Math 224 - Representations of Reductive Lie Groups

Taught by Dennis Gaitsgory
Notes by Dongryul Kim

Spring 2018

This was a course taught by Dennis Gaitsgory. The class met on Tuesdays and Thursdays from 11:30am to 1pm, and the main textbook was Springer's *Linear Algebraic Groups*. There were a few assignments near the beginning of the course, and a final paper.

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This is a course on linear algebraic groups. We are going to try and follow Springer's *Linear algebraic groups*. We will be working over a field k, but we are not going to restrict to varieties.

1.1 Linear algebraic groups

Definition 1.1. An **algebraic group** is a group object in the category $\mathsf{Sch}_{\mathrm{ft}}$ of schemes of finite type.

Then for every scheme S, $\mathrm{Hom}(S, G)$ is a group, and is contravariant in S.

Definition 1.2. A **linear algebraic group** is a algebraic group that is affine.

Example 1.3. For V a vector space, we define $\mathrm{GL}(V)$ in the following way:

$$\mathrm{Hom}(S, \mathrm{GL}(V)) = \text{group of automorphisms of } V \otimes O_S \text{ as an } O_S\text{-module.}$$

This is something you have actually seen before. Let W be a finite-dimensional vector space, and suppose we want to find $\mathbb{W}$ such that

$$\mathrm{Hom}(S, \mathbb{W}) = W \otimes O_S.$$

Then $\mathbb{W}$ is actually $\operatorname{Spec}\operatorname{Sym}(W^*)$. This is because

$$\mathrm{Hom}(S, \operatorname{Spec}\operatorname{Sym}(W^*)) = \mathrm{Hom}_{\mathsf{ComAlg}}(\operatorname{Sym}(W^*), O_S) = \mathrm{Hom}_{\mathsf{Vect}}(W^*, O_S) = O_S \otimes W.$$

Let us generalize this and consider

$$\mathrm{Hom}(S, M(V)) = \text{monoid of all } O_S\text{-linear endomorphisms of } V \otimes o_S.$$

Then $M(V)$ is obtained from the vector space $\mathrm{End}(V)$ in the previous construction.

Example 1.4. For $V = k$, we have $\mathrm{GL}(k) = \mathbb{G}_m$. Here, we have

$$\mathrm{Hom}(S, \mathbb{G}_m) = O_S.$$

Example 1.5. The group $\mathbb{G}_a$ is going to be $\mathrm{Hom}(S, \mathbb{G}_a) = O_S$ under addition.

A map between algebraic groups is a natural transformation of functors. Now let me try to explain what a determinant is. For each V, we can define $\det V = \bigwedge^{\mathrm{top}}(V)$. We want to construct a map $M(V) \to M(\det(V))$. To construct this, we need to find a map between monoids. If α acts on $V \otimes O_S$, its determinant $\det(\alpha)$ acts on $\bigwedge^{\mathrm{top}}_{O_S}(V \otimes O_S)$. This is the determinant map. If V is of dimension 1, we have $O_S \cong \mathrm{End}_{O_S}(V)$. So we have the map

$$\det : M(V) \to M(\det(V)) \cong M(k) \cong \mathbb{A}^1.$$

So we have a map $\mathrm{GL}(V) \to \mathbb{G}_m$, and $\mathrm{SL}(V)$ is the kernel of this map. This is a scheme because can take the fiber product.

Now fix a flag $V = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_0 = 0$. Now we want to define a subgroup $P \subseteq \mathrm{GL}(V)$ by

$$\mathrm{Hom}(S, P) \subseteq \mathrm{Hom}(S, \mathrm{GL}(V))$$

consisting of endomorphisms that send each $V_i \otimes O_S$ to itself. This is going to be the intersection of some M_P and $\mathrm{GL}(V)$ in $M(V)$. We want to show that P is a closed subscheme of $M(V)$. To do this, it suffices to show that M_P is closed in $M(V)$. Each $V_i \otimes O_S$ preserved by α means that

$$V_i \otimes O_s \to V \otimes O_s \xrightarrow{\alpha} V \otimes O_S \to (V/V_i) \otimes O_S$$

is zero. This is really $\mathbb{H}\mathrm{om}(V_i, V/V_i)$. So this M_P fits in the fiber product

$$\begin{tikzcd}
M_P \arrow[r] \arrow[d] & M(V) \arrow[d] \\
* \arrow[r] & \prod_i \mathbb{H}\mathrm{om}(V_i, V/V_i).
\end{tikzcd}$$

Now we have a map

$$1 \to U \to P \to \prod_i \mathrm{GL}(V_i/V_{i-1}) \to 1.$$

Here, the map $P \to \prod_i \mathrm{GL}(V_i/V_{i-1})$ is given in the following way. For each $\alpha : V \otimes O_s \to V \otimes O_S$, we have $V_i \otimes O_S \to V_i \otimes O_S$ so $(V_i/V_{i-1}) \otimes O_S \to (V_i/V_{i-1}) \otimes O_S$. So we can consider its kernel.

Take a bilinear form $V \otimes V \to k$. We can also define a $H \subseteq \mathrm{GL}(V)$ given by $\mathrm{Hom}(S, H) \subseteq \mathrm{Hom}(S, \mathrm{GL}(V))$ consisting of α such that

$$\begin{tikzcd}
(V \otimes O_S) \otimes_{O_S} (V \otimes O_S) \arrow[r] \arrow[d, "\alpha\otimes\alpha"] & O_S \arrow[d, equal] \\
(V \otimes O_S) \otimes_{O_S} (V \otimes O_S) \arrow[r] & O_S
\end{tikzcd}$$

commute.

Now let O_G be the algebra of regular functions on G. This O_G is going to be a commutative Hopf algebra, with maps $\Delta : O_G \to O_G \otimes O_G$ and $\gamma : O_G \to O_G$ and $\epsilon : O_G \to k$.

There is also a notion of an algebraic group acting on a scheme. This is going to be a morphism $G \times X \to X$. This is going to be an action of $\mathrm{Hom}(S, G)$ on $\mathrm{Hom}(S, X)$ for each S that are compatible with maps on S.

Example 1.6. Any G acts on itself on the left and on the right. Also $\mathrm{GL}(V)$ acts on $\mathbb{V}$ for a finite-dimensional V. This is tautological, because $\mathrm{Hom}(S, \mathrm{GL}(V))$ is the automorphisms of $V \otimes O_S$ and $\mathrm{Hom}(S, \mathbb{V})$ is $V \otimes O_S$.

The projective space is defined as

$$\mathrm{Hom}(S, \mathbb{P}(V)) = \{\text{line bundles } \mathcal{L} \text{ on } S \text{ with } 0 \to \mathcal{L} \to V \otimes O_S \text{ with locally free cokernel}\}.$$

Example 1.7. $\mathrm{GL}(V)$ acts on $\mathbb{P}(V)$ because α is an automorphism of $V \otimes O_S$.

More generally, consider $n = n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq n_0 = 0$ where $\dim V = n$. We can define the **flag variety** as the following: $\mathrm{Hom}(S, \mathrm{Fl}(V)^{n_k,\dots,n_0})$ is the collection of locally free sheaves $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k$ such that $\mathrm{rk}(M_i) = n_i$ and an identification $M_k \cong V \otimes O_S$ such that M_i/M_{i-1} are locally free.

1.2 Representations of linear algebraic groups

Definition 1.8. Let V be a vector space. A **representation** of G on V is, for every S, an action of $\mathrm{Hom}(S, G)$ on $V \otimes O_S$ by automorphisms of O_S-modules.

Lemma 1.9. *To define on V a structure of G-representations is equivalent to define on V a structure of an O_G-comodule.*

The idea is that $V \otimes O_G$ is the functions on G with values in V. So a comodule structure $V \to V \otimes O_G$ is something we want to send v to $g \mapsto gv$.

Proof. Take S to be G, and the identity element on $\mathrm{Hom}(G, G)$. Then there is a canonical automorphism α on $V \otimes O_G$. Then we can compose $V \to V \otimes O_G \xrightarrow{\alpha} V \otimes O_G$. For the other direction, we start with a comodule structure $V \to V \otimes O_G$ to a O_G-linear map $\alpha : V \otimes O_G \to V \otimes O_G$. This turns out to be an automorphism. Then for any test scheme S with $S \to G$, restrict $V \otimes O_G \to V \otimes O_G$ to S. $\square$

Corollary 1.10. *If V is a G-representation, then for every finite-dimensional $W \subseteq V$, there exists a $W \subseteq V'$ such that V' is finite-dimensional and preserved by the G-action.*

Proof. Given W, let V' be the minimal subspace in V such that $\mathrm{im}(W) \subseteq V' \otimes O_G$. We'll continue next time. $\square$

2 January 25, 2018

We'll continue talking about representations. Let G be an algebraic group. A representation of G by V is an action of $\mathrm{Hom}(S, G)$ on $V \otimes O_S$. For $g \in \mathrm{Hom}(S, G)$, the map $V \otimes O_S \to V \otimes O_S$ is just a map $V \to V \otimes O_S$ of vector spaces. So this should be send an element $v \in V$ to a function $S \to V$. So we had an equivalence $\mathsf{Rep}(G) \cong O_G-\mathsf{coMod}$.

2.1 Embedding of linear algebraic groups

Proposition 2.1. *Let $V \in \mathsf{Rep}(G)$, then for any finite-dimensional $W \subseteq V$ there exists a $W \subseteq V' \subseteq V$ where V' is finite-dimensional and a subrepresentation.*

Proof. We have a coaction $V \to V \otimes O_G$ and let V' be the minimal subspace in V such that the co-action of W is contained in $V' \otimes O_G$. So given a $W' \subseteq V \otimes O_G$, what is the minimal V' such that $W' \subseteq V' \otimes O_G$? This can be computed by the span of $(\mathrm{id} \otimes \xi)(W)$ for $\xi \in U^*$.

Now our claim is that V' is a subrepresentation. We need to show that the coaction of V' is contained in $V' \otimes O_G$. By definition, we need to show that $W \to V \otimes O_G \to V \otimes O_G \otimes O_G \to V$ always lies in V'. This is clear by interpreting the map $V \otimes O_G \to V \otimes O_G \otimes O_G$ as the comultiplication on O_G. $\square$

Corollary 2.2. *Any algebraic group G has a closed embedding to $\mathrm{GL}(V)$ for some finite-dimensional V.*

Proof. We know that O_G is a representation. Because G is finite type, there exists a $W \subseteq O_G$ such that $\mathrm{Sym}(W) \to O_G$, and so we can replace W with a $V \subseteq O_G$ that is a representation. From this we get a map $G \to \mathrm{GL}(V)$. We would like to show that this is a closed embedding, and it suffices to show that

$$G \hookrightarrow \mathrm{GL}(V) \hookrightarrow M(V)$$

is a closed embedding, where $\mathrm{GL}(V) \hookrightarrow M(V)$ is an open embedding.

Now $M(V) = \mathbb{M}(V)$ is the variety associated to the vector space $V \otimes V^*$. I'm trying to show that $\mathrm{Sym}((V \otimes V^*)) \to O_G$ is surjective. But we started with a surjection $\mathrm{Sym}(V) \twoheadrightarrow O_G$. But how is $V \to O_G$ related to $V \otimes V^* \to O_G$? There is a natural covector $V \to O_G \to k$ by evaluation on 1. It turns out that the original $V \to O_G$ is

$$V \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_1} V \otimes V^* \to O_G.$$

So $\mathrm{Sym}(V \otimes V^*) \to O_G$ is surjective. $\square$

Corollary 2.3. *If G acts on X and X is affine, there exists a closed embedding of $X \hookrightarrow \mathbb{V}$ for V a finite-dimensional representation on which G acts.*

Proof. We have $G \times X \to X$ which becomes $O_X \to O_X \otimes O_G$. Let $W \subseteq O_X$ be a subspace that generates it as an algebra. We can replace W by V that is a subrepresentation. We have a surjection $\mathrm{Sym}(V) \twoheadrightarrow O_X$, or a closed embedding $X \hookrightarrow \mathbb{V}$. $\square$

Proposition 2.4. $\mathrm{Hom}_{\mathsf{Rep}(G)}(V, O_G) = V^*$.

Proof. The map in one direction is $V \to O_G \xrightarrow{\mathrm{ev}_1} k$. For the other direction, a representation is given by $V \otimes V^* \to O_G$. So any element of V^* gives a map $V \to O_G$. $\square$

Actually, the map $V \otimes V^* \to O_G$ is a map of $G \times G$-representations.

2.2 The regular representation

We would now like to construct the regular representations from the representations. A map between representations $\varphi : V \to W$ will give a commutative diagram

$$\begin{tikzcd} V \otimes V^* \arrow[r] & O_G \\ V \otimes W^* \arrow[u, "\mathrm{id} \otimes \varphi^*"] \arrow[r, "\varphi \otimes \mathrm{id}"] & W \otimes W^* \arrow[u] \end{tikzcd}$$

For a category $\mathcal{C}$, we define the **twisted arrow category** $\mathsf{TwArr}(\mathcal{C})$ consisting of objects $C_0 \to C_1$ and morphisms

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C_1 \\ C_0' \arrow[r] & C_1'. \arrow[u] \end{tikzcd}$$

If we have two functors $F, G : \mathcal{C} \to \mathcal{D}$, then we can look at the natural transformations from F to G. These are collections of $\alpha_C \in \mathrm{Hom}(F(C), G(C))$ satisfying some compatibility conditions. Now my claim is that

$$\mathrm{Nat}(F, G) = \varprojlim_{\mathsf{TwArr}(\mathcal{C})} H$$

where H is

$$H(\varphi : C_0 \to C_1) = \mathrm{Hom}_{\mathcal{D}}(F(C_1), G(C_0)).$$

This is because a limit is just a family satisfying some compatibility conditions.

Definition 2.5. Let $F : \mathcal{C} \to \mathsf{Set}$ and $G : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ be the functors. We can define a functor $H : \mathsf{TwArr}(\mathcal{C}) \to \mathsf{Set}$ defined by

$$H(\alpha : C_0 \to C_1) = F(C_0) \times G(C_1).$$

Then we define the **coend** as

$$\mathrm{coend}(F, G) = \varinjlim_{\mathsf{TwArr}(\mathcal{C})} H.$$

So let $\mathcal{C} = \mathsf{Rep}(G)^{\mathrm{fd}}$ and $\mathcal{D} = \mathsf{Rep}(G \times G)$. Then we define

$$H(\alpha : V \to W) = V \otimes W^*.$$

Then the claim we want to show is that

$$\varinjlim_{\mathsf{TwArr}(\mathsf{Rep}(G))^{\mathrm{fd}}} H = O_G.$$

3 February 1, 2018

Last time I sold the product of thinking of representations as comodules.

3.1 Examples of groups and classification of representations

Let's consider the group $\mathbb{G}_a$. This is $\mathbb{A}_1$ with addition, so it is given by $O_G = k[t]$ with

$$k[t] \to k[t] \otimes k[t]; \quad t \mapsto t \otimes 1 + 1 \otimes t.$$

I would like to give a complete description of representations. These are going to be

$$V \to V \otimes k[t],$$

and so we can dualize it to

$$k[s] \otimes V \to V.$$

How are we doing this? In general, if we have $V_1 \to V_2 \otimes W$ then we can dualize it to $W^* \otimes V_1 \to V_2$. This will have the condition that any $v_1 \in V_1$ is killed by a sufficiently large quotient of W^*. So this $k[s] \otimes V \to V$ should be a map such that any $v \in V$ is killed by a sufficiently large power of s.

Let Γ be a finite group. This is an algebraic variety given by

$$O_\Gamma = \bigoplus_{\gamma \in \Gamma} k\delta_\gamma,$$

with comultiplication

$$\Delta(\delta_\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} \delta_{\gamma_1} \otimes \delta_{\gamma_2}.$$

So we can truly dualize $V \to V \otimes O_\Gamma$, and get $k[\Gamma] \otimes V \to V$. There are no more representations other than the normal ones.

Now let $G = \mathbb{G}_m$. Here, $O_G = k[t, t^{-1}]$ and $\Delta(t) = t \otimes t$. There are representations $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^n$, which we can think of as nth characters.

Proposition 3.1. *Every representation of $\mathbb{G}_m$ is a direct sum $V \cong \oplus_n V^{(n)}$ where $V^{(n)}$ is a direct sum of copies of the nth characters.*

Proof. Look at the map $V \to V \otimes k[t, t^{-1}]$ and write it as

$$v \mapsto \sum_n v_n \otimes t^n.$$

Now let us apply ◻ todo

Let us look at $\Lambda \cong \mathbb{Z}^n$ a lattice. We have

$$k[\Lambda] \cong k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}],$$

and then the **torus**

$$T = \operatorname{Spec} k[\Lambda] \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m.$$

It can be checked that $\Delta(\lambda) = \lambda \otimes \lambda$.

Proposition 3.2. $\mathsf{Rep}(T)$ *is equivalent to* Vect^Λ.

You can do the exact same thing. Note that this discussion is contravariant in Λ.

Proposition 3.3. *There is an isomorphism* $\operatorname{Hom}(\Lambda_1, \Lambda_2) \cong \operatorname{Hom}_{\mathsf{AlgGrp}}(T_2, T_1)$.

Proof. By definition, $\operatorname{Hom}(\Lambda_1, \Lambda_2)$ isomorphic to $\operatorname{Hom}(k[\Lambda_1], k[\Lambda_2])$ where Hom is as Hopf algebras. Given any $\lambda_1 \in \Lambda_1$, we want to show that $phi(\lambda_1)$ is primitive. This is because primitive is equivalent to the statement that $\Delta(\lambda) = \lambda \otimes \lambda$. So we get map $\Lambda_1 \to \Lambda_2$. $\square$

3.2 Tannaka duality

Representations is a tensor category. That is, you can tensor representations. For a test scheme S, there is an action of $\operatorname{Hom}(S, G)$ on $V \otimes O_S$. Then if I have V_1 and V_2, we can take

$$V_1 \otimes V_2 \otimes O_S = (V_1 \otimes O_S) \otimes_{O_S} (V_2 \otimes O_S).$$

At the level of comodules, we can describe this in the following way. If A is a Hopf algebra, we can define the tensor product of $V_1 \to V_1 \otimes A$ and $V_2 \to V_2 \otimes A$ then

$$V_1 \otimes V_2 \to V_1 \otimes A \otimes V_2 \otimes A \to V_1 \otimes V_2 \otimes A \otimes A \to V_1 \otimes V_2 \otimes A.$$

Here, we're using that A as an algebra is commutative, when we're saying that $V_1 \otimes V_2 \cong V_2 \otimes V_1$.

Corollary 3.4. *(i) We have an isomorphism* $\Lambda \cong \operatorname{Hom}(T, \mathbb{G}_m)$.

(ii) We have an isomorphism $\Lambda^\vee = \operatorname{Hom}(\Lambda, \mathbb{Z}) = \operatorname{Hom}(\mathbb{G}_m, T)$.

Now here is Tannaka's theorem. There is a forgetful functor

$$F : \mathsf{Rep}(G) \to \mathsf{Vect}$$

which is a tensor functor, which means that it sends tensor products to tensor products. Sometimes I will write $F(V) = \underline{V}$. Tannaka's theorem says that we can recover G by this functor.

Given any element $g \in G$, for every V the element g acts on $F(V)$ as $\alpha_V : F(V) \to F(V)$. So Tannaka's theorem is going to say that the automorphism group of this functor is going to be G. But it is going to be a tensor automorphism. In particular, we have

$$\begin{tikzcd} F(V_1 \otimes V_2) \arrow[r, "\alpha_{V_1 \otimes V_2}"] \arrow[d, "\cong"] & F(V_1 \otimes V_2) \arrow[d, "\cong"] \\ F(V_1) \otimes F(V_2) \arrow[r, "\alpha_{V_1} \otimes \alpha_{V_2}"] & F(V_1) \otimes F(V_2). \end{tikzcd}$$

I would like to recover S-points of G.

So we can consider the functor

$$F_S^{\mathrm{fd}} : \mathsf{Rep}^{\mathrm{fd}}(G) \to \mathsf{Coh}(S); \quad V \mapsto V \otimes O_S.$$

There is an obvious map

$$\mathrm{Hom}(S, G) \to \mathrm{Aut}_{\mathrm{tensor}}(F_S)$$

with the same imposed condition.

Theorem 3.5 (Tannaka)**.** *This map* $\mathrm{Hom}(S, G) \to \mathrm{Aut}_{\mathrm{tensor}}(F_S)$ *is an isomorphism.*

Proof. We are going to construct the map in the other direction. Every tensor automorphism of F_S^{fd} canonically extends to a tensor automorphism of $F_S : \mathsf{Rep}(G) \to \mathsf{QCoh}(S)$. Now we can consider maps $A \otimes A \to A$ for $A \in \mathsf{Rep}(G)$. So it makes sense to talk about commutative algebras in $\mathsf{Rep}(G)$. If F is a tensor category, every $\mathrm{Aut}_{\mathrm{tensor}}(F_S)$ should be respect the algebra structure.

Now let us take $A = O_G$. Then $F_S(A) = O_S \otimes O_G$, and we obtain an automorphism of $O_S \otimes O_G$ as an O_S-algebra. So we get an automorphism $\alpha : S \times G \to S \times G$ over S. We need to check that this is given by left multiplication, and it suffices to show that the diagram

$$\begin{tikzcd} S \times G \times G \arrow[r, "\alpha \otimes \mathrm{id}_G"] \arrow[d, "\mathrm{id} \times m"] & S \times G \times G \arrow[d, "\mathrm{id} \times m"] \\ S \times G \arrow[r, "\alpha"] & S \times G \end{tikzcd}$$

commutes. To show this, we need to go back to the functor. This comes from the commutative diagram

$$\begin{tikzcd} F(O_G \otimes \underline{O_G}) \arrow[r, "\alpha_{O_G \otimes \underline{O_G}}"] & F(O_G \otimes \underline{O_G}) \\ F(O_G) \arrow[r, "\alpha_{O_G}"] \arrow[u, "\Delta"] & F(O_G) \arrow[u, "\Delta"] \end{tikzcd}$$

and taking spectra. $\square$

What this also tells us is that if $g_1, g_2 \in \mathrm{Aut}_{\mathrm{tensor}}(F_S)$ acts in the same way on O_G, then it acts in the same way on each V. Here is the reason. Every V can be embedded in $\underline{V} \otimes O_G$, which is a bunch of copies of O_G. Here is why.

We saw from last time that we have a map $V \otimes V^* \to O_G$ as $G \times G$-representation. Then we can move V^* to the other side and get $V \to O_G \otimes V$ and an O_G-linear map

$$V \otimes O_G \to V \otimes O_G.$$

Considering $O_G \otimes V$ as functions on G with values in V, this is $f \mapsto \tilde{f}(g) = g^{-1} f(g)$. Now we consider a $G \times G$-action on the domain $V \otimes O_G$, given by the first G acting on $V \otimes O_G$ and the second G acting on O_G. Consider the $G \times G$-action on the target $V \otimes O_G$, given by the first G acting on the second O_G and the second G acting on $V \otimes G$. Then the map $V \otimes O_G \to V \otimes O_G$ intertwines the $G \times G$-action. This is a wonderful property of the regular representation. For the first G, you lose the representation, and for the second G, you gain this information. Now if we ignore the second G-action, we get an intertwining

$$V \otimes O_G \to \underline{V} \otimes O_G$$

that is an isomorphism of G-representations, where G acts on the left side diagonally. This will prove that the two construction of the Tannaka duality are inverses.

4 February 6, 2018

Let G be an algebraic group.

4.1 Connected component

Theorem 4.1. *There exists an irreducible component $G^0 \subseteq G$ such that*

(a) it is a subgroup,

(b) it is a connected component,

(c) it is normal with finite index.

Proof. Suppose I have to components X, Y containing e. Then we can look at the closure $\overline{XY}$. Then we get $X \subseteq \overline{XY}$ and $Y \subseteq \overline{XY}$ and so $X = \overline{XY} = Y$. Likewise, if $e \in X$ then $e \in X^{-1}$ and so $X = X^{-1}$. This shows (a).

Now consider the decomposition

$$\coprod_{g \in G} gG^0 = G$$

as a scheme for $g \in G(k)$ (if k is algebraically closed). The claim is that all $g \in G^0$ are connected. This can be seen by checking $g_1G^0 \cap g_2G^0 \neq 0$ implies $g_1G^0 = g_2G^0$.

To see that it is normal, we look at the conjugation on G. It sends connected components to connected components, and the identity to the identity. So it sends G^0 to G^0. It being finite index means that $\mathrm{Hom}(S, G^0)$ is of finite index in $\mathrm{Hom}(S, G)$, which is because G is quasi-compact. $\square$

The groups $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n, B_n, U_n, SL_n are all connected. SL_n is because

$$\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{SL}_n; \quad (g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$$

is surjective at the level of k-points. The source is connected, and so the target should be connected. But O_n is not connected for $\mathrm{char}\, k \neq 2$ because

$$\begin{tikzcd} \mathrm{O}_n \arrow[r] \arrow[d] & \pm 1 \arrow[d] \\ \mathrm{GL}_n \arrow[r, "\det"] & \mathbb{G}_m. \end{tikzcd}$$

We also have that SO_n and Sp_{2n} are connected. To show this, we use the fact that these groups are generated by connected subgroups.

Proposition 4.2. *Let Y_i by irreducible with maps $\phi_i : Y_i \to G$. Assume that $e \in \mathrm{im}(\phi_i)$. Let H be the (closed) subgroup of G generated by them. Then H is irreducible.*

Proof. Let $I = \{1, \ldots, n\}$, and consider the multiplication (with inversions) map

$$Y_1 \times \cdots \times Y_n \times Y_1 \times \cdots \times Y_n \times \cdots \times Y_n \to G.$$

Let Z_m be the closure of the image. Then $Z_m \subseteq Z_{m+1} \subseteq G$ are irreducible varieties, and because G has finite Krull dimension, it stabilizes. Then $Z_m \cdot Z_m \subseteq Z_{m+m}$ and also $Z_m^{-1} \subset Z_m$, so Z_m is an irreducible subgroup. $\square$

4.2 Image and surjectivity

If $\Phi : G' \to G$ is a homomorphism, we can look at its image $G' \to \overline{\mathrm{im}(\Phi)}$. This is a subgroup, as you can see by the functor of points.

Proposition 4.3. *The map $G' \to \overline{\mathrm{im}(\Phi)}$ is surjective on k-points.*

Proof. In general, if we have a dominant morphism $X \to Y$ of reduced varieties, there exists some open dense $Y^0 \subseteq Y$ such that $Y^0(k) \subseteq \mathrm{im}(X(k))$. (This is some kind of Chevalley.) Let's use the following lemma on $U = Y^0$ and $V = \mathrm{im}(G'(k))$. Then we get

$$G(k) = U(k) \cdot V \subseteq \mathrm{im}(G'(k)) \cdot \mathrm{im}(G'(k)) \subseteq \mathrm{im}(G'(k)).$$

This finishes the proof. $\square$

Lemma 4.4. *If $U \subseteq G$ is open and $V \subseteq G(k)$ is dense, then $U \cdot V = G$.*

Proof. For $g \in G(k)$, we have

$$g^{-1}V \cap U(k) \neq \emptyset$$

by definition of density. This means that $g \in V \cdot U(k)$. $\square$

Let G act on a variety X. Consider the action map $G \to X$ and look at the closure $\overline{Gx}$.

Definition 4.5. An action of G on X is **transitive** if $G(k)$ acts transitively on $X(k)$.

This is not good if X is not reduced, but is reasonable if X is reduced.

Proposition 4.6. *Let G be reduced. Then $\overline{Gx}$ contains a unique open G-stable subvariety with a transitive action that contains the point x.*

Proof. The map $G \to \overline{Gx}$ is dominant, and therefore there exists an open $Y^\circ \subseteq \overline{Gx}$ such that $Y^\circ(k) \subseteq \mathrm{im}(G(k))$. Define the desired open subvariety by $Z = \bigcup_g gY^\circ$. This is transitive and G-stable by construction. Suppose $x \notin Z(k)$, so that $x \notin \overline{Gx} - Z$. But the right hand side is closed and G-stable, so $\overline{Gx} \subseteq \overline{Gx} - Z$, which is a contradiction. $\square$

Proposition 4.7. *Let X be reduced and $G(k)$ act on $X(k)$ act transitively. Then for all $x \in X(k)$,*

$$G \to X; \quad g \mapsto gx$$

is faithfully flat.

Proof. In general, if I have a dominant map $\varphi : Z_1 \to Z_2$ with Z_2 reduced, there exists a nonempty open $Z_2^\circ \subseteq Z_2$ over which φ is faithfully flat. Applying this, you see that there exists a nonempty open $X^\circ \subseteq X$ over which $G \to X$ is faithfully flat. Then the same is true for gX° for all $g \in G(k)$, and then $X = \bigcup_g gX^\circ$. □

Using a similar idea, you can show that if the kernel $\ker(G_1 \to G_2)$ is finite, then the morphism $G_1 \to G_2$ is finite. Quasi-finite does not imply finite, but you can take a open on which it is finite and then translate it around.

Example 4.8. Let X be a point and G be any non-reduced algebra group. Then $G \to X$ is not smooth. But if G is any non-reduced algebraic group then the kernel of Frob : $G \to G$ is never reduced. For instance, if $G = \mathbb{G}_a$ then $\ker(\mathrm{Frob}) = \operatorname{Spec} k[t]/t^p$.

5 February 8, 2018

We were talking about the Frobenius $X \to X$.

5.1 Fun with Frobenius

If $X = \mathbb{A}^n$, the preimage of the Frobenius would be

$$k[t_1, \ldots, t_n]/t_1^q \cdots t_n^q.$$

So if X is smooth of dimension n then $\mathrm{Frob}_q^{-1}(x)$ is a scheme of length q^n. So the inverse image is non-reduced unless the point is isolated.

Consider $X = \mathbb{G}_a$. If we look at $\ker(\mathrm{Frob}_p$, then it looks like $\mathrm{Spec}(k[t]/t^p)$. Let us look at its representations. Its dual is $O_G^* = k[x]/x^p$, and so these are endomorphisms whose p-th power is zero.

Also consider $X = \mathbb{G}_m$, and this is

$$\ker(\mathrm{Frob}_p) = \mu_p = \mathrm{Spec}(k[t, t^{-1}]/t^p = 1).$$

Again, let us look at representations of G. These are modules for the dual O_G^*. There is an element called $\xi = t\partial_t \in O_G^*$ that maps $t^i \mapsto i$. (The motivation is taking the Lie derivative $t\partial_t$.)

The claims is that $\xi^p = \xi$. This is becasue

$$\xi^p(t^i) = \langle \xi \otimes \cdots \otimes \xi, t^i \otimes \cdots \otimes t^i \rangle = i^p = i$$

for any i. Then you can show that $O_G^* = k[\xi]/\xi^p - \xi$. Now representations $\mathsf{Rep}(G)$ are vector spaces graded by $\mathbb{Z}/p\mathbb{Z}$. You can see this in the following way: we have a short exact sequence

$$1 \to G \to \mathbb{G}_m \to \mathbb{G}_m \to 1,$$

where $\mathbb{G}_m \to \mathbb{G}_m$ is given by taking the pth power. Representations of $\mathbb{G}_m$ are given by $\mathsf{Vect}^{\mathbb{Z}}$, so representations pull back by a factor-p scaling of indices.

5.2 More on orbits

We defined a transitive action of a reduced G on X to be an action such that X is reduced and $G(k)$ acts transitively on $X(k)$.

Proposition 5.1. *If G acts on X transitively, then for all $x \in X$, the action map $G \to X$ is faithfully flat.*

So we can deduce the strongest notion from the weakest notion.

Theorem 5.2. *Let G reduced act on X. The for all $x \in X(k)$, there exists a unique locally closed subset $Y \subseteq X$ such that $x \in Y$ and the action of G on Y is transitive. This Y is called the **orbit** of x in X.*

Proposition 5.3. *Let G act on X. There exist closed orbits.*

Proof. Take an orbit of smallest possible dimension Y. Call it Y. If Y is not $\bar{Y}$, there is going to a point in $\bar{Y}$ outside Y, and then consider the orbit of anything in Z. $\square$

5.3 Jordan decomposition

Let V be a vector space and T be a endomorphism.

Theorem 5.4 (Jordan decomposition)**.**

(1) There exists a decomposition $T = T_n + T_{ss}$ such that they commute.

(2) If T is an automorphism, there exists a decomposition $T = T_u \cdot T_{ss}$ such that $T_u T_{ss} = T_{ss} T_u$. Moreover,

(a) there exist $p_u, p_{ss} \in k[t]$ such that $T_u = p_u(T)$ and $T_{ss} = pss(T)$,

(b) $[S,T] = 0$ implies $[S,T_u] = [S,T_s] = 0$,

(c) if $T = T'_u + T'_{ss}$ and $[T'_u, T'_{ss}] = 0$ then $T'_u = T_u$ and $T'_{ss} = T_{ss}$,

(d) if $S : V_1 \to V_2$ intertwines T_1 and T_2 so that $S \circ T_1 = T_2 \circ S$, then $ST_{1,u} = T_{2,u}S$ and $ST_{1,ss} = T_{1,ss}S$,

(e) If T_1 acts on V_1 and T_2 acts on V_2 so that $T_1 \otimes T_2$ acts on $V_1 \otimes V_2$, then $(T_1 \otimes T_2)_u = T_{1,u} \otimes T_{2,u}$ and $(T_1 \otimes T_2)_{ss} = T_{1,ss} \otimes T_{2,ss}$.

Proof. By Jordan decomposition, we know that $V = \bigoplus_\lambda V^{(\lambda)}$ so that $(T - \lambda\,\mathrm{id})|_{V^{(\lambda)}}$ are nilpotent. Then we can set $T_{ss}|_{V^{(\lambda)}} = \lambda\,\mathrm{id}$. Similarly, we can define T_u and T_{ss}.

(a) Let π_λ be the projector of V onto $V^{(\lambda)}$. By the Chinese remainder theorem, there exists a p_λ such that $\pi_\lambda = p_\lambda(T)$. Then we can set $\sum_\lambda \lambda p_\lambda(T)$. Then the others can be deduced. □

The theorem is false if the field is perfect. Suppose $\mathrm{char}(k) = p$ and not perfect, so that there exists an element $a \in k$ that does not admit a pth root. Consider the vector space

$$V = k' = k[t]/(t^p - a),$$

with T multiplication by t. If we tensor with the algebraic closure,

$$V \otimes_k \overline{k} = \overline{k}[t]/(t^p - a) = \overline{k}[t]/(t - b)^p.$$

In this case, $T_n = t - b$ and $T_{ss} = b$, and this cannot be done in k.

Let G be reduced and $g \in G(k)$.

Definition 5.5. g is **semi-simple/unipotent** if the following equivalent conditions hold:

(1) g acts on O_G unipotently on the left (this is on finite-dimensional subspaces)

(2) g acts on O_G unipotently on the right

(3) g acts unipotently on any finite-dimensional representation.

Lemma 5.6. *If $g \in G$ is unipotent/semi-simple and $\gamma : G \to G'$ is a homomorphism, then $\varphi(g)$ is unipotent/semi-simple.*

Proof. Given any G'-action on V, we can consider it as a G-action. □

Theorem 5.7. *For every g there exists a $g = g_u \cdot g_{ss}$ with $g_{ss}g_u = g_u g_{ss}$ with g_u unipotent and g_{ss} semi-simple. Furthermore,*

(a) the decomposition survives homomorphisms $G \to G'$,

(b) the decomposition is unique.

Proof. This can be done using Tanaka, but we can do this in an elementary way. Consider the decomposition of right multiplication $r(g) = r(g)_u r(g)_{ss}$ in $\mathrm{End}(O_G)$. You can do this by the locally finite action of G on O_G. If we look at O_G, we have

$$O_G \otimes O_G \to O_g,$$

and the action on the left with $r_g \otimes r_g$ is the action on the right with r_g. That is, we have

$$(r(g) \otimes r(g))_u = r(g)_u \otimes r(g)_u.$$

But these are also algebra homomorphisms. But to show that these actually come from right multiplication, it suffices to show that they commute with left multiplication. This can be checked. □

Definition 5.8. A group is called **unipotent** if every element in it is unipotent.

Example 5.9. $\mathbb{G}_a$ is unipotent. Take an element $a \in \mathbb{G}_a$, and let's see how it acts on the regular representation $k[t]$. It acts as $t \mapsto t - a$, and on the span of $\{1, \dots, t^n\}$, it is an upper triangular matrix.

Theorem 5.10. *Take $G = \mathrm{GL}_n$. A point g is semi-simple/unipotent in this algebraic group sense if and only if it is semi-simple/unipotent as an automorphism of k^n.*

We'll prove this later.

Lemma 5.11. *If G is unipotent, and G' is a subgroup of G, then G' is unipotent.*

Proof. For $g \in G' \subseteq G$, we get $O_G \twoheadrightarrow O_{G'}$ and it preserves ideals by actions of g. □

Theorem 5.12. *Let $g \in G(k)$ be of finite order. Then it is finite order. (As long as the order is coprime to the characteristic of k.)*

Proof. Assume $T^m = \mathrm{id}$. Then $t^m - 1$ has distinct roots. □

For example, the subgroup $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{G}_a$ is semi-simple if and only if characteristic of k is not p.

6 February 13, 2018

Let G be an algebraic group.

Definition 6.1. A $g \in G(k)$ is **semi-simple/unipotent** if its action on any finite-dimensional representation of G is semi-simple/unipotent. This is equivalent to its action on O_G being semi-simple/unipotent.

We showed that any homomorphism $\phi : G_1 \to G_2$ sends semi-simple/unipotent elements to semi-simple/unipotent elements.

Example 6.2. Any element in $\mathbb{G}_m$ is semi-simple. Any element in $\mathbb{G}_a$ is unipotent.

Theorem 6.3. *An element $g \in \mathrm{GL}_n(k)$ is semi-simple/unipotent if and only if its action on k^n is semi-simple/unipotent.*

Proof. Assume that g acts semi-simple/unipotently on $k^n = V$. If g is semi-simple, we can write

$$V = \bigoplus_i V^i, \quad g|_{V^i} = \lambda_i.$$

Then we can consider $g \in T(k)$ where $T \hookrightarrow \mathrm{GL}_n$ is the torus $\prod \mathbb{G}_m$. Now assume that g is nilpotent in $\mathrm{GL}_n(k)$. We want to define $\varphi : \mathbb{G}_a \to \mathrm{GL}_n$ with $g = \varphi(1)$. But defining φ is the same as defining a representation of $\mathbb{G}_a$ on k^n, and this is the same as the datum of a unipotent operator on k^n. □

There was a fire alarm. But we moved to the next building.

6.1 Representations of unipotent groups

Proposition 6.4. *(i) If G is a finite group and $(|G|, p) = 1$, then any element of $G(k)$ is semi-simple.*

(ii) If $|G| = p^k$, then any element of $G(k)$ is unipotent.

Theorem 6.5 (Jordan decomposition)**.** *For every $g \in G(k)$, there exists a unique decomposition $g = g_s g_u$ where g_s, g_u are semi-simple and unipotent, and $g_s g_u = g_u g_s$.*

Theorem 6.6. *Let G be unipotent and let V be an irreducible representation. Then V is trivial.*

If G acts on V, then $G(k)$ acts on V. If V_1 is a subrepresentation in the first sense, then V_1 is a subrepresentation in the second sense. But the other direction is true in the reduced case as well.

Lemma 6.7. *If $V_1 \subseteq V$ is preserved by $G(k)$, then it is a subrepresentation.*

Proof. We know that $G(k)$ is Zariski dense in G. So

$$G \times V_1 \to G \times V \to G \times V/V_1$$

□ todo

Let V be irreducible as a representation of $G(k)$.

Theorem 6.8 (Burnside)**.** *Let T_α be a collection of endomorphisms of V such that there does not exist V_1, $0 \subsetneq V_1 \subsetneq V$ such that $T_\alpha|_{V_1} \subseteq V_1$. Then the span of T_α and their compositions is* $\mathrm{End}(V)$*.*

So for instance, $G(k)$ span $\mathrm{End}(V)$.

Corollary 6.9. *Let G be unipotent and let V be a finite-dimensional representation. Then there exists a flag*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = V$$

such that V_i are subrepresentations such that V_i/V_{i-1} is trivial.

Proof. By induction. $\square$

Corollary 6.10. *Let $\varphi : G \to \mathrm{GL}_n$ be a homomorphism. Then there exists an element $g \in \mathrm{GL}_n(k)$ such that conjugation by g gives*

$$G \to U(n).$$

Proof. g is what takes the standard flag to the flag of the previous corollary. $\square$

Corollary 6.11. *G has a filtration by normal group subschemes*

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

such that G_i/G_{i-1} is a subgroup of $\mathbb{G}_a$.

This is not very informative in characteristic p. Let G' be the Zariski closure of the group generated by $G \times G \to G$ with $(g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$.

Corollary 6.12. *If G is unipotent, then $G' \subsetneq G$.*

Proof. This follows from the corresponding property of $U(n)$. $\square$

Suppose unipotent G acts on an affine variety X. We have a notion of the orbit $Gx \subseteq \overline{Gx} \subseteq X$.

Proposition 6.13. *Let G be unipotent. Then Gx is closed.*

Example 6.14. Consider $\mathbb{G}_a$ acting on $\mathbb{P}^1$. Then the orbit of 0 is $\mathbb{A}^1$, which is not closed.

Proof. Let $I \subseteq O_{\overline{Gx}}$ be the ideal of $\overline{Gx} - Gx$. Then I is a representation of G, and every representation has a nonzero $0 \neq f \in I^G$. Then $f|_{Gx}$ is constant, and because Gx is dense in $\overline{Gx}$, we have $f|_{\overline{Gx}}$ is constant. But $f|_{\overline{Gx}-Gx} = 0$, so $f = 0$. $\square$

6.2 Groups of dimension 1

Theorem 6.15. *Let G be a connected algebraic group of dimension 1. Then either $G = \mathbb{G}_a$ or $G = \mathbb{G}_m$.*

Proof. Because it is smooth, there exists a canonical curve $\overline{G}$ such that G is an open subset. The claim I want to make is that the action of G on itself by left translations extends to an action of G on $\overline{G}$.

For now let's just use it to prove the theorem. Consider $S = \overline{G} - G$. This consists of finitely many points and nonempty because G is not projective. $G(k)$ acts on $\overline{G}$ because it is functorial in the naïve sense, and it preserves S because G is connected.

Now I claim that if a proper curve X admits an infinite group of automorphisms that preserve a nonempty subset $S \subseteq X$, then $X = \mathbb{P}^1$. The only other case possible is an elliptic curve, but this is not possible because the only automorphisms are translations up to a finite group.

Now we show that G is commutative. We'll get back to this later. $\square$

Theorem 6.16. *Let G be a commutative algebraic group. Then $G \to G_u \times G_s$ where G_u is unipotent and all elements in G_s are semi-simple.*

7 February 15, 2018

Theorem 7.1. *If G is unipotent and V is an irreducible representation, then $V \cong k$.*

Corollary 7.2. *If G is unipotent, every representation admits a flag such that the action of G on V_i/V_{i-1} is trivial.*

Corollary 7.3. *If G is unipotent, it embeds into $U(n)$.*

Corollary 7.4. *If G is unipotent, there exist $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n$ such that G_i is reduced and $G_i' \subseteq G_{i-1}$.*

Proof. Take $G_i = (G \cap U(n)_i)_{\mathrm{red}}$. □

Theorem 7.5. *If G is connected and reduced of dimension 1, then $G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.*

Proof. We first prove that any G as above is commutative. Let $\varphi_g = \mathrm{Ad}_g$ for $g \in G(k)$. We want to show that $\varphi_g = \mathrm{id}$. We will do this later.

So G is commutative. We want to show that we can write $G = G_u \times G_s$ where G_u is unipotent and all elements of G_s are semi-simple. First note that the set of unipotent elements is Zariski-closed because on every representation V, it is $\{T \in \mathrm{End}(V) : (T - \mathrm{id})^{\dim V} = 0\}$. Also, if g_1, g_2 are unipotent and commute, then $g_1 g_2$ is unipotent. So G_u is a closed subgroup of G whose k-points are exactly the unipotent elements.

Likewise, we have $G \hookrightarrow \mathrm{GL}(V)$ and because G commutes, we can diagonalize all semi-simple elements of $G(k)$ so that

$$V \cong \bigoplus_i V_i.$$

Then we can take $G_s = G \cap T$ where T is the torus that acts on each V_i. Then every semi-simple element of G will belong to G_s.

We naturally have a map $G_s \times G_u \to G$. We will construct a map $\varphi : G \to G_s$ that at the level of k-points exacts the semi-simple part of the Jordan decomposition. To construct this, we note that with the diagonalization $V = \bigoplus_i V_i$, we have

$$G \to B_n \twoheadrightarrow T_n$$

where B_n is the upper-triangular matrices and $B_n \to T_n$ is the projection. At the level of k-points, this map extracts the semi-simple part, and also it lands in G_s.

So $G = G_s \times G_u$, and so either $G_u = G$ or $G_s = G$. □

7.1 Commutative groups with semi-simple points

Theorem 7.6. *Let G be reduced. The following are equivalent:*

(1) G is commutative and all of points are semi-simple.

(2) *Every representation splits as a direct sum of* 1*-dimensional representations.*

(3) *There exists an embedding* $G \hookrightarrow T_n$.

(4) *Characters span* O_G.

(5) *There exists a finite abelian group* Λ *such that* $(|\Lambda^{\text{tors}}|, \operatorname{char} k) = 1$ *and* $O_G \cong k[\Lambda]$.

Definition 7.7. A **character** is a homomorphism $G \to \mathbb{G}_m$, which is the same as $k[t, t^{-1}] \to O_G$, which is the same as an element $f \in O_G^\times$ such that $\Delta(f) = f \otimes f$.

Here is an application.

Corollary 7.8. *If* G *is a commutative connected group of dimension* 1 *all of whose points are semi-simple, then* $G \cong \mathbb{G}_m$.

Proof. Let us write $G = \operatorname{Spec} k[\Lambda]$. Then $\Lambda = \Lambda^{\text{tors}} \oplus \mathbb{Z}^n$ and so $G = \operatorname{Spec} k[\Lambda^{\text{tors}}] \times (\mathbb{G}_m)^{\times n}$, and the reducedness, connectivity and dimension conditions force $G = \mathbb{G}_m$. □

Now let us prove the theorem.

Proof. (1) ⇒ (2) is just simultaneous diagonalization. (2) ⇒ (3) follows from the embedding $G \hookrightarrow \mathrm{GL}(V)$. For (3) ⇒ (4), We have $O_{T_n} \twoheadrightarrow O_G$, and $O_{T_n} = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. So characters span O_G. For (4) ⇒ (5) we define Λ to be the group formed by the characters of G. Then we have a surjection

$$O_G \twoheadleftarrow k[\Lambda]$$

of Hopf algebras. We need to show that the characters are linearly independent in O_G. Suppose $\sum_i a_i f_i = f_1$ with minimal cardinality. Then applying comultiplication gives

$$\sum_i a_i f_i \otimes f_i = f_1 \otimes f_1 = \sum_{i,j} a_i a_j f_i \otimes f_j.$$

Because f_i are linearly independent, $a_i a_j = 0$ and $a_i^2 = a_i$. For (5) ⇒ (1), we just use

$$G = \operatorname{Spec} k[\Lambda^{\text{tors}}] \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m.$$

Then it suffices to show that $\operatorname{Spec} k[\Lambda^{\text{tors}}]$ have semi-simple points, but this is something we have seen. It is going to be a finite product of μ_n. □

Now let us show that if G is a unipotent group of dimension 1 then $G = \mathbb{G}_a$. We showed that

$$G = X \hookrightarrow \overline{X} \cong \mathbb{P}^1.$$

Also, $\overline{X} - X = S \neq \emptyset$ is finite and nonempty.

Proposition 7.9. *(1)* $\mathrm{PGL}_2 \cong \mathrm{Aut}(\mathbb{P}^1)$*, i.e.,* $\mathrm{Hom}(S, \mathrm{PGL}_2)$ *is isomorphic to* $\mathrm{Aut}(S \times \mathbb{P}^1/S)$.

(2) The action of G on X by left multiplication extends to an action of G on $\overline{X}$.

If we have the proposition, we obtain a map $G \to \mathrm{PGL}(2)$. Then $G \to \mathrm{Perm}(S)$, and because G is connected, G preserves elements of S. Choose one element S, and call it ∞. The subgroup of PGL_2 that preserves ∞ is $\mathbb{G}_m \ltimes \mathbb{G}_a$:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_a \arrow[r] & \mathbb{G}_a \ltimes \mathbb{G}_a \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
 & & G \arrow[u] & &
\end{tikzcd}$$

But the composite $G \to \mathbb{G}_m$ is trivial, because G is unipotent and the only unipotent element of $\mathbb{G}_m$ is 1. (We will prove that $G \to \mathrm{PGL}_2$ is a closed embedding.) So we get $G \hookrightarrow \mathbb{G}_a$, and because both are 1-dimensional, $G \to \mathbb{G}_a$ is an isomorphism.

Now let us show that $G \to \mathrm{PGL}_2$ is a closed embedding. We know that $\varphi : G_1 \to G_2$ being a homomorphism with finite kernel implies that φ is finite. A finite map is a closed embedding if it is injective on S-points. So it is enough to show that $G \to \mathrm{PGL}_2$ is injective at the level of S-points, and this is because any nontrivial automorphism $S \times X/S$ induces a nontrivial automorphism $S \times \overline{X}/S$.

To prove this proposition, we are going to use the following general fact. For X a proper scheme and Y a quasi-projective scheme, we want to have

$$\mathrm{Hom}(S, \mathrm{Map}(X, Y)) = \mathrm{Hom}(S \times X, Y).$$

Theorem 7.10. $\mathrm{Map}(X, Y)$ *is representable.*

Proof. If $Y_1 \hookrightarrow Y_2$ is a closed embedding, $\mathrm{Map}(X, Y_1) \to \mathrm{Map}(X, Y_2)$ is a closed embedding, and likewise for open embeddings. So it suffices to show this for $Y = \mathbb{P}(V)$.

We have $S \times X \to \mathbb{P}(Y)$ corresponding to a line bundle $\mathscr{L}$ on $S \times X$ such that $\mathscr{L} \hookrightarrow V \otimes O_{S\times X}$ with flat quotient. This means $V^* \otimes O_{S\times X} \twoheadrightarrow \mathscr{L}^{-1}$.

Recall that if X is proper and $\mathscr{E}$ is a coherent sheaf on X, then the functor that sends S to the to the set of all quotient coherent sheaves

$$\mathscr{E} \otimes O_S \twoheadrightarrow \mathscr{F}$$

on $X \times S$ that are S-flat is representable. This representing scheme is called the **Hilbert scheme** of $\mathscr{E}$. Now take $\mathscr{E} = V^* \otimes O_S$. In our functor, $\mathscr{F}$ has to be a line bundle on $X \times S$, and this is an open condition. So it is representable. □

We are actually interested in $\mathrm{Aut}(\mathbb{P}^1)$, not $E = \mathrm{Map}(\mathbb{P}^1, \mathbb{P}^1)$. Then we have to look at the image of the inverse image of the identity of $E \times E \to E$.

8 February 20, 2018

We are still trying to prove that if G is connected, reduced, 1-dimensional, then $G = \mathbb{G}_m$ or $G = \mathbb{G}_a$.

Theorem 8.1. *Such G is commutative.*

Proof. If there exists a $g \in G(k)$ of infinite order, then G is commutative. This is because $\mathrm{Ad}_g = \mathrm{id}$ coincides on g^n. Then g is in the center, and also g^k is in the center, and this is also Zariski dense.

Now let us assume that there is no element of infinite order after any field extension. Now I claim that the order is bounded, i.e., $g^N = 1$ for all $g \in G(k)$. To show this, let us consider the map $G \to G$ given by $g \mapsto g^N$, and look at the kernel G^N. If we assume that the order is unbounded, all subschemes G^N are proper. Let K be the field of rational functions on G. Then the generic point η can be considered as a closed point in the base change $O_G \otimes_k \overline{K}$. Now if $Z \subseteq G$ is any proper subvariety, then $\eta \in G(\overline{K})$ is not in $Z(\overline{K})$. So $\eta \notin G^N(\overline{K})$ for all N, and this shows that $\eta \in G(\overline{K})$ is of infinite order. (Here, commutativity can be checked after a base change, because it's a variety.)

Now take a embedding $\phi : G \hookrightarrow \mathrm{GL}(n)$. We know that $\phi(g)^N = \mathrm{id}$. Therefore the characteristic polynomial of $\phi(g)$ divides $(1 - t^N)^n$. Therefore there exist finitely many options for the characteristic polynomial $\mathrm{ch}(\phi(g))$. Now we can consider the characteristic polynomial as

$$G \xrightarrow{\phi} \mathrm{GL}(n) \xrightarrow{\mathrm{ch}} \mathbb{A}^n.$$

But we know that the image lies in finitely many points, and G is connected. So it should all map to 1. Now matrices that have the characteristic polynomial $(x-1)^n$ is unipotent by definition. So $\phi(G)$ consists of unipotent elements, and thus G is unipotent. This shows that $G' \subsetneq G$, and also G' is connected. So $G' = \{e\}$ and G is commutative. $\square$

8.1 Continuous family of automorphisms

Then we showed that if G is a commutative algebra group then $G = G_s \times G_u$ where G_s have all elements semi-simple and G_u is unipotent. Then we had the following theorem.

Theorem 8.2. *The following are equivalent:*

(1) G is commutative and the elements are semi-simple.

(2) Every representation splits as a direct sum of 1-dimensional representations.

(3) $G \subseteq \mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

(4) O_G is spanned by characters.

(5) $G = \operatorname{Spec} k[\Lambda]$ where Λ is a finitely generated abelian group whose torsion is coprime to p.

Now it can be shown that $\operatorname{Spec} k[\Lambda_1] \to \operatorname{Spec} k[\Lambda_2]$ correspond to maps $\Lambda_2 \to \Lambda_1$. (We showed this for Λ torsion-free, and the torsion case can be proved in the same way.) This is very different from the unipotent case. There is a continuous family of automorphisms $\mathbb{G}_a \to \mathbb{G}_a$, for instance, given by

$$\mathbb{G}_m \times \mathbb{G}_a \to \mathbb{G}_a.$$

This is actually the universal example.

Proposition 8.3. *Let $\phi : S \times G_1 \to G_2$ be a morphism where G_1 and G_2 are diagonalizable. If S is connected, then ϕ factors through $S \times G_1 \to G_1 \to G_2$.*

Proof. It suffices to show that $G_2 = \mathbb{G}_m$, because G_2 embeds into some torus. Let us denote $G_1 = G$. Then we have a map

$$S \times G \to \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$$

where t maps to $\sum_i f_i \otimes \chi_i$, where χ_i are characters on G. The condition that this is a group homomorphism is

$$\left(\sum_i f_i \otimes \chi_i\right) \otimes_{O_S} \left(\sum_i f_j \otimes \chi_j\right) = \sum_k f_k \otimes \chi_k \otimes \chi_k.$$

Then

$$\sum_{i,j} f_i f_j \otimes \chi_i \otimes \chi_j = \sum_k f_k \otimes \chi_k \otimes \chi_k.$$

This shows that only f_i is 1 and all other f_i are 0, because S connected means that there are only two idempotents 0 and 1. $\square$

Now comes the fun part. We looked at $\overline{G} = \mathbb{P}^1$. Then $\mathbb{P}^1 - G = S$ is a finite set. We are assuming that G is unipotent and trying to show that $G \cong \mathbb{G}_a$.

Theorem 8.4. *(1) There is an action $\mathrm{PGL}_2 \to \operatorname{Aut}(\mathbb{P}^1)$.*

(2) There exist a homomorphism $G \to \operatorname{Aut}(\mathbb{P}^1, S)$ such that the action restricts to the left multiplication on G.

Here, if X and Y are algebraic varieties, then we can define a prestack

$$\operatorname{Hom}(S, \operatorname{Map}(X, Y)) = \operatorname{Hom}(S \times X, Y).$$

Theorem 8.5. *If Y is quasi-projective and X is proper, then $\operatorname{Map}(X, Y)$ is representable.*

Before proving Theorem 8.4, we finish the proof of $G = \mathbb{G}_a$. We may assume that $\infty \in S$, and then we get

$$G \to \operatorname{Aut}(\mathbb{P}^1, \times) \cong \mathbb{G}_m \ltimes \mathbb{G}_a.$$

Because G is unipotent, $G \to \mathbb{G}_m \ltimes \mathbb{G}_a \to \mathbb{G}_m$ is trivial. So G maps to $\mathbb{G}_a$. If we can show that this is a closed embedding, we are done.

Lemma 8.6. *A homomorphism of algebra groups is a closed embedding if it is a categorical injection, i.e., injective S-points for all S.*

Proof. We can factor $G_1 \to \overline{\phi(G_2)} \hookrightarrow G_2$. Now it suffices to show that $G_1 \to \overline{\phi(G_2)}$ is a closed embedding. But it is a general fact that a finite morphism of schemes that is a categorical embedding is a closed embedding. So it suffices to show that $G_1 \to \overline{\phi(G_1)}$ is finite. But we know that it is quasi-finite because its kernel is a point, and then a quasi-finite morphism of algebraic group is finite. $\square$

Now let us show that $G \to \mathrm{Aut}(\mathbb{P}^1, S)$ is a categorical injection. This is actually a tautology because if something acts trivially on $\mathrm{Aut}(\mathbb{P}^1, S)$, then it acts trivially on G.

8.2 Automorphisms of projective space

We have this group PGL_n, and this has the universal property

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}(V) \arrow[r] \arrow[dr] & \mathrm{PGL}(V) \arrow[r] \arrow[d, dashed] & 1 \\ & & & H. & \end{tikzcd}$$

So we have a map

$$\mathrm{PGL}(V) \to \mathrm{Aut}(\mathbb{P}(V))$$

and want to show that this is an isomorphism.

But first let us look at $\mathrm{Maps}(\mathbb{P}(V_1), \mathbb{P}(V_2))$. The k-points of this should be polynomial functions $\mathrm{poly}^n(V_1, V_2) \cong \mathrm{Sym}^n(V_1^*) \otimes V_2$, but with some nonvanishing condition.

Proposition 8.7. $\mathrm{Maps}(\mathbb{P}(V_1), \mathbb{P}(V_2)) \cong \coprod_{n \geq 0} \mathbb{P}(\mathrm{poly}^n(V_1, V_2))^\circ$.

9 February 22, 2018

We were trying to compute $\mathrm{Maps}(\mathbb{P}(V_1), \mathbb{P}(V_2))$.

Proposition 9.1. $\coprod_{n\geq 0} \mathbb{P}(\mathrm{Sym}(V_1^*) \otimes V_2)^\circ \cong \mathrm{Maps}(\mathbb{P}(V_1), \mathbb{P}(V_2))$.

Proof. We use the functor of points. A map $S \times \mathbb{P}(V_1) \to \mathbb{P}(V_2)$. Recall that a map $Y \to \mathbb{P}(V)$ is the same as a line bundle $\mathscr{L}$ on Y with an embedding $\mathscr{L} \hookrightarrow V \otimes \mathscr{O}_Y$, with $V^* \otimes \mathscr{O}_Y \to \mathscr{L}^{\otimes -1}$ surjective.

So we take $Y = S \times \mathbb{P}(V_1)$. Now it is a fact in algebraic geometry that every line bundle on $S \times \mathbb{P}(V)$ is of the form

$$\mathscr{L}_S^{\otimes -1} \boxtimes \mathscr{O}(n)$$

for some $\mathscr{L}_S^{\otimes -1}$ on S. So the data is

$$V_2^* \otimes \mathscr{O}_{S \times \mathbb{P}(V_1)} \to \mathscr{L}_S^{\otimes -1} \boxtimes \mathscr{O}(n),$$

which is the same as

$$\pi^*(\mathscr{L}_S \otimes V_2^*) \to O_S \boxtimes \mathscr{O}(n),$$

where $\pi : S \times \mathbb{P}(V) \to \mathbb{P}(V)$. Then this is the same as

$$\mathscr{L}_S \otimes V_2^* \to O_S \otimes \mathrm{Sym}^n(V_1^*),$$

which is just a $\mathscr{L}_S \to \mathrm{Sym}^n(V_1^*) \otimes V_2$. This is the same as a map $S \to \mathbb{P}(\mathrm{Sym}^n(V_1^* \otimes V_2))$. Now surjectivity will give us the condition that the image is nondegenerate. $\square$

Corollary 9.2. $\mathrm{Isom}(\mathbb{P}(V), \mathbb{P}(V)) = \mathbb{P}(\mathrm{Hom}(V, V))^\circ = \mathrm{PGL}(V)$.

So let us finish this proof that we have been working on for two weeks. For $S \subseteq \mathbb{P}^1$ a finite set, we have

$$\mathrm{Aut}(\mathbb{P}^1, S) = \mathrm{Aut}(\mathbb{P}^1) \times_{\mathrm{Map}(S,\mathbb{P}^1)} \mathrm{Map}(S, S).$$

Theorem 9.3. *The action of G on itself by left multiplication extends to a map $G \to \mathrm{Aut}(\mathbb{P}^1, S)$.*

9.1 Ind-schemes

Definition 9.4. An **ind-scheme** is a contravariant functor on the category of affine schemes that can be written as

$$\mathrm{colim}_n\, Y_n$$

where $Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \cdots$ is a closed embedding. (Or you can also allow a filtered indexing set.) Then $\mathrm{Hom}(S, Y) = \bigcup_n \mathrm{Hom}(S, Y_n)$.

Example 9.5. Consider $\mathbb{A}^\infty$ that can be written $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \hookrightarrow \cdots$. This is different from $\mathbb{A}^{\infty,\mathrm{pro}} = \operatorname{Spec} k[t_1, \ldots] = \varprojlim \mathbb{A}^n$.

There are two senses in which we can take colimit. First, we can take

$$\operatorname{Hom}(S, \text{“}\operatorname{colim}\text{”} Y_i) = \operatorname{colim} \operatorname{Hom}(S, Y_i).$$

The other thing we can do is to take the colimit $\operatorname{colim} Y_n$ in the category of affine schemes. If $Y_i = \operatorname{Spec} A_i$, then we have

$$\operatorname{colim} Y_i = \operatorname{Spec}(\varprojlim_i A_i).$$

For instance, if $A_i = \operatorname{Sym}(V_i^*)$, then

$$\operatorname{Hom}(S, \operatorname{colim} \underline{V}_i) = \operatorname{Hom}(\varinjlim(\operatorname{Sym}(V_1^*)), O_S)$$

and the set $\operatorname{Hom}(S, \text{“}\operatorname{colim}\text{”} V_i)$ are only those that are continuous.

Example 9.6. Consider the sequence

$$\operatorname{Spec} k[t]/t \to \operatorname{Spec} k[t]/t^2 \to \cdots$$

and there is a notion of

$$\operatorname{Spf} k[[t]] = \text{“}\operatorname{colim}\text{”} \operatorname{Spec} k[t]/t^n,$$

which is an ind-scheme. There is also the scheme

$$\operatorname{Spec} k[[t]] = \operatorname{colim} \operatorname{Spec} k[t]/t^i = \operatorname{Spec} k[[t]].$$

By definition,

$$\begin{aligned}
\operatorname{Hom}(S, \operatorname{Spec} k[[t]]) &= \operatorname{Hom}(k[[t]], O_S), \\
\operatorname{Hom}(S, \operatorname{Spf} k[[t]]) &= \operatorname{colim}_n (k[t]/t^n, O_S) = (\text{nilpotents in } O_S).
\end{aligned}$$

Definition 9.7. Let X be an affine scheme and $Y \subseteq X$ a closed subscheme. We define

$$\operatorname{Hom}(S, X_Y^\wedge) = \operatorname{Hom}(S, X) \times_{\operatorname{Hom}(S_{\mathrm{red}}, X)} \operatorname{Hom}(S_{\mathrm{red}}, Y).$$

Example 9.8. We claim that $(\mathbb{A}^1)_0^\wedge = \operatorname{Spf} k[[t]]$. This is because

$$\operatorname{Hom}(S, \mathbb{A}^1) \times_{\operatorname{Hom}(S_{\mathrm{red}}, \mathbb{A}^1)} \operatorname{Hom}(S_{\mathrm{red}}, \operatorname{Spec} k[t]/(t)) = \{f \in O_S : f \text{ is in the nilradical}\}.$$

Similarly, we can prove that

$$\hat{X_Y} = \text{“}\operatorname{colim}_n\text{”} \operatorname{Spec} A/I^n.$$

Proposition 9.9. $\operatorname{Map}(X, Y)$ *is an ind-affine ind-scheme if* Y *is affine. (This just means that the* Y_i *can be taken to be affine.)*

Proof. Let us first take $Y = \mathbb{A}^1$. Here, we claim

$$\mathrm{Map}(X, \mathbb{A}^1) \cong \underline{\Gamma(X, \mathscr{O}_X)}.$$

To show this, we compare functor of points. Then

$$\mathrm{Hom}(S, \mathrm{Map}(X, \mathbb{A}^1)) = \mathrm{Hom}(S \times X, \mathbb{A}^1) = \Gamma(S \times X, \mathscr{O}_{S\times X}) = O_S \otimes \Gamma(X, \mathscr{O}_X).$$

Then $\underline{V}$ is a ind-scheme for a vector space V. For $Y = \mathbb{A}^n$, we can do the same thing. Or we can use the fact that the product of ind-schemes is an ind-scheme. Also, it is clear that $\mathrm{Map}(X, Y_1 \times Y_2) = \mathrm{Map}(X, Y_1) \times \mathrm{Map}(X, Y_2)$.

Now let us look at arbitrary finite type Y. Then Y fits in the fiber product

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{A}^m. \end{array}$$

Now it suffices to show that the fiber product of ind-schemes is an ind-scheme. It can be checked that

$$(\operatorname{colim} Y_i) \times_{(\operatorname{colim} Z_j)} (\operatorname{colim} W_k) = \operatorname{colim}_{i,k} Y_i \times_{Z_j} W_k,$$

where j is supposed to sufficiently large. This is because X mapping into them should come as something into Y_i and into W_k and into Z_j for some large enough indices. By filteredness, we can find a large enough Z_j such that $Y_i \to Z_j$ and $W_k \to Z_j$. □

So what is $\mathrm{Aut}(\mathbb{A}^1)$? We have

$$\mathrm{Hom}(\mathrm{Spec}(k), \mathrm{Aut}(\mathbb{A}^1)) = \mathrm{Isom}(\mathbb{A}^1, \mathbb{A}^1) = (\mathbb{G}_m \ltimes \mathbb{G}_a)(k).$$

We have a map $\mathbb{G}_m \ltimes \mathbb{G}_a \to \mathrm{Aut}(\mathbb{A}^1)$ defined by the action, but is this an isomorphism? A priori, $\mathrm{Aut}(\mathbb{A}^1)$ is only an ind-scheme.

Let us evaluate at R. We are looking at automorphisms $R[t] \to R[t]$. If R is a field, the only ones come from scaling and translation. But if R has nilpotents, like $R = k[\epsilon]$, then

$$R[t] \to R[t]; \quad t \mapsto t + \epsilon t^{100}$$

is an automorphism.

Proposition 9.10. $\mathbb{G}_m \ltimes \mathbb{G}_a \cong (\mathrm{Aut}(\mathbb{A}^1))_{\mathrm{red}}$.

The reduced ind-scheme is just defined as taking "colim"$(Y_n)_{\mathrm{red}}$. This is independent of the choice of the Y_n. Given $X \to$ "colim"Y_i, the closure of the image makes sense because all the connecting maps are closed embeddings. Moreover,

Lemma 9.11. *Let $G \to G_1$ be a homomorphism, where G is a group scheme and G_1 is a group ind-scheme, and φ is injective on S-points, and φ is bijective on k-points. Then φ induces an isomorphism $G \to (G_1)_{\mathrm{red}}$.*

Proof. So consider the closure $G \xrightarrow{\bar{\varphi}} \overline{G} \hookrightarrow G_1$. Then $\bar{\varphi}$ is injective on S-points, so it is an isomorphism. Now I claim that if $X \hookrightarrow Y$ where X is reduced, and $X(k) \cong Y(k)$, then $X \cong Y_{\mathrm{red}}$. We have $X \hookrightarrow Y_i$ for some Y_i, and then $X(k) \cong Y_i(k)$ implies $X \cong (Y_i)_{\mathrm{red}}$ by Nullstellensatz. $\square$

10 February 27, 2018

Last time we define this notion of an ind-scheme. Then we proved that if Y is (finite-type) affine, $\mathrm{Maps}(X, Y)$ is an ind-scheme. Then for X affine, $\mathrm{Aut}(X)$ was a group ind-scheme. We had

$$\mathrm{Aut}(\mathbb{A}^1) \supseteq \mathrm{Aut}(\mathbb{A}^1)_{\mathrm{red}} \cong \mathbb{G}_m \ltimes \mathbb{G}_a.$$

10.1 Comparing reduced parts of automorphism functors

We had a group $G = X$ acting on $\overline{X}$ fixing S.

Theorem 10.1. *We have a homomorphism $G \to \mathrm{Aut}(\overline{X}, S)$.*

We can look at $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}})$ and $\mathrm{Aut}(\overline{X}, S) = \mathrm{Aut}(\overline{X}) \times_{\mathrm{Map}(S,X)} \mathrm{Aut}(S)$. Also, given any functor, we can consider its reduced functor by a left Kan extension. That is, we are taking $F_{\mathrm{red}}(S) = \mathrm{colim}_{S' \text{ rem}} F(S')$ for $S \to S'$. Then I have

$$\begin{tikzcd}
 & \mathrm{Aut}(X) & \mathrm{Aut}(\overline{X}, S_{\mathrm{set}}) \arrow[l] & \mathrm{Aut}(\overline{X}, S) \arrow[l] \\
G \arrow[r] \arrow[ur] & \mathrm{Aut}(X)_{\mathrm{red}} \arrow[u] & \mathrm{Aut}(\overline{X}, S_{\mathrm{set}})_{\mathrm{red}} \arrow[l] \arrow[u] & \mathrm{Aut}(\overline{X}, S)_{\mathrm{red}}. \arrow[l] \arrow[u]
\end{tikzcd}$$

We want to map $G \to \mathrm{Aut}(\overline{X}, S)$. We will do this by showing that the bottom two maps are isomorphisms.

But let me torture you a bit first. I claim that $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}}) \leftarrow \mathrm{Aut}(\overline{X}, S)$ is not an isomorphism. If we look at a T-point of $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}})$, this is a diagram

$$\begin{tikzcd}
T \times \overline{X} \arrow[r, "\phi"] \arrow[d] & \overline{X} \arrow[d] \\
T \times S \arrow[r] & S'
\end{tikzcd}$$

with the set-theoretic image of $T \times S$ being S. This means that the scheme-theoretic image is a nilpotent thickening S' of S. But if we look at a T-point of $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}})_{\mathrm{red}}$, we can just test at T reduced. Then $T \times S$ is reduced, and so $T \times S \to S'$ factors through S. Therefore we the isomorphism $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}})_{\mathrm{red}} \cong \mathrm{Aut}(\overline{X}, S)_{\mathrm{red}}$.

Also, we need to show that $\mathrm{Aut}(X)_{\mathrm{red}} \leftarrow \mathrm{Aut}(\overline{X}, S_{\mathrm{set}})_{\mathrm{red}}$ is an isomorphism.

Lemma 10.2. *Let $G_1 \to G_2$ be a homomorphism where G_1 a group scheme and G_2 a group ind-scheme. If ϕ is injective on T-points and bijective on k-points, then ϕ is an isomorphism.*

todo

Recall that for $X \supseteq Y$, we have defined

$$\mathrm{Hom}(T, X^{\wedge}_{Y}) = \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T_{\mathrm{red}}, X)} \mathrm{Hom}(T_{\mathrm{red}}, Y).$$

For $S \supseteq H$ is a subgroup, then we can also define $G^\wedge_H$ as a group ind-scheme. Take $\overline{X} = \mathbb{P}^1$ and $S = \infty$, we have

$$\mathrm{Aut}(\overline{X}, S_{\mathrm{set}}) \supseteq \mathrm{Aut}(\overline{X}, S) = \mathbb{G}_m \ltimes \mathbb{G}_a.$$

Proposition 10.3. $\mathrm{Aut}(\overline{X}, S_{\mathrm{set}}) = \mathrm{Aut}(\overline{X})^\wedge_{\mathrm{Aut}(\overline{X},S)}$.

10.2 Lie algebras

Definition 10.4. For G an algebraic group, we define its **Lie algebra** as $\mathfrak{g} = T_e G$. The **tangent space** is defined as

$$T_x X = \mathrm{Hom}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X) \times_{\mathrm{Hom}(\mathrm{Spec}\, k, X)} \{x\}.$$

This is going to be a vector space.

Theorem 10.5 (Lie)**.** *$\mathfrak{g}$ has a structure of a Lie algebra.*

I'll use Tannaka's theorem to give a different description.

Theorem 10.6. *An element $\xi \in \mathfrak{g}$ is the same thing as*

- *for every $V \in \mathsf{Rep}(G)$, an element $T_\xi \in \mathrm{End}(V)$ that is*
- *functorial in the sense that for $\phi : V_1 \to V_2$ we have $\phi \circ T_\xi = T_x \circ \phi$ and*
- *T_ξ acts on $V_1 \otimes V_2$ as $T_\xi \otimes \mathrm{id} + \mathrm{id} \otimes T_\xi$.*

Proof. By Tanaka, an element of $\mathfrak{g}$ is the same as, for every V we have automorphisms $\tilde{T}_\epsilon$ on $V \otimes k[\epsilon]/\epsilon^2$ which is the identity modulo ϵ. Then if we write $\tilde{T}_\xi = \mathrm{id} + \epsilon T_\xi$, then $\tilde{T}_\xi|_{V_1 \oplus V_2} = \tilde{T}_\xi|_{V_1} \otimes \tilde{T}_\xi|_{V_2}$ gives the last condition. □

Now let us prove Lie's theorem. For ξ_1 and ξ_2, we can define $[\xi_1, \xi_2]$ as

$$T_{[\xi_1,\xi_2]} = [T_{\xi_1}, T_{\xi_2}].$$

Corollary 10.7. *Every element $\xi \in \mathfrak{g}$ admits a unique Jordan decomposition $\xi = \xi_n + \xi_{ss}$ such that ξ_n acts nilpotently on every representation and ξ_{ss} acts semi-simply on every representation and $[\xi_n, \xi_{ss}] = 0$.*

Proof. You do Jordan decomposition on every T_ξ. □

Corollary 10.8. *$T_e G$ is in bijection with endomorphisms of O_G that are*

- *compatible with the left left action of G on itself by right translations,*
- *and with the map $O_G \otimes O_G \to O_G$.*

So these are vector fields that are invariant under right translations.

Theorem 10.9. *If k has characteristic 0, then every algebraic group is reduced.*

So in characteristic 0, every algebraic group is smooth. This is because every reduced scheme has a nonempty open on which the scheme is smooth, and then we can translate.

Proof. Note that $\mathcal{O}_{X,x}$ is regular if and only if $\hat{\mathcal{O}}_{X,x}$ is regular. So regular is equivalent to $\mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$ being an isomorphism. We are going to construct the map $\hat{\mathcal{O}}_{G,e} \to \hat{\mathcal{O}}_{\mathfrak{g},0}$ given by

$$\mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1} \to \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

The composition is the identity, and this shows that all the maps are isomorphism.

So we want to construct his map. First we will define an exponential map $\mathfrak{g}_0^\wedge \to G_e^\wedge$ and then induce $\mathrm{Spec}\,\hat{\mathcal{O}}_{\mathfrak{g},0} \to \mathrm{Spec}\,\hat{\mathcal{O}}_{G,e}$. Now we need to define

$$\mathrm{Hom}(\mathrm{Spec}\,R, \tilde{\mathfrak{g}}_0^\wedge) \to \mathrm{Hom}(\mathrm{Spec}\,R, G_0^\wedge).$$

The left hand side is $\xi \in R \otimes \mathfrak{g}$ such that $\overline{\xi} = 0$ in the reduced $\overline{R}$ of R. The right hand side is compatible family of automorphisms of $R \otimes V$ that are identity on $\overline{R} \otimes V$. The left hand side are endomorphims of $R \otimes V$ that vanishes on $\overline{R} \otimes V$. Then we can define

$$\tilde{T}_\xi = \exp(T_\xi) = 1 + T_\xi + \frac{T_\xi^2}{2} + \cdots,$$

because this is actually finite. $\square$

In characteristic 0, we can define

$$\hat{\mathcal{O}}_{X,x} = \varprojlim \mathcal{O}_{X,x}/\mathfrak{m}^i.$$

Then we can define **distributions**

$$\mathrm{Dist}_{X,x} = \mathrm{colim}(\mathcal{O}_{X,x}/\mathfrak{m}^i)^*.$$

We will later prove that in characteristic 0, $\mathrm{Dist}_{G,e} \cong \mathcal{U}(\mathfrak{g})$. For characteristic positive, $\mathcal{U}(\mathfrak{g})$ will looks something more like the wrong symmetric power.

11 March 1, 2018

Let G be an algebraic group. We defined

$$\mathfrak{g} = T_e G = \mathrm{Hom}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, G) \times_{\mathrm{Hom}(\mathrm{Spec}\, k, G)} \{e\}.$$

Theorem 11.1. *An element $\xi \in \mathfrak{g}$ is equivalent to the functor taking $V \in \mathsf{Rep}(G)$ to $T_V(\xi) \in \mathrm{End}(V)$ that is compatible with the tensor structure*

$$T_{V_1 \otimes V_2}(\xi) = T_{V_1}(\xi) \otimes \mathrm{id} + \mathrm{id} \otimes T_{V_2}(\xi).$$

There is the regular representation $O_G \otimes O_G \to O_G$, and then we get an endomorphism Lie^l_ξ of O_G. This commutes with right translations, and so Lie^l_ξ is a derivation of O_G. We can think of this as a right-invariant vector field.

Now let me do this same construction with a different perspective. For each $x \in X$, we can take $T_x X = \mathfrak{m}_x/\mathfrak{m}_x^2$, and $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. We can also say $\Omega_{X,x} = T_x^* X$. But we can also define the sheaf

$$\mathscr{T}_X = \mathscr{H}om_X(\Omega_X, \mathcal{O}_X).$$

These are the tangent vectors, informally. Now we can take

$$\mathcal{T}_{X,x} \to T_x X.$$

This is a special case of doing $(\mathscr{F}^\vee)_x \to (\mathscr{F}_x)^\vee$. This is not in general an isomorphism, but is an isomorphism if $\mathscr{F}$ is locally free.

Proposition 11.2. *There exists an canonical isomorphism $\Omega_G \cong O_G \otimes \mathfrak{g}^*$ and $\mathscr{T}_G \simeq O_G \otimes \mathfrak{g}$ such that sections of Ω_G (repsectively $\mathscr{T}_G$) that is right-invariant corresponds to $\mathfrak{g}^* \hookrightarrow O_G \otimes \mathfrak{g}^*$ and $\mathfrak{g} \hookrightarrow O_G \otimes \mathfrak{g}$.*

But we want to refine this.

11.1 Equivariant quasicoherent sheaf

Let G act on Y, and $\mathscr{F} \in \mathsf{QCoh}(Y)$.

Definition 11.3. A structure of **G-equivariance** on $\mathscr{F}$ is, for all S consider $\mathscr{F} \boxtimes O_S \in \mathsf{QCoh}(Y \times S)$ and make $\mathrm{Hom}(S, G)$ act on it, compatibly with pullbacks with respect to $S' \to S$.

This has all functorial properties built in.

- If G acts on Y_1 and on Y_2, and if $Y_1 \to Y_2$ is an equivariant morphism, then pullbacks and pushforwards of equivariant sheafs are equivariant.
- If Y is a point, then $\mathsf{QCoh}(\mathrm{pt})^G = \mathsf{Rep}(G)$.

Most textbooks will define this differently. A structure of an equivariance on $\mathscr{F}$ is an isomorphism

$$\varphi : \mathrm{pr}^*(\mathscr{F}) \to \mathrm{act}^*(\mathscr{F}).$$

Here, $\mathrm{pr}, \mathrm{act} : G \times Y \to Y$ are the projection and the action. This φ should satisfy some associativity condition.

Lemma 11.4. *Let G act on Y. Then Ω_Y has a canonical structure of a G-equivariance.*

Proof. For any $g \in \mathrm{Hom}(S, G)$, this defines $Y \times S \to Y \times S$. Then it immediately gives an automorphism of $\Omega_Y \boxtimes O_S = \Omega_{Y \times S/S}$. □

Let $Y = G$, with an action by G by left translations.

Lemma 11.5. *There exists an equivalence of categories* $\mathsf{QCoh}(G)^G \simeq \mathsf{Vect}$.

Proof. You take the vector space V and take this to $V \otimes O_G$. This is sending $\mathsf{Vect} \to \mathsf{QCoh}(\mathrm{pt})^G \to \mathsf{QCoh}(G)^G$. The functor in the opposite direction is going to be $\mathscr{F} \mapsto \mathscr{F}_e$. □

Now this immediately shows that $\Omega_G \cong \mathfrak{g}^* \otimes O_G$.

11.2 Distributions

For a point $x \in X$ of a scheme, we defines the distributions as

$$\mathrm{Dist}(X)_x = \text{topological dual } (\hat{\mathscr{O}}_{X,x})^*.$$

Her is another definition. Take $X_i \subseteq X$ be points so that $X_{i,\mathrm{red}} = \{x\}$ by defining $X_i = \mathrm{Spec}(\mathscr{O}_x/\mathfrak{m}_x^i)$. Then we have the ind-scheme

$$X_x^\wedge = \mathrm{colim}_i\, X_i.$$

There, we have

$$\mathrm{Dist}(X)_x = \mathrm{colim}_i (\mathscr{O}_{X_i})^*.$$

This is a vector space, and it is going to be a cocommutative coalgebra. If you know about differential operators, there is a sheaf $\mathscr{D}(X)$ such that $\mathscr{D}(X)_x = \mathrm{Dist}(X)_x$.

For example, let us take $X = \underline{V}$ and $x = 0$. Then we have

$$O_X = \mathrm{Sym}(V^*), \quad O_{X_i} = \mathrm{Sym}^{\leq i}(V^*).$$

Now consider

$$(\mathrm{Sym}^n(V))^* = \widetilde{\mathrm{Sym}}^n(V^*).$$

This normally, $\mathrm{Sym}^n(V)$ is a quotient of $V^{\otimes n}$, and so $\widetilde{\mathrm{Sym}}^n(V)$ is going to be a subspace of $V^{\otimes n}$. If characteristic 0, we are going to get the invariants, but for other characteristic, we have to be a bit careful. $\widetilde{\mathrm{Sym}}^2(V)$ is going to be the vector space spanned by $v \otimes v$. Then we have

$$\widetilde{\mathrm{Sym}^n}(V) = (\widetilde{\mathrm{Sym}^2}(V) \otimes V^{\otimes(n-2)}) \cap (V \otimes \widetilde{\mathrm{Sym}}^2(V) \otimes V^{\otimes(n-3)}) \cap \cdots.$$

Then we can write

$$\mathrm{Dist}(\underline{V})_0 = \widetilde{\mathrm{Sym}}(V).$$

Note that if $f : X \to Y$ is a map sending $x \in X$ to $y \in Y$, then we have a map in the other direction of rings, and so on duals we get

$$\mathrm{Dist}(X)_x \to \mathrm{Dist}(Y)_y.$$

Also, we can check that

$$\mathrm{Dist}(X_1 \times X_2)_{x_1 \times x_2} = \mathrm{Dist}(X_1)_{x_1} \otimes \mathrm{Dist}(X_2)_{x_2}.$$

This shows that we have a map

$$\mathrm{Dist}(G)_e \otimes \mathrm{Dist}(G)_e \to \mathrm{Dist}(G)_e$$

coming from $m : G \times G \to G$. This gives an additional algebra structure, so $\mathrm{Dist}(G)_e$ has a structure of a cocommutative Hopf algebra on $\mathrm{Dist}(G)_e$.

There is something called the **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. This is an algebra $\mathcal{U}(\mathfrak{g})$ such that a map $\mathcal{U}(\mathfrak{g}) \to A$ of algebras is the same as a map $\mathfrak{g} \to A$ of Lie algebras (with the Lie algebra structure given by the commutator.) We can construct this explicitly as

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/(\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 - [\xi_1, \xi_2]).$$

There is a natural filtration given on the tensor algebra, which descends to $\mathcal{U}(\mathfrak{g})$.

Theorem 11.6 (Poincaré–Birkhoff–Witt)**.** *The map* $\mathrm{Sym}(\mathfrak{g}) \to \mathrm{gr}\,\mathfrak{U}(\mathfrak{g})$ *is an isomorphism.*

Note that there is a natural map $\mathfrak{g} \to \mathrm{Dist}(G)_e$. This is because there is generally a map $T_x \to \mathrm{Dist}(X)_x$ given by $\mathscr{O}/\mathfrak{m}^2 \to k$. You can check that this map commutes with the bracket. So we get a map

$$\mathcal{U}(\mathfrak{g}) \to \mathrm{Dist}(G)_e.$$

Consider the case $G = \underline{V}$. Then we have

$$\begin{tikzcd}
U(V) \arrow[r] \arrow[d, equal] & \mathrm{Dist}(\underline{V})_0 \arrow[d, equal] \\
\mathrm{Sym}(V) \arrow[r] & \widetilde{\mathrm{Sym}}(V) \\
T(V) \arrow[u] \arrow[ur, dashed] &
\end{tikzcd}$$

Now this map is defined as

$$T^n(V) \to \widetilde{\mathrm{Sym}}^n(V); \quad v_1 \otimes \cdots \otimes v_n \to \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

This is not necessarily an isomorphism if characteristic is not zero. In particular, for $V = k$ we get

$$k[t] \to k[t]; \quad t^n \to n! t^n.$$

Theorem 11.7. *The map* $\mathcal{U}(\mathfrak{g}) \to \mathrm{Dist}(G)_e$ *is an isomorphism in characteristic* 0.

This will also prove the Poincaré–Birkhoff–Witt theorem.

Proof. It is enough to show that $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \to \mathrm{gr}\,\mathrm{Dist}(G)_e$ is an isomorphism. But we have

$$\begin{tikzcd} \mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \arrow[r] & \mathrm{gr}\,\mathrm{Dist}(G)_e \arrow[d, equal] \\ \mathrm{Sym}(\mathfrak{g}) \arrow[u, two heads] \arrow[r, "\cong"] & \widetilde{\mathrm{Sym}}(T_e G) \end{tikzcd}$$

and so we should have and isomorphism. $\square$

Note that we have an exponential map $\mathfrak{g}_0^\wedge \to G_e^\wedge$, and this induces a map

$$\mathrm{Dist}(\mathfrak{g})_0 \to \mathrm{Dist}(G)_e$$

of only cocommutative coalgebras. We can interpret this as

$$\begin{tikzcd} \mathrm{Dist}(\mathfrak{g})_0 \arrow[r, "\exp"] \arrow[d, equal] & \mathrm{Dist}(G)_e \arrow[d, equal] \\ \widetilde{\mathrm{Sym}}(\mathfrak{g}) \arrow[r] \arrow[d, hook] & \mathcal{U}(\mathfrak{g}) \\ T(\mathfrak{g}) \arrow[ur] & \end{tikzcd}$$

where $T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is dividing by $n!$. This is what Baker–Campbell–Hausdorff really is.

12 March 6, 2018

For a Lie algebra $\mathfrak{g}$, we have the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We also have for each point $x \in X$ the distributions $\mathrm{Dist}(X)_x$, and $\mathrm{Dist}(G)_e$ becomes a cocommutative Hopf algebra by $T_x X \to \mathrm{Dist}(X)_x$. Then we had a map

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[r] & \mathrm{Dist}(G)_e \\ \mathfrak{g} \arrow[u] \arrow[r] & T_e G. \arrow[u] \end{tikzcd}$$

Theorem 12.1. *In characteristic* 0*, the map* $\mathcal{U}(\mathfrak{g}) \to \mathrm{Dist}(G)_e$ *is an isomorphism.*

Corollary 12.2. $\widehat{\mathscr{O}}_{G,e} = (U(\mathfrak{g}))^*$ *and* $G_e^\wedge = \mathrm{Spf}(\mathcal{U}(\mathfrak{g}))^*$.

Today I want to say a few words about what happens in characteristic p.

12.1 Lie algebra over characteristic p

There is a map $\mathfrak{g} \to \mathfrak{g}$ written as $x \to x^{[p]}$ defined in the following way. Any ξ on X is a differential operator of order 1. Then ξ^n is also a differential operator of order 1 because

$$\xi^p(fg) = \xi^{p-1}(\xi(f,g)) = \xi^p(f)g + p(\cdots) + f\xi^p(g).$$

Let's look at some examples.

Example 12.3. Take $G = \mathbb{G}_a$. Then $\mathfrak{g} = \mathrm{span}(\frac{\partial}{\partial t})$. Then we have

$$\left(\frac{\partial}{\partial t}\right)^{[p]}(t) = 0$$

and so $(\frac{\partial}{\partial t})^{[p]} = 0$.

Example 12.4. Take $G = \mathbb{G}_m$. Then we have $\mathfrak{g} = \mathrm{span}(t\frac{\partial}{\partial t})$ and

$$\left(t\frac{\partial}{\partial t}\right)^{[p]}(t) = t$$

and so $(t\frac{\partial}{\partial t})^{[p]} = t\frac{\partial}{\partial t}$.

Now we can define the **restricted universal enveloping algebra** as

$$\mathcal{U}(\mathfrak{g})_{\mathrm{restr}} = \mathcal{U}(\mathfrak{g})/(\xi^{[p]} - \xi^p).$$

Corollary 12.5. *The map* $\mathcal{U}(\mathfrak{g}) \to \mathrm{Diff}(G)^e$ *factors through* $U(\mathfrak{g})_{\mathrm{restr}}$.

Recall that we had $\mathrm{Sym}(\mathfrak{g}) \cong \mathrm{gr}\,\mathcal{U}(\mathfrak{g})$. If we define $\mathrm{Sym}(V)_{\mathrm{restr}} = \mathrm{Sym}(V)/(v^p)$ then we will have

$$\mathrm{Sym}(\mathfrak{g})_{\mathrm{restr}} \twoheadrightarrow \mathrm{gr}\,\mathcal{U}(\mathfrak{g})_{\mathrm{restr}}.$$

If you choose a basis $V = \mathrm{span}\{e_1, \dots, e_n\}$, then this is going to look like

$$\mathrm{Sym}(V)_{\mathrm{rest}} = \bigotimes_i k[t]/t^p.$$

Recall that there is a Frobenius map $\mathrm{Frob} : G \to G$, and its kernel if nilpotent, and the same is true for $\ker(\mathrm{Frob}^n)$. Because the Frobenius kills all tangent vectors, $\mathrm{Spec}(O/\mathfrak{m}^2) \subseteq \ker(\mathrm{Frob})$ and then

$$\mathrm{Spec}(O/\mathfrak{m}^{\cdots}) \subseteq \ker(\mathrm{Frob}^n).$$

This shows that

$$G_e^\wedge = \mathrm{colim}_n \ker(\mathrm{Frob}^n).$$

Here

$$\ker(\mathrm{Frob} : G_e^\wedge \to G_e^\wedge) = \ker(\mathrm{Frob} : G \to G)$$

but $\ker(\mathrm{Frob})$ are non-isomorphic for $\mathbb{G}_a$ for $\mathbb{G}_m$. This shows that there is no way to express this purely in terms of the Lie algebra.

We can consider $(O_{\ker(\mathrm{Frob})})^* \subseteq \mathrm{Dist}(G)_e$.

Lemma 12.6. *The map $\mathcal{U}(\mathfrak{g}) \to \mathrm{Dist}(G)_e$ factor through $(O_{\ker(\mathrm{Frob})})^*$.*

Proof. It suffices to show that generators are mapped into $(O_{\ker(\mathrm{Frob})})^*$. But $\mathfrak{g}$ is mapped to $(O/\mathfrak{m}^2)^*$ and this is contained in $(O_{\ker(\mathrm{Frob})})^*$. □

We thus obtain a map

$$\mathcal{U}(\mathfrak{g})_{\mathrm{rest}} \to (O_{\ker(\mathrm{Frob})})^*.$$

Theorem 12.7. *This is an isomorphism.*

Example 12.8. Last time we had $\widetilde{\mathrm{Sym}}^n(V) = (\mathrm{Sym}^n(V^*))^*$ and then

$$\widetilde{\mathrm{Sym}}(V)_{\mathrm{rest}} \subseteq \widetilde{\mathrm{Sym}}(V).$$

Then we get

$$\widetilde{\mathrm{Sym}}(V)_{\mathrm{rest}} = \bigotimes_i (k[s]^{<p}).$$

You can see that the map $\mathrm{Sym}(V) \to \mathrm{Dist}(V)_e$ factors as

$$\mathrm{Sym}(V) \to \mathrm{Sym}(V)_{\mathrm{rest}} \xrightarrow{\cong} \widetilde{\mathrm{Sym}}(V)_{\mathrm{rest}} \to \mathrm{Dist}(V)_e$$

where the isomorphism is explicitly described as $k[t]/t^p \to k[s]^{<p}$ with $t^n \mapsto n! s^n$.

Proof. We prove this on gr. We have

$$\begin{tikzcd} \operatorname{gr}\mathcal{U}(\mathfrak{g})_{\mathrm{rest}} \arrow[r] & \operatorname{gr}(O_{\ker(\mathrm{Frob})})^* \\ \operatorname{Sym}(\mathfrak{g})_{\mathrm{rest}} \arrow[u, two heads] \arrow[r, "\cong"] & \widetilde{\operatorname{Sym}}(\mathfrak{g})_{\mathrm{rest}} \arrow[u, "\cong"'] \end{tikzcd}$$

where the right vertical map is seen to be an isomorphism by choosing etale coordinates. ☐

12.2 Quotients

Given $H \subseteq G$, we want to define the quotient. That is, we want to find G/H such that

$$\operatorname{Hom}_G(G/H, Y) = Y(k)^H.$$

Definition 12.9. A **G-homogeneous space** is a scheme X equipped with an action of G such that for any $x \in X(k)$, the map $G \to X$ given by $g \mapsto gx$ is faithfully flat.

Proposition 12.10. *Suppose X is a G-homogeneous space and $h = \operatorname{Stab}_G(x)$. Then X has the universal property.*

Proof. We want to show that $\operatorname{Hom}_G(X, Y) \to Y(k)^H$ is an isomorphism. I want to construct the map in the other direction. There are two maps $G \times H \to G$, first projection and multiply. Then the two maps $G \times H \to G \to Y$ agree, and so by faithfully flat descent, we get

$$\begin{tikzcd} G \times H \arrow[d, shift left] \arrow[d, shift right] & \\ G \arrow[r] \arrow[d] & Y \\ X \arrow[ur, dashed] & \end{tikzcd}$$

the map $X \to Y$. ☐

Theorem 12.11. *Let G be reduced and $H \subseteq G$. Then there exists a G-scheme X and $x \in X$ such that $\operatorname{Stab}_G(x) = H$.*

We need to construct this from somewhere.

Theorem 12.12. *There exists a representation V and a subrepresentation $V' \subseteq V$ such that the stabilizer $\operatorname{Stab}_G(V') = H$. Here, by definition,*

$$\operatorname{Hom}(S, \operatorname{Stab}_G(V')) \subseteq \operatorname{Hom}(S, G)$$

consists of automorphisms of $V \otimes O_S$ that maps $V' \otimes O_S$ to itself.

Theorem 12.13. *We can take V' to be* 1*-dimensional.*

If we assume this, here is how you prove it. For $0 \le k \le \dim V$, there is the **Grassmannian** $\mathrm{Gr}^k(V)$ with $\mathrm{Hom}(S, \mathrm{Gr}^k(V))$ given by the locally free subsheaves of $O_S \otimes V$ of rank k. Take $Y = \mathrm{Gr}^k(V)$. Then $G \to \mathrm{GL}(V)$ acts on Y, and V' is a k-point of $Y = \mathrm{Gr}^k(V)$. The stabilizer of this is going to be H. So we get the first theorem from the second.

Here is how you prove the third theorem from the second. We have a natural transformation

$$\mathrm{Gr}^k(V) \to \mathbb{P}(\textstyle\bigwedge^k(V)),$$

given by sending $\mathscr{E} \subseteq O_S \otimes V$ to $\bigwedge^k(\mathscr{E}) \subseteq O_S \otimes \bigwedge^k(V)$, called the Plücker embedding.

Proof of second theorem. Let H correspond to the ideal $I \subseteq O_G$. Then there exist a subrepresentation $V \subseteq O_G$ such that $V \cap I$ generates I. Then we first note $\mathrm{Stab}_G(I) = H$. Also, $\mathrm{Stab}_G(V \cap I) = \mathrm{Stab}_G(I)$. So we take $V' = V \cap I$. ☐

Corollary 12.14. *There exists a homogeneous space such that* $\mathrm{Stab}_G(x) = H$.

Proof. The homogeneous space is the orbit Gx. ☐

Corollary 12.15. *If G is reduced, then G/H is reduced.*

Also, note that G/H is quasi-projective because we took it from the Grassmannian. We are assuming that G is reduced.

Proposition 12.16. *The map $G \to G/H$ is smooth if and only if H is reduced.*

Proof. Because G/H is reduced and $G \to G/H$ is faithfully flat, we can check smoothness of fibers. But then H is smooth if it is reduced. ☐

Proposition 12.17. *Let $H \subseteq G$ be normal. Then G/H has a unique group structure such that $G \to G/H$ is a group homomorphism and G/H is affine.*

Proof. We can use faithfully flat descent to construct $G/H \times G/H \to G/H$. To prove that G/H is affine, we first take $G^\circ/G' \cap H$. This is a connected component, so we may assume that G is connected. Take V a G-representation such that $\mathrm{Stab}_G(kv) = H$. H acts on v by a character χ. But characters are rigid, so the action of G and H be conjugation preserves the character. Let

$$V_1 = \{v_1 \in V_1 : h(v_1) = \chi(h)v_1\}.$$

This a subrepresentation, and so we can assume that $H|_V$ acts via χ. This gives a map

$$G/H \to \mathrm{PGL}(V).$$

This is categorically injective, so G/H is isomorphic to a closed subscheme of $\mathrm{PGL}(V)$. Now it suffices to show that $\mathrm{PGL}(V)$ is affine. This sat inside $\mathbb{P}(\mathrm{End}(V))$ as an open subset, with nonzero determinant. ☐

13 March 8, 2018

Last time we introduced quotients. For $H \subseteq G$, we characterized G/H with as the universal property.

Definition 13.1. Let H acts on X. We say that $X \to Y$ is a **faithfully flat quotient** of X by H if

(1) π is H-invariant,

(2) π is faithfully flat,

(3) $H \times X \cong X \times_Y X$.

Lemma 13.2. $\mathrm{Hom}(Y, Z) = \mathrm{Hom}(X, Z)^H$.

Proof. The proof is basically faithfully flat descent. The two maps $X \times_Y X \rightrightarrows X \to Z$ agree if $X \to Z$ is H-invariant, so we get $Y \to Z$. □

Observe that if X is a G-homogeneous space, and $H = \mathrm{Stab}_G(x)$, then the map $G \to Y$ is a faithfully flat quotient of G with respect to the action of H by right translations. We showed last time that for any H there exists a homogeneous space with this property. It sufficed to construct a G-space Y and $y \in Y$ such that $\mathrm{Stab}_G(y) = H$.

Theorem 13.3. *Let a* 1*-dimensional connected group scheme G act on a complete variety. Then it has a fixed point.*

Proof. Take an arbitrary point. Then we get a map $G \to X$, and then we can compactify $G \hookrightarrow \overline{G}$. Because X is proper, we can extend $G \to X$ to $\overline{G} \to X$ using the valuative criterion, and this is a map of G-varieties. Then the points of $\overline{G} \setminus G$ are fixed. □

13.1 Parabolic subgroups and solvable groups

Definition 13.4. A subgroup $P \subseteq G$ is **parabolic** if G/P is complete.

Lemma 13.5. *If P is parabolic in G and P_1 is parabolic in P, then P_1 is parabolic in G.*

Note that base being proper and the fiber being proper does not imply proper.

Proof. Note that being proper is local in the faithfully flat topology. So we can take

$$\begin{array}{ccccc} G/P' & \longleftarrow & G \times P/P' & \longrightarrow & P/P' \\ \downarrow & & \downarrow & & \downarrow \\ G/P & \longleftarrow & G & \longrightarrow & \mathrm{pt} \end{array}$$

□ todo

Lemma 13.6. *If $P \subseteq G$ is parabolic and $P \subseteq Q$ then Q is parabolic.*

Proof. We have a surjective $G/P \to G/Q$. Note that if X is complete and $\pi : X \to Y$ is surjective, then Y is complete. To show this, we need to show that $Y \times Z \to Z$ is closed. But we have $X \times Z \to Y \times Z$ surjective, so we can take image of $V \subseteq Y \times Z$ by looking at the inverse image in $X \times Z$ and then looking at the image in Z. $\square$

Definition 13.7. A group G is **solvable** if the following equivalent conditions hold:

(1) If we define $G_0 = G$ and $G_i = [G_{i-1}, G_{i-1}]$ then $G_n = \{1\}$ for $n \gg 0$.
(2) There exists $\{1\} = \tilde{G}_n \subseteq \cdots \subseteq \tilde{G}_1 \subseteq \tilde{G}_0 = G$ of normal reduced subgroups such that $\tilde{G}_i/\tilde{G}_{i-1}$ are abelian.
(3) (2) without assumption on reducedness.

It is clear that (1) implies (2) implies (3). For (3) implies (1), we note that $G_i \subseteq \tilde{G}_i$.

Lemma 13.8. *The following conditions are equivalent:*

(1) G is solvable.
(2) Any representation contains a nonzero G-fixed line.
(3) Any representation V has a G-stable flag.
(4) $G \hookrightarrow B_n$ where B_n is the upper-triangular matrices.

Proof. (2) implies (3) implies (4) implies (1) is clear. We have to show (1) implies (2). We induct on the dimension of the group. Then we know that $V^{G_1} \neq 0$. Then G/G_1 acts on this, and the quotient is abelian so we find something that is invariant. $\square$

Lemma 13.9. *Let G be solvable and connected. Then it has a filtration by normal subgroups such that the subquotients are either $\mathbb{G}_m$ and $\mathbb{G}_a$. G is unipotent if and only if all these subquotients are $\mathbb{G}_a$.*

Note that all unipotent groups are solvable. Let me postpone the proof.

Theorem 13.10 (Borel)**.** *If G is solvable and connected, acting on X complete, then it has a fixed point.*

Proof. For $G_1 \subseteq G$, we have $X^G = (X^{G_1})^{G/G_1}$. $\square$

Lemma 13.11. *$P \subseteq G$ is parabolic if and only if $P^0 \subseteq G^0$ is parabolic.*

Proof. We note that $(G/P)^0 = G^0/G^0 \cap P$. So G/P is proper if and only if $(G/P)^0$ is proper if and only if $G^0/G^0 \cap P$ is proper. But $P^0 \subset G^0 \cap P$ is finite index. $\square$

Theorem 13.12. *(1) If G has no proper parabolics, then G is solvable.*

(2) If G is connected and solvable, then it has no proper parabolics.

Proof. For (2), suppose that G has a proper parabolic. Then G acts on G/P, which is proper, but it does not have a fixed point. This contradicts Borel's theorem. For (1), let V be a representation. We want to show that there exists a G-stable line. Consider the action of G on $\mathbb{P}(V)$ and take a closed orbit. Either this is a single point and we get a fixed point, or this is a not a point and we get a proper parabolic. ☐

Let us now prove that lemma. First reduce to the case when G is abelian. Then we have $G = G_u \times G_{ss}$. We know that G_{ss} is a product of a bunch of $\mathbb{G}_m$s. So we can assume that G is unipotent. By induction, it suffices to find a subgroup isomorphic to $\mathbb{G}_a$. Embed G into U_m. We know that there exists a filtraion

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = U_m$$

such that $\dim(G_i/G_{i-1}) = 1$. Now take i such that $\dim(G \cap G_i) > 0$. Then we get $G \cap G_i / G \cap G_{i-1} \hookrightarrow G_i/G_{i-1}$ so $\dim(G \cap G_i) = 1$. So $((G \cap G_i)_{\mathrm{red}})^0 = \mathbb{G}_a$.

Proposition 13.13. *In characteristic* 0*, a commutative unipotent group is* $\underline{V}$.

Proof. Recall that we had G^u as a Zariski closed subgroup, because we were able to phrase this as a condition on characteristic polynomials. In the same way, we similarly define $\mathfrak{g}^n$. Now the claim is that

$$\mathfrak{g}^n \xrightarrow{\exp} G^u$$

is an isomorphism. Note that the S-points on the right are the automorphisms of $V \otimes O_S$ that are unipotent. On the other hand, $\mathfrak{g}^n$ are endomorphisms on $V \otimes O_S$ that are nilpotent. So we get this from

$$\mathrm{End}_R^{\mathrm{nil}}(V \otimes R) \xrightarrow{\exp} \mathrm{Aut}_R^{\mathrm{uni}}(V \otimes R).$$

Note that this is a homomorphism because $\exp(A + B) = \exp(A)\exp(B)$ by commutativity. ☐

Note that in characteristic p, we have

$$\mathbb{G}_a(k) = k = (\mathbb{Z}/p\mathbb{Z})^{\oplus \cdots}.$$

So we can take any finite subgroup and the quotient is going to be $\mathbb{G}_a$ because it is 1-dimensional reduced.

For G a group over characteristic p, we can define

$$L : G \to G; \quad L(g) = \mathrm{Frob}(g)g^{-1}.$$

Theorem 13.14 (Lang)**.** *L is always étale. If G is connected, then it is surjective. Also* $\ker(L) = G(\mathbb{F}_p)$.

Proof. For étale, we need to show that the differential is an isomorphism. First reduce to the case $g = e$. This is because given an infinitesimal g_0, we can take

$$\mathrm{Frob}(gg_0)g_0^{-1}g^{-1} = \mathrm{Frob}(g)(\mathrm{Frob}(g_0)g_0^{-1})g^{-1}.$$

Now note that the map is the composition of

$$G \xrightarrow{\mathrm{id} \times \mathrm{Frob}} G \times G \xrightarrow{\mathrm{inv} \times \mathrm{id}} G \times G \xrightarrow{\mathrm{mult}} G.$$

But its differential at e is then $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ given by -1, because the derivative of the Frobenius is 0.

Now let us compute its kernel. Consider $g \in G(k)$ such that $\mathrm{Frob}(g) = g$. Note that a point in $X(\mathbb{F}_q)$ satisfies $\mathrm{Frob}(x) = x$ if and only if $x \in X(\mathbb{F}_p)$. todo

For the second part, consider the G action given by G with $g * g_1 = \mathrm{Frob}(g)g_1 g^{-1}$. But

$$L'(g) = g_1^{-1}\,\mathrm{Frob}(g)g_1 g^{-1}$$

is étale because we can use the same argument on the tangent spaces. So all orbits are opens, and because G has one connected component, there is only one orbit. □

14 March 20, 2018

We were talking about solvable groups. A group G is solvable if there exist

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

such that G_i/G_{i-1} are commutative. You can impose the condition that G_i are reduced, and even let $G_i = [G_{i-1}, G_{i-1}]$.

Theorem 14.1. *The following are equivalent:*

(1) G is solvable.

(2) Every representation has an invariant line.

(3) Every representation has a G-stable flag.

(4) Every G can be realized as a subgroup of B_n.

Theorem 14.2. *Let G be a solvable connected group. Then there exists $1 = G_0 \subseteq \cdots \subseteq G_n = G$ such that G_i/G_{i-1} is either $\mathbb{G}_a$ or $\mathbb{G}_m$. Moreover, G is unipotent if and only if all subquotients are $\mathbb{G}_a$.*

Corollary 14.3. *If G is unipotent, there exist $1 = G_0 \subseteq \cdots \subseteq G_n = G$ with each of the normal subgroups such that the adjoint action of G on G_i/G_{i-1} is trivial.*

Definition 14.4. $P \subseteq G$ is parabolic if G/P is complete.

Lemma 14.5. *(a) If P is parabolic and $P \subseteq Q$ then Q is parabolic.*

(b) If P is parabolic in G and P' is parabolic in P then P' is parabolic in G.

(c) If $\varphi : G \twoheadrightarrow G'$ is a surjective morphism, and P is parabolic, then $\varphi(P)$ is parabolic.

Theorem 14.6. *Let G be connected. Then G contains a proper parabolic if and only if G is not solvable.*

To prove this, we used the Borel fixed point theorem.

Theorem 14.7. *If G is connected solvable, and acts on X a complete variety, then it has a fixed point.*

14.1 Borel subgroup

Definition 14.8. A subgroup $B \subseteq G$ is **Borel** if it is connected, solvable, and maximal with these properties.

Theorem 14.9. *(a) D is a parabolic if and only if it contains a Borel.*

(b) A Borel is parabolic.

(c) Any given Borel can be conjugated into any given parabolic.

(d) Any two Borels are conjugate.

Proof. (c) We have B acting on G/P, and this has a fixed point. If we pick something in this fixed point and then conjugate by this, B goes inside P.

(b) If G is solvable, then this is trivial. If G is not solvable, it has a proper parabolic P, and we may assume that $B \subseteq P \subseteq G$. Clearly, B is a Borel of P. By induction on the dimension, we have that B is parabolic of P, and then parabolic is transitive.

(a) If it contains a Borel, it contains a parabolic and is parabolic. If it is parabolic, some Borel can be conjugated into it. (d) is a consequence of (b) and (c). □

Corollary 14.10. *$Z(G)^\circ \subseteq B$.*

14.2 Structure theory of connected solvable groups

Unless said otherwise, we are going to assume that G is solvable and connected.

Proposition 14.11. *(a) There exists a*

$$1 \to U \to G \to T \to 1$$

where U is connected and unipotent, and T is a torus. Here, $g \in G$ is unipotent if and only if $g \in U$.

(b) There exists a non-canonical splitting, so that $G = T \ltimes U$.

Proof. (a) We first have

$$1 \to U_n \to B_n \to T_n \to 1.$$

Also, we know that G embeds into B_n. So we can take $U = G \cap U_n$. These are precisely going to be the unipotent elements of G. Then $G/(G \cap U_n)$ embeds into T_n, and hence is a torus.

For (b), we have $G \twoheadrightarrow T$. Assume first that $k \neq \overline{\mathbb{F}}_q$. Then find an element $\lambda \in T$ such that λ generates a dense subset of T. Find a semi-stable element $g \in GG$ that projects to λ. Define $G' = \overline{\langle g \rangle}$. Then G' surjects to T. Here, G' is a semi-simple commutative group, because under $G \hookrightarrow \mathrm{GL}_n$, G' should lie inside some torus T_n. But the kernel of $G' \twoheadrightarrow \mathbb{G}_m$ should be trivial because it unipotent as well (as it lies in U). So $G' \cong \mathbb{G}_m$ and we get a section.

If $k \neq \overline{\mathbb{F}}_q$, we use the fact that the fiber product

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Map}_{\mathrm{group}}(T, G) \\ \downarrow & & \downarrow \\ \mathrm{id} & \longrightarrow & \mathrm{Map}_{\mathrm{group}}(T, T). \end{array}$$

We want to show that $Y(k)$ is nonempty, but we know that $Y(k')$ is nonempty for other algebraically closed k'. So the ind-scheme Y should be nonempty as a scheme, and then $Y(k)$ is nonempty by Nullstellensatz. □

Proposition 14.12. $\mathrm{Map}_{\mathrm{group}}(T, G)$ *is a scheme.*

Note that we have proved before that $\mathrm{Map}_{\mathrm{group}}(G, T)$ is a discrete scheme.

Proof. Because we can embed $\mathrm{Map}_{\mathrm{group}}(T, G) \to \mathrm{Map}_{\mathrm{group}}(T, \mathrm{GL}_n)$, we reduce to the case of GL_n. Note that the k-points of this is going to be the ways of $\mathbb{G}_m$ acting on V, which is the ways to write $V = \bigoplus_i V_i$ with $\mathbb{G}_m$ acting on V_i by the n_i-th power. So we will have

$$\coprod_{\dim V = \sum_i \dim V_i} \mathrm{GL}(V) / \prod_i \mathrm{GL}(V_i)$$

This is because the S-points are locally free sheaves $\mathcal{E}$ of rank $\dim V$ over S, equipped with an action of $\mathbb{G}_m$, which is equal to to splittings $\mathcal{E} = \bigoplus_i \mathcal{E}_i$. $\square$

14.3 Maximal tori

Definition 14.13. A **maximal torus** in G is $T' \subseteq G$ that projects isomorphically to T.

We just showed that maximal tori exists for connected solvable group G.

Theorem 14.14. *Let G be a connected solvable.*

(a) All maximal tori are conjugate.

(b) If s_1, s_2 are two semisimple elements that project to the same elements in T, then they are conjugate.

(c) Centralizers of semisimple elements are connected.

Proof. (b) We first reduce to when $U = \mathbb{G}_a$. Let us induct on the dimension of $\mathbb{G}_a$. Let us pick $\mathbb{G}_a \hookrightarrow U$ and take the quotient of G by $\mathbb{G}_a$ and let $G' = G/\mathbb{G}_a$ and $U' = G'/\mathbb{G}_a$. Then $1 \to U' \to G' \to T \to 1$. Now any two maximal tori are conjugate in G'. So given any two elements in G, we can make them have equal image in G'. That is, we may now assume that G is the inverse image of T in the map $G \to G'$. That is, we can assume that $G = T \ltimes \mathbb{G}_a$.

But this is now completely explicit. Every action of $\mathbb{G}_a$ by automorphisms of $\mathbb{G}_a$ is the n-th power of the standard characters. Consider $G = \mathbb{G}_m \ltimes \mathbb{G}_a$ for simplicity. Take two elements (s, a) and $(s, 0)$. Assume that $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by n-th powers. If $n = 0$, we are done. If $n \neq 0$, we need to solve the equation $bsb^{-1} = bs$, which is $b - \lambda^n b = \lambda^n$. This is solvable if $\lambda^n \neq 1$, which is the case because if $as = sa$ then we get a contradiction to that (s, a) is semisimple. $\square$

15 March 27, 2018

We talked about connected solvable groups. We should that any solvable group fits in a short exact sequence

$$1 \to U \to G \to T \to 1$$

with a splitting $T \to G$, and moreover U has a filtration with subquotients $\mathbb{G}_a$.

Theorem 15.1. *Let g_1 and g_2 be two semisimple elements that project to the same element in T. Then they are conjugate.*

Proof. By induction, we reduce to the case when $U : \mathbb{G}_a$. Now G is $T \ltimes \mathbb{G}_a$, and so T Acts on $\mathbb{G}_a$. This is given by some character χ. Say that our two points are $(t, 0)$ and (t, a). If $\chi(t) = 1$, then $(t, a) = (t, 0)(1, a)$ and so they can't both be semisimple unless $a \neq 0$. If $\chi(t) \neq 1$, then we want to solve $(1, b)(t, 0)(1, -b) = (t, a)$. Then this is the same as solving $\chi(b) = a + b$, which is $a = \chi(t)b - b$. Because $\chi(t) \neq 1$, we can solve it. □

Recall that a maximal torus in G is a $T' \subseteq G$ that projects isomorphically to T.

Corollary 15.2. *Any torus can be conjugated into any given maximal torus. So any two maximal tori are conjugate.*

Proof. Consider $\{g : \mathrm{Ad}_g(S) \subseteq T\}$. We want to say that this has a point, so we have to show that it has a point in some field extension. □ todo

Theorem 15.3. *Let $t \in T \subseteq G$ be semisimple. Then $Z_G(t)$ is connected.*

Proof. By induction, we can take out a $\mathbb{G}_a$-factor from U. By Let this by $0 \to U' \to G' \to T \to 1$. Then we assume $Z_{G'}(t)$ is connected. So we now have

$$1 \to \mathbb{G}_a \to G \to G' \to 1$$

where $t \in T \subseteq G'$, and moreover that the image of t in G' is central. Now T acts on $\mathbb{G}_a$, by some character χ.

First suppose that $\chi(t) \neq 1$. Then, consider the map $Z_G(t) \to Z_{G'}(t)$, and we want to show that this is an isomorphism. This map is injective becasue the kernel should be in $\mathbb{G}_a$ but $\mathbb{G}_a$ acts non-trivially. (If characteristic is 0, then we can conclude this by looking at Lie algebras and showing that $(\mathfrak{g})^{\mathfrak{t}} \twoheadrightarrow (\mathfrak{g}')^{\mathfrak{t}}$.) Given any $g' \in G'$, we want to lift it to g that is in the centralizer. Let us first lift to $g \in G'$ to $g \in G$. Then we can write

$$\mathrm{Ad}_t(g) = ag$$

for some $a \in \mathbb{G}_a$. Then we find a b such that $\mathrm{Ad}_t(bg) = bg$, by solving the same equation we did.

Now consider the case when $\chi(t) = 1$. Here, we have $G = T \ltimes \mathbb{G}_a$. Let us write $G = T \ltimes U$ in general, where U is a unipotent group. Then U has a lift such that the adjoint action of t on the subquotients is trivial. This means that we have a map

$$T \to \mathrm{Aut}^{\mathrm{flag}}(U)$$

of flag-preserving automorphisms. Here, you can show that $(\mathrm{Aut}^{\mathrm{flag}}(U))^\circ_{\mathrm{red}}$ is an algebraic group, and that the kernel

$$1 \to \mathrm{Aut}^{\circ,\mathrm{flag}}(U) \to \mathrm{Aut}^{\mathrm{flag}}(U) \to \prod \mathbb{G}_m$$

of the map given by taking the subquotients has $(\mathrm{Aut}^{\circ,\mathrm{flag}})^\circ_{\mathrm{red}}$ unipotent. So t maps to something that is semisimple, but it lies in a unipotent group so it is the identity. This means that t acts as the identity on U. So $Z_G(t) = G$. □

Corollary 15.4. *Let G be solvable, and consider a maximal torus T. The normalizer is $N_G(T) = Z_G(T)$ and is connected.*

Proof. After field extension, there is going to be a point that generates the torus. So $Z_G(T) = Z_G(t)$ for some t, and is connected. To show that $N_G(T) = Z_G(T)$, we look at $ntn^{-1}t^{-1}$ for any normalizing n. Then $ntn^{-1}t^{-1} \in T$ so it is semisimple. But if we project to the torus, this should be 1, so the commutator should lie in U. This means that is is unipotent, and so 1. □

We also talked about Borel subgroups. For G a connected group, $B \subseteq G$ is Borel if it is solvable and maximal with this property.

Theorem 15.5. *(1) If B is a Borel subgroup, then G/B is complete.*

(2) Any connected solvable can be conjugated into any parabolic.

Corollary 15.6. *Any Borel can be conjugated into any parabolic. Any solvable can be conjugated into any Borel.*

Corollary 15.7. *Any two Borels are conjugate.*

Lemma 15.8. *$Z(G)^\circ \subseteq Z(B) \subseteq Z(G)$.*

Proof. Take any $g \in Z(B)$. Then we have $gxg^{-1}x^{-1} : G \to G$. This factors though G/B, and G/B is proper and G is affine. □

Lemma 15.9. *If B is normal, then $B = G$.*

Proof. Consider G/B. This is proper, and affine. □

15.1 Reductive groups

Definition 15.10. The **unipotent radical** $R_u(G)$ is the maximal normal unipotent connected subgroup. We say that G is **reductive** if $R_u(G) = 1$.

Lemma 15.11. *$R_u(G) = (\bigcap_B B_U)^\circ$ where B runs over all Borels and B_U is the unipotent part.*

Proof. One direction is by definition, because the right hand side is a normal unipotent connected. In the other direction, R_u is in some B_U and then conjugate them around. $\square$

Consider $B = T \ltimes B_U$.

Proposition 15.12. *Suppose that the adjoint action of T on B_U is trivial. Then $G = B$.*

Proof. If $B = T \times B_U$, then there exists some $\mathbb{G}_a \subseteq Z(B_U) \subseteq Z(B)$. Then $\mathbb{G}_a \subseteq Z(G)$ and then we can quotient out $G' = G/\mathbb{G}_a$. Then we can proceed by induction. $\square$

Corollary 15.13. *$B = T$ implies $G = B = T$.*

Corollary 15.14. *$B \subseteq (N_G(B))^\circ$ is an equality.*

Proof. Let $H = (N_G(B))^\circ$. Because $B \subseteq H$, we have $B = H$. $\square$ todo

Theorem 15.15. *Every element of G is contained in a Borel.*

Proof. We define

$$\tilde{G} = (G \times B)/(g, b_1) \sim (gb^{-1}, bb_1b^{-1}).$$

Then we can consider

$$\begin{tikzcd}
\tilde{G} \arrow[r, hook] \arrow[dr, "\pi"'] & G \times G/B \arrow[d] \\
& G.
\end{tikzcd}$$

This is going to be (g, b_1) being sent to $(gb_1b^{-1}, g \bmod B)$. We'll do this next time. $\square$

16 March 29, 2018

Let G be a connected group.

Lemma 16.1. *Maximal tori are conjugate.*

Theorem 16.2. *$Z_G(T)$ is connected. (Later, if G is reductive, then $T \subseteq Z_G(T)$ is an equality.)*

Theorem 16.3. *Every element of G belongs to a Borel, and every semisimple element of G belongs to a maximal torus.*

Theorem 16.4. *$B \hookrightarrow N_G(B)$ is an equality.*

Before proving, I will play around with these theorems.

Corollary 16.5. *If P is a parabolic, then P is connected and $P \hookrightarrow N_G(P)$ is an equality.*

Proof. We will show that $P^\circ \to N_G(P)$ is an equality. Suppose $x \in N_G(P)$. Then for $B \subseteq P^\circ$, we have $\mathrm{Ad}_x(B) \subseteq P^\circ$. Because both are Borels in P°, there exists a $y \in P^\circ$ such that $\mathrm{Ad}_x(B) = \mathrm{Ad}_y(B)$. Then if we write $x' = y^{-1}x$, we get $\mathrm{Ad}_{x'}(B) = B$. So $x' \in B \subseteq P^\circ$ by the third theorem. □

Corollary 16.6. *If $B \subseteq P^1$ and $B \subseteq P^2$ are conjugate parabolics, then $P^1 = P^2$.*

Corollary 16.7. *Let T be a torus, and let S be the Borels containing T. Then $N_G(T)$ acts on S.*

(i) This action is transitive.

(ii) $Z_G(T)$ acts trivially.

(iii) $N_G(T)/Z_G(T)$ acts simply.

Proof. (1) Suppose $T \subseteq B^1, B^2$. Then we can find $g \in G$ such that $\mathrm{Ad}_g(B^1) = B^2$. But inside the solvable group, any two maximal tori are conjugate. So we can assume $\mathrm{Ad}_g(T) = T$.

(2) If $T \subseteq B$, it suffices to show that $Z_G(T) \subseteq B$. We will prove unconditionally that $Z_G(T)^\circ \subseteq B$. Let $C = Z_G(T)^\circ$. Last time we proved that if a maximal torus is central in the Borel, then $B_C = C$. So C is connected solvable, so $T \subseteq C \subseteq B'$ for some other Borel B'. Now we can play our usual game.

(3) For $T \subseteq B$, assume that $x \in N_G(T)$ and $x \in N_G(B)$. By the third theorem, we have $x \in N_B(T)$. But then recall that $N_B(T) = Z_G(T)$. (This was proved by $ntn^{-1}t^{-1}$ being both semisimple and unipotent.) □

16.1 Cartan subgroup

Definition 16.8. The definition of the **Cartan** is $C = Z_G(T)^\circ$.

Proposition 16.9. *There exists a dense open subset in G such that all of its elements can be conjugated into C.*

Corollary 16.10. *Every element of G belongs to a Borel.*

Proof. The theorem implies that there exists a dense open of elements that belong to a Borel. But we introduced this Grothendieck alteration $\tilde{G} \subseteq G/B \times G$ such that the fiber over G/B is the conjugate of the Borel B. The image is dense open, and because it is closed, the image is the whole thing. □

Theorem 16.11. *For $S \subseteq G$ a torus, $Z_G(S)$ is connected.*

Proof. Let $g \in Z_G(S)$, and let B be a Borel such that $g \in B$. Then let $X \subseteq G/B$ consisting of those x such that $\mathrm{Ad}_x(B) \ni g$. This is nonempty, and S acts on X. This means that the action has a fixed point. Let x be the fixed point, and $B' = \mathrm{Ad}_x(B)$. Then $g \in B'$, so we have $S \subseteq (N_G(B'))^\circ$ implies $S \subseteq B'$. Then $g \in Z_{B'}(S)$. But we have proved that inside a solvable group, centralizers of tori are connected. So $g \in Z_{B'}(S)^\circ \subseteq Z_G(S)^\circ$. □

Proof. Consider similarly the

$$G/C \times G \supseteq \tilde{\tilde{G}} \xrightarrow{\pi} G.$$

We want to show that π is dominant. It is enough to show that there exist $g \in G$ such that π has a finite nonempty fiber over G, by semi-continuity.

We use our trick of enlarging the field, and take $t \in T$ such that the closure of $\langle t \rangle$ is T. We want to see that there are finitely many (modulo C) elements g such that $t \in \mathrm{Ad}_g(C)$. Here, $t \in \mathrm{Ad}_g(C)$ implies $\mathrm{Ad}_{g'}(t) \subseteq C$, which implies $\mathrm{Ad}_{g'}(T) = T$. So $g \in N_G(T)$, and we want to see that $N_G(T)/Z_G(T)$ is finite. This is because the action on a torus is given by a discrete character, and so $(N_G(T))^\circ \subseteq Z_G(T)$. □

Now we show that the normalizer of a Borel is itself.

Proposition 16.12. *Let S is a torus and $B \supseteq S$, then $Z_G(S) \cap B$ is a Borel in $Z_G(S)$.*

Using this we can prove the theorem.

Proof. Let $x \in N_G(B)$. Then we can assume that $\mathrm{Ad}_x(T) = T$, because we can move T back. Consider the following strange map

$$\psi : T \to T; \quad t \mapsto \mathrm{Ad}_x(t)t^{-1}.$$

Let $S = (\ker \psi)^\circ$. First consider the case $S \neq 1$. We know that $x \in Z_G(S)$ and also it normalizes $B \cap Z_G(S)$, which is a Borel inside $Z_G(S)$. If $Z_G(S) \subsetneq G$, then by the induction hypothesis on $Z_G(S)$, we get $x \in B \cap Z_G(S) \subseteq B$. If $Z_G(S) = G$, then we can replace G by G/S, and we are again done by induction.

Now assume that $S = 1$. Then $\psi : T \to T$ is surjective. Let $H = N_G(B)$. Then there exists a representation V and $v \in V$ such that

$$H = \{g \in G : gv \subseteq \mathrm{span}\{v\}\}.$$

(This came up in our construction of the quotient.) Then we get a map

$$G \to V; \quad g \mapsto gv.$$

The claim is that this factors through $G \to G/B \to V$. To show this, we need to show that B acts trivially on v. A priori, the action of $N_G(B)$ on V is given by a character $\chi : N_G(B) \to \mathbb{G}_m$. We want $\chi|_B$ to be trivial. But we know that $B = B_u \rtimes T$, and we need to see that $\gamma|_T$ is trivial. But $T \subseteq [N_G(H), N_G(H)]$ because ψ was surjective. $\square$

So we are done, up to that proposition.

17 April 3, 2018

Today we are going to do more examples. But last time, we needed something.

Proposition 17.1. *Let G be connected, and $S \subseteq B \subseteq G$ where S is a torus. Then $Z_G(S) \cap B \subseteq Z_G(B)$ is a Borel subgroup.*

Proof. We want to show that $Z_G(S)/Z_G(S) \cap B \to G/B$ is closed. This means that $Z(G) \times B$ has close image in G. Let Y be the image, so that we want to show $\overline{Y} = Y$.

Consider the map

$$\overline{Y} \times S \to B; \quad (y, s) \mapsto \mathrm{Ad}_{y^{-1}}(s).$$

But we can map further to $\overline{Y} \times S \to B \to B/B_U = T$. This is a family of maps of tori, parametrized by $\overline{Y}$. So $\mathrm{Ad}_y^{-1}(s) = s$ modulo B_U, for $y \in \overline{Y}$. □ todo

By the centralizer of S, I could have meant two things. I could have meant the scheme-theoretic centralizer, or I could have meant the reduced subscheme of the k-points.

Proposition 17.2. *Scheme-theoretic $Z_G(S)$ is reduced.*

Proof. It is enough to prove that for $s \in G$ a semi-simple element, its centralizer $Z_G(s)$ is reduced. Let s be a semi-simple automorphism of a smooth affine algebraic variety Y. (This means that it is semi-simple on the finite stables of the ring of functions.) Now we can take Y^s as

$$\mathrm{Hom}(Z, Y^s) = (\mathrm{Hom}(Z, Y))^s.$$

If $Y = \operatorname{Spec} A$, then $Y^s = \operatorname{Spec} A/(f - s(f))$. The claim is that Y^s is smooth. Smoothness is equivalent to every n-jet extending to an $n + 1$-jet.

$$\begin{tikzcd} \operatorname{Spec} k[t]/t^n \arrow[r] \arrow[d, hook] & Y^s \arrow[d] \\ \operatorname{Spec} k[t]/t^{n+1} \arrow[r, "\varphi_0"] \arrow[ur, dashed] & Y \end{tikzcd}$$

Here, we have $\varphi : A \to k[t]/t^{n+1}$, and we want to find $\varphi = \varphi_0 + t^n l$ so that φ is s-invariant. The claim is that if $\varphi' = \varphi_0 + t^n l'$ where l' is the s-invariant projection of l, then φ' is an algebra homomorphism. □

17.1 Nonsolvable groups with maximal torus of dimension 1

Let G be connected and T be a maximal torus. Assume $\dim T = 1$. (This is called a group of rank 1.) If G is solvable, it is a semi-direct product and I can't say much about this.

Theorem 17.3. *If G is not solvable, then*

$$1 \to K \to G \to \mathrm{PGL}_2 \to 1,$$

where K° is unipotent, so that it is the unipotent radical.

Proof. Consider G/B. We'll prove that $\dim(G/B) = 1$ as a variety. If we have this, B acts on G/B which is proper, and fixes a point. So G/B should be $\mathbb{P}^1$. Because the automorphisms is PGL_2, we get a map $G \to \mathrm{PGL}_2$. Then $G \to \mathrm{PGL}_2$ is surjective because it acts transitively on the flag variety. Because K° is a connected algebra group, $B(K^\circ)$ is unipotent. Then $B(K^\circ) = K^\circ$.

So $\dim(G/B) = 1$ is what we want to prove. We have $W = N_G(T)/Z_G(T)$ acts on T. So $|W| \leq 2$ because $T = \mathbb{G}_m$. Because W acts simply transitively on the set of Borels containing T, there are at most 2 Borels. Take a representation $v \in V$ such that $B = \{g \in G : gv \in kv\}$. I can restrict to subrepresentations containing v, so we can assume that G-translates of v span V.

Choose a basis $e_1, \dots, e_m$ such that $\mathbb{G}_m$ acts on e_i by n_i-th power of the standard character. Assume that $n_1 \geq n_2 \geq \cdots \geq n_m$. We know that T acts on G/B, and we are interested in the T-fixed points. But G/B sites inside $\mathbb{P}(V)$, and so let's study the $\mathbb{G}_m$-fixed points in $\mathbb{P}(V)$. A line $\sum_i a_i e_i$ can be a fixed point if and only if $a_i, a_j \neq 0$ implies $n_i = n_j$.

But we also wan to look at attractors. Take an arbitrary element $v' = \sum a_i e_i$ and look at the map $\mathbb{G}_m \to \mathbb{P}(V)$ given by $\lambda \mapsto \lambda v'$. By the valuative criterion, this extends to $\mathbb{P}^1 \to \mathbb{P}(V)$. The image of 0 and ∞ will be something like chucking away all a_i except for the minimal or maximal n_i.

Without restriction of generality, we can assume that $G \hookrightarrow \mathrm{GL}(V)$. Then I claim that $n_1 > n_m$. If it wasn't, T acts on V by scalars and $T \subseteq Z(G)$, then G would be solvable. Now I will be playing with the fact that there are only two Borels. There exists a translate $g_1 v$ such that

$$g_1 v = a_1 e_1 + \sum_{n \geq 2} a_i e_i$$

with $a_1 \neq 0$. Here, we will have

$$\lim_{\lambda \to 0} (\lambda g v) = \text{line in the space of } \{e_i\} \text{ with eigenvalue } n_1.$$

Because G/B is the space of Borel, and this is a fixed point in projective space with the T-action, it is a Borel. Now by the similar logic, we take g_2 such that $g_2 v = a_m e_m + \sum_{n \leq m-1} a_n e_n$ and then $\lim_{\lambda \to \infty}$ has eigenvalue n_m. This is another Borel.

Now take $V' = \mathrm{span}\{e_1, \dots, e_{m-1}\}$. Then $\mathbb{P}(V') \subseteq \mathbb{P}(V)$. We have $\mathbb{P}(V') \cap G/B \subseteq G/B$, and we will prove that $\mathbb{P}(V') \cap (G/B) = B_1$. If we show this, hitting a projective variety by a hyperplane only reduces the dimension by at most 1, so $\dim(G/B) = 1$. STo show this, take $\sum_i a_i e_i \in G/B$ with $a_m = 0$. Now $\lim_{\lambda \to \infty} \lambda \sum a_i e_i$ must be B_1 because it cannot be B_2. Then it actually should be B_1. $\square$

Theorem 17.4. *Let G be connected reductive with* $\dim(T) = 1$*. Then* $G = \mathrm{PGL}_2$ *or* $G = \mathrm{SL}_2$*.*

Proof. Let $G' = \mathrm{PGL}_2$. There is a map $G \to G'$ and this should induce an isomorphism $G/B \cong G'/B'$ because both are $\mathbb{P}^1$. There is

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & G \arrow[r] & G' \arrow[r] & 1 \\ 1 \arrow[r] & K \arrow[r] \arrow[u, equal] & B \arrow[r] \arrow[u] & B' \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

This gives $B = T \ltimes B_U \to T \ltimes B'_U = B'$. The claim is that $B_U \to B'_U$ is an isomorphism. If we believe this, we have

$$K = \ker(T \to T') = \ker(\mathbb{G}_m \to \mathbb{G}_m; t \mapsto t^n).$$

The claim is that either $n = 1$ or $n = 2$. We have

$$1 \to \mu_n \to G \to G' \to 1.$$

Even if characteristic is nonzer, the representation theory of μ_n is governed by characters. So G being connected implies that $\mu_n \in Z(G)$. Take n a nontrivial element in W. Then $\mathrm{Ad}_n(t) = t^{-1}$. Because μ_n lives in the center, this is trivial. Therefore inversion is trivial, and so $n = 1$ or $n = 2$.

For $n = 1$, this is easy. For $n = 2$, we want to show that G is SL_2. Take

$$\begin{tikzcd} G''' = (\mathrm{SL}_2 \times_{G'} G)^\circ_{\mathrm{red}} \arrow[r] \arrow[d] & \mathrm{SL}_2 = G'' \arrow[d] \\ G \arrow[r] & G'. \end{tikzcd}$$

If we can show that the left vertical is an isomorphism, we have $B \to B'' \to B'$, and then $T \to T'' \to T'$ because it is an isomorphism on the unipotent. Then $T \to T'$ is covering of degree 2 and T'' is degree 2, so $T \to T''$ is an isomorphism. Then $B \to B''$ is an isomorphism, so $G \to G''$ is an isomorphism. To see that $G''' \to G$ is an isomorphism, we can again take Borels and tori, and see that $T''' \to T$ is an isomorphism. This again can be seen by looking at degree. □ todo

18 April 5, 2018

Last time we did the following.

Theorem 18.1. *Let G be a connected nonsolvable and $T \cong \mathbb{G}_m$. Then there exists a nonsolvable map $G \to \mathrm{PGL}_2$ that is surjective and induces an isomorphism $G/B \twoheadrightarrow \mathbb{P}^1$.*

The only nontrivial case is when $\dim G = 2$.

Lemma 18.2. *If $\dim(G) = 2$ then G is solvable.*

Proof. Consider $B \subseteq G$. Either $G = B$ and G is solvable. If $\dim(B) \leq 1$, then B is commutative and is contained in the center of G. So we can quotient by it and G is solvable as well. □

Theorem 18.3. *If G is reductive and $\dim T = 1$, then G is either SL_2 or PGL_2.*

We looked at $G \to \mathrm{PGL}_2$ and looked at $T \to \mathbb{G}_m$. Then the Borels are going to be $B = T \ltimes B_U \to B' = T' \ltimes B'_U$. The claim is that $B_U \to B'_U$ is an isomorphism. We know that $\dim(B_U) = 1$ and so $B_U = \mathbb{G}_a$. So we have a morphism $\varphi : \mathbb{G}_a \to \mathbb{G}_a$. This shows that

$$\phi = \sum a_i \operatorname{Frob}^i .$$

But this has to be $\mathbb{G}_m$-equivariant, with some $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^n$. This will show that there cannot be higher Frobenius.

18.1 Groups of semi-simple rank 1

Today we will tighten our control. Let G be reductive. The **rank** of G is defined as $\dim T$ and the **semi-simple rank** is the rank of $G/Z(G)^\circ$.

Theorem 18.4. *Let G be reductive of semi-simple rank 1.*

(1) Either $[G, G]$ is SL_2 or PGL_2.

(2) If $[G, G] = \mathrm{PGL}_2$, then $G = \mathrm{PGL}_2 \times T_0$. If $[G, G] = \mathrm{SL}_2$, then $G = \mathrm{SL}_2 \times T_0$ or $G = \mathrm{GL}_2 \times T_0^1$.

Theorem 18.5. *Let G be reductive.*

(1) $[G, G] \cap Z(G)$ is finite.

(2) $[G, G] \times Z(G)^\circ/([G, G] \cap Z(G)^\circ) \to G$ is an isomorphism.

Using the short exact sequences

$$1 \to [G, G] \to G \to Z(G)^\circ/[G, G] \cap Z(G)' \to 1,$$

$$1 \to Z(G)' \to G \to [G, G]/[G, G] \cap Z(G)' \to 1,$$

we get the following.

Corollary 18.6. *(1)* $[G,G]$ *is reductive.*

(2) $G = [G,G]$ *if and only if* $Z(G)^\circ = 1$ *if and only if* $Z(G)$ *is finite.*

(3) $Z([G,G]) \subseteq Z(G)$.

(4) $Z(G)' \to G \to G/[G,G]$ *is surjective with finite kernel. (It is an isogeny.)*

Proof. What we really have to prove is (4). To prove that the kernel is finite, we need to map $G \to T$ so that $Z(G)' \to G \to T$ has finite kernel. To do this, find a faithful representation G on V such that $G \hookrightarrow \mathrm{GL}(V)$. Inside there, there is a torus $Z(G)^\circ$ and it acts on V by a finite number of characters. Therefore we can write V as $V = \bigoplus_i V_i$. Then G respects this so that so that $Z(G)' \to G \to \prod_i \mathrm{GL}(V_i)$. Then we can take the determinant

$$Z(G)' \hookrightarrow G \to \prod_i \mathrm{GL}(V_i) \to \prod_i \mathbb{G}_m.$$

This has finite kernel.

Now we show that it is surjective. Define the adjoint quotient $G_{\mathrm{ad}} = G/Z(G)$ and take

$$Z(G) \to G \to G/[G,G] \to G_{\mathrm{ad}}/[G_{\mathrm{ad}}, G_{\mathrm{ad}}].$$

The claim is that $[G_{\mathrm{ad}}, G_{\mathrm{ad}}] = G_{\mathrm{ad}}$ implies that the map is surjective. If we have this fact, then $Z(G)$ surjects onto $G/[G,G]$. But because $G/[G,G]$ is connected, $Z(G)'$ already surjects. We will prove that $[G_{\mathrm{ad}}, G_{\mathrm{ad}}] = G_{\mathrm{ad}}$ later. □

Now let us prove the classification theorem of semi-simple rank 1 groups.

Proof. (1) We know that $G = [G,G] \times Z(G)'/[G,G] \cap Z(G)'$.

(2) First if $[G,G] = \mathrm{PGL}_2$, then we have $[G,G] \cap Z(G)' = 1$ so we are taking the product with $Z(G)'$. If $[G,G] = \mathrm{SL}_2$, then either $G/Z(G)' = \mathrm{SL}_2$ or PGL_2. In the first case, we also have $[G,G] \cap Z(G)' = 1$ and we get $G = \mathrm{SL}_2 \times Z(G)'$. If $G/Z(G)' = \mathrm{PGL}_2$, then $[G,G] \cap Z(G)' = \mu_2$. Then we have

$$(\mathrm{SL}_2 \times T^\circ)/\mu_2 = (\mathrm{SL}_2 \times \mathbb{G}_m/\mu_2) \times T^{\circ\circ}$$

because $\mu_2 \to T^\circ$ looks like $\mu_2 \to \mathbb{G}_m \times T^{\circ\circ}$ with the μ_2 action on $\mathbb{G}_m$ part. □

18.2 Roots

We don't have the ABC of the structure theory of reductive groups, but we will develop them with roots. Let G be a connected group and let T be a maximal torus.

Definition 18.7. A **fake root** ζ is a nonzero eigenvalue of the adjoint action of T on $\mathfrak{g}$.

Let α be a fake root. We write

$$Z_G((\ker(\alpha))^\circ) = G_\alpha.$$

Suppose G is reductive. If we appeal to your picture of reductive groups, this will look like

$$\mathrm{Lie}(G_\alpha) = \mathfrak{g}_\alpha \oplus \mathfrak{t} \oplus \mathfrak{g}_{-\alpha}.$$

In general, your group is the semi-direct product

$$G = R_u(G) \rtimes G_{\mathrm{red}}.$$

There are several scenarios for the fake root.

1. α occurs as a root on G_{red}. In this case, we get

$$G_\alpha = U_\alpha \rtimes G_{\alpha,\mathrm{red}},$$

   where U_α is the part of $R_u(G)$ centralized by $\ker(\alpha)^\circ$.

2. α occurs as a root on G_{red}, but its rational multiple appears in $R_u(G)$. In this case, we again have $G_\alpha = U_\alpha \rtimes G_{\alpha,\mathrm{red}}$.
3. α occurs on $R_u(G)$, but its rational multiple α' occurs in G_{red}. Then $G_\alpha = U_\alpha \rtimes G_{\alpha',\mathrm{red}}$.
4. No rational multiple of α occurs in G_{red}. Then $G_\alpha = U_\alpha \times T$ so that G_α is solvable.

Definition 18.8. A **true root** is a fake root that occurs on the reductive part of G_α.

Lemma 18.9. *For α a true root, either G_α is solvable or $G_\alpha/R_u(G_\alpha)$ is of semi-simple rank 1.*

Proof. Consider $G_\alpha/R_u(G_\alpha) \supseteq T$ such that $\ker(\alpha)^\circ$ is contained in the center. Consider $G_\alpha/R_u(G_\alpha)\ker(\alpha)^\circ$. □ todo

We can consider $R \subseteq X^*(T)$ the set of true roots.

Definition 18.10. Let Λ be a lattice and $\Lambda^\vee$ be the dual roots. For α, we consider $s_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha}\rangle\alpha$. A **finite root system** is a pair of sets

$$R \subseteq \Lambda, \quad R^\vee \subseteq \Lambda^\vee$$

and a correspondence $R \leftrightarrow R^\vee$ such that

- $\langle \alpha, \check{\alpha}\rangle = 2$,
- s_α preserve $R \subseteq \Lambda$,
- the subgroup that they generate $\mathrm{Aut}(\Lambda)$ is finite.

The claim is that the set of true roots form a root system. I need to produce coroots for you. Recall that

$$1 \to R_u(G_\alpha) \to G_\alpha \to G_\alpha/R_u(G_\alpha) \to 1$$

where $G_\alpha/R_u(G_\alpha)$ is a reductive group of semi-simple rank 1. Whichever case, there is a canonical map

$$\mathrm{SL}_2 \to G_\alpha/R_u(G_\alpha).$$

Then there is a corresponding map $\mathbb{G}_m \to T$ of tori. This we define to be $\check{\alpha} \in \Lambda^\vee = X_*(T)$. We can check $\langle \alpha, \check{\alpha} \rangle = 2$ inside SL_2.

We can now consider the **Weyl group**

$$W = N_G(T)/Z_G(T).$$

The action of W on $X^*(T)$ preserves the set of true roots. Also, $s_\alpha \subseteq W(G_\alpha) \subseteq W$ by pushing $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in SL_2.

Proposition 18.11. *W is generated by the reflections s_α.*

Proof. We do this by induction on the dimension of G. Let $w \in W$. Because T is commutative, consider the map $\varphi(t) = w(t)t^{-1}$ from T to T. Then either $\dim(\ker(\varphi)) \geq 1$ or $\ker(\varphi)$ is finite.

If $\dim(\ker) \geq 1$, let $S = \ker(\varphi)^\circ$. Take $G' = Z_G(S)$ so that $w \in G'$. If $G' < G$, then we are done by induction. If $G' = G$, then S is central in G and so we can take $G'' = G/S$. Then we are also done by induction.

If $\ker(\varphi)$ is finite, it is an isogeny $\varphi : T \to T$. Consider $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$. φ defines an isomorphism, and is defined by $v \mapsto w(v) - v$. Let $\alpha \in R$ and let $v \in \Lambda_\mathbb{Q}$ be such that $w(v) - v = \alpha$. Now we claim that $\langle \check{\alpha}, v \rangle = -1$. If we know that,

$$s_\alpha w(v) = s_\alpha(v + \alpha) = v - \langle \check{\alpha} v \rangle \alpha - \alpha = v.$$

Then we can set $w' = s_\alpha W$ and w' has eigenvalue 1. Then we can peel this off and do induction again.

Now let's prove that $\langle \check{\alpha}, v \rangle = -1$. I have an invariant form, and I can average over W. Then we have

$$\langle \check{\alpha}, x \rangle = \frac{2(\alpha, x)}{(\alpha, \alpha)}$$

and $(v + \alpha, v + \alpha) = (w(v), w(v)) = (v, v)$ implies that $\langle \check{\alpha}, v \rangle = -1$. $\square$

19 April 10, 2018

Miraculously, we are developing the theory of roots without the structure theory of reductive groups. Let G be connected and $T \subseteq G$ be a maximal torus. Fake roots are just the eigenvalues of T on $\mathfrak{g}$. For γ a character, we can take

$$G_\gamma = Z_G(\ker(\gamma)^\circ).$$

If you look at the reductive of G_γ, we can write

$$1 \to \ker(\gamma)^\circ \to G_{\gamma,\mathrm{red}} \to G_{\gamma,\mathrm{red}}/\ker(\gamma)^\circ \to 1.$$

By the classification from before, there are two scenarios: either $G_{\gamma,\mathrm{red}}$ is $\mathbb{G}_m$ or it is PGL_2 or SL_2.

We say that γ is a true root if both of the following are satisfied:

(i) the second scenario occurs,

Look at the adjoint action of $G_\gamma/\ker(\gamma)^\circ$ on $\mathfrak{g}_\gamma$, which is a 1-dimensional torus because $G/\ker(\gamma)^\circ$. Then we get a short exact

$$0 \to \mathrm{Lie}(R_u(G_\gamma)) \to \mathfrak{g}_\gamma \to \mathrm{Lie}(G_{\gamma,\mathrm{red}}) \to 0.$$

(ii) γ appears as a character of T on $\mathrm{Lie}(G_{\gamma,\mathrm{red}})$.

We consider the Weyl group $W = N_G(T)/Z_G(T)$. For α a true root, we get an element $s_\alpha \in W(G_\alpha) \subseteq w$. Let $X^*(T)$ be the lattice of characters of T and $X_*(T)$ be the lattice of cocharacters of T. For all α, there exists a $\check{\alpha} \in X_*(T)$ such that

$$s_\alpha(\lambda) = \lambda - \langle \check{\alpha}, \lambda \rangle \alpha, \quad s_\alpha(\check{\lambda}) = \check{\lambda} - \langle \alpha, \check{\lambda} \rangle \check{\alpha}.$$

for $\lambda \in X^*(T)$ and $\check{\lambda} \in X_*(T)$.

19.1 Examples of root systems

We defined root systems last class.

Theorem 19.1. *s_α generate W.*

Example 19.2. Consider $\Lambda = \mathbb{Z}$ and $\check{\Lambda} = \mathbb{Z}$. Then we can have roots ± 1 and coroots ± 2. Or we can have roots ± 2 and coroots ± 1. The first corresponds to PGL_2 and the second corresponds to SL_2. There are no others, because we need $\langle \alpha, \check{\alpha} \rangle = 2$.

Example 19.3. Consider $(\mathbb{G}_a \times \mathbb{G}_a) \rtimes \mathbb{G}_m$ with the action given by $(5, 7)$ times the standard character. Then $5, 7$ are fake roots in $X^*(T) \cong \mathbb{Z}$, because in both cases, G_γ is the entire group.

Example 19.4. Consider $(\mathbb{G}_a \times \mathbb{G}_a) \rtimes \mathrm{SL}_2$. Then $X^*(T) = \mathbb{Z}$ and the action is given by $(1, -1)$. Then the eigenvalues are these two and ± 2. In all the cases (except for $\alpha = 0$), we have $G_\alpha = G$ and so the true roots are only those coming from SL_2, which are ± 2. For $\alpha = 0$, we have $G_\alpha = Z_G(\mathbb{G}_m)$ so this is a fake root. You can do this for other representations, like $(-2, 0, 2) \rtimes \mathrm{SL}_2$.

Example 19.5. Consider A_2, which is $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -\alpha_1-\alpha_2\}$. Consider the dual basis $\check{\omega}_i$ such that $\langle \check{\omega}_i, \alpha_j \rangle = \delta_{ij}$. Then we will have $\check{\alpha}_1 = 2\check{\omega}_1 - \check{\omega}_2$ and $\check{\alpha}_2 = -\check{\omega}_1 + 2\check{\omega}_2$. This is PGL_3 and the dual is the SL_3.

Example 19.6. Consider B_2. Here, $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and the roots are $\pm\omega_i$ and $\pm\omega_1 \pm \omega_2$. Then for $\alpha = \omega_1$ and $\beta = -\omega_1 - \omega_2$ and we have $\check{\alpha} = 2\check{\omega}_1$ and $\check{\beta} = \check{\omega}_2$. In this case, $\Lambda = \mathrm{Span}\{\alpha, \beta\}$ and $\mathrm{Span}\{\check{\alpha}, \check{\beta}\} \subseteq \check{\Lambda}$ has index 2. The corresponding group and its dual are SO_5 and $\mathrm{Sp}_4 \cong \mathrm{Spin}_5$. (Index 2 corresponds to this being a double cover.)

Example 19.7. We consider G_2. For this, both the group and its dual are generated by the roots.

Definition 19.8. A set of **positive roots** is a choice of $R^+ \subseteq R$ such that there exists a $\xi \in \check{\Lambda}$ or $\xi \in \check{\Lambda}_{\mathbb{Q}}$ such that $\langle \alpha, \check{\xi} \rangle \neq 0$ for every $\alpha \in R$ and $R_+ = \{\alpha : \langle \alpha, \check{\xi} \rangle > 0\}$.

The Weyl group acts on the choices of positive roots. We will see that it does so simply transitively.

Theorem 19.9. *Let G be a connected group and $T \subseteq B \subseteq G$. Consider the action of true roots that is contained in B. Then it is a system of positive roots.*

Proof. There exists a representation V and a vector $v \in V$ such that $\{g : gv \subseteq kv\} = B$. Then the action is given by some character $tv = \lambda(t)v$ for $\lambda \in X^*(T)$. We will show that $\alpha \in B$ if and only if $\langle \check{\alpha}, \lambda \rangle > 0$. $\square$

20 April 12, 2018

We talked about root systems. We said that $R^+ \subseteq R$ is a set of positive roots if there exists a $\check{\lambda} \in \Lambda^\vee$ such that $\alpha \in R^+$ if and only if $\langle \check{\lambda}, \alpha \rangle > 0$.

Theorem 20.1. *Choose $T \subseteq B \subseteq G$. Note that we have shown that $G_\alpha \cap B$ is a Borel in G_α. Then for any root α, either α or $-\alpha$ is in the Borel. The claim is that $\{\alpha : \alpha \in B\}$ is a system of positive roots.*

Proof. Let V be a representation and v such that $gv \subseteq kv$ if and only if $g \in B$. Then $tv = \lambda(t)v$ for some $\lambda \in \Lambda = X_*(T)$. We will show that $\alpha \in B$ implies $\langle \check{\alpha}, \lambda \rangle > 0$.

We can replace G by G_α. Then we have the uniponent radical $R_u(G_\alpha) \subseteq G_\alpha$ and this acts trivially on v. So we can replace V by $V^{R_u(G_\alpha)}$. Then we can replace G_α with $G_\alpha/R_u(G_\alpha)$. This will come from $\mathrm{SL}_2 \to G_\alpha$ and my $\check{\alpha}$ will be in SL_2.

Now we assume that $G = \mathrm{SL}_2$, and V is generated by gv for $g \in G$. Let $n_1 \geq n_2 \geq \cdots \geq n_m$ be the eigenvalues. Because $\sum_i n_i = 0$, we have $n_i > 0$. Then we can look at $G/B \subseteq \mathbb{P}(V)$. We can look at $G/B \to \mathbb{P}(V)$ given by $g \mapsto gv$, and v should be a $\mathbb{G}_m$-attractor. This shows that v has eigenvalue n_1, which is positive. □

Here is another proof. For $\xi : V \to k$, an equivariant line bundle on a point is the same as a G-equivariant line bundle $\mathscr{L}$ on G/B. Now the condition that gv generates V is equivalent to the condition that the G-equivariant $V \hookrightarrow \Gamma(G/B, \mathscr{L})$ is injective. todo

20.1 Structure of reductive groups

Theorem 20.2. *Fix $T \subseteq G$.*

(1) $\bigcap_{B \supseteq T} R_u(B) = R_u(G)$.

(2) $\mathrm{Lie}(R_u(G))$ *is the span of the fake roots.*

For (1), it suffices to show that the left hand side is normal.

Corollary 20.3. *Let S be a torus. If G is reductive then $Z_G(S)$ is reductive.*

Proof. Take $S \subseteq T$. For B' Borel subgroup of $Z_G(S)$, we have

$$\bigcap_{B' \supseteq T} R_u(B') = \bigcap_{B \supseteq T} (R_u(B) \cap Z_G(S)) \subseteq \bigcap_{B \supseteq T} R_u(B) = \{1\},$$

where B' runs over Borel subgroups of $Z_G(S)$. □

Corollary 20.4. *For G reductive, $Z_G(T) = T$.*

Proof. We know that $Z_G(T)$ is reductive. So $Z_G(T) = T \times R_u(Z_G(T)) = T$. □

Corollary 20.5. *If α is a root, then*

1. $\dim \mathfrak{g}_\alpha = 1$,

2. $c\alpha$ is root if and only if $c = \pm 1$.

Proof. Take α to be a root and take $G_\alpha = Z_G(\ker(\alpha)^\circ)$. We know that $\mathfrak{g}_\alpha$ lies in the Lie algebra of G_α but we know that this is reductive. So $G_\alpha / \ker(\alpha)^\circ$ is either SL_2 or PGL_2. It suffices to check in this case. □

Let's no prove the theorem.

Proof. We'll first prove (2). The inclusion $\mathrm{Lie}(R_u(G)) \subseteq$ span is obvious. Take the eigenspace $\mathfrak{g}^\gamma$ and consider the corresponding G_γ. Then $\mathfrak{g}^\gamma$ is in the Lie algebra of $R_u(G)$ and is contained in G_γ. In the other direction, if γ is a fake root then $\mathfrak{g}^\gamma \subseteq \mathrm{Lie}(R_u(G_\gamma))$ and is contained in R_u or every Borel of G_γ.

For (1), we need to show that $\bigcap_{T\subseteq B} R_u(B)$ is normal in G. The claim is that G is generated by G_γ. This is because $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}^\gamma$ and $\mathfrak{g}^\gamma \subseteq \mathrm{Lie}(G_\gamma)$. So it suffices to show that $\bigcap_{T\subseteq B} R_u(B)$ is preserved under conjugation by each G_γ.

If G_γ is solvable, then $G_\gamma \cap B = G_\gamma$ for some B. Then $G_\gamma \subseteq \bigcap B \subseteq \bigcap_{T\subseteq B} B$. Then it is uninteresting. Suppose G_γ is not reductive. Then

$$1 \to R_u(G_\alpha) \to G_\alpha \to G_{\alpha,\mathrm{red}} \to 1$$

and $G_{\alpha,\mathrm{red}}$ is generated by its two Borel subgroups that contains a given maximal torus. Then for $T \subseteq B_\alpha \subseteq G_\alpha$ we want to show that B_α preserves $\bigcap_{T\subseteq B} R_u(B)$.

We want to create a unipotent subgroup R_α such that $R_\alpha \supseteq R = \bigcap_{T\subseteq B} R_u(B)$ that contains $R_u(G_\alpha)$ and has $\dim(R_\alpha) - \dim(R) = 1$. If such R_α exists, then R is necessarily normal in it by the following lemma. We define

$$R = \bigcap_{T\subseteq B} R_u(B), \quad R_\alpha = \bigcap_{T\subseteq B, \alpha\in B} R_u(B).$$

Then we need to check $\dim(R_\alpha) - \dim(R) = 1$. But $\mathrm{Lie}(R_\alpha)/\mathrm{Lie}(R)$ is spanned by those true roots β for which the following happen: if $\alpha \in B$ then $\beta \in B$.

In this case, we show that this actually implies $\alpha = \beta$. The Weyl group W acts on the set of positive root systems. A combinatorial fact is that W acts simply transitivity. So every system of positive roots is of the form $\{\alpha \in B\}$ for some Borel B. The next proposition solves this. □

Lemma 20.6. *If H is connected and unipotent, and $H' \subseteq H$ is such that $\dim(H) - \dim(H') = 1$, then H' is normal.*

Lemma 20.7. *If H is a connected unipotent with $H' \subseteq H$, if $\dim(H') < \dim(H)$ then $\dim(N_H(H')) > \dim(H')$.*

Proof. We have some $\mathbb{G}_a \subseteq Z_G(H)$. If H' does not contain $\mathbb{G}_a$, we can ad it. Else, quotient out. □

21 April 17, 2018

We proved the following structural theorem. Let G be connected.

Theorem 21.1. *$(\bigcap_{T\subseteq B} R_u(B))^\circ = R_u(G)$. Also, we have*

$$\mathrm{Lie}(R_u(G)) = \bigoplus_{\alpha\neq 0 \text{ fake}} \mathfrak{g}_\alpha \oplus \mathrm{Lie}(R_u(Z_G(T))).$$

Here, $Z_G(T) = T \times R_u(Z_G(T))$.

Corollary 21.2. *If $S \subseteq G$ is a torus and G is reductive, then $Z_G(S)$ is reductive. If G is reductive, then $Z_G(T) = T$.*

From now on, assume G is reductive.

Corollary 21.3. *$Z(G) \subseteq T$.*

Corollary 21.4. *If $\Gamma \subseteq G$ is finite normal unipotent then $\Gamma = \{1\}$.*

Proof. G acts on Γ by conjugation, so Γ is central. So Γ is in a torus, which means that Γ is semi-simple. □

Let's look at root systems again. Note that if R_1 and R_2 are two root systems, we can make a stupid root system $R = R_1 \amalg R_2$ and $\Lambda = \Lambda_1 \oplus \Lambda_2$. So there actually is another root system or rank 2, namely $A_1 \times A_1$. These four $A_1 \times A_1, A_2, B_2, G_2$ are the all the rank 2 root systems.

Here is how you see it. Take α, β a basis. Then we can take $\langle\check{\alpha}, \beta\rangle \leq 0$. In this case $s_\beta \circ s_\alpha$ acts on Λ and preserves orientation. So the matrix

$$\begin{pmatrix} -1 & -\langle\check{\alpha},\beta\rangle \\ \langle\check{\beta},\alpha\rangle & \langle\check{\alpha},\beta\rangle\langle\check{\beta},\alpha\rangle - 1 \end{pmatrix}$$

has trace strictly less than 2. This shows that $\langle\check{\alpha},\beta\rangle\langle\check{\beta},\alpha\rangle \leq 3$.

21.1 Structure of reductive groups II

For G a reductive group, its Lie algebra is

$$\mathfrak{g} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha \oplus \mathfrak{t} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}.$$

Then $G^\alpha = Z_G(\ker(\alpha)^0$ and

$$\mathrm{Lie}(G^\alpha) = \mathfrak{g}_\alpha \oplus \mathfrak{t} \oplus \mathfrak{g}_{-\alpha}.$$

Its commutator subgroup will have

$$\mathrm{Lie}([G^\alpha, G^\alpha]) = \mathfrak{g}_\alpha \oplus \mathrm{Span}(\check{\alpha}) \oplus \mathfrak{g}_{-\alpha}.$$

Lemma 21.5. *$t \in Z(G)$ if and only if $\alpha(t) = 1$ for all $\alpha \in R^+$.*

Proof. Because G^α generate G, it is enough to show this for G^α. But we know what these look like. □

Corollary 21.6. *If $Z(G)^0 = \{1\}$ if and only if $[G, G] = G$.*

Proof. It is enough to show that $\mathrm{Lie}([G, G])$ is all of $\mathfrak{g}$. But $\mathfrak{g}_\alpha \subseteq \mathrm{Lie}([G^\alpha, G^\alpha])$. So it suffices to show that $Z(G)^\circ = 1$ implies that $\mathfrak{t}$ is spanned by $\check{\alpha}$. This follows from $\mathrm{Lie}(Z(G)) = (\mathrm{span}\{\alpha\})^\perp$. □

Definition 21.7. G is **semi-simple** if it is reductive and $Z(G)^\circ = 1$.

Theorem 21.8. *If G is reductive, then*

$$Z(G)^\circ \to G \to G/[G, G]$$

is a isogeny, i.e., it is a surjective map of tori with finite kernel.

Corollary 21.9. *$Z(G) \cap [G, G]$ is finite. Also,*

$$\frac{[G, G] \cap Z(G)^\circ}{[G, G] \cap Z(G)^\circ} \to G$$

is an isomorphism.

Corollary 21.10. *$Z([G, G]) \subseteq Z(G)$. $Z([G, G])$ finite implies that $[G, G]$ is semi-simple.*

We proved that $Z(G) = \{t \in T : \alpha(t) = 1 \text{ for all } \zeta \in R\}$.

Corollary 21.11. *$X_*(Z(G)) = X_*(Z(G)')$ as a subset of $X_*(T)$ is the set $\{\check{\lambda} \in X_*(T) : \langle \check{\lambda}, \alpha \rangle = 0\}$.*

Lemma 21.12. *The map $T \to G \to [G, G]$ is a surjection and $X^*(G/[G, G])$ as a subset of $X^*(T)$ is $\{\lambda : \langle \lambda, \check{\alpha} \rangle = 0\}$.*

Corollary 21.13. *The following are equivalent:*

(1) $Z(G)^\circ = \{1\}$
(2) $[G, G] = G$
(3) $\mathrm{Span}_{\mathbb{Q}}\{\alpha\} = \Lambda_{\mathbb{Q}}$
(4) $\mathrm{Span}_{\mathbb{Q}}\{\check{\alpha}\} = \check{\Lambda}_{\mathbb{Q}}$.

Definition 21.14. G is said to be of **adjoint type** if $Z(G) = 1$.

Lemma 21.15. *G is of adjoint type if* $\mathrm{Span}_{\mathbb{Z}}\{\alpha\} = X^*(T)$.

For G reductive, $G/Z(G)$ is actually adjoint type.

Definition 21.16. G is said to be **simply-connected** if $\mathrm{Span}\{\check{\alpha}\} = X_*(T)$.

Lemma 21.17. *For every G reductive there exists*

$$G_{\mathrm{sc}} \to [G,G] \hookrightarrow G$$

which is the dual picture of $G \twoheadrightarrow G/Z(G)^\circ \twoheadrightarrow G/Z(G)$.

Definition 21.18. For G_1, G_2 reductive, we say that the map $G_1 \to G_2$ is an **isogeny** if it is surjective and has finite kernel on the torus.

Theorem 21.19. *Let G be semi-simple and $G_1 \subseteq G$ be a connected normal subgroup. Then there exists a unique $G_2 \subseteq G$ such that $G_1 \times G_2 \to G$ is an isogeny.*

Proof. Take a Cartan T and $T_1 = (T \cap C_1)^\circ \subseteq T$. We can take $G_2 = [Z_G(T_1), Z_G(T_1)]$. This is the right think to look at, because $Z_{G_1 \times G_2}(T_1) = T_1 \times G_2$. □

Let's now look at Borels. Take $B \subseteq G$ and look at the unipotent part U. Then we have

$$\mathrm{Lie}(U) \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha.$$

For each α, we actually have $\mathrm{SL}_2 \to G^\alpha \to G$, and we can look at the image of $\mathbb{G}_a$ and write U_α.

Proposition 21.20. *The multiplication map*

$$\prod_{R^+} U_\alpha \to U$$

is an isomorphism of varieties equipped with a T-action.

Proof. Assume that the characters of T on $\mathrm{Lie}(U)$ appear with multiplicity 1. Then there exists a $\mathbb{G}_a \subseteq Z(U)$ such that the action of T on $\mathbb{G}_a$ is given by α_0. Then we can look at

$$\begin{array}{ccc} \prod_\alpha U_\alpha & \longrightarrow & U \\ \downarrow & & \downarrow \\ \prod_{\alpha \neq \alpha_0} U_\alpha & \longrightarrow & U/U_{\alpha_0}. \end{array}$$

Now we can use the following fact. □

Suppose we have

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\pi_x"'] & Y \arrow[d, "\pi_y"] \\ & \operatorname{Spec} A, \end{tikzcd}$$

where everything is equipped with a $\mathbb{G}_m$-action, and the grading on A is positive. (This means that $A_0 = k$ and $A_n = 0$ for $n < 0$.) If π_x, π_y are smooth, ϕ

defines an isomorphism on the fiber $\pi_x^{-1}(0) \to \pi_y^{-1}(0)$, and the action on Y is contractible, then ϕ is an isomorphism.

Let G be reductive. To each B, we have $B/R_u(B)$. Note that if g' and g'' both conjugates B_1 to B_2, they define the same map $B_1/R_u(B_1) \to B_2/R_u(B_2)$. This is because the normalizer of B is itself and then the action of $b \in B$ on $B/R_u(B)$ is trivial. So all $B/R_u(B)$ are canonically isomorphic, and this is called the **abstract Cartan**. Likewise, you can define the **abstract Weyl group**.

Theorem 21.21 (Bruhat decomposition)**.** *Let $X = G/B$ be the flag variety. Then the orbits of the diagonal action of G on $X \times X$ are in bijection with W.*

Given an element $w \in W$, we are going to take $(B, B^w) \subseteq X \times X$.

22 April 19, 2018

We had this notion of an abstract Cartan that is canonically attached to G. If G is reductive, we look at a Borel B and take $B/R_u(B)$. If g_1, g_2 conjugates B_1 to B_2, they give the same isomorphism $B_1/R_u(B_1) \to B_2/R_u(B_2)$. Similarly we can define the abstract Weyl group. For T, we look at $N_G(T)/T$. If g conjugates $(T_1 \subseteq B_1) \to (T_2 \subseteq B_2)$, we will get map $\mathrm{Ad}_g : N_G(T_1)/T_1 \to N_G(T_2)/T_2$. But then any normalizer of $(T_1 \subseteq B_1)$ is in B_1, so it is T_1 itself. So we can define the abstract Weyl group like this.

22.1 Bruhat decomposition

Let X be the flag variety, the variety of Borels.

Theorem 22.1. *G-orbits of $X \times X$ are in bijection with W.*

Proof. We need to construct the maps. Given $T \subseteq B$ and $w \in N_G(T)/T$, we consider the orbit of $(B, \mathrm{Ad}_w(B)) \in X \times X$. This is independent of the choice of $T \subseteq B$, because we are going to conjugate anyways. The construction in the other way is more remarkable. Consider B_1, B_2 two Borels. Then we get

$$\begin{tikzcd}
(B_1 \cap B_2)/R_u(B_1 \cap B_2) \arrow[r] \arrow[d] & B_2/R_u(B_2) \arrow[d, "\cong"] \\
B_1/R_u(B_1) \arrow[r, "\cong"] & T
\end{tikzcd}$$

that doesn't necessarily commute. Then by the following lemma, $B_1 \cap B_2$ contains some maximal torus, so all the maps are isomorphisms. The noncommutativity is going to be given by some automorphism of T, and this is going to be the w we want. $\square$

Lemma 22.2. *The intersection of any two Borels contains a maximal torus.*

Now choose $T \subseteq B$.

Corollary 22.3. *The B-orbits on G/B are in bijection with $W = N_G(T)/T$.*

Consider $X^w = Bw$ inside G/B. Then this looks like

$$X^w = B/B \cap \mathrm{Ad}_w(B).$$

These are called **Schubert cells**. This is a homogeneous space, but we can say more things. Recall that $B = T \ltimes U$ and we proved last time that $U = \prod_{\alpha \in R^+} U_\alpha$ where $U_\alpha = \mathbb{G}_a$. Then we can also write $X^w = U/U \cap \mathrm{Ad}_w(U)$.

Lemma 22.4. *(a) $U \cap \mathrm{Ad}_w(U) \subseteq U$ can be described as $\prod_{\alpha, w(\alpha) \in R^+} U_\alpha \subseteq \prod_{\alpha \in R^+} U_\alpha$.*

(b) $U/U \cap \mathrm{Ad}_w(U) \cong \prod_{\alpha \in R^+, w(\alpha) \in R^-} U_\alpha$.

This lemma is highly believable. At least it is evident at the level of Lie algebras.

Corollary 22.5. *$B \times B$-orbits on G (with left and right multiplication) are in bijection with W:*

$$w \mapsto BwB = G^w.$$

These are called ***Bruhat cells****.*

We can also write this as $G^w = (\prod_{\alpha \in R^+, w(\alpha) \in R^-} U_\alpha) wB$. Let $w_0 \in W$ be the element such that $X^{w_0} \subseteq W$ is open. Then w_0 satisfies

$$\dim(X) = \dim(G) - \dim(B) = \dim(\mathfrak{g}) - \dim(\mathfrak{b}) = |R^+|.$$

So $\dim(X^{w_0}) = \dim(X)$ means that w_0 satisfies $w_0(R^+) = R^-$. This is called the longest element of the Weyl group. In this case,

$$G^{w_0} = Uw_0B, \quad X^{w_0} = Uw_0.$$

22.2 Simple roots

Suppose we have a root system R and pick a system of positive roots $R^+ \subseteq R$.

Definition 22.6. $\alpha \in R^+$ is **simple** if it cannot be written as a sum of positive roots with nonnegative integral coefficients $\alpha \neq \alpha_1 = \alpha_2$.

Let $I \equiv R^+$ be the set of simple roots.

Lemma 22.7. *(1) α_i form a basis for* $\mathrm{Span}_{\mathbb{Q}}(\alpha) = (\Lambda_{[G,G]})_{\mathbb{Q}}$.

(2) s_i generates W.

In the semi-simple case, we can look at

$$\Lambda_{\mathrm{ad}} = \mathrm{Span}\{\alpha\} \subseteq \Lambda.$$

We already had $\check{\Lambda}_{\mathrm{sc}} = \mathrm{Span}\{\check{\alpha}\}$ and so we have

$$\Lambda_{\mathrm{ad}} \subseteq \Lambda \subseteq \Lambda_{\mathrm{sc}}.$$

So there are only finitely many choices for Λ.

Lemma 22.8. Λ_{ad} *can be uniquely recovered by the Cartan matrix* $(I, \langle \alpha_i, \check{\alpha}_j \rangle)$.

Proof. We have $R^+ \subseteq \Lambda_{\mathrm{ad}} = \mathrm{Span}\{\alpha\}$. Then we can just reflect $s_i(\alpha_j) = \alpha_j - \langle \alpha_j, \check{\alpha}_i \rangle \alpha_i$. Then you can look at the set generated by all these reflections. $\square$

Theorem 22.9. *Let B be a Borel. Parabolics that contain B are in bijection with subsets of I.*

Let $J \subseteq I$. We will find a parabolic P_J with

$$\mathrm{Lie}(P_J) = \mathrm{Lie}(B) \oplus \bigoplus_{\beta} \mathfrak{g}_{-\beta}$$

where β lies in the positive integral span of α_j for $j \in J$. This is going to be given by

$$P_J = R_u(P_J) \rtimes M_J, \quad M_J = Z_G((\textstyle\bigcap_{j \in J} \ker(\alpha_j))^\circ).$$

Then $\mathrm{Lie}(M_J) = \bigoplus_\beta \mathfrak{g}_\beta \oplus \mathfrak{t} \oplus \bigoplus_\beta \mathfrak{g}_{-\beta}$ and $\mathrm{Lie}(R_u(P_J)) = \bigoplus_{\alpha \neq \beta} \mathfrak{g}_\alpha$.

Let's now show that these are all the parabolics. If $\mathbb{G}_m$ acts on Z, then we can think of $Z^+ = \mathrm{Maps}_{\mathbb{G}_m}(\mathbb{A}^1, Z)$, which is the $z \in Z$ such that $\lim_{t \to 0} tz$ exists. There is a natural map $Z^+ \subseteq Z$. We can also think of $Z^0 = \mathrm{Maps}_{\mathbb{G}_a}(\mathrm{pt}, Z)$. Then there is a natural map $Z^+ \to Z^0$ by taking the attractor, and a $Z^0 \to Z^+$ by looking at the constant map. If $Z = \operatorname{Spec} A$, with grading, we have $Z^+ = \operatorname{Spec} A^+$ where A^+ is A quotiented out by all the negative degree things. Then A^0 is Aquotiented out by all nonzero degree things.

Lemma 22.10. *(a) If Z is smooth then Z^+, Z^0 are smooth.*

(b) If $z \in Z^0$, then $T_z(Z^0)$ is the direct sum of $T_z Z$ on which $\mathbb{G}_m$ acts trivially. $T_z(Z^+)$ is the direct sum of $T_z Z$ on which $\mathbb{G}_m$ acts with nonnegative eigenvalues.

So here is the construction. For $J \subseteq I$, we look at $\check{\lambda} \in X_+(t)$ such that $\langle \check{\lambda}, \alpha_j \rangle = 0$ for $j \in J$ and $\langle \check{\lambda}, \alpha_i \rangle > 0$ for $j \notin J$. Then $\check{\lambda} : \mathbb{G}_m \to T$. Now we define $P_J = G^+$ and $M_J = G^0$.

23 April 24, 2018

I want to classify parabolic subgroups. Fix a Borel $T \subseteq B$ and we want to classify the parabolics containing B.

23.1 Classification of parabolics

Theorem 23.1. *There is a bijection between $J \subseteq I$ and P_J a parabolic containing B, such that*

$$\operatorname{Lie}(P_J) = \operatorname{Lie}(B) \oplus \bigoplus_{\alpha \in \operatorname{Span}\{\alpha_j : j \in J\}} \mathfrak{g}_\alpha,$$

and

$$P_J = M_J \ltimes R_u(P_J), \quad M_J = Z_G((\textstyle\bigcap_{j\in J} \ker(\alpha_j))^0),$$
$$\prod_{\alpha \in R^+, \alpha \notin \operatorname{Span}\{\alpha_j\}} G_\alpha = R_u(P_J) \subseteq R_u(B) = \prod_{\alpha \in R^+} G_\alpha.$$

Our method was to look at the $\mathbb{G}_m$ action on Z and look at $Z^0 \to Z^+ \hookrightarrow Z$. For example, on $Z = V^{>0} \times V^0 \times V^{<0}$ an affine space, we will have $Z^+ = V^{>0} \times V^0$ and $Z^0 = V^0$.

Lemma 23.2. *If Z is smooth, then Z^+ and Z^0 are smooth. If $z \in Z^0$, then $T_z Z^0 = (T_z Z)^0$ and $T_z Z^+ = (T_z Z)^{>0}$.*

Now consider a coroot $\check{\lambda} : \mathbb{G}_m \to T$ such that $\langle \check{\lambda}, \alpha_j \rangle = 0$ if $j \in J$ and $\langle \check{\lambda}, \alpha_j \rangle > 0$ if $j \notin J$. Then we define $P_J = G^+$. This P_J is going to contain B, because we have

$$P_J B = B^+ \subseteq B$$

and $T_e B^+ = T_e B$ by design. Now consider $M_J = G^0$. We have maps $P_J \to M_J$ and $M_J \to P_J$. Then we have $M_J = Z_G(\check{\lambda}(\mathbb{G}_m))$. The claim is that

$$\ker(P_J \to M_J) = R_u(P_J).$$

This can be shown again by comparing Lie algebras.

Let us now do this other direction. Let $B \subseteq P$ be a parabolic.

Lemma 23.3. *On G/P, there is a very ample G-equivariant line bundle $\mathscr{L}$ on G/P.*

Proof. We look at a representation V of G and look at $G/P \to \mathbb{P}(V)$. $\square$

Let $V = \Gamma(G/P, \mathscr{L})$. Then we have a P-invariant functional $V \to \mathscr{L}_1$ and dualizing gives $(\mathscr{L}_1)^* \hookrightarrow V^*$. The stabilizer of a line/coline is exactly P.

Let us be given $\ell \subseteq V$ and $\operatorname{Stab}_G(\ell) = P \supseteq B$. Let B act on ℓ via some character λ and consider

$$J = \{j : \langle \lambda, \check{\alpha}_j \rangle = 0\}.$$

I then claim that $P = P_J$. Let us first do the inclusion $P_J \subseteq P$. We know that $P_J = R_u(P_J) \rtimes M_J$ where M_J is generated by G^α for α in the span of J. Here, G^α acts on V with B^α acing on v via a character. Because

$$1 \to [G^\alpha, G^\alpha] \to G^\alpha \to G^\alpha/[G^\alpha, G^\alpha] \to 1,$$

$[G^\alpha, G^\alpha]$ receives a surjection from SL_2, and $G^\alpha/[G^\alpha, G^\alpha]$ receives an action from a torus, it suffices to show that SL_2 acts trivially on v. This can be done.

Now because $P_J \subseteq P$, it is enough to show that $\mathrm{Lie}(P_J) = \mathrm{Lie}(P)$. For $\alpha \in R^+$ such that $\mathfrak{g}_{-\alpha} \subseteq \mathrm{Lie}(P)$, we want to show that $\mathfrak{g}_{-\alpha} \subseteq \mathrm{Lie}(P_J)$. But G^α stabilizes kv and the same proof on G^α works.

We talked about the Bruhat decomposition. We said that orbits in are in bijection with W. Now we can be interested in G-orbits of $(G/P_1) \times (G/P_2)$. These each correspond to $J_1 \subseteq I$ and $J_2 \subseteq I$.

Proposition 23.4. *The Weyl group $W_J = W(M_J) = \langle s_j : j \in J \rangle$ is naturally a subgroup of W. Then the G-orbits are in bijection with $W_{J_1} \setminus W/W_{J_2}$.*

23.2 Length

Let $w \in W$. We define its **length** as

$$l(w) = \#\{\alpha \in R^+, w(\alpha) \in R^-\}.$$

There is a different way to define length. For any $w \in W$, we can write

$$w = s_{i_1} \cdots s_{i_n}.$$

Definition 23.5. We say that this is a reduced decomposition if $l(w) = n$.

We will talk about the interplay between length in the Bruhat decomposition. Suppose $w = w_1 w_2$ be such that $l(w_1) + l(w_2) = l(w)$. Consider $(X \times X)_w$ the corresponding G-orbit on $X \times X$. Then there is a map

$$(X \times X)_{w_1} \times_X (X \times X)_{w_2} \to X \times X.$$

Theorem 23.6. *This is an isomorphism onto $(X \times X)_w$.*

There is something called the Bruhat order. Then for the Schubert cells, you can prove that

$$X_w \subseteq B \cdots w \subseteq X, \quad \overline{X}_w = \bigcup_{w' \leq w} X_{w'}.$$

Suppose $w = s_{i_1} \cdots s_{i_n}$. According to the previous theorem, we have an isomorphism

$$(X \times X)_{s_{i_1}} \times_X (X \times X)_{s_{i_2}} \times_X \cdots \times_X (X \times X)_{s_{i_n}} \to (X \times X)_{s_{i_n}}.$$

Then you can take closures on both sides. Then we get a resolution of singularities $\overline{(X \times X)_w}$. Here, $(\overline{X \times X})_{s_i}$ is going to be some $\mathbb{P}^1$ fibered over the flag variety.

We know how to get a root data from a reductive group. We want to see to what extent this classifies reductive groups. Consider the data of $T \subseteq B \subseteq G$ and a pinning, i.e., a choice of a nonzero vector in $\mathfrak{g}_{\alpha_i}$ for $i \in I$. This category is called $\mathcal{C}_1$. There is another category $\mathcal{C}_2$ of just $R \subseteq \Lambda$ and $R^\vee \equiv \Lambda^\vee$. There is a functor

$$\mathcal{C}_1 \to \mathcal{C}_2.$$

Theorem 23.7. *This is an equivalence of categories.*

The pinning is needed because the group has more automorphisms than the root system. $T/Z(G)$ will act nontrivially on G, and this can be identified with $\prod_{i \in I} \mathbb{G}_m$ given by $\{\alpha_i\}$.

Because we don't want to see pinnings, which seem to be artificial, we can define the following category. Objects are reductive groups, and morphism $\mathrm{Mor}(G_1, G_2)$ are isomorphisms up to inner automorphisms. You can also define a category with G with some enhancement.

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