Solutions to *Introduction to Commutative Algebra* by M.F Atiyah and I.G. Macdonald

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Contents

Chapter 1

Exercise 1.1. Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. Suppose that x is nilpotent in A. That is, there is some $n > 0$ such that $x^n = 0$. We can directly compute the inverse element of $(1+x)$.

$$(1+x)(1-x+x^2+\cdots+(-1)^n x^n) = 1+(-1)^n x^{n+1} = 1$$

Thus $(1+x)$ is a unit by definition. Now, to show that the sum of a nilpotent element is a unit, consider a unit u and a nilpotent element x. Then, write

$$(x+u) = u(1+u^{-1}x)$$

Recall that the product of units is a unit, and so our result will follow if we show that $1+u^{-1}x$ is a unit. Now, since $x^n = 0$, we know that $(u^{-1}x)^n = 0$. By the first part of the problem this implies that $1+u^{-1}x$ is a unit, and so we are done. □

Exercise 1.2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1 x + ... + a_n x^n \in A[x]$. Prove that
i) f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, ..., a_n$ are nilpotent in A.
ii) f is nilpotent if and only if $a_0, ..., a_n$ are nilpotent
iii) f is a zero divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.
iv) f is said to be primitive if $(a_0, ..., a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.

Solution. i) First, suppose f is a unit in $A[x]$. That is, there is some $g \in A[x]$ such that $fg = 1$. We can compute that $a_0 b_0 = 1$, and so a_0 is a unit in A. To show that each a_i is nilpotent we must show that $a_i \in p$ for each prime ideal p of A. This is due to a consequence of Zorn's lemma, namely that in a nonzero commutative ring the intersection of all the prime ideals is the set of nilpotent elements in the ring. Now, fix an arbitrary p and consider $a + p \in A/p$. Define a homomorphism $\phi : A[x] \to (A/p)[x]$ by

$$\phi(a_0 + ... + a_n x^n) = (a_0 + p) + ... + (a_n + p)x^n$$

Since $f = a_0 + a_1 x + ... + a_n x^n$ is a unit in A, $\phi(f)$ is a unit in $(A/p)[x]$. Since A/p is an integral domain all units of $(A/p)[x]$ are of degree 0. It follows that $a_i + p$ is zero in A/p for all i which in turn implies that $a_i \in p$ for all i. This proves that a_i is nilpotent for each i.

For the other direction, assume that a_0 is a unit and $a_1, ..., a_n$ are all nilpotent. Using the fact that a_i^m for a large enough m and all $i = 1, 2, ..., n$ we can write

$$(a_1 x + ... + a_n x^n)^{n(m-1)+1} = 0$$

Since a_0 is a unit, and the sum of a unit and a nilpotent element is a unit, it follows that f is a unit.

ii) Suppose that f is nilpotent. Take some $x \in A$ and construct another nilpotent element of $A[x]$: $xf = a_0 x + ... + a_n x^{n+1}$. From the result of Exercise 1.1 we know that $1 + xf$ is a unit in $A[x]$. The fact that $a_0, ..., a_n$ are all nilpotent then follows immediately from part i.

Conversely, if $a_0, ..., a_n$ are all nilpotent, there exists some large $m \in \mathbb{N}$ such that $a_i^m = 0$ for all $i = 1, ..., n$. We can compute the following

$$f^{(n+1)(m-1)+1} = 0$$

Which shows that f is nilpotent.

iii) First, note that the backwards direction follows immediately from the definition of a zero divisor. For the forward direction, begin by supposing that $f = a_0 + ... + a_n x^n$ is a zero divisor. Then there exists some nonzero $g \in A[x]$ such that $fg = 0$. Take $g = b_0 + ... + b_m x^m$ to be of minimal degree, and assume this degree is greater than 0. The minimality of $\deg(g)$ forces $b_m f \neq 0$, and so $b_m a_i \neq 0$ for some i. Then, for that same i, $ga_i \neq 0$. Consider the maximal i which satisfies this relation. We can remove each term with index $j > i$, since in the product fg those terms will zero. In other words, consider the following

$$fg = (a_0 + ... + a_i x^i)(b_0 + ... b_m x^m) = 0$$

Since $b_m a_i = 0$ the polynomial given by the product ga_i is of smaller degree than g. We can check and see that ga_i annhilates f,

$$(ga_i)f = a_i(fg) = 0,$$

which contradicts the minimality of g. It follows that g must be of degree 0, and so f is annhilated by a constant in A. □

Exercise 1.3. Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \cdots, x_r]$ in several indeterminates.

Solution. We will state the results without proof, noting that each follows as a result of induction on r and applications of the corresponding results in Exercise 1.2. □

Exercise 1.4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical

Solution. Let J be the jacobson radical and let N be the nilradical. Note that for any ring R since N is the intersection of prime ideals in R and J is the intersection of maximal ideals in R, N is always contained in J. Now, consider $A[x]$ and some $j \in J$. By Proposition 1.9 in the book, $1 + xj$ is a unit. Then, write

$$1 + xj = 1 + a_0 x + \cdots + a_n x^{n+1}$$

It follows from Exercise 1.2 that each a_i is nilpotent and thus j is nilpotent. This shows that $j \in N$ and thus $N = J$ in $A[x]$. □

Exercise 1.6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution. Recall that the nilradical is always contained in the Jacobson radical. By hypothesis, every ideal not contained in the nilradical contains a non-zero idempotent. The result will follow from showing that the Jacobson radical cannot contain a non-zero idempotent.

We know that x is in the Jacobson radical if and only if $1 + ax$ is invertible for every a. Let e be an idempotent in the Jacobson radical and write

$$(1-e)^2 = 1 - 2e + e^2 = 1 - e$$

Which implies that

$$(1-e)^2(1-e)^{-1} = 1$$

And so $e = 0$. This completes the proof. □

Exercise 1.7. Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution. To show that every prime ideal p is maximal it suffices to show that A/p is a field. We will make use of the fact that A/p is an integral domain by default. Factor out a prime and consider some $x \in A/p$ such that $x \neq 0$. Then, since $x^n = x$, we have $x^n - x = 0$ and thus $x(x^{n-1} - 1) = 0$. Since $x \neq 0$ this implies that $x^{n-1} = 1$. We can now give an inverse element,

$$x(x^{n-2}) = 1$$

and so A/p is a field and all prime ideals are maximal. □

Exercise 1.8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution. Let S be the set of prime ideals. This set is partially ordered with respect to inclusion, and it is nonempty since every maximal ideal is prime. There is at least one maximal ideal since R is taken to be nonzero. Consider $\{C_i\}_{i \in I}$, a chain in S indexed by some totally ordered set I. Then, we want to show that $C = \bigcap_{i \in I} P_i$ is a prime ideal of R. The intersection of ideals is an ideal, and so we must show that it is prime. Consider $ab \in C$. Then, for all i, either $a_i \in C_i$ or $b_i \in C_i$. Choose some i and suppose without loss of generality that $a \notin C_i$. This forces $b \in C_i$. Also, $a \notin C_j$ for all $j \leq i$, and so we can say that $b \notin C_j$ for all $j \leq i$. We must also have $b \in C_j$ for $j > i$, else we would have a contradiction, and so $b \in C$. It follows that C is prime, and so every chain in S is bounded below. We can use Zorn's lemma to conclude that S has minimal elements. □

Exercise 1.12. A local ring contains no idempotent $\neq 0, 1$.

Solution. Consider an idempotent $e \neq 0, 1$. We have that e is in the Jacobian and thus $1 - e$ is invertible. This, however, tells us that $e(1-e) = 0$, which is a contradiction. □

Exercise 1.14. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Solution. We can apply Zorn's lemma to see that Σ has maximal elements. Then, considering some maximal element m, we want to show that m is prime.

If $x \notin m$ and $y \notin m$ then $m \subset m + (x)$ and $m \subset m + (y)$ as ideals. Note that both $m + (x)$ and $m + (y)$ must contain non-zero-divisors, else it would contradict the maximality of m. We can construct another ideal, $(m + (x))(m + (y)) \subseteq m + (xy)$, which must also contain non-zero-divisors. This implies that $m \subset m + (xy)$ and thus $xy \notin m$, which shows that m is prime. $\square$

Exercise 1.15. Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that
i) if a is the ideal generated by E, then $V(E) = V(a) = V(r(a))$
ii) $V(0) = X$, $V(1) = \varnothing$.
iii) if $(E_t)_{i \in I}$ is any family of subsets of A, then

$$V\left(\cup_{i \in I} E_i\right) = \cap_{i \in I} V(E_i)$$

iv) $V(a \cap b) = V(ab) = V(a) \cup V(b)$ for any ideals a, b of A.

Solution. i) Consider a prime ideal p of A and note that

$$E \subseteq p \iff a \subseteq p \iff r(a) \subseteq p$$

ii) For every prime ideal p of A we have $0 \subseteq p$ and so $V(0) = X$. There is no prime ideal of A such that $1 \subseteq p$, and so $V(1) = \varnothing$.

iii) Consider the following chain of implications, which hold for any prime ideal of A and for all i

$$p \in V\left(\bigcup_{i \in I} E_i\right) \iff \bigcup_{i \in I} E_i \subseteq p \iff E_i \subseteq p \iff p \in V(E_i) \iff p \in \bigcap_{i \in I} V(E_i)$$

iv) Consider some $p \in V(a \cap b)$. By definition, $(a \cap b) \subseteq p$. Since $ab \subseteq a \cap b$, this implies that $p \in V(ab)$. By the primality of p either $a \subseteq p$ or $b \subseteq p$, and so $p \in V(a) \cup V(b)$.

If we consider some $p \in V(a) \cup V(b)$ Then $a \subseteq p$ or $b \subseteq p$, which implies that $a \cap b \subseteq p$ and therefore $p \in V(a \cap b)$. $\square$

Exercise 1.17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that
i) $X_f \cap X_g = X_{fg}$
ii) $X_f = \varnothing \iff f$ is nilpotent iii)$X_f = X \iff f$ is a unit iv) $X_f = X_g \iff r((f)) = r((g))$
v) X is quasi-compact (that is, every open covering of X has a finite sub-covering.
vi) More generally, each X_f is quasi-compact.
vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f. The sets X_f are called basic open sets of $X = \mathrm{Spec}(A)$

Solution. First, we show that these sets form a basis of open sets for the Zariski topology. Consider an open set S of X. There exists some $E \subseteq A$ such that $S = X \setminus V(E)$. We can write $V(E) = \cap_{f\in E} V(f)$, and then

$$S = \bigcup_{f\in E} (X \setminus V(f)) = \bigcup_{f\in E} X_f$$

i) Consider a prime ideal p and note the following chain of implications

$$p \in X_f \cap X_g \iff p \in X_f, X_g \iff f, g \notin p \iff fg \notin p$$

ii) We know that X_f is empty precisely when $V(f) = X$, which occurs if and only if $f \in p$ for all prime ideals p. The final condition is equivalent to f being nilpotent.
iii) This follows from iv, which we are about to prove.
iv) Suppose that $X_f = X_g$. Then, $V(f) = V(g)$, and so $r(f) = r(g)$, since $r(f) = \cap_{p\in V(f)} p$ and $r(g) = \cap_{p\in V(g)} p$. The other direction is the same reasoning but backwards.
v) This follows from vi, which we are about to prove.
vi) ♣finish typing this♣ vii) Suppose that an open set S is a finite union of X_f. Each X_f is quasi-compact, and so S must be as well. Now, suppose that an open set S is quasi-compact. Then, since the collection of X_f forms a basis of open sets, there exists a subset E of A such that

$$S = \bigcup_{f\in E} X_f$$

Finiteness follows immediately from the quasi-compactness. □

Exercise 1.18. When thinking about a prime ideal of A as a point of $X = \mathrm{Spec}(A)$ denote it by x, and when thinking of x as a prime ideal of A denote it by p_x. Show that
i) the set $\{x\}$ is closed in $\mathrm{Spec}(A)$ if and only if p_x is maximal
ii) $\overline{\{x\}} = V(p_x)$, where $V(x)$ denotes the set of all prime ideals of A which contain x.
iii) $y \in \overline{\{x\}}$ if and only if $p_x \subseteq p_y$
iv) X is a T_0 space

Solution. i) By part ii), which we will prove in a moment, $\{x\}$ is closed if and only if it equals $V(p_x)$, which happens if and only if $p_x = x$ is the only prime ideal containing p_x, which implies that p_x is maximal. The final implication goes in the other direction as well.

ii) Note that $x = p_x \in V(p_x)$, since we do not require proper containment. This implies that $\overline{\{x\}} \subseteq V(p_x)$. By the definition of closed sets in $\mathrm{Spec}(A)$ we have $\overline{\{x\}} = V(E)$ for some subset E. Then, $p_x = x \in \overline{\{x\}} = V(E)$ implies that $E \subseteq p_x$. It follows that for every $p \in V(p_x)$ we have $E \subseteq p_x \subseteq p$ and so $p \in V(E) = \overline{\{x\}}$. This shows that $\overline{\{x\}} = V(p_x)$.

iii) This follows from part ii and the definition of $V(E)$.

$$y \in \overline{\{x\}} \iff y \in V(p_x) \iff p_x \subseteq p_y$$

iv) Consider two distinct points $x, y \in X$. We either have $p_x \not\subseteq p_y$ or $p_y \not\subseteq p_x$. If we suppose without loss of generality that $p_x \not\subseteq p_y$ then $y \notin \overline{\{x\}}$ by part iii. This means that the neighborhood $X \setminus \overline{\{x\}}$ is a neighborhood of y which does not contain x. □

Exercise 1.19. A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalent if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Solution. For the following argument recall the results of Exercise 1.17 as we will use them freely. Let N denote the nilradical of R.

Suppose that $\mathrm{Spec}(A)$ is irreducible and consider arbitrary elements $a, b \in A$. If $f, g \notin N$ then X_a and X_b are non-empty open sets. This implies that $X_a \cap X_b = X_{ab} \neq \varnothing$, and thus $ab \notin N$. It follows that N is a prime ideal.

For the other direction, suppose that N is a prime ideal p. Consider two nonempty open subsets of $\mathrm{Spec}(A)$, S_1 and S_2. We have a basis of open sets in $\mathrm{Spec}(A)$ given by X_f, and so there exists $f_i \in A$ for $i \in \{1, 2\}$ such that $\varnothing \neq X_{f_i} \subseteq S_i$. It follows that $f_1 \notin N$, $f_2 \notin N$, and $f_1 f_2 \notin N$. Then, $X_{f_1} \cap X_{f_2} = X_{f_1 f_2} \neq \varnothing$, and so $S_1 \cap S_2 \neq \varnothing$. This shows that the space is irreducible. $\square$

Chapter 2

Exercise 2.1. Show that $(\mathbb{Z}/m/Z) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z) = 0$ if and only if $(n, m) = 1$.

Solution. We will show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z) \cong \mathbb{Z}/d/\mathbb{Z})$ where $d = (m, n)$. To do so, we will construct a non-zero bilinear map from $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ to $\mathbb{Z}/d\mathbb{Z}$. Then we will invoke the universal property. Note that $1 \otimes 1$ spans $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z)$, since 1 spans the constituent R-modules. Now we check the order of $1 \otimes 1$.

$$n(1 \otimes 1) = n \otimes 1 = 0 \otimes 1 = 0m(1 \otimes 1) \qquad\qquad = 1 \times m = 1 \otimes 0 = 0$$

And so the additive order of $1 \otimes 1$ divides d. This implies that $|(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z)| \leq d$.

We will now construct a map from $(\mathbb{Z}/m/Z) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z)$ to $(\mathbb{Z}/d\mathbb{Z})$. Consider the map defined by $f(x \mod n, y \mod m) = nm \mod d$. This map is bilinear, and so we can apply the universal property, which says there exists a map $g : \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \to \mathbb{Z}/d\mathbb{Z}$ which makes the appropriate commutative diagram commute. That means that $g(n \otimes m) = nm$ and $g(n \otimes 1) = n$. This shows that g is onto, and so $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z)$ has size atleast d. Combining this with the previous part and we see that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n/Z)$ has size exactly d, which means we're done. □

Exercise 2.2. Let A be a ring, a an ideal, M an $A-$module. Show that $(A/a) \otimes_A M$ is isomorphic to M/aM. Hint: Tensore the exact sequence $0 \to a \to A \to A/a \to 0$ with M.

Solution. Following the hint we tensor the given exact sequence with M. This gives the following exact sequence

$$a \otimes_A M \xrightarrow[i \otimes 1]{A} \otimes_A M \to A/a \otimes_A M \to 0$$

where is the inclusion map. It follows that $A/a \otimes_A M \cong (A \otimes_A M)/\mathrm{Im}(i \otimes 1)$. Consider the unique isomorphism $\phi : A \otimes_A M \to M$ given by $a \otimes x \to ax$. We can then write

$$A/a \otimes_A M \cong (A \otimes_A M)/\mathrm{Im}(i \otimes 1) \cong \phi(A \otimes_A M)/\phi(\mathrm{Im}(i \otimes 1)) = M/\phi(\mathrm{Im}(1 \otimes 1))$$

Then, since $\phi(\mathrm{Im}(i \otimes 1)) = aM$, we have $A/a \otimes_A M \cong M/\phi(\mathrm{Im}(i \otimes 1)) \cong M/aM$. □

Exercise 2.3. Let A be a local ring, M and N finitely generated $A-$modules. Prove that if $M \otimes_A N = 0$ then $M = 0$ or $N = 0$.

Solution. We will follow the hint given in the book. Suppose m is a maximal ideal of A and let $k = A/mA$ be its residue field. Per the result of exercise 2 we have $M_k = k \otimes_A M \cong M/mM$. This implies that $M_k \otimes_A N_k = (M \otimes_A N)_k \otimes_A k = 0$. However, note that M_k and N_k are vector fields over k, and so $M_k \otimes_A N_k = 0$ implies that either $M_k = 0$ or $N_k = 0$. Suppose that $M_k = 0$. Then, since $M/mM = M_k = 0$ implies $M = mM$ we have $M = 0$ by Nakayama's lemma $M = 0$. □

Exercise 2.4. Let M_i $(i \in I)$ be any family of $A-$modules and let M be their direct sum. Prove that M is flat if and only if each M_i is flat.

Solution. Let M, N, P be $A-$modules. We have that $(M \oplus N) \otimes P = (M \otimes P) \oplus (N \otimes P)$. Then, $M = \bigoplus_{i \in I} M_i$ is flat if and only if the exact sequence

$$0 \to M' \to M \to M'' \to 0$$

is preserved after tensoring.

$$0 \to M' \otimes \bigoplus_{i \in I} M_i \to M \otimes \bigoplus_{i \in I} M_i \to M'' \otimes \bigoplus_{i \in I} M_i \to 0$$

Equivalently,

$$0 \to \bigoplus_{i \in I} (M' \otimes M_i) \to \bigoplus_{i \in I} (M \otimes M_i) \to \bigoplus_{i \in I} (M'' \otimes M_i) \to 0$$

The above sequence is exact if and only if each component is exact, which is true if and only if each M_i is flat. □

Exercise 2.5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat $A-$algebra.

Solution. As an $A-$module, A is flat from the canonical isomorphism $M \otimes_A A \cong M$. Now, note that $A[x]$ seen as a module over A is isomorphic to $\bigoplus_{i=0}^{\infty} A$ defined by

$$a_0 + a_1 x + ... + a_n x^n \leftrightarrow (a_0, a_1, ..., a_n, 0, 0, ...)$$

It follows that $A[x]$ is a flat $A-$module by Exercise 4 and thus that $A[x]$ is a flat $A-$algebra. □

Exercise 2.6. For any $A-$module let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + ... + m_r x^r$$

with $m_i \in M$. Defining the product of an elements of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]-$module. Show that $M[x] \cong A[x] \otimes_A M$.

Solution. Again, note that $M[x]$ is a module over $A[x]$ with the normal product of polynomials. Then, since $M = AM = \{am/a \in A, m \in M\}$, $M[x]$ matches up with $(AM)[x]$. Define the homormohpsim $\varphi : A[x] \otimes_A M \to (AM)[x]$ given by $a(x) \otimes_A m \to a(x)m$. This map has the inverse $\varphi^{-1} : a(x)m \to a(x) \otimes_A m$ and so there is an induced isomorphism of tensor products. □

Exercise 2.7. Let p be a prime ideal in A. Show that $p[x]$ is a prime ideal in $A[x]$. If m is a maximal ideal in a, is $m[x]$ a maximal ideal in $A[x]$?

Solution. Let a be an ideal of A. Then, $A[x]/a[x] \cong (A/a)[x]$. Also, recall that $a[x]$ is the kernal of the projection map $\pi : A[x] \to (a/a)[x]$. Suppose that p is a prime ideal of A. Then, $(A/p)[x]$ is an integral domain, and so $A[x]/p[x]$ is an integral domain as well. It follows that $p[x]$ is a prime ideal in $A[x]$. If p is maximal, $p[x]$ is not always maximal in $A[x]$. Consider the case when A is a field. □

Exercise 2.8. i) If M and N are flat $A-$modules, then so is $M \otimes_A N$.
ii) If B is a flat $A-$algebra and N is a flat $B-$module, then N is flat as an $A-$module

Solution. i) If M and N are flat $A-$modules then so is $M \otimes_A N$ since tensoring an exact sequence by M and then N is the same as tensoring by $M \otimes_A N$.

ii) Since B is a flat $A-$algebra we have the following exact sequence of $B-$modules.

$$0 \to N_1 \otimes_A B \to N_2 \otimes_A B \to N_3 \otimes_A B \to 0$$

Also, since N is flat over B, we have another exact sequence

$$0 \to (N_1 \otimes_A B) \otimes_B N \to (N_2 \otimes_A B) \otimes_B N \to (N_3 \otimes_A B) \otimes_B N \to 0$$

Now, recall that $(N_i \otimes_A B) \otimes_B N$ is isomorphic to $N_i \otimes_A (B \otimes_B N)$ for $i = 1, 2, 3$. It follows that the following sequence is exact:

$$0 \to N_1 \otimes_A (B \otimes_B N) \to N_2 \otimes_A (B \otimes_B N) \to N_3 \otimes_A (B \otimes_B N) \to 0$$

It follows that $B \otimes_B N$ is flat as an $A-$module. It follows that N is flat as an $A-$module. $\square$

Exercise 2.9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A-$modules. If M' and M'' are finitely generated, then so is M.

Solution. Suppose that $g_1, ..., g_n$ are generators of M'', which is finitely generated by assumption. Then, consider the cokernal of the map from M to M'' and the exactness of the given sequence. $\square$

Exercise 2.10. Let A be a ring, a an ideal contained in the Jacobson radical of A; let M be an $A-$module and N a finitely generated $A-$module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/aM \to N/aN$ is surjective, then u is surjective.

Solution. Consider $C =$Coker$(u) = N/\text{Im}(u)$ and the exact sequence $M \to N \to C \to 0$. Use the exact functor $(A/a)\otimes_A$ to get the next exact sequence

$$(A/a) \otimes_A M \to (A/a) \otimes_A N \to (A/a) \otimes_A C \to 0$$

Then, by the canonical isomorphism, the following sequence is exact

$$M/aM \to N/aN \to C/aC \to 0$$

The map from $M/aM \to N/aN$ is surjective, and so $C = aC$ and thus $L = 0$ by Nakayama's lemma. This implies that $N = \text{Im}(u)$ and so u is surjective. $\square$

Exercise 2.13. Let $f : A \to B$ be a ring homomorphism, and let N be a $B-$module. Regarding N as an $A-$module by restriction of scalars, form the $B-$module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps $y \to 1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution. ♣Type this♣ $\square$

Chapter 3

Exercise 3.1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated $A-$module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution. First, suppose that $S^{-1}M = 0$. Let $x_1, ..., x_n$ be the generated of M as an $A-$module. Then, $x_i/1 = 0$ in $S^{-1}M$ for each $i = 1, ..., n$. It follows that there exists some $s_i \in S$ such that $s_i x_i = 0 \in M$. Take $s = s_1 s_2, ..., s_n \in S$ and then note that $sx_i = 0$ for all $i = 1, .., n$. It follows that $xM = 0$.

For the other direction, suppose that there exists some $s \in S$ such that $sM = 0$. Then, for any $m/t \in S^{-1}M$, with $m \in M$ and $t \in S$, we have that $m/m = 0/(st) = 0 \in S^{-1}M$. it follows that $S^{-1}M = 0$. □

Exercise 3.2. Let a be an ideal of a ring A and let $S = 1 + a$. Show that $S^{-1}a$ is contained in the Jacobson radical of $S^{-1}A$.

Solution. Consider $a'/s \in S^{-1}a$ with $s \in S$ and $a' \in a$. We want to show that $1 + (r/t)(a'/s)$ is a unit in $S^{-1}A$. We can easily find the inverse $(ts)/(ts + ra) \in S^{-1}A$, and so it is a unit and we are done. □

Exercise 3.5. Let A be a ring. Suppose that, for each prime ideal p, the local ring A_p has no nilpotent elements $\neq 0$. Show that A has no nilpotent element $\neq 0$. IF each A_p is an integral domain is A necessarily an integral domain.

Solution. Suppose that A has no non-zero nilpotent elements. That is, the nilradical of A_p is 0. This implies that the nilradical of A is 0

Let $A = k \times k$ where k is a field. There are two prime ideals, $p = 0 \times k$ and $q = k \times 0$. It can be checked that A_p and A_q are integral domains but A is not. □

Exercise 3.7. A multiplicatively closed subset S of a ring A is said to be saturated if

$$xy \in S \iff x \in S \text{ and } y \in S$$

Prove that
i) S is saturated if and only if $A \setminus S$ is a union of prime ideals
ii) If S is any multiplicatively closed subset of A, there is a unique smallest multiplicatively closed subset $\overline{S}$ containing S, and that $\overline{S}$ is the complement in A of the union of the prime ideals which do not meet S. Note that $\overline{S}$ is called the saturation of S.
iii) If $S = 1 + a$, where a is an ideal of A, find $\overline{S}$.

Solution. i) Suppose that $A \setminus S$ is a union of prime ideals and that S is not saturated. Then, there exists $xy \notin S$ with $x \notin S$. Then, $x \in A \setminus S$, a union of prime ideals, and so $x \in p \subseteq A \setminus S$. This is a contradiction, since then $xy \in p$, and thus $xy \in A \setminus S$.

For the other direction, suppose that S is saturated. Consider $x \in A \setminus S$. Consider (x) and note that this ideal does not intersect S. The set of ideals containing (x) which do not intersect S

has a maximal element, and this maximal element must be prime. Taking the union of these prime ideals for each choice of x gives the complement of S.

ii) Consider the complement of the union of the prime ideals not meeting S, $\overline{S}$. By part i, $\overline{S}$ is saturated. We will show that this set is minimal among saturated sets containing S. Consider another such set, $\tilde{S}$, which is contained in $\overline{S}$. The complement is $\tilde{S}$ is a union of prime ideals not intersecting $\tilde{S}$, however these ideals also do not intersect S, and so $\tilde{S} = \overline{S}$. This minimal set is unique since if we were to suppose that two different minimal sets existed the union of their complement would be a union of prime ideals. Then, their intersection would be saturated, which would then be contained in both of them and be strictly smaller.

iii) Let a and p be prime ideals of A. A prime p meets S precisely when $v \in a$ and $x \in p$ such that $x = 1 + v$ or $1 = v - x$, so that $(1) = a + p$. So, we can see that the union in $A \setminus \overline{S}$ is over all prime ideals not coprime to a. For each such p there is a maximal ideal $m \subseteq a + p$. Then, since maximal ideals are prime and prime ideals are contained in maximal ideals, we can take the following union of maximal ideals

$$\overline{S} = A \setminus \bigcup_{m \in \mathrm{Max}(A): a \subseteq m} m$$

□

Exercise 3.13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent.
i) M is torsion free.
ii) M_p is torsion free for all prime ideals p.
iii) M_m is torsion free for all maximal ideals m.

Solution. To see that (i) implies (ii) let p be a prime ideal of A and let $x/s \in T(M_p)$. Then, there exists $a/t \neq 0 \in A_p$ such that $0 = (ax)/(ts) \in M_p$. In turn this tells us that there exists some $r \in A \setminus p$ such that $0 = (ra)x$. This implies that $x = 0$ and thus M is torsion free. It follows that $x/s = 0$, and so M_p is torsion free. The implication of $(ii) \to (iii)$ is straightforward. For $(iii) \to (i)$, begin by supposing that M is not torsion free. Then, selects $x \neq 0 \in T(M)$ so that $0 \subset \mathrm{Ann}(x) \subset (1)$. There exists $a \neq 0 \in A$ and a maximal ideal m in A such that $ax = 0$ and $\mathrm{Ann}(x) \subset m$. This implies that $x/1 \neq 0$ in M_m. Also, $(a/1)(x/1) = 0/1 = 0$ in M_m. We can see that $x/1$ is a torsion element in M_m, and so we have a contradiction. □

Exercise 3.14. Let M be an $A-$module and a an ideal of A. Suppose that $M_m = 0$ for all maximal ideals $a \subseteq m$. Prove that $M = aM$.

Solution. Note that M/aMW is naturally an A/a-module. We know that $(M/aM)_m = 0$ for all maximal ideals m of A/a. Using proposition 3.8 we see that $m/aM = 0$ or $M = aM$ □

Exercise 3.15. Let A be a ring, and let F be the $A-$module A^n. Show that every set of n generators of F is a basis of F

Solution. The hint given in the book is essentially the entire proof. □

Chapter 4

Exercise 4.1. if an ideal a has a primary decomposition then spec(A/a) has finitely many irreducible components.

Solution. We can write the primary decomposition as $a = \cap_{i=1}^{n} q_i$. The irreducible components of spec(A/a) are the sets $V(p)$ where p is a minimal prime ideal of A/a. There is a finite number of minimal primes over a. □

Exercise 4.2. If $a = r(a)$, then a has no embedded prime ideals.

Solution. I believe this exercise needs the added assumption that a is decomposable, though perhaps I am missing something. It does not seem obvious that radical ideals are always decomposable (for instance, the zero ideal), but embedded primes are defined only for decomposable ideals. □

Exercise 4.3. If A is absolutely flat, every primary ideal is maximal.

Solution. Let q be a primary ideal in A. Fix some $x \in A \setminus p$ and then note that $\overline{x} \neq 0$ in A/p. Since A is absolutely flat we know that every principal ideal is idempotent. That is, there exists $a \in A$ such that $x(ax - 1) = 0 \in p$. It follows that $\overline{x}(\overline{ax} - \overline{1}) = \overline{0}$ and we see that $\overline{ax} - 1$ is nilpotent in A/p. Then, by Exercise 1.1, $\overline{ax}$ is a unit and so is $\overline{x}$. This means that A/p is a field and thus p is maximal. □

Exercise 4.6. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X. Is the zero ideal decomposable in this ring?

Solution. No, the zero ideal is not decomposable. We will give a proof by contradiction. Consider the prime ideals belonging to the zero ideal, $P_1, ..., P_n$. Each of these is contained in a unique maximal ideal $P_i \subseteq M_i$. We can write $M_i = \{f \in C(X) : f(p_i) = 0\}$ for $p_i \in X$ and $i = 1, ..., n$. Now, since X is an infinite space, we can find $p \in x$ such that $p \notin \{p_1, ..., p_n\}$. Then, there exists some Urysohn function such that $f(p) = 0$ and $f(p_i) = 1$. This f is zero divisor, but it is not contained in M_i, which is a contradiction. □

Exercise 4.16. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Solution. Every proper ideal of $S^{-1}A$ can be seen as an extended ideal $S^{-1}a$ for a in A with $S \cap a = \varnothing$. Consider some proper ideal $S^{-1}a$ and write the decomposition of a as $\cap q_i$. Then,

$$S^{-1}a = S^{-1}(\cap q_i) = \cap S^{-1}q_i$$

We can then clearly see that $\cap S^{-1}q_i$ is a primary decomposition of $S^{-1}a$. □

Chapter 5

Exercise 5.2. Let A be a subring of a ring B such that B is integral over A, and let $f : A \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that f can be extended to a homomorphism of B into Ω.

Solution. Consider the ideal $p = \mathrm{Ker}(f)$ in A. This ideal is prime, and from the universal property f factors as

$$A \to A/p \to \omega \to \Omega$$

where ω is the field of fractions of A/p. From Lying-Over there exists a prime $q \subset B$ such that $q \cap A = p$. Note now that B/q is integral over A/p. The natural inclusion map $A/p \to B/q$ gives an inclusion from $\omega \to \psi$ where ψ is the field of fractions of B/q. It follows that ψ/ω is an algebraic extension. We can lift the map $\omega \to \Omega$ to $\psi \to \Omega$, and we have the map $B \to B/q \to \psi$, and so we have a lift of f. □

Exercise 5.3. Let $f : B \to B'$ be a homomorphism of $A-$algebras, and let C be an $A-$algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral.

Solution. Consider $b' \otimes c \in B' \otimes_A C$. Since f is integral there exists $b_0, ..., b_{n-1} \in B$ such that

$$(b')^n + f(b_{n-1}(b')^{n-1} + \cdots + f(b_0) = 0$$

Now, $b' \otimes c$ satisfies this polynomial, and every coefficient is in $f \otimes_A 1_C(B \otimes C)$, and so $f \otimes_A 1_C$ is integral. □

Exercise 5.4. Let A be a subring of a ring B such that B is integral over A. Let n be a maximal ideal of B and let $m = n \cap A$ be the corresponding maximal ideal of A. Is B_n necessarily integral over A_m.

Solution. The hint given in the book constitutes a counter example. We can consider the subring $k[x^2 - 1]$ of $k[x]$, where k is a field, and let $n = (x - 1)$. We will see that the element $1/(x + 1)$ cannot be integral. Now, note that $n^c = (x^2 - 1)$. Since we have integrability we can write $\sum_{i=0}^{n} a_i s_i (x + 1)^{n-1} = 0$ with $a_i \in A$, $a_n = 1$, and $s_i \in A \setminus n^c$. Then, $x + 1 | s_n$, and so $s_n \in (x + 1)B \cap A = (x^2 - 1)$, which is a contradiction. □

Exercise 5.5. Let $A \subseteq B$ be rings, B integral over A
i) If $x \in A$ is a unit in B then it is a unit in A.
ii) The Jacobson radical of A is the contraction of the Jacobson radical of B.

Solution. i) Consider $x, x^{-1} \in B$, $xx^{-1} = 1$. Since B is integral over A we know that x^{-1} satisfies the polynomial

$$y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0 = 0$$

for $a_0, ..., a_{n-1} \in A$. Multiplying through by x^{n-1},

$$y + a_{n-1} + \cdots + a_1 x^{n-2} + a_0 x^{n-1} = 0$$

Which shows that $y \in A$.
ii) This follows immediately from Lying-Over and the definitions. □

Exercise 5.7. Let A be a subring of a ring B such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Solution. Let $b \in B$ be inetgral over A. Then it satisfies some polynomial

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$

with $a_0, ..., a_n \in A$. Then,

$$-a_0 = b(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1)$$

Since $B \setminus A$ is closed under multiplication one of the factors in the above is in A. If $b \in A$ we are done, and if $(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1)$ is in A we can continue this factoring process and eventually we will get $b \in A$. $\square$

Chapter 6

Exercise 6.1. i) Let M be a Noetherian $A-$module and $u : M \to M$ a module homomorphism. If u is surjective, then u is an isomorphism.
ii) If M is Artinian and u is injective, then again u is an isomorphism.

Solution. i) Since the kernal of an $R-$module homomorphism is a submodule we can write the following ascending chain

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f^2) \subset \mathrm{Ker}(f^3) \subset ...$$

Since M is Noetherian by assumption this chain becomes constant, and so for some $n \in N$

$$\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{n+1}) = ...$$

We want to show that f is injective, and so consider $x \in \mathrm{Ker}(f^n) \cap \mathrm{Im}(f^n)$. For $f^n(x) = 0$ there exists $y \in M$ such that $x = f^n(y)$, and so $f^{2n}(y) = 0$, which shows that $y \in \mathrm{Ker}(f^{2n})$. Since $f^n = f^{2n}$ we have that x is zero, and so $\mathrm{Ker}(f^n) \cap \mathrm{Im} f^n = \{0\}$. It now follows from the surjectivity of f that f is injective, as there is a trivial kernel, and so f is an isomorphism.

ii) This exercise is similar, except we will argue starting from the descending chain of submodules

$$\mathrm{Im}(f) \supset \mathrm{Im}(f^2) \supset ...$$

□

Exercise 6.2. Let M be an $A-$module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Solution. Suppose that M is not Noetherian and that there exists a submodule M' of M which is not finitely generated. Then, suppose that the set of finitely generated submodules of M' has a maximal element N. Now, $N \subset M'$, and N is finitely generated, which means there exists some $m \in M'$ such that $m \notin N$. Then, $N' = N + Am$ is a finitely generated submodule of M' which contains N, and thus is a contradiction. □

Exercise 6.3. Let M be an $A-$module and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Solution. Define the map $f : M \to M/N_1 \oplus N/N_2$ by $f(x) = (x + N_1, x + N_2)$ for $x \in M$. This map is a linear $A-$homomorphism and so there is an induced injective $A-$homomorphism $f' : M/\mathrm{Ker}(f) \to M/N_1 \oplus M/N_2$. By hypothesis, M/N_1 and M/N_2 are Noetherian, and so $M/N_1 \oplus M/N_2$ and $M/\mathrm{Ker}(f)$ are as well. We can see that $\mathrm{Ker}(f) = N_1 \cap N_2$ since $f(x) = (0, 0) \in M/N_1 \oplus M/N_2$ if and only if $x \in N_1$ and $x \in N_2$. This argument goes through in exactly the same way if we replace Noetherian with Artinian. □

Exercise 6.4. Let M be a Noetherian $A-$module and let a be the annihilator of M in A. Prove that A/a is a Noetherian ring.

Solution. Let $x_1, ..., x_n$ be the generators of M. Define the map from A to the $n-$fold tensor $f : A \to M \oplus \cdots \oplus M$ by $f(a) = (yx_1, ..., yx_n)$ for $y \in A$.This map is $A-$linear, and since M is Noetherian so is M^n, and so $A/\text{Ker}(f)$ is as well. Our goal is now to show that $\text{Ker}(f) = a$. We can see that $f(a) = (0, ..., 0) \in M^n$ if and only if $aM = 0$, and we're done.

Now, if we were to replace Noetherian with Artinian, the claim would not be true. Consider the $p-$primary sybgroup of Q/Z, an Artinian $Z-$module with 0 annihilator. Note that $Z = Z/\{0\}$ is not Artinian. $\square$

Exercise 6.5. A topological space is said to be Noetherian if the open subsets of X satisfy the ascending chain condition. Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition. Show that, if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

Solution. Consider a subspace $Y \subseteq X$. The open sets of Y are given by intersecting Y with the open sets of X. Consider a chain of open subsets of Y, given by $\{\mathrm{S}_{i \in I}\} and write \mathrm{S}_i = Y \cap O_i$ where O_i is open in X. There exists some n such that $O_n = O_{n+1}$, and so the chain must stabilize and Y must be Noetherian.

Now, take an open cover U of X and consider the set S of finite unions of elements in U. The set S has a maximal element V. If there exists an element in $X \setminus V$ then we can find a $u \in U$ such that $u \cup V$ contains V, which contradicts maximality, and so X is comapct. $\square$

Exercise 6.6. Prove that the following are equivalent: i) X is Noetherian
ii) Every open subspace of X is quasi-compact
iii) Every subspace of X is quasi-compact

Solution. First, we can see that (i) implies (iii) straight from Exercise 6.5: each subspace is Noetherian, and each Noetherian space is compact. Also, (iii) implies (ii) since each open subspace is a subspace. To see that (ii) implies (iii) consider an ascending chain of open subsets of X, $U_1 \subseteq U_2 \subseteq \cdots$. Let $U = \bigcup_{n \in N} U_n$ and note that since U is compact it is a union of a finite set $\{U_{n_1}, ..., U_{n_m}\}$. If we choose $n = \max(n_j)$ we see that $U = U_n$, and we're done. $\square$

Exercise 6.7. A Noetherian space is a finite union of irreducible closed subspaces. Hence the set of irreducible components of a Noetherian space is finite.

Solution. ♣type this set of exercises♣ $\square$

Chapter 7

Exercise 7.1. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime ideals.

Solution. This solution will make use of Zorn's lemma. Consider a subset X of Σ, ordered by containment. Consider $y = \sup_{x \in X} x$ and note that y is an ideal. Now, y cannot be finitely generated, and so it is an upper bound, which means Σ has a maximal element.

Suppose that there exists a maximal element of Σ which is not prime. That is, there exists $x, y \in A \setminus a$ such that $xy \in a$. We know that $a + (x)$ is finitely generated, say with generators $(g_1, ..., g_n)$. Let $g_i = a_i + b_i x$ with $a_i \in a$ and $b_i \in A$. Let $a_0 = (a_1, ..., a_n)$ and note that this forces $a_0 + (x) = a + (x)$. ♣finish typing this♣ □

Exercise 7.2. Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Solution. First, suppose that f is nilpotent. Let N be the nilradical of A. There exists some $n \in N$ such that $N^n = 0$. Then, we can see that $f^k = 0 \in A[[x]]$ when each $a_n \in N$. The other direction, which is more involved, is very similar to the proof in Exercise 1.2. □

Exercise 7.8. If $A[x]$ is Noetherian, is A necessarily Noetherian?

Solution. Yes, A is Noetherian if and only if $A[x]$ is Noetherian. This follows from the Hilbert Basis Theorem and that the quotient ring of a Noetherian ring is Noetherian. □

Exercise 7.11. Let A be a ring such that each local ring A_p is Noetherian. Is A necessarily Noetherian?

Solution. This is not true. Consider an infinite product of copies of $\mathbb{Z}/2\mathbb{Z}$. This ring is non-Noetherian but locally Noetherian everywhere □

Exercise 7.12. Let A be a ring and B a faithfully flat $A-$algebra. If B is Noetherian, show that A is Noetherian.

Solution. Since we have a faithfully flat map $A \to B$, we can construct in ascending chain of ideals in A, extend it to B, which will stabilize by hypothesis, and then move it back to A, which will show that the original chain must stabilize as well. □

Chapter 8

Exercise 8.2. Let A be a Noetherian ring. Prove that the following are equivalent:
i) A is Artiniam;
ii) $\mathrm{Spec}(A)$ is discrete and finite;
iii) $\mathrm{Spec}(A)$ is discrete.

Solution. To show that $(i) \to (ii)$, note that if A is Artinian then every $p \in \mathrm{Spec}(A)$ is maximal, and thus closed under the topology. There are also finitely many maximal ideals, and so $\mathrm{Spec}(A)$ is finite and discrete.

Now, $(ii) \to (iii)$ is clear, and to see that $(iii) \to (i)$ note that $\mathrm{Spec}(A)$ is discrete, every prime ideal is maximal. This implies that the Krull dimension of A is 0, which, in combination with the fact that A is Noetherian, shows that A is Artinian. $\square$

Exercise L. et k be a field and A a finitely generated $k-$algebra. Prove that A is Artinian if and only if A is a finite k-algebra

Solution. We will suppose that A is Artinian and follow the hint in the book. By the classification theorem, we know that A is isomorphic to the product of local Artinian rings,

$$A \cong \prod_{i=1}^{n} A_i$$

Assume that A is a local Artinian ring with a maximal ideal $m \subset A$. Then, from Hilbert's Nullstellensatz, A/m is a finite exntesion of k. Also, $m^n = 0$ for some n, and we can write $A/m \otimes_A m^i \cong m^i/m^{i+1}$ for $1 \leq i \leq n$. We know that $A/m \otimes_A m^i$ is finitely generated as a $A/m-$vector space and thus as a $k-$vector space. The following sequence is exact,

$$m^i/m^{i+1} \to A/m^{i+1} \to A/m^i$$

It is clear that A/m^2 is finite dimensional as a $k-$vector space, and so we can see from induction on n that $A/m^n = A$ is a finite dimensional as a $k-$vector space.

For the other direction, the hint in the book is essentially all there is to it. If we suppose that A is a finite $k-$algebra then we know it is finitely generated $k-$module, and so it satisfies the decending chain condition on $k-$submodules. Every $A-$module can be seen as a $k-$module, and so A satisfies the descending chain condition on ideals, which means it is an Artinian ring. $\square$

Exercise L. et $f : A \to B$ be a ring homomorphism of finite type. Consider the following statements:
i) f is finite
ii) the fibres of f^* are discrete subspaces of $\mathrm{Spec}(B)$
iii) for each prime ideal p of A, the ring $B \otimes_A k(p)$ is a finite $k(p)-$algebra
iv) the fibres of f^* are finite
Prove that $i) \to ii)) \iff iii) \to iv)$. If f is integral and the fibres of f^* are finite, is f finite?

Solution. For the final part, note that if we assume f is of finite type, then it is true, from a remark on page 60 of the book.

If we do not have that assumption, then it is not true. Consider $\operatorname{Spec}(\overline{\mathbb{Q}}) \to \operatorname{Spec}(\mathbb{Q})$ for a counter example: it is integral with finite fibers but it is not finite. $\square$

Chapter 9

Exercise A. valuation ring (other than a field) is Noetherian if and only if it is a discrete valuation ring.

Solution. Suppose that R is a discrete valuation ring. Every DVR is Euclidean, and every Euclidean domain is a PID, and every PID is Noetherian.

For the other direction, suppose that R is a Noetherian valuation ring. We know that then R is local and an integral domain. Consider an ideal r of R. This ideal must be finitely generated since R is Noetherian, say by $(g_1, ..., g_n)$, and these ideals will be totally ordered. This implies that r is principal. Now, since R is a Noetherian PID, every fraction ideal, $x^{-1}(y)$, automatically has an inverse $y^{-1}(x)$, with $x, y \in R$. This is enough to prove the claim. $\square$

Exercise 9.4. Let A be a local domain which is not a field and in which the maximal ideal m is principal and $\bigcap_{n=1}^{\infty} m^n = 0$. Prove that A is a discrete valuation ring.

Solution. There exists some $x \in m$ such that $m = (x)$, and our goal is to show that every non-zero ideal a of A is of the form (x^k) for some $k \geq 0$. There exists some $k \geq 0$ such that $(x^{k+1}) \not\supseteq a \subseteq (x^k)$. Now, $(a : x^k)$ is either contained in $m = (x)$ or is the unit ideal. If $(a : x^k) \subseteq (x)$ then $a = (a : x^k)(x^k) \subseteq (x^{k+1})$, a contradiction. This forces $(a : x^k) = (1)$, and so $a = (x^k)$. $\square$

Exercise 9.5. Let M be a finitely-generated module over a Dedekind domain. Prove that M is flat if and only if M is torsion-free.

Solution. Let D be a Dedekind domain. Following the hint, we use Exercise 3.13 to see that since D is an integral domain, M is torsion-free if M_p is torsion free over the discrete valuation ring A_p for each prime p in A. Each M_p is finitely generated over a PID and thus can be written as a direct sum of the free module and torsion submodule. That is, each M_p is torsion free if and only if it is free. Now, we use Exercise 7.16 to see that since D is Noetherian and M is finitely generated, M is flat if each of the M_p is free. The result follows: M is flat if and only if it is torsion free. $\square$

Exercise 9.9. Let $a_1, ..., a_n$ be ideals and let $x_1, ..., x_n$ be elements in a Dedekind domain A. Then the system of congruences $x = x_i \mod a_i$ with $1 \leq i \leq n$ has a solution $x \in A$ if and only if $x_i = x_j \mod a_i + a_j$ whenever $i \neq j$.

Solution. We will follow the hint in the book. Define the map $f : A \to \bigoplus_{i=1}^{n} A/a_i$ by $f(x)_i = x + a_i$ and the map $g : \bigoplus_{i=1}^{n} A/a_i \to \bigoplus_{i<j} A/(a_i + a_j)$ by $g(\langle x_i + a_i \rangle)_{\langle i,j \rangle} = x_i - x_j + a_i + a_j$. The claim is equivalent to showing the following sequence is exact

$$A \to \bigoplus A/a_i \to \bigoplus A/(a_i + a_j)$$

That is, we want to show that $\mathrm{Im}(f) = \mathrm{Ker}(g)$. We can show this is true by reducing it to the case of localizing at each prime p, which further reduces to showing it for ideals $a_i = p^{k_i}$ of a DVR A. Assume that $k_i \leq k_j$ for $i < j$ and then note that if $\langle x_i + p^{k_i} \rangle \in \mathrm{Ker}(g)$ we have $x_i - x_j \in p^{k_i} + p^{k_j} = p^{k_i}$. Then, $f(x_n) = \langle x_i + p^{k_i} \rangle$. That is, $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. We clearly see that $g \circ f = 0$, and so we are done. $\square$

Chapter 10

Exercise 10.3. Let A be a Noetherian ring, a an ideal and M a finitely generated $A-$module. Using Krull's Theorem and Exercise 14 of Chapter 3, prove that

$$\bigcap_{n=1}^{\infty} a^n M = \bigcap_{a \subseteq m} \operatorname{Ker}(M \to M_m)$$

where m runs over all maximal ideals containing a. Deduce that

$$\hat{M} = 0 \iff \operatorname{Supp}(M) \cap V(a) = \varnothing$$

in $\operatorname{Spec}(A)$.

Solution. By Krull's Theorem we know that the submodule $E = \bigcap_{n=1}^{\infty} a^n M$ is annihilated by an element $1 + a'$ with $a' \in a$. If $a \subseteq m$, $1 + a'$ is a unit in A_m, and so $E_m = 0$ for all maximal ideals containing a. It follows that we have containment in one direction, $E \subseteq \bigcap_{a \subseteq m} \operatorname{Ker}(M \to M_m)$. For the other direction, let $K = \bigcap_{a \subseteq m} \operatorname{Ker}(M \to M_m)$. We know that $K_m = 0$ for each maximal ideal containing a, and so $K = aK$. This implies containment in the other direction.

For the other part of the exercise, use Nakamura's to see that $\hat{M} = 0$ if and only if $\hat{M} = \hat{a}\hat{M}$. This is true if and only if $M = aM$, which is true if and only if $M = \bigcap_{n=1}^{\infty} a^n M$. From the above proof, that is the same as saying that $M_m = 0$ for all maximal ideals m containing a. That is, if and only if $\operatorname{Supp}(M) \cap V(a)$ contains no maximal ideal, which is true if and only if the intersection is empty. □

Exercise 10.4. Let A be a Noetherian ring, a an ideal in A, and $\hat{A}$ the $a - adic$ completion. For any $x \in A$, let $\hat{x}$ be the image of x in $\hat{A}$. Show that x not being a zero divisor in A implies that $\hat{x}$ is not a zero divisor in $\hat{A}$. Does this imply that if A is an integral domain then $\hat{A}$ is an integral domain?

Solution. By assumption the following sequence is exact

$$0 \to A \xrightarrow{x} A$$

Since $\hat{A}$ is flat over A we have another exact sequence

$$0 \to \hat{A} \xrightarrow{x} \hat{A}$$

Which says that x is not a zero divisor over $\hat{A}$.

It is not true that the completion of an integral domain is an integral domain. There is a counterexample on page 187 of Eisenbud's book on Commutative Algebra. □

Exercise 10.6. Let A be a Noetherian ring and a an ideal in A. Prove that a is contained in the Jacobson radical of A if and only if every maximal ideal of A is closed for the $a-$topology.

Solution. First, suppose that a is contained in the Jacobson radical of A. Let m be a maximal ideal in A. This means we have $a \subseteq m$. Consider some $x \notin m$ and note that $(x+a) \cap m = \varnothing$. This means that x is not in the closure of m, since $x + a$ is open, and so m closed.

For the other direction, suppose that a is not contained in the Jacobson radical of A. There exists a maximal ideal m of A such that $a \not\subseteq m$. It follows that $a^n \not\subseteq m$ for all n, and so $a^n + m = (1)$. That is, there exists $a_m \in a$ and $m_n \in m$ such that $a_n + m_n = 1$, which implies that $(1 + a^n) \cap m \neq \varnothing$ for all n. It follows that m is not closed. $\square$

Exercise i.) With the notation of Exercise 9, deduce from Hensel's lemma that if $\overline{f}(x)$ has a simple root $\alpha \in A/m$, then $f(x)$ has a simple root $a \in A$ such that $\alpha = a \mod m$
ii) Show that 2 is a square in the ring of 7−adic integers
iii) Let $f(x, y) \in k[x, y]$, where k is a field, and assume that $f(0, y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x, y(x)) = 0$.

Solution. i) We can write $\overline{f} = (x - a)\overline{g}$ where $\overline{g}$ is coprime to $x - \alpha$. Since $\overline{f}$ is monic, so is $\overline{g}$. We can lift $x - \alpha$ and $\overline{g}$ to $A[x]$. Call these lifts ϕ and ψ. Then write $\phi = x - a$ for some $a \in A$ and send $a \to \alpha$ with the reduction map. Now, this forces a to be a simple root of f, otherwise $(x - \alpha)^2$ would devide $\overline{f}$.
ii) Note that $x^2 - a$ has simple roots (± 3) in $\mathbb{Z}_7/7\mathbb{Z}_7$. This means that 2 has two square roots in $\mathbb{Z}_7$.
iii) We have $k[[x]]/(x) = k$. Consider $f(x, y)$ and note that the image of f in $k[y]$ is $f(0, y)$, which has a simple root in k. Then $f(x, y)$ has a simple root $y(x)$ in $k[[x]]$. $\square$

Exercise 10.12. If A is Noetherian, then $A[[x_1, ..., x_n]]$ is a faithfuly flat A−algebra.

Solution. We know that $A \to A[x_1, .., x_n]$ is flat, and that $A[x_1, .., x_n] \to A[[x_1, ..., x_n]]$ is flat, and so $A \to A[[x_1, ..., x_n]]$ is flat. Consider some ideal a in A and note that $a^e = a + a \cdot (x_1, ..., x_n)$. It follows that $A[[x_1, ..., x_n]]$ is faithfully flat over A. $\square$

Chapter 11