

IMS
MATHS
BOOK-01

Set - IX (i)* Functions of Several Variables *Introduction:-

Most measurable quantities in the real world do not depend on one single factor but on many factors. This indicates that functions of several variables are natural entities in the world of mathematics.

So far we have studied the concepts of limits, continuity, differentiability etc. for functions of a single variable.

Now we introduce the concept of limit, continuity and differentiability of functions of several variables. Mainly we study these concepts for real valued functions of two variables which can be generalised to functions of several variables.

Euclidean Space:

For a fixed $n \in \mathbb{N}$, let \mathbb{R}^n be the set of all ordered n -tuples $x = (x_1, x_2, \dots, x_n)$ where $x_1, x_2, \dots, x_n \in \mathbb{R}$ are called the coordinates of x .

The elements of \mathbb{R}^n are called points or vectors and denoted by x, y, z etc.

\rightarrow we define the addition of vectors and multiplication of a vector by real number (called scalar) as follows:

$$\text{Let } x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

$$y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n; \alpha \in \mathbb{R}$$

$$\text{then } x+y = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in \mathbb{R}^n$$

and $\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in \mathbb{R}^n$. These two operations make \mathbb{R}^n a vector space over the real field \mathbb{R} .

The zero element of \mathbb{R}^n (sometimes called the origin or null vector) is the point $0 = (0, 0, \dots, 0)$.

We define the scalar product (or inner product) of two vectors x and y by

$$x \cdot y = \sum_{i=1}^n x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

and the norm of x by $\|x\| = (x \cdot x)^{1/2}$

$$= \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$

The vector space \mathbb{R}^n with the above inner product and norm is called n -dimensional Euclidean space.

In particular, we get $\mathbb{R}^2, \mathbb{R}^3$ for $n=1, 2, 3$ respectively we write

$$x = (x_1, x_2) \text{ if } x \in \mathbb{R}^2$$

$$x = (x_1, x_2, x_3) \text{ if } x \in \mathbb{R}^3$$

Functions of Several Variables:

Let $f: X \rightarrow \mathbb{R}$, if $x \in \mathbb{R}^n$ then f is called a function of n variables.

$\rightarrow f$ is a function of several variables if $n > 1$.

$\rightarrow f: X \rightarrow \mathbb{R}$ is a function of two variables if $x \in \mathbb{R}^2$.

$f: X \rightarrow \mathbb{R}$ is a function of three variables if $x \in \mathbb{R}^3$.

* Neighbourhood of a point:

\rightarrow spherical neighbourhood of a point:

Let \mathbb{R}^n be the Euclidean space and $a \in \mathbb{R}^n$ (i.e. $a = (a_1, a_2, \dots, a_n)$).

If δ is any +ve real number, then the set $\{x \in \mathbb{R}^n / \|x-a\| < \delta\}$ is called an open sphere.

$$\text{(since } \|x-a\| < \delta \Rightarrow \left[\sum_{i=1}^n (x_i - a_i)^2 \right]^{\frac{1}{2}} < \delta \text{)}$$

$$\Rightarrow \sum_{i=1}^n (x_i - a_i)^2 < \delta^2$$

$$\Rightarrow (x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2 < \delta^2$$

The point a is called the centre and δ the radius of the sphere.

This open sphere is denoted by $S(a, \delta)$.

A closed sphere is denoted by

$S[a, \delta]$ and is defined by

$$S[a, \delta] = \{x \in \mathbb{R}^n / \|x-a\| \leq \delta\}$$

Any open sphere with a as its

Centre is called a spherical

neighbourhood of the point a .

The open sphere with centre

$a = (a_1, a_2) \in \mathbb{R}^2$ and radius δ is

$$S(a, \delta) = \{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}$$

\therefore the open sphere

in this case consists

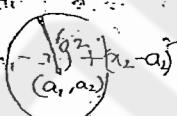
of all points of the

Cartesian plane

which lie within

the circle

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2. \quad \{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}$$



* Rectangular Neighbourhood of a point:

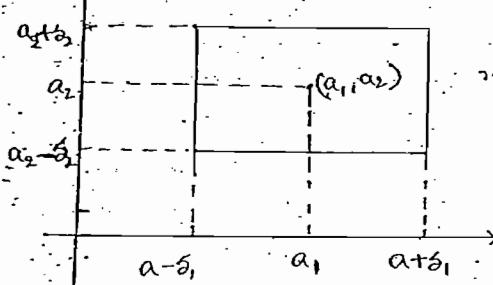
Rectangular neighbourhood of a point

$a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ is defined to be the set $\{x \in \mathbb{R}^n / |x_i - a_i| < \delta_i, i=1, 2, \dots, n\}$

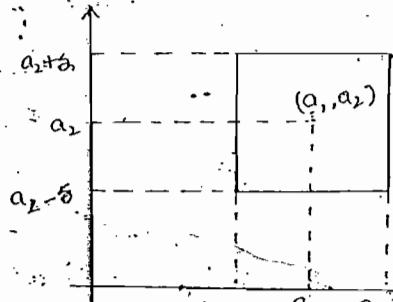
The rectangular neighbourhood of

(a_1, a_2) is $\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1 \text{ and } |x_2 - a_2| < \delta_2\}$

If in particular, $\delta_1 = \delta_2 = \delta$ then such a neighbourhood is referred to as a square neighbourhood of side 2δ .



$$\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2\}$$



$\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta \text{ and } |x_2 - a_2| < \delta\}$
Note: (1) Every spherical neighbourhood of a point in \mathbb{R}^n contains a rectangular neighbourhood of a point and vice versa.

* δ -neighbourhood of a point:

Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and δ be a +ve real number. The set of points $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

where $|a_i - x_i| < \delta, i=1, 2, \dots, n$

i.e. $|x_1 - a_1| < \delta, |x_2 - a_2| < \delta, \dots, |x_n - a_n| < \delta$ is called a δ -neighbourhood of the point a and is denoted by $N(a, \delta)$.

If we exclude the point a from $N(a, \delta)$ then it is called a deleted neighbourhood of a and is denoted by $N'(a, \delta)$.

* Limit of a function:

Let $f: X \rightarrow \mathbb{R}, X \subset \mathbb{R}^n$. Then f is said to tend to limit $l \in \mathbb{R}$ as x approaches a .

i.e. If $f(x) = l$

i.e. for a given $\epsilon > 0, \exists \delta > 0$.

such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$ (or) $0 < |x_i - a_i| < \delta_i$.

Note: - (1) The limit of a function is unique as $x \rightarrow a$, if it exists at all, is unique.

(2) If $x \rightarrow a$ where $n \geq 2$ then x approaches a' along infinitely many ways unlike the case of $n=1$ when x approaches a' along two ways only (i.e. $x \rightarrow a^-$ & $x \rightarrow a^+$)

Further for $n \geq 2$ x may approaches along straight lines (or) along different curves.

In the case of $n=1$, the existence of limit of $f(x)$ as $x \rightarrow a$ is independent of the two approaches.

In the case $n \geq 2$ also, the existence of limit is independent of infinitely many approaches.

(3) For a function of two variables i.e. in the case of $n=2$, two approaches are of special importance. These are

(i) x approaches a' first along a line parallel to the first axis, then along a line parallel to the second axis.

(ii) x approaches a' first along a line parallel to the second axis and then along a line parallel to the first axis.

The limits, if they exist, in these approaches are called repeated limits (or) iterated limits.

(i.e. the function $f(x, y)$ of two variables x and y and (x_0, y_0) is the limiting point of a set of values in two dimensional space.)

If $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = g(x)$ then $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = g(x)$

is defined by $\lim_{x \rightarrow x_0} g(x)$.

If $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = h(y)$ then $\lim_{\substack{y \rightarrow y_0 \\ x \rightarrow x_0}} f(x, y) = h(y)$

is defined by $\lim_{y \rightarrow y_0} h(y)$.

The limits (if exist) $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y)$

and $\lim_{\substack{y \rightarrow y_0 \\ x \rightarrow x_0}} f(x, y)$ are the repeated limits.

The limit defined above that is independent of the different approaches is referred to as the double limit (or) simultaneous limit to distinguish the two approaches.

i.e. we say that the simultaneous limit exists and is equal to l as $(x, y) \rightarrow (x_0, y_0)$.

Symbolically written as

$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = l$ if for given $\epsilon > 0$

(however small) \exists a $\delta > 0$ (depending on ϵ)

such that $|f(x, y) - l| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$
 $|x-x_0| < \delta, |y-y_0| < \delta$

Note!

(i) The double (simultaneous) limit may exist but the repeated limits may not exist, but if they exist they must be equal to the double limit.

(ii) The repeated limits may exist but the double limit may not exist.

(iii) If the repeated limits are not equal, the simultaneous limit cannot exist.

*Non-Existence of Simultaneous limit:

for the existence of simultaneous limit, not only must we have same limiting value if the variable point (x, y) approaches the limiting point (x_0, y_0) through any set of values dense at the point, but we must also have the same limiting value as the variable point approaches its limiting position along any curve what so ever.

thus, if we can find two methods of approach to the limiting point, which give different limiting values then we can conclude that the simultaneous limit does not exist.

Problems:

→ show that the simultaneous limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2+y^6}$$

does not exist.

$$\text{Soln: Let } f(x,y) = \frac{xy^3}{x^2+y^6}; (x,y) \neq (0,0)$$

If we approach the origin along any axis then $\lim_{x \rightarrow 0} f(x,0) = 0 = \lim_{y \rightarrow 0} f(0,y)$

$$f(x,y) \rightarrow 0 \text{ as }$$

$(x,y) \rightarrow (0,0)$ along coordinate axes.

If we approach $(0,0)$ along a straight line path $y=mx$.

$$f(x, mx) = \frac{xm^3x^3}{x^2+m^6x^3} = \frac{m^3x^2}{1+m^6x}$$

$$\lim_{x \rightarrow 0} f(x, mx) = 0$$

$\therefore f(x,y) \rightarrow 0$ as $(x,y) \rightarrow (0,0)$ along a straightline path.

If we approach $(0,0)$ along the curve $x=ny^3$

$$f(ny^3, y) = \frac{ny^3y^3}{n^2y^6+y^6}$$

$$= \frac{m}{1+m^2}$$

$$\therefore \lim_{y \rightarrow 0} f(ny^3, y) = \frac{m}{1+m^2} \neq 0.$$

Since the limit dependence upon the value of m .

$f(x,y)$ approaches different values along the different curves.

The limit at the origin does not exist.

Note: The existence of the simultaneous limit

$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) \Rightarrow$ the single limits

$$\lim_{x \rightarrow x_0} f(x, y_0), \lim_{y \rightarrow y_0} f(x_0, y)$$

also exist.

However, it does not follow the single limits $\lim_{x \rightarrow x_0} f(x, y), \lim_{y \rightarrow y_0} f(x, y)$ exist

for $y \neq y_0, x \neq x_0$ respectively.

→ show that the simultaneous

limit $\lim_{(x,y) \rightarrow (0,0)} y \sin(\frac{1}{x})$ exists and equals 0

but the single limit $\lim_{x \rightarrow 0} y \sin(\frac{1}{x})$ does not exist.

Soln: Let $\epsilon > 0$ be given,

$$\text{Now we have } |y \sin(\frac{1}{x}) - 0| = |y \sin(\frac{1}{x})|$$

$$= |y| |\sin(\frac{1}{x})|$$

$$= |y| (\because |\sin(\frac{1}{x})| \leq 1)$$

$$< \epsilon$$

$$\text{whenever } 0 < |y| < \frac{\epsilon}{1}$$

$$= \delta \text{ (choosing)}$$

$$\therefore |y \sin(\frac{1}{x}) - 0| < \epsilon \text{ whenever } 0 < |x| < \delta$$

$$0 < |y| < \delta$$

$$\lim_{(x,y) \rightarrow (0,0)} y \sin(\frac{1}{x}) = 0.$$

but for any constant value of $y = y_1 \neq 0$, we get

$$\underset{x \rightarrow 0}{\lim} y_1 \sin \frac{1}{x} = y_1, \underset{x \rightarrow 0}{\lim} \sin \frac{1}{x} \text{ does not exist.}$$

→ show that $\underset{(x,y) \rightarrow (0,0)}{\lim} \frac{xy}{x^2+y^2}$ does not exist.

Sol'n: If we put $x=mx$ and let $y \rightarrow 0$

$$\therefore \underset{y \rightarrow 0}{\lim} \frac{2mxy^2}{(m^2+1)} = \frac{2m}{(m^2+1)} \text{ does not exist}$$

C: the limit dependence upon the value of m .

→ show that $\underset{(x,y) \rightarrow (0,0)}{\lim} xy \frac{x^2-y^2}{x^2+y^2} = 0$.

Sol'n: Put $x = r \cos \theta$; $y = r \sin \theta$

$$\left| xy \frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \cos \theta \sin \theta (\cos 2\theta) \right|$$

$$= \left| \frac{r^2}{2} \sin 2\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} 2 \sin 2\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin(4\theta) \right|$$

$$\leq \frac{r^2}{4} = \frac{x^2+y^2}{4}$$

$$= \frac{x^2}{4} + \frac{y^2}{4}$$

$\leq \epsilon$

$$\text{if } \frac{x^2}{4} < \epsilon \quad \frac{y^2}{4} < \epsilon$$

i.e. if $|x| < \sqrt{2}\epsilon = \delta$, $|y| < \sqrt{2}\epsilon = \delta$

∴ for $\epsilon > 0$, $\exists \delta > 0$.

$$\text{such that } \left| xy \frac{x^2-y^2}{x^2+y^2} - 0 \right| < \epsilon$$

whenever $0 < |x| < \delta$

$$0 < |y| < \delta$$

$$\therefore \underset{(x,y) \rightarrow (0,0)}{\lim} xy \frac{x^2-y^2}{x^2+y^2} = 0$$

→ show that repeated limits exists when $(x,y) \rightarrow (0,0)$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0); \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$\text{sol'n: } \underset{y \rightarrow 0}{\lim} \underset{x \rightarrow 0}{\lim} f(x,y) = \underset{y \rightarrow 0}{\lim} (0) = 0$$

$$\text{and } \underset{x \rightarrow 0}{\lim} \underset{y \rightarrow 0}{\lim} f(x,y) = \underset{x \rightarrow 0}{\lim} (0) = 0$$

∴ the repeated limit exists and are equal. But the simultaneous limit does not exist by putting $y=mx$.

$$\therefore f(x,y) = \frac{y-x}{y+x} \cdot \frac{1+x}{1+y} \text{ then }$$

$$\underset{x \rightarrow 0}{\lim} \underset{y \rightarrow 0}{\lim} f(x,y) = \underset{y \rightarrow 0}{\lim} \left[(-1) \left(\frac{1+y}{1} \right) \right] = -1$$

$$\underset{y \rightarrow 0}{\lim} \underset{x \rightarrow 0}{\lim} f(x,y) = \underset{y \rightarrow 0}{\lim} \left(\frac{1}{1+y} \right) = 1$$

∴ the repeated limits exist but are not equal.

∴ the simultaneous limit does not exist.

→ show that the simultaneous limit exists at the origin

$$f(x,y) = \begin{cases} x \left(\sin \frac{1}{y} \right) + y \sin \left(\frac{1}{x} \right) & ; xy \neq 0 \\ 0 & ; xy = 0 \end{cases}$$

Sol'n: Here $\underset{y \rightarrow 0}{\lim} f(x,y)$, $\underset{x \rightarrow 0}{\lim} f(x,y)$ do not exist.

$\therefore \lim_{x \rightarrow 0} f(x,y); \lim_{y \rightarrow 0} f(x,y)$ do not exist.

$$\text{Now } |f(x,y) - 0| = |x \sin \frac{1}{y} + y \sin \frac{1}{x}| \\ \leq |x| + |y| < \epsilon$$

whenever $0 < |x| < \epsilon_1/2$; $0 < |y| < \epsilon_2/2$

choosing $\delta = \epsilon_1/2$

$$\therefore |f(x,y) - 0| < \epsilon \text{ whenever } 0 < |x| < \delta; \\ 0 < |y| < \delta \\ \therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0.$$

→ Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist.

$$\text{where } f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

$$\underline{\text{sol'n: }} \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = 1 = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y)$$

$$\therefore \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y) = 1 = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$$

∴ Repeated limits exist and are equal.

Let $(x,y) \rightarrow (0,0)$ along the coordinate axes.

$$\lim_{x \rightarrow 0} f(x,0) = 0 \stackrel{y \rightarrow 0}{=} \lim_{y \rightarrow 0} f(0,y) = 0$$

$\therefore f(x,y) \rightarrow 0$ as.

$(x,y) \rightarrow (0,0)$ along the coordinate axes.

Let $(x,y) \rightarrow (0,0)$ along any other path.

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 1$$

$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

→ Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ and $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y)$ exist but $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$ does not exist.

$$\text{where } f(x,y) = \begin{cases} y + x \sin(\frac{1}{y}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

sol'n: Here $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = \lim_{y \rightarrow 0} y = 0$

$$\text{Now } \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y) = \lim_{x \rightarrow 0} y = 0$$

$\therefore \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$ exists and is equal to 0.

$$\text{and now } |f(x,y) - 0| = |y + x \sin(\frac{1}{y})|$$

$$\leq |y| + |x| \quad (\because |\sin(\frac{1}{y})| \leq 1)$$

$< \epsilon$ whenever

$$0 < |x| < \epsilon_1/2; 0 < |y| < \epsilon_2/2$$

choosing $\epsilon_2 = \delta$.

$$\therefore |f(x,y) - 0| < \epsilon \text{ whenever } 0 < |x| < \delta; \\ 0 < |y| < \delta.$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0.$$

$$(x,y) \rightarrow (0,0)$$

* Algebra of Limits:-

If f, g are two functions defined on some neighbourhood of a point (a,b) .

such that $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = l$.

$$(x,y) \rightarrow (a,b)$$

$$\lim_{(x,y) \rightarrow (a,b)} g(x,y) = m \text{ then } (1) \lim_{(x,y) \rightarrow (a,b)} f+g = \lim_{(x,y) \rightarrow (a,b)} f + \lim_{(x,y) \rightarrow (a,b)} g = l+m$$

$$(2) \quad \text{Lt} (f \cdot g) = \text{Lt} f \cdot \text{Lt} g \\ = l \cdot m$$

$$(3) \quad \text{Lt} \left(\frac{f}{g} \right) = \frac{\text{Lt} f}{\text{Lt} g} = \frac{l}{m} \text{ provided } m \neq 0 \\ \text{when } (x,y) \rightarrow (a,b)$$

Problems

$$\rightarrow \text{Lt}(x^2 + 2y) \\ (x,y) \rightarrow (1,2) \quad \text{Lt} x^2 + \text{Lt} 2y \\ (x,y) \rightarrow (1,2) \quad (x,y) \rightarrow (1,2) \\ = 1 + 2(2) \\ = 1 + 4 \\ = 5$$

$$\rightarrow \text{Lt} \frac{x \sin(x^2+y^2)}{x^2+y^2} \\ (x,y) \rightarrow (0,0) \\ = \text{Lt} x \quad \text{Lt} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \\ (x,y) \rightarrow (0,0) \quad (x,y) \rightarrow (0,0) \\ = 0 \cdot 1 = 0$$

$$\rightarrow \text{Lt} \frac{\sin(xy-2)}{\tan(xy-6)} \\ (x,y) \rightarrow (2,1)$$

$$= \text{Lt} \frac{\sin(t)}{\tan(3t)} \quad [\text{Put } xy-2=t \text{ and} \\ t \rightarrow 0] \quad (x,y) \rightarrow (2,1) \\ \Rightarrow t \rightarrow 0 \\ = \text{Lt} \frac{1}{\sqrt{1-t^2}} \\ t \rightarrow 0 \\ = \frac{1}{\sqrt{1+9t^2}} \\ = \text{Lt} \frac{1}{3} \cdot \frac{1+9t^2}{\sqrt{1+t^2}} \\ t \rightarrow 0 \\ = \frac{1}{3}$$

* Continuity:

→ A function $f(x,y)$ is said to be continuous at a point (a,b) if

$$\text{Lt} f(x,y) = f(a,b) \\ (x,y) \rightarrow (a,b)$$

i.e. if for given $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x,y) - f(a,b)| < \epsilon$ whenever $|x-a| < \delta$ and $|y-b| < \delta$

→ If f is not continuous at (a,b) , $(a,b) \in D \subset \mathbb{R}^2$ then f is said to be discontinuous at (a,b) .

→ f is said to be continuous on the domain D , if f is continuous at each point of D .

Note:-

Let $D \subset \mathbb{R}^2$ and $f: D \rightarrow \mathbb{R}$ be continuous function at $(a,b) \in D$.

Let $f_1(x) = f(x,b)$, then f_1 is a function of single variable x .

Since $f(x,y)$ is continuous at (a,b) ,

$$\rightarrow \text{Lt}_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

i.e. given $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x,y) - f(a,b)| < \epsilon$ whenever $|x-a| < \delta$ and $|y-b| < \delta$; $(x,y) \in D$.

$\Rightarrow |f(x,b) - f(a,b)| < \epsilon$ whenever $|x-a| < \delta$; $(x,b) \in D$.

$\Rightarrow |f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$; $(x,b) \in D$.

$\Rightarrow f_1$ is continuous at a .

Similarly, we show that $f_2(y) = f(a,y)$ is continuous at b .

If $f(x,y)$ is continuous at (a,b)

then

- (i) $f(x,b)$ is continuous at $x=a$ and
- (ii) $f(a,y)$ is continuous at $y=b$.

But the converse of above is not true.

i.e. if $f(x,b)$ is continuous at $x=a$ and $f(a,y)$ is continuous at $y=b$. then $f(x,y)$ need not be continuous at (a,b) .

Problem :-

- (1) Examine the continuity at $(1,2)$ of the function

$$f(x,y) = \begin{cases} x^2+4y & \text{when } (x,y) \neq (1,2) \\ 0 & \text{when } (x,y) = (1,2) \end{cases}$$

Sol'n: Let $\lim_{(x,y) \rightarrow (1,2)} f(x,y) = \lim_{(x,y) \rightarrow (1,2)} (x^2+4y)$

$$\stackrel{(x,y) \rightarrow (1,2)}{=} \lim_{(x,y) \rightarrow (1,2)} x^2 + \lim_{(x,y) \rightarrow (1,2)} 4y$$

$$= 1^2 + 4(2)$$

$$= 1 + 8 = 9 \text{ and}$$

$\therefore \lim_{(x,y) \rightarrow (1,2)} f(x,y) \neq f(1,2)$, $f(1,2) = 0$

$\therefore f(x,y)$ is not continuous at $(1,2)$.

→ Show that the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{(y^2-x^2)yx}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Sol'n: Let $\epsilon > 0$ given, Now we have

$$|f(x,y) - f(0,0)| = \left| \frac{(y^2-x^2)yx}{x^2+y^2} \right| - 0$$

$$= \left| \frac{(y^2-x^2)}{x^2+y^2} \right| yx$$

$$= \left| \frac{y^2-x^2}{x^2+y^2} \right| |xy|$$

$$\leq |xy| \quad \left\{ \because \left| \frac{y^2-x^2}{x^2+y^2} \right| \leq 1 \text{ for } f(x,y) \neq 0 \right.$$

$$= |x| |y|$$

$$< \epsilon \text{ whenever } |x| < \sqrt{\epsilon} \text{ & } |y| < \sqrt{\epsilon}$$

choosing $\sqrt{\epsilon} = \delta$

$$\therefore |f(x,y) - f(0,0)| < \epsilon \text{ whenever}$$

$$|x| < \delta, |y| < \delta$$

$\therefore f(x,y)$ is continuous at $(0,0)$.

→ Discuss the continuity of the function

$$(i) f(x,y) = \begin{cases} \frac{2xy^2}{x^3+3y^3} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Sol'n: $f(0,0) = 0$

Let $(x,y) \rightarrow (0,0)$ along the coordinate axes then $\lim_{x \rightarrow 0} f(x,0) = 0 = \lim_{y \rightarrow 0} f(0,y)$.

$\therefore f(x,y) \rightarrow 0$ as $(x,y) \rightarrow (0,0)$ along the coordinate axes.

Let $(x,y) \rightarrow (0,0)$ along straight line $y=x$ then $\lim_{x \rightarrow 0} f(x,x) = \lim_{x \rightarrow 0} \frac{2x \cdot x^2}{x^3+3x^3} = \lim_{x \rightarrow 0} \frac{2x^3}{4x^3} = \frac{1}{2}$

$$= \lim_{x \rightarrow 0} \frac{2}{4} = \frac{1}{2}$$

$\therefore f(x,y) \rightarrow \frac{1}{2}$ as $(x,y) \rightarrow (0,0)$ along the straight line path.

Since the two methods of approach to the limiting points give different limiting values.

∴ the simultaneous limits do not exist.

i.e., If $f(x,y)$ does not exist $(x,y) \rightarrow (0,0)$

$\therefore f(x,y)$ is not continuous at $(0,0)$.

→ Show that function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$:

Soln! Let $\epsilon > 0$ be given

Now we have $|f(x,y) - f(0,0)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right|$

$$= \left| \frac{xy}{\sqrt{x^2+y^2}} \right|$$

$$= \left| xy \right| \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$\leq \frac{1}{2} \sqrt{x^2+y^2} \quad [\because |xy| \leq \frac{1}{2}(x^2+y^2)]$$

$$\Rightarrow \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}$$

$$\text{if } (xy) \neq (0,0)$$

$$< \sqrt{x^2+y^2}$$

$$< \epsilon$$

whenever $x^2+y^2 < \epsilon^2 = \delta$ (choosing)

$|f(x,y) - f(0,0)| < \epsilon$ whenever $x^2+y^2 < \epsilon^2$

$\therefore f(x,y)$ is continuous at $(0,0)$ (or)

Let $x = r\cos\theta$; $y = r\sin\theta$.

$$|f(x,y) - f(0,0)| = \frac{|xy|}{\sqrt{x^2+y^2}}$$

$$= \frac{|r^2 \sin\theta \cos\theta|}{r}$$

$$= r |\sin\theta| |\cos\theta|$$

$$\leq r \quad (\because |\sin\theta| \leq 1 \text{ & } |\cos\theta| \leq 1)$$

$$= \sqrt{x^2+y^2}$$

$$< \epsilon \text{ whenever } x^2+y^2 < \epsilon^2 = \delta.$$

$$\therefore |f(x,y) - f(0,0)| < \epsilon \text{ whenever } x^2+y^2 < \delta$$

$\therefore f(x,y)$ is continuous at $(0,0)$.

→ Show that the following functions are discontinuous at $(0,0)$

$$(i) f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$(ii) f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & ; (x,y) \neq 0 \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$(iii) f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

→ Show that the following functions are continuous at the origin.

$$(i) f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$(ii) f(x,y) = \begin{cases} \frac{x^3y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$(iii) f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

→ Discuss the following function for

continuity at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & ; x^2+y^2 \neq 0 \text{ i.e. } (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function w.r.t the variable.

→ Partial derivative of $f(x,y)$ w.r.t x denoted by $\frac{\partial f}{\partial x}$ or $f_x = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$, while those w.r.t y denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x, y)$.

$$\therefore \frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$\text{and } \frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

→ The partial derivatives at a particular point (a, b) are often denoted by $(\frac{\partial f}{\partial x})_{(a,b)}$, $\frac{\partial f(a,b)}{\partial x}$ or $f_x(a,b)$ and

$$(\frac{\partial f}{\partial y})_{(a,b)}, \frac{\partial f(a,b)}{\partial y}, f_y(a,b)$$

$$f_x(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\text{and } f_y(a,b) = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

Note: we have, by definition

$$f_x(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$= \lim_{x \rightarrow a} \frac{f(x, b) - f(a, b)}{x - a}$$

and

$$f_y(a, b) = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$= \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}$$

problem:Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function given by

$$f(x, y) = x^2 + xy + y^3$$

find $f_x(a, y)$ and $f_y(a, y)$.Soln: By definition,

$$f_x(a, y) = \lim_{h \rightarrow 0} \frac{f(a+h, y) - f(a, y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a+h)^2 + (a+h)y + y^3 - a^2 - ay - y^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2ab + b^2 + axy + aby + y^3 - a^2 - ay - y^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2ah + h^2 + aby}{h}$$

$$= \lim_{h \rightarrow 0} 2a + h + aby$$

$$= \underline{\underline{2a + by}}$$

Similarly

$$f_y(a, y) = \lim_{k \rightarrow 0} \frac{f(a, y+k) - f(a, y)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{x^2 + xy + y^3 + (y+k)^3 - x^2 - xy - y^3}{k}$$

$$= \lim_{k \rightarrow 0} \frac{ak + 3yk^2 + 3ky + k^3}{k}$$

$$= \lim_{k \rightarrow 0} x + 3y^2 + 3ky + k^2$$

$$= \underline{\underline{x + 3y^2}}$$

$\mathbb{R}^3 \rightarrow \mathbb{R}$ be a function defined by

$$(x, y, z) = xy + yz + zx.$$

Find the partial derivatives f_x, f_y, f_z at (a, b, c) .

$$\text{Given, } f(x, y, z) = xy + yz + zx.$$

definition (i)

$$f_x(a, b, c) = \lim_{h \rightarrow 0} \frac{f(a+h, b, c) - f(a, b, c)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a+h)b + bc + (a+h)a - ab - bc - ca}{h}$$

$$= \lim_{h \rightarrow 0} \frac{hb + ch}{h}$$

$$= b+c$$

$$f_y(a, b, c) = \lim_{k \rightarrow 0} \frac{f(a, b+k, c) - f(a, b, c)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{a(b+k) + (b+k)c + ca - ab - bc - ca}{k}$$

$$= a+c$$

$$f_z(a, b, c) = \lim_{l \rightarrow 0} \frac{f(a, b, c+l) - f(a, b, c)}{l}$$

$$= \lim_{l \rightarrow 0} \frac{ab + b(c+l) + (c+l)a - ab - bc - ca}{l}$$

$$= b+a$$

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function defined by

$$f(x_1, x_2, \dots, x_n) = x_1^n + x_2^n + \dots + x_n^n$$

Find the f_{x_i} at the point (a_1, a_2, \dots, a_n)

$$(a_1, a_2, \dots, a_n)$$

Sol: To find the partial derivative of f w.r.t x_i at the point (a_1, a_2, \dots, a_n)

we write

$$\begin{aligned}
 f_{x_i}(a_1, a_2, \dots, a_n) &= \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, a_2, \dots, a_n)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a_1^r + a_2^r + \dots + a_{i-1}^r + (a_i + h)^r + a_{i+1}^r + \dots + a_n^r - (a_1^r + a_2^r + \dots + a_n^r)}{h} \\
 &\stackrel{\text{h}}{=} \lim_{h \rightarrow 0} \frac{2a_i h + h^r}{h} = 2a_i^r.
 \end{aligned}$$

Note: $f_{x_i}(x, y)$ is nothing but the derivative of $f(x, y)$ considered as a function of a single variable x , treating y as a constant.

Similarly $f_{y_j}(x, y)$ is nothing but the derivative of $f(x, y)$, considering it as a function of the single variable y , and treating x as a constant.

In general, $(\frac{d}{dx_i}) f(x_1, x_2, \dots, x_n)$, i.e., the derivative of $(f(x_1, x_2, \dots, x_n))$ w.r.t x_i treating all the other variables as constants.

Let us find the partial derivatives of the following functions.

(i) $f = x^3 - 4xy^4 + 8y^2$

(ii) $f = x^2 \sin y + y \cos x$

(iii) $f = x e^y + y e^x$

Sol: In all the three cases, the functions involved are either polynomials or

trigonometric or exponential functions.

This method ensures that the partial derivatives exist.

(\because the polynomial, trigonometric and exponential functions

of single variable are differentiable).

we get.

$$(i) \frac{\partial f}{\partial x} = 3x^2 - 8xy^2$$

$$\text{and } \frac{\partial f}{\partial y} = -8x^2y + 16y$$

$$(ii) \frac{\partial f}{\partial x} = \sin y - y \sin x$$

$$\text{and } \frac{\partial f}{\partial y} = x \cos y + \cos x$$

$$(iii) \frac{\partial f}{\partial x} = e^y + ye^x$$

$$\text{and } \frac{\partial f}{\partial y} = xe^y + e^x$$

\rightarrow if $f(x,y) = 2x^2 - xy$ then
then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(1,2)$.

$$\text{Sol: } \frac{\partial f}{\partial x} = 4x - y = 2 \text{ at } (1,2)$$

$$\frac{\partial f}{\partial y} = -x + 4y = 7 \text{ at } (1,2)$$

Note: The calculation of partial derivatives is not always as simple as in these examples. In some exceptional cases, we have to use the limiting process.

Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Find the two partial derivatives at the points $(0,0)$, $(a,0)$, $(0,b)$ and (a,b) where $a \neq 0$, $b \neq 0$.

Sol: By definition

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,0+k) - f(0,0)}{k}$$
$$= \lim_{k \rightarrow 0} \frac{0-0}{k} = 0$$

$$f_x(a,0) = \lim_{h \rightarrow 0} \frac{f(a+h,0) - f(a,0)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{(0-h)}{h} = 0$$

$$f_y(a,0) = \lim_{k \rightarrow 0} \frac{f(a,0+k) - f(a,0)}{k} = \lim_{k \rightarrow 0} \frac{f(a,k) - f(a,0)}{k}$$
$$= \lim_{k \rightarrow 0} \frac{\frac{ak}{a^4+k^4}}{k} = \lim_{k \rightarrow 0} \frac{-a}{a^4+k^4}$$
$$= \frac{a}{a^4} = \frac{1}{a^3}$$

$$f_x(0,b) = \lim_{h \rightarrow 0} \frac{f(0+h,b) - f(0,b)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\frac{bh}{b^4+h^4}}{h} = \frac{1}{b^3}$$

$$f_x(0, b) = \lim_{k \rightarrow 0} \frac{f(0, b+k) - f(0, b)}{k}$$
$$= \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0$$

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{(a+h)b - ab}{h} = \frac{ab}{a^4 + b^4}$$
$$= \lim_{h \rightarrow 0} \frac{(ab + h)b - ab}{h} = \frac{b^2}{a^4 + b^4}$$
$$= \lim_{h \rightarrow 0} \frac{(a^4 + b^4) - (a^4 + b^4)}{h} = 0$$

$$f_y(a, b) = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$
$$= \lim_{k \rightarrow 0} \frac{ab + k - ab}{k} = \frac{ab}{a^4 + b^4}$$
$$= \frac{a^5 - 3ab^4}{a^4 + b^4}$$

Notes In the above problem, by direct differentiation, we could have obtained $f_x(a, b)$ and $f_y(a, b)$ correctly, but not $f_x(0, 0)$ or $f_y(0, 0)$.

because f is defined as a quotient of two polynomial functions for all $(x, y) \neq (0, 0)$; we can use direct differentiation to calculate the partial derivatives at these points. But to calculate $f_x(0, 0)$ or $f_y(0, 0)$ we need to use $f(0, 0)$, which is not defined.

→ If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{x}{y} + \frac{y}{x}, & y \neq 0, x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then show that $f_x(0,1)$ and $f_y(1,0)$ do not exist.

Soln: By definition

$$\begin{aligned} f_x(0,1) &= \lim_{h \rightarrow 0} \frac{f(0+h,1) - f(0,1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h,1) - f(0,1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h + \frac{1}{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{h + \frac{1}{h}}{h} \\ &= \lim_{h \rightarrow 0} \left(1 + \frac{1}{h}\right) = \infty \end{aligned}$$

and $f_y(1,0) = \lim_{k \rightarrow 0} \frac{f(1,0+k) - f(1,0)}{k}$

$$\begin{aligned} &= \lim_{k \rightarrow 0} \frac{\frac{1}{k} + k - 0}{k} \\ &= \lim_{k \rightarrow 0} \frac{\frac{1}{k} + 1}{k} \\ &\Rightarrow -\infty \end{aligned}$$

$f_x(0,1)$ and $f_y(1,0)$ do not exist

→ Note:

The existence of partial derivatives at a point need not imply continuity at that point.

for example:

Ques. If $f(x,y) = \begin{cases} \frac{xy}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that both the partial derivatives exist at $(0,0)$ but the function is not continuous at $(0,0)$.

$$\text{Sol: } f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0 - 0}{h}$$

$$= 0$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{0 - 0}{k}$$

\therefore if f possesses both the partial derivatives at $(0,0)$.

Now let $(x,y) \rightarrow (0,0)$ along the straight line $y=mx$.

$$\lim_{n \rightarrow \infty} f(n, mx) = \lim_{n \rightarrow \infty} \frac{mn^2}{2(1+m^2)}$$

$= \frac{m}{1+m^2}$
which depends upon m .

$\therefore f(x,y)$ does not exist.

$$(x,y) \rightarrow (0,0)$$

$\therefore f(x,y)$ is not continuous at $(0,0)$.

Ques Find $f_x(0,0)$ and $f_y(0,0)$ & $f_{xy}(0,0)$

$$\text{Ans: } f(x,y) = \begin{cases} \frac{x-y}{x+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$$\text{If } f(x,y) = \begin{cases} x^3 + y^3 & ; x \neq y \\ 0 & ; x = y \end{cases}$$

then show that f is discontinuous at the origin but the partial derivatives exist at the origin.

Soln: Let $(x,y) \rightarrow (0,0)$ along the curve $y = x - mx^3$.

Ques Show that the given below is not continuous at $(0,0)$

$$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$$

Soln: Let $(x,y) \rightarrow (0,0)$ along the coordinate axes.

$$\therefore \lim f(x,0) = \lim 0 = 0$$

$$\text{and } \lim f(0,y) = \lim 0 = 0$$

$$\therefore \lim f(x,0) = 0 = \lim f(0,y)$$

$\therefore f(x,y) \rightarrow 0$ as $(x,y) \rightarrow (0,0)$ along the coordinate axes.

Let $(x,y) \rightarrow (0,0)$ along any other path

$$\lim f(x,y) = 1$$

$$(x,y) \rightarrow (0,0)$$

for example:

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|.$$

Show that f is continuous at $(0, 0, 0)$ but does not possess any of the three first order partial derivatives at $(0, 0, 0)$.

Soln:

Now at the point $(0, 0, 0)$,

we have

$$\frac{f(0+h, 0, 0) - f(0, 0, 0)}{h} = \frac{|h|}{h} = f_1(h) \text{ (say)}$$

$$\therefore \lim_{h \rightarrow 0^+} f_1(h) = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1.$$

$$\text{but } \lim_{h \rightarrow 0^-} f_1(h) = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1.$$

Hence $\lim_{h \rightarrow 0} f_1(h)$ does not exist.

Similarly, $\lim_{h \rightarrow 0} f_2(h)$ and $\lim_{h \rightarrow 0} f_3(h)$ also do not exist.

So if f does not possess any of the three first order partial derivatives at the point $(0, 0, 0)$, But the function is continuous,

at $\underline{(0, 0, 0)}$.

Ans. b.

Since the two methods of approach to the limiting point give different limiting values.

\therefore $f(x,y)$ does not exist.

$$(x,y) \rightarrow (0,0)$$

12.11 $\therefore f(x,y)$ is not continuous at $(0,0)$.

2006 \rightarrow Show that the function given by

$$f(x,y) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

(i) is continuous at $(0,0)$

(ii) possesses partial derivatives $f_x(0,0)$ & $f_y(0,0)$

2007 \rightarrow Show that the function given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$ but partial derivatives $f_x(0,0)$, $f_y(0,0)$ exist at $(0,0)$.

\rightarrow Examine the continuity of the function

$$f(x,y) = |xy|$$
 at the origin.

to be done

to do

Note We know that a real valued continuous function of a real variable need not be differentiable. The same is true for functions of several variables.

i.e., functions of several variables which are continuous at a point need not have any of the partial derivatives at the point.

We have seen that the existence of partial derivatives does not imply continuity. However, if partial derivatives satisfy some more conditions, then we can ensure continuity. In order to prove this theorem we need a simple result which follows easily from Lagrange's mean value theorem.

Mean value theorem:
If f_x exists throughout a nbd of a point (a, b) and $f_y(a, b)$ exists then for any point $(ath, b+k)$ of this nbd,

$$f(ath, b+k) - f(a, b) = h f_x(a + \theta h, b + k) + k(f_y(a, b) + \eta), \quad (1)$$

where $0 < \theta < 1$ and η is a function of k , which tends to 0 as $k \rightarrow 0$.

Note: we can see that this is an extension of Lagrange's mean value theorem to functions from $\mathbb{R}^n \rightarrow \mathbb{R}$.

Interchanging x and y in the above theorem, the theorem can be written as follows.

If f_x, f_y exist throughout a nbd of a point (a, b) and $f_x(a, b)$ exists then for any point $(ath, b+k)$ of the nbd,

$$f(ath, b+k) - f(a, b) = k f_y(a + \theta k, b) + f_x(a, b) + \eta, \quad (2)$$

where $0 < \theta < 1$ and η is a function of k and tends zero as $k \rightarrow 0$.

A Sufficient Condition for continuity:

A sufficient condition that a function

If f be continuous at (a, b) & that one of the partial derivatives exists and is bounded in a nbd of (a, b) and that the other exists at (a, b) .

५०८

Let f be cont. and bounded in a nbd of x_0

(a, b) - and let $f_y(a, b)$ exist.

then for any point $(a+b, b+k)$ of this nbhd

we have

$$f(a+b, b+k) - f(a, b) = h \int_a^b f_x(a+th, b+k) dt \\ + k \left[f_y(a, b) + \eta_1 \right] \quad (1)$$

where $\eta \neq 0$, and $\eta \rightarrow 0$ as $k \rightarrow \infty$
 proceeding to limits as $(k, \eta) \rightarrow (0, 0)$,

Since the boundary in a neighborhood of x_0

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it follows that

$$\text{L}(\text{h}_x^f, \text{Cat}(\theta h, h+k)) = \emptyset$$

$$(h, k) \rightarrow (0, 0)$$

Consequently, from ①, we get

$$f(a+bk) = f(a, b)$$

$$(h,k) \rightarrow (0,0)$$

f is continuous at (a, b)

Note: A sufficient condition that a function be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

of the mean value theorem

$$f(a+kh, b+k) - f(a, b) = f(a+kh, b+k) - f(a, b+k) + f(a, b+k) - f(a, b)$$

(1)

Since f_a exists throughout a nbhd of

a point (a, b) ,
therefore by Lagrange's mean value
theorem,

$$f_a(a+kh, b+k) = \frac{f(a+kh, b+k) - f(a, b+k)}{h} \quad 0 < h < 1$$

$$\Rightarrow f(a+kh, b+k) - f(a, b+k) = h \cdot f_a(a+kh, b+k) \quad 0 < h < 1$$

(2)

Also $f_y(a, b)$ exists, so the

$$\lim_{K \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{K} = f_y(a, b).$$

$$\Rightarrow \lim_{K \rightarrow 0} [f(a, b+k) - f(a, b)] = K \lim_{K \rightarrow 0} [f_y(a, b) + \gamma(K)]$$

where $\gamma(K)$ is a
function of K .

$$\therefore f(a, b+k) - f(a, b) = K f_y(a, b) + K \gamma(K)$$

(3)

where γ is a function of K .

and tends to zero as $K \rightarrow 0$.

From (1), (2) and (3), we have

$$f(a+kh, b+k) - f(a, b) = h f_a(a+kh, b+k) + K [f_y(a, b) + \gamma(K)]$$

which is the required result.

Differentiability

Let f be a real valued function defined

in a nbhd N of a point (a, b) :

If we say that the function f is differentiable at (a, b) , if

$$(i) f(a+h, b+k) - f(a, b) = Ah + BK + h\phi(h, k) + k\psi(h, k)$$

where

- h and k are real numbers such that $(a+h, b+k) \in N$
- A and B are constants independent of h and k but dependent on the function f and the point (a, b)
- ϕ and ψ are two functions tending to zero as $(h, k) \rightarrow (0, 0)$.

(or)

Let f be a real valued function defined in a nbhd of a point (a, b) , then the function is said to be differentiable at the point (a, b) , if there exist two constants A and B (depending on

and the point (a, b) only) such that

$$f(a+h, b+k) - f(a, b) = Ah + BK + \sqrt{h^2 + k^2} \phi(h, k);$$

where $\phi(h, k)$ is real valued function such that $\phi(h, k) \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$.

→ Let $f(x, y) = x^m y^n$ s.t. f is differentiable

at any point (a, b) .

so for any two real numbers h and k ,

we have

$$\begin{aligned} f(a+h, b+k) - f(a, b) &= (a+h)^m + (b+k)^n - (a^m + b^n) \\ &= 2ah + \frac{1}{2}h^2 + 2bk + k^2 \end{aligned}$$

$$= 2ah + 2bk + hh + kk.$$

If we let $A = 2a$, $B = 2b$, $\phi(h, k) = h$ and
 $\psi(h, k) = k$.

then

$$f(a+h, b+k) = f(a, b) + Ah + Bk + \phi(h, k) + k\psi(h, k)$$

where A and B are constants independent of h & k :

$$\phi(h, k) \rightarrow 0 \text{ and } \psi(h, k) \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0).$$

f is differentiable at the point (a, b) .

Let $f(x, y) = \frac{a}{y}$. Then show that f is differentiable at all points (x, y) in the domain of definition of the function.

Given $f(x, y) = \frac{a}{y}$.
Since f is not defined for $y = 0$,

we take $b \neq 0$.

Let a and b be two real numbers such that $(a+h, b+k)$ is a point in a nbhd N of (a, b) which is contained in the domain of f .

Then $b+k \neq 0$.

$$\begin{aligned} \text{and } f(a+h, b+k) - f(a, b) &= \frac{ah}{b+k} - \frac{a}{b} \\ &= \frac{a}{b+k} - \frac{a}{b} + \frac{ab}{b+k} \\ &= \frac{-ak}{b(b+k)} + \frac{b}{b+k} \\ &= -\frac{ak}{b^2} \left[1 - \frac{k}{b+k} \right] + \frac{b}{b+k} \\ &= \frac{1}{b}h - \frac{a}{b^2}k + h \left[\frac{-k}{b(b+k)} \right] + \\ &\quad k \left(\frac{ak}{b^2(b+k)} \right) \end{aligned}$$

Thus set $A = \frac{1}{b}$, $B = -\frac{a}{b^2}$, $\phi(h, k) = -\frac{k}{b(b+k)}$ and $\psi(h, k) = \frac{ak}{b^2(b+k)}$.

$$(b; k) \text{ if } f(a, b) = Ah + BK + h\phi(h, k) + k\psi(h, k)$$

where A and B are constants independent of h and k, $\phi(h, k) \rightarrow 0$ and $\psi(h, k) \rightarrow 0$ as

$$(h, k) \rightarrow (0, 0)$$

Hence f is differentiable at (a, b).

→ Prove that the function given by $f(x, y) = |x| + |y|$ is not differentiable at (0, 0).

Sol: Suppose, if possible that f is differentiable at (0, 0).

$$\text{Then } f(0+h, 0+k) - f(0, 0) = Ah + BK + h\phi(h, k) + k\psi(h, k)$$

where A and B are constants.

$$\phi(h, k), \text{ and } \psi(h, k) \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0)$$

$$Ah + BK + h\phi(h, k) + k\psi(h, k) \quad (1)$$

Let $h=0$ and $k>0$, then from (1),

$$K = BK + K\psi(0, k) \Rightarrow 1 = B + \psi(0, k)$$

Taking limits on both sides as $(h, k) \rightarrow (0, 0)$

$$\text{we get } B=1, \text{ because } \psi(0, k) \rightarrow 0$$

Now, let $h=0$ and $k<0$, then

$$-K = BK + K\psi(0, k)$$

$$-1 = B + \psi(0, k)$$

Taking limits on both sides as $(h, k) \rightarrow (0, 0)$

$$\text{we get } B=-1; \text{ because } \psi(0, k) \rightarrow 0$$

The assumption that the given function is differentiable at (0, 0) leads us to the contradiction $B=1=-1$.

$|x| + |y|$ is not differentiable at (0, 0).

Theorem Let f be real valued function defined in a nbd' N of a point (a, b) . If f is differentiable at (a, b) , then f is continuous at (a, b) .

The above theorem shows that continuity in the two variables is a necessary condition for differentiability. However, it is not a sufficient condition.

For example,

Show that the function defined by

$$\text{Ans} \quad f(x, y) = \begin{cases} xy & ; (x, y) \neq (0, 0) \\ 0 & ; (x, y) = (0, 0) \end{cases}$$

is continuous at the origin but not differentiable.

Sol: we were already told that f is continuous in the pg no. 50 back page. Now we prove that $f(x, y)$ is not differentiable at the origin.

We have

$$f(0+h, 0+k) = \frac{hk}{\sqrt{h^2+k^2}}$$

$$\therefore f(0+h, 0+k) - f(0, 0) = h+k + \sqrt{h^2+k^2} \cdot \frac{hk}{\sqrt{h^2+k^2}}$$

$$\text{So that } A=0, B=0 \text{ and } \phi(h, k) = \frac{hk}{\sqrt{h^2+k^2}}$$

i.e. A and B are independent of h, k .

If we put $k=mh$, then we have

$$\lim_{(h, k) \rightarrow (0, 0)} \phi(h, k) = \lim_{h \rightarrow 0} \frac{hk}{\sqrt{h^2+m^2h^2}} = \lim_{h \rightarrow 0} \frac{hm}{\sqrt{1+m^2}}$$

$$= \lim_{h \rightarrow 0} \frac{mh^2}{h^2(1+m^2)} = \frac{m}{1+m^2}$$

\therefore This limit does not exist since it depends.

upon m :

$\therefore \lim_{(h,k) \rightarrow (0,0)} \phi(h, k) \neq 0$ as $(h, k) \rightarrow (0, 0)$.

It follows that the given function is not differentiable at $(0, 0)$.

Note: To show that the function is not differentiable, it enough to show that it is not continuous at $(0, 0)$.

for example, $(0, 0) \neq f(0, 0)$:

The function f :

$$\text{where } f(x, y) = \begin{cases} xy & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

is not differentiable at the origin

because it is discontinuous there.

Show that the function f ,

$$\text{where } f(x, y) = \begin{cases} \frac{xy}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$$

is not differentiable.

Show that the following functions are not differentiable at $(0, 0)$ by showing that they are discontinuous at $(0, 0)$.

$$(1) f(x, y) = \begin{cases} \frac{x^4 + y^4}{x-y} & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases} \quad (2) f(x, y) = \begin{cases} \frac{xy}{x+y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$(3) f(x, y) = \begin{cases} \frac{x^5}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$[\text{Hint: } x^4 \geq x^2 \Rightarrow |f(x, y)| \leq \left| \frac{x^5}{x^4 + y^4} \right| \leq \left| \frac{x^5}{x^4} \right| = |x|]$$

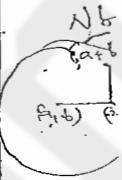
If f is differentiable at (a,b) then f possesses both the partial derivatives at (a,b) .

Sol Since f is differentiable at (a,b)

$$f(a+h, b+k) - f(a,b) = Ah+Bk + h\phi(h,k) \quad \text{--- (1)}$$

where A and B are constants depend on f & the point (a,b) .
and independent of h & k .

$\phi(h,k) \rightarrow 0, \psi(h,k) \rightarrow 0$ as $(h,k) \rightarrow (0,0)$.



If $(a+h, b+k)$ belonging to the neighborhood of (a,b)
then $(a+h, b)$ and $(a, b+k)$ also belong to neighborhood of (a,b) .

If we take $k=0$ in (1) then, we have

$$f(a+h, b+0) - f(a,b) = ah + \phi(h,0).$$

$$\Rightarrow \frac{f(a+h, b) - f(a,b)}{h} = A + \phi(h,0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a,b)}{h} = A \quad (\because \lim_{h \rightarrow 0} \phi(h,0) = 0)$$

$$\Rightarrow [f_x(a,b)] = A \quad \left(\frac{\partial f}{\partial x}(a,b) \right)$$

Similarly we can prove that $B = \left(\frac{\partial f}{\partial y}(a,b) \right)$

From this we see that for small values of h and k we can approximate $f(a+h, b+k) - f(a,b)$ by the expression.

$$h f_x(a,b) + k f_y(a,b)$$

$$\text{i.e. } f(a+h, b+k) - f(a,b) = h f_x(a,b) + k f_y(a,b).$$

Defn. Let $f(x,y)$ be a real-valued function defined in a neighborhood of the point (a,b) . Then

If $f(x,y)$ is differentiable at (a,b) then

the linear function $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$T(h,k) = h f_x(a,b) + k f_y(a,b)$$

is called the differential of f at (a,b) and it is denoted by $df(a,b)$.

Note!

The converse of the above need not be true. i.e. if f is continuous and possesses partial derivative at a point then f need not be differentiable at that point.

for example:-

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous and possesses partial derivatives but is not differentiable at the origin.

Sol

put $x = r\cos\theta$, $y = r\sin\theta$

$$\begin{aligned} \left| \frac{x^2-y^2}{x+y} - 0 \right| &= \left| r(\cos^2\theta - \sin^2\theta) \right| \\ &\leq |r| [|\cos^2\theta| + |\sin^2\theta|] \\ &\leq |r| [1+1] \\ &= 2|r|. \end{aligned}$$

$$= 2\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} < \epsilon$$

$$\text{whenever } r < \frac{\epsilon}{2},$$

$$r < \frac{\epsilon}{2} \quad (\text{or})$$

$$\begin{aligned} |x| &< \frac{\epsilon}{2}, |y| &< \frac{\epsilon}{2} \\ |x| &< \frac{\epsilon}{2r}, |y| &< \frac{\epsilon}{2r} \end{aligned}$$

$$\Rightarrow \left| \frac{x^2-y^2}{x+y} - 0 \right| < \epsilon \text{ whenever } |x| < \frac{\epsilon}{2r}, |y| < \frac{\epsilon}{2r}$$

$\therefore f(x,y)$ is conti. in $(0,0)$.

$$\begin{aligned} & \text{If } f(0+h, 0) - f(0, 0) \\ & h \rightarrow 0 : \frac{f(0+h, 0) - f(0, 0)}{h} \\ & = \lim_{h \rightarrow 0} \frac{\frac{h^3}{h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h^3} = 1 \\ & f(0, k) = \lim_{k \rightarrow 0} \frac{f(0, 0+k) - f(0, 0)}{k} \\ & = \lim_{k \rightarrow 0} \frac{\frac{-k^3 - 0}{h^2}}{k} = \lim_{k \rightarrow 0} \frac{-k^3}{k^3} = -1 \end{aligned}$$

f possesses partial derivatives at $(0, 0)$.

Now we prove that f is not differentiable at $(0, 0)$.

If possible suppose that f is differentiable at $(0, 0)$.

$$\text{Then } f(0+h, 0+k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + \sqrt{h^2+k^2} \phi(h, k)$$

where $\phi(h, k) \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$.

where $A = f_x(0, 0) = 1$

and $B = f_y(0, 0) = 1$.

$$\Rightarrow f(h, k) = h + k + \sqrt{h^2+k^2} \phi(h, k)$$

$$\Rightarrow \phi(h, k) = \frac{f(h, k) - h - k}{\sqrt{h^2+k^2}} \quad \text{where } \phi(h, k) \rightarrow 0$$

$$\text{i.e., } \lim_{(h, k) \rightarrow (0, 0)} \frac{f(h, k) - h - k}{\sqrt{h^2+k^2}} = 0$$

Now if $h = r \cos \theta$, $k = r \sin \theta$:

then

$$\begin{aligned} \frac{f(h, k) - h - k}{\sqrt{h^2+k^2}} &= \frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta - r \cos \theta + r \sin \theta}{r^2 (\cos^2 \theta + \sin^2 \theta)} \\ &= \frac{r(\cos^3 \theta - \sin^3 \theta - \cos \theta + \sin \theta)}{r^2} \\ &= \cos^3 \theta - \sin^3 \theta - \cos \theta + \sin \theta \end{aligned}$$

$$0 = \lim_{(h,k) \rightarrow (0,0)} \frac{f(h,k) - h+k}{\sqrt{h^2+k^2}} = \lim_{r \rightarrow 0} \frac{(\cos^3 \theta - \sin^2 \theta + \cos \theta + \sin \theta)}{\sqrt{2r^2}} \quad (1)$$

Since the expression $\cos^3 \theta - \sin^2 \theta + \cos \theta + \sin \theta$ is

independent of θ and (1) implies that

$$\cos^3 \theta - \sin^2 \theta + \cos \theta + \sin \theta = 0 \quad \forall \theta$$

which is impossible for arbitrary θ .

Our assumption that f is differentiable

is wrong.

f is not differentiable at $(0,0)$

Sufficient condition for differentiability

Theorem: If (a,b) be a point of the domain of definition

of a function f such that

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b)

then f is differentiable at (a,b) .

Similarly, the statement that f is differentiable at (a,b) if f_x exists at (a,b) and f_y is continuous at (a,b) is true.

i.e., the continuity of one of partial derivatives and the existence of others guarantees the differentiability of the function under consideration.

Note: The conditions of the theorem are not necessary for differentiability i.e. a function can be differentiable at a point even when none of the partial derivatives is continuous at that point.

However, if the function is continuous, then the conditions of the theorem will be sufficient.

for example:

Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y} & \text{if } xy \neq 0 \\ x \sin \frac{1}{x} & \text{if } x \neq 0 \text{ and } y=0 \\ y \sin \frac{1}{y} & \text{if } x=0 \text{ and } y \neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$$

Prove that f is differentiable at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$.

Sol: Here the partial derivatives at $(0,0)$ are given by

$$f_x(x,y) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

$$\text{and } f_y(x,y) = \begin{cases} 2y \sin \frac{1}{y} - \cos \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y=0. \end{cases}$$

Since $\lim_{x \rightarrow 0} \cos \frac{1}{x}$ does not exist and $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

$\therefore \lim_{(x,y) \rightarrow (0,0)} f_x(x,y)$ and $\lim_{(x,y) \rightarrow (0,0)} f_y(x,y)$ do not exist.

i.e., f_x and f_y are discontinuous at $(0,0)$.

Both the partial derivatives exist at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$.

Now show that the function is differentiable at $(0,0)$.

$$\begin{aligned} f(h,k) - f(0,0) &= h \sin \frac{1}{h} + k \sin \frac{1}{k} \\ &= oh + ok + h \cdot h \sin \frac{1}{h} + k \cdot k \sin \frac{1}{k} \end{aligned}$$

Now $\lim_{(h,k) \rightarrow (0,0)} h \sin \frac{1}{h} = 0$ and $\lim_{(h,k) \rightarrow (0,0)} k \sin \frac{1}{k} = 0$.

$\therefore f$ is differentiable at $(0,0)$.

Note

→ A real valued function f of two variables is said to be continuously differentiable at a point (a, b) if both the first order partial derivatives exist in a nbd of (a, b) and are continuous at the point (a, b) .

→ A function, which is continuously differentiable at a point is differentiable at that point.

→ Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function given by

$$f(x, y) = \begin{cases} xy + \frac{x-y}{x+y}, & \text{if } x+y \neq 0 \\ 0, & \text{if } x+y=0. \end{cases}$$

Show that f is differentiable at $(0, 0)$.

Now $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$

$$= 0$$

Similarly, $f_y(0, 0) = 0$

and for $(x, y) \neq (0, 0)$

$$f_x(x, y) = \frac{x^4 y + 4x^3 y^3 - y^5}{(x^2 + y^2)^2}$$

Using polar co-ordinates.

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\begin{aligned} \text{we get } |f_x(x, y) - f_x(0, 0)| &= r \left| \cos^4 \theta \sin \theta + 4 \cos^3 \theta \sin^3 \theta - \sin^5 \theta \right| \\ &\leq r \left[|\cos^4 \theta \sin \theta| + 4 |\cos^3 \theta \sin^3 \theta| + |\sin^5 \theta| \right] \\ &= 6r \quad (\because \sin \theta \leq 1 \Rightarrow \cos \theta \leq 1) \\ &= 6\sqrt{x^2 + y^2}. \end{aligned}$$

$$\leq C \quad \text{if } |x| < \frac{C}{\sqrt{2}} \text{ and } |y| < \frac{C}{\sqrt{2}}$$

$$\therefore f_x(x, y) = f_x(0, 0)$$

$$(x, y) \rightarrow (0, 0)$$

we have f_x is continuous at $(0, 0)$ and f_y exists at $(0, 0)$.

f is differentiable at $(0, 0)$.

→ S.t. $f(x, y) = e^{xy} \sin x + x^2 + xy$ is differentiable everywhere.

Since $f_x(x, y) = e^{xy} \sin x + e^{xy} y \cos x + 2x + 2y$ and $f_y(x, y) = e^{xy} + x^2 + 2x + 2y$ is differentiable everywhere.

PARTIAL DERIVATIVES OF HIGHER ORDER

we studied partial derivatives of first order and differentiability. We must have seen often that partial derivatives of first order again define functions.

for example:

$$\text{if } f(x,y) = 3x^3 + 2xy^2 + 5y^5 + 6,$$

$$\text{then } f_x(3,y) = 9x^2 + 2y$$

and $f_y(x,y) = 4xy + 10y^4$ are again real valued functions of two variables.

with the domain \mathbb{R}^2 .

Now we can talk of first order partial derivatives of these new functions.

If we consider a function of two variables

there are two first order partial derivatives, which may give rise to four more partial derivatives.

which might again turn out to be functions.

If this chain continues, then we obtain higher order partial derivatives.

In general, let $D \subset \mathbb{R}^2$ and let $f: D \rightarrow \mathbb{R}$ have a first order partial derivative $f_{xx}(x)$ at every point of D . This new function f_x , which is defined on D may or may not possess first order partial derivatives.

In case it does, then f_{xx} and f_{yy} are called

the second order partial derivatives of f .

Similarly, if the function f has a first order partial derivative f_y at every point of D , then f_y defines a new function and if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{xy} and f_{yy} .

\therefore If $f(x, y)$ is a real-valued function defined in a nbd of (a, b) having both the partial derivatives at all the points of the nbd, then

$$f_{xx}(a, b) = \lim_{h \rightarrow 0} \frac{f_x(a+h, b) - f_x(a, b)}{h} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) (\text{say})$$

$$f_{yx}(a, b) = \lim_{k \rightarrow 0} \frac{f_y(a, b+k) - f_y(a, b)}{k} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) (\text{say})$$

$$f_{xy}(a, b) = \lim_{h \rightarrow 0} \frac{f_y(a+h, b) - f_y(a, b)}{h} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) (\text{say})$$

$$f_{yy}(a, b) = \lim_{k \rightarrow 0} \frac{f_y(a, b+k) - f_y(a, b)}{k} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) (\text{say})$$

provided each one of these limits exists.

The second order partial derivatives of f are also denoted by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}; \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}; \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, \frac{\partial^2 f(a,b)}{\partial x^2}, f_{xx}(a,b)$ or $f_{xx}(a,b)$;

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}, \frac{\partial^2 f(a,b)}{\partial x \partial y}$ or $f_{xy}(a,b)$ and so on.

In a similar manner partial derivatives of orders higher than two are defined.

for example

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial y \partial z} \right) = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) \right]$$

(i.e.) f_{xzy} or f_{zyx}

i.e. $\frac{\partial^3 f}{\partial x \partial y \partial z}$ stands for the partial derivative of f with respect to x .

$\frac{\partial^2 f}{\partial y \partial z}$ with respect to y

→ Find all the second order partial derivatives of the following function.

(i) $f(x, y) = x^3 + y^3 + 3axy$, ~~a~~ constant

(ii) $f(x, y, z) = x^2y^2 + xyz^3$.

(i) $f(x, y) = x^3 + y^3 + 3axy$
~~a~~
 $\frac{\partial f}{\partial x} = 3x^2 + 3ay$ and $\frac{\partial f}{\partial y} = 3y^2 + 3ax$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (3x^2 + 3ay) = 6x$$

$$\text{and } \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3y^2 + 3ax) = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 3a$$

$$\text{and } \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 3a$$

(iv) For $f(x,y,z) = x^2 + yz + xz^3$,

$$\frac{\partial f}{\partial x} = 2x + z^2 ; \quad \frac{\partial f}{\partial y} = z ; \quad \frac{\partial f}{\partial z} = y + 3xz^2$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2 ; \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 0 ;$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) = 3z^2.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 0 ; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) = 0$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right) = 3z^2 ; \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) = 1$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) = 6xz$$

\rightarrow if $f(x,y) = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$, $x \neq 0, y \neq 0$

$$\text{Show that } \frac{\partial^2 f}{\partial x^2} = \frac{x^2 - y^2}{x^2 y^2} - \frac{2xy}{x^2 + y^2}$$

$$\text{Solt: } f(x,y) = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$$

$$\begin{aligned} \text{Now, } \frac{\partial f}{\partial y} &= x^2 \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1} \frac{x}{y} - y \cdot \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2} \right) \\ &= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^3}{x^2 + y^2} \\ &= \frac{x(x^2 + y^2) - 2y^2 x}{(x^2 + y^2)^2} \tan^{-1} \frac{x}{y} \\ &= x - 2y \tan^{-1} \frac{x}{y} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \\
 &= \frac{\partial}{\partial x} \left(x - 2y \tan^{-1} \frac{x}{y} \right) \\
 &= 1 - 2y \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(\frac{1}{y} \right) \\
 &= 1 - \frac{2y^2}{x^2 + y^2} \\
 &= \frac{x^2 - y^2}{x^2 + y^2}
 \end{aligned}$$

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\rightarrow If $f(x,y,z) = e^{xyz}$, then show that

$$\frac{\partial^2 f}{\partial x \partial y \partial z} = (1 + 2xyz + xy^2z^2) e^{xyz}.$$

\rightarrow Find all the second order partial derivatives of the following functions.

(a) $f(x,y) = \cos \frac{y}{x}$ (b) $f(x,y) = x^5 + y^4 \sin x$

(c) $f(x,y,z) = \sin xy \tan yz^2 + \cos xz$

(d) $f(x,y,z) = xy^2 + xz^2 + x^3y$

\rightarrow If $f(x,y,z) = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

Show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

\rightarrow Verify that $\frac{\partial f}{\partial x y} = \frac{\partial f}{\partial y x}$ for each of the following functions

(a) $f(x,y) = x^3y + e^{xy^2}$ (b) $f(x,y) = \tan(xy^3)$

(c) $f(x,y) = \frac{xy}{x+y}$ (d) $f(x,y) = x \tan xy$

→ Already we have seen that it is not always possible to find first order partial derivatives by direct differentiation.
The same is true for higher order partial derivatives of some functions.

(1) for example:

Consider the function

$$f(x, y) = \begin{cases} xy & (x-y) \\ x+y & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Find the second order partial derivatives
- of f at (0,0).

Sol: Since $f_{xx}(0,0) = \lim_{h \rightarrow 0} \frac{f_x(h,0) - f_x(0,0)}{h}$ ①

we first evaluate $f_x(h,0)$ and $f_x(0,0)$.

$$\text{now } f_x(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0-0}{t} = 0$$

$$f_x(h,0) = \lim_{t \rightarrow 0} \frac{f(h+t,0) - f(h,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0-0}{t} = 0 \quad \& f_x(h,0) \text{ is } ①$$

Substitute the value of $f_x(0,0)$

$$\therefore f_{xx}(0,0) = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

Since $f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{f_y(h,0) - f_y(0,0)}{h}$ ②

now let $f_y(h,0) = \lim_{t \rightarrow 0} \frac{f(h,t) - f(h,0)}{t}$

$$\begin{aligned} & \underset{t \rightarrow 0}{\lim} \frac{f(h, t) - f(h, 0)}{t} \\ &= \underset{t \rightarrow 0}{\lim} \frac{ht(h-t)}{t(h+t)} = h \end{aligned}$$

Now let $f_{xy}(0, 0) = \underset{t \rightarrow 0}{\lim} \frac{f(0, t) - f(0, 0)}{t}$

$$= \underset{t \rightarrow 0}{\lim} \frac{0 - 0}{t} = 0$$

$$\therefore f_{xy}(0, 0) = \underset{h \rightarrow 0}{\lim} \frac{h - 0}{h} = 1$$

Since $f_{y_2}(0, 0) = \underset{k \rightarrow 0}{\lim} \frac{f_x(0, 0+k) - f_x(0, 0)}{k}$

now let $f_x(0, k) = \underset{t \rightarrow 0}{\lim} \frac{f(t, k) - f(0, k)}{t}$

$$= \underset{t \rightarrow 0}{\lim} \frac{tk(t-k)}{t^2 + k^2} = 0$$

$$\therefore \underset{t \rightarrow 0}{\lim} \frac{k(t-k)}{t^2 + k^2} = -k$$

and $f_x(0, 0) = 0$

$$\therefore f_x(0, 0) = \underset{k \rightarrow 0}{\lim} \frac{-k - 0}{k} = -1$$

Since $f_{yy}(0, 0) = \underset{k \rightarrow 0}{\lim} \frac{f_y(0, k) - f_y(0, 0)}{k}$

now let $f_y(0, k) = \underset{t \rightarrow 0}{\lim} \frac{f(0, k+t) - f(0, k)}{t}$

$$= \underset{t \rightarrow 0}{\lim} \frac{0 - 0}{t} = 0$$

and $f_y(0, 0) = 0$

$$\therefore f_y(0, 0) = \underset{k \rightarrow 0}{\lim} \frac{0 - 0}{t} = 0$$

Ex Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$, for the function

given by

$$f(x,y) = \begin{cases} (x^2+y^4)\tan^{-1}\left(\frac{y}{x^2}\right), & x \neq 0 \\ \frac{\pi y^4}{2}, & x=0 \end{cases}$$

Note: Now we will give an example of a function whose first order partial derivatives exist, but higher order ones do not exist and also see that the existence of a partial derivative of a particular order does not imply the existence of other partial derivatives of the same order.

(IV)

for example:

Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{\ln xy}, & xy \neq 0 \\ 0, & xy=0 \end{cases}$$

Examine whether the second order partial derivatives of f at $(0,0)$ exist or not.

Sol: Now $\frac{f_x(0,0)}{h} = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h}$

$$\text{let } f_x(h,0) = \lim_{t \rightarrow 0} \frac{f(h+t,0) - f(h,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0-0}{t} = 0$$

$$\text{and } f_{xx}(0,0) = \lim_{t \rightarrow 0} \frac{f(0+t,0) - f(0,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0-0}{t} = 0$$

$$\therefore f_{xx}(0,0) = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

$$\text{Now } f_{xy}(0,0) = \lim_{k \rightarrow 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$\text{let } f_x(0,k) = \lim_{t \rightarrow 0} \frac{f(t,k) - f(0,k)}{t}$$

$$\begin{aligned} &= \lim_{t \rightarrow 0} \frac{\frac{tK^2}{\sqrt{t+K^2}} - 0}{t} \\ &= \lim_{t \rightarrow 0} \frac{K^2}{\sqrt{t+K^2}} = \frac{K^2}{K} = \pm K. \end{aligned}$$

and $f_{xx}(0,0) = 0$. i.e. limit exists.

$$\text{Now, } f_{xy}(0,0) = \lim_{k \rightarrow 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \rightarrow 0} \frac{|k|}{k},$$

which does not exist.

$\therefore f_{xy}(0,0)$ does not exist at $(0,0)$.

$$\text{Now } f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$\text{Let } f_y(h,0) = \lim_{t \rightarrow 0} \frac{f(h,t) - f(h,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{ht^2}{\sqrt{h+t^2}} = 0$$

$$= \lim_{t \rightarrow 0} \frac{ht^2}{\sqrt{h+t^2}} = 0$$

$$\text{and } f_y(0,0) = \lim_{t \rightarrow 0} \frac{f(0,t) - f(0,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$\text{Now } f_{yy}(0,0) = \lim_{k \rightarrow 0} \frac{f_y(0,k) - f_y(0,0)}{k}$$

$$\text{let } f_y(0,k) = \lim_{t \rightarrow 0} \frac{f(0,k+t) - f(0,k)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0$$

$$\text{and } f_y(0,0) = 0.$$

$$\therefore f_{yy}(0,0) = \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0$$

f_{xx}, f_{xy} and f_{yy} exist at $(0,0)$ and are equal to 0 , while $f_{xy}(0,0)$ does not exist.

The study of above examples must have convinced that we have to be careful about the order of variables w.r.t. which higher order derivatives are taken. For instance, from example ⑩ it is clear that f_{xy} need not be equal to f_{yx} .

Example ⑪ goes a step further, where f_{xy} exists at $(0,0)$, while f_{yx} does not, showing that the question of their equality doesn't arise at all.

If we look at the definitions of f_{xy} and f_{yx} at a point (a,b) more carefully, we would see why the expectation of the equality $f_{xy}(a,b) = f_{yx}(a,b)$ is farfetched (very difficult to believe).

Now by definition

$$\begin{aligned} f_{xy}(a,b) &= \lim_{h \rightarrow 0} \frac{f_y(a+h, b) - f_y(a, b)}{h} \\ &= \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \left[\lim_{k \rightarrow 0} \frac{f(a+h, b+k) - f(a+h, b)}{k} \right] \\ &\quad \left[\lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k} \right] \\ &= \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk} \end{aligned}$$

$$\text{where } \phi(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$

Similarly

$$f_{yx}(a,b) = \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} \frac{\phi(h,k)}{hk}$$

The expressions $f_{xy}(a,b)$ and $f_{yx}(a,b)$ are the repeated limits of the same expression taken in different orders. and we have already seen that repeated limits are not equal, in general.

Now we will give the conditions under which these mixed partial derivatives become equal.

Sufficient conditions for the equality of f_{xy} and f_{yx}

Theorem: Let $f(x,y)$ be a real valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a,b) :

$$\text{Then } f_{xy}(a,b) = f_{yx}(a,b)$$

Schwarz's theorem:

Let $f(x,y)$ be a real valued function defined in a nbhd of (a,b) such that

- (i) f_{xy} exists on a certain nbhd. of (a,b) .
- (ii) f_{xy} is continuous at (a,b) .

Then f_{yx} exists at (a,b)
and $f_{yx}(a,b) = f_{xy}(a,b)$.

Note: The conditions in Schwarz's theorem are less restrictive than those in theorem (i).

→ evaluate f_{xy} at a point (x,y) for the function f defined by $f(x,y) = x^4 + 2xy^2 + y^6$. Use Schwarz's theorem to evaluate f_{yx} at the point (x,y) .

Sol: By direct differentiation

$$f_y(x,y) = 2x^3y + 6y^5$$

$$\Rightarrow f_{xy}(x,y) = 4x^2y$$

Since $4xy$ is a polynomial.

$\therefore f_{xy} = 4xy$ is a continuous function.

Further $f_{xy} = 4x^2 + 2xy^2$ exist.

$\therefore f$ satisfies the conditions of Schwarz's theorem.

$$\therefore f_{xy} = f_{yx} = 4xy$$

Note: In theorem ① we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous and that f_x exists.

But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives.

In otherwords, we can have functions whose mixed partial derivatives at some point are equal, but which do not satisfy the requirements of Schwarz's theorem.

for example:

Consider the function f defined by-

$$(i) f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x=y=0 \end{cases}$$

1. Show that $f_{xy}(0,0) = f_{yx}(0,0)$; even though f does not fulfill the requirements of Schwarz's theorem.

Solⁿ: Since $f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

$$\text{now, } f_y(h,0) = \lim_{t \rightarrow 0} \frac{f(h,t) - f(h,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{\frac{ht^2}{h^2+t^2} - 0}{t}$$

$$= \lim_{t \rightarrow 0} \frac{\frac{ht^2}{h^2+t^2}}{t} = 0.$$

$$\text{and } f_y(0,0) = \lim_{t \rightarrow 0} \frac{f(0,t) - f(0,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

$$\therefore f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0.$$

$$\text{Since } f_{yx}(0,0) = \lim_{k \rightarrow 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$\text{Now, let } f_x(0,k) = \lim_{t \rightarrow 0} \frac{f(t,k) - f(0,k)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{\frac{t^2 k^2}{t+k} - 0}{t}$$

$$= \lim_{t \rightarrow 0} \frac{t^2 k^2}{t^2 + t k} = 0$$

$$\text{and } f_x(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{0-0}{t} = 0$$

$$\therefore f_{yx}(0,0) = 0.$$

$$\therefore f_{xy}(0,0) = f_{yx}(0,0).$$

Now we show that the conditions of Schwarz's theorem are not satisfied.

For $(x,y) \neq (0,0)$, we can find the partial derivatives of f at (x,y) by differentiating directly.

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \\ &= \frac{2(x^2+y^2)xy - 2x^3y^2}{(x^2+y^2)^2} \end{aligned}$$

$$\begin{aligned} \text{Further } f_{xy} &= \frac{\partial}{\partial x} \left(\frac{2x^2y}{(x^2+y^2)^2} \right) \\ &= \frac{8x^2y(x^2+y^2)^2 + 8x^5y(x^2+y^2)}{(x^2+y^2)^4} \\ &= \frac{(8x^3y(x^2+y^2) - 8x^5y)(x^2+y^2)}{(x^2+y^2)^4} \end{aligned}$$

$$= \frac{8x^3y(x^2+y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

$\lim_{(x,y) \rightarrow (0,0)}$ does not exist.

for, if (x,y) , approaches $(0,0)$ along the line $y = mx$,

we get

$$\lim_{(x,y) \rightarrow (0,0)} \frac{8x^3y^3}{(x^2+y^2)^3} \underset{x \rightarrow 0}{=} \lim_{x \rightarrow 0} \frac{78x^3m^3x^3}{(x^2+m^2x^2)^3}$$

$$\underset{x \rightarrow 0}{=} \lim_{x \rightarrow 0} \frac{8m^3}{(1+m^2)^3}$$

The limit is different for different values of m . i.e., the limit does not exist.

$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) \neq f(0,0) = 0$.

$(x,y) \rightarrow (0,0)$

which implies that f_{xy} is not continuous at $(0,0)$.

Young's theorem:

Let $f(x,y)$ be a real-valued function defined in a nbd of a point (a,b) such that both the first order partial derivatives f_x and f_y are differentiable at (a,b) .

Then $f_{xy}(a,b) = f_{yx}(a,b)$.

Note: As in the case of Schwarz's theorem the conditions stated in Young's theorem are less strict than in theorem (2).

However, these are not necessary for the equality of mixed partial derivatives.

→ Consider the function f defined by

$$(i) f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x=y=0 \end{cases}$$

S.T. $f_{xy}(0,0) \neq f_{yx}(0,0)$, even though the conditions of Young's theorem are not satisfied.

Sol: Already we have seen that

$$f_{xy}(0,0) = \lim_{h \rightarrow 0} f_{xy}(0,h) = 0$$

Let us now show that the conditions of Young's theorem are not satisfied.

Now we prove that f_x is not differentiable at $(0,0)$.

For this, assume that f_x is differentiable at $(0,0)$.

Then there exist functions $\phi(b,k)$ and $\psi(b,k)$

such that

$$f_x(b,k) - f_x(0,0) = k\phi(b,k) + K\psi(b,k) + \phi(b,k) + \psi(b,k)$$

and $\phi(b,k) \rightarrow 0$ as $(b,k) \rightarrow (0,0)$.

$\psi(b,k) \rightarrow 0$ as $(b,k) \rightarrow (0,0)$.

$$\text{Now let } f_{xx}(0,0) = \lim_{h \rightarrow 0} \frac{f_x(h,0) - f_x(0,0)}{h} \quad (2)$$

$$\text{Let } f_{xx}(h,0) = \lim_{t \rightarrow 0} \frac{f_x(h+t,0) - f_x(h,0)}{t}$$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\frac{0-0}{t}}{t} = 0$$

$$\text{and } f_{xx}(0,0) = \lim_{t \rightarrow 0} \frac{f_x(t,0) - f_x(0,0)}{t}$$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\frac{0-0}{t}}{t} = 0$$

$$\text{Eq } \textcircled{2} \Rightarrow f_{xx}(0,0) = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

\therefore eqn $\textcircled{1}$ becomes

$$f_x(h,k) - 0 = h \phi(h,k) + k \psi(h,k) \quad \text{--- } \textcircled{3}$$

$(\because f_{xy}(0,0) = 0 \text{ & } f_{xx}(0,0) = 0)$

for $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{2xy^2(x^2+y^2) - 2x^2(y^2)}{(x^2+y^2)^2}$$

$$= \frac{2xky^4}{(x^2+y^2)^2}$$

$$\Rightarrow f_x(h,k) = \frac{2hk^4}{(h^2+k^2)^2}$$

$\therefore \text{Eq } \textcircled{3} \Rightarrow$

$$\frac{2hk^4}{(h^2+k^2)^2} = h \phi(h,k) + k \psi(h,k).$$

Putting $h = r \cos \theta, k = r \sin \theta$.

we get

$$\frac{2r^5 \cos \theta \sin^4 \theta}{r^4} = r \cos \phi(r \cos \theta, r \sin \theta) + r \sin \psi(r \cos \theta, r \sin \theta)$$

$$2r^5 \cos \theta \sin^4 \theta = \cos \phi(r \cos \theta, r \sin \theta) + \sin \psi(r \cos \theta, r \sin \theta) \quad \text{--- } \textcircled{4}$$

for arbitrary θ , $(h,k) = (r \cos \theta, r \sin \theta) \rightarrow (0,0)$ as $r \rightarrow 0$.

and $\phi(h,k) \rightarrow 0$ and $\psi(h,k) \rightarrow 0$.

Taking the limit of $\textcircled{4}$ as $r \rightarrow 0$, we get

$$2r^5 \cos \theta \sin^4 \theta = 0.$$

which is impossible for arbitrary θ .

$\therefore f$ is not differentiable at $(0,0)$.

Similarly, we can show that f_y is not differentiable at $(0,0)$.

The function f does not satisfy the conditions

of Young's theorem, even though

$$\text{we have } \underline{f_{xy}(0,0) = f_{yx}(0,0)}$$

Differentials of higher order:

Let $z = f(x, y)$ be a function of two independent variables x and y , defined in a domain N and let it be differentiable at a point (x, y) of the domain.

The first differential of z at (x, y) , denoted by dz is given by

$$- dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{--- (1)}$$

If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x, y) then dz is a function of x and y and itself is differentiable at (x, y) . Then the differential of dz , called the second differential of z is denoted by d^2z .

$$\begin{aligned} \therefore d^2z &= d(dz) \\ &= d\left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right) \\ &= d\left(\frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2\right) \quad (\because dx \text{ and } dy \text{ are constants}) \end{aligned} \quad \text{--- (2)}$$

$$\begin{aligned} \text{Now let } d\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y \partial x} dy \quad (\text{from (1)}) \\ &= \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y \partial x} dy \end{aligned} \quad \text{--- (3)}$$

$$\text{Similarly } d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} dx + \frac{\partial^2 z}{\partial y^2} dy$$

Also by Young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable

$$\text{we have } \frac{\partial z}{\partial x \partial y} = \frac{\partial z}{\partial y \partial x} \quad \text{--- (4)}$$

from (2), (3) & (4)

$$d^2z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 \quad \text{--- (5)}$$

$$\text{where } dx^2 = dx \cdot dx = (dx)^2 \\ dy^2 = (dy)^2$$

Eqn (5) can be written as

$$dz = \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right) dz \quad \text{--- (6)}$$

Again dz is differentiable at (x, y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ are differentiable at (x, y) .

$$\begin{aligned} d^2z &= d(dz) = d \left[\frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 \right] \\ &= d \left(\frac{\partial^2 z}{\partial x^2} \right) dx^2 + 2 \cdot d \left(\frac{\partial^2 z}{\partial x \partial y} \right) dx dy + d \left(\frac{\partial^2 z}{\partial y^2} \right) dy^2 \\ &= \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) dy \right] dx^2 \\ &\quad + 2 \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x \partial y} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x \partial y} \right) dy \right] dx dy \\ &\quad + \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial y^2} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial y^2} \right) dy \right] dy^2 \\ &= \frac{\partial^3 z}{\partial x^3} dx^3 + \frac{\partial^3 z}{\partial y \partial x^2} dy dx^2 + 2 \frac{\partial^3 z}{\partial x^2 \partial y} dx dy^2 \\ &\quad + 2 \frac{\partial^3 z}{\partial y \partial x \partial y} dy dx + \frac{\partial^3 z}{\partial x^2} dx dy^2 + \frac{\partial^3 z}{\partial y^3} dy^3 \\ &= \frac{\partial^3 z}{\partial x^3} dx^3 + 3 \frac{\partial^3 z}{\partial x^2 \partial y} dx^2 dy + 3 \frac{\partial^3 z}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 z}{\partial y^3} dy^3 \\ &= \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) dy \right] dz \end{aligned}$$

In general, $d^2z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right) dz$ exists if d^2z is differentiable.

Note: In the above discussion, x & y are independent variables and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \Delta x$, $dy = \Delta y$.

Functions of Functions:

So far we have considered functions of the form
 $z = f(x, y, \dots)$

where the variables x, y, \dots are the independent variables.

NOW we consider functions

$$z = f(x, y, \dots)$$

where x, y, \dots are not independent variables
but are themselves functions of other independent
variables u, v, \dots , so that

$$x = g(u, v, \dots) \text{ and } y = h(u, v, \dots)$$

Theorem:

If $z = f(x, y)$ is a differentiable function of x, y
and $x = g(u, v), y = h(u, v)$ are themselves differential
functions of the independent variables u, v , then
 z is a differentiable function of u, v and

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

just as though x, y were the independent variables

- Note:
- (1) The theorem establishes a fact of fundamental importance that the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.
 - (2) The differential dz is sometimes referred to as the total differential.

Differentials of Higher Order of a function of functions -

If $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are differentiable functions of x, y .
so that they are also differentiable functions of

u, v and dx, dy are differentiable functions of u, v , then from the above theorem
we have

$$\begin{aligned} d^2z &= d(dz) \\ &= d\left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right) \quad \left(\because dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right) \\ &= d\left(\frac{\partial z}{\partial u}\right) du + \frac{\partial z}{\partial u} d(du) + d\left(\frac{\partial z}{\partial v}\right) dv + \frac{\partial z}{\partial v} d(dv) \\ &= \left[\frac{\partial^2 z}{\partial u^2} du^2 + \frac{\partial^2 z}{\partial u \partial v} dv\right] du + \frac{\partial^2 z}{\partial u^2} \underbrace{d\left(\frac{\partial z}{\partial u}\right) du}_{\text{here}} + \left[\frac{\partial^2 z}{\partial v^2} dv^2 + \frac{\partial^2 z}{\partial u \partial v} du\right] dv \\ &\quad + \frac{\partial^2 z}{\partial v^2} dv^2. \end{aligned}$$

$$\begin{aligned} d^2z &= \frac{\partial^2 z}{\partial u^2} du^2 + 2 \frac{\partial^2 z}{\partial u \partial v} dudv + \frac{\partial^2 z}{\partial v^2} dv^2 + \frac{\partial^2 z}{\partial u^2} dv^2 + \frac{\partial^2 z}{\partial v^2} du^2 \\ &= \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) du^2 + \frac{\partial^2 z}{\partial u \partial v} dv^2. \end{aligned}$$

The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given. The introduction of more than two variables, which are functions of independent variables causes no difficulty. Thus, when $z = f(x_1, x_2, x_3)$, and x_1, x_2, x_3 are not the independent variables,

$$\begin{aligned} d^2z &= \left(\frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \frac{\partial z}{\partial x_3} dx_3\right)^2 + \frac{\partial^2 z}{\partial x_1^2} dx_1^2 \\ &\quad + \frac{\partial^2 z}{\partial x_2^2} dx_2^2 + \frac{\partial^2 z}{\partial x_3^2} dx_3^2. \end{aligned}$$

Note: If x, y are linear functions of independent variables u and v , i.e., x and y of the form $x = a + bu + cv$, $y = d + bu + cv$, then dx and dy are constants, and so d^2x, d^2y and all higher differentials of x and y are zero, and

$$\therefore d^2z = \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right)^2$$

the form being same as for independent x & y .

The Derivation of composite functions (The chain Rule)

Let $Z = f(x, y)$ possess continuous first order partial derivatives.

Let $x = \phi(t)$, $y = \psi(t)$ possess continuous derivatives

Then the composite function

$Z = f(\phi(t), \psi(t))$ has derivative given by

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}$$

($\frac{dZ}{dt}$ is called the total derivative)

Because:

x, y are differentiable functions of

single variable t

$$\therefore dx = \frac{dx}{dt} dt \text{ and } dy = \frac{dy}{dt} dt$$

$\therefore Z$ is differentiable function of x and y

Since Z is differentiable function of x and y

x, y are differentiable functions of t .

$\therefore Z$ is a differentiable function of t .

$$dZ = \frac{\partial Z}{\partial t} dt \quad (1)$$

$$\text{Also, } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} dt + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} dt \quad (2)$$

from (1) & (2) we have

$$\boxed{\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}}$$

Corollary: If $Z = f(x, y)$ possesses first order partial derivatives, x, y one linear functions of single variable t , i.e. $x = at + b$, $y = bt + k$, where a, b, k are constants then

$$\frac{d^n z}{dt^n} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n z$$

SOL

$$\text{Now } \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y} \quad \text{where } \begin{aligned} \frac{dx}{dt} &= h \\ \frac{dy}{dt} &= k \end{aligned}$$

$$= \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) z \quad \text{①}$$

$$\text{Now } \frac{d^2 z}{dt^2} = \frac{d}{dt} \left(\frac{dz}{dt} \right)$$

$$= \frac{d}{dt} \left(h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y} \right)$$

$$= h \frac{\partial}{\partial x} \left(h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y} \right) + k \frac{\partial}{\partial y} \left(h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y} \right)$$

$$= h^2 \frac{\partial^2 z}{\partial x^2} + 2hk \frac{\partial^2 z}{\partial x \partial y} + k^2 \frac{\partial^2 z}{\partial y^2}$$

$$= \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 z$$

In general,-

$$\frac{d^n z}{dt^n} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n z$$

Let $Z = f(x, y)$ be posses \exists continuous first order partial derivatives.

Let $x = \phi(u, v)$
 $y = \psi(u, v)$ possess continuous first order partial derivatives

$$\text{Then } \frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial u} \text{ and}$$

$$\frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Derivatives of an implicit function:
 Let y be the a function of x
 defined implicitly by the equation
 $f(x, y) = 0$.

By the above

$$\frac{\partial f}{\partial x} \frac{dy}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$$

$$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$$

provided $f_y \neq 0$.

Problem:

$$\text{Let } f(x, y) = ax + bxy = 0$$

then find $\frac{dy}{dx}$.

$$\text{Ans: } \frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\text{find } \frac{dy}{dx} \text{ for } u = \sin(xay)$$

where x and y satisfy the eqn

$$ax + bxy = 0$$

$$\begin{aligned} \text{Sol: } \frac{dy}{dx} &= \frac{\partial y}{\partial x} \frac{dx}{dx} + \frac{\partial y}{\partial y} \frac{dy}{dx} \\ &= a \cos(xay) + b \cos(xay) \frac{dy}{dx} \\ &= \cos(xay) \left[a + b \frac{dy}{dx} \right] \end{aligned} \quad (1)$$

$$\text{Now let } \phi(x, y) = ax + bxy - C = 0$$

$$\begin{aligned} \text{then, } \frac{dy}{dx} &= -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \\ &= -\frac{a + b^2x}{b + ax} \end{aligned}$$

$$\therefore \text{Q} \equiv \frac{dy}{dx} = \cos(x-y) \left[\frac{a_1 + b_1 y}{a_2 + b_2 y} \right]$$

$$= 2a \cos(x-y) \left[1 - \frac{a_1}{a_2} \right].$$

→ find $\frac{dy}{dx}$, for each of the following
functions.

(a) $u = x^m - ay + by^2$, $y = x^2 + 2$

(b) $u = x^m - y^3$, $y = \ln x$

(c) $u = x \ln xy$ where $x^3 - y^3 + 2xy = 1$.

Homogeneous function :-

A function $Z = f(x, y)$ is called a homogeneous function of degree 'n' if it is expressible as $Z = x^n g\left(\frac{y}{x}\right)$.

Ex:- $a_1 x^m + a_2 x^m y + a_3 x^m y^2 = x^m [a_1 + a_2 \left(\frac{y}{x}\right) + a_3 \left(\frac{y}{x}\right)^2]$

$$= x^m g\left(\frac{y}{x}\right).$$

∴ It is a homogeneous function degree 'm'.

→ The following functions are homogeneous functions:

(i) $f(x, y) = \tan\left(\frac{y}{x}\right)$, degree '0'

(ii) $f(x, y) = \sqrt[m]{x^m + y^m}$, degree $\frac{m}{2}$

(iii) $f(x, y) = \frac{\sin\left(\frac{x+y}{x-y}\right)}{\ln\left(\frac{x+y}{x-y}\right)}$, degree '0'.

Because:

Since x, y are differentiable functions
of two independent variables u and

$$\left. \begin{aligned} \therefore dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \end{aligned} \right\} \quad (1)$$

Since Z is differentiable function of x and
 y and x, y are differentiable
functions of u and v
 $\therefore Z$ is a differentiable function of u and v

$$\therefore dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \quad (2)$$

$$\text{Also } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \quad (\text{from } 1)$$

$$= \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \right) du + \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \right) dv \quad (3)$$

Hence from (2) & (3), we

$$\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \right) du + \left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \right) dv$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}}$$

$\rightarrow \text{if } z = e^{xy}, x = t \cos t, y = t \sin t. \text{ compute } \frac{\partial z}{\partial t}$

$\therefore x + t = \frac{\pi}{2}$

Sol $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

 $= (y^2 e^{xy}) (\cos t - t \sin t) + (2xy e^{xy}) (\sin t + t \cos t)$

$\therefore t = \frac{\pi}{2}, x = 0, y = \frac{\pi}{2}$

$\therefore \left[\frac{\partial z}{\partial t} \right]_{t=\frac{\pi}{2}} = \frac{\pi^2}{4} (-1) = -\frac{\pi^3}{8}$

$\rightarrow \text{if } z = x^3 - xy + y^3, x = r \cos \theta, y = r \sin \theta.$

find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

Sol $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = (3x^2 - y) \cos \theta + (y^2 - x) \sin \theta$

$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$
 $= -(3x^2 - y) (-r \sin \theta) + (y^2 - x) r \cos \theta$

\rightarrow Show that $z = f(xy)$, where f is differentiable, satisfies $\Rightarrow \left(\frac{\partial z}{\partial x} \right) = y \left(\frac{\partial z}{\partial y} \right)$

Sol Let $xy = u$ then $z = f(u)$.

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = f'(u) \cdot 2xy$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = f'(u) \cdot x^2$

$\therefore x \frac{\partial z}{\partial x} = f'(u) 2xy = 2y \frac{\partial z}{\partial y}$

Altter: $\frac{\partial z}{\partial x} = f'(u)u_1 = f'(xy) (2xy + x + y)$ ①

Also $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ ②

from ① & ②, we have

$\frac{\partial z}{\partial x} = f'(xy), \frac{\partial z}{\partial y} = f'(xy)$

The result now follows Q.E.D.

Euler Theorem on Homogeneous functions

If $Z = f(x, y)$ is homogeneous function
of x, y of degree 'n', then
 $x \frac{\partial Z}{\partial x} + y \frac{\partial Z}{\partial y} = nZ$.

cor. If $Z = f(x, y)$ is a homogeneous function
of x, y of degree 'n'
then $x \frac{\partial^2 Z}{\partial x^2} + ny \frac{\partial^2 Z}{\partial x \partial y} + y^2 \frac{\partial^2 Z}{\partial y^2} = n(n-1)Z$

problem:-

If $u = \cot^{-1} \left(\frac{x+y}{\sqrt{x+y}} \right)$, show that
 $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{4}$ is zero.

Sol Let $u = \cot^{-1} \left(\frac{x+y}{\sqrt{x+y}} \right)$.

then $\cot u = \frac{x+y}{\sqrt{x+y}} (= Z)$ say

clearly Z is homo. function
of x and y of degree $\frac{1}{2}$.

By Euler's theorem

$$x \frac{\partial Z}{\partial x} + y \frac{\partial Z}{\partial y} = \frac{1}{2}(Z) \quad (1)$$

$$\text{from } (1), \frac{\partial Z}{\partial x} = -\csc^2 u \frac{\partial u}{\partial x}$$

$$\frac{\partial Z}{\partial y} = -\csc^2 u \frac{\partial u}{\partial y}$$

$$(1) \Rightarrow x \left(-\csc^2 u \frac{\partial u}{\partial x} \right) + y \left(-\csc^2 u \frac{\partial u}{\partial y} \right) = \frac{1}{2}(Z) \quad (2)$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\csc^2 u \cdot \left(\frac{1}{2} \frac{\partial Z}{\partial u} \right) = -\frac{1}{4} \sin 2u$$

$\tan u = \frac{x^3 + y^3}{x - y}$, $x \neq y$ show that-

$$(i) x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

$$(ii) x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4 \sin^2 u) \sin 2u.$$

(iii) Here, $u = \tan \left(\frac{x^3 + y^3}{x - y} \right)$ is not a homogeneous function.

However, we write

$$\tan u = \frac{x^3 + y^3}{x - y} (= z) \text{ say.} \quad (1)$$

$$\Rightarrow z = x^3 \left[\frac{1 + (\frac{y}{x})^3}{1 - (\frac{y}{x})} \right]$$

$\therefore z$ is a homogeneous function of x, y of degree 2.

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z \quad (2)$$

$$\text{But } \frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} - 2 \frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y} \quad (3)$$

$\therefore (1) \equiv$

$$x \left[\sec^2 u \frac{\partial u}{\partial x} \right] + y \sec^2 u \frac{\partial u}{\partial y} = 2z = 2 \tan u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u} \cdot \cos^2 u \\ = 2 \sin u \cos u = \sin 2u. \quad (4)$$

(ii) From (3)

$$\frac{\partial^2 z}{\partial x^2} = \sec^2 u \frac{\partial^2 u}{\partial x^2} + 2 \sec^2 u \tan u \left(\frac{\partial u}{\partial x} \right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \sec^2 u \frac{\partial^2 u}{\partial y^2} + 2 \sec^2 u \tan u \left(\frac{\partial u}{\partial y} \right)^2$$

$$\text{and } \frac{\partial^2 z}{\partial x \partial y} = \sec^2 u \frac{\partial^2 u}{\partial x \partial y} + 2 \sec^2 u \tan u \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}$$

By corollary of Euler's theorem,

we have

$$x^v \frac{\partial z}{\partial x} + 2xy \frac{\partial z}{\partial xy} + y^v \frac{\partial z}{\partial y^v} = 2(2-1)z.$$

$$\Rightarrow \sec^u \left(x^v \frac{\partial u}{\partial x} + 2xy \frac{\partial u}{\partial xy} + y^v \frac{\partial u}{\partial y^v} \right)$$

$$+ 2 \sec^u \tan u \left[x^v \left(\frac{\partial u}{\partial x} \right)^2 + 2xy \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + y^v \left(\frac{\partial u}{\partial y} \right)^2 \right] = 2 \tan u$$

$$\Rightarrow x^v \frac{\partial u}{\partial x} + 2xy \frac{\partial u}{\partial xy} + y^v \frac{\partial u}{\partial y^v} + \cancel{2 \sec^u \tan u} + 2 \tan u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)^2 = 2 \sin u \cos u$$

$$\Rightarrow x^v \frac{\partial u}{\partial x} + 2xy \frac{\partial u}{\partial xy} + y^v \frac{\partial u}{\partial y^v} = \sin u - 2 \tan u \sin u$$

$$= (1 - 2 \tan u \sin u) \sin^2 u$$

$$= (1 - 4 \sin^2 u) \sin^2 u$$

\rightarrow if $z = (x+y) \phi(y/x)$, where ϕ is any arbitrary function prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$

$$\text{Soln: } \frac{\partial z}{\partial x} = \phi(y/x) + (x+y) \phi'(y/x) \cdot \left(-\frac{y}{x^2}\right)$$

$$\Rightarrow x \frac{\partial z}{\partial x} = x \phi(y/x) - \frac{y}{x} (x+y) \phi'(y/x) \quad \text{--- (1)}$$

$$\text{Also, } \frac{\partial z}{\partial y} = \phi(y/x) + (x+y) \phi'(y/x) \left(\frac{1}{x}\right)$$

$$\Rightarrow y \frac{\partial z}{\partial y} = y \phi(y/x) + \frac{x}{y} (x+y) \phi'(y/x) \quad \text{--- (2)}$$

Adding (1) & (2)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = (x+y) \phi(y/x)$$

$$= z$$

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

→ If $u = x^m e^{ay}$, where z is a homogeneous function of x and y of degree n , prove that
 $\alpha \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = (m+ny)u.$

→ If $z = x^m (y)_n + x^n g(y)_n$. prove that

$$\alpha \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + m n z = (m+n-1) \left(n \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right)$$

→ If $z = \log \left(\frac{xy+y^2}{2x+y} \right)$, then $\alpha \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3$

→ If $u = \sec^{-1} \left(\frac{x^2+y^2}{xy} \right)$, then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u$

→ If $u = f(x+2y) + g(x-2y)$. Show that

$$\therefore \alpha \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

→ If $u = \phi(x+a y) + \phi(x-a y)$. Show that

$$\frac{\partial u}{\partial x} = a^m \frac{\partial^m u}{\partial y^m}$$

→ If $z = f \left[\frac{(ny-mz)}{(mx-lz)} \right]$. prove that

$$(mx-lz) \frac{\partial z}{\partial x} + (ny-mz) \frac{\partial z}{\partial y} = 0$$

2006: If f is diff. $(y/x) + g(y/x)$

Show that $x^2 \frac{\partial z}{\partial x^2} + 2xy \frac{\partial z}{\partial xy} + y^2 \frac{\partial z}{\partial y^2} = 0$.

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$$\rightarrow \text{if } z = x^m f(y/x) + x^n g(x/y),$$

$$\text{prove that } x^r \frac{\partial^r z}{\partial x^r} + 2xy \frac{\partial^r z}{\partial x \partial y} + y^r \frac{\partial^r z}{\partial y^r} + mnz \\ = (m+n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right)$$

$$\underline{\text{Sol}}: \text{ Let } u = x^m f(y/x) \text{ and } v = x^n g(x/y)$$

$$\text{Then } z = u+v \quad \text{--- (1)}$$

Now $u = x^m f(y/x)$ is a homogeneous function in x and y of degree m .

Therefore by Euler's theorem, we have

$$x^r \frac{\partial^r u}{\partial x^r} + 2xy \frac{\partial^r u}{\partial x \partial y} + y^r \frac{\partial^r u}{\partial y^r} = m(m-1)u. \quad \text{--- (2)}$$

Also, $v = x^n g(x/y)$ is a homogeneous function in x and y of degree n .

so we have

$$x^r \frac{\partial^r v}{\partial x^r} + 2xy \frac{\partial^r v}{\partial x \partial y} + y^r \frac{\partial^r v}{\partial y^r} = n(n-1)v. \quad \text{--- (3)}$$

Adding (2) & (3), we have

$$x^r \frac{\partial^r (u+v)}{\partial x^r} + 2xy \frac{\partial^r (u+v)}{\partial x \partial y} + y^r \frac{\partial^r (u+v)}{\partial y^r} \\ = m(m-1)u + n(n-1)v.$$

$$\Rightarrow x^r \frac{\partial^r z}{\partial x^r} + 2xy \frac{\partial^r z}{\partial x \partial y} + y^r \frac{\partial^r z}{\partial y^r} = m(m-1)u + n(n-1)v \quad \text{--- (4)}$$

Now let $m(m-1)u + n(n-1)v$

$$= (m^r u + n^r v) - (mu + nv) \\ = m^r u + n^r v - mn u + mn v \\ - mn u + mn v - (mu + nv)$$

$$= mu(m+n) + nv(m+n) - mn(u+v)$$

$$-(mu+nv)$$

$$= (mu+nv)(m+n) - mn(u+v) - (mu+nv)$$

$$= \cancel{(mu+nv)}(m+n-1) - mn(u+v).$$

$$\cancel{+ mn^2} + (mu+nv)(m+n-1)$$

(from ①
 $z = u+v$)

Again from Euler's theorem we have for u & v ,
which are homogeneous functions in x
& y of degree m and n respectively,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = mu \text{ and}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv.$$

Adding these we get,

$$x \frac{\partial (u+v)}{\partial x} + y \frac{\partial (u+v)}{\partial y} = mu+nv.$$

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = mu+nv \quad (\because z = u+v)$$

∴ from ⑤

we have

$$m(m+n)v + n(m+n-1)v = (m+n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) - mn^2$$

∴ from ④ we have

$$x^m \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^m \frac{\partial^2 z}{\partial y^2} = (m+n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) - mn^2$$

$$\Rightarrow x^m \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^m \frac{\partial^2 z}{\partial y^2} + mn^2 = (m+n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right)$$

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$$\rightarrow \text{If } z = f\left(\frac{ny-mz}{nx-lz}\right)$$

$$\text{prove that } (nx-lz) \frac{\partial z}{\partial x} + (ny-mz) \frac{\partial z}{\partial y} = 0$$

Sol: Given $z = f\left(\frac{ny-mz}{nx-lz}\right)$.

$$\frac{\partial z}{\partial x} = f'\left(\frac{ny-mz}{nx-lz}\right) \frac{\partial}{\partial x} \left(\frac{ny-mz}{nx-lz}\right)$$

$$= f'\left(\frac{ny-mz}{nx-lz}\right) \cdot (ny-mz) \cdot \frac{n}{(nx-lz)^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{ny-mz}{nx-lz}\right) \frac{\partial}{\partial y} \left(\frac{ny-mz}{nx-lz}\right)$$

$$= f'\left(\frac{ny-mz}{nx-lz}\right) \frac{n}{nx-lz} \quad \text{--- (2)}$$

Multiplying (1) by $(nx-lz)$ and (2) by $(ny-mz)$
and adding these, we get

$$(nx-lz) \frac{\partial z}{\partial x} + (ny-mz) \frac{\partial z}{\partial y} = (nx-lz) f'\left(\frac{ny-mz}{nx-lz}\right) \frac{-n(ny-mz)}{(nx-lz)^2}$$

$$+ (ny-mz) f'\left(\frac{ny-mz}{nx-lz}\right) \frac{n}{nx-lz}$$

$$= f'\left(\frac{ny-mz}{nx-lz}\right) \left[\frac{-n(ny-mz)}{nx-lz} + \frac{n(ny-mz)}{nx-lz} \right]$$

$$= f'\left(\frac{ny-mz}{nx-lz}\right) \cdot 0$$

$$= 0$$

$$(nx-lz) \frac{\partial z}{\partial x} + (ny-mz) \frac{\partial z}{\partial y} = 0$$

Taylor's theorem for function of two variablesMonomial:

→ Defn: Let x & y denote two variables. Then an expression of the form $a_{jk}x^jy^k$, where j and k are non-negative integers and $a_{jk} \in \mathbb{R}$, is called a monomial. The integer $j+k$ is called the degree of the monomial.

for example:

x^3y^2 is a monomial of degree 5.

x^4 is a monomial of degree 4.

xy^7 is a monomial of degree 7.

→ A polynomial in two variables x & y with coefficients in \mathbb{R} is an expression of the type

$$P(x, y) = a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \dots + (a_{10}x^i + a_{(i-1)}x^{i-1}y + \dots + a_{0i}y^i) + \dots + (a_{nn}x^n + a_{(n-1)}x^{n-1}y + \dots + a_{0n}y^n)$$

where a_{ij} 's are real numbers.

In the first bracket, each term is a monomial of degree 1.

In the second, each is a monomial of degree 2.
and so on.

for example:

$P(x, y) = 1 + 2xy + x^2y$ is a polynomial in two variables.

This polynomial is a sum of three monomials, having degree 0, 2 and 3 respectively.

The number 3, which is the maximum of these numbers is called the degree of this polynomial.

- The highest degree of the monomials present in a polynomial $p(x, y)$ is called the degree of $p(x, y)$.
- n^{th} Taylor polynomial of a function of two variables

Defn: Let $f(x, y)$ be a real-valued function of two variables. Assume that it has continuous partial derivatives of all types of orders less than or equal to n in some nbhd. of a point (x_0, y_0) .

Then $T_n(x, y) = \sum_{\substack{i+j \leq n \\ i, j=0}} \frac{1}{i! j!} \left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_0, y_0) \right] (x-x_0)^i (y-y_0)^j$

is called the n^{th} Taylor polynomial of f at (x_0, y_0) .

In particular, if $f(x, y)$ is a polynomial of degree n , then all partial derivatives of order m for $m > n$ will be zero.

$$\therefore T_m(x, y) = T_n(x, y) \text{ for all } m \geq n.$$

further, as in the case of one variable, we can see that $T_n(x, y)$ at $(0, 0)$ is equal to $f(0, 0)$.

From the defn, we can see that:

$$T_{n+1}(x, y) = T_n(x, y) + \sum_{\substack{i+j=n+1 \\ i, j \geq 0}} \frac{1}{i! j!} \left[\frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j} \right] (x-x_0)^i (y-y_0)^j$$

- Find the Taylor polynomials of the function

$$P(x, y) = 1 + 2xy + x^2y \text{ at } (1, 1).$$

Sol: Given $P(x, y) = 1 + 2xy + x^2y$

By defn $P(1, 1) = 4$.
 $T_n(x, y) = \sum_{i,j=0}^{i+j=n} \frac{1}{i! j!} \left[\frac{\partial^{i+j} P(x_0, y_0)}{\partial x^i \partial y^j} \right] (x-x_0)^i (y-y_0)^j$ ①

put $n=0$ in ①

$$T_0(x,y) = \frac{1}{0!0!} \left[\frac{\partial^0}{\partial x^0 \partial y^0} p(x_0, y_0) \right] (x-x_0)^0 (y-y_0)^0$$

$$= p(x_0, y_0)$$

$$= p(1,1) = 4.$$

$$\therefore T_0(x,y) = 4$$

put $n=1$ in ①

$$T_1(x,y) = \sum_{i,j=0}^{i+j \leq 1} \frac{1}{i!j!} \left[\frac{\partial^{i+j}}{\partial x^i \partial y^j} p(x_0, y_0) \right] (x-x_0)^i (y-y_0)^j$$

$$= p(x_0, y_0) + \frac{1}{1!0!} \frac{\partial p(x_0, y_0)}{\partial x} (x-x_0) + \frac{1}{0!1!} \frac{\partial p(x_0, y_0)}{\partial y} (y-y_0)$$

$$= T_0(x,y) + \frac{\partial p}{\partial x}(1,1)(x-1) + \frac{\partial p}{\partial y}(1,1)(y-1) \quad (6)$$

$$\therefore p(1,1) = T_0(x,y)$$

NOW

$$\frac{\partial p}{\partial x} = 2y + 2xy \Rightarrow \left(\frac{\partial p}{\partial x} \right)_{(1,1)} = 4$$

$$\frac{\partial p}{\partial y} = 2x + x^2 \Rightarrow \left(\frac{\partial p}{\partial y} \right)_{(1,1)} = 3.$$

∴ ②

$$T_1(x,y) = 4 + 4(x-1) + 3(y-1)$$

put $n=2$ in ① $\therefore T_1(x,y) = (1,1)$

$$T_2(x,y) = \sum_{i,j=0}^{i+j \leq 2} \frac{1}{i!j!} \left[\frac{\partial^{i+j}}{\partial x^i \partial y^j} p(x_0, y_0) \right] (x-x_0)^i (y-y_0)^j$$

$$= T_1(x,y) + \frac{(x-1)^2}{2!} \frac{\partial^2 p}{\partial x^2}(1,1) + \frac{(x-1)(y-1)}{1!1!} \frac{\partial^2 p}{\partial x \partial y}(1,1)$$

$$+ \frac{(y-1)^2}{2!} \frac{\partial^2 p}{\partial y^2}(1,1) \quad (3)$$

NOW $\frac{\partial p}{\partial x^2} = 2y \Rightarrow \left(\frac{\partial p}{\partial x^2} \right)_{(1,1)} = 2$

$$\frac{\partial^2 p}{\partial x \partial y} = 2x^2 \Rightarrow \left(\frac{\partial^2 p}{\partial x \partial y} \right)_{(1,1)} = 4$$

$$\frac{\partial^3 p}{\partial y^2} = 0 \Rightarrow \left(\frac{\partial^3 p}{\partial y^2}\right)_{(0,0)} = 0$$

Substituting these values in ②;

we get

$$T_2(x, y) = 4 + 4(x-1) + 3(y-1) + (x-1)^2 + 4(x-1)(y-1)$$

$$\text{since } \frac{\partial^3 p}{\partial x^3} = 0, \frac{\partial^3 p}{\partial x \partial y^2} = 0, \frac{\partial^3 p}{\partial x^2 \partial y} = 0 \text{ and } \frac{\partial^3 p}{\partial y^3} = 0$$

we get

$$T_3(x, y) = T_2(x, y) + (x-1)^2(y-1)$$

$$\text{and } T_3(x, y) = T_3(x, y) \text{ for all } r > 3.$$

\rightarrow find the Taylor polynomial $T_3(x, y)$ for the function $\sin(x+y)$ at $(0, 0)$.

Sol: let $f(x, y) = \sin(x+y)$

Clearly f has continuous partial derivatives of all orders.

Also $f(0, 0) = 0$.

Now $\frac{\partial f}{\partial x} = \cos(x+y) = \frac{\partial f}{\partial y}$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+y) = \frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 0$$

and $\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$$\Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = -\cos(x+y) \Big|_{(0,0)} = -1$$

The third Taylor polynomial of $\sin(x+y)$ at $(0,0)$ is

$$T_3(x,y) = \sum_{i,j=0}^{i+j \leq 3} \frac{1}{i!j!} \left[\frac{\partial^{i+j} f(0,0)}{\partial x^i \partial y^j} \right] x^i y^j$$

$$\begin{aligned} &= \frac{1}{0!0!} f(0,0) + \frac{x}{1!0!} \frac{\partial f(0,0)}{\partial x} + \frac{y}{0!1!} \frac{\partial f(0,0)}{\partial y} + \\ &\quad \frac{x^2}{2!0!} \frac{\partial^2 f(0,0)}{\partial x^2} + \frac{xy}{1!1!} \frac{\partial^2 f(0,0)}{\partial x \partial y} + \frac{y^2}{0!2!} \frac{\partial^2 f(0,0)}{\partial y^2} \\ &+ \frac{1}{3!0!} \left(\frac{\partial^3 f(0,0)}{\partial x^3} \right) x^3 + \frac{1}{2!1!} \left(\frac{\partial^3 f(0,0)}{\partial x^2 \partial y} \right) x^2 y + \frac{1}{3!1!} \left(\frac{\partial^3 f(0,0)}{\partial x \partial y^2} \right) x y^2 + \frac{1}{1!2!} y^3 \\ &= 0 + \frac{x}{1!} + \frac{y}{1!} - 0 - \frac{1}{3!} x^3 - \frac{1}{2!1!} x^2 y - \frac{1}{1!2!} x y^2 - \frac{1}{3!} y^3 \end{aligned}$$

$$\begin{aligned} \Rightarrow T_3(x,y) &= (x+y) - \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3) \\ &= (x+y) - \underline{\underline{\frac{(x+y)^3}{3!}}} \end{aligned}$$

- HW: → Find the second Taylor polynomial of e^{xt+y} at $(0,0)$
HW: → find the Taylor polynomials of $f(x,y) = 2+x^2+y^3$ at $(0,0)$

Now let us consider a function $f(x,y)$ of two variables. Assume that f has continuous partial derivatives of all orders less than or equal to n , for some integer n ; in a nbd of a point (x_0, y_0) .

Then the n th Taylor polynomial

$$T_n(x,y) = \sum_{i,j=0}^{i+j \leq n} \frac{1}{i!j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(x_0, y_0)} (x-x_0)^i (y-y_0)^j$$

has the same value as $f(x,y)$ at (x_0, y_0) , and the same partial derivatives of all orders $\leq n$ as f at (x_0, y_0) . As in the case of one variable, we would naturally like to know whether we can approximate f by the corresponding Taylor polynomials.

Put differently, we would like to have some information about the function

$$R_{n+1}(x, y) = f(x, y) - T_n(x, y).$$

An analogue of Taylor's theorem which we state now, provides us some information about the function $R_{n+1}(x, y)$.

Taylor's theorem:

Let f be a real-valued function of two variables x and y with continuous partial derivatives of orders $\leq n+1$ in some nbhd of

$S(\bar{x}, \bar{y})$ of $\bar{x} = (x_0, y_0)$. Then for a given $(x, y) \neq (x_0, y_0)$ in $S(\bar{x}, \bar{y})$, there exists a point (c_1, c_2) on the line segment joining (x_0, y_0) and (x, y)

such that

$$f(x, y) = T_n(x, y) + R_{n+1}(x, y) \quad (1)$$

$$\text{where, } T_n(x, y) = \sum_{i+j=n} \frac{1}{i! j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(x_0, y_0)} (x-x_0)^i (y-y_0)^j$$

$$\text{and } R_{n+1}(x, y) = \sum_{i+j=n+1} \left(\frac{1}{i! j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(c_1, c_2)} (x-x_0)^i (y-y_0)^j \right)$$

$$\text{i.e., } R_{n+1}(x, y) = \frac{1}{(n+1)!} \left(\frac{\partial^{n+1} f}{\partial x^{n+1} \partial y^n} \right)_{(c_1, c_2)} (x-x_0)^{n+1} +$$

$$\frac{1}{n! (n+1)!} \left(\frac{\partial^{n+1} f}{\partial x^n \partial y^{n+1}} \right)_{(c_1, c_2)} (x-x_0)^n (y-y_0) +$$

$$\frac{1}{(n-1)! n!} \left(\frac{\partial^{n+1} f}{\partial x^{n-1} \partial y^n} \right)_{(c_1, c_2)} (x-x_0)^{n-1} (y-y_0)^2 + \dots$$

$$+ \frac{1}{(n+1)!} \left(\frac{\partial^{n+1} f}{\partial x^n \partial y^n} \right)_{(c_1, c_2)} (y-y_0)^{n+1}.$$

i.e., $R_{n+1}(x, y)$ involves all the $(n+1)^{th}$ order partial derivatives of f evaluated at the point (x_0, y_0) .

The RHS of ① is called the n^{th} Taylor expansion of f at (x_0, y_0) .

Now we consider only the second Taylor expansion of functions.

If we look at the expression for $R_2(x, y)$, we will see that it contains powers of $(x-x_0)$ and $(y-y_0)$. Now if we take the point (x, y) close enough to (x_0, y_0) , then $(x-x_0)$ and $(y-y_0)$ will be very small.

Therefore, we can get a good enough approximation of $f(x, y)$ by a second degree polynomial. Of course, $f(x, y)$ can be approximated as closely as we like by a polynomial by choosing n sufficiently large.

We write the expression for $T_2(x, y)$ and the second Taylor expansion of $f(x, y)$ at (x_0, y_0) explicitly:

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0) (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) (y - y_0) \right] \\ &\quad + \frac{1}{2} \left[\frac{\partial^2 f}{\partial x^2}(x_0, y_0) (x - x_0)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) (x - x_0)(y - y_0) \right. \\ &\quad \left. + \frac{\partial^2 f}{\partial y^2}(x_0, y_0) (y - y_0)^2 \right] + R_2(x, y) \\ &= f(x_0, y_0) + \left[(x - x_0) \frac{\partial f}{\partial x} + (y - y_0) \frac{\partial f}{\partial y} \right] f(x_0, y_0) \\ &\quad + \frac{1}{2} \left[(x - x_0) \frac{\partial^2 f}{\partial x^2} + (y - y_0) \frac{\partial^2 f}{\partial y^2} \right] f(x_0, y_0) + \underline{R_2(x, y)} \end{aligned}$$

→ find the second Taylor expansion of the function $f(x, y) = \log(1+x+2y)$. for points close to $(2, 1)$.

Sol: Given $f(x, y) = \log(1+x+2y)$
 $f(2, 1) = \log 5.$

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+2y} \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(2,1)} = \frac{1}{5}$$

$$\frac{\partial f}{\partial y} = \frac{2}{1+x+2y} \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(2,1)} = \frac{2}{5}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(2,1)} = -\frac{1}{25}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-4}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(2,1)} = -\frac{4}{25}$$

$$\frac{\partial^2 f}{\partial xy} = \frac{-2}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial xy}\right)_{(2,1)} = -\frac{2}{25}$$

The second Taylor expansion is given by

$$f(x, y) = \log 5 + \left[\frac{1}{5}(x-2) + \frac{2}{5}(y-1) \right] + \frac{1}{2} \left[\left(-\frac{1}{25}\right)(x-2)^2 + \left(-\frac{2}{25}\right)(x-2)(y-1) + \left(-\frac{4}{25}\right)(y-1)^2 \right]$$

→ find the second Taylor expansion for the function $f(x, y) = xy^2 + \cos xy$ about $(1, \pi/2)$.

→ find an approximation to the function $f(x, y) = e^{xy}$ by a second degree polynomial near $(0, 0)$.

EXTREME VALUES

Maxima and minima

Defn: A function $f: D \rightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^n$, is said to have maximum at 'a' if there exists a nbd of 'a' for every point x of which $f(x) \leq f(a)$.
i.e., if $f(x, y)$ be a real valued function of two variables, we say that the function f has a maximum at (a, b) if $f(x, y) \leq f(a, b)$ for every $(x, y) \in N_\delta(a, b)$ for some $\delta > 0$.

Similarly, $f: D \rightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^n$ is said to have a minimum at 'a' if there exists a nbd of 'a' at every point of which $f(x) \geq f(a)$.

i.e., if $f(x, y)$ be a real valued function of two variables, we say that the function f has a minimum at (a, b) if $f(x, y) \geq f(a, b)$ for every $(x, y) \in N_\delta(a, b)$ for some $\delta > 0$.

(OR)

Let (a, b) be a point of the domain of definition of function f . Then $f(a, b)$ is an extreme value of f , if for every point (x, y) , (other than (a, b)) of some nbd of (a, b) , the difference

$$f(x, y) - f(a, b) \quad \text{--- (1)}$$

keeps the same sign.

The extreme value $f(a, b)$ is called a max or min. value according as the sign of (1) is -ve or +ve.

A function f is said to have an extreme value at (a, b) , if $f(a, b)$ is either a maximum or minimum.

value of the function.

- A necessary condition for $f(x,y)$ to have an extreme value at (a,b) is that $f_x(a,b)=0$, $f_y(a,b)=0$, provided the partial derivatives exist.

(or)

Let f be a function of two variables. Suppose f has an extremum at some point (a,b) and the partial derivatives of f exist at that point. Then

$$f_x(a,b) = 0 = f_y(a,b).$$

- To check whether a given function has an extremum at some point or not, we can use above theorem. All we have to do is to see whether the partial derivatives vanish at that point (if they exist).

- ① Ex: Check whether the function given by

$$f(x,y) = x^2 - 2x + \frac{y^2}{4}, \text{ has maximum or minimum values.}$$

Sol: The given function $f(x,y) = x^2 - 2x + \frac{y^2}{4}$ is differentiable everywhere.

First we have to find out the points (x,y) such that $f_x(x,y) = 0 = f_y(x,y)$:

$$\text{Now } f_x(x,y) = 2x - 2.$$

$$f_y(x,y) = \frac{y}{2}.$$

$f_x(x,y)$ and $f_y(x,y)$ will vanish only when $x=1$ and $y=0$.

The point $(1,0)$ is the only possible point where f can have a max. or min. value.

Now, let us see whether $(1, 0)$ is a max or min. point for f .

$$\begin{aligned}f(x, y) &= x^2 - 2x + \frac{y^2}{4} \\&= x^2 - 2x + 1 + \frac{y^2}{4} - 1 \\&= (x-1)^2 + \frac{y^2}{4} - 1\end{aligned}$$

$$\begin{aligned}f(1, 0) &= 1 - 1 = 0 \\f(x, y) - f(1, 0) &= x^2 - 2x + \frac{y^2}{4} - 1 \\&= (x-1)^2 + \frac{y^2}{4} - 1 \\&\geq 0 \\&\therefore f(x, y) - f(1, 0) \geq 0 \\&\therefore f(x, y) \geq f(1, 0)\end{aligned}$$

This shows that $f(x, y) \geq -1 = f(1, 0)$.

\therefore The function f has minimum at $(1, 0)$.

The minimum value is $f(1, 0) = -1$.

and the function has no maximum value.

Note: If $f_x \neq 0$ or $f_y \neq 0$ at some point, then we can't straightaway say that the function does not have an extremum at that point.

But if $f_x = f_y = 0$ at some point, then this does not imply that the function has extremum at that point.

It is possible that all the first order partial derivatives of a function are zero at some point (a, b) , but still, that point is not an extremum point for that function.

i.e., the converse of the above theorem is not true.

for example

Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = 1 - x^2 - y^2$$

Sol: We have $f_x = -2x$ & $f_y = -2y$

$$\therefore f_x(0,0) = 0 = f_y(0,0).$$

Now, let us check whether the function f has an extremum at $(0,0)$.

We have $f(0,0) = 1$.

$f(ax, 0) < 1$ and $f(0, b) > 1$ for all non-zero a and b .

In the nbd of $(0,0)$, we can find points of the type $(a,0)$ and $(0,b)$.

\therefore There exists no nbd N of $(0,0)$ for which $f(x,y) < f(0,0)$ or $f(x,y) > f(0,0)$.
 $\Rightarrow f(x,y) \in N$.

$(0,0)$ is neither a maximum nor a minimum point for f , even though both the partial derivatives of f vanish at $(0,0)$.

Q7. If $f(x,y) \geq 0$ if $x=0$ or $y=0$
= 1 elsewhere

then both the partial derivatives exist (each equal to zero) at the origin, but $f(0,0)$ is not an extreme value.

Thus the conditions obtained in the above theorem are only necessary and not sufficient.

Some times it may happen that the partial derivatives of a function do not exist at a point, but, still the function has an extremum at that point.

for example:

Consider the function given by

$$f(x, y) = 1 + \sqrt{x+y}.$$

Soln: Since $f(0, 0) = 1$

$$f(x, y) = 1 + \sqrt{x+y} > 1 = f(0, 0)$$

i.e., $f(x, y) > f(0, 0)$ for every point (x, y) in the nbd of $(0, 0)$.

It follows that f has a minimum at $(0, 0)$.

$$\begin{aligned} \text{Now, } f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + \sqrt{h^2} - 1}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{h^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} \end{aligned}$$

which does not exist.

Similarly f_x does not exist at $(0, 0)$.

Similarly f_y does not exist at $(0, 0)$.

Ex 2: The function $f(x, y) = |x| + |y|$, has an extreme value at $(0, 0)$ even though the partial derivatives f_x and f_y do not exist at $(0, 0)$.

Soln: Since $f(0, 0) = 0 < |x| + |y| = f(x, y)$

i.e., $f(x, y) > f(0, 0)$ for every point (x, y) in the nbd of $(0, 0)$.

It follows that f has a minimum at $(0, 0)$.

Now for the existence of partial derivatives of f at $(0, 0)$:

$$\begin{aligned} f_x(0,0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h| - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} \end{aligned}$$

which does not exist.

f_x does not exist at $(0,0)$.

Similarly f_y does not exist at $(0,0)$.

Defn: Let f be a function of two variables. A point (a,b) is said to be a stationary point of f if both the partial derivatives are zero at (a,b) .

Sufficient Condition for $f(x,y)$ to have an extreme value at (a,b) :

Theorem If $f(x,y)$ has an extreme value at (a,b) and second order partial derivatives of $f(x,y)$ are continuous at (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ and $(f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$ then $f(a,b)$ is a max or min according as f_{xx} (or f_{yy}) is -ve or +ve at (a,b)

i.e., $f_{xx}f_{yy}(a,b) - f_{xy}^2(a,b) > 0$

and $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$

then f has a maximum at (a,b) .

$\rightarrow f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$

and $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$

then f has minimum at (a,b) .

Note 1: Further investigation is necessary, if $f_{xx}(a,b) - f_{yy}(a,b) > 0$ and $f_{xy}(a,b) = 0$

Note 2: $f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b) < 0$, then f has neither max nor min.

(1) Find the maxima and minima of the function

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

Sol: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$$f_x(x,y) = 3x^2 - 3$$

$$f_y(x,y) = 3y^2 - 12$$

Equating to zero these f_x and f_y

we get $3x^2 - 3 = 0$
 $\Rightarrow x = \pm 1$

and $3y^2 - 12 = 0$

$$y = \pm 2$$

\therefore the function f has four stationary points

$$(1,2), (-1,2), (1,-2), (-1,-2)$$

Now $f_{xx}(x,y) = 6x$

$$f_{xy}(x,y) = 0$$

$$f_{yy}(x,y) = 6y$$

At $(1,2)$ $f_{xx} = 6 > 0$, $f_{yy} = 12 > 0$ and $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = 6 \times 12 - 0$
 $= 72 > 0$

Hence $(1,2)$ is the minimum point

i.e., $f(x,y)$ has minimum of $f(x,y)$.

At $(-1,2)$ $f_{xx} = -6$, $f_{yy} = 12$ and $f_{xy} = 0$

$$= -72 < 0.$$

\therefore The $f''(x,y)$ has neither max. nor min. at $(-1,2)$

At $(1,-2)$

$$f_{xx} = 6, f_{yy} = 0 \text{ and } f_{xy} = -12$$

$$\text{and } f_{xx}f_{yy} - f_{xy}^2 = -12 > 0$$

\therefore The function $f(x,y)$ has neither max. nor min. at $(1,-2)$.

At $(-1,-2)$

$$f_{xx} = -6, f_{yy} = -12 < 0 \text{ and } f_{xy} = 0$$

$$\text{and } f_{xx}f_{yy} - f_{xy}^2 = 72 > 0$$

$\therefore (x,y) = (-1,-2)$ is the max pt of $f(x,y)$.
i.e., $f(x,y)$ has maximum at $(-1,-2)$.

Note: Stationary points like $(-1,2), (1,-2)$ which are not extreme (neither max. nor min.) points are called the saddle points.

①

Find the all the maximum and minimum of $f(x,y) = x^3 + y^3 - 6x(y)$

$$\text{Soln} \quad f_{xx}(x,y) = x^3 + 4y^2 - 6y - 6x$$

$$f_x(x,y) = 3x^2 - 6y + 12y$$

$$\text{but } f_x(x,y) = 0$$

$$\Rightarrow 3x^2 - 6y + 12y = 0$$

$$\Rightarrow 3(x^2 - 2y + 4y) = 0$$

$$\Rightarrow x^2 + 4y = 21 \quad \text{--- (1)}$$

$$\text{and } f_y(x,y) = 3y^2 - 6x + 12x$$

$$\text{and } f_y(x,y) = 0$$

$$\Rightarrow 3y^2 - 6x + 12x = 0$$

$$\Rightarrow y^2 + 4x = 21 \quad \text{--- (2)}$$

$$\text{①} - \text{②} \Rightarrow x^2 - y^2 + 4(y - x) = 0$$

$$\Rightarrow (x-y)(x+y) + 4(y-x) = 0$$

$$\Rightarrow (x-y)[x+y-4] = 0$$

$$\Rightarrow x-y=0; x+y=4$$

$$\Rightarrow \boxed{x=y}; \boxed{y=4-x}$$

Now sub $x=y$ in ①

$$x^2 + 4x - 21 = 0$$

$$\Rightarrow (x-3)(x+7) = 0$$

$$\Rightarrow x=3, -7$$

$$\Rightarrow y=3, -7$$

$\therefore (3, 3), (-7, -7)$ are stationary points.

Now sub $y=4-x$ in ①

$$\Rightarrow x^2 + 4(4-x) = 21$$

$$\Rightarrow x^2 - 4x + 16 = 21$$

$$\Rightarrow x^2 - 4x - 5 = 0$$

$$\Rightarrow (x-5)(x+1) = 0$$

$$\Rightarrow x=5, -1$$

sub $x=5$ in $y=4-x$; sub $x=-1$ in $y=4-x$

$$\Rightarrow y= -1; \quad \Rightarrow y=5$$

$$(x, y) = (5, -1) \quad (x, y) = (-1, 5).$$

$\therefore (3, 3), (-7, -7), (5, -1), (-1, 5)$ are stationary points.

Ans (x, y)

Now $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 12$

$$\text{put } D = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot 6y - (12)^2$$

$$= 36xy - 144$$

$$\boxed{D = 36xy - 144}$$

At (3, 3)

$$\begin{aligned} f_{xx} &= 18 > 0 \quad \text{and} \quad D = 36x^2 - 144 \\ &\qquad\qquad\qquad = 324 - 144 = 180 > 0 \end{aligned}$$

\therefore the function $f(x,y)$ is minimum at (3, 3).

At (-7, -7)

$$\begin{aligned} f_{xx} &= -42 < 0 \quad \text{and} \quad D = 36(-7)(-7) = 36 \times 49 = 180 > 0 \\ &\therefore \text{the function } f(x,y) \text{ is maximum at } (-7, -7) \end{aligned}$$

At (-1, 5)

$$f_{xx} = -6 < 0 \quad \text{and} \quad D = 36(-1)(5) = 180 < 0$$

\therefore the function $f(x,y)$ has neither max nor min at (-1, 5).

At (1, -5)

$$f_{xx} = 6 > 0 \quad \text{and} \quad D = 36(1)(-5) = -180 < 0$$

\therefore the function has neither max nor min at (1, -5).

$\therefore (3, 3), (-7, -7)$ are called extreme points

Q) Set $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a max nor a minimum at (0, 0).

where $f_{xx}, f_{yy}, f_{xy} = 0$.

$$f(x,y) = 2x^4 - 3x^2y + y^2$$

$$f_x(x,y) = 8x^3 - 6xy$$

$$f_y(x,y) = -3x^2 + 2y$$

$$\text{also } f_{xx}(0,0) = 0 \quad \text{and} \quad f_{yy}(0,0) = 0.$$

$$\text{and } f_{xy} = 2x^2 - 6y \quad f_{xy} = -6x \quad \text{and} \quad f_{yy} = 2$$

$$\text{At } (0,0) \therefore f_{xy} = 0, f_{yy} = 0 \Rightarrow f_{xy} = 0.$$

$$\therefore f_{xy} = 0.$$

so that it is a doubtful case, and so requires

$$\begin{aligned} \text{Again } f(x,y) &= 2x^4 - 3x^2y + y^2 \\ &= 2x^2(2x^2 - 3xy + y^2) \\ &\geq 2x^2(x^2 - 4) - y(x^2 - 4) \\ &= (2x^2 - y)(x^2 - 4) \end{aligned}$$

$$\text{and } f(0,0) = 0$$

$$\text{now let } f(x,y) - f(0,0) = (x^2 - y)(2x^2 - y)$$

≥ 0 if for $y < 0$, or $x^2 > y \geq 0$.

$$< 0 \text{ for } y > x^2 > \frac{y}{2} \geq 0.$$

\therefore $f(x,y) - f(0,0)$ does not keep the same sign near the origin.

Hence f has neither maximum nor minimum at the origin.

Ques. See the function $f(x,y) = (y-x)^4 + (x-2)^4$ has ∞ minima at $(2,2)$.

\Rightarrow Let $f(x,y) = y^4 + x^4 - 4xy + 2x^2$ has a minimum at $(0,0)$.

$$\text{Soln } f(x,y) = y^4 + x^4 - 4xy + 2x^2$$

$$f_{xx}(x,y) = -4y + 4x^3 ; f_{yy} = 2y + 12x^2$$

$$f_{yy}(x,y) = 2y + x^2 ; f_{yy} = 2.$$

$$\text{and } f_{xx}(0,0) = 0 \quad f_{yy}(0,0) = 0$$

$$\therefore f_{xy}(0,0) = 0$$

At $(0,0)$ $f_{xx} = 0$, $f_{yy} = 2$ and $f_{xy} = 0$

$$\therefore f_{xx} f_{yy} - f_{xy}^2 = 0$$

So that it is a doubtful case and requires further investigation.

$$\text{Ans. Now } f(x,y) = y^4 + x^4 - 4xy + 2x^2$$

$$= (y + \frac{1}{2}x^2)^4 + \frac{3}{4}x^4$$

$$\text{and } f(x,y) - f(0,0) = (y + \frac{1}{2}x^2)^4 + \frac{3}{4}x^4 \geq 0 \quad \forall (x,y) \text{ in the neighborhood of } (0,0)$$

$\therefore f$ has a minimum at the origin.

Extreme values of a function of n variables.

A point (a_1, a_2, \dots, a_n) is said to be an extreme point, and $f(a_1, a_2, \dots, a_n)$ an extreme point value of a function f , if for every point (x_1, x_2, \dots, x_n) , other than (a_1, a_2, \dots, a_n) , of some nbhd. of (a_1, a_2, \dots, a_n) , the difference,

$$f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n) \text{ keeps the same sign.}$$

The extreme value is a maximum or a minimum value according as the sign is -ve or +ve.

→ The necessary conditions for $f(a_1, a_2, \dots, a_n)$ to be an extreme value of the function f are that all the partial derivatives $f_{x_1}, f_{x_2}, f_{x_3}, \dots, f_{x_n}$, in case they exist, vanish at (a_1, a_2, \dots, a_n) .

Since there are only necessary and not sufficient conditions therefore points which satisfy these conditions may not be extreme points. A point (a_1, a_2, \dots, a_n) is called a stationary point if all the first order partial derivatives of the function vanish at that point. Thus the stationary points are determined by solving the following n equations

simultaneously -

$$f_{x_1}(a_1, a_2, \dots, a_n) = 0$$

$$f_{x_2}(a_1, a_2, \dots, a_n) = 0$$

$$\vdots \vdots \vdots \vdots \vdots \vdots \vdots$$

$$f_{x_n}(a_1, a_2, \dots, a_n) = 0$$

for a function of n independent variables x_1, x_2, \dots, x_n . The condition can be given in a more compact form i.e. if (a_1, a_2, \dots, a_n) is a stationary point, then $\nabla f(a_1, a_2, \dots, a_n) = 0$.

$$\begin{cases} \therefore df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots \\ \therefore df = f_x dx_1 + f_y dx_2 + \dots \end{cases}$$

stationary point for, at the stationary point all the partial derivatives vanishes and therefore

$$df(a_1, a_2, \dots, a_n) = f_x(a_1, a_2, \dots, a_n) dx + f_y(a_1, a_2, \dots, a_n) dy + \dots + f_z(a_1, a_2, \dots, a_n) dz = 0$$

Conversely, when $df = 0$, the coefficients of the differentials dx, dy, \dots, dz of independent variables, are separately equal to zero.

Rule for a function $f(x, y, z)$ of three independent variables, sufficient conditions for (a, b, c) to be an extreme point are that

i) $df(a, b, c) = f_x da + f_y dy + f_z dz = 0$, so that
 $f_x = f_y = f_z = 0$

and

ii) $d^2f(a, b, c) = f_{xx}(da)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy} da dy + 2f_{yz} dy dz + 2f_{zx} dz dx$

Keeps the same sign for arbitrary values of da, dy, dz ; the extreme point being a maxima or a minima according as the sign of d^2f is ve or pos. The point will be neither a maxima nor a minima if d^2f does not keep the same sign; and requires further investigation, if d^2f keeps the same sign but vanishes at some points of a nbd of (a, b, c) .

$$\begin{aligned}
 df &= f_x da + f_y dy + f_z dz \\
 d^2f &= d(df) \\
 &= d(f_x da + f_y dy + f_z dz) = d(f_x da + d f_y) \\
 &\quad + d(f_y dy + f_z dz) + d(f_z dz) \\
 &= (f_{xx} da + f_{xy} dy + f_{xz} dz) da + f_x da \\
 &\quad + (f_{yx} da + f_{yy} dy + f_{yz} dz) dy + f_y dy \\
 &\quad + (f_{zx} da + f_{zy} dy + f_{zz} dz) dz + f_z dz
 \end{aligned}$$

but $da = dy = dz = 0$

$$\therefore d^2f(a, b, c) = f_{xx}(da)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2$$

The conditions that Df keeps the same sign may be stated in terms of matrices, as follows.

Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Df will always be +ve iff all the three principal

minors

$$\begin{vmatrix} f_{xx} \end{vmatrix},$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix},$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all +ve.

then f has minimum at (a, b, c) .

and Df will always negative iff their signs are alternatively negative and positive; then f has maximum at (a, b, c) .

(OR)

Let $f: D \rightarrow \mathbb{R}$, $D \subset \mathbb{R}^n$ be a function possessing continuous partial derivatives upto the second order partial derivatives in a nbd of a point α at which all the first order partial derivatives vanish,

then
 i) if f has minimum at α , if D_1, D_2, \dots, D_n are all +ve
 ii) f has maximum at α if D_1, D_2, \dots, D_n are alternatively +ve and -ve.

where $D_1 = \begin{vmatrix} d_{11} \end{vmatrix}$, $D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$, $D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$,

and $d_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

Examine the following function for extreme values:

$$f(x, y, z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$$

Soln Given $f(x, y, z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$

$$f_x = 4x - 3y$$

$$f_y = 6y - 3x$$

$$f_z = 8z + 8$$

Equating f_x, f_y, f_z to zero.

we get $4x - 3y = 0$

$$x - 2y = 0$$

$$z + 1 = 0$$

$$\therefore \Rightarrow x = 0, y = 0 \text{ and } z = -1$$

\therefore we get the only stationary point of f as $(0, 0, -1)$.

Now at $(0, 0, -1)$,

we have

$$d_{11} = f_{xx} = 4; \quad d_{12} = f_{xy} = -3; \quad d_{13} = f_{xz} = 0$$

$$d_{21} = f_{yx} = -3; \quad d_{22} = f_{yy} = 6; \quad d_{23} = f_{yz} = 0$$

$$d_{31} = f_{zx} = 0; \quad d_{32} = f_{zy} = 0; \quad d_{33} = f_{zz} = 8$$

$$\text{and } D_1 = d_{11} = 4 > 0, \quad D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix}$$

$$= 24 - 9 = 15 > 0$$

$$D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = \begin{vmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix}$$

$$= 8(24 - 9) = 8 \times 15 = 120 > 0$$

Hence $f(x, y, z)$ has a minimum at $(0, 0, -1)$.

in common and the maximum values of
 $f(x, y, z) = (ax + by + cz) e^{-\alpha x^2 - \beta y^2 - \gamma z^2}$ are

$$\pm \sqrt{\frac{1}{2} (a^2 \alpha^2 + b^2 \beta^2 + c^2 \gamma^2) / e} \text{ and } \sqrt{\frac{1}{2} (a^2 \alpha^2 + b^2 \beta^2 + c^2 \gamma^2) / e}$$

Soln $f(x, y, z) = (ax + by + cz) e^{-\alpha x^2 - \beta y^2 - \gamma z^2}$

$$f_x = (ax + by + cz) e^{-\alpha x^2 - \beta y^2 - \gamma z^2} \cdot (-2\alpha x) + a e^{-\alpha x^2 - \beta y^2 - \gamma z^2}$$

$$= [a - (ax + by + cz)(+2\alpha x)] e^{-\alpha x^2 - \beta y^2 - \gamma z^2}$$

$$= [a - 2\alpha x \sum ax] e^{-\sum ax^2}$$

$$\text{Sly } f_y = [b - 2\beta y \sum ax] e^{-\sum ax^2}$$

$$\text{Sly } f_z = [c - 2\gamma z \sum ax] e^{-\sum ax^2}$$

Evaluating f_x, f_y, f_z to zero.

$$f_x = (a - 2\alpha x \sum ax) e^{-\sum ax^2} = 0$$

$$\Rightarrow a - 2\alpha x \sum ax = 0, \quad \left. \begin{array}{l} \text{since } e^{-\sum ax^2} \neq 0 \\ \end{array} \right\}$$

$$f_y = 0 \quad \Rightarrow b - 2\beta y \sum ax = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (1)$$

$$f_z = 0 \quad \Rightarrow c - 2\gamma z \sum ax = 0$$

$$\therefore x \sum ax = \frac{a}{2\alpha} \quad (2)$$

$$y \sum ax = \frac{b}{2\beta} \quad (3)$$

$$z \sum ax = \frac{c}{2\gamma} \quad (4)$$

Multiplying (2) by a, (3) by b, (4) by c

and adding:

$$(ax + by + cz) \sum ax = \frac{a^2}{2\alpha} + \frac{b^2}{2\beta} + \frac{c^2}{2\gamma}$$

$$\sum ax \sum ax = \frac{1}{2} \sum ax^2$$

$$\Rightarrow (\sum ax)^2 = \frac{1}{2} \sum ax^2 + \Gamma \sum ax^2 = \pm k \text{ (say)}$$

$$a - 2\alpha^2 x \sum \alpha x = 0$$

$$a - 2\alpha^2 x (-k) = 0$$

$$\Rightarrow 2\alpha^2 x k = a$$

$$\Rightarrow x = \frac{a}{2\alpha^2 k}$$

$$\text{Also } y = \frac{b}{2\beta^2 k}, z = \frac{c}{2\gamma^2 k}.$$

putting $\sum \alpha x = -k$, in (1)

$$x = \frac{a}{2\alpha^2 k}, y = \frac{b}{2\beta^2 k}, z = \frac{c}{2\gamma^2 k}$$

∴ the stationary points are

$$\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k} \right), \left(\frac{-a}{2\alpha^2 k}, \frac{-b}{2\beta^2 k}, \frac{-c}{2\gamma^2 k} \right)$$

Again, we have

$$f_{xx} = \left[0 - 2\alpha^2 x a - 2\alpha^2 \sum \alpha x \right] e^{-\sum \alpha x^2} + \left[a - 2\alpha^2 x \sum \alpha x \right] \frac{-2\alpha^2 x}{-2\alpha^2 e^{-\sum \alpha x^2}}$$

$$= -2\alpha^2 [a - 2\alpha^2 x \sum \alpha x] e^{-\sum \alpha x^2} - 2\alpha^2 [\sum \alpha x + a x] e^{-\sum \alpha x^2}$$

$$f_{xy} = \left[b - 2\beta^2 y \sum \alpha x \right] e^{-\sum \alpha x^2} \left(-2\alpha^2 x \right)$$

$$+ (-2\beta^2 y a) e^{-\sum \alpha x^2}$$

$$= -2\alpha^2 (b - 2\beta^2 y \sum \alpha x) e^{-\sum \alpha x^2} - 2\beta^2 y a e^{-\sum \alpha x^2}$$

and similar expressions for f_{yy} , f_{zz} , f_{yz} , f_{zx} .

At the stationary point $\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k} \right)$,

$$\text{we have } \sum \alpha x^2 = \frac{1}{2}.$$

$$f_{xx} = 0 - 2d^2 \left[\frac{k + a}{2k} \right] e^{-2}$$

$$= -\frac{1}{k^2} \left[2d^2 k + a^2 \right] = -\left(\frac{2d^2 k + a^2}{k^2} \right)$$

$$f_{yy} = -\left(\frac{2d^2 k + b^2}{k^2} \right), f_{zz} = -\left(\frac{2d^2 k + c^2}{k^2} \right)$$

$$f_{xy} = 0 - \frac{ab}{k^2}, \quad f_{yz} = -\frac{bc}{k^2}, \quad f_{zx} = -\frac{ca}{k^2}$$

$$\therefore df = -\frac{1}{k^2} \left[(2d^2 k + a^2) dx^2 + (2d^2 k + b^2) dy^2 + (2d^2 k + c^2) dz^2 \right. \\ \left. - \frac{2}{k^2} (abd dx dy + bc dy dz + ca dz dx) \right]$$

$$\text{Now } f_{xx} = -\left(\frac{2d^2 k + a^2}{k^2} \right) < 0$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{2d^2 k + a^2}{k^2} & \frac{ab}{k^2} \\ \frac{ab}{k^2} & -\frac{2d^2 k + b^2}{k^2} \end{vmatrix} \\ = \frac{1}{k^4} \begin{vmatrix} 2d^2 k + a^2 & ab \\ ab & 2d^2 k + b^2 \end{vmatrix}$$

$$= \frac{1}{k^4} (4d^2 \beta^2 k^4 + 2d^2 k^2 b^2 + 2\beta^2 k^2 a^2 + ab^2 - a^2 b^2) \\ = \frac{2}{k^2} (2d^2 \beta^2 k^2 + d^2 b^2 + \beta^2 a^2) > 0$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{k^2} \begin{vmatrix} 2d^2 k + a^2 & ab & ac \\ ab & 2d^2 k + b^2 & bc \\ ac & bc & 2d^2 k + c^2 \end{vmatrix}$$

$$\begin{aligned}
 &= -\frac{1}{k^3 b^2} \left[(2r^v k^v + c^v) [4a^v \beta^v k^v + 2a^v k^v b^v + 2\beta^v k^v a^v + \gamma^v] \right. \\
 &\quad \left. - bc [(bc)(2a^v k^v + \alpha^v) - \alpha^v bc] + ac [\alpha^v bc - (2\beta^v k^v + b^v)] \right] \\
 &= -\frac{1}{k^3 b^2} \left[(2r^v k^v + c^v) 2k^v (2\beta^v k^v + \alpha^v b^v + a^v \beta^v) - 2b^v (2r^v k^v)^2 \right. \\
 &\quad \left. - 2a^v c^v \beta^v k^v \right] \\
 &\geq -\frac{4k}{b^2} (2a^v \beta^v k^v + \alpha^v b^v c^v + a^v \beta^v c^v + a^v b^v \gamma^v) \leq 0
 \end{aligned}$$

Thus the three principal minors have alternatively +ve and -ve signs and so d^2f is always -ve.

$\therefore \left(\frac{a}{2r^v k^v}, \frac{b}{2r^v k^v}, \frac{c}{2r^v k^v} \right)$ is a point of maxima.

and the maximum value = $ke^{-\frac{1}{2}}$

$$\begin{aligned}
 &\approx \sqrt{\frac{1}{2} \sum a_i^2} \sqrt{\frac{1}{e}} \\
 &\approx \sqrt{\frac{1}{2} \sum a_i^2} / e
 \end{aligned}$$

At the point $\left(\frac{-a}{2r^v k^v}, \frac{-b}{2r^v k^v}, \frac{-c}{2r^v k^v} \right)$, it may be shown as above

that $\sum a_i^2 = \frac{1}{2}$ and

$$f_{xx} = \frac{2a^v k^v + a^v}{k^3 b^2}, f_{yy} = \frac{2r^v k^v + b^v}{k^3 b^2}, f_{zz} = \frac{2r^v k^v + c^v}{k^3 b^2}$$

$$f_{xy} = \frac{ab^v}{k^3 b^2}, f_{yz} = \frac{bc^v}{k^3 b^2}, f_{zx} = \frac{ca^v}{k^3 b^2}$$

and the three principal minors are of +ve signs.

so that d^2f is +ve.

$\left(\frac{-a}{2r^v k^v}, \frac{-b}{2r^v k^v}, \frac{-c}{2r^v k^v} \right)$ is a point of minima

and the minimum value of the function = $-ke^{-\frac{1}{2}}$

$$= -\sqrt{\frac{1}{2} \sum a_i^2} / e$$

$\rightarrow S.O.T f(x, y, z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$.
has a minima at $(1, 1, 1)$ and a maxima at $(-1, -1, -1)$.

Given $f(x, y, z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$.

$$f_x = 3(x+y+z)^2 - 24yz - 3$$

$$f_y = 3(x+y+z)^2 - 24zx - 3$$

$$f_z = 3(x+y+z)^2 - 24xy - 3$$

\therefore the stationary points are given by

$$(x+y+z)^2 - 8yz - 3 = 0 \quad \text{--- (1)}$$

$$(x+y+z)^2 - 8zx - 3 = 0 \quad \text{--- (2)}$$

$$(x+y+z)^2 - 8xy - 3 = 0 \quad \text{--- (3)}$$

$$\text{(2)-(1)} \quad z(x-y) = 0 \Rightarrow z=0 \text{ or } y=x$$

$$\text{(3)-(2)} \quad x(y-z) = 0 \Rightarrow x=0 \text{ or } y=z$$

$$\text{(3)-(1)} \quad y(z-x) = 0 \Rightarrow y=0 \text{ or } z=x$$

i.e., either $x=y=z=0$ or $x=y=z$.

\therefore the stationary points are $(1, 1, 1)$ and $(-1, -1, -1)$

Again we have

$$f_{xx} = 6(x+y+z) = f_{yy} = f_{zz}$$

$$f_{xy} = 6(x+y+z) - 24z = f_{yx}$$

$$f_{yz} = 6(x+y+z) - 24x = f_{zy}$$

$$f_{zx} = 6(x+y+z) - 24y = f_{xz}$$

At $(1, 1, 1)$

$$f_{xx} = f_{yy} = f_{zz} = 18$$

$$f_{xy} = f_{yz} = f_{zx} = -6$$

$$df = 18(f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}dxdy + 2f_{yz}dydz + 2f_{zx}dzdx)$$

$$+ 2f_{yz}dydz + 2f_{zx}dzdx$$

$$\frac{\partial^2 f}{(1,1,1)} = 18(dx^2 + dy^2 + dz^2) - 12(dxdy + dydz + dzdx)$$

$$= 6[3(dx^2 + dy^2 + dz^2) - 2(dxdy + dydz + dzdx)]$$

$$df = 6[(dx^2 + dy^2 + dz^2) + (d_x - d_y)^2 + (d_y - d_z)^2 + (d_z - d_x)^2]$$

which is +ve for all values of d_x, d_y, d_z
and does not vanish for $(d_x, d_y, d_z) \neq (0, 0, 0)$.

$\therefore (1, 1, 1)$ is a point of minima of the function
i.e. f has minimum at $(1, 1, 1)$.

At $(-1, -1, -1)$:

$$f_{xx} = f_{yy} = f_{zz} = -18 \quad ; \quad f_{xy} = f_{yz} = f_{zx} = 6$$

$$\begin{aligned} df &= -18[dx^2 + dy^2 + dz^2] + 12(d_x dy + dy dz + dz dx) \\ &= -6[3(dx^2 + dy^2 + dz^2) + 2(dady + dydz + dzdx)] \\ &= -6[(dx^2 + dy^2 + dz^2) + (d_x - d_y)^2 + (d_y - d_z)^2 + (d_z - d_x)^2] \end{aligned}$$

which is -ve for all d_x, d_y, d_z and never vanishes.

Hence the function has maximum at $(-1, -1, -1)$.

\rightarrow S.T. the following functions have a minima at the points indicated

i) $x^2y^2z^2 + 2xyz$ at $(0, 0, 0)$

ii) $x^4 + y^4 + z^4 - 4xyz$ at $(1, 1, 1)$.

\rightarrow S.T. the function

$$f(x, y, z) = 2xyz - 4x^2 - 2y^2 + z^2 + 4x^2 - z^2 - 2x - 4y + 4z$$

has 5 stationary points but has a minimum value

only at $(0, 0, 0)$ or $(1, 2, 0)$

\rightarrow S.T. the function

$$g(x, y, z) = 2x^3 - 2y^3 - 2z^3, (x, y, z) \neq (0, 0, 0).$$

has only one extreme value, $\log(3/e^2)$

→ find a point with in a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Sol: Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the vertices of the triangle and (x, y) be a point inside the triangle. Let $f(x, y)$ denotes the sum of the squares of the distances of (x, y) from three vertices,

then

$$f(x, y) = [(x-x_1)^2 + (y-y_1)^2] + [(x-x_2)^2 + (y-y_2)^2] \\ + [(x-x_3)^2 + (y-y_3)^2]$$

$$\Rightarrow f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3) \text{ & } f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3)$$

for maximum or minimum,

we have

$$f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3) = 0$$

$$\Rightarrow 3x - (x_1 + x_2 + x_3) = 0$$

$$\Rightarrow x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly

$$f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3) = 0$$

$$\Rightarrow 3y - (y_1 + y_2 + y_3) = 0$$

$$\Rightarrow y = \frac{y_1 + y_2 + y_3}{3}$$

also $f_{xx} = 2+2+2=6$

$$f_{yy} = 0 \text{ & } f_{yy} = 6$$

$$f_{xx} f_{yy} - (f_{xy})^2 = (6)(6) - 0 = 36 > 0$$

and $f_{xx} = 6 > 0$

$\therefore f$ is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, y = \frac{y_1 + y_2 + y_3}{3}$$

The required point is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$

→ Show that the function

$$f(x, y, z) = 2xyz - 4xz - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$$

has 5 stationary points but has a minimum value only at (1, 2, 0).

Solⁿ: Given that

$$f(x, y, z) = 2xyz - 4xz - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$$

$$f_x = 2yz - 4z + 2x - 2$$

$$f_y = 2xz - 2z + 2y - 4$$

$$f_z = 2xy - 4x - 2y + 2z + 4.$$

∴ the stationary points are given by

$$f_x = 0 \Rightarrow 2yz - 4z + 2x - 2 = 0 \Rightarrow yz - 2z + x - 1 = 0 \quad \text{--- (1)}$$

$$f_y = 0 \Rightarrow 2xz - 2z + 2y - 4 = 0 \Rightarrow xz - z + y - 2 = 0 \quad \text{--- (2)}$$

$$f_z = 0 \Rightarrow 2xy - 4x - 2y + 2z + 4 = 0 \Rightarrow xy - 2x - y + 2 + z = 0 \quad \text{--- (3)}$$

Adding the last three equations, we see

that system of equations to

$$yz - 2z + x - 1 = 0$$

$$xz - z + y - 2 = 0$$

$$xy - 2x - y + 2 + z = 0$$

These stationary points are given by the two systems of equations

$$\begin{cases} yz - 2z + x - 1 = 0 \\ xz - z + y - 2 = 0 \\ xy - 2x - y + 2 + z = 0 \end{cases}$$

$$\begin{cases} yz - 2z + x - 1 = 0 \\ xz - z + y - 2 = 0 \\ z + y - 2 = 0 \end{cases}$$

These give $(0, 3, 1)$, $(0, 1, -1)$, $(1, 2, 0)$, $(2, 1, 1)$
 $(2, 3, -1)$ as the stationary points
of the function.

Again, we have at any point (x, y, z)

$$f_{xx} = 2, f_{yy} = 2, f_{zz} = 2;$$

$$f_{yz} = 2z, f_{xz} = 2x - 2, f_{xy} = 2y - 4 = f_{yx}$$

for $(0, 3, 1)$ the matrix of the quadratic form

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

the principal minors

$$2, \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

are 2, 0, and -32 respectively.

∴ the function is neither a maximum nor

a minimum at $(0, 3, 1)$.

It may similarly be shown that the
function is neither a max nor
a minimum at the stationary
points $(0, 1, -1)$, $(2, 1, 1)$ and $(2, 3, -1)$.

∴ it has minimum
of principal minors
are all +ve

& it has maximum
of principal minors
are alternating
+ve & -ve

At $(1, 2, 0)$ the matrix of the quadratic

form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minors are

$$(2), \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8$$

and 2, 4 & 8 respectively.

i.e., the principal minors are all +ve.

$\therefore f(x, y, z)$ has a minimum at $(1, 2, 0)$.

→ Show that the function

$$3 \log(x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3, \quad (x, y, z) \neq (0, 0, 0)$$

has only one extreme value, $\log(3/e^2)$.

Sol:

→ find the extreme value of xyz if $x+y+z=a$.

Sol: Let $f = xyz$

$$\phi = x+y+z-a.$$

- Now consider the function F of three independent variables x, y, z such that

$$F = xyz + \lambda(x+y+z-a)$$

where λ is a constant

$$dF = (yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz.$$

At stationary points $df=0$

$$f_x=0 \Rightarrow yz+\lambda=0 \quad \text{---(1)}$$

$$f_y=0 \Rightarrow xz+\lambda=0 \quad \text{---(2)}$$

$$f_z=0 \Rightarrow xy+\lambda=0 \quad \text{---(3)}$$

multiplying ① by x , ② by y & ③ by z
and adding,

we get

$$3xyz + \lambda(x+y+z) = 0$$

$$\Rightarrow 3xyz + \lambda(a) = 0 \quad (\because x+y+z=a)$$

$$\Rightarrow \boxed{\lambda = \frac{-3xyz}{a}}$$

From ①,

we have

$$yz+\lambda=0 \Rightarrow yz - \frac{3xyz}{a} = 0$$

$$\Rightarrow yz\left(1 - \frac{3x}{a}\right) = 0$$

$$\Rightarrow 1 - \frac{3x}{a} = 0 \text{ or } yz = 0 \quad (\text{ignoring } \lambda)$$

$$\Rightarrow x = \frac{a}{3}$$

from ②

$$x_2 + 20 \Rightarrow x_2 + \left(-\frac{3x_1}{a}\right) = 20$$

$$\Rightarrow x_2 \left(1 - \frac{3}{a}\right) = 20$$

$$\Rightarrow y = \frac{a}{3}.$$

Similarly, we get $z = \frac{9}{2}$.

The stationary point is $(\frac{a}{3}, \frac{a}{3}, \frac{a}{3})$.

$$\therefore f = xyz$$

$$= \left(\frac{a}{3} \right) \left(\frac{a}{3} \right) \left(\frac{a}{3} \right)$$

$$= -\frac{a^2}{2} \text{ at } \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right).$$

$$\text{Hence } f = xyz = \frac{a^3}{27} \quad (4)$$

$$= d(\alpha F) \\ = d[(y^2 + z)dx + (\alpha x + z)dy + (\beta y + \lambda)dz]$$

$$= \int [(y_2 + z)dx + 2dy + ydz] dx +$$

$$[(xz+2)dy + 2dz + xde] dy +$$

$$[(x^2y + \frac{1}{2}y^2)dx^2 + (ydx^2 + xdy)] dx^2.$$

$$= 2(z \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} + y \frac{\partial}{\partial x}),$$

(as $f_x \geq 0, f_y = 0, f_z = 0$)

Here $f_{11} = 0$, $f_{12} = 0$, $f_{22} = 0$ -

~~1. front Fixy
Lynn Fixy | Fixy Fixy Fixy~~
~~and~~ ~~Fix Fixy Fixy~~ ~~Fixy Fixy Fixy~~
~~Fixy Fixy Fixy~~ ~~Fixy Fixy Fixy~~

R. D. H. C. S. regarding further investigation

Here $f_{xx} = 0$.

∴ we require further investigation.

Treating z as function of x and y ,

we get from

$$\text{if } f(x, y, z) = xy^2 - \frac{a^3}{27} = 0$$

$$y^2 + 2xy \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{y^2}{2x}$$

$$\text{Similarly } \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

$$\text{Also } \frac{\partial^2 z}{\partial x^2} = -\left[x \frac{\partial^2 z}{\partial x^2} - z \right]$$

$$= -\left[x \left(-\frac{y^2}{2x} \right) - z \right]$$

$$\text{Similarly } \frac{\partial^2 z}{\partial y^2} = \frac{2z}{y^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{\left(y \frac{\partial z}{\partial x} - 2z \right)}{y^2}$$

$$= -\left[y \left(-\frac{z}{y} \right) - z \right] = -\frac{z}{y}.$$

At $(\frac{a}{3}, \frac{a}{3}, \frac{a}{3})$

$$z_{xx} = \frac{2 \cdot \frac{a}{3}}{\left(\frac{a}{3}\right)^2} = \frac{2 \cdot 3}{a^2} = \frac{6}{a^2} > 0 \text{ if } a > 0$$

$$z_{yy} = \frac{6}{a^2} > 0$$

$$z_{xy} = \frac{a/3}{(a/3)^2} = \frac{3}{a^2} > 0$$

$$z_{xx} > 0 \text{ and } z_{xx} z_{yy} - z_{xy}^2 = \left(\frac{6}{a^2}\right)\left(\frac{6}{a^2}\right) - \left(\frac{3}{a^2}\right)^2$$

$$= \frac{36}{a^4} - \frac{9}{a^4}$$

$$= \frac{27}{a^4} > 0$$

$\therefore f(x, y, z)$ has a minimum value at $(\frac{a}{3}, \frac{a}{3}, \frac{a}{3})$ and the minimum value is $3abc$.

Set - IX (ii)

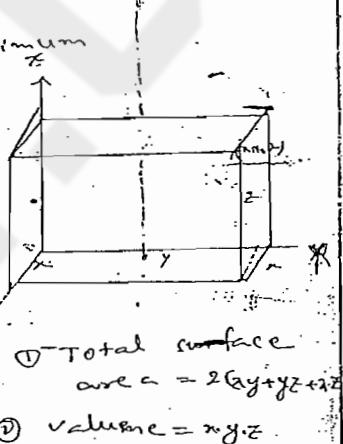
* Lagrange's method of undetermined multipliers. (for several independent variables)

Lagrange's method of multipliers, enable us to locate the stationary points when the variables are not free but are subject to some additional conditions.

Suppose we want to construct a closed box in the form of a parallelopiped of maximum volume using a piece of tins of area A . Let x, y, z denote the length, width and height of the box, respectively. Then the problem reduces to finding the maximum of the function $f(x, y, z) = xyz$ given that $2xy + 2xz + 2yz = A$ ①

We shall now discuss such problems for functions of two variables, where the variables satisfy some side conditions as in ①. That is, we will discuss a method to find out the maximum and minimum values of a function $Z = f(x, y)$, given that x and y are connected by an equation $g(x, y) = 0$.

If we eliminate one variable from the equation $Z = f(x, y)$ with the help of the relation $g(x, y) = 0$, then Z becomes a function of one variable. Then we can easily find its extreme values. So, the problem reduces to finding the maximum and minimum values for a function of one variable.



① Total surface area $A = 2(xy + xz + yz)$

② Volume $= xyz$

for example:

find the extreme values of the function
 $f(x,y) = x^2 + xy - x$ on the unit circle $x^2 + y^2 = 1$.

Sol

Given that $f(x,y) = x^2 + xy - x$ (1)

subject to condition

$$x^2 + y^2 = 1 \quad (2)$$

$$\Rightarrow \boxed{y^2 = 1 - x^2}$$

$$\therefore f(x,y) = f(x, \sqrt{1-x^2})$$

$= x^2 + 2(1-x^2) - x$ (which is clearly say

$$= -x^2 + 2 - x \quad \text{(one variable)}$$

$$= g(x) \text{ say}$$

$$\text{i.e. } g(x) = -x^2 + 2 - x \quad (3)$$

Now we shall find out the points of extreme for $g(x)$.

For this we find $g'(x)$ and equating it to zero.

$$g'(x) = -2x - 1$$

$$\Rightarrow -2x - 1 = 0$$

$\Rightarrow \boxed{x = -\frac{1}{2}}$ is a stationary point of $g(x)$.

Now we check whether it is a maximum or minimum point.

For this, $g''(x) = -2$.

$$\therefore g''(-\frac{1}{2}) = -2 < 0$$

$g(x)$ has minimum at $x = -\frac{1}{2}$.

$$g(-\frac{1}{2}) = -\frac{1}{4} + 2 + \frac{1}{2}$$

$= \frac{9}{4}$, which is the reqd maximum value of the function $f(x,y)$ in the unit circle.

$g(x)$ has maximum value at $x = \pm \frac{1}{2}$

so that

$$\text{from } ②, y^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore y = \pm \frac{\sqrt{3}}{2}$$

Therefore, we conclude that the function has a maximum at two points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

$$\text{Also } f(x,y) = f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{1}{2}$$

$$= \frac{9}{4}$$

$$= f\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Thus, the maximum value of the function on the unit circle is $\frac{9}{4}$.

Note:- we must have found this example quite easily to follow but it is not always feasible to use this procedure.

The reduction of the given function to a function of one variable using the given constraints might prove to be cumbersome or sometimes might not be possible at all.

we now present an alternative method which is often more convenient. This method is known as the method of Lagrange's multipliers:

Suppose we want to minimize or maximize a function $Z = f(x,y)$ subject to the condition $g(x,y) = 0$.

Theoretically, Z is a function of a single variable say x and the extreme values $\frac{dZ}{dx} = 0$.

$$\text{Since } \frac{\partial Z}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial y} = 0 \quad \text{--- (1)}$$

From the relation $g(x, y) = 0$, we find that at

the extrema, we have $\frac{\partial g}{\partial x} + \lambda \frac{\partial g}{\partial y} \cdot \frac{dy}{dx} = 0$.

Multiplying eqn (2) by an undetermined multiplier λ and adding this to equation (1), we get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} \right) + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} \right) \frac{dy}{dx} = 0 \quad \text{--- (2)}$$

Choosing λ so that the coefficient of $\frac{dy}{dx} = 0$ (eqn (1)),

Hence at the points of extrema, we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad \text{--- (3)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad \text{--- (4)}$$

$$g(x, y) = 0 \quad \text{--- (5)}$$

From these equations we can determine
the three unknowns x, y, λ .

The values of x, y give us the coordinates
of the stationary points.

The role of λ is over and we don't need
it any more.

We may add here that each stationary point
so determined need to be a max. or min.

Sometimes, we can determine their nature
by simple observation of equation $Z = f(x, y)$.

In some cases we can apply the second
derivative test, by eliminating the dependent
variable.

In fact, we can observe that equations (4), (5) and (6) are obtained by equating the partial derivatives of the function.

$$f(x,y,z) = f(x,y) + z g(x,y) \text{ to zero,}$$

treating x, y and z as independent variables.

Suppose we are given the function $f(x,y)$, whose extrema are to be found subject to the constraint $g(x,y)=0$. we form the auxiliary function

$$F(x,y,z) = f(x,y) + z g(x,y) \quad (7)$$

where z is to be determined.

Then we find the three partial derivatives of $F(x,y,z)$ and equate them to zero.

Then we solve three equations. Thus obtained are the stationary points of the given function under the given constraint.

The number z is called Lagrange's multiplier.

for example:

find the largest and the smallest values of $f(x,y) = xy$ on the circle $x^2+y^2=1$.

sol. $f(x,y) = xy \quad (1)$

and $g(x,y) = x^2+y^2-1 \quad (2)$

Now the auxiliary function is

$$F(x,y,z) = (x+zy) + z(x^2+y^2-1)$$

partially diff (7) w.r.t x, y and z ,

and equating to zero, we get

$$1 + \lambda (2x) = 0 \quad (4)$$

$$2 + \lambda (2y) = 0 \quad (5)$$

$$x + y = 1 \quad (6)$$

solving (4) and (5), we get

$$x = -\frac{1}{2\lambda}, y = \frac{-1}{\lambda} \quad (7)$$

$$(6) \Rightarrow \left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 = 1$$

$$\Rightarrow \lambda^2 = \frac{5}{4}$$

$$\lambda = \pm \frac{\sqrt{5}}{2}$$

It is given

from (7), $x = -\frac{1}{\sqrt{5}}$

$$y = \frac{-2}{\sqrt{5}} \quad (\because \lambda = \pm \frac{\sqrt{5}}{2})$$

and $\left[f(x,y) = -\frac{5}{\sqrt{5}} = -\sqrt{5} \right]$

from (7), $x = \frac{1}{\sqrt{5}}, y = \frac{2}{\sqrt{5}} \quad (\because \lambda = -\frac{\sqrt{5}}{2})$

and $\left[f(x,y) = \sqrt{5} \right]$

thus, the stationary points are

$$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \text{ and } \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

Since $f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \sqrt{5}$ and $f\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = -\sqrt{5}$

we get the largest value is $\sqrt{5}$ and
smallest value is $-\sqrt{5}$.

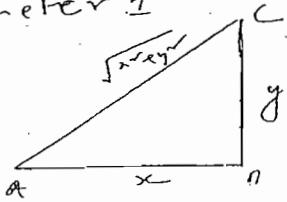
Ques

max value = $\sqrt{5}$
min value = $-\sqrt{5}$

if find the extreme values of the function $f(x,y) = xy$ on the surface $g(x,y)$
where $g(x,y) = \frac{xy}{2} + \frac{y^2}{2} - 1 = 0$.

→ find the right angled triangle of perimeter 1 with largest area. (51)

Sol suppose ABC is a right angled triangle with perimeter 1



Let the sides of triangle be $x, y, \sqrt{x^2+y^2}$.

then $f(x,y) = \text{area of } \triangle ABC$
 $= \frac{1}{2}xy \dots \text{(i)}$

and the perimeter of the triangle ABC
 $= x+y+\sqrt{x^2+y^2}$

but given that $x+y+\sqrt{x^2+y^2} = 1$.

Now we have to find the maximum of

$$f(x,y) = \frac{1}{2}xy \quad \text{(i)}$$

subject to the condition

$$g(x,y) = x+y+\sqrt{x^2+y^2}-1 = 0. \quad \text{(ii)}$$

Now let us form the system of the equations for this f and g: we get

$$\frac{\partial}{\partial x} [y + \lambda \left(1 + \frac{x}{\sqrt{x^2+y^2}} \right)] = 0 \quad \text{(iii)}$$

$$\frac{\partial}{\partial y} [x + \lambda \left(1 + \frac{y}{\sqrt{x^2+y^2}} \right)] = 0 \quad \text{(iv)}$$

$$x+y+\sqrt{x^2+y^2}-1 = 0 \quad \text{(v)}$$

from the first two equations, we have

$$\frac{\frac{1}{2}y}{1 + \frac{a}{\sqrt{a^2+yz}}} = \frac{\frac{1}{2}x}{1 + \frac{y}{\sqrt{a^2+yz}}}$$

(or)

$$\frac{y}{\sqrt{a^2+yz+x}} = \frac{yc}{\sqrt{a^2+yz+xy}}$$

(or)

$$\frac{y}{1-y} = \frac{a}{1-x}$$

$$\Rightarrow y-a y = x-a y$$

$$\Rightarrow \boxed{a = y - x}$$

∴ The sides are a, x and $\sqrt{a^2+x^2}$ & t

$$a+x+\sqrt{a^2+x^2}=1$$

$$\text{i.e. } (\sqrt{a^2+x^2})_a = 1$$

$$\Rightarrow \boxed{\lambda = \frac{1}{2+\sqrt{2}}}$$

Thus, the reqd sides are

$$\frac{1}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}} \text{ and } \frac{1}{1+\sqrt{2}}$$

(58)

To find the stationary points of the function
 $f(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m)$ of $n+m$ variables which
are connected by the equations

$$\phi_x(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m) = 0, x=1, 2, 3, \dots, n. \quad (1)$$

If the m variables u_1, u_2, \dots, u_m are determinate as functions of x_1, x_2, \dots, x_n from the system of ' m ' equation (2), then f can be regarded as function of ' n ' independent variables x_1, x_2, \dots, x_n .

For stationary values, $df = 0$

$$\therefore 0 = df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n + \frac{\partial f}{\partial u_1} du_1 + \dots + \frac{\partial f}{\partial u_m} du_m. \quad (2)$$

Differentiating equations (2), we get

$$\begin{aligned} & \frac{\partial \phi_1}{\partial x_1} dx_1 + \dots + \frac{\partial \phi_1}{\partial x_n} dx_n + \frac{\partial \phi_1}{\partial u_1} du_1 + \dots + \frac{\partial \phi_1}{\partial u_m} du_m = 0 \\ & \frac{\partial \phi_2}{\partial x_1} dx_1 + \dots + \frac{\partial \phi_2}{\partial x_n} dx_n + \frac{\partial \phi_2}{\partial u_1} du_1 + \dots + \frac{\partial \phi_2}{\partial u_m} du_m = 0 \\ & \vdots \\ & \frac{\partial \phi_m}{\partial x_1} dx_1 + \dots + \frac{\partial \phi_m}{\partial x_n} dx_n + \frac{\partial \phi_m}{\partial u_1} du_1 + \dots + \frac{\partial \phi_m}{\partial u_m} du_m = 0 \end{aligned} \quad (4)$$

Multiplying the equations (4) by $\lambda_1, \lambda_2, \dots, \lambda_m$ respectively

and adding to the eqn(3) we get

$$\begin{aligned} 0 = df &= \left(\frac{\partial f}{\partial x_1} + \sum \lambda_i \frac{\partial \phi_i}{\partial x_1} \right) dx_1 + \dots + \left(\frac{\partial f}{\partial x_n} + \sum \lambda_i \frac{\partial \phi_i}{\partial x_n} \right) dx_n \\ &+ \left(\frac{\partial f}{\partial u_1} + \sum \lambda_i \frac{\partial \phi_i}{\partial u_1} \right) du_1 + \dots + \left(\frac{\partial f}{\partial u_m} + \sum \lambda_i \frac{\partial \phi_i}{\partial u_m} \right) du_m \end{aligned} \quad (5)$$

$\lambda_i = 1, 2, \dots, m$

Let the ' m ' multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ be so chosen

that the coefficients of m differentials
 du_1, du_2, \dots, du_m all vanish.

$$\begin{aligned} \text{i.e., } & \frac{\partial f}{\partial u_1} + \sum \lambda_r \frac{\partial \phi_r}{\partial u_1} = 0, \\ & \frac{\partial f}{\partial u_2} + \sum \lambda_r \frac{\partial \phi_r}{\partial u_2} = 0 \\ & \vdots \\ & \frac{\partial f}{\partial u_m} + \sum \lambda_r \frac{\partial \phi_r}{\partial u_m} = 0 \end{aligned} \quad \left. \right\} \quad (6)$$

then eqn (5) becomes

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_1} \right) dx_1 + \dots + \left(\frac{\partial f}{\partial x_n} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_n} \right) dx_n$$

so that the differential df is expressed in terms of the differentials of independent variables only.

Hence

$$\begin{aligned} \frac{\partial f}{\partial x_1} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_1} &= 0, \quad \frac{\partial f}{\partial x_2} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_2} = 0, \dots \\ & \quad \vdots \\ \frac{\partial f}{\partial x_n} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_n} &= 0. \end{aligned} \quad \left. \right\} \quad (7)$$

Equations (2), (6), (7) form a system of $m+n = 2m+n$ equations which may be simultaneously solved to determine the 'm' multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ and the $n+m$ coordinates $x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m$ of the stationary points of f .

Important Rules:

for practical purposes, the process of obtaining equations (6) and (7) of the above, may be put in a precise form as follows:

Define a function

$$F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \dots + \lambda_m \phi_m$$

and consider all the variables $x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m$

as independent.

At a stationary point of f , $df = 0$.

$$\therefore 0 = df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n + \frac{\partial f}{\partial u_1} du_1 + \dots + \frac{\partial f}{\partial u_m} du_m$$

$$\therefore \frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \dots, \frac{\partial f}{\partial x_n} = 0, \frac{\partial f}{\partial u_1} = 0, \dots, \frac{\partial f}{\partial u_m} = 0$$

which are same as equations (6) & (7).

The stationary points of f may be found by determining the stationary points of the function F , where $F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \dots + \lambda_m \phi_m$ and considering all the variables as independent variable.

A stationary point will be an extreme point of f if dF keeps the same sign, and will be maxima or minima according as dF is -ve or +ve.

Find the shortest distance from the origin to the

$$\text{hyperbola } x^2 + 8xy + 7y^2 = 225, z = 0.$$

Sol: We have to find the minimum value of $x^2 + y^2$ (the square of the distance from the origin to any point in the xy plane) subject to the constraint,

$$x^2 + 8xy + 7y^2 = 225.$$

$$\text{Let } f = x^2 + y^2; \phi = x^2 + 8xy + 7y^2 - 225 = 0$$

Consider the function

$$F = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$$

where x, y are independent variables and λ is a constant.

$$df = (2x + 2y\lambda + 8y\lambda) dx + (2y + 8x\lambda + 14y\lambda) dy$$

$$\therefore \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \quad (\because df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy)$$

$$\Rightarrow 2(1+\lambda)x + 8y\lambda = 0 \quad \Rightarrow 2(1+\lambda)y + 8x\lambda = 0$$

$$\Rightarrow (1+\lambda)x + 4y\lambda = 0 \quad \underline{(1)} \quad \Rightarrow 4x + (1+7\lambda)y = 0 \quad \underline{(2)}$$

Now eliminating x and y from (1) & (2)

we get

$$\begin{vmatrix} 1+\lambda & 4\lambda \\ 4\lambda & 1+7\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(1+7\lambda) - 16\lambda^2 = 0$$

$$\Rightarrow 1 + 8\lambda + 7\lambda^2 - 16\lambda^2 = 0$$

$$\Rightarrow -9\lambda^2 + 8\lambda + 1 = 0$$

$$\Rightarrow 9\lambda^2 - 8\lambda - 1 = 0$$

$$\Rightarrow 9\lambda(\lambda - 1) + 1(\lambda - 1) = 0$$

$$\Rightarrow (9\lambda + 1)(\lambda - 1) = 0$$

$$\therefore \lambda = -\frac{1}{9}; \lambda = 1$$

If $\lambda = 1$, from (1) $x = -2y$.

NOW from $2x + 8xy + 7y^2 = 225$

we get $y^2 = -45$, for which no real solution exists.

If $\lambda = -\frac{1}{9}$, from (1)

we get $y = 2x$.

NOW from $x^2 + 8xy + 7y^2 = 225$

we get $x^2 = 5$, $y^2 = 20$.

$$\therefore x^2 + y^2 = 25.$$

$$\begin{aligned}
 \text{Now } d^2F &= d(df) \\
 &= d[(2x+2x^2+8y\lambda)dx + (2y+8x\lambda+14y\lambda)dy] \quad (6) \\
 &= [(2+2\lambda)dx + 8\lambda dy]dx + [8\lambda dx + (2+14\lambda)dy \\
 &\quad + (2x+2x^2+8y\lambda)d^2x + (2y+8x\lambda+14y\lambda)d^2y] \\
 &= 2(1+\lambda)(dx)^2 + 2(1+7\lambda)(dy)^2 + 16\lambda dx dy
 \end{aligned}$$

$$\begin{aligned}
 \because \frac{\partial f}{\partial x} = 0 \\
 \therefore 2x+2x^2+8y\lambda = 0 \\
 \text{and } \frac{\partial f}{\partial y} = 0 \\
 \Rightarrow 2y+8x\lambda+14y\lambda = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{At } \lambda = -\frac{1}{9}, d^2F &= \frac{16}{9}(dx)^2 - \frac{16}{9}dx dy + \frac{4}{9}(dy)^2 \\
 &= \frac{4}{9}[4(8x)^2 - 4dx dy + (dy)^2] \\
 &= \frac{4}{9}(2dx - dy)^2 \\
 &\rightarrow 0, \text{ and cannot vanish because } (dx, dy) \neq (0, 0)
 \end{aligned}$$

Hence the function x^2+y^2 has a minimum value 25.

→ find the maximum and minimum of $x^2+y^2+z^2$
 Subject to the conditions $\frac{x}{4} + \frac{y}{5} + \frac{z}{25} = 1$ and $x = y = z$.

So let $f = x^2+y^2+z^2$
 $\phi_1 = \frac{x}{4} + \frac{y}{5} + \frac{z}{25} - 1 = 0$

$$\text{and } \phi_2 = x + y - z = 0$$

Consider a function F of independent variables x, y, z . Where:

$$f = x^2+y^2+z^2 + \lambda_1 \left(\frac{x}{4} + \frac{y}{5} + \frac{z}{25} - 1 \right) + \lambda_2 (x+y-z)$$

$$\begin{aligned}
 df &= \left(2x + \frac{2}{4}\lambda_1 + \lambda_2 \right) dx + \left(2y + \frac{2}{5}\lambda_1 + \lambda_2 \right) dy \\
 &\quad + \left(2z + \frac{2}{25}\lambda_1 - \lambda_2 \right) dz
 \end{aligned}$$

$$df = f_1 dx + f_2 dy + f_3 dz$$

As x, y, z are independent variables,

$$\therefore \frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 2\lambda_1 + \lambda_2 = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + \frac{5}{5}\lambda_1 + \lambda_2 = 0$$

$$\frac{\partial f}{\partial z} = 0 \Rightarrow 2z + \frac{25}{25}\lambda_1 - \lambda_2 = 0$$

$$\therefore x = \frac{-2\lambda_2}{\lambda_1+4}, y = \frac{-5\lambda_2}{5\lambda_1+10}, z = \frac{25\lambda_2}{2\lambda_1+50} \quad \textcircled{1}$$

Substituting in $x+y+z$, we get

$$-\frac{2\lambda_2}{\lambda_1+4} + \frac{(-5\lambda_2)}{5\lambda_1+10} + \frac{25\lambda_2}{2\lambda_1+50} = 0$$

$$\lambda_2 \left[\frac{2}{\lambda_1+4} + \frac{5}{5\lambda_1+10} + \frac{25}{2\lambda_1+50} \right] = 0$$

$$\text{if } \lambda_2 \neq 0, \frac{2}{\lambda_1+4} + \frac{5}{5\lambda_1+10} + \frac{25}{2\lambda_1+50} = 0 \quad \textcircled{2}$$

if $\lambda_2 = 0$

then from $\textcircled{1}$

$$x=0, y=0, z=0, \text{ but } (x, y, z) = (0, 0, 0) \text{ does not satisfy the other condition of the constraint}$$

$$\therefore \text{from } \textcircled{2}, 17\lambda_1 + 245\lambda_1 + 750 = 0$$

$$\Rightarrow \lambda_1 = -10, \lambda_2 = -75/17$$

for $\lambda_1 = -10$, from $\textcircled{1}$

$$x = \frac{1}{3}\lambda_2, y = \frac{1}{5}\lambda_2, z = \frac{5}{6}\lambda_2 \quad \textcircled{3}$$

Now substituting $\textcircled{3}$ in $\frac{x}{4} + \frac{y}{5} + \frac{z}{25} = 1$

$$\text{we get } \lambda_2 \left[\frac{1}{86} + \frac{1}{20} + \frac{1}{36} \right] = 1$$

$$\frac{19\lambda_2}{180} = 1 \Rightarrow \lambda_2 = \frac{180}{19} \quad \text{or } \lambda_2 = \pm 6$$

putting $\lambda_2 = \pm 6\sqrt{19}$ in (3)
 from (1) the corresponding stationary points are

$$\text{at } (2\sqrt{519}, 3\sqrt{519}, 5\sqrt{519}) \text{ and } (-2\sqrt{519}, -3\sqrt{519}, -5\sqrt{519})$$

The value of $x^2 + y^2 + z^2$ corresponding to these points

is 10.

for $\lambda_2 = -7\sqrt{17}$

from (1) $x = \frac{34}{7} \lambda_2, y = -\frac{17}{4} \lambda_2, z = \frac{17}{28} \lambda_2 \quad \text{(2)}$
 which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{28} = 1$ give

$$\lambda_2 = \pm \frac{140}{17\sqrt{646}}$$

substituting $\lambda_2 = \pm \frac{140}{17\sqrt{646}}$ in (2)

Then the corresponding stationary points are

$$\left(\frac{40}{1646}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}} \right) \text{ and } \left(-\frac{40}{1646}, \frac{35}{\sqrt{646}}, -\frac{5}{\sqrt{646}} \right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these points
 is $75/17$.

\therefore the maximum value is 10 and the minimum
 value is $75/17$.

Note:

1. We have not theoretically established the existence
 of maximum or minimum value. we have simply
 shown that of all possible values, 10 is maximum
 and $75/17$ the minimum.

2. Using constraint conditions, $dZ = dxdy; \frac{x}{4}dx + \frac{y}{5}dy + \frac{z}{28}dz$
 so that $dx = dz$ and consequently dF may be expressed
 in terms of dF (or dz) alone. It can, then, be easily
 verified that 10 is a maximum value and $75/17$ the minimum.

Determine the extreme values of $b(x+y+z)$
 Subject to the conditions $xyz = abc$. where $x \geq 0, y \geq 0, z \geq 0$

Let $f = b(x+y+z) + \lambda(xyz - abc)$

Now consider a function f of three independent variables x, y, z , where.

$$f = b(x+y+z) + \lambda(xyz - abc) \quad \text{where } \lambda \text{ is constant.}$$

$$df = (bx+by+bz)\lambda dx + (cx+ay+az)\lambda dy + (ab+xy)\lambda dz.$$

$\therefore df = f_x dx + f_y dy + f_z dz$

As x, y, z are independent variables

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \Rightarrow bx+by+bz\lambda = 0 \\ &\Rightarrow xyz = -bc \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= 0 \Rightarrow cx+ay+az\lambda = 0 \\ &\Rightarrow xz\lambda = ca \quad \text{--- (2)} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial z} &= 0 \Rightarrow ab+xy\lambda = 0 \\ &\Rightarrow xy\lambda = -ab. \quad \text{--- (3)} \end{aligned}$$

Multiplying (1), (2), (3)

$$\lambda^3 xyz = -abc$$

$$\lambda^3 (abc) = -abc \quad (\because xyz = abc)$$

$$\Rightarrow \lambda^3 = -1$$

Sub $\lambda = 1$ in (1), (2), (3)
 \therefore ~~from (1)~~ $\Rightarrow yz = bc$; ignoring the imaginary values of λ .

$$xz = ca$$

$$xy = ab.$$

Solving we get the stationary point as (a, b, c)
 i.e., $x=a, y=b, z=c$

$$\therefore f = b(x+y+z) + \lambda(xyz - abc)$$

at (a, b, c)

$$= abc + abc - abc$$

$$= 3abc$$

$$\text{Now } d^2F = d(df) = d[(bx+yz)dx + (ax+yz)dy + (bx+2xy)dz]$$

$$\therefore d^2F = -2[ydx dz + zdy dx + xdy dz] \quad \begin{array}{l} \text{the coefficients of} \\ \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x} \\ \text{are } f_{xy}=0, f_{yz}=0, f_{zx}=0 \end{array}$$

Here $f_{xx}=0, f_{yy}=0, f_{zz}=0$

$$\begin{vmatrix} f_{xy} & f_{yz} \\ f_{yx} & f_{yy} \end{vmatrix} = 0$$

$$\text{and } \begin{vmatrix} f_{xy} & f_{yz} & f_{zx} \\ f_{yx} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zx} \end{vmatrix} = \begin{vmatrix} 0 & f_{yz} & f_{zx} \\ f_{yx} & 0 & f_{zy} \\ f_{xz} & f_{yz} & 0 \end{vmatrix} = 0$$

$$\therefore d^2F = 0.$$

we require further investigation.

Treating z as function of x and y , we get from

$$g(x, y, z) = xyz - abc = 0$$

$$yz + xy \frac{\partial z}{\partial x} = 0 \quad (\text{or}) \quad \frac{\partial z}{\partial x} = -\frac{y}{x}$$

$$\text{Similarly } \frac{\partial z}{\partial y} = -\frac{x}{y} \quad \text{--- (6)}$$

$$\text{Also, } \frac{\partial^2 z}{\partial x^2} = -\left[\frac{x \frac{\partial z}{\partial x} - z}{x^2} \right] = \frac{2z}{x^2}$$

$$\text{Similarly } \frac{\partial^2 z}{\partial y^2} = \frac{2z}{y^2}, \frac{\partial^2 z}{\partial x \partial y} = 0$$

Hence at (a, b, c) ,

$$\text{we have } z_{xx} = \frac{2z}{a^2} = \frac{2c}{a^2} > 0; z_{yy} = \frac{2z}{b^2} < 0$$

$$z_{xy} = 0$$

$$z_{xx} > 0 \text{ and } z_{xx} z_{yy} = z_{xy}^2 \\ = \frac{2c}{a^2} \cdot \frac{2c}{b^2} - 0 > 0$$

$f(x, y, z)$ has a minimum value at (a, b, c) and the minimum value is abc .

P.T. the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{8abc}{3\sqrt{3}}$.

Sol. If $P(x, y, z)$ is the vertex of a parallelopiped inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then the part that lies in the same quadrant as P will have volume xyz . for the largest parallelopiped such parts in all the quadrants must be similar and hence the volume will be $8xyz$.

So, we are to find the maximum value of $8xyz$ subject to the conditions.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, x \geq 0, y \geq 0, z \geq 0 \quad (1)$$

$$\text{Let } f = 8xyz$$

$$\text{and } g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Let us consider a function F of three independent variables x, y, z . where

$$F = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

$$\therefore df = \left(8yz + \frac{2x\lambda}{a^2} \right) dx + \left(8xz + \frac{2y\lambda}{b^2} \right) dy + \left(8xy + \frac{2z\lambda}{c^2} \right) dz$$

At stationary points

$$f_x = 0 \Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0$$

$$f_y = 0 \Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0$$

$$f_z = 0 \Rightarrow 8xy + \frac{2z\lambda}{c^2} = 0$$

}

(2)

Multiplying (2) by x, y, z respectively and adding

$$24xyz + 2\lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right] = 0$$

$$\Rightarrow 24xyz + 2\lambda(1) = 0$$

$$\Rightarrow 12xyz + \lambda = 0$$

$$\Rightarrow \lambda = -12xyz$$

\therefore from (2)

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

$$\text{and so } \lambda = -12 \left(\frac{a}{\sqrt{3}} \right) \left(\frac{b}{\sqrt{3}} \right) \left(\frac{c}{\sqrt{3}} \right)$$

$$\Rightarrow \frac{-12abc}{3\sqrt{3}} = -\frac{4abc}{\sqrt{3}}$$

Again,

$$df = 2\lambda \left(\frac{dx}{a^2} + \frac{dy}{b^2} + \frac{dz}{c^2} \right) + 16z dxdy + 16x dydz + 16y dxdz. \quad \begin{matrix} \text{(as coefficient of } dz \text{ is } 0) \\ (\because f_x = 0, f_y = 0, f_z = 0) \end{matrix}$$

$$df = -\frac{8abc}{\sqrt{3}} \sum \frac{1}{a^2} dx + \frac{16}{\sqrt{3}} \sum c dy. \quad \rightarrow (B)$$

Now from (1)

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0$$

at $(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}})$

$$\frac{a}{\sqrt{3}} \frac{da}{a} + \frac{b}{\sqrt{3}} \frac{dy}{b} + \frac{c}{\sqrt{3}} \frac{dz}{c} = 0$$

$$\Rightarrow \frac{da}{a} + \frac{dy}{b} + \frac{dz}{c} = 0 \quad \text{--- (4)}$$

Squaring, we get

$$\sum \frac{da^2}{a^2} + 2 \sum \frac{dadx}{ab} = 0$$

$$(or) \sum \frac{da^2}{a^2} + 2 \times \frac{1}{abc} \sum (adx dy) = 0$$

$$\Rightarrow abc \sum da^2 + 2 \sum c dx dy = 0$$

$$\Rightarrow \underbrace{abc \sum da^2}_{= -2 \sum c dx dy} = 0 \quad \text{--- (5)}$$

$$\text{from (3)} \quad dF = -\frac{8abc}{\sqrt{3}} \sum \frac{da^2}{a^2} + \frac{16}{\sqrt{3}} \left(\frac{abc}{2} \right) \sum \frac{dx^2}{a^2}$$

(from)

$$dF = -\frac{8abc}{\sqrt{3}} \sum \frac{da^2}{a^2} - \frac{8abc}{\sqrt{3}} \sum \frac{dx^2}{a^2}$$

$$= -\frac{16abc}{\sqrt{3}} \sum \frac{dx^2}{a^2}$$

$$< 0$$

Hence $(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}})$ is a point of maximum and the maximum value of dF is

$$\frac{8abc}{3\sqrt{3}}$$

S.T. that the lengths of axes of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx+my+nz=0$ are the roots of the quadratic in r^2 ,

$$\frac{l^2}{x^2-a^2} + \frac{m^2}{y^2-b^2} + \frac{n^2}{z^2-c^2} = 0$$

Sol: we have to find the stationary values of the function r^2 , where $r = \sqrt{x^2 + y^2 + z^2}$ subject to the two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

$$lx + my + nz = 0 \quad (2)$$

Let us consider a function F of independent variables x, y, z .

$$\text{Let } f = x^2 + y^2 + z^2.$$

$$\phi_1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\phi_2 = lx + my + nz = 0$$

$$\therefore F = (x^2 + y^2 + z^2) + \lambda_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) + 2\lambda_2 (lx + my + nz)$$

$$\therefore dF = \left(2x + \frac{2x\lambda_1}{a^2} + 2\lambda_2 l \right) dx + 2 \left(2y + \frac{2y\lambda_1}{b^2} + 2\lambda_2 m \right) dy + \left(2z + \frac{2z\lambda_1}{c^2} + 2\lambda_2 n \right) dz$$

At Stationary points

$$-x + \frac{\partial}{\partial x} \lambda_1 + l \lambda_2 = 0, \quad y + \frac{\partial}{\partial y} \lambda_1 + m \lambda_2 = 0$$

$$z + \frac{\partial}{\partial z} \lambda_1 + n \lambda_2 = 0 \quad (3)$$

Multiplying by x, y, z respectively and adding,

we get

$$\lambda_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) + x^2 + y^2 + z^2 + \lambda_2 (lx + my + nz) = 0$$

$$\Rightarrow \lambda_1 (1) + x^2 + y^2 + z^2 + \lambda_2 (0) = 0. \quad (\because \text{from (1&2)})$$

$$\Rightarrow x^2 + y^2 + z^2 + \lambda_1 = 0$$

$$\Rightarrow \lambda_1 = -(x^2 + y^2 + z^2)$$

$$\Rightarrow \lambda_1 = -r^2$$

From (3)

$$x = \frac{a^2 \lambda_2}{x^2 - a^2}, \quad y = \frac{b^2 \lambda_2}{y^2 - b^2}, \quad z = \frac{c^2 \lambda_2}{z^2 - c^2}$$

Substituting these values in $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$$\Rightarrow \lambda_2 \left\{ \frac{a^m}{x-a} + \frac{b^m}{x-b} + \frac{c^m}{x-c} \right\} = 0$$

and since $\lambda_2 \neq 0$, we get the quadratic in x giving the stationary values:

$$\frac{a^m}{x-a} + \frac{b^m}{x-b} + \frac{c^m}{x-c} = 0$$

\rightarrow If $(x^2 + y^2 + z^2) = ax^2 + by^2 + cz^2$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$, s.t. the maximum values or minimum values of $x^2 + y^2 + z^2$ are given by the equation.

$$\frac{1}{x-a} + \frac{m}{x-b} + \frac{n}{x-c} = 0$$

sol^{ns} Let $f = x^2 + y^2 + z^2$
 $\phi_1 = ax^2 + by^2 + cz^2 - f^2$
 $\phi_2 = \lambda_1 + \lambda_2 + \lambda_3 - 0$

Set us consider a function f of independent variables x, y, z

$$\therefore f = (x^2 + y^2 + z^2) + \lambda_1 (ax^2 + by^2 + cz^2 - f^2) + 2\lambda_2 (\lambda_1 + \lambda_2 + \lambda_3)$$

At Stationary points
 $df = (2x + 2a^2\lambda_1 + 2\lambda_2)dx + (2y + 2b^2\lambda_1 + 2m\lambda_2)dy + (2z + 2c^2\lambda_1 + 2n\lambda_2)dz$

At Stationary points

$$2x + 2a^2\lambda_1 + 2\lambda_2 = 0$$

$$2y + 2b^2\lambda_1 + 2m\lambda_2 = 0$$

$$2z + 2c^2\lambda_1 + 2n\lambda_2 = 0$$

Multiplying by x, y, z respectively and adding we get

$$\lambda_1(a\tilde{x}^2 + b\tilde{y}^2 + \tilde{z}^2) + \lambda_2(1x + my + nz) + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 0$$

$$\lambda_1(\tilde{x}^2) + \lambda_2(0) + \tilde{z}^2 = 0$$

$$\Rightarrow \tilde{x}^2 + \lambda_1 \tilde{z}^2 = 0$$

$$\Rightarrow \boxed{\lambda_1 = -\frac{1}{\tilde{z}^2}}$$

$$\text{So } x = \frac{\lambda_2 \tilde{x}^2}{a^2 - \tilde{y}^2}, \quad y = \frac{\lambda_2 m \tilde{y}^2}{b^2 - \tilde{y}^2}, \quad z = \frac{\lambda_2 n \tilde{y}^2}{c^2 - \tilde{y}^2}$$

Substituting these values in $1x + my + nz = 0$

we get

$$\frac{1}{a^2 - \tilde{y}^2} + \frac{m^2}{b^2 - \tilde{y}^2} + \frac{n^2}{c^2 - \tilde{y}^2} = 0 \quad \text{since } \lambda_2 \neq 0.$$

2007 → find the rectangular parallelopiped of greatest volume for a given total surface area S ; using its Lagrange's method of Multipliers.

2008 → Determine the maximum and minimum distances of the origin from the curve given by the equation $3x^2 + 4xy + 5y^2 = 140$.



→ find the maxima and minima of $x^a y^b z^c$ subject to
the conditions $ax+by+cz=1$ and $lx+my+nz=0$.

Sol "Given that

$$f(x, y, z) = x^a y^b z^c \quad \text{--- (1)}$$

subject to

$$\text{conditions } ax+by+cz=1 \quad \text{--- (2)}$$

$$\text{and } lx+my+nz=0 \quad \text{--- (3)}$$

Let us consider a function F of independent variables x, y, z :

where

$$F = x^a y^b z^c + \lambda_1 (ax+by+cz-1) + \lambda_2 (lx+my+nz) \quad \text{--- (4)}$$

$$\therefore dF = (a+2ax\lambda_1 + l\lambda_2)dx + (by+2by\lambda_1 + m\lambda_2)dy$$

$$+ (2z+2cz\lambda_1 + n\lambda_2)dz \quad (\because dF = F_x dx + F_y dy + F_z dz)$$

At stationary points, $dF=0$.

$$\therefore F_x = 0 \Rightarrow a+2ax\lambda_1 + l\lambda_2 = 0$$

$$F_y = 0 \Rightarrow by+2by\lambda_1 + m\lambda_2 = 0$$

$$F_z = 0 \Rightarrow 2z+2cz\lambda_1 + n\lambda_2 = 0$$

respectively and multiplying (1) by $\lambda_1, \lambda_2, \lambda_3$

$$\text{adding, we get } 2(a+by+cz) + 2(ax+my+nz) = 0$$

$$\Rightarrow 2u + 2(1)\lambda_1 + (0)\lambda_2 = 0 \quad \text{where } u = ax+by+cz$$

$$\Rightarrow \lambda_1 = -u$$

from (1), we have

$$2x+2ax(-u) + l\lambda_2 = 0 \Rightarrow x = \frac{-l\lambda_2}{2(1-u)}$$

$$2y+2by(-u) + m\lambda_2 = 0 \Rightarrow y = \frac{-m\lambda_2}{2(1-u)}$$

$$2z+2cz(-u) + n\lambda_2 = 0 \Rightarrow z = \frac{-n\lambda_2}{2(1-u)}$$

$$\textcircled{1} \Rightarrow l \left(\frac{-\lambda x_2}{2(1-ay)} \right) + m \left(\frac{-\lambda x_2}{2(1-bx)} \right) + n \left(\frac{-\lambda x_2}{2(1-cz)} \right) = 0.$$

$$\Rightarrow \lambda x_2 \left[\frac{l}{1-ay} + \frac{m}{1-bx} + \frac{n}{1-cz} \right] = 0.$$

If $\lambda_2 = 0$ then we get $xyz=0$ ①

but $(x, y, z) = (0, 0, 0)$ does not satisfy one of the condition of the constraint $\textcircled{1}$.

So that $\lambda_2 \neq 0$.

from $\textcircled{1}$, we have

$$\frac{l}{1-ay} + \frac{m}{1-bx} + \frac{n}{1-cz} = 0.$$

which gives the maxima and minima of u i.e $u = \underline{\underline{xyz}}$

Show that the maximum and minimum of radius vectors of the section of the surface

$(x^2 + y^2 + z^2)^{1/2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$; by the plane

$ax+by+cz=0$ are given by the equation

$$\frac{a^2 x}{1-ay} + \frac{b^2 y}{1-bx} + \frac{c^2 z}{1-cz} = 0.$$

sol. we have to find the maximum and minimum of the radius vector

\vec{r} , where $r^2 = x^2 + y^2 + z^2$.

Let it be $f(x, y, z)$

i.e $f(x, y, z)$ consider

subject to the conditions ①

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \textcircled{2}$$

$$\text{and } ax+by+cz=0 \quad \textcircled{3}$$

Let us consider a function F of the independent variables x, y, z .

where

$$f(x, y, z) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - r^2 \right) + \lambda_1 \left(\frac{2x}{a^2} + \frac{2y}{b^2} + \frac{2z}{c^2} - 2r \right) + \lambda_2 (rx + my + nz)$$

$$\therefore dF = \left(x_1 + \frac{2x}{a^2} \lambda_1 + \lambda_2 z \right) dx + \left(y_1 + \frac{2y}{b^2} \lambda_1 + \lambda_2 y \right) dy + \left(z_1 + \frac{2z}{c^2} \lambda_1 + \lambda_2 z \right) dz \quad (\because dF = f_x dx + f_y dy + f_z dz)$$

At stationary points, $dF = 0$

$$\begin{cases} F_x = 0 \Rightarrow 2x + \frac{2x}{a^2} \lambda_1 + \lambda_2 z = 0 \\ F_y = 0 \Rightarrow 2y + \frac{2y}{b^2} \lambda_1 + \lambda_2 y = 0 \\ F_z = 0 \Rightarrow 2z + \frac{2z}{c^2} \lambda_1 + \lambda_2 z = 0 \end{cases} \quad (6)$$

Multiplying each (6) by x, y, z respectively and adding, we get

$$\begin{aligned} & 2(x^2 + y^2 + z^2) + 2\lambda_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) + \lambda_2 (rx + my + nz) = 0 \\ & 2r^2 + 2\lambda_1 (r^2) + \lambda_2 (0) = 0 \\ & \Rightarrow -r^2 + \lambda_1 r^2 = 0 \\ & \Rightarrow r^2 (1 + \lambda_1) = 0 \\ & \text{If } r \neq 0, 1 + \lambda_1 = 0 \\ & \Rightarrow \boxed{\lambda_1 = -\frac{1}{r^2}} \end{aligned}$$

From (6), we have

$$\begin{aligned} 2x + \frac{2x}{a^2} \left(-\frac{1}{r^2} \right) + \lambda_2 z = 0 & \Rightarrow 2x \left(1 - \frac{1}{a^2 r^2} \right) = -\lambda_2 z \\ & \Rightarrow x = -\frac{\lambda_2 z}{(a^2 r^2 - 1)} \\ & \Rightarrow x = \frac{a^2 r^2 \lambda_2}{1 - a^2 r^2} \end{aligned}$$

Similarly, $y = \frac{b^2 r^2 \lambda_2}{1 - b^2 r^2}$

and $z = \frac{c^2 r^2 \lambda_2}{1 - c^2 r^2}$

Substituting the values of x, y and z in eqn (6).

we get

$$\begin{aligned} & \lambda \left(\frac{\alpha^r \lambda \lambda_2}{1-\alpha^r} \right) + \mu \left(\frac{\beta^r \mu \lambda_2}{1-\beta^r} \right) + \nu \left(\frac{\gamma^r \gamma \lambda_2}{1-\gamma^r} \right) = 0 \\ \Rightarrow & \lambda_2 r \left(\frac{\lambda \alpha^r}{1-\alpha^r} + \frac{\mu \beta^r}{1-\beta^r} + \frac{\nu \gamma^r}{1-\gamma^r} \right) = 0 \\ \Rightarrow & \lambda_2 \left[\frac{\lambda \alpha^r}{1-\alpha^r} + \frac{\mu \beta^r}{1-\beta^r} + \frac{\nu \gamma^r}{1-\gamma^r} \right] = 0, \text{ if } r \neq 0 \end{aligned}$$

If $\lambda_2 = 0$, then we get $x = y = z = 0$.

but $(x, y, z) = (0, 0, 0)$ does not satisfy the other condition of the constraint.

$\therefore \lambda_2 \neq 0$

from (P), we have

$$\frac{\lambda \alpha^r}{1-\alpha^r} + \frac{\mu \beta^r}{1-\beta^r} + \frac{\nu \gamma^r}{1-\gamma^r} = 0$$

which gives the minimum and maximum of $x^r + y^r + z^r$.

Now find the maximum and minimum values of

$$\frac{x^r}{a^r} + \frac{y^r}{b^r} + \frac{z^r}{c^r}, \text{ where } \lambda_2 + \mu y + \nu z = 0 \text{ and }$$

$$\frac{x^r}{a^r} + \frac{y^r}{b^r} + \frac{z^r}{c^r} = 1.$$

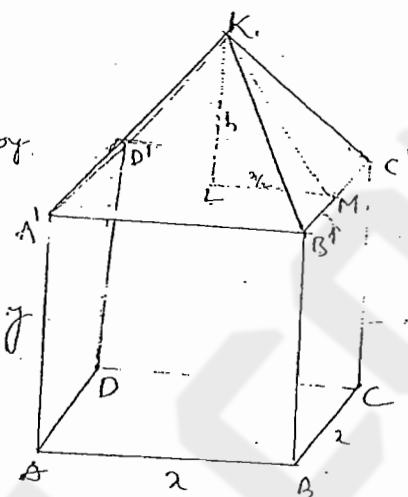
A tent on a square base of side ' x ', has its sides vertical of height ' y ' and the top is a regular pyramid of height ' h '. Find $2\pi y$ in terms of h , if the canvas required for its construction is to be minimum for the tent to have a given capacity.

Sol:

Let ' V ' be the volume enclosed by the tent and ' S ' be its surface area.

Surface area -

$$\begin{aligned} \text{Then } V &= \text{cuboid } (ABCD, A'B'C'D') + \\ &\quad + \text{pyramid } (K, A'B'C'D') \\ &= xy + \frac{1}{3}x^2h \\ &= xy\left(y + \frac{h}{3}\right). \end{aligned}$$



$$S = 4(ABGF) + 4\Delta KGH = 4xy + 4(\frac{1}{2}xh)$$

$$= 4xy + x\sqrt{x^2 + 4h^2} \quad (\because \sqrt{KL^2 + LM^2} = \sqrt{h^2 + (\frac{x}{2})^2})$$

for constant V , we have

$$\delta V = 2x\left(y + \frac{h}{3}\right)(\delta x) + x(2y) + \frac{2}{3}(2h)\delta h = 0 \quad (\delta V = 0)$$

for minimum S , we have

$$\begin{aligned} \delta S &= \left[4y + \sqrt{x^2 + 4h^2} + x \cdot \frac{1}{2} (x^2 + 4h^2)^{\frac{1}{2}} \cdot \frac{x}{2} \right] \delta x \quad (\delta S = 0) \\ &\quad + 4x\delta y + x \cdot \frac{1}{2} (x^2 + 4h^2)^{\frac{1}{2}} \cdot 2h \delta h = 0 \end{aligned}$$

By Lagrange's method of multipliers

$$\begin{aligned} dF &= \left[2xy + \sqrt{x^2 + 4h^2} + \frac{x^2}{2} (x^2 + 4h^2)^{\frac{1}{2}} + \lambda(2x(y + \frac{h}{3})) \right] dx \\ &\quad + (4x + 2x^2) \delta y + \left(4h x (x^2 + 4h^2)^{\frac{1}{2}} + \lambda \frac{x}{3} \right) \delta h = 0 \end{aligned}$$

At stationary points $dF = 0$.

$$\text{Fy} \Rightarrow \text{sum of forces in vertical direction} \\ 1_2 \Rightarrow \text{sum}((x+4h)^2 + a^2) = L$$

$$\text{sum}(x+4h)^2 + a^2 = L \\ \Rightarrow x(4+a^2) = 0 \\ \Rightarrow x = -\frac{a^2}{4}$$

then (3) becomes

$$4h^2(x^2 + 4h^2) - \frac{a^2}{4} = 0 \\ \Rightarrow x = \sqrt{5}h$$

now putting $x = \sqrt{5}h$, $x = -\frac{a^2}{4}$, in (1),

we get

$$4y + 3h + \frac{5h}{3} - \frac{a^2}{4} 2x(y + \frac{h}{3}) = 0 \\ \Rightarrow 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0 \\ \Rightarrow y = h/2$$

if the variables x, y, z satisfy the condition
 $\phi(x) \cdot f(y) \leq k$ and $\phi(z) = k$, then
show that the function $f(x) + f(y) + f(z)$

has a maximum when $x=y=z=a$, precisely
that $\left\{ \begin{array}{l} f''(x) = -\frac{\phi''(x)}{\phi'(x)} \\ f'(x) = \phi'(x) \end{array} \right\} \Rightarrow f''(x) < 0$.

Sol: Let $F = f(x_1) + f(y_1) + f(z_1) + \lambda (\phi(x_1) \phi(y_1) \phi(z_1) - k^2)$

$$\therefore dF = f'_x dx + f'_y dy + f'_z dz$$

$$\Rightarrow dF = \sum \{ f'(x_1) + \lambda \phi'(x_1) \phi(y_1) \phi(z_1) \} dx$$

At stationary points

$$f'(x_1) + \lambda \phi'(x_1) \phi(y_1) \phi(z_1) = 0$$

$$f'(y_1) + \lambda \phi'(x_1) \phi'(y_1) \phi(z_1) = 0$$

$$f'(z_1) + \lambda \phi'(x_1) \phi(y_1) \phi'(z_1) = 0$$

Since the function has a maximum

at (a, a, a) :

$$\text{Therefore } f'(a_1) + \lambda \phi'(a_1) \phi(a_1) \phi(a_1) = 0$$

$$\Rightarrow \lambda = -\frac{f'(a_1)}{\phi'(a_1) [\phi(a_1)]}$$

$$= -\frac{f'(a_1)}{K \phi'(a_1)} \quad (\because \phi'(a_1) \neq 0, \phi(a_1) \neq 0)$$

Now $dF = \sum f_{xx} dx^2 + \sum f_{yy} dy^2 + \sum f_{zz} dz^2$

$$\Rightarrow dF = \sum \{ f''(x_1) + 2 \phi''(x_1) \phi'(y_1) \phi'(z_1) \} dx^2 + 2 \lambda \sum \phi'(x_1) \phi'(y_1) \phi'(z_1) dx dy dz$$

All the following points are now

$$dF = \{f''(w) + \lambda K^* \psi''(w)\} \leq d\lambda \leq -\lambda K[\psi'(w)] \leq 0, \quad \text{for } w \in \mathbb{C}$$

Now $d(\gamma) \cdot d(g) \cdot \phi(\gamma) = 1$

$$\Rightarrow \int \psi(x) \phi(y) \phi(u) dx = 0$$

$$\Rightarrow k^*\psi(a)(d_2 + d_4 + d_7) = 0 \quad , \quad \exists (a, b, c)$$

$$\Rightarrow dx + dy + dz = 0 \quad (c \neq 0, \phi'(c) \neq 0)$$

$$\Rightarrow (dx + dy + dz)^2 = 0$$

$$\Rightarrow 2 \sum dxdy = - \sum da^r \quad \dots \quad (3)$$

from ② and ③ we obtain

$$\tilde{dF} = \left\{ f''(a) + \lambda K^2 \phi''(a) \right\} \sum dx^r - \lambda K [\phi'(a)]^2 \sum d\sigma^r$$

$$= \left[f''(a) - f'(a) \frac{\phi''(a)}{\phi'(a)} + f'(a) \frac{\phi'(a)}{K} \right] \sum d\sigma^r$$

$$= -f''(a) - f'(a) \left[\frac{\phi''(a)}{\phi'(a)} + \frac{\phi'(a)}{\phi(a)} \right] \quad (\because \phi(a) = k)$$

- For a maximum value at (a, a, a) ,

we have $d^T F < 0$

$$\Rightarrow f'(a) - f'(c) \left\{ \frac{\phi'(a)}{\phi'(c)} - \frac{\phi'(c)}{\phi(a)} \right\} < 0$$

$$\Rightarrow f^q(a) < f'(a) \left\{ \frac{\phi'(a)}{\phi(a)} - \frac{\phi'(a)}{\phi(a)} \right\}$$

~~✓~~ Hence

$$f'(a) \left\{ \frac{\phi'(a)}{\phi(a)} - \frac{\phi''(a)}{\phi(a)} \right\} > f''(a)$$

→ If $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, show that the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ is given by

$$x = \frac{a+b+c}{a}, \quad y = \frac{a+b+c}{b}, \quad z = \frac{a+b+c}{c}.$$

$$\text{SOL}: \text{Let } f = (a^3x^2 + b^3y^2 + c^3z^2) + \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}).$$

$$\therefore f_x = 2a^3x - \lambda(\frac{1}{x^2}), \quad f_y = 2b^3y - \lambda(\frac{1}{y^2})$$

$$f_z = 2c^3z - \lambda(\frac{1}{z^2}) \quad \text{①}$$

$$\text{At stationary value } f_x = f_y = f_z = 0 \quad \text{②}$$

$$\therefore x f_x + y f_y + z f_z = 0 \text{ gives that}$$

$$2(a^3x^2 + b^3y^2 + c^3z^2) - \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) = 0$$

$$\Rightarrow 2f - \lambda(1) = 0$$

$$\Rightarrow 2f = \lambda \quad \text{where } f = a^3x^2 + b^3y^2 + c^3z^2$$

from ① & ②,

$$2a^3x^3 = 2b^3y^3 = 2c^3z^3 (\because f_n \Rightarrow 2a^3x^2 - \frac{\lambda}{x} = 0)$$

$$\Rightarrow a^3x^3 = b^3y^3 = c^3z^3 \Rightarrow \lambda = 2a^3x^3 \quad \text{③}$$

$$\Rightarrow \frac{1}{x} = \frac{a}{K}; \quad \frac{1}{y} = \frac{b}{K}; \quad \frac{1}{z} = \frac{c}{K} \quad \text{④}$$

$$\Rightarrow 1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{K}(a+b+c)$$

$$\Rightarrow K = a+b+c$$

putting this value of K in ③, we get

$$x = \frac{a+b+c}{a}, \quad y = \frac{a+b+c}{b}, \quad z = \frac{a+b+c}{c}$$

JACOBIANS

→ If u and v are functions of two independent variables x and y , then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ is called the } \underline{\text{Jacobian}} \text{ of } u, v$$

with respect to x, y and is denoted by

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ or } J. \left(\frac{u, v}{x, y} \right).$$

Similarly the Jacobian of u, v, w with respect to x, y, z

$$\text{is } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

In general, the Jacobian of $u_1, u_2, u_3, \dots, u_n$ with respect to x_1, x_2, \dots, x_n is

$$\frac{\partial(u_1, u_2, u_3, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Note: An important application of Jacobians is in connection with the change of variables in multiple integrals.

Properties of Jacobians

We give below two of the important properties of Jacobians.

For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

$$(5) J \cdot J' = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J' = \left(\begin{array}{c} \frac{\partial(x,y)}{\partial(u,v)} \\ \frac{\partial(y,u)}{\partial(u,v)} \end{array} \right)$$

then $J \cdot J' = 1.$

Proof Let $u = f(x,y)$ and $v = g(x,y)$

suppose, on solving for x and y ,

we get $x = \phi(u,v)$ and $y = \psi(u,v)$

Then

$$\frac{\partial u}{\partial v} = 1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\text{and } \frac{\partial v}{\partial u} = 1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\therefore J \cdot J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(After changing the rows
and columns of the
second determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad (\because \text{by (1)})$$

$$J \cdot J' = 1$$

(i) Note:-
 If $J = \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$ and $J^{-1} = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}$
 then $J J^{-1} = 1$.

(ii) Chain rule for Jacobian (Jacobian of function of function)
 If u, v are functions of r, s and x, y

are functions of x, y then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

$$\text{Proof } \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$

(Interchanging
rows and columns
of determinants)

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial s}{\partial x} & \frac{\partial u}{\partial x} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial s}{\partial y} \\ -\frac{\partial v}{\partial x} \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial s}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial r}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial s}{\partial y} \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}$$

Note:- If u_1, u_2, \dots, u_n are functions of x_1, x_2, \dots, x_n
 and y_1, y_2, \dots, y_n are functions of x_1, x_2, \dots, x_n

then

$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \frac{\partial(u_1, u_2, \dots, u_n)}{\partial(y_1, y_2, \dots, y_n)} \cdot \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)}$$

A particular case of Jacobian:

If the functions u_1, u_2, \dots, u_n of x_1, x_2, \dots, x_n are of the form

$$u_1 = f_1(x_1), u_2 = f_2(x_1, x_2), u_3 = f_3(x_1, x_2, x_3), \dots \\ \dots \dots u_n = f_n(x_1, x_2, \dots, x_n)$$

Then it clearly seen that

$$\frac{\partial u_1}{\partial x_2} = 0 = \frac{\partial u_1}{\partial x_3} = \frac{\partial u_1}{\partial x_4} = \dots = \frac{\partial u_1}{\partial x_n}$$

$$\text{and } \frac{\partial u_2}{\partial x_3} = 0 = \frac{\partial u_2}{\partial x_4} = \frac{\partial u_2}{\partial x_5} = \dots = \frac{\partial u_2}{\partial x_n} \text{ etc.}$$

and therefore

$$\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \dots & \frac{\partial u_2}{\partial x_n} \\ \vdots & & & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \dots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ \frac{\partial u_{n-1}}{\partial x_1} & \frac{\partial u_{n-1}}{\partial x_2} & \dots & \dots & \frac{\partial u_{n-1}}{\partial x_n} \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \dots \dots \frac{\partial u_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial u_n}{\partial x_n}$$

i.e. the Jacobian reduces to its leading terms.

problems:

→ If $x = r \cos \theta, y = r \sin \theta$, then find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Q) prove that $\frac{\partial(r, \theta)}{\partial(x, y)}, \frac{\partial(x, y)}{\partial(r, \theta)} = 1$.

$$\underline{\text{Sol}} : \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\cos\theta}{r} & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}(\cos^2\theta + \sin^2\theta) = \frac{1}{r}$$

$x = r\cos\theta$
 $\theta = \tan^{-1}(y/x)$
 $\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$
 $\text{and } \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$
 $\frac{\partial r}{\partial y} = \frac{\cos\theta}{r}$

Also $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{r} \cdot r = 1$

~~If~~ $x = r\cos\theta \cos\phi, y = r\cos\theta \sin\phi, z = r\sin\theta$,
then find Jacobian of x, y and z w.r.t r, θ, ϕ
i.e. $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

~~If~~ $x = r\sin\theta \cos\phi, y = r\sin\theta \sin\phi, z = r\cos\theta$ then show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

~~If~~ $x = u(1-v), y = uv, \text{ prove that}$

$$JJ' = 1.$$

Sol since $J = \frac{\partial(x,y)}{\partial(u,v)} \rightarrow J' = \frac{\partial(u,v)}{\partial(x,y)}$

$$\text{Now } \begin{cases} \frac{\partial x}{\partial u} = 1-v \\ \frac{\partial x}{\partial v} = -u \end{cases} \quad \begin{cases} \frac{\partial y}{\partial u} = v \\ \frac{\partial y}{\partial v} = u \end{cases}$$

$$\text{Since } x = u(1-v), \quad y = uv \quad \textcircled{2}$$

$$\begin{aligned} \textcircled{1} &= x = u - uv \\ \Rightarrow x &= u - y \quad (\text{from } \textcircled{1}) \end{aligned}$$

$$\Rightarrow u = x + y$$

$$\textcircled{2} \Rightarrow y = (x+y)v$$

$$\Rightarrow v = \frac{y}{x+y}$$

$$\begin{aligned} \text{Now } \frac{\partial x}{\partial u} &= 1 \quad \frac{\partial v}{\partial u} = -\frac{y}{(x+y)^2} = -\frac{v}{u} \\ \frac{\partial x}{\partial v} &= 1 \quad \frac{\partial v}{\partial v} = \frac{(x+y)(1)-y}{(x+y)^2} \\ &= \frac{x+y}{(x+y)^2} + \frac{y^2}{(x+y)^2} = \frac{1}{u} + \frac{v}{u} \\ &= 1 + \frac{v}{u} \end{aligned}$$

$$\text{Now } J J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} \times \begin{vmatrix} -\frac{v}{u} & 1 \\ 1 & \frac{1-v}{u} \end{vmatrix}$$

$$= (u(1-v) + uv) \times \left(\frac{1-v}{u} + \frac{v}{u} \right)$$

$$= (u)(\frac{1}{u})$$

$$= 1$$

$$\rightarrow \text{If } x = a \cosh \theta \cos \phi, \quad y = a \sinh \theta \sin \phi \\ \text{then show that } \frac{\partial(x,y)}{\partial(\theta, \phi)} = h^2 a^2 (\cosh^2 \theta - \sin^2 \phi).$$

Jacobian of Implicit functions:

If y_1, y_2 and x_1, x_2 are implicitly connected by two equations as:

$$f_1(y_1, y_2, x_1, x_2) = 0$$

$$f_2(y_1, y_2, x_1, x_2) = 0$$

$$\text{then } \frac{\partial(f_1, f_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}$$

Sol: Given that

$$f_1(y_1, y_2, x_1, x_2) = 0$$

$$f_2(y_1, y_2, x_1, x_2) = 0$$

now differentiating the relations w.r.t x_1 & x_2

we get

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} = 0$$

$$\Rightarrow \frac{\partial f_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_1} \quad \text{--- (1)}$$

$$\text{Similarly, } \frac{\partial f_1}{\partial x_2} \frac{\partial y_1}{\partial x_2} + \frac{\partial f_1}{\partial y_2} \frac{\partial y_2}{\partial x_2} = -\frac{\partial f_1}{\partial x_2} \quad \text{--- (2)}$$

$$\text{and } \frac{\partial f_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial f_2}{\partial y_2} \frac{\partial y_2}{\partial x_1} = -\frac{\partial f_2}{\partial x_1} \quad \text{--- (3)}$$

$$\text{and } \frac{\partial f_2}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial f_2}{\partial y_2} \frac{\partial y_2}{\partial x_2} = -\frac{\partial f_2}{\partial x_2} \quad \text{--- (4)}$$

NOW

$$\begin{aligned} \frac{\partial(f_1, f_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} &= \left| \begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right| \times \left| \begin{array}{cc} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{array} \right| \\ &= \left| \begin{array}{cc} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{array} \right| \times \left| \begin{array}{cc} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{array} \right| \end{aligned}$$

(Exchanging the rows & columns of determinant)

$$\begin{aligned}
 &= \left| \begin{array}{cc} \frac{\partial f_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial f_1}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial f_1}{\partial y_2} \frac{\partial y_2}{\partial x_2} \\ \frac{\partial f_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial f_2}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial f_2}{\partial y_2} \frac{\partial y_2}{\partial x_2} \end{array} \right| \\
 &= \left| \begin{array}{cc} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} \end{array} \right| \\
 &= (-1)^2 \left| \begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right| \\
 &= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(y_1, y_2)} = R.H.S.
 \end{aligned}$$

Note: If y_1, y_2, \dots, y_n and x_1, x_2, \dots, x_n are implicitly connected by 'n' equations as

$$f_1(y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_n) = 0$$

$$f_2(y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_n) = 0$$

$$\dots$$

$$f_n(y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_n) = 0$$

then

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(y_1, y_2, \dots, y_n)} \cdot \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(y_1, y_2, \dots, y_n)}$$

Given the roots of the equation in λ .

$$(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0 \text{ are } u, v, w, \text{ Then}$$

$$\text{show that } \frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Sol: Given that

$$(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$$

$$\Rightarrow 3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0.$$

Let the roots of the above equation be u, v, w .

Then

$$u+v+w = x+y+z$$

$$uv+vw+wu = x^2+y^2+z^2$$

$$uvw = \frac{x^3+y^3+z^3}{3}$$

$\therefore a, b, r$ are the roots of
equation $ax^3+bx^2+cx+d=0$

$$a+b+r = -\frac{b}{a}$$

$$ab+br+cr = \frac{c}{a}$$

$$abc = -\frac{d}{a}$$

Now these relations can be written as

$$f_1 = u+v+w-x-y-z = 0$$

$$f_2 = uv+vw+wu-x^2-y^2-z^2 = 0$$

$$f_3 = uvw - \frac{1}{3}(x^3+y^3+z^3) = 0.$$

$$\text{Since } \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$\Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(-1)^3 \partial(f_1, f_2, f_3)}{\partial(f_1, f_2, f_3)} / \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}$$

$$\begin{aligned} \text{Now } \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} \\ &= (-1)^3 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \end{aligned}$$

$$= (-1)^3 \begin{vmatrix} 1 & 0 & 0 & G_1 C_2 - C_1 \\ 0 & x & z & G_2 C_3 - C_2 \\ x^2 & y^2 & z^2 & G_3 C_1 - C_3 \end{vmatrix}$$

$$= (-2)(y-z)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ z & 1 & 1 \\ z^2 & y+z & z+x \end{vmatrix}$$

$$= (-2)(y-z)(z-x)[1(z+x) - (y+z)]$$

$$= (-2)(y-z)(z-x)(z-y)$$

$$= -2(x-y)(y-z)(z-x) \quad \textcircled{2}$$

$$\text{Let } \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ u+w & u+w & u+w \\ vw & vw & vw \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ u+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \begin{matrix} C_2 \sim C_2 - C_1 \\ C_3 \sim C_3 - C_1 \end{matrix}$$

$$= (u-v)(u-w) \begin{vmatrix} 1 & 0 & 0 \\ u+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$$= (u-v)(u-w) [1(v-w)]$$

$$= -(u-v)(v-w)(w-u) \quad \textcircled{3}$$

Substituting \textcircled{2} & \textcircled{3} in \textcircled{1},

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{(-2)(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

→ If $u^3 + v^3 = x+y$, $u^2+v^2 = x^2+y^2$

$$\text{then prove that } \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{3} \frac{(y-x)}{uv(u-v)}$$

→ If $u^3 + v^3 = x+y$, $u^2+v^2 = x^2+y^2$, show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2} \frac{(y-x)}{uv(u-v)}$$

→ If $u^2+v^2+w = x^2+y^2+z^2$

$$u+v+w = x+y+z$$

$$u+v+w^2 = x+y+z^2$$

$$\text{then prove that } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1-4(x^2+y^2+z^2) + 16xyz}{2-3(u^2+v^2+w^2) + 27uvw}$$

→ If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^2+y^2+z^2$,

$$u+v+w = x+y+z$$

$$\text{then show that } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$

→ If $u = \frac{x}{(1-x)^2}$, $v = \frac{y}{(1-y)^2}$, $w = \frac{z}{(1-z)^2}$

$$\text{where } x^2 = x^2+y^2+z^2$$

$$\text{then show that } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{(1-x)^5}$$

Sol: Given $u = \frac{x}{(1-x)^2}$

$$\Rightarrow x = u(1-x)^2 \Rightarrow x^2 = u(1-x^2)$$

$$\Rightarrow x^2 = u^2(1-x^2-y^2-z^2)$$

$$\text{Let } f_1 = x^2 - u^2(1-x^2-y^2-z^2) \geq 0$$

$$\text{Similarly } f_2 = y^2 - v^2(1-x^2-y^2-z^2) \geq 0$$

$$f_3 = z^2 - w^2(1-x^2-y^2-z^2) \geq 0$$

Ques II → If α, β, γ are the roots of the equation in t ,

$$\text{such that } \frac{u}{\alpha+t} + \frac{v}{\beta+t} + \frac{w}{\gamma+t} = 1$$

$$\text{then prove that } \frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = \frac{-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)}$$

* Particular case of Jacobian of implicit functions:

If the implicit relations are given as

$$f_1(x_1, x_2, \dots, x_n, y_1) = 0$$

$$f_2(x_1, x_2, \dots, x_n, y_1, y_2) = 0$$

$$f_3(x_1, x_2, \dots, x_n, y_1, y_2, y_3) = 0$$

$$\vdots$$

$$f_n(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = 0$$

then it is easily seen that

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(y_1, y_2, \dots, y_n)} = \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}$$

$$\text{and } \frac{\partial(f_1, f_2, f_3, \dots, f_n)}{\partial(x_1, x_2, x_3, \dots, x_n)} = \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$

$$\therefore \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}}{\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(y_1, y_2, \dots, y_n)}} \\ = (-1)^n \frac{\frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}}{\frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}}$$

Note: Let y_1, y_2, \dots, y_n be functions of n independent variables x_1, x_2, \dots, x_n . The necessary and sufficient condition that the functions be connected by a relation $f(y_1, y_2, \dots, y_n) = 0$ is that the Jacobian $\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)}$ vanishes identically.

(or)

Let y_1, y_2, \dots, y_n be functions of n independent variables x_1, x_2, \dots, x_n are functionally related (i.e., $f(y_1, y_2, \dots, y_n) = 0$) iff $\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} = 0$

→ P.T. the functions $u = x+y-z$, $v = x-y+z$ $\rightarrow \text{Q}$
 $w = x^2+y^2+z^2 - 2yz$ are not independent of one another
 find a relation between them.

Soh:

we have

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2(z-y) \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ x & y-2 & z-y \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ y-2-x & z-y+x \end{vmatrix} \quad c_2 \rightarrow c_2 - c_1 \\ c_3 \rightarrow c_3 + c_1$$

$$= 2 [-2(z-y+x) - 2(y-2-x)]$$

$$= -4(x-y+z+y-2-x)$$

$$= 0$$

Since $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

The given functions are not independent.
 i.e., the functions u, v, w are functionally related.

Now we have to find the relation.

From Q&G

$$u+v = 2x \quad \text{--- (1)}$$

$$u-v = 2(y-z)$$

$$\text{from (3), } w = x^2+y^2+z^2 - 2yz$$

$$= x^2 + (y-z)^2$$

$$= \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 \quad (\text{by (1)})$$

$$\begin{aligned}
 &= \frac{1}{4} [(u+v)^2 + (u-v)^2] \\
 &= \frac{1}{4} [2(u^2 + v^2)] \\
 w &= \frac{u^2 + v^2}{2}
 \end{aligned}$$

$$\Rightarrow u^2 + v^2 = 2w.$$

which is the required relation
between the given functions
 u, v & w .

- If $u = x^3 + x^2y + xy^2 - z^2(2x+y+z)$, $v = x+z$,
 $w = x^2 - z^2 + xy - zy$. Prove that u, v and w are
connected by a functional relation
- If $u = y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)}$, $v = \sqrt{(1-x^2)(1-y^2)} - xy$.
Prove that u and v are not independent and
find the relation between them.
- If $u = x+xy+z^2$, $v = x-2y+3z$, $w = 2xy - xz + 4yz - z^2$
Show that they are not independent. and find
the relation between them.
- Prove that the functions $u = x(y+z)$, $v = y(x+z)$,
 $w = z(xy)$ are not independent. and find
the relation between them.
- If $x = \cos u$, $y = \sin u \cos v$, $z = \sin u \sin v \cos w$
then find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

Since $x = f(u)$, $y = f(u, v)$, $z = f(u, v, w)$

$$\begin{aligned}
 \therefore \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} \\
 &= (-\sin u) (-\tan u \sec v) (\sin u \sin v \cos w) \\
 &= -\sin^3 u \sin^2 v \sin w.
 \end{aligned}$$

2005 → If $x+y+z=u$, $y+z=uv$, $z=uvw$ then
 Show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Soln. Since $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Given that $x+y+z=u$ ①, $y+z=uv$ ②, $z=uvw$ ③

from ① & ②

$$\begin{aligned} x+uv &= u \\ \Rightarrow x &= u-uv. \end{aligned}$$

② & ③

$$\begin{aligned} y+uvw &= uv \\ \Rightarrow y &= uv-uvw \end{aligned}$$

$$\text{and } z = uvw.$$

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} 1-u & -u & 0 \\ v-uvw & uw-wvw & -uv \\ -vw & uw & uv \end{vmatrix} \\ &= \begin{vmatrix} 1-u & -u & 0 \\ v & u & 0 \\ vw-wvw & uv & uv \end{vmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_3} \\ &= uv[u(-v)+uv] \\ &= uv[u-u^2+v^2] \\ &= u^2v. \end{aligned}$$

→ If $u_1 = x_1+x_2+x_3+x_4$

$$u_1u_2 = x_1+x_2+x_3$$

$$u_1u_2u_3 = x_1+x_2+x_3+x_4, \quad u_1u_2u_3u_4 = x_4.$$

$$\text{Show that } \frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3u_2^2u_3.$$

→ If $y_{m+1}, y_{m+2}, \dots, y_n$ are constant w.r.t x_1, x_2, \dots, x_m (or) (ii) y_1, y_2, \dots, y_m are constant w.r.t $x_{m+1}, x_{m+2}, \dots, x_n$, then

$$\frac{\partial(y_1, y_2, \dots, y_m, \dots, y_n)}{\partial(x_1, x_2, \dots, x_m, \dots, x_n)} = \frac{\partial(y_1, y_2, \dots, y_m)}{\partial(x_1, x_2, \dots, x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \dots, y_n)}{\partial(x_{m+1}, x_{m+2}, \dots, x_n)}$$

(i) Given $y_{m+1}, y_{m+2}, \dots, y_n$ are constant w.r.t to x_1, x_2, \dots, x_m

i.e., $\frac{\partial y_r}{\partial x_s} = 0$, where $r = m+1, m+2, \dots, n$ — (1)
 $s = 1, 2, \dots, m$.

$$\begin{aligned} \frac{\partial(y_1, y_2, \dots, y_m, y_n)}{\partial(x_1, x_2, \dots, x_m, x_n)} &= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \frac{\partial y_1}{\partial x_{m+2}} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \frac{\partial y_2}{\partial x_{m+2}} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \frac{\partial y_m}{\partial x_{m+2}} & \dots & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \dots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \frac{\partial y_{m+1}}{\partial x_{m+2}} & \dots & \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \frac{\partial y_n}{\partial x_{m+2}} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \frac{\partial y_1}{\partial x_{m+2}} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \frac{\partial y_m}{\partial x_{m+2}} & \dots & \frac{\partial y_m}{\partial x_n} \\ 0 & 0 & \dots & 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \dots & \frac{\partial y_{m+1}}{\partial x_n} \\ 0 & 0 & \dots & 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \dots & \frac{\partial y_{m+2}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{\partial y_n}{\partial x_{m+1}} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \end{aligned}$$

$$= \frac{\partial(y_1, y_2, \dots, y_m)}{\partial(x_1, x_2, \dots, x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \dots, y_n)}{\partial(x_{m+1}, x_{m+2}, \dots, x_n)}$$

(ii) may also be proved similarly

Multiple IntegralsS010

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Multiple Integrals: The process of integration for one variable can be extended to the functions of more than one variable.

The generalisation of definite integrals is known as multiple integrals.

Double Integrals:

A double integral is the counterpart, in two dimensions, of the definite integral of a function of a single variable. Let A be, a finite region of the xy -plane, and let $f(x, y)$ be a function of the independent variables x, y defined at every point in A . Divide the region A into n parts, of areas $\delta A_1, \delta A_2, \dots, \delta A_n$.

Let (x_r, y_r) be any point inside the r^{th} elementary area δA_r .

Form the sum

$$f(x_1, y_1) \delta A_1 + f(x_2, y_2) \delta A_2 + \dots + f(x_n, y_n) \delta A_n \quad (1)$$

$$\text{i.e. } \sum_{r=1}^n f(x_r, y_r) \delta A_r \quad (1)$$

Increase the number of subdivisions taking smaller and smaller elementary areas. Then the limit of the sum (1), if it exists, as n tends infinity and the dimension of each subdivision tend to zero, is called the double integral of

$f(x, y)$ over the region A ; and be denoted by

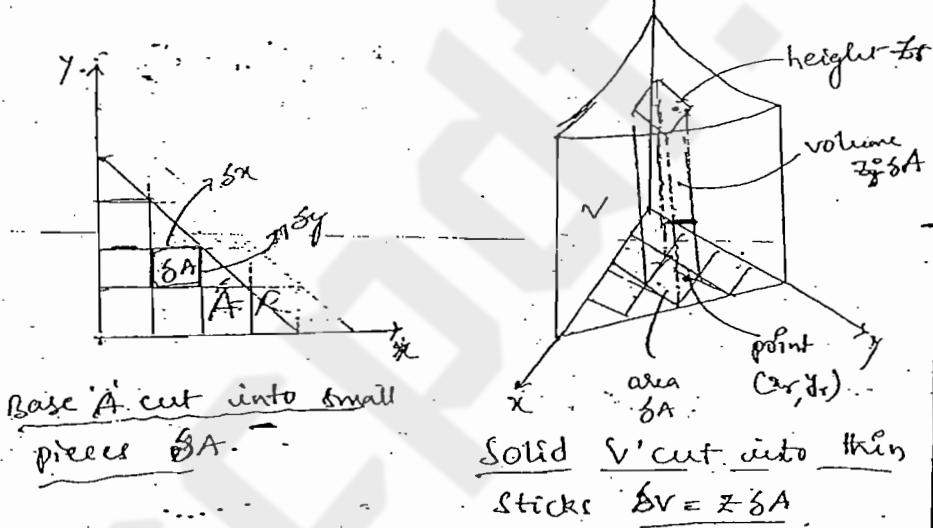
$$\iint_A f(x, y) dA \quad (2)$$

$$\text{This } \iint_A f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(x_r, y_r) \Delta A_r \quad (3)$$

This definition corresponds to the definition

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(x_r) \Delta x_r \quad (4)$$

for the definite integral of a single variable.
Just as the definite integral (4) can be interpreted as an area, similarly the double integral (2) can be interpreted as a volume.



- For single integrals, the interval $[a, b]$ is divided into short pieces of length Δx .
- For double integrals, A is divided into small

rectangles of area $\delta A = (\delta x)(\delta y)$.

Above the δA rectangle is a thin stick with small volume. That volume is the base area δA times the height above it - except that this height $z = f(x, y)$ varies from point to point. Therefore we select a point (x_r, y_r) in the δA rectangle and compute the volume from the height above that point.

volume of one stick $= f(x_r, y_r) \delta A$

volume of all sticks $= \sum f(x_r, y_r) \delta A$.

This is the crucial step for any integral -

to see it as a sum of small pieces.

Now take limits $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$; the height $z = f(x, y)$ is nearly constant over each rectangle (assume that f is continuous function).

The sum approaches a limit, which depends only on the base A and the surface above it.

The limit is the volume of the solid, and

it is the double integral of $f(x, y)$ over A .

$$\iint_A f(x, y) dA = \lim_{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}} \sum f(x_r, y_r) \delta A$$

for purposes of evaluation, ② is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$.

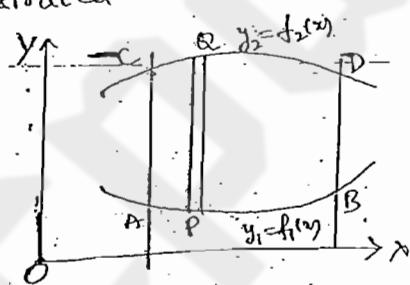
→ Its value is found as follows:-

① When y, y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated w.r.t y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t x with in the limits x_1, x_2 . i.e.

$$I_1 = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y) dy \right] dx$$

where integration is carried from the inner to outer rectangle; which is geometrically

illustrated as shown below-



Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

PA is a vertical strip of width dx .

Then the inner rectangle integral means that the integration is along one edge of

the strip PQ from P to Q (x remaining constant), while the outer rectangle corresponds to the sliding of the edge from AC to BD .

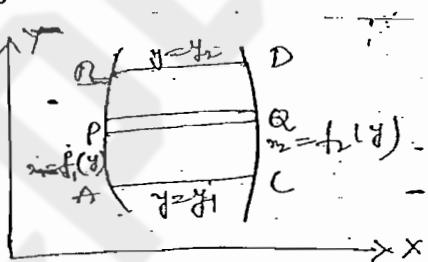
Thus the whole region of integration is the area $ABDC$.

- (ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r.t x keeping y fixed, with in the limits x_1, x_2 and the resulting expression is integrated w.r.t y between the limits y_1, y_2 , i.e.,

$$I_2 = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right] dy$$

which is geometrically illustrated as

shown below.



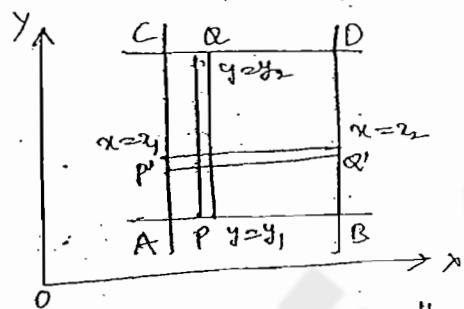
Here AB and CD are the curves $u = f_1(y)$ and $u = f_2(y)$. PQ is a horizontal strip of width dy .

The inner rectangle indicates that the integration is along one edge of the strip from P to Q while the outer rectangle

corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area ABDC.

(iii) when both pairs of limits are constants, the region of integration is the rectangle ABDC.



In 8₁, we integrate along the vertical strip PQ and then slide it from AC to BD.

In 8₂, we integrate along the horizontal strip P'Q' and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether first integrate w.r.t x and then w.r.t y or vice versa.

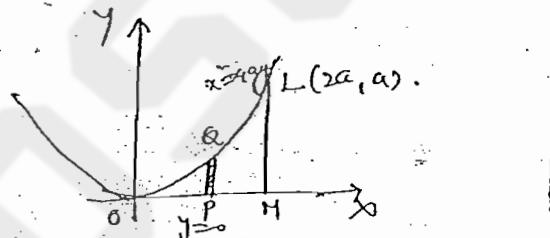
→ Evaluate $\int \int x(x^2+y^2) dx dy$

$$\begin{aligned}
 \text{Sol:} \quad & \text{Let } I = \int_0^5 \int_0^{x^2} x(x^2+y^2) dy dx \\
 & = \int_0^5 dx \int_0^{x^2} (x^3+y^2x) dy \\
 & = \int_0^5 \left[x^3y + \frac{y^3}{3} \right]_0^{x^2} dx \\
 & = \int_0^5 x^5 + \frac{x^6}{3} dx \\
 & = \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^5 \\
 & = \frac{5^6}{6} + \frac{5^8}{24} = \frac{5^6}{6} \left[\frac{1}{6} + \frac{25}{24} \right] \\
 & = \frac{5^6}{24} \left[\frac{29}{24} \right]
 \end{aligned}$$

→ Evaluate $\iint_A xy dy dx$, where A is the domain bounded by x-axis, ordinate $x=2a$ and the curve $x=4ay^2$.

Sol: The line $x=2a$ and the parabola $x=4ay^2$ intersect at $L(2a, a)$.

The figure shows the domain A which is the area OML.



Integrating first over a vertical strip PQ, i.e,

w.r.t y from $y=0$ to $y=\sqrt{4a}$ on the

parabola and then w.r.t x from $x=0$ to $x=2a$,

- we have

$$\begin{aligned} \iint_A xy \, dx \, dy &= \int_0^{2a} dx \int_0^{\sqrt{4a}} xy \, dy \\ &= \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{\sqrt{4a}} \, dx \\ &= \frac{1}{32a^2} \int_0^{2a} x^5 \, dx \\ &= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} \\ &\approx \frac{a^4}{3}. \end{aligned}$$

Otherwise integrating first over a horizontal

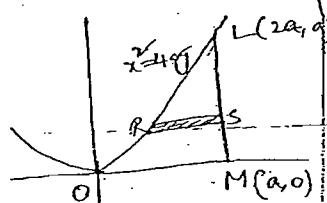
strip RS, i.e, w.r.t x from

$x=2\sqrt{ay}$ on the parabola

and then w.r.t y from

$y=0$ to $y=a$, we get

$$\begin{aligned} \iint_A xy \, dx \, dy &= \int_0^a dy \int_{2\sqrt{ay}}^{2a} xy \, dx \\ &= \int_0^a y \left[\frac{x^2}{2} \right]_{2\sqrt{ay}}^{2a} \, dy \\ &= \int_0^a y \left[\frac{(4a)^2 - (4ay)^2}{2} \right] \, dy = 2a \int_0^a (a - y^2) \, dy \\ &= 2a \left[\frac{ay^2}{2} - \frac{y^3}{3} \right]_0^a \\ &= \frac{a^4}{3}. \end{aligned}$$



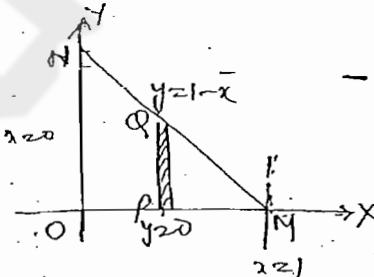
→ Evaluate $\iint_D \frac{dx dy}{1+x^2+y^2}$

$$\begin{aligned} \iint_D \frac{dx dy}{1+x^2+y^2} &= \int_0^{\sqrt{\pi}} dx \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \\ &\approx \int_0^{\sqrt{\pi}} \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx \\ &= \int_0^{\sqrt{\pi}} \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) dx \\ &= \int_0^{\sqrt{\pi}} \left(\frac{\pi}{4} - 0 \right) \frac{1}{\sqrt{1+x^2}} dx \\ &= \frac{\pi}{4} \left[\log \{x + \sqrt{1+x^2}\} \right]_0^{\sqrt{\pi}} \\ &= \frac{\pi}{4} \log (1+\sqrt{2}). \end{aligned}$$

→ Evaluate $\iint_D xy dy dx$ over the region in the positive quadrant for which $x+xy \leq 1$.

Sol:

The region of integration is the area A bounded by the two axes and the straight line $x+xy=1$.



Consider a strip parallel to y-axis.
It has its extremities on $y=0$ and $y=1-x$.
Hence limits of y are from $y=0$ to $y=1-x$.
The limits of x are from $x=0$ to $x=1$.

Hence the given integral

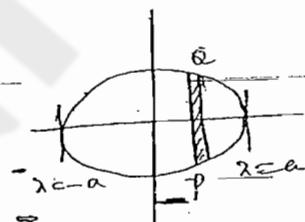
$$\begin{aligned}
 \iint xy \, dxdy &= \int_0^1 \int_0^{1-x} xy \, dy \, dx \\
 &= \int_0^1 x \left(\frac{y^2}{2} \right) \Big|_0^{1-x} \, dx \\
 &= \frac{1}{2} \int_0^1 x(1-x)^2 \, dx \\
 &= \frac{1}{2} \left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right) \Big|_0^1 \\
 &= \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\
 &= \frac{1}{24}.
 \end{aligned}$$

→ Evaluate $\iint (x+y)^2 \, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

for:

for the ellipse

$$\begin{aligned}
 \frac{y}{b} &= \pm \sqrt{1 - \frac{x^2}{a^2}} \\
 \Rightarrow y &= \pm b \sqrt{1 - \frac{x^2}{a^2}}
 \end{aligned}$$



Integrating first w.r.t y along a vertical strip
 w.r.t which extends from $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ to

$$y = +b\sqrt{1 - \frac{x^2}{a^2}}.$$

To cover the region, we then integrate
 w.r.t x from $x=-a$ to $x=a$.

The given integral

$$\iint (x+y)^2 dx dy = \int_{-a}^a \int_{-\sqrt{1-\frac{x^2}{a^2}}}^{\sqrt{1-\frac{x^2}{a^2}}} (x^2 + 2xy + y^2) dy dx$$

$$= \int_{-a}^a \left[x^2 y + 2xy^2 + \frac{y^3}{3} \right]_{-\sqrt{1-\frac{x^2}{a^2}}}^{\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \int_{-a}^a \left\{ 2bx\sqrt{1-\frac{x^2}{a^2}} + 0 + \frac{2}{3}b^3 \left(\sqrt{1-\frac{x^2}{a^2}} \right)^3 \right\} dx$$

$$= 2 \int_{-a}^a \left\{ bx\sqrt{1-\frac{x^2}{a^2}} + \frac{b^3}{3} \left(1 - \frac{x^2}{a^2} \right)^{\frac{3}{2}} \right\} dx$$

$$= 4b \int_0^a \left\{ x\sqrt{1-\frac{x^2}{a^2}} + \frac{b^3}{3} \left(1 - \frac{x^2}{a^2} \right)^{\frac{3}{2}} \right\} dx$$

~~substituting~~ putting $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$

Units: when $x=0 ; \theta=0$
 $x=a ; \theta=\pi/2$

$$= 4b \int_0^{\pi/2} \left\{ a^2 \sin^2 \theta \cos \theta + \frac{1}{3} b^3 \cos^3 \theta \right\} da \cos \theta d\theta$$

$$= 4ab \int_0^{\pi/2} \left\{ a^2 \sin^2 \theta \cos^2 \theta + \frac{b^3}{3} \cos^4 \theta \right\} d\theta$$

$$= 4ab \left\{ \int_0^{\pi/2} \frac{a^2}{4} \sin^4 \theta d\theta + \frac{b^3}{3} \int_0^{\pi/2} \cos^4 \theta d\theta \right\}$$

$$= 4ab \left[\int_0^{\pi/2} \frac{a^2}{4} \cdot \frac{(1-\cos 8\theta)}{2} d\theta + \frac{b^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} \right]$$

$$= 4ab \left[\frac{a^2}{8} \left(\theta + \frac{\sin 8\theta}{8} \right) \Big|_0^{\pi/2} + \frac{b^3}{8} \right]$$

$$= 4ab \left[\frac{a^2}{8} \left(\frac{\pi}{2} + 0 \right) + \frac{b^3}{16} \right]$$

$$= \frac{1}{4} \pi ab (a^2 + b^2)$$

$$\Rightarrow \text{Show that } \int_0^1 dx \int_0^x \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^{x-y} \frac{1}{(x+y)^3} dx \quad (7)$$

Find the values of the two integrals.

$$\begin{aligned}
 \text{Sol: LHS} &= \int_0^1 dx \int_0^x \frac{x-y}{(x+y)^3} dy \\
 &= \int_0^1 dx \int_0^x \frac{2x-(x+y)}{(x+y)^3} dy \\
 &= \int_0^1 dx \left[\left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy \right] \\
 &= \int_0^1 dx \left[\left[\frac{-x}{(x+y)^2} + \frac{1}{x+y} \right] \right] \quad \left(\int \frac{1}{x^n} dx = \frac{1}{n-1} x^{n-1} \right) \\
 &= \int_0^1 \left[\left[\frac{-x}{(1+x)^2} + \frac{1}{x+1} + \frac{1}{1+x} - \frac{1}{x} \right] \right] dx \\
 &= \int_0^1 \frac{dx}{(1+x)^2} \\
 &= \left[\frac{-1}{1+x} \right] \Big|_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \text{RHS} &= \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx \\
 &= \int_0^1 dy \int_0^1 \frac{x+y-2y}{(x+y)^3} dx \\
 &= \int_0^1 dy \left\{ \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] \right\} dx \\
 &= \int_0^1 \left\{ \left[\frac{-1}{x+y} + \frac{y}{(x+y)^2} \right] \right\} \Big|_0^1 dy \\
 &= \int_0^1 \left[\frac{1}{1+y} + \frac{1}{y} + \frac{y}{1+y} - \frac{1}{y} \right] dy \\
 &= \int_0^1 \frac{dy}{(1+y)^2} = \left[\frac{1}{1+y} \right] \Big|_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}
 \end{aligned}$$

Evaluate the following double integrals:

$$(1) \int_0^a \int_0^b (x+y) dy dx ; \text{ Ans: } 5$$

$$(2) \int_0^a \int_0^b (x+y^2) dy dx ; \text{ Ans: } \frac{1}{3} ab(a^2+b^2)$$

$$(3) \int_1^a \int_1^b \frac{dy dx}{xy} ; \text{ Ans: } \log a \log b$$

$$(4) \int_1^2 \int_0^x \frac{dy dx}{x+y^2} ; \text{ Ans: } \frac{1}{4} \log e^2$$

$$(5) \int_1^2 \int_0^{y^2} y dy dx ; \text{ Ans: } \frac{1}{6}$$

$$(6) \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos(x+y) dy dx ; \text{ Ans: } -2$$

$$(7) \int_0^a \int_0^{\sqrt{a-x^2}} x^2 y dy dx ; \text{ Ans: } a^5/15$$

$$(8) \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx ; \text{ Ans: } 4/3$$

$$(9) \int_0^1 \int_0^{x^2} (x^2+y^2) dy dx ; \text{ Ans: } 3/35$$

$$(10) \int_0^a \int_0^{\sqrt{a-x^2}} e^{x^2+y^2} dy dx ; \text{ Ans: } \frac{1}{2}$$

$$(11) \int_0^a \int_0^{\sqrt{a-x^2}} \sqrt{a-x^2-y^2} dy dx ; \text{ Ans: } \frac{\pi a^3}{6}$$

$$(12) \int_0^{\pi/2} \int_0^{\sin x} x dy dx ; \text{ Ans: } \pi/2$$

(13) Evaluate $\iint x^2 y^2 dy dx$ over the region $x^2+y^2 \leq 1$. Ans: $\pi/24$.

(14) Evaluate $\iint (x+y)^2 dy dx$ over the region in the positive quadrant for which $x+y \leq 1$. Ans: $\pi/6$.

(15) Evaluate $\iint \frac{dy}{x+y} dx dy$ over the positive quadrant of the circle $x^2+y^2=1$. Ans: $\pi/6$.

Area enclosed by plane curves:

Consider the area enclosed by the curves

$y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=x_1$,

$x=x_2$ as shown in the figure.

Divide this area into vertical strips of width δx .

If $P(x, y)$, $Q(x+\delta x, y+\delta y)$

be two neighbouring points,

then the area of the small rectangle $PQ = \delta x \delta y$

$$\therefore \text{Area of Strip KL} = \lim_{\delta y \rightarrow 0} \sum \delta x \delta y.$$

Since for all rectangles in this strip δx is the same and y varies from $y=f_1(x)$ to $y=f_2(x)$.

$$\therefore \text{Area of the strip KL} = \delta x \lim_{\delta y \rightarrow 0} \sum_{y=f_1(x)}^{f_2(x)} dy$$

$$= \delta x \int_{f_1(x)}^{f_2(x)} dy$$

Now adding up all such strips from $x=x_1$ to $x=x_2$

we get the area $ABCD$

$$= \lim_{\delta x \rightarrow 0} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

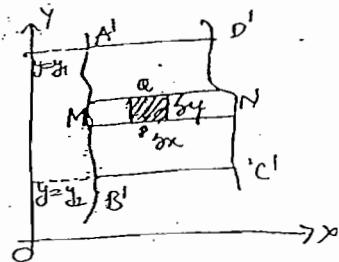
$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dy dx$$



Similarly dividing the area $A'B'C'D'$ into

horizontal strips of width Δy , we get the area

$$A'B'C'D' = \int_{y_1}^{y_2} f(y) dy$$



→ find the area lying between the parabola $y = 4x - x^2$ and the line $y = 2x$.

Sol: The equation of the parabola

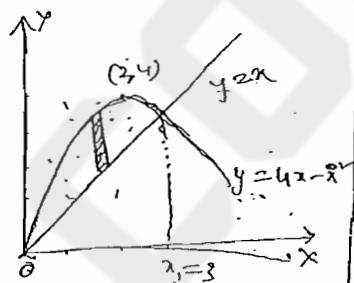
$y = 4x - x^2$ may be written

$$\text{as } (x-2)^2 = -(y-4)$$

i.e., this parabola has the

vertex at the point $(2, 4)$ and

its concavity is downwards.



The points of intersection of two curves are given as follows

$$4x - x^2 = x$$

$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x-3) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 3$$

and hence from $y = x$, we get....

$$y = 0 \text{ at } x = 0$$

$$y = 3 \text{ at } x = 3$$

∴ The points of intersection of the two curves are $(0, 0), (3, 3)$.

The area can be considered as lying between the curves $y=x$, $y=4x-x^2$, $x=0$ and $x=3$ so integrating along a vertical strip first, i.e. y from $y=x$ to $y=4x-x^2$ and then w.r.t x from $x=0$ to $x=3$.

$$\begin{aligned}\therefore \text{The required area} &= \int_{0}^{3} \int_{y=x}^{4x-x^2} dy dx \\ &= \int_{0}^{3} [y]_{x}^{4x-x^2} dx \\ &= \int_{0}^{3} (4x-x^2-2) dx \\ &= \int_{0}^{3} (3x-x^2) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 \\ &= \frac{27}{2} - 9 = \frac{9}{2}\end{aligned}$$

→ Show that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16a^2}{3}$.

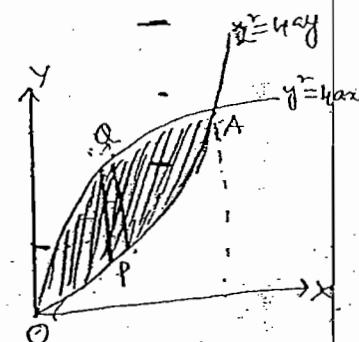
Sol: Solving the equations

$$y^2=4ax \text{ and } x^2=4ay,$$

the parabolas intersect at $O(0,0)$ and $A(4a, 4a)$. As such, for the shaded area between these parabolas (as shown in the figure)

x varies from 0 to $4a$ and y varies from

$$y=\sqrt{4ax} \text{ to } y=2\sqrt{ax}.$$



Hence the required area

$$= \int_0^{4a} \int_{\frac{2\sqrt{ax}}{x^2+4a^2}}^{2\sqrt{ax}} dy dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{ax} \cdot \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4a} \cdot \frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

- find by double integration, the area enclosed by the curves $y = 32/x^2 + 2$ and $4y = x^2$.

Ans: $\frac{2}{3} \log e - \frac{2}{3}$

- find by double integration the area of the region enclosed by the following curves:

(1) $x+y^2=a^2$ and $x+y=a$ (in the first quadrant)

Ans: $\frac{(\pi-2)a^2}{4}$

(2) $y^2=x^3$ and $y=x^2$ Ans: $\frac{1}{10}a^5$

(3) $9xy=4$ and $2x+y=2$ Ans: $\frac{1}{3} - \frac{4}{9} \log e^2$

(4) $(x^2+4a^2)y=8a^3$, $2y=x$ and $x \geq 0$ Ans: $(\pi-1)a^2$

Volume as double integral:

Consider a surface $z = f(x, y)$.

Let the orthogonal projection
on xy -plane of its portion
be the area S .

Divide S into elementary
rectangles of area $dx dy$
by drawing lines parallel to
 x and y axes. With each
of these rectangles as base,
erect a prism having its
length parallel to oz .

\therefore volume of this prism between S and the given
surface $z = f(x, y)$ is $z dx dy$.

Hence the volume of the solid cylinder on S
as base, bounded by the given surface with
generators parallel to the z -axis:

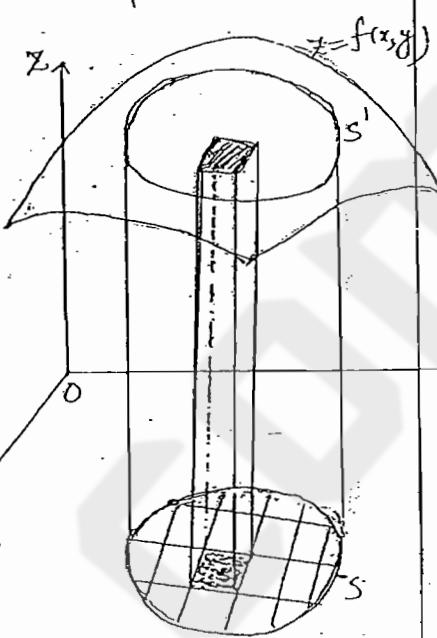
$$= \lim_{\substack{Sx \rightarrow 0 \\ Sy \rightarrow 0}} \sum z dx dy$$

$$= \iint z dxdy$$

$$= \iint f(x, y) dxdy$$

where the integration is carried
over the area S .

i.e., if the region S may be considered
as enclosed by the curves $y = f_1(x)$, $y = f_2(x)$,



$x=a$ and $x=b$, we can write volume as:

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dy dx.$$

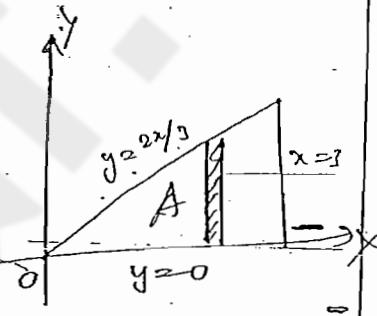
Note: When writing the integral for the volume, the integrand $f(x, y)$ is taken from the surface $z = f(x, y)$ which covers the top of the volume while the limits a, b, f_1, f_2 are taken from the base area S in the xy -plane.

- find the volume under the plane $x+y+z=6$ and above triangle in the xy -plane bounded by $2x=3y, y \geq 0, x=3$.

Sol: The required volume V

$$= \iint_A z dA$$

$$= \iint_A (6-x-y) dA,$$



where A is the region shown in the figure.

Integrating along a vertical strip first, we have

$$V = \iint_0^{2x/3} (6-x-y) dy dx$$

$$= \int_0^3 \left(6y - xy - \frac{y^2}{2} \right) \Big|_0^{2x/3} dx$$

$$= \int_0^3 \left(4\pi x - \frac{2}{3}x^2 - \frac{2}{9}x^3 \right) dx$$

$$= \int_0^3 \left(4\pi x - \frac{8}{9}x^2 \right) dx$$

$$= \left(2\pi x^2 - \frac{8}{27}x^3 \right)_0^3$$

$$= 18\pi - 8 = 10.$$

2006

P-I

P-II

find the volume in the positive octant of the

$$\text{ellipsoid } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solⁿ: The required volume lies between the

ellipsoid $z = c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ and the plane xoy , and is bounded on the sides by the planes $x=0, y=0$.

The given ellipsoid cuts xoy

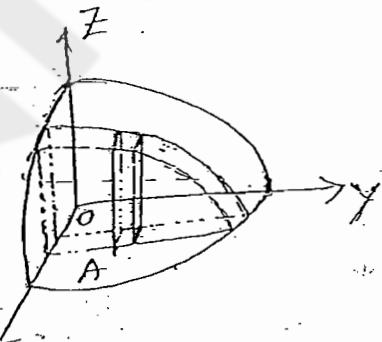
plane in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z=0.$$

Therefore the region A, above which the required volume lies, is bounded by curves

$$y=0, y = b \sqrt{1 - \frac{x^2}{a^2}},$$

$x=0$, and $x=a$.



Hence, the required volume

$$\begin{aligned}
 &= \int_A z \, dA \\
 &= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c \sqrt{\left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} \, dy \, dx \\
 &= c \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{y^2}{b^2} - \frac{y^2}{b^2}} \, dy \, dx \quad \text{on putting } \sqrt{1 - \frac{x^2}{a^2}} = \frac{y}{b} \\
 &= \frac{c}{b} \int_0^a \left[\frac{1}{2} y \sqrt{y^2 - y^2} + \frac{1}{2} y^2 \sin^{-1} \frac{y}{b} \right]_0^Y \, dx \\
 &= \frac{c}{b} \int_0^a \frac{1}{2} y^2 \frac{\pi}{2} \, dx \\
 &= \frac{\pi c}{4b} \int_0^a y^2 \, dx \\
 &= \frac{\pi c}{4b} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) \, dx \quad (\because y = b\sqrt{1 - \frac{x^2}{a^2}}) \\
 &= \frac{1}{4} \pi b c \left[x - \frac{x^3}{3a^2} \right]_0^a \\
 &= \frac{1}{4} \pi b c \left[a - \frac{a^3}{3a^2} \right] \\
 &= \frac{1}{4} \pi b c \frac{2a}{3} \\
 &= \underline{\underline{\frac{1}{6} \pi b c a}}
 \end{aligned}$$

→ find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y+z=4$ and $z=0$.

Sol: from the figure, it is self-evident that $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy -plane. To cover the shaded

half of this circle, x varies from 0 to $\sqrt{4-y^2}$
and y varies from -2 to 2.

\therefore Required volume

$$= 2 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} z dz dy$$

$$= 2 \int_{-2}^2 \int_0^{(4-y)} z dz dy$$

$$= 2 \int_{-2}^2 (4-y) \left[\frac{z^2}{2} \right]_0^{(4-y)} dy$$

$$= 2 \int_{-2}^2 (4-y) \sqrt{4-y^2} dy$$

$$= 2 \int_{-2}^2 y \sqrt{4-y^2} dy - 2 \int_{-2}^2 y \sqrt{4-y^2} dy$$

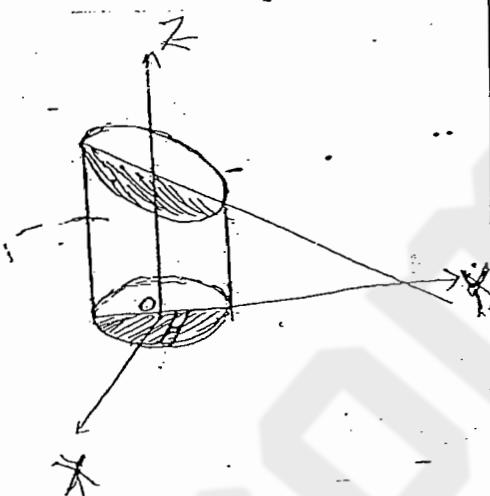
$$= 8 \int_{-2}^2 y \sqrt{4-y^2} dy \quad (\text{Here the second term vanishes as the integrand is an odd function})$$

$$= 8 \left[\frac{y \sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \frac{y}{2} \right]_0^2$$

$$= 8 [0 + 2 \sin(1) - 0 - 2 \sin(-1)]$$

$$= 8 \left[2 \frac{\pi}{2} + 2 \frac{\pi}{2} \right]$$

$$= 16\pi$$



Find the volume of the cylinder $x^2+y^2=ax$ bounded by the planes $z=0$ and $z=2$. Ans: $\pi a^2/8$

Find the volume under the plane $x+z=2$, above $z=0$ and within the cylinder $x^2+y^2=a^2$. Ans: 8π

- Find the volume under the plane $z = x + y$ and above the area cut from the first quadrant by ellipse $4x^2 + 9y^2 = 36$. Ans: 10
- Find the volume bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Ans: $\frac{abc}{6}$
- Find the volume bounded by $x^2 + 16z^2 = y^2$ and the plane $z = 0$. Ans: 16π
- Find the volume enclosed by the cylinders $y^2 = x$ and $x^2 + y^2 = a^2$ and the plane $z = 0$. Ans: $\frac{\pi a^4}{4}$
- Find the volume in the first octant bounded by the parabolic cylinders $z = 9 - x^2$, $z = 3 - y^2$. Ans: $\frac{102\sqrt{2}}{35}$
- Find the volume in the first octant bounded by $z = x^2 + y^2$ and $y = 1 - x^2$. Ans: $2/7$
- Find the volume inside the paraboloid $x^2 + 4z^2 + 8y = 16$ and on the positive side of $x = 0$ -plane

Polar co-ordinates:-

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To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) dr d\theta$, we first integrate w.r.t r between limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r.t θ from θ_1 to θ_2 . In this integral

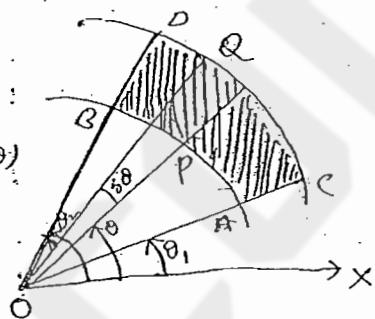
Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$. PAQ is a wedge of angular thickness $d\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta) dr$ indicates that the integration is along PA from P to A while integration w.r.t θ corresponds to the turning of PA from AC to BD .

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

→ Calculate $\iint r^3 dr d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

Sol: Given circles $r=2\sin\theta$ and $r=4\sin\theta$ are as shown in the figure



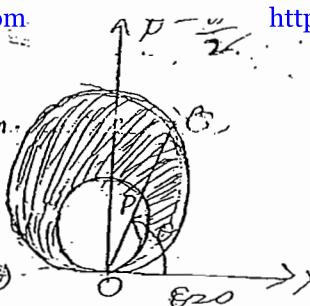
The shaded area between three circles is the region of integration.

If we integrate first w.r.t x ,

then its limits are from $P(r=2\sin\theta)$

to $Q(r=4\sin\theta)$ and to cover the

whole region θ varies from 0 to π .



Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr$$

$$= \int_0^{\pi} d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= \frac{1}{4} \int_0^{\pi} (256 - 16) \sin^4 \theta d\theta$$

$$= \frac{240}{A} \int_0^{\pi} \sin^4 \theta d\theta$$

$$= -60 \times 2 \int_0^{\pi/2} \sin^4 \theta d\theta$$

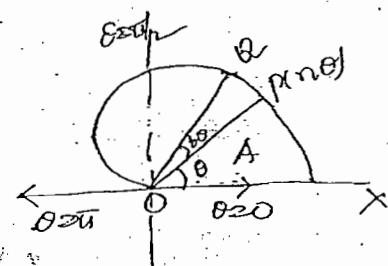
$$= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= 22.5\pi$$

→ Integrate $r \sin \theta$ over the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line.

Soln:

Here the region of integration
A can be covered by



radial strips whose ends are $\theta=0$ and $r=a(1+\cos\phi)$ i.e,

The strips start from $\theta=0$ and end at $\theta=\pi$.

Therefore the required integral

$$\begin{aligned}
 &= \iint_A r \sin\theta \, dA = \int_0^{\pi} \int_{r=0}^{r=a(1+\cos\phi)} r \sin\theta \, dr \, d\theta \\
 &= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos\phi)} \, d\theta \\
 &= \frac{1}{3} a^3 \int_0^{\pi} \sin\theta (1+\cos\phi)^3 \, d\theta \\
 &= \frac{1}{3} a^3 \int_0^{\pi} 2\sin\theta \cos\phi (2\cos^2\phi)^3 \, d\theta \\
 &= \frac{16}{3} a^3 \int_0^{\pi} \sin\theta \cos^7\phi \, d\theta
 \end{aligned}$$

putting $\theta=2\phi \Rightarrow d\theta=2d\phi$

limits: $\phi=0$ when $\theta=0$
 $\phi=\pi/2$ when $\theta=\pi$

$$\begin{aligned}
 &= \frac{16}{3} a^3 \int_0^{\pi/2} \sin\phi \cos^7\phi \cdot 2d\phi \\
 &= \frac{32}{3} a^3 \left[-\frac{\cos^8\phi}{8} \right]_0^{\pi/2} \\
 &= \frac{4}{3} a^3
 \end{aligned}$$

Evaluate $\iint_A r^2 \sin\theta \, d\theta \, dr$ over the area of
 Cartesian $\sqrt{a^2(1+\cos\phi)}$ above the initial line.

Ans: $\frac{4}{3} a^3$

radial strips whose ends are $r=0$ and $r=a(\cos\theta + \cos\phi)$ i.e,

The strips start from $\theta=0$ and end at $\theta=\pi$.

Therefore the required integral

$$\pi a^2 (\cos\theta + \cos\phi)$$

$$= \iint_A r \sin\theta \, dA = \int_0^\pi \int_{r=0}^{r=a(\cos\theta + \cos\phi)} r \sin\theta \, r \, dr \, d\theta.$$

$$= \int_0^\pi \sin\theta \left[\frac{r^3}{3} \right]_0^{a(\cos\theta + \cos\phi)} d\theta$$

$$= \frac{1}{3} a^3 \int_0^\pi \sin\theta (\cos\theta + \cos\phi)^3 d\theta$$

$$= \frac{1}{3} a^3 \int_0^\pi 2 \sin\theta \cos\theta (\cos^2\theta) d\theta$$

$$= \frac{16}{3} a^3 \int_0^\pi \sin\theta \cos^7\theta d\theta$$

$$\text{putting } \theta = 2\phi \Rightarrow d\theta = 2d\phi$$

$$\text{limits: } \phi = 0 \text{ when } \theta = 0$$

$$\phi = \pi/2 \text{ when } \theta = \pi$$

$$= \frac{16}{3} a^3 \int_0^{\pi/2} \sin\phi \cos^7\phi 2d\phi$$

$$= \frac{32}{3} a^3 \left[-\frac{\cos^8\phi}{8} \right]_0^{\pi/2}$$

$$= \frac{4}{3} a^3$$

Evaluate $\iint_A r \sin\theta \, dA$ over the area of

Cardsoid $r = a(\cos\theta + \cos\phi)$ above the initial line.

$$\text{Ans: } \frac{4}{3} a^3$$

Area enclosed by plane curves

polar co-ordinates:

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points.

Mark circular areas of radii r and $r+\delta r$ meeting $\partial\Omega$ in R and QP in S.

$$\text{Since } \text{arc PR} = r\delta\theta \quad (\because l = r\theta)$$

$$\text{and PS} = \delta r$$

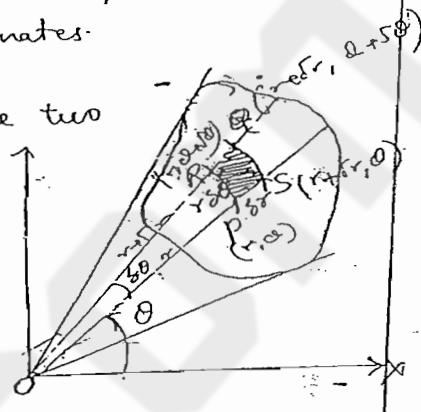
∴ Area of the curvilinear rectangle PRQS is approximately $= PR \cdot PS$
 $= r\delta\theta \cdot \delta r$

If the whole area is divided into such curvilinear rectangles, the sum $\sum \sum r\delta\theta \delta r$ taken for all these rectangles, gives in the limit the area

$$\text{Hence } A = \lim_{\substack{\delta r \rightarrow 0 \\ \delta\theta \rightarrow 0}} \sum \sum r\delta\theta \delta r = \iint r d\theta dr.$$

where the limits are to be so chosen as to cover the entire area.

→ Calculate the area included between the curve $r = a(\theta + \cos\theta)$ and its asymptote



Soln: The curve is symmetrical about the initial line and has an asymptote $r = a \sec \theta$.

Draw any line op- cutting the curve at P and its asymptote at P'.

along this line, θ is constant and r varies from $a \sec \theta$ at P' to $a \sec(\theta + \pi)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

\therefore The required area

$$= 2 \int_{0}^{\pi/2} r dr d\theta$$

$$= 2 \int_{0}^{\pi/2} \left[\frac{r^2}{2} \right]_{a \sec \theta}^{a \sec(\theta + \pi)} d\theta$$

$$= 2 \frac{a^2}{2} \int_0^{\pi/2} [(\sec(\theta + \pi))^2 - \sec^2 \theta] d\theta$$

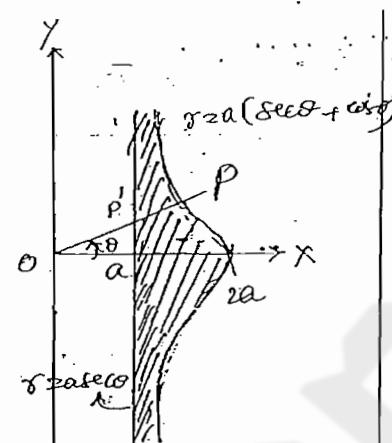
$$= a^2 \int_0^{\pi/2} (2 + \cos^2 \theta) d\theta = a^2 \int_0^{\pi/2} [2\theta + \frac{1}{2} \sin 2\theta] d\theta$$

$$= \underline{\underline{\frac{5\pi a^2}{4}}}$$

→ find by double integration, the area lying inside the circle $r = a(1 - \cos \theta)$ and outside the cardioid.

$$r = a(1 - \cos \theta)$$

Ans:



Change of order of integration

The integral $\iint u dx dy$ is first integrated with respect to the variable 'y', then limits of 'y' are substituted (which in general may be function of 'x'), and the result is integrated with respect to 'x'. But if we want to change $\iint u dx dy$ to $\iint u dy dx$ then we have to find the new limits of 'x' as functions of 'y'.

i.e., in a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral.

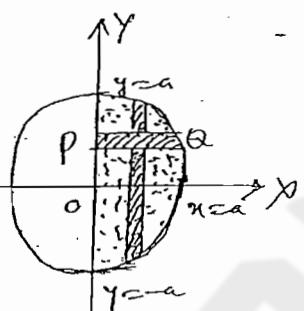
→ Change

→ Change the order of integration in the integral $I = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} f(x, y) dy dx$.

SOL:

Here the elementary strip is parallel to x -axis (such as PQ) and extends from $x=0$ to $x=\sqrt{a^2-y^2}$ (i.e., the circle $x^2+y^2=a^2$) and this strip slides from $y=-a$ to $y=a$. This shaded semi-circular area is, therefore, the region of integration.

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from $R [y=-\sqrt{a^2-x^2}]$ to $S [y=\sqrt{a^2-x^2}]$. To cover the given region, we then integrate w.r.t. x from $x=-a$ to $x=a$.

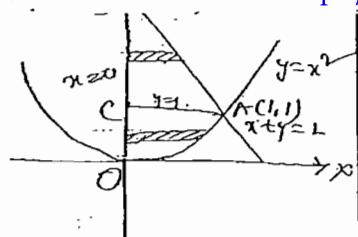


$$\text{Thus, } I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y) dy dx \\ = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y) dy dx.$$

→ Change the order of integration in $I = \int_0^{x^2} \int_y^{2-x} xy dy dx$ and hence evaluate the same.

Sol:

The given limits show that the region of integration is bounded by the curves $y=x^2$, $y=2-x$, $x=0$, $x=1$.



The first is a parabola with vertex at the origin, and the second the straight line $y=2-x$.

They intersect at the point $(1,1)$.

Therefore the region of integration is OAB .

When we integrate w.r.t x first along a horizontal strip, the strip starts from $x=0$

But some of the strips end on OA while the others end on AB . i.e., At a strip parallel to x -axis change their character.

Hence through the point A , draw a straight line $\overset{(y=1)}{CA}$ parallel to the x -axis. This straight line $\overset{(y=1)}{CA}$ divides the region OAB into two parts namely OAC and ABC .

In the region OAC , the strip parallel to x -axis has its extremities on $x=0$ and $y=x^2$.

Hence limits of x are from $x=0$ to $x=\sqrt{y}$.

As the point A is $(1,1)$, the limits of y are

from $y=0$ to $y=1$.

Again in the region ABC, the strip parallel to x-axis has its extremities on $x=0$ and $y=2x$. Hence limits of x are from $x=0$ to $x=2y$. The limits of y are from $y=1$ to $y=2$.

Hence changing the order of integration
the given integral is

$$\begin{aligned}
 \iint_{0 \leq x \leq 2} xy \, dx \, dy &= \int_0^1 dy \int_0^{2y} xy \, dx + \int_1^2 dy \int_0^{2-y} xy \, dx \\
 &= \int_0^1 \left[\frac{x^2 y}{2} \right]_0^{2y} dy + \int_1^2 \left[\frac{x^2 y^2}{2} \right]_0^{2-y} dy \\
 &= \frac{1}{2} \int_0^1 y^3 dy + \frac{1}{2} \int_1^2 y(2-y)^2 dy \\
 &= \frac{1}{2} \left(\frac{y^4}{4} \right) \Big|_0^1 + \frac{1}{2} \left[\frac{y^3}{3} - \frac{4y^3}{3} + 2y^2 \right] \Big|_1^2 \\
 &= \frac{1}{2} \left(\frac{1}{4} \right) + \frac{1}{2} \left[4 - \frac{32}{3} + 8 - \left(\frac{1}{3} - \frac{4}{3} + 2 \right) \right] \\
 &= \frac{1}{6} + \frac{1}{2} \left[\frac{4}{3} - \frac{9}{4} + \frac{4}{3} \right] \\
 &= \frac{1}{6} + \frac{1}{2} \left(\frac{5}{12} \right) \\
 &= \frac{1}{6} + \frac{5}{24} \approx \frac{3}{8}. \quad \text{Ans.}
 \end{aligned}$$

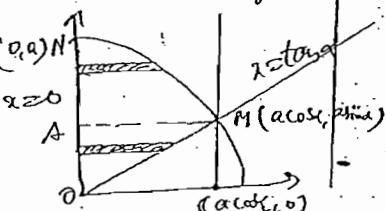
→ Change the order of integration in

$$\iint_{0 \leq y \leq 2x} f(x, y) \, dx \, dy.$$

→ Change the order of integration in the integral

along $\overrightarrow{AB \rightarrow C}$

$$\iint_{\text{triangle}} f(x, y) \, dx \, dy \quad \text{and} \quad \text{verify the result when } f(x, y)=1.$$



Change the order of integration in $\int \int v dxdy$
where v is a function of x and y .

Ex: The limits of integration are given by the parabolas $\frac{x^2}{a} = y$ i.e. $x^2 = ay$,
 $x^2/a = y$ i.e. $ax - x^2 = ay$
and the lines $x=0$, $x=a$.

Also the equation of parabola $ax - x^2 = ay$

may be written as $(x - \frac{a}{2})^2 = a(y - \frac{a}{4})$

i.e. this parabola has the vertex at the point $(\frac{a}{2}, \frac{a}{4})$ and its concavity is downwards.

The points of intersection of two parabolas are given as follows:

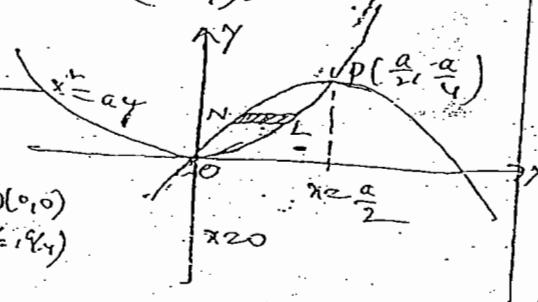
$$ax - x^2 = x^2 \Rightarrow x(a-2x) = 0 \\ \Rightarrow x=0 \text{ or } x = \frac{a}{2}.$$

and hence from $x^2 = ay$,

we get $y=0$ at $x=0$
 $y = \frac{a}{4}$ at $x = \frac{a}{2}$.

Hence the points of intersection of the two parabolas are $(0,0)$ and $(\frac{a}{2}, \frac{a}{4})$.

Draw the two parabolas $x^2 = ay$ and $ax - x^2 = ay$ intersecting at the points $O(0,0)$ and $P(\frac{a}{2}, \frac{a}{4})$.



Now draw the lines $x=0$ and $x=a$.

Clearly the integral extends to the area ONPLO.
Now take strips of the type NL parallel to the

x -axis.

Writing $ay = ax - x^2$

$x - ax + ay = 0$ for x , we get

$$x = \frac{1}{2} \left[a \pm \sqrt{a^2 - 4ay} \right]$$

$$= \frac{1}{2} \left[a - \sqrt{a^2 - 4ay} \right]$$

rejecting the +ve sign before

square root since x is not greater than $\frac{a}{2}$.
for the region of intersection.

For the region ONPLO, the strip NL has the

extremities N and L on $ax - x^2 = ay$ and $x = ay$.

Thus the limits of x are from $x = \frac{1}{2} \left[a - \sqrt{a^2 - 4ay} \right]$

to $x = \sqrt{ay}$.

for limits of y , at O , $y=0$ and at P $y=\frac{a}{4}$.

Hence changing the order of integration

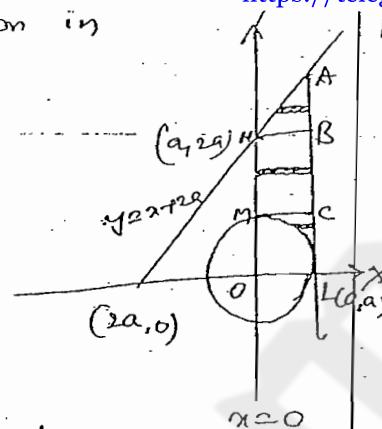
We have

$$\int_0^{a/4} \int_{\sqrt{ay}}^{\sqrt{a}} v dy dx = \int_0^{a/4} \int_{\frac{1}{2}(a - \sqrt{a^2 - 4ay})}^{\sqrt{ay}} v dy dx$$

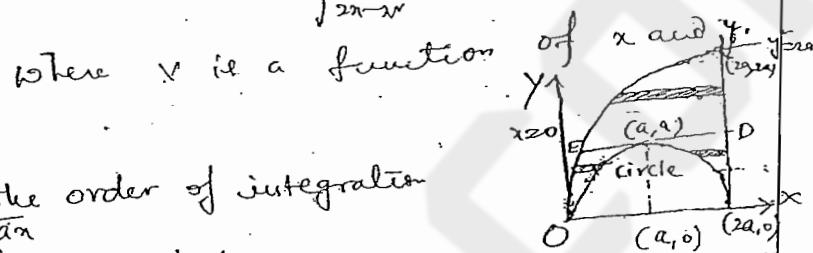
~~$\int_0^{a/4} \int_{\sqrt{ay}}^{\sqrt{a}}$~~

→ Change the order of integration in

$$\int_{\text{sector}}^a \int_0^{x+2a} f(x, y) dy dx$$



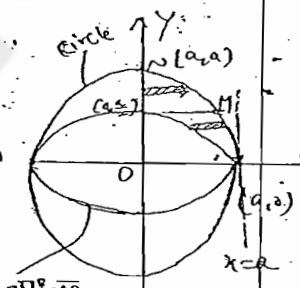
→ Change the order of integration in the double integral $\int_0^{2a} \int_0^y v dy dx$, where v is a function of x and y .



→ Change the order of integration in $\int_0^a \int_0^y f(x, y) dy dx$

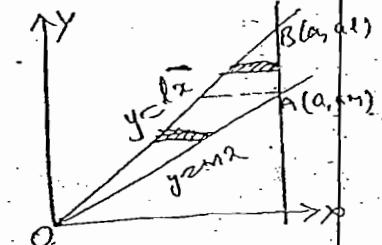
$$\int_0^a \int_0^x f(x, y) dy dx$$

→ Change the order of integration in $\int_0^a \int_0^x v dy dx$, where v is a function of x and y .



→ Change the order of integration in $\int_0^a \int_0^x v dy dx$, where v is a function of x and y .

$$\int_0^a \int_{y/x}^x v dy dx$$



→ Show that

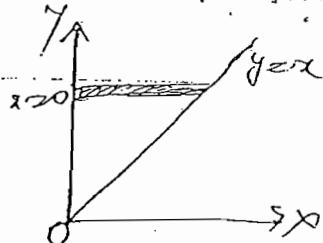
$\int_0^{2a} \int_0^{x+2a} f(x, y) dy dx = \int_0^{2a} \int_0^{y/2a} f(x, y) dy dx$

$$\int_0^{2a} \int_0^{x+2a} f(x, y) dy dx = \int_0^{2a} \int_0^{y/2a} f(x, y) dy dx$$

change the order of integration in the double integral $\int_0^a \int_{y^2}^y e^{-x^2} dy dx$

and hence find the value.

Ans: 1

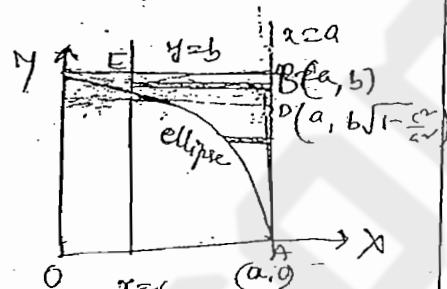


→ Change the order of integration in

$$\int_c^a \int_y^a x dy dx$$

$$c/a \sqrt{a^2 - x^2}$$

where c is less than a.

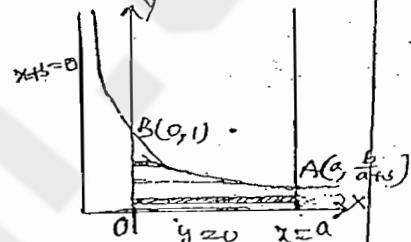


→ Change the order of integration in $\int_0^a \int_0^{x^2} f(x, y) dy dx$.

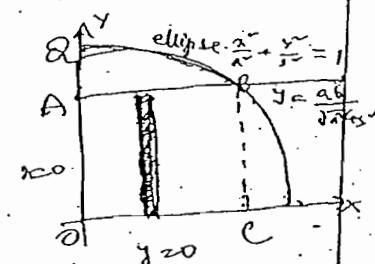
→ Change the order of integration in double integral

$$\int_0^a \int_0^{b/x} v dy dx$$

$$\text{Ans: } I = \int_0^a \int_0^{b/x} v dy dx + \int_{b/x}^a \int_0^{b/(x-a)} v dy dx$$

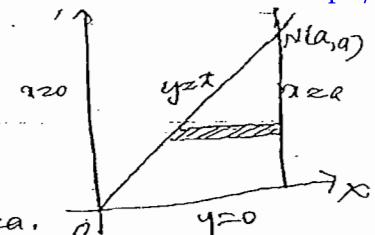


→ Change the order of integration in double integral $\int_0^{ab/\sqrt{a^2+b^2}} \int_a^b \frac{b}{\sqrt{x^2+y^2}} dy dx$.



→ Change the order of integration in double integral $\int_0^a \int_0^a \frac{g(y)}{\sqrt{(a-y)(x-y)}} dy dx$ and hence find its value.

The limits of integration
are given by the straight
lines $y=0$, $y=x$, $x=a$ and $x=a$.



Clearly the region of integration is ΔNM .

Take strips parallel to the x -axis.

The limits of x are from $x=y$ to $x=a$
and the limits of y are from $y=0$ to $y=a$.

Hence we have

$$\iint \frac{\phi(y) dy dx}{\sqrt{(a-y)(x-y)}} = \int_0^a \int_y^a \frac{\phi(y) dy dx}{\sqrt{(a-x)(x-y)}}$$

To find the value:

$$\text{Let } x = a \sin^2 \theta + y \cos^2 \theta$$

$$\Rightarrow dx = 2(a-y) \sin \theta \cos \theta d\theta$$

$$\text{Also } x-y = a \sin^2 \theta + y \cos^2 \theta - y \\ = a \cos^2 \theta - y \sin^2 \theta = (a-y) \cos^2 \theta$$

$$\text{and } x+y = a \sin^2 \theta + y \cos^2 \theta + y$$

$$= a \sin^2 \theta + y \sin^2 \theta = (a+y) \sin^2 \theta$$

for limits of θ , when $x=y$, we have

$$y = a \sin^2 \theta + y \cos^2 \theta$$

$$\Rightarrow (y-a) \sin^2 \theta = 0$$

$$\text{ie } \sin^2 \theta = 0$$

$$\text{ie } \theta = 0$$

and when $x=a$, we have a z axis by cor.

$$\Rightarrow (x-y) \cos \theta = 0$$

$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Thus the limits of θ are from $\theta=0$ to $\theta=\frac{\pi}{2}$

we get

$$\therefore \text{The given integral} = \int_0^a \int_y^a \frac{\phi(y) dy dx}{\sqrt{(a-y)(a-y)}}$$

$$= \int_0^a \int_0^{a-y} \frac{\phi(y) 2(a-y)}{(a-y) \sin \theta \cos \theta} \sin \theta \cos \theta$$

$$= 2 \int_0^a \int_0^y \phi(y) dy d\theta$$

$$= 2 \int_0^a \phi(y) [\theta]_0^{\frac{\pi}{2}} dy$$

$$= 2 \frac{\pi}{2} \int_0^a \phi(y) dy$$

$$= \pi [\phi(y)]_0^a$$

$$= \pi (\phi(a) - \phi(0))$$

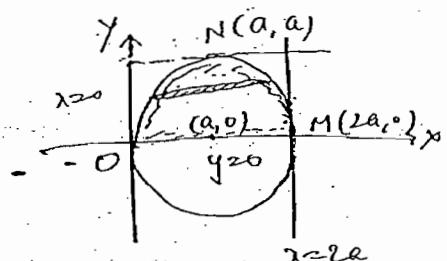
→ Change the order of integration in

$$\int_0^a \int_{\sqrt{a^2-x^2}}^a \phi(y) (x^2+y^2) x dy dx \text{ and hence}$$

$$\int_a^{\sqrt{4a^2-(x+y)^2}} \phi(y) dy$$

evaluate it.

Sol:



~~evaluate $\int \int \sin x \sin y (\sin x \sin y) dx dy$.~~

Solⁿ: Let $\sin x \sin y = \sin \theta$.

Then $\sin x \cos y dy = \cos \theta d\theta$, keeping x constant.

when $y=0$, $\sin \theta=0 \Rightarrow \theta=0$

& when $y=\pi/2$, $\sin \theta=\sin x \Rightarrow \theta=x$.

Hence θ varies from 0 to x .

Given integral

$$\int_0^{\pi/2} \int_0^x \sin x \sin^2 (\sin x \sin y) dx dy$$

$$= \int_0^{\pi/2} \int_0^x \sin x \sin^2 (\sin \theta) dx \cdot \frac{d\sin \theta}{\sin x \cos y}$$

$$= \int_0^{\pi/2} \int_0^x \sin x \cdot d\theta \frac{\cos \theta}{\sin x \cos y} d\theta$$

$$= \int_0^{\pi/2} \int_0^x \frac{\cos x \cos \theta d\theta}{\cos y}$$

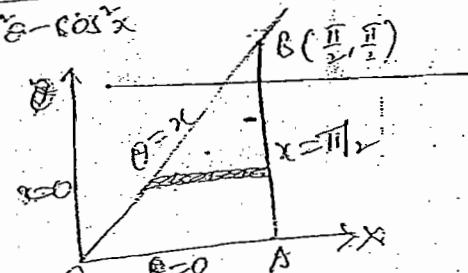
$$= \int_0^{\pi/2} \int_0^x \frac{\cos x \cos \theta d\theta}{\sqrt{1 - \sin^2 \theta}} \quad \left(\begin{array}{l} \sin x \sin y = \sin \theta \\ \Rightarrow \sin y = \frac{\sin \theta}{\sin x} \\ \Rightarrow \cos y = \sqrt{1 - \sin^2 \theta} \end{array} \right)$$

$$= \int_0^{\pi/2} \int_0^x \frac{\cos x \cos \theta \sin x d\theta}{\sqrt{\sin^2 x - \sin^2 \theta}}$$

$$= \int_0^{\pi/2} \int_0^x \frac{\cos x \sin x d\theta}{\sqrt{\cos^2 \theta - \cos^2 x}}$$

Clearly it is convenient
to integrate first w.r.t x .

Therefore we shall change



the order of integration.

The limits of integration are given by the straight lines $\theta=0$, $\theta=x$ and $x=0$, $x=\pi/2$.

Clearly the area of integration is OABO.

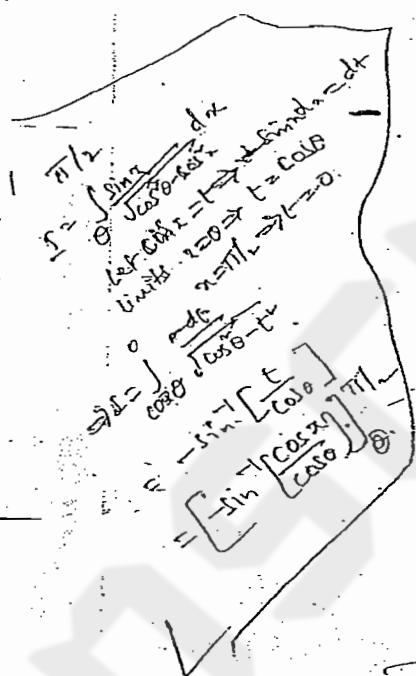
Consider strips parallel to x -axis.

The limits of x are from 0 to $\pi/2$ and limits of θ are from 0 to $\pi/2$.

Hence changing the order of integral,

we have

$$\int_0^{\pi/2} \int_0^x \frac{\theta \cos \theta \sin \theta d\theta dx}{\sqrt{\cos^2 \theta - \cos^2 x}} = \int_0^{\pi/2} \int_0^{\theta} \frac{\theta \cos \theta \sin \theta d\theta dx}{\sqrt{\cos^2 \theta - \cos^2 x}}$$



$$\begin{aligned}
 &= \int_0^{\pi/2} \theta \cos \theta \left[-\sin^{-1} \left(\frac{\cos \theta}{\cos x} \right) \right]_{0}^{\pi/2} d\theta \\
 &= \int_0^{\pi/2} \theta \cos \theta \left[-\tan \frac{\pi}{2} \right] d\theta \\
 &= \frac{\pi}{2} \int_0^{\pi/2} \theta \cos \theta d\theta \\
 &= \frac{\pi}{2} \left[(\theta \sin \theta) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin \theta d\theta \right] \\
 &= \frac{\pi}{2} \left[\frac{\pi}{2} - 0 + [\cos \theta] \Big|_0^{\pi/2} \right] \\
 &= \frac{\pi}{2} \left[\frac{\pi}{2} - 1 \right]
 \end{aligned}$$

Change of order of integration of polar co-ordinates:

→ change the order of integration in double integral
 $r=2\cos\theta$..

$$\int_0^r \int_0^{2\cos\theta} f(r, \theta) d\theta dr.$$

Sol: The limits of integration
 are given by $r=0$ (pole),

$r=2\cos\theta$ (a circle), $\theta=0$ (initial line)
 and $\theta=\pi/2$ (far to initial line at the pole)

Clearly the region of integration is OPMO.

To change the order of integration, we consider circular arc LM on which θ varies and r remains constant.

Now for limits of θ , the arc LM has its extremities on $\theta=0$ (initial line) and $r=2\cos\theta$.

Also the limits of r are from $r=0$ to $r=2\cos\theta$.

Here $r=2\cos\theta$ $\Rightarrow r = 2\cos^{-1}(r/2)$

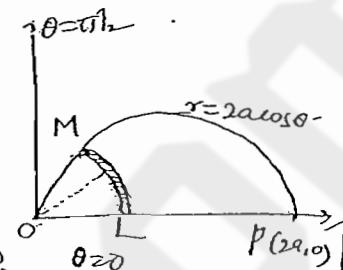
$$\int_0^r \int_0^{2\cos^{-1}(r/2)} f(r, \theta) d\theta dr = \int_0^r \int_0^{2\cos^{-1}(r/2)} f(r, \theta) d\theta dr$$

→ Change the order of integration in the system

of integrals $\int_0^{\pi/2} \int_{r=2\cos\theta}^{r=2} f(r, \theta) r dr d\theta + \int_{\pi/2}^{\pi} \int_{r=2\cos\theta}^{r=2} f(r, \theta) r dr d\theta$

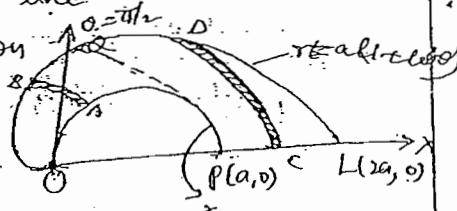
Sol: The limits of integration are given by from $r=2\cos\theta$ to $r=2\sin(\theta/2)$, $\theta=0$ to $\theta=\pi/2$ and $\theta=\pi/2$ to $\theta=\pi$.

i.e., the region of integration is bounded by



upper half circle $r=a\cos\theta$, upper half cardioid $r=a(1+\cos\theta)$ and the initial line.

Clearly $OAPQBQ$ is the region of integration.



Now to change the order of integration consider elementary circular arcs (lines AB and CD) about pole 'O' as a centre.

These arcs change their character at P.

Hence the region is divided into two parts namely $OAPQBQ$ and $OPLQ$.

In the region $OAPQBQ$, the extremities of the arc AB lie on $r=a\cos\theta$ and $r=a(1+\cos\theta)$. Hence θ varies from $-\theta = \cot^{-1}(1/a)$ to $\theta = \cot^{-1}(r/a)$. Also r varies from $r=0$ to $r=a$ as $OP=a$.

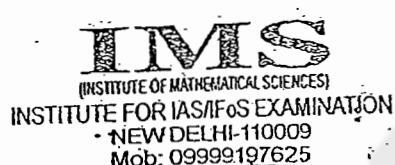
In the region $OPLQ$, the extremities of the arc CD lie on $\theta=0$ and cardioid $r=a(1+\cos\theta)$.

Hence θ varies from $\theta=0$ to $\theta = \cot^{-1} \frac{r-a}{a}$ and r varies from $r=a$ to $r=2a$ as $OL=2a$.

Hence the given integral becomes

$$\int_0^{\cot^{-1}(r/a)} f(r, \theta) r d\theta + \int_{\theta=0}^{\cot^{-1}(r-a/a)} f(r, \theta) r d\theta$$

Set - X



Multiple Integrals and Their Applications

7.1. DOUBLE INTEGRALS

The definite integral $\int_a^b f(x) dx$ is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \dots + f(x_n)\delta x_n,$$

where $n \rightarrow \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy -plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \dots, \delta A_n$. Let (x_r, y_r) be any point within the r th elementary area δA_r . Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \dots + f(x_n, y_n)\delta A_n, \text{ i.e. } \sum_{r=1}^n f(x_r, y_r)\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral* of $f(x, y)$ over the region R and is written as $\iint_R f(x, y)dA$.

Thus

$$\iint_R f(x, y)dA = \lim_{\substack{n \rightarrow \infty \\ \delta A \rightarrow 0}} \sum_{r=1}^n f(x_r, y_r)\delta A_r \quad \dots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purposes of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dx dy$.

Its value is found as follows:

(i) When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated w.r.t. y , keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2 i.e.

$$I_1 = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y) dy \right] dx$$

where integration is carried from the inner to the outer rectangle.

Fig. 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx .

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD .

Thus the whole region of integration is the area $ABDC$.

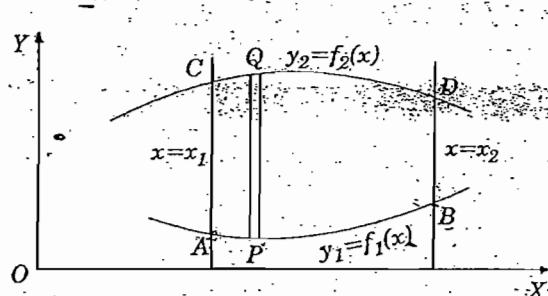


Fig. 7.1.

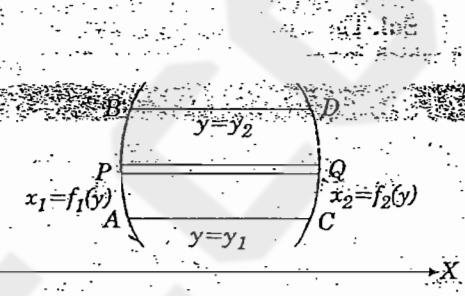


Fig. 7.2.

(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2 , i.e.

$$I_2 = \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x, y) dx \right] dy$$

which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy .

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD .

Thus the whole region of integration is the area $ABDC$.

(iii) When both pairs of limits are constants, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1 , we integrate along the vertical strip PQ and then slide it from AC to BD .

In I_2 , we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD .

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

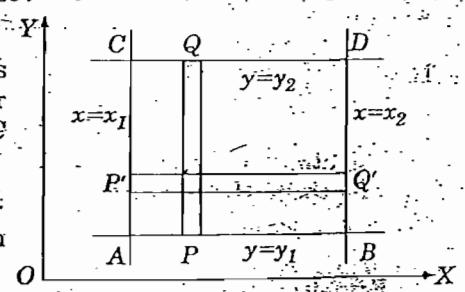


Fig. 7.3.

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

2

Example 7.1. Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy$.

Sol. $I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^2) dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3} \right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{x^6}{3} \right] dx$
 $= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5 = 5^6 \left[\frac{1}{6} + \frac{5^2}{24} \right] = 18880.2 \text{ nearly.}$

Example 7.2. Evaluate $\iint_A xy dx dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. (Gulbarga, 1999 S)

Sol. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Fig. 7.4 shows the domain A which is the area OML.

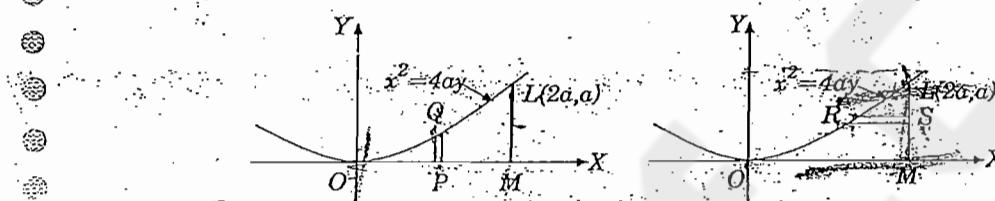


Fig. 7.4.

Integrating first over a vertical strip PQ , i.e. w.r.t. y from $P(y=0)$ to $Q(y=x^2/4a)$ on the parabola and then w.r.t. x from $x=0$ to $x=2a$, we have

$$\begin{aligned} \iint_A xy dx dy &= \int_0^{2a} dx \int_{0}^{x^2/4a} xy dy = \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{x^2/4a} dx \\ &= \frac{1}{32a^2} \int_0^{2a} x^5 dx = \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} = \frac{a^4}{3}. \end{aligned}$$

Otherwise integrating first over a horizontal strip RS , i.e. w.r.t. x from $R(x=2\sqrt{ay})$ on the parabola to $S(x=2a)$ and then w.r.t. y from $y=0$ to $y=a$, we get

$$\begin{aligned} \iint_A xy dx dy &= \int_0^a dx \int_{2\sqrt{ay}}^{2a} xy dx = \int_0^a y \left[\frac{x^2}{2} \right]_{2\sqrt{ay}}^{2a} dy \\ &= 2a \int_0^a (ay - y^2) dy = 2a \left[\frac{ay^2}{2} - \frac{y^3}{3} \right]_0^a = \frac{a^4}{3}. \end{aligned}$$

7.2. CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.3. Change the order of integration in the integral

$$I = \int_{-a}^a \int_0^{\sqrt{a^2 - y^2}} f(x, y) dx dy.$$

Sol. Here the elementary strip is parallel to x -axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{a^2 - y^2}$ (i.e. to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7-5).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{a^2 - x^2}$] to S [$y = \sqrt{a^2 - x^2}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$:

$$\text{Thus } I = \int_0^a dx \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} f(x, y) dy$$

$$\text{or } = \int_0^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} f(x, y) dy dx$$

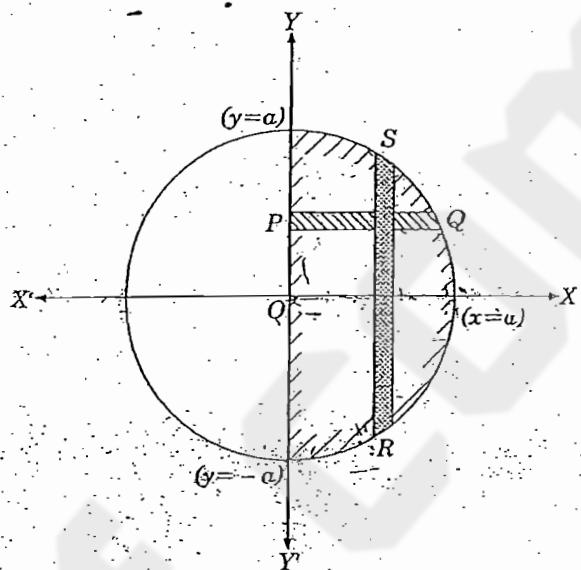


Fig. 7-5.

Example 7.4. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy dx dy$ and hence evaluate the same. (Andhra, 1999; Gauhati, 1999)

Sol. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7-6.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), i.e. the curvilinear triangle OAC and the triangle ABC .

For the region OAC , the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy dx.$$

For the region ABC , the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy dx.$$

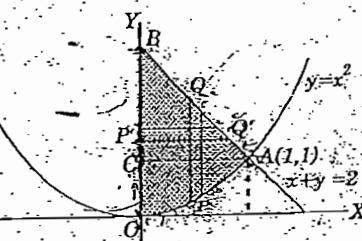


Fig. 7-6.

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

Hence, on reversing the order of integration,

$$\begin{aligned}
 I &= \int_0^1 dy \int_0^{y} xy \, dx + \int_1^2 dy \int_0^{2-y} xy \, dx \\
 &= \int_0^1 dy \left[\frac{x^2}{2} \cdot y \right]_0^y + \int_1^2 dy \left[\frac{x^2}{2} \cdot y \right]_0^{2-y} = \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y(2-y)^2 dy \\
 &= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.
 \end{aligned}$$

7.3 DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) dr d\theta$, we first integrate w.r.t. r between

limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2 . In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Fig. 7.7 illustrates the process geometrically.

Here AB and CD are the curves $r_1=f_1(\theta)$ and $r_2=f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$. PQ is a wedge of angular thickness $d\theta$.

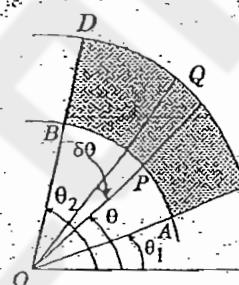


Fig. 7.7:

Then $\int_{r_1}^{r_2} f(r, \theta) dr$ indicates that the integration is along PQ from P to Q , while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD .

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.5. Calculate $\iint r^3 dr d\theta$ over the area included between the circles $r=2 \sin \theta$ and $r=4 \sin \theta$. (J.N.T.U., 1999; Marathwada, 1998)

Sol. Given circles $r=2 \sin \theta$... (i)

and $r=4 \sin \theta$... (ii)

are shown in Fig. 7.8. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r , then its limits are from $P(r=2 \sin \theta)$ to $Q(r=4 \sin \theta)$ and to cover the whole region θ varies from 0 to π . Thus the required integral is

$$\begin{aligned}
 I &= \int_0^\pi d\theta \int_{2 \sin \theta}^{4 \sin \theta} r^3 dr = \int_0^\pi d\theta \left[\frac{r^4}{4} \right]_{2 \sin \theta}^{4 \sin \theta} \\
 &= 60 \int_0^\pi \sin^4 \theta d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4 \theta d\theta \\
 &= 120 \times \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 22.5 \pi.
 \end{aligned}$$

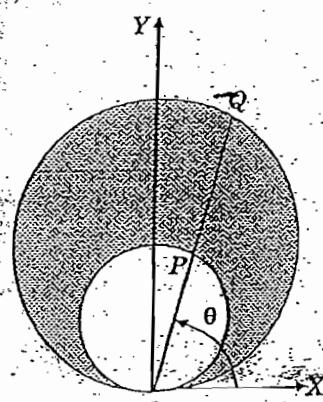


Fig. 7.8.

Problems 7-1

Evaluate the following integrals (1—7) :

$$1. \int_1^2 \int_1^3 xy^2 dx dy \quad (\text{Madras, 1998 S})$$

$$2. \int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy. \quad (\text{V.T.U., 2000})$$

$$3. \int_0^4 \int_0^{x^2} e^{y/x} dy dx. \quad (\text{Osmania, 1999 S})$$

$$4. \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}. \quad (\text{Madras, 2000})$$

$$5. \iint xy dx dy \text{ over the positive quadrant of the circle } x^2 + y^2 = a^2. \quad (\text{V.T.U., 2001; Madras, 2000})$$

$$6. \iint (x+y)^2 dx dy \text{ over the area bounded by the ellipse } x^2/a^2 + y^2/b^2 = 1.$$

$$7. \iint xy(x+y) dx dy \text{ over the area between } y = x^2 \text{ and } y = x.$$

Evaluate the following integrals by changing the order of integration (8—16) :

$$8. \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx dy. \quad (\text{Pondicherry, 1998 S})$$

$$9. \int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy. \quad (\text{Anna, 2003 S; V.T.U., 2003; Delhi, 2002})$$

$$10. \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx.$$

$$11. \int_0^1 \int_x^{\sqrt{(2-x^2)}} -\frac{x dy dx}{\sqrt{x^2+y^2}}. \quad (\text{Rohtak, 2003; I.S.M., 2001})$$

$$12. \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2) dx dy \quad (a > 0).$$

(Bhopal, 1998)

$$13. \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2) dx dy. \quad (\text{Marathwada, 1998})$$

$$14. \int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2 x^2}}.$$

$$15. \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx.$$

(Madras, 2003; V.T.U., 2000)

$$16. \int_0^{\infty} \int_0^x xe^{-x^2/y} dy dx.$$

(V.T.U., 2004; Delhi, 2002)

$$17. \text{Sketch the region of integration of } \int_a^{ae^{\pi/4}} \int_{2 \log(r/a)}^{\pi/2} f(r, \theta) r dr d\theta \text{ and change the order of integration.}$$

$$18. \text{Evaluate } \iint r \sin \theta dr d\theta \text{ over the cardioid } r = a(1 - \cos \theta) \text{ above the initial line.}$$

$$19. \text{Show that } \iint_R r^2 \sin \theta dr d\theta = 2a^2/3, \text{ where } R \text{ is the semi-circle } r = 2a \cos \theta \text{ above the initial line.}$$

$$20. \text{Evaluate } \iint \frac{r dr d\theta}{\sqrt{a^2 + r^2}} \text{ over one loop of the lemniscate } r^2 = a^2 \cos 2\theta.$$

7.4. AREA ENCLOSED BY PLANE CURVES

(1) Cartesian co-ordinates.

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $= x_1, x = x_2$ (Fig. 7-9).

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

Divide this area into vertical strips of width δx . If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

area of strip $KL = \lim_{\delta y \rightarrow 0} \sum \delta x \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

area of the strip $KL = \delta x \lim_{\delta y \rightarrow 0} \sum dy = \delta x \int_{f_1(x)}^{f_2(x)} dy$.

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \lim_{\delta x \rightarrow 0} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx dy$$

Similarly dividing the area $A'B'C'D'$ (Fig. 7.10) into horizontal strips of width δy , we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx dy$$

(2) Polar co-ordinates.

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta \theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting at Q in R and OP (produced) in S (Fig. 7.11).

Since $\text{arc } PR = r\delta\theta$ and $PS = \delta r$.

∴ area of the curvilinear rectangle $PRQS$ is approximately $= PR \cdot PS = r\delta\theta \cdot \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\sum r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A .

$$\text{Hence } A = \lim_{\substack{\delta r \rightarrow 0 \\ \delta\theta \rightarrow 0}} \sum r\delta\theta\delta r = \int \int r d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

Example 7.6. Find the area of a plate in the form of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{V.T.U., 2001; Osmania, 2000 S})$$

Sol. Dividing the area into vertical strips of width

δx , y varies from $K(y=0)$ to $L(y=b\sqrt{1-x^2/a^2})$ and then x varies from 0 to a (Fig. 7.12).

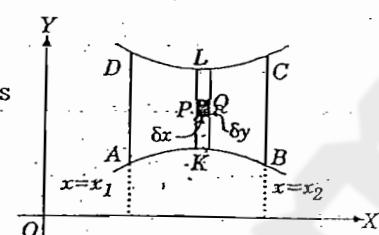


Fig. 7.9.

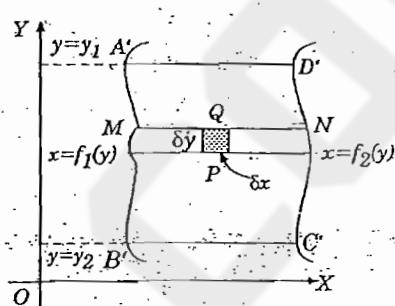


Fig. 7.10.

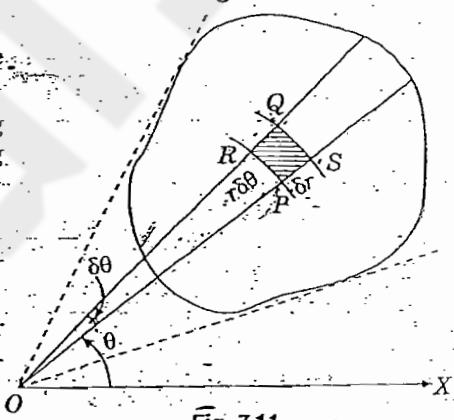


Fig. 7.11.

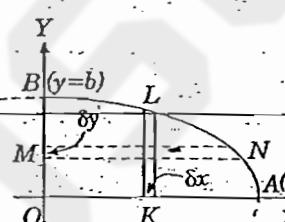


Fig. 7.12.

Sol. Changing to polar spherical coordinates by putting

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

we have $dx dy dz = r^2 \sin \theta dr d\theta d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a , θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

Volume of the sphere

$$\begin{aligned} &= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin \theta dr d\theta d\phi \\ &= 8 \int_0^a r^2 dr \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\phi = 8 \cdot \left[\frac{r^3}{3} \right]_0^a \cdot \left[-\cos \theta \right]_0^{\pi/2} \cdot \frac{\pi}{2} \\ &= 4\pi \cdot \frac{a^3}{3} \cdot (-0+1) = \frac{4}{3}\pi a^3. \end{aligned}$$

Example 7.18. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$. (Rohtak, 2003)

Sol. The required volume is easily found by changing to cylindrical co-ordinates (ρ, ϕ, z) . We therefore, have

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z,$$

$$\text{and } J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin \phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.23 for which z varies from 0 to $\sqrt{a^2 - \rho^2}$, ρ varies from 0 to $a \sin \phi$ and ϕ varies from 0 to π .

Hence the required volume

$$\begin{aligned} &= 2 \int_0^\pi \int_0^{a \sin \phi} \int_0^{\sqrt{a^2 - \rho^2}} \rho dz d\rho d\phi \\ &= 2 \int_0^\pi \int_0^{a \sin \phi} \rho \sqrt{a^2 - \rho^2} d\rho d\phi = 2 \int_0^\pi \left[-\frac{1}{3}(a^2 - \rho^2)^{3/2} \right]_0^{a \sin \phi} d\phi \\ &= \frac{2a^3}{3} \int_0^\pi (1 - \cos^3 \phi) d\phi = \frac{2a^3}{9} (3\pi - 4). \end{aligned}$$

Example 7.19. Evaluate $\int_0^1 \int_0^1 \int_{\sqrt{x^2 + y^2}}^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{x^2 + y^2 + z^2}}$

Sol. We change to spherical polar co-ordinates (r, θ, ϕ) , so that

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

$$\text{and } J = r^2 \sin \theta, x^2 + y^2 + z^2 = r^2.$$

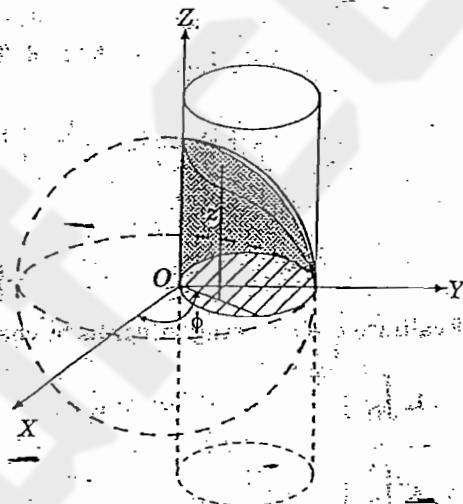


Fig. 7.23.

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The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.24). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec \theta$ and ϕ varies from 0 to $\pi/2$.

Given integral becomes

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec \theta} \frac{1}{r} r^2 \sin \theta dr d\theta d\phi \\ &= \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left[\frac{r^3}{3} \right]_0^{\sec \theta} \sin \theta d\theta \\ &= \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2 \theta}{2} \sin \theta d\theta \\ &= \frac{\pi}{4} \int_0^{\pi/4} \sec \theta \tan \theta d\theta \\ &= \frac{\pi}{4} [\sec \theta]_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4} \end{aligned}$$

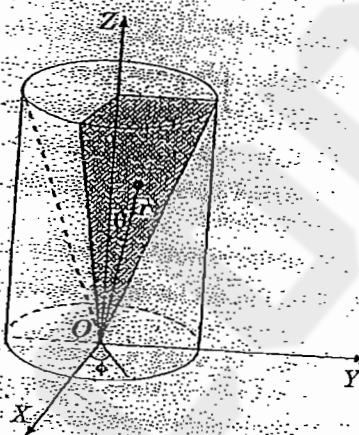


Fig. 7.24

Problems 7.4:

Evaluate the following integrals by changing to polar co-ordinates

1. $\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2) dy dx$

2. $\int_0^a \int_y^a \frac{x^2 dx dy}{\sqrt{x^2 + y^2}}$

3. $\int_0^{4a} \int_{y^2/4a}^y \frac{x^2 - y^2}{x^2 + y^2} dx dy$

4. $\iint xy (x^2 + y^2)^{n/2} dx dy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$

5. Transform the following to cartesian form and hence evaluate $\int_0^{\pi/2} \int_0^a r^3 \sin \theta \cos \theta dr d\theta$

6. By using the transformation $x + y = u, xy = v$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)} dy dx = \frac{1}{2}(e-1)$

Evaluate the following integrals by changing to spherical co-ordinates

7. $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx dy dz}{\sqrt{(1-x^2-y^2-z^2)}}$

(Madras, 1998; Marathwada, 1998 S; Punjab, 1997)

8. $\iiint z^2 dx dy dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$. (J.N.T.U., 1991)
9. Find the volume bounded by the xy -plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. (I.S.M., 2000)
10. Find the volume bounded by the xy -plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. (Kanpur, 1991)
11. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
12. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.
13. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.
14. Find the volume of the cylinder $x^2 + y^2 = 2ax$, intercepted between the paraboloid $x^2 + y^2 = 2az$ at the xy -plane.
15. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
16. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$. (Kanpur, 1991)
17. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos \theta)$ about its axis is $8\pi a^3/3$. (V.T.U., 2000)
18. Using triple integration, find the volume of the sphere $x^2 + y^2 + z^2 = a^2$. (V.T.U., 2000)
19. Evaluate $\iiint (x + y + z) dx dy dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. [See Fig. 7-28] (Bangalore, 1998)
20. Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (Burdwan, 2000)
21. Find the volume of the solid surrounded by the surface $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$.
 [Sol. Changing the variables, x, y, z to X, Y, Z where
 $(x/a)^{1/3} = X, (y/b)^{1/3} = Y, (z/c)^{1/3} = Z$
 i.e. $x = X^3, y = bY^3, z = cZ^3$ so that
 $J = \partial(x, y, z)/\partial(X, Y, Z) = 27abcX^2Y^2Z^2$.
 ∴ Reqd. volume = $\iiint dx dy dz = 27abc \iiint X^2Y^2Z^2 dX dY dZ$.
 taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$.
 Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin \theta \cos \phi$, $Y = r \sin \theta \sin \phi$, $Z = r \cos \theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2 \sin \theta$. To describe the positive octant of the sphere (i), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.
 ∴ Reqd. volume = $27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2 \theta \cos^2 \phi \times r^2 \sin^2 \theta \sin^2 \phi \cdot r^2 \cos^2 \theta \cdot r^2 \sin \theta dr d\theta d\phi$
 $= 216abc \int_0^1 r^8 dr \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta \int_0^{\pi/2} \sin^2 \phi \cos^2 \phi d\phi = 4\pi abc/35]$
22. Work out example 7-12 by changing the variables.

7.8. AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$.

Let its projection on the xy -plane be the region A . Divide it into area elements by drawing lines parallel to the axes of X and Y . (Fig. 7-25).

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS .

As $\delta x \delta y$ is the projection of δS on the xy -plane,

$\therefore \delta x \delta y = \delta S \cdot \cos \gamma$, where γ is the angle between the xy -plane and the tangent plane to S at P , i.e. it is the angle between the Z -axis and the normal to S at P ($= \angle ZPN$).

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$$

the direction cosines of the normal to S [$F = f(x, y) - z$] are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the z -axis are $0, 0, 1$.

Hence $\cos \gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$

Hence $S = \lim_{\delta S \rightarrow 0} \sum \delta S = \iint_A \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dx dy$

Similarly, if B and C be the projections of S on the yz - and zx -planes respectively, then

$S = \iint_B \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1} dy dz$ and $S = \iint_C \sqrt{\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1} dz dx$

Example 7.20. Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.

Sol. Fig. 7-26 shows one-eighth of the required area. Its projection on the xy -plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $x^2 + z^2 = 4$, ... (i)

we have $\frac{dz}{dx} = \frac{x}{z} \frac{dz}{dy} = 0$

so that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}$.

Hence the required surface area = 8

(surface area of the upper portion of (i) lying within the

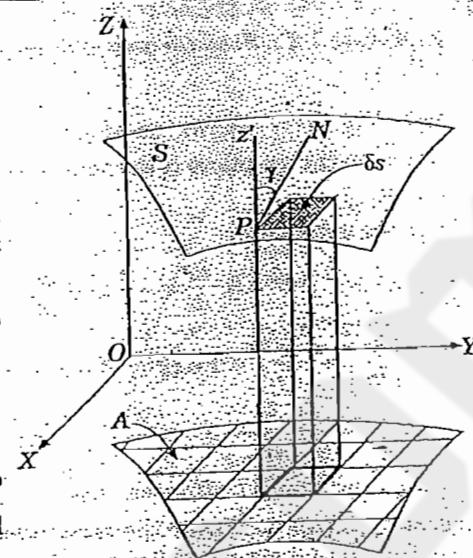


Fig. 7-25

$$\delta S = \frac{\delta x \delta y}{\cos \gamma} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \delta x \delta y$$

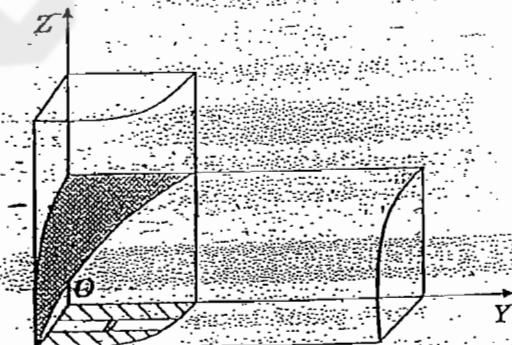


Fig. 7-26

cylinder $x^2 + z^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}} dx dy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$

