Math 99r - Toward Fukaya categories

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This was a undergraduate student seminar led by Hiro Lee Tanaka. Students formed groups of two or three and gave presentations on a certain topic for one ore two weeks. I only attended the seminar near the end, and hence these notes are far from complete.

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1 April 9, 2018

Today we are going to start Floer homology. This is sort of like an infinite-dimensional Morse theory. Given a symplectic manifold (M, ω) and two Largrangian submanifolds $L_0, L_1 \subseteq M$, their dimension are going to be $\frac{1}{2} \dim M$. So if they are transverse, the intersection is going to be a finite number of points. We can define some moduli space of paths

$$\mathcal{M}(p, q, J, [u])$$

for $p, q \in L_0 \cap L_1$, and define some chain complex $CF(L_0, L_1)$.

1.1 Almost-complex structure

Definition 1.1. An **almost-complex structure** on M is a smooth section

$$J : M \to \mathrm{End}(TM)$$

such that $J^2 = -1$. A 1-parameter family of almost complex structures is a smooth map

$$J : [0, 1] \times M \to \mathrm{End}(TM)$$

over M such that each J_t is an almost complex structure, i.e., $J_t^2 = -1$.

Example 1.2. For example $\times i$ is an almost complex structure on $M = \mathbb{C}$.

Any manifold that admits an almost complex structure is even-dimensional, by linear algebra. An almost-complex structure does not give rise to a complex structure, but the other direction can be done.

Definition 1.3. An almost-complex structure J is said to be **ω-compatible** if

$$g(-, -) = \omega(-, J-)$$

is a Riemannian metric.

This is equivalent to the condition

$$\omega(J-, J-) = \omega(-, -), \quad g(v, v) > 0.$$

For every symplectic manifold M, there exists a compatible almost-complex structure, which sends the

Definition 1.4. Let (M, J) and (M', J') be two manifolds with almost-complex structures. A map between them $\phi : (M, J) \to (M, J')$ is called **(J, J')-holomorphic** if

$$d\phi \circ J = J' \circ d\phi'.$$

If M and M' are actually complex manifolds, this is going to be equivalent to the usual notion of holomorphic maps. Note that on $\mathbb{R} \times [0, 1] \subseteq \mathbb{C}$ is a complex manifold, and thus have a natural complex structure.

Definition 1.5. Choose two Lagrangian submanifolds $L_0, L_1 \subseteq M$ that intersect transversely. A ***J*-holomorphic strip** is a J-holomorphic map

$$u : \mathbb{R} \times [0,1] \to M$$

such that $u(s,i) \in L_i$ and $\lim_{s\to\infty} u(s,t) = p$ and $\lim_{s\to-\infty} u(s,t) = q$ for some $p, q \in L_0 \cap L_1$.

We can think of these strips as gradient flows of paths. Note that $\mathbb{R} \times [0,1]$ is biholomorphic to the subset

$$\{|z| = 1\} \setminus \{1, -1\} \subseteq \mathbb{C}.$$

So a J-holomorphic strip is just a J-holomorphic map from a closed disc to M.

Given (M, ω) symplectic manifolds, and L_0 and L_1 are transverse Lagrangian submanifolds, we are going to use this to sort of do Morse theory on

$$\rho(L_0, L_1) = \{(\gamma : [0,1] \to M) : \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

2 April 13, 2018

So we are about to define the Floer complex. Suppose we have transverse Lagrangians L_0, L_1 in (M, ω). Recall that we have our path space

$$\rho(L_0, L_1) = \{\gamma : [0,1] \to M : \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

Let $\tilde{\rho}(L_0, L_1)$ be the universal cover of this space. Explicitly, this the space of $(\gamma, [u])$ where $[u]$ is a homotopy between γ and some fixed path γ_0.

2.1 Floer complex

Definition 2.1. We define the **action functional** as

$$\mathcal{A}(\gamma, [u]) = \int_{I^2} u^*\omega.$$

Note that this does not depend on the choice of u (up to homotopy of u) because $d\omega = 0$ and Stokes's theorem. We want to find the gradient flow is. To find this, we can just formally manipulate, for a vector field v on γ,

$$d_v A(\gamma) = \int_{[0,1]} w(\dot{\gamma}, v)dt = \int_{[0,1]} g(J\dot{\gamma}, v) = \langle J\dot{\gamma}, v\rangle_{L^2}.$$

This shows that critical points are constant paths on $L_0 \cap L_1$, because we need $J\dot{\gamma}$ to vanish. Also, we may take our gradient paths as $\frac{d}{ds}\gamma = J\dot{\gamma}$ if we set the L^2-norm as the metric on the tangent space consisting of vector fields on γ. So gradient flowlines are precisely the holomorphic disks.

Definition 2.2. We define the **Floer complex** as

$$CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} Rp$$

for $R = \mathbb{Z}$ or $R = \mathbb{Z}/2\mathbb{Z}$.

Definition 2.3. For $p, q \in L_0 \cap L_1$ and $[u] \in \pi_2(M, L_0 \cup L_1)$, we consider the **moduli space**

$$\mathcal{M}(p, q, [u], J)$$

of J-holomorphic curves such that $u(s, 0) \in L_0$, $u(s, 1) \in L_1$, and $\lim_{s\to+\infty} u(s,t) = p$, and $\lim_{s\to-\infty} u(s,t) = q$, and also

$$E(u) = \int u^*w < \infty.$$

(This is called the **energy** and only depends on the homotopy class of u.)

Like in Morse theory, we can shift in the s coordinate and get a same thing. So we also define

$$\hat{M} = M(p, q, [u], J)/\mathbb{R}.$$

Here, $\hat{M}$ becomes a smooth manifold of dimension $\mathrm{ind}([u]) - 1$.

2.2 Gromov compactification

Gromov compactification requires that finite energy condition. We are going to use sequential compactness in this context. Given an infinite sequence $\{u_i\}$ consisting of homotopy strips u, there are three things that can happen:

- sphere bubbling: the energy concentrates in the interior of the disk and bubbles off a sphere.
- disk bubbling: the energy concentrates on the side of the disk and bubbles off a disk.
- strip breaking: the disk becomes a concatenation of two disks.

Under some mild conditions, the first two cannot happen. In this case, we can show $\partial^2 = 0$ by counting boundaries.

Definition 2.4. We define the **differential** as

$$\partial : CF^*(L_0, L_1) \to CF^*(L_0, L_1); \quad \partial p = \sum_{q \in L_0 \cap L_1, \mathrm{ind}([u])=1} (\#\hat{M}(p, q, [u], J))q.$$

Theorem 2.5. *If $[\omega] \cdot \pi_2(M, L_i) = 0$, then the Floer differential is well-defined and $\partial^2 = 0$. Moreover, the corresponding homology* $\ker \partial / \operatorname{im} \partial = HF^*(L_0, L_1)$ *is independent of J and the Hamiltonian isotopies of L_0 and L_1.*

3 April 16, 2018

Today we are going to talk about A_∞-categories. Here A stands for associativity and ∞ refers to the relaxation of associativity up to higher homotopy.

3.1 A_∞-categories

Here is a motivating example. Let X be a topological space and pick a basepoint $x_0 \in X$. Now consider the set

$$\Omega X = \{(\gamma : [0,1] \to X) : \gamma(0) = \gamma(1) = x_0\}.$$

You can concatenate two loops as

$$\gamma * \gamma'(t) = \begin{cases} \gamma'(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

The interesting thing is that this operation is not associative, because there is a reparametrizing of the unit interval involved. So what you do when defining the fundamental group is to quotient out by homotopy. But this loses a lot of information. I want to compose stuff in a coherent way, but we don't want to throw away all the extra structure.

Let us write $m_2(\gamma', \gamma) = \gamma' * \gamma$. This is not associative as we have seen, but there is a homotopy. In particular, we have a map

$$m_3 : [0,1] \times \Omega X \times \Omega X \times \Omega X \to \Omega X$$

that satisfies

$$m_3|_{\{0\}\times(\Omega X)^3} = m_2(-, m_2(-,-)), \quad m_3|_{\{1\}\times(\Omega X)^3} = m_2(m_2(-,-),-).$$

Definition 3.1. An **A_∞-category** $\mathcal{C}$ is the data of

(0) a class of objects $\mathrm{ob}(\mathcal{C})$,

(1) for all $X_0, X_1 \in \mathrm{ob}(\mathcal{C})$ a cochain complex $\hom^\bullet(X_0, X_1)$ with the differential map we call m_1,

(2) for all $X_0, X_1, X_2 \in \mathrm{ob}(\mathcal{C})$ a map

$$m_2 : \hom(X_1, X_2) \otimes \hom(X_0, X_1) \to \hom(X_0, X_2)$$

of cochain complexes,

(3) for all $X_0, X_1, X_2, X_3 \in \mathrm{ob}(\mathcal{C})$ a map

$$m_3 : \hom(X_2, X_3) \otimes \hom(X_1, X_2) \otimes \hom(X_0, X_1) \to \hom(X_0, X_3)[-1]$$

of cochain complexes, ...

(k) for all $X_0, \ldots, X_k \in \mathrm{ob}(\mathcal{C})$ a map

$$m_k : \hom(X_{k-1}, X_k) \otimes \cdots \otimes \hom(X_0, X_1) \to \hom(X_0, X_k)[-k+2]$$

of cochain complexes, ...

satisfying

(1) $m_1^2 = 0$,

(2) $m_1 m_2(-,-) = m_2(m_1-,-) \pm m_2(-, m_1-)$,

(3) $m_1 m_3(-,-,-) \pm m_3(m_1-,-,-) \pm m_3(-,m_1-,-) \pm m_3(-,-,{,_1}) = m_2(m_2(-,-),-) \pm m_2(-, m_2(-,-))$, ...

(k) $\sum_{a+b+c=k} \pm m_{a+1+c}(-^a, m_b(-^b), -^c) = 0$.

You can show that m_2 induces a map on cohomology

$$H^*(\hom(X_1, X_2)) \otimes H^*(\hom(X_0, X_1)) \to H^*(\hom(X_0, X_2))$$

and (3) implies that this is associative.

These data of higher m_k is something that we have always used but never realized. In a group, why is $g_1 \cdots g_k$ well-defined? You need to induct on k and use the three-term associativity. In the A_∞-setting, we don't have associativity and so we are tracking all the homotopies.

3.2 Structure on the Floer homologies

Recall that the definition of Floer chain complex is

$$CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}/2\mathbb{Z}p$$

with differential

$$\partial p = \sum_{q \in L_0 \cap L_1, \mathrm{ind}[u]=1} \#\widehat{M}(p, q; [u], J) q.$$

We can define m_2 here. Let $p \in L_0 \cap L_1$ and $q \in L_1 \cap L_2$ and $r \in L_0 \cap L_2$. Consider a J-holomorphic disk $u : D^2 \to M$ that fills in a triangle between L_0, L_1, L_2. This means that $u(1) = r$, $u(j) = p$, $u(j^2) = q$ and the intervals lie on the correct Lagrangian. We may consider the moduli space

$$M(p, q, r, [u], J)$$

of such holomorphic disks. Then we can define m_2 as

$$m_2(q, p) = \sum_{r \in L_1 \cap L_2, \mathrm{ind}[u]=0} \#M(p, q, r, [u], J) r.$$

4 April 20, 2018

We almost defined the Fukaya category. Let (M, ω) be a symplectic manifold, and for $L_0, L_1 \subseteq M$ we can associate a Floer chain complex $CF^\bullet(L_0, L_1)$. These L_0, L_1 are the objects and the morphisms are $CF^\bullet(L_0, L_1)$. Here, we had

$$m_1 p = \partial p = \sum_{q \in L_0 \cap L_1, \mathrm{ind}[u]=1} \#\widehat{M}(p, q, [u], J)q.$$

Then we have

$$m_2(q, p) = \sum_{r \in L_0 \cap L_2, \mathrm{ind}[u]=0} \#M(p, q, r; [u], J)r.$$

The reason we don't need $\widehat{M}$ is because the automorphism group of $D^2 \setminus \{z_0, z_1\}$ is just $\mathbb{R}$. But the automorphism group of $D^2 \setminus \{z_0, z_1, z_2\}$ is trivial, so we don't have to take the quotient by any automorphism group. We can generalize this to higher k by looking at the moduli space $\mathcal{M}$ and quotienting out by $\mathrm{Aut}(D^2)$.

Definition 4.1. Let $p_1, \ldots, p_k$ be such that $p_i \in L_i \cap L_{i-1}$, where $L_0, \ldots, L_k$ are Lagrangian submanifolds. Then we define

$$\begin{aligned} m_k &: CF^\bullet(L_{k-1}, L_k) \otimes \cdots \otimes CF^\bullet(L_0, L_1) \to CF^\bullet(L_0, L_k)[2-k]; \\ m_k(p_k, \ldots, p_1) &= \sum_{q \in L_0 \cap L_k, \mathrm{ind}[u]=0} (\#\widehat{M}(p_0, \ldots, p_k, q; [u], J))q. \end{aligned}$$

Here, $\widehat{M}$ and M are defined as

$$M = \{(z_0, \ldots, z_p, u) : z_i \in \partial D^2, u : D^2 \to M, du \circ J = J \circ du, u(z_i) = p_i, \ldots\}$$

and $\widehat{M} = M/\operatorname{Aut}(D^2)$.

Now we have all our higher-order operations. But we need to check infinitely many more relations to check.

Proposition 4.2. *For all $n \geq 1$, and $p_1, \ldots, p_n$ with $p_i \in L_{i-1} \cap L_i$, we have*

$$\sum_{\substack{k,l \geq 1 \\ k+l=n+1 \\ 0 \leq j \leq l-1}} \pm m_l(p_n, \ldots, p_{j+k+1}, m_k(p_j + 1, \ldots, p_1), p_j, \ldots, p_1) = 0.$$

Proof. If you draw all the ways broken discs can arise, the boundary is precisely this. $\square$

5 April 23, 2018

What is the Fukaya category of $M = \mathrm{pt}$? The objects are going to be

$$\operatorname{ob} \mathsf{Fuk}(M) = \{\emptyset, M = L\}.$$

Given (M, ω), we roughly had that $\operatorname{ob} \mathsf{Fuk}(M)$ is the set of Lagrangians $L \subseteq M$, i.e., the submanifolds $L \subseteq M$ with $\dim L = \frac{1}{2} \dim M$ and $\omega|_L = 0$. The hom was given as

$$\hom^\bullet(L_0, L_1) = CF^*(L_0, L_1) = \bigcap_{p \in L_0 \cap L_1} \mathbb{Z}.$$

The differential counts holomorphic strips

$$m_1 p = \sum_q \#\{\text{strips between } p \text{ and } q\} q$$

and there is a composition map

$$m_2 : \hom(L_1, L_2) \otimes \hom(L_0, L_1) \to \hom(L_0, L_2)$$

given by $x_2 \otimes x_1$ being sent to $\sum_x \#\{\text{strips between } x, x_1, x_2\} x$. The A_∞-category conditions then just become splittings of polygons.

5.1 Homological mirror symmetry conjecture

There is a conjecture that says that this Fukaya category is equal to some other category which is a lot simpler. Here is the conjecture, which is false on the nose but morally true.

Conjecture 5.1 (homological mirror symmetry conjecture)**.** *Let (M, ω) be a symplectic manifold. Then there exists a complex manifold $M^\vee$ such that*

$$\mathsf{Fuk}(M) \simeq D^b \mathsf{Coh}(M^\vee)$$

Here, $\mathsf{Shv}(M^\vee)$ should be something like vector bundles on M.

Actually something stronger holds. Fix a manifold M with a symplectic structure ω and a complex structure J. Then there exists a $(M^\vee, \omega^\vee, J^\vee)$ such that

$$\mathsf{Fuk}(M) \simeq D^b \mathsf{Coh}(M^\vee), \quad D^b \mathsf{Coh}(M) \simeq \mathsf{Fuk}(M^\vee).$$

Here is an example. It turns out that

$$M = T^* S^1, \quad M^\vee = \mathbb{C} \setminus \{0\}$$

is a pair. The sheaves on $\mathbb{C} \setminus \{0\}$ are modules over $\mathbb{C}[x, x^{-1}]$. So we are going to have $D^b(\mathsf{Coh}^\vee) \cong \mathsf{ChCmplxMod}(\mathbb{C}[t, t^{-1}])$.

In this context, take $R = \mathbb{C}[t, t^{-1}]$ and consider R as an R-module. Then

$$\hom_R(R, R) = R = \mathbb{C}[t, t^{-1}],$$

which is a infinite-dimensional $\mathbb{C}$-vector space. This corresponds to a vertical line in the cylinder, $L_0 = T_p S^1$. To compute $CF^*(L_0, L_0)$, we look at a Hamiltonian deformation L_0' and compute $CF^*(L_0, L_0')$. For reasons we haven't talked about, we have to take a special deformation. Let us take the vector field that grows linearly in both sides. Then there are infinitely many intersection points between L_0 and L_0', and also there are no holomorphic strips. Then

$$CF^*(L_0, L_0') = \mathbb{C}^{\mathbb{Z}}$$

and the ring structure is given by $\mathbb{C}[t, t^{-1}]$.

Here is another example. In the Fukaya category $\mathsf{Fuk}(T^*S^1)$, there is the zero section $L_1 = S^1$. Then the hom ring is

$$CF^*(L_1, L_1) = \begin{cases} \mathbb{C} & * = 0 \\ \mathbb{C} & * = 1. \end{cases}$$

This corresponds to the module $\mathbb{C}$ over R. You can see that

$$\mathrm{Ext}^*(\mathbb{C}, \mathbb{C}) = \begin{cases} \mathbb{C} & * = 0 \\ \mathbb{C} & * = 1. \end{cases}$$

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