Physics 253a - Quantum Field Theory I

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You need at least 10 hours a week to take this course. This course will get more difficult as we go into renormalization. Then it will get easier once we pass this and get to applications.

We will start with special relativity and quantum mechanics, put them together and see what happens. We won't start with the axioms, because they are just statements that sound reasonable but cannot be tested.

1.1 Quantum theory of radiation

When you turn on the lights, the number of particles increase. How does this happen? Max Planck in the 1900s observed that discrete energy can explain blackbody radiation. Einstein in 1916 explained spontaneous/stimulated emission, and Paul Dirac in 1927 invented quantum electrodynamics, the microscopic theory of radiation.

We have a box of size L, poke a hole and heat it up. Then light comes out. We know that the wave numbers associated with the box are $\vec{k} = \frac{2\pi}{L}\vec{n}$, and $\omega = |\vec{k}|c$. This is classical prediction. Then the number of modes $\leq n$ is proportional to n^3, and the classical equipartition theorem predicts that each mode has the same energy. Sow we would have

$$dI(\omega) \sim \omega^2 d\omega.$$

This is called the ultraviolet catastrophe. But experimentally, we have exponential decay.

Planck said that energy E is quantized, so that $E_n = \hat{h}\omega_n$. Here, $\omega_n = \frac{2\pi}{L}n$ where $n = |\vec{n}|$. Then each mode gets excited an integer number of times, E_n^{tot} is an integer times E_n. The probability of $E_n^{\mathrm{tot}} \sim e^{-\beta E_n}$. Then

$$\langle E_n \rangle = \frac{\sum_{j=0}^{\infty}(\hbar j \omega_n) e^{-j\hbar\omega_n\beta}}{\sum_{j=0}^{\infty} e^{-j\hbar\omega_n\beta}} = \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1}.$$

Then the total energy up to ω is

$$\begin{aligned} E(\omega) &= \int_0^{\omega} d^3n \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1} = \hbar \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \int_0^{L\omega/2\pi} n^2 dn \frac{\omega_n}{e^{\hbar\omega_n\beta} - 1} \\ &= \hbar \frac{L^3}{(2\pi)^3} 4\pi \int_0^{\omega} \frac{\omega^3}{e^{\hbar\omega\beta - 1}}. \end{aligned}$$

So we get Planck's formula

$$I(\omega) = \frac{K}{2\pi^2} \frac{\omega^3}{e^{\hbar\omega\beta} - 1} \times 2.$$

The point here is that each mode gets excited an integer number of times. This is called **second quantization**. This really is just quantization, because the first quantization refers to $\vec{k} = \frac{2\pi}{L}\vec{n}$, which is just classically solving wave equations with boundary conditions.

Let us now look at a number of atoms, either in the ground state or the excited state with energy difference $E_2 - E_1 = \hbar\omega$. Let n_1, n_2 be the number of atoms with energy E_1, E_2. Also assume that there is a bath of photons of frequency ω, with intensity $I(\omega)$ and number $n_\omega = \frac{\pi^2}{\omega^3}I(\omega)$. If we look at the probability of atoms getting excited or emitting, we get

$$dn_2 = -An_2 - BI(\omega)n_2 + B'I(\omega)n_1.$$

Here, the first term is spontaneous emission, the second is stimulated emission, and the third is stimulated absorption. It's not obvious that the second term should exist, but it turns out to be nonzero. In equilibrium, we have

$$I(\omega)(B'n_1 - Bn_2) = An_2.$$

So we get

$$I(\omega) = \frac{A}{B'\frac{n_1}{n_2} - B} = \frac{A}{B'e^{\beta\hbar\omega} - B}$$

because $n_1 = e^{-\beta E_1}$ and $n_2 = e^{-\beta E_2}$.

Matching with Planck's formula, we get the relations

$$B = B', \quad A = \frac{\hbar}{\pi^2}\omega^3 B,$$

called Einstein's equations. The number B can be calculated by quantum mechanics. So we can calculate A using this relation and quantum mechanics.

This is what got to Dirac. It's great that we can compute the coefficient of spontaneous emission, but it will be good to calculate this without using thermal systems, just from fundamental laws. The second quantization really looks like the simple harmonic oscillator. So we are going to identify

$$|n\rangle = n \text{ photon state} = n\text{th excited state of the oscillator.}$$

Consider $a^\dagger$ the creation operator and a the annihilation operator so that $[a, a^\dagger] = 1$ and $N = a^\dagger a$ is the number operator with $\hat{N}|n\rangle = n|n\rangle$. We can compute

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

This turns out to be a powerful tool.

Now Fermi's golden rules says that the transition rate is $\Gamma \sim |M|^2\delta(E_f - E_i)$. If we use this, we get at the end,

$$|M_{2\to 1}|^2 = |M_0|^2(n_\omega + 1), \quad |M_{1\to 2}|^2 n\omega|M_0|^2.$$

So this algebra of creation and annihilation operation gives us the relation between spontaneous emission and stimulated absorbtion. Then more algebra gives

$$dn_2 = -|M_0|^2\Big(1 + \frac{\pi^2}{\hbar\omega}I(\omega)\Big)n_2 + \frac{\pi^2}{\hbar\omega^3}I(\omega)n_1.$$

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Today we are going to start the systematic development of the field. We let $c = 1$ and $\hbar = 1$.

2.1 Special relativity

There are rotations on the plane,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x_i \to R_{ij} x_j.$$

We can also rotate row vectors as

$$x^i \to x^i (R^T_{ij}),$$

and the rotations satisfy $R^T_{ij} \cdot 1_{jk} R_{kl} = 1_{il}$. This is because rotations should preserve $x^i x_i = x^2 + y^2$. In 3 dimensions, we have $x^2 + y^2 + z^2$, and in 4 dimensions, we have $t^2 - x^2 - y^2 - z^2$. So **Lorentz transformations** satisfy

$$\Lambda^T g \Lambda = g, \quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Examples include

$$\Lambda_{\theta_z} = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}, \quad \Lambda_{\beta_x} = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Four momentum is defined as

$$p^\mu = (E, p_x, p_y, p_z),$$

and it satisfies $p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$. Usually, $\vec{x}$ or x_i denotes a 3-dimensional vector, and x or x^μ denotes a 4-dimensional vector.

Tensors transform as

$$T_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}.$$

We define the **d'Alembertian** as

$$\Box = \partial_\mu^2 = g^{\mu\nu} \partial_\mu \partial_\nu = \partial_t^2 - \vec{\nabla}^2 = \partial_t^2 - \Delta.$$

We say that a vector is **timelike** if $V^2 > 0$, and **spacelike** if $V^2 < 0$, and **lightlike** if $V^2 = 0$.

The proper **orthochronous** Lorentz group has $\det \Lambda = 1$ and $\Lambda_{00} > 0$. There are four components of the Lorentz group, and this is the connected component at the identity. The **Poincaré group** are Lorentz transformations plus translations.

2.2 Quantum mechanics

Remember we had normal modes in a box last time. These frequencies are quantized classically. Then Planck said that the energy should be associated to the frequency $E = j\hbar\omega$. Einstein was the one who interepreted these as particles, which we call photons, and Dirac developed this microscopic theory of $H = H_0 a^\dagger + H_0 a$.

Let us review the simple harmonic oscillator. We have a ball with a spring on it, and its equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

You can solve this, and you get

$$x(t) = \cos\Big(\sqrt{\frac{k}{m}}t\Big).$$

The classical Hamiltonian is given by

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2.$$

Then we quantize this using $[\hat{x}, \hat{p}] = i\hbar$, and define

$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} + \frac{i\hat{p}}{m\omega}\Big), \quad a^\dagger = \cdots, \quad [a, a^\dagger], \quad H = \hbar\omega(N + \tfrac{1}{2}), \quad N = a^\dagger a.$$

We found last time that

$$N|n\rangle = n|n\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

Then in the Heisenberg picture,

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0).$$

Now what can the equation of motion for the scalar field be? It should be Lorentz invariant, so the simplest possible equation is

$$\Box\phi = 0 = (\partial_t^2 - \vec{\nabla}^2)\phi = 0.$$

Take the Fourier transform, and let

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}]$$

Then the equation becomes

$$(\partial_t^2 + \vec{p}^2)a_p(t) = 0.$$

Now each component is just a classical simple harmonic oscillator. So we can quantize each separately, and then put them back together.

Electromagnetic saves are oscillators,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

This concisely encodes Maxwell's equations

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial_\mu F_{\mu\nu} = 0$$

in empty space. It's also helpful to write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This vector potential A_μ is more useful for field theory, because there are only 4 components, and also because it is invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \alpha(x),$$

called gauge invariance.

We can choose $\partial_\mu A_\mu = 0$, and this is called **Lorentz gauge**. When you do that, Maxwell's equations become

$$0 = \partial_\mu F_{\mu\nu} = \Box A_\nu.$$

So then we can make $A_\nu(x,t)$ into a set of harmonic oscillators. We write

$$A_\nu(x,t) = \int \frac{d^3p}{(2\pi)^3}(A_\nu^p(t)e^{i\vec{p}\cdot\vec{x}} + A_\nu^{p*}(t)e^{-i\vec{p}\cdot\vec{x}}), \quad (\partial_t^2 + \vec{p}^2)A_\nu^{\vec{p}} = 0.$$

Then the free electromagnetic field is equivalent to an infinite number of simple harmonic oscillators, labeled by 3 vectors $\vec{p}$ with frequencies $\omega_p = |\vec{p}|$.

Now we quantize as in quantum mechanics. Then

$$H_0 = \int \frac{d^3p}{(2\pi)^3}\omega_p(a_p^\dagger a_p + \tfrac{1}{2}).$$

The relations between these creation and annihilation operators are

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k}), \quad a_p|0\rangle = 0, \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_p}}|p\rangle.$$

What we have done is that we have constructed the Hilbert space

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p,$$

called the **Fock space**.

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Last time we reviewed the simple harmonic oscillator. To quantize this theory, we defined $H = \omega(a^\dagger a + \frac{1}{2})$. For fields, we classically had $\Box A_\mu(x) = 0$ or $(\Box + m^2)\phi(x) = 0$. We do the Fourier transform, and we get something like

$$A(x,t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}].$$

Then the equation becomes $[\partial_t^2 + \omega_p^2]a_p(t) = 0$ and $\omega_p = \sqrt{\vec{p}^2 + m^2}$. Then we quantize and get

$$H = \int \frac{d^3p}{(2\pi)^3}[\omega_p(a_p^\dagger a_p + \frac{1}{2}].$$

3.1 Operators on the Fock space

The Fock space is then

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p = \bigoplus_n \mathcal{H}_n$$

where p is the momentum and n is the number of particles. The creation and annihilation operators then behave as

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We normalize

$$a_p|0\rangle = 0, \quad |p\rangle = \sqrt{2\omega_p}a_p^\dagger|0\rangle.$$

Then we get

$$\langle p|k\rangle = \sqrt{2\omega_p}\sqrt{2\omega_k}\langle 0|a_p a_k^\dagger|0\rangle = 2\omega_p(2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We also have

$$\mathbf{1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}|p\rangle\langle p|.$$

Then you can check $|k\rangle = \mathbf{1}|k\rangle$.

Also, we define

$$A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}].$$

This is like a creation operator in position space. Indeed, we compute

$$\langle p|A(x)|0\rangle = \int d^3k\delta^3(p-k)\langle 0|0\rangle e^{-i\vec{k}\cdot\vec{x}} = e^{-\vec{p}\vec{x}}.$$

But $A(x)A(y)|0\rangle$ is not just particles at x and y.

In quantum field theory, we work with the Heisenberg picture, so we define $a_p^\dagger(t) = e^{i\omega_p t}a_p^\dagger(0)$. Then

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p(0)e^{i\vec{p}\cdot\vec{x} - i\omega_p t} + a_p^\dagger(0)e^{i\omega_p t - i\vec{p}\cdot\vec{x}}].$$

Here, you can interpret the exponent as $p^\mu x_\mu$, because $p^\mu = (\omega_p, \vec{p})$.

3.2 Classical field theory

The main object is the Hamiltonian

$$H(p,x) = \text{energy} = K + V.$$

This is not Lorentz invariant, and generates time translation. On the other hand, the Lagrantian

$$L[x,\dot{x}] = K - V$$

is not a conserved quantity, but it is Lorentz-invariant and the dynamics is determined by minimizing the action $S = \int d\epsilon L$.

For fields, we are going to have

$$L[\phi,\dot{\phi},\vec{\nabla}\phi] = L[\phi,\partial_\mu\phi], \quad H[\phi,\pi,\vec{\nabla}\phi].$$

We still talk about kinetic terms

$$K = \text{things like } \frac{1}{2}\phi\Box\phi, \quad \frac{1}{4}F_{\mu\nu}^2, \frac{1}{2}m^2\phi^2, \phi\partial_\mu A^\mu,$$

and interactions

$$V = \text{things like } A\phi^3, e\bar{\psi}A\psi, e(\partial_\mu\phi)\phi^* A_\mu.$$

Example 3.1. Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi).$$

To minimize the action, we perturb the field a little bit and look at the difference. Then

$$\delta S = \int d^4x \Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\Big] \quad = \int d^4x \Big\{\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big] + \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big]\Big\}.$$

Here, we assume $\phi(\infty) = 0$, so we get

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

This is called the **Euler–Lagrangian equations**.

Example 3.2. In the above example, we get

$$-V'(\phi) = \partial_\mu[\partial_\mu\phi] = \Box\phi.$$

3.3 Noether's theorem

Suppose $\mathcal{L}$ is invariant under some specific continuous variation. For instance, take

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^*$$

which is invariant under $\phi \to e^{i\alpha}\phi$. Then

$$0 = \frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \Big\{ \Big[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\partial \alpha} \Big\}.$$

So if the Euler–Lagrange equations are satisfied, the first term is zero so

$$\partial_\mu J^\mu = 0, \quad J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}.$$

Then if we define $Q = \int d^3x J^0$, we have $\partial_t Q = 0$. This is the statement and proof of **Noether's theorem**.

Let's think about what we get for $\phi \mapsto e^{i\alpha}\phi$. We have

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(-i\phi^*) = i\phi \partial_\mu \phi^* - i\phi^* \partial_\mu \phi.$$

We can check

$$\partial_\mu J^\mu = i\partial_\mu \phi \partial_\mu \phi^* + i\phi \Box \phi^* - i\partial_\mu \phi^* \partial_\mu \phi - i\phi^* \Box \phi = i\phi \Box \phi^* - i\phi^* \Box \phi.$$

This is zero because at the equations of motion, we have $\Box \phi = m^2 \phi$.

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Noether's theorem says that if and action has a continuous symmetry, then there exists a current J^μ with $\partial_\nu J^\mu = 0$ when the equations of motion are satisfied. In this case,

$$Q = \int d^3x J^0$$

satisfies $\partial_t Q = 0$.

Consider translation invariance. When we look at a translate of $\mathcal{L}$, we get

$$\partial_\mu(g_{\mu\nu}\mathcal{L}) = \partial_\nu\mathcal{L} = \Big[\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\Big]\frac{\delta\phi_n}{\partial\xi^\nu} + \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\partial\xi^\nu}\Big].$$

Because the first term vanishes at equation on motion. So we have

$$\partial_\mu T_{\mu\nu}$$

where

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n - g_{\mu\nu}\mathcal{L}.$$

This is called the **energy-momentum tensor**. Here, we note that

$$\mathcal{E} = T_{00} = \sum\frac{\partial\mathcal{L}}{\partial\dot\phi_n}\dot\phi_n - \mathcal{L} = \pi\dot\phi - \mathcal{L} = \mathcal{H}$$

is just the energy. So energy $E = \int d^3x T^{00}$ is conserved over time.

4.1 Coulomb's law

We are going to introduce an external current

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - J_\mu A^\mu.$$

(When I say current, I don't mean Noether current here.) Because $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we have

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - J_\mu A^\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu + \frac{1}{2}\partial_\mu A_\nu\partial_\nu\text{‘}A_\mu - J_\mu A^\mu.\end{aligned}$$

Then $\partial\mathcal{L}/\partial A_\nu = -J_\nu$ and $\partial\mathcal{L}/\partial\partial_\mu A_\nu = -\partial_\mu A_\nu + \partial_\nu A_\mu = -F_{\mu\nu}$. Then the Euler–Lagrange equation is

$$\partial_\mu F_{\mu\nu} = J_\nu,$$

which is Maxwell's equations. If we go to Lorentz gauge, we get

$$\Box A_\nu = J_\nu.$$

We are going solve this by inverting the d'Alembertian $\Box$. Here, note that we have Fourier transform

$$f(x) = \int \frac{dk}{2\pi}\tilde{f}(k)e^{ikx}, \quad \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}.$$

Then the inverse is

$$\tilde{f}(k) = \int dx f(x) e^{-ikx}.$$

We can compute

$$\Box f(x) = \int d^k k \Box \tilde{f}(x) e^{ikx} = \int d^4 k (-k^2)\tilde{f}(k)e^{ikx}.$$

So $\Box$ corresponds to $-k^2$ in Fourier space.

We want to solve the equation when there is a point charge, when $J_0 = \delta^3(x)$ and $\vec{J} = 0$. Then

$$\begin{aligned} A_0(x) &= \frac{e}{\Box}\delta^3(x) = -\frac{e}{\Delta}\delta^3(x) = \int \frac{d^3k}{(2\pi)^3}\frac{e}{\vec{k}^2}e^{i\vec{k}\vec{x}} \\ &= \frac{e}{i4\pi^2}\int_0^\infty dk \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{e}{4\pi r}. \end{aligned}$$

This is the Coulomb potential.

4.2 Green's functions

Let's look at a complicated example,

$$\mathcal{L} = -\frac{1}{2}h\Box h + \frac{1}{3}\lambda h^3 + hJ.$$

This is a toy example for gravity, because gravitons interact with each other. Then the Euler–Lagrange equation is

$$\Box h - \lambda h^2 - J = 0.$$

We now work perturbatively in λ. For $\lambda = 0$, we know

$$h_0 = \frac{1}{\Box}J.$$

If $\lambda \neq 0$, we can write $h = h_0 + h_1$, where $h_1 = O(\lambda)$. If we plug in into the original equation, we get $\Box h_1 = \lambda h_0^2$. So we can write

$$h_1 = \frac{\lambda}{\Box}\Big(\frac{1}{\Box}J\Big)^2.$$

So we get

$$h = \frac{1}{\Box}J + \lambda\Big(\frac{1}{\Box}\Big)\Big(\frac{1}{\Box}J\Big)\Big(\frac{1}{\Box}J\Big) + \cdots.$$

We can interpret each of these in terms of Feynman diagrams. Think of each J as a source, $\frac{1}{\Box}$ as a propagation or a branch coming our from a source, and λ as an interaction between these branches. Then this is something like the Sun emitting a graviton, emitting another graviton, and they interact and become one. There are other diagrams we can draw but are not represented in the solution, and these are purely quantum mechanical effects that we will discuss. What we are doing now is classical.

If we look at the solution for $\Box_x A(x) = J(x)$ again, we have

$$A(x) = -\int d^4y \Pi(x,y) J(y), \quad \Pi(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2}.$$

Then you can check that $\Box_x \Pi(x,y) = -\delta^(x-y)$. We call this a **propagator** or the **Green's function**. (We have $\frac{1}{\Box} = -\Pi$.)

Let us do what we did this above in this context. Then

$$h(x) = \int d^4y \delta^4(x-y) h(y) = -\int d^4\pi(x,y) \Box_y h(y) = -\int d^4y \Pi(x,y) J(y).$$

So this is the propagator of the potential from the source. We can do the same thing on the next order. We have

$$h(x) = -\int d^4y \Pi(x,y) J(y) + \lambda \int d^4w \int d^4y \int d^4z \Pi(x,w) \Pi(w,y) \Pi(w,z) J(y) J(z).$$

Then these have good physical interpretation. In quantum field theory, there will also be interactions in loops and so on.

5 September 18, 2018

This week and next week will be a bit dry. Why do we talk about cross sections in scattering? Scattering is a universal way of probing something that we can't see. We are skipping Chapter 4, which is old-fashioned perturbation theory.

5.1 Scattering

In quantum mechanics, we calculate amplitudes, $\langle f|i\rangle$, and probabilities, $|\langle f|i\rangle|^2$. In field theory, we calculate the same objects.

Let us consider the situation where two particles collide, and two or more particles come out. In the Schrödinger picture, we want to calculate

$$\langle f; t=\infty | i; t=-\infty\rangle.$$

In this Heisenberg picture, we are trying to measure $\langle f|S|i\rangle$. We are interested in this matrix S.

Classically, if we throw a beam of particles on a large particle,s we can consider the cross-section area as

$$\sigma = \frac{\#\text{particles scattered}}{\text{time} \times \text{number density of beam} \times \text{velocity of beam}}.$$

We may think this as $N = L\sigma$, where L is the luminosity.

What we want to do now is to talk about quantum mechanics. Here, σ is just a cross-section. Particles have a probability of scattering: $P = N_{\text{scatter}}/N_{\text{incident}}$. We are going to let $N_{\text{incident}} = 1$, so we are throwing one particles at a time. Then the flux is

$$\text{Flux} = \frac{|\vec{v}|}{V} = \frac{|\vec{v}_1 - \vec{v}_2|}{V}.$$

Now our formula for σ is

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}dP, \quad dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi.$$

The last factor $d\Pi$ is the density of states. On a line of size L, momenta are $p_n = \frac{2\pi}{L}n$ and so $dp = \frac{2\pi}{L}dn$. So we have

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}d^3p_j.$$

The initial and final states are given by

$$|i\rangle = |p_1\rangle|p_2\rangle, \quad |f\rangle = |p_3\rangle\cdots|p_n\rangle.$$

Because we are working in a box, we consider $|p|p\rangle = 2E_p\delta^3(0) = 2E_pV$. Then

$$|i|i\rangle = 2E_1 2E_2 V^2, \quad |f|f\rangle = \prod_{j=3}^{n}(2E_j)V.$$

Then we have

$$d\sigma = \frac{V}{T}\frac{|\langle f|S|i\rangle|^2}{|\vec{v}_1 - \vec{v}_2|\prod_j(2E_i)2E_1 2E_2 V^n}\prod_n \frac{V}{(2\pi)^3}d^3p_i.$$

We write

$$S = 1 + iT,$$

where $T = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - \cdots - p_n)M$, because momentum is conserved. Then

$$|\langle f|S|i\rangle|^2_{f\neq i} = (2\pi)^8\delta^4(\textstyle\sum p)\delta^4(0)|M|^2.$$

If we plug this in, we get

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} \times |M^2| \prod_j \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^4(\textstyle\sum p).$$

This second term is also called the Lorentz-invariant phase space, $d\Pi_{\text{LIPS}}$. So we can write the decay wrate as

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\Pi_{\text{LIPS}}.$$

There is no flux factor, and no $1/2E_2$.

5.2 Two-to-two scattering

Let us look at the example of a $2 \to 2$ scattering. Let us call the four particles p_1, p_2, p_3, p_4. In the center of mass frame, we have

$$|\vec{p}_1| = |\vec{p}_2| = p_i, \quad |\vec{p}_3| = |\vec{p}_4| = p_f.$$

Energy conservation is $E_1 + E_2 = E_3 + E_4 = E_{\text{CM}}$. Now we look at

$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}.$$

But this has a lot of redundancies, so we can express in terms on the direction. If we integrate over $\vec{p}_4$, we get

$$\begin{aligned} d\Pi_{\text{LIPS}} &= \frac{1}{4(2\pi)^2}\frac{dp_3}{E_3E_4}\delta(E_3 + E_4 - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int p_f^2 dp_f \frac{1}{E_3E_4}\delta(\sqrt{m_3^2 + p_f^2} + \sqrt{m_4^2 + p_f^2} - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int_{m_3+m_4-E_{\text{CM}}}^{\infty} dx\delta(x)\frac{p_f}{E_{\text{CM}}} = \frac{d\Omega}{16\pi^2}\frac{p_E}{E_{\text{CM}}}\theta(E_{\text{CM}} - m_3 - m_4). \end{aligned}$$

So we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_1 2E_2|\vec{v}_1 - \vec{v}_2|}\frac{1}{16\pi^2}\frac{p_f}{E_{\text{CM}}}|M|^2 = \frac{1}{64\pi^2E_{\text{CM}}^2}\frac{p_f}{p_i}|M|^2.$$

in the center of mass frame. (This $d\Omega$ is the spherical angle $d\phi d\cos\theta$, so that $d^3p_3 = p_3^2 dp_3 d\Omega$.)

Let us look at the non-relativistic limit. Consider the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}|\tilde{V}(k)|^2.$$

Here, $\tilde{V}(k)$ is the Fourier transformation

$$\tilde{V}(k) = \int d^3x e^{-i\vec{k}\vec{x}} V(x) = \frac{e^2}{\vec{k}^2}$$

in the Coulomb potential. So we have

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}\left(\frac{e^2}{\vec{k}^2}\right)^2.$$

Let us how see this agrees with what we have done so far.

The free theory for the proton and the electron is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 - \phi_e^*(\Box + m_e^2)\phi_e - \phi_p^*(\Box + m_p^2)\phi_p \\ &- ieA_\mu(\phi_e^*\partial_\mu\phi_e - \phi_e\partial_\mu\phi_e^*) + ieA_\mu(\phi_p^*\partial_\mu\phi_p - \phi_p\partial_\mu\phi_p^*).\end{aligned}$$

If we take the non-relativistic limit, we have $p_\mu = (E, \vec{p}) = (\sqrt{m^2 + \vec{p}^2}, \vec{p}) \approx (m, 0)$, and so $\partial_t\phi \approx im\phi$. So we redefine $\phi \to e^{im_e t}\phi$ so that the phases don't rotate. If we do that, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \phi_e\vec{\nabla}^2\phi_e + 2em_e\phi_e^*\phi_e A_0 + \phi_p\vec{\nabla}^2\phi_p - 2em_p\phi_p^*\phi_p A_0.$$

The matrix M is going to be

$$M = \frac{(2em_e)(-2em_p)}{\vec{k}^2}$$

because the coefficient of $\phi_e^*\phi_e A_0$ is $2em_e$ and this is the interaction between the electron, electron, photon, and likewise for $(-2em_p)$, and $1/|\vec{k}^2|$ is the Green's function. Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_p^2}|M|^2 = \frac{1}{64\pi^2 m_p^2}\frac{16e^4 m_e^2 m_p^2}{k^4} = \frac{1}{4\pi^2}\frac{m_e^2 e^4}{|\vec{k}^2|^2}.$$

This is the same formula we had for the Born approximation.

6 September 20, 2018

We also have to need this other technical theorem. We recall that light satisfies $\Box A_\mu = 0$, and so $(\partial_t^2 + |\vec{k}|^2)A_\mu = 0$. We had our operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{i\vec{p}\cdot\vec{x}} + a_p e^{-i\vec{p}\cdot\vec{x}}).$$

Then $i\partial_t a_p = -[H, A_p] = \omega_p a_p$, so we can define even for difference time

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}}^\dagger e^{ipx} + a_{\vec{p}} e^{-ipx}).$$

6.1 LSZ reduction

We also talked about cross sections,

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|M|^2 d\Pi_{\text{LIPS}},$$

where $S = 1 + iT$ and $T = (2\pi)^4\delta^4(\sum p)M$. So again, our initial state is

$$|i\rangle = |p_1p_2\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle$$

We are using $|\Omega\rangle$ because the vacuum is time-dependent. Similarly, in the final state, we can write

$$|f\rangle = |p_3\cdots p_n\rangle = \sqrt{2\omega_3}\cdots\sqrt{2\omega_n}a_{p_3}^\dagger(+\infty)\cdots a_{p_n}^\dagger(+\infty)|\Omega_{+\infty}\rangle.$$

Now the matrix element between the two things is

$$\langle f|S|i\rangle = \sqrt{2\omega_1}\cdots\sqrt{2\omega_n}\langle\Omega_\infty|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle.$$

So how do we create $|p\rangle$ from $\phi(x)$? WE have

$$\langle p|\phi(x)|0\rangle = e^{ipx}, \quad \phi(x)|0\rangle = \int d^3p\frac{1}{2\omega_p}e^{i\vec{p}\cdot\vec{x}}|p\rangle = \int d^4p\delta(p^2-m^2)\theta(p^0)e^{i\vec{p}\vec{x}}|p\rangle.$$

But we have

$$-2\pi i\delta(p^2 - m^2) = \lim_{\epsilon\to 0}\Big[\frac{1}{p^2 + m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\Big].$$

So we roughly have

$$\int e^{-ipx}(\Box + m^2)\phi(x)|0\rangle = |p\rangle.$$

The precise expression is

$$i\int d^4x e^{ipx}(\Box + m^2)\phi(x,t) = \sqrt{2\omega_p}(a_p(\infty) - a_p(-\infty)).$$

Let me try to derive this. If we do spatial integration by parts, we get

$$\begin{aligned}
i\int d^4x e^{ipx}(\Box+m^2)\phi(x,t) &= i\int d^4x e^{ipx}(\partial_t^2+\omega_p^2)\phi(x,t)\\
&= \int dt\partial_t\left[e^{i\omega_p t}\int d^3x e^{-i\vec{p}\vec{x}}(i\partial_t+\omega_p)\phi(x,t)\right]\\
&= \int dt\partial_t\left[e^{i\omega_p t}\sqrt{2\omega_p}a_p e^{-i\omega_p t}\right]\\
&= \sqrt{2\omega_p}[a_p(\infty)-a_p(-\infty)].
\end{aligned}$$

Here, we are assuming that at $t=\pm\infty$, the field behaves like a free field, so we can compute the spatial integral simply. Now if we take the complex conjugate of both sides, we get

$$-i\int d^4x e^{-ipx}(\Box+m^2)\phi(x,t) = \sqrt{2\omega_p}(a_p^\dagger(\infty)-a_p^\dagger(-\infty)).$$

Now can compute $\langle f|i\rangle$. Here, we introduce a **time-ordering operation** T which just puts things in the correct time order. Then we have

$$\begin{aligned}
&\langle\Omega|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle\\
&\quad= \langle\Omega|T\{[a_{p_3}(\infty)-a_{p_3}(-\infty)]\cdots[a_{p_n}(\infty)-a_{p_n}(-\infty)]\\
&\qquad [a_{p_1}^\dagger(\infty)-a_{p_1}^\dagger(-\infty)][a_{p_2}^\dagger(\infty)-a_{p_2}^\dagger(-\infty)]|\Omega\rangle,
\end{aligned}$$

because all other terms become 0. So we get

$$\begin{aligned}
\langle f|i\rangle &= \left[i\int d^4x_1 e^{ip_1x_1}(\Box+m_1^2)\right]\cdots\left[-i\int d^4x_n e^{-ip_nx_n}\right]\\
&\quad\times\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.
\end{aligned}$$

This is called the **LSZ reduction formula**.

6.2 Feynman propagators

The simplest example is

$$D_F(x,y) = \langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle = \lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x-y)}.$$

This is called the **Feynman propagator**, and satisfies $(\Box_x+m^2)D_F(x,y) = \int d^4k e^{ik(x-y)} = \delta^4(x-y)$. So this is the factor you put in when you want to talk about propagation between to interactions.

The first thing we do is to calculate without time ordering. We have

$$\begin{aligned}
\langle 0|\phi(x_1)\phi(x_2)|0\rangle &= \int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{\sqrt{2\omega_1}\sqrt{2\omega_2}}\langle 0|a_{k_1}a_{k_2}^\dagger|0\rangle e^{i(k_2x_2-k_1x_1)}\\
&= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{ik(x_1-x_2)}.
\end{aligned}$$

If we do have time-ordering, we get

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= \langle 0|\phi(x_1)\phi(x_2)|0\rangle\theta(t_1-t_2) + \langle 0|\phi(x_2)\phi(x_1)|0\rangle\theta(t_2-t_1)\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{ik(x_2-x_1)}\theta(\tau) + e^{ik(x_1-x_2)}\theta(-\tau)]\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k\tau}\theta(\tau) + e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{i\omega_k\tau}\theta(-\tau)]\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}[e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau)].\end{aligned}$$

Here, we have

$$\theta(\tau) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{\omega + i\epsilon}\frac{1}{2\pi i}.$$

So we get

$$\begin{aligned}\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\frac{i}{\omega^2 - \vec{p}^2 - m^2 + i\epsilon}e^{ik(x-y)}\\ &= \int d^4k \frac{i}{k^2 - m^2 + i\epsilon}e^{ik(x-y)}.\end{aligned}$$

This ϵ we are going to think of as uncertainty of energy. Feynman's ingenious idea is that you can add the two possible time orderings in one propagator, so that we don't have to think of the two cases separately.

7 September 25, 2018

Last time we showed that

$$\langle P_3 \cdots P_n | P_1 P_2 \rangle = \left[i \int d^4 x e^{-ip_1 x_1} (\Box_1 + m_1^2) \right] \cdots \left[i \int d^4 x_n e^{+ip_n x_n} (\Box_n + m_n^2) \right] \langle \Omega | T\{\phi(x_1) \cdots \phi(x_n) | \Omega \rangle.$$

Then for free field, we got the Feynman propagator as

$$D_F(x,y) = \langle 0 | T\{\phi_0(x)\phi_0(y)\} | 0 \rangle = \lim_{\epsilon \to 0} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{ik(x-y)}.$$

Then

$$(\Box_x + m^2) D_F(x,y) = -i\delta^4(x-y).$$

We will check this later.

7.1 Schwinger–Dyson equations

We want to specify the dynamics and the commutations relations. We can use things like $[x,p] = i\hbar$ and $[\theta, H] = i\partial_t \theta$, but then we need to worry about the separation of space and time.

Assume that ϕ satisfies the equations of motion as an operator,

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \mathcal{L}_{\text{int}}[\phi].$$

Then the Euler–Lagrange equations become

$$(\Box + m^2)\phi = \mathcal{L}'_{\text{int}}[\phi].$$

The commutation relations are

$$[\phi(\vec{x},t), \partial_t \phi(\vec{y},t)] = i\hbar\delta(\vec{x}-\vec{y}), \quad [\phi(\vec{x},t), \phi(\vec{y},t)] = 0.$$

These are equal-time commutations relations. The second relation should be thought of as, the two points $(\vec{x},t)$ and $(\vec{y},t)$ are causally unrelated.

We can check in the free theory that these hold. Here, we have

$$\phi_0(x,t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx}), \quad [a_p, a_q^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

Our equations of motion are

$$(\Box + m^2)\phi_0(x,t) = \int \frac{d^3 p}{(2\pi)^3} (-p^2 + m^2) \frac{1}{\sqrt{2\omega_p}} (\cdots) = 0$$

because $p^2 = m^2$. For the commutation relation, we first compute

$$\partial_t \phi(\vec{y}, t) = -i \int \frac{d^3 q}{(2\pi)^3} \sqrt{\frac{\omega_q}{2}} (a_q e^{-iqy} - a_q^\dagger e^{iqy}).$$

Then

$$\begin{aligned}
[\phi(\vec{x}, t), \partial_t \phi(\vec{y}, t)] &= -i \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \sqrt{\frac{\omega_q}{4\omega_p}} ([a_p^\dagger, a_q] e^{ipx - iqy} - [a_p, a_q^\dagger] e^{-ipx + iqy}) \\
&= -\frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} (-e^{ip(x-y)} - e^{-ip(x-y)}) = i\delta^3(x - y).
\end{aligned}$$

So this is satisfied in the free theory, and we're assuming this holds also in the interacting theory. This is not so surprising, because we are looking at one time slice.

What we are going to do now, is to look at the time ordering operator using these relations. We can compute

$$\begin{aligned}
\partial_t \langle \Omega | T\{\phi(x), \phi(x')\} | \Omega \rangle &= \partial_t [\langle \phi(x)\phi(x') \rangle \theta(t - t') + \langle \phi(x')\phi(x) \rangle \theta(t' - t)] \\
&= \langle T\{\partial_t \phi(x) \phi(x')\} \rangle + \langle \phi(x)\phi(x') \rangle \delta(t - t') - \langle \phi(x')\phi(x) \rangle \delta(t - t') \\
&= \langle T\{\partial) t \phi(x) \phi(x')\} \rangle
\end{aligned}$$

because $\phi(x)$ and $\phi(x')$ commute at $t = t'$. Then the second derivative is

$$\begin{aligned}
\partial_t^2 \langle T\{\phi(x)\phi(x')\} \rangle &= \partial_t \langle T\{\partial_t \phi(x) \phi(x')\} \rangle \\
&= \langle T\{\partial_t^2 \phi(x) \phi(x')\} \rangle + \langle [\partial_t \phi(x), \phi(x')] \rangle \delta(t - t') \\
&= \langle T\{\partial_t^2 \phi(x) \phi(x')\} \rangle - i\hbar \delta^4(x - x').
\end{aligned}$$

So we finally get

$$(\Box_x + m^2) \langle T\{\phi(x)\phi(x')\} \rangle = \langle T\{(\Box + m^2)\phi(x)\phi(x')\} \rangle - i\hbar \delta^4(x - x').$$

This is

$$(\Box_x + m^2) D_F(x, y) = -i\hbar \delta^4(x - y).$$

Now we can assume that we have more than terms. Then we can show that

$$\begin{aligned}
\Box_x \langle T\{\phi(x)\phi(y_1) \cdots \phi(y_n) \rangle &= \langle T\{\Box_x \phi(x) \phi(y_1) \cdots \phi(y_n)\} \rangle \\
&- i\hbar \sum_j \delta^4(x - y_j) \langle T\{\phi(y_1) \cdots \phi(y_{j-1}) \phi(y_{j+1}) \cdots \phi(y_n)\} \rangle.
\end{aligned}$$

This is called the **Schwinger–Dyson equations**. Historically, this was how Schwinger showed that the perturbative way and Feynman diagram way of doing quantum field theory are equivalent.

7.2 Feynman diagrams

Write $\delta_{xi} = \delta^4(x - x_i)$ and $D_{ij} = D_{ji} = D_F(x_i, x_j)$ and $D_{xi} = D_F(x, x_i)$. Then $\Box_x D_{x1} = -i\delta_{x1}$ and we have

$$\langle\phi_1\phi_2\rangle = \int d^4x\delta_{x1}\langle\phi_x\phi_2\rangle = i\int d^4x(\Box_x D_{x1})\langle\phi_x\phi_2\rangle = i\int d^4xD_{x1}\Box\langle\phi_x\phi_2\rangle.$$

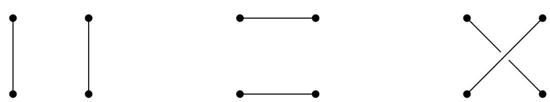
For instance in the free theory, $\Box_x\langle\phi_x\phi_y\rangle = -i\delta_{xy}$ so

$$\langle\phi_1\phi_2\rangle = i\int d^4xD_{x1}(-i\delta_{x2}) = D_{12}.$$

If we have four terms,

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= i\int d^4xD_{x1}\Box_x\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= \int d^4xD_{x1}[\delta_{x2}\langle\phi_3\phi_4\rangle + \delta_{x3}\langle\phi_2\phi_4\rangle + \delta_{x4}\langle\phi_2\phi_3\rangle] \\ &= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.\end{aligned}$$

We can represent these terms as diagrams that connect dots between $1, 2, 3, 4$.



Let us now look at a theory with interactions, with the simplest possible interaction

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \frac{g}{3!}\phi^3, \quad \Box\phi = \frac{g}{2}\phi^2.$$

In this case,

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4D_{1x}\Box_x\langle\phi_x\phi_2\rangle \\ &= i\int d^4xD_{1x}[\langle\tfrac{g}{2}\phi_x^2\phi_2\rangle - e\delta_{x2}] \\ &= D_{12} - \frac{g}{2}\int d^4xd^4yD_{1x}D_{y2}\Box_y\langle\phi_x^2\phi_y\rangle \\ &= D_{12} - \frac{g^2}{4}\int d^4xd^4yD_{1x}D_{2y}\langle\phi_x^2\phi_y^2\rangle + ig\int d^4xD_{1x}D_{2y}\langle\phi_x\rangle.\end{aligned}$$

And we can go on, by removing one x and putting in two x, until we get the order of g we want. For $\langle\phi_x^2\phi_y^2\rangle$, we can just assume the free field, because we already have g, and so

$$\begin{aligned}\langle\phi_x^2\phi_y^2\rangle &= \langle\phi_x\phi_x\phi_y\phi_y\rangle = 2D_{xy}^2 + D_{xx}D_{yy} + O(g), \\ \langle\phi_x\rangle &= i\int d^4yD_{yx}\Box_y\langle\phi_y\rangle = \frac{ig}{2}\int d^4yD_{xy}D_{yy}.\end{aligned}$$

So we finally get

$$\langle\phi_1\phi_2\rangle = D_{12}-g^2\int d^4x d^4y\Big(\frac{1}{2}D_{1x}D_{xy}^2 D_{y2}+\frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2}+\frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}\Big).$$

You can draw the diagrams for each term as well.

We can generalize this process to **position space Feynman rules**.

(1) Points are x for each external position.

(2) Draw a line from each point.

(3) A line can either connect to another line or split due to an interaction.

(4) A vertex is proportional to the coefficient of $\mathcal{L}'_{\text{int}}[\phi] \times i$.

(5) At a given order of perturbation theory, sum all possible diagrams and integrate over internal positions.

But what are the numerical factors like 1/2 or so on? We conventionally normalize $n!$ for each ϕ^n, like

$$\mathcal{L} = \frac{g}{2!6!}\phi_1^2\phi_2^6.$$

If two line connect to each other, we get a symmetry factor. If the diagram has a symmetry, then we need to divide by the number of symmetries. (You actually rarely need symmetry factors.)

8 September 27, 2018

We had the formula for the cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\text{CM}}^2}|\mathcal{M}|^2, \quad S = \mathbf{1} + i\delta^4(p)\mathcal{M},$$

and then we looked at

$$\langle f|S|i\rangle = \int d^4x_1(\Box_1 + m^2)e^{ipx_1}\cdots\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.$$

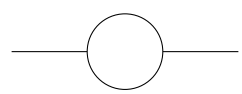
Then we were able to interpret

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$$

as a sum of Feynman diagrams. We are now going to try and do this in momentum space. This way, we don't have to take the Fourier transform and things become more simpler.

8.1 Feynman diagrams in momentum space

Let's take a example of a $\frac{g}{6}\phi^3$ interaction and the following diagram.



Then the corresponding term is

$$T = \frac{(ig)^2}{2}\int d^4x \int d^4y D_F(x,x)D_F(x,y)^2 D_F(y,x_2).$$

By Fourier transforming, we get

$$T = -\frac{g^2}{2}\int d^4x d^4y d^4p_1\cdots d^4p_4 e^{ip_1(x_1-x)}e^{ip_2(y-x_2)}e^{ip_3(x-y)}e^{ip_4(x-y)}$$
$$\cdot\frac{i}{p_1^2+i\epsilon}\cdots\frac{i}{p_4^2+i\epsilon}.$$

If we do the x-integral, we get momentum conservation $\delta^4(p_3+p_4-p_1)$ and if we do the y-integral, we get $\delta^4(p_2-p_3-p_4)$. Then the p_3-integral sets $p_3 = p_1 - p_4$ and so $\delta^4(p_2-p_3-p_4)$ becomes $\delta^4(p_2-p_1)$. So if we set $k = p_4$ and $p_3 = p_1 - k$, we get

$$T = -\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\delta^4(p_1-p_2)(2\pi)^4 e^{ip_1x_1}e^{-ip_2x_2}$$
$$\frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{(p_1-k)^2+i\epsilon}\frac{i}{k^2+i\epsilon}.$$

LSZ says that

$$\langle p_f|S|p_i\rangle = \left[-i\int d^4x_1 e^{-ip_ix_1}p_i^2\right]\left[-i\int d^4x_2 e^{+ip_fx_2}p_f^2\right]T.$$

The integrals over x_1 and x_2 give $\delta^4(p_1-p_i)$ and $\delta^4(p_2-p_f)$, and this becomes

$$\langle p_f|S|p_i\rangle = (2\pi)^4\delta^4(p_i-p_f)\left[-\frac{g^2}{2}\int\frac{d^2k}{(2\pi)^4}\frac{i}{(p_i-k)^2+i\epsilon}\frac{i}{k^2+i\epsilon}\right].$$

So here are the **momentum space Feynman rules**:

(1) Internal lines get $\frac{i}{p^2-m^2+i\epsilon}$.

(2) Vertices come from $\mathcal{L}_{\text{int}}$.

(3) External lines do not get propagators.

(4) 4-momentum is conserved at each vertex.

(5) Integrate over undetermined momentum.

8.2 Hamiltonian derivation

Recall that in the Lagrangian formalism, we have

$$[\phi(x,t),\dot\phi(x,t)] = i\hbar\delta^3(\vec x-\vec y),\quad (\Box+m^2)\phi = \mathcal{L}'_{\text{int}}[\phi].$$

In the Hamiltonian picture, we have

$$[\phi(x,t),\pi(x,t)] = i\hbar\delta^3(\vec x-\vec y),\quad i\partial_t\phi = [H,\phi].$$

Our fields are

$$\phi(\vec x) = \int d^3p\frac{1}{\sqrt{2\omega_p}}(a_pe^{i\vec p\vec x}+a_p^\dagger e^{-i\vec p\vec x}).$$

In the interaction picture, fields have interaction,

$$H = H_0+H_{\text{int}},\quad \phi_0(x,t) = e^{iH_0t}\phi(\vec x)e^{-iH_0t} = \int d^3p(a_pe^{ipx}+a_p^\dagger e^{-ipx}).$$

We now write

$$\phi(x,t) = e^{iHt}\phi(x,0)e^{-Ht} = U^\dagger(t,0)\phi_0(x,t)U(x,0).$$

Here, U can be formally written as

$$U(t,0) = e^{iH_0t}e^{-Ht}\approx e^{-iV}.$$

To be precise, we have

$$U_{21} = U(t_2,T_1) = e^{iH_0(t_2-t_1)}e^{-iH(t_2-t_1)} = T\{\exp(-i\textstyle\int_{t_1}^{t_2}dtV_I(t))\},$$

where V_I is the potential for the interacting theory. For instance, $\phi(x,t_2) = U_{21}\phi(x,t_1)U_{12}$. So we get

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0|T\{U_{\infty 0}U_{01}\phi_0(x_1)U_{01}U_{02}\phi(x_2)U_{20}\cdots U_{0-\infty}|0\rangle}{\langle 0|U_{\infty-\infty}|0\rangle}\\ &= \frac{\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)U_{-\infty\infty}\}|0\rangle}{\langle 0|U_{-\infty\infty}|0\rangle},\end{aligned}$$

because U commute with each other. So we have

$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\}|\Omega\rangle = \Big\langle 0\Big|T\Big\{&\phi_0(x_1)\phi_0(x_2) - ig\int d^4y\phi_0(y)^3\phi_0(x_1)\phi_0(x_2)\\ &+\frac{(-ig)^2}{2}\int d^4x\int d^4y\phi_0(x)^3\phi_0(y)^3\phi_0(x_1)\phi_0(x_2)\}\Big|0\Big\rangle.\end{aligned}$$

You can write out this and see that this is really the same thing as the Feynman diagrams.

8.3 Matrix element for the two-to-two scattering

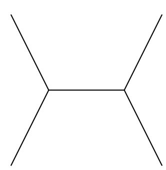
Let us take our favorite example

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3.$$

Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2E^2}|\mathcal{M}|^2.$$

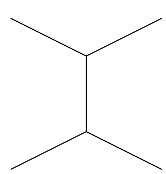
If we look at



we get

$$iM = ig\frac{i}{k^2-m^2+i\epsilon} = \frac{-ig^2}{(p_1+p_2)^2-m^2+i\epsilon},\quad M_t = \frac{g^2}{s-m^2+i\epsilon},\quad s=(p_1+p_2)^2.$$

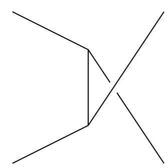
But there are other diagrams. We have



with

$$M_t = \frac{-g^2}{t - m^2 + i\epsilon}, \quad t = (p_1 - p_3)^2,$$

and there is



which is

$$M_u = \frac{-g^2}{t - m^2 + i\epsilon}, \quad u = (p_1 - p_4)^2,$$

Then we can add them up and get

$$M_{\text{tot}} = -g^2\Big(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\Big), \quad s + t + u = \sum_{i=1}^{4} m_i^2.$$

Electron positron $e^- e^+ \to \mu^- \mu^+$ only has an s-channel, Rutherford scattering only has a t channel, electron scattering has t and u channels, etc.

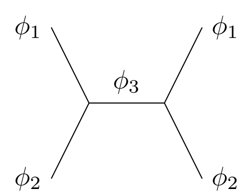
The Lagrangian sometimes have derivative couplings. For instance,

$$\mathcal{L} = \lambda\phi_1(\partial_\mu\phi_2)(\partial_\mu\phi_3).$$

Our field are

$$\phi = \int d^3p \frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ipx} + a_p e^{-ipx}),$$

so we get out a factor of ip_μ if a particle is created, and $-ip_\mu$ if a particle is annihilated. This means that the momentum leaves or enters the vertex. If we look at $\phi_1\phi_2 \to \phi_1\phi_2$, and the diagram



corresponds to

$$iM = (i\lambda)^2 \frac{(-ip_2^\mu)(ik^\mu)(-ik^\mu)(-ip_4^\mu)}{k^2 - m^2 + i\epsilon}.$$

9 October 2, 2018

The course up to this point can be summarized as the momentum space Feynman rules. The example we've been studying is scalar field theory

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3.$$

Then you draw all the Feynman diagrams and internal lines get factors of $\frac{i}{p^2-m^2+i\epsilon}$. Then we get factors of

$$ig$$

from the tree level, and then

$$(ig)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k-p_1)^2-m^2+i\epsilon}\frac{i}{(k-p_2)^2-m^2+i\epsilon}$$

from the next level with one loop. If you have a derivative term, you put in ip for annihilation and $-ip$ for creation.

9.1 Writing down the Lagrangian

But how do we write down the Lagrangian for the theory? We start with symmetries of the theory. There are

- translation invariance $\phi(x) \to \phi(x+a)$,
- Lorentz invariance $x^\mu \to \Lambda^{\mu\nu}x^\nu$, including rotations and boosts,
- unitarity, that is, conservation of probabilities, $\langle\phi|\phi'\rangle = \langle\psi|e^{-iHt}e^{iHt}|\phi'\rangle$, preserved by time-translation and other symmetries,
- internal symmetries, e.g., phase rotation $\phi \to e^{i\alpha}\phi$.

Lorentz invariance and translation invariance are together called Poincaré invariance.

Symmetries mix states in the Hilbert space. For instance, e^- has two states

$$|\uparrow\rangle, J_z = \frac{1}{2}, \quad |\downarrow\rangle, J_z = -\frac{1}{2}.$$

If we rotate our apparatus, we should also be rotating the spins. Another example is polarization of light. We go as far as defining a **particle** as a (minimal) set of states that mix under Poincaré transformations. More mathematically, particles transform as an irreducible unitary representations of the Poincaré group. This is good because we have reduced the physics to a mathematical problem.

Definition 9.1. A **group** $G = \{g_i\}$ with a rule $g_ig_j = g_k$. The conditions are

1. there exists a 1 with $1g = g1 = g$ for all g,
2. there is an inverse g_i^{-1} with $g^{-1}\cdot g = 1$,
3. it is associative, $g_1(g_2g_3) = (g_1g_2)g_3$.

Examples are rotations which are 3×3 matrices with $R_{ij}^T = R_{ij}^{-1}$.

Definition 9.2. A **representation** is an embedding of G into a set of linear operators acting on a vector space.

An example is the trivial representation $g_i \mapsto 1$. Another representation of the 3×3 rotations is the 4-dimensional representation given by just

$$A \mapsto \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 9.3. An **irreducible representation** is a representation where no subspace of vector space is preserved under G.

But there is a problem between Poincaré invariance and unitarity. If P_{ij} is a transformation, then I should have

$$\langle\psi|\psi'\rangle = \langle\psi|P^\dagger P|\psi\rangle, \quad P^\dagger P = 1.$$

In general, this is easy to do, but if we have Lorentz invariance, this is not easy. Let's take a 4-dimensional space for instance, and write

$$|\psi\rangle = c_0|V_0\rangle + c_1|V_1\rangle + c_2|V_2\rangle + c_3|V_3\rangle.$$

Then

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 \geq 0.$$

But this is not Lorentz invariant. So we really want $\langle V_\mu|V_\nu\rangle = g_{\mu\nu}$. But then this can't be interpreted as probability. If you study representations of the Poincaré group, you find the following.

Proposition 9.4. *There is no finite-dimensional unitary representations of the Lorentz group of the Poincaré group.*

However, there exist infinite-dimensional unitary representations of the Poincaré group. Wigner studied this, and it turns out that there are two classes of representations, classified by mass m and spin J where

- $m > 0$ and there are $2J + 1$ states,
- $m = 0$ and there are 2 states.

So how do we interpret this physically? We want $\mathcal{E} > 0$ in a classical theory. This can be computed by the energy-momentum tensor

$$\mathcal{E} = T_{00} = \sum_n \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\dot{\phi}_n - \mathcal{L}.$$

1. In the spin 0 case, there is one degree of freedom, with $J = 0$ and $m \geq 0$ one state. Then
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2, \quad (\Box + m^2)\phi = 0.$$
Then
$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{1}{2}[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] \geq 0.$$
This is satisfies as long as $m^2 \geq 0$.
2. In massive spin 1, we expect $2J + 1 = 3$ degrees of freedom. Minimally, we can embed this in $A_\mu(x)$, which is 4 degrees of freedom, but this splits into $4 = 3 + 1$. Then we can write down
$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial_\nu A_\mu) + \frac{1}{2}m^2 A_\mu^2, \quad (\Box + m^2)A_\mu = 0.$$
Then the energy density is
$$\mathcal{E} = \frac{1}{2}[(\partial_t \vec{A})^2 + (\nabla_i A_j)^2 + m^2\vec{A}]^2 - \frac{1}{2}[(\partial_t A_0)^2 + (\vec{\nabla}A_0)^2 + m^2 A_0^2].$$
This holds generally for any four scalar fields, but if we further impose the condition that the Lorentz group representation corresponding to A_μ is the standard representation, we can write more things like
$$\mathcal{L} = \frac{a}{2}A_\mu \Box A_\mu + \frac{b}{2}A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2.$$
Then the equations of motion becomes
$$a\Box A_\mu + b\partial_\mu\partial_\nu A_\nu + m^2 A_\mu = 0, \quad ((a+b)\Box + m^2)(\partial_\mu A_\mu) = 0.$$
If $a = -b$, then we will get $\partial_\mu A_\mu = 0$. What this is doing is projecting out the 1-dimensional trivial representation. If we choose $a = 1$ and $b = -1$, then we get
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
with energy density
$$\begin{aligned}\mathcal{E} &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}m^2(A_0^2 + \vec{A}^2) - A_0\partial_t(\partial_\mu A_\mu) - A_0(\Box + m^2)A_0 + \partial_i[A_0 F_{0i}], \\ &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}m^2(A_0^2 + \vec{A}^2) + \partial_i[A_0 F_{0i}] \geq 0\end{aligned}$$
with the equations of motion $(\Box + m^2)A_\mu = 0$ and $m^2\partial_\nu A_\mu = 0$. This is called the **Proca Lagrangian**. Note that the positive energy density condition forced our Lagrangian to be this.

10 October 4, 2018

There was this theorem due do Wigner.

Theorem 10.1 (Wigner)**.** *A particle is associated to an irreducible unitary representation of the Poincaré group.*

There are these things $A_\mu, T_{\mu\nu}$ that are finite-dimensional, irreducible, and non-unitary. Then there are fields

$$A_\mu(x), T_{\mu\nu}(x)$$

that are infinite-dimensional, reducible, non-unitary representations. Mathematically, unitary irreducible representations correspond to two invariants m and J. These are

- $J > 0$ and $m > 0$: $2J + 1$ polarizations,
- $J > 0$ and $m = 0$: 2 polarizations,
- $J = 0$: 1 polarization,

where $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. We want to describe the physics. What Wigner did was describe the Hilbert space, with vectors $\epsilon^i(p) = |\epsilon^i; p\rangle$ for $i = 1, \ldots, n$.

10.1 Representations of the Poincaré group

So we want to first construct the representations $\epsilon^i_\mu(p)$, and then construct a Lagrangian so that the extra degree of freedom A_μ does not get produced. Let's review what we did last time. We connected unitarity with $E > 0$. For a scalar field, we can write

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2,$$

and then we got

$$\mathcal{E} = \frac{1}{2}[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] \geq 0$$

if $m^2 \geq 0$. For a massive spin 1 particle, we wrote

$$\mathcal{L} = a(\partial_\mu A_\mu)^2 + b(\partial_\mu A_\nu)^2 + m^2 A_\mu^2,$$

and this constraint $\mathcal{E} \geq 0$ was satisfied for $a = -b$ and $m^2/a > 0$. Then the constraint was $\partial_\mu A_\mu = 0$ and $(\Box + m^2)A_\mu = 0$. So the representation $A_\mu(x)$ splits into a spin 1 and a spin 0 representation.

So how do we quantize this theory? We can write

$$A_\mu(x) = \sum_{j=1}^{3} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (\epsilon^j_\mu(p) e^{ipx} a^{j\dagger}_p + \epsilon^j_\mu(p) e^{-ipx} a^j_p), \quad [a^j_p, a^{k\dagger}_q] = (2\pi)^3 \delta^3(p-k)\delta_{jk}.$$

Then we can write

$$|\epsilon^j; p\rangle = a^{j\dagger}_p |0\rangle, \quad \langle 0|A_\mu(x)|\epsilon_i; p\rangle \epsilon^i_\mu(p) e^{ipx}.$$

But we haven't talked about what the representation looks like. Suppose we have $p^\mu = (m, 0, 0, 0)$, where $\partial_\mu A_\mu = 0$ looks like $p_\mu \epsilon^j_\mu = 0$. Then if we have

$$\epsilon_1^\mu = (0, 1, 0, 0), \quad \epsilon_2^\mu = (0, 0, 1, 0), \quad \epsilon_3^\mu = (0, 0, 0, 1),$$

how do we boost it to $p^\mu = (E, 0, 0, p_z)$ with $p_z^2 + m^2 = E^2$? This uses the method of induced representations.

10.2 Induced representations

How do we construct unitary representations of the Poincaré group?

1. Find a subgroup that stabilizes p^μ. Here, we can take $p^\mu = (m, 0, 0, 0)$ and the little group is SO(3).
2. Then construct finite-dimensional irreducible representations of the little group. For SO(3), we have $0, 1/2, 1, \dots$. In our case, we have $J = 1/2$ and $\epsilon^1_\mu = (0, 1, 0, 0$ and so on.
3. Any g in the Lorentz group can be written as $g = b \cdot r$ where b is the boosts and r is the rotation, where b is in the coset $\mathrm{SO}(1,3)/\mathrm{SO}(3)$.
4. Once we can write this, we find a basis $\epsilon^i_\mu(b \cdot p_\mu)$.
5. Now we define the representation by

$$g \cdot \epsilon = (r \cdot \epsilon)(b \cdot p_\mu).$$

What we really want to get is a photon. But here, we have to deal with the massless case. So how do we want to even construct this theory? Let's just see what happens. Let's take the massive theory and just take the limit. We expect two polarizations. Then natural thing to try is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2.$$

This does two things: we lose the $\partial_\mu A_\mu = 0$ constraint. If we just look at the equations of motion, what we get is

$$\Box A_\mu + \partial_\mu A_\nu A_\nu = 0, \quad \mathcal{E} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \geq 0.$$

So somehow, despite these problems, remarkably things come out naturally. What happened is that because

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

this is invariant under $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ for any $\alpha(x)$. This is called **gauge invariance**, and this is some redundancy in the embedding of the physics of ϵ_μ into $A_\mu(x)$.

Because of this redundancy, I can impose additional condition and choose what A to use. This is called gauge choice. We are going to choose A_μ so that $\partial_i A_i = 0$. We can do this because the gauge transformation is given by

$$\partial_i A_i \to \partial_i A_i + \vec{\nabla}^2 \alpha; \quad A_0 \to A_0 + \partial_t,$$

and then you can always solve this Laplace equation. Moreover, we can even set $A_0 = 0$. At the end, we have

$$A_0 = 0, \quad \partial_i A_i = 0, \quad \partial_\mu A_\mu = 0.$$

This is called the **Coulomb gauge**.

So let's take $p = (E, 0, 0, E)$ some photon. The two constraints on the polarization vectors are $\epsilon_0 = 0$ and $p_\mu \epsilon_\mu = 0$. Then the two polarization that are allowed are

$$\epsilon_1 = (0, 1, 0, 0), \quad \epsilon_2 = (0, 0, 1, 0).$$

In this case, we have this decomposition

$$A_\mu = 1 \oplus 0 \oplus 0$$

of dimension $4 = 2 + 1 + 1$.

Example 10.2. Let me take $p^\mu = (E, 0, 0, E)$, so that

$$\epsilon_1^\mu = (0, 1, 0, 0), \quad \epsilon_2^\mu = (0, 0, 1, 0).$$

If we take

$$\Lambda = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix},$$

we get that Λ is a Lorentz transformation in the little group. You can compute

$$\Lambda \epsilon_1 = (1, 1, 0, 1) = \epsilon_1 + \frac{1}{E} p^\mu.$$

This means that this acts on ϵ_1 to give ϵ_1.

11 October 9, 2018

In a massless spin 1 theory, we can quantize the theory by looking at

$$A_\mu(x) = \sum_{j=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p^{j\dagger} \epsilon_\mu^*(p) e^{ipx} + a_p^j \epsilon_\mu(p) e^{-ipx}).$$

Here, we have

$$|\epsilon_1(p)\rangle = a_p^{1\dagger}|0\rangle, \quad |\epsilon_2(p)\rangle = a_p^{2\dagger}|0\rangle.$$

Under Lorentz transformations, this transforms as $\epsilon_\mu^i \to \epsilon_\mu^i + p_\mu$. So this is Lorentz-invariant only if $p_\mu M^\mu = 0$. This is called the **Ward identity**.

11.1 Scalar quantum electrodynamics

The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$, and this is exactly $\epsilon_\mu \to \epsilon_\mu + p_\mu$ under the Fourier transformation.

If we wanted an interaction with a scalar field, we can write

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + A_\mu \phi \partial_\mu \phi.$$

This is not Lorentz-invariant, and you can check this by

$$A_\mu \phi \partial_\mu \phi + \frac{1}{e}(\partial_\mu \alpha)\phi\partial_\mu \phi.$$

So what we will do is to remove redundancy using this. This only possibly works with more than one field. Let us try ϕ_1 and ϕ_2, with

$$\phi = \phi_1 + i\phi_2 \to e^{i\alpha}\phi.$$

If we define a covariant derivative, it transforms as

$$D_\mu \phi = [\partial_\mu + ieA_\mu]\phi \to e^{-i\alpha} D_\mu \phi.$$

So for scalar QED, we can write down the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{SQED}} &= -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - m^2 |\phi|^2 \\ &= -\frac{1}{4} F_{\mu\nu}^2 + ieA_\mu(\phi\partial_\mu \phi^* - \phi^* \partial_\mu \phi) + e^2 A_\mu^2 \phi\phi^* - m^2 \phi\phi^*.\end{aligned}$$

If you recognize the term

$$J_\mu = \phi\partial_\mu \phi^* - \phi^* \partial_\mu \phi,$$

this is the Noether current associated to phase rotation. So it has $\partial_\mu J^\mu = 0$ on the equation of motion. It's not a coincidence that A_μ couples to a Noether current; the current is something that can be measured.

$$\begin{array}{c} \text{massless} \\ \text{spin 1} \end{array} \Rightarrow \begin{array}{c} \text{gauge invariant} \\ \text{Lagrangian} \end{array} \Rightarrow \begin{array}{c} \text{global} \\ \text{symmetry} \end{array} \Rightarrow \begin{array}{c} \text{conserved} \\ \text{charge} \end{array}.$$

The equations of motion is given by

$$(\Box + m^2)\phi = i(-eA_\mu)\partial_\mu\phi + i\partial_\mu(-eA_\mu\phi) + (-eA_\mu)^2\phi,$$
$$(\Box + m^2)\phi^* = i(eA_\mu)\partial_\mu\phi^* + i\partial_\mu(eA_\mu\phi^*) + (eA_\mu)^2\phi^*.$$

So another consequence of a massless spin 1 particle, is that there is an **antiparticle** associated to a particle. For instance, π^- and π^+ are spin 0 particles that couple to the photon.

11.2 Photon propagator

We want to develop the Feynman rules for scalar QED. So we need the propagator for the photon. Recall that for a scalar, we had

$$D_F = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int d^4p e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}.$$

Then we had $(\Box + m^2)D_F = -i\delta^4(x-y)$. Classically, we can think of this as inverting

$$(\Box + m^2)\phi = J, \quad \phi(x) = \Pi(x,y)J(y),$$

which is the equations of motion to the classical Lagrangian.

In the photon case, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + A_\mu J^\mu, \quad \partial_\mu F_{\mu\nu} = J_\nu.$$

If we try to Fourier transform and invert this, we get

$$(k^2 g_{\mu\nu} - k_\mu k_\nu)\tilde{A}_\mu = \tilde{J}_\nu.$$

But then this matrix is not invertible, which we should have expected because there is gauge-invariance. One choice is to choose a gauge and then substitute into the Lagrangian. But this is annoying, so we are going to deform the Lagrangian and write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + JA.$$

(This deformation is very special, and it's okay for reasons that are not obvious.) Then if we invert this, we get

$$i\Pi_{\mu\nu} = -i\frac{g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 + i\epsilon}.$$

This you can check this explicitly, that

$$-[k^2 g_{\mu\nu}(1-\tfrac{1}{\xi})k_\mu k_\nu]\Pi_{\nu\alpha} = g_{\mu\alpha}.$$

Generically, propagators are like

$$\Pi \sim \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{\sum_s |s\rangle\langle s|}{k^2-m^2+i\epsilon}$$

because we want the propagator to preserve the spin states.

Now we are almost done. Let us quantize the theory

$$\phi_1 = \int \frac{d^3p}{(2\pi)^3}(a_{p,1}e^{-ipx} + a^\dagger_{p,1}e^{ipx}),$$
$$\phi_2 = \int \frac{d^3p}{(2\pi)^3}(a_{p,2}e^{-ipx} + a^\dagger_{p,2}e^{ipx}).$$

Then if we write $a_p = a_{p,1} + ia_{p,2}$ and $b_p = a_{p,1} - ia_{p,2}$, then we get

$$\phi = \int \frac{d^3p}{(2\pi)^3}(a_p e^{-ipx} + b^\dagger_p e^{ipx}),$$
$$\phi^* = \int \frac{d^3p}{(2\pi)^3}(b_p e^{-ipx} + a^\dagger_p e^{ipx}).$$

So ϕ is creating an antiparticle and annihilating a particle, and ϕ^* is creating a particle and annihilating an antiparticle.

11.3 Feynman rules for scalar QED

Now we compute

$$\langle 0|T\{\phi(x)^*\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{ip(x-y)}.$$

There are interaction terms

$$-ieA_\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2 A_\mu A_\mu \phi\phi^*.$$

So let's see. The term ϕ creates π^- with ip^μ, and ϕ^* creates π^+ with $i\pi^\mu$. Or, ϕ annihilates π^+ with $-ip^\mu$ and ϕ^* annihilates π^- with $-ip^\mu$.

If you analyze all the cases, you are going to get Figure 1. But if you look at this, you see that you can package all of this compactly. We can invert the arrows for π^+, and then only the interactions we are allowed to have are arrows continuous thought the vertices. Then all interaction terms are just

$$-ie(p_1^\mu + p_2^\mu).$$

This is called the **Feynman–Stueckelberg interpretation**. You can interpret as a photon decaying to e^-e^+ as a electron bouncing off a photon.

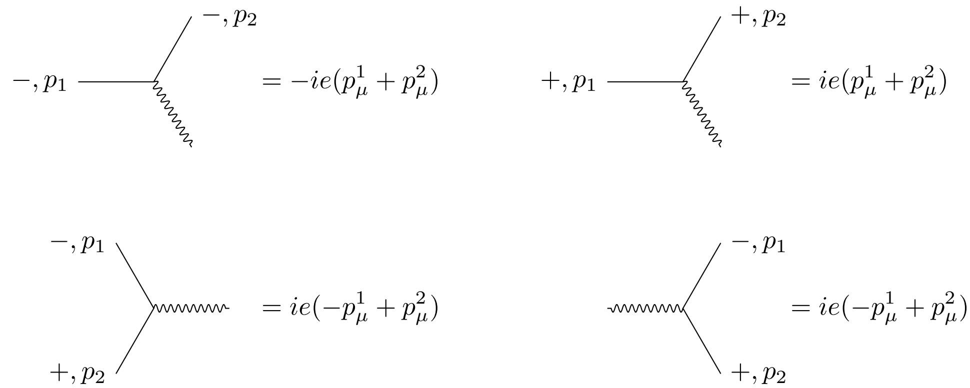


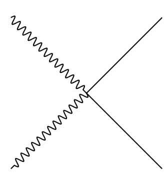
Figure 1: Feynman diagrams for scalar QED: time from left to right

Dirac had this interpretation of thinking of antiparticles as holes in this Dirac sea of negative energy. Dirac didn't like the Klein–Gordan equation

$$(\Box + m^2)\phi = 0$$

having negative energy states, $E_p = \pm\sqrt{\vec{p}^2 + m^2}$. The mathematics is just the same, by looking at creation operators as just dagger of annihilation operators, but this language is totally unnecessary. Also, there is no physical justification for this interpretation.

The other interaction is



and this has contribution $(i2e^2 g^{\mu\nu})$.

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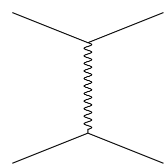
So we have our first theory of scalar QED. There is the Lagrangian

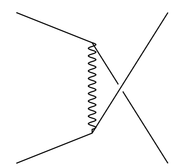
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \phi(\Box + m^2)\phi^* - ieA_\mu \phi^* \partial_\mu \phi + ieA_\mu \phi \partial_\mu \phi^* + e^2 A_\mu^2 \phi^* \phi.$$

with all these Feynman rules. We should be able to check gauge invariance and the Ward identity.

12.1 Gauge invariance and the Ward identity for scalar QED

Let us consider the Mollar scattering $e^- e^- \to e^- e^-$ in scalar QED.





The t-channel contribution is

$$iM_t = (-ie)(p_1^\mu + p_2^\mu) \frac{-i[g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2} (-ie)(p_2^\nu + p_4^\nu).$$

But then

$$k^\mu (p_1^\mu + p_3^\mu) = (p_1^\mu - p_3^\mu)(p_1^\mu + p_3^\mu) = m^2 - m^2 = 0$$

and similarly $k^\nu (p_2^\nu + p_4^\nu) = 0$. So we just have

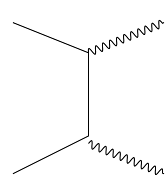
$$M_t = e^2 \frac{s-u}{t}.$$

Likewise, we have

$$M_u = e^2 \frac{s-t}{u}, \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}(M_t + M_u)^2.$$

Here, the fact that the ξ term vanishes shows that the theory is gauge invariant.

Let's now check the Ward identities, using the process $\pi^+ \pi^- \to \gamma\gamma$. Here,



gives the contribution

$$iM_t = (-ie)^2 \frac{(2p_1^\mu - p_3^\mu)\epsilon_3^{*\mu}(p_4^\nu - 2p_2^\nu)\epsilon_4^\nu}{(p_1 - p_3)^2 - m^2} = M_{\mu\nu}\epsilon_\mu^{3*}\epsilon_\nu^{4*}.$$

Now checking the Ward identity is checking if this is zero if $\epsilon_3^{\mu*} = p_3^\mu$. But then we get

$$M_t^w = e^2[p_4^\nu - 2p_2^\nu]\epsilon_4^\nu.$$

This is nonzero because there are other diagrams. If we look at the u-channel, we get

$$M_u^w = e^2[p_4 - 2p_3^\nu]\epsilon_4^\nu, \quad M_4^2 = 2e^2 p_3 \epsilon_4,$$

where M_4 is the contribution of the 4-particle interaction.

12.2 Lorentz invariance and soft photons

Suppose we have a diagram, and there is some p_1 going in, with contribution $\mathcal{M}_0(p_i)$. Here, we can modify the diagram by just adding a photon. The M matrix for this is

$$\mathcal{M}_i = (-ie)\frac{i[2p_1 - q]\epsilon}{(p-q)^2 - m^2}\mathcal{M}_0.$$

If we use $p^2 = m^2$ and $q^2 = 0$ and $\epsilon q = 0$, then we can approximate this as

$$\mathcal{M}_i = -e\frac{p\epsilon}{pq}\mathcal{M}_0(p_i + q) \approx -e\frac{p\epsilon}{pq}\mathcal{M}_0(p_i)Q_i.$$

Likewise, we can add one photon to the diagram at the outgoing edge. Then we get a contribution of

$$\mathcal{M}_i = +e\frac{p\epsilon}{pq}\mathcal{M}_0 Q_i.$$

So adding a photon in some edge gives a contribution of

$$\mathcal{M} = e\mathcal{M}_0\bigg[\sum_{\text{incoming}} Q_i \frac{p_i\epsilon}{p_i q} - \sum_{\text{outgoing}} Q_i \frac{p_i \epsilon}{p_i q}\bigg].$$

Under the Lorentz transformation, we get $\epsilon^\mu \to \epsilon^\mu + \Lambda q^\mu$. So for $\mathcal{M}$ to be Lorentz-invariant, we must have

$$e\mathcal{M}_0\bigg[\sum_{\text{incoming}} Q_i - \sum_{\text{outgoing}} Q_i\bigg] = 0.$$

So we get conservation of charge from this.

This is really universal. Even if the interaction term is arbitrary, say

$$-ie\Gamma_\mu(p,q)\epsilon^\mu = -ie(F_j p^\mu + G_j q^\mu)\epsilon_\mu, \quad F_j(p^2, q^2, pq) = F_j\Big(\frac{pq}{m^2}\Big),$$

we can just look at the $q \to 0$ limit and get

$$-ieF_j\Big(\frac{pq}{m^2}\Big)p \to -ieF_j(0)p.$$

So we can just look at an arbitrary theory and define $F_j(0) = Q_j$ as the charge. So we can think of charge as the interaction with low-energy photons for long distances. In this case, we can do the same thing and get conservation

$$\sum_{\text{in}} F_j(0) = \sum_{\text{out}} F_j(0).$$

This gets more interesting. Consider a massless spin-2 particles, and embed into $\epsilon_{\mu\nu}(p)$ polarization tensors. These are transverse and traceless and symmetric:

$$g_{\mu\nu}\epsilon_{\mu\nu} = 0, \quad q_\mu \epsilon_{\mu\nu} = 0, \quad g_{\mu\nu} = g_{\nu\mu}.$$

But because it is massless, it transforms in the same weird way

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \Lambda_\nu q_\mu + \Lambda_\mu q_\nu \Lambda q_{\mu\nu}.$$

If we look at the soft limit, we some contribution

$$\epsilon_{\mu\nu}\Gamma_{\mu\nu} = \epsilon_{\mu\nu} p^\mu p\nu F\Big(\frac{p \cdot q}{m^2}\Big).$$

Then if we o the same thing,

$$\mathcal{M} = \mathcal{M}_0 \Bigg[\sum_{\text{incoming}} F(0) \frac{p_\mu^i p_\nu^i}{(p_i \cdot q)} - \sum_{\text{outgoing}} \frac{p_i^\mu p_i^\nu}{(p_i \cdot q)}\Bigg] \epsilon_{\mu\nu}.$$

The Ward identity gives

$$0 = \mathcal{M}_0 \Lambda_\nu \Bigg[\sum_{\text{in}} F_i(0) p_i^\nu - \sum_{\text{out}} F_j(0) p_j^\nu\Bigg].$$

We already have momentum conservation, and this lets us solve for p_1 as a function of the other p_j. So this is consistent only when $F_i(0) = G_N$ for some universal constant G_N. This is saying that gravity is universal, and couples with every particle with the same coupling constant. This is the only way to have a consistent theory of a spin-2 particle, and this particle must be unique. What about spin-3? If you do the same thing, we should get

$$\sum_{\text{in}} \gamma_j p_j^\nu p_j^\mu = \sum_{\text{out}} \gamma_j p_j^\nu p_j^\mu.$$

If we only look at the $(0,0)$-component, we get $\sum \gamma_j E^2 = \sum \gamma_j E^2$. This is impossible, and so there are no consistent interacting theories of massless particles with $J > 2$.

12.3 Spinors

You've seen spinors before. In non-relativistic quantum mechanics, there is this $|\psi\rangle = |\uparrow\rangle$ and $|\downarrow\rangle$. The quantum mechanics is governed by the Schrödinger–Pauli equation

$$i\partial_t \psi = H\psi = \left[\left(\frac{p^2}{2m} + V(r) - \mu_B \vec{B} \cdot \vec{L}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}\right]\psi.$$

Here, we can write

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

and then that matrix is just $\vec{\sigma} \cdot \vec{B}$.

Here, $\vec{B}$ is a 3-vector, and $\vec{\sigma} \cdot \psi$ also transforms like a 3-vector. Also, $\partial_i \sigma_i \psi$ is rotation invariant, and then we can guess and check that

$$\partial_t \psi - \partial_i \sigma_i \psi = 0$$

is Lorentz-invariant. In fact, if we define $\sigma^\mu = (1, \vec{\sigma})$, then

$$\sigma^\mu \partial_\mu \psi = 0$$

is the Dirac equation for $m = 0$. Maybe we could guess that $\sigma^\mu \partial_\mu \psi = m\psi$ is the massive case, but this is wrong. So enough guessing.

This follows naturally from representations of the Lorentz group. There are these

$$\Lambda_{R_z} = R(\theta_z) = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}, \quad B_{\beta_x} = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Now we can look at the infinitesimal generators and extract the **Lie algebra**. Then we get

$$R_z = R(\theta_z) = \begin{pmatrix} 0 & & & \\ & & \theta_z & \\ & -\theta_z & & \\ & & & 0 \end{pmatrix}, \quad \beta_x = \begin{pmatrix} & \beta_x & & \\ \beta_x & & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Then we get

$$\Delta V_0 = \beta_i V_i, \quad \Delta V_i = \beta_i V_0 - \epsilon_{ijk}\theta_j V_k.$$

If we write the i times the rotation generators as J_i and the i times the boost generators as K_i, hen we have

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

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