Math 230br - Advanced Differential Geometry

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This course as taught by Shing-Tung Yau. The lectures were given in the CMSA building room G10, at TTh 1-2:30. There were no official textbooks and there were 3 undergraduates and 8 graduate students enrolled. There was one take-home midterm and a take-home final, with no course assistants.

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1.1 Vector bundles

In general, a vector bundle is given by local trivializations and transition maps $g_{\alpha\beta} : O_\alpha \cap O_\beta \to \mathrm{GL}(n, \mathbb{R})$ satisfying the relations $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and the cocycle relation $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{id}$.

We can require the transition maps $g_{\alpha\beta}$ to map to a smaller subset. For instance, we can require $g_{\alpha\beta} : O_\alpha \cap O_\beta \to \mathrm{O}(n)$ and then the vector bundle have a natural inner product structure. In the case n is even, we can require $g_{\alpha\beta} : O_\alpha \cap O_\beta \to \mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})$. If the underlying space is a complex manifold, and $g_{\alpha\beta}$ is holomorphic, i.e., $\bar{\partial} g_{\alpha\beta} = 0$, then the bundle will be called a **holomorphic vector bundle**.

Definition 1.1. Given a vector bundle $p : V \to M$, a **section** is a map $s : M \to V$ such that $p \circ s = \mathrm{id}$. Denote $s(x) = (x, v(x))$. A **holomorphic section** is a section with v holomorphic.

A compact complex manifold has no non-constant holomorphic function by the maximal principle. So the interesting sections are the twisted vector valued functions.

For a manifold M^n, consider its tangent bundle

$$TM = \{(x, v) : x \in M, v \in T_x M\} = \{(x_1, \dots, x_n, a_1(x), \dots, a_n(x))\}.$$

For two coordinate charts x and y, the transition functions are given by

$$a^i = \sum b^j(y) \frac{\partial x^i}{\partial y^j}.$$

So the transition maps are given by $g_{\alpha\beta} = \partial x^\alpha / \partial y^\beta$.

Example 1.2. Recall that

$$\begin{aligned} \mathbb{R}P^n &= S^n/(-1) = \{\{x, -x\} : \|x\| = 1\}, \\ \mathbb{C}P^n &= \{\{e^{i\theta} x\} : \|x\| = 1\}. \end{aligned}$$

So there is a natural projection

$$\tau_n^1 = \{(\{x, -x\}, tx) : \|x\| = 1, t \in \mathbb{R}\} \to \mathbb{R}P^n.$$

This is called the **tautological bundle**. Likewise there is a complex line bundle

$$\tau_n^{\mathbb{C}} = \{(\{e^{i\theta} x\}, tx) : t \in \mathbb{C}\} \to \mathbb{C}P^n.$$

Let us try to visualize the bundle of $\mathbb{R}P^1$. Because $\mathbb{R}P^1$ can be parameterized by $\pm(\cos\theta, \sin\theta)$ for $\theta \in (0, \pi)$, the bundle τ_1^1 will be a Möbius band.

Indeed, there are exactly two line bundles over S^1, the one being the trivial bundle $S^1 \times \mathbb{R}$ and the other being the Möbius bundle. The reason is that S^1 can be covered by two open sets O_1 and O_2 so that the connected components of $O_1 \cap O_2$ are I_1 and I_2. The image of the transition maps lie in $\mathrm{GL}(2, \mathbb{R}) = \mathbb{R}^*$ and essentially there are two choices, ± 1. If they have the same sign, then the resulting bundle is trivial, and if they have different signs, then the bundle is the Möbius bundle.

Proposition 1.3. *For all n, the bundle $\tau_n^1 \to \mathbb{R}P^n$ is non-trivial.*

Proof. We will show that every section $s : \mathbb{R}P^n \to \tau_n^1$ must have a zero. This can be seen using Euler numbers, but we will not do this now. □

Let us look at another example of a vector bundle. Consider the space

$$V_n = \{(x, v) : x \in S^n, v \perp x\}/(x, v) \sim (-x, -v) \to \mathbb{R}P^n$$

which is naturally a vector bundle over $\mathbb{R}P^n$. Then $\tau_n^1 \oplus V_n = \mathbb{R}^{n+1}$. That is, the sum of two non-trivial bundles becomes trivial. There is a complex analogue of this:

$$\begin{tikzcd} 0 \arrow[r] & \tau_n^1 \arrow[r] \arrow[dr] & \mathbb{C}^n \arrow[r] \arrow[d] & T \arrow[r] \arrow[dl] & 0 \\ & & \mathbb{C}P^{n-1} & & \end{tikzcd}$$

This is a diagram that is important in complex geometry.

1.2 Operations on vector bundles

Given two vector bundles $V_1 \to M$ and $V_2 \to M$, these give a vector bundle $V_1 \times V_2 \to M \times M$. The transition functions will take the form of block diagonal matrices.

After this, note that there is a diagonal map $M \to M \times M$. Then you can pull back the bundle to get a bundle over M.

$$\begin{tikzcd} V_1 \oplus V_2 \arrow[d] & V_1 \times V_2 \arrow[d] \\ M \arrow[r] & M \times M \end{tikzcd}$$

This is the direct sum of vector bundles.

Likewise you can define dual bundles, that uses the transition maps $g_{\alpha\beta}^{-1}$. You can also define tensor products $V_1 \otimes V_2$ by taking the tensor products of the two transition maps $g_{\alpha\beta}$ and $g'_{\alpha\beta}$.

So one goal of bundle theory is to construct some kind of and invariant that enjoy nice properties with respect to these operations. These are characteristic classes. We want to study the set

$$S = \{\text{spaces of vector bundles modulo isomorphism}\}.$$

The characteristic class is a map

$$S \xrightarrow{c} H^*(M) = \oplus_{i=0}^{\infty} H^i(M)$$

such that $c(f^*v) = f^*c(v)$, i.e., behaves well with respect to pullbacks. (Recall that a map $f : M_1 \to M_2$ gives a map $f^* : H^*(M_2) \to H^*(M_1)$.) This is the first axiom we want the map c to satisfy.

This map c also satisfies

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2),$$

where multiplication is natural multiplication in the cohomology ring. This is called the Whitney product formula.

These two are the most important properties we want. The rest are just normalizations. We want $c(\text{trivial}) = 1$ and $c(\tau_2^1) = 1 + \alpha \in H^*(\mathbb{R}P^2, \mathbb{Z}/2)$, where $\alpha \in H^1$ is the nonzero element. These are the four axioms we want the characteristic classes to satisfy.

There are essentially two kinds of characteristic classes: Stiefel–Whitney classes in $H^*(M, \mathbb{Z}/2)$ and Chern classes in $H^*(M, \mathbb{Z})$. We will develop the theory of these classes.

A good book is J. Milnor's *Characteristic classes.*

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The set of smooth vector bundles form a ring.

Definition 2.1. We say that two vector bundles V_1 and V_2 are **stably equal** if there exist trivial bundle E_1 and E_2 such that $V_1 \oplus E_1 \cong V_2 \oplus E_2$. In this case, we write $V_1 \equiv V_2$.

Now this is clearly an equivalence relation and so we may look at the quotient space with respect to the equivalence. The trivial bundle is the identity, i.e., $V \oplus E \equiv V$. This group is called the **topological K group** and is written as $K_0(M)$.

2.1 Axioms for characteristic classes

Given a vector bundle $V \to M$, we want to associate some element $H^*(M, \mathbb{Z}/2)$ so that the following axioms hold:

(1) To each bundle V of rank k is associated an element $w(V) = 1 + w_1(V) + \cdots + w_k(V)$, where $w_i(V) \in H^i(M, \mathbb{Z}/2)$.

(2) (naturality) For a map $f : M_1 \to M_2$ and a bundle $V_2 \to M_2$, $w(f^*V) = f^*w(V)$.

(3) (product formula) For two bundles V_1 and V_2, $w(V_1 \oplus V_2) = w(V_1) \cdot w(V_2)$.

(4) (normalization) The Stiefel–Whitney class of the tautological line bundle $\tau_1^1 \to \mathbb{R}P^1$ is $w(\tau_1^1) = 1 + w_1$ where $w_1 \neq 0$ in $H^1(\mathbb{R}P^1, \mathbb{Z}/2)$.

For the Chern class, the fourth axiom is modified as the following way:

(4′) The Chern class of the tautological line bundle is $c(\tilde{\tau}_1^1) = 1 + c_1$, where $c_1 \in H^2(\mathbb{C}P^1, \mathbb{Z})$ is the positive generator.

Note that we are working in the smooth category, but in the holomorphic category, the characteristic class $c(V)$ lies in $H^*(M, \mathbb{Z})$. One interesting aspect of the C^∞ category is that any short exact sequence $1 \to V_1 \to E \to V_2 \to 1$ splits.

Proposition 2.2. *If $1 \to V_2 \to E \to V_2 \to 1$ is a short exact sequence of smooth bundles, then $E \cong V_1 \oplus V_2$.*

Proof. We first give a inner product on E. We know that V_1 lies in E. Its orthogonal complement $V_1^\perp$ in E is a subbundle, and it must be isomorphic to V_2. □

The point is that in the smooth category, axiom (3) implies $w(E) = w(V_1) \cdot w(V_2)$.

If two bundles $V_1, V_2 \to M$ are stably equivalent, then there exist trivial bundles E_1, E_2 such that $V_1 \oplus E_1 \cong V_2 \oplus E_2$. Then

$$w(V_1) \cdot w(E_1) = w(V_1 \oplus E_1) = w(V_2 \oplus E_2) = w(V_2) \cdot w(E_2).$$

But characteristic classes of trivial bundles are 1, and so we conclude $w(V_1) = w(V_2)$. That is, stably equivalent bundles have same Stiefel–Whitney classes. This defines a map

$$w : K_0(M) \to H^*(M, \mathbb{Z}/2).$$

Likewise the Chern classes give a map

$$c : K_{\mathbb{C}}(M) \to \bigoplus_{i=0}^{n} H^{2i}(M, \mathbb{Z}).$$

Later, Hirzebruch defined the Chern characters, which becomes a ring homomorphism

$$\mathrm{ch} : K_{\mathbb{C}}(M) \to \bigoplus_{i=0}^{n} H^{2i}(M, \mathbb{Z}).$$

For holomorphic bundles, the Chern classes actually map to a smaller set:

$$c : K_{\mathrm{hol}}(M) \to \bigoplus_{i=0}^{n} H^{i,i}(M).$$

For a manifold M, we define $w(M) = w(TM)$, where TM is the tangent bundle. Likewise for the Chern class, we define $c(M) = c(T^{1,0}(M))$. The Stiefel–Whitney class turns out to be homotopic invariant. (The Chern class on the other hand depends on the holomorphic structure.)

2.2 Cobordism

There is a very basic construction that is due to Pontryagin (and Thom). Take a manifold M^n and write $w(M) = 1 + w_1 + \cdots$. For any $(i_1, \dots, i_l)$ with $\sum i_j = n$, we have

$$w_{i_1} \smile \cdots \smile w_{i_l} \in H^n(M, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This is called the **Stiefel–Whitney number**. If I take a manifold $M^n = \partial B^{n+1}$, then all Stiefel–Whitney numbers are zero.

Theorem 2.3 (Thom)**.** *Suppose all Stiefel–Whitney numbers of M are zero. Then $M = \partial B$ for some manifold B with boundary.*

Let's see what this gives us. For a tuple $I = (i_1, \dots, i_n)$ with $\sum k i_k = n$, let us write

$$w_I(M) = w_1(M)^{i_1} \cdots w_n(M)^{i_n}.$$

If $w_I(M_1) = w_I(M_2)$ for all I, then this gives $M_1 \cup M_2 = \partial B$. This shows that the Stiefel–Whitney number determines the cobordism class. In fact, we can choose

$$M_2 = \sum_I m_I \prod \mathbb{R}P^I.$$

This basically classify the cobordism classes.

Why is this important? Suppose I have a manifold M_1 that I don't know, but know how to compute its Stiefel–Whitney numbers. Then it is cobordant to something I know, although the manifold B might be complicated. Now I want this B as simple as possible. The ideal situation is $B = M_1 \times I$. Still I simplify the manifold B by doing surgery until $H^*(M_1) = H^*(B)$. Then something called the h-cobordism theorem tells us that it is actually $B = M_1 \times I$.

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Cobordism gives a equivalence relation $M \equiv N$ if $M \cup N = \partial B$ for some B. There is an addition operator on cobordism classes given by $[M] + [N] = [M \cup N]$, and there is a product given by $[M] \times [N] = [M \times N]$. So the set of cobordism classes have a ring structure. There also is a connected sum that satisfies $M \# N \equiv M \cup N$.

3.1 Stiefel–Whitney number of a boundary

Recall that there is a Stiefel–Whitney number given by

$$w^I(M) = w_1^{i_1} \smile \cdots \smile w_k^{i_k} \in H^n(M; \mathbb{Z}/2) = \mathbb{Z}/2.$$

Theorem 3.1 (Thom)**.** *$w^I(M) = 0$ for all I if and only if $M \equiv 0$, i.e., $M = \partial B$.*

One direction of the theorem is easy. Suppose $M = \partial B$. We note that

$$T(B)|_B = T(M) \oplus \nu,$$

where ν is a trivial line bundle. This is because there is a neighborhood retract of M and so there is a non-vanishing section.

Now denote $i : M \hookrightarrow B$. Then from what we have just said,

$$i^*(w(B)) = w(M) \cdot w(\nu) = w(M).$$

We have an exact sequence $H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$ and so

$$(w^I(M), M) = (i^* w^I(B), M) = (i^* w^I(B), \partial B) = (\delta i^* w^I(B), B) = 0.$$

3.2 Stiefel–Whitney number of projective space

Let us compute $w(T(\mathbb{R}P^n))$. Note that

$$T(\mathbb{R}P^n) \oplus \mathbb{R} = S^n \times \mathbb{R}^{n+1}/((x, v) \sim (-x, -v)) = \tau_n^1 \oplus \cdots \oplus \tau_n^1.$$

It follows that $w(T(\mathbb{R}P^n)) = w(\tau_n^1)^{n+1}$. Hence we only need to compute the characteristic class of the tautological line bundle.

Now there is an inclusion $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ and we note that $i^* \tau_n^1 = \tau_1^1$. So

$$i^*(w(\tau_n^1)) = w(i^* \tau_n^1) = w(\tau_1^1) = 1 + \alpha \in H^*(\mathbb{R}P^1).$$

It follows that $w(\tau_n^1) = 1 + \alpha \in H^*(\mathbb{R}P^n)$. So we get

$$w(T(\mathbb{R}P^n)) = (1 + \alpha)^{n+1} = 1 + \sum_{i=1}^{n} \binom{n+1}{i} \alpha^i.$$

Corollary 3.2. *$w(\mathbb{R}P^n) = 1$ if and only if $n + 1$ is a power of 2.*

For example, $\mathbb{R}P^1, \mathbb{R}P^3, \ldots$ have trivial Stiefel–Whitney class. Thus $\mathbb{R}P^n$ has trivial tangent bundle only if $w(\mathbb{R}P^n) = 1$, which is true if and only if $n+1$ is a power of 2.

In fact, $\mathbb{R}P^n$ has trivial tangent bundle if and only if $n = 1, 3, 7$. This was an important problem in topology and it was proved by Adams.

There is another related problem. A map $f : M^n \hookrightarrow \mathbb{R}^{n+k}$ is called a **immersion** if $f_* : T_p M \to T_q \mathbb{R}^{n+k}$ is injective everywhere. If there exists an immersion, then there exists a decomposition

$$T(M) \oplus \nu(M) = \text{trivial},$$

where $\nu(M)$ is the normal bundle. Hence $w(\nu) = 1/w(TM)$. Now note that ν has rank k. This already gives some useful information.

For example, let us suppose that we want to find the least q such that there exists an immersion $\mathbb{R}P^n \hookrightarrow \mathbb{R}^q$. If $n = 2^r$, then we have $w(\mathbb{R}P^n) = (1+\alpha)^{2^r+1} = 1 + \alpha$. Hence

$$w(\nu) = \frac{1}{1+\alpha} = 1 + \alpha + \cdots + \alpha^{n-1}.$$

It follows that $q \geq 2n - 1$, which is the sharp bound given by the Whitney embedding theorem.

Definition 3.3. A **division algebra** of dimension n over $\mathbb{R}$ is an associative bilinear map $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ that has no zero divisor.

We know that all the possible division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the Cayley numbers. This is because the existence of a division algebra implies that $T(\mathbb{R}P^{n-1})$ is trivial. Take a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. We know that for $a \neq 0$, the vectors $\langle a, e_1 \rangle, \ldots, \langle a, e_n \rangle$ are linearly independent. Consider the set

$$\{a \in \mathbb{R}^n : \langle a, e_1 \rangle \subseteq S^{n-1}\}$$

which is homeomorphic to S^{n-1}. For each $\langle a, e_1 \rangle$, we look at the projection of $\langle a, e_2 \rangle, \ldots, \langle a, e_n \rangle$ onto $T_{\langle a, e_1 \rangle}(S^{n-1})$. This gives non-vanishing linearly independent vector fields on S^{n-1} and it is further a vector field on $\mathbb{R}P^{n-1}$.

Now we have done all the introduction. We are going to find a "universal" bundle for M. This is going to be a bundle $U \to B$ such that for any vector bundle $V \to B$ there exists a $f : M \to B$ so that $V = f^* U$.

$$\begin{array}{ccc} V & & U \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

This f is going to be unique up to homotopy.

This is something you know classically. For any manifold $M^n \subseteq \mathbb{R}^{n+1}$, you get a map $M \to S^n$ given by a map pointing to its normal vector. More generally, if $M_n \subseteq \mathbb{R}^{n+k}$ is a manifold, we get a **Gauss map**

$$M^n \to G(n, n+k) = \left\{ \begin{matrix} \text{spaces of rank } n \\ \text{in } \mathbb{R}^{n+k} \end{matrix} \right\}.$$

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Let M^n be a manifold and suppose $M^n \hookrightarrow \mathbb{R}^{n+k}$ is an immersion. Then we get the Gauss map $M \to G_{\mathbb{R}}(n, n+k)$. We can also map each point $p \in M$ to the orthogonal complement of its tangent space. This gives a map $M \to G_{\mathbb{R}}(k, n+k)$.

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & G_{\mathbb{R}}(n, n+k) \arrow[d, shift left] \\
& G_{\mathbb{R}}(k, n+k) \arrow[u, shift left]
\end{tikzcd}$$

There is also a tautological vector bundle γ_n on $G(n, n+k)$ given by

$$\gamma_n = \{(v, x) : v \in G(n, n+k), x \in v\}.$$

Given this construction it is obvious that the pullback bundle of γ_n with respect to the Gauss map is the tangent bundle.

$$\begin{tikzcd}
f^*\gamma_n = TM \arrow[r] \arrow[d] & \gamma_n \arrow[d] \\
M \arrow[r, "f"] & G(n, n+k)
\end{tikzcd}$$

Note that by the Whitney embedding theorem, we always have an immersion $M^n \hookrightarrow \mathbb{R}^{n+k}$. Thus the tangent bundle can be realized as the pullback of some γ_n.

Theorem 4.1. *Any vector bundle can be realized as a pullback of the tautological bundle $\gamma^k \to G(k, N)$ with respect to a map $f : M \to G(k, N)$, for large N. Furthermore, such an f is unique up to homotopy.*

Note that it is clear that if f and g are homotopic then $f^*\gamma \equiv g^*\gamma$ as bundles. Also note that there is an embedding $G(k, N) \hookrightarrow G(k, N+1)$ for any N. So we may define

$$G(k, \infty) = \varinjlim G(k, N)$$

by gluing all the Grassmannians. Then what the theorem asserts is

$$\{\text{vector bundle } V \to M\} \quad \longleftrightarrow \left\{\begin{matrix}\text{homotopy class of} \\ f : M \to G(k, \infty)\end{matrix}\right\} = [M, G(k, \infty)].$$

This reduces the theory of bundles to homotopy theory.

4.1 Pontryagin–Thom construction

Any manifold M^n can be embedded in some big Euclidean space $\mathbb{R}^{n+k}$. Then there is a tubular neighborhood of radius ϵ such that it looks like $M \times B^k$.

Take a vector bundle $V \to M$, in our case the normal bundle, and give a metric on V. Now take the ball bundle

$$B = \{v \in V : |v| \leq 1\} \subseteq V.$$

We now collapse $V \setminus B$ to a single point. This is now not a smooth vector bundle but it is a cell complex. This is called the **Thom complex** and is denoted $T(\nu)$.

There is a topological map $S^{n+k} \to T(\nu)$. This give an element of $\pi_{n+k}(T(\nu))$. Now note that there is a map $T_\nu \to T(\gamma_k \to G(k, n+k))$ and this induces a map

$$\pi_{n+k}(T(\nu)) \to \pi_{n+k}\big(T(\gamma_k \to G(k, n+k))\big).$$

So we get from M an element of $\pi_{n+k}(T(\gamma_k \to G(k, n+k)))$.

This whole construction gives a map

$$\{\text{manifolds } M^n\} \to \lim_{n\to\infty} \pi_{n+k}(T(\gamma_k \to G(k, n+k))).$$

Proposition 4.2. *This map is a bijection between the cobordism classes of manifolds M^n and $\lim_{n\to\infty} \pi_{n+k}(T(\gamma_k))$.*

I won't prove this because it is 10 pages long.

4.2 Almost complex structure

Let us write

$$B(\mathrm{O}(2k)) = \varinjlim_{n\to\infty} G_{\mathbb{R}}(2k, n), \quad B(\mathrm{U}(k)) = \varinjlim_{n\to\infty} G_{\mathbb{C}}(k, n).$$

Then there is a forgetful map $\pi : B(\mathrm{U}(k)) \to B(\mathrm{O}(k))$. Given a map $f : M \to B(\mathrm{O}(2k))$, it is natural to ask whether there exists a map $\tilde{f}$ that makes the following diagram commute.

$$\begin{tikzcd} & B(\mathrm{U}(k)) \arrow[d, "\pi"] \\ M \arrow[ur, dashed, "\tilde{f}"] \arrow[r, hook, "f"'] & B(\mathrm{O}(2k)) \end{tikzcd}$$

This has a important meaning. This is asking if there exists a continuous way of assigning complex k planes to each point. This $\tilde{f}$ exists if and only if there exists a complex structure on TM. That is, whether there exists a section $J_x \in \mathrm{End}(T_x(M^{2k}))$ such that $J_x^2 = -1$. If this J exists, we call that M is an **almost complex manifold**.

Given an f, this existence of $\tilde{f}$ is purely homotopic problem. Note that you can lift to $\tilde{f}$ and $\tilde{\tilde{f}}$ with $\tilde{f}$ and $\tilde{\tilde{f}}$ not homotopic.

4.3 Construction of Stiefel–Whitney classes

From Theorem 4.1 we know that bundles $V \to M$ are classified by homotopy classes of maps $M \to G(k, \infty)$. You can compute the compute the cohomology ring of $G(k, \infty)$, and it is

$$H^*(G(k, \infty); \mathbb{Z}/2) = \{\text{generated by } \gamma_1, \dots, \gamma_k\}.$$

After this, we can simply define the Stiefel–Whitney class as

$$w_i(M) = f^*(\gamma_i).$$

Then the pull back axiom is automatically satisfied, and I have to worry about the product formula.

For a vector bundle $V \to M$, we may take a partition of unity and find a set of sections that generate V. This can be thought of as a surjective bundle map $f : M \times \mathbb{R}^N \to V$. Then we can take the kernel of this vector bundle.

$$\begin{tikzcd}
1 \arrow[r] & \ker f \arrow[r] \arrow[dr] & M \times \mathbb{R}^N \arrow[r, two heads, "f"] \arrow[d] & V \arrow[r] \arrow[dl] & 1 \\
& & M & &
\end{tikzcd}$$

So far we have proved the following.

Proposition 4.3. *Given any vector bundle $V \to M$, there exists a $W \to M$ such that $V \oplus W$ is trivial.*

Now let us look at $f : (\ker f)^\perp \to V$. This is an isomorphism of vector bundles. This gives an embedding of V into a big trivial bundle. This further gives a map $M \to G(k, N)$ and the pullback of the tautological bundle respect to this map is the bundle $V \to M$. This shows the existence part of Theorem 4.1.

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Once we have a surjective bundle map $M \times \mathbb{R}^N \xrightarrow{F} V \to 1$, we get an isomorphism $(\ker F)^\perp \cong V$. This shows the following.

Proposition 5.1. *Any $V \to M$ is a subbundle of a trivial bundle. There also exists a bundle $V' \cong \ker F$ such that $V \oplus V'$ is trivial.*

This shows that the isomorphism classes of vector bundles under stable equivalence forms a group, which is called the topological K-group $K_{\text{top}}(M)$. The Stiefel–Whitney class will be a map

$$w : K_{\text{top}}(M) \to H^*(M, \mathbb{Z}_2).$$

In this procedure we learned something. We have a bundle map $V \hookrightarrow M \times \mathbb{R}^N$ and so we get a map

$$M \to G(k, N); \quad x \mapsto \text{fiber over } x.$$

There is a tautological bundle τ_k over $G(k, N)$ and then V is just the pullback bundle.

Further more, if you have two vector bundles $V \to M$ and $V' \to M$ each coming from $f, g : M \to G(k, \infty)$, then you can prove that $V \cong V'$ if and only if f and g are homotopic. So in fact, $f \in [M, G(k, \infty)]$. This is $\tau_k \to G(k, \infty)$ is called the **classifying space**.

You have to be quite careful if you are working in the holomorphic category. We can take some sections that generate the vector bundle and this gives a surjective map $\mathbb{C}^N \to V$. But we can't take the orthogonal complement of the dual. Instead, we look at the dual, which is $\mathbb{C}^{N*} \leftarrow V^*$. So we can still talk about the Grassmannian.

Now given a map $f : M \to G(k, \infty)$, we get a map

$$f^* : H^*(G(k, \infty)) \to H^*(M)$$

on cohomology rings. This map is called the **characteristic class**.

5.1 Cohomology of the Grassmannian

To understand characteristic classes, we need to understand the cohomology ring $H^*(G(k, N))$. The Grassmanian $G(k, N)$ is the space of $k \times N$ matrices of full rank, up to some elementary row operations. Using Gaussian elimination, we can uniquely make it into a reduced row-echelon form. Let r_i be the number of columns where there are i free entries. Then we have $r_1 + \cdots + r_k \leq N - k$ and this part forms a cell

$$C_{r_1,\ldots,r_k} \cong \mathbb{R}^{r_1+2r_2+\cdots+kr_k} \subseteq G(k, N).$$

Then we have a cell decomposition

$$G(k,N) = \bigcup_{r_1,\ldots,r_k} C_{r_1,\ldots,r_k}.$$

For example, $G(1, N+1) = \mathbb{R}P^N$ is formed by 1 cells, each of dimensions $0, 1, \ldots, N$.

The Stiefel–Whitney class of $\gamma_k \to G(k,N)$ will look like

$$w(\gamma_k) = w_1(\gamma_k) + w_2(\gamma_k) + \cdots + w_k(\gamma_k).$$

Assuming the existence of Stiefel–Whitney classes we claim that

$$H^*(G(k,N), \mathbb{Z}/2) = \mathbb{Z}/2[w_1(\gamma_k), \ldots, w_k(\gamma_k)],$$

the polynomial ring with $\mathbb{Z}/2$ coefficients.

Proposition 5.2. *There are no relations among the $w_i(\gamma_k)$.*

Proof. Suppose $P(w_1(\gamma_k), \ldots, w_k(\gamma_k)) = 0$. Then for any vector bundle $V \to M$, we should get

$$P(w_1(V), \ldots, w_k(V)) = f^*(P(w_1(\gamma_k), \ldots, w_k(\gamma_k))) = 0.$$

So it suffices to exhibit an example of a vector bundle such that this is nonzero.

Take the tautological line bundle $\tau_n \to \mathbb{R}P^n$. Then $w(\tau_n) = 1 + \alpha$ for $\alpha \neq 0 \in H^1(\mathbb{R}P^n, \mathbb{Z}/2)$. Take the projection map $\pi_i : M = \mathbb{R}P^n \times \cdots \times \mathbb{R}P^n \to \mathbb{R}P^n$, we define

$$\bigoplus_i \pi_i^*(\tau_n) = V \to M = \mathbb{R}P^n \times \cdots \times \mathbb{R}P^n.$$

By the Künneth theorem, we have $H^*(M) = \langle \alpha_1, \ldots, \alpha_k \rangle$. Also we have

$$w\Big(\bigoplus_i \pi_i^*(\tau_n)\Big) = \prod (1 + \alpha_i)$$

and so w_i are the elementary symmetric polynomials generated by $\alpha_1, \ldots, \alpha_k$. These are algebraic independent. $\square$

Now we have established that $H^*(G(k,N), \mathbb{Z}/2) \supseteq \mathbb{Z}/2[w_i(\tau_k)]$. Let us show that this is an equality.

The i-dimensional cells of $G(k,N)$ are the nonnegative solutions to

$$r_1 + 2r_2 + 3r_3 + \cdots + kr_k = i, \quad r_1 + \cdots + r_k \leq N - k.$$

This is equal to the dimension of the i-th graded piece of $\mathbb{Z}/2[w_i(\tau_k)]$. This shows that the two sides are equal.

5.2 Construction of Stiefel–Whitney classes 2

For a vector bundle $V \to M$, define

$$V_0 = \{(x, v) : v \neq 0\} \subseteq V.$$

Fiberwise, $(F, F_0 = F \cap V_0)$ is a sphere. So we have

$$H^i(F, F_0; \mathbb{Z}/2) = \begin{cases} 0 & i \neq k \\ \mathbb{Z}/2 & i = k. \end{cases}$$

The interesting thing happens when we look at (V, V_0). We have

$$H^i(V, V_0; \mathbb{Z}/2) = \begin{cases} 0 & i < k \\ H^{i-k}(M) & i \geq k, \end{cases}$$

where k is the fiber dimension.

Theorem 5.3. *(1) $H^i(V, V_0) = 0$ for $i < k$.*
(2) There exists a unique class U in $H^k(V, V_0)$ such that for every $F = \pi^{-1}(x)$ and $j_x : F \hookrightarrow V$, we have $j_x^* U \neq 0$ in $H^k(F, F_0)$.
(3) There are isomorphisms

$$\begin{tikzcd} H^j(M) \arrow[r, "\phi"] \arrow[d, "\pi^*"] & H^{j+n}(V, V_0) \\ H^j(V) \arrow[ur, "\smile U"'] & \end{tikzcd}$$

You can use spectral sequences (1), but you can intuitive see this by cell decomposition. There are only cells of dimension at least k. I won't go into the details because this is topological.

There is a natural operator called the **Steenrod square**,

$$\mathrm{Sq}^i : H^k(X, Y) \to H^{k+i}(X, Y),$$

satisfying

(1) $\mathrm{Sq}^i(\alpha^k) = \begin{cases} 0 & k < i \\ \alpha^k \cup \alpha^k & k = i, \end{cases}$

(2) $\mathrm{Sq}^0 = \mathrm{id}$,

(3) $\mathrm{Sq}^k(\alpha \cup \beta) = \sum_{i+j=k} \mathrm{Sq}^i(\alpha) \smile \mathrm{Sq}^j(\beta)$.

Using this, define the **Stiefel–Whitney class** as

$$w_i(V) = \phi^{-1}(\mathrm{Sq}^i \phi(1)).$$

6 February 9, 2017

We have the classifying space $B(\mathrm{O}(k)) = G(k, \infty)$ with the tautological bundle. We have the Stiefel–Whitney classes $w_i \in H^i(G(k,\infty), \mathbb{Z}/2)$, and proved that the polynomial ring $\mathbb{Z}/2[w_1, \dots, w_k]$ is isomorphic to the cohomology ring $H^*(G(k,\infty), \mathbb{Z}/2)$. Then for any $f : M \to G(k, \infty)$, we have $w_i(V) = f^*(w_i(\tau_k))$. So everything forms a beautiful picture of everything coming from one bundle.

For a vector bundle $V \to M$, we defined $V_0 = \{v : v \neq 0\}$. We want to consider the pair (V, V_0). Homotopically, this is collapsing the boundary points to a single point. This is called the **Thom space**.

Consider the Thom space $T(\tau_k, (\tau_k)_0)_N$ of the tautological bundle $\tau_k \to G(k, N)$. There is a natural inclusion

$$T(\tau_k, (\tau_k)_0)_N \hookrightarrow T(\tau_k, (\tau_k)_0)_{N+1}.$$

This is just a suspension if you think about it.

If you have a manifold M, lying in $\mathbb{R}^N$, you get the Gauss map $M \to G(k, N)$ and so $T(\nu) \to T(\tau_k)$ where ν is the normal bundle. We also have a map $S^N \to T(\nu)$ by collapsing some points. This gives an element of $\pi_N(T(\tau_k))$, and this is indeed a map

$$\{\text{cobordism classes } [M]\} \to \pi_N(T(\tau_k)).$$

This is actually surjective because given a map you can make it transverse and then take the inverse image to be M. And you can also show that it is injective.

Theorem 6.1. *For a rank k vector bundle $V \to M$, the cohomology of the Thom space is $H^{i+k}(V, V_0) \cong H^i(M)$ and $H^i(V, V_0) \cong 0$ for $i < k$.*

Using the Thom isomorphism ϕ, we can define the Stiefel–Whitney class as

$$w_i(V) = \phi^{-1} \operatorname{Sq}^i(\phi(i)).$$

You can prove that $w(V \oplus W) = w(V) \smile w(W)$.

We define the **Euler class** of a vector bundle modulo 2 as

$$w_k(V^k) \in H^k(M, \mathbb{Z}/2).$$

The Euler characteristic of a manifold M is then

$$w_n(TM) = \sum_{i \text{ even}} \dim H^i - \sum_{i \text{ odd}} \dim H^i.$$

6.1 Splitting principle

Suppose $\pi : V_{\mathbb{C}} \to M$ is a complex vector bundle. You can projectivize this bundle and get a fiber bundle

$$\mathbb{C}P^{k-1} \to P(V_{\mathbb{C}}) \xrightarrow{\pi_1} M.$$

Then $P(V_{\mathbb{C}})$ is a manifold, and so we can look at the pullback bundle $\pi_1^*(V_{\mathbb{C}}) \to P(V_{\mathbb{C}})$. We also have a tautological line bundle $\mathcal{L} \to P(V_{\mathbb{C}})$. There is also an natural embedding $\mathcal{L} \hookrightarrow \pi_1^*(V_{\mathbb{C}}^k)$. So take the quotient Q:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{L}^1 \arrow[r, hook] \arrow[dr] & \pi_1^*(V_{\mathbb{C}}^k) \arrow[r] \arrow[d] & Q \arrow[dl] \\ & & P(V_{\mathbb{C}}). & \end{tikzcd}$$

This gives

$$\pi_1^* c(V_{\mathbb{C}}^k) = c(\pi_1^* V_{\mathbb{C}}^k) = c(\mathcal{L}^1) c(Q^{k-1}).$$

You can do this again and get

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{L}_2 \arrow[r] \arrow[dr] & \pi_2^*(Q^{k-1}) \arrow[r] \arrow[d] & Q_2^{k-2} \arrow[r] \arrow[dl] & 1 \\ & & P(Q^{k-1}). & & \end{tikzcd}$$

The rank of the vector bundle decrease at each step.

After doing this all the way up, you finally get a flag bundle $\pi : \mathcal{F}(V) \to M$ such that

$$\pi_* V = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots .$$

You can further show that $\pi^* : H^* M \to H^*(\mathcal{F}V)$ is injective. So

$$\pi^* c(V) = c(\pi^* V) = \prod_i (1 + \alpha_i)$$

for $1 + \alpha_i = c(\mathcal{L}_i)$ so that $\alpha_i \in H^2(\mathcal{F}(V))$. These α_i are called the **Chern roots**. In the case of algebraic manifolds, you can prove that α_i will be actual algebraic cycles.

Why is this an important principle? This says that every bundle is essentially a direct sum of line bundles, if you go up to some manifold. We can look at the **Chern character**

$$\begin{aligned} \mathrm{ch}(V) = \sum_i e^{\alpha_i} &= k + \sum \alpha_i + \frac{1}{2} \sum \alpha_i^2 + \cdots \\ (=) &k + c_1 + \frac{c_1^2 - 2c_2}{2} + \cdots \in H^*(M), \end{aligned}$$

defined by Hirzebruch. This is going to satisfy $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$ and is indeed ring morphism

$$\mathrm{ch} : K_0(M) \to H^{2*}(M, \mathbb{Z}).$$

It turns out that $K_0(M) \otimes \mathbb{Q} \to H^{2*}(M, \mathbb{Z}) \otimes \mathbb{Q}$ is an isomorphism. This is very much related to the Hodge conjecture.

7 February 14, 2017

Last time we talked about the splitting principle. We first look at a complex vector bundle $\mathbb{C}^k \to V^k \to M$. Then take the projectivization $P_{\mathbb{C}}(V)$ and split $\pi^* V$ by the line bundle over $P_{\mathbb{C}}(V)$. Repeating this multiple times, we get a decomposition

$$\pi^* V =_{C^\infty} L_1 \oplus L_2 \oplus \cdots \oplus L_k,$$

where $\pi : \mathcal{F}(V) \to M$ is the projection map.

Now you can show that $H^*(M) \hookrightarrow H^*(\mathcal{F}(V))$ is an injective map (this is basically Thom isomorphism). So we don't lose any information. When we do this only once, we see that the cohomology of $P_{\mathbb{C}}(V)$ is given by

$$H^*(P_{\mathbb{C}}(V)) = H^*(M)[x]/(\textstyle\sum_i c_i(V)x^i),$$

where x corresponds to $c_1(L)$. Proving this statement involves a lot of computation. This gives another way to define the Chern class.

The splitting of $\pi^* V$ gives

$$\pi^* c(V) = c(\pi^* V) = \prod_i c(L_i) = \prod_i (1 + t_i),$$

where $t_i = c_1(L_i)$. You can show that the symmetric polynomials $\sigma_i(t_1, \dots, t_k)$ come from downstairs $H^*(M)$.

Using this, Hirzebruch defined the Chern character as

$$ch(V) = \sum_i e^{t_i} = k + c_1 + \frac{c_1^2 - 2c_2}{2} + \cdots \in H^*(M).$$

Then $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$. We also have $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1)\,\mathrm{ch}(V_2)$. This is because for two line bundles L_1 and L_2, we have $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

7.1 L genus and the Hirzebruch index theorem

There is a power series

$$\frac{\sqrt{t}}{\tanh \sqrt{t}} = \sum_k \frac{2^{2k} B_{2k} t^k}{(2k)!} = 1 + \frac{t}{3} - \frac{t^2}{45} + \cdots.$$

Then we can define the **L genus** as

$$\sum_k L_k(V) = \sum_i \frac{\sqrt{t_i}}{\tanh \sqrt{t_i}}.$$

They can be explicitly written as

$$L_0 = 1, \quad L_1 = \frac{p_1}{3}, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2), \dots$$

where p_i are the Pontryagin classes.

Assuming that M^{4n} is oriented, we get a bilinear form

$$H^{2n}(M) \otimes H^{2n}(M) \xrightarrow{\smile} H^{4n}(M) = \mathbb{Z}.$$

We define $\mathrm{sgn}(M)$ as the signature of this pairing.

Theorem 7.1 (Hirzebruch index theorem)**.** $\mathrm{sgn}(M) = L(p_1, \dots, p_i)(M)$.

How does one prove something like this? You can first prove that the signature is cobordism invariant, additive in the sense that $\mathrm{sgn}(M_1 \amalg M_2) = \mathrm{sgn}(M_1) + \mathrm{sgn}(M_2)$, and multiplicative. So signature is a ring morphism from the cobordism ring to $\mathbb{Z}$. Actually, you can give more structure and make it to a map to some other ring.

Anyways, we know that sgn gives a map, and Pontryagin numbers also satisfies the same condition. (You can show that the Pontryagin numbers of the ∂B are zero similarly as for the Stiefel–Whitney classes.)

Now we need to show that two ring morphism of the form

$$(\text{Cobordism ring}) \otimes \mathbb{Q} \to \mathbb{Q}$$

are equal. (Because the torsion part maps to zero, we don't have to care.) Thom calculated the cobordism ring, using the Pontryagin–Thom construction.

Theorem 7.2. *The cobordism ring (tensored with $\mathbb{Q}$) is generated by $\mathbb{C}P^k$ as a ring.*

Now we can simply check that the two maps agree on $\mathbb{C}P^k$.

8 February 16, 2017

For a complex vector bundle $V_{\mathbb{C}} \to M$, we defined the Chern class $c \in H^{2\bullet}(M, \mathbb{Z})$, which is natural and satisfies the product formula.

Given such a vector bundle, it is given by the pullback of the vector bundle over $G_{\mathbb{C}}(k, N)$. Let $f : M \to G_{\mathbb{C}}(k, N)$ be the map. This gives a map $f^* : H^*(G_{\mathbb{C}}(k, N)) \to H^*(M)$. The cohomology ring of the complex Grassmannian is very nice, because all the Schubert cells have trivial boundary. Then we can map the generators to $H^*(M)$.

8.1 Pontryagin classes

Let $V \to M$ be a real vector bundle. We can take its complexification $V \otimes \mathbb{C} = V_{\mathbb{C}} \to M$. We can look at its Chern class

$$c(V \otimes \mathbb{C}) = 1 + c_1(V_{\mathbb{C}}) + c_2(V_{\mathbb{C}}) + c_3(V_{\mathbb{C}}) + \cdots .$$

If we look at its complex conjugate, we see that

$$c(\overline{V \otimes \mathbb{C}}) = 1 - c_1(V_{\mathbb{C}}) + c_2(V_{\mathbb{C}}) - c_3(V_{\mathbb{C}}) + \cdots .$$

But since $V \otimes \mathbb{C}$ and its complex conjugate are isomorphic, we have $2c_{2i+1}(V) = 0$.

We define the **Pontryagin class** as

$$p_i(V) = (-1)^i c_{2i}(V_{\mathbb{C}}) \in H^{4i}(M).$$

Because $c(V_1 \oplus V_2)_{\mathbb{C}} = c(V_1)_{\mathbb{C}} c(V_2)_{\mathbb{C}}$, we have

$$p(V_1 \oplus V_2) - p(V_1)p(V_2) = 0$$

up to 2-torsion.

Given a complex vector bundle $V_{\mathbb{C}}^k$, we can look it as a rank $2k$ real vector bundle. Then

$$(-1)^i p_i(V_{\mathbb{C}}^k = V_{\mathbb{R}}^{2k}) = (-1)^i p_i(V_{\mathbb{C}}^k \oplus \bar{V}_{\mathbb{C}}^k) = \sum_{k+j=i} (-1)^j c_k(V_{\mathbb{C}}) c_j(V_{\mathbb{C}}).$$

Example 8.1. The Chern class of the tangent bundle of $\mathbb{C}P^n$ is given by

$$c(T(\mathbb{C}P^n)) = (1+\alpha)^{n+1}, \quad c(\bar{T}) = (1-\alpha)^{n+1}.$$

So the Pontryagin classes can be computed as

$$1 - p_1 + p_2 - \cdots = (1 - \alpha^2)^{n+1}.$$

8.2 Other classes defined by a power series

We define the L-genus as

$$\sum_k L_k(V) = \sum_i \frac{\sqrt{t_i}}{\tanh\sqrt{t_i}}.$$

In the case of $\mathbb{C}P^{2k}$, we have $p_i = \binom{2k+1}{i}\alpha^{2i}$. If you compute $L(p_1, \dots, p_n)$, you get 1, essentially because the coefficient of z^k in $(\sqrt{z}/\tanh\sqrt{z})^{2k+1}$ is 1.

We now define the **Todd class**. This is defined by

$$\sum_k T_k(V) = \sum_i \frac{t_i}{1 - e^{-t_i}} = 1 + \frac{1}{2}c_1 + \frac{c_2 + c_1^2}{12} + \frac{c_1 c_2}{24} + \cdots.$$

There are also A-polynomials given by $A_1 = -2/3p_1$, $A_2 = 2/45(-4p_2+7p_1^2)$, $\dots$, which comes from the power series $2\sqrt{z}/(\sinh 2\sqrt{z})$.

Why do we care about all these? There are some important relations, like

$$L(p_1, \dots, p_k)[M^{4k}] = \operatorname{index}(M^{4k}).$$

If you look only at the definition, there is no reason this has to be an integer. For instance, there is a piecewise linear manifold M^8 such that $L(M^8)$ is not an integer. What does this mean? It means that M has no C^∞ structure.

There is another interesting question. Let X be a connected finite simplicial complex, and assume that $\pi_1(X) = 0$ (this conditions is needed to do surgery). Under what conditions is X homotopic to a C^∞ manifold?

- Clearly, Poincaré duality must hold.
- There has to exist a vector bundle $\xi \to X$ such that $L(p_i(\xi)) = I(X)$, if $\dim X$ is even.

If $\dim X \geq 5$, then there is a theorem saying that this suffice.

How about the Todd class? These are equally important in the algebraic geometric setting. Let $M_{\mathbb{C}}$ be a complex manifold, and let $T_{\mathbb{C}}M$ be the complex vector bundle. It turns out the $T_{2n}[M]$ is the arithmetic genus. (This is the first major result of Hirzebruch.) More generally, Hirzebruch–Riemann–Roch says that

$$(\operatorname{ch}(V) \cdot \operatorname{Todd} M)[M] = \chi(M, V).$$

This works for any vector bundle.

9 February 21, 2017

9.1 Exotic sphere

Let M^7 be a 7-dimensional manifold satisfying $H^3(M) = H^4(M) = 0$. The cobordism class of 7-manifolds is trivial, and so we always have $M^7 = \partial B^8$. There is a long exact sequence in homology, and this gives a map

$$\partial : H_8(B^8, M^7) \to H_7(M^7).$$

The fundamental class $\nu \in H_8(B^8, M^7)$ maps to the fundamental class $\mu = \partial\nu \in H_7(M^7)$. Since $H^3(M) = H^4(M) = 0$, we have an isomorphism

$$i : H^4(B^8, M^7) \to H^4(B^8)$$

coming from the long exact sequence. Let $p_1 \in H^4(B^8)$ be the Pontryagin class. Consider the number

$$q(B^8) = \langle \nu, (i^{-1}(p_1))^2 \rangle.$$

On the group $H^4(B, M)$ modulo torsion, there is a quadratic form given by $\alpha \mapsto \langle \nu, \alpha^2 \rangle$. This has an index $\tau(B^8)$. Milnor then claimed that

$$\lambda(M) = 2q(B^8) - \tau(B^8) \bmod 7$$

is independent of B. He later constructed a manifold M^7 homeomorphic to S^7 but $\lambda(M) \neq 0$. Then M is not diffeomorphic to S^7. This is called the **exotic sphere**.

Suppose $M = \partial B_1 = \partial B_2$. Take $C = B_1 \cup B_2$. Then C is a closed manifold and $q(C) = q(B_1) - q(B_2)$. We have

$$\tau(C) = \left\langle \nu, \frac{1}{45}(7p_2(C) - p_1(C)^2) \right\rangle$$

by the index formula, and so

$$45\tau(C) + q(C) = 7\langle \nu, p_2 \rangle \equiv 0 \pmod 7.$$

So $2q(C) - \tau(C) \equiv 0 \pmod 7$. You can also prove that $\tau(M) = \tau(B_1) - \tau(B_2)$. So you conclude that $\lambda(M)$ is independent of the choice of B.

To prove that $\tau(M) = \tau(B_1) - \tau(B_2)$, you look at

$$\begin{tikzcd}
H^4(B_1, M) \oplus H^4(B_2, M) \arrow[d, "i_1 \oplus i_2"] & H^4(C, M) \arrow[l] \arrow[d] \\
H^4(B_1) \oplus H^4(B_2) & H^4(C). \arrow[l]
\end{tikzcd}$$

All arrows are isomorphisms, and you can split

$$\langle \nu, \alpha^2 \rangle = \langle \nu_1 \oplus (-\nu_2), \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle.$$

Now let us construct the exotic sphere. A sphere bundle $S^3 \to M^7 \to S^4$ has classifying space is the oriented Grassmannian $B\,\mathrm{SO}(4) = Gr(4, N)$. So by obstruction theory, the bundle space can be computed as $H^4(M, \pi_3(\mathrm{SO}(4)))$.

The homotopy group can be computed as $\pi_3(\mathrm{SO}(4)) = (\alpha, \beta)$ because $\mathrm{SO}(4) \cong S^3 \times S^3$. Milnor gives an explicit construction. Given two integers (h, j), we construct the map

$$f_{hj}(u) \cdot (v) = u^h v u^j,$$

for $v \in \mathbb{R}^4$ as quaternions. Then given S^4, you take the rank 4 trivial bundle on the upper hemisphere and the trivial bundle on the lower hemisphere and glue the two along the boundary with this map. Denote this bundle by $\xi_{h,j} \to S^4$. The Pontryagin class turns out to be

$$p_1(\xi_{h,j}) = \pm(h - j)[S^4].$$

If we choose h, j so that $h + j = 1$ and $h - j$, and look at the sphere bundle manifold $M_{h,j} = M_k$, we can compute $\lambda(M_k) = k^2 - 1$.

Another interesting fact is that M_k has a Morse function such that there exists exactly 2 critical points. By Morse theory, this implies that M_k is homeomorphic to a sphere. To construct this function, note that M_k^7 is covered by $\mathbb{R}^4 \times S^3$ and $\mathbb{R}^4 \times S^3$ with the coordinate transformation being

$$(u, v) \mapsto (u', v') = \Big(\frac{u}{|u|^2}, \frac{u^h v u^r}{|u|}\Big).$$

Now define the function

$$f(x) = \frac{\Re(v)}{(1 + |u|^2)^{1/2}} = \frac{\Re(u'')}{(1 + |u''|^2)^{1/2}},$$

where $u'' = u'/v'$.

There is an interesting open problem in geometry. Is there a metric on Milnor's S^7 with positive sectional curvature? There is a metric with nonnegative curvature everywhere, and positive except on a set with codimension at least 4.

10 February 23, 2017

10.1 Nonnegative curvature on the Milnor sphere

There is a paper by Gromoll and Meyer, 1973, that says that $M_{2,1}$ admits a metric with $K \geq 0$ and $K > 0$ at one point. Let me describe this construction.

Look at the symplectic group $\mathrm{Sp}(n)$ of $n \times n$ quaternion matrices Q with $QQ^* = Q^*Q = 1$. We have $\mathrm{Sp}(1) = \mathrm{SU}(2) = S^3$. There is a $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$-action on $\mathrm{Sp}(2)$ given by

$$(q_1, q_2)Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_1 \end{pmatrix} Q \begin{pmatrix} \bar{q}_2 & 0 \\ 0 & 1 \end{pmatrix}.$$

You can check that this has no fixed points. Then we can take the quotient can it turns out that the quotient is diffeomorphic to S^4, with the map being

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (2\bar{b}d, \|b\|^2 - \|d\|^2) \in S^4.$$

Now there is a diagonal $\Delta \cong \mathrm{Sp}(1) \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Then we get a bundle

$$S^3 \to \mathrm{Sp}(2)/\Delta \to S^3.$$

This is an S^3-bundle over S^4, and so it is classified by $(i, j) \in \pi_3(\mathrm{SO}(4))$. Note that in general, Milnor proved that if $i + j = 1$ and $(i - j)^2 \neq 1 \pmod 7$ then it is a exotic sphere. In this case, we have $(i, j) = (2, -1)$. You can check this using explicit maps

$$h_1(u, q) = \mathrm{orbit}_\Delta \frac{1}{(1 + \|u\|^2)^{1/2}} \begin{pmatrix} q & \bar{u} \\ -uq & 1 \end{pmatrix},$$
$$h_2(v, r) = \mathrm{orbit}_\Delta \frac{1}{(1 + \|v\|^2)^{1/2}} \begin{pmatrix} \bar{v}r & 1 \\ -r & v \end{pmatrix}.$$

Now we have $\mathrm{Sp}(2)$ as a compact Lie group, and so it has a bi-invariant metric. Then this metric is also invariant under the action of Δ. This means that the metric on $\mathrm{Sp}(2)$ induces a metric on $\mathrm{Sp}(2)/\Delta$.

Definition 10.1. A **Riemannian submersion** is a map $N^{n+k} \to M^n$ such that each map on tangent spaces is an isomorphism and is an isometry on the orthogonal complement of the kernel.

Given vectors $X, Y \in T_pM$, we can lift it to $\tilde{X}, \tilde{Y} \in T_{\tilde{x}}N$ in the orthogonal complement. Then I can take the sectional curvature $K_M(X, Y)$. Then **O'Neill's formula** states

$$K_M(X, Y) = \tilde{K}_N(\tilde{X}, \tilde{Y}) + \frac{3}{4}\|[\tilde{X}, \tilde{Y}]^T\|^2.$$

From this, we learn that the curvature of the quotient manifold is always greater than the curvature of the original manifold.

Now we have a quotient $\mathrm{Sp}(2) \to \mathrm{Sp}(2)/\Delta$. We can compute the curvature on a Lie group, which is given by $K(X, Y) = c\|[X, Y]\|^2 \geq 0$ always. So we already get a metric on $\mathrm{Sp}(2)/\Delta$ that has nonnegative curvature.

11 February 28, 2017

Gromoll and Meyer constructed a metric with $K \geq 0$ by looking at the group $\mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(2))$. Here $\mathrm{Sp}(2)$ has a bi-invariant metric given by the Killing form:

$$\langle X, Y \rangle = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y).$$

You can prove that $K(X,Y) = \|[X,Y]\|^2/4 \geq 0$.

The curvature is positive at one point, and later a German geometer, Eschenburg, constructed a metric such that the curvature is positive outside a hypersurface. The idea is similar but useful. Take the manifold $\mathrm{Sp}(2) \times K$ where $K = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, and then quotient it by K with the diagonal action, i.e., acting both on $\mathrm{Sp}(2)$ and K. This is of course topologically homeomorphic to $\mathrm{Sp}(2)$, but the metric is more positive.

This lies in the context of the problem of what manifolds admit a Riemannian structure with positive curvature.

11.1 Brief introduction to Riemannian geometry

A connection is given by the covariant differentiation $\nabla_X : V \to V$ for a vector field X. This has to satisfy $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ and $\nabla_{fX}(Y) = f\nabla_X Y$ for Y a section of V. In the case $V = TM$, we say that the connection is torsion-free if

$$\mathrm{Tor}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

This connection preserves the inner product if $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Given a Riemannian structure on T, the connection ∇ is called **Levi-Civita** if it is torsion free and preserves the metric.

Given a connection and $X, Y \in TM$, we have a map $R(X,Y) : T \to T$ given by

$$Z \mapsto R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

Because this is skew-symmetric, $R(X,Y)$ can be looked as a two-form. So $R(X,Y) \in (\wedge^2 T^* M) \otimes \mathrm{End}(V)$. That is, curvature is an endomorphism-valued two-form.

The curvature operator has some symmetry:

- $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$,
- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

Then R can be considered as a symmetric operator $\Omega^2 \to \Omega^2$. The curvature being positive means that R as an operator is positive.

The curvature contains a lot of information. For instance the Bochner vanishing formula can prove things like the Betti numbers of M is zero or that π_i is finite.

If $\pi_i - 0$ for $i < n$, then basic topology shows that $M \cong S^n$ is an homeomorphism. Is it diffeomorphic to S^n? Hamilton's **Ricci flow** is given by

$$\frac{dg_{ij}(t)}{dt} = -2R_{ij} + cg_{ij},$$

where c is a parameter that makes the volume of M constant. One of Hamilton's important contribution is that the property of having positive curvature is preserved under the Ricci flow.

There is also something called a sphere theorem, proved by Berger and Klingenberg. This states that if $0 < c < K \leq 1$ and $c = 1/4$, then M is homotopic to S^n. The proof only shows that M can be covered by two cells. Only a few years ago using the Ricci flow it was proved by Brendel and Schoen that $1/4 < K \leq 1$ implies that M is diffeomorphic to S^n.

There are indeed metrics with positive sectional manifolds, and there are many constructions. But all the examples we know are of dimension at most 26. A conjecture I have made some years ago is the following:

Conjecture 11.1. *For $n \gg 0$ (possibly for $n > 26$), a manifold M^n must be standard (rank 1 locally symmetric space) if $K > 0$.*

Take a manifold $M = \partial B$. We are interested not only in M but also in B. This is important in the development of AdS/CFT theory. Let f be a nonnegative function on B such that f vanishes only on M. Let us a metric that takes the form of

$$ds_B^2 = \frac{(df)^2 + ds_U^2}{f^2}$$

where ds_U^2 is just a metric near M. We can multiply f by a function ρ with $\nabla\rho \neq 0$, and it turns out this changes ds_U^2 only by a scalar multiplication. This shows that the metric ds_B^2 determines a conformal structure on M.

12 March 2, 2017

12.1 Holonomy group

Given a vector bundle $V \to M$, a connection gives a covariant differentiation $\nabla_X s$ for a section s and $X \in T(M)$. If $\sigma(t)$ is a path on M, then we can uniquely parallel transport a vector s along the path so that it satisfies $\nabla_{\sigma'(t)} s = 0$. But it depends on the choice of σ.

The parallel transport is unique. To see this, set up a coordinate $(x^1, \dots, x^n)$, and take a local frame s_i so that $\langle s_i(x) \rangle = V_x$ is a basis. Then we can write

$$\nabla_{\partial/\partial x^i} s_\alpha = \Gamma^\beta_{\alpha i}(x) s_\beta.$$

This is called a choice of gauge. A general section is given by $s = a^\alpha s_\alpha$, and a general vector is given by $X = b^i(\partial/\partial x^i)$. Then the covariant derivative is given by

$$\nabla_X s = X(a^\alpha) s_\alpha + a^\alpha \nabla_X s_\alpha = b^i \frac{\partial a^\alpha}{\partial x^i} s_\alpha + a^\alpha b^i \Gamma^\beta_{\alpha i} s_\beta.$$

So when $X = \partial/\partial t = b^i(\partial/\partial x^i)$, the parallel transport is the same as

$$\frac{\partial a^\beta}{\partial t} + a^\alpha b^i \Gamma^\beta_{\alpha i} = 0.$$

This is just an ordinary differential equation and so given $a^\beta(0)$ there is a unique solution.

When you move along a closed path σ, if you start with $s(0)$ then you end up with $s(1)$ in the vector space V_x. This means that any closed path σ at x gives a linear endomorphism of V_x. (Linearity follows from linearity of the equation.) We also see that this acts well with composition: the composite of the endomorphism induced by σ_1 and σ_2 is the endomorphism is induced by the concatenation. Hence we get a group morphism $\mathrm{Loop}_x \to \mathrm{End}(V_x)$. The image of this map is called the **holonomy group**.

This holonomy group, which is a subgroup of $\mathrm{End}(V_x) \cong \mathrm{GL}(k)$, contains some information about the connection. This group might not be connected, since the loop space Ω is not connected. The connected component is the contractible loops $\Omega_0 \subseteq \Omega$, with $\Omega/\Omega_0 = \pi_1(M)$. So we may also define the **restricted holonomy group** as the image of Ω_0 in $\mathrm{End}(V_x)$. The restricted holonomy group is quite well understood.

Both of these groups are well-defined up to conjugation, assuming that M is connected. This is because given two points x and y, we can look at the path connecting x and y and use this path as a way to conjugate. So a holonomy group is well-defined concept associated to a connection. Also they will both be Lie subgroups of $\mathrm{GL}(k)$.

Suppose we have a pairing $\langle\, , \rangle : V \otimes V \to \mathbb{R}$. Assume also that ∇_X preserves $\langle\, , \rangle$. Then along a parallel transport, $\langle s_1, s_2 \rangle$ is preserved, by the definition

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle$$

of being preserved. Then the holonomy group is a subgroup of the group $\{x \in \mathrm{GL}(k) \text{ preserving } \langle\, ,\, \rangle\}$. If $\langle\, ,\, \rangle$ is a positive definite pairing, then this group is $\mathrm{O}(k)$, and if it is hermitian, then it is $\mathrm{U}(k)$.

You can do this the other way round. If you know that the holonomy group is in $\mathrm{U}(k)$, then you can first define any hermitian inner product on V_x and then you can move it around using the parallel transport. This gives a hermitian pairing on the whole vector bundle, that is preserved by the connection.

Now assume that $V = T$ and focus on the Levi-Citiva connection. The restricted holonomy group of the Levi-Civita connection can be classified. This was done by Cartan, Berger, and Simons. The possible one are $\mathrm{O}(n), \mathrm{SO}(n)$ in the case of a general Riemannian manifold, and $\mathrm{U}(n)$ in the case of a Kähler manifold, $\mathrm{SU}(n)$ in the case of a Calabi–Yau manifold. There are other exotic ones with holonomy groups G_2, $\mathrm{Spin}(7)$, $\mathrm{Sp}(n)$, $\dots$.

In the non-compact case, for instance, when the holonomy group is in $\mathrm{O}(p, q)$ or $\mathrm{U}(p, q)$, we don't have a very good understanding. These are the problems we need to solve.

12.2 Curvature of a bundle

We have a connection ∇ and a tangent vector x gives a vector $\nabla_x s$ for some section s. So we can think ∇ as a endomorphism-valued 1-form. In other words, we have a connection 1-form ${\omega^\beta}_\alpha$ given by

$$D(s) = D(a^\alpha s_\alpha) = (da^\beta + a^\alpha {\omega^\beta}_\alpha) s_\beta.$$

Let us differentiate it again. We have

$$\begin{aligned} D^2(s) &= (da^\alpha \otimes {\omega^\beta}_\alpha + a^\alpha d{\omega^\beta}_\alpha) s_\beta - (da^\beta + a^\alpha {\omega^\beta}_\alpha){\omega^\gamma}_\beta s_\gamma \\ &= a^\alpha (d{\omega^\gamma}_\alpha - {\omega^\beta}_\alpha {\omega^\gamma}_\beta) s_\gamma. \end{aligned}$$

This is good, because we haven't differentiated the a^α. This means that $D^2(s)$ is a tensor, i.e., $D^2(fs) = fD^2(s)$. We call

$$\Omega^\gamma_\alpha = d{\omega^\gamma}_\alpha - {\omega^\beta}_\alpha {\omega^\gamma}_\beta = {R_{ij}}^\gamma{}_\alpha dx^i \wedge dx^j$$

the **curvature**.

The other way to define this is using the formula

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = R(X, Y)s.$$

You also see that this is an endomorphism-valued 2-form, because it is skew-symmetric.

You can further differentiate this can get

$$d{\Omega^\gamma}_\alpha = -d{\omega^\beta}_\alpha \wedge {\omega^\gamma}_\beta + {\omega^\beta}_\alpha \wedge d{\omega^\gamma}_\beta = -\Omega^\beta_\alpha \wedge {\omega^\gamma}_\beta + {\omega^\beta}_\alpha \Omega^\gamma_\beta.$$

So

$$D\Omega^\gamma_\alpha = d{\Omega^\gamma}_\alpha + \Omega^\beta_\alpha {\omega^\gamma}_\beta - {\omega^\beta}_\alpha \Omega^\gamma_\beta = 0.$$

This is called the Bianchi identity.

Now take

$$\det\Big(1 + \frac{\Omega}{2\pi i}\Big) = \sum \frac{c_k(\Omega)}{(2\pi i)^k}.$$

These $c_k(\Omega)$ turn out to vanish when we take the derivative, and they are well-defined for different connections. They turn out to be the Chern classes.

13 March 7, 2017

A connection of a vector bundle is given by

$$D_X a_i^\alpha = X a_i^\alpha + a_i^\alpha \theta^i{}_j,$$

where $\theta^i{}_j$ is the connection 1-form, dependent on the choice of the framing α. Conversely, if I have a connection 1-form I can define a connection.

If I have a different frame β, related to the original one by $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$, the connection 1-forms related by

$$\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + dg_{\alpha\beta}g_{\alpha\beta}^{-1}.$$

This term $dg_{\alpha\beta}g_{\alpha\beta}^{-1}$ does not behave like a tensor. So this choice of $g_{\alpha\beta}$ is called a gauge choice.

A connection takes a section s of V and gives a vector-valued 1-form. We can extend this, so that if η is a q-form and ξ is a section of V,

$$D(\eta \otimes \xi) = d\eta \otimes \xi + (-1)^q \eta \otimes D\xi.$$

Then

$$D^2(\xi) = \Omega \otimes \xi, \quad \Omega = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha.$$

It turns out that Ω transforms in an gauge-invariant way: $\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}$. In other words, it is a tensor.

If we take its derivative as a 1-form, we get

$$d\Omega = -d\theta \wedge \theta + \theta \wedge d\theta = -\Omega \wedge \theta + \theta \wedge \Omega.$$

Then $D\Omega = 0$, and this is the Bianchi identity.

13.1 Chern classes from the connection

Fix a connection. We can look at

$$\det\Big(I + \frac{i}{2\pi}\Omega\Big) = 1 + c_1 + c_2 + \cdots,$$

where c_i is an $2i$-form. Because $\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}$, the c_i are invariant. These c_i are called **Chern forms**.

Note that c_i are globally defined, because the connection is globally defined. We see that, for instance if the bundle has rank 2,

$$dc_1(\Omega) = \frac{i}{2\pi}(d\Omega_{11} + d\Omega_{22}).$$

To compute the right hand side, we look at the formula $\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + (dg_{\alpha\beta})g_{\alpha\beta}^{-1}$. You can choose a frame e_α so that $\theta_\alpha = 0$ at one point x_0, by locally solving a differential equation for $g_{\alpha\beta}$. Now the Bianchi identity implies $d\Omega_\alpha = 0$ at x_0. Thus we conclude $dc_1(\Omega) = 0$. The same argument works for other c_i. For instance,

$$d(\Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{21}) = (d\Omega_{11})\Omega_{22} + \cdots = 0.$$

Now we know that each $c_i(\Omega)$ is closed. So we have

$$c_i \in H^{2i}(M) = \{\text{closed } 2i\text{-forms}\}/d(\omega^{2i-1}).$$

Thus the connection θ gives a form $\Omega_{2i}(\theta) \in H^{2i}(M)$.

Suppose you have a different connection $\tilde{\theta}$. Their difference $\eta = \theta - \tilde{\theta}$ will be a gauge-invariant matrix-valued 1-form, because of how they transform. This means that $\theta_t = \theta + t\eta$ is a one-parameter family of connections. By the fundamental theorem of calculus,

$$c_i(\tilde{\theta}) - c_i(\theta) = \int_0^1 \frac{d}{dt}[c_i(\theta_t)].$$

The derivative of $\Omega(\theta_t)$ with respect to t is

$$\frac{d}{dt}\Omega(\theta_t) = d\Big(\frac{\partial\theta_t}{\partial t}\Big) - \Big(\frac{\partial\theta_t}{\partial t}\Big) \wedge \theta_t - \theta_t \wedge \Big(\frac{\partial\theta_t}{\partial t}\Big) = d\eta,$$

because we can choose a gauge so that θ_t in the last step. We then obtain stuff like

$$\begin{aligned} c_1(\tilde{\theta}) - c_1(\theta) &= d(\eta_{11} + \eta_{22}), \\ c_2(\tilde{\theta}) - c_2(\tilde{\theta}) &= (d\eta_{11})\Omega_{22} + \Omega_{11}(d\eta_{22}) - (d\eta_{12})\Omega_{21} - \Omega_{12}(d\eta_{12}) \\ &= d(\eta_{11}\Omega_{22} + \Omega_{11}\eta_{22} - \eta_{12}\Omega_{21} - \Omega_{12}\eta_{21}), \end{aligned}$$

again by choosing a gauge and the Bianchi identity. Generally, we have $c_i(\tilde{\Omega}) - c_i(\Omega) = dP(\eta, \Omega)$. This means that

$$c_i(\tilde{\Omega}) = c_i(\Omega) \in H^{2i}(M).$$

We get an element of $H^{2i}(M)$ that is independent of the choice of the connection.

We call a polynomial P **invariant** if $P(A) = P(gAg^{-1})$ for all g. Using this we can define a symmetric polynomial

$$\tilde{P}(A, B) = \frac{1}{2}P(A + B) - P(A) - P(B),$$

and likewise define $\tilde{P}(A_1, A_2, \ldots, A_k)$. Then under the same setting $\theta_t = \theta + t\eta$ and $\tilde{\theta} = \theta + \eta$,

$$d\Big[k\int_0^1 \tilde{P}(\eta, \Omega_t, \ldots, \Omega_t)\Big] = P(\tilde{\Omega}) - P(\Omega).$$

The form in the square bracket is something I call the **Chern–Simons form**.

Anyways given a vector bundle $V \to M$ we can extract a class $c_i \in H^{2i}(M)$ out of it. This satisfies some axioms. Firstly,

$$c_i(f^*V) = f^*c_iV.$$

This is because we can pull back connections by $D_X(f^*s) = D_{f_*X}(s)$. Then $f^*\Omega_V = \Omega_{f^*V}$ and so c_i acts similarly.

Next let us look at $V_1 \otimes V_2$. The connection is just given by the block matrix

$$\theta = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}.$$

Then the curvature is just $d\theta - \theta \wedge \theta$, which is the block matrix with Ω_1 and Ω_2 on the diagonal. If we take the determinant of $I + (i/2\pi)\Omega$, it is clear that we get

$$c(\Omega) = \det\Big(I + \frac{i}{2\pi}\Omega\Big) = \det\Big(I + \frac{i}{2\pi}\Omega_1\Big)\det\Big(I + \frac{i}{2\pi}\Omega_2\Big) = c(\Omega_1)c(\Omega_2).$$

This is the Whitney product formula.

You now have to check the normalization, i.e., compute for the line bundle on $\mathbb{P}^1$.

14 March 9, 2017

For the curvature Ω of a connection D, we defined

$$\det\Big[I + \frac{i}{2\pi}\Omega\Big] = 1 + c_1(\Omega) + c_2(\Omega) + \cdots .$$

Then $c_i(\Omega)$ is a closed form and gives an element $c_i(\Omega) \in H^{2i}(M)$ that is independent of the connection. You can check that $c_i(f^*V) = f^*c_i(V)$. Also the Whitney product formula is satisfied.

14.1 Connection on $\mathbb{C}P^1$

For the tautological line bundle over $\mathbb{C}P^1$, we can explicitly write down a connection. There are two charts:

$$U_1 = \{[z^1, z^2] : z^1 \neq 0\}, \quad U_2 = \{[z^1, z^2] : z^2 \neq 0\},$$

and the line bundle $(z^1, z^2) \mapsto [z^1, z^2]$ has transition map $z^2/z^1 \in \mathbb{C}$. We want to define a connection that is compatible with the metric

$$\|(z^1, z^2)\|^2 = |z^1|^2 + |z^2|^2.$$

Because this is a complex manifold, there will be a decomposition $D \in \Omega^1(\operatorname{End} V)$ into $D^{1,0} \in \Omega^{1,0}(\operatorname{End} V)$ and $D^{0,1} \in \Omega^{0,1}(\operatorname{End} V)$. If the bundle is holomorphic, then for any change of frame $u_\beta = u_\alpha g_{\alpha\beta}$, we have $\bar\partial g_{\alpha\beta} = 0$ and so

$$\bar\partial u_\beta = (\bar\partial u_\alpha) g_{\alpha\beta}.$$

That is, the operator $\bar\partial$ is well-defined. So we may just set $D^{0,1} = \bar\partial$.

These three properties (metric compatibility, being a connection, and $D^{0,1} = \bar\partial$) turn out to uniquely determine D, for a general complex manifold. This connection can be written in terms of the hermitian metric h. If we write $\langle u, v\rangle = uh\bar v$, then

$$D_X\langle u, v\rangle = X(uh\bar v) = (D_X u)h\bar v + uh\overline{D_X v}.$$

After you do some computations, you will find out that $\Omega = \partial\bar\partial \log h$.

Let us come back to our case of $\mathbb{C}P^1$. We have

$$\Omega = \partial\bar\partial \log(|z^1|^2 + |z^2|^2)$$

Then $c_1(L) = (i/2\pi)\Omega \in H^2(\mathbb{C}P^1)$, and integrating Ω over $\mathbb{C}P^1$ is just the Gauss–Bonnet formula. This shows that $c_1(L) \in H^2(\mathbb{C}P^1, \mathbb{Z})$ with the normalization constant.

14.2 Cohomology of the Grassmanian again

There is another way of proving that $c_i(\Omega)$ is indeed the Chern classes. Recall that any vector bundle is identified with a homotopy class of $f : M \to Gr(k, N)$. This means that it suffices to show that the formula is true for the tautological bundle $T^k \to Gr(k, N)$.

There is a Schubert cell decomposition of $Gr(k, N)$. Consider a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n,$$

and define the cells as

$$\sigma_a(V) = \{\Lambda^k : \dim(\lambda^k \cap V_{n-k+i-a_i}) \geq i\},$$

which have codimension $2 \sum a_i$. Then $(-1)^i \sigma^*_{1,\dots,1} = c_\gamma(T)$, where $*$ is the Poincaré dual. We further have that $H^*(Gr(k, n), \mathbb{Z})$ is generated by these algebraic cycle.

You can ask how they intersect in the Grassmannian. Let us write

$$\sigma_a \cdot \sigma_b = \sum \#(\sigma_a \sigma_b \sigma_c) \sigma_c.$$

Then the reduction formulas are given by, if $0 \leq \alpha, \beta, \gamma \leq k$ and $\alpha + \beta + \gamma \leq 2k + 1$, then

$$\#(\sigma_a \sigma_b \sigma_c)_{Gr(k,n)} = \begin{cases} 0 & a_\alpha + b_\beta + c_\gamma > n - k, \\ \#(\sigma_{a-a_\alpha} \sigma_{b-b_\beta} \sigma_{c-c_\gamma})_{Gr(k-1,n-1)} & a_\alpha + b_\beta + c_\gamma = n - k. \end{cases}$$

Also for $a_\alpha + b_\beta + c_\gamma \geq 2(n - k) + 1$,

$$\#(\sigma_a \sigma_b \sigma_c)_{Gr(k,n)} = \begin{cases} 0 & \alpha + \beta + \gamma > k, \\ \#(\sigma_{a_1 \cdots \hat{a_\alpha} \cdots} \sigma_{\cdots \hat{b}_\beta \cdots} \sigma_{\cdots \hat{c}_\gamma \cdots})_{Gr(k,n-1)} & \alpha + \beta + \gamma = k, \alpha_a > \alpha_{a+1}, \dots. \end{cases}$$

If you want to define a Pontryagin class, you need a smooth structure on M. But this is not general enough. The way to go is the first define the L-classes, over $\mathbb{Q}$. If we have a pairing $\langle L \cup \omega_1, M \rangle$ for some ω_1, we get what L is.

15 March 21, 2017

Recall that $L_i(p_1, \dots, p_i)$ is defined as a polynomial in $p_1, \dots, p_i$. The Hirzebruch index formula gives a way to relate the L_i classes to the index of M.

15.1 Pontryagin classes on PL-manifolds

We now want to define the rational Pontryagin classes for piecewise linear manifolds. Our strategy is to first define L_i and then retrieve the Pontryagin classes p_i from it.

Consider a PL-manifold M^n, and for technical reasons, assume $n \geq 8i+2$. (This is not a problem because you can simply take the product with spheres to increase dimension.) We are going to look at the cohomotopy group, i.e., the homotopy classes maps of $f : M^n \to S^{n-4i}$. By Sard's theorem, we can deform f so that f is a submersion at most points.

Now if $y \in S^{n-4i}$ is a point such that $f^{-1}(y)$ is a non-singular submanifold, we have a splitting

$$T(M)|_{f^{-1}(y)} \cong T(f^{-1}(y)) \oplus (\text{trivial}).$$

So we would have

$$L_i(p_1(TM), \dots, p_i(TM))|_{f^{-1}(y)} = L_i(p_1(Tf^{-1}(y)), \dots, p_i(Tf^{-1}(y))).$$

Let $u \in H^{n-4i}(S^{n-4i})$ be the fundamental class. Poincaré duality can the index formula gives us

$$\begin{aligned}\int_{M^n} L_i(p_1(TM), \dots, p_i(TM)) \smile f^*u &= \int_{f^{-1}(y)} L_i(p_1(TM), \dots, p_i(TM))|_{f^{-1}(y)} \\ &= \operatorname{index}(f^{-1}(y)).\end{aligned}$$

So if you think about it, L_i gives a map from $\pi^{n-4i}(M)$ to $\mathbb{Q}$. By Poincaré duality again, this should define the element $L_i \in H^{4i}(M)$. This definition of L_i works even if M is only piecewise linear. Therefore, our conclusion is that rational Pontryagin classes are well-defined on PL-manifolds.

There is a famous theorem due to Whitehead that says that every C^∞ manifold admits a unique PL structure. The smooth definition and the PL definition of Pontryagin classes agree on these two structures.

Theorem 15.1. *There exist two simply connected C^∞ manifolds M_1, M_2, such that M_1 and M_2 are homotopic but but piecewise linear equivalent.*

Let us construct at some vector bundle $V^n \to S^4 = \mathbb{H}P^1$. We are going to construct it so that the first Pontryagin class will be $p_1(V^n) = 2m\sigma$ for arbitrary m, where σ generates $H^4(S^4; \mathbb{Z})$. Consider the tangent bundle $T^8 \to \mathbb{H}P^2$. The Pontryagin class is going to be $p(T) = 1 + 2u + 7u^2$, where $u \in H^8(\mathbb{H}P^2)$ is the generator. Choose a map $f : S^4 \to \mathbb{H}P^2$ so that $f^*u = m\sigma$ (this is possible because $\pi_{1,2,3}(\mathbb{H}P^2) = 0$ and $\pi_4(\mathbb{H}P^2) = \mathbb{Z}$ by Hurewicz), and pull back the tautological bundle.

So we have a vector bundle $\pi : V^n \to S^4$. We have a splitting

$$p(TV) = \pi^*(p(TB)p(V)).$$

Denote by $S(V) \to S^4$ be the sphere bundle in $V \to S^4$. By the Gysin sequence, the map $\pi^* : H^4(S^4;\mathbb{Z}) \to H^4(S(V);\mathbb{Z})$ is an isomorphism. Then $p_1(S(V)) = 2m\pi^*(\sigma)$. So we get an infinite collection of $(n+3)$-dimensional manifolds $S(V_m)$ such that $p_1(S(V_m))$ are all distinct.

Now James and Whitehead proved, using a J-homomorphism argument, that there exist only finitely many homotopy types of $S(V_m)$. This shows that there exist $S(V_m)$ with the same homotopy type.

Theorem 15.2. *There exists a 8-dimensional PL-manifold with does not admit a compatible C^∞ structure.*

Our main idea is to find a PL-manifold such that L_i is not an integer. Let $V^4 \to S^4$ be a vector bundle with $p_1(V) = i\sigma$ and $\chi(V) = j\sigma$, where $i \equiv 2j \pmod 4$. By obstruction theory, (i, j) classifies the bundle over S^4. If $j = 1$ and $i \equiv 2 \pmod 4$, then $S(V)$ is homotopic to the sphere. Now note that the disc bundle $D(V)$ has boundary $\partial D(V) = S(V)$, and then you can glue the cone over $S(V)$ along the boundary of $D(V)$. You then get a 8-dimensional manifold. If you compute the Pontryagin class of this manifold, you will see that it is not an integer.

15.2 Different definitions of Chern classes

Recall that there are algebraic cycles

$$G(k, n) \supseteq \sigma_a(V) = \{\Lambda^k : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}$$

for $0 \leq a_i \leq n-k$ and $0 = V_0 \subseteq \cdots \subset V_n = \mathbb{C}^n$. These cycles $\sigma_a(V)$ generate $H_*(G(k, n))$. The Chern classes of the universal bundle τ^k is explicitly described as

$$c_r(\tau^k) = (-1)^r \sigma^*_{1,\dots,1}.$$

For a vector bundle $V^k \to M$, take a generic set $\{\sigma_1, \dots, \sigma_k\}$ of sections. Define

$$D_i(\sigma) = \{x : \sigma_1(x) \wedge \cdots \wedge \sigma_i(x) = 0\}.$$

Then we get a stratification $\cdots \subseteq D_i \subseteq \cdots$, and $D_{i+1} \setminus D_i$ is a open submanifold of D_{i+1}. Then each D_i defines a cycle. The r-th Chern class $c_r(V)$ is then the Poincaré dual to D_{k-r+1}.

In the case of holomorphic vector bundles, you don't have to worry about genericity because things that are locally cut out by complex equations are automatically cycles.

16 March 23, 2017

Today we are going to talk about the paper *Groups of homotopy spheres* by Kervaire and Milnor.

16.1 Group of homotopy spheres

Consider the set of homotopy spheres, under the equivalence relation of h-cobordism. Let us call this group

$$\Theta_n = \{M^n \simeq S^n\}/\text{h-cobordism},$$

with the connected sum as addition. We are interesting in studying this group Θ_n. Because of the Milnor sphere, we now know $\mathbb{Z}/7 \subseteq \Theta_7$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
$\#\Theta_n$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	$\cdots$

Table 1: Size of the group Θ_n

The main way to study these group is surgery. If $\pi_1 M = 1$ and M is h-cobordant to S^n, then you can show that $M = \partial W$ for some contractible W.

A manifold M is parallelizable if and only if the tangent bundle $T(M)$ is trivial. We say that a manifold is **s-parallelizable** if $T \oplus \nu = \nu'$ for trivial bundles ν and ν'.

Proposition 16.1. *A homotopy sphere is s-parallelizable.*

The obstruction for $T \oplus \nu$ being trivial can be measured by the element

$$\sigma_n(M) \in H^n(M^n; \pi_{n-1}(\mathrm{SO}(n+1))) = \pi_{n-1}(\mathrm{SO}(n+1)) = \pi_{n-1}(SO).$$

This can be seen by dividing M into simplices and picking frames inductively on simplices. By Bott periodicity, we know $\pi_{n-1}(SO)$.

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0.

Table 2: Homotopy groups of SO

So for $n \equiv 3, 5, 6, 7 \pmod 8$ there is no obstruction. For $n \equiv 0, 4 \pmod 8$, you can show that there is a $m \neq 0$ such that

$$m\sigma_n(M) = p_k(\tau \oplus \nu) = p_k(\tau).$$

But because M is a homotopy sphere, the index vanishes, and by the index formula, the L-classes vanish. Thus $p_k(\tau) = 0$.

The remaining case is $n \equiv 1, 2 \pmod 8$. There is a **J-homomorphism**

$$J_{n-1} : \pi_{n-1}(\mathrm{SO}(k)) \to \pi_{n+k-1}(S^k).$$

Here is how you construct it. A map $S^{n-1} \to \mathrm{SO}(k)$ is the same thing as a map $S^{n-1} \times S^{k-1} \to S^{k-1}$. You can extend this map into two ways:

$$S^{n-1} \times B^k \to S^k_+, \quad S^n \times S^{k-1} \to S^k_-,$$

where S^k_+ and S^k_- are upper and lower hemispheres. Then you can glue them to get a map $S^{n+k-1} \to S^k$.

There is an embedding $M^n \hookrightarrow S^{n+k}$ due to Whitney. Then M is s-parallelizable if and only if its normal bundle is trivial. Now in this case, we can look at a tube around M^n, contract everything outside to a point, and project the tube to B^k. This Pontryagin–Thom construction gives a map $S^{n+k} \to S^k$, which lives in $\lim_{k\to\infty} \pi_{n+k}(S^k) = \Pi_n(S)$. So any framing ϕ of the normal bundle gives an element $p(M,\phi) \subseteq \Pi_n$. Consider the subset

$$p(M) = \{p(M,\phi)\} \subseteq \Pi_n.$$

You can show that $0 \in p(M)$ if and only if $M = \partial W$ for W parallelizable. Thus

$$bP_{n+1} \to \Theta_n \to \Pi_n/p(S^n)$$

is an exact sequence, where bP_{n+1} is the boundary of parallelizable $n+1$-manifolds, and $p(S^n)$ is the image of J_n. Let us now look at the groups.

n	1	2	3	4	5	6	7	8
Π_n	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\Pi_n/p(S^n)$	0	$\mathbb{Z}/2$	0	0	0	$\mathbb{Z}/2$	0	0
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	$\mathbb{Z}/2$

Table 3: Table of Π_n, $\Pi_n/p(S^n)$, and Θ_n/bP_{n+1}

Now to conclude that Θ_n is finite, we need to show that bP_{n+1} is finite. If n is even, then it turns out that $bP_{n+1} = 0$. If n is odd, then bP_{2m} is always cyclic, and the order of bP_{4m} is

$$2^{2m-1}(2^{2m-1}-1)B_m j_m a_m/m,$$

where B_m is the Bernoulli number, j_m is the order of $J(\pi_{m-1}(SO)) \subseteq \Pi_{4m-1}$, and $a_m = 1$ if m is even and $a_m = 2$ if m is odd.

17 March 28, 2017

The homotopy group of spheres is defined as the equivalence class of homotopy spheres under h-cobordism. Kervaire and Milnor was quite successful in computing these groups.

Question. *What manifold admits a metric with positive scalar curvature $R > 0$?*

This is a problem I worked on in 1978. In the 2-dimensional case, Gauss–Bonnet implies that $\chi > 0$ and then $M^2 \cong S^2$ or $M^2 \cong \mathbb{R}P^2$. In the 3-dimensional case, if there exists a map $\pi_1(\Sigma_g) \hookrightarrow \pi_1(M^3)$ with $g \geq 2$, then M^3 does not admit a metric with $R > 0$. Assuming the first geometrization conjecture of Thurston, you can show that M^3 are of the form S^3/Γ, $S^2 \times S^2/\Gamma$, ...

17.1 Manifolds with positive scalar curvature and spin structure

Consider a manifold M^n with metric. There is a map $M^n \to B(\mathrm{SO}(n))$. We want to see when M admits a spin structure, i.e., when there is a lift $B(\mathrm{Spin}(n))$.

$$\begin{tikzcd} & B(\mathrm{Spin}(n)) \arrow[d] \\ M^n \arrow[r] \arrow[ur, dashed] & B(\mathrm{SO}(n)) \end{tikzcd}$$

Such a lift exists if and only if $w_2(M) = 0$. This is because there is an exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

Furthermore, you can show that the number of lifts is $H^1(M; \mathbb{Z}/2)$.

The reason we care about spin structure is because they are need to define Clifford multiplication and the Dirac operator. For an elliptic operator L, we define its index as

$$\mathrm{index}(L) = \dim \ker(L) - \dim \mathrm{coker}(L).$$

This depends only on the symbol of L. It can be calculated in terms the Pontryagin classes and Chern classes of the bundle associated. The Hirzebruch–Riemann–Roch tells us about the index of $\bar{\partial} : \bigoplus_i V \otimes \Omega^{0,2i} \to V \otimes \bigoplus_i V \otimes \Omega^{0,2i-1}$ acting on a holomorphic bundle. It says that the index is equal to

$$\int \mathrm{ch}(V) \otimes \mathrm{Todd}(M).$$

We want to generalize it to other elliptic operators, namely the Dirac operator.

If M^n is a spin manifold with positive scalar curvature, then you can show that $\ker D = 0$. This is because if $Ds = 0$ then $\Delta \|s\|^2 \geq 0$ under $R > 0$, but then

$\|s\|$ is constant by the maximum principle. It follows that s is parallelizable. Anyways, we get $\ker D = 0$. But it can be shown that $\dim \ker D \bmod 2$ is an invariant, called the α-invariant, if we change D without changing the symbol. So we get that if $R_M > 0$ then $\dim \ker D \equiv 0 \pmod 2$.

Theorem 17.1. *If $w_2(M) = 0$ and $\alpha(M) \neq 0$, then M does not admit a metric with $R > 0$.*

Lawson and Yau proved that if a manifold admits an effective SU(2) action, then it has a metric with positive scalar curvature.

Corollary 17.2. *If $w_2(M) = 0$ and $\alpha(M) \neq 0$, then M does not admit an effective* SU(2)*-action.*

In 1979 Schoen and I tried to understand positive scalar curvature manifolds more extensively, using surgery. If we do surgery, we will be able to kill some homotopy groups. In M^n, take an embedded S^k with $2k < n$. The condition $2k < n$ is needed to move the sphere around to make it an embedding. We can make the normal bundle to be trivial, and then the neighborhood is $S^k \times D^{n-k}$. Its boundary is

$$\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1}).$$

So we can replace $S^k \times D^{n-k}$ with $D^{k+1} \times S^{n-k-1}$.

Theorem 17.3 (Schoen–Yau, 1979)**.** *Let M^n be a manifold admitting positive scalar curvature. If we do a surgery with $k \leq n - 3$ and get $\hat{M}^n$, then $\hat{M}^n$ still admits a metric with $R > 0$.*

18 March 30, 2017

We were talking about surgery on the category of manifolds with $R > 0$. A manifold M^3 is called **asymptotically flat** if it looks like $M^3 \cong \overline{M}^3 \setminus \text{pts}$, in other words, the space becomes flat as it goes to infinity.

We want to understand asymptotically flat 3-manifolds M. An example of a asymptotically flat 3-manifold is something that looks like $\mathbb{R}^3 \# \mathbb{R}^3$, but with the bridge between them having positive scalar curvature. This is possible because a small S^2 has big positive R. This is called the Einstein–Rosen bridge.

Let us assume that there exists a compact C such that

$$M \setminus C = M_1 \cup \cdots \cup M_k$$

with $M_i \cong \mathbb{R}^3 \setminus B^3$, with the metric looking like

$$ds^2 = dx_i^2 + dy_i^2 + dz_i^2 + O\Big(\frac{1}{\sqrt{x_i^2 + y_i^2 + z_i^2}}\Big).$$

The motivation for this comes from physics. We want gravity to decay as we get far away from the source. When we say $O(1/r)$, we are also assuming that its derivatives decay at the order of the derivative of the error term.

18.1 Surgery on manifolds with nonnegative scalar curvature

We were talking about surgery. Suppose I have a manifold M with $R \geq 0$. Consider a point x_0 with $R > 0$ at $x_0 \in M$. We can deform this conformally by replacing ds^2 with $e^{\nu} ds^2$. Carefully manipulating this, you can make $R > 0$ everywhere.

We can also deform it by $ds_t^2 = g_{ij} - tR_{ij} > 0$. If we compute the scalar curvature $R_t = R(ds_t^2)$. Then you will get

$$\frac{dR_t}{dt} \sim |R_{ij}|^2 \geq 0.$$

This means that under the Ricci flow $dg_{ij}/dt = -2R_{ij}$, the scalar curvature only increases. Anyways, if $R_{ij}(x_0) \neq 0$, then we can deform it so that $R(x_0) > 0$. Using these two processes, we get the following theorem.

Theorem 18.1. *Suppose that a manifold M has $R \geq 0$. Unless $R_{ij} \equiv 0$, there exists a metric with $R > 0$.*

If $R_{ij} \equiv 0$, you might not be able to do anything. In fact, $M = T^3/F$ does not admit a metric with positive scalar curvature.

There is a axially symmetric minimal surface called a **catenoid**. In 2-dimension, the catenoid grows in the axial direction infinitely, but in higher dimensions, you can make it bounded in two planes. This is the ideal thing to glue in while doing surgery.

Suppose I have a manifolds M, N with $R > 0$. Pick $x_0 \in M$ and $y_0 \in N$. We can make a neighborhood of x_0 in M and y_0 in N close to the flat Euclidean space while having $R > 0$. Then we can glue in the catenoid to take the connected sum of M and N. Note that this works only in dimension $n \geq 3$.

Proposition 18.2. *Let M_1, M_2 be manifolds that admit a metric with $R > 0$. Then $M_1 \# M_2$ also admits a metric with $R > 0$.*

Let us look at an example. For a finite group $\Gamma \in \mathrm{SO}(4)$ without fixed points, we can construct the Lens space S^3/Γ. Then we have a manifold

$$(S^3/\Gamma_1)\#(S^3/\Gamma_2)\#\cdots\#(S^3/\Gamma_n)$$

that admits a metric with $R > 0$. Its fundamental group will be $\pi_1 = \Gamma_1 * \cdots * \Gamma_n$. This gives rise to a huge number of other manifolds admitting positive scalar curvature.

In the general case of surgery, we have an embedded manifold $N = S^p \subseteq M^{p+1}$. We take a tubular neighborhood $T_\epsilon(S^p) = S^p \times D^q$. On $N \times S^{q-1} \times (\epsilon, R)$, we take the metric that is the product of any metric on $N = S^p$ and the metric on the catenoid. This can be made to have scalar curvature. Along the catenoid, we deform the metric so that on the other side $N = S^p$ has the standard metric. Then simply glue $D^{p+1} \times S^{q-1}$ on that side. This allows us to do surgery while still preserving $R > 0$. This works if $q \geq 3$.

Proposition 18.3. *Let M be a manifold with $R > 0$. Then any codimension ≥ 3 surgery gives a manifold that admits a metric with $R > 0$.*

You can do other constructions. Suppose you have two S^3 bundles over S^4. You can take out $S^3 \times D^4$ out from each bundle. Then you get holes with boundary $S^3 \times S^3$. You can then glue the two bundles together with while changing the two components. This gives another manifold. In particular, running this construction on Milnor spheres give another (possibly exotic) homotopy sphere.

Definition 18.4. We say that M_1 is **spin cobordant** to M_2 if $M_1 \cup (-M_2) = \partial W$ with $w_2(W) = 0$.

Two manifolds are spin cobordant if they are obtained from each other by surgery of codimension ≥ 3. In fact, the converse is true: if cobordant manifolds can be obtained from surgery with codimension ≥ 3.

You can also define the spin cobordism group.

Theorem 18.5. *For a manifold M with $\pi_1 M = 0$, whether it admits a metric with $R > 0$ is determined by the KO-characteristic number, $\hat{A}$-genus, and the $\mathbb{Z}/2$-invariant.*

For $\pi_1 \neq 0$, it is more complicated.

19 April 4, 2017

Let us consider a manifold with metric ds^2. We want to change the metric conformally to $u^{\frac{4}{n-2}}ds^2$ and make it have constant. This conformal metric change can be thought of as minimizing

$$\int_M R(ds^2)$$

subject to $\mathrm{vol}(ds^2) = \mathrm{vol}(ds_0^2)$. The equation for this turns out to be

$$Lu = \Delta u + \alpha_n R_0 u = cu^{\frac{n+2}{n-2}}.$$

Here L, also called the **conformally invariant operator** is a self-adjoint unbounded operator. So it has a discrete spectrum given by $\lambda_1 < \cdots$. It can be shown that the eigenfunction of the least eigenstate φ_1 is everywhere strictly positive. Then this gives a conformally same manifold with $R \geq 0$.

Proposition 19.1. *Let L be the conformally invariant operator. If the first eigenvalue of L is nonnegative, then ds_0^2 can be conformally changing to a metric with $R \geq 0$.*

19.1 Minimal submanifold and positive scalar curvature

Consider a manifold M^n with $R_M > 0$, and consider a submanifold $N^{n-1} \subseteq M^n$. We can choose N to have minimal volume so that N is stable. This means that for the vector field ν and any function $f : N \to \mathbb{R}$, the volume of $X_t = X + tf\nu$ has derivative

$$\frac{d}{dt}\mathrm{vol}(X_t)\Big|_{t=0} = 0, \quad \frac{d^2}{dt^2}\mathrm{vol}(X_t)\Big|_{t=0} \geq 0.$$

It turns out that we can compute

$$0 \leq \frac{d^2}{dt^2}\mathrm{vol}(X_t)\Big|_{t=0} = \int_N (|\nabla f|^2 - \mathrm{Ric}_M(\nu,\nu)f^2).$$

Consider the orthonormal frame $e_1, \dots, e_{n-1}, e_n = \nu$. Then

$$\mathrm{Ric}(\nu,\nu) = \sum_{i\neq n} K(e_n, e_i), \quad R_M = \sum_i K_M(e_n, e_i) + \sum_{j\neq i} K_M(e_i, e_j).$$

Gauss's curvature formula computes the difference

$$K_M(e_i, e_j) - K_N(e_i, e_j) = \sum h_{ij}^2,$$

where h_{ij} is the second fundamental form. So we get

$$R_M = \mathrm{Ric}_M(\nu,\nu) + R_N + \sum h_{ij}^2.$$

So we conclude that, if N is a minimal submanifold, then

$$\int_N |\nabla f|^2 + (R_M - R_N - \textstyle\sum h_{ij}^2) f^2 \geq 0$$

for every function f.

Let us come back to the conformally invariant operator

$$L_N f = -\Delta_N f - \alpha_n R_N f.$$

The first eigenvalue is simply $\lambda_1 = \inf\{\langle f, L_N f\rangle : \|f\|^2 = 1\}$. Here

$$\langle f, L_N f\rangle = \int f L_N f = -\int f\Delta f - \alpha R_N f^2 = \int |\nabla f|^2 - \alpha_n R_N f^2$$

by partial integration. You can use the inequality to prove the following theorem.

Theorem 19.2. *Let M be a manifold with $R_M > 0$. If N is a minimal hypersurface, then N can be conformally deformed to ds^2 with $R_N > 0$.*

Let us look at an example. Consider a surface Σ sitting in T^3, and take a minimal surface. Then we can make $R_\Sigma > 0$ and so $\chi(\Sigma) \geq 0$ by Gauss–Bonnet. This implies that $\Sigma \cong S^2$ or $\Sigma \cong \mathbb{R}P^2$.

Let us look at a more general situation. Take homology classes $H_1, \dots, H_n$ of hypersurfaces in M^n such that $H_1 \cap \dots \cap H_n \neq \emptyset$. Assume that M has a metric with $R_M > 0$. We can first take a minimal surface $\tilde{H}_1$ such that $R_{\tilde{H}_1} > 0$. Then we can make $H_2, \dots, H_{n-1}$ transversal to $\tilde{H}_1$, and then define $\tilde{H}_2 = H_2 \cap \tilde{H}_1$ and so forth. This gives a manifold $R_{\tilde{H}_1} > 0$ with $\tilde{H}_2 \cap \dots \cap \tilde{H}_n \neq \emptyset$. We can inductively go down to dimension $n = 2$, in which case there is no such homology classes. This is a contradiction.

Theorem 19.3. *Let M^n be a manifold with homology classes $H_1, \dots, H_n$ with $H_1 \cap \dots \cap H_n \neq 0$. Then there does not exist a metric ds^2 that makes $R_M > 0$.*

For instance, $T^6 \# M^6$ does not admit a metric with positive R.

Conjecture 19.4. *If M is $K(\pi, 1)$, i.e., $\pi_i(M) = 0$ for $i \geq 2$, then there is no metric so that $R_M > 0$.*

This is an interesting question because every manifold M has a classifying map

$$F \to M \to K(\pi_1(M), 1)$$

and the scalar curvature of M is related to the scalar curvature $K(\pi, 1)$ and F. Another remark is that if $K(\pi, 1)$ is a C^∞ manifold then π is torsion-free because $H_*(M)$ has to be finitely generated. The conjecture can be proved for $n = 3, 4$.

20 April 6, 2017

If $N^{n-1} \subseteq M^n$ is a minimal submanifold, we get from $(d/dt)^2 \operatorname{vol}(N_t)|_{t=0}$,

$$\int_N |\nabla f|^2 \geq \int_N \operatorname{Ric}(\nu, \nu) f^2 + \int_N \sum_{i,j} h_{ij}^2 f^2$$

for any $f \in C^\infty(N)$. Gauss' formula gives

$$\tilde{R}^M_{ijij} - R^N_{ijij} = h_{ii} h_{jj} - h_{ij}^2$$

and so summing this over $i, j < n$ gives

$$R^M = 2 \operatorname{Ric}^M(n, n) + \tilde{R}^N + \sum h_{ij}^2,$$

because $\sum h_{ii} = 0$. This implies that

$$\int_N |\nabla f|^2 \geq \int_N \frac{R^M f^2}{2} - \int_N \Big(\frac{\tilde{R}^N f^2}{2} + \frac{1}{2} \Big(\sum h_{ij}^2 \Big) f^2 \Big) \geq - \int_N \frac{\tilde{R}^N f^2}{2},$$

because M is a positive curvature manifold.

Consider the differential operator

$$L^N = \Delta_N - \frac{n-3}{4(n-2)} \tilde{R}^N,$$

and consider an eigenfunction $L^N \phi = \lambda \phi$ with $\lambda \geq 0$. We want to show that $\phi = 0$. By definition,

$$\frac{2(n-2)}{n-3} \int |\nabla \phi|^2 = -\frac{1}{2} \int \tilde{R} \phi^2 - \frac{2\lambda(n-2)}{n-3} \int \phi^2 \leq \int |\nabla \phi|^2.$$

But $2(n-2)/(n-3) > 1$, and thus $\phi = 0$. This shows that the spectrum $-L^N$ is positive. So we can conformally deform the metric on N to make it have positive scalar curvature.

To see this, take the first eigenfunction, $-L^N u = \lambda_1 u$. Then $\lambda_1 > 0$, and also $u > 0$. Now the scalar curvature of $u^{4/(n-3)} ds_N^2$ can be computed as

$$u^{-\frac{4}{n-3} - 1} \Big(\tilde{R}^N u - \frac{4(n-2)}{n-3} \nabla_N u \Big) > 0.$$

Let C_3' be a compact 3-manifold which does not admit a metric with positive scalar curvature. Then you can prove that$M = T^3$ or there is a map such that $\pi_1 \Sigma_g \to \pi_1 N^3$ is injective for some $g \geq 1$.

Theorem 20.1. *Suppose C_n' is a manifold with a codimension 1 homology class with can be represented by C_{n-1}'. If C_{n-1}' admits no metric with $R > 0$, then C_n' also admits no metric with $R > 0$.*

Schoen and I showed that if you obtain M_2 from M_1 by codimension 3 surgery, (which is equivalent to M_1 and M_2 are spin cobordant) then M_1 admits a metric with $R > 0$ if and only if M_2 admits a metric with $R > 0$. Stolz proved the following theorem.

Theorem 20.2 (Stolz)**.** *Let M be a manifold with $\pi_1(M) = 0$. Then $R_M > 0$ if and only if $\hat{A} = 0$ and the α invariant is zero.*

Milnor and Kervaire studied the group of homotopy spheres Θ_n. There is a subgroup of boundaries of parallelizable manifolds, bP_n, and there is also a subgroup of boundaries of spin manifolds, $bSpin$. It can be shown that Θ/bP_n is the stable homotopy group of spheres. The group $\Theta/bSpin$ also can be computed, and

$$\Theta_n/bSpin = \mathbb{Z}/2, \qquad \text{for } n = 8k+1, 8k+2.$$

But we don't understand very well the manifolds that bound a spin manifold but not a parallelizable manifold.

20.1 Manifold with $\mathrm{SU}(2)$ action

Lawson–Yau proved that if M admits an effective $\mathrm{SU}(2)$ action, then it admits a metric with $R > 0$. Let me tell you how to prove this.

First consider the case when $G = \mathrm{SU}(2)$ acts freely. Then all the orbits are $\mathrm{SU}(2)$s. In other words, this is a foliation. So there is a fiber bundle

$$G \to M \to M/G.$$

We can construct an invariant metric on M by simply averaging over the G-action. Now for $t > 0$ consider the metric

$$ds_t^2 = f^*(ds^2(M/G)) + t ds^2|_G.$$

You can compute this curvature using O'Neill's formula. As $t \to 0$, the curvature coming from the small sphere will be very large and dominate everything. So as $t \to 0$, we would get $R(ds_t^2) > 0$.

The problem is when the action has a fixed point. Let F be the fixed points of the action. Take a normal bundle $N(F) \times \mathbb{R}^m$ where $\mathbb{R}^m$ is a representation of G by isometries. Then $N(F) \times \mathbb{R}^m$ has a free G action. Then we get a G-invariant metric on $N(F) \times \mathbb{R}^m$ and we can push it down to $N(F)$. This is a G-invariant metric with $R > 0$.

Now we can construct metrics with $R > 0$ near fixed points, and also away from fixed points. So you will be able to glue these two.

If G is a non-compact group, there is a left invariant metric on G. In this case, $R \leq 0$ unless there is a cover $\mathbb{R}^n \to G$. So this argument won't work in general.

21 April 11, 2017

We showed that if M admits an effective SU(2) action, then it admits a metric with $R > 0$. This is not true for noncompact groups, for instance, on $\mathrm{SL}(2, \mathbb{R})$ there is no left invariant metric with $R > 0$. In fact, a Lie group G has left invariant metric $R \geq 0$ only if G is not topologically covered by $\mathbb{R}^n$.

21.1 Conformal structure on a boundary

This is a paper by Witten–Yau in 1999.

Take a manifold M^n, and I want to write it as $M^n = \partial N^{n+1}$. We want to find a metric ds^2 on N such that $R_{ij} = -g_{ij}$. So I want the manifold to asymptotically look like hyperbolic space with

$$ds^2 = \frac{\sum (dx^i)^2}{(1 - |x|^2)^2}.$$

Pick a function y on N such that $M = \{y = 0\}$ and $\nabla y \neq 0$ along M. Then we can write the metric as

$$\frac{dy^2 + \sum g_{ij} dx^i dx^j + \sum g_{i0} dx^i dy}{y^2}.$$

If we change the coordinate as $\tilde{y} = \rho y$ with $\rho \neq 0$ smooth, then we get a change of metric in the form

$$\sum g_{ij} dx^i dx^j \to \frac{1}{\rho^2} \sum g_{ij} dx^i dx^j.$$

That is, the metric M is conformal invariant under this coordinate change.

Question. *Given a conformal structure on M, can we find a metric ds^2 on N so that the conformal structure is induced from ds^2 and also $R_{ij}(ds^2) = -ds^2$?*

This is an important question is general relativity, and we don't know anything about this except when the structure is a small perturbation of S^n?

Let us call the class of such manifolds $\mathscr{C}$. Given manifolds $M_1, M_2 \in \mathscr{C}$ with boundaries N_1 and N_2, you can make a tunnel between M_1, M_2 and simultaneously N_1, N_2 carefully. This shows that $M_1 \# M_2 \in \mathscr{C}$. It will be a nice project to do this for higher dimensional surgery.

21.2 Killing homotopy groups via surgery

Let me talk about the Kervaire–Milnor paper for the last time. Recall that bP_{2k+1} is the group of exotic spheres $S^{2k} = \partial M^{2k+1}$ where TM_{2k+1} is trivial. Then Θ / bP is going to be related to the stable homotopy groups and thus finite. The main part is bP.

You can show that $bP_{2k+1} = 0$. Why is this? Take $S^{2k} \in bP_{2k+1}$ so that $S^{2k} = \partial M$ and M is parallelizable. Let $\varphi : S^p \times D^{q+1} \hookrightarrow M$ be an embedding, and let M' be the manifold obtained by surgery. The homotopy groups of M' is

$$\pi_i(M') = \begin{cases} \pi_i(M) & \text{if } i < \min(p,q) \\ \pi_p(M)/\Lambda & \text{if } i = p < q. \end{cases}$$

This allows us to kill the homotopy groups $\pi_p(M)$ for $p \leq n/2 - 1$ with the following lemma.

Lemma 21.1. *If M^n is stably parallelizable, then for $p < n/2$, $\lambda \in \pi_p(M)$ can be represented by an embedding $S^p \times D^{n-p} \to M$.*

Also if M is stably parallelizable, then M' is also stably parallelizable. So at the very end, from a stably parallelizable manifold M^n with $n \geq 2k$, you get a stable parallelizable M' with $\pi_i(M') = 0$ for $i \leq k-1$ by surgery.

We now use homology. Let M be $k-1$-connected, and $\varphi : S^k \times D^{k+1} \hookrightarrow M$ be an embedding. Denote $M_0 = M \setminus \varphi(S^k \times D^{k+1})$. Let λ be the element in $H_k(M)$ corresponding to φ and likewise let λ' be the element in $H_k(M')$ corresponding to φ. Then we get the following diagram:

$$\begin{tikzcd}
 & & H_{k+1}(M') \arrow[d] & & \\
 & & \mathbb{Z} \arrow[d, "\epsilon"] \arrow[dr, "\lambda"] & & \\
H_{k+1}(M) \arrow[r] & \mathbb{Z} \arrow[r, "\epsilon'"] \arrow[dr, "\lambda'"'] & H_k(M_0) \arrow[r, "i"] \arrow[d, "i'"] & H_k(M) \arrow[r] & 0 \\
 & & H_k(M') \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

Then we get isomorphisms

$$H_k(M)/\lambda(\mathbb{Z}) \cong H_k(M_0)/(\epsilon(\mathbb{Z}) + \epsilon'(\mathbb{Z})) \cong H_k(M')/\lambda'(\mathbb{Z}).$$

We make a choice so that $H_k(M_0) \to H_k(M)$ is an isomorphism. Then we can make $H_k(M') \cong H_k(M)/\lambda(\mathbb{Z})$. This way we can make the manifold so that middle dimension homology is torsion.

For W an oriented manifold of dimension $2r$, define its **semi-characteristic**

$$\mathcal{C}^*(bW, F) = \sum_{i=0}^{r-1} \operatorname{rank} H_i(bW, F) \bmod 2.$$

There is a bilinear pairing $H_r(W,F) \otimes H_r(W,F) \to F$ and the intersection number is $\mathcal{C}^*(bW,F) + \mathcal{C}(2) \bmod 2$.

Suppose M is a manifold with $\partial M = \emptyset$ and let M' be a manifold obtained by surgery. Then the cobordism looks like $W = (M \times [0,1]) \cup (D^{k+1} \times D^{k+1})$. Its Euler characteristic can be computed as $\mathcal{C}(W) = \mathcal{C}(M) + (-1)^{k+1} = (-1)^{k+1}$.

If k is a even, then $H_{k+1}(W,\mathbb{Q}) \otimes H_{k+1}(W,\mathbb{Q}) \to \mathbb{Q}$ is skew symmetric and has even rank. Then

$$\mathcal{C}^*(M+M',\mathbb{Q}) + (-1)^{k+1} \equiv 0 \pmod 2$$

and so $\mathcal{C}^*(M,\mathbb{Q}) \not\equiv \mathcal{C}^*(M',\mathbb{Q})$. That is, $\operatorname{rank} H_k(M,\mathbb{Q}) \neq \operatorname{rank} H_k(M',\mathbb{Q})$.

22 April 13, 2017

22.1 Curvature of holomorphic vector bundles

Consider a holomorphic bundle over a complex manifold. Recall that there is a well-defined operator $D^{0,1} = \bar{\partial}$. Given a local holomorphic frame e_i and a hermitian metric $h_{ij} = \langle e_i, e_j \rangle$, we can define a connection 1-form given by

$$\theta = (\partial h)h^{-1}$$

so that

$$De_i = \sum_j \theta_{ij} e_j.$$

Because we require $d\langle e_i, e_j \rangle = 0$, we have $\theta_{ij} = \bar{\theta}_{ij}$.

Let $V \to M$ be a holomorphic bundle, and let $F \hookrightarrow V$ be a subbundle. The connection on V induces a connection on F by

$$D^F = \pi^F \circ D^V,$$

where π^F is the projection with respect to the hermitian metric.

We can define the curvature by

$$D^2 e_i = \sum_j \Theta_{ij} e_j$$

with respect to a frame, and it can be checked that Θ is a tensor. In terms of the connection, it is given by

$$\Theta = d\theta_e - \theta_e \wedge \theta_e.$$

It can be checked that $\Theta \in \Omega^{1,1}(\operatorname{End} V)$. So if we take any polynomial and look at its trace, we would have $P(\Theta) \in H^{k,k}(M)$. In particular, the Chern classes live in

$$c_i(V) \in H^{i,i}(M, \mathbb{C}) \cap H^{2i}(M; \mathbb{Z}).$$

Given a submanifold $F \hookrightarrow V$ and its orthogonal complement Q, let us define

$$A = D_V|_{a^0(F)} - D_F : a^0(F) \to a^0(Q).$$

This is the second fundamental form, and lives in $a^{1,0}(\operatorname{Hom}(F, Q))$. If you choose a nice frame, you can check that the connection on V takes the form of

$$\theta_V = \begin{pmatrix} \theta_F & {}^t\overline{A} \\ A & \theta_Q \end{pmatrix},$$

and then the curvature is

$$\Theta_V = d\theta_V - \theta_V \wedge \theta_V = \begin{pmatrix} d\theta_F - \theta_F \wedge \theta_F - {}^t\overline{A} \wedge A & * \\ * & d\theta_Q - \theta_Q \wedge \theta_Q - A \wedge {}^t\overline{A} \end{pmatrix}.$$

So we get

$$\Theta_F = \Theta_V|_F + {}^t\overline{A} \wedge A, \quad \Theta_Q = \Theta_V|_Q + A \wedge {}^t\overline{A}.$$

It makes sense to talk about positivity of Θ, because the matrix given by

$$-\sqrt{-1}\Theta\langle v, \bar{v}\rangle$$

for $v \in T^{1,0}(M)$ is a hermitian matrix.

We can write the matrix 1-form A as

$$A = \sum_{\substack{1 \le j \le \operatorname{rank} F \\ \operatorname{rank} F \le \lambda \le \operatorname{rank} V}} a^{\alpha}_{\lambda j} dz_\alpha \otimes e_\lambda e_j^*.$$

Then we have

$$A \wedge {}^t\overline{A} = \sum \alpha^{\alpha}_{ik} \bar{a}^{\beta}_{\mu\kappa} dz_\alpha \wedge dz_\beta \otimes e_\lambda \otimes e_j^*.$$

So we get

$$\left\langle A \wedge {}^t\overline{A}; \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha} \right\rangle = \sum a^{\alpha}_{ik} \bar{a}^{\alpha}_{jk} e_i \otimes e_j^* = A^\alpha \overline{A}^\alpha \ge 0.$$

From this we conclude that

$$\Theta_F \le \Theta_V|_F, \quad \Theta_Q \ge \Theta_V|Q.$$

In particular, if $TM \le 0$ and $N \subseteq M$ then $TN \le 0$.

22.2 Stability of bundles with respect to a Kähler form

Take a Kähler form $\omega \in H^{1,1}(M)$. Consider the first Chern class

$$c_1(V) = \frac{\sqrt{-1}}{2\pi} \sum R_{ij}{}^{\alpha}{}_{\alpha} dz^i \wedge d\bar{z}^j.$$

Now we can take the form $c_1(V) \wedge \omega^{n-1}$, which will be a top dimensional form. Then we can integrate and define

$$\deg(V) = \int_M c_1(V) \wedge \omega^{n-1}.$$

We also define the slope of V to be $\deg(V)/\operatorname{rank}(V)$.

Now there is something called manifold slope stability. A bundle can be, locally, viewed as a map $U \to G(k, N)$, where U is a small open set. Take a coherent subsheaf F, meaning that there is a meromorphic (defined outside a (complex) codimension 2) map $U \to G(r, N)$ such that

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & G(k,N) \\ & G(r,N) \arrow[u, hook] \end{tikzcd}$$

commutes. In this situation, you can again define the Chern class $c_1(F)$ and degree, and it turns out that

$$\operatorname{slope}(F) < \operatorname{slope}(V).$$

Slope stability implies that there exists a hermitian Yang–Mills connection on V. This was proved by Uhlenbeck–Yau.

Definition 22.1. A connection is called **hermitian Yang–Mills connection** if, when we write

$$\Theta^{1,1} \wedge \omega^{n-1} = E \otimes \omega^n$$

for $E \in \operatorname{End}(V)$, then $E = \alpha \cdot \mathrm{id}$.

Eventually I will give a proof of this fact.

22.3 Chern classes in complex geometry

Let $V^k \to M$ be a holomorphic vector bundle and let $\sigma_1, \dots, \sigma_n$ be sections that generate V. Then there is a map

$$M \times \mathbb{C}^n \to V \to 0,$$

and V is a quotient of a trivial bundle. This shows that V has nonnegative curvature.

Definition 22.2. A line bundle $L \to M$ is called **ample** if $R_{ij} > 0$.

Because curvature of tensor products adds up you can show that $V \otimes L^\gamma$ is positive for $\gamma \gg 0$. The bundle $V \otimes L^\gamma$ gives enough holomorphic sections $\sigma_1, \dots, \sigma_n$, and then there is a map

$$f : M \to G(k, n)$$

such that V is the pullback of the tautological bundle. In particular, we would have

$$c_i(V \otimes L^\gamma) = f^* c_i(G(k, n)).$$

Recall that the Chern classes $c_i(G(k, n))$ are represented by algebraic cycles, namely Schubert cells. So the pullbacks $c_i(V \otimes L^\gamma)$ are also represented by algebraic cycles. In fact, it is not hard to show that $c_1(L)$ is represented by a algebraic cycle and so $c_i(V)$ are represented by algebraic cycles. This was proved by Chern.

Because we have the formula for curvature,

$$\Theta^\alpha{}_\beta(V) = \sum A^\alpha{}_\mu \wedge \overline{A^\beta{}_\mu},$$

we can compute the Chern class as

$$c_q(\Theta) = \left(\frac{i}{2\pi}\right)^q \sum \frac{1}{q!} \sum_{\alpha_1 < \cdots < \alpha_q} \operatorname{sgn} \pi (\Theta^{\alpha_1}_{\alpha_{\pi(1)}} \wedge \cdots \wedge \Theta^{\alpha_q}_{\alpha_{\pi(q)}} = (\sqrt{-1})^q \sum_\mu \eta_\mu \wedge \bar{\eta}_\mu,$$

where

$$\eta_\mu = \left(\frac{1}{2\pi}\right)^q \frac{1}{q!} \sum_{\alpha_1 < \cdots < \alpha_q, \pi} \mathrm{sgn}(\pi) A_{\mu_1}{}^{\alpha_{\pi(1)}} \wedge \cdots \wedge A_{\mu_q}{}^{\alpha_{\pi(q)}}.$$

This immediately shows that $c_q(\Theta) \geq 0$. If you have a cycle Z, it follows that $\int_Z c_q(V) \geq 0$, and things like the Cauchy–Schwarz inequality tells us

$$\int_Z (c_1(\Theta)^2 - 2c_2(\Theta)^2) \geq 0.$$

23 April 18, 2017

Last time I have shown that if

$$0 \to F \to V \to Q \to 0,$$

there are inequalities $\mathrm{curv}(F) \le \mathrm{curv}(V) \le \mathrm{curv}(Q)$. In particular, if a vector bundle is generated by finitely many sections, it is a quotient of a trivial bundle and thus has nonnegative curvature.

23.1 Chern classes of ample vector bundles

Definition 23.1. A vector bundle V is called **ample** if, the line bundle $\mathcal{O}(1) \to \mathbb{P}V$ is ample.

Recall that a line bundle is ample if its first Chern class

$$c_1(L) = -\frac{i}{2\pi}\partial\bar{\partial}\log(\det h) > 0.$$

Note that $\partial\bar{\partial}\log(\det h)$ is well-defined globally.

Here is one of the most fundamental results in algebraic geometry.

Theorem 23.2. *Let $L > 0$. Then there exists a $k \gg 0$ such that there exist global sections $s_1, \dots, s_N \in \Gamma(M, L^k)$ such that $z \mapsto [s_0(z), \dots, s_N(z)] \in \mathbb{C}P^k$ is one-to-one and non-singular. That is, there is an embedding L^k into $\mathbb{C}P^N$.*

Note that if there is an embedding, we can write $L^k = f^*H$ with H the hyperplane bundle. Then $c_1(H) > 0$ and so $c_1(L^k) = f^*c_1(H) > 0$. This implies that $L > 0$.

This raises an important question. Is $c_k(V) > 0$ for any ample vector bundle V? (This means $\int_Z c_k > 0$ for all cycles Z.) People tried to express the Chern classes in terms of the curvature, but it is not easy. We are going to look at the 1971 paper of Bloch and Gieseker.

Consider the graded algebra $A^*(M)$ of cycles. For an ample line bundle $\ell \in A^1(M)$, the strong Lefschetz theorem says that the map $\ell^{n-2i} : A^i(M) \to A^{n-i}(M)$ is injective.

For $V^r \to M^n$ and $s = \min(r, n)$, we want to show that $c_s : A^i(M) \to A^{i+s}(M)$ is an injection if $i \le (n-s)/2$. The cohomology of the projecitivized bundle was computed by Thom, Hersch, Chern:

$$A^*(\mathbb{P}(V)) \cong A^*(M)[T]/T^r - c_1T^{r-1} + \cdots + (-1)^s c_s T^{r-s},$$

where T corresponds to $\xi = c_1(\mathcal{O}(1))$. Suppose that exists a $b \in A^i(M)$ such that $c_s b = 0$. Then

$$\eta = b(\xi^{s-1} - c_1\xi^{s-2} + \cdots + (-1)^{s-1}c_{s-1}) \ne 0 \in A^{i+s-1}(\mathbb{P}(V)).$$

On the other hand, we have $\eta \cdot \xi^{r-s+1} = 0$. But because $\mathcal{O}(1) \to \mathbb{P}(V)$ is ample, the map

$$A^{i+s-1}(\mathbb{P}(V)) \xrightarrow{\xi^{r-s+1}} A^{i+r}(\mathbb{P}(V))$$

is injective. This is a contradiction.

In general, if you define

$$P^j(M,V) = \{a \in H^j(M,\mathbb{C}) : c_k a = 0 \text{ for every } k > m-j\}$$

then $c_{n-j} : P^j(M,V) \to H^{2n-j}(M,\mathbb{C})$ is an injection. If $r \le n$, then

$$H^{n-r}(M,\mathbb{C}) \xrightarrow{c_r} H^{n+r}(M,\mathbb{C})$$

is an isomorphism. This is an higher analogue of the strong Lefschetz theorem.

Lemma 23.3. *Let $L \to M$ be an ample line bundle and $d > 0$. Then there exists an algebraic manifold with a finite surjection $f : M' \to M$ such that $f^*L = (\mathcal{L})^d$.*

We want to show that if $V^r \to M^n$ with $r \ge n$, then $c_n(V) > 0$. We previously showed that this is not equal to zero. Now we want to show that it is strictly positive.

Proposition 23.4. *If $V^r \to M^n$ is an ample bundle with $r \ge n$, then $c_n(V) > 0$.*

Proof. We use induction on n. Suppose that $c_n(V) < 0$. Consider a generic hyperplane section $N^{n-1} = H \cap M^n \hookrightarrow M^n$. You can check that $V|_{N^{n-1}}$ is also ample. Then

$$c_{n-1}(V|_N) = c_{n-1}(V)(N) > 0.$$

By our assumption, there exists a rational number $d > 0$ such that

$$c_{n-1}(V) \cdot (N) + dc_n(V) = 0.$$

Now there exists a map $f : M' \to M$ such that $f^*L = \mathcal{L}^d$ and $\mathcal{L} \to M'$ is a line bundle. It can then be shown that $f^*L, \mathcal{L}, f^*V \oplus \mathcal{L}$ are all ample. But now

$$c_n(f^*V \oplus \mathcal{L}) = c_n(f^*V) + c_{n-1}(f^*(V))c_1(\mathcal{L}) = f^*\Big(c_n(V) + \frac{1}{d}c_{n-1}(V)(N)\Big) = 0.$$

This is a contradiction, since the top Chern class is never zero. $\square$

Therefore if $T^*(M)$ is ample, then $(-1)^n\chi(M) > 0$. So for instance, if a Kähler manifold has negative curvature, then $(-1)^n\chi(M^n) > 0$. What happens if I just assume non-positive curvature? Then is $(-1)^n\chi(M^n) \ge 0$? The whole argument doesn't work, and also doing a perturbation is also not easy. Another interesting question is what happens when $\chi(M^n) = 0$? You would expect some degeneracy, and for instance, for $n = 1$ you get $M \cong T^2$. For $n = 2$, it has to covered by some $T^2 \times \Sigma_g$.

Here is an interesting question. Let M^k be a manifold sitting inside a large abelian variety $A^n = T^n_{\mathbb{C}}$, with $T^*(M^k) \ge 0$. If $\chi(M^k) = 0$, what happens? I think this implies that the Kobayashi metric is zero.

24 April 20, 2017

A holomorphic vector bundle $V^r \to M^n$ is defined to be ample if and only if $\mathcal{O}(1) \to \mathbb{P}V$ is ample.

24.1 Stable bundles and hermitian Yang–Mills connections

For a Calabi–Yau manifold, if $c_1 = 0$ then $c_2(M) \wedge \omega^{n-2} > 0$. There is an inequality

$$\frac{i(n+1)}{n} c_2(M) \wedge (-c_1)^{n-2} \geq (-c_1)^n$$

by looking at the L^2 norm of the curvature R. So for $n = 2$, you get $3c_2 \geq c_1^2$. In fact, $3c_2 = c_1^2$ implies that there is a Kähler Einstein metric with constant holomorphic sectional curvature. In the case of $c_1 < 0$, M^2 having $3c_2 = c_1^2$ is equivalent to there being a cover $B^2 \to M^2$.

Actually, if there is a Kähler Einstein metric on $T(M)$, then the bundle $T(M)$ is stable, i.e., for any subbundle $F \subseteq T(M)$,

$$\frac{c_1(F) \wedge \omega^{n-1}}{\operatorname{rank} F} < \frac{c_1(T(M)) \wedge \omega^{n-1}}{\operatorname{rank} M}.$$

So if $c_1 < 0$ then TM is stable with respect to $\omega = (-c_1)$.

You can try to generalize this to general vector bundles $(V^k, h) \to (M, \omega)$. You can look at its curvature $\Omega(H) = R_{ij}{}^{\alpha}{}_{\beta} \in \Omega^{1,1}(\operatorname{End} V)$, and define

$$\Lambda\Omega = \omega^{n-1} \wedge \Omega = A \otimes \omega^n$$

for $A \in \operatorname{End}(V)$. In the case $V = T(M)$ is Kähler Einstein, then A is the Ricci tensor, and so $A = cI$. This connection $\Lambda\Omega = c \operatorname{id} \otimes \omega$ is called the hermitian Yang–Mills connection.

The name Yang–Mills connection comes from physics. On a 4-manifold, there is a dual map $* : \bigwedge^2 M \to \bigwedge^2 M$, and so there is a self-dual part $\bigwedge^2_+ M$ and an anti-self-dual part $\bigwedge^2_- M$. For a curvature Ω, we can likewise define $\Omega = \Omega_+ \oplus \Omega_-$. This connection is called **anti-self-dual** if $\Omega_+ = 0$.

Recall that $D\Omega = 0$ because $\Omega = d\omega - \omega \wedge \omega$. If either $\Omega_+ = 0$ or $\Omega_- = 0$, we get $D(*\Omega) = 0$. This implies the Yang–Mills equation $DD^*\Omega = 0$. C. N. Yang observed that $\Omega_+ = 0$ is equivalent to some Cauchy–Riemann equations, and it becomes $\Lambda\Omega$. So our equation $\Lambda\Omega = 0$ is the right generalization for higher dimensions.

Now we are interested, for M a Kähler manifold, in the equation $\omega^{n-1} \wedge \Omega^{1,1}(\operatorname{End} V) = 0$. In the case that V is polystable, i.e., it splits into a direct sum of stable vector bundles, there is the inequality

$$\left(c_2 - \frac{k-1}{2k} c_1^2\right) \wedge \omega^{n-2} \geq 0.$$

If there is an hermitian Yang–Mills connection, it is polystable and so we have this inequality. Further, if equality holds, then the connection is projective flat.

Given vector bundles V_i, we can take the tensor product, and connections on each V_i induce a connection on $V_1 \otimes \cdots \otimes V_m$. Then if V_i are stable, then their tensor product also satisfies the equation and thus polystable.

Consider a complex manifold M^n with a hermitian $(1,1)$ form. Assume that $\partial\bar{\partial}(\omega^{n-1}) = 0$. This always exists after doing a conformal change. Assume that V is a stable bundle with $c_1(V) = 0$. We want to construct a metric with $\omega^{n-1} \wedge \Omega(h) = 0$.

Take an arbitrary metric (V, h_0). Take a self-adjoint hermitian operator h. Using this we want to deform h_0 into what we want. Define h_1 as $(u, v)_1 = (hu, v)_0$. Then

$$A^1 = h_1^{-1}\partial h_1 = A^0 + h^{-1}\partial^0 h.$$

So we want to find a self-adjoint h such that

$$H = H_0 - \bar{\partial}(h^{-1}\partial^0 h) + \epsilon \log h - \sigma h^{-1} H_0 h^{1/2} = 0$$

where $\epsilon > 0$ and $0 \leq \sigma \leq 1$. The auxiliary factors ϵ and σ are there to control the norm of h. So we are going to show that this equation can be solved, and then see what happens when $\epsilon, \sigma \to 0$.

25 April 25, 2017

Given a holomorphic bundle $V \to M$, there is a hermitian connection with D and $D^{0,1} = \bar{\partial}$ and $D\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$. Then the metric h on V is determined by D.

V is stable means that $\operatorname{slope}(F) < \operatorname{slope}(V)$ for all proper subbundles F. This stability implies $\operatorname{End} V = c\,\mathrm{id}$. Given the metric, we can define

$$\Omega^{1,1}(h) \wedge \omega^{n-1} = A \otimes \omega^n$$

for $A \in \operatorname{End}(V)$. We say that the connection is hermitian Yang–Mills if $A = c\,\mathrm{id}$. We can calculate c in terms of the degree as

$$c = \frac{\deg V}{r \operatorname{vol}(M)}.$$

If h_1, h_2 are solutions of the hermitian Yang–Mills equation, $h_1 = \alpha h_2$ for a constant α.

Uhlenbeck–Yau proved that if M^n is Kähler and V is stable, then there exists a unique hermitian Yang–Mills connection up to a constant. So you can study the moduli space of stable bundle, by looking at hermitian Yang–Mills connections. This is similar to every Kähler manifold with $c_1 < 0$ admitting a unique Kähler–Einstein metric. Given a geometric object, like an algebraic curve, you can associate to it a geometric structure, like the Poincaré metric.

A metric ds^2 induces a Laplacian Δ_{pq} that acts on $\Omega^{p,q}$. Its kernel $\ker \Delta_{p,q}$ are the harmonic functions, and its dimension in H^{p+q} only depends on the topology, by Hodge theory. You can further look at the eigenfunctions $\Delta_{p,q}\varphi_i^{p,q} = \lambda_i^{p,q}\varphi_i^{p,q}$. Then $\{\lambda_i^{p,q}\}$ are all the holomorphic invariants of M.

If there is an Kähler–Einstein metric, and $R < 0$, the metric completely determines the complex structure. For $R = 0$, this may fail, in the case when the metric is hyperKähler. The next question is, whether $\{\lambda_i^{p,q}\}$ determines the complex structure. This is open in full generality. This is even interesting in the algebraic plane curve case.

You can form a zeta function out of this spectrum:

$$\sum \frac{1}{(\lambda_i^{p,q})^s} = \zeta^{p,q}(s)$$

for $\Re(s) \gg 0$ is well-defined, and can be meromorphically extended. This is related to the Ray–Singer invariant and the Chern–Simons invariant.

Classically, there is the heat kernel

$$\operatorname{tr} h(t)^{p,q} = \sum_i e^{-t\lambda_i^{p,q}}(\text{multiplicity of } \lambda_i^{p,q}).$$

As $t \to \infty$, we get global information about $\ker \Delta^{p,q}$. As $t \to 0$ we get some local information on $h^{p,q}(t)$, where $\partial h/\partial t = \Delta^{p,q} h$.

25.1 Convergence of the hermitian Yang–Mills metric

Given a metric h_0 on $V \to M$, we want to define a new metric h_1 as

$$\langle s, t\rangle_1 = \langle hs, t\rangle_0,$$

for an endomorphism h. As said last time, we want the solve the equation

$$H_1 - H_0 + \bar{\partial}(h^{-1}\partial^0 \log) = \epsilon \log h - \sigma h^{-1/2} H_0 h^{1/2}.$$

Here, $0 < \epsilon$ and $0 \leq \sigma \leq 1$. This equation can be solved, and if we write $m = \max|\log h| \leq \epsilon^{-1} \max_M |H_0 - \tilde{H}_0|$ then m can be estimated by $\int_M |h|^2$.

The curvature cannot be controlled pointwise, but its L^2 norm can be:

$$\int_M \|F_\epsilon\|_\epsilon^2 \leq \int_M \|F_0\|^2.$$

Suppose that the hermitian connections we have constructed does not converge: $\int_M |\log h_\epsilon|^2 \to \infty$. Then we can scale it by $\rho_\ell \to 0$ so that $\rho_\ell h(\epsilon_\ell) \to h_\infty \neq 0$ and $h_\ell \leq I$. In this case, we can show that $I - h_\infty^\sigma \to \pi$ for some $\pi \in \Gamma(\mathrm{End}(V))$, and further $\pi^2 = \pi$. Then π represents a holomorphic subsheaf F of V.

It can be shown that $0 < \operatorname{rank} F < \operatorname{rank} V$ and that

$$\frac{\deg F}{\operatorname{rank} F} \geq \frac{\deg V}{\operatorname{rank} V}.$$

This contradicts our assumption that the vector bundle is stable.

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