Math 99r - Representations and Cohomology of Groups

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This course was taught by Meng Guo, a graduate student. The lectures were given at TF 4:30-6 in Science Center 232. There were no textbooks, and there were 7 undergraduates enrolled in our section. The grading was solely based on the final presentation, with no course assistants.

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1.1 Chain complexes

In the context of and abelian category, we can define a chain complex.

Definition 1.1. An **abelian category** $\mathscr{A}$ is a category such that:

- $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is an abelian group,
- there exists a zero object 0 that is both initial and terminal, i.e., $\mathrm{Hom}(A, 0)$ and $\mathrm{Hom}(0, B)$ are always trivial,
- composition $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is bilinear,
- it has finite products and coproducts (in this case they agree),
- for any $\phi : A \to B$ there exists a kernel $\sigma : K \to A$ with the required universal property,
- for any $\phi : A \to B$ there exists a cokernel $\sigma : B \to Q$,
- every monomorphism is the kernel of its cokernel,
- every epimorphism is the cokernel of its kernel,
- any morphism $\phi : A \to B$ factors like $A \to C \to B$ so that $A \to C$ is an epimorphism and $C \to B$ is a monomorphism.

Example 1.2. The category of R-modules is an abelian category.

Definition 1.3. A functor $F : \mathscr{A} \to \mathscr{B}$ is called **additive** if $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is a group homomorphism.

Definition 1.4. A **chain complex** $\{C_n\}_{n\in\mathbb{Z}}$ is a sequence of objects $C_n \in \mathscr{A}$ with boundary maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

$$\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \xrightarrow{d_{-2}} \cdots$$

Because we have the notion of a kernel, we can say that in a chain complex, there is a monomorphism $\operatorname{im} d_{n+1} \hookrightarrow \ker d_n$. We then define the **homology** as

$$H_n(C_\bullet, d_\bullet) = \operatorname{coker}(\operatorname{im} d_{n+1} \to \ker d_n).$$

Definition 1.5. A **cochain complex** is $(C^\bullet, d^\bullet)$ has maps $d^n : C^n \to C^{n+1}$ that satisfy $d^{n+1} \circ d^n = 0$. We likewise define the **cohomology**.

A **morphism** $f_\bullet : (C_\bullet, d_\bullet) \to (D_\bullet, \delta_\bullet)$ is a collection of maps $f_n : C_n \to D_n$ such that the diagram

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\delta_n"] & D_{n-1} \end{tikzcd}$$

commute for all n. In this case, the chain map gives rise to maps

$$H_\bullet(f_\bullet) : H_\bullet(C_\bullet, d_\bullet) \to H_\bullet(D_\bullet, \delta_\bullet)$$

on homology.

Also there is an obvious equivalence between categories of chain complexes and cochain complexes given by $C_n \mapsto C^{-n}$.

Definition 1.6. A **chain homotopy** between two maps $f_\bullet, g_\bullet : (C, d) \to (D, \delta)$ is a collection of maps $h_n : C_n \to D_{n+1}$ such that

$$h_{n-1}d + dh_n = f_n - g_n.$$

1.2 Projective resolution

Definition 1.7. An object $P \in \mathscr{C}$ is called **projective** if you can always (maybe not uniquely) lift f to $\tilde{f}$ making the diagram commute:

$$\begin{tikzcd} & X \arrow[d, two heads] \\ P \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

A **projective R-module** is a projective object in the category of R-modules. Note that this is equivalent to saying that $\mathrm{Hom}_R(P, -)$ is an exact functor. This is because $\mathrm{Hom}_R(P, -)$ is automatically a left exact functor and this says projectivity says that it preserves epimorphisms.

Definition 1.8. A **projective resolution** of A is a long exact sequence

$$\cdots \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to A \to 0,$$

or equivalently a chain complex $\cdots \to P_1 \to P_0$ with homology

$$H_n(P_\bullet) = \begin{cases} 0 & \text{if } n > 0 \\ A & \text{if } n = 0. \end{cases}$$

Definition 1.9. A **positive complex** is a complex $(C_\bullet, d_\bullet)$ with $C_n = 0$ for all $n < 0$.

Definition 1.10. An positive **acyclic** chain complex is a complex (C, d) with $C_n = 0$ for $n < 0$ and $H_n(C) = 0$ for $n > 0$.

So a projective resolution is positive chain complex that is both acyclic and projective (i.e., consisting of projective objects).

Proposition 1.11. *If (C, d) and (D, δ) are two positive chain complexes, (C, d) is projective, and (D, d_0) is acyclic, then*

$$(C_\bullet \xrightarrow{\varphi} D_\bullet) \quad \mapsto \quad H_0(C_\bullet) \xrightarrow{H_0(\varphi)} H_0(D_\bullet)$$

induces a bijection

$$\left\{\begin{matrix} \textit{homotopy classes of} \\ \textit{chain maps } C_\bullet \to D_\bullet \end{matrix}\right\} \quad \longleftrightarrow \quad \{H_0(C_\bullet) \to H_0(D_\bullet)\}.$$

Proof. So we have to prove two things: first that we can get maps on chains from $H_0(C) \to H_0(D)$, and that if two maps give the same maps then they are chain homotopic.

We are inductively going to lift maps.

$$\begin{tikzcd} C_n \arrow[r] \arrow[d, dashed] & C_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[d] \\ D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r, "d_{n-1}"] & D_{n-2} \end{tikzcd}$$

Note that the composition $C_n \to C_{n-1} \to D_{n-1}$ has image lying in the $\ker d_{n-1} = \operatorname{im} d_n$. Then using that $D_n \to \operatorname{im} d_n$ is an epimorphism and that C_n is projective, we can lift this map.

Now we show that two chain maps giving the same maps on homology is homotopic. We may assume that $f_\bullet$ gives the zero map $H_0(C) \to H_0(D)$ and show that $f_\bullet$ is null-homotopic. You can check this by lifting maps similarly. □

2 February 8, 2017

Last time we defined abelian categories, chain complexes and cochain complexes, associated homology and cohomology. Also we defined a projective resolution of some object $A \in \mathscr{A}$, which is basically a long exact sequence ending with $\cdots \to A \to 0$ consisting of projective objects. For two objects $A, B \in \mathscr{A}$, we saw that the set of morphisms $A \to B$ is the homotopy classes of a chain map from a projective complex to an acyclic complex.

Corollary 2.1. *Any two projective resolutions $P_\bullet, Q_\bullet$ of A are homotopic to each other. In other words, there are chain maps $\phi : P_\bullet \to Q_\bullet$ and $\psi : Q_\bullet \to P_\bullet$ such that $\phi \circ \psi \sim \mathrm{id}_Q$ and $\psi \circ \phi \sim \mathrm{id}_P$.*

Definition 2.2. We say that $\mathscr{A}$ has **enough projectives** if for any $A \in \mathscr{A}$ there exists a projective $P \in \mathscr{A}$ and an epimorphism $P \to A$.

This is equivalent to saying that any $A \in \mathscr{A}$ has a projective resolution, because we can inductively choose the projective that surjects onto the kernel.

Theorem 2.3. *The category $R-\mathsf{Mod}$ has enough projectives.*

2.1 Left derived functors

Suppose we have an additive functor $F : \mathscr{A} \to \mathscr{B}$, i.e., functor that preserves the additive group structure of Hom. If we apply F to a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$

we get a chain complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0.$$

This is not in general exact, but it is a chain complex and so we can compute its homology. We define the **n-th left derived functor** associated to $P_\bullet$ as

$$L_n^{P_\bullet} F(A) = H_n(F(P_\bullet)).$$

Why is it a functor? If there is a map $A \to B$ then we get a map between projective resolutions, and then we get a map between homology.

Theorem 2.4. *Suppose we have an additive functor $F : \mathscr{A} \to \mathscr{B}$. Let P, Q be two assignment of projective resolutions to objects of $\mathscr{A}$. There exists a canonical natural isomorphism $L_n^P F \simeq L_n^Q F$ between the two functors.*

The main idea is that there is a chain map $P_A \to Q_A$ given by the identity id_A and then this induces a $H_n(P_A) \to H_n(Q_A)$.

Lemma 2.5. *An additive functor $F : \mathscr{A} \to \mathscr{B}$ induces an additive functor $F : \mathsf{Ch}\mathscr{A} \to \mathsf{Ch}\mathscr{B}$, given by $F(C_\bullet)_n = F(C_n)$ and $F(\psi_\bullet)_n = F(\psi_n)$. This satisfies the following:*

(i) If $\Sigma_\bullet : \psi_\bullet \to \varphi_\bullet$ is a homotopy in two maps $\psi_\bullet, \varphi_\bullet \in \mathrm{Hom}_{\mathsf{Ch}\mathscr{A}}(C_\bullet, D_\bullet)$, then $F(\Sigma_\bullet) : F(\psi_\bullet) \to F(\varphi_\bullet)$ is also a homotopy between $F(\psi_\bullet), F(\varphi_\bullet) \in \mathrm{Hom}_{\mathsf{Ch}\mathscr{B}}(F(C_\bullet), F(D_\bullet))$.

(ii) If $C_\bullet \simeq D_\bullet$ are homotopic in $\mathsf{Ch}\mathscr{A}$, then $F(C_\bullet) \simeq F(D_\bullet)$ in $\mathsf{Ch}\mathscr{B}$.

(iii) If $F : \mathscr{A} \to \mathscr{B}$ is an exact functor, then the induced functor $F : \mathsf{Ch}\mathscr{A} \to \mathsf{Ch}\mathscr{B}$ is also exact.

Proof. It is clear that $F : \mathsf{Ch}\mathscr{A} \to \mathsf{Ch}\mathscr{B}$ is an additive functor, because we can add vertical maps one by one.

(i) This is clear, because F is additive and so the conditions translates exactly.

(ii) follows from (i).

(iii) A sequence of chain complexes being exact is equivalent to the sequence at each level being exact. So using exactness of F, we get the exactness of the sequences at each level after applying F. □

Proof of Theorem 2.4. We already have maps id_A between A. So we get two chain maps

$$\begin{tikzcd} P_A \arrow[r] \arrow[d, shift left, "\psi_\bullet"] & A \arrow[d, shift left, "\mathrm{id}_A"] \\ Q_A \arrow[r] \arrow[u, shift left, "\varphi_\bullet"] & A \arrow[u, shift left, "\mathrm{id}_A"] \end{tikzcd}$$

with $\varphi \circ \psi \simeq \mathrm{id}_{P_A}$ and $\psi \circ \varphi \simeq \mathrm{id}_{P_A}$. Then applying F gives

$$F(\varphi) \circ F(\psi) \simeq \mathrm{id}_{F(P_A)}, \quad F(\psi) \circ F(\varphi) \simeq \mathrm{id}_{F(P_A)}.$$

This shows that $H_\bullet(P_A) \cong H_\bullet(Q_A)$, and this does not depend on φ and ψ because homotopic maps give exactly the same map on homology.

Naturality can be checked. □

This theorem shows that the left derived functor L_nF can be defined independent of the choice of the projective resolutions.

Example 2.6. For a fixed R-module B, take the functor

$$F : R{-}\mathsf{Mod} \to R{-}\mathsf{Mod}; \quad A \mapsto A \otimes_R B.$$

This gives left derived functors

$$\mathrm{Tor}_R^n(A, B) = L_nF(A),$$

and these are called the **Tor functors**.

Example 2.7. Let R be a ring and G be a (multiplicative) group. We define the **group ring** as the ring

$$RG = \left\{ \sum_{g_i \in G} r_i g_i : \text{only finitely many } g_i \text{ are nonzero} \right\}$$

with the multiplication structure coming from the group relations $(g_i) \cdot (g_j) = (g_i g_j)$.

Let M be an RG-module. This can also be thought of as an R-module M with an action of G, or an R-linear representation of G. We define the **group homology**

$$H_n(G, M) = \mathrm{Tor}_{RG}(R, M),$$

where the RG-module structure on R is trivial, i.e., $(\sum r_i g_i)r = (\sum r_i)r$.

Proposition 2.8. *If $F : \mathscr{A} \to \mathscr{B}$ is a right exact functor, then there is a canonical natural isomorphisms $L_0 F \cong F$.*

Proof. The 0-th derived functor is defined as

$$L_0 F(A) = \mathrm{coker}(F(P_1) \to F(P_0)).$$

If F is right exact, it preserves cokernels and so $L_0 F(A) \cong F(A)$. □

Proposition 2.9. *If P is projective, $L_n F(P) = 0$ for $n > 0$.*

Proof. $0 \to P \to 0$ is a resolution. □

2.2 Injective resolutions and right derived functors

There is a dual of all the stuff we did so far. Recall that projective means

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

So we may define:

Definition 2.10. An **injective object** I is an object such that there is always a lift

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed] \\ & P & \end{tikzcd}$$

This is same as saying that $\mathrm{Hom}(-, I)$ is exact.

Definition 2.11. An **injective resolution** of A is an exact sequence

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots .$$

We can now define a right derived functor. For an additive functor $F : \mathscr{A} \to \mathscr{B}$, its **$n$-th right derived functor** is defined as

$$R^n F(A) = H^n(F(I_\bullet)).$$

If F is left exact, then $R^0F \cong F$. Also if I is injective then $R^nF(I) = 0$ for $n > 0$.

If F is a contravariant functor, then this can be thought of as a covariant functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$. So you can do similar things. One thing to be careful is that if $P \in \mathscr{A}$ is projective, then $P \in \mathscr{A}^{\mathrm{op}}$ is injective. For instance, to define left derived functors, you first need to pick an injective resolution and apply F and take the homology.

Example 2.12. Let M be a fixed R-module, and consider the functor

$$\mathrm{Hom}_R(M, -) : R\text{−}\mathsf{Mod} \to R\text{−}\mathsf{Mod}; \quad N \mapsto \mathrm{Hom}_R(M, N).$$

The **Ext functor** is defined as

$$\mathrm{Ext}^n_R(M, N) = R^n(\mathrm{Hom}_R(M, -))(N).$$

Example 2.13. Similarly, the **group cohomology** for an R-module M is defined as

$$H^n(G, M) = \mathrm{Ext}^n_{RG}(R, M).$$

Example 2.14. There is also a contravariant functor $\mathrm{Hom}_R(-, N)$. So we may also define

$$\mathrm{Ext}^n_R(M, N) = R^n(\mathrm{Hom}_R(-, N))(M).$$

What is the different between the two definitions of $\mathrm{Ext}^n_R(M, N)$? The first definition is first finding an injective resolution $0 \to N \to I^\bullet$ and compute the cohomology of

$$0 \to \mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \cdots,$$

and the second definition is find a projective resolution $P_\bullet \to M \to 0$ and take the cohomology of

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots.$$

They turn out the be equivalent definitions. People use the second definition more often because they like projective resolutions.

3 February 14, 2017

3.1 Long exact sequence

Lemma 3.1. *Suppose C is a chain complex in $\mathscr{A}$. There is a unique morphism $\delta_n : \operatorname{coker}\partial_{n+1} \to \ker\partial_{n-1}$ such that $C_n \to C_{n-1}$ factors as*

$$C_n \to \operatorname{coker}\partial_{n+1} \xrightarrow{\delta_n} \ker\partial_{n-1} \to C_{n-1}$$

and there exists a exact sequence

$$0 \to H_n(C) \to \operatorname{coker}\partial_{n+1} \xrightarrow{\delta_n} \ker\partial_{n-1} \to H_{n-1}(C) \to 0.$$

Proof. Let us first construct this map. We have an epimorphism $\operatorname{coker}\partial_{n+1} \to \operatorname{im}\partial_n$, and a monomorphism $\operatorname{im}\partial_n \to \ker\partial_{n-1}$. So composing these give the map δ_n. This construction also shows that

$$\begin{aligned}
\ker\delta_n &= \ker(\operatorname{coker}\partial_{n+1} \to \operatorname{im}\partial_n) = H_n(C), \\
\operatorname{coker}\delta_n &= \operatorname{coker}(\operatorname{im}\partial_n \to \ker\partial_{n-1}) = H_{n-1}(C).
\end{aligned}$$

□

Theorem 3.2. *If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of chain complexes in an abelian category, then there is a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(A) \arrow[r] & H_n(A'') \arrow[dll] \\
& H_{n-1}(A') \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(A'') \arrow[r] & \cdots .
\end{tikzcd}$$

Proof. This follows from the snake lemma on

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& H_n(A') \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(A'') \arrow[d] & \\
& \operatorname{coker}\partial'_n \arrow[r] \arrow[d] & \operatorname{coker}\partial_n \arrow[r] \arrow[d] & \operatorname{coker}\partial''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\partial'_n \arrow[r] \arrow[d] & \ker\partial_n \arrow[r] \arrow[d] & \ker\partial''_n \arrow[d] & \\
& H_{n-1}(A') \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(A'') \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Then we immediately get the desired sequence. □

Theorem 3.3. *Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathscr{A}$ and $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Suppose that $\mathscr{A}$ has enough projectives. Then there exists a map $w_n : L_nF(A) \to L_{n-1}F(A')$ such that*

$$\begin{tikzcd}
 & \cdots \arrow[r] & L_nF(A) \arrow[r] & L_nF(A'') \arrow[dll, "w_n"'] \\
L_{n-1}F(A') \arrow[r] & L_{n-1}F(A) \arrow[r] & \cdots &
\end{tikzcd}$$

$$\cdots \longrightarrow L_0F(A) \longrightarrow L_0F(A'') \longrightarrow 0$$

is a long exact sequence.

Proof. We first look at the projective resolutions

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0.
\end{tikzcd}$$

But if we apply F, there is no reason for the sequence to remain exact. The key idea is to make $P_n = P'_n \oplus P''_n$. We look at the first level.

$$\begin{tikzcd}
0 \arrow[r] & P'_1 \arrow[r] \arrow[d, "\partial'"] & P'_1 \oplus P''_1 \arrow[r] \arrow[d, dashed, "\partial"] & P''_1 \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "\epsilon'"] & P'_0 \oplus P''_0 \arrow[r] \arrow[d, dashed, "\epsilon"] & P''_0 \arrow[r] \arrow[d, "\epsilon''"] \arrow[dl, dashed, "\epsilon_0"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We first lift the map $P''_0 \to A$ using the fact that P''_0 is projective and $A \to A''$ is surjective. Then take $\epsilon(a', a'') = f\epsilon'(a') + \epsilon_0(a'')$. Let us look at the next step. Because $g \circ \epsilon_0 \circ \partial'' = 0$, we see that there is some $\nu : P''_1 \to A'$ such that $f \circ \nu = \epsilon_0 \circ \partial''$. Then using surjectivity of $P'_0 \to A'$ we can lift the map to $\lambda : P''_1 \to P'_0$ such that $f\epsilon'\lambda = \epsilon_0\partial''$. Define

$$\partial(a', a'') = (\partial'(a') - \lambda(a''), \partial''(a'')).$$

Then we check that

$$\epsilon\partial(a', a'') = f\epsilon'(\partial'(a') - \lambda(a'')) + \epsilon_0\partial''(a'') = -f\epsilon'\lambda(a'') - \epsilon_0\partial''(a'') = 0$$

and further that this is exact. Continue this. $\square$

Likewise we have the long exact sequence coming from right derived functors.

Theorem 3.4. *We have a long exact sequence*

$$0 \to R_0F(A') \to \cdots \to R_nF(A) \to R_nF(A'') \to R_{n+1}F(A') \to R_{n+1}F(A) \to \cdots .$$

Corollary 3.5. *L_0F is right exact and R_0F is left exact.*

For example, we have the exact sequence

$$\begin{aligned}\cdots \to \operatorname{Tor}_1^R(A', B) \to \operatorname{Tor}_1^R(A, B) \to \operatorname{Tor}_1^R(A'', B)& \\ \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0.&\end{aligned}$$

So Tor measures the non-exactness of the $(- \otimes B)$ functor.

Likewise we have

$$\begin{aligned}0 \to \operatorname{Hom}_R(B, A') \to \operatorname{Hom}_R(B, A) \to \operatorname{Hom}_R(B, A'')& \\ \to \operatorname{Ext}_R^1(B, A') \to \operatorname{Ext}_R^1(B, A) \to \operatorname{Ext}_R^1(B, A'') \to \cdots .&\end{aligned}$$

We also have

$$\begin{aligned}0 \to \operatorname{Hom}_R(A'', B) \to \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A', B)& \\ \to \operatorname{Ext}_R^1(A'', B) \to \operatorname{Ext}_R^1(A, B) \to \operatorname{Ext}_R^1(A', B) \to \cdots .&\end{aligned}$$

One property is that

$$\operatorname{Ext}_R^n\Big(\bigoplus A_i, B\Big) = \bigoplus \operatorname{Ext}_R^n(A_i, B), \quad \operatorname{Tor}_n^R\Big(\bigoplus A_i, B\Big) = \bigoplus \operatorname{Tor}_n^R(A_i, B).$$

4 February 17, 2017

Today we are going to talk more about the Ext functor. We will give a interpretation

$$\mathrm{Ext}^n_R(M,N) = \left\{ \begin{matrix} \text{equivalence class of } n\text{-fold extensions} \\ 0 \to N \to M_{n-1} \to \cdots \to M_0 \to M \to 0. \end{matrix} \right\}$$

4.1 Ext as extensions

We say that two extension are **equivalent** if there are maps

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & M_{n-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, equal] \arrow[d, equal] & P_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[u] \arrow[d] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M'_{n-1} \arrow[r] & \cdots \arrow[r] & M'_0 \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where P_i are projective. Reflexivity and symmetry are clear. To check transitivity, you look at the pullback of $P_k \to M_k \leftarrow P'_k$.

Recall that

$$\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(P_\bullet, N)),$$

where $P_\bullet \to M$ is a projective resolution. Note that if you have a projective resolution, then you can construct maps and then the map $P_n \to N$ represents a cycle in $\mathrm{Hom}(P_n, N)$.

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M_{n-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Note that this element is independent of the resolution $P_\bullet$ and so this gives a map $\{\text{equivalence classes}\} \to \mathrm{Ext}^n_R(M,N)$.

We now check that this map is bijective. Given an element $[f] \in H^n(\mathrm{Hom}_R(P_\bullet, N))$, how do we construct the extension? First we may assume that $P_n \to N$ is surjective by direct summing R^k for sufficiently large k to the projective resolution. Then we can set

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, two heads, "f"] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, hook] & P_{n-1}/\ker f \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

In the same way, if two extensions come from the same element in $\mathrm{Ext}^n_R(M,N)$, then we can exhibit a equivalence.

Take a short exact sequence $0 \to N \to M_0 \to M \to 0$. The equivalence class of this can be seen to be the image of id_M in the map

$$0 \to \mathrm{Hom}(N,N) \to \mathrm{Hom}(M,M_0) \to \mathrm{Hom}(M,M) \to \mathrm{Ext}^1_R(M,N) \to \cdots .$$

Then we get an element of $\mathrm{Ext}^1_R(M,N)$.

Let us see how this generalizes to the higher order. Given an exact sequence $0 \to N \to M_1 \xrightarrow{\beta} M_0 \to M \to 0$, we get two short exact sequences $0 \to N \to M_1 \to E \to 0$ and $0 \to E \to M_0 \to M \to 0$, where $E = \operatorname{im}\beta$. Then we get long exact sequences coming from $\mathrm{Hom}(-,N)$,

$$\cdots \to \mathrm{Hom}(N,N) \to \mathrm{Ext}^1_R(E,N) \to \cdots ,$$
$$\cdots \to \mathrm{Ext}^1_R(E,N) \to \mathrm{Ext}^2_R(M,N) \to \cdots .$$

The image of id_N under the two maps gives an element of $\mathrm{Ext}^2_R(M,N)$. This directly generalizes to higher n.

There is an R-linear map

$$\mathrm{Ext}^n_R(M,N) \times \mathrm{Ext}^m_R(N,L) \to \mathrm{Ext}^{n+m}_R(M,L)$$

which is called the **Yoneda multiplication**. How is this defined? Given two extensions

$$0 \to N \to M_{n-1} \to \cdots \to M_0 \to M \to 0,$$
$$0 \to L \to Q_{m-1} \to \cdots \to Q_0 \to N \to 0,$$

we define the product as

$$0 \to L \to Q_{m-1} \to \cdots \to Q_0 \to M_{n-1} \to \cdots \to M_0 \to M \to 0.$$

This gives a graded ring structure on $\mathrm{Ext}^\bullet_R(M,M)$. Then $\mathrm{Ext}^\bullet_R(M,N)$ has a $\mathrm{Ext}^\bullet_R(M,M)$-module structure. Using this multiplication, we can say that in the long exact sequence

$$0 \to \mathrm{Hom}(L,N) \to \mathrm{Hom}(L,M_0) \to \mathrm{Hom}(L,M) \to$$
$$\cdots \to \mathrm{Ext}^n(L,M) \to \mathrm{Ext}^{n+1}(L,N) \to \cdots ,$$

the connecting maps are multiplication by x, where $x \in \mathrm{Ext}^1_R(M,N)$ represents the sequence $0 \to N \to M_0 \to M \to 0$.

4.2 Low-dimensional group cohomology

Recall that

$$H^*(G,M) = \mathrm{Ext}^*_{RG}(R,M).$$

Taking $R = \mathbb{Z}$, we have

$$H^0(G,M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},M) = \mathrm{Hom}_G(\mathbb{Z},M) = M^G.$$

Theorem 4.1. *The first cohomology is*

$$H^1(G,M) = \left\{ \begin{matrix} M\text{-conjugate classes of splitting group extensions} \\ 1 \to M \to M \rtimes G \to G \to 1 \end{matrix} \right\}.$$

5 February 21, 2017

Last time we showed that $\mathrm{Ext}^n_R(M,N)$ is the equivalence classes of n-fold extensions $0 \to N \to M_{n-1} \to \cdots \to M_0 \to N \to 0$.

5.1 Computing the first cohomology and homology

Today we are going to compute the first cohomology of groups. We claim that

$$H^1(G,M) = \begin{Bmatrix} M\text{-conjugacy classes of group extensions} \\ 1 \to M \to E \to G \to 1 \text{ that are splitting} \end{Bmatrix}.$$

A group extension means that we have a short exact sequence, i.e., $E/M \cong G$. The extension is said to **split** if there is a section $s : G \to E$ that is also a group homomorphism. In this case, we can write $E = M \rtimes G$ as a semi-direct product. The multiplicative structure is something like

$$(m_1, g_1)(m_2, g_2) = (m_1 + g_1 m_2, g_1 g_2).$$

Definition 5.1. A map $d : G \to M$ is a **derivation** if $d(gh) = gd(h) + d(g)$.

Proposition 5.2. *For any section $s : G \to M \rtimes G$ of the form $s(g) = (d(g), g)$, the map s is a group homomorphism if and only if d is a derivation.*

Proof. Just compute $s(g)s(h) = (d(g) + gd(h), gh)$. □

Now we need to compute the first cohomology. Note that there is a natural $\mathbb{Z}G$-linear map $\mathbb{Z}G \to \mathbb{Z}$, and we can look at its kernel:

$$0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

This gives a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \to \mathrm{Hom}_{\mathbb{Z}G}(IG, M) \\ &\to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, M) \to \cdots . \end{aligned}$$

Since $\mathbb{Z}G$ is a free $\mathbb{Z}G$-module, we see that

$$0 \to M^G \to M \to \mathrm{Hom}_{\mathbb{Z}G}(IG, M) \to H^1(G,M) \to 0.$$

Proposition 5.3. *A map of sets $d : G \to M$ defines a group homomorphism $\delta : IG \to M$ given by $\delta(g-1) = d(g)$. This δ is a $\mathbb{Z}G$-linear if and only if d is a derivation.*

Proof. The map δ being G-equivariant means $\delta(h(g-1)) = h\delta(g-1)$. This means $d(hg) - d(h) = hd(g)$. □

Now this shows that the $\mathrm{Hom}_{\mathbb{Z}G}(IG, M) = \mathrm{Der}(G, M)$. ($\mathrm{Der}(G, M)$ has G action given by $(gd)(h) = gd(h^{-1}hg)$.) So we now have to study that map $M \to \mathrm{Der}(G, M)$. By definition, any element $m \in M$ maps to $(-)m : IG \to M$. So an element $m \in M$ maps to the derivation $d(g) = gm - m$. So the conclusion is

$$H^1(G, M) = \frac{\mathrm{Der}(G, M)}{\{(g \mapsto gm - m) : m \in M\}}.$$

Definition 5.4. We say that two section $s_1, s_2; G \to M \rtimes G$ are M-**conjugate** if there exists an $m \in M$ such that $(m, 1)s_1(g)(m, 1)^{-1} = s_2(g)$.

This is equivalent to $d_1(g) = d_2(g) + m - gm$. So we get our theorem. Think about how this relates to the interpretation

$$H^1(G, M) = \{0 \to M \to E \to \mathbb{Z} \to 0\}.$$

Recall that $H_n(G, M) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$.

Proposition 5.5. $H_1(G, \mathbb{Z}) \cong G/[G, G] = G^{\mathrm{ab}}$.

Proof. We again take the sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$. Then we get

$$\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}G} IG \to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0.$$

Then

$$\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}G} IG \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0.$$

Now note that $\mathbb{Z} \otimes_{\mathbb{Z}G} IG = IG/(IG)^2$. This can be seen to be isomorphic to the abelianization $G/[G, G]$. $\square$

Next time we are going to construct a space $K(G, 1)$ such that $\pi_1(K) = G$ and $\pi_i(K) = 0$ for $i > 1$. Then we get $H_1(G, \mathbb{Z}) = H_1(K(G, 1))$. So H_1 is the abelianization of π_1.

6 February 24, 2017

6.1 Bar resolution

We note that free R-modules are always projective. Let

$$F_n = \text{free } R\text{-module on } (n+1)\text{-tuples } (g_0, \dots, g_n) \in G^{n+1},$$

and we define the maps

$$\partial_n(g_0, \dots, g_n) = \sum_{i=1}^{n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n).$$

Then we get a sequence

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to R \to 0.$$

We first claim that this is acyclic. Define the homotopy $h_n : F_n \to F_{n+1}$ by

$$h_n(g_0, \dots, g_n) = (1, g_0, \dots, g_n).$$

Then you can check that $\partial h_n + h_{n-1}\partial = \mathrm{id}$ for $n \geq 1$. So this acyclic.

We can further give a free RG-module structure on each F_n. We are going to give the action $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$. We use the **bar notation**

$$\begin{aligned} (g_0, \dots, g_n) &= g_0[g_0^{-1}g_1 | g_1^{-1}g_2 | \cdots | g_{n-1}^{-1}g_n], \\ [g_1|g_2|\cdots|g_n] &= (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n). \end{aligned}$$

Then F_n is a free RG-module generated by $[g_1|\cdots|g_n]$. Using this notation, we can write

$$\begin{aligned} \partial_n[g_1|\cdots|g_n] = g_1[g_2|\cdots|g_n] &+ \sum_{i=1}^{n-1}(-1)^i [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &+ (-1)^n[g_1|\cdots|g_{n-1}]. \end{aligned}$$

Now we have a free (and hence projective) resolution $F_\bullet \to R \to 0$ and so we have

$$\begin{aligned} H_i(G, M) &= H_i(M \otimes_{RG} F_\bullet), \\ H^i(G, M) &= H^i(\mathrm{Hom}_{RG}(F_\bullet, M)). \end{aligned}$$

Here, note that the cochain complex for cohomology is

$$0 \to \mathrm{Hom}_{RG}(F_0, M) \xrightarrow{d_0} \mathrm{Hom}_{RG}(F_1, M) \xrightarrow{d_1} \cdots$$

where the cochain complex are

$$C^i(G, M) = \mathrm{Hom}_{RG}(F_i, M) = \mathrm{Fun}(G^n, M)$$

and the coboundary maps

$$\begin{aligned}(d^n f)([g_1|\cdots|g_{n+1}]) &= f\circ\partial_n[g_1|\cdots|g_{n+1}] \\ &= g_1 f([g_2|\dots|g_{n+1}]) + \sum_{i=1}^{n}(-1)^i f([g_1|\dots|g_i g_{i+1}|\dots|g_{n+1}]) \\ &\quad + (-1)^{n+1} f([g_1|\dots|g_n]).\end{aligned}$$

6.2 Computing cohomology using the bar resolution

In particular, we have

$$C^0(G,M) = \mathrm{Hom}_{RG}(RG\langle[]\rangle, M) \cong M.$$

Then for the generator $a \in C^0(G,M)$, we have

$$(d^0 a)([g]) = ga - a.$$

Given a function $f \in C^1(G,M)$, we have

$$(d^1 f)([g_1|g_2]) = g_1 f([g_2]) - f([g_1 g_2]) + f([g_1]).$$

Note that $d^1 f = 0$ is equivalent to $f([g_1 g_2]) = g_1 f([g_2]) + f([g_1])$. This is exactly saying that $f \in \mathrm{Der}(G,M)$. This again shows

$$H^1(G,M) = \frac{\mathrm{Der}(G,M)}{\{ga - a\}}.$$

Let us now compute $H^2(G,M)$. We have for $f \in C^2(G,M)$,

$$(d^2 f)([g_1|g_2|g_3]) = g_1 f([g_2|g_3]) - f([g_1 g_2|g_3]) + f([g_1|g_2 g_3]) - f([g_1|g_2]).$$

So we can write

$$H^2(G,M) = \frac{\{(f : G\times G \to M) : d^2 f = 0\}}{\{f(g_1,g_2) = g_1\alpha(g_2) - \alpha(g_1 g_2) + \alpha(g_1) : (\alpha : G \to M)\}}.$$

Theorem 6.1. *$H^2(G,M)$ is the equivalence classes of group extensions $1 \to M \to E \to G \to 1$ with the given G-action.*

Let us first describe the bijection. Suppose we are given a group extension

$$1 \longrightarrow M \xrightarrow{i} E \underset{s}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 1.$$

If g is a group homomorphism, then it splits. In general, we can find a function $f : G \times G \to M$ such that

$$i(f(g,h)) = s(gh)s(g)^{-1}s(h)^{-1}.$$

This function f must satisfy

$$\begin{aligned}s(ghk) &= i(f(gh,k))s(gh)s(k) = i(f(gh,k))i(f(g,h))s(g)s(h)s(k) \\ &= i(f(g,k))s(g)s(hk) = i(f(g,hk))s(g)i(f(hk))s(h)s(k).\end{aligned}$$

7 February 28, 2017

A group extension of G by M is a short exact sequence

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1.$$

Since M is a normal subgroup of E, the group E acts on M by conjugation. If M is abelian, we further get an action on G on M, induced from E, by lifting $g \in G$ to $\tilde{g} \in E$ and letting $i(g \cdot m) = \tilde{g} i(m) \tilde{g}^{-1}$. If we are already given an $\mathbb{Z}G$-module structure, we require the two actions to be the same.

We pick up from last time. The first cohomology $H^1(\mathbb{Z}/2, \mathbb{F}_2)$ with the trivial action is $H^1(\mathbb{Z}/2, \mathbb{F}_2) = \mathbb{F}_2$. Indeed there are two splitting extensions

$$1 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1,$$

with two sections $s_1 : 1 \mapsto (1,1)$ and $s_2 : 1 \mapsto (0,1)$.

7.1 Computing the second cohomology

We claim that $H^2(G, M)$ is the equivalence classes of $1 \to M \to E \to G \to 1$. Let us choose any (set) section s. (If s is a group homomorphism, then the group splits.) Then we should be able to write

$$s(gh) i(f(g,h)) = s(g)s(h)$$

for some $i : G \times G \to M$. Then the group extension structure can be completely recovered by f and the $\mathbb{Z}G$-module structure on M, because

$$i(m)s(g)i(n)s(h) = i(m + gn)s(g)s(h) = i(m + gn + f(g,h))s(gh).$$

But an arbitrary f does not work, because it needs to satisfy associativity. We have

$$\begin{aligned}
((a,g)(b,h))(c,k) &= (a + gb + f(g,h), gh)(c,k) \\
&= (a + gb + f(g,h) + ghc + f(gh,k), ghk), \\
(a,g)((b,h)(c,k)) &= (a,g)(b + hc + f(h,k), hk) \\
&= (a + gb + ghc + gf(h,k) + f(g,hk), ghk).
\end{aligned}$$

So the condition we get is

$$gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0.$$

This is precisely $d^2 f = 0$, that is, f is in the 2-cocycle.

We now need to quotient out by equivalence. Suppose we have a diagram

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, shift left, "\pi"] \arrow[d, equal] & G \arrow[r] \arrow[d, equal] \arrow[l, shift left, hook', "s_1"] & 1 \\
1 \arrow[r] & M \arrow[r] & E \arrow[r, shift left] & G \arrow[r] \arrow[l, shift left, hook', "s_2"] & 1.
\end{tikzcd}$$

Since $\pi \circ s_1 = \pi \circ s_2 = \mathrm{id}_G$, there is a function $c : G \to M$ such that

$$s_1(g) = i(c(g))s_2(g).$$

Then we have

$$\begin{aligned} i(f_1(g,h) + c(gh))s_2(gh) &= i(f_1(g,h))s_1(gh) = s_1(g)s_1(h) \\ &= i(c(g))s_2(g)i(c(h))s_2(h) = i(c(g) + gc(h))s_2(g)s_2(h) \\ &= i(c(g) + gc(h) + f_2(g,h))s_2(gh). \end{aligned}$$

So we get $(f_1 - f_2)(g,h) = gc(h) - c(gh) + c(g)$. This means that the two are equivalent if and only if they differ by a coboundary.

Theorem 7.1. *The second group cohomology is*

$$H^2(G, M) = \{\textit{equivalence classes of } 1 \to M \to E \to G \to 1\}.$$

Actually I lied, because there is also the condition $f(1,g) = 0 = f(g,1)$. But this is fine because you can use the normalized bar resolution defined by

$$\bar{F}_n = \frac{RG\langle [g_1|\cdots|g_n]\rangle}{RG\langle [g_1|\cdots|g_n] : g_i = 1 \text{ for some } i\rangle}.$$

8 March 3, 2017

8.1 Group action on a CW-complex

We are now going to talk about group homology via topology. Suppose we have a cellular G-action on a CW-complex $\tilde{X}$. Assume that the action completely discontinuous and has no fixed points. Associated to $\tilde{X}$ is a (singular or cellular) chain complex

$$\cdots \to C_2(\tilde{X};\mathbb{Z}) \to C_1(\tilde{X};\mathbb{Z}) \to C_0(\tilde{X};\mathbb{Z}) \to 0,$$

of $\mathbb{Z}G$-modules. If $\tilde{X} \simeq *$, then the two chain complexes $C_*(\tilde{X};\mathbb{Z})$ and $\mathbb{Z}(0)$ are quasi-isomorphic. This means that $C_*(\tilde{X};\mathbb{Z})$ is a free $\mathbb{Z}G$-module resolution of $\mathbb{Z}$.

To compute $H_*(G;\mathbb{Z})$, we need to take $\mathbb{Z} \otimes_{\mathbb{Z}G} (-)$ on the chain complex and look at the homology. We see that

$$\mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\tilde{X};\mathbb{Z}) = C_n(\tilde{X};\mathbb{Z})/\langle gx - x : x \in C_n(\tilde{X};\mathbb{Z})\rangle \cong C_n(\tilde{X}/G;\mathbb{Z}).$$

The second equality holds because simplices are simply connected and so there always is a lifting. We give a name for this:

$$K(G,1) = BG^{\text{discrete}} = \tilde{X}/G.$$

We have then have an exact sequence $G \to \tilde{X} \to \tilde{X}/G$ and so $\pi_1(X) = G$ and $\pi_i(X) = 0$ for $i > 1$. Whitehead's theorem then tells us that $K(G,1)$ is unique up to homotopy.

Let us now give a construction for $\tilde{X}$. For each $g_0, \dots, g_n \in G$, you make a simplicial complex so that the faces of the n-simplex $[g_0, \dots, g_n] \in G^{n+1}$ are

$$\partial[g_0, \dots, g_n] = \sum_{i=0}^{n} (-1)^i [g_0, \dots, \hat{g}_i, \dots, g_n].$$

This is indeed homotopy equivalent to a point, because you can just shrink everything to $[e]$ linearly. The G-action is trivially $g[g_0, \dots, g_n] = [gg_0, \dots, gg_n]$.

Example 8.1. We have $K(\mathbb{Z},1) = S^1$ and $K(\mathbb{Z}/2,1) = \mathbb{R}P^\infty$.

We have $K(G \times H, 1) = K(G,1) \times K(H,1)$ and

$$H_*(G,R) = H_*(K(G,1),R).$$

So for instance, $H_n(\mathbb{Z}/2,\mathbb{F}_2) = \mathbb{F}_2$ for all n. There is also a cup product

$$H^*(G,M) \times H^*(G,M) \xrightarrow{\smile} H^*(G,M).$$

9 March 7, 2017

We might have a presentation during the last lectures. Today we are going to talk about induction and restriction.

9.1 Induction and restriction

Consider an ring morphism $\Gamma \to \Lambda$, so that Λ is an Γ-algebra. There are three things we can do:

- Reduction: If M is a left Λ-module, then we can just consider M as a left Γ-module. We write this as $M \downarrow_\Gamma$ or $\operatorname{Res}_\Gamma^\Lambda M$.
- Induction: If N is a left Γ-module, then we can construct tensor product $\Lambda \otimes_\Gamma M$ as a left Λ-module and a right Γ-module. We write this as $N \uparrow^\Gamma N$ or $\operatorname{Ind}_\Gamma^\Lambda N$.
- Coinduction: If N is a left Γ-module, then we can construct $\operatorname{Hom}_\Gamma(\Lambda, N)$, which is a right Λ-module. We denote this as $N \Uparrow^\Lambda$ or $\operatorname{Coind}_\Gamma^\Lambda N$.

Lemma 9.1. *For Γ and Λ rings, M a left Λ-module, A a Λ-Γ bimodule, and N a left Γ-module, there is a natural isomorphism*

$$\operatorname{Hom}_\Gamma(N, \operatorname{Hom}_\Lambda(A, M)) \cong \operatorname{Hom}_\Lambda(A \otimes_\Gamma N, M).$$

In other words, $A \otimes_\Gamma (-)$ is left adjoint to $\operatorname{Hom}_\Gamma(A, -)$.

Proof. We construct the maps $\phi : \operatorname{Hom}_\Gamma \to \operatorname{Hom}_\Lambda$ as

$$\phi(\alpha)(a \otimes n) = \alpha(n)(a),$$

and $\psi : \operatorname{Hom}_\Lambda \to \operatorname{Hom}_\Gamma$ as

$$(\psi(\beta)(n))(a) = \beta(a \otimes n).$$

You can check that this is a bijection and isomorphism as left Γ-modules. □

Proposition 9.2. *Let $\Gamma \hookrightarrow \Lambda$ be a ring morphism, M be a left Λ-module, N a left Γ-module.*

(i) $\operatorname{Hom}_\Gamma(N, \operatorname{Res}_\Gamma^\Lambda M) \cong \operatorname{Hom}_\Lambda(\operatorname{Ind}_\Gamma^\Lambda, M)$

(ii) $\operatorname{Hom}_\Gamma(\operatorname{Res}_\Gamma^\Lambda M, N) \cong \operatorname{Hom}_\Lambda(M, \operatorname{Coind}_\Gamma^\Lambda N)$

So $\operatorname{Res}_\Gamma^\Lambda$ has a left adjoint $\operatorname{Ind}_\Gamma^\Lambda$ and a right adjoint $\operatorname{Coind}_\Gamma^\Lambda$.

Proof. For (i) use $A = \Lambda$, and for (ii) use switch the roles of Λ and Γ and use $A = \Lambda$. □

Proposition 9.3. *Suppose $\Gamma \hookrightarrow \Lambda$ and Λ is a projective Γ-module. Let M and N be as above.*

(i) $\operatorname{Ext}_\Gamma^n(N, \operatorname{Res}_\Gamma^\Lambda M) \cong \operatorname{Ext}_\Lambda^n(\operatorname{Ind}_\Gamma^\Lambda N, M)$

(ii) $\operatorname{Ext}^n_\Gamma(\operatorname{Res}^\Lambda_\Gamma M, N) \cong \operatorname{Ext}^n_\Lambda(M, \operatorname{Coind}^\Lambda_\Gamma N)$

Proof. (i) Suppose we have a projective resolution $P_\bullet \to N \to 0$ over Γ. Then by definition, $\operatorname{Ext}^n_\Gamma(N, \operatorname{Res}^\Lambda_\Gamma M) = H^n(\operatorname{Hom}(P_\bullet, M))$. Because Λ is projective, tensoring with Λ is exact, and we get a projective resolution $\Lambda \otimes_\Gamma P_\bullet \to \Lambda \otimes_\Gamma N \to 0$. Then

$$\operatorname{Ext}^n_\Lambda(\operatorname{Ind}^\Lambda_\Gamma N, M) = H^n(\operatorname{Hom}(\Lambda \otimes_\Gamma P_\bullet, M)) = H^n(\operatorname{Hom}(P_\bullet, \operatorname{Res}^\Lambda_\Gamma M)).$$

(ii) You do the same thing. Restrict the projection resolution of M to over Γ. This is easier. $\square$

Now suppose you have a group G and a subgroup $H \leq G$. Then there is an inclusion $RH \hookrightarrow RG$. We see that RG is a free RH-module.

If G is finite, then there is an isomorphism

$$RG \cong \operatorname{Hom}_R(RG, R) = (RG)^*$$

as RG-RG bimodules. Then

$$\operatorname{Ind}^G_H N = RG \otimes_{RH} N = (RG)^* \otimes_{RH} N = \operatorname{Hom}_{RH}(RG, N) = \operatorname{Coind}^G_H N.$$

We further have, where we denote $M^* = \operatorname{Hom}_R(M, R)$,

$$\operatorname{Res}^G_H M^* = (\operatorname{Res}^G_H M)^*, \quad \operatorname{Ind}^G_H N^* = (\operatorname{Ind}^G_H N)^*.$$

That is, the dual functor commutes with restriction and induction.

Proposition 9.4. *For finite groups $H \leq G$,*

(i) $\operatorname{Hom}_{RH}(N, \operatorname{Res}^G_H M) \cong \operatorname{Hom}_{RG}(\operatorname{Ind}^G_H N, M)$,

(ii) $\operatorname{Hom}_{RH}(\operatorname{Res}^G_H M, N) \cong \operatorname{Hom}_{RG}(M, \operatorname{Ind}^G_H N)$,

(iii) $\operatorname{Ext}^n_{RH}(N, \operatorname{Res}^G_H M) \cong \operatorname{Ext}^n_{RG}(\operatorname{Ind}^G_H N, M)$,

(iv) $\operatorname{Ext}^n_{RH}(\operatorname{Res}^G_H M, N) \cong \operatorname{Ext}^n_{RG}(M, \operatorname{Ind}^G_H M)$.

If we apply (ii) to $M = R$, then we get

$$H^n(H, N) = \operatorname{Ext}^R_{RH}(\operatorname{Res} R, N) = \operatorname{Ext}^n_{RG}(R, \operatorname{Ind}^G_H N) = H^n(G, \operatorname{Ind}^G_H N).$$

Example 9.5. Let us $H^*(\mathbb{Z}/2; \mathbb{F}_2 \oplus \mathbb{F}_2)$ with the action being switching the two components. Note that this is simply $\mathbb{F}_2[\mathbb{Z}/2] \otimes_{\mathbb{F}_2} \mathbb{F}_2$. Then

$$H^*(\mathbb{Z}/2, \mathbb{F}_2 \oplus \mathbb{F}_2) = H^*(\{1\}, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & * = 0, \\ 0 & * > 0. \end{cases}$$

10 March 21, 2017

Last time we proved

$$\mathrm{Hom}_\Gamma(N, \mathrm{Res}^\Lambda_\Gamma M) \cong \mathrm{Hom}_\Lambda(\mathrm{Ind}^\Lambda_\Gamma N, M),$$
$$\mathrm{Hom}_\Gamma(\mathrm{Res}^\Lambda_\Gamma M, N) \cong \mathrm{Hom}(M, \mathrm{Coind}^\Lambda_\Gamma N),$$

and if Λ is projective over Γ, then the same thing is true for Ext. If $H \subseteq G$ is a subgroup, then

$$H^*(H, M) \cong H^*(G, \mathrm{Coind}^G_H M), \quad H_*(H, M) \cong H_*(G, \mathrm{Ind}^G_H M).$$

10.1 Double coset formula

Consider two subgroups $H, K \le G$. Let us see what happens if we induce from K to G and then restrict from G to H.

Theorem 10.1 (Mackey decomposition)**.** *Let M be an RK-module. Then*

$$\mathrm{Res}^G_H \mathrm{Ind}^G_K M = \bigoplus_{HgK} \mathrm{Ind}^H_{H\cap gKg^{-1}} \mathrm{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM.$$

Here gM is same as M as abelian groups, but has the gKg^{-1}-action $(gkg^{-1})(gm) = gkm$. Furthermore, if H is normal in G, then

$$\mathrm{Res}^G_H \mathrm{Ind}^G_H M = \bigoplus_{g\in G/H} gM.$$

Proof. We have a decomposition

$$RG = \bigoplus_{HgK} R(HgK)$$

as a RH-RK bimodule. Then

$$\begin{aligned}\mathrm{Res}^G_H \mathrm{Ind}^G_K M = RG \otimes_{RK} M &= \bigoplus_{HgK} R(HgK) \otimes_{RK} M \\ &= \bigoplus_{HgK} RH \otimes_{R(H\cap gKg^{-1})} R(gK) \otimes_{RK} M \\ &= \bigoplus_{HgK} \mathrm{Ind}^H_{H\cap gKg^{-1}} \mathrm{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM. \qquad \square\end{aligned}$$

Corollary 10.2. *If M is an RK-module and N an RH-module, then*

(i) $\mathrm{Ind}^G_H M \otimes_R \mathrm{Ind}^G_H N = \bigoplus_{HgK} \mathrm{Ind}^G_{H\cap gKg^{-1}}((\mathrm{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM) \otimes_R (\mathrm{Res}^H_{H\cap gKg^{-1}} N))$

(ii) $\mathrm{Hom}_R(\mathrm{Ind}^G_K M, \mathrm{Ind}^G_H N) = \bigoplus_{HgK} \mathrm{Ind}^G_{H\cap gKg^{-1}} \mathrm{Hom}_R(\mathrm{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM, \mathrm{Res}^H_{H\cap gKg^{-1}} N)$

(iii) $\operatorname{Ind}_K^G M \otimes_{RG} \operatorname{Ind}_H^G N = \bigoplus_{HgK} \operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM \otimes_{R(H\cap gKg^{-1})} \operatorname{Res}^H_{H\cap gKg^{-1}} N$

(iv) $\operatorname{Hom}_{RG}(\operatorname{Ind}_H^G N, \operatorname{Ind}_K^G M) = \bigoplus_{HgK} \operatorname{Hom}_{R(H\cap gKg^{-1})}(\operatorname{Res}^H_{H\cap gKg^{-1}} N, \operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM)$.

Lemma 10.3. *For N an RH-module and M and RG-module,*

(i) $\operatorname{Ind}_H^G(N \otimes_R \operatorname{Res}_H^G M) = \operatorname{Ind}_H^G N \otimes_R M$,

(ii) $\operatorname{Ind}_H^G \operatorname{Hom}_R(N, \operatorname{Res}_H^G M) = \operatorname{Hom}_R(\operatorname{Ind}_H^G, M)$.

Proof of Corollary 10.2. (i) We have, by the decomposition,

$$\begin{aligned}
\operatorname{Ind}_K^G M \otimes_R \operatorname{Ind}_H^G N &= \operatorname{Ind}_H^G(\operatorname{Res}_H^G \operatorname{Ind}_K^G M \otimes_R N) \\
&= \bigoplus_{HgK} \operatorname{Ind}_H^G(\operatorname{Ind}^H_{H\cap gKg^{-1}} \operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} M \otimes_R N) \\
&= \bigoplus_{HgK} \operatorname{Ind}^G_{H\cap gKg^{-1}}(\operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM \otimes_R \operatorname{Res}^H_{H\cap gKg^{-1}} N).
\end{aligned}$$

The other ones can be done similarly. □

The third and fourth formulas work for Ext and Tor.

Corollary 10.4. *For M and RK-module and N and RH-module,*

(v) $\operatorname{Tor}_*^{RG}(\operatorname{Ind}_K^G M, \operatorname{Ind}_H^G N) = \bigoplus_{HgK} \operatorname{Tor}_*^{R(H\cap gKg^{-1})}(\operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM, \operatorname{Res}^H_{H\cap gKg^{-1}} N)$,

(vi) $\operatorname{Ext}^*_{RG}(\operatorname{Ind}_H^G N, \operatorname{Ind}_K^G M) = \bigoplus_{HgK} \operatorname{Ext}^*_{R(H\cap gKg^{-1})}(\operatorname{Res}^H_{H\cap gKg^{-1}} N, \operatorname{Res}^{gKg^{-1}}_{H\cap gKg^{-1}} gM)$.

10.2 Transfer maps

Given any group homomorphism $\alpha : H \to G$, and M a both RG-module and an RG-module, there are induced maps

$$H_*(H, M) \xrightarrow{\alpha_*} H_*(G, M), \quad H^*(G, M) \xrightarrow{\alpha^*} H^*(H, M).$$

If $H \leq G$ has finite index, there are canonical maps in the other direction:

$$\operatorname{Tr}_H^G : H_*(G, M) \to H_*(H, M), \quad \operatorname{Tr}_H^G : H^*(H, M) \to H^*(G, M).$$

These are called **transfer maps**.

Let us see how to construct this. Let us name our inclusion map $\alpha : H \hookrightarrow G$. There is a canonical surjection $RG \otimes_{RH} M \to M$ and a canonical injection $M \hookrightarrow \operatorname{Hom}_{RH}(RG, M)$. Let us first see how the maps α_* and α^* are defined.

$$\begin{tikzcd}
H_*(G, RG \otimes_{RH} M) \arrow[r] \arrow[d, equal] & H_*(G, M) \\
H_*(H, M) \arrow[ur, "\alpha_*"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(G, M) \arrow[r] \arrow[dr, "\alpha^*"'] & H^*(G, \operatorname{Hom}_{RH}(RG, M)) \arrow[d, equal] \\
& H^*(H, M)
\end{tikzcd}$$

Now in the case when the index $H \le G$ is finite, coinduction is the same as induction. So we can apply the homology or cohomology in the other way.

$$\begin{tikzcd} H^*(G, RG \otimes_{RH} M) \arrow[r] & H^*(G,M) \\ H^*(H,M) \arrow[u, equal] \arrow[ur, "\mathrm{Tr}_H^G"'] & \end{tikzcd} \qquad \begin{tikzcd} H_*(G,M) \arrow[r] \arrow[dr, "\mathrm{Tr}_H^G"'] & H_*(G, \mathrm{Hom}_{RH}(RG, M)) \arrow[d, equal] \\ & H_*(H,M) \end{tikzcd}$$

11 March 24, 2017

Last time we defined the transfer map for $[G : H] < \infty$ using induction. These maps also can be defined directly.

11.1 Another way of defining transfer maps

For RG-modules M and M', there is a map

$$\mathrm{Tr}_H^G : \mathrm{Hom}_{RH}(M, M') \to \mathrm{Hom}_{RG}(M, M'); \quad \phi \mapsto \sum_{g \in G/H} (g\phi : m \mapsto g\phi(g^{-1}m)).$$

Proposition 11.1. *Let $H \le G$ be a finite index subgroups.*

(i) *Let M_1 be a RH-module and M_2, M_3 be RG-modules. If $\alpha \in \mathrm{Hom}_{RH}(M_1, M_2)$ and $\beta \in \mathrm{Hom}_{RG}(M_2, M_3)$, then $\beta \circ \mathrm{Tr}_H^G(\alpha) = \mathrm{Tr}_H^G(\beta \circ \alpha)$.*

(ii) *Under a similar condition, $\mathrm{Tr}_H^G(\beta) \circ \alpha = \mathrm{Tr}_H^G(\beta \circ \alpha)$.*

(iii) *If $H \le K \le G$ are finite index subgroups, $\mathrm{Tr}_K^G \mathrm{Tr}_H^K = \mathrm{Tr}_H^G$.*

(iv) *If $H, K \le G$ and $\alpha \in \mathrm{Hom}_{RH}(M_1, M_2)$, then*

$$\mathrm{Tr}_K^G(\alpha) = \sum_{HgK} \mathrm{Tr}_{H \cap gKg^{-1}}^H (g\alpha).$$

Given any projective resolution $P_\bullet \to M$ as an RG-module, it is also a projective resolution as an RH-module. This gives a transfer map

$$\mathrm{Hom}_{RH}(P_\bullet, M') \to \mathrm{Hom}_{RG}(P_\bullet, M').$$

Applying cohomology gives a map $\mathrm{Ext}^n_{RH}(M, M') \to \mathrm{Ext}^n_{RG}(M, M')$. In particular, letting $M = R$ gives a map $H^*(H, M) \to H^*(G, M)$.

You can do a similar thing for homology. For M a right RG-module and M' a left RG-module, there is a map

$$\mathrm{Tr}_H^G : M \otimes_{RG} M' \to M \otimes_{RH} M'; \quad m \otimes m' \mapsto \sum_{g \in G/H} mg \otimes g^{-1}m'.$$

Proposition 11.2. *Let $H \le G$ be a subgroup of finite index.*

(i) $\beta \circ \mathrm{Tr}_H^G(\alpha) = \mathrm{Tr}_H^G(\beta \circ \alpha)$

(ii) $\mathrm{Tr}_H^G(\beta) \circ \alpha = \mathrm{Tr}_H^G(\beta \circ \alpha)$

(iii) $\mathrm{Tr}_K^G \mathrm{Tr}_H^K = \mathrm{Tr}_H^G$

(iv) $\mathrm{Res}_H^G \mathrm{Tr}_K^G(\alpha) = \sum_{HgK} \mathrm{Tr}_{H \cap gKg^{-1}}^H \mathrm{Res}_{H \cap gKg^{-1}}^{gKg^{-1}}(g\alpha)$ *for $\alpha \in H^*(K, M)$.*

(v) $\mathrm{Tr}_H^G \mathrm{Res}_K^G(\alpha) = \sum_{HgK} \mathrm{Res}_{H \cap gKg^{-1}}^H \mathrm{Tr}_{H \cap gKg^{-1}}^{gKg^{-1}}(g\alpha)$ *for $\alpha \in H_*(K, M)$.*

(vi) $\operatorname{Tr}_H^G \operatorname{Res}_H^G(\alpha) = [G:H]\alpha$ *for* $\alpha \in H^*(G, M)$.

On the zeroth homology and cohomology, there is a very explicit description. Recall that $H_0(G, M) = M/(gm - m)$. Then the transfer map is given by

$$\operatorname{Tr}_H^G : [m]_G \mapsto \sum_{g \in H\backslash G} [gm].$$

In the other direction, $H^0(G, M) = M^H$. We have

$$\operatorname{Tr}_H^G : m \mapsto \sum_{g \in G/H} gm.$$

There is also a topological description. If $H \leq G$ is a finite index, there is a covering map $BH \to BG$ of index $[G : H]$. Using the theory of covering spaces, you can construct the transfer maps.

Corollary 11.3. *Let* M *be an* RG*-module with* $[G : H] < \infty$*, and assume that* $H^n(H, M) = 0$ *for some* n*. Then* $H^n(G, M)$ *is annihilated by* $[G : H]$*. In particular, if* $[G : H]$ *is invertible in* M *then* $H^n(G, M) = 0$.

Proof. For the first part, simply note that $\operatorname{Tr}_H^G \operatorname{Res}_H^G(\alpha) = [G : H]\alpha$. For the second part, multiplication by $[G : H]$ induces an isomorphism $M \to M$ and hence an isomorphism $H^n(G, M) \to H^n(G, M)$. □

Corollary 11.4. *If* $|G|$ *is finite, then* $H^n(G, M)$ *is annihilated by* $|G|$ *for all* $n > 0$*. If* $|G|$ *is invertible in* M*, then* $H^n(G, M) = 0$.

Proposition 11.5. *If* $[G : H]$ *is invertible in* R*, then*

$$\operatorname{Res}_H^G : \operatorname{Ext}^n_{RG}(M, M') \hookrightarrow \operatorname{Ext}^n_{RH}(M, M')$$

is injective. In particular, $H^*(G, M) \hookrightarrow H^*(H, M)$.

Denote by $H^n(G, M)_{(p)}$ the p-primary component that vanishes under multiplication by some power of p. If G, is finite, we get an embedding

$$H^n(G, M)_{(p)} \hookrightarrow H^n(P, M),$$

where $P \leq G$ is a p-Sylow subgroup.

Definition 11.6. An element $z \in H^n(H, M)$ is G**-invariant** if

$$\operatorname{Res}^H_{H \cap gHg^{-1}} z = \operatorname{Res}^{gHg^{-1}}_{H \cap gHg^{-1}} gz$$

for all $g \in G$.

In the case when H is normal in G, there is an action of G/H on $H^*(H, M)$. Then being G-invariant is exactly being invariant under this action.

Proposition 11.7. $H^n(G, M)_{(p)}$ *is isomorphic to the set of* G*-invariant elements in* $H^n(P, M)$.

12 March 28, 2017

12.1 Spectral sequence from a filtration

A spectral sequence consist of a collection the E^r-page $E^r_{p,q}$ with differentials

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

So the total degree decrease by 1 as $p+q \to p+q-1$. The filtration degree p decreases by r. The next page can be computed as

$$E^{r+1}_{p,q} = H(E^r, d^r).$$

For cohomology, we will get

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad E_{r+1}^{p,q} = H(E_r, d_r).$$

There are two ways to construct a spectral sequence. The first one is from a filtration and the second is from a exact couple.

For an R-module M, consider an increasing **filtration** on M,

$$\cdots \subseteq F_{p-1}M \subseteq F_pM \subseteq F_{p+1}M \subseteq \cdots \subseteq M,$$

where F_pM are submodules of M and $\bigcup_p F_pM = M$. To this filtration is an **associated graded module**

$$\operatorname{Gr} M = \bigoplus_p \operatorname{Gr}_p M = \bigoplus_p (F_pM/F_{p-1}M).$$

Lemma 12.1. *If $f : M \to M'$ preserves filtration, and it induces $\operatorname{Gr} M \cong \operatorname{Gr} M'$, then f is an isomorphism.*

Let us now assume that M is graded independent to the filtration. In this case, we require F_pM to be a graded submodule. Then for fixed n, $\{F_pM_n\}$ gives a filtration on M_n. Then we define

$$\operatorname{Gr}_{p,q} = F_pM_{p+q}/F_{p-1}M_{p+q}.$$

We call p the filtration degree, $p+q$ the total degree, and q the complementary degree.

Now consider a chain complex $\{C_\bullet, \partial\}$ with F_pC a filtration that is also a subcomplex of C. There is an induced filtration

$$F_pH(C) = \operatorname{im}\{H(F_pC) \to H(C_\bullet)\} = \frac{F_pC \cap Z}{F_pC \cap B}.$$

We can identify

$$\operatorname{Gr}_p H(C) = \frac{F_pH(C)}{F_{p-1}H(C)} = \frac{F_pC \cap Z}{(F_pC \cap B) + (F_{p-1}C \cap Z)}.$$

Let us define

$$Z^r_{p,q} = F_pC_{p+q} \cap \partial^{-1}F_{p-r}C_{p+q-1}, \quad Z^\infty_{p,q} = F_pC_{p+q} \cap Z$$

and

$$B^r_{p,q} = F_pC_{p+q} \cap \partial F_{p+r-1}C_{p+q+1} = \partial Z^{r-1}_{p+r-1}, \quad B^\infty_{p,q} = F_pC_{p+q} \cap B.$$

Then we have

$$F_pC = Z^0_p \supseteq Z^1_p \supseteq \cdots \supseteq Z^\infty_p \supseteq B^\infty_p \supseteq \cdots \supseteq B^1_p \supseteq B^0_p.$$

Assume that each $\{F_pC_n\}$ is finite, i.e., there exists a $\ell(n)$ depending on n such that $F_{\ell(n)}C_n = C_n$ and $F_{-\ell(n)}C_n = 0$. Then for each p and q, there exists a r (actually something like $r = \ell(p+q-1)+p$ does the job) such that

$$Z^r_{p,q} = Z^{r+1}_{p,q} = \cdots = Z^\infty_{p,q}.$$

Likewise we have

$$B^\infty_{p,q} = \cdots = B^{r+1}_{p,q} = B^r_{p,q}$$

for some r.

Let us now define

$$E^r_{p,q} = \frac{Z^r_{p,q}}{B^r_{p,q} + Z^{r-1}_{p-1,q+1}}.$$

Because both Z^r_p and B^r_p stabilizes, we can say that $E^r_{p,q}$ converges (i.e., stabilizes) to

$$\frac{Z^\infty_p}{B^\infty_p + Z^\infty_{p-1}} = \mathrm{Gr}_p\, H(C)$$

as $r \to \infty$.

On the 0-th page, we have

$$E^0_{p,q} = \frac{Z^0_{p,q}}{B^0_{p,q} + Z^0_{p-1,q}} = \frac{F_pC_{p+q}}{(\text{sth in } F_{p-1}C_{p+q}) + F_{p-1}C_{p+q}} = \mathrm{Gr}_{p,q}\, C.$$

The 0-th differential d^0 is induced from ∂ as

$$d^0 : E^0_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} \to \frac{F_pC_{p+q-1}}{F_{p-1}C_{p+q-1}} = E^0_{p,q-1}.$$

If you try hard, you will check that

$$E^1_{p,q} = \frac{F_pC \cap \partial^{-1}F_{p-1}C}{\partial F_pC + F_{p-1}C} = H(E^0, d^0).$$

Later we will define $d^r : E^r \to E^r$ and see that $E^{r+1} = H(E^r, d^r)$.

There is another way of getting a spectral sequence, from an exact couple

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B. \arrow[ul, "k"] & \end{tikzcd}$$

We have a differential $d : B \to B$ defined by jk, and it is a differential because $jkjk = j0k = 0$.

13 March 31, 2017

Last time we defined a spectral sequence $\{E^r_{p,q}, d^r\}$ and discussed how it arises from an increasing filtration $F_pC_\bullet$ of a chain complex $C_\bullet$. We can define a spectral sequence $E^r_{p,q}$ that computes $H_\bullet(C_\bullet)$. Actually the spectral sequence converges to

$$E^\infty_{p,q} \cong \mathrm{Gr}_{p,q} H_*(C_\bullet)$$

in the case when the filtration is dimension-wise finite. The 0-th page is given by

$$E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

There is a nice application. For X a finite-dimensional CW-complex, we would like to show that $H^{\mathrm{sing}}_* \cong H^{\mathrm{cell}}_*$. Consider the cellular chain complex C^{sing}_* and give a filtration given by

$$F_pC^{\mathrm{sing}}_*(X) = C^{\mathrm{sing}}_*(X^{(p)}).$$

We have

$$E^0_{p+q} = \frac{F_pC^{\mathrm{sing}}_{p+q}(X)}{F_{p-1}C^{\mathrm{sing}}_{p+q}(X)} = \frac{C^{\mathrm{sing}}_{p+q}(X^{(p)})}{C^{\mathrm{sing}}_{p+q}(X^{(p-1)})} = C^{\mathrm{sing}}_{p+q}(X^{(p)}, X^{(p-1)}).$$

Then we see that

$$E^1_{p,q} = H^{\mathrm{sing}}_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} 0 & \text{if } q > 0, \\ C^{\mathrm{cell}}_p(X) & \text{if } q = 0. \end{cases}$$

The spectral sequence then looks like Figure 1, where no dot means that there is a 0 there.

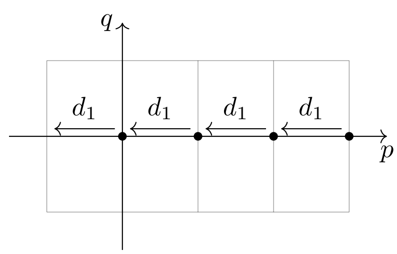

Figure 1: Spectral sequence coming from $F_pC^{\mathrm{sing}}_*(X) = C^{\mathrm{sing}}_*(X^{(p)})$

This sequence has

$$E^2_{p,0} = H^{\mathrm{cell}}_p(X)$$

and then all $d^r = 0$ for $r > 1$. So the spectral sequence stays the same from the E^2 page and we get

$$H^{\mathrm{sing}}_*(X) \cong H^{\mathrm{cell}}_*(X),$$

because the only nonzero graded associated in the filtration of $H^{\mathrm{sing}}_*(X)$ is $E^2_{*,0} = H^{\mathrm{cell}}_*(X)$.

13.1 Spectral sequence from an exact couple

Consider a series of maps

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B. \arrow[ul, "k"] & \end{tikzcd}$$

We want this to be exact, i.e.,

$$\ker j = \operatorname{im} i, \quad \ker k = \operatorname{im} j, \quad \ker i = \operatorname{im} k.$$

Such a thing is called an **exact couple**. If we define $d = jk$, then $d^2 = jkjk = 0$. So we may take the homology $B_1 = H(B)$ of B with respect to d. Let us define $A_1 = i(A)$.

Proposition 13.1. *Under this construction,*

$$\begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

is again an exact couple.

For example, for a filtration $F_pC_\bullet$ of a chain complex $C_\bullet$, you can take

$$A = \bigoplus_p F_pC_\bullet, \quad i : F_{p-1}C_\bullet \hookrightarrow F_pC_\bullet, \quad B = \bigoplus_p \frac{F_pC_\bullet}{F_{p-1}C_\bullet}.$$

This induces a long exact sequence

$$\cdots \to H_k(A) \xrightarrow{i_1} H_k(A) \xrightarrow{j_1} H_k(B) \xrightarrow{k_1} H_{k-1}(A) \to \cdots .$$

We then get a exact couple

$$\begin{tikzcd} H_*(A) \arrow[rr, "i_1"] & & H_*(A) \arrow[dl, "j_1"] \\ & H_*(B). \arrow[ul, "k_1"] & \end{tikzcd}$$

So $B_1 = E^1_{*,*}$, and if you run the construction iteratively, you are going to get $B_r = E^r_{*,*}$ for $r \geq 0$.

13.2 Spectral sequence from a double complex

As another example, let us consider a **double complex** $\{C_{p,q}\}$ with differentials ∂' and ∂'' such that

$$\begin{tikzcd} C_{p,q} \arrow[r, "\partial'"] \arrow[d, "\partial''"] & C_{p-1,q} \arrow[d, "\partial''"] \\ C_{p,q-1} \arrow[r, "\partial'"] & C_{p-1,q-1} \end{tikzcd}$$

commutes and $\partial'^2 = \partial''^2 = 0$. This can also be considered as a chain complex in the category of chain complexes.

Given this data, we can define a total complex

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}, \quad D = \partial' + (-1)^p \partial''.$$

We can filter this total complex by

$$F_p(TC)_n = \bigoplus_{i \le p} C_{i,n-i}.$$

If we assume that $C_{p,q}$ has finitely many nonzero entries for every fixed $p+q$, (e.g., when $C_{p,q} \neq 0$ only when $p, q > 0$) then our filtration is going to be dimension-wise finite. Then $\{E^r_{p,q}\}$ is going to compute $H_*(TC)$.

The zeroth page is

$$E_{p,q} = \frac{F_p(TC_{p+q})}{F_{p-1}(TC_{p+q})} = \frac{\bigoplus_{i \le p} C_{i,p+q-i}}{\bigoplus_{i \le p-1} C_{i,p+q-i}} = C_{p,q}$$

and the 0-th differentials is going to be simple ∂''. Then $E^1_{p,q} = H_q(C_{p,*})$ and so on.

There is an application. Let k be a field and consider the double complex

$$C_{p,q} = C_p(X) \otimes_k C_q(Y), \quad C_*(-) = C_*(-; k).$$

Then

$$E^1_{p,q} = H_p(C_p(X) \otimes C_*(Y)) = H_q(Y; C_p(X)) = H_q(Y) \otimes C_p(X)$$

and it follows that

$$E^2_{p,q} = H_p(X) \otimes H_q(Y).$$

If you dig in, you will be able to check that $d^r = 0$ for $r \ge 2$. Then we get

$$H_n(X \times Y) = H_n(TC) = \bigoplus_{p+q=n} E^\infty_{p,q} = \bigoplus_{p+q=n} E^2_{p,q} = \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y).$$

This is the Künneth formula for over a field.

14 April 4, 2017

14.1 Hochschild–Serre spectral sequence

This is about homology of a group with coefficient in a chain complex. Recall that
$$H_n(G, M) = H_*(F_\bullet \otimes_{RG} M),$$
where $F_\bullet$ is a projective resolution of R. We have computed $H_0(G, M) = M_G = M/\{gm - m\}$.

Given a chain complex $C_\bullet$, we can define
$$H_*(G, C_\bullet) = H_*(F_\bullet \otimes C_\bullet)$$
where the homology on the right side is considered as a homology as a double complex $(F \otimes C)_n = \bigoplus_{p+q=n} F_p \otimes C_q$.

Let us compute this homology using the spectral sequence. We have
$$E^1_{p,q} = H_q(F_p \otimes_{RG} C_\bullet) = F_p \otimes_{RG} H_q(C_\bullet)$$
since F_q is projective. The next page will be
$$E^2_{p,q} = H_p(F_\bullet \otimes H_q(C_\bullet)) = H_p(G, H_q(C_\bullet)).$$
This spectral sequence, which is called the **Hochschild–Serre spectral sequence** is going to converges to $H_{p+q}(G, C_\bullet)$.

Proposition 14.1. *If $f : C_\bullet \to C'_\bullet$ is a quasi-isomorphism, then $H_*(G, C_\bullet) \cong H_*(G, C'_\bullet)$.*

There is another direction of running the spectral sequence. We can set
$$E^1_{p,q} = H_q(F_\bullet \otimes_{RG} C_p) = H_q(G, C_p).$$
This spectral sequence should also compute the same homology $H_*(G, C_\bullet)$.

Let us put in an assumption $(*)$ that $H_*(G, C_p) = 0$ for $* > 0$. (For instance, set each C_p to be a free RG-module.) Then
$$E^1_{p,0} = H_0(G, C_p) = (C_p)_G = C_p/\{gm - m\},$$
and all other things in the first page vanish. All d^r for $r > 2$ should vanish because either the source or the target is zero. So we get
$$H_*(C_G) = E^2 = E^\infty = H_*(G, C_\bullet).$$

Proposition 14.2. *Under the assumption $(*)$ on $C_\bullet$, the spectral sequence*
$$E^2_{p,q} = H_p(G, H_q(C_\bullet))$$
computes $H_{p+q}(C_G)$.

Suppose we have a short exact sequence

$$1 \to H \to G \to Q = G/H \to 1.$$

Let $F_\bullet$ be a projective resolution of R as an RG-module. Then

$$F \otimes_{RG} M = (F \otimes_R M)_G = ((F \otimes_R M)_H)_Q = (F \otimes_{RH} M)_Q.$$

In the spectral sequence, take $C_\bullet = F_\bullet \otimes_{RH} M$. To apply the proposition, we need to check that $C_p = F_p \otimes M$ satisfies the condition $(*)$. It suffices to check that $RG \otimes_{RH} M$ satisfies this condition. But recall that

$$RG \otimes_{RH} M = \operatorname{Ind}_H^G \operatorname{Res}_H^G M = R[G/H] \otimes_R M = \operatorname{Ind}_{\{1\}}^Q M.$$

So

$$H_*(Q, \operatorname{Ind}_{\{1\}}^Q M) = H_*(\{1\}, M) = 0 \text{ for } * > 0.$$

Now we can finally apply the proposition and get the spectral sequence

$$E^2_{p,q} = H_p(Q, H_q(H, M)) \quad \Rightarrow \quad H_{p+q}(G, M).$$

To make sense of this, we need to specify the Q-action on $H_q(H, M)$. The action

$$c_g : h \mapsto ghg^{-1}, \quad m \mapsto gm$$

gives an action

$$c_{g*} : H_*(H, M) \to H_*(gHg^{-1}, M).$$

Proposition 14.3. *If $g \in H$, then $c_{g*} : H_*(M) \to H_*(M)$ is the identity.*

Proof. Even on the chain level, we have

$$c_{g*} : F \otimes_{RH} M \to F \otimes_{R[gHg^{-1}]} M; \quad x \otimes m \mapsto xh^{-1} \otimes hm = x \otimes m.$$

So it is the identity on homology. □

So this indeed gives an RQ-module structure on $H_*(H, M)$. There is also a spectral sequence in homology that gives

$$E_2^{p,q} = H^p(Q, H^q(H, M)) \quad \Rightarrow \quad H^{p+q}(G, M).$$

Corollary 14.4. *There are exact sequences*

$$H_2(G, M) \to H_2(G, M_H) \to H_1(H, M)_Q \to H_1(G, M) \to H_1(Q, M_H) \to 0,$$

$$0 \to H^1(Q, M^H) \to H^1(G, M) \to H^1(H, M)^Q \to H^2(Q, M^H) \to H^2(G, M).$$

Proof. Look at the E^2 page of the homology spectral sequence.

Because the differentials d^r at $E^2_{1,0}$ are all zero for $r \geq 2$, we see that $E^2_{1,0} = E^\infty_{1,0}$. On the other hand, we have d^2 that can possibly do something to $E_{0,1}$, which is given by

$$0 \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to 0.$$

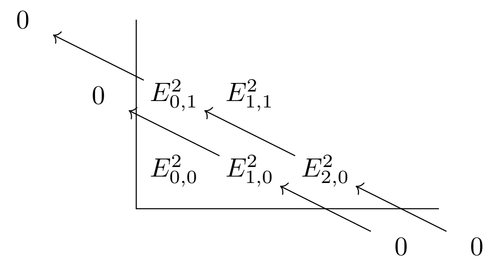

Figure 2: E^2 page of the Hochschild spectral sequence

Thus after taking the homology, we will get

$$0 \to E^3_{2,0} \to E^2_{2,0} \to E^2_{0,1} \to E^3_{0,1} \to 0.$$

Furthermore, there are no more differential in and out of $E^3_{0,1}$ and $E^3_{2,0}$. Thus $E^3_{0,1} = E^\infty_{0,1}$ and $E^3_{2,0} = E^\infty_{2,0}$. That is,

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \to E^2_{0,1} \to E^\infty_{0,1} \to 0.$$

On the other hand, one of the basic theorem we had is that the homology of the total chain complex is given by the filtration. Then we have a short exact sequence

$$0 \to E^\infty_{0,1} \to H_1(G, M) \to E^\infty_{1,0} \to 0.$$

We can then concatenate the two exact sequences to get

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \to E^2_{0,1} \to H_1(G, M) \to E^\infty_{1,0} \to 0.$$

Finally note that $E^\infty_{0,2}$, $E^\infty_{1,1}$, and $E^\infty_{2,0}$ give a filtration on $H_2(G, M)$. So $E^\infty_{2,0}$ appear as a quotient of $H_2(G, M)$. Hence we get

$$H_2(G, M) \to E^2_{2,0} \to E^2_{0,1} \to H_1(G, M) \to E^\infty_{1,0} \to 0.$$

This is precisely the exact sequence we want. $\square$

15 April 7, 2017

15.1 Homology and cohomology of cyclic groups

Let us compute the group cohomology or homology with $G = \mathbb{Z}$. We know that $\mathbb{Z}[G] = \mathbb{Z}[x, x^{-1}]$ as a ring. We have a resolution

$$0 \to \mathbb{Z}[G] \xrightarrow{\cdot(1-x)} \mathbb{Z}[G] \xrightarrow{x \mapsto 1} \mathbb{Z} \to 0.$$

This is related to the fact that $B\mathbb{Z} = S^1$ which has finite homology or cohomology. Anyways, we can compute homology $H_*(\mathbb{Z}, A)$ for a $\mathbb{Z}[G]$-module A as

$$0 \to A \xrightarrow{f} A \to 0.$$

Then

$$H_0(\mathbb{Z}, A) = \operatorname{coker} f = A_G, \quad H_1(\mathbb{Z}, A) = \ker f = A^G.$$

In the other direction, we are going to have

$$0 \leftarrow \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = A \xleftarrow{f^*} \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = A \leftarrow 0$$

and so

$$H^0(\mathbb{Z}, A) = A^G, \quad H^1(\mathbb{Z}, A) = A_G.$$

We can make a similar computation for $G = \mathbb{Z}/p$. Let x be the generator of $G = \mathbb{Z}/p$. The resolution is given by

$$\cdots \to \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{1-x} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{1+x+\cdots+x^{n-1}} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{1-x} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{x \mapsto 1} \mathbb{Z}.$$

Indeed, $\mathbb{R}P^\infty = B(\mathbb{Z}/2)$ has cohomology ring $\mathbb{Z}/2[x]$ with coefficients $\mathbb{Z}/2$.

Algebraically, we need to compute the homology of

$$\cdots \to A \xrightarrow{1-x} A \xrightarrow{1+x+\cdots+x^{n-1}} A \xrightarrow{1-x} A \xrightarrow{x \mapsto 1} \mathbb{Z}.$$

This immediately shows

$$H_0 = A/\langle a - ax \rangle = A_G, \quad H_{2i+1} = H_1 = A^G/NA, \quad H_{2i} = H_2.$$

where $N = 1 + x + x^2 + \cdots + x^{n-1}$. This doesn't have a very good description. For cohomology, we can do the same thing and get

$$H^0 = A^G, \quad H^{2i+1} = H^1 = \ker N / \operatorname{im}(1-x), \quad H^{2i} = H^2.$$

We can compute $H^i(\mathbb{Z}/p; \mathbb{F}_p)$ with trivial action on $\mathbb{F}_p$ using this. Because the N map is going to be zero, all maps are zero and thus we get

$$\dim_{\mathbb{F}_p} H^i(\mathbb{Z}/p; \mathbb{F}_p) = 1$$

for all i. The cup product structure (which comes from the comultiplication map $B_\bullet \to B_\bullet \otimes B_\bullet$) can be computed as

$$H^*(\mathbb{Z}/p;\mathbb{F}_p) = \begin{cases} \mathbb{F}_2[x] & p=2, \\ \mathbb{F}_p[x,y^2]/(x^2) & p>2. \end{cases}$$

There is another relation between x and y. There is a connected homomorphism

$$H^1(\mathbb{Z}/p,\mathbb{Z}) \to H^1(\mathbb{Z}/p,\mathbb{Z}) \to H^1(\mathbb{Z}/p,\mathbb{F}_p) \xrightarrow{\alpha} H^2(\mathbb{Z}/p,\mathbb{Z}) \to \cdots$$

and y can be taken to be the image of $\alpha(x)$ under the map $H^2(\mathbb{Z}/p,\mathbb{Z}) \to H^2(\mathbb{Z}/p,\mathbb{F}_p)$ coming from the reduction map $\mathbb{Z} \to \mathbb{F}_p$.

This procedure has a name. There is always a map

$$H^*(X;\mathbb{F}_p) \to H^{*+1}(X;\mathbb{F}_p)$$

and this is called a **Bockstein homomorphism**. In the case $p=2$ we have $y = \mathrm{Sq}^1(x)$, but $\mathrm{Sq}^1(x) = x^2$ if $x \in H^1$. This is why we can take $y = x^2$.

Finally, let us try to compute the cohomology with $G = \Sigma_3$ the permutation group. We have an exact sequence

$$1 \to \mathbb{Z}/3 \to \Sigma_3 \to \mathbb{Z}/2 \to 1.$$

Take $k = \mathbb{F}_2$. Then the pages are given by

$$E^2_{p,q} = H^p(\mathbb{Z}/2, H^q(\mathbb{Z}/3,\mathbb{F}_2)) = \begin{cases} H^p(\mathbb{Z}/2,\mathbb{F}_2) & q=0 \\ 0 & q>0. \end{cases}$$

This shows that

$$H^p(\Sigma_3,\mathbb{F}_2) \cong H^p(\mathbb{Z}/2,\mathbb{F}_2).$$

This can also been seen from the fact that $\Sigma_3 = \mathrm{GL}_2(\mathbb{F}_2)$.

16 April 11, 2017

16.1 Wreath product

If you have a group G, we define the **wreath product** as

$$G \wr \mathbb{Z}/p = G^{\times p} \rtimes \mathbb{Z}/p,$$

where the action is given by

$$x \cdot (g_1, \dots, g_p) = (g_2, \dots, g_p, g_1).$$

Then there is a short exact sequence

$$1 \to G^p \to G^p \wr \mathbb{Z}/p \to \mathbb{Z}/p \to 1.$$

We want to compute $H^*(G \wr \mathbb{Z}/p, \mathbb{F}_p)$ over $\mathbb{F}_p$. There is a spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}/p, H^t(G^p, \mathbb{F}_p)) \quad \Rightarrow \quad H^{s+t}(G \wr \mathbb{Z}/p, \mathbb{F}_p).$$

To compute this, we need to know the $\mathbb{F}_p[\mathbb{Z}/p]$-module structure on $H^*(G^p, \mathbb{F}_p)$. First note that by the Künneth formula,

$$H^*(G^p, \mathbb{F}_p) = \bigotimes_p H^*(G, \mathbb{F}_p).$$

Because the action of $\mathbb{Z}/p$ on G^p is given by shifting the elements, the induced action of $\mathbb{Z}/p$ on $H^*(G^p, \mathbb{F}_p)$ is also going to be just shifting the tensors.

Lemma 16.1. *Let $\Lambda = \bigoplus_i \Lambda_i$ be a graded $\mathbb{F}_p$-vector space. Give a $\mathbb{Z}/p$-action on $\Lambda^{\otimes p}$ by cyclic shifting. Then as a $\mathbb{F}_p[\mathbb{Z}/p]$-module, $\Lambda^{\otimes p}$ is a direct sum of a free $\mathbb{F}_p[\mathbb{Z}/p]$-module and the trivial module ($\mathbb{F}_p$-vector space with trivial action) generated by some of $\lambda_i \otimes \cdots \otimes \lambda_i$.*

Using this lemma, we can decompose

$$H^*(G^p, \mathbb{F}_p) = \bigoplus \mathbb{F}_p[\mathbb{Z}/p] \oplus \bigoplus \mathbb{F}_p.$$

Then we see that

$$E_2 = H^*(\mathbb{Z}/p, H^*(G^p, \mathbb{F}_p)) = \bigoplus H^*(\mathbb{Z}/p, \mathbb{F}_p[\mathbb{Z}/p]) \oplus \bigoplus H^*(\mathbb{Z}/p, \mathbb{F}_p).$$

We know hat $H^*(\mathbb{Z}/p, \mathbb{F}_p[\mathbb{Z}/p]) = 0$ for $* > 0$. For the second factor, we see that $\lambda_i \otimes \cdots \otimes \lambda_i$ lives in the pi-th graded pieces and so will come out in the pi-th cohomology. This shows that,

$$E_2^{s,t} = H^s(\mathbb{Z}/p, H^t(G^p, \mathbb{F}_p)) = \begin{cases} H^t(G^p, \mathbb{F}_p)^{\mathbb{Z}/p} & s = 0 \\ 0 & s > 0, p \nmid t \\ H^s(\mathbb{Z}/p, H^{ip}(G^p, \mathbb{F}_p)) & t = ip. \end{cases}$$

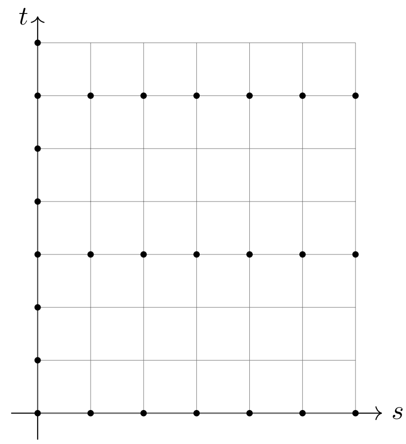

Figure 3: Spectral sequence from $H^s(\mathbb{Z}/p, H^t(G^p, \mathbb{F}_p))$

In particular, if $t = ip$ then

$$E_2^{s,ip} = \bigoplus_{\substack{x \text{ a basis of} \\ H^n(G,\mathbb{F}_p)}} H^s(\mathbb{Z}/p, \mathbb{F}_p).$$

So our spectral sequences looks something like Figure 3.

Proposition 16.2. *The E_2 page collapses, i.e., $E_2 = E_\infty$.*

Note that if there is a map between exact sequence of groups

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & M \arrow[r] & H \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

you get a map between the Hochschild–Serre spectral sequences.

$$\begin{tikzcd}
E_2 = H^*(Q, H^*(N,k)) \arrow[r, Rightarrow] & H^*(G,k) \\
F_2 = H^*(P, H^*(M,k)) \arrow[r, Rightarrow] \arrow[u] & H^*(H,k) \arrow[u]
\end{tikzcd}$$

We remark that this is also true for fibrations and the Serre spectral sequence.

17 April 14, 2017

Recall that the wreath product is defined as $G \wr \mathbb{Z}/p = G^p \rtimes \mathbb{Z}/p$. Then there is an exact sequence and so $G^p \to G \wr \mathbb{Z}/p \to \mathbb{Z}/p$. So we have a spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}/p, H^s(G^p, \mathbb{F}_p)) \quad \Rightarrow \quad H^{s+t}(G \wr \mathbb{Z}/p, \mathbb{F}_p).$$

Because $H^t(G^p, \mathbb{F}_p)$ is a direct sum of $\mathbb{F}_p[\mathbb{Z}/p]$ and $\mathbb{F}_p$, we get some description of $E_2^{s,t}$.

17.1 Cohomology of wreath products

For $s = 0$, we have

$$E_2^{0,t} = H^s(G^p, \mathbb{F}_p)^{\mathbb{Z}/p} = \{\lambda^{\otimes p} : \lambda \in H^i(G, \mathbb{F}_p)\} \oplus (\text{free } \mathbb{F}_p[\mathbb{Z}/p]).$$

Recall that if $A \leq B$ is of finite index, there is a transfer map $\mathrm{tr} : H^*(A, M) \to H^*(B, M)$, and we have

$$\mathrm{res} \circ \mathrm{tr} = \sum_{g \in B/A} g.$$

So applying this to $G^p \hookrightarrow G \wr \mathbb{Z}/p$ gives

$$\mathrm{res} \circ \mathrm{tr} = \sum_{g \in \mathbb{Z}/p} g = (1 + T + \cdots + T^{p-1}) : H^*(G^p, \mathbb{F}_p) \to H^*(G^p, \mathbb{F}_p).$$

Then we see that actually

$$E_2^{0,t} = \{\lambda^{\otimes p} : \lambda \in H^i(G^p, \mathbb{F}_p)\} \oplus (\text{image of } \mathrm{res} \circ \mathrm{tr}).$$

Lemma 17.1. *If $x \in E_2^{0,*}$ comes from $x \in \mathrm{im}(H^*(Q, k) \xrightarrow{\mathrm{res}} H^*(P, k))$, then x is a permanent cycle, i.e., $d_* x = 0$.*

Proof. Consider the map

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, equal] & Q \arrow[r] & Q/P \arrow[d] \\ P \arrow[r] & P \arrow[u, "i"] \arrow[r] & \{1\}. \end{tikzcd}$$

Then there is a map of spectral sequences

$$\begin{tikzcd} H^*(Q/P, H^*(P, k)) \arrow[r, Rightarrow] \arrow[d] & H^*(Q, k) \arrow[d, "\mathrm{res}"] \\ H^*(\{1\}, H^*(P, k)) \arrow[r, Rightarrow] & H^*(P, k). \end{tikzcd}$$

Since the bottom spectral sequence collapses at zero, all differentials are zero. So in order for anything in the image of res to live, it has to be a permanent cycle. ∎

We note that

$$H^s(\mathbb{Z}/p, \mathbb{F}_p\langle\lambda^{\otimes p}\rangle) = H^s(\mathbb{Z}/p, \mathbb{F}_p) \otimes \langle\lambda^{\otimes p}\rangle.$$

But recall that $H^*(\mathbb{Z}/p, \mathbb{F}_p) \cong \mathbb{F}_p[x, y]/(x^2)$. So the basis are $y^j \otimes \lambda^{\otimes p}$ and $xy^j \otimes \lambda^{\otimes p}$. Furthermore, this multiplicative structure is preserved on the differential:

$$d(x \otimes \lambda^{\otimes p}) = d(x) \otimes \lambda^{\otimes p} \pm x \otimes d(\lambda^{\otimes p}).$$

So in most cases, knowing the differential on the $t = 0$ column is enough to find out all the differentials.

Let us go back into the case of the wreath product. The image of the restriction map is a permanent cycle, and so we may as well just ignore that and look at the other parts.

Lemma 17.2. *The differential on $\lambda_i^{\otimes p} \in E_r^{0,ip}$ has to be zero.*

Proof. If we have a short exact sequence $H \to G \to G/H$, this gives a fibration $BH \to BG \to B(G/H)$. This gives a Serre spectral sequence

$$H^s(B(G/H), \mathcal{H}^t(BH, k)) \quad \Rightarrow \quad H^{s+t}(BG, k),$$

where the calligraphic $\mathcal{H}$ means the local coefficients.

Given an element $\lambda_n \in H^n(BG, \mathbb{Z}/p)$, it corresponds to a map $f_n : BG \to K(\mathbb{Z}/p, n)$. Then this induces a map

$$f_n^* : H^*(K(\mathbb{Z}/p), n) \to H^*(BG),$$

and the fundamental class $\iota_n \in H^n(K(\mathbb{Z}/p, n))$ corresponding to id is going to map to λ_n. Then we have a sequence of fibrations.

$$\begin{tikzcd}
BG^p \arrow[r] \arrow[d, "f_i^p"] & BG \wr \mathbb{Z}/p \arrow[r] \arrow[d] & B\mathbb{Z}/p \arrow[d, equal] \\
K(\mathbb{Z}/p, i)^p \arrow[r] & K(\mathbb{Z}/p, i)^p \times_{\mathbb{Z}/p} E\mathbb{Z}/p \arrow[r] & B\mathbb{Z}/p
\end{tikzcd}$$

This gives a map of spectral sequence

$$\begin{tikzcd}
H^s(B\mathbb{Z}/p; \mathcal{H}^t(K(\mathbb{Z}/p, n)^p; \mathbb{F}_p) \arrow[r, Rightarrow] \arrow[d] & H^{s+t}(K(\mathbb{Z}/p, i)^p \times_{\mathbb{Z}/p} E\mathbb{Z}/p; \mathbb{F}_p) \arrow[d] \\
H^s(B\mathbb{Z}/p; \mathcal{H}^t(BG^p; \mathbb{F}_p)) \arrow[r, Rightarrow] & H^{s+t}(BG \wr \mathbb{Z}/p; \mathbb{F}_p)
\end{tikzcd}$$

Here, note that for $s = 0$ we are going to get

$$H^t(K(\mathbb{Z}/p, n)^p; \mathbb{F}_p) \to H^t(BG^p; \mathbb{F}_p).$$

Since $f^*(\iota_p^{\otimes n}) = \lambda_n^{\otimes p}$, we can use functoriality to obtain data of the differential on $\lambda_n^{\otimes p}$.

We want to show that all differentials on $\lambda_n^{\otimes p}$ are zero. It suffices to check that all differentials on $\iota_n^{\otimes p}$ are zero, by the above argument. But by the Hurewicz theorem and the universal coefficients theorem, we have $H^t(K(\mathbb{Z}/p, n), \mathbb{F}_p) = 0$ for $t \leq n-1$. So we have $E_2^{s,t} = 0$ for $0 < t < np$. So the only possible nonzero differential is $d_{np+1} : E^{0,np} \to E^{np+1,0}$.

Here is a general fact for Serre spectra sequences (or Hochschild–Serre spectral sequences). The bottom row of the infinity page is

$$E_\infty^{*,0} = \operatorname{im}\bigl(H^*(B\mathbb{Z}/p; \mathbb{F}_p) \to H^*(K(\mathbb{Z}/p, i)^p \times_{\mathbb{Z}/p} E\mathbb{Z}/p)\bigr).$$

If show that this map is injective, then nothing on the bottom row can be killed and so $d_{np+1} : E^{0,np} \to E^{np+1,0}$ has to be zero. This can be shown by exhibiting a section of the fibration, and this can be done. $\square$

Example 17.3. We have $D_8 = \mathbb{Z}/2 \wr \mathbb{Z}/2$ and so

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, e]/(xy).$$

Let us define

$$W_1 = \mathbb{Z}/p, \quad W_r = W_{r-1} \wr \mathbb{Z}/p.$$

If $n = \sum a_i p^i$ and $0 \leq a_i < p$, then the p-Sylow subgroups of S_n can be written as

$$W_1^{a_1} \times W_2^{a_2} \times \cdots \times W_k^{a_k}.$$

So knowing about the cohomology of the wreath products is going to tell us something about the cohomology of symmetric groups.

18 April 18, 2017

18.1 Atiyah completion theorem

This was a presentation of mine. This is based on Atiyah's 1961 paper *Characters and cohomology of finite groups*. For a finite group G, what is $H^*(G,\mathbb{Z})$? We have

$$H^1(G,\mathbb{Z}) = \frac{\{(f : G \to \mathbb{Z}) : f(xy) = f(x) + f(y)\}}{0} = 0.$$

The next cohomology can be computed as

$$H^2(G,\mathbb{Z}) = \{1 \to \mathbb{Z} \to E \to G \to 1\} = \mathrm{Hom}_{\mathsf{Grp}}(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathsf{Grp}}(G, \mathrm{U}(1)).$$

But if you think about it, these are 1-dimensional complex representations of G, or equivalently 1-dimensional characters of G. So there seems to be some connection between the finite complex representations of G and $H^*(G,\mathbb{Z})$.

Here is the big idea. We want to find a relation between

$$H^*(G,\mathbb{Z}) = H^*(BG;\mathbb{Z}) \quad \longleftrightarrow \quad \{\text{representations of } G\}.$$

This is an algebraic problem, but it turns out the connection can be established easier if we use topology. If you think about it a bit, complex representations of G are somewhat like complex vector bundles over BG. Rank k complex vector bundles over BG are $[BG, BU(n)]$, and this is quite similar to group homomorphisms $G \to \mathrm{U}(n)$. So it seems reasonable to expect representations of G to be related to $K^*(BG)$. Now K^* is a cohomology theory. There is something called an Atiyah–Hirzebruch spectral sequence that computes a cohomology theory in terms of its coefficient ring.

For a space X, the set of complex vector bundles over X has a additive structure, coming from $\oplus$, and a multiplicative structure, coming from $\otimes$.

Definition 18.1. For a finite CW-complex X, we define

$$K^0(X) = \text{groupification of } \{\text{complex vector bundles } /X\}.$$

This has a natural ring structure. If X has a base point x_0, there is a ring morphism $K^0(X) \to \mathbb{Z}$ given by mapping to the dimension of the fiber over x_0. We further define the ideal

$$\widetilde{K}^0(X) = \ker(K^0(X) \to \mathbb{Z}).$$

Theorem 18.2 (Bott periodicity)**.** *For any finite CW-complex X,*

$$\widetilde{K}^0(\Sigma^2 X) = \widetilde{K}^0(X).$$

$\widetilde{K}^0(-)$ is a representable functor, and this allows us to define a cohomology theory $\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X)$, which will further extend to the positive side using Bott periodicity as

$$\widetilde{K}^n(X) = \begin{cases} \widetilde{K}^0(X) & 2 \mid n, \\ \widetilde{K}^0(\Sigma X) & 2 \nmid n, \end{cases} \qquad K^*(X) = \widetilde{K}^*(X_+).$$

Definition 18.3. For a CW-complex X such that each skeleton X^n is finite, define

$$\mathscr{K}^*(X) = \varprojlim K^*(X^n).$$

Just like complex vector bundles, there is an additive structure, coming from $\oplus$, and a multiplicative structure, coming from $\otimes$, on the set of complex representations of G. So we can again make this into a ring $R(G)$. In this case, we have a pretty interpretation

$$R(G) \subseteq (\text{ring of class functions } G \to \mathbb{C})$$

by looking at the characters.

Given a complex representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$, we can glue this into the principal G-bundle $EG \to BG$ by

$$\frac{\{(v,x) \in EG \times \mathbb{C}^n\}}{(v,x) \sim (gv, gx)} \to BG,$$

which is going to be a complex vector bundle over BG. This gives a ring map

$$\alpha : R(G) \to \mathscr{K}^0(BG) \hookrightarrow \mathscr{K}^*(BG).$$

There is an augmentation map $R(G) \to \mathbb{Z}$ that sends a representation to its dimension. Let $I(G)$ be its kernel. This gives a filtration

$$R(G) \supseteq I(G) \supseteq I(G)^2 \supseteq \cdots,$$

and we can can complete $R(G)$ with respect to the filtration. Then the map α extends to

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(BG).$$

Theorem 18.4. *For every finite G, $\hat{\alpha}$ is an homeomorphism.*

Let us just check this for cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$.

Example 18.5. Let us look at this map for cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$. We know all the representations: they are simply the 1-dimensional ones $1, \rho, \rho^2, \ldots, \rho^{n-1}$. So the ring $R(G)$ is given by

$$R(G) \cong \mathbb{Z}[\rho]/(\rho^n - 1) \cong \mathbb{Z}[\sigma]/((\sigma+1)^n - 1),$$

and $I(G) = (\sigma)$. So the completion is going to look like

$$\widehat{R(G)} = \mathbb{Z}[[\sigma]]/((\sigma+1)^n - 1)$$

But how do you compute $\mathscr{K}^*(BG)$? If you recall, there is a Serre spectral sequence

$$H^s(B; H^t(F;\mathbb{Z})) \quad \Rightarrow \quad H^{s+t}(E;\mathbb{Z})$$

and for group cohomology this corresponds to the Hochschild–Serre spectral sequence

$$H^s(G/H, H^t(H, M)) \quad \Rightarrow \quad H^{s+t}(G, M).$$

For a general cohomology theory, there is the Atiyah–Hirzebruch spectral sequence

$$H^s(B; K^t(F)) \quad \Rightarrow \quad K^{s+t}(E)$$

and if you have some faith, you will believe that there is a spectral sequence

$$H^s(G/H, \mathscr{K}^t(BH)) \quad \Rightarrow \quad \mathscr{K}^{s+t}(BG).$$

In particular, $H^s(G, K^t(*)) \Rightarrow \mathscr{K}^{s+t}(BG)$.

Example 18.6. We can try to compute $\mathscr{K}^*(B(\mathbb{Z}/n\mathbb{Z}))$. This doesn't immediately give what the ring exactly is, but you at least see what the associated graded is. This can be checked to be exactly the associated graded for $\widehat{R(G)}$. . Actually, the map $\hat{\alpha}$ will induce a map on the associated graded $G\widehat{R(\mathbb{Z}/n\mathbb{Z})} \to G\mathscr{K}^*(B(\mathbb{Z}/n\mathbb{Z}))$ and by following the generators, it can be checked that this map is an isomorphism. It follows from this that $\hat{\alpha}$ is a homeomorphism.

Let me make a remark about how Theorem 18.4 is proved. We checked that this is true for finite groups. Using representation theory, you can check that if $H \triangleleft G$ has quotient $G/H \cong \mathbb{Z}/q\mathbb{Z}$ for some prime q, then the statement for H implies the statement for G. This proves the theorem for solvable G. Next use the Brauer's theorem on elementary groups to show that this proves the general case.

Corollary 18.7. *For any finite G, $\mathscr{K}^{2k+1}(BG) = 0$.*

Corollary 18.8. *There is a spectral sequence*

$$H^s(G/H, \widehat{R(H)}) \quad \Rightarrow \quad \widehat{R(G)}.$$

In particular, $H^s(G, \mathbb{Z}) \Rightarrow \widehat{R(G)}$.

18.2 Skew-commutative graded differential algebras

This was a presentation by Carlos Albors-Riera. This is based on a paper published by John Tate, *Homology of Noetherian rings and local rings.* First thing I should talk about is what this thing is.

Definition 18.9. An associative algebra X and a R-linear map $d : X \to X$ constitute an **skew-commutative graded differential R-algebra** (I will write these as just R-algebras) if these satisfy:

(1) X is graded by $X = \bigoplus_{\lambda=-\infty}^{\lambda} X_\lambda$ such that $X_i X_j \subseteq X_{i+j}$,

(2) $X_{<0} = 0$, $1 \in X_0$, X_λ is finitely generated,

(3) $xy = (-1)^{\lambda\mu} yx$ where $x \in X_\lambda$ and $y \in X_\mu$, $x^2 = 0$ if λ is odd,

(4) $dX_\lambda \subseteq X_{\lambda-1}$, $d^2 = 0$,

(5) $d(xy) = (dx)y + (-1)^\lambda x(dy)$ for $x \in X_\lambda$ and $y \in X_\mu$.

From this you produce a chain complex

$$\cdots \xrightarrow{d} X_\lambda \xrightarrow{d} X_{\lambda-1} \xrightarrow{d} \cdots .$$

Then $H_\bullet(X)$ is going to have a natural structure of an algebra. If X is free and acyclic, and $R = X_0$, we have a free resolution

$$\cdots \to X_2 \to X_1 \to R \to B/M,$$

where $M = B_0$ is the boundaries.

Proposition 18.10. *Let $p > 0$. There exists a canonical extension $Y \supseteq X$ of an R-algebra such that*

(0) t has degree $p-1$,

(1) $Y_\lambda = X_\lambda$ for $\lambda < p$,

(2) $B_{p-1}(Y) = B_{p-1}(X) \oplus Rt$.

Proof. When p is odd, consider the free X-module XT with basis $1, T$ where T is supposed to be in degree p. Let us define $Y = X \oplus XT$. Then $Y_\lambda = X_\lambda + X_{\lambda-p}T$. Because $\deg T$ is odd, we are right to set $T^2 = 0$. Define $t = dT$. Then $Tx = (-1)^\lambda xT$ for $x \in X_\lambda$ and

$$d(xT) = (dx)T + (-1)^\lambda x(dT).$$

You can check that this satisfies all the conditions.

For p even, let Y be the free module on $1, T, T^{(2)}, \dots$ where $T^{(k)}$ is supposed to be in degree pk. Then you can consider

$$Y_\lambda = X_\lambda \oplus X_{\lambda-p}T \oplus X_{\lambda-2p}T^{(2)} \oplus \cdots$$

with the relations

$$T^{(i)}T^{(j)} = \frac{(i+j)!}{i!j!}T^{(i+j)}, \quad T^{(i)}x = xT^{(i)}, \quad dT^{(i)} = tT^{(i-1)}.$$

This does to trick. □

In general, for $\tau_i \in H_{p-1}(X)$, you can construct $Y = X\langle T_1, \dots, T_n\rangle$ such that

(1) $Y \supseteq X$ and $Y_\lambda = X_\lambda$ for $\lambda < p$,

(2) $H_{p-1}(Y) = H_{p-1}(X)/(R\tau_1 + \cdots + R\tau_n)$.

Theorem 18.11. *For $M \subseteq R$ and ideal, there exists a free acyclic R-algebra X such that $H_0(X) = R/M$.*

Proof. Let X be the union $X^0 \subseteq X^1 \subseteq \cdots$ where we construct them in the following way. Write $M = (t_1, \ldots, t_n)$. We let X^0 and $X^1 = X^0\langle T_1, \ldots, T_n\rangle$. Now $H_0(X^1) = R/M$.

Now let $s_1, \ldots, s_m \in Z_1(X^1)$ be such that s_i generate $H_1(X^1)$. Then we can set $X^2 = X^1\langle S_1, \ldots, S_m\rangle$ so that $H_1(X^2) = 0$ and $H_0(X^2) = R/M$. You can go on. □

Proposition 18.12. *Let X, Y be free acyclic algebras such that $H(X) = R/M$ and $H(Y) = R/N$. Let $j : X \to R/M$ and $k : Y \to R/N$ be the induced maps. Then*

$$(R/M) \otimes Y \xleftarrow{j\otimes 1} X \otimes Y \xrightarrow{1\otimes k} X \otimes R/N$$

induce isomorphisms

$$H(R/M \otimes Y) \cong H(X \otimes Y) \cong H(X \otimes R/N).$$

Definition 18.13. We define $\mathrm{Tor}^R(R/M, R/N) = H(X \otimes Y)$.

Theorem 18.14. *Let M, N be ideals and let $a \in MN$ be a non-zero divisor. Let $\overline{R} = aR$ and set $K = R/M$, $L = R/N$. Then*

$$\mathrm{Tor}^{\overline{R}}(K, L) = \mathrm{Tor}^R(K, L)\langle u\rangle$$

with u in degree 2.

Proof. Let X be a free, acyclic R-algebra such that $H(X) = K_\lambda = (R/M)_\lambda$. Then

$$d(NX_1) = NdX_1 = NM$$

and so we can choose $s \in NX_1$ such that $ds = a$. Let $\overline{X} = X/aX$, with $\overline{s} \in \overline{X}_1$. You can show that $H(\overline{X}) = K\langle\sigma\rangle$ where σ is the class of s. We can make $\overline{X}\langle s\rangle$ with $dS = \overline{s}$ be the free resolution of K.

Now we can actually compute Tor.

$$\begin{aligned}\mathrm{Tor}^{\overline{R}}(K, L) &= H((X \otimes L)\langle s\rangle) = H(\textstyle\sum (X \otimes L)S^{(i)})\\ &= \sum_{i=1}^{\infty} H(X \otimes L)U^i = H(X \otimes L)\langle u\rangle.\end{aligned}$$

□

19 April 21, 2017

19.1 Cohomology of D_{2n}

This was a talk by Max Hopkins. We are going to talk about the cohomology of $D_{2n} = \langle \sigma, \tau | \sigma^n = \tau^2 = 1, \sigma\tau\sigma = \sigma^{-1} \rangle$. The basic method is the Hochschild–Serre spectral sequence.

Example 19.1. We are going to have

$$\mathrm{Ext}^i_{\mathbb{Z}D_{2n}}(\mathbb{Z}[\tau], A) = \mathrm{Ext}^i_{\mathbb{Z}[\mathbb{Z}/n]}(\mathbb{Z}, A) = H^i(\mathbb{Z}/n, A).$$

Lemma 19.2. *Let $1 \to H \to G \to G/H \to 1$ be a short exact sequence and consider a resolution $\cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$ over $\mathbb{Z}H$. Then*

(1) $\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} = \mathbb{Z}G/H$,

(2) $\cdots \to \mathbb{Z}G \otimes_{\mathbb{Z}H} C_1 \to \mathbb{Z}G \otimes_{\mathbb{Z}H} C_0 \to \mathbb{Z}G/H \to 0$ is a resolution over $\mathbb{Z}G$.

Proof. (1) can be checked by the coset decomposition. (2) can also be checked because this tensoring is exact. □

Corollary 19.3. *Let A be a $\mathbb{Z}G$-module, regarded as a $\mathbb{Z}H$-module. Then*

$$\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}[G/H], A) = \mathrm{Ext}^i_{\mathbb{Z}H}(\mathbb{Z}, A) = H^i(H, A).$$

We first take a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G/H]$-module, and the take the projective resolution of of them as $\mathbb{Z}G$-modules. Then we get

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
 & C_{01} \arrow[d, "\partial"] & C_{11} \arrow[l, "d_{10}"'] \arrow[d, "\partial"] & C_{21} \arrow[l, "d_{11}"'] \arrow[d, "\partial"] & \cdots \arrow[l, "d_{21}"'] \\
 & C_{00} \arrow[d, "\partial"] & C_{10} \arrow[l, "d_{00}"'] \arrow[d, "\partial"] & C_{20} \arrow[l, "d_{10}"'] \arrow[d, "\partial"] & \cdots \arrow[l, "d_{20}"'] \\
\mathbb{Z} & B_0 \arrow[l, "\epsilon"'] & B_1 \arrow[l, "d_0"'] & B_2 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"']
\end{tikzcd}$$

Here we can lift the maps to get $d_{00}, d_{10}, \ldots$, but we don't know if the lifted maps are going to be chain complexes. But this can be fixed.

Anyways, we can pass the complex through $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ and take the vertical cohomology. Then we get

$$\begin{aligned}
\mathrm{Ext}^*_{\mathbb{Z}G}(B_*, A) &= \mathrm{Ext}^*_{\mathbb{Z}G}(B_* \otimes_{\mathbb{Z}[G/H]} \mathbb{Z}[G/H], A) \\
&= \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}[G/H], A) \otimes_{\mathbb{Z}[G/H]} B_* = H^i(H, A) \otimes_{\mathbb{Z}[G/H]} B_i.
\end{aligned}$$

So we get the Hochschild–Serre spectral sequence $H^*(G/H, H^*(H, M)) \Rightarrow H^*(G)$.

Anyways, we are going to apply it to

$$1 \to \mathbb{Z}/n \to D_{2n} \to \mathbb{Z}/2 \to 1.$$

We already know a nice resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/n]$, and so we can explicitly construct the chain complex. You can track all the differentials, because we can make our maps explicit.

Theorem 19.4. *The cohomology ring of D_{2^n} with coefficients $\mathbb{F}_2$ is*

$$H^*(D_{2^n}, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/(xy)$$

where x, y are in dimension 1 and w is in dimension 2.

19.2 The third cohomology

This was a presentation by Stefan Spataru.

Definition 19.5. A **cross module** is a pair of groups (N, E) with a map $\alpha : N \to E$ and an action ${}^e n$ such that

(1) $\alpha({}^e n) = e\alpha(n)e^{-1}$,
(2) ${}^{\alpha(n)}n' = nn'n^{-1}$.

Theorem 19.6. *For a group G and an RG-module A, there is a bijection between $H^3(G, A)$ and the equivalence classes of short exact sequences*

$$0 \to A \to N \to G \to E \to 1.$$

Here, the equivalence relation is the one generated by to sequence equivalent if there are maps φ_N and φ_E such that

$$\begin{array}{ccccccc} A & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G \\ \| & & \downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \varphi_E} & & \| \\ A & \longrightarrow & N' & \longrightarrow & E' & \longrightarrow & G. \end{array}$$

From the bar resolution, $H^3(G, A)$ is the set of some maps $f : G \times G \times G \to A$ quotiented out by some subgroup.

Now given an exact sequence

$$0 \to A \xrightarrow{i} N \xrightarrow{\alpha} E \xrightarrow{\pi} G \to 1,$$

with a section $s_0 : G \to E$, we can define a function

$$f(g, h) = s_0(g)s_0(h)s_0(gh)^{-1}.$$

This f is going to satisfy

$$f(g, h)f(gh, k) = {}^{s(g)}f(h, k)f(g, hk).$$

We can lift this to a map $F : G \times G \to N$, again put in a modifying factor $k : G \times G \times G \to N$ to make

$$F(g, h)F(gh, k) = k(g, h, k) \cdot {}^{s(g)}F(h, k) \cdot F(g, hk).$$

The image of k has to be in the image of i. So there is a map $z : G \times G \times G \to A$ such that $k = i \circ z$. You can check that $d^3 z = 0$.

Now let us show that z is independent of the choice of F, in as a cohomology class. If we choose another

$$F_2(g,h) = F_1(g,h)c(g,h),$$

then we would get

$$c(g,h)c(gh,k) = {}^{s(g)}c(h,k)c(g,hk).$$

Then $d^2 c = 0$ and so $z' = z + \partial c$. A similar computation shows that z is independent on the choice of the first section s_0. You can further check that equivalent sequences give the same element in $H^3(G,A)$. Then we get a well-defined map from equivalence classes to $H^3(G,A)$.

We now want to show that this map is bijective. Consider any function $z : G \times G \times G \to A$. Let F be the free group generated by G. There is a natural embedding $s_0 : G \to F$, and this association gives an epimorphism $\pi : F \to G$. Then we get an exact sequence

$$0 \to A \xrightarrow{i} A \times \ker \pi \xrightarrow{\alpha} F \xrightarrow{\pi} G \to 1.$$

Here α is given by $(a,x) \mapsto x$.

Proposition 19.7. *(1) The kernel of π is generated by the image of T.*
(2) $A \times \ker \pi$ is generated by the image of i and the image of T.

Here, T is the section that was named F before.

20 April 25, 2017

20.1 Swan's theorem

This was a talk by Aaron Slipper. This version is related to ranks of projective modules. We want to associate to each module a rank, but this is hard because we are trying to do this over an arbitrary ring, in particular, the group ring $\mathbb{Z}\Gamma$.

We can define the **rank** as

$$\operatorname{rk} P = \operatorname{rk}_{\mathbb{Z}}(P \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}).$$

Now rank of $\mathbb{Z}$-modules are well-defined, and so we get a good notion of rank. Then $\operatorname{rk}(\mathbb{Z}\Gamma)^n = n$. Brown calls this

$$\epsilon(P) = \operatorname{rk}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} P).$$

There is also more simple definition

$$\rho(P) = \operatorname{rk}_{\mathbb{Z}}(P)/|\Gamma|.$$

It is not a priori clear that this is even an integer. Also, this is defined only for finite Γ.

Theorem 20.1 (Swan)**.** *$\epsilon(P) = \rho(P)$ for all projective P.*

This was proved by Swan in 1960 and by Bass in 1976 and 1979. One motivation for looking at this is to define Euler characteristics of groups.

The idea is to find another definition of rank and show that all three are the same. If F is a free $\mathbb{Z}$-module, then $\operatorname{tr}_{\mathbb{Z}}(\mathrm{id}_F) = n$. But because our ring is noncommutative, and so trace can't be defined in an invariant way. Even in the 1-dimensional case, $x \neq axa^{-1}$.

The idea of Bass is to look at the abelianization $T(A) = A/[A, A]$, where $[A, A]$ is the ideal generated by $ab - ba$. Then we can define the trace map

$$\operatorname{tr} : \operatorname{Mat}_n(A) \to A/[A, A].$$

Then for α an $n \times m$ matrix and β an $m \times n$ matrix, we have

$$\operatorname{tr}(\alpha\beta) = \operatorname{tr}(\beta\alpha).$$

So we have a well-defined trace for endomorphisms of free modules.

Now we need to define it for general projective modules. Let $\alpha : P \to P$ be a map, and let $\pi : F \twoheadrightarrow P$ be a surjection from a free module of finite rank. We can lift

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "i"] \arrow[r, "\mathrm{id}"] & P. \end{tikzcd}$$

Then we have

$$\operatorname{tr}(\alpha) = \operatorname{tr}(\alpha\pi i) = \operatorname{tr}(i\alpha\pi),$$

where $i\alpha\pi : F \to F$. We can take this as a definition. We still have to show that this well-defined. Consider

$$F \underset{i}{\overset{\pi}{\rightleftarrows}} P \underset{\alpha}{\overset{\beta}{\rightleftarrows}} P' \underset{\pi'}{\overset{j'}{\rightleftarrows}} F'.$$

Then the identity $\mathrm{tr}(i\alpha\pi' i'\beta\pi) = \mathrm{tr}(i'\beta\pi i\alpha\pi')$ implies

$$\mathrm{tr}(\alpha\beta) = \mathrm{tr}(i\alpha\beta\pi) = \mathrm{tr}(i'\beta\alpha\pi') = \mathrm{tr}(\beta\alpha).$$

This also implies well-definedness.

Now let $\varphi : \alpha \to \beta$ be a ring morphism. This induces a map $T\varphi : T(A) \to T(B)$. Let $\alpha : P \to P$ be a map. This induces a map $B \otimes_A \alpha : B \otimes_A P \to B \otimes_A P$. Then we get the identity

$$\mathrm{tr}_B(B \otimes_A \alpha) = T(\varphi)(\mathrm{tr}_A(\alpha)).$$

This follows from the free case.

We can do the analogous thing for restriction. Let $\varphi : A \to B$ be a ring homomorphism such that B is a finitely generated left A-module. Then there exists a map $\mathrm{tr}_{B/A} : T(B) \to T(A)$ such that for any $\alpha : P \to P$,

$$\mathrm{tr}_A(\alpha) = \mathrm{tr}_{B/A}(\mathrm{tr}_B(\alpha)).$$

Let me try to define this map. For any $b \in B$, there is a right multiplication map $\mu_b : B \to B$. This gives a map $\tilde{\varphi} : B \to T(A)$, and we have

$$\tilde{\varphi}(bb' - b'b) = \mathrm{tr}_A(\mu_b \mu_{b'}) - \mathrm{tr}_A(\mu_{b'}\mu_\beta) = 0.$$

So the map factors through $B \to T(B) \to T(A)$. This is going to be the map $\mathrm{tr}_{B/A}$.

Definition 20.2. The **Hattori–Stallings rank** is defined as $R(P) = \mathrm{tr}(\mathrm{id}_P)$.

If Γ is a finite group, then there is an augmentation map $\varphi : \mathbb{Z}\Gamma \to \mathbb{Z}$ and

$$\epsilon_\Gamma(P) = T(\varphi)(R_{\mathbb{Z}\Gamma}(P)).$$

Likewise, we have

$$\rho_\Gamma(P) = \tau(R_{\mathbb{Z}\Gamma}(P))$$

for the map $\tau : T(\mathbb{Z}\Gamma) \to \mathbb{Z}$ with $\tau : 1 \mapsto 1$ and $\gamma \mapsto 0$.

Note that if $P \neq 0$, then $\rho_\Gamma(P) > 0$. In particular, $\rho_\Gamma(\mathbb{Z}\Gamma) = 1$, and this shows that $\mathbb{Z}\Gamma$ is not decomposable into two projective $\mathbb{Z}\Gamma$-modules.

20.2 Grothendieck spectral sequence

This was a talk by George Torres. This computes the right derived functors of the composition of two functors. Let $\mathscr{A}$ be an abelian category.

Definition 20.3. Given a (co)chain complex $C^\bullet$ we define an **injective resolution** of $C^\bullet$ as $I^{p,\bullet} \to I^{p+1,\bullet}$ with commuting maps.

Any injective resolution induces exact sequences

$$0 \to Z^p(C^\bullet) \to Z^p(I^{\bullet,0}) \to Z^p(I^{\bullet,1}) \to \cdots,$$
$$0 \to B^p(C^\bullet) \to B^p(I^{\bullet,0}) \to B^p(I^{\bullet,1}) \to \cdots.$$

This further gives an exact sequence

$$0 \to H^p(C^\bullet) \to H^p(I^{\bullet,0}) \to H^p(I^{\bullet,1}) \to \cdots.$$

Definition 20.4. An injective resolution is called **fully injective** if the previous exacts sequences are injective resolutions.

Proposition 20.5. *If $\mathscr{A}$ has enough injectives, then every cochain complex admits a fully injective resolution.*

Proof. We can first find an injective resolution of $0 \to B^n(C^\bullet) \to Z^n(C^\bullet) \to H^n(C^\bullet) \to 0$. Then using the exact sequence, $0 \to Z^n(C^\bullet) \to C^n \to B^{n+1}(C^\bullet) \to 0$, you get an injective resolution of C^n. Then you can put them together and get an injective resolution of $C^\bullet$. ☐

Definition 20.6. A fully injective resolution of $C^\bullet$ of this form is called a **Cartan–Eilenberg resolution**.

These are used in defining hyper-derived categories, etc.

Definition 20.7. A **bicomplex** is a lattice of objectives $C^{ij} \in \mathscr{A}$ with maps $\partial_1^{ij} : C^{ij} \to C^{(i+1)j}$ and $\partial_2^{ij} : C^{ij} \to C^{i(j+1)}$ satisfying $\partial_1^2 = \partial_2^2 = 0$ and $\partial_1\partial_2 = \partial_2\partial_1$.

Definition 20.8. The **totalization** of the bicomplex is the complex

$$\operatorname{Tot}(C^{ij})^n = \bigoplus_{i+j=n} C^{ij}.$$

There are two canonical filtrations of the $\operatorname{Tot}(C)$. There is

$$F^p_{(1)}(\operatorname{Tot}(C))^n = \bigoplus_{r\geq p} C^{r,n-r}, \quad F^p_{(2)}(\operatorname{Tot}(C))^n = \bigoplus_{r\geq p} C^{n-r,r}.$$

These two induce two spectral sequences, which we will call ${}^{(1)}E$ and ${}^{(2)}E$. There are two differentials, and so there are two homologies:

$$H^{ij}_{(1)} = \ker(\partial_1^{ij})/\operatorname{im}(\partial_1^{i(j-1)}), \quad H^{ij}_{(2)} = \ker(\partial_2^{ij})/\operatorname{im}(\partial_2^{(i-1)j}).$$

Proposition 20.9. *Let $\mathscr{A}$ be cocomplete and let C^{ij} be a bicomplex. Then*

$${}^{(1)}E_2^{p,q} = H^p(H^{\bullet,q}_{(1)}(C)), \quad {}^{(2)}E_2^{p,q} = H^p(H^{q,\bullet}_{(2)}(C)),$$

and they both converges to $H^{p+q}(\operatorname{Tot}(C))$.

Definition 20.10. Let $\mathscr{A}$ have enough injectives and let $T : \mathscr{A} \to \mathscr{B}$ be additive. Then $Q \in \mathscr{A}$ is **right T-acyclic** if $R^iT(Q) = 0$ for $i > 0$.

Theorem 20.11. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be additive functors, and assume $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Assume that $F(Q)$ is G-acyclic for all Q injective. Then for all $A \in \mathscr{A}$, there exists a spectral sequence $E_r^{p,q}$ such that*

$$E_2^{p,q} = R^pG(R^qF(A)) \quad \Longrightarrow \quad R^{p+q}(FG)(A).$$

Proof. Let $A \to C^\bullet$ be an injective resolution. Then take a fully injective resolution of $F(C^\bullet)$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{01} & \longrightarrow & I^{11} & \longrightarrow & I^{21} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{00} & \longrightarrow & I^{10} & \longrightarrow & I^{20} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots
\end{array}$$

Now I is a bicomplex and so GI is a bicomplex. Then we have two spectral sequences ${}^{(1)}E$ and ${}^{(2)}E$. WE have

$${}^{(1)}E_2^{p,q} = H^p(H_{(1)}^{\bullet,q}(GI)) = H^p(R^qG(FC^\bullet)) = \begin{cases} 0 & q > 0 \\ H^p(GF(C^\bullet)) & q = 0. \end{cases}$$

So this collapses at this point, and so

$$H^n(\mathrm{Tot}(GI)) = R^n(GF)(A).$$

On the other hand, let us look at ${}^{(2)}E$. There is a short exact sequence

$$0 \to Z^p(I^{\bullet,q}) \to I^{p,q} \to B^{p+1}(I^{\bullet,q}) \to 0,$$

that splits, and so you can show $G(H_{(2)}^{p,q}(I)) \cong H_{(2)}^{p,q}(GI)$. Then we see that

$${}^{(2)}E_2 = H^p(H_{(2)}^{q,\bullet}(GI)) = H^p(G(H_{(2)}^{q,\bullet}(I))) \cong R^pG(R^qF(A)).$$

This finishes the proof. $\square$

Example 20.12 (Leray spectral sequence)**.** Consider X a topological space and $\mathscr{F}$ an abelian sheaf. Consider a continuous map $f : X \to Y$ and take $F = f_*(-)$ and $G = \Gamma(Y, -)$. Then we have a spectral sequence

$$H^p(Y, R^qf_*(\mathscr{F})) \quad \Longrightarrow \quad H^{p+q}(X, \mathscr{F}).$$

20.3 Čech cohomology

This was a talk by Julian Salazar.

Definition 20.13. A **presheaf** on a category $\mathscr{C}$ with $\mathscr{S}$-values is a functor $\mathscr{C}^{\text{op}} \to \mathscr{S}$. For a topological space, a **presheaf** on it is one for $\mathscr{C} = \text{Op}(X)$, the open sets of X with inclusion maps.

Definition 20.14. A **sheaf** is a presheaf $\mathscr{F}$ such that the following hold: given any open cover $\{U_i\}_{i\in I}$ and sections $\sigma_i \in \mathscr{F}(U_i)$ satisfying $\sigma_\alpha|_{U_\alpha\cap U_\beta} = \sigma_\beta|_{U_\alpha\cap U_\beta}$ for all $\alpha, \beta \in I$, there exists a unique $\sigma \in \mathscr{F}(X)$ such that $\sigma|_{U_\alpha} = \sigma_\alpha$.

Example 20.15. For X a smooth manifold, the space of p-form $\Omega^p(-)$ is a sheaf of $\mathbb{R}$-algebras. Moreover, $\Omega^*(-)$ is a sheaf of skew-commutative differential algebras.

Definition 20.16. A **ringed space** is a pair $(X, \mathscr{O}_X)$ of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X.

Proposition 20.17. *The category* $\mathsf{Sh}(X) = \mathsf{Sh}(X, \mathscr{O}_X-\mathsf{Mod})$ *has enough injectives.*

Definition 20.18. $\Gamma(X, -) : \mathsf{Sh}(X) \to \Gamma(X, \mathscr{O}_X)-\mathsf{Mod}$ is left exact, and we define **sheaf cohomology** as

$$H^i(\mathscr{F}) = R^i\Gamma(\mathscr{F}).$$

It's nice conceptually, but it is really hard to compute.

Proposition 20.19. *Let X be compactly-generated weakly-Hausdorff space. Then*

$$H^n_{\text{sing}}(X; A) = H^n(X, A_X),$$

where A_X is the constant sheaf.

Proposition 20.20. $H^{p,q}(X) \cong H^q(X, \Omega^p)$ *for complex analytic spaces X.*

Now let us defined Čech cohomology. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover. Let $\mathscr{P}$ be a presheaf on X.

Definition 20.21. The **Čech complex** is defined with objects

$$\check{C}^p(\mathfrak{U}, \mathscr{P}) = \prod_{i_0<\cdots<i_p} \mathfrak{P}(U_{i_0} \cap \cdots \cap U_{i_p})$$

and differentials

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}.$$

The **Čech cohomology** $\check{H}(\mathfrak{U}, \mathscr{P})$ is the cohomology coming from this complex.

If you think about this, this is actually looking at the simplicial cohomology of the nerve of the category coming from the open cover.

Example 20.22. With respect to the standard open cover of S^1 with two contractible sets, the Čech cohomology is $\check{H}^0(S^1, \mathbb{Z}_{S^1}) = \mathbb{Z}$ and $\check{H}^1(S^1, \mathbb{Z}_{S^1}) = \mathbb{Z}$.

Now let us apply the Grothendieck spectral sequence. This says that if $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are additive functors with F taking injectives to G-acyclics, then there is a spectral sequence

$$E_2^{p,q} = R^p G(R^q F(-)) \quad \Longrightarrow \quad R^{p+q}(G \circ F)(-).$$

Note that the Hochschild–Serre spectral sequence is an instance of this, with the functors $(-)^N$ and $(-)^{G/N}$.

Consider the diagram

$$\begin{tikzcd} \mathsf{Sh}(X) \arrow[r, hook, "i"] \arrow[dr, "\Gamma(X,-)"'] & \mathsf{PSh}(X) \arrow[d, "\check{H}^0(X,-)"] \\ & \mathscr{O}_X-\mathsf{Mod}. \end{tikzcd}$$

This gives a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathscr{U}, \mathcal{H}^q(\mathscr{F})) \quad \Longrightarrow \quad H^{p+q}(X, \mathscr{F}).$$

Here $\mathcal{H}^q(\mathscr{F})$ is the sheaf $U \mapsto H^q(U, \mathscr{F})$.

Example 20.23 (Mayer–Vietoris)**.** Consider $\mathfrak{U} = \{U, V\}$. Let's draw the E_2 page. There are going to be two columns, because there are two open sets. Then you recover the Mayer–Vietoris sequence with sheaves as coefficients. If $\mathscr{F}$ is the constant sheaf, then you recover Mayer–Vietoris for ordinary cohomology.

Theorem 20.24 (Leray)**.** *If $\mathscr{F} \in \mathsf{Sh}(X)$ and $\mathscr{F}$ is acyclic on every open in $\mathfrak{U}$, we have*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \cong H^p(X, \mathscr{F}).$$

This is used in computing things like cohomology of line bundles over projective space. There is the canonical affine open cover $\mathfrak{U} = \{U_0, U_1, U_2, \ldots\}$ and this allows us to to compute $H(\mathbb{P}^n_k, \mathcal{O}(n))$.

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