Math 341 Group Project: Solid Angle Polynomials

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1 Introduction

This report will give an overview of the theory developed in chapter 8 of Sinai's book regarding solid angle polynomials.

The project of solid angle polynomials is to give a discrete account of volume. Given a polytope P we may center spheres at all integer points in $\mathbb{Z}^d$. Measuring the intersection of these spheres with P and summing will give us a discrete account of volume. This is the main intuition that we will be starting with.

We begin by generalizing the notion of an angle to higher dimensions. This will let us define angles in terms of arbitrary dimensional polytopes. After summing these angles, and placing certain restrictions on our polytope, we get what is known as a solid angle polynomial.

From here we will define a convolution between the indicator function of our polytope and a gaussian. We will explore the connection between this convolution and our angle polynomial. This will ultimately allow us to use poisson summation to uncover information about the behavior of our polynomial by relating it to Fourier analysis.

Using this theory we will go on to state select results and deliver a proof of a theorem that connects the k-tiling of $\mathbb{R}^d$ to solid angles. Finally, we will give a brief account of how one would extend the notion of the sum of the angles of a triangle to higher dimensions by defining what are known as Gram relations.

2 Angles in Higher Dimensions

To extend the notion of an angle to higher dimensions, we can consider a cone $\mathcal{K} \subseteq \mathbb{R}^d$ and a sphere centered at the apex of the cone. The angle is then the proportion of the sphere that intersects the cone.

We can denote the angle by $\omega_{\mathcal{K}}$. In $\mathbb{R}^d$, we know that S^{d-1} is the unit sphere. More precisely,

$$S^{d-1} = \{x \in^d \mid ||x|| = 1\} \tag{1}$$

So, mathematically, the angle of the cone $\mathcal{K}$ is given by

$$\omega_{\mathcal{K}} = \frac{\mathrm{vol}\,(\mathcal{K} \cap S^{d-1})}{\mathrm{vol}\,(S^{d-1})}$$

So,

$$\omega_{\mathcal{K}} = \frac{\int_{\mathcal{K}\cap S^{d-1}} 1\, dx}{\int_{S^{d-1}} 1\, dx}$$

Now, changing to polar cooridnates, we get

$$\omega_{\mathcal{K}} = \frac{\int_{r=0}^{1} \int_{\theta\in S^{d-1}\cap\mathcal{K}} r^{d-1}\, dr d\theta}{\int_{r=0}^{1} \int_{\theta\in S^{d-1}} r^{d-1}\, dr d\theta}$$

So,

$$\omega_{\mathcal{K}} = \frac{\int_{r=0}^{1} r^{d-1}\, dr\ \int_{\theta\in S^{d-1}\cap\mathcal{K}}\ d\theta}{\int_{r=0}^{1} r^{d-1}\, dr\ \int_{\theta\in S^{d-1}}\ d\theta}$$

So,

$$\omega_{\mathcal{K}} = \frac{\int_{S^{d-1}\cap\mathcal{K}}\ d\theta}{\int_{S^{d-1}}\ d\theta}$$

Multiplying the numerator and denominator by $\int_{r=0}^{\infty} r^{d-1}\, dr$,

$$\omega_{\mathcal{K}} = \frac{\int_{r=0}^{\infty} e^{-\pi r^2} r^{d-1}\, dr \int_{S^{d-1}\cap\mathcal{K}}\ d\theta}{\int_{r=0}^{\infty} e^{-\pi r^2} r^{d-1}\, dr \int_{S^{d-1}}\ d\theta}$$

So,

$$\omega_{\mathcal{K}} = \frac{\int_{S^{d-1}\cap\mathcal{K}} \int_{r=0}^{\infty} e^{-\pi r^2} r^{d-1}\, dr d\theta}{\int_{S^{d-1}} \int_{r=0}^{\infty} e^{-\pi r^2} r^{d-1} dr d\theta}$$

Changing back to the original coordinates,

$$\omega_{\mathcal{K}} = \frac{\int_{S^{d-1}\cap\mathcal{K}} e^{-\pi||x||^2} dx}{\int_{\mathbb{R}^d} e^{-\pi||x||^2} dx}$$

We know that the denominator is 1. So,

$$\boxed{\omega_{\mathcal{K}} = \int_{\mathcal{K}} e^{-\pi||x||^2} dx}$$

3 Local Solid Angles for a Polytope and Gaussian Smoothing

We now want to connect solid angles and polytopes. Let $P \subset \mathbb{R}^d$ be a polytope and fix some point $x \in^d$. Construct a $d-1$ dimensional sphere centered at x with radius ϵ. The following definition will give us the fraction of our sphere that intersects with P.

$$\omega_{\mathcal{P}}(x) = \lim_{\epsilon\to 0} \frac{\mathrm{vol}(S^{d-1})(x,\epsilon)\cap\mathcal{P}}{\mathrm{vol}(S^{d-1}(x,\epsilon)}$$

By definition this quantity, the **solid angle** fraction, is trapped between 0 and 1 and will be maximized if x lies in the interior of P and minimized if x is outside of P. If x exists on the boundary of P we have $\omega_{\mathcal{P}}(x) > 0$. For instance, if we place x on the meeting point of two faces then $\omega_{\mathcal{P}}(x)$ will measure the dihedral angle subtended by P at x.

Using this definition we are able to define a new measure of discrete volume in which we assign a weight of $\omega_{\mathcal{P}}(x)$ to all integer points and take the sum, known as the **solid angle-sum** of P. Formally this looks like,

$$\mathcal{A}_{\mathcal{P}}(t) = \sum_{n \in \mathbb{Z}} \omega_{t\mathcal{P}}(n)$$

where $t \in \mathbb{Z}^+$ and $t\mathcal{P}$ represents the corresponding dilation of P.

If we restrict ourselves to integer polytopes and positive integer dilations we can express the above sum has the following polynomial,

$$\mathcal{A}_{\mathcal{P}}(t) = (\text{vol})\mathcal{P}t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + ... + \begin{cases} a_1 t, & \text{if d is odd} \\ a_2 t^2, & \text{if d is even} \end{cases}$$

which will be denoted as the **angle-polynomial** of P.

Definition 1. Heat Kernel
For each real $\epsilon > 0$, we define a **heat kernel**:

$$G_\epsilon(x) = \epsilon^{-\frac{d}{2}} e^{-\frac{\pi}{\epsilon}||x||^2}$$

for all $x \in \mathbb{R}^d$.

Definition 2. Gaussian Smoothing
For any polytope P, the convolution of its indicator function $1_{\mathcal{P}}$ by the heat kernel G_ϵ is called the **Gaussain smoothing** of $1_{\mathcal{P}}$.

So, the Gaussian smoothing of $1_{\mathcal{P}}$ is

$$(1_{\mathcal{P}} * G_\epsilon)(x) = \int_{\mathbb{R}^d} 1_{\mathcal{P}}(y) G_\epsilon(x-y) dy = \int_{\mathcal{P}} G_\epsilon(y-x) dy = \epsilon^{-\frac{d}{2}} \int_{\mathcal{P}} e^{-\frac{\pi}{\epsilon}(||y-x||^2)} dy$$

This is a C^∞ function for $x \in \mathbb{R}^d$ and is also a Schwartz function.

Lemma 3. *Let P be a full-dimensional polytope in $\mathbb{R}^d$. Then for each point $x \in \mathbb{R}^d$, we have*

$$\lim_{\epsilon \to 0} (1_{\mathcal{P}} * G_\epsilon)(x) = \omega_P(x). \tag{2}$$

Lemma 4. *Let P be a full-dimensional polytope in $\mathbb{R}^d$. Then,*

$$A_{\mathcal{P}}(t) = \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{t\mathcal{P}} * G_\epsilon)(n). \tag{3}$$

Proof. (From [3])
First, recall that $A_{\mathcal{P}}(t) = \sum_{n \in \mathbb{Z}} \omega_{t\mathcal{P}}(n)$.

Now, note that

$$(1_{t\mathcal{P}} * G_\epsilon)(x) = \int_{tP} G_\epsilon(y-x) dy \le \int_{tP} \sup_{y \in tP} G_\epsilon(y-x) dy = \sup_{y \in tP} G_\epsilon(y-x) \int_{tP} 1 \, dy$$

(The last equality follows because of the fact that the supremum is simply a function of x, and can be treated as a constant in the integral)

So, we get

$$(1_{tP} * G_\epsilon)(x) \le \sup_{y \in tP} G_\epsilon(y-x) \mathrm{vol}\, tP$$

For ease of notation, we denote $\sup_{y\in tP} G_\epsilon(y-x)$ by $l_\epsilon(x, tP)$. So, we have

$$(1_{tP} * G_\epsilon)(x) \le l_\epsilon(x, tP) \mathrm{vol}\, tP$$

Now, recall that the heat kernel G_ϵ is given by

$$G_\epsilon(x) = \epsilon^{-\frac{d}{2}} e^{-\frac{\pi}{\epsilon}||x||^2}$$

and so for a fixed x, the heat kernel is a function can be thought of as a function of ϵ. With respect to ϵ, its derivative is (by the product rule)

$$-\frac{d}{2}\epsilon^{-\frac{(d+2)}{2}} \cdot e^{-\frac{\pi}{\epsilon}||x||^2} + \epsilon^{-\frac{d}{2}} \cdot e^{-\frac{\pi}{\epsilon}||x||^2} \cdot (\frac{\pi}{\epsilon^2}||x||^2) = e^{-\frac{\pi}{\epsilon}||x||^2}\epsilon^{-\frac{d}{2}}\left(-\frac{d}{2}\cdot\frac{1}{\epsilon} + \frac{\pi}{\epsilon^2}||x||^2\right)$$

We can see that this derivative will be positive if

$$-\frac{d}{2}\cdot\frac{1}{\epsilon} + \frac{\pi}{\epsilon^2}||x||^2 > 0$$

which is equivalent to

$$\frac{\pi}{\epsilon}||x||^2 > \frac{d}{2} \iff ||x||^2 > \frac{d\epsilon}{2\pi}$$

So, if $||x||^2 > \frac{d\epsilon}{2\pi}$, then $G_\epsilon(x)$ is a strictly increasing function of x. Now, if we choose ϵ such that $0 < \epsilon < 1$, then $G_\epsilon(x) < G_1(x)$.

Now, for any ϵ, define a positive number natural number N to be the ceiling function of $d\epsilon$ (the largest integer greater than or equal to $d\epsilon$).

Now, for any positive real number R such that $2\pi R > N$, it turns out that for any $x \in \mathbb{R}$ such that $||x||^2 > R$ the inequality $G_\epsilon(x) < G_1(x)$ holds (this is because $2\pi R > N$ is equivalent to $R > \frac{N}{2\pi}$, and since we know that $N \ge d\epsilon$, this ensures that $R > \frac{d\epsilon}{2\pi}$). So, for any $x \in \mathbb{R}^d$ such that $||x||^2 > R$, it turns out that $||x||^2 > \frac{d\epsilon}{2\pi}$, which ensures that the inequality that $G_\epsilon(x) < G_1(x)$ holds.

Now, for any positive number R such that $2\pi R > N$, we can define a set

$$\Omega(R) := \{x \in \mathbb{R}^d \,|\, \sup_{y\in tP} ||x-y||^2 \le R\}$$

We already showed that

$$(1_{tP} * G_\epsilon)(x) \le l_\epsilon(x, tP)\mathrm{vol}tP$$

and so

$$\sum_{x\in\mathbb{Z}^d\setminus\Omega(R)} (1_{tP} * G_\epsilon)(x) \le \mathrm{vol}tP \sum_{x\in\mathbb{Z}^d\setminus\Omega(R)} l_\epsilon(x, tP) \le \mathrm{vol}tP \sum_{x\in\mathbb{Z}^d\setminus\Omega(R)} l_1(x, tP)$$

(becasue $x \notin \Omega(R)$, we know that the heat kernel is a strictly increasing function of ϵ). Now, note that

$$\sum_{x \in \mathbb{Z}^d \setminus \Omega(R)} l_1(x, tP) \le \int_{\mathbb{R}^d \setminus (\Omega(R - r^2)} l_1(x, tP) dx$$

where r is the diameter of the unit cell. So, we have

$$0 \le \sum_{x \in \mathbb{Z}^d \setminus \Omega(R)} (1_{tP} * G_\epsilon)(x) \le \int_{\mathbb{R}^d \setminus (\Omega(R - r^2)} l_1(x, tP) dx$$

For ease of notation, we can denote the term on the right side of the inequality by $g(R)$. So,

$$g(R) := \int_{\mathbb{R}^d \setminus (\Omega(R - r^2)} l_1(x, tP) dx$$

When $R \to \infty$, $g(R) \to 0$.

We know that

$$0 \le \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} (1_{t\mathcal{P}} * G_\epsilon)(n) \le g(R)$$

Taking the limit as $R \to \infty$, the term in the middle is squeezed to 0, so

$$\lim_{R \to \infty} \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} (1_{t\mathcal{P}} * G_\epsilon)(n) = 0$$

Now, note that

$$\sum_{n \in \mathbb{Z}^d} (1_{tP} * G_\epsilon)(n) = \sum_{n \in \mathbb{Z}^d \setminus \Omega(R)} (1_{tP} * G_\epsilon)(n) + \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} (1_{tP} * G_\epsilon)(n)$$

Taking the limit as $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{tP} * G_\epsilon)(n) = \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \setminus \Omega(R)} (1_{tP} * G_\epsilon)(n) + \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} (1_{tP} * G_\epsilon)(n)$$

Since the sum on the rightmost side is finite, we can pull the limit inside the sum.

$$\lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{tP} * G_\epsilon)(n) = \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \setminus \Omega(R)} (1_{tP} * G_\epsilon)(n) + \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} \lim_{\epsilon \to 0} (1_{tP} * G_\epsilon)(n)$$

Now, taking the limit as $R \to \infty$,

$$\lim_{R \to \infty} \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{tP} * G_\epsilon)(n) = \lim_{R \to \infty} \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \setminus \Omega(R)} (1_{tP} * G_\epsilon)(n) + \lim_{R \to \infty} \sum_{n \in \mathbb{Z}^d \cap \Omega(R)} \lim_{\epsilon \to 0} (1_{tP} * G_\epsilon)(n)$$

The left side does not depend on R, so it doesn't change. On the right side, we showed that the first term goes to 0 as $R \to \infty$. For the other term, note that as $R \to \infty$, $\Omega(R)$ becomes very large, so the last term goes to $\sum_{n \in \mathbb{Z}^d} (1_{tP} * G_\epsilon)(x)$. So, we get

$$\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}(1_{tP}*G_\epsilon)(n) = 0 + \lim_{R\to\infty}\sum_{n\in\mathbb{Z}^d}\lim_{\epsilon\to 0}(1_{tP}*G_\epsilon)(n)$$

Using Lemma 3, we get

$$\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}(1_{tP}*G_\epsilon)(n) = 0 + \lim_{R\to\infty}\sum_{n\in\mathbb{Z}^d}\omega_{tP}(n)$$

The right side is now, by definition, $A_P(t)$. So, we have

$$\lim_{\epsilon\to 0}\sum_{n\in\mathbb{Z}^d}(1_{tP}*G_\epsilon)(n) = A_P(t)$$

$\square$

We can see that by the Poisson summation formula,

$$\sum_{x\in\mathbb{Z}^d}(1_{tP}*G_\epsilon)(x) = \sum_{\xi\in\mathbb{Z}^d}\widehat{1_{tP}*G}_\epsilon(\xi) = \sum_{\xi\in\mathbb{Z}^d}\hat{1}_{tP}(\xi)\hat{G}_\epsilon(x) = \sum_{\xi\in\mathbb{Z}^d}\hat{1}_{tP}e^{-\pi\epsilon||\xi||^2}$$

Taking the limits on both sides as $\epsilon\to 0$,

$$\lim_{\epsilon\to 0}\sum_{x\in\mathbb{Z}^d}(1_{tP}*G_\epsilon)(x) = \lim_{\epsilon\to 0}\sum_{\xi\in\mathbb{Z}^d}\hat{1}_{tP}e^{-\pi\epsilon||\xi||^2} = t^d\lim_{\epsilon\to 0}\sum_{\xi\in\mathbb{Z}^d}\hat{1}_P(t\xi)e^{-\pi\epsilon||\xi||^2}$$

Using the lemma just proved,

$$A_P(t) = t^d\lim_{\epsilon\to 0}\sum_{\xi\in\mathbb{Z}^d}\hat{1}_P(t\xi)e^{-\pi\epsilon||\xi||^2}$$

Lemma 5. *For any $x\in\mathbb{R}$, we have*

$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\sum_{\xi\in\mathbb{Z}-\{0\}}\frac{e^{-2\pi ix\xi-\epsilon\pi\xi^2}}{\xi} = x - \lfloor x\rfloor - \frac{1}{2} \tag{4}$$

Theorem 6. *Let $\mathcal{P}$ be an integer polygon. Then the angle polynomial of $\mathcal{P}$ is:*

$$A_{\mathcal{P}}(t) = (\operatorname{area}\mathcal{P})t^2$$

for all positive integer dilations t.

For $t=1$, this theorem is equivalent to Pick's formula:

Theorem 7. ***Pick's Formula, 1899***
Let $\mathcal{P}$ be an integer polygon. Then

$$\operatorname{Area}\mathcal{P} = I + \frac{1}{2}B - 1$$

where I is the number of interior integer points in $\mathcal{P}$, and B is the number of boundary integer points in $\mathcal{P}$.

We can also use solid angle sums to characterize the polytopes that k-tile $\mathbb{R}^d$ by translations.

Theorem 8. *A polytope $\mathcal{P}$ k-tiles $\mathbb{R}^d$ by integer translations if and only if*

$$\sum_{\lambda \in \mathbb{Z}^d} \omega_{\mathcal{P}+v}(\lambda) = k$$

Proof. (From [1]:

Recall two results: 1. If a convex polytope k-tiles $\mathbb{R}^d$ by translation then it and it's facets are centrally symmetric. 2. A convex polytope $-1 \cdot P$ k-tiles $\mathbb{R}^d$ by translations with a multiset Λ if and only if $(\Lambda \cap \{P+v\}) = k$ for every v in general position.

Using the first of these results we can select a polytope that k-tiles $\mathbb{R}^d$ and know that $-P$ $k-$tiles as well. By the second we have that $\#(\Lambda \cap \{P+x\}) = k$ for almost every x. With this we can construct a d-dimensional ball centered at v and with radius R and use it to integrate with respect to x.

$$\begin{aligned}
k \cdot V(B(v,R)) = \int_{B(v,R)} k\, dx &= \int_{B(v,R)} \#(\Lambda \cap \{P+x\})\, dx \\
&= \int_{B(v,R)} \sum_{\lambda \in \Lambda} 1_{\lambda - P}(x)\, dx \\
&= \sum_{\lambda \in \Lambda} \int_{B(v,R)} 1_{\lambda - P}(x)\, dx \\
&= \sum_{\lambda \in \Lambda} V(B(v,R) \cap \{\lambda - P\}) \\
&= \sum_{\lambda \in \Lambda} V(\{\lambda - B_R\} \cap \{P+v\})
\end{aligned}$$

From this we have the eqaulity,

$$k = \sum_{\lambda \in \Lambda} \frac{V(\{\lambda - B_R\} \cap \{P+v\})}{V(B(v,R))}$$

Take $R \to 0$ and note that the right hand side approaches $\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda)$.

Going in the other direction note that $\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda)$ is an equivalent notion to $\#(\Lambda \cap \{P+x\}) = k$. Then, invoking the result (2) followed by result (1) we have that P k-tiles $\mathbb{R}^d$ with the multiset λ

4 Gram Relations for Solid Angles

We know that the sum of the angles of a triangle is π radians. We can extend this identity to higher dimensions using solid angles. These are called the Gram relations. First, for each face $\mathcal{F}$ of a polytope $\mathcal{P}$, we define the **solid angle of** $\mathcal{F}$:

Definition 9. Solid Angle of the Face of a Polytope
The solid angle of the face $\mathcal{F}$ of a polytope $\mathcal{P}$ is given by

$$\omega_{\mathcal{F}} = \omega_{\mathcal{P}}(x_0)$$

for any $x_0 \in \mathbb{R}^d$.

Theorem 10. ***Gram Relations***
If P is a d-dimensional polytope in $\mathbb{R}^d$, then

$$\sum_{\mathcal{F}\subset\mathcal{P}} (-1)^{\dim\mathcal{F}} \omega_{\mathcal{F}} = 0.$$

Example 11. In the case of a triangle, we have one face of 2 dimensions (the whole triangle), 3 faces of 1 dimension (the three edges) each with solid angle $\frac{1}{2}$, and 3 faces of 0 dimension (the three vertices). We can call the solid angles of the three vertices θ_1, θ_2, and θ_3. Now,

$$\sum_{\mathcal{F}\subset\mathcal{P}} (-1)^{\dim\mathcal{F}} \omega_{\mathcal{F}} = (-1)^2 \cdot 1 + 3 \cdot (-1) \cdot \frac{1}{2} + (-1)^0 \cdot \theta_1 + (-1)^0 \cdot \theta_2 + (-1)^0 \cdot \theta_3 = 1 - \frac{3}{2} + \theta_1 + \theta_2 + \theta_3$$

By the Gram relations, we have

$$1 - \frac{3}{2} + \theta_1 + \theta_2 + \theta_3 = 0$$

So,

$$\theta_1 + \theta_2 + \theta_3 = \frac{1}{2}$$

References

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[2] Beck M., Robins S.: Computing the continuous discretely: integer-point enumeration in polyhedra, 2'nd edition, Springer, New York, (2015), 1-285

[3] Diaz R., Le Q., Robins S.: Fourier Transforms of Polytopes, Solid Angle Sums, and Discrete Volume