Physics 253a - Quantum Field Theory I

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Contents

1 September 4, 2018

You need at least 10 hours a week to take this course. This course will get more difficult as we go into renormalization. Then it will get easier once we pass this and get to applications.

We will start with special relativity and quantum mechanics, put them together and see what happens. We won't start with the axioms, because they are just statements that sound reasonable but cannot be tested.

1.1 Quantum theory of radiation

When you turn on the lights, the number of particles increase. How does this happen? Max Planck in the 1900s observed that discrete energy can explain blackbody radiation. Einstein in 1916 explained spontaneous/stimulated emission, and Paul Dirac in 1927 invented quantum electrodynamics, the microscopic theory of radiation.

We have a box of size L, poke a hole and heat it up. Then light comes out. We know that the wave numbers associated with the box are $\vec{k} = \frac{2\pi}{L}\vec{n}$, and $\omega = |\vec{k}|c$. This is classical prediction. Then the number of modes $\leq n$ is proportional to n^3, and the classical equipartition theorem predicts that each mode has the same energy. Sow we would have

$$dI(\omega) \sim \omega^2 d\omega.$$

This is called the ultraviolet catastrophe. But experimentally, we have exponential decay.

Planck said that energy E is quantized, so that $E_n = \hat{h}\omega_n$. Here, $\omega_n = \frac{2\pi}{L}n$ where $n = |\vec{n}|$. Then each mode gets excited an integer number of times, E_n^{tot} is an integer times E_n. The probability of $E_n^{\text{tot}} \sim e^{-\beta E_n}$. Then

$$\langle E_n \rangle = \frac{\sum_{j=0}^{\infty}(\hbar j\omega_n)e^{-j\hbar\omega_n\beta}}{\sum_{j=0}^{\infty}e^{-j\hbar\omega_n\beta}} = \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta}-1}.$$

Then the total energy up to ω is

$$\begin{aligned}E(\omega) &= \int_0^{\omega} d^3n \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta}-1} = \hbar\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \int_0^{L\omega/2\pi} n^2 dn \frac{\omega_n}{e^{\hbar\omega_n\beta}-1} \\ &= \hbar\frac{L^3}{(2\pi)^3}4\pi\int_0^{\omega}\frac{\omega^3}{e^{\hbar\omega\beta-1}}.\end{aligned}$$

So we get Planck's formula

$$I(\omega) = \frac{K}{2\pi^2}\frac{\omega^3}{e^{\hbar\omega\beta}-1}\times 2.$$

The point here is that each mode gets excited an integer number of times. This is called **second quantization**. This really is just quantization, because the first quantization refers to $\vec{k} = \frac{2\pi}{L}\vec{n}$, which is just classically solving wave equations with boundary conditions.

Let us now look at a number of atoms, either in the ground state or the excited state with energy difference $E_2 - E_1 = \hbar\omega$. Let n_1, n_2 be the number of atoms with energy E_1, E_2. Also assume that there is a bath of photons of frequency ω, with intensity $I(\omega)$ and number $n_\omega = \frac{\pi^2}{\omega^3} I(\omega)$. If we look at the probability of atoms getting excited or emitting, we get

$$dn_2 = -An_2 - BI(\omega)n_2 + B'I(\omega)n_1.$$

Here, the first term is spontaneous emission, the second is stimulated emission, and the third is stimulated absorption. It's not obvious that the second term should exist, but it turns out to be nonzero. In equilibrium, we have

$$I(\omega)(B'n_1 - Bn_2) = An_2.$$

So we get

$$I(\omega) = \frac{A}{B'\frac{n_1}{n_2} - B} = \frac{A}{B'e^{\beta\hbar\omega} - B}$$

because $n_1 = e^{-\beta E_1}$ and $n_2 = e^{-\beta E_2}$.

Matching with Planck's formula, we get the relations

$$B = B', \quad A = \frac{\hbar}{\pi^2}\omega^3 B,$$

called Einstein's equations. The number B can be calculated by quantum mechanics. So we can calculate A using this relation and quantum mechanics.

This is what got to Dirac. It's great that we can compute the coefficient of spontaneous emission, but it will be good to calculate this without using thermal systems, just from fundamental laws. The second quantization really looks like the simple harmonic oscillator. So we are going to identify

$$|n\rangle = n \text{ photon state} = n\text{th excited state of the oscillator.}$$

Consider $a^\dagger$ the creation operator and a the annihilation operator so that $[a, a^\dagger] = 1$ and $N = a^\dagger a$ is the number operator with $\hat{N}|n\rangle = n|n\rangle$. We can compute

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

This turns out to be a powerful tool.

Now Fermi's golden rules says that the transition rate is $\Gamma \sim |M|^2\delta(E_f - E_i)$. If we use this, we get at the end,

$$|M_{2\to1}|^2 = |M_0|^2(n_\omega + 1), \quad |M_{1\to2}|^2 n\omega|M_0|^2.$$

So this algebra of creation and annihilation operation gives us the relation between spontaneous emission and stimulated absorbtion. Then more algebra gives

$$dn_2 = -|M_0|^2\Big(1 + \frac{\pi^2}{\hbar\omega}I(\omega)\Big)n_2 + \frac{\pi^2}{\hbar\omega^3}I(\omega)n_1.$$

2 September 6, 2018

Today we are going to start the systematic development of the field. We let $c = 1$ and $\hbar = 1$.

2.1 Special relativity

There are rotations on the plane,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x_i \to R_{ij} x_j.$$

We can also rotate row vectors as

$$x^i \to x^i (R^T_{ij}),$$

and the rotations satisfy $R^T_{ij} \cdot 1_{jk} R_{kl} = 1_{il}$. This is because rotations should preserve $x^i x_i = x^2 + y^2$. In 3 dimensions, we have $x^2 + y^2 + z^2$, and in 4 dimensions, we have $t^2 - x^2 - y^2 - z^2$. So **Lorentz transformations** satisfy

$$\Lambda^T g \Lambda = g, \quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Examples include

$$\Lambda_{\theta_z} = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}, \quad \Lambda_{\beta_x} = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Four momentum is defined as

$$p^\mu = (E, p_x, p_y, p_z),$$

and it satisfies $p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$. Usually, $\vec{x}$ or x_i denotes a 3-dimensional vector, and x or x^μ denotes a 4-dimensional vector.

Tensors transform as

$$T_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}.$$

We define the **d'Alembertian** as

$$\Box = \partial_\mu^2 = g^{\mu\nu} \partial_\mu \partial_\nu = \partial_t^2 - \vec{\nabla}^2 = \partial_t^2 - \Delta.$$

We say that a vector is **timelike** if $V^2 > 0$, and **spacelike** if $V^2 < 0$, and **lightlike** if $V^2 = 0$.

The proper **orthochronous** Lorentz group has $\det \Lambda = 1$ and $\Lambda_{00} > 0$. There are four components of the Lorentz group, and this is the connected component at the identity. The **Poincaré group** are Lorentz transformations plus translations.

2.2 Quantum mechanics

Remember we had normal modes in a box last time. These frequencies are quantized classically. Then Planck said that the energy should be associated to the frequency $E = j\hbar\omega$. Einstein was the one who interepreted these as particles, which we call photons, and Dirac developed this microscopic theory of $H = H_0 a^\dagger + H_0 a$.

Let us review the simple harmonic oscillator. We have a ball with a spring on it, and its equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

You can solve this, and you get

$$x(t) = \cos\Big(\sqrt{\frac{k}{m}}t\Big).$$

The classical Hamiltonian is given by

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2.$$

Then we quantize this using $[\hat{x}, \hat{p}] = i\hbar$, and define

$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} + \frac{i\hat{p}}{m\omega}\Big), \quad a^\dagger = \cdots, \quad [a, a^\dagger], \quad H = \hbar\omega(N + \tfrac{1}{2}), \quad N = a^\dagger a.$$

We found last time that

$$N|n\rangle = n|n\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

Then in the Heisenberg picture,

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0).$$

Now what can the equation of motion for the scalar field be? It should be Lorentz invariant, so the simplest possible equation is

$$\Box\phi = 0 = (\partial_t^2 - \vec{\nabla}^2)\phi = 0.$$

Take the Fourier transform, and let

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}]$$

Then the equation becomes

$$(\partial_t^2 + \vec{p}^2)a_p(t) = 0.$$

Now each component is just a classical simple harmonic oscillator. So we can quantize each separately, and then put them back together.

Electromagnetic saves are oscillators,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

This concisely encodes Maxwell's equations

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial_\mu F_{\mu\nu} = 0$$

in empty space. It's also helpful to write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This vector potential A_μ is more useful for field theory, because there are only 4 components, and also because it is invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \alpha(x),$$

called gauge invariance.

We can choose $\partial_\mu A_\mu = 0$, and this is called **Lorentz gauge**. When you do that, Maxwell's equations become

$$0 = \partial_\mu F_{\mu\nu} = \Box A_\nu.$$

So then we can make $A_\nu(x,t)$ into a set of harmonic oscillators. We write

$$A_\nu(x,t) = \int \frac{d^3p}{(2\pi)^3} (A_\nu^p(t) e^{i\vec{p}\cdot\vec{x}} + A_\nu^{p*}(t) e^{-i\vec{p}\cdot\vec{x}}), \quad (\partial_t^2 + \vec{p}^2) A_\nu^{\vec{p}} = 0.$$

Then the free electromagnetic field is equivalent to an infinite number of simple harmonic oscillators, labeled by 3 vectors $\vec{p}$ with frequencies $\omega_p = |\vec{p}|$.

Now we quantize as in quantum mechanics. Then

$$H_0 = \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + \tfrac{1}{2}).$$

The relations between these creation and annihilation operators are

$$[a_k, a_p^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}), \quad a_p|0\rangle = 0, \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_p}}|p\rangle.$$

What we have done is that we have constructed the Hilbert space

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p,$$

called the **Fock space**.

3 September 11, 2018

Last time we reviewed the simple harmonic oscillator. To quantize this theory, we defined $H = \omega(a^\dagger a + \frac{1}{2})$. For fields, we classically had $\Box A_\mu(x) = 0$ or $(\Box + m^2)\phi(x) = 0$. We do the Fourier transform, and we get something like

$$A(x,t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}].$$

Then the equation becomes $[\partial_t^2 + \omega_p^2]a_p(t) = 0$ and $\omega_p = \sqrt{\vec{p}^2 + m^2}$. Then we quantize and get

$$H = \int \frac{d^3p}{(2\pi)^3}[\omega_p(a_p^\dagger a_p + \frac{1}{2}].$$

3.1 Operators on the Fock space

The Fock space is then

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p = \bigoplus_n \mathcal{H}_n$$

where p is the momentum and n is the number of particles. The creation and annihilation operators then behave as

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We normalize

$$a_p|0\rangle = 0, \quad |p\rangle = \sqrt{2\omega_p}a_p^\dagger|0\rangle.$$

Then we get

$$\langle p|k\rangle = \sqrt{2\omega_p}\sqrt{2\omega_k}\langle 0|a_p a_k^\dagger|0\rangle = 2\omega_p(2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We also have

$$\mathbf{1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}|p\rangle\langle p|.$$

Then you can check $|k\rangle = \mathbf{1}|k\rangle$.

Also, we define

$$A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}].$$

This is like a creation operator in position space. Indeed, we compute

$$\langle p|A(x)|0\rangle = \int d^3k\delta^3(p-k)\langle 0|0\rangle e^{-i\vec{k}\cdot\vec{x}} = e^{-\vec{p}\vec{x}}.$$

But $A(x)A(y)|0\rangle$ is not just particles at x and y.

In quantum field theory, we work with the Heisenberg picture, so we define $a_p^\dagger(t) = e^{i\omega_p t}a_p^\dagger(0)$. Then

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p(0)e^{i\vec{p}\cdot\vec{x} - i\omega_p t} + a_p^\dagger(0)e^{i\omega_p t - i\vec{p}\cdot\vec{x}}].$$

Here, you can interpret the exponent as $p^\mu x_\mu$, because $p^\mu = (\omega_p, \vec{p})$.

3.2 Classical field theory

The main object is the Hamiltonian

$$H(p,x) = \text{energy} = K + V.$$

This is not Lorentz invariant, and generates time translation. On the other hand, the Lagrantian

$$L[x,\dot{x}] = K - V$$

is not a conserved quantity, but it is Lorentz-invariant and the dynamics is determined by minimizing the action $S = \int d\epsilon L$.

For fields, we are going to have

$$L[\phi,\dot{\phi},\vec{\nabla}\phi] = L[\phi,\partial_\mu\phi], \quad H[\phi,\pi,\vec{\nabla}\phi].$$

We still talk about kinetic terms

$$K = \text{things like } \frac{1}{2}\phi\Box\phi, \quad \frac{1}{4}F_{\mu\nu}^2, \frac{1}{2}m^2\phi^2, \phi\partial_\mu A^\mu,$$

and interactions

$$V = \text{things like } A\phi^3, e\bar{\psi}A\psi, e(\partial_\mu\phi)\phi^* A_\mu.$$

Example 3.1. Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi).$$

To minimize the action, we perturb the field a little bit and look at the difference. Then

$$\delta S = \int d^4x \Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\Big] \quad = \int d^4x \Big\{\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big] + \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big]\Big\}.$$

Here, we assume $\phi(\infty) = 0$, so we get

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

This is called the **Euler–Lagrangian equations**.

Example 3.2. In the above example, we get

$$-V'(\phi) = \partial_\mu[\partial_\mu\phi] = \Box\phi.$$

3.3 Noether's theorem

Suppose $\mathcal{L}$ is invariant under some specific continuous variation. For instance, take

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^*$$

which is invariant under $\phi \to e^{i\alpha}\phi$. Then

$$0 = \frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \Big\{ \Big[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\partial \alpha} \Big\}.$$

So if the Euler–Lagrange equations are satisfied, the first term is zero so

$$\partial_\mu J^\mu = 0, \quad J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}.$$

Then if we define $Q = \int d^3x J^0$, we have $\partial_t Q = 0$. This is the statement and proof of **Noether's theorem**.

Let's think about what we get for $\phi \mapsto e^{i\alpha}\phi$. We have

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(-i\phi^*) = i\phi \partial_\mu \phi^* - i\phi^* \partial_\mu \phi.$$

We can check

$$\partial_\mu J^\mu = i\partial_\mu \phi \partial_\mu \phi^* + i\phi \Box \phi^* - i\partial_\mu \phi^* \partial_\mu \phi - i\phi^* \Box \phi = i\phi \Box \phi^* - i\phi^* \Box \phi.$$

This is zero because at the equations of motion, we have $\Box \phi = m^2 \phi$.

4 September 13, 2018

Noether's theorem says that if and action has a continuous symmetry, then there exists a current J^μ with $\partial_\nu J^\mu = 0$ when the equations of motion are satisfied. In this case,

$$Q = \int d^3 x J^0$$

satisfies $\partial_t Q = 0$.

Consider translation invariance. When we look at a translate of $\mathcal{L}$, we get

$$\partial_\mu (g_{\mu\nu}\mathcal{L}) = \partial_\nu \mathcal{L} = \Big[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\Big] \frac{\delta \phi_n}{\partial \xi^\nu} + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\partial \xi^\nu}\Big].$$

Because the first term vanishes at equation on motion. So we have

$$\partial_\mu T_{\mu\nu}$$

where

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu}\mathcal{L}.$$

This is called the **energy-momentum tensor**. Here, we note that

$$\mathcal{E} = T_{00} = \sum \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \dot{\phi}_n - \mathcal{L} = \pi \dot{\phi} - \mathcal{L} = \mathcal{H}$$

is just the energy. So energy $E = \int d^3 x T^{00}$ is conserved over time.

4.1 Coulomb's law

We are going to introduce an external current

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - J_\mu A^\mu.$$

(When I say current, I don't mean Noether current here.) Because $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - J_\mu A^\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\partial_\mu A_\nu \partial_\nu`A_\mu - J_\mu A^\mu. \end{aligned}$$

Then $\partial \mathcal{L}/\partial A_\nu = -J_\nu$ and $\partial \mathcal{L}/\partial \partial_\mu A_\nu = -\partial_\mu A_\nu + \partial_\nu A_\mu = -F_{\mu\nu}$. Then the Euler–Lagrange equation is

$$\partial_\mu F_{\mu\nu} = J_\nu,$$

which is Maxwell's equations. If we go to Lorentz gauge, we get

$$\Box A_\nu = J_\nu.$$

We are going solve this by inverting the d'Alembertian $\Box$. Here, note that we have Fourier transform

$$f(x) = \int \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}, \quad \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}.$$

Then the inverse is

$$\tilde{f}(k) = \int dx f(x) e^{-ikx}.$$

We can compute

$$\Box f(x) = \int d^k k \Box \tilde{f}(x) e^{ikx} = \int d^4 k (-k^2) \tilde{f}(k) e^{ikx}.$$

So $\Box$ corresponds to $-k^2$ in Fourier space.

We want to solve the equation when there is a point charge, when $J_0 = \delta^3(x)$ and $\vec{J} = 0$. Then

$$\begin{aligned} A_0(x) = \frac{e}{\Box}\delta^3(x) = -\frac{e}{\Delta}\delta^3(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{e}{\vec{k}^2} e^{i\vec{k}\vec{x}} \\ = \frac{e}{i4\pi^2} \int_0^\infty dk \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{e}{4\pi r}. \end{aligned}$$

This is the Coulomb potential.

4.2 Green's functions

Let's look at a complicated example,

$$\mathcal{L} = -\frac{1}{2} h \Box h + \frac{1}{3} \lambda h^3 + hJ.$$

This is a toy example for gravity, because gravitons interact with each other. Then the Euler–Lagrange equation is

$$\Box h - \lambda h^2 - J = 0.$$

We now work perturbatively in λ. For $\lambda = 0$, we know

$$h_0 = \frac{1}{\Box} J.$$

If $\lambda \neq 0$, we can write $h = h_0 + h_1$, where $h_1 = O(\lambda)$. If we plug in into the original equation, we get $\Box h_1 = \lambda h_0^2$. So we can write

$$h_1 = \frac{\lambda}{\Box} \Big(\frac{1}{\Box} J \Big)^2.$$

So we get

$$h = \frac{1}{\Box} J + \lambda \Big(\frac{1}{\Box} \Big) \Big(\frac{1}{\Box} J \Big) \Big(\frac{1}{\Box} J \Big) + \cdots .$$

We can interpret each of these in terms of Feynman diagrams. Think of each J as a source, $\frac{1}{\Box}$ as a propagation or a branch coming our from a source, and λ as an interaction between these branches. Then this is something like the Sun emitting a graviton, emitting another graviton, and they interact and become one. There are other diagrams we can draw but are not represented in the solution, and these are purely quantum mechanical effects that we will discuss. What we are doing now is classical.

If we look at the solution for $\Box_x A(x) = J(x)$ again, we have

$$A(x) = -\int d^4y \Pi(x,y) J(y), \quad \Pi(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2}.$$

Then you can check that $\Box_x \Pi(x,y) = -\delta^(x-y)$. We call this a **propagator** or the **Green's function**. (We have $\frac{1}{\Box} = -\Pi$.)

Let us do what we did this above in this context. Then

$$h(x) = \int d^4y \delta^4(x-y) h(y) = -\int d^4\pi(x,y) \Box_y h(y) = -\int d^4y \Pi(x,y) J(y).$$

So this is the propagator of the potential from the source. We can do the same thing on the next order. We have

$$h(x) = -\int d^4y \Pi(x,y) J(y) + \lambda \int d^4w \int d^4y \int d^4z \Pi(x,w) \Pi(w,y) \Pi(w,z) J(y) J(z).$$

Then these have good physical interpretation. In quantum field theory, there will also be interactions in loops and so on.

5 September 18, 2018

This week and next week will be a bit dry. Why do we talk about cross sections in scattering? Scattering is a universal way of probing something that we can't see. We are skipping Chapter 4, which is old-fashioned perturbation theory.

5.1 Scattering

In quantum mechanics, we calculate amplitudes, $\langle f|i\rangle$, and probabilities, $|\langle f|i\rangle|^2$. In field theory, we calculate the same objects.

Let us consider the situation where two particles collide, and two or more particles come out. In the Schrödinger picture, we want to calculate

$$\langle f; t=\infty | i; t=-\infty\rangle.$$

In this Heisenberg picture, we are trying to measure $\langle f|S|i\rangle$. We are interested in this matrix S.

Classically, if we throw a beam of particles on a large particle,s we can consider the cross-section area as

$$\sigma = \frac{\#\text{particles scattered}}{\text{time} \times \text{number density of beam} \times \text{velocity of beam}}.$$

We may think this as $N = L\sigma$, where L is the luminosity.

What we want to do now is to talk about quantum mechanics. Here, σ is just a cross-section. Particles have a probability of scattering: $P = N_{\text{scatter}}/N_{\text{incident}}$. We are going to let $N_{\text{incident}} = 1$, so we are throwing one particles at a time. Then the flux is

$$\text{Flux} = \frac{|\vec{v}|}{V} = \frac{|\vec{v}_1 - \vec{v}_2|}{V}.$$

Now our formula for σ is

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}dP, \quad dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi.$$

The last factor $d\Pi$ is the density of states. On a line of size L, momenta are $p_n = \frac{2\pi}{L}n$ and so $dp = \frac{2\pi}{L}dn$. So we have

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}d^3p_j.$$

The initial and final states are given by

$$|i\rangle = |p_1\rangle|p_2\rangle, \quad |f\rangle = |p_3\rangle\cdots|p_n\rangle.$$

Because we are working in a box, we consider $|p|p\rangle = 2E_p\delta^3(0) = 2E_pV$. Then

$$|i|i\rangle = 2E_1 2E_2 V^2, \quad |f|f\rangle = \prod_{j=3}^{n}(2E_j)V.$$

Then we have

$$d\sigma = \frac{V}{T}\frac{|\langle f|S|i\rangle|^2}{|\vec{v}_1 - \vec{v}_2|\prod_j(2E_i)2E_1 2E_2 V^n}\prod_n \frac{V}{(2\pi)^3}d^3p_i.$$

We write

$$S = 1 + iT,$$

where $T = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - \cdots - p_n)M$, because momentum is conserved. Then

$$|\langle f|S|i\rangle|^2_{f\neq i} = (2\pi)^8\delta^4(\textstyle\sum p)\delta^4(0)|M|^2.$$

If we plug this in, we get

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} \times |M^2| \prod_j \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^4(\textstyle\sum p).$$

This second term is also called the Lorentz-invariant phase space, $d\Pi_{\text{LIPS}}$. So we can write the decay wrate as

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\Pi_{\text{LIPS}}.$$

There is no flux factor, and no $1/2E_2$.

5.2 Two-to-two scattering

Let us look at the example of a $2 \to 2$ scattering. Let us call the four particles p_1, p_2, p_3, p_4. In the center of mass frame, we have

$$|\vec{p}_1| = |\vec{p}_2| = p_i, \quad |\vec{p}_3| = |\vec{p}_4| = p_f.$$

Energy conservation is $E_1 + E_2 = E_3 + E_4 = E_{\text{CM}}$. Now we look at

$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}.$$

But this has a lot of redundancies, so we can express in terms on the direction. If we integrate over $\vec{p}_4$, we get

$$\begin{aligned} d\Pi_{\text{LIPS}} &= \frac{1}{4(2\pi)^2}\frac{dp_3}{E_3E_4}\delta(E_3 + E_4 - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int p_f^2 dp_f \frac{1}{E_3E_4}\delta(\sqrt{m_3^2 + p_f^2} + \sqrt{m_4^2 + p_f^2} - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int_{m_3+m_4-E_{\text{CM}}}^{\infty} dx\delta(x)\frac{p_f}{E_{\text{CM}}} = \frac{d\Omega}{16\pi^2}\frac{p_E}{E_{\text{CM}}}\theta(E_{\text{CM}} - m_3 - m_4). \end{aligned}$$

So we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_1 2E_2|\vec{v}_1 - \vec{v}_2|}\frac{1}{16\pi^2}\frac{p_f}{E_{\text{CM}}}|M|^2 = \frac{1}{64\pi^2 E_{\text{CM}}^2}\frac{p_f}{p_i}|M|^2.$$

in the center of mass frame. (This $d\Omega$ is the spherical angle $d\phi d\cos\theta$, so that $d^3p_3 = p_3^2 dp_3 d\Omega$.)

Let us look at the non-relativistic limit. Consider the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}|\tilde{V}(k)|^2.$$

Here, $\tilde{V}(k)$ is the Fourier transformation

$$\tilde{V}(k) = \int d^3x e^{-i\vec{k}\vec{x}} V(x) = \frac{e^2}{\vec{k}^2}$$

in the Coulomb potential. So we have

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}\Big(\frac{e^2}{\vec{k}^2}\Big)^2.$$

Let us how see this agrees with what we have done so far.

The free theory for the proton and the electron is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 - \phi_e^*(\Box + m_e^2)\phi_e - \phi_p^*(\Box + m_p^2)\phi_p \\ &- ieA_\mu(\phi_e^*\partial_\mu\phi_e - \phi_e\partial_\mu\phi_e^*) + ieA_\mu(\phi_p^*\partial_\mu\phi_p - \phi_p\partial_\mu\phi_p^*).\end{aligned}$$

If we take the non-relativistic limit, we have $p_\mu = (E, \vec{p}) = (\sqrt{m^2 + \vec{p}^2}, \vec{p}) \approx (m, 0)$, and so $\partial_t\phi \approx im\phi$. So we redefine $\phi \to e^{im_e t}\phi$ so that the phases don't rotate. If we do that, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \phi_e\vec{\nabla}^2\phi_e + 2em_e\phi_e^*\phi_e A_0 + \phi_p\vec{\nabla}^2\phi_p - 2em_p\phi_p^*\phi_p A_0.$$

The matrix M is going to be

$$M = \frac{(2em_e)(-2em_p)}{\vec{k}^2}$$

because the coefficient of $\phi_e^*\phi_e A_0$ is $2em_e$ and this is the interaction between the electron, electron, photon, and likewise for $(-2em_p)$, and $1/|\vec{k}^2|$ is the Green's function. Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_p^2}|M|^2 = \frac{1}{64\pi^2 m_p^2}\frac{16e^4 m_e^2 m_p^2}{k^4} = \frac{1}{4\pi^2}\frac{m_e^2 e^4}{|\vec{k}^2|^2}.$$

This is the same formula we had for the Born approximation.

6 September 20, 2018

We also have to need this other technical theorem. We recall that light satisfies $\Box A_\mu = 0$, and so $(\partial_t^2 + |\vec{k}|^2)A_\mu = 0$. We had our operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{i\vec{p}\cdot\vec{x}} + a_p e^{-i\vec{p}\cdot\vec{x}}).$$

Then $i\partial_t a_p = -[H, A_p] = \omega_p a_p$, so we can define even for difference time

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}}^\dagger e^{ipx} + a_{\vec{p}} e^{-ipx}).$$

6.1 LSZ reduction

We also talked about cross sections,

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|M|^2 d\Pi_{\mathrm{LIPS}},$$

where $S = 1 + iT$ and $T = (2\pi)^4\delta^4(\sum p)M$. So again, our initial state is

$$|i\rangle = |p_1 p_2\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle$$

We are using $|\Omega\rangle$ because the vacuum is time-dependent. Similarly, in the final state, we can write

$$|f\rangle = |p_3\cdots p_n\rangle = \sqrt{2\omega_3}\cdots\sqrt{2\omega_n}a_{p_3}^\dagger(+\infty)\cdots a_{p_n}^\dagger(+\infty)|\Omega_{+\infty}\rangle.$$

Now the matrix element between the two things is

$$\langle f|S|i\rangle = \sqrt{2\omega_1}\cdots\sqrt{2\omega_n}\langle\Omega_\infty|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle.$$

So how do we create $|p\rangle$ from $\phi(x)$? WE have

$$\langle p|\phi(x)|0\rangle = e^{ipx}, \quad \phi(x)|0\rangle = \int d^3p\frac{1}{2\omega_p}e^{i\vec{p}\cdot\vec{x}}|p\rangle = \int d^4p\delta(p^2 - m^2)\theta(p^0)e^{i\vec{p}\vec{x}}|p\rangle.$$

But we have

$$-2\pi i\delta(p^2 - m^2) = \lim_{\epsilon\to 0}\Big[\frac{1}{p^2 + m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\Big].$$

So we roughly have

$$\int e^{-ipx}(\Box + m^2)\phi(x)|0\rangle = |p\rangle.$$

The precise expression is

$$i\int d^4x e^{ipx}(\Box + m^2)\phi(x,t) = \sqrt{2\omega_p}(a_p(\infty) - a_p(-\infty)).$$

Let me try to derive this. If we do spatial integration by parts, we get

$$\begin{aligned}
i\int d^4x e^{ipx}(\Box+m^2)\phi(x,t) &= i\int d^4x e^{ipx}(\partial_t^2+\omega_p^2)\phi(x,t)\\
&= \int dt\partial_t\left[e^{i\omega_p t}\int d^3x e^{-i\vec{p}\vec{x}}(i\partial_t+\omega_p)\phi(x,t)\right]\\
&= \int dt\partial_t\left[e^{i\omega_p t}\sqrt{2\omega_p}a_p e^{-i\omega_p t}\right]\\
&= \sqrt{2\omega_p}[a_p(\infty)-a_p(-\infty)].
\end{aligned}$$

Here, we are assuming that at $t=\pm\infty$, the field behaves like a free field, so we can compute the spatial integral simply. Now if we take the complex conjugate of both sides, we get

$$-i\int d^4x e^{-ipx}(\Box+m^2)\phi(x,t)=\sqrt{2\omega_p}(a_p^\dagger(\infty)-a_p^\dagger(-\infty)).$$

Now can compute $\langle f|i\rangle$. Here, we introduce a **time-ordering operation** T which just puts things in the correct time order. Then we have

$$\begin{aligned}
&\langle\Omega|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle\\
&\quad=\langle\Omega|T\{[a_{p_3}(\infty)-a_{p_3}(-\infty)]\cdots[a_{p_n}(\infty)-a_{p_n}(-\infty)]\\
&\qquad[a_{p_1}^\dagger(\infty)-a_{p_1}^\dagger(-\infty)][a_{p_2}^\dagger(\infty)-a_{p_2}^\dagger(-\infty)]|\Omega\rangle,
\end{aligned}$$

because all other terms become 0. So we get

$$\begin{aligned}
\langle f|i\rangle &= \left[i\int d^4x_1 e^{ip_1x_1}(\Box+m_1^2)\right]\cdots\left[-i\int d^4x_n e^{-ip_nx_n}\right]\\
&\quad\times\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.
\end{aligned}$$

This is called the **LSZ reduction formula**.

6.2 Feynman propagators

The simplest example is

$$D_F(x,y)=\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle=\lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x-y)}.$$

This is called the **Feynman propagator**, and satisfies $(\Box_x+m^2)D_F(x,y)=\int d^4k e^{ik(x-y)}=\delta^4(x-y)$. So this is the factor you put in when you want to talk about propagation between to interactions.

The first thing we do is to calculate without time ordering. We have

$$\begin{aligned}
\langle 0|\phi(x_1)\phi(x_2)|0\rangle &= \int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{\sqrt{2\omega_1}\sqrt{2\omega_2}}\langle 0|a_{k_1}a_{k_2}^\dagger|0\rangle e^{i(k_2x_2-k_1x_1)}\\
&= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{ik(x_1-x_2)}.
\end{aligned}$$

If we do have time-ordering, we get

$$\begin{aligned}
\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= \langle 0|\phi(x_1)\phi(x_2)|0\rangle\theta(t_1-t_2) + \langle 0|\phi(x_2)\phi(x_1)|0\rangle\theta(t_2-t_1) \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{ik(x_2-x_1)}\theta(\tau) + e^{ik(x_1-x_2)}\theta(-\tau)] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k\tau}\theta(\tau) + e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{i\omega_k\tau}\theta(-\tau)] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}[e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau)].
\end{aligned}$$

Here, we have

$$\theta(\tau) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{\omega + i\epsilon}\frac{1}{2\pi i}.$$

So we get

$$\begin{aligned}
\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\frac{i}{\omega^2 - \vec{p}^2 - m^2 + i\epsilon}e^{ik(x-y)} \\
&= \int d^4k \frac{i}{k^2 - m^2 + i\epsilon}e^{ik(x-y)}.
\end{aligned}$$

This ϵ we are going to think of as uncertainty of energy. Feynman's ingenious idea is that you can add the two possible time orderings in one propagator, so that we don't have to think of the two cases separately.

7 September 25, 2018

Last time we showed that

$$\langle P_3 \cdots P_n | P_1 P_2 \rangle = \left[i \int d^4 x e^{-ip_1 x_1} (\Box_1 + m_1^2) \right] \cdots \left[i \int d^4 x_n e^{+ip_n x_n} (\Box_n + m_n^2) \right] \langle \Omega | T\{\phi(x_1) \cdots \phi(x_n) | \Omega \rangle.$$

Then for free field, we got the Feynman propagator as

$$D_F(x,y) = \langle 0 | T\{\phi_0(x)\phi_0(y)\} | 0 \rangle = \lim_{\epsilon \to 0} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{ik(x-y)}.$$

Then

$$(\Box_x + m^2) D_F(x,y) = -i\delta^4(x-y).$$

We will check this later.

7.1 Schwinger–Dyson equations

We want to specify the dynamics and the commutations relations. We can use things like $[x,p] = i\hbar$ and $[\theta, H] = i\partial_t \theta$, but then we need to worry about the separation of space and time.

Assume that ϕ satisfies the equations of motion as an operator,

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \mathcal{L}_{\text{int}}[\phi].$$

Then the Euler–Lagrange equations become

$$(\Box + m^2)\phi = \mathcal{L}_{\text{int}}'[\phi].$$

The commutation relations are

$$[\phi(\vec{x},t), \partial_t \phi(\vec{y},t)] = i\hbar\delta(\vec{x} - \vec{y}), \quad [\phi(\vec{x},t), \phi(\vec{y},t)] = 0.$$

These are equal-time commutations relations. The second relation should be thought of as, the two points $(\vec{x},t)$ and $(\vec{y},t)$ are causally unrelated.

We can check in the free theory that these hold. Here, we have

$$\phi_0(x,t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx}), \quad [a_p, a_q^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

Our equations of motion are

$$(\Box + m^2)\phi_0(x,t) = \int \frac{d^3 p}{(2\pi)^3} (-p^2 + m^2) \frac{1}{\sqrt{2\omega_p}} (\cdots) = 0$$

because $p^2 = m^2$. For the commutation relation, we first compute

$$\partial_t \phi(\vec{y}, t) = -i \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_q}{2}} (a_q e^{-iqy} - a_q^\dagger e^{iqy}).$$

Then

$$\begin{aligned}
[\phi(\vec{x}, t), \partial_t \phi(\vec{y}, t)] &= -i \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_q}{4\omega_p}} ([a_p^\dagger, a_q] e^{ipx - iqy} - [a_p, a_q^\dagger] e^{-ipx + iqy}) \\
&= -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} (-e^{ip(x-y)} - e^{-ip(x-y)}) = i\delta^3(x - y).
\end{aligned}$$

So this is satisfied in the free theory, and we're assuming this holds also in the interacting theory. This is not so surprising, because we are looking at one time slice.

What we are going to do now, is to look at the time ordering operator using these relations. We can compute

$$\begin{aligned}
\partial_t \langle \Omega | T\{\phi(x), \phi(x')\} | \Omega \rangle &= \partial_t [\langle \phi(x)\phi(x') \rangle \theta(t - t') + \langle \phi(x')\phi(x) \rangle \theta(t' - t)] \\
&= \langle T\{\partial_t \phi(x)\phi(x')\} \rangle + \langle \phi(x)\phi(x') \rangle \delta(t - t') - \langle \phi(x')\phi(x) \rangle \delta(t - t') \\
&= \langle T\{\partial) t \phi(x)\phi(x')\} \rangle
\end{aligned}$$

because $\phi(x)$ and $\phi(x')$ commute at $t = t'$. Then the second derivative is

$$\begin{aligned}
\partial_t^2 \langle T\{\phi(x)\phi(x')\} \rangle &= \partial_t \langle T\{\partial_t \phi(x)\phi(x')\} \rangle \\
&= \langle T\{\partial_t^2 \phi(x)\phi(x')\} \rangle + \langle [\partial_t \phi(x), \phi(x')] \rangle \delta(t - t') \\
&= \langle T\{\partial_t^2 \phi(x)\phi(x')\} \rangle - i\hbar\delta^4(x - x').
\end{aligned}$$

So we finally get

$$(\Box_x + m^2) \langle T\{\phi(x)\phi(x')\} \rangle = \langle T\{(\Box + m^2)\phi(x)\phi(x')\} \rangle - i\hbar\delta^4(x - x').$$

This is

$$(\Box_x + m^2) D_F(x, y) = -i\hbar\delta^4(x - y).$$

Now we can assume that we have more than terms. Then we can show that

$$\begin{aligned}
\Box_x \langle T\{\phi(x)\phi(y_1) \cdots \phi(y_n) \rangle &= \langle T\{\Box_x \phi(x)\phi(y_1) \cdots \phi(y_n)\} \rangle \\
&\quad - i\hbar \sum_j \delta^4(x - y_j) \langle T\{\phi(y_1) \cdots \phi(y_{j-1})\phi(y_{j+1}) \cdots \phi(y_n)\} \rangle.
\end{aligned}$$

This is called the **Schwinger–Dyson equations**. Historically, this was how Schwinger showed that the perturbative way and Feynman diagram way of doing quantum field theory are equivalent.

7.2 Feynman diagrams

Write $\delta_{xi} = \delta^4(x - x_i)$ and $D_{ij} = D_{ji} = D_F(x_i, x_j)$ and $D_{xi} = D_F(x, x_i)$. Then $\Box_x D_{x1} = -i\delta_{x1}$ and we have

$$\langle\phi_1\phi_2\rangle = \int d^4x\delta_{x1}\langle\phi_x\phi_2\rangle = i\int d^4x(\Box_x D_{x1})\langle\phi_x\phi_2\rangle = i\int d^4x D_{x1}\Box\langle\phi_x\phi_2\rangle.$$

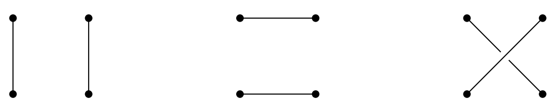
For instance in the free theory, $\Box_x\langle\phi_x\phi_y\rangle = -i\delta_{xy}$ so

$$\langle\phi_1\phi_2\rangle = i\int d^4x D_{x1}(-i\delta_{x2}) = D_{12}.$$

If we have four terms,

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= \int d^4x D_{x1}[\delta_{x2}\langle\phi_3\phi_4\rangle + \delta_{x3}\langle\phi_2\phi_4\rangle + \delta_{x4}\langle\phi_2\phi_3\rangle] \\ &= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.\end{aligned}$$

We can represent these terms as diagrams that connect dots between $1, 2, 3, 4$.



Let us now look at a theory with interactions, with the simplest possible interaction

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \frac{g}{3!}\phi^3, \quad \Box\phi = \frac{g}{2}\phi^2.$$

In this case,

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4 D_{1x}\Box_x\langle\phi_x\phi_2\rangle \\ &= i\int d^4x D_{1x}[\langle\tfrac{g}{2}\phi_x^2\phi_2\rangle - e\delta_{x2}] \\ &= D_{12} - \frac{g}{2}\int d^4x d^4y D_{1x}D_{y2}\Box_y\langle\phi_x^2\phi_y\rangle \\ &= D_{12} - \frac{g^2}{4}\int d^4x d^4y D_{1x}D_{2y}\langle\phi_x^2\phi_y^2\rangle + ig\int d^4x D_{1x}D_{2y}\langle\phi_x\rangle.\end{aligned}$$

And we can go on, by removing one x and putting in two x, until we get the order of g we want. For $\langle\phi_x^2\phi_y^2\rangle$, we can just assume the free field, because we already have g, and so

$$\begin{aligned}\langle\phi_x^2\phi_y^2\rangle = \langle\phi_x\phi_x\phi_y\phi_y\rangle = 2D_{xy}^2 + D_{xx}D_{yy} + O(g), \\ \langle\phi_x\rangle = i\int d^4y D_{yx}\Box_y\langle\phi_y\rangle = \frac{ig}{2}\int d^4y D_{xy}D_{yy}.\end{aligned}$$

So we finally get

$$\langle\phi_1\phi_2\rangle = D_{12} - g^2\int d^4x d^4y\Big(\frac{1}{2}D_{1x}D_{xy}^2D_{y2} + \frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2} + \frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}\Big).$$

You can draw the diagrams for each term as well.

We can generalize this process to **position space Feynman rules**.

(1) Points are x for each external position.

(2) Draw a line from each point.

(3) A line can either connect to another line or split due to an interaction.

(4) A vertex is proportional to the coefficient of $\mathcal{L}'_{\text{int}}[\phi] \times i$.

(5) At a given order of perturbation theory, sum all possible diagrams and integrate over internal positions.

But what are the numerical factors like 1/2 or so on? We conventionally normalize $n!$ for each ϕ^n, like

$$\mathcal{L} = \frac{g}{2!6!}\phi_1^2\phi_2^6.$$

If two line connect to each other, we get a symmetry factor. If the diagram has a symmetry, then we need to divide by the number of symmetries. (You actually rarely need symmetry factors.)

8 September 27, 2018

We had the formula for the cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\text{CM}}^2}|\mathcal{M}|^2, \quad S = \mathbf{1} + i\delta^4(p)\mathcal{M},$$

and then we looked at

$$\langle f|S|i\rangle = \int d^4x_1(\Box_1 + m^2)e^{ipx_1}\cdots\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.$$

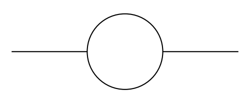
Then we were able to interpret

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$$

as a sum of Feynman diagrams. We are now going to try and do this in momentum space. This way, we don't have to take the Fourier transform and things become more simpler.

8.1 Feynman diagrams in momentum space

Let's take a example of a $\frac{g}{6}\phi^3$ interaction and the following diagram.



Then the corresponding term is

$$T = \frac{(ig)^2}{2}\int d^4x \int d^4y D_F(x,x)D_F(x,y)^2 D_F(y,x_2).$$

By Fourier transforming, we get

$$T = -\frac{g^2}{2}\int d^4x d^4y d^4p_1\cdots d^4p_4 e^{ip_1(x_1-x)}e^{ip_2(y-x_2)}e^{ip_3(x-y)}e^{ip_4(x-y)} \\ \cdot\frac{i}{p_1^2+i\epsilon}\cdots\frac{i}{p_4^2+i\epsilon}.$$

If we do the x-integral, we get momentum conservation $\delta^4(p_3+p_4-p_1)$ and if we do the y-integral, we get $\delta^4(p_2-p_3-p_4)$. Then the p_3-integral sets $p_3 = p_1 - p_4$ and so $\delta^4(p_2-p_3-p_4)$ becomes $\delta^4(p_2-p_1)$. So if we set $k = p_4$ and $p_3 = p_1 - k$, we get

$$T = -\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\delta^4(p_1-p_2)(2\pi)^4 e^{ip_1x_1}e^{-ip_2x_2} \\ \frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{(p_1-k)^2+i\epsilon}\frac{i}{k^2+i\epsilon}.$$

LSZ says that

$$\langle p_f|S|p_i\rangle = \left[-i\int d^4x_1 e^{-ip_ix_1}p_i^2\right]\left[-i\int d^4x_2 e^{+ip_fx_2}p_f^2\right]T.$$

The integrals over x_1 and x_2 give $\delta^4(p_1-p_i)$ and $\delta^4(p_2-p_f)$, and this becomes

$$\langle p_f|S|p_i\rangle = (2\pi)^4\delta^4(p_i-p_f)\left[-\frac{g^2}{2}\int\frac{d^2k}{(2\pi)^4}\frac{i}{(p_i-k)^2+i\epsilon}\frac{i}{k^2+i\epsilon}\right].$$

So here are the **momentum space Feynman rules**:

(1) Internal lines get $\frac{i}{p^2-m^2+i\epsilon}$.

(2) Vertices come from $\mathcal{L}_{\text{int}}$.

(3) External lines do not get propagators.

(4) 4-momentum is conserved at each vertex.

(5) Integrate over undetermined momentum.

8.2 Hamiltonian derivation

Recall that in the Lagrangian formalism, we have

$$[\phi(x,t),\dot\phi(x,t)] = i\hbar\delta^3(\vec x-\vec y),\quad (\Box+m^2)\phi = \mathcal{L}'_{\text{int}}[\phi].$$

In the Hamiltonian picture, we have

$$[\phi(x,t),\pi(x,t)] = i\hbar\delta^3(\vec x-\vec y),\quad i\partial_t\phi = [H,\phi].$$

Our fields are

$$\phi(\vec x) = \int d^3p\frac{1}{\sqrt{2\omega_p}}(a_pe^{i\vec p\vec x}+a_p^\dagger e^{-i\vec p\vec x}).$$

In the interaction picture, fields have interaction,

$$H = H_0+H_{\text{int}},\quad \phi_0(x,t) = e^{iH_0t}\phi(\vec x)e^{-iH_0t} = \int d^3p(a_pe^{ipx}+a_p^\dagger e^{-ipx}).$$

We now write

$$\phi(x,t) = e^{iHt}\phi(x,0)e^{-Ht} = U^\dagger(t,0)\phi_0(x,t)U(x,0).$$

Here, U can be formally written as

$$U(t,0) = e^{iH_0t}e^{-Ht}\approx e^{-iV}.$$

To be precise, we have

$$U_{21} = U(t_2,T_1) = e^{iH_0(t_2-t_1)}e^{-iH(t_2-t_1)} = T\{\exp(-i\textstyle\int_{t_1}^{t_2}dtV_I(t))\},$$

where V_I is the potential for the interacting theory. For instance, $\phi(x, t_2) = U_{21}\phi(x, t_1)U_{12}$. So we get

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0|T\{U_{\infty 0}U_{01}\phi_0(x_1)U_{01}U_{02}\phi(x_2)U_{20}\cdots U_{0-\infty}|0\rangle}{\langle 0|U_{\infty-\infty}|0\rangle} \\ &= \frac{\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)U_{-\infty\infty}\}|0\rangle}{\langle 0|U_{-\infty\infty}|0\rangle},\end{aligned}$$

because U commute with each other. So we have

$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\}|\Omega\rangle = \bigg\langle 0\bigg|T\bigg\{&\phi_0(x_1)\phi_0(x_2) - ig\int d^4y\phi_0(y)^3\phi_0(x_1)\phi_0(x_2) \\ &+ \frac{(-ig)^2}{2}\int d^4x\int d^4y\phi_0(x)^3\phi_0(y)^3\phi_0(x_1)\phi_0(x_2)\}\bigg|0\bigg\rangle.\end{aligned}$$

You can write out this and see that this is really the same thing as the Feynman diagrams.

8.3 Matrix element for the two-to-two scattering

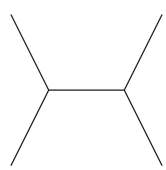
Let us take our favorite example

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3.$$

Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2E^2}|\mathcal{M}|^2.$$

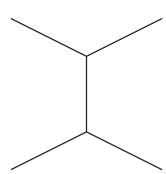
If we look at



we get

$$iM = ig\frac{i}{k^2 - m^2 + i\epsilon} = \frac{-ig^2}{(p_1 + p_2)^2 - m^2 + i\epsilon}, \quad M_t = \frac{g^2}{s - m^2 + i\epsilon}, \quad s = (p_1+p_2)^2.$$

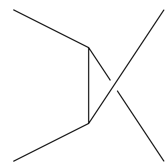
But there are other diagrams. We have



with

$$M_t = \frac{-g^2}{t - m^2 + i\epsilon}, \quad t = (p_1 - p_3)^2,$$

and there is



which is

$$M_u = \frac{-g^2}{t - m^2 + i\epsilon}, \quad u = (p_1 - p_4)^2,$$

Then we can add them up and get

$$M_{\text{tot}} = -g^2\Big(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\Big), \quad s + t + u = \sum_{i=1}^{4} m_i^2.$$

Electron positron $e^-e^+ \to \mu^-\mu^+$ only has an s-channel, Rutherford scattering only has a t channel, electron scattering has t and u channels, etc.

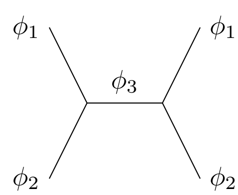
The Lagrangian sometimes have derivative couplings. For instance,

$$\mathcal{L} = \lambda\phi_1(\partial_\mu\phi_2)(\partial_\mu\phi_3).$$

Our field are

$$\phi = \int d^3p \frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ipx} + a_p e^{-ipx}),$$

so we get out a factor of ip_μ if a particle is created, and $-ip_\mu$ if a particle is annihilated. This means that the momentum leaves or enters the vertex. If we look at $\phi_1\phi_2 \to \phi_1\phi_2$, and the diagram



corresponds to

$$iM = (i\lambda)^2 \frac{(-ip_2^\mu)(ik^\mu)(-ik^\mu)(-ip_4^\mu)}{k^2 - m^2 + i\epsilon}.$$

9 October 2, 2018

The course up to this point can be summarized as the momentum space Feynman rules. The example we've been studying is scalar field theory

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3.$$

Then you draw all the Feynman diagrams and internal lines get factors of $\frac{i}{p^2-m^2+i\epsilon}$. Then we get factors of

$$ig$$

from the tree level, and then

$$(ig)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k-p_1)^2-m^2+i\epsilon}\frac{i}{(k-p_2)^2-m^2+i\epsilon}$$

from the next level with one loop. If you have a derivative term, you put in ip for annihilation and $-ip$ for creation.

9.1 Writing down the Lagrangian

But how do we write down the Lagrangian for the theory? We start with symmetries of the theory. There are

- translation invariance $\phi(x) \to \phi(x+a)$,
- Lorentz invariance $x^\mu \to \Lambda^{\mu\nu}x^\nu$, including rotations and boosts,
- unitarity, that is, conservation of probabilities, $\langle\phi|\phi'\rangle = \langle\psi|e^{-iHt}e^{iHt}|\phi'\rangle$, preserved by time-translation and other symmetries,
- internal symmetries, e.g., phase rotation $\phi \to e^{i\alpha}\phi$.

Lorentz invariance and translation invariance are together called Poincaré invariance.

Symmetries mix states in the Hilbert space. For instance, e^- has two states

$$|\uparrow\rangle, J_z = \frac{1}{2}, \quad |\downarrow\rangle, J_z = -\frac{1}{2}.$$

If we rotate our apparatus, we should also be rotating the spins. Another example is polarization of light. We go as far as defining a **particle** as a (minimal) set of states that mix under Poincaré transformations. More mathematically, particles transform as an irreducible unitary representations of the Poincaré group. This is good because we have reduced the physics to a mathematical problem.

Definition 9.1. A **group** $G = \{g_i\}$ with a rule $g_ig_j = g_k$. The conditions are

1. there exists a 1 with $1g = g1 = g$ for all g,
2. there is an inverse g_i^{-1} with $g^{-1}\cdot g = 1$,
3. it is associative, $g_1(g_2g_3) = (g_1g_2)g_3$.

Examples are rotations which are 3×3 matrices with $R_{ij}^T = R_{ij}^{-1}$.

Definition 9.2. A **representation** is an embedding of G into a set of linear operators acting on a vector space.

An example is the trivial representation $g_i \mapsto 1$. Another representation of the 3×3 rotations is the 4-dimensional representation given by just

$$A \mapsto \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 9.3. An **irreducible representation** is a representation where no subspace of vector space is preserved under G.

But there is a problem between Poincaré invariance and unitarity. If P_{ij} is a transformation, then I should have

$$\langle\psi|\psi'\rangle = \langle\psi|P^\dagger P|\psi\rangle, \quad P^\dagger P = 1.$$

In general, this is easy to do, but if we have Lorentz invariance, this is not easy. Let's take a 4-dimensional space for instance, and write

$$|\psi\rangle = c_0|V_0\rangle + c_1|V_1\rangle + c_2|V_2\rangle + c_3|V_3\rangle.$$

Then

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 \geq 0.$$

But this is not Lorentz invariant. So we really want $\langle V_\mu|V_\nu\rangle = g_{\mu\nu}$. But then this can't be interpreted as probability. If you study representations of the Poincaré group, you find the following.

Proposition 9.4. *There is no finite-dimensional unitary representations of the Lorentz group of the Poincaré group.*

However, there exist infinite-dimensional unitary representations of the Poincaré group. Wigner studied this, and it turns out that there are two classes of representations, classified by mass m and spin J where

- $m > 0$ and there are $2J + 1$ states,
- $m = 0$ and there are 2 states.

So how do we interpret this physically? We want $\mathcal{E} > 0$ in a classical theory. This can be computed by the energy-momentum tensor

$$\mathcal{E} = T_{00} = \sum_n \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\dot{\phi}_n - \mathcal{L}.$$

1. In the spin 0 case, there is one degree of freedom, with $J = 0$ and $m \geq 0$ one state. Then
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2, \quad (\Box + m^2)\phi = 0.$$
Then
$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{1}{2}[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] \geq 0.$$
This is satisfies as long as $m^2 \geq 0$.

2. In massive spin 1, we expect $2J + 1 = 3$ degrees of freedom. Minimally, we can embed this in $A_\mu(x)$, which is 4 degrees of freedom, but this splits into $4 = 3 + 1$. Then we can write down
$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial_\nu A_\mu) + \frac{1}{2}m^2 A_\mu^2, \quad (\Box + m^2)A_\mu = 0.$$
Then the energy density is
$$\mathcal{E} = \frac{1}{2}[(\partial_t \vec{A})^2 + (\nabla_i A_j)^2 + m^2\vec{A}]^2 - \frac{1}{2}[(\partial_t A_0)^2 + (\vec{\nabla} A_0)^2 + m^2 A_0^2].$$
This holds generally for any four scalar fields, but if we further impose the condition that the Lorentz group representation corresponding to A_μ is the standard representation, we can write more things like
$$\mathcal{L} = \frac{a}{2}A_\mu \Box A_\mu + \frac{b}{2}A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2.$$
Then the equations of motion becomes
$$a\Box A_\mu + b\partial_\mu\partial_\nu A_\nu + m^2 A_\mu = 0, \quad ((a+b)\Box + m^2)(\partial_\mu A_\mu) = 0.$$
If $a = -b$, then we will get $\partial_\mu A_\mu = 0$. What this is doing is projecting out the 1-dimensional trivial representation. If we choose $a = 1$ and $b = -1$, then we get
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
with energy density
$$\begin{aligned}\mathcal{E} &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}m^2(A_0^2 + \vec{A}^2) - A_0\partial_t(\partial_\mu A_\mu) - A_0(\Box + m^2)A_0 + \partial_i[A_0 F_{0i}], \\ &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}m^2(A_0^2 + \vec{A}^2) + \partial_i[A_0 F_{0i}] \geq 0\end{aligned}$$
with the equations of motion $(\Box + m^2)A_\mu = 0$ and $m^2\partial_\nu A_\mu = 0$. This is called the **Proca Lagrangian**. Note that the positive energy density condition forced our Lagrangian to be this.

10 October 4, 2018

There was this theorem due do Wigner.

Theorem 10.1 (Wigner)**.** *A particle is associated to an irreducible unitary representation of the Poincaré group.*

There are these things $A_\mu, T_{\mu\nu}$ that are finite-dimensional, irreducible, and non-unitary. Then there are fields

$$A_\mu(x), T_{\mu\nu}(x)$$

that are infinite-dimensional, reducible, non-unitary representations. Mathematically, unitary irreducible representations correspond to two invariants m and J. These are

- $J > 0$ and $m > 0$: $2J + 1$ polarizations,
- $J > 0$ and $m = 0$: 2 polarizations,
- $J = 0$: 1 polarization,

where $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. We want to describe the physics. What Wigner did was describe the Hilbert space, with vectors $\epsilon^i(p) = |\epsilon^i; p\rangle$ for $i = 1, \ldots, n$.

10.1 Representations of the Poincaré group

So we want to first construct the representations $\epsilon^i_\mu(p)$, and then construct a Lagrangian so that the extra degree of freedom A_μ does not get produced. Let's review what we did last time. We connected unitarity with $E > 0$. For a scalar field, we can write

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2,$$

and then we got

$$\mathcal{E} = \frac{1}{2}[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] \geq 0$$

if $m^2 \geq 0$. For a massive spin 1 particle, we wrote

$$\mathcal{L} = a(\partial_\mu A_\mu)^2 + b(\partial_\mu A_\nu)^2 + m^2 A_\mu^2,$$

and this constraint $\mathcal{E} \geq 0$ was satisfied for $a = -b$ and $m^2/a > 0$. Then the constraint was $\partial_\mu A_\mu = 0$ and $(\Box + m^2)A_\mu = 0$. So the representation $A_\mu(x)$ splits into a spin 1 and a spin 0 representation.

So how do we quantize this theory? We can write

$$A_\mu(x) = \sum_{j=1}^{3}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(\epsilon^j_\mu(p)e^{ipx}a^{j\dagger}_p + \epsilon^j_\mu(p)e^{-ipx}a^j_p), \quad [a^j_p, a^{k\dagger}_q] = (2\pi)^3\delta^3(p-k)\delta_{jk}.$$

Then we can write

$$|\epsilon^j; p\rangle = a^{j\dagger}_p|0\rangle, \quad \langle 0|A_\mu(x)|\epsilon_i; p\rangle\epsilon^i_\mu(p)e^{ipx}.$$

But we haven't talked about what the representation looks like. Suppose we have $p^\mu = (m, 0, 0, 0)$, where $\partial_\mu A_\mu = 0$ looks like $p_\mu \epsilon^j_\mu = 0$. Then if we have

$$\epsilon_1^\mu = (0,1,0,0), \quad \epsilon_2^\mu = (0,0,1,0), \quad \epsilon_3^\mu = (0,0,0,1),$$

how do we boost it to $p^\mu = (E, 0, 0, p_z)$ with $p_z^2 + m^2 = E^2$? This uses the method of induced representations.

10.2 Induced representations

How do we construct unitary representations of the Poincaré group?

1. Find a subgroup that stabilizes p^μ. Here, we can take $p^\mu = (m, 0, 0, 0)$ and the little group is SO(3).
2. Then construct finite-dimensional irreducible representations of the little group. For SO(3), we have $0, 1/2, 1, \ldots$. In our case, we have $J = 1/2$ and $\epsilon^1_\mu = (0, 1, 0, 0$ and so on.
3. Any g in the Lorentz group can be written as $g = b \cdot r$ where b is the boosts and r is the rotation, where b is in the coset $\mathrm{SO}(1,3)/\mathrm{SO}(3)$.
4. Once we can write this, we find a basis $\epsilon^i_\mu(b \cdot p_\mu)$.
5. Now we define the representation by

$$g \cdot \epsilon = (r \cdot \epsilon)(b \cdot p_\mu).$$

What we really want to get is a photon. But here, we have to deal with the massless case. So how do we want to even construct this theory? Let's just see what happens. Let's take the massive theory and just take the limit. We expect two polarizations. Then natural thing to try is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2.$$

This does two things: we lose the $\partial_\mu A_\mu = 0$ constraint. If we just look at the equations of motion, what we get is

$$\Box A_\mu + \partial_\mu A_\nu A_\nu = 0, \quad \mathcal{E} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \geq 0.$$

So somehow, despite these problems, remarkably things come out naturally. What happened is that because

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

this is invariant under $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ for any $\alpha(x)$. This is called **gauge invariance**, and this is some redundancy in the embedding of the physics of ϵ_μ into $A_\mu(x)$.

Because of this redundancy, I can impose additional condition and choose what A to use. This is called gauge choice. We are going to choose A_μ so that $\partial_i A_i = 0$. We can do this because the gauge transformation is given by

$$\partial_i A_i \to \partial_i A_i + \vec{\nabla}^2 \alpha; \quad A_0 \to A_0 + \partial_t,$$

and then you can always solve this Laplace equation. Moreover, we can even set $A_0 = 0$. At the end, we have

$$A_0 = 0, \quad \partial_i A_i = 0, \quad \partial_\mu A_\mu = 0.$$

This is called the **Coulomb gauge**.

So let's take $p = (E, 0, 0, E)$ some photon. The two constraints on the polarization vectors are $\epsilon_0 = 0$ and $p_\mu \epsilon_\mu = 0$. Then the two polarization that are allowed are

$$\epsilon_1 = (0, 1, 0, 0), \quad \epsilon_2 = (0, 0, 1, 0).$$

In this case, we have this decomposition

$$A_\mu = 1 \oplus 0 \oplus 0$$

of dimension $4 = 2 + 1 + 1$.

Example 10.2. Let me take $p^\mu = (E, 0, 0, E)$, so that

$$\epsilon_1^\mu = (0, 1, 0, 0), \quad \epsilon_2^\mu = (0, 0, 1, 0).$$

If we take

$$\Lambda = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix},$$

we get that Λ is a Lorentz transformation in the little group. You can compute

$$\Lambda \epsilon_1 = (1, 1, 0, 1) = \epsilon_1 + \frac{1}{E} p^\mu.$$

This means that this acts on ϵ_1 to give ϵ_1.

11 October 9, 2018

In a massless spin 1 theory, we can quantize the theory by looking at

$$A_\mu(x) = \sum_{j=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p^{j\dagger} \epsilon_\mu^*(p) e^{ipx} + a_p^j \epsilon_\mu(p) e^{-ipx}).$$

Here, we have

$$|\epsilon_1(p)\rangle = a_p^{1\dagger}|0\rangle, \quad |\epsilon_2(p)\rangle = a_p^{2\dagger}|0\rangle.$$

Under Lorentz transformations, this transforms as $\epsilon_\mu^i \to \epsilon_\mu^i + p_\mu$. So this is Lorentz-invariant only if $p_\mu M^\mu = 0$. This is called the **Ward identity**.

11.1 Scalar quantum electrodynamics

The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$, and this is exactly $\epsilon_\mu \to \epsilon_\mu + p_\mu$ under the Fourier transformation.

If we wanted an interaction with a scalar field, we can write

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + A_\mu \phi \partial_\mu \phi.$$

This is not Lorentz-invariant, and you can check this by

$$A_\mu \phi \partial_\mu \phi + \frac{1}{e}(\partial_\mu \alpha)\phi \partial_\mu \phi.$$

So what we will do is to remove redundancy using this. This only possibly works with more than one field. Let us try ϕ_1 and ϕ_2, with

$$\phi = \phi_1 + i\phi_2 \to e^{i\alpha}\phi.$$

If we define a covariant derivative, it transforms as

$$D_\mu \phi = [\partial_\mu + ieA_\mu]\phi \to e^{-i\alpha} D_\mu \phi.$$

So for scalar QED, we can write down the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{SQED}} &= -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - m^2 |\phi|^2 \\ &= -\frac{1}{4} F_{\mu\nu}^2 + ieA_\mu(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) + e^2 A_\mu^2 \phi \phi^* - m^2 \phi \phi^*. \end{aligned}$$

If you recognize the term

$$J_\mu = \phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi,$$

this is the Noether current associated to phase rotation. So it has $\partial_\mu J^\mu = 0$ on the equation of motion. It's not a coincidence that A_μ couples to a Noether current; the current is something that can be measured.

$$\begin{array}{c}\text{massless}\\ \text{spin 1}\end{array} \Rightarrow \begin{array}{c}\text{gauge invariant}\\ \text{Lagrangian}\end{array} \Rightarrow \begin{array}{c}\text{global}\\ \text{symmetry}\end{array} \Rightarrow \begin{array}{c}\text{conserved}\\ \text{charge}\end{array}.$$

The equations of motion is given by

$$\begin{aligned}(\Box + m^2)\phi &= i(-eA_\mu)\partial_\mu\phi + i\partial_\mu(-eA_\mu\phi) + (-eA_\mu)^2\phi,\\ (\Box + m^2)\phi^* &= i(eA_\mu)\partial_\mu\phi^* + i\partial_\mu(eA_\mu\phi^*) + (eA_\mu)^2\phi^*.\end{aligned}$$

So another consequence of a massless spin 1 particle, is that there is an **antiparticle** associated to a particle. For instance, π^- and π^+ are spin 0 particles that couple to the photon.

11.2 Photon propagator

We want to develop the Feynman rules for scalar QED. So we need the propagator for the photon. Recall that for a scalar, we had

$$D_F = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int d^4p e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}.$$

Then we had $(\Box + m^2)D_F = -i\delta^4(x-y)$. Classically, we can think of this as inverting

$$(\Box + m^2)\phi = J, \quad \phi(x) = \Pi(x,y)J(y),$$

which is the equations of motion to the classical Lagrangian.

In the photon case, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + A_\mu J^\mu, \quad \partial_\mu F_{\mu\nu} = J_\nu.$$

If we try to Fourier transform and invert this, we get

$$(k^2 g_{\mu\nu} - k_\mu k_\nu)\tilde{A}_\mu = \tilde{J}_\nu.$$

But then this matrix is not invertible, which we should have expected because there is gauge-invariance. One choice is to choose a gauge and then substitute into the Lagrangian. But this is annoying, so we are going to deform the Lagrangian and write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + JA.$$

(This deformation is very special, and it's okay for reasons that are not obvious.) Then if we invert this, we get

$$i\Pi_{\mu\nu} = -i\frac{g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 + i\epsilon}.$$

This you can check this explicitly, that

$$-[k^2 g_{\mu\nu}(1-\tfrac{1}{\xi})k_\mu k_\nu]\Pi_{\nu\alpha} = g_{\mu\alpha}.$$

Generically, propagators are like

$$\Pi \sim \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{\sum_s |s\rangle\langle s|}{k^2 - m^2 + i\epsilon}$$

because we want the propagator to preserve the spin states.

Now we are almost done. Let us quantize the theory

$$\phi_1 = \int \frac{d^3p}{(2\pi)^3}(a_{p,1}e^{-ipx} + a^\dagger_{p,1}e^{ipx}),$$
$$\phi_2 = \int \frac{d^3p}{(2\pi)^3}(a_{p,2}e^{-ipx} + a^\dagger_{p,2}e^{ipx}).$$

Then if we write $a_p = a_{p,1} + ia_{p,2}$ and $b_p = a_{p,1} - ia_{p,2}$, then we get

$$\phi = \int \frac{d^3p}{(2\pi)^3}(a_p e^{-ipx} + b^\dagger_p e^{ipx}),$$
$$\phi^* = \int \frac{d^3p}{(2\pi)^3}(b_p e^{-ipx} + a^\dagger_p e^{ipx}).$$

So ϕ is creating an antiparticle and annihilating a particle, and ϕ^* is creating a particle and annihilating an antiparticle.

11.3 Feynman rules for scalar QED

Now we compute

$$\langle 0|T\{\phi(x)^*\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{ip(x-y)}.$$

There are interaction terms

$$-ieA_\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2 A_\mu A_\mu \phi\phi^*.$$

So let's see. The term ϕ creates π^- with ip^μ, and ϕ^* creates π^+ with $i\pi^\mu$. Or, ϕ annihilates π^+ with $-ip^\mu$ and ϕ^* annihilates π^- with $-ip^\mu$.

If you analyze all the cases, you are going to get Figure 1. But if you look at this, you see that you can package all of this compactly. We can invert the arrows for π^+, and then only the interactions we are allowed to have are arrows continuous thought the vertices. Then all interaction terms are just

$$-ie(p_1^\mu + p_2^\mu).$$

This is called the **Feynman–Stueckelberg interpretation**. You can interpret as a photon decaying to e^-e^+ as a electron bouncing off a photon.

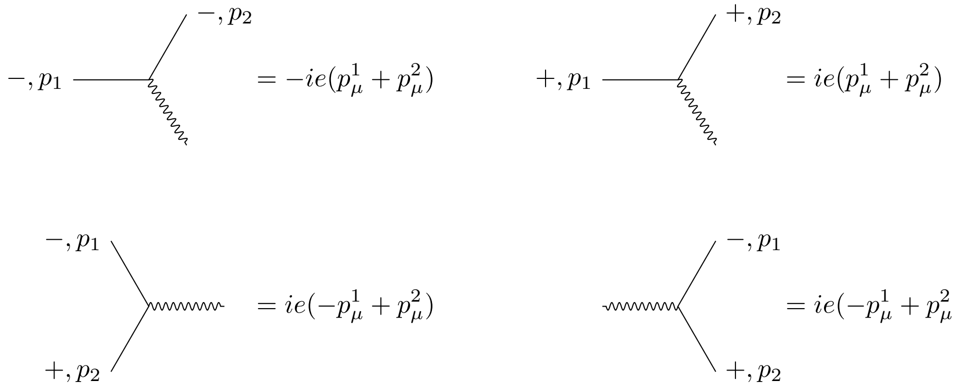


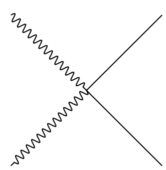
Figure 1: Feynman diagrams for scalar QED: time from left to right

Dirac had this interpretation of thinking of antiparticles as holes in this Dirac sea of negative energy. Dirac didn't like the Klein–Gordan equation

$$(\Box + m^2)\phi = 0$$

having negative energy states, $E_p = \pm\sqrt{\vec{p}^2 + m^2}$. The mathematics is just the same, by looking at creation operators as just dagger of annihilation operators, but this language is totally unnecessary. Also, there is no physical justification for this interpretation.

The other interaction is



and this has contribution $(i2e^2 g^{\mu\nu})$.

12 October 11, 2018

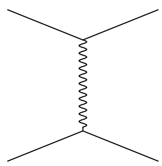
So we have our first theory of scalar QED. There is the Lagrangian

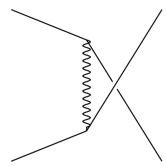
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \phi(\Box + m^2)\phi^* - ieA_\mu\phi^*\partial_\mu\phi + ieA_\mu\phi\partial_\mu\phi^* + e^2A_\mu^2\phi^*\phi.$$

with all these Feynman rules. We should be able to check gauge invariance and the Ward identity.

12.1 Gauge invariance and the Ward identity for scalar QED

Let us consider the Mollar scattering $e^-e^- \to e^-e^-$ in scalar QED.





The t-channel contribution is

$$iM_t = (-ie)(p_1^\mu + p_2^\mu)\frac{-i[g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2}(-ie)(p_2^\nu + p_4^\nu).$$

But then

$$k^\mu(p_1^\mu + p_3^\mu) = (p_1^\mu - p_3^\mu)(p_1^\mu + p_3^\mu) = m^2 - m^2 = 0$$

and similarly $k^\nu(p_2^\nu + p_4^\nu) = 0$. So we just have

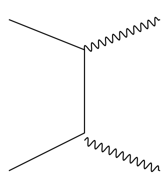
$$M_t = e^2\frac{s-u}{t}.$$

Likewise, we have

$$M_u = e^2\frac{s-t}{u}, \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}(M_t + M_u)^2.$$

Here, the fact that the ξ term vanishes shows that the theory is gauge invariant.

Let's now check the Ward identities, using the process $\pi^+\pi^- \to \gamma\gamma$. Here,



gives the contribution

$$iM_t = (-ie)^2\frac{(2p_1^\mu - p_3^\mu)\epsilon_3^{*\mu}(p_4^\nu - 2p_2^\nu)\epsilon_4^\nu}{(p_1 - p_3)^2 - m^2} = M_{\mu\nu}\epsilon_\mu^{3*}\epsilon_\nu^{4*}.$$

Now checking the Ward identity is checking if this is zero if $\epsilon_3^{\mu *} = p_3^\mu$. But then we get
$$M_t^w = e^2[p_4^\nu - 2p_2^\nu]\epsilon_4^\nu.$$
This is nonzero because there are other diagrams. If we look at the u-channel, we get
$$M_u^w = e^2[p_4 - 2p_3^\nu]\epsilon_4^\nu, \quad M_4^2 = 2e^2 p_3 \epsilon_4,$$
where M_4 is the contribution of the 4-particle interaction.

12.2 Lorentz invariance and soft photons

Suppose we have a diagram, and there is some p_1 going in, with contribution $\mathcal{M}_0(p_i)$. Here, we can modify the diagram by just adding a photon. The M matrix for this is
$$\mathcal{M}_i = (-ie)\frac{i[2p_1 - q]\epsilon}{(p-q)^2 - m^2}\mathcal{M}_0.$$
If we use $p^2 = m^2$ and $q^2 = 0$ and $\epsilon q = 0$, then we can approximate this as
$$\mathcal{M}_i = -e\frac{p\epsilon}{pq}\mathcal{M}_0(p_i + q) \approx -e\frac{p\epsilon}{pq}\mathcal{M}_0(p_i)Q_i.$$
Likewise, we can add one photon to the diagram at the outgoing edge. Then we get a contribution of
$$\mathcal{M}_i = +e\frac{p\epsilon}{pq}\mathcal{M}_0 Q_i.$$
So adding a photon in some edge gives a contribution of
$$\mathcal{M} = e\mathcal{M}_0\biggl[\sum_{\text{incoming}} Q_i \frac{p_i\epsilon}{p_i q} - \sum_{\text{outgoing}} Q_i \frac{p_i \epsilon}{p_i q}\biggr].$$

Under the Lorentz transformation, we get $\epsilon^\mu \to \epsilon^\mu + \Lambda q^\mu$. So for $\mathcal{M}$ to be Lorentz-invariant, we must have
$$e\mathcal{M}_0\biggl[\sum_{\text{incoming}} Q_i - \sum_{\text{outgoing}} Q_i\biggr] = 0.$$
So we get conservation of charge from this.

This is really universal. Even if the interaction term is arbitrary, say
$$-ie\Gamma_\mu(p,q)\epsilon^\mu = -ie(F_j p^\mu + G_j q^\mu)\epsilon_\mu, \quad F_j(p^2, q^2, pq) = F_j\Bigl(\frac{pq}{m^2}\Bigr),$$
we can just look at the $q \to 0$ limit and get
$$-ieF_j\Bigl(\frac{pq}{m^2}\Bigr)p \to -ieF_j(0)p.$$

So we can just look at an arbitrary theory and define $F_j(0) = Q_j$ as the charge. So we can think of charge as the interaction with low-energy photons for long distances. In this case, we can do the same thing and get conservation

$$\sum_{\text{in}} F_j(0) = \sum_{\text{out}} F_j(0).$$

This gets more interesting. Consider a massless spin-2 particles, and embed into $\epsilon_{\mu\nu}(p)$ polarization tensors. These are transverse and traceless and symmetric:

$$g_{\mu\nu}\epsilon_{\mu\nu} = 0, \quad q_\mu \epsilon_{\mu\nu} = 0, \quad g_{\mu\nu} = g_{\nu\mu}.$$

But because it is massless, it transforms in the same weird way

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \Lambda_\nu q_\mu + \Lambda_\mu q_\nu \Lambda q_{\mu\nu}.$$

If we look at the soft limit, we some contribution

$$\epsilon_{\mu\nu}\Gamma_{\mu\nu} = \epsilon_{\mu\nu} p^\mu p \nu F\Big(\frac{p \cdot q}{m^2}\Big).$$

Then if we o the same thing,

$$\mathcal{M} = \mathcal{M}_0 \Bigg[\sum_{\text{incoming}} F(0) \frac{p^i_\mu p^i_\nu}{(p_i \cdot q)} - \sum_{\text{outgoing}} \frac{p^\mu_i p^\nu_i}{(p_i \cdot q)}\Bigg] \epsilon_{\mu\nu}.$$

The Ward identity gives

$$0 = \mathcal{M}_0 \Lambda_\nu \Bigg[\sum_{\text{in}} F_i(0) p^\nu_i - \sum_{\text{out}} F_j(0) p^\nu_j\Bigg].$$

We already have momentum conservation, and this lets us solve for p_1 as a function of the other p_j. So this is consistent only when $F_i(0) = G_N$ for some universal constant G_N. This is saying that gravity is universal, and couples with every particle with the same coupling constant. This is the only way to have a consistent theory of a spin-2 particle, and this particle must be unique. What about spin-3? If you do the same thing, we should get

$$\sum_{\text{in}} \gamma_j p^\nu_j p^\mu_j = \sum_{\text{out}} \gamma_j p^\nu_j p^\mu_j.$$

If we only look at the $(0,0)$-component, we get $\sum \gamma_j E^2 = \sum \gamma_j E^2$. This is impossible, and so there are no consistent interacting theories of massless particles with $J > 2$.

12.3 Spinors

You've seen spinors before. In non-relativistic quantum mechanics, there is this $|\psi\rangle = |\uparrow\rangle$ and $|\downarrow\rangle$. The quantum mechanics is governed by the Schrödinger–Pauli equation

$$i\partial_t \psi = H\psi = \Bigg[\Big(\frac{p^2}{2m} + V(r) - \mu_B \vec{B} \cdot \vec{L}\Big)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}\Bigg]\psi.$$

Here, we can write

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

and then that matrix is just $\vec{\sigma} \cdot \vec{B}$.

Here, $\vec{B}$ is a 3-vector, and $\vec{\sigma} \cdot \psi$ also transforms like a 3-vector. Also, $\partial_i \sigma_i \psi$ is rotation invariant, and then we can guess and check that

$$\partial_t \psi - \partial_i \sigma_i \psi = 0$$

is Lorentz-invariant. In fact, if we define $\sigma^\mu = (1, \vec{\sigma})$, then

$$\sigma^\mu \partial_\mu \psi = 0$$

is the Dirac equation for $m = 0$. Maybe we could guess that $\sigma^\mu \partial_\mu \psi = m\psi$ is the massive case, but this is wrong. So enough guessing.

This follows naturally from representations of the Lorentz group. There are these

$$\Lambda_{R_z} = R(\theta_z) = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}, \quad B_{\beta_x} = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Now we can look at the infinitesimal generators and extract the **Lie algebra**. Then we get

$$R_z = R(\theta_z) = \begin{pmatrix} 0 & & & \\ & & \theta_z & \\ & -\theta_z & & \\ & & & 0 \end{pmatrix}, \quad \beta_x = \begin{pmatrix} & \beta_x & & \\ \beta_x & & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Then we get

$$\Delta V_0 = \beta_i V_i, \quad \Delta V_i = \beta_i V_0 - \epsilon_{ijk}\theta_j V_k.$$

If we write the i times the rotation generators as J_i and the i times the boost generators as K_i, hen we have

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

13 October 16, 2018

We were talking about representations of the Lorentz group. The Lorentz group is defined as

$$L = \{\Lambda^F : (\Lambda^F)^\dagger g \Lambda^F = g\}.$$

Once we have these matrices, there is a group operation, so we get a group $\mathrm{O}(1,3)$. There is the obvious 4-vector representation

$$V_\mu \to \Lambda_{\mu\nu} V_\nu.$$

To find a representation, we could instead look at the representations of the Lie algebra, which is generated by iJ_1, iJ_2, iJ_3 and iK_1, iK_2, iK_3.

13.1 Representations of the Lorentz group

The Lie algebra structure is given by

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_i] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k,$$

which is called $\mathfrak{o}(1,3)$. How do we find representations of this? We observe that there is a convenient linear combination,

$$J_i^+ = \frac{1}{2}(J_i + iK_i), \quad J_i^- = \frac{1}{2}(J_i - iK_i).$$

Then the relations become

$$[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+, \quad [J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-, \quad [J_i^+, J_j^-] = 0.$$

This means that we really have $\mathfrak{o}(1,3) = \mathfrak{su}(2) \times \mathfrak{su}(2)$.

But we know the finite-dimensional representations of $\mathfrak{su}(2)$: they are just $(2j+1)$-dimensional vector spaces. So irreducible representations of $\mathfrak{so}(1,3)$ are just indexed by (A,B), which is a $(2A+1)(2B+1)$-dimensional representation obtained by taking the tensor (box) product.

Now given any representation of $\mathfrak{so}(1,3)$, we will also get a representations of $\mathfrak{so}(3)$ by looking at $\vec{J} = \vec{J}^+ + \vec{J}^-$. Now we can think about which spin representations appear in each (A,B). Here is a table of this for small values:

(A,B)	dim	spin of $\mathfrak{so}(3)$-representations
$(0,0)$	1	0
$(\frac{1}{2},0)$	2	$\frac{1}{2}$
$(0,\frac{1}{2})$	2	$\frac{1}{2}$
$(\frac{1}{2},\frac{1}{2})$	4	1, 0
$(1,0)$	3	1

Table 1: Representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ restricted to $\mathfrak{so}(3)$

13.2 Dirac spinors

So what are these representations? If we look at $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, these different representations with spin $\frac{1}{2}$. These are called **right-handed spinors** ψ_L

$$J^+ = \frac{\sigma_i}{2}, \quad J^- = 0, \quad \vec{J} = J^+ + J^- = \frac{\vec{\sigma}}{2}, \quad \vec{K} = i(J^- - J^+) = -i\frac{\vec{\sigma}}{2},$$

and **left-handed spinors** ψ_R

$$J^+ = 0, \quad J^- = \frac{\sigma_i}{2}, \quad \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = +i\frac{\vec{\sigma}}{2}.$$

The representations then can be written as

$$\begin{aligned} \delta\psi_L &= \frac{1}{2}(i\theta_i - \beta_i)\sigma_i\psi_L, \quad \delta\psi_L^\dagger = \frac{1}{2}(-i\theta_i - \beta_i)\psi_L^\dagger\sigma_i, \\ \delta\psi_R &= \frac{1}{2}(i\theta_i + \beta_i)\sigma_i\psi_R, \quad \delta\psi_R^\dagger = \frac{1}{2}(-i\theta_i + \beta_i)\psi_R^\dagger\sigma_i. \end{aligned}$$

Now what would the Lagrangian be? We could try and write down

$$\mathcal{L} = \psi_L^\dagger\psi_L,$$

but then

$$\delta(\psi_L^\dagger\psi_L) = -\beta_i\psi_L^\dagger\sigma_i\psi_i \neq 0.$$

This is somewhat expected, because we're using the Hermitian norm on something that transforms under Lorentz transformations. It turns out that the expression

$$\mathcal{L} = m[\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L]$$

is Lorentz invariant.

This is the mass term. What about the kinetic term? It turns out that the Lagrangian

$$\mathcal{L} = \psi_L^\dagger\Box\psi_R$$

is invariant if ψ_L is a spinor or if ψ_L is two scalars. Let us define

$$\sigma_\mu = (\mathbf{1}, \vec{\sigma}), \quad \bar{\sigma}_\mu = (\mathbf{1}, -\vec{\sigma}).$$

Then the Lagrangian

$$\mathcal{L} = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

turns to be Lorentz invariant.

To simplify notation, let us define

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} = \psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\psi}\psi = \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L.$$

This is called the **Dirac spinor**. If we moreover define

$$\gamma^\mu_{4\times 4} = \begin{pmatrix} 0 & \sigma^\mu \\ -\bar\sigma^\mu & 0 \end{pmatrix},$$

then we can write the **Dirac Lagrangian** in the following compact form:

$$\mathcal{L}_{\text{Dirac}} = i\bar\psi\gamma^\mu\partial_\mu\psi + m\overline{\psi}\psi.$$

This representation really is different from just the vector representation. If we look at J_3 with the Dirac representation, we get

$$J_3^{\text{Dirac}} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad J_3^{\text{vector}} = i\begin{pmatrix} 0 & & & \\ & & -1 & \\ & 1 & & \\ & & & 0 \end{pmatrix}.$$

When we look at the eigenvalues, the first one has $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$, and the second one has $-1, 0, 0, 1$. So we see that the first one has two spin $\frac{1}{2}$ representations, while the second one has one spin 0 and one spin 1.

What happens if we rotate by 2π? We get

$$\Lambda_3 = \exp[i\theta J_e^{\text{D}}] = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

So what we really get is that if we rotate by 2π, then we get the -1 of itself. This is not really a representations of the Lorentz group, because 1 is not mapped to 1. Instead, we have constructed a **projective representation**, where group multiplication holds up to a phase. Projective representations of a group G are equivalent to representations of the universal cover of $G = \mathrm{SO}(1,3)$, which is in this case $\mathrm{SL}(2,\mathbb{C})$.

Here is another interesting consequence. If we rotate by π, we get the matrix

$$\Lambda_3(\theta = \pi) = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix}.$$

So if we have $|\uparrow\uparrow\rangle$ and rotate around by π, we get $-(|\uparrow\uparrow\rangle)$, so it is indistinguishable from itself. So this is somehow expressing the Pauli exclusion principle.

14 October 18, 2018

Last time we constructed the Dirac spinors. We looked at irreducible finite-dimensional representations of the Lorentz group, characterized by two half-integers, and then put ψ_L and ψ_R together. They transform as

$$\psi_L \to \exp\Big[\frac{i\theta_j}{2}\sigma_j + \frac{\beta_j}{2}\sigma_j\Big]\psi_L, \quad \psi_R \to \exp\Big[\frac{i\theta_j}{2}\sigma_j - \frac{\beta_j}{2}\sigma_j\Big]\psi_R.$$

14.1 The Dirac equation

Then we wanted the Lagrangian, and we got

$$\mathcal{L} = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R + i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_L^\dagger \psi_R - m\psi_R^\dagger \psi_L,$$

where $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma})$. Then we defined $\psi = \left(\begin{smallmatrix}\psi_L \\ \psi_R\end{smallmatrix}\right)$ and $\gamma^\mu = \left(\begin{smallmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{smallmatrix}\right)$ and then wrote

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi.$$

The equations of motion is then given by

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

We can think of these matrices γ_μ as algebraic objects, independent of their representations. They satisfy the anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad \frac{i}{4}[\gamma_\mu, \gamma_\nu] = S_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}.$$

There are different choices of γ, for instance, the Majorana representations

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & \\ & -i\sigma^1 \end{pmatrix}$$

that is purely imaginary.

Let's look at the Dirac equation and multiply it by $(i\gamma^\mu \partial_\mu + m)$. Then we have

$$\begin{aligned} 0 &= (i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m)\psi = (-\gamma_\mu \gamma_\nu \partial_\mu \partial_\nu - m^2)\psi \\ &= (-\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} - m^2)\psi = -(\Box + m^2)\psi. \end{aligned}$$

So anything satisfying the Dirac equations also satisfy the Klein–Gordan equations.

If we want photon interactions, we can write

$$D_\mu \psi = [\partial_\mu + ieA_\mu]\psi, \quad \mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi,$$

so that there is a cubic interaction term, and the equations of motion become

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0.$$

If we do the same trick of multiplying $(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu + m)$ on the left, we get

$$0 = \left[-\frac{e}{2}F_{\mu\nu}\frac{i}{2}[\gamma_\mu, \gamma_\nu] + D_\mu^2 - m^2\right]\psi.$$

First, this tells us that this is not just the Klein–Gordan equation. If we try to figure out what $\sigma^{\mu\nu}$ is, you can calculate that

$$\sigma_{0i} = -\frac{i}{2}\begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}, \quad \sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix}.$$

Then we get

$$\left[(\partial_\mu - imA_\mu)^2 + m^2 - e\begin{pmatrix} (\vec{B} + i\vec{E})\vec{\sigma} & 0 \\ 0 & (\vec{B} - i\vec{E})\vec{\sigma} \end{pmatrix}\right]\psi = 0.$$

You might remember doing this in quantum mechanics. The electron has some spin, and this interacts with the magnetic field. But we did not know how strong this is going to be. But if we incorporate Lorentz invariant, this exactly fixes the strength of the interaction. In fact, this prediction is that the electron's magnetic dipole moment is given by

$$\mu_B = \frac{e}{2m}.$$

The measured values is about $1.002\mu_B$, so this shows that the spinor business is really going on in nature.

This theory is symmetric under interchanging L and R, and so we say that it is not **chiral**. But nature is actually chiral because of weak interactions, as discovered in the 50s.

Spin is a vector, and a spin operator $\vec{J}$ is an appropiate representation of $SO(1,3)$. For spinors, we have

$$\vec{J} = \frac{\vec{\sigma}}{2} = \vec{S}.$$

So ψ has spin s along the $\vec{v}$ axis if $\vec{v}\cdot\vec{S}\psi = s\psi$. **Helicity** is defined as the projection of the spin on the direction of motion. So we have

$$\frac{\vec{p}\cdot\vec{S}}{|\vec{p}|s} = h\psi,$$

and this is ± 1. This is a useful concept in the massless limit. In this case, we get look at

$$i\gamma^\mu \partial_\mu \psi_L = 0, \quad i\gamma^\mu \partial_\mu \psi_R = 0,$$

so we get $(E - \vec{p}\cdot\vec{\sigma})\psi_R = 0$ and $(E + \vec{p}\cdot\vec{\sigma})\psi_L = 0$. Then ψ_R has helicity always $+1$ and ψ_L has helicity always -1.

14.2 Quantum electrodynamics

Now we want to quantize the theory. Remember how this was done. We had

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + b_p^\dagger e^{ipx}),$$
$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (b_p e^{-ipx} + a_p^\dagger e^{ipx}).$$

Then we can just do the same thing and write

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p^s u_p^s e^{-ipx} + b_p^{s\dagger} v_p^s e^{ipx}),$$
$$\bar{\psi}(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (b_p^s \bar{v}_p e^{-ipx} + a_p^\dagger \bar{u}_p^s e^{ipx}).$$

Here, u_p^s and v_p^s are the wavefunctions for the spin, for the particle and the anti-particle.

To get the spin, we need to solve

$$\begin{pmatrix} -m & p^\mu \sigma_\mu \\ p^\mu \bar{\sigma}^\mu & -m \end{pmatrix} u_s(p) = 0, \quad \begin{pmatrix} -m & -p^\mu \sigma_\mu \\ -p^\mu \bar{\sigma}_\mu & -m \end{pmatrix} v_s(p) = 0.$$

If we solve this, the solutions are

$$u_1(p) = \begin{pmatrix} \sqrt{p\sigma} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \\ \sqrt{p\bar{\sigma}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \end{pmatrix}, \quad u_2(p) = \begin{pmatrix} \sqrt{p\sigma} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \\ -\sqrt{p\bar{\sigma}} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \end{pmatrix}.$$

If we choose $p = (E, 0, 0, p_E)$, then we can show that these are

$$\begin{pmatrix} \sqrt{E - p_E} \\ 0 \\ \sqrt{E - p_E} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \sqrt{E + p_E} \\ 0 \\ \sqrt{E + p_E} \end{pmatrix}.$$

But most of the time we will only care about the sum of these, the average over the spins. Then we get

$$\bar{u}_s u_{s'} = u_s^\dagger \gamma_0 u_{s'} = 2m\delta_{ss'}, \quad \bar{v}_s v_{s'} = -2m\delta_{ss'}.$$

So we can define the inner product

$$\langle u_s | u_{s'} \rangle = \bar{u}_s u_{s'} = 2m\delta_{ss'},$$

and also the outer product

$$\sum_{s=1}^{2} u_s(p) \bar{u}_s(p) = \gamma^\mu p_\mu + m.$$

We will calculate this later, but it will turn out that the propagator is given by

$$\langle \bar{\psi} \psi \rangle = \frac{i(\gamma^\mu \partial_\mu + m)}{p^2 - m^2 + i\epsilon}.$$

15 October 23, 2018

We derived the Lagrangian of QED, (we write $\not{X}$ for $\gamma^\mu X_\mu$)

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi, \quad (i\not{\partial} - e\not{A} - m)\psi = 0.$$

In the non-relativistic limit, we get

$$i\partial_t \psi = H\psi = \left(\frac{\vec{p}^2}{2m} + \frac{2e}{m}\vec{B}\cdot\vec{\sigma} + 0\frac{e}{m}\vec{E}\cdot\vec{\sigma}\right)\psi,$$

so we expect $2e/m$ to be the electron's dipole moment.

To quantize the theory, we had to introduce polarizations u_p^s and v_p^s, and they satisfied

$$\not{p}u = mu, \quad -\not{p}v = mv.$$

So we solved this and got

$$u_p^1 = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ 0 \\ \sqrt{p\cdot\bar{\sigma}} \\ 0 \end{pmatrix}, u_p^2 = \begin{pmatrix} 0 \\ \sqrt{p\cdot\sigma} \\ 0 \\ \sqrt{p\cdot\bar{\sigma}} \end{pmatrix}, v_p^1 = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ 0 \\ -\sqrt{p\cdot\bar{\sigma}} \\ 0 \end{pmatrix}, v_p^2 = \begin{pmatrix} 0 \\ \sqrt{p\cdot\sigma} \\ 0 \\ -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}.$$

In the non-relativistic limit, these are the four polarizations of the particle and the antiparticle.

15.1 Identical particles

Let us look at creation and annihilation operators for every particle. There are

$$a^\dagger_{\vec{p}_i, s_i, n_i}, \quad s_i, \quad n_i,$$

where s_i is the spin of the particle and n_i are the "internal numerators" of the particles (so there are many particles with the same state). States created with the same n_i are called identical particles. So when we fix n, we get

$$|\psi_{12}\rangle = |s_1 p_1 n s_2 p_2 n\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}a^\dagger_{p_1 s_1}a^\dagger_{p_2 s_2}|0\rangle.$$

If we act in a different order, we will get

$$|\psi_{21}\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}a^\dagger_{p_2 s_2}a^\dagger_{p_1 s_1}|0\rangle.$$

We should have that $|\psi_{12}\rangle$ is the same physical state as $|\psi_{21}\rangle$. So we only have a difference in phase, so

$$|\psi_{12}\rangle = \alpha|\psi_{21}\rangle.$$

What α can this be? This should be a representation of the Lorentz group, and there are no one-dimensional representations except for the trivial representation. But α might depend on the path in $|\psi_{12}\rangle \to |\psi_{21}\rangle$.

Example 15.1. In 2-dimensions, if we have two particles at A and B, there are $\mathbb{Z}$-many topological ways to start with $\{A, B\}$ and end with $\{A, B\}$. This is saying that the fundamental group of SO(2) is $\mathbb{Z}$, so we could get $|\psi_{12}\rangle \mapsto \alpha|\psi_{12}\rangle \mapsto \alpha^2|\psi_{12}\rangle \mapsto \cdots$.

But in 3-dimensions, we have too much room, so there is only one way of interchanging A and B. So we have

$$|\psi_{12}\rangle = \alpha|\psi_{21}\rangle,$$

with $\alpha = \pm 1$. If $\alpha = 1$, then the particle is called a **boson** and we get

$$[a^\dagger_{p_1 s_1}, a^\dagger_{p_2 s_2}] = 0.$$

On the other hand, if $\alpha = -1$ then we call it a **fermion** and we have

$$\{a^\dagger_{p_1 s_1}, a^\dagger_{p_2 s_2}\} = 0.$$

If α is not ± 1 (which may happen only in 2-dimensions), then the particle is called an **anyon**. We haven't seen how this connects to the spin-$\frac{1}{2}$ particle, but this at least shows that we can have a consistent Lorentz-invariant theory with $\alpha \neq -1$.

If we had a J^s_z, then we would have

$$J^s_z = \begin{pmatrix} -s & & & \\ & -s+1 & & \\ & & \ddots & \\ & & & s \end{pmatrix}.$$

So if we rotate about the z-axis by angle π, then we would have

$$|AB\rangle = (e^{i\pi J_z})^2 |BA\rangle = e^{2\pi i s}|BA\rangle,$$

where s is the spin. This shows that if s is an integer, this is a boson, and if s is a half-integer, it is a fermion. For a fermion, we have $0 = \{a^\dagger_p, a^\dagger_p\} = 2(a^\dagger_p)^2$, so there are no two fermions in the same state.

This spin statistics is related to stability. If you choose the right statistics and compute the energy density, you get

$$\mathcal{E} = \sum_s \int d^3q \omega_q (a^\dagger_q a_q + b^\dagger_q b_q)$$

which is positive. But if you choose the wrong statistics, you get something like

$$\mathcal{E} = \sum_s \int d^3q \omega_q (a^\dagger_q a_q - b^\dagger_q b_q)$$

which implies that the universe is not stable. Also, to get causality $[\phi(x), \phi(y)] = 0$ or $\{\phi(x), \phi(y)\} = 0$, we need to choose the correct statistics.

15.2 Propagator for QED

This is also important in getting a Lorentz-invariant S matrix. Recall that

$$T\{\phi(x)\phi(y)\} = \phi(x)\phi(y)\theta(x^0 - y^0) + \phi(y)\phi(x)\theta(y^0 - x^0)$$

for scalar fields. If we try to do this for fermions, then we have to define

$$T\{\psi(x)\chi(y)\} = \psi(x)\chi(y)\theta(x^0 - y^0) - \chi(y)\psi(x)\theta(y_0 - x_0)$$

because otherwise $T\{\psi(x)\chi(y)\} \neq -T\{\chi(y)\psi(x)\}$. If we use this definition and try to calculate the time-ordered product, we get

$$\langle 0|T\{\psi(0)\bar{\psi}(x)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{ipx},$$

which is Lorentz-invariant. If we use the wrong statistics, we would have gotten something that is not Lorentz-invariant.

We can also see what the propagator is, directly from the Dirac equation. Recall that the free Dirac equation is

$$(i\not{\partial} - m)\psi = 0, \quad (\not{p} - m)u = 0.$$

Because $\not{p}^2 = \gamma^\mu\gamma^\nu p_\mu p_\nu = p^2$, we can write

$$\langle 0|T\{\psi(x)\bar{\psi}(0)\}|0\rangle = \frac{i}{\not{p} - m} \frac{\not{p} + m}{\not{p} + m} = \frac{i(\not{p} + m)}{p^2 - m^2},$$

which is what we had.

16 October 25, 2018

Let us now write down the Feynman rules for quantum electrodynamics.

16.1 Feynman rules for quantum electrodynamics

We have the photon propagator

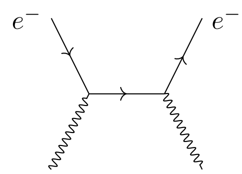
$$\frac{i}{p^2 + i\epsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

and the electron propagator

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon},$$

and interactions $-i\epsilon\gamma^\mu$. Then for photons coming out from a vertex, there is a ϵ^*_μ, for a photon going in, there is ϵ_μ, and for an outgoing u-spinor, $\bar{u}(p)$, for an incoming u-spinor, $u(p)$, and for an outgoing v-spinor, $\bar{v}$, and for an outgoing v-spinor, $\bar{v}(p)$.

So if we have a diagram like



becomes

$$\epsilon_4^{*\mu}\frac{\bar{u}(p_3)\gamma^\mu(\not{q} + m)\gamma^\nu u(p_1)}{q^2 - m^2 + i\epsilon}\epsilon_2^\nu \times (-ie)^2 i.$$

But electrons are fermions, so there are extra minus signs coming from

- each fermion loop,
- each time you swap two external fermions.

To see why this is the case, consider $e^-e^- \to e^-e^-$ in the free theory. Then to compute

$$\langle 0|T\{\psi(x_3)\bar{\psi}(x_1)\psi(x_4)\bar{\psi}(x_2)\}|0\rangle.$$

Then when we try to move things around, we need to put minus signs. For instance,

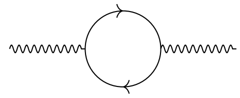
$$D_F(1,3)D_F(2,4) = \langle 0|a_3 a_1^\dagger|0\rangle\langle 0|a_4 a_2^\dagger|0\rangle,$$
$$D_F(1,4)D_F(2,3) = \langle 0|a_3 a_2^\dagger 0\rangle\langle 0|a_4 a_1^\dagger|0\rangle.$$

have a relative minus sign. There is always a minus sign for loops. When I have a loop, I am doing something like writing $a_1 a_5^\dagger a_5 a_3^\dagger$ and then identify a_3 and a_1. Then we have

$$a_1 a_5^\dagger a_5 a_1^\dagger = -a_1^\dagger a_1 a_5^\dagger a_5$$

and so we need a minus sign.

Example 16.1. Let us write down the matrix element for the fermion loop.



We have

$$\begin{aligned} i\mathcal{M} &= (-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\epsilon_1^\mu \epsilon_2^{*\nu}\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon}\gamma^\mu_{\alpha\beta}\left[\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\right]\gamma^\nu_{\beta\alpha} \\ &= e^2\epsilon_1^\mu\epsilon_2^{*\nu}\int\frac{d^4k}{(2\pi)^4}\frac{\operatorname{Tr}[(\not{p}+\not{k}+m)\gamma^\mu(\not{k}+m)\gamma^\nu]}{((k+p)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}. \end{aligned}$$

We are not going to compute this integral, but maybe we want to know that this trace is. The nice property we have is that traces are cyclic. If we write $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ then $\gamma_5^2 = 1$ and $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. Moreover, $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$ and so we can do stuff like

$$\operatorname{Tr}[\gamma^\mu] = \operatorname{Tr}[\gamma_5\gamma_5\gamma^\mu] = \operatorname{Tr}[\gamma_5\gamma^\mu\gamma_5] = -\operatorname{Tr}[\gamma_5\gamma_5\gamma^\mu] = -\operatorname{Tr}[\gamma^\mu].$$

So this is zero. Similarly you can calculate these things without choosing representations.

Example 16.2. If you calculate $e^+e^- \to \mu^+\mu^-$, then we can approximate $m_e = 0$ because muon is heavier, and then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E}\right)\cos^2\theta\right).$$

In the ultrarelativistic $E \gg m_\mu$ limit, we get $\frac{\alpha^2}{16E^2}(1+\cos^2\theta)$.

Example 16.3. Let us look at Rutherford scattering $e^-p^+ \to e^-p^+$. Here, we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4v^2p^2\sin^4\frac{\theta}{2}}\left(1-v^2\sin^2\frac{\theta}{2}\right)$$

in the p^+ rest frame, where $m_p \gg m_e$. This is called the **Mott formula**.

Example 16.4. There is also Compton scattering $\gamma e^- \to \gamma e^-$, and the cross section is given by

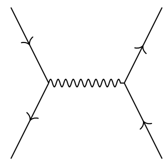
$$\frac{d\sigma}{d\cos\theta}\frac{\pi_\alpha^2}{m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right], \quad \omega' = \frac{\omega}{1+\frac{\omega}{m_e}(1-\cos\theta)}.$$

This is called the **Klein–Nishna formula**. If $m_e \gg \omega$, we get

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta),$$

which is Thompson scattering. The Klein–Nishna formula was discovered in 1929, so this was the first achievements in quantum electrodynamics.

Let's just wee how we write down the tree level diagram for $e^+e^- \to \mu^+\mu^-$.



Then we have

$$i\mathcal{M} = (-ie)^2 \frac{\bar{u}(p_3)\gamma^\mu v(p_2) v(p_4)\bar{v}(p_2)\gamma^\nu u(p_1)}{k^2} i\left(-g^{\mu\nu} + (1-\epsilon)\frac{k^\mu k^\nu}{k^2}\right).$$

Here, we see that the gauge term vanishes because $\not{k} = \not{p}_3 + \not{p}_4$. Then we can see that

$$\mathcal{M} = \frac{e^2}{s}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)].$$

If we take the conjugate transpose, we get

$$\mathcal{M}^\dagger = \frac{e^2}{s}[\bar{v}(p_4)\gamma^\mu u(p_3)][\bar{u}(p_1)\gamma_\mu v(p_2)].$$

So to get the spin sums, we should add them

$$\begin{aligned}\sum_{\text{spins}} \mathcal{M}\mathcal{M}^\dagger &= \sum_{\text{spins}} [\bar{v}(p_4)\gamma^\nu u(p_3)][\bar{u}(p_3)\gamma^\mu v(p_4)][\bar{u}(p_1)\gamma^\nu v(p_2)][\bar{v}(p_2)\gamma^\mu u(p_1)] \\ &= \operatorname{Tr}[(\not{p}_3 + m)\gamma^\mu(\not{p}_4 - m)\gamma^\nu] \operatorname{Tr}[\cdots]\end{aligned}$$

and then use the trace formulas.

17 October 30, 2018

Today we are going to look at path integrals. They are a beautiful we to think about quantum field theory, or quantum mechanics in general.

17.1 Path integrals

Consider the double slit experiment. Here, you add the two wavefunctions that passes through the two holes. But suppose there are three hole, or many slits with many holes. Then you add up all the possible paths. So we have something like

$$\langle f|i\rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t) e^{iS[x(t)]}.$$

In quantum field theory, we get

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\int D\phi e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int D\phi e^{iS[\phi]}}.$$

This is actually a really remarkable formula, because the right hand side is classical. First we are going to derived this formula, in two ways, non-perturbatively using the Hamiltonian and also perturbatively using Feynman rules.

Let me first think of the one-dimensional example

$$Z_0 = \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}a\phi^2} = \sqrt{\frac{2\pi}{a}}.$$

Then

$$\langle\phi\phi\rangle \sim \frac{1}{Z_0}\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}a\phi^2}\phi^2 = \frac{1}{a}.$$

So in the free theory, we are going to get

$$\frac{1}{Z_0}\int d\phi e^{-\frac{1}{2}\phi(\Box+m^2)\phi} = \frac{1}{\Box+m^2}$$

the propagator back. If we do this for ϕ^4, we get something like

$$\frac{1}{Z_0}\int d\phi e^{-\frac{1}{2}\phi(\Box+m^2)\phi}\phi^4 = \frac{3}{(\Box+m^2)^2},$$

which is just the three Feynman diagrams.

First, let us get this mathematics out of the way. We have

$$\int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2+Jp} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}.$$

For many p_i, we can look at a general quadratic form $ap^2 \to p_i A_{ij} p_j$, and then

$$\int dp_1\cdots dp_n e^{-\frac{1}{2}\vec{p}\cdot A\vec{p}+\vec{J}\cdot\vec{p}} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2}\vec{J}A^{-1}\vec{J}}$$

after diagonalization.

Now let us go back to the quantum mechanics. Consider a typical non-relativistic system $H = \frac{p^2}{2m} + V(x,t)$, and some initial state $|i\rangle$ and final state $|f\rangle$. Then we can break the time interval $[t_i, t_f]$ to small time slices. Then

$$\langle f|i\rangle = \int dx_1 \cdots dx_n \langle x_f|e^{-iH(x_f,t_t)\delta t}|x_n\rangle\langle x_n|\cdots|x_1\rangle\langle x_1|e^{-iH(x_1,t_1)\delta t}|x_i\rangle.$$

But then, I can just write

$$\begin{aligned}\langle x_{j+1}|e^{-iH(x_j,t_j)\delta t}|x_j\rangle &= \int \frac{dp}{2\pi}\langle x_{j+1}|p\rangle\langle p|e^{-i[\frac{p^2}{2m}+V(x_j,t_j)]\delta t}|x_j\rangle \\ &= e^{-iV_j\delta t}\int \frac{dp}{2\pi}e^{-i\frac{p^2}{2m}\delta t - ip(x_{j+1}-x_j)} \\ &= Ne^{[-iV_j\delta t + i\frac{(\delta t)^2}{2\delta t}m]} = Ne^{i[(\frac{\delta x}{\delta t})^2\frac{m}{2}-V_j]\delta t} = Ne^{iL[\dot{x},x]\delta t}\end{aligned}$$

where $J = -i\delta x$ and $a = i\delta t/m$. The magic is that we get the Legendre transform at the end. So we have

$$\begin{aligned}\langle f|i\rangle &= \int dx_1\cdots dx_n e^{iL[\dot{x}_f,x_f]\delta t}\cdots e^{iL[\dot{x}_1,x_1]\delta t} \\ &= \int Dx(t)e^{i\int dtL[\dot{x}(t),x(t)]},\end{aligned}$$

where the integral in the exponent is the action S.

In quantum mechanics, $|x\rangle$ is an eigenstates of $\hat{x}$: we have $\hat{x}|x\rangle = x|x\rangle$. In quantum field theory, we have these operators

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{i\vec{p}\vec{x}} + a_p e^{-i\vec{p}\vec{x}}).$$

These are the analogue of $\hat{x}$, and the analogue of $\hat{p}$ is just

$$\hat{p} \sim \hat{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\hat{\phi}}} \sim \dot{\phi} = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}(a_p^\dagger e^{ipx} - a_p e^{-ipx}).$$

We then need eigenstates $\hat{\phi}(x)|\Phi\rangle = \Phi(x)|\Phi\rangle$. Similarly, $\hat{\pi}(x)|\Pi\rangle = \pi(x)|\Pi\rangle$. There are also the completeness relations we need:

$$\mathbf{1} = \int D\Phi|\Phi\rangle\langle\Phi| = \int D\Pi|\Pi\rangle\langle\Pi|.$$

We would also have

$$\langle\Pi|\Phi\rangle = \exp\left[-i\int d^3x\Pi(x)\Phi(x)\right] \sim e^{-ipx}.$$

Now we can use this to do the same path integral formulation. We have

$$\langle 0;t_f|0;t_i\rangle = \int D\Phi_1(x)\cdots D\Phi_n(x)\langle 0|e^{-iH[\hat{\phi},\hat{\pi}]\delta t}|\Phi_n\rangle\langle\Phi_n|\cdots|\Phi_1\rangle\langle\Phi_1|\cdots|0\rangle.$$

Here, we have

$$H[\hat{\phi}, \hat{\pi}] = \int d^3x[\tfrac{1}{2}\hat{\pi}^2 + V(\hat{\phi}) + \tfrac{1}{2}(\vec{\nabla}\hat{\phi})^2].$$

If we do this, we get

$$\begin{aligned}\langle \Phi_{j+1}|e^{-iH(t_j)\delta t}|\Phi_j\rangle &= \int d^3\Pi \langle \Phi|\Pi\rangle\langle \Pi|e^{-iH_j\delta t}|\Phi_j\rangle \\ &= N\exp\biggl[i\delta t \int d^3x \mathcal{L}[\phi_j, \dot{\phi}_j]\delta t\biggr].\end{aligned}$$

At the end, we get

$$\langle 0; t_f|0; t_i\rangle = \int D\Phi(x,t)e^{iS[\Phi]}.$$

Here, $D\Phi$ is choosing a configuration of the field for each t. This is the denominator of the path integral. You can see from this that the dominating term is the when S is slowly moving, which is when S is extremized. In the classical limit $\hbar \to 0$, we have really the normalization $\frac{i}{\hbar}S[\phi]$, and we recover the Euler–Lagrange equations.

Let us now look at the time-ordered product. Note that

$$f(x_j, t_j) = \int D\Phi(x,t)e^{iS[\Phi]}\Phi(x_j, t_j) = \langle 0|\hat{\phi}(t_j, x_j)|0\rangle,$$

because I can just stick in $\hat{\phi}(x_j, t_j)$ inside thie expansion we had, and then we are picking up the eigenvalue $\Phi(x_j, t_j)$ of $|\Phi_j\rangle\langle\Phi_j|$. If we have two fields, we need to order the two fields, so we really have

$$\int D\Phi e^{iS[\Phi]}\Phi(x_1, t_1)\Phi(x_2, t_2) = \langle 0|T\{\hat{\phi}(x_1, t_1), \hat{\phi}(x_2, t_2)\}|0\rangle.$$

This is why we get the correlation functions in the form we wanted.

17.2 Generating functionals

Consider

$$Z[J] = \int D\phi e^{iS[\phi] + i\int d^4x \phi(x)J(x)}.$$

This is called the **generating functional**. One useful thing is $Z[0]$, which is the denominator of the path integral. If I take the derivative with respect to J, this is

$$\frac{1}{Z[0]}(-i)\frac{dZ}{dJ(x_1)} = \int D\phi e^{iS[\phi] + i\int J\phi}\phi(x_1).$$

Then if I take the derivative n times, with respect to $J(x_j)$, we get

$$\frac{1}{Z[0]}(-i)^n \frac{d^nZ}{dJ(x_1)\cdots dJ(x_n)}\Big|_{J=0} = \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle.$$

So if we know everything about the generating functionals J, we can obtain everything about the theory.

Let us try to calculate this in the free theory. Here, we have $\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi$ as we all know. Then

$$\begin{aligned} Z[J] &= \int D\phi e^{i\int d^4x[-\frac{1}{2}\phi(\Box+m^2)\phi+J\phi]} \\ &= Ne^{\frac{1}{2}J\frac{-i}{\Box+m^2}J} = Ne^{i\int d^4xd^4yJ(x)\Pi(x-y)J(y)}. \end{aligned}$$

What this means is that

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{1}{Z[0]}\frac{\partial^2 Z}{\partial J(x)\partial J(y)}\Big|_{J=0} = i\Pi(x-y),$$

where $(\Box + m^2)\Pi(x-y) = -\delta(x-y)$.

18 November 1, 2018

Today we are going to finish the deriving the Feynman rules. In quantum mechanics, the path integral was another way of calculating $\langle x_f|e^{-iHt}|x_i\rangle = \int_x Dxe^{iS[x]}$. In quantum field theory, we have

$$\langle\phi_f|e^{-iHt}|\phi_i\rangle = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS[\phi]}.$$

Here, $|\phi_f\rangle$ are the eigenvectors, $\hat{\phi}(x)|\phi_f\rangle = \phi_f(x)|\phi_f\rangle$. Then we had the master formula

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\int D\phi e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int D\phi e^{iS[\phi]}}.$$

Then we had the generating functional

$$Z[J] = \int D\phi e^{i\int d^4x[\mathcal{L}+J\phi]}.$$

Then we were able to recover everything by taking

$$(-i)^n \frac{\partial^n Z[J]}{\partial J(x_1)\cdots\partial J(x_n)}\bigg|_{J=0} = \langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.$$

18.1 Feynman rules from the path integral

In the free theory, this is given by

$$\begin{aligned} Z[J] &= \int D\phi e^{i\int d^4x[-\frac{1}{2}\phi(\Box+m^2)\phi+J\phi]} \\ &= N\exp\left[i\int d^4xd^4yJ(x)\Pi(x,y)J(y)\right]. \end{aligned}$$

For example, a 4-point function in free theory is given by

$$\begin{aligned} \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle &= \frac{1}{Z}\frac{d^4Z}{dJ_1dJ_2dJ_3dJ_4}e^{-\frac{1}{2}\int dxdyJ_xD_{xy}J_y}|_{J=0} \\ &= \frac{1}{Z}\frac{d^4}{dJ_1dJ_2dJ_3}(-J_zD_{z4})Z|_{J=0} \\ &= \frac{1}{Z}\frac{d^2}{dJ_1dJ_2}(-D_{34}+J_zD_{z3}J_wD_{w4})Z \\ &= D_{34}D_{12}+D_{23}D_{14}+D_{13}D_{24}. \end{aligned}$$

So that's for the free theory. Let us now look at interactions, for instance,

$$\mathcal{L} = -\frac{1}{2}\phi(\Box+m^2)\phi + \frac{g}{3!}\phi^3.$$

Then we have

$$Z[J] = \int D\phi \exp\Big[i \int d^4x(-\tfrac{1}{2}\phi(\Box + m^2)\phi + J\phi)\Big]$$
$$\left(1 + i\int d^4x \frac{g}{3!}\phi^3 - \int d^4x d^4y \left(\frac{g}{3!}\right)^2 \phi(x)^3\phi(y)^3 + \cdots\right).$$

But then we have

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \langle 0|T\{\phi_0(x_1)\phi_0(x_2)\}|0\rangle$$
$$+ \frac{g}{3!}\int d^4x \langle 0|T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)^3\}|0\rangle - \cdots .$$

So you get the Feynman rules.

In the propagator, we didn't get an $i\epsilon$. This should come from the constraint on fields. We want to evaluate the time evolution operator

$$_\infty\langle\Omega|S|\Omega\rangle_{-\infty} = \int_{\phi(-\infty)=\phi_0}^{\phi(\infty)=\phi_0} D\phi e^{iS} = \int D\phi e^{iS}{}_{+\infty}\langle\Omega|\phi(t=+\infty)\rangle\langle\phi(t=-\infty)|\Omega\rangle_{-\infty}.$$

Remember how we quantized the theory. Here, the ground state or vacuum for each $\vec{p}$ is given by

$$\phi_{\text{ground}}(x) = \langle x|0\rangle = e^{-\frac{1}{2}x^2 m\omega}.$$

In quantum field theory, the ground state is a wave functional $\Psi_0[\phi]$, given by

$$\Psi_0[\phi] = \exp\Big[-\int d^3p \phi_p^*\phi_p\omega_p\Big] = \exp\Big[-\int d^3pd^3xd^3y e^{i\vec{p}(\vec{x}-\vec{y})}\phi(x)\phi(y)\omega_p\Big].$$

To evaluate this, we can use the following slick trick due to Weinberg:

$$\begin{aligned}
\langle\Omega|\phi_\infty\rangle\langle\phi_{-\infty}|\Omega\rangle &= \lim_{\epsilon\to 0}\exp\Big[-\epsilon\int_{-\infty}^{\infty} d\epsilon d^3pd^3xd^3y\phi(x,t)\phi(y,t)e^{ip_1(x-y)}\omega_p e^{-\epsilon|t|}\Big]\\
&= \exp\Big[-\int d^3pd^3xd^3y\omega_p e^{ip(x-y)}[\phi(x,\infty)\phi(y,\infty) + \phi(x,-\infty)\phi(y,-\infty)]\Big]\\
&= \Psi_0[\phi(x,-\infty)]\Psi_0^*[\phi(y,-\infty)]\\
&= \lim_{\epsilon\to 0}\exp\Big[-\int d^4xd^3yd^3pe^{i\vec{p}(\vec{x}-\vec{y})}\phi(x,t)\phi(y,t)\epsilon\omega_p\Big]\\
&= \lim_{\epsilon\to 0}\exp\Big[-\int d^4x\epsilon\phi(x)\phi(x)\Big].
\end{aligned}$$

So we really have $\int D\phi \exp[-\int d^4x[-\frac{1}{2}\phi[\Box + m^2 - i\epsilon]\phi]$ and so we get $\Pi = (\Box + m^2 - i\epsilon)^{-1}$ when we calculate the propagator. I don't think this is mathematically rigorous, but this should be morally correct.

18.2 Gauge invariance from path integrals

Remember we took the deformation

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}(\not{D} - m)\psi - \frac{1}{2\xi}(\partial_\mu A_\mu)^2.$$

Let me start by defining a function

$$f(\xi) = \int D\pi e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2}.$$

Now I am going to change variables $\pi \to \pi - \frac{1}{\Box}\partial_\mu A_\mu$. This is what you would choose to put your field in Lorentz gauge. If I look at scalar QED, define

$$F[\mathcal{O}] = \int D\phi D\phi^* DA e^{i\int d^4x[-\frac{1}{4}F_{\mu\nu}^2 + D_\mu\phi^2]}\mathcal{O}\frac{1}{f(\xi)}\int D\pi e^{-\frac{i}{2\xi}\int d^4x(\Box\pi - \partial_\mu A_\mu)^2},$$

so that $\langle\Omega|T\{\mathcal{O}\}|\Omega\rangle = F[\mathcal{O}]/F[\mathbf{1}]$. If we make a change of variables $A_\mu \to A_\mu + \partial_\mu\pi$ and $\phi \to e^{i\pi}\phi$ and $\phi^* \to e^{-i\pi}\phi^*$, then we get

$$F[\mathcal{O}] = \frac{\int D\pi}{f(\xi)}\int DAD\phi D\phi^* \exp\left[i\int -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right].$$

So we see that this cancels out when we take the ratio.

For fermions, we know that spinors have to anticommute, $\psi_1(x)\psi_2(y) = -\psi_2(y)\psi_1(x)$. Then the path integral for fermions is

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} \exp\left[i\int d^4x i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi\right].$$

So we want a things that all anticommute. These are called **Grassmann numbers**, $\mathcal{Y} = \{\theta_i\}$ over $\mathbb{C}$ such that $\theta_i\theta_j = -\theta_j\theta_i$. So these are like differential forms, but it is not a useful analogy.

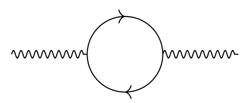
19 November 6, 2018

Now we want to start renormalization, which is the heart of quantum field theory.

19.1 Renormalization

If you try to look at the photon propagator, you can look at the ones with one fermion loop and ones with two fermion loops or so on. Then you try to add all these loop corrections, and you get infinity. You can do this with other things like mass correction of the electron, or splitting of the energy levels of the hydrogen atom. This was so bad that Dirac tried to get rid of quantum field theory, even though he invented it. The idea is that we can be very careful about calculating things that are only physical. Maybe the energy potential is infinite, but the force or energy difference is finite.

If we consider



then we the integral

$$\int d^4p \frac{i}{p^2+i\epsilon} \frac{i(k+2p)^\mu (2p+k)^\mu}{(2p+k)^2+i\epsilon} \approx \int p^3 dp \frac{p^2}{p^4} \sim \int^\Lambda pdp \sim \Lambda^2$$

if we cut off at Λ. This is a slight complicated, so we are not going to start with this.

A simpler example is the vacuum energy, which has to do with the zero modes of the harmonic oscillators. Then we have

$$\text{vacuum energy} = \langle 0|H|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2} = 4\pi \int k^2 dk \frac{k}{2} \sim \Lambda^4,$$

which is quartically divergent.

So let us look at the what we are doing. We can first consider this box of size r, and restrict things to this box. Then we get

$$\omega_n = \frac{\pi}{r} n, \quad E(r) = \sum_{n=1}^{\infty} \frac{\omega_n}{2} = \infty, \quad F(r) = -\frac{dE}{dr} = -\frac{\pi}{2r^2} \sum_{n=1}^{\infty} = \infty.$$

This didn't help much. This first missing the things outside the box, but it is still infinite after putting in these terms. So we take a cut off Λ so that only low frequency $\omega < \pi\Lambda$ are quantized. Then we get

$$E(r) = \frac{\pi}{4r} \lfloor \Lambda r \rfloor (\lfloor \Lambda r \rfloor + 1).$$

So define $x = \Lambda a - \lfloor \Lambda a \rfloor$ and if we add the energy for a and $L-a$ we get

$$E_{\text{tot}}(a) = \frac{\pi}{4}\bigg(\Lambda^2 L - \frac{x(1-x)}{a} - \frac{x(1-x)}{L-a} \bigg).$$

Now if we take the limit $L \gg a$, and then look at F, we get

$$F = -\frac{d}{da}E_{\text{tot}} = -\frac{\pi}{24a^2} = -\frac{\pi\hbar c}{24a^2}.$$

If we did the same exercise in three dimensions, we would get

$$F(a) = -\frac{\pi\hbar c}{240a^4}A = -1.3 \times 10^{-27}\mathrm{N}\cdot\mathrm{m}^2\frac{1}{a^4}.$$

This is the **Casmir effect**, and was experimentally measured in 2002 with constant $(-1.22 \pm 0.16) \times 10^{-27}$.

This seems like a pretty arbitrary way to cut off and compute the number, so let us look at a different regularization, called the **heat kernel regularization**. Here, we assume a damping

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-\omega_n/\pi\Lambda} = \frac{e^{\frac{1}{r\Lambda}}}{(e^{\frac{1}{r\Lambda}}-1)^2} \approx \frac{\pi}{2}\Lambda^2 r - \frac{\pi}{24r} + \cdots, \quad \omega_n = \frac{n\pi}{r}.$$

Now if we look at the difference, we get

$$E_{\text{tot}} = E(a) + E(L-a) = \frac{\pi^2}{2}\Lambda^2 L - \frac{\pi}{24}\left(\frac{1}{a} + \frac{1}{L-a}\right), \quad F = -\frac{\pi}{24a^2}.$$

You could do a **Gaussian regulator**

$$E(r) = \sum \frac{\omega_n}{2}e^{-(\frac{\omega_n}{\pi n})^2}, \quad \Lambda \to \infty,$$

or a **ζ-function regulator**

$$E(r) = \frac{1}{2}\sum \omega_n\left(\frac{\omega_n}{M}\right)^{-s}, \quad s \to 0,$$

and all of these will give the same prediction for Casmir effect.

But you should be skeptical of this, because what if you get something else by using some other cut off? Here is a regulartor-independent calculation, given by Casmir himself in 1948. Let us consider

$$E(r) = \frac{\pi}{2}\sum_n \frac{n}{r}f\left(\frac{n}{r\Lambda}\right),$$

where f is something like e^{-x} or $\theta(|x|-1)$ or e^{-x^2} or something that doesn't affect small x but kills off large x. Then we have

$$E(L-a) = \frac{\pi}{2}(L-a)\Lambda^2\sum_{n=1}^{\infty}\frac{n}{(L-a)^2\Lambda^2}f\left(\frac{n}{(L-a)\Lambda}\right) = \frac{\pi}{2}(L-a)\Lambda^2\int xdxf(x)$$

because we can write $x = \frac{n}{(L-a)\Lambda}$ and assume that the spacing is getting close. Then we can write

$$E_{\text{tot}}(0) = E(a) + E(L-a) = \rho L + \frac{\pi}{2a}\left[\sum_n n f\left(\frac{n}{a\Lambda}\right) - \int n dn f\left(\frac{n}{a\Lambda}\right)\right].$$

This can be written as a Euler–MacLaurin series

$$\sum_{n=1}^{N} F(n) - \int_0^N F(n)dn = \frac{F(0)+F(N)}{2} + \frac{F'(N)-F'(0)}{12} + \cdots.$$

If you plug in $F(n) = nf(\frac{n}{a\Lambda})$ then we see that only the term that survives is

$$E_{\text{tot}} = \rho L - \frac{\pi}{24a} f(0) = \rho L - \frac{\pi}{24a}.$$

This is universal, in the sense that you can see this effect as long as the box can hold photons.

20 November 8, 2018

Let me start by summarizing the principles of renormalization. The basic idea is that

- long-distance / infrared(IR) / low-energy physics is independent of short-distance / ultraviolet(UV) / high-energy physics.
- observables are finite.
- infinites such as UV divergences appear at intermediate stages of the calculation.
- a regulator like Λ or ϵ is introduced to cut off UV divergence, and the limit as $\Lambda \to \infty$ and $\epsilon \to 0$ exists for the observable.

20.1 Examples of renormalization—ϕ^4

Let's take the ϕ^4 theory,

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4.$$

If we look at $\phi\phi \to \phi\phi$, the leading order is just $i\mathcal{M} = -i\lambda$. Then the next term is

$$i\mathcal{M}_2 = (-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\frac{1}{2}p - k)^2(\frac{1}{2}p + k)^2} \sim \int k^3 \frac{dk}{(k^2 - \frac{s}{4})^2} \sim \log \Lambda,$$

where $s = p^2$ and $p = p_1 + p_2$. There is this trick due to Feynman, which is to look at

$$\frac{\partial \mathcal{M}_2(s)}{\partial s} = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-\frac{1}{2}}{(\frac{s}{4} - k^2)^3} = -\frac{\lambda^2}{32\pi}\frac{1}{s}.$$

So we can integrate this and write

$$\mathcal{M}_2(s) = -\frac{\lambda^2}{32\pi^2} \log s + \text{const} = -\frac{\lambda^2}{32\pi^2} \log \frac{s}{\Lambda^2}.$$

Even if we can't make sense out of $\mathcal{M}(s)$, we can still make sense out of their relative difference

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) = -\frac{\lambda^2}{32\pi^2} \log \frac{s_1}{s_2}.$$

But $\mathcal{M}$ should be a measurable quantity, because we can scatter pions and get a number. What I measure is the sum of all the Fyenman diagrams, so I can measure for s_0 and then write this as

$$\mathcal{M}(s_0) = -\lambda_R.$$

This λ_R is the renormalized coupling, and it is defined at a reference scale s_0 to all orders in perturbation theory. Now if we try to expand λ out in λ_R, we will see that

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2} \log \frac{s_0}{\Lambda^2} + \cdots .$$

If we take this λ and try to plug this into s, we get

$$M(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log\frac{s_0}{s}.$$

20.2 Examples of renormalization—vacuum polarization I

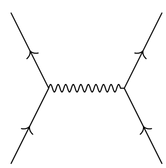
Now let us look at the one-fermion loop given by

$$(-ie)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\,\mathrm{Tr}[\gamma^\mu(\not{p}-\not{k}-m)\gamma^\nu(\not{k}-m)].$$

Here, we can write

$$\begin{aligned} i\Pi^{\mu\nu} &= -ie^2(p^2g^{\mu\nu}-p^\mu p^\nu)\Pi(p^2), \\ \Pi(p^2) &= \frac{1}{2\pi^2}\int_0^1 dx x(1-x)\log\frac{\Lambda^2}{m^2-p^2x(1-x)}. \end{aligned}$$

Using this, let us try to compute the interaction between the proton and the electron in the hydrogen atom,



Then if we add them together, we get

$$-i\frac{g^{\mu\nu}}{p^2}(1-e^2\Pi_2(p^2)+O(e^4)), \quad \tilde{V}(p^2) = e^2\frac{1-e^2\Pi_2(p^2)}{p^2}.$$

This again is going to be infinite. So we now consider e as a measurement, though $V(r)$ or at some r. Then we define the renormalized coupling as

$$e_p^2 = p_0^2\tilde{V}(p_0^2)$$

exactly at some p_0^2. In that case, can solve perturbatively, $e_R^2 = p_0^2\tilde{V}(p_0^2) = e^2 - e^4\Pi_2(p_0^2)$ and so

$$e^2 = e_R^2 + e_R^4\Pi_2(p_0)^2 + \cdots.$$

Now we see that

$$p^2\tilde{V}(p^2) = e_R^2 - e_R^4\Pi_2(p^2) + e_R^4\Pi_2(p_0^2) = e_R^2\frac{e_R^4}{2\pi}\int dx x(1-x)\log\frac{m^2-p^2x(1-x)}{m^2-p_0^2x(1-x)}.$$

Now if we choose $p_0 = 0$, we see that $e_R^2/4\pi = \frac{1}{137} = \alpha$. Then we can compute things like

$$\tilde{V}(p^2) = -\frac{e_R^2}{p^2}\left[1+\frac{e_R^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(1-\frac{p^2}{m^2}x(1-x)\right)+\cdots\right],$$

and so if we take the Fourier transform, we get

$$V(r) = -\frac{e^2}{4\pi r}\left(1 + \frac{e_R^2}{6\pi}\int_0^1 dx e^{-2mrx}\frac{2x^2+1}{2x^4}\sqrt{x^2-1} + \cdots\right).$$

If we numerically integrate, we see that

$$\Delta E_{2s_{1/2}} = -27\text{MHz}, \quad \Delta E_{2p_{1/2}} = 0.$$

You can also calculate other loops in Feynman diagrams, and then we see that ΔE between the two orbitals are 1024MHz. This is in excellent agreement with measurements.

Consider the scale $p^2 \gg m^2$ and then take the reference scale $p_0^2 = -m^2$. In this case, we have

$$\int dx x(1-x)\log\left(\frac{-p^2}{m^2}\right) = \frac{1}{6}\log\left(-\frac{p^2}{m^2}\right)$$

so we have

$$\tilde{V}(p^2) = \frac{e_R^2}{p^2}\left(1 + \frac{e_R^2}{12\pi^2}\log\frac{-p^2}{m^2}\right) = \frac{e_R^2}{p^2}(1 + 0.0077\log\frac{Q^2}{m^2}).$$

This is really telling us that perturbative QED is breaking down when Q is very very large, for instance, when 10^{270}eV. If we add all these diagrams involving one loop and two loops and so on, we see that we have

$$\Pi = \frac{e_R^2}{p^2}\left(1 + \frac{e_R^2}{12\pi^2}\log\frac{p^2}{m^2} + \left(\frac{e_R^2}{12\pi^2}\log\frac{p^2}{m^2}\right)^2 + \cdots\right) = \frac{1}{p^2}\left(\frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\log\frac{-p^2}{m^2}}\right).$$

So when we approach $Q \approx 10^{270}$eV, also called the **Landau pole**, we see that the energy really is going to infinity. So QED is breaking down at this crazy energy level. One way to understand this is that there is some screening for vacuum, given by all these diagrams which are like virtual electron-positron pairs. In smaller scales, this effect becomes small so the force becomes stronger.

21 November 13, 2018

Last time we looked at the 1-loop contribution in QED. We didn't compute it, but today we are going to do this. We had

$$\tilde{V}(p^2) = \frac{e^2}{p^2}\frac{e^4}{p^2}\Pi_2(\vec{p}^2) \to \cdots,$$

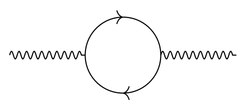
and we renormalized this to $e_R^2 = \tilde{V}(p_0^2)p_0$. Then we were able to predict

$$p^2\tilde{V}(p^2) = e_R^2 + \frac{e_R^4}{2\pi}\int_0^1 dx x(1-x)\log\frac{m^2-p^2x(1-x)}{m^2-p_0^2x(1-x)}$$

which is finite. If you Fourier transform, we get an exponentially decaying correction term in the Coulomb potential.

21.1 Examples of renormalization—vacuum polarization II

Let's actually compute



Here, we can write this as

$$\begin{aligned}-(-ie)^2\int\frac{d^4k}{(2\pi)^4}&\frac{i}{(p-k)^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\operatorname{Tr}[\gamma^\mu(\not{k}-\not{p}+m)\gamma^\nu(\not{k}+m)]\\ &= -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + g^{\mu\nu}(-k^2+pk+m^2)]}{((p-k)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}\\ &= \Delta_1\left(\frac{p^2}{m^2}\right)p^2 g^{\mu\nu} + \Delta_2\left(\frac{p^2}{m^2}\right)p^\mu p^\nu.\end{aligned}$$

Here, by the Ward identity we should get that $\Delta_1 = -\Delta_2$. So we can only compute Δ_1. Then we can drop the first two terms in the numerator, because they only contribute to Δ_2.

First we are going to regulate the integral, combine denominators, complete the squares to make everything in terms of k^2, drop the odd terms, and the Wick rotate $k_0 \to ik_0$ to integrate. To combine the denominators, we use the trick

$$\frac{1}{AB} = \int_0^1\frac{dx}{[A+(B-A)x]^2}.$$

Then for $A = (p-k)^2 + m^2 + i\epsilon$ and $B = k^2 - m^2 + i\epsilon$, we get

$$A + (B-A)x = [k-p(1-x)]^2 + p^2x(1-x) - m^2 + i\epsilon.$$

Now our integral is,

$$\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{f(k,p)}{[[k-p(1-x)]^2+p^2x(1-x)-m^2+i\epsilon]^2}$$
$$= \int \frac{f(k+p(1-x),p)}{[k^2+p^2x(1-x)-m^2+i\epsilon]^2}.$$

Now if we drop all terms that are odd in k, we are left with

$$i\mathcal{M} = -4e^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{2k^\mu k^\nu - g^{\mu\nu}[k^2-x(1-x)p^2-m^2]}{[k^2+p^2x(1-x)-m^2+i\epsilon]^2}.$$

So we should know how to integrate something like

$$\int \frac{d^4k}{(2\pi)^4}\frac{1 \text{ or } k^2 \text{ or } k^\mu k^\nu}{(k^2-\Delta+i\epsilon)^n}.$$

Here, we note that the denominator looks like

$$k^2-\Delta+i\epsilon = [k_0-(\sqrt{\vec{k}^2+\Delta}-i\epsilon)][k_0-(-\sqrt{\vec{k}^2+\Delta}+i\epsilon)],$$

and as function of k_0, has zeros at a (negative plus small imaginary) and a (positive minus small imaginary). This tells us that if we do a contour along a figure-eight figure, we can change the integral along the real axis to an integral along the imaginary axis. So when we **Wick rotate** this becomes

$$i\int\frac{d^4k}{(2\pi)^4}\frac{1}{-k_E^2-\Delta)^n} = i\int\frac{d\Omega_4}{(2\pi)^4}\int_0^\infty k_E^3 dk_E(-1)^n\frac{1}{(k_E^2+\Delta)^n}.$$

For instance, for $n=3$ this integral is $-i/32\pi^2\Delta$.

For $n=2$, it is infinite, and we should expect this because we have to regulate. Define

$$I_2(\Delta) = \int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{8\pi^2}\int_0^\infty k^3\frac{dk}{(k^2+\Delta)^2}.$$

Then the derivative is

$$I_2'(\Delta) = -2I_3(\Delta) = \frac{i}{16\pi^2\Delta}, \quad I_2(\Delta) = -\frac{i}{16\pi^2}\log\frac{\Delta}{\Lambda^2}.$$

But this only gives a feel for the answer, and we need to be careful because we have lots of different integrals and be consistent about cutoffs.

We are going to do dimensional regularization, but let me first talk about Pauli–Villars. Here, we introduce a fictitious **ghost particles** of mass Λ. We say this PV ghost has all the same interactions but with a minus sign. In this case,

$$I(\Delta) = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-\Delta+i\epsilon)^2}-\frac{1}{(k^2-\Lambda^2+i\epsilon)^2}\right] = -\frac{i}{16\pi^2}\log\frac{\Delta}{\Lambda^2}.$$

Here we get a convergent integral because the $1/k^4$ cancels and then the leading term is $1/k^6$. Here, this is a systematic change of the theory. But people don't use this anymore.

21.2 Dimensional regularization

The regulator that is used universally is **dimensional regularization**. Here, the key is that the integral is continuous in less than 4 dimensions. So we calculate it for d dimensions and analytically continue to $d = 4 - \epsilon$. We note that

$$\int \frac{d^d k}{(k^2+\Delta)^2} = \Omega_d \frac{k^{d-1}dk}{(k^2+\Delta)^2},$$

and this is finite if $d < 4$.

We first note that we have

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A.$$

If we do dimensional analysis, we see that

$$[A] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}, \quad [m] = 1, \quad [e] = \frac{4-d}{2}.$$

But it is good to have the interaction constants dimensional. So we introduce a scale μ of $[\mu] = 1$ and then write

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A\mu^{\frac{4-d}{2}}.$$

Also note that $g^{\mu\nu}g - \mu\nu = \mathrm{tr}(g^2) = d$.

Now we can do the integrals. Write

$$\int d^d k k^\mu k^\nu f(k^2) = g^{\mu\nu} X.$$

If we contract with $g^{\mu\nu}$, then we get $\int d^d k k^2 f(k^2) = dX$. So we have

$$\int d^d k k^\mu k^\nu f(k^2) = g^{\mu\nu}\frac{1}{d}\int k^2 f(k^2).$$

Also, we have

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

This $\Gamma(x)$ is the analytic continuation of $\Gamma(x) = (x-1)!$ and has poles at $x = 0, -1, -2, \ldots$. At ϵ, we have $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E$. So if we do the integral, we have

$$e^2 \int \frac{d^d k}{(k^2 - \Delta + i\epsilon)^2} = \mu^{4-d}\frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-d/2}}\Gamma\left(\frac{4-d}{2}\right).$$

Near $d = 4$, we will get

$$\frac{i}{16\pi^2}\left[\frac{2}{\epsilon} + \log\frac{\mu^2 4\pi e^{-\gamma_E}}{\Delta} + O(\epsilon)\right].$$

The final answer is going to be

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx x(1-x)\left[\frac{2}{\epsilon} + \log\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right], \quad \tilde{\mu}^2 = \mu^2 4\pi e^{-\gamma_E}.$$

If we take the difference $\Pi_2(p_0^2) - \Pi_2(p^2)$, both ϵ and $\tilde{\mu}^2$ drop out and we get the right answer.

22 November 15, 2018

We looked at the Pauli–Villars ghost particle, which is physical but not very useful, and also the dimensional regularization. The main thing here is to keep the e dimensionless, and so we introduce a dimensionful μ. At the end, we got

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx x(1-x)\left[\frac{2}{\epsilon} + \log\frac{4\pi\mu^2 e^{-\gamma_E}}{m^2 - p^2 x(1-x)} + O(\epsilon)\right].$$

Only the difference is meaningful, so we look at

$$\tilde{V}(p^2) = \frac{e^2}{p^2} - \frac{e^4}{p^2}\Pi_2(p^2), \quad e_R^2 = \tilde{V}(p_0^2)p_0^2 = e^2 - e^4\Pi_2(p_0^2), \quad e^2 = e_R^2 + e_R^4\Pi_2(p_0^2).$$

Then we get

$$\begin{aligned} p^2\tilde{V}(p^2) &= e_R^2 + e_R^4\Pi_2(p_0^2) - e_R^4\Pi_2(p^2) \\ &= e_R^2 + \frac{e_R^4}{2\pi}\int_0^1 dx x(1-x)\log\frac{m^2 - p^2x(1-x)}{m^2 - p_0^2 x(1-x)}. \end{aligned}$$

22.1 Examples of renormalization—anomalous magnetic moment

This was done around 1940, and it got people really excited because there was data that we can compare theory to. Remember we had this non-relativistic Hamiltonian

$$H_{\text{scalar}} = \frac{\vec{p}^2}{2m} + V(r)e + \frac{e}{2m}\vec{B}\cdot\vec{L}, \quad H_{\text{spinor}} = \frac{\vec{p}^2}{2m} + V(r)e + \frac{e}{2m}\vec{B}\cdot\vec{L} + \frac{e}{2m}g\vec{B}\cdot\vec{S}.$$

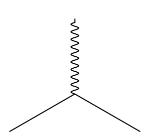
Here, only $g = 2$ agrees with Lorentz-invariance, which is the Dirac equation. But is really $g = 2$? The challenge here is to disentangle the corrections to e and the corrections to g.

Remember that we had the Dirac equation $(i\not{D} - m)\psi = 0$, and multiplied this with $(i\not{D} + m)$ to get

$$(D_\mu^2 + \tfrac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2)\psi = 0.$$

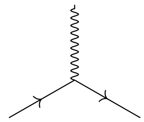
For a scalar, we would only have $(D_\mu^2 - m^2)\phi = 0$.

How do we interpret this in QED? For the Feynman diagrams in the tree-level, we have, for a scalar,



$$= -ie(q_1^\mu + q_2^\mu)$$

and for a spinor, we have



$$= -ie\bar{u}(q_2)\gamma^\mu u(q_1).$$

Here, the Gordon identity gives

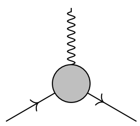
$$\bar{u}\gamma^\mu u = \frac{1}{2m}\bar{u}(q_1^\mu + q_2^\mu)u - \frac{i}{2m}\bar{u}\sigma^{\mu\nu}u(q_1^\nu - q_2^\nu),$$

and so we can write the $\mathcal{M}$-matrix as

$$-ie\bigg[\frac{q_1^\mu + q_2^\mu}{2m}\bar{u}u + \frac{i}{2m}\bar{u}\sigma^{\mu\nu}up_\nu\bigg]\epsilon_\mu.$$

The first term can be considered as what we have for the scalar particle, because $\bar{u}_s u_{s'} = 2m\delta_{ss'}$ and we are looking at the limit $q_1 - q_2 \to 0$. The second term is what predicts $g = 2$.

So suppose we computed the loop correction and got



$$= -ia\bar{u}(q_2)\gamma^\mu u(q_1) + \frac{e}{2m}b\bar{u}\sigma^{\mu\nu}up_\nu.$$

This is making the shift

$$e \to e + a, \quad g \to g + 2b.$$

So suppose we calculate this out and get

$$\epsilon_\mu \bar{u}(q_2)\Gamma^\mu u(q_1), \quad \Gamma^\mu = f_1\gamma^\mu + f_2 p^\mu + f_3 q_1^\mu + f_4 q_2^\mu.$$

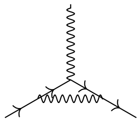
Then using $p = q_1 - q_2$, we can first eliminate f_2, and then the Ward identities tell us that $f_1\not{p} + f_3 pq_1 + f_4 pq_2 = 0$. So we can eliminate f_4 and have two independent terms. So now we can use the Gordon identity to choose the two terms as

$$\Gamma^\mu = -ie\bigg[F_1\bigg(\frac{p^2}{m_e^2}\bigg)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_e}p_\nu F_2\bigg(\frac{p^2}{m_e^2}\bigg)\bigg].$$

This F_1 can be interpreted as the scale-dependent electric charge, and F_2 gives us the correction to the electron's magnetic dipole moment. In the tree level, $F_1 = 1$ and $F_2 = 0$ and so $g = 2$. Then after we calculate the correction, we will have

$$g = 2 + 2F_2(0).$$

If we look at the 1-loop correction, there are four diagrams, and the only diagram where we can get some $\sigma^{\mu\nu}$ between the u is only the following diagram.



$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4}\frac{-ig^{\nu\alpha}}{(k-q_1)^2}\bar{u}\gamma^\nu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2}\gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2}\gamma^\alpha u$$

Now first we combine the denominators and complete the square. At the end, we get

$$i\mathcal{M} = \int d^4k \int dxdydz\delta(1-x-y-z)\frac{N^\mu}{(k^2-\Delta)^3}, \quad \Delta = -xyp^2 + (1-z)^2m^2,$$
$$\begin{aligned} N^\mu = {} & [-\tfrac{1}{2}k^2 + (1-x)(1-y)p^2 + (1-4z+z^2)m^2]\bar{u}\gamma^\mu u \\ & + imz(1-z)p_\nu \bar{u}\sigma^{\mu\nu}u + mz(1-z)(x-y)p^\mu \bar{u}u. \end{aligned}$$

Actually, the last term in N^μ vanishes after integration, so we can use

$$N^\mu = [-\tfrac{1}{2}k^2 + (1-x)(1-y)p^2 + (1-4z+z^2)m^2]\bar{u}\gamma^\mu u + imz(1-z)p_\nu \bar{u}\sigma^{\mu\nu}u.$$

This really is as expected.

We are only interested in the F_2 term, so we don't really have to care about the other term. Here, the growth is like $1/k^6$, so there is no divergence as well. So we get

$$F_2 = \frac{2m}{e}(4ie^3m)\int dxdydz\delta(1-x-y-z)\int \frac{d^4k}{(2\pi)^4}\frac{z(1-z)}{(k^2-\Delta+i\epsilon)^2}\cdots.$$

We can Wick rotate, and then we get

$$F_2(p^2) = \frac{8e^2}{32\pi^2}m^2\int d^3x\delta(x+y+z-1)\frac{z(1-z)}{(1-z)^2m - xyp^2}.$$

Because g is defined as $F_2(0)$, we can integrate

$$\begin{aligned} F_2(0) &= \frac{8e^2}{32\pi^2}\int_0^1 dz \int_0^1 dy \int_0^1 dx\delta(1-x-y-z)\frac{z}{1-z} \\ &= \frac{\alpha}{\pi}\int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z} = \frac{\alpha}{\pi}\int_0^1 z = \frac{\alpha}{2\pi}. \end{aligned}$$

Therefore

$$g = 2 + \frac{\alpha}{\pi} \approx 2.00232.$$

23 November 20, 2018

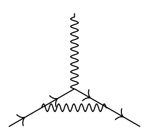
Today we are going to do mass renormalization. The main renormalization we talked about was the fermion loop. We defined α_R so that $\alpha_R = \lim_{r\to\infty} rV(r)$, and then we got

$$V(r) = \frac{\alpha_R}{r}\left[1 + \frac{\alpha_R}{(mr)^{3/2}}e^{-mr} + \cdots\right].$$

Here the renormalization condition was a specification of a parameter in the theory through some observable. So we can define a **renomalizatble theory** as a theory of which a finite number of renormalization conditions are needed to define the theory completely. A theory can be renormalized, even if it is not renormalizable.

23.1 Renormalizability of QED

We will show that QED is renormalizable. Last time we looked at



$$= \bar{u}\sigma^{\mu\nu}up_\mu + \bar{u}\gamma^\mu u$$

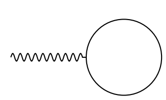
We saw that $\sigma^{\mu\nu}$ was finite, but this other term $\bar{u}\gamma^\mu u$ is going to be UV-divergent. Similarly, the electron self-energy is UV-divergent.

But there can be other many different graphs we can draw. So we want to consider showing that all Green's functions or S-matrix elements can be renormalized in an increasing number of fields/extenral momenta. For 0-point functions, we want

$$\langle\Omega|\Omega\rangle = 1.$$

But then we draw all diagrams with no external legs, and this in fact is strongly divergent. But this doesn't really affect the physics, so we can just set this to Λ_R and really, set this to whatever we want. This is important in general relativity because the measure $\sqrt{g}$ involves the constant.

Now we look at 1-point functions. Actually, all 1-point functions vanish in QED. We could have something like



which is called a tadpole graph, but this is zero because if we have

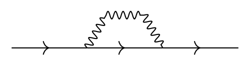
$$\langle\Omega|\psi|\Omega\rangle \neq 0 \text{ or } \langle\Omega|\psi|\Omega\rangle \neq 0$$

this violates Lorentz invariance. On the other hand, if we have a scalar field, for instance like

$$\mathcal{L} = c\phi - \frac{1}{2}\phi(\Box - m^2)\phi,$$

this can have nonzero contribution for 1-point functions. We can think of as there being a quadratic potential $V(\phi)$, and so if we shift ϕ so that $\phi = 0$ is the minimal energy state, there won't be such a problem. So these tadpoles indicate instability of the theory. For instance, $e^- e^- = \phi^{2-}$ interaction in some materials can be attractive. Here, $c = \langle 0|\phi|0\rangle \neq 0$, and then the ground state should be charged. If you do this, you see that photon gets a mass, and then we get screening of magnetic fields in type II superconductors. This is the BCS theory of superconductivity.

Anyways, let's now talk about 2-point functions. There is $\langle A_\mu A_\nu \rangle$ we talked about, and we also have $\langle \bar\psi \psi \rangle$ given by



given by

$$\begin{aligned} i\Sigma_2(\not p) &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not k + m)\gamma^\mu)}{k^2 - m^2 + i\epsilon} \frac{-i}{(p-k)^2 + i\epsilon} \\ &= \frac{\alpha}{4\pi} \int_0^1 dx[(2-\epsilon)x\not p - (4-\epsilon)m]\left[\frac{2}{\epsilon} + \log \frac{\tilde\mu^2}{(1-x)(m^2 - p^2 x)}\right] \\ &= \frac{\alpha}{\pi}\left(\frac{\not p - 4m}{2\epsilon} + \text{finite}\right). \end{aligned}$$

So now we would like to remove UV divergence by redefining m. If we take m_0 as the bare mass, and define $m_0 = m_R + m_R \delta_m$ up to order e^2 for m_R the renormalized mass, we get

$$\begin{aligned} \langle \bar\psi\psi \rangle &= \frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}[i\Sigma_2(\not p)]\frac{i}{\not p - m_0} + \cdots \\ &= \frac{i}{\not p - m_R} + \frac{i}{\not p - m_R}[i\Sigma_2(\not p) - i\delta_m m_R]\frac{i}{\not p - m_R} + \cdots . \end{aligned}$$

To make this finite, we choose $\delta_m = -\frac{\alpha}{\pi}\frac{4}{2\epsilon}$.

Here is how you can think about this. We have the bare Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + i\bar\psi_0 \not\partial \psi_0 - m_0 \bar\psi_0 \psi_0 + ie_0 \bar\psi_0 \not A \psi_0,$$

and we renormalized this to

$$\psi_0 = \sqrt{Z_2}\psi_R, \quad A_0 = \sqrt{Z_3} A_R,$$

for $Z_2 = 1 + \delta_2$ and $Z_3 = 1 + \delta_3$. Then the renormalized Lagrangian can be written as

$$\mathcal{L} = Z_3(-\tfrac{1}{4}(\partial_\mu A_R^\mu - \partial_\nu A_R^\nu)^2) + iZ_2 \bar\psi_R \not\partial \psi_R - m_R(1+\delta_m)Z_2 \bar\psi_R \psi + ie_R(1+\delta_e)Z_2\sqrt{Z_3}\bar\psi_R \not A_R \psi_R.$$

So coming back to the calculation, we can now write

$$\begin{aligned}\langle\psi_R\psi_R\rangle &= \frac{1}{Z_2}\langle\psi^0\psi^0\rangle = \frac{1}{1+\delta_2}\frac{i}{\not{p}-m_R-\delta_m m_R}\\ &= \frac{i}{\not{R}-m_R}+\frac{i}{\not{p}-m_R}[i(\delta_2\not{p}-(\delta_2+\delta_m)m_R+\Sigma(\not{p}))]\frac{i}{\not{p}-m_R}+O(e^4).\end{aligned}$$

At the end we will have

$$\delta_2 = \frac{\alpha}{2\pi}\frac{1}{\epsilon}, \quad \delta_m = -\frac{3\alpha}{2\pi}\frac{1}{\epsilon}+\text{finite}.$$

How do we compute this renormalization condition? Recall that the mass was defined from the representation of the Poincaré group. We should have $\langle\bar{\psi}_R\psi_R\rangle = (i\not{p}+m)/(p^2-m^2)$ has a pole at $p^2=m^2$. It seems that there are too many poles, but actually we are adding over many different corrections and this does shift the pole. So you can get

$$\langle\psi^R\psi^R\rangle = \frac{i}{\not{p}-m_R+\Sigma_R(\not{p})}, \quad \Sigma_R(\not{p}) = \Sigma_2(\not{p})+\delta_2\not{p}-(\delta_m+\delta_2)m_R+O(e^4).$$

Then the renormalization condition is that there is a pole at $\not{p}=m_R$ with residue i,

$$\langle\psi^R\psi^R\rangle = \frac{i}{\not{p}-m_R}+\text{regular}.$$

This means that $\Sigma_R(m_R)=0$ and $\Sigma'_R(m_R)=0$.

We can solve this for δ_2 and δ_m. Because we have this explicit form

$$\Sigma_R(\not{p}) = \Sigma_2(\not{p})+\delta_2\not{p}-(\delta_m+\delta_2)m_R,$$

evaluating this at m_R gives

$$\delta_m = \frac{\alpha}{2\pi}\left(-\frac{3}{\epsilon}-\frac{3}{2}\log\frac{\tilde{\mu}^2}{m_R^2}-2\right), \quad \delta_2 = -\frac{\alpha}{2\pi}\left(\frac{1}{\epsilon}+\frac{1}{2}\log\frac{\tilde{\mu}^2}{m_R}+2+\frac{m_\gamma^2}{m_R^2}\right),$$

where m_γ^2 is the photon mass in the IR regulator. This is called the **on-shell renormalization**. An alternative is minimal subtraction. Here, we drop all finite contributions to δ_2 and δ_m in dimensional regularization.

24 November 27, 2018

We wanted to show that all Green's functions can be made UV-finite through renormalization. We encountered infinities and we redefined the mass $m_0 = m_R(1+\delta_m)$ and also $\psi_0 = \frac{1}{Z_2}\psi_R$ where $Z_2 = 1+\delta_2$. The finite parts of δ_2 and δ_m are not unique, so we need a convention for defining. This is called the subtraction scheme or the **renormalization scheme**. This is the prescription for fixing the finite parts of δs. We used the on-shell scheme, so that $m_R = m_{pole}$ is the location of the pole in the propagator. Once we did that we got

$$\delta_m = \frac{e^2}{8\pi}\biggl(-\frac{3}{\epsilon} - 4 - \frac{3}{2}\log\frac{\tilde{\mu}^2}{m_p^2}\biggr).$$

The other scheme we used was minimal subtraction, and this means that the finite part is zero. This is simpler but not physical. Then we get the MS-mass m_R^{MS}. There is also that MS-bar where the finite part is $\log e^{-\gamma_E}4\pi$. In this scheme we have

$$\delta_m = \frac{e^2}{8\pi^2}\biggl(-\frac{3}{\epsilon}\biggr).$$

The differences between them are just finite corrections. Then

$$m_0 = m_R(1+\delta_m^{\mathrm{MS}}) = m_p(1+\delta_m^{\mathrm{OS}}),$$

and so they are related by

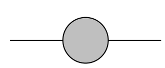
$$m_R = m_p\biggl(1 - \frac{\alpha}{4\pi}\biggl(4 + 3\log\frac{\mu^2}{m_p^2}\biggr)\biggr).$$

For example, the top quark has

$$m_p = 174\text{GeV}, \quad m_{\overline{MS}} = 163\text{GeV}.$$

24.1 1PI and amputation

The other concept we introduced was **one-particle irreducibility** or just 1PI. A 1PI graph cannot be split into two disconnected graphs by cutting a single internal line. The good thing about this is that we have



$$= \frac{1}{1-(1\mathrm{PI})}.$$

If we can show that all 1PI graphs are finite, then all graphs are finite. So this reduces the program of showing renormalizability to only 1PI graphs.

In S-matrix elements, external momenta are on-shell. But then, we really are including contributions of graphs where the external legs have some bubbles. So the LSZ formula gives

$$\begin{aligned}\langle p_3\cdots p_n|p_1p_2\rangle = \int d^4x_1\cdots d^4x_n e^{ip_1x_1}\cdots e^{-ip_nx_n}(\not{p}_1 - m_p)\cdots(\not{p}_n - m_p)\\ \times\langle\Omega|T\{\psi_R(x_1)\cdots\psi_R(x_n)\}|\Omega\rangle.\end{aligned}$$

So this is just the sum of the amputated diagrams, diagrams that don't contain bubbles on external legs. Here, we are actually having

$$\psi_R^{\mathrm{OS}} = \sqrt{\frac{Z_2^{\mathrm{MS}}}{Z_2^{\mathrm{OS}}}}\psi_R^{\mathrm{MS}}$$

and so we really have

$$\langle p_3 \cdots p_n | p_1 p_2 \rangle = (\text{Amputated}) \times \left(\sqrt{\frac{Z_2^R}{Z_2^{\mathrm{OS}}}} \right)^n.$$

This is the renormalized version of the LSZ reduction formula, and it is cleaner in some sense.

24.2 Renormalized perturbation theory

We have the bare Lagrangian

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^0 - \partial_\nu A_\mu^0)^2 + \bar{\psi}^0 (i\partial\!\!\!/ - e_0 A\!\!\!/_0 - m_0)\psi^0.$$

Now we renormalize $\psi^0 = \sqrt{Z_2}\psi^R$ and $A_\mu^0 = \sqrt{Z_3}A_\mu^R$ with $e_0 = Z_e e_R$ and $m_0 = Z_m m_R$. Then

$$\mathcal{L} = -\frac{1}{4}Z_3(\partial_\mu A_\nu^R - \partial_\nu A_\mu^R)^2 + iZ_2\bar{\psi}_R \partial\!\!\!/ \psi_R - Z_2 Z_m m_R \bar{\psi}\psi - e_R Z_e Z_2 \sqrt{Z_3}\bar{\psi}_R A\!\!\!/^R \psi_R.$$

Now we define

$$Z_e Z_2 \sqrt{Z_3} = Z_1 = 1 + \delta_1, \quad Z_2 = 1 + \delta_2, \quad Z_3 = 1 + \delta_3, \quad Z_m = 1 + \delta_m.$$

We have that δ are formally of order at least 2 in e_R even if they are infinite. So we can do perturbation theory in e_R and then write

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\partial\!\!\!/\psi - m_R\bar{\psi}\psi - e_R\bar{\psi}A\!\!\!/\psi \\ &-\frac{1}{4}\delta_3 F_{\mu\nu}^2 + i\delta_2\bar{\psi}^2\partial\!\!\!/\psi - (\delta_m + \delta_2 + \delta_m\delta_2)m_R\bar{\psi}\psi - e_R\delta_1\bar{\psi}A\!\!\!/\psi\end{aligned}$$

where all these fields are now renormalized. Then we can read off the Feynman rules for these graphs, and for instance,

 $= i(p\!\!\!/\delta_2 - (\delta_m + \delta_2 + \delta_2\delta_m)m_R).$

We can think of this as these loops contracted to a point. We now want to choose δ so that we cancel out all the infinite parts.

Now let us expand the electron self energy for instance. The we get

$$\frac{i}{p\!\!\!/ - m_R} + \frac{i}{p\!\!\!/ - m_R}\Sigma_2(p\!\!\!/)\frac{i}{p\!\!\!/ - m_R} + \frac{i}{p\!\!\!/ - m_R}(ip\!\!\!/\delta_2 - (\delta_m + \delta_2)m_R)\frac{i}{p\!\!\!/ - m_R}.$$

Similarly, we see that the photon self energy is

$$-ie_R^2(p^2g^{\mu\nu}-p^\mu p^\nu)\Pi_2(p^2)-i(p^2g^{\mu\nu}-p^\mu p^\nu)\delta_3,$$
$$\Pi_2=\frac{1}{2\pi}\int_0^1 dxx(1-x)\left[\frac{2}{\epsilon}+\log\frac{\tilde{\mu}^2}{m_R^2-p^2x(1-x)}\right].$$

To see the on-shell renormalization condition, we need to look at the sum of all the loop corrections and get

$$\frac{ig^{\mu\nu}}{p^2(1+\Pi(p^2))},\quad \Pi(p^2)=e_R^2\Pi_2(p^2)+\delta_3.$$

This has to look like $ig^{\mu\nu}/p^2$, so we get

$$\delta_3^{\mathrm{OS}}=-\frac{e_R^2}{6\pi^2}\frac{1}{\epsilon}-\frac{e_R^2}{12\pi^2}\log\frac{\tilde{\mu}^2}{m_R^2}.$$

So we have dealt with all graph with two external legs. Now let us go to three legs. We have

$$G_3=\langle\Omega|T\{\bar{\psi}A_\mu\psi\}|\Omega\rangle.$$

Here, we can look at the 1PI graphs and write

$$\Gamma_\mu(p^2)=F_1(p^2)\gamma^\mu+i\frac{\sigma^{\mu\nu}}{2m}p_\nu F_2(p^2),$$
$$F_1(p^2)=1+e_R^2(\cdots),\quad F_2(p^2)=0+\frac{\alpha}{\pi}+O(p^2).$$

The loop graph contributing to this is just the graph we used to correct $g-2$, and then we see that we get

$$F_1(p^2)=1-2ie_R^2\int d^3x\delta(1-x-y-z)\int\frac{d^4k}{(2\pi)^4}\frac{k^2-2(1-x)(1-y)p^2+\cdots}{[k^2-m_R^2(1-x)^2\cdots]}.$$

As $p\to 0$, we should get the regular QED interaction, so we should have the δ_1 correction to be cancel out F_1, that is,

$$\delta_1=-F_1^{(2)}(0)=-\frac{\alpha}{2\pi}\left(\frac{1}{\epsilon}+\frac{1}{2}\log\frac{\tilde{\mu}^2}{m_R^2}+2+\log\frac{m_\gamma^2}{m_R^2}\right).$$

In fact, this is equal to δ_2 and so $Z_1=Z_2$. The reason for this is essentially charge conservation.

25 November 29, 2018

Today we will prove that QED is renormalizable. We introduced these renormalizations for the photon self-interaction, the electron self-interaction, and the interaction term. Then we had

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu}^2 + i Z_2 \bar{\psi} \partial\!\!\!/ \psi - e_R Z_1 \bar{\psi} A\!\!\!/ \psi - m_R Z_2 Z_m \bar{\psi} \psi.$$

Then we had renormalization conditions that say that things look like tree-level in long distances. This fixes the 1-loop terms.

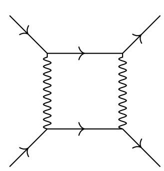
The reason we have $\delta_1 = \delta_2$ is because of the conservation of charge. This implies that $Z_e = 1/\sqrt{Z_3}$. This is saying that the correction to the interaction term is equal to the correction to the photon propagator term. So in some sense, the photon interaction is universal. This is because if we a symmetry about charge, for instance, if we have a quark in our theory and there is this symmetry

$$\psi \to e^{i\alpha} \psi, \quad \psi_q \to e^{-\frac{2}{3} i\alpha} \psi_q,$$

then $\mathcal{L}$ is invariant under this symmetry, and this is true for all renormalizations. So even if the interaction terms get renormalized, the symmetries stay the same.

25.1 Renormalizability of QED

So far, we found 3 UV-divergent graphs in QED, and then removed those by Z_1, Z_2, Z_m, Z_3. But are there more 1-loop UV-divergences? For 2-point and 3-point diagrams, we don't have anything more. You can worry about three photon legs, but this is not possible. For 4-point diagrams, we have



$$\int d^4k \frac{1}{k^2} \frac{1}{k\!\!\!/} \frac{1}{k^2} \frac{1}{k\!\!\!/} \sim \frac{1}{\Lambda^2}$$

and this is UV finite. Similarly, we can compute a similar thing for the diagram with two external photons and two external legs. Finally, for four external photons, we get zero because the divergent term should look like

$$c \log \Lambda \epsilon^\mu \epsilon^\nu \epsilon^\alpha \epsilon^\beta (g_{\mu\nu} g_{\alpha\beta} + g_{\mu\alpha} g_{\nu\beta} g_{\mu\beta} g_{\nu\alpha}).$$

Then the Ward identities show that this is only possible if $c = 0$. This diagram is still going to be nonzero, but it is finite because the divergent part is zero. More external lines are less UV-divergent, so we don't have to worry about this. This shows that QED is renormalizable in 1-loop.

What about 2-loops? There can be a 1PI two-loop electron self-interaction, but then we can renormalize this using the terms that are higher-order in e_R. You can also realize that adding more loops don't make things more divergent, because you can do dimensional analysis and see that adding more loops doesn't change the order of divergence. So we need to use the same parameters and just correct the parameters in higher order.

The formal proof can be found in Weinberg, and the proof is due to BPHZ. To prove renormalizability of the Standard Model, you need take more care, but it is a nice thing to have because this says that you can do a finite number of measurements and then start calculating.

Here is the dimensional analysis part. We can superficially calculate the degree of divergence D, which makes $\mathcal{M} \sim \Lambda^D$. Indeed, we have

$$[\mathcal{M}] = 4 - n_b - \tfrac{3}{2} n_f = D,$$

where n_b is the number of external bosons and n_f is the number of external fermions. So there is only a finite number of (n_b, n_f) such that the superficial degree of divergence D is nonnegative. This really is all there is to the proof of renormalizability of QED.

25.2 Nonrenormalizable theories

QED is special because the only coupling constant e_R is dimensionless. But if there are other scales in the problem, for instance,

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi + G\bar{\psi}\psi\bar{\psi}\psi,$$

then we have $[G] = -2$. So how does this screw up our power counting? If we have a $\psi\bar{\psi} \to \psi\bar{\psi}$ with two Fermi interactions, we get

$$G^2 \int d^4k \frac{1}{k\!\!\!/ k\!\!\!/} \sim G^2\Lambda^2, \quad [\mathcal{M}] = -2.$$

Moreover, if I have any diagram, I can start adding in edges and then we get UV-divergent graphs. So any small mass dimension -1 interaction screws up the renormalizability of the theory.

A graph with n_j factors of a coupling of dimension Δ_j has dimension

$$D = 4 - n_b - \tfrac{3}{2} n_f - \sum_j n_j \Delta_j.$$

If some coupling has $\Delta_j < 0$ then there are an infinite number of UV divergent amplitudes. We call a theory

- **renormalizable** if we have $\Delta_j \geq 0$ for all j,
- **non-renormalizable** if we have $\Delta_j < 0$ for all j,
- **super-renormalizable** if we have $\Delta_j > 0$ for all j.

As we will see, some non-renormalizable theories can be renormalized, and super-renormalizable theories are horrible.

Let's take a scalar theory

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + G\phi^2(\partial_\mu\phi)(\partial_\mu\phi), \quad [G] = -2$$

for instance. At the tree level, we just have a interaction $iGp^i_\mu p^j_\mu$ and everything is fine. At 1-loop, we can have things like

$$G^2(c_1\Lambda^4 + c_2\Lambda^2 p^2 + c_3 p^4 \log\Lambda + c_4\Lambda^4 \log p^2 + \text{finite}).$$

So we can first renormalize G using $G_R Z_G$, $Z_G = 1 + \delta_G$. Then we can set $\delta_G = c_2\Lambda^2 G$ to cancel the c_2 term. But then there are three more terms. For instance, to cancel out $p^4\log\Lambda$, we really need something with 4 derivatives. So we write

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + G_R Z_G \phi^2\Box\phi^2 + \kappa_R\delta_R\phi^2\Box^2\phi + \lambda_R z_\lambda \phi^4$$

and set something like

$$\delta_\kappa = -\frac{G^2}{\kappa_R}c_3\log\Lambda, \quad \delta_\lambda = -\frac{1}{\lambda}G^2\Lambda^4 G.$$

So we get rid of c_1, c_2, c_3. It turns out that c_4 actually never arises.

Theorem 25.1. *The UV-divergences are always polynomials in external momenta.*

The Weinberg proves this is to differentiate. We can differentiate with respect to external momenta until the integral is convergent, and then integrate. We can look at something like

$$I(p) = \int dk \frac{k}{k+p} \sim \Lambda$$

and then integrate this. So we get

$$I'(p) \sim \log\Lambda, \quad I''(p) = \frac{1}{p}.$$

Now if we try to integrate this, we get

$$\begin{aligned} I'(p) &= \log p + \text{const} = \log\frac{p}{\Lambda}, \\ I(p) &= p\log\frac{p}{\Lambda} - p + c_1\Lambda = p\log p - p\log\Lambda - p + c_1\Lambda. \end{aligned}$$

So any divergences are only polynomial in p.

It's really important that we don't get terms like $\log\Box$ that are not local. In non-renormalizable theories,

(1) we must add every local term to the Lagrangian that are consistent with symmetries,

(2) then we need an infinite number of renormalization conditions to define the theory,

(3) but the theory is *still* predictive because the new terms are suppressed at low energy.

Index