Artificial Intelligence

Week 4

- First Order Logic
- Syntax and semantic
- Quantifiers
- Inference First Order Logic
- Conversion to CNF

LO 3: Apply various knowledge representations for reasoning purpose

Rosen, K. H. (2012). Discrete mathematics and its applications (7th ed). McGraw-Hill.
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https://www.cs.jhu.edu/~phi/ai/slides/lecture-first-order-logic.pdf

First Order Logic (FOL)

- **Some kind of logic**

Language	Ontological	Epistemological
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	Degree of belief ϵ [0, 1]
Fuzzy logic	Degree of truth ϵ [0, 1]	known interval value

**the truth of what can be said about a sentence*

- **Pros and Cons of Propositional Logic**
  - Propositional logic is **declarative**: *pieces of syntax correspond to facts*
  - Propositional logic is **compositional**: *meaning of* $B_{1,1} \wedge P_{1,2}$ *is derived from meaning of* $B_{1,1}$ *and of* $P_{1,2}$
  - Meaning in propositional logic is **context-independent**
  - Propositional logic **has very limited** expressive power
    E.g., cannot say "**pits cause breezes in adjacent squares**" except by writing one sentence for each square

- **Introduction to First-Order Logic**
  Whereas **propositional logic** assumes the world contains **facts**,
  first-order logic (like **natural language**) assumes the world contains
  - **Objects**: people, houses, numbers, colors, baseball games, wars, …
  - **Relations**: red, round, prime, brother of, bigger than, part of, …
  - **Functions**: father of, best friend, one more than, plus, …

- **English natural language, Propositional Logic, First-Order Logic**
  - English: **Squares adjacent to pits are breezy**
  - Propositional Logic: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$, $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$, etc
    We should define all possible facts that satisfy the "English"
  - First-Order Logic (FOL): $\forall s\ Breezy(s) \Leftrightarrow \exists r\ Adjacent(r,s) \wedge Pit(r)$
    We can satisfy the "English" with only a sentence

Standard Logic Symbols

$\forall$ = **For all**
[e.g : every one, every body, any time, etc]

$\exists$ = **There exists**
[e.g : some one, some time, etc]

$\Rightarrow$ = **Implication**
[if … then ….]

$\Leftrightarrow$ = **Equivalent; biconditional**
[if … and … only … if …]

$\neg$ = **Not ; negation**

$\vee$ = **OR ; disjunction**

$\wedge$ = **AND ; conjunction**

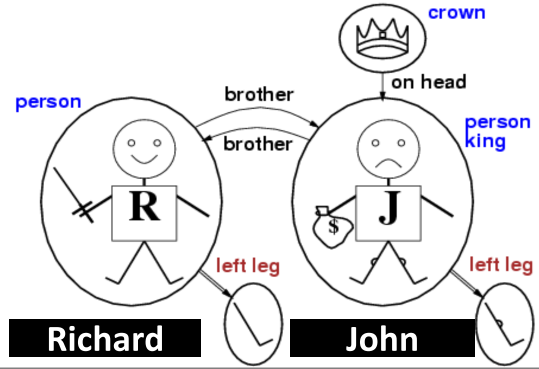
Syntax and Semantic of First-Order Logic

- **Basic elements**
  - Constants : KingJohn, 2, Binus, … (Objects)
  - Predicates : Brother, >, loves, … (Relations)
  - Functions : Sqrt, LeftLegOf, … (Functions)
  - Variables : x, y, a, b,…
  - Connectives : ¬, ⇒, ∧, ∨, ⇔
  - Equality : =
  - Quantifiers : ∀, ∃

- **Term** = function($term_1$,…,$term_n$) or constant or variable
  - A logical expression that refers to an object
  - "Richard the Lionheart is the brother of King John".
    e.g., *Brother(Richard, John)* ➔ A term
  - "King John's left leg"
    e.g., LeftLeg(John) ➔ A term

- **Atomic sentence** formed from a predicate symbol optionally followed by parenthesized list of terms
  e.g.: *Brother(Richard, John)*
  - **Complex terms** as arguments in atomic sentences is :
    married (Father(Richard), Mother(John))
  - An atomic sentence is true in a given model.
    if the relation referred to by the predicate symbol holds among the objects referred to by the arguments.

- **Complex sentence**
  are made from atomic sentences using connectives
  Examples:
  ¬ Brother(LeftLeg(Richard), John)
  Brother(Richard, John) ∧ Brother(John, Richard)
  King(Richard) ∨ King(John)
  ¬ King(Richard) ⇒ King(John)



Quantifiers

First-order logic contains two standard quantifiers, called ***universal*** quantifiers and ***exixtential*** quantifiers.

- **Universal quantifiers ∀**

Attention! Don't use ∧ for ∀

Sentence: **All Kings are persons**

Variable x = {Richard, King John, the crown}

FOL: *∀x King(x) ⇒ Person(x)* ~~*∀x King(x) ∧ Person(x)*~~

Richard is a King → Richard is a person

King John is a King → King John is a person

The crown is a King → the crown is a person

- **Existential quantifiers ∃**

Attention! Don't use ⇒ for ∃

∃x : "There exists an x such that …" ***or*** *"For some x …"*

*∃x P says that P is true in **at least one***

extended interpretation that assigns x to a domain element

Sentence: **The King John has a crown on his head**

Variable x = {Richard, King John, the crown}

FOL: *∃x Crown(x) ∧ OnHead(x, John)*

*The crown is a Crown ∧ the crown is on John's head **(True)***

then the sentence is true, at least one

Nested quantifiers

- Sentence: **Brothers are siblings**
  $\forall x\ \forall y$ Brother(x, y) $\Rightarrow$ Sibling(x, y)
- Consecutive quantifiers of the same type
  can be written as one quantifier with several variables
- To say that **siblinghood** is a symmetric relationship:
  $\forall x,y$ Sibling(x,y) $\Leftrightarrow$ Sibling(y,x)
- A **mixture**:
  "**Everybody loves somebody**":
  $\forall x\ \exists y$ Loves(x, y)
  "**There is someone who is loved by everyone**" :
  $\forall y\ \exists x$ Loves(x, y)

$\forall x\ \forall y$ is the same as $\forall y\ \forall x$ (why??)

$\exists x\ \exists y$ is the same as $\exists y\ \exists x$ (why??)

$\exists x\ \forall y$ is **not** the same as $\forall y\ \exists x$

$\exists x\ \forall y\ \ Loves(x,y)$
"There is a person who loves everyone in the world"

$\forall y\ \exists x\ \ Loves(x,y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x\ \ Likes(x, IceCream) \qquad \neg\exists x\ \ \neg Likes(x, IceCream)$

$\exists x\ \ Likes(x, Broccoli) \qquad \neg\forall x\ \ \neg Likes(x, Broccoli)$

Inference in First-Order Logic

- We begin with some simple inference rules (**Instantiation**) that can be applied to sentences with quantifiers to obtain sentences without quantifiers.
- These rules lead naturally to the idea that *first-order* inference can be done by converting the knowledge base to *propositional* logic and using *propositional* inference

- **Universal Instantiation (UI)**
  For any **sentence** α, **variable** *v*, and **constant** symbol *g*

$$\frac{\forall v\ \ \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

Axiom that **all greedy king are evil** :
e.g., ***$\forall x$ King(x) $\wedge$ Greedy(x) $\Rightarrow$ Evil(x)*** yields:
King(John) $\wedge$ Greedy(John) $\Rightarrow$ Evil(John)
King(Richard) $\wedge$ Greedy(Richard) $\Rightarrow$ Evil(Richard)
King(Father(John)) $\wedge$ Greedy(Father(John)) $\Rightarrow$ Evil(Father(John))

- **Existential Instantiation (EI)**
  For any **sentence** α, **variable** *v*, and **constant** symbol *k* that does not appear elsewhere in the knowledge base.

$$\frac{\exists v\ \ \alpha}{\textsc{Subst}(\{v/k\}, \alpha)}$$

e.g., ***$\exists x$: Crown (x) $\wedge$ OnHead (x, John)*** yields:
Crown (C_1) $\wedge$ OnHead (C_1, John)
provided ***C_1*** is a new constant symbol,
called a Skolem constant

Unification

We can get the inference immediately if we can find a substitution θ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\text{UNIFY}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, OJ)$	$\{x/OJ, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, OJ)$	$fail$

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

To unify ***Knows (John, x)*** *and* ***Knows (y, z)***, could return:

θ = {y/John, x/z}

or

θ = {y/John, x/John, z/John}

The first unifier is more general than the second.

There is a single Most General Unifier (**MGU**) that is unique up to renaming of variables.

MGU *= { y/John, x/z }*

Generalized Modus Ponens (GMP)

$$\frac{p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}',\ \ (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q)}{q\theta}$$

where $p_i{}'\theta = p_i\theta$ for all i

$p_1{}'$ is $King(John)$ $\quad$ p_1 is $King(x)$
$p_2{}'$ is $Greedy(y)$ $\quad$ p_2 is $Greedy(x)$
θ is $\{x/John, y/John\}$ $\quad$ q is $Evil(x)$
$q\theta$ is $Evil(John)$

GMP used with KB of definite clauses (**exactly** one positive literal)
All variables assumed universally quantified

Example of FOL

The law says that it is a crime
for an American to sell weapons to hostile nations.
The country Nono, an enemy of America,
has some missiles,
and all of its missiles were sold to it
by Colonel West, who is American.

Prove that **Colonel West is a criminal**
using resolution

- ...it is a crime for an American to sell weapons to hostile nations:
  $\forall x,y,z$: $American(x) \wedge Weapon(y) \wedge Sells(x,y,z) \wedge Hostile(z) \Rightarrow Criminal(x)$
- Nono … has some missiles, i.e., $\exists x\ Owns(Nono, x) \wedge Missile(x)$:
  $Owns(Nono,M_1)$
  $Missile(M_1)$
- … all of its missiles were sold to it by Colonel West
  $\forall x$: $Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West,x,Nono)$
- Missiles are weapons:
  $\forall x$: $Missile(x) \Rightarrow Weapon(x)$
- An enemy of America counts as "hostile":
  $\forall x$: $Enemy(x, America) \Rightarrow Hostile(x)$
- West, who is American …
  $American(West)$
- The country Nono, an enemy of America …
  $Enemy(Nono, America)$
- Colonel West is a criminal
  $Criminal(West)$

Resolution

Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{(\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)\theta}$$

where $\text{UNIFY}(\ell_i, \neg m_j) = \theta$.

For example,

$$\frac{\begin{array}{l}\neg Rich(x) \vee Unhappy(x) \\ Rich(Ken)\end{array}}{Unhappy(Ken)}$$

with $\theta = \{x/Ken\}$

Apply resolution steps to $CNF(KB \wedge \neg\alpha)$; complete for FOL

Conversion to CNF (Conjunctive Normal Form)

The **procedure** for **conversion** to CNF is similar to **the propositional case**

Example of Conversion to CNF: $\mathbf{B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})}$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   $$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}) .$$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$:
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1}) .$$
3. CNF requires $\neg$ to appear only in literals, so we "move $\neg$ inwards" by repeated application of the following equivalences from Figure 7.11:
   $\neg(\neg\alpha) \equiv \alpha$ (double-negation elimination)
   $\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$ (De Morgan)
   $\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$ (De Morgan)

   In the example, we require just one application of the last rule:
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1}) .$$
4. Now we have a sentence containing nested $\wedge$ and $\vee$ operators applied to literals. We apply the distributivity law from Figure 7.11, distributing $\vee$ over $\wedge$ wherever possible.
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1}) .$$

Example of Conversion FOL to CNF

We illustrate the procedure by translating the sentence
"**Everyone who loves all animals is loved by someone**"

$$\forall x\ [\forall y\ \ Animal(y) \Rightarrow\ Loves(x,y)] \Rightarrow [\exists y\ \ Loves(y,x)]$$

The steps are as follows:

- **Eliminate implications**: Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \lor \beta$

$$\forall x\ [\neg\forall y\ \neg Animal(y) \lor Loves(x,y)] \lor [\exists y\ \ Loves(y,x)]$$

- **Move $\neg$ inwards**: In addition to the usual rules for negated connectives, we need rules for negated quantifiers. Thus, we have

$\neg\forall x\ p$ becomes $\exists x\ \neg p$
$\neg\exists x\ p$ becomes $\forall x\ \neg p$

Our sentence goes through the following transformations:

$$\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y\ \ Loves(y,x)]$$
$$\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ \ Loves(y,x)]$$
$$\forall x\ [\exists y\ \ Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ \ Loves(y,x)]$$

- **Standardize variables**: For sentences like $(\exists x\, P(x)) \lor (\exists x\, Q(x))$ which use the same variable name twice, change the name of one of the variables. This avoids confusion later when we drop the quantifiers. Thus, we have

$$\forall x\ [\exists y\ \ Animal(y) \land \neg Loves(x,y)] \lor [\exists z\ \ Loves(z,x)]$$

- **Skolemize**: **Skolemization** is the process of removing existential quantifiers by elimination. In the simple case, it is just like the Existential Instantiation rule of Section 9.1: translate $\exists x\ P(x)$ into $P(A)$, where A is a new constant. However, we can't apply Existential Instantiation to our sentence above because it doesn't match the pattern $\exists v\ \alpha$; only parts of the sentence match the pattern. If we blindly apply the rule to the two matching parts we get

$$\forall x\ [Animal(A) \land \neg Loves(x,A)] \lor Loves(B,x)\ ,$$

which has the wrong meaning entirely: it says that everyone either fails to love a particular animal A or is loved by some particular entity B. In fact, our original sentence allows each person to fail to love a different animal or to be loved by a different person. Thus, we want the Skolem entities to depend on x and z:

$$\forall x\ [Animal(F(x)) \land \neg Loves(x,F(x))] \lor Loves(G(z),x)\ .$$

- **Drop universal quantifiers**: At this point, all remaining variables must be universally quantified. Moreover, the sentence is equivalent to one in which all the universal quantifiers have been moved to the left. We can therefore drop the universal quantifiers:

$$[Animal(F(x)) \land \neg Loves(x,F(x))] \lor Loves(G(z),x)\ .$$

- **Distribute $\lor$ over $\land$**:

$$[Animal(F(x)) \lor Loves(G(z),x)] \land [\neg Loves(x,F(x)) \lor Loves(G(z),x)]\ .$$

Sentence :
"**Everyone who loves all animals is loved by someone**":
$\forall$x [$\forall$y Animal(y) $\Rightarrow$ Loves(x, y)] $\Rightarrow$ [$\exists$y Loves(y, x)]

1. Eliminate implications
   $\forall x\ [\neg\forall y\ \neg Animal(y) \lor Loves(x, y)] \lor [\exists y\ Loves(y, x)]$
2. Move $\neg$ inwards:
   $\neg\forall x\ p \equiv \exists x\ \neg p$, $\neg\exists x\ p \equiv \forall x\ \neg p$
   $\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x, y))] \lor [\exists y\ Loves(y, x)]$
   $\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x, y)] \lor [\exists y\ Loves(y, x)]$
   $\forall x\ [\exists y\ Animal(y) \land \neg Loves(x, y)] \lor [\exists y\ Loves(y, x)]$
3. Standardize variables:
   each quantifier should use a different one
   $\forall x\ [\exists y\ Animal(y) \land \neg Loves(x, y)] \lor [\exists z\ Loves(z, x)]$
4. Skolemize: a more general form of existential instantiation
   Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:
   $\forall x\ [Animal(F(x)) \land \neg Loves(x, F(x))] \lor Loves(G(z),x)$
5. Drop universal quantifiers:
   $[Animal(F(x)) \land \neg Loves(x, F(x))] \lor Loves(G(z),x)$
6. Distribute $\lor$ over $\land$:
   $[Animal(F(x)) \lor Loves(G(z),x)] \land [\neg Loves(x, F(x)) \lor Loves(G(z), x)]$

Example of FOL, CNF, and Resolution

The law says that it is a crime
for an American to sell weapons to hostile nations.
The country Nono, an enemy of America,
has some missiles,
and all of its missiles were sold to it
by Colonel West, who is American.

Based on given premises above, create:

a) FOL
b) Convert FOL in part a) to CNF
c) Prove by Resolution, that **Colonel West is a criminal**

The **procedure** for **conversion** to CNF

Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$

Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$

"move $\neg$ inwards" $\neg(\neg\alpha) \equiv \alpha$ (double-negation elimination)
$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$ (De Morgan)
$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$ (De Morgan)

Standardize variables: each quantifier should use a different one.

Skolemize: a more general form of existential instantiation.

Drop universal quantifiers.

sentence containing nested $\wedge$ and $\vee$ operators applied to literals.

a) FOL

- ...it is a crime for an American to sell weapons to hostile nations:
  $\forall x,y,z$: *American(x) $\wedge$ Weapon(y) $\wedge$ Sells(x,y,z) $\wedge$ Hostile(z) $\Rightarrow$ Criminal(x)*
- Nono … has some missiles, i.e., $\exists x$ *Owns(Nono, x) $\wedge$ Missile(x)*:
  Owns(Nono,M_1)
  Missile(M_1)
- … all of its missiles were sold to it by Colonel West
  $\forall x$: *Missile(x) $\wedge$ Owns(Nono, x) $\Rightarrow$ Sells(West,x,Nono)*
- Missiles are weapons:
  $\forall x$: *Missile(x) $\Rightarrow$ Weapon(x)*
- An enemy of America counts as "hostile":
  $\forall x$: *Enemy(x, America) $\Rightarrow$ Hostile(x)*
- West, who is American …
  American(West)
- The country Nono, an enemy of America …
  Enemy(Nono, America)
- Colonel West is a criminal
  Criminal(West)

b) Conversion of FOL in part a) to CNF

$\forall x,y,z$: American(x) $\wedge$ Weapon(y) $\wedge$ Sells(x,y,z) $\wedge$ Hostile(z) $\Rightarrow$ Criminal(x)
$\neg$American(x) $\vee$ $\neg$Weapon(y) $\vee$ $\neg$Sells(x, y, z) $\vee$ $\neg$Hostile(z) $\vee$ Criminal(x)

Owns(Nono,M_1) and Missile(M_1)
Owns (Nono, M1)
Missile(M1)

$\forall x$: Missile(x) $\wedge$ Owns(Nono, x) $\Rightarrow$ Sells(West, x, Nono)
$\neg$Missile(x) $\vee$ $\neg$Owns(Nono, x) $\vee$ Sells(West, x, Nono)

$\forall x$: Missile(x) $\Rightarrow$ Weapon(x)
$\neg$Missile(x) $\vee$ Weapons(x)

$\forall x$: Enemy(x, America) $\Rightarrow$ Hostile(x)
$\neg$Enemy(x, America) $\vee$ Hostile(x)

American(West)
American(West)

Enemy(Nono, America)
Enemy(Nono, America)

Apply resolution steps to $CNF(KB \wedge \neg\alpha)$; complete for FOL

Resolution

c) Prove by Resolution, that Colonel West is a criminal