cdots + e^{3n/n}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}e^{3r/n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}e^{3\left(\frac{r}{n}\right)}$$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{3(r/n)}$

$= \int_0^1 e^{3x}dx = \left[\frac{e^{3x}}{3}\right]_0^1 = \frac{e^3}{3} - \frac{e^0}{3}$

$= \frac{e^3}{3} - \frac{1}{3}$

→ show that the function f defined by $f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1 & \text{otherwise} \end{cases}$ is integrable on $[0, m]$, m being a +ve integer.

Sol'n: $f(x) = \begin{cases} 0 & \text{if } x = 0,1,2 \ldots m \\ 1 & \text{if } r-1 < x < r \\ & r = 1,2 \ldots m \end{cases}$

⇒ f is bounded and has only m+1 points of finite discontinuity at $0, 1, 2, \ldots m-1, m$.

Since the points of discontinuity of f in $[0,m]$ are finite in number.

∴ f is integrable on $[0, m]$.

Now $\int_0^m f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \ldots + \int_{m-1}^{m} f(x)dx$

$= \int_0^1 1dx + \int_1^2 1dx + \ldots + \int_{m-1}^{m} 1dx$

$= [1-0] + [2-1] + \ldots + [m-(m-1)]$

$= [1+1+1+ \ldots + 1]$ (m times)

$= m$.

→ Let f be a function on $[0,1]$ defined by $f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}$ show that $f \in R[0,1]$ and find $\int_0^1 f$.

Sol'n: clearly $0 \le f(x) \le 1 \; \forall x \in [0,1]$

⇒ f(x) is bounded on $[0,1]$ and the function has only point of discontinuity $\frac{1}{2}$

∴ $f \in R[0,1]$

Now $\int_0^1 f(x)dx = \int_0^{1/2} f(x)dx + \int_{1/2}^1 f(x)dx$

$= \int_0^{1/2} 1dx + \int_{1/2}^1 1dx = \frac{1}{2} + 1 - \frac{1}{2} = 1$

H.W.
→ A function f is defined on $[-1,1]$ as follows $f(x) = \begin{cases} k \text{ (constant) when } x \neq 0 \\ 0 \text{ when } x = 0 \end{cases}$

show f is integrable on $[-1,1]$ and $\int_{-1}^{1} f(x)dx = 2k$.

→ If f is continuous on $[a,b]$, $f(x) \ge 0 \; \forall x \in [a,b]$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0 \; \forall x \in [a,b]$.

sol'n: If possible suppose that $f(x) \neq 0 \; \forall x \in [a,b]$.

Then $\exists \, c \in [a,b]$ such that $f(c) \neq 0 \Rightarrow f(c) > 0$

Let $\epsilon = \frac{1}{2} f(c)$.

Suppose $a < c < b$

Since f is continuous at c $\exists$ a $\delta > 0$ such that

$|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$

$\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$

$\Rightarrow f(x) > f(c) - \frac{1}{2}f(c) = \frac{1}{2}f(c)$

$\therefore f(x) > \frac{1}{2} f(c)$

Now choosing δ such that

$a < c - \delta < c < c + \delta < b$

$\int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx$

$\ge \int_{c-\delta}^{c+\delta} f(x)dx \quad \because f(x) \ge 0 \; \forall x \in [a,b]$

$\ge \int_{c-\delta}^{c+\delta} \frac{1}{2} f(c)\, dx = \frac{1}{2} f(c)(2\delta)$

$= \delta f(c) > 0$

$\therefore \int_a^b f(x)dx > 0$

which is contradiction to $\int_a^b f(x)dx = 0$

If $c = a$ or $c = b$ then also we get a contradiction.

$\therefore f(x) \neq 0 \; \forall x \in [a,b]$ is wrong.

$\therefore f(x) = 0 \; \forall x \in [a,b]$.

✻ Uniform Convergence ✻

[Sequences and Series of functions]

✻ Just as sequence and series of real numbers play a fundamental role in analysis.

Sequence and series of functions are also important elements of modern analysis.

In many situations, we come across these elements, particularly in connection with convergence.

→ **Sequences of real-valued functions:-**

Let f_n be a real valued function defined on an interval I (or on a subset D of $\mathbb{R}$) and for each $n \in N$.

Then $\{f_1, f_2, f_3, \cdots f_n, \cdots\}$ is called a sequence of real-valued functions on I. It is denoted by $\{f_n : I \to \mathbb{R}, n \in N\}$ (or) by $\{f_n\}$ (or) $\langle f_n \rangle$ (or) (f_n).

For example:

(i) If f_n is a real valued function defined by $f_n(x) = x^n$, $0 \le x \le 1$, then $\{f_1(x), f_2(x), f_3(x), \cdots\}$

$= \{x^1, x^2, x^3, \cdots\}$

is a sequence of a real valued functions on $[0,1]$.

Example (2):-

If f_n is a real-valued function defined by

$$f_n(x) = \frac{\sin nx}{n}, \quad 0 \le x \le 1$$

then $\{f_1(x), f_2(x), f_3(x), \cdots\}$

$= \left\{\sin x, \frac{\sin 2x}{2}, \frac{\sin 3x}{3}, \cdots\right\}$ is a [sequence] of real valued functions on $[0,1]$

→ If $\{f_n\}$ is a sequence of functions defined on I, then for $c \in I$,

$\{f_n(c)\} = \{f_1(c), f_2(c), \cdots f_n(c), \cdots\}$ is a sequence of real numbers.

For example:

If $\{f_n\}$ is a sequence of functions defined by $f_n(x) = x^n$, $0 \le x \le 1$, then

$\{f_n(\frac{1}{2})\} = \{f_1(\frac{1}{2}), f_2(\frac{1}{2}), f_3(\frac{1}{2}), \cdots f_n(\frac{1}{2}), \cdots\}$

$= \left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots \frac{1}{2^n}, \cdots\right\}$

is a sequence of real numbers corresponding to $\frac{1}{2} \in [0,1]$.

∴ to each $x \in I$, we have a sequence of real numbers.

→ **Pointwise convergence of a sequence of functions:-**

Let $\{f_n\}$ be a sequence of functions on I and $c \in I$, then the sequence of real numbers $\{f_n(c)\}$ may be convergent.

In fact for each $c \in I$, the corresponding sequence of real numbers may be convergent.

If $\{f_n\}$ is a sequence of real-valued functions on I and for each $x \in I$, the corresponding sequence of real numbers is convergent, then we say the sequence $\{f_n\}$ converges pointwise. The limiting values of the sequences of real numbers corresponding to $x \in I$ define a function 'f' called the limit function (or) simply the limit of the sequence $\{f_n\}$ of functions on I.

*** Definition:**

Let $\{f_n\}$ be a sequence of functions on I. If to each $x \in I$ and to each $\epsilon > 0$, there corresponds to positive integer 'm' such that

$|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$ then we say that $\{f_n\}$ converges pointwise to the function f on I.

Note(1): $\{f_n\}$ converges pointwise to the function f on $I \Rightarrow \lim_{n\to\infty} f_n(x) = f(x) \ \forall x$

$f(x)$ is called the limit function (or) simply the limit (or) the pointwise limit of $\{f_n(x)\}$ on I

Note(2): The positive integer 'm' depends on $x \in I$ and given $\epsilon > 0$ i.e. $m = m(\epsilon, x)$

Ex:- ① Let $f_n(x) = x^n, x \in [0,1]$.

Since $\lim_{n\to\infty} x^n = 0$ for $0 \leq x < 1$.

$$\boxed{\therefore \lim_{n\to\infty} f_n(x) = 0 \text{ for } 0 \leq x < 1}$$

When $x = 1$, the corresponding sequence $\{f_n(x)\} = \{1, 1, 1, 1, \ldots\}$ is a constant sequence converging to 1.

$$\therefore \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$

Hence $\{f_n\}$ converges pointwise on $[0,1]$.

$f(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ 1 & \text{when } x = 1 \end{cases}$ is the limit function of $\{f_n(x)\}$ on $[0,1]$.

→ Let $\epsilon = \frac{1}{2}$ be given

Then for each $x \in [0,1]$, there exists a +ve integer 'm' such that

$|f_n(x) - f(x)| < \frac{1}{2} \ \forall n \geq m$ —①

If $x = 0$, $f(x) = 0$ and $f_n(x) = 0 \ \forall n$

$\therefore |f_n(x) - f(x)| = |0 - 0|$

$= 0 < \frac{1}{2} \;\forall\, n \geq 1$

$\therefore$ ① is true for $m=1$ when $x=0$.

Similarly ① is true for $m=1$ when $x=1$.

If $x = 3/4$, $f(x) = 0$ and $f_n(x) = (3/4)^n$

$\therefore |f_n(x) - f(x)| = |(3/4)^n - 0| = (3/4)^n < \frac{1}{2} \;\forall\, n \geq 3$.

$\therefore$ ① is true for $m=3$ when $x = 3/4$.

Similarly ① is true for $m=7$ when $x = 9/10$

$\therefore$ There is no single value of m for which ① holds $\forall\, x \in [0,1]$.

i.e. m is depending on x & ϵ.

Example ②: Let $f_n(x) = \frac{x}{1+nx}$, $x \geq 0$.

Then for $x > 0$, $\lim_{n\to\infty} f_n(x) = 0$

Also $f_n(0) = 0 \;\forall\, n \in N$ so that $\{f_n(0)\}$ converges to 0

$\therefore \lim_{n\to\infty} f_n(x) = 0 \;\forall\, x \geq 0$.

Hence $\{f_n(x)\}$ converges to zero pointwise on $[0,\infty)$ and $f(x) = 0$ is the limit function of $\{f_n(x)\}$ on $[0,\infty)$.

Example(3): Let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

For $x \neq 0$, $f_n(x) = \frac{1/nx}{\frac{1}{n^2x^2} + 1} \to 0$ as $n \to \infty$

Also $f_n(0) = 0 \;\forall\, n \in N$

$\therefore \lim_{n\to\infty} f_n(x) = 0 \;\forall\, x \in \mathbb{R}$.

Hence $\{f_n(x)\}$ converges to zero pointwise on $\mathbb{R}$ and $f(x) = 0$ is the function of $\{f_n(x)\}$ on $\mathbb{R}$.

Note (3): For a sequence $\{f_n\}$ of functions, an important question

If each function of a sequence has a certain property such as continuity, differentiability (or) integ... then to what extent is this pr... transferred to the limit functi...

In fact pointwise convergence is strong enough to transfer any the properties mentioned above the terms f_n of $\{f_n\}$ to the the limit function.

Let us consider a few examples:-

(i) A sequence of continuous fun... with a discontinuous limit function.

Consider the sequence $\{f_n\}$ where

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \; x \in \mathbb{R}$$

$$\text{then } f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x|<1 \\ \frac{1}{2} & \text{if } |x|=1 \\ 1 & \text{if } |x|>1 \end{cases}$$

Here each f_n is continuous on $\mathbb{R}$ but f is discontinuous at $x = \pm 1$.

(ii) A sequence of differentiable f... in which the limit of the derivat... is not equal to the derivative ...

the limit function.

Consider the sequence $\{f_n\}$ where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}},\ x \in \mathbb{R}$$

Then $f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0 \ \forall x \in \mathbb{R}$

i.e. $f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in \mathbb{R}$.

Now $f'(x) = 0 \ \forall x \in \mathbb{R} \Rightarrow f'(0) = 0$.

and $f_n'(x) = \sqrt{n}\cos nx$.

$\Rightarrow f_n'(0) = \infty$ as $n \to \infty$ $\left(\because \cos 0 = 1 \text{ and } \sqrt{n} \to \infty\right)$

$\therefore$ At $x = 0$ $\lim_{n\to\infty} f_n'(x) \neq f'(0)$.

(iii) A sequence of functions in which the limit of integrals is not equal to the integral of the limit function.

Consider the sequence $\{f_n\}$ where

$f_n(x) = nx(1-x^2)^n;\ x \in [0,1]$

then $f_n(x) = 0$ when $x = 0$ or $x = 1$

Also if $0 < x < 1$ then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n \quad \left|\text{form } \infty \times 0\right.$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \quad \left|\text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x^2)^{-n}\log(1-x^2)} = 0$$

$\left[\frac{d}{dx} a^x = a^x \log a,\ \frac{d}{dx} a^{-x} = -a^{-x}\log a\right]$

$\therefore f(x) = 0 \ \forall x \in [0,1]$

i.e. $f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in [0,1]$.

Now $\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n dx$

$$= -\frac{n}{2}\int_0^1 (1-x^2)^n(-2x)\,dx$$

$$= \frac{-n}{2}\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1$$

$$= \frac{-n}{2}\left[0 - \frac{1}{n+1}\right]$$

$$= \frac{n}{2(n+1)}$$

$\Rightarrow \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$

Also $\int_0^1 f(x)\,dx = \int_0^b 0\,dx = 0$

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$.

The above few examples show that we need to investigate under what supplementary conditions these (or other) properties of the terms f_n of $\{f_n\}$ are transformed to the limit function f. A concept of great importance in this respect, is that known as uniform convergence.

* Uniform Convergence of Sequence of functions:-

Let $\{f_n\}$ be a sequence of functions on I then $\{f_n\}$ is said to be uniformly convergent to a function f on I, if to each $\epsilon > 0$, there exists a positive integer 'm' (depending on ϵ only) such that $|f_n(x) - f(x)| < \epsilon \ \forall \ n \geq m$ and $\forall x \in I$.

The function 'f' is called uniform limit of the sequence $\{f_n\}$ on I.

Ex:- Now consider the sequence $\{f_n\}$ defined by $f_n(x) = \frac{x}{1+nx}$, $x \geq 0$.

It converges pointwise to zero i.e. $f(x) = 0 \ \forall x \geq 0$.

Now $\because 0 \leq f_n(x) = \frac{x}{1+nx} \leq \frac{x}{nx} = \frac{1}{n}$

$\therefore$ for any $\epsilon > 0$,

$|f_n(x) - f(x)| = |f_n(x)| \leq \frac{1}{n} < \epsilon$

whenever $n > \frac{1}{\epsilon}$ $\forall x \in [0, \infty)$.

If m is a +ve integer $> \frac{1}{\epsilon}$, then

$|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$ and $x \in [0, \infty)$

Thus, in ~~this example~~, we can find an 'm' which depends only on ϵ and not on $x \in [0, \infty)$.

We say that the sequence $\{f_n\}$ is uniformly convergent to f on $[0, \infty)$.

Note (1) :- If a uniform 'm' is found for all a values of x, the sequence $\{f_n\}$ is uniformly converges to f on I.

Note (2) :- A uniformly convergent sequence is a pointwise convergent sequence i.e.

uniform convergence $\Rightarrow$ pointwise c[onvergence]

However, the converse is not t[rue]

For example: If $f_n(x) = x^n \ \forall x \in$ [...] then the sequence $\{f_n\}$ con[verges] pointwise to the function f on [...] where $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

but $\{f_n\}$ does not converge uni[formly] on $[0, 1]$.

A sequence $\{f_n\}$ of func[tions] defined on I does not converge uni[formly] to f on I iff there exists some [ϵ] such that there is no positive in[teger] 'm' for which the statement $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$ and $\forall x$ [...] holds.

Definition:

* Uniformly Bounded Sequence functions:—

A sequence $\{f_n\}$ of funct[ions] defined on I is said to be uniformly bounded on I if $\exists$ real number K such that

$|f_n(x)| < K \ \forall n \in N$ and $\forall x \in I$.

The number K is called a uniform bound for $\{f_n\}$ on I.

Ex:- If $f_n(x) = \sin nx$, $\forall x \in \mathbb{R}$ then

$|f_n(x)| = |\sin nx| \leq 1 \; \forall n \in N$ and $x \in \mathbb{R}$

$\therefore$ The sequence $\{f_n\}$ is uniformly bounded on $\mathbb{R}$.

Theorem (Cauchy's Criterion for Uniform Convergence) :—

A sequence $\{f_n\}$ of functions defined on I is uniformly convergent on I iff for each $\epsilon > 0$ and for all $x \in I$, $\exists$ a +ve integer m such that for any integer $p \geq 1$, $|f_{n+p}(x) - f_n(x)| < \epsilon$ $\forall n \geq m$.

Problems:

→ Show that the sequence $\{f_n\}$ where $f_n(x) = x^n$ is uniformly convergent on $[0,k]$, $k<1$ but only pointwise convergent on $[0,1]$.

Sol'n: Here $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x = 1 \end{cases}$

The sequence $\{f_n\}$ converges pointwise to f on $[0,1]$.

To see whether the sequence $\{f_n\}$ is uniformly convergent.

Let $\epsilon > 0$ be given.

For $0 < x < 1$, $|f_n(x) - f(x)| = |x^n - 0|$

$= x^n < \epsilon$

whenever $\frac{1}{x^n} > \frac{1}{\epsilon}$

i.e whenever $n \log \frac{1}{x} > \log \frac{1}{\epsilon}$

i.e whenever $n > \dfrac{\log 1/\epsilon}{\log 1/x}$ (Note $0<x<1 \Rightarrow \frac{1}{x} > 1 \Rightarrow \log \frac{1}{x} > 0$)

The numbers $\dfrac{\log 1/\epsilon}{\log 1/x}$ increases with x having maximum value $\dfrac{\log 1/\epsilon}{\log 1/k}$ on $[0,k]$, $k<1$.

Choose a positive integer $m > \dfrac{\log 1/\epsilon}{\log 1/k}$

then $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $0 < x < 1$

At $x=0$, $|f_n(x) - f(x)| = |0-0| = 0$ $\forall n \geq 1$.

$\therefore \exists$ a +ve integer m such that

$|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in [0,k]$, $k<1$.

$\Rightarrow \{f_n\}$ is uniformly convergent on $[0,k]$, $k<1$, when $x \to 1$, the numbers $\dfrac{\log 1/\epsilon}{\log 1/x} \to \infty$.

Thus it is not possible to find a positive integer m such that

$|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in [0,1]$. Hence the sequence $\{f_n\}$ is not uniformly convergent on any interval containing 1 and in particular on $[0,1]$.

→ **A Test For Uniform Convergence of Sequences of Functions:—**

To determine whether a given sequence $\{f_n\}$ is uniformly convergent (or) not in a given interval, we have been using the definition of uniform convergence. Thus, we find a +ve integer 'm', independent of 'x' which is not easy in most of the cases. The following test is more convenient in practice and does not involve the computation of 'm'.

*** Theorem (M_n Test):—**

Let $\{f_n\}$ be a sequence of functions on I such that $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in I$

and let $M_n = \sup\{|f_n(x) - f(x)| / x \in I\}$

Then $\{f_n\}$ converges uniformly on I iff $\lim_{n\to\infty} M_n = 0$.

Note (1): M_n = the maximum value of $|f_n(x) - f(x)|$ for fixed 'n' and $x \in I$.

Note (2): If M_n does not tend to '0', then the sequence $\{f_n\}$ is not uniformly convergent.

Note (3): $F(x)$ is maximum at $x = c \in I$ if (i) $F'(c) = 0$ and (ii) $F''(c) < 0$

Problems

→ Show that the sequence of func where $f_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$ conv uniformly on any closed inter

Sol'n:- Here $f(x) = \lim_{n\to\infty} f_n(x)$.

$= \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$

Now $|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2} - 0\right|$

Let $y = \frac{x}{1+nx^2}$ then, → ②

$$\frac{dy}{dx} = \frac{(1+nx^2)\cdot 1 - x(2nx)}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{(1+nx^2)^2}$$

For max or min. $\frac{dy}{dx} = 0$

$\Rightarrow 1 - nx^2 = 0$

$\Rightarrow x = \frac{1}{\sqrt{n}}$

Also $\frac{d^2y}{dx^2} = \frac{\cancel{(1+nx^2)^2(-2nx) - (1-nx^2)2(1+nx^2)2nx}}{\cancel{(1+nx^2)^4}}$

$$= \frac{-2nx(1+nx^2) - 4nx(1-nx^2)}{(1+nx^2)^3}$$

Now $\left[\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}\,(1+1)}{(1+1)^3}$

$$= \frac{-\sqrt{n}}{2} < 0$$

$\Rightarrow y$ is maximum when $x = \frac{1}{\sqrt{n}}$ and the maximum value of y from ②

$$y = \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}}$$

$\therefore M_n = \underset{x\in[a,b]}{Max} |f_n(x) - f(x)|$

$= \underset{x\in[a,b]}{Max} \left|\frac{x}{1+nx^2}\right|$ (from (i))

$= \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$

$\{f_n\}$ converges uniformly to f on $[a,b]$

$\rightarrow$ show that if $f_n(x) = \frac{n^2x}{1+n^4x^2}$, then $\{f_n\}$ converges non-uniformly on $[0,1]$.

Sol^n:- Here $f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$= \underset{n\to\infty}{Lt} \frac{n^2x}{1+n^4x^2}$

$= \underset{n\to\infty}{Lt} \frac{x/n^2}{\frac{1}{n^4}+x^2}$

$= 0 \;\forall x\in[0,1]$.

Now $|f_n(x) - f(x)| = \left|\frac{n^2x}{1+n^4x^2} - 0\right|$

$= \left|\frac{n^2x}{1+n^4x^2}\right|$ ——(1)

Let $y = \frac{n^2x}{1+n^4x^2}$ ——(2)

$\frac{dy}{dx} = \frac{(1+n^4x^2).n^2 - n^2x.2n^4x}{(1+n^4x^2)^2}$

$= \frac{n^2[1+n^4x^2-2n^4x^2]}{(1+n^4x^2)^2}$

$= \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$

For max or min $\frac{dy}{dx} = 0$

$\Rightarrow \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2} = 0$

$\Rightarrow n^2(1-n^4x^2) = 0$

$\Rightarrow 1-n^4x^2 = 0$

$\Rightarrow n^4x^2 = 1 \Rightarrow x = 1/n^2$.

Also

$\frac{d^2y}{dx^2} = \frac{n^2(1+n^4x^2)^2(-2n^4x) - (1-n^4x^2).2(1+n^4x^2).2n^4x}{(1+n^4x^2)^4}$

$= \frac{n^2[1+n^4x^2][(1+n^4x^2)(-2n^4x) - 4n^4x(1-n^4x^2)]}{(1+n^4x^2)^4}$

$= \frac{n^2(-2n^4x)[1+n^4x^2+2(1-n^4x^2)]}{(1+n^4x^2)^3}$

$= \frac{-2n^6x[3-n^4x^2]}{(1+n^4x^2)^3}$

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n^2}} = \frac{-2n^6.\frac{1}{n^2}\left[3-n^4.\frac{1}{n^4}\right]}{\left(1+n^4.\frac{1}{n^4}\right)^3}$

$= \frac{-2n^4(3-1)}{(1+1)^3} = \frac{-4n^4}{8}$

$= \frac{-n^4}{2} < 0$.

$\therefore y$ is max when $x = \frac{1}{n^2}$ and the max value of

$y = \frac{n^2.1/n^2}{1+n^4.\frac{1}{n^4}} = \frac{1}{1+1} = \frac{1}{2}$ [from (2)]

$\therefore M_n = \underset{x\in[0,1]}{max} |f_n(x) - f(x)|$

$= \underset{x\in[0,1]}{max} \left|\frac{n^2x}{1+n^4x^2}\right| = 1/2$

i.e. $M_n \to \frac{1}{2}$ as $n \to \infty$

which does not tend to zero as $n\to\infty$

$\therefore \{f_n\}$ Converges non-uniformly on $[0,1]$.

→ show that the sequence $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing zero.

$\boxed{x=\frac{1}{n} \to 0 \text{ as } n\to\infty \quad 0 \in \text{any interval}}$

→ show that the sequence $\{f_n\}$, where $f_n(x) = \frac{n^2x}{1+n^3x^2}$ is not uniformly convergent on $[0,1]$.

→ show that the sequence of functions $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$ converges uniformly on any closed interval $[a,b]$.

→ show that the sequence of functions $\{f_n\}$, where $f_n(x) = nx(1-x)^n$, is not uniformly convergent on $[0,1]$.

sol^n: For $0<x<1$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} \quad \left[\frac{d}{dx}a^{-x} = -a^{-x}\log a\right]$$

$$= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)} = 0$$

Also when $x=0$, $f_n(x)=0 \ \forall n$

when $x=1$, $f_n(x) = 0 \ \forall n$

$\therefore f(x) = 0 \ \forall x \in [0,1]$

Now $|f_n(x) - f(x)| = |nx(1-x)^n - 0|$

$= |nx(1-x)^n|$.

Let $y = nx(1-x)^n$ —(2)

then $\frac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$

$= n(1-x)^{n-1}[1-x-nx]$

$= n(1-x)^{n-1}[1-(n+1)x]$

For max or min $\frac{dy}{dx}=0 \Rightarrow x = \frac{1}{n+1}$

Also $\frac{d^2y}{dx^2} = n(n-1)(1-x)^{n-2}[1-(n+1)x] - ($

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n+1}} = 0 - n(n+1)\left(\frac{n}{n+1}\right)^{n-1}$$

$$= -n(n+1)\left(\frac{n}{n+1}\right)^{n-1} < 0.$$

$\Rightarrow y$ is maximum at $x = \frac{1}{n+1}$ and

max value of y is $n \cdot \frac{1}{(n+1)}\left(1-\frac{1}{n+1}\right)^n$

$$= \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$$

$$\therefore M_n = \max_{x\in[0,1]} |f_n(x) - f(x)|$$

$$= \max_{x\in[0,1]} \left|\left(1-\frac{1}{n+1}\right)^{n+1}\right|$$

$$= \frac{1}{e} \text{ as } n\to\infty$$

$\therefore M_n \to \frac{1}{e}$ as $n\to\infty$

i.e., M_n does not tend to zero as $n \to \infty$.

$\therefore$ the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

→ Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0,\pi]$.

Sol^n:- Here $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sin nx = 0 \quad \forall x \in [0,\pi]$$

Now $|f_n(x) - f(x)| = \left|\frac{\sin nx}{\sqrt{n}} - 0\right| = \left|\frac{\sin nx}{\sqrt{n}}\right|$

Let $y = \frac{\sin nx}{\sqrt{n}}$ —— (1)

then $\frac{dy}{dx} = \sqrt{n}\cos nx$

For max or min, $\frac{dy}{dx} = 0$

$$\Rightarrow \sqrt{n}\cos nx = 0$$
$$\Rightarrow nx = \pi/2$$
$$\Rightarrow x = \pi/2n$$

Also $\frac{d^2y}{dx^2} = -n^{3/2}\sin nx$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pi/2n} = -n^{3/2}\sin \pi/2 = -n^{3/2} < 0.$$

$\Rightarrow y$ is max when $x = \pi/2n$

and the max value of y is

$\frac{\sin \pi/2}{\sqrt{n}} = \frac{1}{\sqrt{n}}$ [from (1)]

$$\therefore M_n = \max_{x\in[0,\pi]} |f_n(x) - f(x)|$$

$$= \max_{x\in[0,\pi]} \left|\frac{1}{\sqrt{n}}\right|$$

$$= \frac{1}{\sqrt{n}} \to 0 \text{ as } n\to\infty.$$

$\therefore$ The sequence $\{f_n\}$ converges uniformly to '0' on $[0,\pi]$.

* Series of real valued functions

Definition: If $\{f_n\}$ is a sequence of real valued functions on an interval I, then $f_1+f_2+f_3+f_4+f_5+ \cdots +f_n+ \cdots$ is called a series of real-valued functions defined on I.

This series is denoted by $\sum_{n=1}^{\infty} f_n$ (or $\sum f_n$).

For example: (i) If $f_n : [0,\infty) \to \mathbb{R}$ is defined by $f_n(x) = \frac{1}{n+x}$, then the series is $\sum f_n = f_1+f_2+f_3+ \cdots$

$$= \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \cdots$$

(ii) If $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, then the series is

$$\sum f_n = f_1 + f_2 + f_3 + \cdots$$

$$= \sin x + \frac{\sin 2x}{\sqrt{2}} + \frac{\sin 3x}{\sqrt{3}} + \cdots$$

* Convergence (or Pointwise Convergence) of a series of functions:-

Let $\sum f_n$ be a series of a function defined on an interval I.

Let $S_1 = f_1$, $S_2 = f_1 + f_2$ ...

$S_n = f_1 + f_2 + f_3 + \cdots + f_n$

then the sequence $\{S_n\}$ is a sequence of partial sums of the series $\sum f_n$.

If the ~~series~~ seq $\{S_n\}$ converges pointwise on I, then the series $\sum f_n$ is said to converge pointwise on I. The limit function f of $\{S_n\}$ is called the pointwise sum (or) simply the sum of the series $\sum f_n$ and we write $\sum_{n=1}^{\infty} f_n(x) = f(x)\ \forall x \in I$

(or) simply $\sum f_n = f$.

For example: Consider the series $\sum f_n$ defined by $f_n(x) = x^n, -1 < x < 1$.

then $\sum f_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$

where $-1 < x < 1$

$$S_n(x) = x + x^2 + \cdots + x^n$$

$$= \frac{x(1-x^n)}{1-x}$$

Now $\lim_{n\to\infty} S_n = \frac{x}{1-x}$ ($\because -1<x<1 \Rightarrow x^n \to 0$ as $n\to\infty$)

$\therefore$ The sequence $\{S_n\}$ of partial sums converges pointwise to $\frac{x}{1-x}$ on $(-1, 1)$

$\Rightarrow$ The series $\sum f_n$ converges pointwise to $f(x) = \frac{x}{1-x}$ on $(-1,1)$

$\Rightarrow \sum f_n(x) = \frac{x}{1-x}$ on $(-1, 1)$

* Uniform Convergence of Series of functions:—

Definition: Let $\sum f_n$ be a series of functions defined on an interval I & $S_n = f_1 + f_2 + f_3 + \cdots + f_n$.

If the sequence $\{S_n\}$ of partial sums converges uniformly on I, then the series $\sum f_n$ is said to be uniformly convergent on I.

Thus, a series of functions converges uniformly to a function f on an interval I, if for each ϵ and for each $x \in I$, $\exists$ +ve integer m (depending only on ϵ and not on x) such that

$$|S_n(x) - f(x)| < \epsilon \quad \forall n \geq m.$$

The uniform limit function f of $\{S_n\}$ is called the sum of the series $\sum f_n$ and we write

$$\sum f_n = f \ \forall\ x \in I.$$

* Theorem: [Cauchy's Criterion for uniform convergence of a series of functions]:—

A series of functions $\sum f_n$ is uniformly convergent on an interval I iff for each $\epsilon > 0$ and for all x

$\exists$ a +ve integer 'm' (depending on ϵ) such that $|S_{n+p} - S_n| < \epsilon \;\forall\; n \geq m, p \geq 0$

where $S_n = f_1(x) + f_2(x) + \cdots + f_n(x)$.

i.e. $|f_{n+1}(x) + f_{n+2}(x) + \cdots f_{n+p}(x)| < \epsilon$ $\forall\, n \geq m$.

Note(1): Uniform convergence $\Rightarrow$ Pointwise convergence

Note(2): The method of testing the uniform convergence of a series $\sum f_n$ by definition, involves finding S_n which is not always easy. The following test avoids S_n.

Theorem [Weierstrass's M-Test]:

A series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely) on an interval I.

if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of non-negative terms (i.e. $M_n \geq 0 \;\forall n \in \mathbb{N}$) such that

$|f_n(x)| \leq M_n \;\forall n \in \mathbb{N}$ and $x \in I$.

$\rightarrow$ Show that if $0 < r < 1$, then each of the following series is uniformly convergent on $\mathbb{R}$.

(i) $\sum_{n=1}^{\infty} r^n \cos nx$ (ii) $\sum_{n=1}^{\infty} r^n \sin nx$

(iii) $\sum_{n=1}^{\infty} r^n \cos n^2 x$ (iv) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

Sol'n: (i) $\sum_{n=1}^{\infty} r^n \cos nx$

Let $f_n(x) = r^n \cos nx \;\forall x \in \mathbb{R}$ then

$|f_n(x)| = |r^n \cos nx|$

$= r^n |\cos nx|$

$\leq r^n \quad (\because r > 0)$

$= M_n \;\forall x \in \mathbb{R}$ —— (1)

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric series with $0 < r < 1$, it is convergent.

Hence by Weierstrass's M-Test, the given series is convergent uniformly on $\mathbb{R}$.

$\rightarrow$ Show that the following series are uniformly convergent for all real 'x'

(i) $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2 x)}{n(n^2+2)}$

Sol'n:- (i) Here $f_n(x) = \frac{\sin(x^2+n^2x)}{n(n+2)}$

$|f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+2)}\right|$

$= \frac{|\sin(x^2+n^2x)|}{|n(n+2)|} \leq \frac{1}{n(n+2)} \;(\because)$

$\leq \frac{1}{n^2} = M_n \;\forall x \in \mathbb{R}$.

Since $\sum_{n=1}^{\infty} M_n = \sum \frac{1}{n^2}$ is convergent (by P-Test).

$\therefore$ By Weierstrass's M-test, the given series is uniformly convergent for all real 'x'.

→ Show that the following series are uniformly and absolutely convergent for all real values of x and $p>1$.

(i) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$

[2002 P-II]

Sol'n (i): Here $f_n(x) = \frac{\sin nx}{n^p}$

$$|f_n(x)| = \left|\frac{\sin nx}{n^p}\right| = \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n \quad \forall x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is Convergent for $p>1$ (by p-Test)

∴ By Weierstrass's M-test, the given series converges uniformly and absolutely for all real values of x.

→ Test for uniform convergence series (i) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ (ii) $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$

Sol'n:- (i) Here $f_n(x) = \frac{x}{(n+x^2)^2}$ —①

$$\Rightarrow \frac{df_n(x)}{dx} = \frac{(n+x^2)^2 \cdot 1 - x\cdot 2(n+x^2)\cdot 2x}{(n+x^2)^4}$$

$$= \frac{(n+x^2)[n+x^2-4x^2]}{(n+x^2)^4}$$

$$= \frac{n-3x^2}{(n+x^2)^3}$$

For max or min $\frac{df_n(x)}{dx} = 0$

$\Rightarrow n-3x^2 = 0$

$\Rightarrow x = \sqrt{n/3}$

Also $\frac{d^2 f_n(x)}{dx^2} = \frac{(n+x^2)^3(-6x) - (n-3x^2)\ldots}{(n+x^2)^6}$

$$= \frac{(n+x^2)^2\left[(n+x^2)(-6x) - 6x(n-3x^2)\right]}{(n+x^2)^6}$$

$$= \frac{-6x\left[(n+x^2)+(n-3x^2)\right]}{(n+x^2)^4}$$

$\frac{d^2 f_n(x)}{dx^2}$ at $x=\sqrt{n/3}$ $= \frac{-6\sqrt{n/3}\left(n+\frac{n}{3}+\ldots\right)}{\left(n+\frac{n}{3}\right)^4}$

$$= \frac{-27\sqrt{3}}{32 n^{5/2}} < 0$$

$\Rightarrow f_n(x)$ is maximum at $x=\sqrt{\frac{n}{3}}$ and maximum value of $f_n(x)$ is

$$\frac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16 n^{3/2}} \quad (\text{from ①})$$

$$|f_n(x)| \le \frac{3\sqrt{3}}{16 n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent (by p-Test)

∴ by Weierstrass M-test, the given series is uniformly convergent for all values of x.

(ii) Here $f_n(x) = \frac{x}{n(1+nx^2)}$ —①

$$\Rightarrow \frac{df_n(x)}{dx} = \frac{1}{n}\cdot\frac{(1+nx^2)\cdot 1 - x\cdot 2\ldots}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{n(1+nx^2)^2}$$

For max or min $\frac{df_n(x)}{dx} = 0$.

$\Rightarrow 1-nx^2=0$

$\Rightarrow x = 1/\sqrt{n}$

Also $\frac{d^2 f_n(x)}{dx^2} = \frac{1}{n}\frac{(1+nx^2)^2(-2nx)-(1-nx^2)2(1+nx^2)2nx}{(1+nx^2)^4}$

$$= \frac{-2x\left[(1+nx^2)+2(1-nx^2)\right]}{(1+nx^2)^3}$$

$$\left.\frac{d^2 f_n(x)}{dx^2}\right|_{x=\frac{1}{\sqrt{n}}} = \frac{-2\frac{1}{\sqrt{n}}\left[1+1+0\right]}{(1+1)^3}$$

$$= \frac{-1}{2\sqrt{n}} < 0$$

$\Rightarrow f_n(x)$ is max at $x = \frac{1}{\sqrt{n}}$ and the max value of $f_n(x)$ is $\frac{\frac{1}{\sqrt{n}}}{n(1+1)} = \frac{1}{2n^{3/2}}$ [from ①]

$\Rightarrow |f_n(x)| \leq \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}} = M_n$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent (by p-Test).

∴ By weierstrass's M-test, the given series is uniformly convergent for all values of x.

→ show that the series $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges in $[1,\infty)$.

Sol^n:- Here $f_n(x) = \frac{1}{1+n^2x}$

$|f_n(x)| = \left|\frac{1}{1+n^2x}\right| \leq \frac{1}{1+n^2} < \frac{1}{n^2} = M_n \ \forall x \in [1,\infty)$.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent by weierstrass M-test, the given series is uniformly convergent $\forall x \in [1,\infty)$.

→ show that the series (i) $\sum_{n=1}^{\infty}\frac{a_n x^{2n}}{1+x^{2n}}$ (ii) $\sum_{n=1}^{\infty}\frac{a_n x^n}{1+x^{2n}}$ converge uniformly for all real values of x, if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Sol^n:- Here $f_n(x) = \frac{a_n x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}} < 1 \ \forall x \in \mathbb{R}$

$\therefore |f_n(x)| = \left|\frac{a_n x^{2n}}{1+x^{2n}}\right| = |a_n|\frac{x^{2n}}{1+x^{2n}} < |a_n| = M_n \ \forall x \in \mathbb{R}$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

$\therefore \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}|a_n|$ is convergent.

Hence by weierstrass M-test, the given series is uniformly convergent.

(ii) Here $f_n(x) = \frac{a_n x^n}{1+x^{2n}}$.

Let $y = \frac{x^n}{1+x^{2n}}$

$\Rightarrow \frac{dy}{dx} = \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$

For max or min $\frac{dy}{dx} = 0$

$\Rightarrow x = 1$

Also $\frac{d^2y}{dx^2}$

$= \frac{(1+x^{2n})\left[n(n-1)x^{n-2}(1-x^{2n}) - 2n^2x^{3n-2}\right] - 4n^2x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2\left[0-2n^2\right]-0}{(2)^3}$$

$$= \frac{-n^2}{2} < 0$$

$\Rightarrow y = \frac{x^n}{1+x^{2n}}$ is max at $x=1$ and the max value of y is $\frac{1}{2}$.

$$\therefore |f_n(x)| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{x^n}{1+x^{2n}}\right| |a_n|$$

$$\leq \frac{1}{2}|a_n| < |a_n| = M_n \quad \forall x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\therefore \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent

Hence by weierstrass M-test, the given series is uniformly convergent for all real x.

→ If the series $\sum a_n$ converges absolutely, then prove that : (i) $\sum a_n \cos nx$ and (ii) $\sum a_n \sin nx$ are uniformly convergent on $\mathbb{R}$.

Sol^n (i) $\sum a_n \cos nx$

Here $f_n(x) = a_n \cos nx$

$$|f_n(x)| = |a_n \cos nx| = |a_n|\,|\cos nx|$$

$$\leq |a_n| = M_n$$

$$[\because |\cos nx| \leq 1 \ \forall x \in \mathbb{R}]$$

Since $\sum a_n$ is absolutely convergent

$\therefore \sum M_n = \sum |a_n|$ is convergent.

Hence by weierstrass's M-test series $\sum a_n \cos nx$ is uniformly Co

→ Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$ absolutely and uniformly conver for all real x if $p>1$.

Sol^n :- Here $f_n(x) = \frac{(-1)^n}{n^p} \frac{x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}} < 1 \ \forall x \in \mathbb{R}$

$$\therefore |f_n(x)| = \left|\frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{n^p} \quad \forall x \in$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is conve $p>1$.

by Weierstrass's M-test, the given series is absolutely and uniformly convergent $\forall x \in \mathbb{R}$ if $p>1$.

* Uniform Convergence and Continuity :-

→ Theorem 1 : If a sequence of con functions $\{f_n\}$ is uniformly conver to a function f on $[a,b]$, then continuous on $[a,b]$.

→ Theorem 2 : If a series $\sum_{n=1}^{\infty} f_n$ o continuous functions is uniformly convergent to a function f on $[a$ then the sum function f is also continuous on $[a,b]$.

Note :- The above theorems converse is not true. i.e. uniform convergence of the sequence $\{f_n\}$ is only a sufficient but not a necessary condition for the continuity of the limit function f, i.e. if the limit function f is continuous on $[a,b]$, then it is not necessary that the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$.

Theorem 1 shows that if the limit function f is discontinuous then the sequence $\{f_n\}$ of continuous functions cannot be uniformly convergent on $[a,b]$. Thus the theorem provides a very good negative test for uniform convergence of a sequence.

Similarly, if the sum function f is discontinuous then the series $\sum f_n$ of continuous functions cannot be uniformly convergent.

Problems: ① Test for uniform convergence and continuity the sequence $\{f_n\}$, where $f_n(x) = x^n$, $0 \le x \le 1$.

Sol'n: Here $f_n(x) = x^n$, $0 \le x \le 1$.

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} x^n$$

$$= \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Clearly f is discontinuous at $x=1$ and hence f is discontinuous on $[0,1]$.

Also $f_n(x) = x^n$, $0 \le x \le 1$ is continuous on $[0,1]$ $\forall n \in N$.

Since $\{f_n\}$ is a sequence of continuous functions and its limit function f is discontinuous on $[0,1]$

$\therefore$ The sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

→ Test the uniform convergence and continuity of $\{f_n\}$ where $f_n(x) = \frac{1}{1+nx}$, $0 \le x \le 1$.

Sol'n:- The limit function f is given by $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx}$

$$= \begin{cases} 0 & \text{for } 0 < x \le 1 \\ 1 & \text{for } x = 0 \end{cases}$$

Clearly f is discontinuous at $x=0$ and hence f is discontinuous on $[0,1]$.

Also $f_n(x) = \frac{1}{1+nx}$, $0 \le x \le 1$ is continuous on $[0,1]$ $\forall n \in N$.

Since $\{f_n\}$ is a sequence of continuous function and its limit function f is discontinuous on $[0,1]$.

$\therefore$ The sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$

H.W. → Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx$ is not uniformly convergent on $[0,1]$.

[Hint:- $f(x) = \begin{cases} \pi/2 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$ is discontinuous at $x=0$.

→ If $f_n(x) = \frac{1}{x+n}$, $x \geq 0$ then show that $\{f_n(x)\}$ converges uniformly to the continuous function zero.

Sol'n:- Here $f_n(x) = \frac{1}{x+n}$, $x \geq 0$.

which is continuous $\forall\ n \in N$ and $x \geq 0$.

Now $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall\ x \geq 0$$

f being a constant function is continuous for all $x \geq 0$.

But continuity of f is no guarantee for uniform convergence of $\{f_n\}$.

Now $|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right|$

$$= \left|\frac{1}{x+n}\right|$$

$$= \frac{1}{x+n}$$

$\therefore |f_n(x) - f(x)| = \frac{1}{n+x} < \epsilon$ if $n+x > \frac{1}{\epsilon}$

if $n > \frac{1}{\epsilon} - x$

choose any +ve integer $\geq \frac{1}{\epsilon}$

then $|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq m$

$\therefore \{f_n(x)\}$ is uniformly convergent

H.W. → Examine for uniform convergence and continuity of the limit fun of the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \leq x \leq 1.$$

→ show that the series $\sum_{n=1}^{\infty} (1-$ not uniformly convergent on $[0$

Sol'n:- $f_n(x) = (1-x)x^n \ \forall x \in$

$\Rightarrow f_1(x) = (1-x)x$,

$f_2(x) = (1-x)x^2$

$f_3(x) = (1-x)x^3$

$\vdots \qquad \vdots$

$f_n(x) = (1-x)x^n$

$\therefore S_n(x) = x(1-x) + x^2(1-x) + \cdots + x^n($

$= (1-x)[x + x^2 + x^3 + \cdots + x$

$= (1-x)\frac{x(1-x^n)}{1-x} = x(1-x^n$

$\therefore S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} x(1-x^n)$

$= \begin{cases} 0 & \text{when } x=0 \text{ (or} \\ x & \text{when } 0<x< \end{cases}$

Since the sum function $S(x)$ is discontinuous at $x=0 \in [0,1]$.

$\therefore$ the given series not uniformly convergent on $[0,1]$.

→ Show that the series $\sum_{n=1} f_n(x)$ where $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$ is not uniformly convergent on $[0,1]$ though the sum function is continuous on $[0,1]$.

Sol'n :- Here $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$

$$\therefore f_1(x) = \frac{x}{1+x^2} - 0$$

$$f_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$

$$f_3(x) = \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}$$

$$\vdots$$

$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

Now $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$$= \frac{nx}{1+n^2x^2}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x/n}{\frac{1}{n^2}+x^2} = 0 \quad \forall x \in [0,1]$$

$\therefore$ The sum $S(x)$ is continuous $\forall x \in [0,1]$

But the continuity of $S(x)$ is no guarantee for uniform convergence of $\sum f_n(x)$.

Now $|S_n(x) - S(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right|$

$$= \left|\frac{nx}{1+n^2x^2}\right| \quad \text{--- (1)}$$

Let $y = \frac{nx}{1+n^2x^2}$ then y is maximum when $x = 1/n$ and the maximum value of $y = 1/2$ (Prove it yourself).

Now $M_n = \max_{x\in[0,1]} |S_n(x) - S(x)|$

$$= \max_{x\in[0,1]} |S_n(x) - S(x)|$$

$$= 1/2$$

$\therefore$ which does not tend to '0' as $n\to\infty$

$\therefore$ By M_n-Test for sequences, the sequence $\{S_n(x)\}$ of partial sums is not uniformly convergent.

$\therefore$ The series is not uniformly convergent.

→ * Uniform Convergence and Integration :-

Theorem (1): If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$ then 'f' is integrable on $[a,b]$ and the sequence $\left\{\int_a^b f_n(x)\,dx\right\}$ converges uniformly to $\int_a^b f(x)\,dx$.

Theorem (2) :- If a series of a function $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$ and $\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx$ converges uniformly to $\int_a^b f(x)\,dx$.

i.e $\sum_{n=1}^{\infty} \int_a^b f(x)dx = \int_a^b f(x)dx$

i.e. the series is term by term integrable.

Note (1): the uniform convergence of the sequence $\{f_n\}$ (or series $\sum_{n=1}^{\infty} f_n$) is only sufficient but not a necessary condition for the validity of term by term integration.

Note(2): If $\{f_n\}$ is a sequence of integrable functions converging to f on $[a,b]$ and if $\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b f(x)dx$ then $\{f_n(x)\}$ cannot converge uniformly to f.

Problems :-

→ show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$, $n\in N$ is not uniformly convergent on $[0,1]$.

Sol'n: Here $f(x) = \lim_{n\to\infty} f_n(x)$

$= \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$

$= \lim \frac{nx}{1+\frac{nx^2}{1!}+\frac{n^2x^4}{2!}+\cdots}$

$=0$ for $x\in[0,1]$

Also $\int_0^1 f(x) = 0$

and $\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx^2}dx$

$= \int_0^n \frac{1}{2} e^{-t} dt$ where $t = nx^2$; if $x=0$, $t=0$ and at $x=1$, $t=n$

$= -\frac{1}{2}\left[e^{-t}\right]_0^n$

$= -\frac{1}{2}\left[e^{-n}-1\right]$

$= \frac{1}{2}\left[1-e^{-n}\right]$

Now $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{2}(1-e^{-n})$

$= \frac{1}{2} \neq \int_0^1 f(x)dx$

⇒ the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

→ Prove that $\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right)dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

Sol'n:- Let $f_n(x) = \frac{x^n}{n^2}$

$|f_n(x)| = \left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n$ for $0\le x\le 1$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test)

∴ By weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent for $0\le x\le 1$.

∴ the series can be integrated term by term.

⇒ $\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2}dx$

$= \sum_{n=1}^{\infty} \left[\frac{x^{n+1}}{n^2(n+1)}\right]_0^1$

$= \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

→ Show that the series $1-x+x^2-x^3+ \cdots$, $0 \le x \le 1$, admits of term by term integration on $[0,1]$, though it is not uniformly convergent on $[0,1]$.

Sol'n:- The given series is

$$1-x+x^2-x^3+\cdots$$

When $x=1$, the series $1-1+1-1+\cdots$ oscillates.

For $0 \le x < 1$,

$$1-x+x^2-x^3+\cdots = \frac{1}{1-(-x)} = \frac{1}{1+x} \quad \text{--- (1)}$$

∴ The series is not uniformly convergent on $[0,1]$.

However, integrating L.H.S of (1) term by term over the interval $[0,1]$, we have

$$\int_0^1 1\,dx - \int_0^1 x\,dx + \int_0^1 x^2 dx - \int_0^1 x^3 dx + \cdots$$

$$= 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots = \log 2.$$

R.H.S.

Also $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \Big[\log(1+x)\Big]_0^1$

$$= \log 2$$

i.e. the two sides are equal.

∴ Term by term integration is possible over $[0,1]$, eventhough the given series is not uniformly convergent on $[0,1]$.

→ Test for uniform convergence or term by term integration the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$. Also show that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}\right) dx = \frac{1}{2}$$

Sol'n:- We know that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{(1+x^2)^2}$ is uniformly convergent.

Hence it is integrable term by term between any finite limits.

$$\Rightarrow \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}\right) dx = \lim_{n\to\infty} \int_0^1 \sum_{n=1} \frac{x}{(n+x^2)^2}$$

$$= \lim_{n\to\infty} \sum_{n=1}^{\infty} \int_0^1 x(n+x^2)^{-2} dx$$

$$= \lim_{n\to\infty} \sum_{n=1}^{\infty} \left[\frac{(n+x^2)^{-1}}{-2}\right]_0^1$$

$$= \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1-\frac{1}{n+1}\right)$$

$$= \frac{1}{2}$$

* Uniform Convergence and Differentiation :-

Theorem 1 :- If a sequence of functions $\{f_n\}$ is such that

(i) each f_n is differentiable on $[a,b]$

(ii) each f_n' is continuous on $[a,b]$

(iii) $\{f_n\}$ converges to f on $[a,b]$

(iv) $\{f_n'\}$ converges uniformly to g on $[a,b]$.

then f is differentiable and $f'(x)=g(x)$ $\forall x\in[a,b]$.

Theorem 2: If a series of functions $\sum_{n=1}^{\infty} f_n$ is such that

(i) each f_n is differentiable on $[a,b]$

(ii) each f_n' is continuous on $[a,b]$

(iii) $\sum_{n=1}^{\infty} f_n$ converges to f on $[a,b]$.

(iv) $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a,b]$.

then f is differentiable on $[a,b]$ and $f'(x)=g(x)$ $\forall x\in[a,b]$

Problems:

(1) show that the sequence $\{f_n\}$ where $f_n(x)=\dfrac{nx}{1+n^2x^2}$, $0\le x\le 1$, cannot be differentiated term by term at $x=0$.

Sol'n:- Here $f(x)=\lim_{n\to\infty} f_n(x)=0$ $\forall x\in[0,1]$

$\therefore f'(0)=0$

Also $f_n'(0)=\lim_{h\to 0}\dfrac{f_n(0+h)-f_n(0)}{h}$

$=\lim_{h\to 0}\dfrac{\frac{nh}{1+n^2h^2}-0}{h}$

$=\lim_{h\to 0}\dfrac{n}{1+n^2h^2}=n$

$\therefore f_n'(0)\to\infty$ as $n\to\infty$

$\Rightarrow f'(0)\neq \lim_{n\to\infty} f_n'(0)$

$\therefore \{f_n\}$ cannot be differentiat term by term at $x=0$.

→ show that for the seque $\{f_n\}$ where $f_n(x)=\dfrac{x}{1+nx^2}$ formula $\lim_{n\to\infty} f_n'(x)=f'(x)$ is tr $x\neq 0$ and false if $x=0$. wh

Sol'n:- we know that the se $\{f_n\}$ converges uniformly to for all real x.

$\Rightarrow f(x)=0$ $\forall x\in\mathbb{R}$

$\Rightarrow f'(x)=0$ $\forall x\in\mathbb{R}$ —— (1)

when $x\neq 0$

$f_n'(x)=\dfrac{(1+nx^2)\cdot 1-2nx\cdot x}{(1+nx^2)^2}$

$=\dfrac{1-nx^2}{(1+nx^2)^2}$

$\lim_{n\to\infty} f_n'(x)=\lim_{n\to\infty}\dfrac{1-nx^2}{(1+nx^2)^2}$ (fo

$=\lim_{n\to\infty}\dfrac{-x^2}{2(1+nx^2)\cdot x^2}$

$=0$

$=f'(x)$ (from (1)

So that if $x\neq 0$, the formu $\lim_{n\to\infty} f_n'(x)=f'(x)$ true At $x=$

$$f_n'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h}{1+nh^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{1}{1+nh^2} = 1$$

so that $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$ [from (*)]

Hence at $x=0$, the formula

$\lim_{n\to\infty} f_n'(x) = f'(x)$ is false.

It is so because the sequence $\{f_n'\}$ is not uniformly convergent in any interval containing zero.

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Improper Integrals *

Introduction:

The Concept of Riemann integrals requires that the range of integration is finite and the integrand 'f' remains bounded in that domain.

If either (or both) of these assumptions is not satisfied, it is necessary to attach a new interpretation to the integral.

In Case the integrand 'f' becomes infinite in the interval $a \le x \le b$; i.e. f has points of infinite discontinuity (Singular points) in $[a,b]$ (or) the limits of integration a or b (or both) become infinite, the symbol $\int_a^b f(x)\,dx$ is called an improper integral or (infinite (or) generalised) integral.

Ex:-

$$\int_1^\infty \frac{1}{x^2}dx,\ \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx,\ \int_0^1 \frac{1}{x(1-x)}dx,\ \int_{-1}^{\infty}\frac{1}{x^2}dx$$

are all improper integrals.

The integrals which are not improper are called proper integrals.

Ex: $\int_0^1 \frac{\sin x}{x}dx$ is a proper integral

$(\because \text{As } x\to 0\ f(x)=\frac{\sin x}{x}\to 1)$

→ In the definite integral $\int_a^b f(x)\,dx$, if either a or b (or both) are infinite so that the interval of integration unbounded (i.e. the range of the integration is unbounded) ~~but f is~~ bounded then $\int_a^b f(x)\,dx$ is called a improper integral of the first kind.

Ex:- $\int_1^\infty \frac{dx}{x^2},\ \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx,\ \int_{-1}^{\infty}\frac{1}{x^2}dx,\ \int_{-\infty}^{0}e^{2x}dx$, are improper integrals of the first kind.

→ In the definite integral $\int_a^b f(x)\,dx$ if both a and b are finite so that the interval of integration is finite but has one or more points of infinite discontinuity. i.e. f is not bounded on $[a,b]$ then $\int_a^b f(x)\,dx$ is called an improper integral of the second kind ⇒

Ex: $\int_0^1 \frac{1}{x(1-x)}dx,\ \int_0^1 \frac{1}{x^2}dx,\ \int_1^2 \frac{1}{2-x}dx,$

$\int_1^4 \frac{1}{(x-1)(4-x)}dx$ are improper integrals of the second kind

→ In the ~~definite integral $\int_a^b f(x)dx$,~~ if the interval of the integration is unbounded (so that 'a' or 'b' or both are infinite) and f is also unbounded then $\int_a^b f(x)dx$ is called an improper integral of the third kind

Ex:- $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$ is an improper integral of the third kind.

Improper Integral as the limit of a proper Integral:

→ when the improper integral is of the first kind, either a (or) b (or) both a and b are infinite but f is bounded.

we define (i) $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \quad (t>a)$

The improper integral $\int_a^\infty f(x)dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to divergent if the limit is $+\infty$ (or) $-\infty$.

— If the integral is neither convergent nor divergent then it is said to be oscillating.

ii) $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx \quad (t<b)$

~~the~~ improper integral $\int_{-\infty}^b f(x)dx$ is said to be convergent if the limit on the right handside exists finitely and the integral is said to be divergent of the limit $+\infty$ (or) $-\infty$

(iii) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$

where c is any real number.

$$= \lim_{t_1\to-\infty}\int_{t_1}^c f(x)dx + \lim_{t_2\to\infty}\int_c^{t_2} f(x)dx$$

The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be convergent if both the limits on the right handside exist finitely and independent of eachother. otherwise, it is said to be divergent.

Note! $\int_{-\infty}^\infty f(x)dx \neq \lim_{t\to\infty}\left[\int_{-t}^c f(x)dx + \int_c^t f(x)dx\right]$

(b) when the improper integral is of the second kind; both a and b are finite but f has one (or more) points of infinite discontinuity on [a,b].

(i) If f(x) becomes infinite at x=a only;

we define $\int_a^b f(x)dx = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b f(x)dx$

$0<\epsilon<b-a$

The improper integral $\int_a^b f(x)dx$ converges if the limit on the right handside exists finitely, otherwise it is said to be divergent.

(ii) If $f(x)$ becomes infinite at $x=b$ only; we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0+} \int_a^{b-\epsilon} f(x)dx \qquad 0<\epsilon<b-a$$

the improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand exists finitely and the integral is said to be divergent if the limit is $+\infty$ (or) $-\infty$.

(iii) If $f(x)$ becomes infinite at $x=a$ & $x=b$ only.

we define $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$= \lim_{\epsilon_1\to 0+} \int_{a+\epsilon_1}^{c} f(x)\,dx + \lim_{\epsilon_2\to 0+} \int_c^{b-\epsilon_2} f(x)dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limits on the right hand exist finitely and independent of each other, otherwise it is said to be divergent.

Note: The improper integral is also defined as $\int_a^b f(x)dx = \lim_{\substack{\epsilon_1\to 0+ \\ \epsilon_2\to 0+}} \int_{a+\epsilon_1}^{b-\epsilon_2} f(x)dx.$

The improper integral exists if the limit exists.

(iv) If $f(x)$ becomes infinite at x only where $a<c<b$ and c is an interior point.

we define $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)$

$$= \lim_{\epsilon_1\to 0} \int_a^{c-\epsilon_1} f(x)dx + \lim_{\epsilon_2\to 0+} \int_{c+\epsilon_2}^b f(x)$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other otherwise it is said to be divergent.

Similarly, —

If the function has a finite number of points of infinite discontinuity,

$c_1, c_2, c_3 ---- c_m$ with in $[a,b]$.

where $a\le c_1<c_2<c_3<----c_m\le b$.

we define the improper integral $\int_a^b f(x)dx$ as

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + ---- + \int_{c_m}^b f(x)dx.$$

and is said to be convergent if all the integrals on the R.H.S are convergent, otherwise it is divergent.

Note(1): If f has infinite discontinuity at an end point of the interval of the integration then the point of infinite discontinuity is approached from within the interval.

i.e., if the interval of integration is $[a, b]$ and

, f has infinite discontinuity at 'a' then we consider $[a+\epsilon, b]$ as $\epsilon \to 0+$.

, f has infinite discontinuity at 'b' then we consider $[a, b-\epsilon]$ as $\epsilon \to 0+$.

Note(2):

A proper integral is always convergent.

Note(3):

If $\int_a^b f(x)\,dx$ is convergent then

(i) $\int_a^b k\,f(x)\,dx$ is convergent ; $k \in \mathbb{R}$.

(ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where $a < c < b$. and each integral on right hand side is convergent.

Note(4):

For any point c between a & b i.e. $a < c < b$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $\int_c^a f(x)\,dx$ is a proper integral then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^c f(x)\,dx$ converge or diverge together i.e, while testing the integral $\int_a^b f(x)\,dx$ for convergence at 'a' it may be replaced by $\int_a^c f(x)\,dx$ for any convenient 'c' such that $a < c < b$.

Problems:—

Examine the convergence of the improper integral.

(i) $\int_1^\infty \frac{1}{x}\,dx$ (ii) $\int_0^\infty \frac{1}{1+x^2}\,dx$.

Sol'n: i) By definition

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} [\log x]_{x=1}^{t} = \lim_{t\to\infty} (\log t - \log 1)$$

$$= \lim_{t\to\infty} (\log t) = \infty$$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

(ii) By definition,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} [\tan^{-1} x]_0^t$$

$$= \lim_{t\to\infty} [\tan^{-1} t - \tan^{-1}(0)]$$

$$= \lim_{t\to\infty} [\tan^{-1} t - 0]$$

$= \pi/2$ which is finite.

$\therefore \int_0^\infty \frac{1}{1+x^2}\,dx$ is convergent.

(i) $\int_0^\infty e^{-mx}dx \quad (m>0)$

(ii) $\int_a^\infty \frac{x}{1+x^2}dx$ (iii) $\int_0^\infty \sin x\,dx$

(iv) $\int_0^\infty \frac{1}{x^2+4a^2}dx$ (v) $\int_0^\infty e^{2x}dx$

(ii) By definition

$$\int_a^\infty \frac{x}{1+x^2}dx = \lim_{t\to\infty}\int_a^t \frac{x}{1+x^2}dx \quad (a<t)$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_a^t \frac{2x}{1+x^2}dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\log(1+x^2)\right]_a^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\log(1+t^2)-\log(1+a^2)\right]$$

$$= \frac{1}{2}\left[\infty - \log(1+a^2)\right] = \infty$$

$\therefore \int_a^\infty \frac{x}{1+x^2}dx$ is divergent.

(iii) $\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left[-\cos x\right]_0^t$

$$= \lim_{t\to\infty}\left[-\cos t + \cos 0\right]$$

$$= \lim_{t\to\infty}\left[-\cos t + 1\right]$$

$$= l \quad (\because -1 \le \cos t \le 1)$$

which does not exist uniquely.

Since l is finite but not fixed because $\cos t$ oscillates between -1 & $+1$.

$\int_0^\infty \sin x\,dx$ oscillates.

(iv) $\int_0^\infty \frac{1}{x^2+4a^2}dx = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+(2a)^2}$

$$= \lim_{t\to\infty}\left[\frac{1}{2a} + \tan^{-1}\left(\frac{x}{2a}\right)\right]_0^t$$

→ (i) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$ (ii) $\int_2^\infty \frac{2x^2}{x^4-1}dx$

(iii) $\int_1^\infty \frac{x}{(1+2x)^3}dx$ (iv) $\int_1^\infty \frac{x}{(1+x)^3}dx$

Sol'n: (i) $\int_{\sqrt{2}}^\infty \frac{1}{x\sqrt{x^2-1}}dx = \lim_{t\to\infty}\int_{\sqrt{2}}^t \frac{1}{x\sqrt{x^2-1}}dx$

$$= \lim_{t\to\infty}\left[\sec^{-1}x\right]_{\sqrt{2}}^t$$

(ii) $\int_2^\infty \frac{2x^2}{x^4-1}dx = \lim_{t\to\infty}\int_2^t \frac{2x^2}{x^4-1}dx$

$$= \lim_{t\to\infty}\int_2^t \frac{2x^2}{(x^2)^2-1}dx = \lim_{t\to\infty}\int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2-1)(x^2+1)}dx$$

$$= \lim_{t\to\infty}\int_2^t\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)+\tan^{-1}x\right]_2^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\left(\frac{t-1}{t+1}\right)+\tan^{-1}t-\frac{1}{2}\log\left(\frac{1}{3}\right)-\tan^{-1}(2)\right]$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\left(\frac{1-1/t}{1+1/t}\right)+\tan^{-1}t-\frac{1}{2}\log\left(\frac{1}{3}\right)-\tan^{-1}(2)\right]$$

$$= \frac{1}{2}(0)+\frac{\pi}{2}-\frac{1}{2}\log\frac{1}{3}-\tan^{-1}(2)$$

$$= \frac{\pi}{2}-\frac{1}{2}\log\left(\frac{1}{3}\right)-\tan^{-1}(2)$$

which is finite.

$\int_2^\infty \frac{2x^2}{x^4-1}dx$ is convergent.

$$\int_1^\infty \frac{x}{(1+2x)^3}dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}dx$$

$$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}dx$$

$$, \int_1^\infty \frac{x}{(1+x)^3}dx = \lim_{t\to\infty}\int_1^t \frac{(1+x)-1}{(1+x)^3}dx$$

$\rightarrow$ (i) $\int_1^\infty xe^{-x}dx$ (ii) $\int_0^\infty x^2e^{-x}dx$

(iii) $\int_0^\infty xe^{-x^2}dx$ (iv) $\int_0^\infty x^3e^{-x^2}dx$ (v) $\int_0^\infty x\sin x\,dx$

Sol: (ii) $\int_0^\infty x^2e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}dx$

(Integrating by parts)

$$= \lim_{t\to\infty}\left[x^2(-e^{-x}) - \int 2x(e^{-x})dx\right]_0^t$$

$$= \lim_{t\to\infty}\left[-x^2e^{-x} + 2(-xe^{-x} - e^{-x})\right]_0^t$$

$$= \lim_{t\to\infty}\left[-x^2e^{-x} - 2e^{-x}(x+1)\right]_0^t$$

$$= \lim_{t\to\infty}\left[-t^2e^{-t} - 2e^{-t}t - 2e^{-t} + 2\right]$$

$$= \lim_{t\to\infty}(-t^2e^{-t}) - 2\lim_{t\to\infty}e^{-t}t - \lim_{t\to\infty}e^{-t} + \lim_{t\to\infty}(2)$$

$$= -\lim_{t\to\infty}\frac{t^2}{e^t} - 2\lim_{t\to\infty}\frac{t}{e^t} - 0 + 2$$

(Applying L-Hospital's rule)

$$= -\lim_{t\to\infty}\frac{2t}{e^t} - 2\lim_{t\to\infty}\frac{1}{e^t} + 2$$

$$= -\lim_{t\to\infty}\frac{2(1)}{e^t} - 2(0) + 2$$

$= -2(0) + 2 = 2$ which is finite.

$\therefore \int_0^\infty x^2e^{-x}dx$ is convegent.

(iii) $\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx$ — (1)

put $x^2 = z$, $2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

and if $x=0$ then $z=0$ of $x=t$ then $z=t^2$

$\therefore$ (1) $\equiv$

$$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_{z=0}^t e^{-t}\frac{dz}{2}$$

(iv) $\int_0^\infty x^3e^{-x^2}dx = \lim_{t\to\infty}\int_0^\infty x.x^2e^{-x^2}dx$

put $x^2 = z$, $2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

$\rightarrow$ (i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_0^\infty e^{-x}\sin x\,dx$

(iii) $\int_0^\infty e^{ax}\cos bx\,dx$

Sol^n: (i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty}\int_1^t \frac{1}{(1+x)\sqrt{x}}dx$ — (1)

put $\sqrt{x} = z \Rightarrow \frac{1}{2\sqrt{x}}dx = dz$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dz$

when $x=1 \Rightarrow z=1$; when $x=t \Rightarrow z=\sqrt{t}$

$\therefore$ (1) $\equiv$

$$\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{1}{(1+z^2)}(2dz)$$

$$= \lim_{t\to\infty}\left[2\tan^{-1}z\right]_{z=1}^{\sqrt{t}}$$

(ii) $\int_0^\infty e^{-x}\sin x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin x\,dx$

$= \lim_{t\to\infty}\left[\frac{e^{-x}}{(-1)^2+1^2}\left(-1.\sin x - 1.\cos x\right)\right]_0^t$

$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

(iii) by using the formula

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

→ (i) $\int_1^\infty \frac{dx}{x(x+1)}$ (ii) $\int_1^\infty \frac{dx}{x^2(x+1)}$

(By using partial fractions)

(iii) $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$ (iv) $\int_0^\infty e^{-\sqrt{x}}\,dx$

Sol^n (iii): $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}\,dx$ —— (1)

put $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\,d\theta$

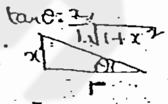
Now $\int \frac{\tan^{-1}x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta$

$= \int \theta\,\mathrm{cosec}^2\theta\,d\theta$

$= \theta(-\cot\theta) - \int 1\,(-\cot\theta)\,d\theta$

$= -\theta\cot\theta + \log\sin\theta$

$= \frac{-\tan^{-1}x}{x} + \log\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\left[\tan\theta = \frac{x}{1}\right.$  $\left., \sin\theta = \frac{x}{\sqrt{1+x^2}}\right]$

$\therefore (1) = \int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$

$= \lim_{t\to\infty}\left[\frac{-\tan^{-1}x}{x} + \log\frac{x}{\sqrt{1+x^2}}\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{-\tan^{-1}t}{t} + \log\frac{t}{\sqrt{1+t^2}} + \frac{\tan^{-1}(1)}{1} - \log\left(\frac{1}{\sqrt{2}}\right)\right]$

$= \frac{-\pi/2}{\infty} + \log\left(\frac{1}{\sqrt{1+\frac{1}{\infty}}}\right) + \pi/4 - \log$

$= 0 + 0 + \pi/4 + \log(\sqrt{2})$

$= \pi/4 + \frac{1}{2}\log 2$ which is finit

$\therefore \int_1^\infty \frac{\tan^{-1}x}{x^2}$ is convergent.

(iv) $\int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{t\to\infty}\int_0^t e^{-\sqrt{x}}\,dx$

put $\sqrt{x} = z$

$\Rightarrow x = z^2$

$\Rightarrow dx = 2z\,dz$

when $x = 0 \Rightarrow z = 0$

$x = t \Rightarrow z = \sqrt{t}$

$\therefore \int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{t\to\infty}\int_{z=0}^{\sqrt{t}} e^{-z}(2z\,dz)$

→ (i) $\int_{-\infty}^0 e^{2x}\,dx$ (ii) $\int_{-\infty}^1 \frac{dx}{1+x^2}$

(iii) $\int_{-\infty}^0 \cosh x\,dx$ $\left[\text{using } \cosh x = \frac{e^x+e^{-x}}{2}\right]$

(iv) $\int_{-\infty}^0 \sinh x\,dx$ $\left[\text{using } \sinh x = \frac{e^x-e^{-x}}{2}\right]$

Sol^n: $\int_{-\infty}^0 e^{2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\,dx.$

→ (i) $\int_{-\infty}^\infty e^{-x}\,dx$ (ii) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ (iii) $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

(iv) $\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx$

Sol'n: $\int_{-\infty}^{\infty} e^{-x}dx = \int_{-\infty}^{0} e^{-x}dx + \int_{0}^{\infty} e^{-x}dx$

$= \lim_{t_1\to-\infty}\int_{t_1}^{0} e^{-x}dx + \lim_{t_2\to\infty}\int_{0}^{t_2} e^{-x}dx$

, $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx = \lim_{t_1\to-\infty}\int_{t_1}^{0}\frac{1}{(x+1)^2+1}dx + \lim_{t_2\to\infty}\int_{0}^{t_2}\frac{1}{(x+1)^2+1}dx$

$= \lim_{t_1\to-\infty}\left[\tan^{-1}(x+1)\right]_{t_1}^{0} + \lim_{t_2\to\infty}\left[\tan^{-1}(x+1)\right]_{0}^{t_2}$

$= \lim_{t_1\to-\infty}\left[\frac{\pi}{4} - \tan^{-1}(t_1+1)\right] + \lim_{t_2\to\infty}\left[\tan^{-1}(t_2+1) - \frac{\pi}{4}\right]$

$= \frac{\pi}{4} - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \pi/4$

$= 2\tan^{-1}(\infty)$

$= 2\,\pi/2 = \pi$ (finite).

$\therefore \int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}dx$ is convergent.

→ (i) $\int_0^1 \log x\,dx$ (ii) $\int_0^{1/e}\frac{dx}{x(\log x)^2}$

(iii) $\int_0^{e}\frac{1}{x(\log x)^3}dx$ (iv) $\int_1^{2}\frac{dx}{x(\log x)}$

Sol'n: (i) 0 is the only point of infinite discontinuity of the integrand 'f' on $[0,1]$.

$\therefore \int_0^1 \log x\,dx = \lim_{\epsilon\to0+}\int_{0+\epsilon}^{1}(\log x)\cdot 1\,dx$

$= \lim_{\epsilon\to0_+}\left[x\log x - x\right]_{\epsilon}^{1}$

$= \lim_{\epsilon\to0_+}\left[1(0) - 1 - \epsilon\log\epsilon + \epsilon\right]$

$= \lim_{\epsilon\to0_+}\left[-1 - \epsilon\log\epsilon + \epsilon\right]$

$= -1 \quad (\because \lim_{x\to0+} x^n\log x = 0;\ n>0)$

$\therefore \int_0^1 \log x\,dx$ is convergent.

(ii) Since $\lim_{x\to0+}(x\log x)^n = 0;\ n>0$.

$\therefore$ 0 is the only point of infinite discontinuity of the integrand on $[0,1/e]$

$\therefore \int_0^{1/e}\frac{1}{x(\log x)^2}dx = \lim_{\epsilon\to0+}\int_{0+\epsilon}^{1/e}\frac{1}{x(\log x)^2}dx$

$= \lim_{\epsilon\to0_+}\int_{\epsilon}^{1/e}(\log x)^{-2}\cdot\frac{1}{x}dx$

$= \lim_{\epsilon\to0+}\left[\frac{(\log x)^{-1}}{-1}\right]_{\epsilon}^{1/e}$

$= \lim_{\epsilon\to0+}\left(-\left[\frac{1}{\log 1/e} - \frac{1}{\log\epsilon}\right]\right)$

$= \left(-\left[-\frac{1}{\log e} - 0\right]\right)$

$= -[-1-0]$

$= 1$

$\therefore \int_0^{1/e}\frac{dx}{x(\log x)^2}$ is convergent.

(i) $\int_0^a \frac{dx}{\sqrt{a-x}}$ (ii) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(iii) $\int_0^{\pi/2} \tan\theta\, d\theta$.

(i) 'a' is the only point of infinite discontinuity of the Integrand f on [0,a]

$$\therefore \int_0^a \frac{dx}{\sqrt{a-x}} = \lim_{\epsilon\to 0+} \int_0^{a-\epsilon} (a-x)^{-1/2}\,dx$$

$$= \lim_{\epsilon\to 0+} \left[\frac{-(a-x)^{-1/2+1}}{-1/2+1}\right]_0^{a-\epsilon}$$

$$= \lim_{\epsilon\to 0+} \left[-2(a-x)^{1/2}\right]_0^{a-\epsilon}$$

$$= \lim_{\epsilon\to 0+} -\left[2(\epsilon)^{1/2} - 2(a)^{1/2}\right]$$

$$= -\left[0-2\sqrt{a}\right]$$

$$= 2\sqrt{a} \quad \text{(finite)}$$

$\therefore \int_0^a \frac{dx}{\sqrt{a-x}}$ is Convergent.

(i) $\int_{-1}^{1} \frac{dx}{x^2}$ (ii) $\int_a^{3a} \frac{dx}{(x-2a)^2}$

(iii) $\int_0^{2a} \frac{dx}{(x-a)^2}$

Sol:- (i) The integrand f becomes infinite at $x=0$ and $-1<a<1$.

0 is the only point of infinite discontinuity of the integrand f on [-1,1].

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$$

$$= \lim_{\epsilon_1\to 0+} \int_{-1}^{0-\epsilon_1} \frac{1}{x^2}\,dx + \lim_{\epsilon_2\to 0+} \int_{0+\epsilon_2}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{\epsilon_1\to 0+} \left[\frac{-1}{x}\right]_{-1}^{-\epsilon_1} + \lim_{\epsilon_2\to 0+} \left[\frac{-1}{x}\right]_{\epsilon_2}^{1}$$

$$= \lim_{\epsilon_1\to 0+} \left(\frac{1}{\epsilon_1}-1\right) + \lim_{\epsilon_2\to 0+} \left(-1+\frac{1}{\epsilon_2}\right)$$

$$= (\infty-1) + (-1+\infty) = \infty.$$

$\therefore \int_{-1}^{1} \frac{1}{x^2}\,dx$ is divergent.

(ii) The integrand f becomes infinite at $x=2a$ and $a<2a<3a$.

(i) $\int_0^4 \frac{1}{x(4-x)}\,dx$ (ii) $\int_0^2 \frac{1}{2x-x^2}\,dx$

(iii) $\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\,dx$ (iv) $\int_0^{\pi} \frac{1}{\sin x}\,dx$ (v) $\int_0^{\pi} \frac{1}{1+\cos x}\,dx$

(i) Both the end points 0 & 4 are the points of infinite discontinuity of the integrand f on [0,4].

$$\therefore \int_0^4 \frac{1}{x(4-x)}\,dx = \lim_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \int_{0+\epsilon_1}^{4-\epsilon_2} \frac{1}{x(4-x)}\,dx \quad \left[(\text{or}) \int_0^1 + \int_1^4 \right.$$

$$\left. = \lim_{\epsilon_1\to 0} \int_{0+\epsilon_1}^{1} + \lim_{\epsilon_2\to 0} \int_1^{4-\epsilon_2} \right]$$

$$= \lim_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \int_{\epsilon_1}^{4-\epsilon_2} \frac{1}{4}\left(\frac{1}{x}+\frac{1}{4-x}\right)dx$$

$$= \lim_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \frac{1}{4}\left[\log x - \log(4-x)\right]_{\epsilon_1}^{4-\epsilon_2}$$

$$= \lim_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \frac{1}{4}\left[\log(4-\epsilon_2) - \log\epsilon_2 - \log\epsilon_1 + \log(4-\epsilon_1)\right]$$

$$= 2\log 4 - (-\infty) - (-\infty) = \infty$$

Comparision Tests for convergence at 'a' of $\int_a^b f(x)\,dx$:—

→ Let 'a' be the only point of infinite discontinuity of f on [a,b]

The case when 'b' is the only point of infinite discontinuity can be dealt with in the same way.

When the integrand 'f' keeps the same sign, +ve (or) −ve in a small neighbourhood of 'a', we may suppose that f is non-negative there in, for, if negative it can be replaced by (−f), for testing the convergence of $\int_a^b f(x)dx$ the case f=0 being trivial so there is no loss of generality to suppose that f is +ve throughout.

Theorem

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ at the point 'a', where f is +ve on [a,b], is that there exists a +ve number M (i.e M>0), independent of ∈>0 such that

$\int_{a+\epsilon}^b f(x)\,dx < M$; $0<\epsilon<b-a$.

i.e. the improper integral $\int_a^b f(x)dx$ converges iff ∃ a M>0 and independent of ∈>0 such that $\int_{a+\epsilon}^b f(x)dx < M$

Note:– If for every M>0 and some ∈ in (0,b−a);

$\int_{a+\epsilon}^b f(x)\,dx > M$, then $\int_{a+\epsilon}^b f(x)dx$ is not bounded above.

$\therefore \int_{a+\epsilon}^b f(x)dx \longrightarrow \infty$ as $\epsilon \to 0_+$ and hence the improper integral $\int_a^b f(x)dx$ diverges to +∞.

Comparison Test-I :—

If f and g are two +ve functions with $f(x) \le g(x)$ $\forall x \in [a,b]$ and 'a' is the only point of infinite discontinuity on [a,b] then

(i) $\int_a^b g(x)\,dx$ is convergent ⇒ $\int_a^b f(x)dx$ is convergent

(ii) $\int_a^b f(x)\,dx$ is divergent ⇒ $\int_a^b g(x)dx$ is divergent.

Comparison Test II (Limit Form):

→ If f and g be two +ve functions on (a,b], 'a' is the only point of infinite discontinuity and

$\lim_{x\to a+} \frac{f(x)}{g(x)} = l$ where 'l' is a non-zero finite number, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge (or) diverge together.

→ Let f and g be two +ve functions on (a,b], 'a' is the point of infinite discontinuity, then

(i) $\lim_{x\to a+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ converges

$\Rightarrow \int_a^b f(x)\,dx$ converges.

(ii) $\lim_{x\to a+} \frac{f(x)}{g(x)} = +\infty$ and $\int_a^b g(x)\,dx$ diverges.

$\Rightarrow \int_a^b f(x)\,dx$ diverges.

→ Useful Comparison Integral

The improper integral $\int_a^b \frac{1}{(x-a)^n}dx$ is convergent if and only if $n<1$.

Proof: If $n\le 0$ then the integral $\int_a^b \frac{1}{(x-a)^n}$ is proper.

If $n>0$, the integral is improper and 'a' is the only point of infinite discontinuity of the integrand on $[a,b]$

Case (i): when $n=1$

$$\int_a^b \frac{1}{(x-a)^n}dx = \int_a^b \frac{1}{(x-a)}dx$$

$$= \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon\to 0+} \left[\log(x-a)\right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon\to 0+} \left[\log(b-a) - \log\epsilon\right]$$

$$= \log(b-a) - \infty$$

$$= \infty$$

$\int_a^b \frac{dx}{(x-a)^n}$ diverges if $n=1$.

Case-II: when $n\neq 1$.

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b (x-a)^{-n}dx$$

$$= \lim_{\epsilon\to 0+} \left[\frac{(x-a)^{1-n}}{1-n}\right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon\to 0+} \left(\frac{1}{1-n}\right)\left[(b-a)^{1-n} - \epsilon^{1-n}\right]$$

Subcase I when $n>1$

$\Rightarrow (n-1)>0$

$$\therefore ① \equiv \int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0+} \left(\frac{1}{1-n}\right)\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\epsilon^r}\right]$$

$$= \left(\frac{1}{1-n}\right)\left[\frac{1}{(b-a)^{n-1}} - \infty\right]$$

$$= \left(\frac{1}{1-n}\right)(-\infty) = \infty \quad (\because 1-n<0)$$

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n>1$.

$$\boxed{\therefore \int_a^b \frac{dx}{(x-a)^n} \text{ diverges if } n\geq 1}$$

Subcase-II when $0<n<1$

$\Rightarrow 1-n>0$

$$\therefore ① \equiv \int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \epsilon^{1-n}\right]$$

$$= \left(\frac{1}{1-n}\right)(b-a)^{1-n}$$

which is finite.

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ converges if $n<1$.

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ is convergent iff $n<1$.

Note: The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent iff $n<1$.

→ (2) If a is the only point of infinite discontinuity of f on $[a,b]$ and $\lim_{x\to a+} (x-a)^n f(x)$ exists and is non-zero finite then $\int_a^b f(x)dx$ converges iff $n<1$.

) If b is the only point of infinite discontinuity of f on $[a,b]$ and $\lim_{x\to b-} (b-x)^n f(x)$ exists and is non-zero finite then $\int_a^b f(x)dx$ converges iff $n<1$.

) If f is +ve on $(a,b]$ 'a' is the only point of infinite discontinuity then the integral $\int_a^b f(x)dx$ converges at 'a'.

If $\exists$ +ve number $n<1$ and a fixed +ve number M such that

$f(x) \leq \frac{M}{(x-a)^n}$ $\forall\, x\in(a,b]$.

Also $\int_a^b f(x)dx$ diverges if $\exists$ a number $n\geq 1$ and a fixed +ve number G such that $f(x) \geq \frac{G}{(x-a)^n}$ $\forall\, x\in(a,b]$

Problems:

Examine the convergence of the integrals (i) $\int_0^1 \frac{1}{\sqrt{x^2+x}}dx$ (ii) $\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ (iii) $\int_0^1 \frac{1}{\sqrt{1-x^3}}dx$ (iv) $\int_0 \frac{1}{x^{1/2}(1+x^2)}dx$

(v) $\int_0^{\pi/2} \frac{\sin x}{x^p}dx$.

Sol^n (i) :- Method ①:

Let $f(x) = \frac{1}{\sqrt{x^2+x}}$

$= \frac{1}{\sqrt{x}\,(\sqrt{x+1})}$

Let $g(x) = \frac{1}{\sqrt{x}}$

∴ f, g are +ve on $(0,1]$ and '0' is the only point of infinite discontinuity

Now $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{\sqrt{x+1}} = 1$ (non-zero finite number)

∴ By Comparison Test $\int_0^1 f(x)dx$ & $\int_0^1 g(x)dx$ are convergent (or) divergent together.

Since $\int_0^1 g(x)dx = \int_0^1 \frac{1}{\sqrt{x}}dx$

$= \int_0^1 \frac{1}{(x-0)^{1/2}}dx$ is of the form $\int_a^b \frac{1}{(x-a)^n}dx$.

Here $n = 1/2 < 1$

∴ $\int_0^1 g(x)\,dx$ is convergent.

$\int_0^1 f(x)\,dx$ is convergent.

Method ②: Let $f(x) = \frac{1}{\sqrt{x^2+x}}$

∴ f is +ve on $(0,1]$.

and '0' is the only point of infinite discontinuity of f on $[0,1]$.

Now $f(x) = \frac{1}{\sqrt{x}(\sqrt{x+1})}$

Clearly $\frac{1}{\sqrt{x+1}}$ is bounded function on $[0,1]$.

$\therefore \exists$ a +ve number M as an upperbound such that $\frac{1}{\sqrt{x+1}} \le M \;\forall x \in [0,1]$

$\therefore f(x) \le \frac{M}{\sqrt{x}} \;\forall x \in (0,1]$

$\Rightarrow f(x) \le \frac{M}{(x-0)^{1/2}} \;\forall x \in (0,1]$.

Also $\int_0^1 \frac{1}{(x-0)^{1/2}}$ is convergent $(\because n = \frac{1}{2} < 1)$

$\therefore$ By Comparison test.

$\int_0^1 \frac{1}{\sqrt{x^2+x}}\,dx$ is convergent.

(ii) Let $f(x) = \frac{1}{(1+x)(\sqrt{2-x})}$

$\therefore f$ is +ve on $[1,2)$.

and 2 is the only point of infinite discontinuity of 'f' on $[1,2]$.

Now $f(x) = \frac{1}{(1+x)(\sqrt{2-x})}$

clearly $\frac{1}{1+x}$ is bounded on $[1,2]$

Let M be the upperbound

$\therefore \frac{1}{1+x} \le M \;\forall x \in [1,2]$.

$\therefore f(x) \le \frac{M}{\sqrt{2-x}} \;\forall x \in [1,2)$

$\Rightarrow f(x) \le \frac{M}{(2-x)^{1/2}} \;\forall x \in [1,2)$

Also $\int_1^2 \frac{1}{(2-x)^{1/2}}\,dx$ is convergent $(\because n = \frac{1}{2} < 1)$.

$\therefore$ By Comparison test

$\int_1^2 \frac{1}{(1+x)(\sqrt{2-x})}\,dx$ is convergent.

(iii) Let $f(x) = \frac{1}{\sqrt{1-x^3}}$

$= \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$

$= \frac{1}{(\sqrt{1-x})(\sqrt{1+x+x^2})}$

$\therefore$ 1 is the only point of infinite discontinuity of 'f' on $[0,1)$.

Clearly $\frac{1}{\sqrt{1+x+x^2}}$ is bounded on $[0,$

Let M be the upper bound

$\therefore \frac{1}{\sqrt{1+x+x^2}} \le M \;\forall x \in [0,1]$

$\therefore f(x) \le \frac{M}{(1-x)^{1/2}} \;\forall x \in [0,1]$.

Also $\int_0^1 \frac{1}{(1-x)^{1/2}}\,dx$ is convergent $(\because n = \frac{1}{2} < 1)$

$\therefore \int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$ is convergent.

(V) For $P \le 1$ it is a proper integral

For $P > 1$, it is an improper integral and '0' is the only point of infinite discontinuity.

Now let $f(x) = \frac{\sin x}{x^P}$

$$= \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x}$$

Let $g(x) = \frac{1}{x^{p-1}} \ \forall \ x \in \left(0, \pi/2\right]$

$\therefore \lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\sin x}{x} = 1$ (a non-zero finite number)

$\therefore$ By Comparison Test

$\int_0^{\pi/2} f(x)dx$ & $\int_0^{\pi/2} g(x)dx$ are convergent (or) divergent together.

Since $\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{1}{(x-0)^{p-1}}$ is

Convergent if $p-1<1$ i.e. $p<2$

$\therefore \int_0^{\pi/2} \frac{\sin x}{x^p} dx$ is Convergent for $p<2$

and $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$ divergent $p \geq 2$.

$\rightarrow$ (i) $\int_0^1 \frac{dx}{x^3(2+x^2)^5}$ (ii) $\int_0^1 \frac{dx}{\sqrt{x}(1+x)^2}$

(iii) $\int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$ (iv) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

Sol (iv) :-

Let $f(x) = \frac{1}{(\sqrt{x})(\sqrt{1-x})}$

Both the end points 0 & 1 are the points of infinite discontinuity of f on $[0,1]$.

Now $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx = \int_0^a \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

where $0<a<1$.

To examine the convergence at $x=0$.

Let $I_1 = \int_0^a \frac{dx}{\sqrt{x(1-x)}}$

0 is the only point of infinite discontinuity of f on $[0,a]$.

Let $g(x) = \frac{1}{\sqrt{x}} \ \forall x \in (0,a]$

then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{\sqrt{1-x}}$

$= 1$ (a non-zero finite number)

$\therefore$ By Comparison Test

$I_1 = \int_0^a f(x)\,dx$ & $\int_0^a g(x)\,dx$ are convergent (or) divergent together.

But $\int_0^a g(x)dx = \int_0^a \frac{dx}{(x-0)^{1/2}}$ is convergent ($\because n = \frac{1}{2} < 1$)

$\therefore I_1$ is convergent.

To examine the convergence at $x=1$

Let $I_2 = \int_a^1 \frac{1}{\sqrt{x(1-x)}} dx$

1 is the only point of discontinuity of f on $[a,1]$

Let $g(x) = \frac{1}{\sqrt{1-x}} \ \forall x \in [a,1)$

then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{\sqrt{x}} = 1$

(which is finite and non-zero)

$\therefore$ By Comparison test I_2 & $\int_a^1 g(x)dx$ convergent (or) divergent together.

But $\int_a^1 g(x)dx = \int_a^1 \frac{1}{\sqrt{1-x}} dx$ is convergent ($\because n = \frac{1}{2} < 1$)

$\therefore I_2$ is convergent.

Since I_1 & I_2 are both convergent

$\therefore$ from ①,

$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$ is convergent.

Note: If I_1 or I_2 is divergent then

$\int_0^1 f(x)dx$ is divergent.

$\Longrightarrow$ (i) $\int_2^3 \frac{dx}{(x-2)^{1/4}(3-x)^2}$

(ii) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$ (iii) $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$

(iv) $\int_0^1 \frac{x^n}{1-x}\,dx$ (v) $\int_0^1 \frac{x^n}{1+x}\,dx$

(vi) $\int_1^2 \frac{x^\lambda}{x-1}\,dx$ (vii) $\int_2^3 \frac{x^2+1}{x^2-4}\,dx$

Sol'n: (iv) Let $f(x) = \frac{x^n}{1-x}$

Case(i): if $n \geq 0$ then 1 is the only point of infinite discontinuity on $[0,1]$

Let $g(x) = \frac{1}{1-x}$ $\forall x \in [0,1)$.

Then $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} x^n = 1$ (a non-zero finite number)

$\therefore$ By comparison test

$\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ are convergent (or) divergent together

But $\int_0^1 g(x)dx = \int_0^1 \frac{1}{(1-x)}\,dx$ is divergent ($\because n=1$)

$\therefore \int_0^1 f(x)dx$ is divergent.

Case(ii) If $n<0$;

Let $n=-m$ where $m>0$.

then $f(x) = \frac{1}{x^m(1-x)}$

$\therefore$ 0 & 1 both are the points of infinite discontinuity of f on $[0,1]$.

Now $\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx$ ——(

where $0<a<1$

Next please try yourself.

(v) Here $f(x) = \frac{x^n}{1+x}$ of $n \geq 0$ then $\int_0^1 f(x)$ is a proper integral and hence it is convergent.

if $n<0$ then let $n=-m$; where $m>0$

$\therefore f(x) = \frac{1}{x^m(1+x)}$

Here '0' is the point of infinite discontinuity of on $[0,1]$

Let $g(x) = \frac{1}{x^m}$ proceed Next.

$\longrightarrow$ (i) $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$ (ii) $\int_0^1 \frac{-\log x}{\sqrt{x}}\,dx$

(iii) $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$

Sol'n: (i) Let $f(x) = \frac{\log x}{\sqrt{2-x}}$

Clearly 0 & 2 are only the points of infinite discontinuity of f on $[0,2]$

Now

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx \quad \text{--- (1)}$$

To test the convergence of $\int_0^1 f(x)\,dx$ at $x=0$:

Since $f(x)$ is $-$ve on $[0,1]$

we consider $-f(x)$

Take $g(x) = \frac{1}{x^n}$

Now $\lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{\sqrt{2-x}}$

$= 0$ if $n>0$

$(\because \lim_{x\to 0^+} x^n \log x = 0$ if $n>0.)$

$\therefore$ Taking n b/w 0 & 1 —

$\therefore \int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By Comparison Test

$\int_0^1 f(x)dx$ is Convergent.

To test the convergence of $\int_1^2 f(x)dx$ at $x=2$:—

Take $g(x) = \frac{1}{\sqrt{2-x}}$ $\forall x \in [1,2)$

$\therefore \lim_{x\to 2^-} \frac{f(x)}{g(x)} = \lim_{x\to 2^-} \log x$

(a non-zero finite number).

$\therefore$ By Comparison test

$\int_1^{2} f(x)\,dx$ & $\int_1^2 g(x)dx$

converge or diverge together.

But $\int_1^2 g(x)dx = \int_1^2 \frac{1}{(2-x)^{1/2}}$ is convergent

$(\because n = \frac{1}{2} < 1)$

$\therefore \int_1^2 f(x)dx$ is also convergent.

$\therefore$ from (1) —

$\int_0^2 f(x)dx$ is convergent.

(iii) Let $f(x) = \frac{\sqrt{x}}{\log x}$

1 is the only point of infinite discontinuity of 'f' on $[1,2]$

Take $g(x) = \frac{1}{(x-1)^n}$

$\therefore \lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)^n \sqrt{x}}{\log x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 1^+} \frac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x}$

$= \lim_{x\to 1^+} (x-1)^{n-1}\left[n x^{3/2} + \left(\frac{x-1}{2}\right)\cdot\sqrt{x}\right]$

$= 1$ if $n=1$

$(\because$ a non-zero finite number$)$

$\therefore$ By comparison test

$\int_1^2 f(x)\,dx$ & $\int_1^2 g(x)\,dx$ are convergent (or) divergent together. But

$\int_1^2 g(x)\,dx$ diverges. $(\because n=1)$

$\int_1^2 f(x)\,dx$ diverges.

(i) $\int_0^1 \frac{\log x}{1+x}dx$ (ii) $\int_0^1 \frac{\log x}{1+x^2}dx$

(iii) $\int_0^2 \frac{\log x}{2-x}dx$ (iv) $\int_0^1 \frac{\log x}{1-x^2}dx$

Sol^n: (iii) Let $f(x) = \frac{\log x}{2-x}$

clearly 0 & 2 are only the points of infinite discontinuity of 'f' on [0,2].

$\therefore \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)dx$ —— (1)

To test the convergence of $\int_0^1 f(x)dx$

at $x=0$:—

Since $f(x)$ is $-$ve in (0,1].

we consider $-f(x)$ which is +ve in (0,1].

Take $g(x) = \frac{1}{x^n}$ $\forall x \in (0,1]$

$\therefore \lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{2-x}$

$= 0$ if $n>0$.

$(\because \lim_{x\to 0^+} x^n \log x = 0$ if $n>0)$

Taking n b/w 0 & 1,

$\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 f(x)dx$ is convergent.

To test the convergent of $\int_1^2 f(x)dx$

at $x=2$:

Take $g(x) = \frac{1}{2-x}$

$\therefore \lim_{x\to 2^-} \frac{f(x)}{g(x)} = \lim_{x\to 2^-} \log(x)$

$= \log 2$ (a non-zero finite number)

By Comparison Test.

$\int_1^2 f(x)dx$ & $\int_1^2 g(x)dx$ are convergent (or) divergent together.

but $\int_1^2 f(x)\,dx = \int_1^2 \frac{1}{(2-x)}dx$ is divergent

$(\because n=1)$

$\therefore \int_1^2 f(x)dx$ is divergent.

$\therefore$ from (1),

$\int_0^2 f(x)\,dx$ is divergent.

(iv) Since $\frac{\log x}{1-x^2}$ is $-$ve in (0,1] then

Let $f(x) = \frac{-\log x}{1-x^2}$

Now $\lim_{x\to 1^-} -f(x) = \lim_{x\to 1^-} \frac{-\log x}{1-x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 1^-} \frac{-1/x}{-2x} = \frac{1}{2}$

$\therefore$ 0 is the only point of infinite discontinuity of 'f' on [0,1].

Take $g(x) = \frac{1}{x^n}$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{1-x^2}$

$= 0$ if $n>0$

Take n b/w 0 & 1, the integral $\int_0^1 g(x)dx$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 f(x)dx$

is convergent.

$\int_0^1 \frac{\log x}{1-x^2}\,dx$ is convergent.

→ (i) $\int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ (ii) $\int_0^1 \frac{x^p \log x}{1+x^2}\,dx$

(iii) $\int_0^1 \frac{(x^p+x^{-p})\log(1+x)}{x}\,dx$

(iv) $\int_0^1 x^{n-1}\log x\,dx$

Sol^n (i) Let $f(x) = \frac{x^n \log x}{(1+x)^2}$

$\lim_{x\to 0+} \frac{x^n \log x}{(1+x)^2} = 0$ if $n>0$

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is a proper integral and hence it is convergent.

If $n=0$: Let $f(x) = \frac{-\log x}{(1+x)^2}$

0 is the only point of infinite discontinuity.

Take $g(x) = \frac{1}{x^p}$

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^p \log x}{(1+x)^2}$

$= 0$ if $p>0$

Taking p b/w 0 & 1.

$\int_0^1 g(x)\,dx$ is convergent.

$\Rightarrow \int_0^1 f(x)\,dx$ is convergent.

$\Rightarrow \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is convergent.

If $n<0$, let $n=-m$, $m>0$

Let $f(x) = \frac{-x^n \log x}{(1+x)^2}$

$= \frac{-\log x}{x^m (1+x)^2}$

Take $g(x) = \frac{1}{x^q}$

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^{q-m}\log x}{(1+x)^2}$

$= 0$ if $q-m>0$

Taking $0<q<1$ and also $q-m>0$.
i.e. $q>m$.

$\Rightarrow 0<m<q<1$

$\Rightarrow m<1$

$\Rightarrow -n<1$

$\Rightarrow n>-1$

$\therefore \int_0^1 g(x)\,dx$ is convergent and hence $\int_0^1 f(x)\,dx$ is convergent.

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is convergent for all $n>-1$.

(iii) Let $p>0$ and

$f(x) = \left(x^p + \frac{1}{x^p}\right)\frac{\log(1+x)}{x}$

Here '0' is the point of infinite discontinuity.

Take $g(x) = \frac{1}{x^p}$

Now

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} (x^{2p}+1)\frac{\log(1+x)}{x}$

$$= \lim_{x\to 0+} (x^{2P}+1) \lim_{x\to 0+} \frac{\log(1+x)}{x}$$

$$= 2 \lim_{x\to 0+} \frac{\frac{1}{1+x}}{2}$$

$= (1)(1) = 1$ (a non-zero finite number)

Since $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^P} dx$ is convergent if $0<P<1$.

$\therefore \int_0^1 f(x)dx$ is convergent if $0<P<1$.

If $P=0$

$$f(x) = \frac{2\log(1+x)}{x}$$

Since $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{2\log(1+x)}{x}$ $\left|\frac{0}{0}\text{form}\right.$

$$= \lim_{x\to 0+} \frac{2\left(\frac{1}{1+x}\right)}{(1)}$$

$$= 2$$

$\therefore \int_0^1 f(x)dx$ is a proper integral and hence convergent.

If $P<0$:

Let $g(x) = \frac{1}{x^{-P}}$

$$\therefore \lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \left(1+\frac{1}{x^{2P}}\right)\frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0+} \left(1+\frac{1}{x^{2P}}\right) \left[\because \lim_{x\to 0+} \frac{\log(1+x)}{x} = 1\right]$$

$= 1$ (Since $P<0$).

which is non-zero and finite.

Since $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{-P}} dx$ is convergent if $-P<1$ i.e. if $P>-1$

$\therefore \int_0^1 f(x)dx$ is convergent if $P>-$

$\therefore \int_0^1 f(x)dx$ is convergent if $P>-$

$\therefore -\int_0^1 f(x)$ is convergent if $-1<P<$

(iv) We know that $\lim_{x\to 0+} x^r \log x = 0$ when $r>$

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is a proper integral when $(n-1)>0$ i.e. when $n>1$.

When $n=1$ 0 is the point of infinite discontinuity.

$$\therefore \int_0^1 \log x\,dx = \lim_{\epsilon\to 0+} \int_{0+\epsilon}^1 \log x\,dx$$

$$= \lim_{\epsilon\to 0+} \left[x\log x - x\right]_\epsilon^1$$

$$= \lim_{\epsilon\to 0+} \left[0-1-\epsilon\log\epsilon+\epsilon\right]$$

$= -1$ $\left(\because \lim_{\epsilon\to 0+} \epsilon\log\epsilon = 0\right)$

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is convergent if $n=1$.

When $n<1$:

Let $f(x) = -x^{n-1}\log x$

($\because x^{n-1}\log x$ is $-$ve in $(0,1]$)

Here 0 is the point of infinite discontinuity.

Take $g(x) = \frac{1}{x^P}$

$$\therefore \lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^{(n-1)}\log x}{1/x^P}$$

$$= \lim_{x\to 0+} -x^{P+n-1}\log x$$

$= 0$ if $p+n-1>0$

$= \infty$ if $p+n-1 \leq 0$

Taking $0<p<1$ and $p>1-n$

$\Rightarrow 1-n<p<1$

$\Rightarrow 1-n<1$

$\Rightarrow n>0$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^p}$ is Convergent. $(\because 0<p<1)$

$\therefore \int_0^1 f(x)\,dx$ is also convergent if $n>0$ (ie $0<n<1$)

when $p=1$ and $p \leq 1-n$

$\Rightarrow 1 \leq 1-n$

$\Rightarrow n \leq 0$.

Since $\int_0^1 g(x)\,dx$ is divergent $(\because p=1)$

$\therefore \int_0^1 f(x)\,dx$ is divergent

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is divergent $\forall n \leq 0$.

$\rightarrow$ (i) $\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ (ii) $\int_0^1 \frac{\operatorname{cosec} x}{x}\,dx$

(iii) $\int_0^1 \frac{\sec x}{x}\,dx$

Sol^n: (i) Let $f(x) = \frac{\cos x}{x^n}$

If $n \leq 0$ then the integral

$\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ is a proper integral

If $n>0$ then 0 is the only point of infinite discontinuity of f on $(0, \pi/2]$

Let $g(x) = \frac{1}{x^n}$ $\forall x \in (0, \pi/2]$

$\therefore \lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \cos x$

$= 1$ (a non-zero finite number)

$\therefore$ By Comparison test

$\int_0^{\pi/2} f(x)\,dx$ & $\int_0^{\pi/2} g(x)\,dx$ are Convergent or divergent together.

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^n}\,dx$ is convergent if $n<1$.

$\therefore \int_0^{\pi/2} f(x)$ is convergent.

and $\int_0^{\pi/2} g(x)\,dx$ is divergent if $n \geq 1$

$\therefore \int_0^{\pi/2} f(x)\,dx$ is divergent if $n \geq 1$.

(ii) Let $f(x) = \frac{\operatorname{cosec} x}{x}$

0 is the only point of infinite discontinuity of f on $[0,1]$.

Since $|\sin x| \leq 1$ $\forall x \in \mathbb{R}$.

$\Rightarrow \left|\frac{1}{\sin x}\right| \geq 1$

$\Rightarrow |\operatorname{cosec} x| \geq 1$

$\Rightarrow \left|\frac{\operatorname{cosec} x}{x}\right| \geq \frac{1}{1 \cdot x}$ $\forall x \in (0,1]$

$= \frac{1}{x}$

$\Rightarrow f(x) \geq \frac{1}{x}$ $\forall x \in (0,1]$

Since $\int_0^1 \frac{1}{(x-0)^1}\,dx$ is divergent $(\because n=1)$

$\therefore$ By Comparison test, $\int_0^1 f(x)\,dx$ is divergent.

P-II [2001] Show that $\int_0^{\pi/2} x^m \operatorname{cosec}^n x\,dx$ exists iff $n<m+1$.

Let $f(x) = x^m \operatorname{cosec}^n x$

$= \frac{x^m}{\sin^n x}$

$= \left(\frac{x}{\sin x}\right)^n \cdot x^{(m-n)}$

$\therefore \lim_{x\to 0^+} f(x) = \begin{cases} 0 & \text{if } m-n>0 \\ 1 & \text{if } m-n=0 \\ \infty & \text{if } m-n<0 \end{cases}$

$\Rightarrow$ The given integral is proper integral if $m-n\geq 0$ i.e. if $m\geq n$.

AND the given integral is improper integral if $m-n<0$ i.e. if $m<n$.

$\therefore$ 0 is the only point of infinite discontinuity of f on $[0,\pi/2]$.

when $m-n<0 \Rightarrow n-m>0$.

$\therefore f(x) = \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}} \;\forall x\in(0,\pi/2]$

Let $g(x) = \frac{1}{x^{n-m}} \;\forall x\in(0,\pi/2)$

Now $\lim_{x\to 0^+}\frac{f(x)}{g(x)} = 1$ (a non-zero finite number)

By Comparison test

$\int_0^{\pi/2} f(x)dx$ & $\int_0^{\pi/2} g(x)dx$ are convergent or divergent together.

But $\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2}\frac{1}{x^{n-m}}$ is convergent iff $n-m<1$ i.e. iff $n<m+1$.

$\int_0^{\pi/2} f(x)dx$ is convergent iff $n<m+1$.

$\rightarrow$ Show that $\int_0^{\pi/2}\frac{\sin^m x}{x^n}dx$ exists iff $n<m+1$.

Sol^n: Let $f(x) = \frac{\sin^m x}{x^n}$

$= \left(\frac{\sin x}{x}\right)^m \cdot x^{m-n}$

$\rightarrow$ (i) $\int_0^{\pi/4}\frac{1}{\sqrt{\tan x}}dx$ (ii) $\int_0^1\left(\log\frac{1}{x}\right)^n dx$

Sol^n:- (i) 0 is the only point of infinite discontinuity of f on $[0,\pi/4]$ -

Let $f(x) = \frac{1}{\sqrt{\tan x}}$

$= \sqrt{\frac{\cos x}{\sin x}}$

Let $g(x) = \frac{1}{\sqrt{x}} \;\forall x\in(0,\pi/4]$

then $\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}\cdot\sqrt{\cos x}$

$= \lim_{x\to 0^+}\sqrt{\frac{x}{\sin x}}\cdot\lim_{x\to 0^+}\sqrt{\cos x}$

$= 1$

$\therefore$ By Comparison test

$\int_0^{\pi/4} f(x)dx$ & $\int_0^{\pi/4} g(x)dx$ are convergent or divergent together.

But $\int_0^{\pi/4} g(x)dx = \int_0^{\pi/4}\frac{1}{x^{1/2}}dx$ is convergent ($\because n=\frac{1}{2}<1$).

$\therefore \int_0^{\pi/4} f(x)$ is also convergent.

Let $f(x) = \left(\log \frac{1}{x}\right)$

Since 0 & 1 are the only points of infinite discontinuity on [0,1].

Now we write

$$\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx \quad \text{--- (1)}$$

where $0<a<1$

To test the convergence of $\int_0^a f(x)dx$ at $x=0$.

Now $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \left(\log \frac{1}{x}\right)^n$

$$= \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n<0 \end{cases}$$

∴ The integral is proper if $n\le 0$.

If $n>0$: 0 is the only point of infinite discontinuity.

Let $g(x) = \frac{1}{x^p}$ $\forall x \in (0,a]$

Now $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \left(\log \frac{1}{x}\right)^n \cdot x^p$

$= 0$.

Since $\int_0^a f(x)dx = \int_0^a \frac{1}{x^p}$ is convergent if $0<p<1$.

∴ By Comparison test

$\int_0^a f(x)dx$ is convergent.

To test the convergence of $\int_a^1 f(x)dx$ at $x=1$:

The integral is proper if $n\geq 0$.

If $n<0$ then 1 is the only point of infinite discontinuity.

For $n<0$:

Let $g(x) = \frac{1}{(1-x)^{-n}}$

Now $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \left(\frac{\log 1/x}{1-x}\right)^n$

$$= \left[\lim_{x\to 1-} \frac{\log 1/x}{1-x}\right]^n$$

$$= \lim_{x\to 1-} \frac{x\left(-\frac{1}{x^2}\right)}{-1}$$

$= 1$ (which is non-zero and finite)

But $\int_a^1 g(x)dx = \int_a^1 \frac{1}{(1-x)^{-n}}$ is convergent if $-n<1$ i.e. if $n>-1$

∴ By Comparison test,

$\int_a^1 f_1(x)dx = \int_a^1 \left(\log \frac{1}{x}\right)^n dx$ is Convergent if $-1<n<0$.

∴ From (1),

$\int_0^1 \left(\log \frac{1}{x}\right)^n dx$ is Convergent if $-1<n<0$.

→ Find the values of m & n for which the integral $\int_0^1 e^{-mx} \cdot x^n dx$ converges.

Sol'n: Irrespective of the values of 'm', when $n\geq 0$.

the given integral is proper and hence it is convergent.

when $n<0$:

whatever 'm' may be, '0' is the only point of infinite discontinuity.

Let $f(x) = e^{-mx} \cdot x^n$

Let $g(x) = x^n = \frac{1}{x^{-n}}$

$\therefore \lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-mx} = 1$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{-n}}$ converges if $-n<1$

i.e. if $n>-1$

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ also Converges. if $-1<x<0$.

$\therefore \int_0^1 e^{-mx} x^n\,dx$ converges only for $-1<n<0$.

irrespective of the value of m.

→ Show that $\int_0^{\pi/2} \log \sin x\,dx$ is Convergent.

Sol'n:- Let $f(x) = \log \sin x$

0 is the point of infinite discontinuity.

Since f is $-$ve on $[0, \pi/2]$

We consider

Take $g(x) = \frac{1}{x^n}$; $n>0$

$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} -x^n \log \sin x$

$= \lim_{x\to 0+} \frac{-\log \sin x}{\frac{1}{x^n}}$ $\left[\frac{\infty}{\infty}\right]$

$= \lim_{x\to 0+} \frac{\frac{\cot x}{n}}{x^{n+1}}$

$= \lim_{x\to 0+} \frac{x^n}{n} \cdot \frac{x}{\tan x}$

$= 0$

Taking n b/w 0 & 1,

$\int_0^{\pi/2} g(x)\,dx$ is convergent

By Comparison test.

$\int_0^{\pi/2} -f(x)\,dx$ is convergent.

$\Rightarrow \int_0^{\pi/2} f(x)\,dx$ is convergent.

→ Show that $\int_0^1 \frac{\operatorname{cosec} x}{x^n}\,dx$ is divergent if $n\geq 1$.

Sol'n: Let $f(x) = \frac{\operatorname{cosec} x}{x^n}$

Since $|\sin x| \leq 1 \;\forall x\in\mathbb{R}$

$\Rightarrow |\operatorname{cosec} x| \geq 1 \;\forall x\in\mathbb{R}$

$\Rightarrow \left|\frac{\operatorname{cosec} x}{x^n}\right| \geq \frac{1}{|x^n|}$ for all $x\in(0,1]$.

$\therefore f(x) \geq \frac{1}{x^n} \;\forall x\in(0,1]$.

Since $\int_0^1 \frac{1}{x^n}\,dx$ is divergent if $n\geq 1$.

$\therefore$ By Comparison Test

$\int_0^1 \frac{\text{Cosec}x}{x^n}dx$ is divergent if $n \geq 1$.

$\rightarrow$ Test for Convergence the integral $\int_0^1 \frac{\sin x}{x^{3/2}}dx$.

Sol^n: Let $f(x) = \frac{\sin x}{x^{3/2}}$

$= \left(\frac{\sin x}{x}\right) \cdot \frac{1}{\sqrt{x}}$

$\forall x \in (0,1]$

Let $g(x) = \frac{1}{\sqrt{x}} \;\forall\; x \in [0,1]$.

$\because$ 0 is the only point of infinite discontinuity of f on $[0,1]$.

$\underset{x \to 0_+}{Lt} \frac{f(x)}{g(x)} = \underset{x \to 0_+}{Lt} \frac{\sin x}{x}$

$= 1$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/2}}dx$ is Convergent ($\because n = \frac{1}{2} < 1$)

$\therefore$ By Comparison test

$\int_0^1 f(x)\,dx$ is also Convergent.

* General Test For Convergence (Integrand may change Sign):

This test for convergence of an improper integral (finite limits of integration ~~but~~ discontinuous integrand) hold whether or not the integrand keeps the same sign.

Cauchy's Test:

The improper integral $\int_a^b f(x)\,dx$, 'a' is the only the point of infinite discontinuity, converges at 'a' iff to each $\epsilon>0$, $\exists$ a $\delta>0$ such that $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \epsilon$ $\forall\, 0<\lambda_1,\lambda_2<\delta$

Note: $\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \to 0$ as $\lambda_1,\lambda_2\to 0$.

Definition:

Absolute Convergence: The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

→ Every absolutely convergent integrand is convergent.

i.e. $\int_a^b |f(x)|\,dx$ exists. $\Rightarrow \int_a^b f(x)\,dx$ exists.

Note: (1) The converse of the above is not true.

i.e. Every convergent integral need not be absolutely convergent.

→ A convergent integral which is not absolutely convergent is called a conditional convergent integral.

Problems:

2000 P-I Test the convergence of $\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$

Sol^n: Let $f(x) = \frac{\sin 1/x}{\sqrt{x}}$

clearly f does not keep the same sign in a bounded of '0'.

Now $|f(x)| = \left|\frac{\sin 1/x}{\sqrt{x}}\right|$

$= \frac{|\sin 1/x|}{|\sqrt{x}|} \le \frac{1}{\sqrt{x}}\ \forall x\in($

$[\because |\sin 1/x|\le 1]$

Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent at '0' $(\because n=1/2<1)$

∴ By Comparison Test $\int_0^1 |f(x)|\,dx$ is convergent at '0'.

since absolutely convergence ⇒ convergence

∴ $\int_0^1 f(x)\,dx$ is convergent.

→ show that $\int_0^1 \frac{\sin 1/x}{x^p}$; $p>0$ Converges absolutely for $p<1$.

Sol^n Let $f(x) = \frac{\sin 1/x}{x^p}$; $p>0$

clearly f doesnot keep the same sign in a neighbourhood of '0'.

Now $|f(x)| = \left|\frac{\sin 1/x}{x^p}\right|$

$= \frac{|\sin 1/x|}{x^p} \le \frac{1}{x^p}$

$\forall x \in (0,1]$.

Since $\int_0^1 \frac{1}{x^p}dx$ is convergent iff $p<1$.

By Comparison test $\int_0^1 |f(x)|dx$ is convergent if $p<1$.

$\therefore \int_0^1 f(x)\,dx$ Converges absolutely for $p<1$.

Convergence at ∞, the integrand f being +ve :

→ A necessary and sufficient condition for the convergence of $\int_a^\infty f(x)\,dx$, where $f(x)>0$ $\forall x \in [a,t]$ is that there exists a +ve number M, independent of t, such that $\int_a^t f(x)\,dx < M \;\forall\; t \ge a$.

→ **Comparison test I :-**

If f & g are two functions such that $0 < f(x) \le g(x) \forall x \in [a,\infty)$

then (i) $\int_a^\infty g(x)\,dx$ is convergent

$\Rightarrow \int_a^\infty f(x)dx$ is Convergent.

$\int_a^\infty f(x)\,dx$ is divergent

$\Rightarrow \int_a^\infty g(x)dx$ is divergent.

Comparison test-II :-

If f and g are +ve functions on $[a,\infty)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ then

if (i) 'l' is non-zero and finite, then the two integrals $\int_a^\infty f(x)dx$ & $\int_a^\infty g(x)dx$ converge (or) diverge together.

(ii) If $l=0$ and $\int_a^\infty g(x)\,dx$ Converges then $\int_a^\infty f(x)\,dx$ Converges.

(iii) If $l=\infty$ and $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)\,dx$ diverges.

A useful Comparison integrals

→ The improper integral $\int_a^\infty \frac{dx}{x^n}$, $(a>0)$ is convergent iff $n>1$.

→ $\int_a^\infty \frac{dx}{x^n}$ $(a>0)$ is divergent iff $n \le 1$.

Problems:

Examine the Convergence of the following:

(i) $\int_1^\infty \frac{x^3}{(1+x)^5}dx$ (ii) $\int_1^\infty \frac{dx}{(2+x)\sqrt{x}}$

(iii) $\int_0^\infty \frac{x}{x^3+1}dx$ (iv) $\int_1^\infty \frac{x^3+1}{x^4}dx$

Sol'n :- (i) Let $f(x) = \frac{x^3}{(1+x)^5}$

$= \frac{x^3}{x^5(1+1/x)^5} = \frac{1}{x^2(1+\frac{1}{x})^5}$

Take $g(x) = 1/x^2$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{(1+1/x)^5}$

$= 1$ (finite & non-zero)

$\therefore$ By Comparison test

$\int_0^\infty f(x)dx$ & $\int_0^\infty g(x)dx$ are convergent (or) divergent together.

But $\int_0^\infty g(x)dx = \int_0^\infty \frac{1}{x^2} dx$ is Convergent. ($\because n = 2 \geq 1$)

$\therefore \int_0^\infty f(x)dx$ is convergent.

(i) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx$ $m, n > 0$

(ii) $\int_0^\infty \frac{x^{p-1}}{1+x} dx$

Sol'n :- (i) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \int_0^a \frac{x^{2m}}{1+x^{2n}} dx + \int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$ — (1)

where $0 < a < \infty$

Since $\int_0^a \frac{x^{2m}}{1+x^{2n}} dx$ is a proper integral.

hence it is a Convergent.

The given integral $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx$ is Convergent or divergent according as

$\int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$ is Convergent or divergent

Let $f(x) = \frac{x^{2m}}{1+x^{2n}}$

$= \frac{x^{2m}}{x^{2n}(1+\frac{1}{x^{2n}})}$

$= \frac{x^{2m-2n}}{(1+\frac{1}{x^{2n}})}$

Let $g(x) = x^{2m-2n} = \frac{1}{x^{2n-2m}}$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^{2n}}} = 1$ ($\because n > 0$)

By Comparison test

$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ Convergent or divergent together.

But $\int_a^\infty g(x)dx = \int_a^\infty \frac{dx}{x^{2n-2m}}$ Converges

iff $2n-2m > 1$ i.e. iff $n-m > 1/2$

$\therefore \int_a^\infty f(x)dx$ converges iff $n-m > 1/2$

$\boxed{2003}\boxed{P-I}$

(i) $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx$ (ii) $\int_0^\infty \frac{\sin^2 x}{x^2} dx$

(iii) $\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$ (iv) $\int_e^\infty \frac{dx}{x(\log x)^{n+1}}$

Sol'n : (i) $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx =$

$\int_0^a \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx + \int_a^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx$ — (1)

where $0 < a < \infty$

Since $\int_0^a \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx$ is a proper integral.

and hence it is convergent.

The given integral is $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx$ Convergent (or) divergent according as $\int_a^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} dx$ is Convergent (or) divergent.

Let $f(x) = \dfrac{x\tan^{-1}x}{(1+x^4)^{1/3}}$

$= \dfrac{\tan^{-1}x}{x^{4/3-1}\left(1+\frac{1}{x^4}\right)}$

$= \dfrac{\tan^{-1}x}{x^{1/3}\left(1+\frac{1}{x^4}\right)}$

Let $g(x) = 1/x^{1/3}$

i, Put $\log x = t$

so that $\frac{1}{x}dx = dt$

when $x = e \Rightarrow t = 1$

when $x \to \infty \Rightarrow t \to \infty$

$\therefore \int_e^\infty \frac{dx}{x(\log x)^{3/2}} = \int_1^\infty \frac{dt}{t^{3/2}}$ is convergent

$(\because n = 3/2 > 1)$

$\therefore \int_e^\infty \frac{dx}{x(\log x)^{3/2}}$ is convergent.

→ (i) $\int_0^\infty e^{-x^2}dx$ (ii) $\int_1^\infty x^n e^{-x}dx$

(iii) $\int_1^\infty \frac{\log x}{x^2}dx$ (iv) $\int_0^\infty \frac{\cos x}{1+x^2}dx$

Sol'n: (i) $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ —— (1)

Since $\int_0^1 e^{-x^2}dx$ is a proper integral.

hence it is convergent.

Let us consider $\int_1^\infty e^{-x^2}dx$

we know that $e^{x^2} > x^2 \ \forall x \in \mathbb{R}$

$\Rightarrow e^{-x^2} < \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}dx$ converges at ∞

$\therefore \int_1^\infty e^{-x^2}dx$ also convergent

$\therefore$ from (1), $\int_0^\infty e^{-x^2}dx$ is convergent.

Note: $\int_0^\infty e^{-x^2}dx$ is called the Euler-poisson integral and its value is $\frac{\sqrt{\pi}}{2}$.

(ii) Let $f(x) = x^n e^{-x}$

Take $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0 \ \forall n$

Since $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2}dx$ is convergent

$(\because n = 2 > 1)$

$\therefore$ By comparison test

$\int_1^\infty f(x)dx = \int_1^\infty x^n e^{-x}dx$ is convergent.

(iv) Since $|\cos x| \le 1 \ \forall x \in \mathbb{R}$

$\Rightarrow \left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2} \ \forall x \in [0,\infty)$

Since $\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}dx$ $0<t$

$= \lim_{t\to\infty} \left[\tan^{-1}x\right]_0^t$

$= \lim_{t\to\infty} (\tan^{-1}t)$

$= \pi/2$

$\therefore \int_0^\infty \frac{1}{1+x^2}dx$ is convergent

$\therefore$ By comparison test,

$\int_0^\infty \frac{\cos x}{1+x^2}$ is convergent.

→ Show that

$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ is convergent.

Sol^n: Let $f(x) = \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}$

$$= \left(\frac{1}{1+x} - \frac{1}{e^x}\right)\frac{1}{x}$$

$$= \frac{e^x - 1 - x}{e^x(1+x)x}$$

$$= \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1-x}{x(1+x)e^x}$$

$$= \frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{x(1+x)e^x} > 0 \quad \forall x > 0.$$

Now $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\frac{x}{2!}+\frac{x^2}{3!}+\cdots}{(1+x)e^x}$

$$= 0$$

$\therefore$ 0 is not point of infinite discontinuity.

$$\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx \quad \text{--- (1)}$$

Here

$\int_0^1 f(x)dx$ is proper integral and hence it is convergent.

Now $f(x) = \frac{e^x - 1 - x}{e^x(1+x)x}$

$$= \frac{e^x-1-x}{e^x(1+x)x}\cdot\frac{x}{1+x}\cdot\frac{1}{x^2}$$

Let $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{e^x-1-x}{e^x}\right)\cdots$

$$= \lim_{x\to\infty}\frac{e^x-1-x}{e^x}\cdot\lim_{x\to\infty}\cdots$$

$$= \lim_{x\to\infty}\frac{e^x-1}{e^x}\cdot\lim_{x\to\infty}\cdots$$

$$= \lim_{x\to\infty}(1-e^{-x})$$

$$= 1$$

Since $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2}dx$ is conver[gent] $(\because n = 2 > 1)$

$\therefore$ By comparison test

$\int_1^\infty f(x)dx$ is also convergent

$\therefore$ The given integral

$\int_0^\infty f(x)dx$ is convergent.

* General Test for Convergence at ∞ (Integrand may change sign):-

Cauchy's Test: The improper integral $\int_a^\infty f(x)dx$ converges at ∞ iff to each $\epsilon > 0$, $\exists$ a +ve real number K such that $\left|\int_{t_1}^{t_2} f(x)dx\right| < \epsilon \;\forall\, t_1, t_2 > K$.

Absolute Convergence:

Definition: The improper integral

$\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if $\int_a^\infty |f(x)|\,dx$ is convergent.

→ Every absolutely convergent integral is convergent.

i.e. $\int_a^\infty |f(x)|\,dx$ exists.

$\Rightarrow \int_a^\infty f(x)\,dx$ exists.

Note(1): The converse of above is not true.

→ A convergent integral which is not absolutely convergent is called a conditionally convergent integral.

→ Tests for Convergence of the integral of a product of two functions:

→ Abel's Test:-

If $\int_a^\infty f(x)\,dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \geq a$ then

$\int_a^\infty f(x)g(x)\,dx$ converges at ∞.

→ Dirichlet's Test:

If $\int_a^t f(x)\,dx$ is bounded for all $t \geq a$ and $g(x)$ is a bounded and monotonic function for $x \geq a$, tending to 0 as $x \to \infty$, then

$\int_a^\infty f(x)g(x)\,dx$ is convergent at ∞.

Problems

→ Examine the convergence of the integrals:

(i) $\int_0^\infty \frac{\sin x}{x}\,dx$ (ii) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$

(iii) $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$ $\left[\int_0^\infty = \int_0^1 + \int_1^\infty\right.$ ∵ 0 is the point of infinite discontinuity$\left.\right]$

(iv) $\int_a^\infty \frac{\sin x}{x^m}\,dx$ where a & m are +ve.

(v) $\int_0^\infty \frac{\sin kx}{x}\,dx$ (vi) $\int_1^\infty \frac{\sin x^m}{x^n}\,dx$.

Sol'n: (i) Let $f(x) = \frac{\sin x}{x}$

clearly f does not keep the same sign in $(0,\infty)$

Since $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$

∴ 0 is not a point of infinite discontinuity.

Now $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$ —— (1)

Since $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral and hence it is convergent.

To test the convergence of $\int_1^\infty \frac{\sin x}{x}dx$ at ∞ :-

Let $f(x) = \sin x$; $g(x) = 1/x$

Now
$$\left|\int_1^t f(x)dx\right| = \left|\int_1^t \sin x\, dx\right|$$
$$= |\cos 1 - \cos t|$$
$$\leq |\cos 1| + |\cos t|$$
$$\leq 1+1 \quad (\because |\cos x| \leq 1)$$
$$= 2$$

$\therefore \left|\int_1^t f(x)\,dx\right| \leq 2.$

$\therefore \int_1^t f(x)dx$ is bounded for all $t \geq 1$.

Now $g(x)$ is bounded and monotonically decreasing function and tending to '0' as $x \to \infty$

$\therefore$ By Dirichlet's test

$\int_1^\infty f(x)g(x)dx = \int_1^\infty \frac{\sin x}{x}dx$ is convergent.

From (1), $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

(v) Since $\lim_{x\to 0} \frac{k\sin kx}{kx}$

$= \lim_{x\to 0} k\left(\frac{\sin kx}{kx}\right) = k$

$\therefore 0$ is not point of infinite discontinuity.

(vi) For $m=0$ the given integral $\int_1^\infty \frac{\sin x^m}{x^n}dx$ reduces to $\sin 1 \int_1^\infty \frac{1}{x^n}dx$ is convergent at ∞ of $n>1$.

For $m \neq 0$

put $x^m = t \Rightarrow x = t^{1/m}$

$dx = \frac{1}{m}t^{1/m - 1}dt$

$$\therefore \int_1^\infty \frac{\sin x^m}{x^n}dx = \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{n/m}} \cdot t^{1/m - 1}dt$$
$$= \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{n/m - 1/m + 1}}dt$$
$$= \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{\frac{n-1}{m}+1}}dt$$

Let $f(t) = \sin t$; $g(t) = \frac{1}{t^{\frac{n-1}{m}+1}}$

$\rightarrow$ (i) $\int_0^\infty \sin x^2 dx$ (ii) $\int_0^\infty \frac{x}{1+x^2}\sin x\, dx$

(iii) $\int_0^\infty \cos x^3 dx$ (iv) $\int_0^\infty \frac{\cos x}{\sqrt{x+x^2}}dx$

Sol'n :- we have

(i) $\int_0^\infty \sin x^2 dx = \int_0^1 \sin x^2 dx + \int_1^\infty \sin x^2 dx$ — (1)

Since $\int_0^1 \sin x^2 dx$ is a proper integral.

$\therefore$ It is convergent.

To test the Convergence of $\int_1^\infty \sin x^2 dx$ at ∞ :-

Let $f(x) = 2x\sin x^2$ & $g(x) = \frac{1}{2x}$

Since $\left|\int_1^t f(x)dx\right| = \left|\int_1^t 2x\sin x^2 dx\right|$

$$= \left| \left(-\cos x^2\right)_1^t \right|$$

$$= |\cos 1 - \cos t^2|$$

$$\leq 2$$

$\therefore \int_1^t f(x)\,dx$ is bounded for all $t \geq 1$.

iv) Since $x=0$ is a point of infinite discontinuity.

$\therefore$ we have to test the convergence of the given integral both at 0 & ∞

Now $\int_0^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx = \int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx + \int_a^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$ — ① $a>0$.

To test the convergence of $\int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx$ at '0' :—

Let $f(x) = \frac{\cos x}{\sqrt{x+x^2}} = \frac{\cos x}{\sqrt{x}\sqrt{1+x}}$

$\forall x \in (0,a]$

Let $g(x) = \frac{1}{\sqrt{x}}$ $\forall x \in (0,a]$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$

Since $\int_0^a g(x)\,dx = \int_0^a \frac{1}{\sqrt{x}}\,dx$ is convergent.

$(\because n = \frac{1}{2} < 1)$

By Comparison test.

$\int_0^a f(x)\,dx = \int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx$ is

Convergent.

To test the convergence of

$\int_a^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$ at '∞'

— Let $f(x) = \cos x$

$g(x) = \frac{1}{\sqrt{x+x^2}}$

$\rightarrow$ (i) $\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx$; $a \geq 0$

(ii) $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\,dx$; $a>0$

Sol^n (i): Let $f(x) = \frac{\sin x}{a}$ and

$g(x) = e^{-ax}$; $a \geq 0$.

Since $\int_0^\infty f(x)\,dx$ is convergent.

(By known method)

and $g(x)$ is bounded and monotonically $\downarrow$ function for $x>0$.

$\therefore$ By abels test

$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx$

is convergent

:d function

$\frac{\sin x}{x}\,dx$ is

convergent]

Beta and Gamma Functions

12·1. BETA FUNCTION (M.D.U. 1981; K.U. 1982; G.N.D.U. 1981 S, 82 S; Kanpur 1987; Meerut 1988, 90)

Definition. If $m>0$, $n>0$ then the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, which is obviously a function of m and n, is called a **Beta function** and is denoted by $B(m, n)$.

Thus $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ \forall m>0, n>0$. Beta function is also called the **First Eulerian Integral.**

For example,

(i) $\int_0^1 x^3(1-x)^5\,dx=B(3+1, 5+1)$

$=B(4, 6)$

(ii) $\int_0^1 \sqrt{x}\,(1-x)^3\,dx=B\left(\frac{1}{2}+1, 3+1\right)$

$=B\left(\frac{3}{2}, 4\right)$

(iii) $\int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{2}}\,dx=B\left(-\frac{2}{3}+1, -\frac{1}{2}+1\right)$

$=B\left(\frac{1}{3}, \frac{1}{2}\right)$

(iv) $\int_0^1 x^{-3}(1-x)^5\,dx$ is not a Beta function since $m=-3+1$

$=-2<0.$

12·2. CONVERGENCE OF BETA FUNCTION

Theorem. *Show that* $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ *exists if and only if m and n are both positive.* (M.D.U. 1991)

Proof. The integral is proper if $m\geqslant 1$ and $n\geqslant 1$. 0 is the only point of infinite discontinuity if $m<1$ and 1 is the only point of infinite discontinuity if $n<1$.

For $m<1$ and $n<1$.

Take a number, $\frac{1}{2}$ (say), between 0 and 1 and examine the convergence of the improper integrals

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx, \qquad \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$$

at 0 and 1 respectively.

Convergence at 0, when $m<1$

Let $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{(1-x)^{n-1}}{x^{1-m}}$.

Take $g(x)=\dfrac{1}{x^{1-m}}$

Then $\underset{x\to 0+}{\text{Lt}} \dfrac{f(x)}{g(x)}=\underset{x\to 0+}{\text{Lt}} (1-x)^{n-1}=1$ which is non-zero, finite.

Also $\displaystyle\int_0^{\frac{1}{2}} g(x)\,dx=\int_0^{\frac{1}{2}} \frac{dx}{x^{1-m}}$

is convergent if and only if $1-m<1$ *i.e.*, $m>0$.

$$\left[\because \int_a^b \frac{dx}{(x-a)^n} \text{ is convergent iff } n<1\right]$$

$\therefore$ By comparison test,

$$\int_0^{\frac{1}{2}} f(x)\,dx=\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$$

is convergent at $x=0$ if $m>0$.

Convergence at 1, when $n<1$

Let $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$

Take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\underset{x\to 1-}{\text{Lt}} \dfrac{f(x)}{g(x)}=\underset{x\to 1-}{\text{Lt}} x^{m-1}=1$ which is non-zero, finite.

Also $\displaystyle\int_{\frac{1}{2}}^1 g(x)\,dx=\int_{\frac{1}{2}}^1 \frac{dx}{(1-x)^{1-n}}$ is convergent if and only if $1-n<1$, *i.e.*, $n>0$

$$\left[\because \int_a^b \frac{dx}{(b-x)^n} \text{ is convergent iff } n<1\right]$$

$\therefore$ By comparison test,

$$\int_{\frac{1}{2}}^1 f(x)\,dx=\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$$

is convergent at $x=1$ if $n>0$.

Hence $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m>0$, $n>0$.

12·3. PROPERTIES OF BETA FUNCTION

Property I. Symmetry of Beta function i.e. B(m, n)=B(n, m). *(M.D.U. 1983 ; K.U. 1982 ; G.N.D.U. 1981)*

Proof. By definition,

$$B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0$$

Changing x to $1-x$ $\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$$B(m, n)=\int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx$$

$$=\int_0^1 (1-x)^{m-1}x^{n-1}\,dx$$

$$=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx=B(n, m)$$

Hence $B(m, n)=B(n, m)$.

Property II. If m, n are positive integers, then

$$B(m, n)=\frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

(M.D.U. 1983 S, 84 ; K.U. 1981 ; G.N.D.U. 1982 S)

Proof. $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Integrating by parts

$$=\left[x^{m-1}\cdot\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1 (m-1)x^{m-2}\cdot\frac{(1-x)^n}{n(-1)}\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}(1-x)\,dx$$

$$=\frac{m-1}{n}\int_0^1 [x^{m-2}(1-x)^{n-1}-x^{m-1}(1-x)^{n-1}]\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}dx-\frac{m-1}{n}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\frac{m-1}{n}B(m-1, n)-\frac{m-1}{n}B(m, n)$$

$$\Rightarrow \left(1+\frac{m-1}{n}\right)B(m, n)=\frac{m-1}{n}B(m-1, n)$$

$$\Rightarrow B(m, n)=\frac{m-1}{m+n-1}B(m-1, n) \qquad \ldots(1)$$

Changing m to $(m-1)$, we have

$$B(m-1, n)=\frac{m-2}{m+n-2}\,B(m-2, n)$$

Putting this value of $B(m-1, n)$ in (1), we have

$$B(m, n)=\frac{(m-1)(m-2)}{(m+n-1)(m+n-2)}\,B(m-2, n) \qquad \ldots(2)$$

Generalising from (1) and (2)

$$B(m, n)=\frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)}\,B(1, n) \qquad \ldots(3)$$

But $B(1, n)=\int_0^1 x^0(1-x)^{n-1}\,dx=\left[\frac{(1-x)^n}{n(-1)}\right]_0^1=\frac{1}{n}$

$\therefore$ From (3), we get

$$B(m, n)=\frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)n}$$

$$=\frac{(m-1)!}{(m+n-1)(m+n-2)\ldots\ldots(n+1)n}$$

Multiplying the num. and denom. by $(n-1)!$, we have

$$B(m, n)=\frac{(m-1)!\,(n-1)!}{(m+n-1)(m+n-2)\ldots(n+1)n.(n-1)!}$$

$$=\frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Property III.

$$\mathbf{B(m, n)}=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathbf{dx},\ \mathbf{m>0,\ n>0.}$$

(M.D.U. 1982, Rohilkhand 1984)

Proof. $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $\qquad x=\frac{z}{1+z}$

then $\qquad dx=\frac{(1+z).1-z.1}{(1+z)^2}\,dz=\frac{dz}{(1+z)^2}$

$$1-x=1-\frac{z}{1+z}=\frac{1}{1+z}$$

Also $\quad x(1+z)=z \quad\Rightarrow\quad x=z(1-x)$

or $\qquad z=\frac{x}{1-x}$

when $\qquad x=0,\ z=0.$

When $\qquad x\to 1,\ z\to\infty$

$$\therefore \quad B(m, n)=\int_0^\infty \left(\frac{z}{1+z}\right)^{m-1}\left(\frac{1}{1+z}\right)^{n-1} \frac{dz}{(1+z)^2}$$

$$=\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\, dz$$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

[Second Method]

Put $\dfrac{x}{1+x}=z$ then $x=z(1+x)$

or $x(1-z)=z$

$$\therefore \quad x=\frac{z}{1-z}$$

$$dx=\frac{(1-z).1-z(-1)}{(1-z)^2}\, dz=\frac{dz}{(1-z)^2}$$

$$1+x=1+\frac{z}{1-z}=\frac{1}{1-z}$$

When $x=0$, $z=0$

When $x\to\infty$,

$$z=\underset{x\to\infty}{\text{Lt}} \frac{x}{1+x} \qquad \text{Form } \frac{\infty}{\infty}$$

$$=\underset{x\to\infty}{\text{Lt}} \frac{1}{1}=1$$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$=\int_0^1 \left(\frac{z}{1-z}\right)^{m-1} . (1-z)^{m+n} . \frac{dz}{(1-z)^2}$$

$$=\int_0^1 z^{m-1}(1-z)^{n-1}\, dz=B(m, n)$$

Cor. We have proved that

$$B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$\therefore \quad B(n, m)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

But $B(m, n)=B(n, m)$

$$\therefore \quad B(m, n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Hence $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Express the following integrals in terms of Beta functions:*

(i) $\int_0^1 x^m(1-x^2)^n\,dx$ if $m>-1$, $n>-1$

(ii) $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ (G.N.D.U. 1982 S)

(iii) $\int_0^2 (8-x^3)^{-1/3}\,dx.$

Sol. (i) Put $x^2=z$ i.e. $x=z^{1/2}$ so that $dx=\frac{1}{2}z^{-1/2}\,dz$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore \int_0^1 x^m(1-x^2)^n\,dx=\int_0^1 z^{\frac{m}{2}}(1-z)^n\cdot\frac{1}{2}z^{-1/2}\,dz$$

$$=\frac{1}{2}\int_0^1 z^{\frac{m-1}{2}}(1-z)^n\,dz$$

$$=\frac{1}{2}B\left(\frac{m-1}{2}+1, n+1\right)$$

$$=\frac{1}{2}B\left(\frac{m+1}{2}, n+1\right).$$

(ii) Put $x^5=z$, i.e. $x=z^{1/5}$ so that $dx=\frac{1}{5}z^{-4/5}\,dz$

When $x=0$, $z=0$; when $x=1$, $z=1$.

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx=\int_0^1 x^2(1-x^5)^{-1/2}\,dx$$

$$=\int_0^1 z^{2/5}(1-z)^{-1/2}\,\frac{1}{5}z^{-4/5}\,dz$$

$$=\frac{1}{5}\int_0^1 z^{-2/5}(1-z)^{-1/2}\,dz$$

$$=\frac{1}{5}B\left(-\frac{2}{5}+1, -\frac{1}{2}+1\right)$$

$$=\frac{1}{5}B\left(\frac{3}{5}, \frac{1}{2}\right)$$

so that
when

(iii) Put $x^3=8z$, *i.e.* $x=2z^{1/3}$ so that $dx=\frac{2}{3}z^{-2/3}\,dz$

When $x=0,\ z=0$; when $x=2,\ z=1$

$$\therefore \int_0^2 (8-x^3)^{-1/3}\,dx=\int_0^1 (8-8z)^{-1/3}\cdot\frac{2}{3}\,z^{-2/3}\,dz$$

$$=\int_0^1 \frac{2}{3}\,z^{-2/3}\,\frac{1}{2}\,(1-z)^{-1/3}\,dz$$

$$=\frac{1}{3}\int_0^1 z^{-2/3}(1-z)^{-1/3}\,dz$$

$$=\frac{1}{3}\,B\left(-\frac{2}{3}+1,\ -\frac{1}{3}+1\right)$$

$$=\frac{1}{3}\,B\left(\frac{1}{3},\ \frac{2}{3}\right).$$

Example 2. *Express the following as Beta functions :*

(i) $\int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx$

(ii) $\int_0^1 x^3\,(1-x^2)^{3/2}\,dx$

(iii) $\int_0^2 x^3\,(8-x^3)^{-1/3}\,dx$

(iv) $\int_0^1 x^{m-1}\,(1-x^2)^{n-1}\,dx.$

Sol. (i) Put $x^2=4z$ *i.e.* $x=2z^{1/2}$

so that $dx=z^{-1/2}\,dz$

when $x=0,\ z=0$; when $x=2,\ z=1$

$$\therefore \int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx$$

$$=\int_0^1 2^{1/2}\,z^{1/4}\,(4-4z)^{-1/4}\,z^{-1/2}\,dz$$

$$=\int_0^1 2^{1/2}\,z^{-1/4}\,.\,4^{-1/4}\,(1-z)^{-1/4}\,dz$$

$$=\int_0^1 z^{-1/4}\,(1-z)^{-1/4}\,dz$$

$$=B\left(-\frac{1}{4}+1,\ -\frac{1}{4}+1\right)$$

$$=B\left(\frac{3}{4},\ \frac{3}{4}\right)$$

(ii) Please try yourself. $\left[\textbf{Ans. } B\left(4,\ \frac{5}{2}\right)\right]$

(iii) Please try yourself. $\left[\text{Ans. } \frac{8}{3} B\left(\frac{4}{3}, \frac{2}{3}\right)\right]$

(iv) Please try yourself. $\left[\text{Ans. } \frac{1}{2} B\left(\frac{1}{2} m, n\right)\right]$

Example 3. *Show that*

$$\int_0^p x^m (p^q-x^q)^n \, dx=\frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

if $p>0, q>0, m>-1, n>-1.$ (K.U. 1981 S)

Sol. Put $x^q=p^q \cdot z$, *i.e.* $x=pz^{\frac{1}{q}}$

so that $dx=\frac{p}{q} z^{\frac{1}{q}-1} dz$

When $x=0, z=0$; when $x=p, z=1$

$$\therefore \int_0^p x^m (p^q-x^q)^n \, dx=\int_0^1 p^m z^{\frac{m}{q}} (p^q-p^q z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1} dz$$

$$=\int_0^1 p^m \cdot z^{\frac{m}{q}} \cdot p^{qn} (1-z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1} dz$$

$$=\frac{p^{qn+m+1}}{q} \int_0^1 z^{\frac{m+1}{q}-1} (1-z)^n dz$$

$$=\frac{p^{qn+m+1}}{q} B\left(\frac{m+1}{q}, n+1\right)$$

$$=\frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

$[\because B(m, n)=B(n, m)]$

Example 4. *Prove that*

$$\int_0^a (a-x)^{m-1} \cdot x^{n-1} \, dx=a^{m+n-1} B(m, n).$$

(M.D.U. 1984)

Sol. Please try yourself. (Put $x=az$)

Example 5. *Show that*

$$\int_0^n \left(1-\frac{x}{n}\right)^n \cdot x^{t-1} \, dx=n^t B(t, n+1) \text{ when } t>0, n>-1.$$

Sol. Put $\frac{x}{n}=z$ so that $dx=ndz$

When $x=0, z=0$; when $x=n, z=1$

$$\therefore \int_0^n \left(1-\frac{x}{n}\right)^n . x^{t-1}\, dx = \int_0^1 (1-z)^n .(nz)^{t-1} . n\, dz$$

$$= n^t \int_0^1 z^{t-1} (1-z)^n\, dz$$

$$= n^t\, B(t, n+1).$$

Example 6. *Show that if $m>0$, $n>0$, then*

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1}\, dx = (b-a)^{m+n-1}\, B(m, n).$$

Sol. Put $x=a+(b-a)\,z$; so that $dx=(b-a)\,dz$

When $x=a$, $z=0$; when $x=b$, $z=1$

$$\therefore \int_a^b (x-a)^{m-1} (b-x)^{n-1}\, dx$$

$$= \int_0^1 [(b-a)z]^{m-1} [b-a-(b-a)z]^{n-1} (b-a)\, dz$$

$$= \int_0^1 (b-a)^{m-1} . z^{m-1} . (b-a)^{n-1} . (1-z)^{n-1} . (b-a)\, dz$$

$$= (b-a)^{m+n-1} \int_0^1 z^{m-1} (1-z)^{n-1}\, dz$$

$$= (b-a)^{m+n-1}\, B(m, n).$$

Example 7. *Show that*

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{(a+b)^m . a^n} B(m, n).$$

(K.U. 1981)

Sol. Put $\dfrac{x}{a+bx} = \dfrac{z}{a+b . 1}$

so that $\dfrac{(a+bx).1 - x.b}{(a+bx)^2}\, dx = \dfrac{dz}{a+b}$

or $\dfrac{a}{(a+bx)^2}\, dx = \dfrac{dz}{a+b}$ $\quad\therefore\quad \dfrac{dx}{(a+bx)^2} = \dfrac{dz}{a(a+b)}$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} . \left(\frac{1-x}{a+bx}\right)^{n-1} . \frac{1}{(a+bx)^2}\, dx$$

$$= \int_0^1 \left(\frac{z}{a+b}\right)^{m-1} . \left(\frac{1-z}{a}\right)^{n-1} . \frac{dz}{a(a+b)}$$

$$\left[\because (a+b)x = az + bxz \text{ or } x = \frac{az}{a+b-bz} \quad \therefore \frac{1-x}{a+bx} = \frac{1-z}{a}\right]$$

$$=\frac{1}{(a+b)^m \cdot a^n}\int_0^1 z^{m-1}(1-z)^{n-1}\,dz$$

$$=\frac{1}{(a+b)^m \cdot a^n}\,B(m, n).$$

Example 8. *Show that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx=\frac{B(m, n)}{a^n(1+a)^m}$$

Sol. Please try yourself. (same as Ex. 7 with $b=1$).

Example 9. *Express* $\int_0^1 x^m(1-x^n)^p\,dx$ *in terms of Beta function and hence evaluate* $\int_0^1 x^5(1-x^3)^3\,dx$. (G.N.D.U. 1981)

Sol. Put $x^n=z$, *i.e.* $x=z^{\frac{1}{n}}$

so that $dx=\frac{1}{n}z^{\frac{1}{n}-1}\,dz$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore \int_0^1 x^m(1-x^n)^p\,dx=\int_0^1 z^{\frac{m}{n}}(1-z)^p\cdot\frac{1}{n}z^{\frac{1}{n}-1}\,dz$$

$$=\frac{1}{n}\int_0^1 z^{\frac{m+1}{n}-1}(1-z)^p\,dz$$

$$=\frac{1}{n}B\left(\frac{m+1}{n}, p+1\right) \qquad \ldots(1)$$

Comparing $\int_0^1 x^5(1-x^3)^3\,dx$ with $\int_0^1 x^m(1-x^n)^p\,dx$,

we have $m=5, n=3, p=3$

$$\therefore \text{ From (1), } \int_0^1 x^5(1-x^3)^3\,dx=\frac{1}{3}B\left(\frac{5+1}{3}, 3+1\right)$$

$$=\frac{1}{3}B(2, 4)=\frac{1}{3}\int_0^1 x^1(1-x)^3\,dx$$

$$=\frac{1}{3}\int_0^1 (1-x)[1-(1-x)]^3\,dx=\frac{1}{3}\int_0^1 (1-x)x^3\,dx$$

$$=\frac{1}{3}\int_0^1 (x^3-x^4)\,dx=\frac{1}{3}\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1$$

$$=\frac{1}{3}\left[\frac{1}{4}-\frac{1}{5}\right]=\frac{1}{60}$$

Example 10. *Prove that*

$$\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta=\frac{B(m,n)}{2a^m b^n}.$$

Sol. Let $I=\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta$

$$=\int_0^{\pi/2} \frac{\cos^{2m-2}\theta \sin^{2n-2}\theta \cdot \cos\theta\sin\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta$$

$$=\int_0^{\pi/2} \frac{(\cos^2\theta)^{m-1}(\sin^2\theta)^{n-1}\cos\theta\sin\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta$$

Put $\sin^2\theta=x$ so that $2\sin\theta\cos\theta\, d\theta=dx$

and $\cos^2\theta=1-\sin^2\theta=1-x$

When $\theta=0, x=0$; when $\theta=\pi/2, x=1$

$$\therefore \quad I=\int_0^1 \frac{(1-x)^{m-1}x^{n-1}\cdot\frac{1}{2}\,dx}{[a(1-x)+bx]^{m+n}}$$

$$=\frac{1}{2}\int_0^1 \frac{(1-x)^{m-1}x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

Put $\frac{x}{a+(b-a)x}=\frac{z}{a+(b-a)\cdot 1}=\frac{z}{b}$

$$\therefore \quad \frac{[a+(b-a)x]\cdot 1-x\cdot(b-a)}{[a+(b-a)x]^2}\,dx=\frac{dz}{b}$$

$$\Rightarrow \quad \frac{dx}{[a+(b-a)x]^2}=\frac{dz}{ab}$$

When $x=0, z=0$; when $x=1, z=1$

Also $\frac{x}{a+(b-a)x}=\frac{z}{b} \Rightarrow bx=az+(b-a)xz$

$$\Rightarrow [b-(b-a)z]=azx \Rightarrow x=\frac{az}{b-(b-a)z}$$

$$1-x=1-\frac{az}{b-(b-a)z}=\frac{b(1-z)}{b-(b-a)z}$$

$$a+(b-a)x=\frac{bx}{z}=\frac{abz}{b-(b-a)z}$$

so that $\frac{1-x}{a+(b-a)x}=\frac{1-z}{az}$

$$\therefore \quad I=\frac{1}{2}\int_0^1 \frac{(1-x)^{m-1}x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

$$=\frac{1}{2}\int_0^1 \left[\frac{1-x}{a+(b-a)x}\right]^{m-1}\left[\frac{x}{a+(b-a)x}\right]^{n-1}\cdot\frac{dx}{[a+(b-a)x]^2}$$

$$=\frac{1}{2}\int_0^1 \left(\frac{1-z}{az}\right)^{m-1}\left(\frac{z}{b}\right)^{n-1}\frac{dz}{ab}$$

$$=\frac{1}{2a^m b^n}\int_0^1 z^{n-1}(1-z)^{m-1}dz=\frac{B(n, m)}{2a^m b^n}$$

$$=\frac{B(m, n)}{2a^m b^n} \qquad [\because \ B(n, m)=B(m, n)]$$

Example 11. *Prove that if p, q are positive then*

(i) $\dfrac{B(p, q+1)}{q}=\dfrac{B(p+1, q)}{p}=\dfrac{B(p, q)}{p+q}$

(M.D.U. 1982 S ; K.U. 1983)

(ii) $B(p, q)=B(p+1, q)+B(p, q+1)$.

Sol. (i) $\dfrac{B(p, q+1)}{q}=\dfrac{1}{q}\displaystyle\int_0^1 x^{p-1}(1-x)^q\, dx$

$$=\frac{1}{q}\int_0^1 (1-x)^q.\ x^{p-1}\, dx$$

Integrating by parts

$$=\frac{1}{q}\left[\left\{(1-x)^q.\frac{x^p}{p}\right\}_0^1-\int_0^1 q(1-x)^{q-1}(-1).\ \frac{x^p}{p}\ dx\right]$$

$$=\frac{1}{q}\cdot\frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\, dx \qquad \text{...(I)}$$

$$=\frac{B(p+1, q)}{p} \qquad \text{...(II)}$$

Also from (I)

$$\frac{B(p, q+1)}{q}=\frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\, dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}.\ x(1-x)^{q-1}\ dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}[1-(1-x)]\ (1-x)^{q-1}\ dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\ dx-\frac{1}{p}\int_0^1 x^{p-1}(1-x)^q\ dx$$

$$=\frac{1}{p}\ B(p, q)-\frac{1}{p}\ B(p, q+1)$$

or $$\frac{B(p, q+1)}{q}+\frac{B(p, q+1)}{p}=\frac{B(p, q)}{p}$$

or $$\frac{p+q}{pq}\ B(p, q+1)=\frac{1}{p}\ B(p, q)$$

$$\therefore \quad \frac{B(p, q+1)}{q}=\frac{B(p, q)}{p+q} \qquad \text{...(III)}$$

From (II) and (III)

$$\frac{B(p, q+1)}{q}=\frac{B(p+1, q)}{p}=\frac{B(p, q)}{p+q}$$

Note. For another method, see Gamma function.

(*ii*) R.H.S. $=B(p+1, q)+B(p, q+1)$

$$=\int_0^1 x^p(1-x)^{q-1}\,dx+\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$=\int_0^1 [x^p(1-x)^{q-1}+x^{p-1}(1-x)^q]\,dx$$

$$=\int_0^1 x^{p-1}(1-x)^{q-1}[x+(1-x)]\,dx$$

$$=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$=B(p, q)=$L.H.S.

Example 12. *Prove that*

$$\frac{B(m+1, n)}{B(m, n)}=\frac{m}{m+n},\ m>0,\ n>0.$$

Sol. $B(m+1, n)=\int_0^1 x^m(1-x)^{n-1}\,dx$

Integrating by parts

$$=\left[x^m\cdot\frac{(1-x)^n}{-n}\right]_0^1-\int_0^1 mx^{m-1}\cdot\frac{(1-x)^n}{-n}\,dx$$

$$=\frac{m}{n}\int_0^1 x^{m-1}(1-x)^n\,dx$$

$$=\frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}(1-x)\,dx$$

$$=\frac{m}{n}\left[\int_0^1 x^{m-1}(1-x)^{n-1}dx-\int_0^1 x^m(1-x)^{n-1}\,dx\right]$$

$$=\frac{m}{n}\,[B(m, n)-B(m+1, n)]$$

$$\Rightarrow\quad \left(1+\frac{m}{n}\right)B(m+1, n)=\frac{m}{n}\,B(m, n)$$

$$\Rightarrow\quad \frac{B(m+1, n)}{B(m, n)}=\frac{m}{m+n}.$$

Note. For another method, see Gamma function.

Example 13. *Using the property $B(m, n)=B(n, m)$, evaluate*

$$\int_0^1 x^3(1-x)^{4/3}\,dx.$$

Sol. $\int_0^1 x^3(1-x)^{4/3}\,dx = B\left(3+1, \frac{4}{3}+1\right)$

$$= B\left(4, \frac{7}{3}\right) = B\left(\frac{7}{3}, 4\right)$$

$$= \int_0^1 x^{4/3}(1-x)^3\,dx$$

$$= \int_0^1 x^{4/3}(1-3x+3x^2-x^3)\,dx$$

$$= \int_0^1 (x^{4/3}-3x^{7/3}+3x^{10/3}-x^{13/3})\,dx$$

$$= \left[\frac{x^{7/3}}{\frac{7}{3}} - 3\cdot\frac{x^{10/3}}{\frac{10}{3}} + 3\cdot\frac{x^{13/3}}{\frac{13}{3}} - \frac{x^{16/3}}{\frac{16}{3}}\right]_0^1$$

$$= \frac{3}{7} - \frac{9}{10} + \frac{9}{13} - \frac{3}{16} = \frac{243}{7280}$$

Example 14. *Prove that*

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta. \quad \text{(M.D.U. 1983)}$$

Sol. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$

$$\therefore \quad B(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,.\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\,\cos^{2n-2}\theta\,.\,\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$$

Example 15. *Show that*

$$\int_0^{\pi/2} \sin^p\theta\,\cos^q\theta\,d\theta = \frac{1}{2}\,B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

where $p > -1$, $q > -1$. *Deduce that*

$$\int_0^2 x^4(8-x^3)^{-1/3}\,dx = \frac{16}{3}\,B\left(\frac{5}{3}, \frac{2}{3}\right).$$

Sol. Put $\sin^2\theta = z$ so that $2\sin\theta\cos\theta\,d\theta = dz$

When $\theta=0$, $z=0$; when $\theta=\frac{\pi}{2}$, $z=1$

Also $\cos^2\theta=1-\sin^2\theta=1-z$

$$\therefore \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta=\int_0^{\pi/2}(\sin^{p-1}\theta\cos^{q-1}\theta)\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}(\sin^2\theta)^{\frac{p-1}{2}}(\cos^2\theta)^{\frac{q-1}{2}}\cdot\sin\theta\cos\theta\,d\theta$$

$$=\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\cdot\tfrac{1}{2}\,dz$$

$$=\frac{1}{2}\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$=\frac{1}{2}\,B\left(\frac{p-1}{2}+1,\ \frac{q-1}{2}+1\right)$$

$$=\frac{1}{2}\,B\left(\frac{p+1}{2},\ \frac{q+1}{2}\right) \quad ...(I)$$

Second Part. Put $x^3=8z$ *i.e.*, $x=2z^{1/3}$

so that $dx=\frac{2}{3}z^{-2/3}\,dz$

When $x=0,\ z=0$; when $x=2,\ z=1$

$$\therefore \int_0^2 x^4(8-x^3)^{-1/3}\,dx=\int_0^1 16z^{4/3}(8-8z)^{-1/3}\cdot\frac{2}{3}z^{-2/3}\,dz$$

$$=\int_0^1\frac{32}{3}\times 8^{-1/3}z^{2/3}(1-z)^{-1/3}\,dz$$

$$=\frac{32}{3\times 2}\int_0^1 z^{2/3}(1-z)^{-1/3}\,dz$$

$$=\frac{16}{3}\int_0^{\pi/2}\sin^{4/3}\theta\,(\cos^2\theta)^{-1/3}\times 2\sin\theta\cos\theta\,d\theta$$

where $z=\sin^2\theta$

$$=\frac{32}{3}\int_0^{\pi/2}\sin^{7/3}\theta\cos^{1/3}\theta\,d\theta$$

$$=\frac{32}{3}\cdot\frac{1}{2}\,B\left(\frac{\frac{7}{3}+1}{2},\ \frac{\frac{1}{3}+1}{2}\right)$$

| Here $p=\frac{7}{3}$, $q=\frac{1}{3}$ [using I]

$$=\frac{16}{3}\,B\left(\frac{5}{3},\ \frac{2}{3}\right)$$

Example 16. *By putting* $\frac{x}{1-x}=\frac{at}{1-t}$, *where the constant* a *is suitably selected, show that*

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx=\frac{1}{9^{1/3}}B\left(\frac{2}{3},\frac{1}{3}\right).$$

Sol. $\frac{x}{1-x}=\frac{at}{1-t}$

$\Rightarrow \quad x-tx=at-atx$

$\Rightarrow \quad x[1-(1-a)t]=at$

$\Rightarrow \quad x=\frac{at}{1-(1-a)t}$

$\therefore \quad 1-x=1-\frac{at}{1-(1-a)t}=\frac{1-t}{1-(1-a)t}$

$1+2x=1+\frac{2at}{1-(1-a)t}=\frac{1-(1-3a)t}{1-(1-a)t}$

Also $dx=\frac{[1-(1-a)t]\cdot a-at[-(1-a)]}{[1-(1-a)t]^2}\,dt$

$=\frac{a\,dt}{[1-(1-a)t]^2}$

when $x=0$, $t=0$; when $x=1$, $\frac{at}{1-(1-a)t}=1$ so that $t=1$

$\therefore \int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$

$$=\int_0^1\left[\frac{at}{1-(1-a)t}\right]^{-\frac{1}{3}}\cdot\left[\frac{1-t}{1-(1-a)t}\right]^{-\frac{2}{3}}\times\left[\frac{1-(1-3a)t}{1-(1-a)t}\right]^{-1}\cdot\frac{a\,dt}{[1-(1-a)t]^2}$$

$$=a^{2/3}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}[1-(1-3a)t]^{-1}\,dt$$

Choosing $1-3a=0$ i.e. $a=\frac{1}{3}$, we get

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$=\left(\frac{1}{3}\right)^{2/3}\int_0^1 t^{-1/3}(1-t)^{-2/3}\,dt$$

$$=\frac{1}{9^{1/3}}B\left(-\frac{1}{3}+1,-\frac{2}{3}+1\right)$$

$$=\frac{1}{9^{1/3}}B\left(\frac{2}{3},\frac{1}{3}\right).$$

Example 17. *Show that*

$$\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx=\frac{1}{4(2)^{1/4}}\,B\left(\frac{7}{4},\frac{1}{4}\right).$$

Sol. Put $\frac{1-x^4}{1+x^4}=z$ so that $x^4=\frac{1-z}{1+z}$

i.e.
$$x=\left(\frac{1-z}{1+z}\right)^{1/4}$$

$$dx=\frac{1}{4}\left(\frac{1-z}{1+z}\right)^{-3/4}\times\frac{(1+z)(-1)-(1-z).1}{(1+z)^2}\,dz$$

$$=\frac{1}{4}\left(\frac{1+z}{1-z}\right)^{3/4}\cdot\frac{-2}{(1+z)^2}\,dz$$

$$=\frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

Also $1-x^4=1-\frac{1-z}{1+z}=\frac{2z}{1+z}$

$$1+x^4=1+\frac{1-z}{1+z}=\frac{2}{1+z}$$

When $x=0$, $z=1$;

When $x=1$, $z=0$

$$\therefore \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx=\int_1^0 \frac{\left(\frac{2z}{1+z}\right)^{3/4}}{\left(\frac{2}{1+z}\right)^2}\times\frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

$$=\int_0^1 \frac{1}{4(2)^{1/4}}\,z^{3/4}(1-z)^{-3/4}\,dz$$

$$=\frac{1}{4(2)^{1/4}}\,B\left(\frac{7}{4},\frac{1}{4}\right).$$

Example 18. *Show that*

$$\int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}\,B\left(\frac{1}{2}n,\frac{1}{2}n\right),$$

if $a^2>b^2$

Sol. Let $I=\int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx$

$$=\int_0^{\pi} \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{n-1} dx}{\left[a+b\left(1-2\sin^2\frac{x}{2}\right)\right]^n}$$

$$=2^{n-1}\int_0^{\pi} \frac{\sin^{n-1}\frac{x}{2}\cos^{n-1}\frac{x}{2}}{\left(a+b-2b\sin^2\frac{x}{2}\right)^n}\, dx$$

Put $\frac{x}{2}=\theta$ then $dx=2d\theta$.

when $x=0$, $\theta=0$; when $x=\pi$, $\theta=\frac{\pi}{2}$

$$\therefore \quad I=2^{n-1}\int_0^{\pi/2} \frac{\sin^{n-1}\theta\cos^{n-1}\theta}{(a+b-2b\sin^2\theta)^n}\cdot 2d\theta$$

$$=2^{n-1}\int_0^{\pi/2} \frac{\sin^{n-2}\theta\cos^{n-2}\theta . 2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}\, d\theta$$

Put $\sin^2\theta=t$ so that $2\sin\theta\cos\theta\, d\theta=dt$

when $\theta=0$, $t=0$, when $\theta=\frac{\pi}{2}$, $t=1$.

$$\therefore \quad I=2^{n-1}\int_0^{\pi/2} \frac{(\sin^2\theta)^{\frac{n-2}{2}}(1-\sin^2\theta)^{\frac{n-2}{2}} . 2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}\, d\theta$$

$$=2^{n-1}\int_0^1 \frac{t^{\frac{n-2}{2}}(1-t)^{\frac{n-2}{2}}}{(a+b-2bt)^n}\, dt$$

Put $\frac{1-t}{a+b-2bt}=\frac{z}{a+b}$ i.e. $t=\frac{(a+b)(1-z)}{a+b-2bz}$

$$dt=\frac{a+b}{(a+b-2bz)^2}\left[(a+b-2bz)(-1)-(1-z)(-2b)\right] dz$$

$$=\frac{(a+b)(-a+b)}{(a+b-2bz)^2}dz=-\frac{a^2-b^2}{(a+b-2bz)^2}\, dz$$

Also $1-t=1-\frac{(a+b)(1-z)}{a+b-2bz}=\frac{(a-b)z}{a+b-2bz}$

and $a+b-2bt=a+b-\frac{2b(a+b)(1-z)}{a+b-2bz}$

$$=\frac{a^2-b^2}{a+b-2bz}$$

When $t=0$, $z=1$, when $t=1$, $z=0$

$$\therefore \; I = -2^{n-1}\int_1^0 \frac{\left[\dfrac{(a+b)(1-z)}{a+b-2bz}\right]^{\frac{n-2}{2}}\left[\dfrac{(a-b)z}{a+b-2bz}\right]^{\frac{n-2}{2}}}{\left[\dfrac{a^2-b^2}{a+b-2bz}\right]^{n}} \times \frac{a^2-b^2}{(a+b-2bz)^2}\, dz$$

$$=2^{n-1}\int_0^1 \frac{(a^2-b^2)^{\frac{n-2}{2}}(1-z)^{\frac{n-2}{2}} z^{\frac{n-2}{2}}}{(a^2-b^2)^{n-1}}\, dz$$

$$=2^{n-1}\int_0^1 \frac{z^{n/2-1}(1-z)^{n/2-1}}{(a^2-b^2)^{n/2}}\, dz$$

$$=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}\, B\left(\frac{n}{2}, \frac{n}{2}\right).$$

Example 19. *Prove that* $\displaystyle\int_0^\infty \frac{t^3}{(1+t)^7}\, dt = \frac{1}{60}$

Sol. $$\int_0^\infty \frac{t^3}{(1+t)^7}\, dt = \int_0^\infty \frac{t^{4-1}}{(1+t)^{4+3}}\, dt$$

$$\left[\text{Form} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = B(m, n)\right]$$

$$=B(4, 3) = \int_0^1 t^3(1-t)^2\, dt$$

$$=\int_0^1 t^3(1-2t+t^2)dt = \int_0^1 (t^3-2t^4+t^5)\, dt$$

$$=\left[\frac{t^4}{4}-\frac{2t^5}{5}+\frac{t^6}{6}\right]_0^1$$

$$=\frac{1}{4}-\frac{2}{5}+\frac{1}{6}=\frac{1}{60}.$$

Example 20. *Express* $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx$ *in terms of Beta function, where* $m>0, n>0$; $a>0, b>0$.

Sol. Put $bx=az$ or $x=\dfrac{az}{b}$ so that $dx=\dfrac{a}{b}\, dz$.

When $x=0$, $z=0$ and when $x \to \infty$, $z \to \infty$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \int_0^\infty \left(\frac{az}{b}\right)^{m-1} \cdot \frac{1}{(a+az)^{m+n}} \cdot \frac{a}{b}\, dz$$

$$=\int_0^\infty \frac{a^{m-1}\cdot z^{m-1}\cdot a}{b^{m-1}\cdot a^{m+n}(1+z)^{m+n}\cdot b}\, dz$$

$$=\frac{1}{a^n b^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz,$$

$$=\frac{1}{a^n b^m}\,B(m, n).$$

Example 21. *Show that* $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=2B(m, n),$ *where* $m>0, n>0.$ (M.D.U. 1981 S)

Sol. $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=B(m, n)$

Also $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx=B(m, n)$

Adding $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=2B(m, n).$

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Example 22. *Prove that* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=B(m, n),$ *where m, n are both positive.*

Sol. $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad ...(i)$$

In the second integral on R.H.S. of (*i*), put $x=\frac{1}{t}$, so that

$$dx=-\frac{1}{t^2}\,dt$$

When $x=1$, $t=1$; when $x\to\infty$, $t=0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right)dt$$

$$=\int_0^1 \frac{1}{t^{m-1}}\cdot\frac{t^{m+n}}{(1+t)^{m+n}}\cdot\frac{1}{t^2}\,dt$$

$$=\int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx=\int_a^b f(z)\,dz\right]$$

$\therefore$ From (*i*),

$$B(m, n)=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Example. 23. *For $m>0$, $n>0$, show that*

$$\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx=0.$$

Sol. $$\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx-\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$=\text{B}(m,n)-\text{B}(n,m)=0.$$

12·4. GAMMA FUNCTION

(*M.D.U. 1981 ; G.N.D.U. 1981 S ; Kanpur 1987 ; Meerut 1988, 90*)

Definition. If $n>0$, then the integral $\int_0^\infty x^{n-1}e^{-x}\,dx$, which is obviously a function of n, is called a **Gamma function** and is denoted by $\Gamma(n)$.

Thus $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx,\ \forall\ n>0$

Gamma function is also called the **Second Eulerian Integral.**

For example,

(*i*) $\int_0^\infty x^3 e^{-x}\,dx=\Gamma(3+1)=\Gamma(4)$

(*ii*) $\int_0^\infty x^{2/3} e^{-x}\,dx=\Gamma\left(\frac{2}{3}+1\right)=\Gamma\left(\frac{5}{3}\right).$

12·5. CONVERGENCE OF GAMMA FUNCTION

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Theorem. *Show that* $\int_0^\infty x^{n-1}e^{-x}\,dx$ *converges iff* $n>0$.

(M.D.U. 1990 ; Meerut 1981)

Proof. If $n\geq 1$, the integrand $x^{n-1}e^{-x}$ is continuous at $x=0$.

If $n<1$, the integrand $\frac{e^{-x}}{x^{1-n}}$ has infinite discontinuity at $x=0$.

Thus we have to examine the convergence at 0 and ∞ both. Consider any positive number, say 1, and examine the convergence of

$$\int_0^1 x^{n-1}e^{-x}\,dx \quad\text{and}\quad \int_1^\infty x^{n-1}e^{-x}\,dx$$

at 0 and ∞ respectively.

Convergence at 0, when $n<1$

Let $f(x)=\frac{e^{-x}}{x^{1-n}}$

Take $g(x)=\dfrac{1}{x^{1-n}}$

Then $\underset{x\to 0+}{\text{Lt}}\ \dfrac{f(x)}{g(x)}=\underset{x\to 0+}{\text{Lt}}\ e^{-x}=1$

which is non-zero, finite.

Also $\displaystyle\int_0^1 g(x)\,dx=\int_0^1 \frac{dx}{x^{1-n}}$

is convergent iff $1-n<1$ i.e. $n>0$

$\therefore$ By comparison test

$$\int_0^1 f(x)dx=\int_0^1 \frac{e^{-x}}{x^{1-n}}\,dx=\int_0^1 x^{n-1}e^{-x}dx$$

is convergent at $x=0$ if $n>0$.

Convergence at ∞

We know that $e^x>x^{n+1}$ whatever value n may have

$\therefore$ $e^{-x}<x^{-n-1}$

and $x^{n-1}e^{-x}<x^{n-1}\cdot x^{-n-1}=\dfrac{1}{x^2}$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent at ∞.

$\therefore$ $\displaystyle\int_1^\infty x^{n-1}e^{-x}\,dx$ is convergent at ∞ for every value of n.

Now $\displaystyle\int_0^\infty x^{n-1}e^{-x}\,dx=\int_0^1 x^{n-1}e^{-x}\,dx+\int_1^\infty x^{n-1}e^{-x}\,dx$

$\therefore$ $\displaystyle\int_0^\infty x^{n-1}e^{-x}dx$ converges iff $n>0$.

12·6. RECURRENCE FORMULA FOR GAMMA FUNCTION

Prove that $\Gamma(n)=(n-1)\,\Gamma(n-1)$, when $n>1$.

Proof. By def. $\Gamma(n)=\displaystyle\int_0^\infty x^{n-1}e^{-x}\,dx$

Integrating by parts

$$=\left[x^{n-1}\cdot\frac{e^{-x}}{-1}\right]_0^\infty-\int_0^\infty (n-1)\,x^{n-2}\cdot\left(\frac{e^{-x}}{-1}\right)dx$$

$$=-\left[\underset{x\to\infty}{\text{Lt}}\ \frac{x^{n-1}}{e^x}-0\right]+(n-1)\int_0^\infty e^{-x}\cdot x^{n-2}\,dx$$

$$=(n-1)\int_0^\infty e^{-x}\cdot x^{n-2}\,dx\ \left[\because\ \underset{x\to\infty}{\text{Lt}}\ \frac{x^{n-1}}{e^x}=0 \text{ for } n>0\right]$$

$$=(n-1)\,\Gamma(n-1)$$

Hence $\Gamma(n)=(n-1)\,\Gamma(n-1)$.

Cor. If n is a positive integer, then

$$\Gamma(n)=(n-1)\,!$$

When n is a +ve integer, then by repeated application of above formula, we get

$$\Gamma(n)=(n-1)\,\Gamma(n-1)$$
$$=(n-1)\,.\,(n-2)\,\Gamma(n-2)$$
$$=(n-1)(n-2)\,.\,(n-3)\,\Gamma(n-3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=(n-1)(n-2)\ldots\ldots 1\,\Gamma(1)$$
$$=(n-1)\,!\,\Gamma(1)$$

But $\quad \Gamma(1)=\int_0^\infty x^0 e^{-x}\,dx$ (By def.)

$$=\int_0^\infty e^{-x}\,dx=\left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$=-\left[\underset{x\to\infty}{\text{Lt}}\frac{1}{e^x}-e^0\right]=-[0-1]=1.$$

Hence $\quad \Gamma(n)=(n-1)\,!$ when n is a +ve integer.

Note. (*i*) If n is a +ve fraction, then

$\Gamma(n)=(n-1)\times$ go on decreasing by 1 ....,

the series of factors being continued so long as the factors remain positive, the last factor being Γ (last factor).

For example, $\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$$=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

(*ii*) If n is a +ve integer, $\Gamma(n)=(n-1)\,!$

12·7. RELATION BETWEEN BETA AND GAMMA FUNCTIONS

To show that

$$B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \text{ where } m>0,\ n>0.$$

(*Agra 1984 ; Meerut 1986, 87, 88 ; Kanpur 1986*)

Proof. We know that for $n>0$,

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$$

Putting $\quad x=az$ so that $dx=a\,dz$, we have

$$\Gamma(n)=\int_0^\infty (az)^{n-1}e^{-az}\,.\,a\,dz$$

$$=\int_0^\infty a^n z^{n-1}e^{-az}\,dz$$

Replacing z by x,

$$=\int_0^{\infty} a^n x^{n-1} e^{-ax}\, dx$$

Replacing a by z, we have

$$\Gamma(n)=\int_0^{\infty} z^n x^{n-1} e^{-zx}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma(n)\,.\, e^{-z} z^{m-1}=\int_0^{\infty} x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx$$

Integrating both sides w.r.t. z between the limits 0 to ∞, we have

$$\Gamma(n)\int_0^{\infty} e^{-z} z^{m-1}\, dz=\int_0^{\infty}\int_0^{\infty} x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx\, dz$$

$$=\int_0^{\infty}\int_0^{\infty} x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dz\, dx$$

$$\Rightarrow \quad \Gamma(n)\Gamma(m)=\int_0^{\infty} x^{n-1}\left[\int_0^{\infty} z^{m+n-1} e^{-z(1+x)}\, dz\right] dx$$

Putting $\quad z(1+x)=y$

so that $\quad dz=\dfrac{dy}{1+x}$

$$\Gamma(n)\Gamma(m)=\int_0^{\infty} x^{n-1}\left[\int_0^{\infty}\left(\frac{y}{1+x}\right)^{m+n-1} e^{-y}\frac{dy}{1+x}\right] dx$$

$$=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\left[\int_0^{\infty} y^{m+n-1} e^{-y}\, dy\right] dx$$

$$=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,[\Gamma(m+n)]\, dx$$

$$=\Gamma(m+n)\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$=\Gamma(m+n)\, B(m, n)$$

$$\left[\because \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\, dx=B(m, n)\right]$$

Hence $B(m, n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

12·8. PROVE THAT $\Gamma(\frac{1}{2})=\sqrt{\pi}$

(Meerut 1986 ; Kanpur 1985, 87 ; K.U. 1983 ; G.N.D.U. 1982)

Proof. We know that

$$B(m, n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Taking $m=n=\frac{1}{2}$,

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}=\frac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)}$$

or $$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=[\Gamma(\tfrac{1}{2})]^2 \qquad [\because \Gamma(1)=1] \qquad ...(i)$$

Now $$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$=\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

Putting $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$

$$\therefore \quad B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^{\pi/2}\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}d\theta=2\Big[\theta\Big]_0^{\pi/2}=2\left(\frac{\pi}{2}-0\right)=\pi$$

$\therefore$ From (i), $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=\pi$$

or $$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

12·9. PROVE THAT $\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$. *(M.D.U. 1983)*

Proof. Put $x^2=z$ so that $2x\,dx=dz$ or $dx=\dfrac{dz}{2\sqrt{z}}$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty e^{-x^2}\,dx=\int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}}=\frac{1}{2}\int_0^\infty e^{-z}z^{-\frac{1}{2}}\,dz$$

$$=\frac{1}{2}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}\,dz$$

$$=\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx,\text{ here } n=\frac{1}{2}\right]$$

$$=\frac{1}{2}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

Cor. 1. Prove that $\int_{-\infty}^0 e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

Put $x=-z$ so that $dx=-dz$

When $x\to-\infty$, $z\to\infty$; when $x=0$, $z=0$

$$\therefore \int_{-\infty}^{0} e^{-x^2}\,dx=\int_{\infty}^{0} e^{-z^2}(-dz)=-\int_{\infty}^{0} e^{-z^2}\,dz$$

$$=\int_{0}^{\infty} e^{-z^2}\,dz=\int_{0}^{\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

[By Art. 12·9]

Cor. 2. Prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$.

$$\int_{-\infty}^{\infty} e^{-x^2}dx=2\int_{0}^{\infty} e^{-x^2}\,dx$$

[$\because$ e^{-x^2} is an even function of x and

$\int_{-a}^{a} f(x)\,dx=2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is an even function of x]

$$=2\cdot\frac{\sqrt{\pi}}{2}$$

[By Cor. 1]

$$=\sqrt{\pi}.$$

12·10. TO EVALUATE $\int_{0}^{\pi/2} \sin^p x \cos^q x\, dx$

where $p>-1,\ q>-1$. (*K.U. 1980, 82 S*)

Put $\sin^2 x=z$ so that $2\sin x\cos x\,dx=dz$.

When $x=0,\ z=0$; when $x=\frac{\pi}{2},\ z=1$

Also $\cos^2 x=1-\sin^2 x=1-z$

$$\therefore \int_{0}^{\pi/2} \sin^p x\cos^q x\,dx$$

$$=\int_{0}^{\pi/2}(\sin^{p-1} x\cos^{q-1} x)\sin x\cos x\,dx$$

$$=\int_{0}^{\pi/2}(\sin^2 x)^{\frac{p-1}{2}}(\cos^2 x)^{\frac{q-1}{2}}\cdot\sin x\cos x\,dx$$

$$=\int_{0}^{1} z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\cdot\tfrac{1}{2}\,dz$$

$$=\frac{1}{2}\int_{0}^{1} z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$=\frac{1}{2}\,B\left(\frac{p-1}{2}+1,\ \frac{q-1}{2}+1\right)$$

$$=\frac{1}{2}\cdot B\left(\frac{p+1}{2},\ \frac{q+1}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

12·9]

$$\left[\because B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

Hence $\displaystyle\int_0^{\pi/2}\sin^p x\cos^q x\,dx=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$.

Example 1. *Express the following in terms of Gamma functions :*

(i) $\displaystyle\int_0^1 x^p(1-x^q)^n\,dx$ where $p>0,\ q>0,\ n>0$

or. 1]

(ii) $\displaystyle\int_0^1 x^{p-1}(1-x^2)^{q-1}\,dx$ where $p>0,\ q>0$

(iii) $\displaystyle\int_0^a x^{p-1}(a-x)^{q-1}\,dx$ where $p>0,\ q>0$.

Sol. (i) Put $x^q=z$ or $x=z^{\frac{1}{q}}$

so that $dx=\frac{1}{q}z^{\frac{1}{q}-1}\,dz=\frac{1}{q}z^{\frac{1-q}{q}}\,dz$

When $x=0,\ z=0$ and when $x=1,\ z=1$

$$\therefore\ \int_0^1 x^p(1-x^q)^n\,dx=\int_0^1 z^{\frac{p}{q}}(1-z)^n\cdot\frac{1}{q}\,z^{\frac{1-q}{q}}\,dz$$

$$=\frac{1}{q}\int_0^1 z^{\frac{p+1-q}{q}}(1-z)^n\,dz$$

$$=\frac{1}{q}\,B\left(\frac{p+1-q}{q}+1,\ n+1\right)$$

$$=\frac{1}{q}\,B\left(\frac{p+1}{q},\ n+1\right)$$

$$=\frac{1}{q}\,\frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma(n+1)}{\Gamma\left(\frac{p+1}{q}+n+1\right)}$$

(ii) Please try yourself. (Put $x^2=z$)

$$\left[\text{Ans. } \frac{\Gamma(p/2)\,\Gamma(q)}{2\,\Gamma(p/2+q)}\right]$$

(iii) Please try yourself. (Put $x=az$).

$$\left[\text{Ans. } a^{p+q-1}\cdot\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right]$$

Example 2. *Show that* $\int_0^1 \frac{dx}{\sqrt{1-x^n}}=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$.

(*Meerut 1989*)

Sol. Put $x^n=z$ i.e. $x=z^{\frac{1}{n}}$

so that $dx=\frac{1}{n}\,z^{\frac{1}{n}-1}\,dz=\frac{1}{n}\,z^{\frac{1-n}{n}}\,dz$

When $x=0,\ z=0$; when $x=1,\ z=1$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^n}}=\int_0^1 \frac{\frac{1}{n}\,z^{\frac{1-n}{n}}}{\sqrt{1-z}}\,dz$$

$$=\frac{1}{n}\int_0^1 z^{\frac{1}{n}-1}\,(1-z)^{-1/2}\,dz$$

$$=\frac{1}{n}\,B\left(\frac{1}{n},\frac{1}{2}\right)=\frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}\qquad\left[\because\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

Example 3. *Show that*

$$\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}}\,dx=\frac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Sol. $\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}}\,dx=\frac{B(m,n)}{a^n(1+a)^m}$ [See Beta functions]

$$=\frac{\Gamma(m)\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Example 4. *Evaluate*

(i) $\int_0^\infty e^{-x^3}\,dx$ (ii) $\int_0^\infty x^2\,e^{-x^3}\,dx$.

(G.N.D.U. 1981 S)

(iii) $\int_0^\infty \sqrt{x}.e^{-x^3}\,dx$ (iv) $\int_0^\infty e^{-a^2x^2}\,dx,\ a>0.$

Sol. (*i*) Put $x^3=z$ or $x=z^{1/3}$

so that $dx=\frac{1}{3}z^{-\frac{2}{3}}dz$

When $x=0, z=0$; when $x\to\infty, z\to\infty$

$$\therefore \int_0^\infty e^{-x^3}\,dx=\int_0^\infty e^{-z}\cdot\tfrac{1}{3}\,z^{-\frac{2}{3}}\,dz$$

$$=\frac{1}{3}\int_0^\infty e^{-z}\cdot z^{\frac{1}{3}-1}\,dz=\tfrac{1}{3}\Gamma(\tfrac{1}{3})$$

Note. $\int_0^\infty e^{-x^3}\,dx=\frac{1}{3}\Gamma\left(\frac{1}{3}\right)=\Gamma\left(\frac{4}{3}\right)$

(G.N.D.U. 1980 S)

$[\because\ (n-1)\,\Gamma(n-1)=\Gamma(n)]$

(*ii*) Proceeding as in part (*i*)

$$\int_0^\infty x^3\,e^{-x^3}\,dx=\int_0^\infty ze^{-z}\cdot\tfrac{1}{3}\,z^{-\frac{2}{3}}\,dz$$

$$=\frac{1}{3}\int_0^\infty e^{-z}\cdot z^{1/3}\,dz=\frac{1}{3}\int_0^\infty e^{-z}\cdot z^{\frac{4}{3}-1}\,dz=\frac{1}{3}\,\Gamma\left(\frac{4}{3}\right)$$

$$=\frac{1}{3}\cdot\frac{1}{3}\,\Gamma\left(\frac{1}{3}\right)\qquad[\because\ \Gamma(n)=(n-1)\,\Gamma(n-1)]$$

$$=\frac{1}{9}\,\Gamma\left(\frac{1}{3}\right)$$

(*iii*) Proceeding as in part (*i*)

$$\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx=\int_0^\infty z^{1/6}\,e^{-z}\cdot\tfrac{1}{3}z^{-\frac{2}{3}}\,dz$$

$$=\frac{1}{3}\int_0^\infty e^{-z}\cdot z^{-\frac{1}{2}}\,dz$$

$$=\frac{1}{3}\int_0^\infty e^{-z}\cdot z^{\frac{1}{2}-1}\,dz=\frac{1}{3}\,\Gamma\left(\frac{1}{2}\right)=\frac{1}{3}\sqrt{\pi}$$

(*iv*) Put $a^2x^2=z$

i.e. $x=\frac{\sqrt{z}}{a}$

so that $dx=\frac{z^{-\frac{1}{2}}}{2a}\,dz.$

When $x=0, z=0$; when $x\to\infty, z\to\infty$

$$\therefore \int_0^\infty e^{-a^2x^2}\,dx=\int_0^\infty e^{-z}\cdot\frac{z^{-\frac{1}{2}}}{2a}\,dz=\frac{1}{2a}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}\,dz$$

$$=\frac{1}{2a}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2a} \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

Example 5. *Show that*

$$\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\pi. \qquad \text{(M.D.U. 1981 S)}$$

Sol. $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$

$$=\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta\times\int_0^{\pi/2}\sin^{-\frac{1}{2}}\theta\cos^0\theta\,d\theta$$

$$=\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\times\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(5/4)}\times\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})}$$

$$=\frac{1}{4}\cdot\frac{[\Gamma(\frac{1}{2})]^2\,\Gamma(\frac{1}{4})}{\Gamma(5/4)}=\frac{1}{4}\cdot\frac{(\sqrt{\pi})^2\cdot\Gamma(\frac{1}{4})}{\frac{1}{4}\Gamma(\frac{1}{4})}=\pi.$$

Example 6. *Prove that if $n>-1$,*

$$\int_0^\infty x^n e^{-a^2x^2}\,dx=\frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right) \qquad \text{(Agra 1984)}$$

Hence or otherwise show that $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{a}$. (M.D.U. 1982)

Sol. Put $a^2x^2=z$, *i.e.* $x=\frac{\sqrt{z}}{a}$

so that $\qquad dx=\frac{z^{-1/2}}{2a}$

When $\qquad x=0, \quad z=0$; $\quad$ When $x\to\infty, z\to\infty$.

$$\therefore \int_0^\infty x^n e^{-a^2x^2}\,dx=\int_0^\infty\frac{z^{n/2}}{a^n}\cdot e^{-z}\cdot\frac{z^{-1/2}}{2a}\,dz$$

$$=\frac{1}{2a^{n+1}}\int_0^\infty e^{-z}\cdot z^{\frac{n}{2}-1/2}\,dz$$

$$=\frac{1}{2a^{n+1}}\Gamma\left(\frac{n-1}{2}+1\right)=\frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right) \qquad ...(1)$$

Putting $n=0$, $\int_0^\infty e^{-a^2x^2}\,dx=\frac{1}{2a}\,\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2a}$

$$\therefore \int_{-\infty}^{\infty} e^{-a^2x^2}\,dx=2\int_0^\infty e^{-a^2x^2}\,dx=2\cdot\frac{\sqrt{\pi}}{2a}=\frac{\sqrt{\pi}}{a}$$

[$\because$ $e^{-a^2x^2}$ is an even function of x and

$\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ if $f(x)$ is an even function.]

Example 7. *Show that if $a>1$,*

$$\int_0^\infty \frac{x^a}{a^x}\,dx=\frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$ (K.U. 1983 S)

Sol. $\because$ $a=e^{\log a}$ $\therefore$ $a^x=e^{x\log a}$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx=\int_0^\infty \frac{x^a}{e^{x\log a}}\,dx$$

$$=\int_0^\infty e^{-x\log a}\cdot x^a\,dx$$

Put $x\log a=z$, *i.e.* $x=\frac{z}{\log a}$

so that $dx=\frac{dz}{\log a}$

When $x=0,\ z=0$; when $x\to\infty,\ z\to\infty$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx=\int_0^\infty e^{-z}\cdot\frac{z^a}{(\log a)^a}\cdot\frac{dz}{\log a}$$

$$=\frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}\cdot z^{(a+1)-1}\,dz$$

$$=\frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$

Example 8. *Prove that* $\int_0^\infty e^{-ax}x^{n-1}\,dx=\frac{\Gamma(n)}{a^n}$, *where a, n are positive. Hence show that*

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx=\frac{\Gamma(n)}{r^n}\cos n\theta$

(ii) $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx=\frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2=a^2+b^2$ *and* $\theta=\tan^{-1}\frac{b}{a}$.

Sol. Put $ax=z$ so that $dx=\frac{dz}{a}$

When $x=0$, $z=0$, when $x \to \infty$, $z \to \infty$

$$\therefore \int_0^\infty e^{-ax} x^{n-1} dx = \int_0^\infty e^{-z} \cdot \left(\frac{z}{a}\right)^{n-1} \cdot \frac{dz}{a}$$

$$= \frac{1}{a^n} \int_0^\infty e^{-z} z^{n-1} dz = \frac{\Gamma(n)}{a^n}$$

Replacing a by $a+ib$, we have

$$\int_0^\infty e^{-(a+ib)x} x^{n-1} dx = \frac{\Gamma(n)}{(a+ib)^n} \qquad \ldots(1)$$

Now $e^{-(a+ib)x} = e^{-ax-ibx} = e^{-ax} (\cos bx - i \sin bx)$

$[\because e^{p-iq} = e^p (\cos q - i \sin q)]$

Also putting $a = r \cos \theta$ and $b = r \sin \theta$

$$a^2+b^2=r^2 \text{ and } \frac{b}{a} = \tan\theta \text{ i.e. } \theta = \tan^{-1}\frac{b}{a}$$

and $$(a+ib)^n = (r \cos\theta + i \cdot r \sin\theta)^n = r^n(\cos\theta + i \sin\theta)^n$$

$$= r^n (\cos n\theta + i \sin n\theta)$$

[De Moivre's Theorem]

From (1),

$$\int_0^\infty e^{-ax} (\cos bx - i \sin bx) x^{n-1} dx$$

$$= \frac{\Gamma(n)}{r^n(\cos n\theta + i \sin n\theta)}$$

$$= \frac{\Gamma(n)}{r^n} (\cos n\theta + i \sin n\theta)^{-1}$$

$$= \frac{\Gamma(n)}{r^n} (\cos n\theta - i \sin n\theta)$$

Equating real parts

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx \, dx = \frac{\Gamma(n)}{r^n} \cos n\theta$$

Equating imaginary parts

$$\int_0^\infty e^{-ax} x^{n-1} \sin bx \, dx = \frac{\Gamma(n)}{r^n} \sin n\theta.$$

Example 9. *Evaluate*

(i) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x \, dx$ (ii) $\int_0^{\pi/2} \sin^7 x \, dx$

(iii) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$.

Sol. (i) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5/2+1}{2}\right)}{\Gamma\left(\frac{3+1}{2}+\frac{5/2+1}{2}\right)}$

$$\left[\because \int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}\right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{15}{4}\right)} = \frac{1}{2} \cdot \frac{1\,!\,.\,\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}\cdot\frac{7}{4}\,\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{8}{77}$$

$[\because$ $\Gamma(n)=(n-1)\,!$ if n is a +ve integer

and $\Gamma(n)=(n-1)\,\Gamma(n-1)$

$\therefore \Gamma\left(\frac{15}{4}\right) = \frac{11}{4}\,\Gamma\left(\frac{11}{4}\right) = \frac{11}{4}\cdot\frac{7}{4}\,\Gamma\left(\frac{7}{4}\right)]$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx = \int_0^{\pi/2} \sin^7 x \cos^0 x \, dx$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{(4-1)\,!\,.\,\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{6}{\frac{105}{8}} = \frac{16}{35}$$

(iii) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{-\frac{1}{2}+1}{2}\right)}$$

$$=\frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(1)}$$

$$=\frac{1}{2}\cdot\frac{\sqrt{2}\pi}{1}\quad\left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\sqrt{2}\pi\right.$$

[See Cor. with Duplication Formula]

$$=\frac{\pi}{\sqrt{2}}.$$

Example 10. *Show that*

$$\int_0^{\pi/2}\sin^n\theta\,d\theta=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\ \textit{where } n>-1.$$

(M.D.U. 1981)

Sol. $\int_0^{\pi/2}\sin^n\theta\,d\theta=\int_0^{\pi/2}\sin^n\theta\cos^0\theta\,d\theta$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)}\quad[\because\ \Gamma(\tfrac{1}{2})=\sqrt{\pi}]$$

$$=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

Example 11. *Prove that*

$$\int_0^1\frac{1}{\sqrt{1-x^4}}\,dx=\frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2\qquad\text{(K.U. 1981 S)}$$

Sol. Put $x^4=z$ *i.e.* $x=z^{1/4}$ so that $dx=\frac{1}{4}z^{-3/4}\,dz$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore\ \int_0^1\frac{1}{\sqrt{1-x^4}}\,dx=\int_0^1\frac{1}{\sqrt{1-z}}\cdot\frac{1}{4}z^{-3/4}\,dz$$

$$=\frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{-1/2}\,dz=\frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$=\frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}$$

$$=\frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\cdot\sqrt{\pi}}{\Gamma(\frac{3}{4})}$$

$$=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi/\Gamma(\frac{1}{4})}\left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\sqrt{2}\pi\right]$$

$$=\frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Example 12. *Prove that* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

Sol. $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=B(m,n)$

[See examples with Beta Function]

$$=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 13. *Evaluate* $\int_0^1 x^3(1-x)^{4/3}\,dx.$ (K.U. 1982)

Sol. Put $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

When $x=0,\ \theta=0$; when $x=1,\ \theta=\pi/2$

$$\therefore\ \int_0^1 x^3(1-x)^{4/3}\,dx=\int_0^{\pi/2}\sin^6\theta\cos^{8/3}\theta\,.\,2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^7\theta\cos^{11/3}\theta\,d\theta$$

$$=2\cdot\frac{1}{2}\frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{11/3+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{11/3+1}{2}\right)}$$

$$=\frac{\Gamma(4)\,\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{19}{3}\right)}$$

$$=\frac{(4-1)!\,.\,\Gamma\left(\frac{7}{3}\right)}{\frac{16}{3}\cdot\frac{13}{3}\cdot\frac{10}{3}\cdot\frac{7}{3}\,\Gamma\left(\frac{7}{3}\right)}$$

$$=\frac{6\times 81}{16\times 13\times 10\times 7}=\frac{243}{7280}.$$

Example 14. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ *in terms of Beta and Gamma functions; where* $m>0,\ n>0,\ a>0,\ b>0.$

Sol. Put $bx=az$ so that $dx=\frac{a}{b}\,dz$.

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx=\int_0^\infty \frac{\left(\frac{az}{b}\right)^{m-1}}{(a+az)^{m+n}}\cdot\frac{a}{b}\,dz$$

$$=\int_0^\infty \frac{a^{m-1}\cdot a}{b^{m-1}\cdot b\cdot a^{m+n}}\cdot\frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$=\frac{1}{a^n b^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$=\frac{1}{a^n b^m}\,B(m,n) \qquad |\text{ By def.}$$

$$=\frac{1}{a^n b^m}\cdot\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 15. *Prove that* $\Gamma(n)=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy.$

(Meerut 1986; Kanpur 1985, 86; Agra 1981)

Sol. Put $\log\frac{1}{y}=z$, i.e. $\frac{1}{y}=e^z$

or $y=e^{-z}$ so that $dy=-e^{-z}\,dz$

When $y=0$, $z\to\infty$; when $y=1$, $z=0$

$$\therefore \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy=\int_\infty^0 z^{n-1}(-e^{-z})\,dz$$

$$=\int_0^\infty z^{n-1}e^{-z}\,dz=\Gamma(n).$$

Example 16. *Show that*

(i) $\int_0^\infty \sqrt{x}\,e^{-x^2}\,dx\times\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\frac{\pi}{2\sqrt{2}}$

(ii) $\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy\times\int_0^\infty y^2 e^{-y^4}\,dy=\frac{\pi}{4\sqrt{2}}$.

Sol. (i) Put $x^2=z$ i.e. $x=z^{\frac{1}{2}}$ so that $dx=\frac{1}{2}z^{-\frac{1}{2}}\,dz$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \sqrt{x}\,e^{-x^2}\,dx=\int_0^\infty z^{\frac{1}{4}}e^{-z}\cdot\tfrac{1}{2}z^{-\frac{1}{2}}\,dz$$

$$=\tfrac{1}{2}\int_0^\infty z^{-\frac{1}{4}}e^{-z}\,dz=\tfrac{1}{2}\,\Gamma\left(\frac{3}{4}\right)$$

and $\int_0^\infty$

(ii)

Put

Exa

Sol.

2Γ

Sol.

and $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\int_0^\infty \frac{e^{-z}}{z^{1/4}}\cdot\frac{1}{2}z^{-\frac{1}{2}}\,dz$

$$=\frac{1}{2}\int_0^\infty z^{-\frac{3}{4}}e^{-z}\,dz=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$\therefore \int_0^\infty \sqrt{x}\,e^{-x^2}\,dx\times\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$=\frac{1}{4}\cdot\sqrt{2}\,\pi \qquad \left[\because \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\sqrt{2}\pi\right]$$

$$=\frac{\pi}{2\sqrt{2}}$$

(ii) Put $y^2=z$, $\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)$ [As in part (i)]

Put $y^4=z$ i.e. $y=z^{\frac{1}{4}}$ so that $dy=\frac{1}{4}z^{-\frac{3}{4}}\,dz$

When $y=0$, $z=0$; when $y\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty y^2\,e^{-y^4}\,dy=\int_0^\infty z^{\frac{1}{2}}e^{-z}\cdot\frac{1}{4}z^{-\frac{3}{4}}\,dz$$

$$=\frac{1}{4}\int_0^\infty z^{-\frac{1}{4}}e^{-z}\,dz$$

$$\therefore \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy\times\int_0^\infty y^2\,e^{-y^4}\,dy=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{8}\times\sqrt{2}\pi=\frac{\pi}{4\sqrt{2}}$$

Example 17. *Prove that*

$$\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \qquad \text{(Kanpur 1987)}$$

Sol. Please try yourself. [See Art. 12·10]

Example 18. *Show that*

$$2^n\Gamma\left(n+\frac{1}{2}\right)=1.3.5\ldots\ldots(2n-1)\sqrt{\pi},$$

where n is a positive integer.

Sol. $\Gamma\left(n+\frac{1}{2}\right)=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

BETA

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\dots\dots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\dots\dots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$=\frac{1}{2^n}(2n-1)(2n-3)\dots\dots3.1\sqrt{\pi}$$

$$\Rightarrow 2^n\,\Gamma\left(n+\frac{1}{2}\right)=1.3.5.\dots\dots(2n-1)\sqrt{\pi}$$

(writing the factors in reverse order)

Example 19. *Show that*

(*i*) $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}\times\int_0^1 \frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$

(Kanpur 1980)

(*ii*) $\displaystyle\int_0^\infty x\,e^{-x^8}\,dx\times\int_0^\infty x^2e^{-x^4}\,dx=\frac{\pi}{16\sqrt{2}}$

(*iii*) $\displaystyle\int_0^{\pi/2} \sin^p x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\,dx=\frac{\pi}{2(p+1)}$

Sol. (*i*) Put $x^2=\sin\theta$ *i.e.* $x=\sqrt{\sin\theta}$

so that $dx=\dfrac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\dfrac{\pi}{2}$

$$\therefore\ \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)}=\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{4\times\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$=\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

so tha

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Now put $x^2 = \tan\phi$ i.e. $x = \sqrt{\tan\phi}$

so that $dx = \dfrac{\sec^2\phi}{2\sqrt{\tan\phi}}\, d\phi$

When $x=0$, $\phi=0$; when $x=1$, $\phi=\dfrac{\pi}{4}$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{1}{\sec\phi} \cdot \frac{\sec^2\phi}{2\sqrt{\tan\phi}}\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin\phi\cos\phi}} = \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{2\sin\phi\cos\phi}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin 2\phi}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dt}{\sqrt{\sin t}}$$

where $t = 2\phi$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t \cos^0 t\, dt$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$= \sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \cdot \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\pi}{4\sqrt{2}}$$

(ii) Put $x^8 = z$ i.e. $x = z^{1/8}$

so that $dx = \dfrac{1}{8} z^{-7/8}\, dz$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty x e^{-x^8}\, dx = \int_0^\infty z^{1/8} e^{-z} \cdot \frac{1}{8} z^{-7/8}\, dz$$

$$= \frac{1}{8}\int_0^\infty z^{-3/4} e^{-z}\, dz = \frac{1}{8}\,\Gamma\left(\frac{1}{4}\right)$$

Now put $x^4=t$ i.e. $x=t^{1/4}$

so that $dx=\frac{1}{4}t^{-3/4}\,dt$

When $x=0$, $t=0$; when $x\to\infty$, $t\to\infty$

$$\therefore \int_0^\infty x^2 e^{-x^4}\,dx=\int_0^\infty t^{1/2}e^{-t}\cdot\frac{1}{4}e^{-3/4}\,dt$$

$$=\frac{1}{4}\int_0^\infty t^{-1/4}e^{-t}\,dt=\frac{1}{4}\,\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \int_0^\infty xe^{-x^8}\,dx\times\int_0^\infty x^2e^{-x^4}dx$$

$$=\frac{1}{8}\Gamma\left(\frac{1}{4}\right)\times\frac{1}{4}\,\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{1}{32}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{32}\times\sqrt{2}\pi.$$

[See cor. with Art. 12·11]

$$=\frac{\pi}{16\sqrt{2}}$$

(iii) $\int_0^{\pi/2}\sin^p x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\,dx$

$$=\int_0^{\pi/2}\sin^p x\cos^0 x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\cos^0 x\,dx$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}+\frac{0+1}{2}\right)}\times\frac{\Gamma\left(\frac{p+1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+1+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{p+3}{2}\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma\left(\frac{p+1}{2}+1\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)(\sqrt{\pi})^2}{\frac{p+1}{2}\,\Gamma\left(\frac{p+1}{2}\right)}=\frac{\pi}{2(p+1)}.$$

Example 20. *Show that*

$$\int_0^1\sqrt{1-x^4}\,dx=\frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Sol. Put $x^4=z$ i.e. $x=z^{1/4}$

so that $dx=\frac{1}{4}z^{-3/4}\,dz$

When $x=0$, $z=0$, when $x=1$, $z=1$

$$\therefore \int_0^1 \sqrt{1-x^4}\,dx=\int_0^1 (1-z)^{1/2}\cdot\tfrac{1}{4}z^{-3/4}\,dz$$

$$=\frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{1/2}\,dz$$

$$=\frac{1}{4}B\left(\frac{1}{4},\frac{3}{2}\right)=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{2}\right)}$$

$$=\frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}=\frac{1}{8}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)}$$

$$=\frac{\sqrt{\pi}}{6}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}=\frac{\sqrt{\pi}}{6}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)}}$$

$$\left[\because\ \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\sqrt{2}\,\pi\right]$$

$$=\frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2=\frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Example 21. *Prove that*

(i) $B(p,q)=B(p+1,q)+B(p,q+1)$

(ii) $B(p,q)B(p+q,r)=B(q,r)B(q+r,p)$

$\qquad=B(r,p)B(r+p,q)$

(iii) $B(p,q)\,B(p+q,r)\,B(p+q+r,s)=\dfrac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{\Gamma(p+q+r+s)}$

Sol. (i) R.H.S. $=B(p+1,q)+B(p,q+1)$

$$=\frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+q+1)}+\frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$=\frac{p\,\Gamma(p)\,\Gamma(q)+\Gamma(p)\,q\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

$$=\frac{(p+q)\,\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$=B(p, q)=\text{L.H.S.}$$

(ii) $$B(p, q)\,B(p+q, r)=\frac{B(p)\,B(q)}{B(p+q)}\cdot\frac{B(p+q)\,B(r)}{B(p+q+r)}$$

$$=\frac{B(p)\,B(q)\,B(r)}{B(p+q+r)}$$

Similarly for others.

(iii) Please try yourself.

Example 22. *Prove that*

(i) $\dfrac{B(p, q+1)}{q}=\dfrac{B(p+1, q)}{p}=\dfrac{B(p, q)}{p+q}$

(ii) $\dfrac{B(m+1, n)}{B(m, n)}=\dfrac{m}{m+n}$

(iii) $\dfrac{B(m+2, n-2)}{B(m, n)}=\dfrac{m(m+1)}{(n-1)(n-2)}$

Sol. (i) $$\frac{B(p, q+1)}{q}=\frac{\Gamma(p)\,\Gamma(q+1)}{q\,\Gamma(p+q+1)}$$

$$=\frac{\Gamma(p)\,q\Gamma(q)}{q(p+q)\,\Gamma(p+q)}$$

$$=\frac{\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}=\frac{B(p, q)}{p+q}$$

Similarly $\dfrac{B(p+1, q)}{q}=\dfrac{B(p, q)}{p+q}$

Hence the result.

(ii) $$B(m+1, n)=\frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$$

$$=\frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$=\frac{m}{m+n}\,B(m, n)$$

$$\Rightarrow \frac{B(m+1, n)}{B(m, n)}=\frac{m}{m+n}$$

(iii) $$B(m+2, n-2)=\frac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m+n)}$$

$$=\frac{(m+1)m\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)}$$

then

wher

$$B(m, n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \quad \frac{B(m+2, n-2)}{B(m, n)}=\frac{m(m+1)\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)}\cdot\frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$=\frac{m(m+1)\,\Gamma(n-2)}{\Gamma(n)}$$

$$=\frac{m(m+1)\,\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)}$$

$$=\frac{m(m+1)}{(n-1)(n-2)}.$$

Example 23. *Evaluate the following integrals :*

(i) $\int_0^\infty x^6 e^{-2x}\,dx$ (ii) $\int_0^\infty e^{-4x} x^{3/2}\,dx$

(iii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ (iv) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

(v) $\int_0^\infty \frac{x^3(1-x^6)}{(1+x)^{24}}\,dx$ (vi) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx.$

(Meerut 1989 ; Kanpur 1985)

Sol. (i) Put $2x=z$ i.e., $x=\frac{1}{2}z$

then $dx=\frac{1}{2}dz$

when $x=0, z=0$; when $x\to\infty, z\to\infty$

$$\therefore \quad \int_0^\infty x^6 e^{-2x}\,dx=\int_0^\infty (\tfrac{1}{2}z)^6 e^{-z}\cdot\tfrac{1}{2}\,dz$$

$$=\frac{1}{128}\int_0^\infty z^6 e^{-z}\,dz=\frac{1}{128}\,\Gamma(7)$$

$$=\frac{1}{128}(6\,!) \qquad \left|\; \because \Gamma(n)=(n-1)\,! \text{ where } n\in N\right.$$

$$=\frac{6\times5\times4\times3\times2}{128}=\frac{45}{8}$$

(ii) Please try yourself. $\left[\text{Ans. } \frac{3\sqrt{\pi}}{128}\right]$

(iii) Put $x=2z$ then $dx=2dz$

When $x=0, z=0$; when $x=2, z=1$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx=\int_0^1 \frac{4z^2}{\sqrt{2(1-z)}}\cdot 2dz$$

$$=4\sqrt{2}\int_0^1 z^2(1-z)^{-1/2}\,dz$$

$$=4\sqrt{2}B\left(3, \frac{1}{2}\right)$$

$$=4\sqrt{2}\,\frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}$$

$$=4\sqrt{2}\,.\,\frac{(2\,!)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$=8\sqrt{2}\,.\,\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}=\frac{64\sqrt{2}}{15}$$

(iv) $I=\int_0^3 \frac{dx}{\sqrt{3x-x^2}}=\int_0^3 \frac{dx}{\sqrt{x}\sqrt{3-x}}$

Put $x=3z$ then $dx=3dz$

When $x=0,\ z=0$; when $x=3,\ z=1$

$$\therefore\quad I=\int_0^1 \frac{3dz}{\sqrt{3z}\sqrt{3(1-z)}}=\int_0^1 z^{-1/2}(1-z)^{-1/2}\,dz$$

$$=B\left(\frac{1}{2}, \frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$[\because\ \Gamma(1)=1]$

$$=(\sqrt{\pi})^2=\pi$$

(v) $I=\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

$$=\int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx-\int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$=\int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx-\int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$=B(9, 15)-B(15, 9)$$

$\left[\because\ \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=B(m, n)\right]$

$=0$ $\qquad [\because\ B(m, n)=B(n, m)]$