Math 231a - Algebraic Topology

Taught by Peter B. Kronheimer
Notes by Dongryul Kim

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This course was taught by Peter Kronheimer. We met on Mondays, Wednesdays, and Fridays from 2:00pm to 3:00pm at Science Center 507 and used Allen Hatcher's book *Algebraic Topology*. There were 19 students enrolled and there was one final paper. The course assistance for this course was Meng Guo.

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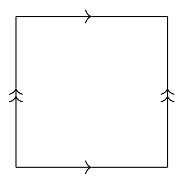
1 August 31, 2016

This is a introduction to algebraic topology, and the textbook is going to be the one by Hatcher. The book really tries to bring the material to life by lots examples and the pdf is available from the author's website. Chapter 1 is about fundamental groups and covering spaces, and is dealt in Math 131. We will start with homology, but we need are going refer to the Chpater 1 material if needed.

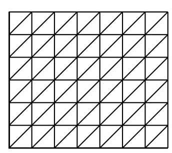
1.1 The first homology of T^2

What I am going to do today is to talk about homology, beginning with examples. Consider the 2-dimensional torus, which we will going to think of as a glued square.



We are going to talk about the first homology of this 2-torus.

In the development of the subject, this was divided into little triangles like this:



This triangulation has vertices, edges, triangles, which are respectively 0-simplices, 1-simplices, and 2-simplices. The **standard k-simplex** Δ^k (in $\mathbb{R}^{k+1}$) is the convex hull of the basis vectors $e_0, \dots, e_k$. Sometimes you would want to talk about $\Delta^{-1} \subseteq \mathbb{R}^0$. This is the convex hull of $\{\}$ and is thus $\emptyset$ by convention.

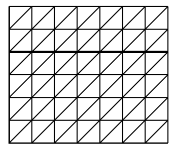
Let S_k be the set of all k-simplices in our triangulated T^2. A **1-chain** (with $\mathbb{Z}/2$ coefficient) is a subset of S_1. Likewise, a 0-chain is a subset of S_0, and an i-chain is a subset of S_i. Let us denote by C_i the set of i-chains. This is called "with $\mathbb{Z}/2$ coefficeints" because C_i is the vector space over $\mathbb{Z}/2$ with basis S_i. That is, the element of C_i looks like

$$\sum_{\sigma \in S_i} \lambda_\sigma \sigma$$

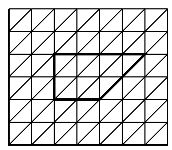
with $\lambda_\sigma \in \{0, 1\}$ and it then corresponds to the subset $\{\sigma : \lambda_\sigma = 1\} \subseteq S_i$. This alternative viewpoint allows us to add 1-chains.

Definition 1.1. A 1-chain is a 1**-cycle** if every vertex is an enedpoint of an *even* number of 1-simplices in the chain.

For instance, this is a 1-cycle that is interesting:



The reason I call this interesting is because it is not a boundary. This 1-cycle, for instance is a boundary of a 2-chain.



On the other hand, you can easily convince yourself that the first 1-cycle is not a boundary.

Definition 1.2. For a 2-simplex, the **boundary** is defined as $\partial T = \sigma_1 + \sigma_2 + \sigma_3 \in C_1$ where σ_i are the three sides of T. For a 2-chain $T_1 + \cdots + T_l$, its boundary is defined as

$$\partial(T_1 + \cdots + T_l) = \sum_{i=1}^{l} \partial T_i.$$

We have thus defined a linear map $\partial : C_2 \to C_1$. The **boundaries** in C_1 are the image $\operatorname{im} \partial = B_1 \subseteq C_1$. We also note that 1-cycles are the kernel of the map $\partial : C_1 \to C_0$ defined similarly as $\sigma \mapsto v_1 + v_2$ and

$$\sigma_1 + \sigma_2 + \cdots + \sigma_l \mapsto \sum \text{all endpoints}.$$

Confusingly both maps $C_2 \to C_1$ and $C_1 \to C_0$ are called ∂, and if you want to distinguish them, you can call them $\partial_2 : C_2 \to C_1$ and $\partial_1 : C_1 \to C_0$.

Definition 1.3. We define the **first homology** H^1 as the quotient vector space $\ker \partial_1 / \operatorname{im} \partial_2$.

That is, I am intereseted in the 1-cycles, but I am not interested in the 1-cycles that are boundaries. It is not so hard to compute the first homology of the torus.

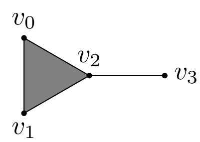
Proposition 1.4. *For T^2, H_1 is a vector space of dimension 2 over $\mathbb{Z}/2$.*

2 September 2, 2016

Today I want to define homology in the context of a simplicial complex.

2.1 Simplicial complex

There are two slightly different ways of looking at it. A **simplicial complex** is a topological space, obtained by gluing simplices together.



This has one 2-simplex $\{v_0, v_1, v_2\}$, and four 1-simplices $\{v_2, v_3\}$, $\{v_0, v_2\}$, $\{v_0, v_1\}$, $\{v_1, v_2\}$. There are four 0-simplex $\{v_0\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$.

Now an **abstract simplicial complex** consisting of a set V of "vertices" and a set S of "simplices" such that

- each simplex $s \in S$ is a finite subset of V and $s \neq \emptyset$, (If s has $k+1$ elements, call s a k-simplex and write $\dim(s) = k$. We denote $S_k = \{s \in S : \dim(s) = k\}$.)
- if $s \in S$ and $\emptyset \neq t \subseteq s$ then $t \in S$.

We can make the definition shorter, if we allow $\emptyset$ also to be a simplex and just do not give V.

An **abstract simplicial complex** is a set S such that every $s \in S$ is finite and $s \in S$ and $t \subseteq s$ implies $t \in S$.

Now given an abstract simplicial complex $K = (V, S)$, we can construct the corresponding topological space. If V is finite, then take the vector space $\mathbb{R}V \cong \mathbb{R}^n$ with V as basis, if $n = \#V$:

$$\mathbb{R}V = \sum_{v \in V} \lambda_v v$$

for $\lambda_v \in \mathbb{R}$. For $s \in S$ with $s = \{v_0, \ldots, v_k\}$, we set $|s|$ to be the convex hull of $v_0, \ldots, v_k$ in $\mathbb{R}V$. Then set

$$|K| = \bigcup_{s \in S} |s| \subseteq \mathbb{R}^n.$$

Now a slight modification is needed if V is infinite. Then $\mathbb{R}V$ becomes

$$\mathbb{R}V = \left\{ \sum_{v \in V} \lambda_v v \right\}$$

with $\lambda_v = 0$ for all but finitely many v. We can similarly define $|K| \subset \mathbb{R}V$. This embeds $|K|$ into some huge Euclidean space, but we still need to get a topological sapce out of it. We do it so that a map $f : |K| \to Y$ is continuous if and only if the restriction to every simplex $|s|$ is. Alternatively, we can set $C \subseteq |K|$ to be closed if and only if $C \cap |s|$ is closed in $|s|$ for every $s \in S$.

2.2 Homology for an abstract simplicial complex over $\mathbb{Z}/2$

Consider an abstract simplicial complex $K = (V, S)$. Write $\mathbb{F} = \mathbb{Z}/2$.

For $k \geq 0$, let S_k be the set of the k-simplices, and let

$$C_k = \mathbb{F}S_k = \mathcal{P}_{\text{finite}}(S_k).$$

These are the **k-chains**.

We are now going to define the boundary map $\partial_k : C_k \to C_{k-1}$. First define it on a basis vector $s \in S_k$ as

$$\partial_k s = \sum_{t \subseteq s,\ \dim(t)=s} t.$$

In other words, if $s = \{v_0, \dots, v_k\}$ then

$$\begin{aligned}\partial_k s &= \{v_1, \dots, v_k\} + \{v_0, v_2, \dots, v_k\} + \cdots + \{v_0, \dots, v_{k-1}\} \\ &= \sum_{i=0}^{k} \{v_0, \dots, \hat{v}_i, \dots, v_k\}.\end{aligned}$$

Also, for several simplices, we define

$$\partial_k(x_1 + \cdots + s_l) = \sum_{j=1}^{l} \partial_k s_j.$$

Definition 2.1. A k-chain $\gamma \in C_k$ is a **k-cycle** if $\partial_k \gamma = 0$. We denote

$$Z_k = \ker \partial_k = \{k\text{-cycles}\} \subseteq C_k.$$

Definition 2.2. A k-chaing $\gamma \in C_k$ is a **k-boundary** if $\gamma = \partial_{k+1}\beta$ for some $\beta \in C_{k+1}$. We denote

$$B_k = \operatorname{im} \partial_{k+1} = \{k\text{-boundary}\} \subseteq C_k.$$

Lemma 2.3. *Boundaries are cycles, i.e., $B_k \subseteq Z_k$, i.e., $\partial_k \partial_{k+1} \beta = 0$ for all $\beta \in C_{k+1}$.*

Proof. We can check $\partial_k \partial_{k+1} \beta = 0$ for a basis vector in C_{k+1}, which is a $(k+1)$-simplex $s = \{v_0, \dots, v_{k+1}\}$.

We have

$$\partial_{k+1} s = \sum_{i=0}^{k+1} \{v_0, \dots, \hat{v}_i, \dots, v_{k+1}\}$$

and then

$$\begin{aligned}
\partial_k \partial_{k+1} s &= \sum_{i=0}^{k+1} \partial_k \{v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}\} \\
&= \sum_{i=0}^{k+1} \sum_{j=0,\ j\neq i}^{k+1} \{v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}\} \\
&= 2 \sum_{0 \leq i < j \leq k+1} \{v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}\} = 0
\end{aligned}$$

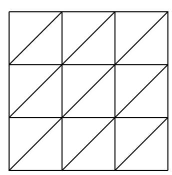
because $2 = 0$ in $\mathbb{F} = \mathbb{Z}/2$. $\square$

One remark to make is that for $k < 0$ we define $C_k = 0$. So $\partial_0 : C_0 \to C_{-1}$ is the zero map.

Definition 2.4. For an abstract simplicial complex K, we define its **k-th homology with $\mathbb{Z}/2$ coefficients** as

$$H_k(K; \mathbb{F}) = Z_k / B_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

Let us go back to our torus with 9 vertices.



The 0-cycles are all the 0-chains, and the 0-boundaries are 0-chains $\{v_1\} + \cdots + \{v_l\}$ with l even. So in this case,

$$H_0 = \frac{\mathbb{F}^9}{B_0} \cong \mathbb{F}$$

is 1-dimensional. The first homology H^1 is 2-dimensional. For H^2, the boundary is $B_2 = 0$ because there are no 3-simplices in this simplicial complex, and Z_2 is isomorphic to $\mathbb{F}$ because it has two elements: 0 and the sum of all 1-simplices.

3 September 7, 2016

In doing $\mathbb{Z}$-coefficients, or more generally G coefficients for an abelian group G, we need an ordering $\leq$ on V for an abstract simplicial complex $K = (V, S)$. This ordering has to be a total ordering, or at least, restricts to a total order on each $s \in S$.

3.1 Simplicial homology with G coefficients

We also use a slightly different notation. For a k-simplex $s = \{v_0, \dots, v_k\}$. We write

$$s = [v_0, \dots, v_k]$$

to mean $s = \{v_0, \dots, v_k\}$ and $v_0 < v_1 < \cdots < v_k$. We need this because we want to take care of the orientation of the simplices. For instance, we want $\partial[v_0, v_1] = \{v_1\} - \{v_0\}$.

The **chain groups** with $\mathbb{Z}$ coefficients will be

$$C_k = C_k(K; \mathbb{Z}) = \mathbb{Z}S_k$$

which is the free abelian group with generators S_k. For coefficients in G, we define

$$C_k(K; G) = \Big\{ \sum_{\text{finite}} a_s s \Big\}$$

for $a_s \in G$.

Now for every k, we are going to define the boundary map $\partial_k : C_k \to C_{k-1}$ as

$$\partial_k(as) = \partial_k(a[v_0, \dots, v_k]) = \sum_{i=0}^{k} (-1)^i a[v_0, \dots, \hat{v}_i, \dots, v_k].$$

We don't need G to be a ring because we aren't actually multiplying anything.

Lemma 3.1. *The map $\partial_k \circ \partial_{k+1} : C_{k+1} \to C_{k-1}$ is zero.*

Proof. Let us use $\mathbb{Z}$ coefficients. Let $s = [v_0, \dots, v_{k+1}]$. Then

$$\partial_{k+1} s = \sum_{i=0}^{k+1} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_{k+1}]$$

and then

$$\begin{aligned}
\partial_k \partial_{k+1} s &= \sum_{i=0}^{k+1} (-1)^i \partial_k [v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \\
&= \sum_i \sum_{j<i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{k+1}] \\
&\quad + \sum_i \sum_{j>i} (-1)^i (-1)^{j+1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}] \\
&= \sum_{i<j} ((-1)^{i+j} + (-1)^{i+j+1}) [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{k+1}] = 0. \qquad \square
\end{aligned}$$

For example if we took $[v_0, v_1, v_2]$ then

$$\begin{aligned}
\partial &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1], \\
\partial\partial &= [v_2] - [v_1] - [v_2] + [v_0] + [v_1] - [v_0] = 0.
\end{aligned}$$

Define $\gamma \in C_k$ to be a **cycle** if $\partial_k \gamma = 0$ and a **boundary** if $\gamma = \partial_{k+1}\beta$ for some β. The lemma is then equivalent to $B_k \subseteq Z_k$.

Definition 3.2. The kth **homology group** is

$$H_k(K; G) = \ker \partial_k / \operatorname{im} \partial_{k+1} = Z_k / B_k.$$

We further call two cycles γ and γ' are **homologous** if $\gamma - \gamma'$ is a boundary. A cycle γ gives a **homology class** $[\gamma] = \gamma + B_k \in Z_k/B_k = H_k(K; G)$.

3.2 Singular homology

In the book, Hatcher uses the notion of a Δ-complex instead of a simplicial complex. This gives him more flexibility to build spaces from a fewer number of simplices.

A **triangulation** of a topological space X is an abstract simplicial complex K and a homeomorphism $|K| \to X$. A lot of simplicial complexes admit a triangulation, but there are topological spaces that are not homeomorphic to any realization of a simplicial complex. One can ask, if K and K' are abstract simplicial complexes that both triangulate X, i.e., there are homeomorphisms $|K| \to X$ and $|K'| \to X$, is it true that $H_k(K; G) \cong H_k(K'; G)$? For instance, are the homologies of the tetrahedron and the octahedron the same? The answer is yes, although it is not obvious.

Also, which topological spaces X are triangulable? It is known that smooth manifolds are, and this was proved by Whitehead. But not all topological manifolds are triangulable. It was known for 30 years that not all 4-manifolds are triangulable, and it was recently shown by C. Manolescu that this is also the case for dimensions at least five.

So you do not want to restrict yourselves to simplicial complexes. Singular homology defines $H_k(X;G)$ for a topological space X not dependent on a choice of triangulation.

Let $\Delta^k \subseteq \mathbb{R}^{k+1}$ be a standard k-simplex. Consider the set

$$\Sigma_k(X) = \{\sigma : \Delta^k \to X\}$$

of all continuous maps. We this the set of **singular k-simplices**.

Now for an abelian group G, we construct the singular k-chains

$$C_k(X;G) = \Big\{ \sum_{\sigma \in \Sigma_k(X)} a_\sigma \sigma \Big\}$$

where $a_\sigma = 0$ except for a finite number of σ and this is just a formal sum. If $G = \mathbb{Z}$, then $C_k(K;\mathbb{Z}) = \mathbb{Z}\Sigma_k$.

We define the group homomorphisms $\partial_k : C_k(X;\mathbb{Z}) \to C_{k-1}(X;\mathbb{Z})$. Given an element $\sigma \in \Sigma_k(X)$, i.e., $\sigma : \Delta^k \to X$, we want to define $\partial_k \sigma$. The intuition is that you would want it to make like the boundary of a simplex.

Given $v_0, \dots, v_n \in \mathbb{R}^N$, there is a unique affine-linear map $\Delta^n \to \mathbb{R}^N$ that sends e_i to v_i. Call this map $[v_0, \dots, v_n] : \Delta^n \to \mathbb{R}^N$. Using this notation, given $\sigma : \Delta^k \to X$ the **faces** of σ are

$$\sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_k] : \Delta^{k-1} \to X.$$

Then we define

$$\partial_k \sigma = \sum (-1)^i \sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_k].$$

4 September 9, 2016

It is worth abstracting what we are doing and looking at it algebraically.

A **chain complex** is a collection of abelian groups C_k with $k \in \mathbb{Z}$ and morphisms $\partial_k : C_k \to C_{k-1}$ satisfying $\partial_k \partial_{k+1} = 0$. Let us all write this (C_*, ∂). The **homology** of (C_*, ∂) is defined as

$$H_k(C_*, \partial) = \ker \partial_k / \operatorname{im} \partial_{k+1} = Z_k / B_k.$$

We will call them cycles and boundaries even if it is just algebra.

4.1 Singular homology

Recall that the singular chain complex for a topological space X is defined as

$$C_k(X; G) = \bigoplus_{\sigma \in \Sigma_k(X)} G, \qquad \Sigma_k = \{\sigma : \Delta^k \to X\}$$

and $C_k = 0$ for $k < 0$. The boundary map is given as

$$\partial_k(a\sigma) = \sum_i (-1)^i a\sigma \circ [e_0, \dots, \hat{e}_i, \dots, e_k].$$

Lemma 4.1. *For $k \geq 1$, $\partial_k \circ \partial_{k+1} = 0$.*

Proof. Let us compute $\partial_k \partial_{k+1}(a\tau)$. It is

$$\partial_k \partial_{k+1}(a\tau) = \sum_i \sum_j (\text{sign}) a\tau[e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{k+1}]$$

with the sign canceling out. So we are done. □

Definition 4.2. The $H_k(X; G)$ is the kth homology of the singular chain complex of X. We also denote $H_k(X) = H_k(X; \mathbb{Z})$.

Usually the chain complex groups are enormous. A point is one of the rare cases we can actually compute the homology just from the definition.

Example 4.3. Let X be a single point. Then Σ is a set with a single element, and so $C_k(X; G) = \oplus_{\sigma \in G} \cong G$. Then the chain complex will look like

$$\cdots \longrightarrow G \xrightarrow{\partial_3} G \xrightarrow{\partial_2} G \xrightarrow{\partial_1} G \xrightarrow{\partial_0} 0 \xrightarrow{\partial_{-1}} 0 \longrightarrow \cdots.$$

The boundary map will be given by

$$\partial_k(a\sigma_k) = \sum_{i=0}^{k} (-1)^k a\sigma_{k-1} = \begin{cases} 0 & \text{if } k \text{ odd}, \\ a & \text{if } k \text{ odd}. \end{cases}$$

Then the chain complex is

$$\cdots \longrightarrow G \xrightarrow{\partial_3=0} G \xrightarrow{\partial_2=1} G \xrightarrow{\partial_1=0} G \xrightarrow{\partial_0} 0 \xrightarrow{\partial_{-1}} 0 \longrightarrow \cdots.$$

Now we can compute the homology. The 0th homology is given as

$$H_0(\mathrm{pt}, G) = \ker \partial_0 / \operatorname{im} \partial_1 = G/0 = G.$$

For $k \geq 1$ odd,

$$H_k(\mathrm{pt}, G) = \ker \partial_k / \operatorname{im} \partial_{k+1} = G/G = 0.$$

For $k \geq 2$ even,

$$H_k(\mathrm{pt}, G) = \ker \partial_k / \operatorname{im} \partial_{k+1} = 0/0 = 0.$$

So the conclusion is that

$$H_k = \begin{cases} G & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

If $X = \bigcup_m X^m$ are the path components of X, then $\Sigma_k(X) = \coprod_m \Sigma_k(X^m)$. Then $C_k(X) = \bigoplus C_k(X^m)$, and we also have $Z_k = \bigoplus_m Z_k(X^m)$ and $B_k = \bigoplus_m B_k(X^m)$. Likewise the obvious thing $H_k(X) = \bigoplus_m H_k(X^m)$ happens for homology.

Using this, we can compute the homology for a finite topological space, or a space whose path components are points.

Let us compute $H_0(X)$ in general. Assume that X is path-connectedd first. We see that

$$\Sigma_0(X) = \{\text{maps } \Delta^0 \to X\} = X$$

and so $C_0(X)$ is the collections of points in X with $\mathbb{Z}$ multiplicities. The cycles are

$$Z_0(X) = \ker(\partial_0 : C_0 \to C_{-1} = 0) = C_0.$$

A 0-chain is a boundary if and only if the sum of the coefficients is zero. This is because we can connect any two points with a path. So

$$B_0(X) = \{\sum a_x x : \sum a_x = 0\}.$$

Then the homology is

$$H_0(X; G) = Z_0(X)/B_0(X) \cong G.$$

So in the general case, $H_0(X; G) = \bigoplus_{\text{path components}} G$.

4.2 Functorial aspects of homology

One good thing about singular homology is that the construction is functorial. For two topological spaces X and Y and a continuous map $f : X \to Y$, this induces a map

$$\begin{aligned} f \circ (-) : \Sigma_k(X) &\to \Sigma_k(Y) \\ \sigma &\mapsto f \circ \sigma. \end{aligned}$$

These maps gives us a chain map $C_*(X) \to C_*(Y)$, i.e., a map for each k satisfying

$$f_\# \circ \partial_k^X = \partial_k^Y \circ f_\#.$$

This gives a map $f_* : H_k(X) \to H_k(Y)$ because $f_\# : Z_k(X) \to Z_k(Y)$ and $f_\# : B_k(X) \to B_k(Y)$.

Two evident facts are that $(f \circ g)_* = f_* \circ g_*$ and if $f = \mathrm{id}_X$ then $f_* : H_k(X) \to H_k(X)$ is also the identity map. This is what allows us to say that homology is functorial.

5 September 12, 2016

5.1 Exact sequences

Let A_i be abelian groups for $i \in \mathbb{Z}$, and let $f_i : A_i \to A_{i-1}$ be homomorphisms.

$$\cdots \longrightarrow A_3 \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \longrightarrow \cdots$$

This sequence is called **exact** if $\ker f_i = \operatorname{im} f_{i+1}$ (in A_i) for all i. If it is true for one particular i, we call that it is **exact at** i. For example, the sequence

$$\cdots \longrightarrow \mathbb{R}^2 \xrightarrow{f_{i+1}} \mathbb{R}^2 \xrightarrow{f_i} \mathbb{R}^2 \longrightarrow \cdots, \qquad f_i = f_{i+1} = \cdots = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is exact. Also, the sequence

$$\mathbb{Z}/2 \xrightarrow{\alpha} \mathbb{Z}/4 \xrightarrow{\beta} \mathbb{Z}/2$$

with $\alpha : x \mapsto 2x$ and $\beta : y \to y$ is exact at the middle group.

The sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

being exact at A means that $\ker \alpha = \operatorname{im} 0 = 0$, i.e., that α is injective. Likewise the exactness of

$$A \longrightarrow B \xrightarrow{\beta} C \xrightarrow{0} 0$$

at C means $\operatorname{im} \beta = \ker 0 = 0$, i.e., β is onto. So if

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact, then α is an isomorphism.

A **short exact sequence** is a sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

That is exact at A, B, C. Then $A \to B$ is injective and $B \to C$ is surjective. In this case we can replace A by $A \cong \alpha(A) \subseteq B$. We can also identify $C = \operatorname{im}(B \to C) = B/A$. So up to isomorphism, the short exact sequence can be also written as

$$0 \longrightarrow A \overset{i}{\hookrightarrow} B \overset{j}{\twoheadrightarrow} B/A \longrightarrow 0.$$

5.2 Exact sequence of chain complexes

Consider a chain complex (C_*, ∂^C). This is not an exact sequence because $\partial_k \partial_{k+1} = 0$ only means that $\ker \partial_k \supseteq \operatorname{im} \partial_{k+1}$. A **chain map** $f : B_* \to C_*$

between chain complexes B_*, C_* means, for every k a map $f_k : B_k \to C_k$ such that $\partial_k^C \circ f_k = f_{k-1} \circ \partial_k^B$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & B_k & \xrightarrow{\partial_k^B} & B_{k-1} & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle f_k} & & \Big\downarrow{\scriptstyle f_{k-1}} & & \\ \cdots & \longrightarrow & C_k & \xrightarrow{\partial_k^C} & C_{k-1} & \longrightarrow & \cdots \end{array}$$

The formula is simply saying that "both routes are equal", i.e., the diagram commutes.

A **short exact sequence of chain complexes**

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

consists of three chain complexes, two chain maps (i, j) with i injective (i_k injective for all k) and j surjective and $\operatorname{im} i = \ker j$ (i.e., $\operatorname{im} i_k = \ker j_k$ for all k). This can be drawn as

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ \cdots & \longrightarrow & A_{k+1} & \xrightarrow{\partial} & A_k & \xrightarrow{\partial} & A_{k-1} & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} & & \\ \cdots & \longrightarrow & B_{k+1} & \xrightarrow{\partial} & B_k & \xrightarrow{\partial} & B_{k-1} & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle j} & & \\ \cdots & \longrightarrow & C_{k+1} & \xrightarrow{\partial} & C_k & \xrightarrow{\partial} & C_{k-1} & \longrightarrow & \cdots \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The rows are chain complexes of abelian groups and the columns are short exact sequences, and the diagram commutes.

Remember that $H_k(A) = \ker \partial_k / \operatorname{im} \partial_{k+1}$. If $\alpha \in \ker \partial_k$ is a cycle, we denote its homology class as $[\alpha] \in H_k(A)$. In the diagram, we can construct a map

$$H_k(C) \xrightarrow{\partial_*} H_{k-1}(A)$$

as follows. Have $[\gamma] \in H_k(C)$ represented by $\gamma \in C_k$ with $\partial\gamma = 0$. Because j is surjective, ther eexists a $\beta \in B_k$ such that $\gamma = j\beta$. Because $j\partial\beta = \partial j\beta = \partial\gamma = 0$. So $\partial\beta \in \ker j = \operatorname{im} i$. So there exists an α with $i\alpha = \partial\beta$. Also α is a cycle: $i\partial\alpha = \partial i\alpha = \partial\partial\beta = 0$ implies $\partial\alpha = 0$ since i is injective. So there exists a homology class $[\alpha] \in H_{k-1}(A)$. We are going to define

$$\partial_* : [\gamma] \mapsto [\alpha].$$

We need to show that this is independent of choice. First we chose γ. Suppose $[\gamma] = [\gamma']$ so that $\gamma' = \gamma + \partial\Gamma$. We made a second choice β with $j\beta = \gamma$. If $j\beta' = j\beta = \gamma$ then $\beta - \beta' \in \ker j = \operatorname{im} i$ so $\beta - \beta' = i(a)$. Then $\partial\beta - \partial\beta' = \partial(ia) = i\partial a$. Then there exists a unqiue α and α' with $i\alpha = \partial\beta$ and $i\alpha = \partial\beta'$. Then $\partial\beta - \partial\beta' = i(\alpha - \alpha') = i\partial a$ so $\alpha - \alpha' = \partial a$. So $[\alpha] = [\alpha']$. I'll leave the independence of the choice of γ.

Now note that i induces the map $i_* : H_k(A) \to H_k(B)$ and j induces the map $j_* : H_k(B) \to H_k(C)$.

Proposition 5.1. *The following sequence is exact:*

$$\begin{tikzcd}
H_k(A) \arrow[r, "i_*"] & H_k(B) \arrow[r, "j_*"] & H_k(C) \arrow[dll, "\partial_*"'] \\
H_{k-1}(A) \arrow[r, "i_*"] & H_{k-1}(B) \arrow[r, "j_*"] & H_{k-1}(c) \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

This is called the ***long exact sequence****.*

6 September 14, 2016

Given a short exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{\ i\ } B_* \xrightarrow{\ j\ } C_* \longrightarrow 0,$$

it induces a long exact sequence

$$\begin{tikzcd} H_{k+1}(A) \arrow[r, "i_*"] & H_{k+1}(B) \arrow[r, "j_*"] & H_{k+1}(C) \arrow[dll, "\partial_*"'] \\ H_k(A) \arrow[r, "i_*"] & H_k(B) \arrow[r, "j_*"] & H_k(C) \arrow[dll, "\partial_*"'] \\ H_{k-1}(A) \arrow[r, "i_*"] & \cdots . \end{tikzcd}$$

We want to show that im $\subseteq$ ker and im $\supseteq$ ker for (i_*, j_*), (j_*, ∂_*), and (∂_*, i_*).

For instance, let us prove $\operatorname{im}(i_*) \supseteq \ker(j_*)$. Given $[\beta] \in H_k(B)$ with $\beta \in B_k$, we have $\partial\beta = 0$. Suppose that $j_*[\beta] = [j\beta] = 0$. We must show that $[\beta] \in \operatorname{im}(i_*)$. Note that $[j\beta] = 0$ means that $j\beta = \partial\gamma$ for some $\gamma \in C_{k+1}$. Because j is onto, there exists an $\omega \in B_{k+1}$ with $j\omega = \gamma$. Then $\partial j\omega = j\partial\omega = \partial\gamma$. So $j(\beta - \partial\omega) = 0$ and then $\beta - \partial\omega \in \operatorname{im} i$ because $\ker j = \operatorname{im} i$, and so $\beta - \partial\omega = i\alpha$. Now α is a cycle because $i\partial\alpha = \partial\beta - \partial\partial\omega = 0$ and so $\partial\alpha = 0$. Because $\beta - \partial\omega = i\partial$, this shows that $[\beta] \in \operatorname{im}(i_k)$. The other ones can be proved similarly.

6.1 Relative homology

Let us temporarily get back to topology. Let X be a topological space and A be a topological subspace with the inclusion $i : A \to X$. This induces a injection $i_\sharp : C_*(A) \hookrightarrow C_*(X)$. This is a starting of a short exact sequence

$$0 \longrightarrow C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X)/C_*(A) \longrightarrow 0$$

of chain complexes. We denote

$$C_*(X, A) = C_*(X)/C_*(A).$$

A chain in this complex will be a chain γ in X and ignores chains $\sigma : \Delta \to X$ whose images are in A. The γ represents a cycle in $C_*(X, A)$ if $\partial\gamma$ consists of simplices in A. The **relative homology groups** are defined as

$$H_*(X, A) = H_*(C_*(X, A), \partial).$$

Because we have a short exact sequence of chain complexes and we have done all the algebra, we have a long exact sequence.

$$\begin{tikzcd} H_k(A) \arrow[r, "i_*"] & H_k(X) \arrow[r, "j_*"] & H_k(X, A) \arrow[dll, "\partial_*"'] \\ H_{k-1}(A) \arrow[r] & H_{k-1}(X) \arrow[r] & \cdots \end{tikzcd}$$

In this context, the map ∂_* has a more geometric interpretation. Start with a relative cycle in (X, A), and choose a chain β in X representing the cycle. Take $\partial\beta$ which lines in A. That is the image of the cycle in ∂_*.

7 September 16, 2016

7.1 The five lemma

Lemma 7.1 (Five lemma)**.** *Suppose we have a commutative diagram (of abelian groups, for example)*

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & G_2 \arrow[r, "p_2"] \arrow[d, "f_2"] & G_3 \arrow[r, "p_3"] \arrow[d, "f_3"] & G_4 \arrow[r, "p_4"] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\ H_1 \arrow[r, "q_1"] & H_2 \arrow[r, "q_2"] & H_3 \arrow[r, "q_3"] & H_4 \arrow[r, "q_4"] & H_5 \end{tikzcd}$$

and suppose that the rows are exact. If the maps f_1, f_2, f_4, f_5 are isomorphism, then f_3 is also an isomorphism.

Proof. We first prove that f_3 is injective. Suppose that $f_3(g_3) = 0$. Consider $g_4 = p_3(g_3)$. Then $f_4(g_4) = f_4(p_3(g_3)) = q_3(f_3(g_3)) = 0$. Because f_4 is an isomorphism, this shows that $g_4 = 0$. So $g_3 \in \ker p_3 = \operatorname{im} p_2$. Let $g_3 = p_2(g_2)$ for some $g_2 \in G_2$. Let us set $h_2 = f_2(g_2)$. Then $q_2(h_2) = 0$ and so there exists and h_1 with $q_1(h_1) = h_2$. In fact, because f_1 is onto there exists $g_1 \in G_1$ such that $f_1(g_1) = h_1$. Consider $p_1(g_1) = g_2'$. Then $f_2(g_2') = f_2(p_1(g_1)) = q_1(f_1(g_1)) = q_1(h_1) = h_2$. So $p_1(g_1) = g_2' = g_2$. Now $g_3 = p_2(g_2) = p_2(p_1(g_1)) = 0$.

You can show surjectivity by yourself. □

If you finish the proof, you will have used that f_1 is onto, f_2, f_4 are isomorphisms, and f_5 is injective.

A **pair** is a pair (X, A) with $A \subseteq X$. We regard X also as a pair $(X, \emptyset)$. A map of pairs $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ with $f(A) \subseteq B$. For example, $(X, \emptyset) \to (X, A)$ is a map of pairs with the identity map, and also the inclusion map induces a map $i : (A, \emptyset) \to (X, \emptyset)$.

A map $f : (X, A) \to (Y, B)$ induces a map $f_* : H_k(X, A) \to H_k(Y, B)$ between homologies. Using the map $j : (X, \emptyset) \to (X, A)$ and $i : (A, \emptyset) \to (X, \emptyset)$, the long exact sequence can be interpreted as

$$\cdots \longrightarrow H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \longrightarrow \cdots .$$

7.2 Eilenberg-Steenrod axioms

In the context of our class, the **Eilenberg-Steenrod axioms** are more like properties of singular homology.

(ES0) H_k assignes to each pair (X, A) an abelian group $H_k(X, A)$.

(ES1) To each $f : (X, A) \to (Y, B)$ there is a map $f_* : H_k(X, A) \to H_k(Y, B)$ for each k.

(ES2) $(g \circ f)_* = g_* \circ f_*$.

(ES3) $(\mathrm{id}_{(X,A)})_* = \mathrm{id}$ on $H_k(X, A)$ for all k.

(ES4) $H_0(\text{point}) = \mathbb{Z}$ and $H_k(\text{point}) = 0$ for $k \neq 0$.

(ES5) For all (X, A) and k, there exist $\partial_* : H_k(X, A) \to H_{k-1}(A)$ that form the long exact sequence.

(ES6) ∂_* is **natural** for all maps of pairs, i.e., for all k and all $f : (X, A) \to (Y, B)$ the following commutes:

$$\begin{array}{ccc} H_k(X, A) & \xrightarrow{\partial_*} & H_{k-1}(A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_k(Y, B) & \xrightarrow{\partial_*} & H_{k-1}(B) \end{array}$$

Consider two maps $f_0, f_1 : X \to Y$. A **homotopy** from f_0 to f_1 is a (continuous) map

$$F : I \times X \to Y, \qquad I = [0, 1]$$

such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$ for all $x \in X$. We also write $F(t, x) = f_t(x)$.

For maps of pairs, consider $f_0, f_1 : (X, A) \to (Y, B)$. A homotopy from f_0 to f_1 is a map $F : (I \times X, I \times A) \to (Y, B)$ such that it is f_0 at $t = 0$ and f_1 at $t = 1$.

(ES7) If f_0 and f_1 are homotopy maps of pairs $(X, A) \to (Y, B)$ then $(f_0)_* = (f_1)*$ as maps $H_k(X, A) \to H_k(Y, B)$ for all k.

We would like to prove this fact for singular homology. We need an algebraic criterion for two chain maps. $f_0 : A_* \to B_*$ and $f_1 : A_* \to B_*$ to give equal maps $(f_0)_* = (f_1)_* : H_k(A) \to H_k(B)$.

Definition 7.2. Two chain maps f_0 and f_1 are called **chain homotopic** if there exist maps $K_k : A_k \to B_{k+1}$ such that

$$\partial^B_{k+1} \circ K_k + K_{k-1} \circ \partial^A_k = f_1 - f_0$$

for every k.

If f_0 and f_1 are chain homotopic, then $(f_0)_* = (f_1)_*$. To see this, consider any $[\alpha] \in H_k(A)$ so $\alpha \in A_k$ with $\partial^A_k \alpha = 0$. Applying the equation, we see that

$$f_1\alpha - f_0\alpha = \partial^B K\alpha.$$

Then $[f_1\alpha] = [f_0\alpha]$ in $H_k(B)$.

8 September 19, 2016

8.1 Homotopy and homology

Proposition 8.1. *If $f_0, f_1 : X \to Y$ are homotopic (to a homotopy $F : I \times X \to Y$), then the chain maps $(f_0)_\#, (f_1)_\# : C_*(X) \to C_*(Y)$ are chain homotopic. That is, there is a $K_k : C_k(X) \to C_{k+1}(Y)$ with $(f_1)_\# - (f_0)_\# = \partial_{k+1}K_k + K_{k-1}\partial_k$ as maps $C_k(X) \to C_k(Y)$. So $(f_1)_* = (f_0)_*$ as maps $H_k(X) \to H_k(Y)$.*

Basically what this proposition is require us to do is to find a way to subdivide $I \times \Delta^k$ into Δ^{k+1}. This way, we can use the homotopy to get a $k+1$-cycle that has $f_1(\gamma) - f_0(\gamma)$ as the boundary. The vertices of $I \times \Delta^k$ are $v_i = (0, e_i)$ and $w_i = (1, e_i)$ for $0 \leq i \leq k$. Then $I \times \Delta^k$ is a unit of $k+1$-simplices:

$$[v_0, w_0, w_1, \dots, w_k], \dots, [v_0, \dots, v_j, w_j, \dots, w_k], \dots, [v_0, \dots, v_k, w_k].$$

Although this is the geometric picture to keep in mind, it is not going to used in the proof.

Proof. We are going to define $K = K_k : C_k(X) \to C_{k+1}(Y)$ on a basis element $\sigma : \Delta^k \to X$. From σ, we get a map $\tilde{\delta} : I \times \Delta^k \to I \times X$ and $F \circ \tilde{\sigma} : I \times \Delta^k \to Y$. Consdier for $j = 0, \dots, k$ the map

$$F \circ \tilde{\sigma}[v_0, \dots, v_j, w_j, \dots, w_k] : \Delta^{k+1} \to Y.$$

Now define

$$K\sigma = \sum_{j=0}^{k} (-1)^j F\tilde{\sigma}[v_0, \dots, v_j, w_j, \dots, w_k].$$

We need to verify $\partial K\sigma + K\partial\sigma = f_1\sigma - f_0\sigma$. First

$$\begin{aligned}\partial K\sigma = \sum_{j=0}^{k} (-1)^j \Bigg(&\sum_{i=0}^{j} (-1)^i F\tilde{\sigma}[v_0, \dots, \hat{v}_i, \dots, v_j, w_j, \dots, w_k] \\ &+ \sum_{i=j}^{k} (-1)^{i+1} F\tilde{\sigma}[v_0, \dots, v_j, w_j, \dots, \tilde{w}_j, \dots, w_k] \Bigg)\end{aligned}$$

and the other way round is

$$\begin{aligned}K\partial\sigma &= K\sum_{i=0}^{k} (-1)^i \sigma[e_0, \dots, \hat{e}_i, \dots, e_k] \\ &= \sum_{i=0}^{k} \sum_{j=0}^{i-1} (-1)^(i+j) F\tilde{\sigma}[v_0, \dots, v_j, w_j, \dots, \hat{w}_i, \dots, w_k] \\ &\quad + \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j-1} F\tilde{\sigma}[v_0, \dots, \hat{v}_i, \dots, v_j, w_j, \dots, w_k].\end{aligned}$$

So when we add them, we get

$$\begin{aligned}
\partial K\sigma + K\partial\sigma &= \sum_{j=0}^{k} F\tilde{\sigma}[v_0, \ldots, v_{j-1}, w_j, \ldots, w_k] \\
&\quad - \sum_{j=0}^{k} F\tilde{\sigma}[v_0, \ldots, v_j, w_{j+1}, \ldots, w_k] \\
&= F\tilde{\sigma}[w_0, w_1, \ldots, w_k] - F\tilde{\sigma}[v_0, v_1, \ldots, v_k] = f_1\sigma - f_0\sigma.
\end{aligned}$$

This finishes the proof. □

So homotopic maps give rise to the same maps on homology.

Definition 8.2. A map $f : X \to Y$ is a **homotopy-equivalence** if there exists a map $g : Y \to X$ such that $g \circ f$ is homotopic to $\mathbf{1}_X$ and $f \circ g$ is homotopic to $\mathbf{1}_Y$.

Corollary 8.3. *If $f : X \to Y$ is a homotopy-equivalence, then the corresponding map f_* on homology is an isomorphism for all k.*

Proof. There exists a g as above. Because $g \circ f$ is homotopic to $\mathbf{1}_X$, we have $g_* \circ f_* = \mathbf{1}_{H_k(X)}$. Smilarly $f_* \circ g_* = \mathbf{1}_{H_k(Y)}$. Thus f_* is an isomorphism. □

Definition 8.4. X is **contractible** if the map $X \to$ point is a homotopy equivalence.

Corollary 8.5. *If X is contractible, then $H_0(X) = \mathbb{Z}$ and $H_k(X) = 0$ for $k > 0$.*

For instance, $\mathbb{R}^N$, the open and closed balls, and cones, are all contractible and so have the same homology to a point.

9 September 21, 2016

If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_*(X) \to H_*(Y)$ is an isomorphism. Likewise, we have the following theorem.

Theorem 9.1. *If $f : (X, A) \to (Y, B)$ is a map of pairs, and $f : X \to Y$ and $f|_A : A \to B$ are both homotopy equivalences, then $f_* : H_*(X, A) \to H_*(Y, B)$ is an isomorphism.*

Proof. Use the five lemma on

$$\begin{tikzcd} H_k(A) \arrow[r] \arrow[d] & H_k(X) \arrow[r] \arrow[d] & H_k(X,A) \arrow[r] \arrow[d] & H_{k-1}(A) \arrow[r] \arrow[d] & H_{k-1}(X) \arrow[d] \\ H_k(B) \arrow[r] & H_k(Y) \arrow[r] & H_k(Y,B) \arrow[r] & H_{k-1}(B) \arrow[r] & H_{k-1}(Y) \end{tikzcd}$$

and we immediately get the result. □

9.1 Reduced homology

We define the **reduced homology** $\tilde{H}_k(X)$. This is going to satisfy

$$H_k(X) \cong \begin{cases} \tilde{H}_k(X) & \text{for } k \neq 0, \\ \tilde{H}_0(X) & \text{for } k = 0 \end{cases}$$

if X is nonempty.

There are two definitions of the reduced homology that turns out to be equivalent. The first one uses augmented chain complexes for X, which are defined as

$$\tilde{C}_k(X) = \begin{cases} C_k(X) & k \geq 0, \\ \mathbb{Z} & k = -1, \\ 0 & k \leq -2, \end{cases}$$

with the maps

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is given by $\epsilon : \sum a_i x_i \mapsto \sum a_i$.

There is a short exact sequence

$$0 \longrightarrow E_* \longrightarrow \tilde{C}_*(X) \longrightarrow C_*(X) \longrightarrow 0,$$

where E_* is the chain complex with $E_{-1} = \mathbb{Z}$ and $E_k = 0$ for $k \neq -1$. Then we have an exact sequence

$$\cdots \to 0\tilde{H}_0(X) \to H_0(X) \to H_{-1}(E_*) = \mathbb{Z} \to \tilde{H}_{-1}(X) = 0 \to \cdots,$$

where $H_{-1}(X) = 0$ because X is nonempty.

So we have an short exact sequence

$$0 \longrightarrow \tilde{H}_0(X) \xrightarrow{i} H_0(X) \xrightarrow{p} \mathbb{Z} \longrightarrow 0.$$

If there exists an q with $p \circ q = \mathrm{id}$, then we say that the sequence **splits** nad $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$. So we need to choose a base point x_0 and send $q : a \mapsto [ax_0]$.

An alternative definition for the reduced homology is $\tilde{H}_0(X) = H_0(X, x_0)$ for a base point x_0. This works because the point x_0 gives a map $H_0(x_0) \cong \mathbb{Z} \to H_0(X)$, which is what we needed.

9.2 Homology of spheres

To compute these, we need the excision axiom. Let (X, A) be a pair and let $Z \subset A$ satisfy $\mathrm{clos}_X(Z) \subset \mathrm{int}_X(A)$. Then $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ in a natural way.

Proposition 9.2 (Excision axiom)**.** *The map i_* on homology $H_*(X \backslash Z, A \backslash Z) \to H_*(X, A)$ is an isomorphism (under the hypothesis* $\mathrm{clos}(Z) \subset \mathrm{int}(A)$*).*

Before proving this, let us see how it is used. Let $B^n = B^n(0, 1) \subseteq \mathbb{R}^n$ be the closed ball. Define $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For instance, S^0 has two points.

Theorem 9.3. *For $n > 0$, $H_n(S^n) = \mathbb{Z}$ and $H_0(S^n)$ and $H_k(S^n) = 0$ for $k \neq n, 0$.*

Over reduced homology this becomes tidier.

Theorem 9.4. *For $n \geq 0$,*

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

By the way, for reduced homology, $\tilde{C}_*(X)/\tilde{C}_*(A) = C_*(X)/C_*(A)$ so there is a long exact sequence with $\tilde{H}_*(A), \tilde{H}_*(X), H_*(X, A)$.

Proof. Let us prove the theorem by induction. The case $n = 0$ is trivial. Assume

$$\tilde{H}_k(S^{n-1} = \begin{cases} \mathbb{Z} & k = n-1 \\ 0 & \text{otherwise}. \end{cases}$$

We must prove $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$. We know that $\tilde{H}_k(B^n) = 0$ for all k because B^n is homotopy equivalent to a point. From the long exact sequence at (B^n, S^{n-1}),

$$\tilde{H}_k(B^n) = 0 \longrightarrow H_k(B^n, S^{n-1}) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \tilde{H}_{k-1}(B^n) = 0.$$

So $H_k(B^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1})$.

Now the map $(B^n, S^{n-1}) \hookrightarrow (B^n(0, 1.5), A)$ is an homotopy equivalence, and so $H_*(B^n, S^{n-1}) \cong H_*(B^n(1.5), A)$. Next we can map B^n around the ball and see

$$(B^n(0, 1.5), A) \cong (\{x_{n+1} \leq -1/2\}, \{-1/2 \leq x_{n+1} \leq 0\}) \subset S^n.$$

By excision, this has the same homology as $(S^n, \{x_{n+1} \leq 0\})$. Because $\{x_{n+1} \leq 0\}$ is homotopy equivalence, we get

$$\begin{aligned} \tilde{H}_{k-1}(S^{n-1}) &\cong H_k(B^n, S^{n-1}) \cong H_k(S^n, \{x_{n+1} \leq 0\}) \\ &\cong H_k(S^n, \text{point}) \cong \tilde{H}_k(S^n). \end{aligned}$$

This proves the theorem. $\square$

We haven't proved excision, and we are going to do it next class. This involves some actual epsilons and deltas, as opposed to what we have been doing so far.

10 September 23, 2016

10.1 Barycentric subdivision

The idea is that every cycle γ is homologous to a cycle γ' whose simplices are "small".

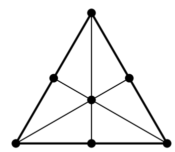


Figure 1: Subdivision of the 2-simplex

Let $\Delta^n = [e_0, e_1, \dots, e_n]$ and $v : \{0, \dots, n\} \to \{e_0, \dots, e_n\}$ be a bijection. For $0 \le j \le n$, let

$$b_v^j = \text{centroid of } [v(j), \dots, v(n)].$$

So b_v^0 is always the center and b_v^n is one of the vertices. Now this bijection v picks out a simplex

$$\underline{b}_v = [b_v^0, \dots, b_v^n]$$

in Δ^n that can be also thought of as an embedding $\underline{b}_v : \Delta^n \to \Delta^n$.

Now define a **subdivision operator** $S : C_n(X) \to C_n(x)$ for all n on generators as

$$S\sigma = \sum_v (-1)^v (-1)^v \sigma \circ \bar{b}_v = \sum_v (-1)^v \sigma[b_v^0, \dots, b_v^n].$$

Proposition 10.1. *The map S is chain-homotopy to the identity $1 : C_*(X) \to C_*(X)$, i.e., there exists an K such that $1 - S = \partial K + K\partial$ and $K : C_n(X) \to C_{n+1}(X)$.*

This will imply that S is a chain map and $[S\gamma] = [\gamma]$ for all cycles γ.

Proof. Define K as

$$K\sigma = \sum_{j=0}^{n} \sum_{\substack{v \\ v(j) < \cdots < v(n)}} (-1)^j (-1)^v \sigma[b_v^0, \dots, b_v^j, v(j), \dots, v(n)].$$

Now we have to verify that scary formula.

Let ∂^i denote "omit the ith vertex counting from 0". Then

$$\begin{aligned}
\partial K\sigma &= \partial \sum_{j=0}^{n} \sum_{v(j)<\cdots<v(n)} (-1)^j(-1)^v \sigma[b_v^0, \ldots, b_v^j, v(j), \ldots, v(n)] \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n+1} \sum_{v(j)<\cdots<v(n)} (-1)^i + j(-1)^v \sigma\partial^i[b_v^0, \ldots, b_v^j, v(j), \ldots, v(n)] \\
&= A + B + C + D
\end{aligned}$$

where

$$\begin{aligned}
A &= \text{terms with } i = j = 0, & B &= \text{terms with } i = n+1, j = n \\
C &= \text{terms with } i = 0, j \geq 0 & D &= \text{terms with } i \geq 1 \text{ that is not } B.
\end{aligned}$$

The terms A and B are

$$\begin{aligned}
A &= \sigma\partial^0[b_v^0, v(0), \ldots, v(n)] = \sigma[v(0), \ldots, v(n)] = \sigma[e_0, \ldots, e_n] = \sigma. \\
B &= \sum_v -(-1)^v \sigma\partial^{n+1}[b_v^0, \ldots, b_v^n, v(n)] = -\sum_v (-1)^v \sigma[b_v^0, \ldots, b_v^n] = -S\sigma.
\end{aligned}$$

C give $K\partial\sigma$, and D cancel in pairs. $\square$

Chain homotopy is preserved under composition, and so there exist L_m such that

$$1 - S^m = \partial L_m + L_m \partial,$$

where $L_m = L_{m-1} + KS^{m-1}$ is defined recursively.

11 September 26,2016

For a k-simplex $\Delta = [v_0, \dots, v_k]$ in $\mathbb{R}^N$, denote by $S\Delta$ be its barycentric subdivision. Let $L = \operatorname{diam}\Delta$ be its longest 1-simplex. Let $\ell = \max\{\operatorname{diam}\delta : \delta \in S\Delta\}$. Then it is an exercise to show

$$\ell \le \frac{k}{k+1}L.$$

So as a corollary, if $\ell_m = \max\{\operatorname{diam}\delta : \delta \in S^m\Delta\}$, then $\ell_m \to 0$ as $m \to \infty$.

Let X be a topological space and $U = \{U_\alpha\}_{\alpha\in A}$ be a cover such that $\operatorname{int} U_\alpha$ cover X. Say a singular simplex $\sigma : \Delta^k \to X$ is **U-small** if $\operatorname{im}(\sigma) \subseteq U_\alpha$ for some $\alpha \in A$. A chain γ is U-fine if all its simplices are U-small. Then there is a subset $\Sigma_k^U(X) \subseteq \Sigma_k(X)$ of small simplices and the corresponding subgroup $C_k^U(X) \subseteq C_k(X)$ of fine chains. Then there is an inclusion of chain maps

$$\varphi : (C_*^U, \partial) \hookrightarrow (C_*, \partial).$$

Proposition 11.1. *The inclusion φ gives isomorphisms*

$$\varphi_* : H_k^U(X) \to H_k(X).$$

Proof. If $[\gamma] \in H_k(X)$ and $\gamma \in C_k(X)$, then there exists an m such that $S^m\gamma$ is U-fine. Why is this? Given $\sigma : \Delta^k \to X$ consider the sets $\sigma^{-1}(U_\alpha)$ for $\alpha \in A$ covering Δ. This is an open cover of a compact subset, and so there exists an $\epsilon > 0$, which is called the Lebesgue number, such that for all balls $K \subseteq \Delta^k$ of diameter ϵ there exists an α with $K \subseteq \sigma^{-1}(\operatorname{int} U_\alpha)$. So there exists an m such that $S^m\sigma \in C_k^U(X)$. Ditto, for every chain γ there exists an $S^m\gamma \in C_k^U(X)$.

Now S^m is a chain map and is chain-homotopic to 1. So in $H_k(X)$, $[\gamma] = [S^m\gamma]$. The latter is in the image of $H_k^U(X)$, and this shows that φ_* is surjective.

To show injectivity, we consider $[\omega] \in H_k^U$ with $\omega \in C_k^U$ that maps to 0 in H_k. Then $\omega \in \partial\beta$ for some $\beta \in C_{k+1}(X)$. Then $S^m\omega = S^m\partial\beta = \partial S^m\beta$. For $m \gg 0$, $S^m\beta$ is U-fine. Also

$$\omega - S^m\omega = (\partial L_m + L_m\partial)\omega = \partial L_m\omega$$

is the boundary of a U-fine chain. So

$$\omega = S^m\omega + \partial L_m\omega = \partial(S^m\beta + L_m\omega) = \partial\zeta,$$

where ζ is U-fine. □

11.1 Excision

Let (X, A) be a pair, and let $Z \subset A$ be a subset with $\operatorname{cl}_X(Z) \subseteq \operatorname{int}_X(A)$. This is simply saying that

$$X = \operatorname{int}(X \setminus Z) \cup \operatorname{int}(A).$$

Consider $U = \{U_1, U_2\}$ with $U_1 = X \setminus Z$ and $U_2 = A$. There is a short exact sequence

$$0 \longrightarrow C_*(A) \longrightarrow C_*^U(X) \longrightarrow \frac{C_*^U(X)}{C_*(A)} \longrightarrow 0.$$

This gives a long exact sequence, and we can compare it with the ordinary long exact seqence.

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(A) \arrow[r] \arrow[d, "\cong"] & H_*^U(A) \arrow[r] \arrow[d, "\varphi_*"] & H_*^U(X, A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X, A) \arrow[r] & \cdots
\end{tikzcd}$$

Applying the five lemma, we see that the chain map

$$\frac{C_*^U(X)}{C_*(A)} \to \frac{C_*(X)}{C_*(A)}$$

gives isomorphisms in homology. The left hand side is

$$\frac{C_*^U(X)}{C_*(A)} = \frac{C_*(A) + C_*(X \setminus Z)}{C_*(A)} = \frac{C_*(X \setminus Z)}{C_*(X \setminus Z) \cap C_*(A)} = \frac{C_*(X \setminus Z)}{C_*(A \setminus Z)}.$$

This shows that

$$H_k(X \setminus Z, A \setminus Z) \cong H_k(X, A).$$

This is called **excision**.

11.2 Mayer-Vietoris sequence

Suppose $X = U_1 \cup U_2$ and furthermore $X = \operatorname{int} U_1 \cup \operatorname{int} U_2$. The chain complex $C_*(U_1)$ computes $H_*(U_1)$, and the chain complex $C_*(U_1) \oplus C_*(U_2)$ computes $H_*(U_1) \oplus H_*(U_2)$. Inside $C_*(X)$, the interior sum $C_*(U_1) + C_*(U_2)$ computes $H_*^U(X) \cong H_*(X)$, where $U = \{U_1, U_2\}$.

Now there is a natural map

$$C_*(U_1) \oplus C_*(U_2) \to C_*(U_1) + C_*(U_2); \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 + \gamma_2.$$

The kernel is pairs with $\gamma_2 = -\gamma_1$, and thus must be a chain in $U_1 \cap U_2$. So we have a short exact sequence

$$0 \longrightarrow C_*(U_1 \cap U_2) \longrightarrow C_*(U_1) \oplus C_*(U_2) \longrightarrow C_*^U(X) \longrightarrow 0.$$

Then we have a long exact sequence

$$\begin{tikzcd}
H_k(U_1 \cap U_2) \arrow[r] & H_k(U_1) \oplus H_k(U_2) \arrow[r] & H_k(X) \arrow[dll] \\
H_{K-1}(U_1 \cap U_2) \arrow[r] & H_{k-1}(U_1) \oplus H_{k-1}(U_2) \arrow[r] & H_{k-1}(X)
\end{tikzcd}$$

This is a very useful sequence in computing homology. It is worth noting that you can make the exactly same exact sequence by looking at the reduced homology instead.

12 September 28, 2016

If $X = \operatorname{int} U_1 \cup \operatorname{int} U_2$, then there is a short exact sequence

$$0 \longrightarrow C_*(U_1 \cap U_2) \longrightarrow C_*(U_1) \oplus C_*(U_2) \longrightarrow C_*^U(X) \longrightarrow 0.$$

This gives a map $\partial_* : H_k(X) = H_k^U(X) \to H_{k-1}(U_1 \cap U_2)$, mapping a k-chain to the $k-1$-chain that "cuts" it into two pieces, one lying in U_1 and the other lying in U_2.

Sometimes $X = A_1 \cup A_2$ but $X \neq \operatorname{int} A_1 \cup \operatorname{int} A_2$. In this case, suppose that $A_1 \subseteq U_1$, $A_2 \subseteq U_2$, and $A_1 \cap A_2 \subset U_1 \cap U_2$ are three homotopy equivalences, and $X = \operatorname{int}(U_1) \cup \operatorname{int}(U_2)$. Then by the homotopy equivalences, we have the same Mayer-Vietoris sequence

$$H_k(A_1 \cap A_2) \longrightarrow H_k(A_1) \oplus H_k(A_2) \longrightarrow H_k(X) \longrightarrow H_{k-1}(A_1 \cap A_2).$$

Let us now compute the homology of the sphere. Consider $S^n \subseteq \mathbb{R}^{n+1}$, with $S^n = B_+^n \cup B_-^n$,

$$B_+^n = \{x_{n+1} \geq 0\}, \quad B_-^n = \{x_{n+1} \leq 0\}.$$

The intersection is $B_+^n \cap B_-^n = S^{n-1} \subseteq \mathbb{R}^n$. Since

$$\tilde{H}_*(B_+^n) \oplus \tilde{H}_*(B_-^n) = 0,$$

the map $\tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1})$ is an isomorphism for all n and k. So again, we obtain

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}.$$

12.1 Retracts and the Brouwer fixed point theorem

A subset $A \subseteq X$ is a **retract** of X if there exists a **retraction** $r : X \to A$ such that $r|_A = \mathrm{id}$. For instance, $\{(0,0)\}$ is a retract of $\{0,1\} \times [0,1]$.

A **deformation retract** is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{id}_X$, $f_1(X) = A$, and $f_1(X) = A$ with $f_1|_A = \mathrm{id}_A$.

Let us now consider the case $X = B^n$ and $A = S^{n-1}$. Is A a retract of X? The answer in this case is no, i.e., there does not exist a map $r : B^n \to S^{n-1}$ such that $r(x) = x$ for all $x \in S^{n-1}$.

Proof. Suppose r exists. Then $r : B^n \to S^{n-1}$ and $i : S^{n-1} \to B^n$ satisfy $r \circ i = \mathrm{id}_{S^{n-1}}$. Apply reduced homology $\tilde{H}_{n-1}$ and get $r_* : \tilde{H}_{n-1}(B^n) \to \tilde{H}_{n-1}(S^{n-1})$ and $i_* : \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(B^n)$. Then the composition

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}$$

must be identity on $\mathbb{Z}$. This is a contradiction. $\square$

Corollary 12.1 (Brouwer fixed point theorem)**.** *If $f : B^n \to B^n$ is a continuous map, then f has a fixed point, i.e., there exists an x with $f(x) = x$.*

Proof. Suppose $f(x) \neq x$ for all x. We construct from f a retraction $r : B^n \to S^{n-1}$ as follows: there exists a unique $t \geq 1$ such that

$$s_t(x) = tx + (1-t)f(x)$$

has norm 1, and define $r(x) = s_t(x)$. This is clearly a tract, and therefore we get a contradiction. $\square$

In the Mayer-Vietoris sequence the map ∂_* is natural. If $f : X \to Y$ is a continuous map with $f(U_i) \subseteq V_i$ for $i = 1, 2$, and $X = \mathrm{int}(U_1) \cup \mathrm{int}(U_2)$, $Y = \mathrm{int}(V_1) \cup \mathrm{int}(V_2)$, then the diagram

$$\begin{tikzcd} H_k(X) \arrow[r, "\partial_*"] \arrow[d, "f_*"] & H_{k-1}(U_1 \cap U_2) \arrow[d] \\ H_k(Y) \arrow[r, "\partial_*"] & H_{k-1}(V_1 \cap V_2) \end{tikzcd}$$

commutes.

Consider $S^n \subseteq \mathbb{R}^{n+1}$ and $f : S^n \to S^n$ given by

$$(x_1, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1}). \tag{$\dagger$}$$

Lemma 12.2. *On $\tilde{H}_n$, $f_* : \mathbb{Z} \to \mathbb{Z}$ is given by $d \mapsto -d$.*

Proof. Recall from the Mayer-Vietoris sequence

$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "f_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "f_*"] \\ \tilde{H}_n(S^n) \arrow[r, "\partial_*", "\cong"'] & \tilde{H}_{n-1}(S^{n-1}) \end{tikzcd}$$

If $f_* = -1$ for S^{n-1}, then it is too for S^n. So it suffices to check for $n = 0$. This is obvious. $\square$

A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ means that $A^T A = I$. Then $\det A = \pm 1$. Denote

$$\begin{aligned} \mathrm{O}(n) &= \text{all orthogonal transformations}, \\ \mathrm{SO}(n) &= \{A \in \mathrm{O}(n) : \det A = 1\}. \end{aligned}$$

Any A_0 in $\mathrm{O}(n)$ can be joined by a path A_t to either I or $\dagger$. This is because you can match each axis one by one.

Corollary 12.3. *An orthogonal matrix A, giving a map $A : S^{n-1} \to S^{n-1}$ applies to homology $A_* : \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(S^{n-1})$ is 1 or -1 according as $\det A = 1$ or -1.*

Corollary 12.4. *If* $\det A = -1$ *then* $A : S^{n-1} \to S^{n-1}$ *is not homotopic to the identity.*

Corollary 12.5. *If* n *is odd, the map* $S^{n-1} \to S^{n-1}$ *sending* $x \mapsto -x$ *is not homotopic to* 1*.*

Corollary 12.6. *If* n *is odd, on* S^{n-1} *there does not exist a nowhere vanishing (continuous) vector field.*

13 September 30, 2016

13.1 Identification spaces and wedge sums

For a closed subset $A \subseteq X$, the **identification space** X/A is defined as identifying all points on A to one point a. Let us call $X/A = \overline{X}$. Then there is a natural map

$$p_* : H_*(X, A) \to H_*(\overline{X}, a).$$

We show that this is an isomorphism in good cases.

Suppose there exists an U such that $A \subseteq \mathrm{int}_X(U)$ and A is a **strong deformation retract** of U. That is, there exists a homotopy $f_t : U \to U$ such that $f_0 = \mathrm{id}_U$ and f_1 is a traction of U to A and $f_t|_A = \mathrm{id}_A$. This gives us a homotopy $\bar{f}_t : \overline{U} \to \overline{U}$ which deformation retracts $\overline{U}$ to $a \in \overline{U}$.

We then have a diagram

$$\begin{tikzcd}
(X \setminus A, U \setminus A) \arrow[r, "p'"] \arrow[d] & (\overline{X} \setminus \{a\}, \overline{U} \setminus \{a\}) \arrow[d] \\
(X, U) \arrow[r, "p"] & (\overline{X}, \overline{U})
\end{tikzcd}$$

By excision, the two vertical maps give isomorphisms on homology, and p' is an homeomorphism. Thus p also gives an isomorphism on homology

$$p_* : H_*(X, U) \to H_*(\overline{X}, \overline{U}).$$

But the left hand side is isomorphic to $H_*(X, A)$ and the right hand side is isomorphic to $H_*(\overline{X}, \{a\})$ because they are homotopic.

For two topological spaces (X, x) and (Y, y) with base points, we define their **wedge sum** as

$$X \vee Y = \frac{X \amalg Y}{\{x, y\}}.$$

Likewise, for a collection of spaces (X_α, x_α), we define

$$\bigvee_\alpha X_\alpha = \frac{\coprod X_\alpha}{\{x_\alpha : \alpha \in A\}}.$$

Suppose for each α there exists a neighborhood U_α of x_α that strongly deformation retracts to x_α. Then

$$\tilde{H}_*\Big(\bigvee X_\alpha\Big) = \tilde{H}_*\Big(\coprod X_\alpha \Big/ \coprod \{x_\alpha\}\Big) \cong H_*\Big(\coprod X_\alpha, \coprod \{x_\alpha\}\Big) = \bigoplus \tilde{H}_*(X_\alpha).$$

The simplest case the bouquet of spheres $X = \bigvee S^{\alpha_n}$. There are pathological cases like attaching two Hawaiian earrings at the accumulation points.

There are natural maps $\pi_\beta : \bigvee X_\alpha \to X_\beta$ given by mapping all other spaces to the base point of X_β and mapping X_β to itself by the identity, and this induces the projection map

$$(\pi_\beta)_* : \tilde{H}_*\Big(\bigvee X_\alpha\Big) \to \tilde{H}_*(X_\beta).$$

Likewise there is a natural inclusion map $i_\beta : X_\beta \to \bigvee X_\alpha$ that induces the map

$$(i_\beta)_* : \tilde{H}_*(X_\beta) \to \tilde{H}_*\Big(\bigvee X_\alpha\Big).$$

13.2 Degree of a map

Any continuous map $f : S^n \to S^n$ gives a map $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$, which is actually $\mathbb{Z} \to \mathbb{Z}$. This is multiplication by d for some d, and this $d = \deg f$ is called the **degree** of f.

The degree of $f : S^n \to S^n$ can be computed by picking a $q \in S^n$ and looking at $f^{-1}(q) \subseteq S^n$ and "counting the points in $f^{-1}(q)$ with multiplicities".

Let $\omega \in \tilde{H}_n(S^n)$ be a generator. Then

$$\omega \in \tilde{H}_n(S^n) = H_n(S^n, \text{point}) = H_n(S^n, U) = H_n(S^n, S^n \setminus q) \cong H_n(B, B \setminus q)$$

by excision. Let ω_q be the generator of $H_n(B, B \setminus q)$ that corresponds to ω.

Now suppose $f^{-1}(q) = P = \{p_1, \dots, p_N\}$ and let $B_i \ni p_i$ be disjoint balls with $f(B_i) \subseteq B$. Like ω_q, there exists an element $\omega_{p_i} \in H_n(B_i, B_i \setminus p_i)$ that is a generator. Then f induces a map $(B_i, B_i \setminus p_i) \to (B, B \setminus q)$, and thus we can let

$$(f_i)_*(\omega_{p_i}) = d_i \omega_q.$$

This d_i is called the **local degree at** p_i.

Proposition 13.1. *In this case, $d = \sum_i d_i$ is the degree of f.*

Proof. Consider $W = S^n \setminus (\bigcup B_i)$, and identify W to one point. Then S^n/W will be a bouquet of spheres, each of one corresponding to the B_i. In the target space, this is the identifying $Z = S^n \setminus B$ to one point. The map between the identification spaces will be adding up the projection maps. It follows that $d = \sum_i d_i$. □

13.3 Attaching cells

Consider a space Y and a map $\varphi : S^{n-1} \to Y$ where S^{n-1} is the boundary of the closed n-ball D^n. Now form

$$Y \cup_\varphi e^n = Y \cup D^n / (x \in D^n \sim \varphi(x) \in Y \text{ if } x \in S^{n-1}).$$

This is the space obtained form Y by attaching an n-cell.

14 October 3, 2016

Recall that for a map $S^{n-1} = \partial D^n \to Y$, we can attach an n-cell to obtain

$$X = Y \cup_\varphi e^n = Y \amalg D^n / \sim .$$

We can also attach many n-cells. For $\alpha \in \mathcal{A}$ where $\mathcal{A}$ may be infinite, consider maps $\varphi_\alpha : S^{n-1} \to Y$. Then we can attach n-cells at the same time by taking

$$X = \frac{Y \amalg (\coprod_\alpha D^n_\alpha)}{\sim} \quad \text{where} \quad x \in \partial D^n_\alpha \sim \varphi_\alpha(x) \in Y.$$

Then X is obtained from Y by attaching n-cells. A subset $K \subseteq X$ is open (respectively closed) if and only if its inverse image in $Y \amalg (\coprod D^n_\alpha)$ is open (respectively closed).

14.1 CW complexes

Definition 14.1. A **CW complex** X is defined by its skeleta $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots$ and $X = \bigcup_n X^n$, where X^n is obtained from X^{n-1} by attaching n-cells, indexed by $\mathcal{A}_n$. In this case, $K \subseteq X$ is closd if and only if $K \cap X^n$ is closed in X^n for all n.

Example 14.2. The sphere S^n is a CW complex because we can attach an n-cell to a point to obtain S^n. This is not the only way to make the sphehre. For example S^2 can be obtained by attaching two 1-cells to two 0-cells to get a S^1 and then attaching two 2-cells.

Consider the pair (X^{n+1}, X^n). For each $\alpha \in \mathcal{A}_{n+1}$, let $o_\alpha \in e^{n+1}_\alpha$ be the center. Then

$$U = X^n \cup (e^n_\alpha \setminus o_\alpha)_\alpha = \text{complement of } \{o_\alpha : \alpha \in \mathcal{A}_{n+1}\}$$

is open in X^{n+1} and moreover U deformation retracts to X^n. Hence $H_k(X^n, X^{n-1}) = \tilde{H}_k(X^n/X^{n-1})$. But X^n/X^{n-1} is just a bouquet of spheres $\bigvee_{\alpha \in \mathcal{A}_n} S^n_\alpha$. Thus

$$\tilde{H}_k(X^n/X^{n-1}) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Now the long exact sequence of the pair (X^n, X^{n-1}) is given by

$$\begin{tikzcd}
\tilde{H}_{m+1}(X^{n-1}) \arrow[r] & \tilde{H}_{m+1}(X^n) \arrow[r] & \tilde{H}_{m+1}(X^n/X^{n-1}) \arrow[dll] \\
\tilde{H}_m(X^{n-1}) \arrow[r] & \tilde{H}_m(X^n) \arrow[r] & \tilde{H}_m(X^n/X^{n-1}) \arrow[dll] \\
\tilde{H}_{m-1}(X^{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, we have the maps $j_n : \tilde{H}_n(X^n) \to \tilde{H}_n(X^n/X^{n-1})$ coming from the pair (X^n, X^{n-1}) and the maps $\partial_n : \tilde{H}_{n+1}(X^{n+1}/X^n) \to \tilde{H}_n(X^n)$. Then the composite map is

$$(j_n \circ \partial_n) : \tilde{H}_{n+1}(X^{n+1}/X^n) \to \tilde{H}_n(X^n/X^{n-1})$$

that will look like $\bigoplus_{\alpha \in \mathcal{A}_{n+1}} \mathbb{Z}_\alpha \to \bigoplus_{\beta \in \mathcal{A}_n} \mathbb{Z}_\beta$.

14.2 Cellular homology

Now define the **cellular chain complex** of X as

$$C_n^{\mathrm{CW}}(X) = \tilde{H}_n(X^n/X^{n-1}) \cong \bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{Z}$$

with

$$\partial_{n+1}^{\mathrm{CW}} = j_n \circ \partial_n : C_{n+1}^{\mathrm{CW}}(X) \to C_n^{\mathrm{CW}}(X).$$

Proposition 14.3. $\partial^{\mathrm{CW}} \circ \partial^{\mathrm{CW}} = 0$ *and*

$$\frac{\ker(\partial_n^{\mathrm{CW}})}{\operatorname{im}(\partial_{n+1}^{\mathrm{CW}})} = \frac{\ker(j_{n-1} \circ \partial_{n-1})}{\operatorname{im}(j_n \circ \partial_n)} \cong H_n(X).$$

For example, consider $S^n = e^0 \cup e_n$ for $n \geq 2$. Then $C_k^{\mathrm{CW}}(X) \cong \mathbb{Z}$ for $k = 0, n$ and $C_k^{\mathrm{CW}}(X) \cong 0$ for $k \neq 0, n$. It follows that $H_k(S^n) \cong \mathbb{Z}$ for $k = 0, n$ and 0 otherwise.

We are going to suppose today that X is finite dimensional, i.e., $X^N = X^{N+1} = \cdots$ for some N. Note that $\tilde{H}_m(X^k/X^{k-1}) \neq 0$ only if $m = k$. Then from the long exact sequence we see that $\tilde{H}_m(X^{k-1}) \to \tilde{H}_m(X^k)$ is injective if $m + 1 \neq k$ and surjective if $m \neq k$. Thus

$$\tilde{H}_n(X^{n+1}) \cong \tilde{H}_n(X^{n+2}) \cong \cdots \cong H_n(X)$$

if X is finite dimensional. Likewise $\tilde{H}_n(X^{n-1}) \cong \tilde{H}_n(X^{n-2}) \cong \cdots \cong 0$.

We have an exact sequence

$$\tilde{H}_{n+1}(X^{n+1}/X^n) \xrightarrow{\partial_{n+1}} \tilde{H}_n(X^n) \longrightarrow \tilde{H}_n(X^{n+1}) \longrightarrow 0$$

so $\tilde{H}_n(X^{n+1}/X^n) \cong \tilde{H}_n(X^n)/\operatorname{im}(\partial_{n+1})$. Likewise

$$0 = \tilde{H}_n(X^{n-1}) \longrightarrow \tilde{H}_n(X^n) \xrightarrow{j} \tilde{H}_n(X^n/X^{n-1})$$

so j is injective.

15 October 5, 2016

Recall that a CW complex is defined by a increasing chain of skeleta

$$X^0 \hookrightarrow X^1 \hookrightarrow \cdots \hookrightarrow X^k \hookrightarrow X^{k+1} \hookrightarrow \cdots .$$

The dimension of X is $\dim X = N$ where N is the minimal number satisfying $X^N = X^{N+1} = \cdots = X$. There is an exact sequence

$$H_{n+1}(\bigvee S^{k+1}) \longrightarrow H_n(X^k) \longrightarrow H_n(X^{k+1}) \longrightarrow H_n(\bigvee S^{k+1})$$

and so $H_n(X^k) \to H_n(X^{k+1})$ is an isomorphism unless $n = k+1$ or $n+1 = k+1$. So for a fixed n, the sequence $H_n(X^0) \to H_n(X^1) \to \cdots$ will look like

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X^{n+2}) \longrightarrow \cdots .$$

So all the interesting stuff happens there in the middle.

15.1 Equivalence of singular and celluar homology

From the pair (X^{n+1}, X^n), we get maps

$$X^n \xrightarrow{I} X^{n+1} \xrightarrow{J} X^{n+1}/X^n$$

and these give an exact sequences

$$H_{n+1}(X^{n+1}) \xrightarrow{j_{n+1}} H_{n+1}(X^{n+1}/X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \xrightarrow{i_{n+1}} H_n(X^{n+1}),$$

$$H_n(X^n) \xrightarrow{j_n} H_n(X^n/X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{i_n} H_{n-1}(X^n),$$

$$H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}/X^{n-2}) \xrightarrow{\partial_{n-1}} H_{n-2}(X^{n-2}) \xrightarrow{i_{n-1}} H_{n-2}(X^{n-1}).$$

Then

$$\begin{aligned} H_n(X) = H_n(X^{n+1}) &= H_n(X^n)/\operatorname{im}\partial_{n+1} \cong j_n(H_n(X^n))/j_n(\operatorname{im}(\partial_{n+1})) \\ &= \operatorname{im} j_n/\operatorname{im}(j_n \circ \partial_{n+1}) = \ker \partial_n/\operatorname{im}(j_n \circ \partial_{n+1}) \\ &= \ker(j_{n+1} \circ \partial_n)/\operatorname{im}(j_n \circ \partial_{n+1}). \end{aligned}$$

The good thing about this is that $j_{n-1} \circ \partial_n$ and $j_n \circ \partial_{n+1}$ are the same up to index shifting. So we can define

$$C_n^{\mathrm{CW}}(X) = H_n(X^n/X^{n-1}), \quad \partial_n^{\mathrm{CW}} = j_{n-1} \circ \partial_n : C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X).$$

This is a chain complex because $\partial_n^{\mathrm{CW}} \circ \partial_n^{\mathrm{CW}} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$. Then what we did can be written as:

Proposition 15.1.

$$H_n(X) = \frac{\ker(\partial_n^{\mathrm{CW}})}{\mathrm{im}(\partial_{n+1}^{\mathrm{CW}})}.$$

This equation is still true for $n = 0$, if we use unreduced throughout, except that we interpret

$$C_0^{\mathrm{CW}}(X) = H_0(X^0, \emptyset) = H_0(X^0).$$

15.2 The infinite dimensional case

What if X is infinite dimensional? It might be the case that $H_n(X^{n+1}) = H_n(X^{n+2}) = \cdots \neq H_n(X)$. In fact, we are okay. Let me give a more general statement. Let Y be an increasing union of $Y_0 \subseteq Y_1 \subseteq \cdots$ and $Y = \bigcup Y_n$. Suppose $K \subseteq Y$ is closed if and only if $K \cap Y_n$ is closed in Y_n for all n, and also that every compact $K \subseteq Y$ is contained in Y_n for some n. How can $H_n(Y)$ be expressed in terms of $H_n(Y_k)$?

We construct the **limit**

$$\varinjlim_k H_n(Y_k) = G.$$

The elements of G are equivalence classes in $\coprod_k H_n(Y_k)$, where $g \in H_n(Y_k)$ and $g' \in H_n(Y'_k)$ satisfy $g \sim g'$ if they become equal in $H_n(Y_{k''})$ for some $k'' > \max\{k, k'\}$.

Proposition 15.2. *In this situation,* $H_n(Y) \cong \varinjlim_k H_n(Y_k)$*.*

Proof. Since there are maps $Y_k \hookrightarrow Y$, there are maps $H_n(Y_k) \to H_n(Y)$. So there is a map

$$\Phi : \varinjlim H_n(Y_k) \to H_n(Y).$$

Now we need to show that Φ is an isomorphism.

First we show that Φ is onto. Think of a $[\gamma] \in H_n(Y)$ represented by

$$\gamma = \sum_{i=1}^{N} a_i \sigma_i$$

for $\sigma_i : \Delta^n \to Y$. Then $\bigcup_i \mathrm{im}\, \sigma_i \subseteq K$ is compact. So there exists an k such that $K \subseteq Y_k$. Regard γ as giving $\gamma_k \in C_n(Y_k)$. Then Φ maps to the corresponding equivalence to $[\gamma]$.

You can show that Φ is injective similarly. $\square$

16 October 7, 2016

16.1 Computing cellular homology

Consider a CW complex X. We proved that the homology of X is the same as the homology coming from the chain complex

$$\partial_n^{\mathrm{CW}} : H_n(X^n/X^{n-1}) \to H_{n-1}(X^{n-1}/X^{n-2})$$

given by $\partial_n^{\mathrm{CW}} = j_{n-1} \circ \partial_n$. But we need to actually describe these maps in terms of X. We know that $H_n(X^n/X^{n-1}) \cong \bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{Z}_\alpha$ and so

$$\partial_n^{\mathrm{CW}} : \bigoplus_{\alpha} \in \mathcal{A}_n \to \bigoplus_{\beta \in \mathcal{A}_{n-1}} \mathbb{Z}_\beta.$$

Take a particular $\alpha_0 \in \mathcal{A}_n$ and $\beta_0 \in \mathcal{A}_{n-1}$. Then there inclusion and projection maps

$$\mathbb{Z} \xrightarrow{q_{\alpha_0}} \bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{Z}_\alpha \longrightarrow \bigoplus_{\beta \in \mathcal{A}_{n-1}} \mathbb{Z}_\beta \xrightarrow{p_{\beta_0}} \mathbb{Z}.$$

The composition will be an entry in the matrix, and we are interested in computing this map, which is represeted by $m_{\alpha\beta} \in \mathbb{Z}$.

The composition can be described as

$$\mathbb{Z} = H_n(D^n_{\alpha_0}, S^{n-1}_{\alpha_0}) \longrightarrow H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1})$$

$$\xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}/X^{n-2}) \xrightarrow{p_{\beta_0}} H_{n-1}(S^{n-1}_\beta) \cong \mathbb{Z}.$$

But there is a commutative diagram

$$\begin{array}{ccc} H_n(D^n_{\alpha_0}, S^{n-1}_{\alpha_0}) & \xrightarrow{\partial} & H_{n-1}(S^{n-1}_{\alpha_0}) \\ \big\downarrow{\scriptstyle q_{\alpha_0}} & & \big\downarrow{\scriptstyle (\varphi_{\alpha_0})_*} \\ H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}). \end{array}$$

On topological spaces, these maps corresponds to

$$S^{n-1}_{\alpha_0} \xrightarrow{\varphi_{\alpha_0}} X^{n-1} \xrightarrow{J_{n-1}} X^{n-1}/X^{n-2} \xrightarrow{p_{\beta_0}} D^{n-1}_{\beta_0}/S^{n-2}_{\beta_0} = S^{n-1}_{\beta_0}.$$

So $\times m_{\alpha\beta}$ is the map

$$\Phi_{\alpha_0\beta_0} = (p_{\beta_0} \circ H_{n-1} \circ \varphi_{\alpha_0}) : S^{n-1} \to S^{n-1}$$

as it acts on $H_{n-1}(S^{n-1})$. That is, $m_{\alpha\beta}$ is the degree of the map $\Phi_{\alpha\beta} : S^{n-1} \to S^{n-1}$.

16.2 Homology of projective space

Using this, let us compute the homology of

$$\mathbb{RP}^n = \{\ell : \ell \text{ is a 1-dimensional subspace of } \mathbb{R}^{n+1}\} = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*.$$
$$\mathbb{CP}^n = \{\ell : \ell \text{ is a 1-dimensional subspace of } \mathbb{C}^{n+1}\} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

Another way of thinking of $\mathbb{RP}^n$ is

$$\mathbb{RP}^n = S^n/(v \sim -v) = \frac{\{v \in \mathbb{R}^{n+1} : \|v\| = 1, v_{n+1} \geq 0\}}{(v_1, \ldots, v_n, 0) \sim (-v_1, \ldots, -v_n, 0)}$$

So there is an onto map $Q : D^n_+ \to \mathbb{RP}^n$ with $Q|_{S^{n-1}} : S^{n-1} \to \mathbb{RP}^{n-1}$. So $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\psi_{n-1}} e^n$, where $\psi_{n-1} : S^{n-1} \to \mathbb{RP}^{n-1}$ is given by $u \mapsto \{u, -u\}$. So

$$\mathbb{RP}^n = \mathbb{RP}^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$

Now we see that

$$C_k^{\mathrm{CW}}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 1, \ldots, n \\ 0 & \text{if } k < 0 \text{ or } k > n. \end{cases}$$

Then the celluar complex looks like

$$\mathbb{Z} \xrightarrow{\partial_n^{\mathrm{CW}}} \mathbb{Z} \xrightarrow{\partial_n^{\mathrm{CW}}} \cdots \xrightarrow{\partial_1^{\mathrm{CW}}} \mathbb{Z}.$$

We now need to compute the boundary maps.

Proposition 16.1. *In $C_*^{\mathrm{CW}}(\mathbb{RP}^n)$, the map ∂_m^{CW} is 0 if m is odd and 2 if m is even.*

Proof. We want to look at the map $S^{m-1} \to \mathbb{RP}^{m-1}/\mathbb{RP}^{m-2} = S^{m-1}$. The degree of this map cna be computed by adding the local degrees. Because the map is 2-to-1, it is either $1 + 1 = 2$ or $1 - 1 = 0$. If m is odd then S^{m-1} is not orientable and it turns out the two have different signs. If m is even then the two has same orientation, so it is 2. $\square$

For example, let us compute $\mathbb{RP}^4$ and $\mathbb{RP}^5$.

$$\begin{array}{lccccccccc} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ H_n(\mathbb{RP}^4): & 0 & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & \mathbb{Z}. \end{array}$$

$$\begin{array}{lccccccccccc} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ H_n(\mathbb{RP}^5): & \mathbb{Z} & & 0 & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & \mathbb{Z}. \end{array}$$

17 October 12, 2016

17.1 Computing more homologies

We have computed the homology of $\mathbb{RP}^n$. We can give a similar argument for $T^n = \mathbb{R}^n/\mathbb{Z}^n$. There is a natural CW complex structure on T^n given by

$$D_I^k = \{\underline{x} \in \mathbb{R}^n : 0 \le x_j \le 1, x_i = 0 \text{ for } i \in I\}$$

with the natural inclusion $D_I^k \to T^n$. The chain complex will be $C_k^{\mathrm{CW}}(T^n) = \mathbb{Z}^{\binom{n}{k}}$, and also $\partial_k^{\mathrm{CW}} = 0$ for all k. So we have

$$H_k(T^n) \cong \mathbb{Z}^{\binom{n}{k}}.$$

We have an infinite sequence $\mathbb{RP}^n \subseteq \mathbb{RP}^{n+1} \subseteq \cdots$. We call the union $\mathbb{RP}^\infty$, where we may consider it as 1-dimensional subspaces of

$$\mathbb{R}^\infty = \{\underline{\mathrm{x}} = (x_1, x_2, \dots) : x_i \in \mathbb{R}, x_i = 0 \text{ for almost all } i\}.$$

Then the k-th homology of $\mathbb{RP}^\infty$ as

$$H_k(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2 & k \text{ odd}, k \ge 1 \\ 0 & k \text{ even}, k \ne 0. \end{cases}$$

When we talked about $\mathbb{RP}^n$, we also talked about $\mathbb{CP}^n$. In this case have

$$\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup e^{2n}$$

because

$$\mathbb{CP}^n = \{\underline{z} : \|\underline{z}\|\}/\underline{z} \sim \lambda\underline{z} = \{\underline{z} : \|\underline{z}\| = 1, z_{n+1} \in \mathbb{R}^{\ge 0}\}/\sim$$

where $(z_1, \dots, z_n, 0) \sim \lambda(z_1, \dots, z_n, 0)$ for $|\lambda| = 1$. So the chain complex looks like

$$\cdots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}.$$

So

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k \text{ is even with } 0 \le k \le 2n \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, we can define $\mathbb{CP}^\infty$ and its homology groups will be

$$H_k(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & k \ge 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

In passing, it is worth saying that if K is an abstract simplicial complex, then there is the abstract simplicial homology $H_k(K)$ and the singular homology $H_k(X)$ for $X = |K|$.

Theorem 17.1. $H_k^{\text{simpl.}}(K) \cong H_k^{\text{sing.}}(X)$.

This is just because simplicial complexes are CW complexes. You need to check that the boundary maps on the simplicial chains and CW chains are actually the same, but you can unravel what the boundary maps actually are.

17.2 Cochains and cohomology

Now I want to talk about cohomology. A homology is something we pick up when we have a chain complex, and a cohomology is something we get from a cochain complex.

Definition 17.2. A **cochain complex** is a collection of abelian groups C^n for $n \in \mathbb{Z}$, a collection of homomorphisms $\delta_n : C^n \to C^{n+1}$ satisfying $\delta_{n+1} \circ \delta_n = 0$.

For example, the De Rahm complex is a cochain complex. If (C_n, ∂) is a chain complex, then we get an uninteresting cochain complex by setting $C^n = C_{-n}$.

18 October 14, 2016

A cochain complex (C^*, δ) is a collection of abelian groups C^n and maps $\delta_n : C^n \to C^{n+1}$ such that $\delta_{n+1} \circ \delta_n = 0$. The **cohomology** of is defined as

$$H^n(C^*) = \frac{\ker \delta_n}{\operatorname{im} \delta_{n-1}}$$

where $\ker \delta_n$ are called the cocycles and $\operatorname{im} \delta_{n-1}$ are called the coboundaries. A **cochain map** $\psi : A^* \to B^*$ is a collection of maps $\psi_n : A^n \to B^n$ satisfying $\psi_n \circ \delta^A_{n-1} = \delta^B_n \psi_n$. Such ψ gives a map $\psi_* : H^n(A^*) \to H^n(B^*)$.

If A and B are abelian groups, then

$$\operatorname{Hom}(A, B) = \{\text{homomorphisms } A \to B\}$$

is an abelian group by the addition $(h + h')(a) = h(a) + h'(a)$. A homomorphism $f : A \to A'$ gives a map $f^* : \operatorname{Hom}(A', B) \to \operatorname{Hom}(A, B)$ by composition $g \mapsto g \circ f$. So this is a "contravariant functor", sending A to $\operatorname{Hom}(A, B)$ and $f : A \to A'$ to $f^* : \operatorname{Hom}(A', B) \to \operatorname{Hom}(A, B)$. It satisfies $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$ and $(1_A)^* = 1_{\operatorname{Hom}(A,B)}$.

Fix an abelian group G and let (C_*, ∂) be a chain complex. Now define a cochain complex

$$C^n = \operatorname{Hom}(C_n, G), \quad \delta = \partial^* : C^n \to C^{n+1}.$$

What is $\ker \delta$? $a \in \operatorname{Hom}(C_n, G)$ means $\delta a = 0$ means $a \circ \partial = 0$ means a annihilates boundaries, i.e., $a|_{B_n} = 0$.

18.1 Singular cohomology

Let X be a topological space, and let $C_n(X)$ be the nth singular chain group with $\mathbb{Z}$ coefficients. These form a singular chain complex $(C_*(X), \partial)$.

For a fixed G, define a cochain complex

$$(C^*(X; G), \delta)$$

using the general $\operatorname{Hom}(-, G)$ idea, i.e., $C^n(X; G) = \operatorname{Hom}(C_n(X), G)$.

What is $a \in C^n(X; G) : C_n(X) \to G$? This is uniquely determined by $a(\sigma) \in G$ for all $\sigma \in \Sigma_n(X)$. Then we would have

$$a(n_1\sigma_1 + \cdots + n_k\sigma_k) = \sum n_i a(\sigma_i).$$

As we have said, a is a cocycle if and only if $a(\gamma) = 0$ whenever γ is a boundary.

A map $f : X \to Y$ gives a chain map $f_\# : C_*(X) \to C_*(Y)$, and $f_* : H_n(X) \to H_n(Y)$. Then $f_\#$ gives a cochain map

$$f^\# : C^*(Y; G) \to C^*(X; G) \quad \text{and} \quad f^* : H^n(Y; G) \to H^n(X; G).$$

We call the cohomology $H^n(X;G)$ coming from $(C^*(X;G),\delta)$ **singular cohomology**. To actually compute singular cohomology, use the CW chain complex (because it's write-downable) and construct the cochain complex–which we hope computes the singular cohomology. (In fact, it does.)

Example 18.1. Consider $S^4 = e_0 \cup e^4$. The chain complex is

$$C_*^{\mathrm{CW}}(S^4): \qquad \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \mathbb{Z}.$$

and so the cochain complex is

$$C^*_{\mathrm{CW}}(S^4;G): \qquad G \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow G.$$

So the cohomology is

$$H^*_{\mathrm{CW}}(S^4;G) = \begin{cases} G, & k = 0,4 \\ 0, & \text{otherwise.} \end{cases}$$

Example 18.2. Let us compute for $\mathbb{RP}^4$. We have

$$C_*^{\mathrm{CW}}: \qquad \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z}$$

$$C^*_{\mathrm{CW}}: \qquad G \xrightarrow{0} G \xrightarrow{\times 2} G \xrightarrow{0} G \xrightarrow{\times 2} G.$$

So in the case of $G = \mathbb{Z}$,

$$H^*_{\mathrm{Cw}}(\mathbb{RP}^4): \qquad \mathbb{Z} \qquad 0 \qquad \mathbb{Z}/2 \qquad 0 \qquad \mathbb{Z}/2.$$

Later we will see from the **universal coefficients theorem** that:

- "If you now $H_*(X)$, then you know $H^*(X)$."
- If $f: X \to Y$ gives isomorphisms $f_*: H_n(X) \to H_n(Y)$, then it also gives isomorphisms $f^*: H^n(Y;G) \to H^n(X;G)$.

For instance, if $f: X \to Y$ is a homotopy-equivalence then f^* is an isomorphism on cohomology.

For pairs, we define

$$C^*(X,A;G) = \mathrm{Hom}(C_*(X,A);G)$$

and define $H^n(X,A;G)$. The excision axiom and the universal coefficients theorem tells us that if (X,A) is a pair with $\mathrm{cl}(Z) \subseteq \mathrm{int}(A)$ then

$$H^n(X,A;G) \to H^n(X \setminus Z, A \setminus Z; G)$$

is an isomorphism.

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We are going to prove the universal coefficients theorem in stages. Begin with a chain complex (C_*, ∂). Assume that C_n is a free abelian group for each n. This is the case for the singular chain complex. Write $H_n = H_n(C_*, \partial)$. We then have a cochain complex $(C^*(G), \delta)$ with $C^n(G) = \mathrm{Hom}(C_n, G)$. Then $H^n(G) = H^n(C^*(G), \delta)$.

In general $H^n(G) \neq \mathrm{Hom}(H_n, G)$ as we have seen in the case of the real projective space, but there is a map $\Phi : H^n(G) \to \mathrm{Hom}(H_n, G)$, given by

$$[a] \mapsto ([\gamma] \mapsto a(\gamma)).$$

Is this well-defined? We have $\partial\gamma = 0$ and $\delta a = 0$, i.e., $a \circ \partial = 0$. If $\gamma' = \gamma + \partial\beta$, then

$$a(\gamma') = a(\gamma + \partial\beta) = a(\gamma) + (a \circ \partial)\beta = a(\gamma).$$

If $a' = a + \delta b = a + b \circ \partial$, then

$$a'(\gamma) = a(\Gamma) + (b \circ \partial)\gamma = a(\gamma) + 0.$$

19.1 Cohomology in terms of homology

Proposition 19.1. *(Still assuming that C_n is free abelian for all n,) Φ is surjective. Also Φ is injective if H_{n-1} is free abelian.*

Proof. Let us show surjectivity first. Let $\psi : H_n \to G$. Then we get a $\bar{\psi} : Z_n \to G$ such that $\bar{\psi}|_{B_n} = 0$. We now want to expend $\bar{\psi} : Z_n \to G$ to all of C_n, to get $\varphi : C_n \to G$ with $\varphi|_{Z_n} = \bar{\psi}$.

We have a short exact sequence

$$0 \longrightarrow Z_n \hookrightarrow C_n \twoheadrightarrow B_{n-1} \longrightarrow 0.$$

This short exact sequence splits, because B_{n-1} is a free abelian group, because $B_{n-1} \subseteq C_{n-1}$. Then $C_n = Z_n \oplus B_{n-1}$ and so we can extend $\bar{\psi}$ to φ by $\varphi(\gamma, \beta) = \bar{\psi}(\gamma)$.

Then $\varphi : C_n \to G$ with $\varphi|_{B_n} = \bar{\psi}|_{B_n} = 0$ is a cocycle, so we have $[\varphi] \in H^n(G)$. An $\Phi([\varphi]) = \psi$ because for all $[\gamma] \in H_n$,

$$\Phi([\varphi])[\gamma] = \varphi(\gamma) = \bar{\psi}(\gamma) = \psi[\gamma].$$

This just means that $\Phi([\varphi]) = \psi$.

Now assume that H_{n-1} is free abelian. We will show that Φ is injective. Consider $[a] \in H^n(G)$ with $\Phi([a]) = 0$, i.e., $a(\gamma) = 0$ for all $[\gamma] \in H_n$, i.e., $a|_{Z_n} = 0$. We want so show that $[a] = 0$, i.e., $a = \delta b$, i.e., $a = b \circ \partial$ for some $b : C_{n-1} \to G$.

$$0 \longrightarrow Z_n \longrightarrow C_n \twoheadrightarrow B_{n-1} \longrightarrow 0$$

Now $a|_{Z_n} = 0$, so a gives $b' : B_{n-1} \to G$, which will be given by $a(\gamma) = (b' \circ \partial)(\gamma)$. We need to extend $b' : B_{n-1} \to G$ to $b : C_{n-1} \to G$. But the short exact sequence

$$B_{n-1} \hookrightarrow Z_{n-1} \twoheadrightarrow H_{n-1}$$

splits and so we can extend from B_{n-1} to Z_{n-1}, and then extend Z_{n-1} to C_{n-1}. $\square$

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Suppose that every chain group in a chain complex (C_n, δ) is free. Define H^n comming from $\mathrm{Hom}(C_*, G) = C^*(G)$. We have shown that

$$H^n(G) \xrightarrow{\Phi} \mathrm{Hom}(H_n, G)$$

is surjective, and injective too if H_{n-1} is free. If H_{n-1} is not free, then what is $\ker \Phi$? To show injectivity, recall that we had to extend a map $B \to G$ to $Z \to G$ and then to $C \to G$.

We have an exact sequence

$$\mathrm{Hom}(Z_{n-1}, G) \xrightarrow{i_*} \mathrm{Hom}(B_{n-1}, G) \xrightarrow{b \mapsto b \circ \partial} H^n(G) \xrightarrow{\Phi} \mathrm{Hom}(H_n, G) \longrightarrow 0.$$

So we get

$$\ker \Phi = \frac{\mathrm{Hom}(B_{n-1}, G)}{i^* \mathrm{Hom}(Z_{n-1}, G)}.$$

20.1 The Ext groups

The general situation is, when we have an exact sequence

$$0 \longrightarrow B \xhookrightarrow{i} Z \longrightarrow H \longrightarrow 0,$$

with B adn Z free abelian groups, examining

$$T = \frac{\mathrm{Hom}(B, G)}{i^* \mathrm{Hom}(Z, G)}.$$

Example 20.1. Suppose $Z = \mathbb{Z}$ and $B = 3\mathbb{Z}$. Let $G = \mathbb{Z}$. In this case, $H = \mathbb{Z}/3$, and

$$T = \frac{\mathrm{Hom}(3\mathbb{Z}, \mathbb{Z})}{i^* \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})} = \frac{\mathbb{Z}}{3\mathbb{Z}} \cong \mathbb{Z}/3.$$

Proposition 20.2. *The group T depends only on H and G, and not on Z and B.*

Consider $\tilde{H}$ and a homorphism $f : \tilde{H} \to H$. Choose generators for $\tilde{H}$ and we get $\tilde{B} \hookrightarrow \tilde{Z} \twoheadrightarrow \tilde{H}$.

$$\begin{tikzcd}
\tilde{B} \arrow[r, hook, "\tilde{i}"] \arrow[d, "h"] & \tilde{Z} \arrow[r, two heads, "\tilde{\pi}"] \arrow[d, "g"] & \tilde{H} \arrow[d, "f"] \\
B \arrow[r, hook, "i"] & Z \arrow[r, two heads, "\pi"] & H
\end{tikzcd}$$

Fine $g : \tilde{Z} \to Z$ so that the square commutes (this is not unique), and this g is going to map $\ker \tilde{\pi}$ into $\ker \pi$. This gives a map $h : \tilde{B} \to B$, and this gives a map

$$h^* : \mathrm{Hom}(B, G) \to \mathrm{Hom}(\tilde{B}, G).$$

Moreover, $h^*(i^* \operatorname{Hom}(Z,G)) \subseteq \tilde{i}^* \operatorname{Hom}(Z,G)$. So we get a map

$$\bar{h}^* : \frac{\operatorname{Hom}(B,G)}{i^* \operatorname{Hom}(Z,G)} = T \to \frac{\operatorname{Hom}(\tilde{B},G)}{i^* \operatorname{Hom}(\tilde{Z},G)} = \tilde{T}.$$

We have made a choice of g here, but you can prove as an exercise that this $\bar{h}^*$ does not depend on the choice of g.

Now take $\tilde{H} = H$ with $f = \mathrm{id}$, but $\tilde{Z}$ and Z still different. Then from the diagram

$$\begin{tikzcd}
B \arrow[r, "i"] \arrow[d, "h'"] & Z \arrow[r, "\pi"] \arrow[d, "g'"] & H \arrow[d, "1"] & T \\
\tilde{B} \arrow[r, "\tilde{i}"] \arrow[d, "h"] & \tilde{Z} \arrow[r, "\tilde{\pi}"] \arrow[d, "g"] & H \arrow[d, "1"] & \tilde{T} \arrow[u] \\
B \arrow[r, "i"] & Z \arrow[r, "\pi"] & H & T \arrow[u]
\end{tikzcd}$$

we see that the composite $T \to \tilde{T} \to T$ is the identity on T and likewise $\tilde{T} \to T \to \tilde{T}$ is the identity on $\tilde{T}$. Therefore $T \cong \tilde{T}$. So we can write

$$T = T(H, G)$$

and further $f : H' \to H$ gives a map $f^* : T(H,G) \to T(H',G)$. This T is usually called Ext, and givng $B \to Z \to H$, we define

$$\operatorname{Ext}(H,G) = \frac{\operatorname{Hom}(B,G)}{i^* \operatorname{Hom}(Z,G)}.$$

Example 20.3. For example,

$$\operatorname{Ext}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n, \quad \operatorname{Ext}(\mathbb{Z},\mathbb{Z}) = 0, \quad \operatorname{Ext}(\mathbb{Z}, G) = 0.$$

Because $\operatorname{Ext}(H_1 \oplus H_2, G) \cong \operatorname{Ext}(H_1, G) \oplus \operatorname{Ext}(H_2, G)$, if H is finitely generated, then

$$\operatorname{Ext}(H, \mathbb{Z}) = \operatorname{Ext}(\mathbb{Z}^r \oplus (\text{torsion}), \mathbb{Z}) \cong (\text{torsion}).$$

Corollary 20.4. *Going to our situation with cohomology, there exists a short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}, G) \longrightarrow H^n(G) \xrightarrow{\Phi} \operatorname{Hom}(H_n, G) \longrightarrow 0.$$

In this particular case, you can prove that the sequence splits. So in fact,

$$H^n(G) \cong \operatorname{Hom}(H_n, G) \oplus \operatorname{Ext}(H_{n-1}, G).$$

Take $G = \mathbb{Z}$, and suppose that H_n, H_{n-1} are finitely generated. Then $\operatorname{Ext}(H_{n-1}, \mathbb{Z}) \cong \operatorname{Tor}(H_{n-1})$. If $H_n = \mathbb{Z}^s \oplus \operatorname{Tor}(H_n)$ then

$$H^n(\mathbb{Z}) \cong (\text{free part of } H_n) \oplus (\text{torsion part of } H_{n-1}).$$

21 October 21, 2016

21.1 Naturality and the universal coefficients theorem

A chain map $f : X \to Y$ induces a map $f_* : H_k(X) \to H_k(Y)$ between homology and $f^* : H^k(Y;G) \to H^k(X;G)$ between cohomology. Then we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X),G) \arrow[r] & H^n(X;G) \arrow[r] & \mathrm{Hom}(H_n(X);G) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, "\cong"] & \mathrm{Ext}(H_{n-1}(Y),G) \arrow[r] \arrow[u, "f^*"] & H^n(Y;G) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y);G) \arrow[r] \arrow[u, "(f_*)^*"] & 0 \arrow[u, "\cong"]
\end{tikzcd}$$

Naturally states that this diagram commutes. This can be checked by mapping elements. In particular, by the five lemma, if $f_* : H_k(X) \to H_k(Y)$ is an isomorphism for $k = n-1$ and $k = n$, then $f^* : H^n(Y;G) \to H^n(X;G)$ is an isomorphism too.

Let X be a CW complex and X^k the skeleta for $k \in \mathbb{N}$.

Proposition 21.1. *The inclusion $i : X^{n+1} \hookrightarrow X$ gives an isomorphism $i^* : H^n(X;G) \to H^n(X^{n+1};G)$ (ditto $H^k \to H^k$ for $k < n$).*

Corollary 21.2. *You can compute $H^k(X)$ from the CW cochain complex $\mathrm{Hom}(C_*^{\mathrm{CW}}(X)mG)$.*

21.2 Relative cohomology groups

Let us omit from now on (sometimes). The **relative cohomology** is defined as the cohomology of a cochain complex

$$\begin{aligned}
C^n(X,A) = \mathrm{Hom}(C_n(X,A),G) &= \mathrm{Hom}\Big(\frac{C_n(X)}{C_n(A)}, G\Big) \\
&= \mathrm{Ann}\, C_n(A)(\text{in } C^n(X) = \mathrm{Hom}(C_n(X),G)).
\end{aligned}$$

Because $C_n(X) = C_n(A) \oplus K$, we have a short exact sequence

$$0 \longrightarrow C^n(X,A) \longrightarrow C^n(X) \longrightarrow C^n(A) \longrightarrow 0.$$

Corollary 21.3. *There exists a long exact sequence in cohomology*

$$\begin{tikzcd}
\arrow[r] & H^n(X) \arrow[r, "i^*"] & H^n(A) \arrow[r, "\delta^*"] & H^{n+1}(X,A) \arrow[dll, "j^*"'] \\
& H^{n+1}(X) \arrow[r, "i^*"] & H^{n+1}(A) \arrow[r, "\delta^*"] & \cdots .
\end{tikzcd}$$

Example 21.4. Consider the space X which is a disjoint union of points, $X = \bigcup_{\alpha\in\mathcal{A}}$ point. Then $H_0(X) = \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}$, and also $H^0(X) = \prod_{\alpha\in\mathcal{A}} \mathbb{Z}$.

Why do we bother about cohomology? If X is a topological space, then the space

$$H^*(X) = \bigoplus_0^\infty H^i(X)$$

is a ring (with coefficients $\mathbb{Z}$ or R a commutative ring). There is the **cup product**

$$H^i \times H^j \to H^{i+j}; \quad (a, b) \mapsto a \smile b$$

given by

$$(a \smile b)[v_0, \dots, v_{i+j}] = a[v_0, \dots, v_i]b[v_i, \dots, v_{i+j}].$$

22 October 24, 2016

22.1 The cup product

As we have defined last time, there is a natural map $\smile: C^l(X) \times C^m(X) \to C^{l+m}(X)$, given by

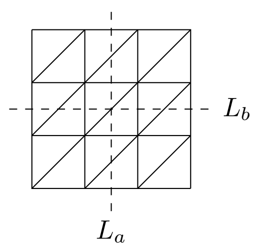
$$(a \smile b)(\sigma) = a(\sigma[e_0, \dots, e_l])b(\sigma[e_l, \dots, e_{l+m}]).$$

If $a \in C^0(X)$ is the one that is given by $a(\sigma) = 1$ for all σ, which is the generator of $H^0(X)$ if X is path-connected, then

$$(a \smile b)(\tau) = a(\tau[e_0])b(\tau[e_0, \dots, e_m]) = b(\tau).$$

So $a \smile b = b$.

Example 22.1. Consider the 2-dimensional torus, with the standard simplicial complex structure. Define the cochain a as $a(\sigma) = 1$ if σ crosses L_a and $a(\sigma) = 0$



otherwise. Likewise define b. Then

$$(a \smile b)[u, x, w] = a[u, x]b[x, w]$$

and it is nonzero just on one 2-simplex.

Lemma 22.2. *For $a \in C^l(X)$ and $b \in C^m(X)$,*

$$\delta(a \smile b) = \delta a \smile b + (-1)^l a \smile \delta b.$$

Proof. We simply evaluate it on a simplex:

$$\begin{aligned}
\delta(a \smile b)\tau = (a \smile b)\partial\tau = (a \smile b) \sum_{i=0}^{n+1} (-1)^i \tau[e_0, \dots, \hat{e}_i, \dots, e_{n+1}] \\
= \sum_{i=0}^{i} a\tau[e_0, \dots, \hat{e}_i, \dots, e_{l+1}] \cdot b\tau[e_{l+1}, \dots, e_{n+1}] \\
+ \sum_{i=l+1}^{n+1} (-1)^i a\tau[e_0, \dots, e_l] b\tau[e_l, \dots, \hat{e}_i, \dots, e_{n+1}] \\
= (\delta a)\tau[e_0, \dots, e_{l+1}] \cdot b\tau[e_{l+1}, \dots, e_{n+1}] + (-1)^l a\tau[e_0, \dots, e_l] \cdot (\delta b) \\
+ (-1)^l a\tau[e_0, \dots, e_l] \cdot (\delta b)\tau[e_l, \dots, e_{n+1}] \\
= \big((\delta a \smile b) + (-1)^l a \smile \delta b\big)\tau.
\end{aligned}$$

$\square$

This lemma tells us that (cocycle) $\smile$ (cocycle) = (cocycle) and also tells us that (cocycle) $\smile$ (coboundary) = (coboundary), i.e., $a \smile (\delta c) = (-1)^l \delta(a \smile c)$. So this is well-defined as a map $H^l(X) \times H^m(X) \to H^{l+m}(X)$.

What about $H^*(X, A)$? Suppose we have $A, B \subseteq X$ and $\mathrm{int}(A) \cup \mathrm{int}(B) = A \cup B$. Then we can define a "$\smile$" as $H^l(X, A) \times H^m(X, B) \to H^{l+m}(X, A \cup B)$. Start with chains $a \in C^l(X, A)$, i.e., $a \in C^l(X)$ and a annihilates $C_l(A)$, i.e., $a(\sigma) = 0$ for σ an l-simplex in A. So

$$(a \smile b)(\tau) = a[e_0, \ldots, e_l]b[e_l, \ldots, e_n] = 0$$

if $\tau(\Delta^n) \subseteq A$ or if $\tau(\Delta^n) \subseteq B$. This means that $(a \smile b)$ is in both in the annihilator of $C_n(A)$ and the annihilator of $C_n(B)$. So $(a \smile b)$ annihilates $C_n(A) + C_n(B) \subseteq C_n(X)$. Temporarily denote $C_n(A + B) = C_n(A) + C_n(B)$, and $C^*(X, A + B)$ be the annihilator of $C_*(A + B)$ in $C^*(X)$. So we have got

$$\smile : C^l(X, A) \times C^m(X, B) \to C^{l+m}(X, A + B).$$

Now we have a cochain map $C^*(X, A \cup B) \to C^*(X, A + B)$. This gives isomorphisms in cohomology groups, because (i) $C^*(A + B) \to C_*(A \cup B)$, by barycentric subdivision, gives isomorphisms in homology, (ii) by the five lemma, gives the same for $C_*(X, A + B)$, (iii) by the universal coefficient theorem, gives isomorphisms on cohomology. Using this isomorphism, we get

$$\smile : H^l(X, A) \times H^m(X, B) \to H^{l+m}(X, A \cup B).$$

23 October 26, 2016

I haven't mentioned last time, but the cup product is associative:

$$(a_1 \smile a_2 \smile \cdots \smile a_n)\sigma = a_1(\sigma[e_0, \dots, e_{l_1}]) \cdot a_2(\sigma[e_{l_1}, \dots, e_{l_1+l_2}]) \cdot \cdots \cdots .$$

23.1 Cohomology of Euclidean space minus the origin

The homology and cohomology of the complex projective space $\mathbb{CP}^n$ is

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots, 2n \\ 0 & \text{otherwise,} \end{cases} \quad H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots, 2n \\ 0 & \text{otherwise,} \end{cases}$$

by the universal coefficients theorem. Let $g_i \in H^*(\mathbb{CP}^n)$ be the generator in dimension $2i$ for $0 \leq i \leq k$.

Proposition 23.1. *$h = g_1 \in H^2(\mathbb{CP}^n)$ generates the ring, i.e., $g_i = h^i = h \smile \cdots \smile h$.*

Proposition 23.2. *For $\mathbb{RP}^n$ with $\mathbb{Z}/2$ coefficients,*

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \frac{(\mathbb{Z}/2)[h]}{\langle h^{n+1} \rangle}$$

for $h \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$.

We have

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = \mathbb{Z}, \quad H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}.$$

Let us look at the case $n = 1$. The map $\sigma : \Delta^1 \to \mathbb{R}$ that passes through 0 represents the generator $[\sigma] \in H_1(\mathbb{R}, \mathbb{R} \setminus 0)$. Then there is a cocycle $a \in C^1(\mathbb{R}, \mathbb{R} \setminus 0)$ with $a(\sigma) = 1$. This $[a]$ generates $H^1(\mathbb{R}, \mathbb{R} \setminus 0) = \mathbb{Z}$.

In $\mathbb{R}^n$, let $p_1 : \mathbb{R}^n \to \mathbb{R}$ be the projection map to the first coordinate. Let $A_1 = \{x_1 \neq 0\} \subseteq \mathbb{R}^n$. So $p_1 : (\mathbb{R}^n, A_1) \to (\mathbb{R}, \mathbb{R} \setminus 0)$. Pull back a by the projection and define $a_1 = p_1^*(a)$. Define p_i and a_i similarly for $i = 2, \dots, n$.

Then

$$b = a_1 \smile a_2 \smile \cdots \smile a_n \in C^n(\mathbb{R}^n, A_1 + \cdots + A_n)$$

and so $[b] \in H^n(\mathbb{R}, A_1 + \cdots + A_n) \cong H^n(\mathbb{R}, A_1 \cup \cdots \cup A_n)$.

Lemma 23.3. *$[b]$ is the generator for $H^n(\mathbb{R}^n, A_1 + \cdots + A_n) \cong \mathbb{Z}$.*

Proof. To prove this, it suffices to find a $[\gamma] \in H_n(\mathbb{R}, A_1 + \cdots + A_n)$ with $b(\gamma) = 1$. Note that we want γ to have boundary in $C_*(A_1) + \cdots + C_*(A_n)$, because it needs to be a cycle. So we try the n-dimensional cube $[-1, 1]$, divided up to simplices in an appropriate way. Explicitly, this can be written as

$$\gamma = \sum_{\pi \in S_n} (\operatorname{sgn} \pi)(\text{some simplex})$$

and if you evaluate at b it becomes 1 for one and 0 for all the other ones. $\square$

Let us now look at $\mathbb{RP}^n$, which is the equivalence classes of nonzero vectors in $\mathbb{R}^{n+1}$, denoted by $[x_0, \dots, x_n]$. Next time we are going to do a similar thing for $\mathbb{RP}^n$.

24 October 28, 2016

24.1 Cohomology of projective space

Let us compute the cohomology ring of $\mathbb{RP}^n$ with $\mathbb{Z}/2$ coefficients. The CW complex has cells in dimension $0, 1, \dots, n$ and the chain complex is given by

$$R \xrightarrow{\times 0} R \xrightarrow{\times 2} R \xrightarrow{\times 0} \cdots \longrightarrow R,$$

where R is a ring. For $\mathbb{Z}/2$, $2 = 0$ and so $H^k(\mathbb{RP}^n, \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq k \leq n$. Let us write $H^k(-)$ for $H^k(-; \mathbb{Z}/2)$ for a while. Where h is the generator for $H^1(\mathbb{RP}^n)$, we would want to show that $H^i(\mathbb{RP}^n)$ is generated by $g_i = h^i$.

For eeach i, the set $\{x_i = 0\} \subseteq \mathbb{RP}^n$ is a hyperplane and is isomorphic to $\mathbb{RP}^{n-1}$. So we get a decomposition

$$\mathbb{RP}^n = (\{x_0 = 0\}) \cup U,$$

where $U \cong \mathbb{R}^n$. For each i, let $A_i = \{x_i \neq 0\}$. Then $A_0 = U$, and $A_i \cong \mathbb{R}^n$ for each i. Also $U \cap A_1, \dots, U \cap A_n$ is just like last time.

We have

$$H^1(\mathbb{RP}^n) = \tilde{H}^1(\mathbb{RP}^n) = H^1(\mathbb{RP}^n, \text{point}) = H^1(\mathbb{RP}^n, A_i).$$

So there exists a class $[a_i] \in H^1(\mathbb{RP}^n, A_i)$ such that $[a_i] \mapsto h \in H^1(\mathbb{RP}^n)$ for $i = 1, \dots, n$. Now take the cup product

$$\begin{aligned}[a_1] \smile \cdots \smile [a_n] \in H^n(\mathbb{RP}^n, A_1 + \cdots + A_n) &\cong H^n(\mathbb{RP}^n, A_1 \cup \cdots \cup A_n) \\ &\cong H^n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \cong \mathbb{Z}/2.\end{aligned}$$

Because $[a_i]$ maps to h in $H^1(\mathbb{RP}^n)$, $[a_1] \smile \cdots \smile [a_n]$ corresponds to $h \smile \cdots \smile h$ in $H^n(\mathbb{RP}^n)$. So $h^n \neq 0$ if $[a_1] \smile \cdots \smile [a_n]$ is nonzero.

From last time, we have the "old" a_i, which now we denote by $\hat{a}_i \in H^1(\mathbb{R}^n, \{x_i \neq 0\})$. From $j : (U, U \cap A_i) \hookrightarrow (\mathbb{RP}^n, A_i)$, we get a map $j^* : H^1(\mathbb{RP}^n, A_i) \to H^1(\mathbb{R}^n, \mathbb{R}^n \cap A_i)$, and this is an isomorphism! So

$$j^* : [a_1] \smile \cdots \smile [a_n] \mapsto [\hat{a}_1] \smile \cdots \smile [\hat{a}_n] \neq 0$$

from last time. This shows that h^k is the generator of $H^k(\mathbb{RP}^n)$. In particular,

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[h].$$

Let us now look at $\mathbb{CP}^n$ with $\mathbb{Z}$ coefficients. We have

$$H^{2k}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

Let $g_1 = h$ be a generator for $H^2(\mathbb{CP}^n)$.

Proposition 24.1. *The element h^k generates $H^{2k}(\mathbb{CP}^n)$.*

Proof. Let $B_i = \{z_i \neq 0\} \subseteq \mathbb{C}^n$. Then

$$H^2(\mathbb{C}^n, B_i) \cong H^2(\mathbb{C}, \mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$$

and so there is a generator $[b_i]$. We want to show that $[b_1] \cup \cdots \cup [b_n]$ is a generator of $H^{2n}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}) \cong \mathbb{Z}$. This is essentially we did for the $\mathbb{RP}^n$ case. $\square$

25 October 31, 2016

Proposition 25.1. *If $[a] \in H^i(X;R)$ and $[b] \in H^j(X;R)$ and R is a commutative ring, then $[a] \smile [b] = (-1)^{ij}[b] \smile [a]$. In other words, $H^*(X;R) = \bigoplus H^i(X;R)$ is a **graded commutative ring**.*

Proof. Let $n = i + j$. Define $\psi : C_*(X) \to C_*(X)$ by, on an n-simplex σ,

$$\psi\sigma = \epsilon(n)\sigma[e_n, \dots, e_0]; \quad \epsilon(n) = (-1)^{n(n+1)/2}.$$

We claim that ψ is a chain map and that ψ is chain homotopic to the identity. Assuming this, we have a cochain map $\psi^* : C^*(X;R) \to C^*(X;R)$ and

$$\begin{aligned}
\psi^*(a \smile b)\sigma &= (a \smile b)(\psi\sigma) = \epsilon(n)(a \smile b)\sigma[e_n, \dots, e_n] \\
&= \epsilon(n)a(\sigma[e_n, \dots, e_{n-i}]) \cdot b(\sigma[e_j, \dots, 0]) \\
&= \epsilon(n)\epsilon(i)\epsilon(j)(\psi^* a)\sigma[e_j, \dots, e_n] \cdot (\psi^* b)\sigma[e_0, \dots, e_j] \\
&= (-1)^{ij}(\psi^*(b) \smile \psi^*(a))\sigma.
\end{aligned}$$

So in cohomology, $[(a \smile b)] = (-1)^{ij}([b] \smile [a])$. ☐

If 2 is invertible in R, e.g. $\mathbb{Z}/m$ for m odd or $\mathbb{Q}$, and $[a] \in H^i$ with i odd, then $[a] \smile [a] = -[a] \smile [a]$ and so $[a] \smile [a] = 0$. In contrast, in $H^*(\mathbb{RP}^n, \mathbb{Z}/2)$ and $h \in H^1$, all $h, h^2, \dots, h^n$ are nonzero.

25.1 Tensor products

Definition 25.2. Let V and W be F-vector spaces. We define $V \otimes_F W$ as the quotient vectors spaces X/Y, where X is the vector space with basis all symbols $v \otimes w$ for $v \in V$ and $w \in W$, and Y is the vector space generated by

$$\begin{gathered}
\lambda v \otimes w = v \otimes \lambda w = \lambda(v \otimes w), \\
(v + v') \otimes w = v \otimes w + v' \otimes w, \\
v \otimes (w + w') = v \otimes w + v \otimes w'.
\end{gathered}$$

Ditto if R is a commutative ring and V and W are R-modules, $V \otimes_R W$ is defined as generators and relations as above. If $R = \mathbb{Z}$ and $\lambda = 1 + \cdots + 1$,

$$(\lambda v) \otimes w = (v + \cdots + v) \otimes w = v \otimes w + \cdots + v \otimes w = \lambda(v \otimes w)$$

and so the first condition is not needed.

There is always a bilinear map $b : H \times G \to H \otimes G$ given by $(h, g) \mapsto h \otimes g$. This has the universal property in the sense that given a bilinear map $\psi : H \times G \to K$ there exists a $\tilde{\psi} : H \otimes G \to K$ such that $\psi = \bar{\psi} \circ b$.

Example 25.3. $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$.

25.2 Simple version of Künneth theorem

We want to look at the cohomology of $X \times Y$ for topological spaces X and Y. We'll only look at the case where $H^*(Y;R)$ is a free R-module of finite rank, where R is a commutative ring. Also we will assume that X is a finite simplicial complex.

There are maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. So there is a bilinear map of R-modules

$$H^*(X) \times H^*(Y) \to H^*(X \times Y); \quad (a_1, a_2) \mapsto p_1^*(a_1) \smile p_2^*(a_2).$$

We can regard this as a map $\kappa : H^*(X) \otimes_R H^*(Y) \to H^*(X \times Y)$.

Theorem 25.4 (Künneth theorem)**.** *If $H^*(Y)$ is free and finitely generated as an R-module, then κ is an isomorphism.*

This means that

$$\kappa : \bigoplus_{i+j=n} H^i(X) \otimes_R H^j(Y) \to H^n(X \times Y)$$

is an isomorphism.

Let me just sketch the proof. Fix an Y and let $L^n(X)$ be the left hand side and $R^n(X)$ be the right hand side. Clearly $\kappa : L^n(X) \to R^n(X)$ is an isomorphism if $X = \emptyset$ and $X =$ point. The main thing is that if $X = U \cup V$ is an open covering, and κ is an isomorphism for U and V and $U \cap V$, then κ is also an isomorphism for X too. This will come from some Mayer-Vietoris sequence and the five lemma. The right hand side is just the cohomology and so it is reasonable. For the left hand side, there is a problem is tensoring because tensoring does not necessarily preserve exactness. This is where finitely generated and free comes in.

26 November 2, 2016

Definition 26.1. A **manifold** of dimension n is a connected Hausdorff space such that for each $x \in M$ there is a neighborhood $U \ni x$ that is homeomorphic to $\mathbb{R}^n$.

Theorem 26.2. *If M is compact and orientable, then $H_n(M) \cong \mathbb{Z}$.*

Theorem 26.3 (Poicaré duality)**.** *If M is compact and orientable, then for every k, $H_k(M) \cong H^{n-k}(M)$.*

26.1 Orientation of manifolds

Note that for every $x \in M$,

$$H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}, \mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Definition 26.4. A generator of $H_n(M, M \setminus \{x\})$ is called a **local orientation** at x.

Note that this notion agrees with the usual notion. An element$u \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ can be represented by $\sigma : \Delta^n \to \mathbb{R}^n$ with $0 \notin \sigma(\partial\Delta^n)$. This is because σ induces a map $\partial\sigma : S^{n-1} \cong \partial\Delta^n \to \mathbb{R}^n \setminus \{0\} \cong S^{n-1}$.

Definition 26.5. The **orientation double cover** is defined as

$$\hat{M} = \{\mu_x \in H_n(M, M \setminus \{x\}) \text{ generator for } x \in M\}.$$

The topology is given by, for a ball $B \subseteq U \cong \mathbb{R}^n$ and $\mu_B \in H_n(M, M \setminus B)$, the basis

$$S_{\mu_B} = \{\mu_x : x \in B \text{ and } H_n(M, M \setminus B) \to H_n(M, M \setminus \{x\}) \text{ maps } \mu_B \text{ to } \mu_x\}.$$

Note that $\hat{M} \to M$ a 2-fold covering map.

Definition 26.6. A **orientation of** M is a continuous map μ such that $\pi \circ \mu = \mathrm{id}$. Equivalently, an orientation is a function $x \mapsto \mu_x \in H_n(M, M \setminus \{x\})$. If M has an orientation, M is called **orientable**.

Proposition 26.7. *Let M be a connected manifold. M is oriented if and only if $\hat{M}$ have two components.*

Proof. If $\hat{M}$ has two components, then $\hat{M} = M \amalg M$ because it is a 2-fold covering. Then it is easy to define μ. If μ is an orientation, then so is $-\mu$. So the images must be the two components of $\hat{M}$. □

Note that a 2-forld cover $N \to M$ is given by a homomorphism $\pi(M) \to \mathbb{Z}/2$. So for $\hat{M} \to M$, we can interpret this as

$$\phi : \pi_1(M) \to \mathbb{Z}/2; \quad \gamma \mapsto \begin{cases} 0 & \text{“}M\text{ is oriented along }\gamma\text{”} \\ 1 & \text{else.} \end{cases}$$

If $\pi_1(M) = 1$ then every 2-fold cover has 2 components and so M is orientable. Also, the n-torus $M = \mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$ is orientable. More generally, every Lie group is orientable because you can push-forward one local orientation to every point. On the other hand, the Möbius band is non-orientable. This is because $\hat{M} \to M$ induces a surjective $\pi_1(M) \to \mathbb{Z}/2$ is surjective.

Theorem 26.8. *Assume M is a manifold with dimension n, with an orientation μ. Then for all compact $K \subseteq M$, there exists a unique $\mu_K \in H_n(M, M \setminus K)$ so that for every $x \in K$, the map $\rho_x : H_n(M, M \setminus K) \to H_n(M, M \setminus \{x\})$ sends $\mu_K \mapsto \mu_x$.*

Corollary 26.9. *Let M be a compact manifold. Take $K = M$. It follows that there exists a $\mu_M \in H_n(M)$ gives a local orientation at each point.*

Definition 26.10. The element $\mu_M \in H_n(M)$ is called the **fundamental class** and is sometimes denoted $[M]$.

Corollary 26.11. *If M is compact and oriented, then $H_n(M) \neq 0$.*

For example, $\mathbb{RP}^2$ is not orientable because $H_2(\mathbb{RP}^2) = 0$. Also the Klein bottle $K = \mathbb{RP}^2 \# \mathbb{RP}^2$ is not orientable because $H_2(K)$. Finally, $\mathbb{RP}^2 \times \mathbb{RP}^2$ is not orientable because

$$H_4(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong H_2(\mathbb{RP}^2) \times H_2(\mathbb{RP}^2) \cong 0$$

by the Künneth formula.

27 November 4, 2016

I missed class. We proved Theorem 26.8.

28 November 7, 2016

Theorem 28.1 (Poincaré duality)**.** *If M is a compact oriented n-manifold, $H_k(M) \cong H^{n-k}(M)$.*

We proved that if M is orientable, $\mu_x \in H_n(M, M \setminus \{x\})$ is an orientation and $K \subseteq M$ is compact, then there exists a $\mu_K \in H_n(M, M \setminus K)$ such that $\mu_K \mapsto \mu_x$ for all $x \in K$. So if M is compact, we can take $K = M$ and get a "fundamental class" $\mu = \mu_M \in H_n(M)$ such that $\mu_M \mapsto \mu_x$ for all $x \in M$.

28.1 The cap product

We define the **cap product** $\frown\colon C_k \times C^l \to C_{k-l}$ as the following, for $k \geq l$:

$$(\sigma, \varphi) \mapsto \varphi(\sigma[e_0, \dots, e_l])\sigma[e_l, \dots, e_k].$$

Let us compute $(\partial\sigma) \frown \varphi$. We have

$$\begin{aligned}
(\partial\sigma) \frown \varphi &= \left(\sum_{i=0}^{n} (-1)^i \sigma[e_0, \dots, \hat{e}_i, \dots, e_n]\right) \frown \varphi \\
&= \sum_{i=0}^{l+1} (-1)^i \varphi(\sigma[e_0, \dots, \hat{e}_i, \dots, e_{l+1}])\sigma[e_{l+1}, \dots, e_n] \\
&\qquad + \sum_{i=l}^{n} (-1)^i \varphi(\sigma[e_0, \dots, e_l])\sigma[e_l, \dots, \hat{e}_i, \dots, e_n] \\
&= \sigma \frown (\delta\varphi) + (-1)^l \partial(\sigma \frown \varphi).
\end{aligned}$$

We can also write this as

$$\partial(\sigma \frown \varphi) = (-1)^l((\partial\sigma) \frown \varphi - \sigma \frown (\delta\varphi)).$$

From this, we again see that cycles times cocycles are cycles, boundaries times cocycles are boundaries, and cycles times coboundaries are boundaries.

Corollary 28.2. *There is a well-defined cap product $\frown\colon H_k(X) \times H^l(X) \to H_{k-l}(X)$.*

28.2 Unimodular pairings

Theorem 28.3 (Poincaré duality)**.** *If M is a compact oriented n-dimensional manifold, and $\mu_M \in H_n(M)$ is a fundamental class, then the map*

$$D : H^l(M) \to H_{n-l}(M); \quad \varphi \mapsto \mu_M \frown \varphi$$

is an isomorphism for all l.

Note that clearly $\frown$ is bilinear. More interesting is the relation between $\frown$ and $\smile$. Recall that there is a map $E : H^k(X) \times H_k(X) \to \mathbb{Z}$. Let us use $\langle \bullet, \bullet \rangle$ for this evaluation map. Then

$$\langle b, \sigma \frown a \rangle = \langle a \smile b, \sigma \rangle = \langle a, \sigma[e_0, \dots, e_l] \rangle \langle b, \sigma[e_l, \dots, e_n] \rangle.$$

Definition 28.4. If V_1, V_2 are free abelian groups of finite rank, a bilinar map $Q : V_1 \times V_2 \to \mathbb{Z}$ is a **unimodular paring** if the corresponding map $V_1 \to \mathrm{Hom}(V_2, \mathbb{Z})$ is an isomorphism.

The univeral coefficients theorem states that

$$E : \frac{H^k(X)}{\text{Torsion}} \to \mathrm{Hom}(H_k(X), \mathbb{Z})$$

is an isomorphism if $H_*(X)$ are finitely generated. In other words, there is a unimodular paring between $H^k(X)/\mathrm{Tor}$ and $H_k(X)/\mathrm{Tor}$. If Poincaré duality is true, then

$$H^k/\mathrm{Tor} \times H^{n-k}/\mathrm{Tor} \to \mathbb{Z}; \quad (b, a) \mapsto \langle b, D(a) \rangle = \langle a \smile b, \mu_M \rangle$$

is unimodluar.

29 November 9, 2016

Recall that $\frown: C_k(X) \times C^l(X) \to C_{k-l}(X)$ is defined by

$$(\sigma, a) \mapsto a(\sigma[e_0, \dots, e_l])\sigma[e_l, \dots, e_k].$$

This induces two kinds of maps:

$$\frac{C_k(X)}{C_k(A)} \times C^l(X) \to \frac{C_{k-l}(X)}{C_{k-l}(A)}, \quad \frac{C_k(X)}{C_k(A)} \times \operatorname{Ann}(C_l(A)) \to C_{k-l}(X).$$

Then there are maps

$$H_k(X, A) \times H^l(X) \to H_{k-l}(X, A), \quad H_k(X, A) \times H^l(X, A) \to H_{k-l}(X).$$

Let $H, K \subseteq G$ be abelian groups. Then there are the following exact sequences

$$0 \longrightarrow H \cap K \longrightarrow H \oplus K \longrightarrow H + K \longrightarrow 0$$
$$0 \longrightarrow \tfrac{G}{H \cap K} \longrightarrow \tfrac{G}{H} \oplus \tfrac{G}{K} \longrightarrow \tfrac{G}{H+K} \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ann}(H + K) \longrightarrow \operatorname{Ann}(H) \oplus \operatorname{Ann}(K) \longrightarrow \operatorname{Ann}(H \cap K) \longrightarrow 0.$$

Let $A, B \subseteq X$ be open subset and let $H = C_*(A)$, $K = C_*(B)$, and $G = C_*(X)$. Then these short exact sequences give Mayer-Vietoris sequences. The first one is the ordinary Mayer-Vietoris sequence. The second one gives an "upside-down" Mayer-Vietoris sequence:

$$H_k(X, A \cap B) \to H_k(X, A) \oplus H_k(X, B) \to H_k(X, A \cup B) \to H_{k-1}(X, A \cap B) \cdots.$$

The third one gives the upside-down Mayer-Vietoris sequence for cohomology:

$$H^k(X, A \cup B) \to H^k(X, A) \oplus H^k(X, B) \to H^k(X, A \cap B) \to H^{k+1}(X, A \cup B) \cdots.$$

29.1 Proof of Poincaré duality for smooth manifolds

Let M be a smooth, compact, oriented, n-dimensional manifold. A **c-pair** is a pair (U^+, U) where $U^+ \subseteq M$ is open, $U \subseteq M$ is closed, and $(U^+, U) \cong (B_1^\circ, B_{1/2})$. A **simple cover** of M means a collection of c-pairs (U_i^+, U_i) for $i = 1, \dots, N$ such that

(a) $M = \bigcup_{i=1}^N U_i$,

(b) for all $I = \{i_1, \dots, i_k\}$, then $U_I^+ = U_{i_1}^+ \cap \cdots \cap U_{i_k}^+$ and $U_I = U_{i_1} \cap \cdots \cap U_{i_k}$ either are both empty sets or form a c-pair.

Sets we are interested in are the sets U_i^+, U_i, their intersections U_I^+, U_I, and *their* unions

$$V^+ = U_{I_1}^+ \cup \cdots \cup U_{I_r}^+, \quad V = U_{I_1} \cup \cdots \cup U_{I_r}.$$

In this case, $H_n(V^+, V^+ \setminus V) \cong H_n(M, M \setminus V)$, which is true by excision, contains a fundamental class μ_V. We have cap product maps

$$\begin{aligned} H_n(V^+, V^+ \setminus V) \times H^k(V^+, V^+ \setminus V) &\to H_{n-k}(V^+), \\ H_n(V^+, V^+ \setminus V) \times H^k(V^+) &\to H_{n-k}(V^+, V^+ \setminus V). \end{aligned}$$

Using μ_V, we get

$$D : H^k(V^+, V^+ \setminus V) \to H_{n-k}(V^+), \quad D' : H^k(V^+) \to H_{n-k}(V^+, V^+ \setminus V).$$

Proposition 29.1. *D and D' are isomorphisms.*

Proof. We prove by induction on r. For $r = 1$, we can check that thing work out fine because the pair is isomorphic to $(B_1^\circ, B_1^\circ \setminus B_{1/2})$.

Now suppose that this is true for r. Let us now prove for $r + 1$. Let $Z = U \cup V$ and $Z = U^+ \cup V^+$ where V is the union of r balls. Then $U \cap V = W$ and $U^+ \cap V^+ = W^+$ are unions of r balls. So by the induction hypothesis, D_U, D_V, D_W and D'_U, D'_V, D'_W are all isomorphisms. We want to show that D_Z and D'_Z are isomorphisms. Let us temporarily write $H^k(X|Z) = H^k(X, X \setminus Z)$.

We have two Mayer-Vietoris sequences. The first one is

$$H^k(M|U \cap V) \longrightarrow H^k(M|U) \oplus H^k(M|U) \longrightarrow H^k(M|U \cup V) \cdots.$$

Applying excision, we get

$$H^k(U^+ \cap V^+|U \cap V) \to H^k(U^+|U) \oplus H^k(V^+|V) \to H^k(U^+ \cup V^+|U \cup V) \cdots.$$

There is also the ordinary Mayer-Vietoris sequence, and together we can write

$$\begin{tikzcd} H^k(W^+|W) \arrow[r] \arrow[d, "D_W"] & H^k(U^+|U) \oplus H^k(V^+|V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H^k(Z^+|Z) \arrow[r] \arrow[d, "D_Z"] & \cdots \\ H_{n-k}(W^+) \arrow[r] & H_{n-k}(U^+) \oplus H_{n-k}(V^+) \arrow[r] & H_{n-k}(Z^+) \arrow[r] & \cdots \end{tikzcd}$$

After verifying that this is a commutative diagram, we conclude from the five lemma that D_Z is an isomorphism. We can also do this for D'.

In the case $V = V^+ = M$, we get the isomorphisms $D : H^k(M) \to H_{n-k}(M)$. ∎

30 November 11, 2016

Let (U_i^+, U_i) be a simple cover of M, an compact orientable manifold. Let $(V^+, V) = \bigcup(U_I^+, U_I)$. One can inductively show using induction, the Mayer-Vietoris sequence, and five lemma that $H_*(V) \to H_*(V^+)$ and $H^*(V^+) \to H^*(V)$ are isomorphisms.

Last time we showed that if V is simple, then

$$H^k(V^+, V^+ \setminus V) \to H_{n-k}(V^+), \quad H^k(V^+) \to H_{n-k}(V^+, V^+ \setminus V)$$

are isomorphisms. Using what we have remarked right before, we get isomorphisms

$$H^k(M, M \setminus V) \to H_{n-k}(V), \quad H^k(V) \to H_{n-k}(M, M \setminus V)$$

by excision.

30.1 Alexander duality

We now want to compare $H^l(K)$ to $H_{n-l}(M, M \setminus K)$ for $K \subseteq M$ compact. We know this when K is a union of finitely many balls. But K might be unpleasant, and to deal with this, we look at the direct limit. Fix a compact set K, and look at all the neighborhoods of K. For $K \subseteq V_2 \subseteq V_1$, consider the maps

$$H^l(V_1) \to H^l(V_2), \quad H_k(M, M \setminus V_1) \to H_k(M, M \setminus V_2).$$

Then we might start looking at the direct limits of these systems.

Formally, let $K \subseteq \mathbb{R}^n$ be a compact set. Consider (V^+, V) a simple pair, with V a neighborhood of K. We have

$$H^l(V) \cong H^l(V^+) \cong H_{n-l}(\mathbb{R}^n, \mathbb{R}^n \setminus V) \cong \tilde{H}_{n-l-1}(\mathbb{R} \setminus V).$$

Is it true that $H^l(K) \cong \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus K)$? The idea is to consider $\mathcal{A} = \{\text{all nbds of } U \text{ of } K\}$. We have

$$\varinjlim_{U \supseteq K} \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus U) = \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus K)$$

becasue the union of $\mathbb{R}^n \setminus U$ is $\mathbb{R}^n \setminus K$ and every compact set in $\mathbb{R}^n \setminus K$ lies in $\mathbb{R}^n \setminus U$ for some U. Furthermore, we have

$$\varinjlim_{\text{simple } V \supset K} \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus K) = \varinjlim_{U \supset K} \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus V)$$

because for each U there exists a simple (V^+, V) with $V \subseteq U$.

On the other side of the duality isomorphism, we also similarly have

$$\varinjlim_{U \supseteq K} H^l(U) = \lim_{\text{simple } V^+ \supset K} H^l(V^+).$$

However, it may not be true that

$$\varinjlim_{U \supseteq K} H^l(U) \cong H^l(K).$$

Certainly, this is an isomorphism if there exists a neighborhood U_0 of K that deformations retracts to K.

Definition 30.1. A compact set $K \subseteq \mathbb{R}^n$ is called **taut** if

$$\varinjlim_{U \supset K} H^l(U) \cong H^l(K)$$

for every l.

So for taut K, we have an isomorphism

$$H^l(K) \cong \varinjlim_{U} H^l(U) \cong \varinjlim_{V^+} H^l(V^+) \cong \varinjlim_{V} \tilde{H}_{n-l-1}(\mathbb{R} \setminus V) \cong \tilde{H}_{n-l-1}(\mathbb{R} \setminus K).$$

If K is compact and locally contractible, then K is taut. Using this, if K is the image of any $S^{n-1} \to \mathbb{R}$. Thus

$$\mathbb{Z} \cong H^{n-1}(S^{n-1}) \cong \tilde{H}_0(\mathbb{R} \setminus S^{n-1}).$$

Therefore $\mathbb{R}^n \setminus \psi(S^{n-1})$ has two path components, and thus two connected components.

31 November 14, 2016

Last time, we showed that if $K \subseteq \mathbb{R}^n$ is compact, then then K being taut implies that

$$H^k(K) \cong \varinjlim_U H^l(U) \cong \tilde{H}_{n-l-1}(\mathbb{R}^n \setminus K).$$

If K is homeomorphic to S^{n-1}, then we see that $\mathbb{R}^n \setminus K$ has two connected components. If $n = 3$ and $K \cong S^1$, e.g., K is a knot, then

$$\mathbb{Z} \cong H^1(K) \cong H_1(\mathbb{R}^3 \setminus K).$$

Even if K is a "wild" knot, this identity holds.

31.1 Traditional Alexander duality

The traditional version of Alexander duality is given by

$$\tilde{H}^l(K) \cong \tilde{H}_{n-l-1}(S^n \setminus K)$$

by adding a point at infinity. Again, this holds when K is taut and nonempty. They are almost equivalent except for $\ell = 0$, where $H^l(K) = \tilde{H}^l(K) \oplus \mathbb{Z}$ and $\tilde{H}_{n-1}(\mathbb{R}^n \setminus K) \cong \tilde{H}_{n-1}(S^n \setminus K) \oplus \mathbb{Z}$.

Lemma 31.1. *$\tilde{H}_{n-1}(\mathbb{R} \setminus K) = \tilde{H}_{n-1}(S^n \setminus K) \oplus \mathbb{Z}$ and the $\tilde{H}_m$s are the same for $m \neq n-1$.*

Proof. Let $x \in K$ and choose $R > 0$ so that $K \subseteq B^n_R = B^n_R(x)$. Then $\tilde{H}_{n-1}(\mathbb{R}^n \setminus K) = \tilde{H}_{n-1}(B^n_R \setminus K)$. Note that $S^{n-1}_r = \partial B^n_R$ is a retract of $B^n_R \setminus K$, given by projecting to the boundary by the ray going out from x.

$$\begin{tikzcd} S^{n-1}_R \arrow[r, hook, shift left, "i"] & B^n_R \setminus K \arrow[l, shift left, "r"] \end{tikzcd}$$

We have a long exact sequence of pairs

$$\begin{tikzcd} \tilde{H}_{n-1}(S^{n-1}_R) \arrow[r, shift left, "i_*"] & \tilde{H}_{n-1}(B^n_R \setminus K) \arrow[l, shift left, "r_*"] \arrow[r] & H_{n-1}(B^n_R \setminus K, S^{n-1}_R) \arrow[r] & 0. \end{tikzcd}$$

This splits, and so we get

$$\begin{aligned} \tilde{H}_{n-1}(\mathbb{R}^n \setminus K) &= \tilde{H}_{n-1}(B^n \setminus K) = H_{n-1}(B^n \setminus K, S^{n-1}) \oplus \mathbb{Z} \\ &= \tilde{H}_{n-1}\Big(\frac{B^n \setminus K}{S^{n-1}}\Big) \oplus \mathbb{Z} = \tilde{H}_{n-1}(S^n \setminus K) \oplus \mathbb{Z}. \end{aligned}$$

□

Sometimes, when you have a notion of homology, and K is compact, you define $\tilde{H}^l(K)$ as simply

$$\tilde{H}^l(K) = \tilde{H}_{n-k-1}(S^n \setminus K)$$

after embedding K in S^n. This look like nuts, but actually happens in algebraic topology.

31.2 Cohomology with compact support

Let us now return to the "other" duality:

$$H^l(M, M \setminus V) \cong H_{n-l}(V^+) \cong H_{n-l}(V)$$

for compact sets V arising in simple pairs. Let $\mathcal{A}$ be all compact ses $K \subseteq M$ directed by inclusions $K_1 < K_2 \Leftrightarrow K_1 \subseteq K_2$. (This is interestint only if M is non-compact.) These V's from simple pairs are cofinal in $\mathcal{A}$. Thus

$$\varinjlim_{V} H_{n-l}(V) = \lim_{K\in\mathcal{A}} H_{n-l}(K) = H_{n-l}(M).$$

On the other hand of the duality, we have

$$\varinjlim_{V} H^l(M, M \setminus V) = \varinjlim_{K\in\mathcal{A}} H^l(M, M \setminus K) \neq H^l(M)$$

in general. For instance, when $M = \mathbb{R}^n$, we have

$$\varinjlim_{K} H^l(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{N\to\infty} H^l(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_N) = \begin{cases} \mathbb{Z} & l = n \\ 0 & \text{otherwise.} \end{cases}$$

Definition 31.2. The **cohomology with compact support** is defined as

$$H^l_c(X) = \varinjlim_{K \text{ compact}} H^l(X, X \setminus K).$$

Why are these called cohomology with compact support? The elements of $H^l_c(X)$ are equivalence classes of elements $[a] \in H^l(X, X \setminus K)$, where $a \in C^l(X, X \setminus K) = \mathrm{Ann}(C^l(X \setminus K))$. If you know de Rham cohomology, then these are cohomology rising from differential forms with compact support.

Anyways, getting back to the duality thing, we get

Theorem 31.3 (Poincaré duality)**.** *If M is n-dimensional and oriented, then*

$$D : H^l_c(M) \to H_{n-l}(M)$$

is an isomorphism.

For example, for $M = \mathbb{R}^n$, as we have seen,

$$\text{l.h.s.} = \begin{cases} \mathbb{Z} & l = n \\ 0 & l \neq n, \end{cases} \qquad \text{r.h.s.} = \begin{cases} \mathbb{Z} & n - l = 0 \\ 0 & n - l \neq 0. \end{cases}$$

Another example is when M is a compact manifold with boundary. If M is a smooth manifold, then M has a collar, i.e., $M \supset C \cong \partial M \times [0, 1)$. Let $M_\epsilon = (M \setminus C) \cup (\partial M \times [\epsilon, 1))$ and $M^\circ = M \setminus \partial M = \bigcup_\epsilon M_\epsilon$. Then

$$H^l_c(M^\circ) = \varinjlim_{\epsilon\to 0} H^l(M^\circ, M^\circ\ M_\epsilon) = H^l(M, \partial M).$$

If M is oriented, then $H^l_c(M^\circ) = H_{n-l}(M^\circ) = H_{n-l}(M)4$. Therefore we get

Theorem 31.4 (Poincaré duality for manifolds with boundary)**.** *If M is an oriented manifold with boundary, then*

$$H^l(M, \partial M) = H_{n-l}(M) \quad \text{and} \quad H^l(M) = H_{n-l}(M, \partial M).$$

32 November 16, 2016

Let us now talk about homotopy. Consider triples of spaces (A, B, C) and define morphisms of triples $f : (A, B, C) \to (A', B', C')$ as a map $f : A \to A'$ with $f(B) \subseteq B'$ and etc. A **homotopy** is a map

$$F : (A, B, C) \times I \to (A', B', C'),$$

i.e., $f_t : (A, B, C) \to (A', B', C')$.

32.1 Homotopy groups

Let us write (X, x_0) for $(X, \{x_0\})$. Denote

$$[(X, A), (Y, B)] = \text{set of homotopy classes of maps}.$$

Often, when the base point is implicit, we are going to write

$$[X, Y] = \text{based homotopy classes} = [(X, x_0), (Y, y_0)].$$

For instance, we define the **fundamental group** as

$$\pi_1(X, x_0) = [(S^1, s), (X, x_0)] = [(I, \{0, 1\}), (X, x_0)].$$

We know want to define higher dimensional analogues of this. We define the **nth homotopy group**

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)] = [(I^n, \partial I^n), (X, x_0)].$$

The two definitions are equivalent because $I^n/\partial I^n \cong S^n$.

For a based pair (X, A, x_0) with $x_0 \in A$, we now want to define its homotopy group. The first idea is to consider the homotopy class of maps

$$(I^n, \partial I^n, \tau) \to (X, A, x_0)$$

where $\tau \in \partial I^n$ is any point. But instead, we can write $\partial_I^n = I^{n-1} \cup J^{n-1}$ where

$$I^{n-1} = \{(t_1, \dots, t_n) \in I^n : t_n = 0\}, \quad J^{n-1} = \{t_n = 1\} \cup (\partial I^{n-1} \times I).$$

Now let us define

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)].$$

Write, for f a map, $[f]$ its homotopy class. Note that there is a map

$$\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0); \quad [f] \mapsto [f|_{\partial I^{n-1}}].$$

Let's look at some special cases. For $n = 1$, $I^0 = \{0\}$ and $J^0 = \{1\}$. So $\pi_1(X, A, x_0)$ is the homotopy classes $f : I \to X$ with $f(0) \in A$ and $f(1) = x_0$.

If $X = \mathbb{R}^2$ then $A \subseteq X$ has n path-components, then there are n homotopy classes.

It is not obvious what $\pi_0(X, A, x_0)$ should be, because we don't know what J^{-1} is. But we do know what $\pi_0(X, x_0)$ is. It is

$$\pi_0(X, x_0) = [(I^0, \partial I^0), (X, x_0)] = [(\mathrm{pt}, \emptyset), (X, x_0)] = [\mathrm{pt}, X]$$

which is the path components of X.

In general $\pi_n(X, x_0)$ is a group if $n \geq 1$ and $\pi_n(X, A, x_0)$ is also a group if $n \geq 2$. Given $f_i : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ for $i = 1, 2$, we define $[f_1] + [f_2] = [f_1 + f_2]$, where

$$(f_1 + f_2)(t_1, \ldots, t_n) = \begin{cases} f_1(2t_1, t_2, \ldots, t_n) & t_1 \leq 1/2 \\ f_2(2t_1 - 1, t_2, \ldots, t_n) & t_1 \geq 1/2. \end{cases}$$

This map is continuous because the values are all x_0 where the two functions f_1 and f_2 meet. Note that when $n = 1$, this doesn't work.

$$\boxed{f_1} + \boxed{f_2} = \boxed{f_1 \mid f_2}$$

Figure 2: Addition law for $\pi_n(X, A, x_0)$

In the non-relative case, this definition works as $\pi_n(X, x_0) = \pi_n(X, x_0, x_0)$ for $n \geq 2$. For $n = 1$, you can define by concatenating paths.

33 November 18, 2016

33.1 Some properties of homotopy groups

Last time we have given a group structure on $\pi_n(X, A, x_0)$ for $n \geq 2$, or $A = \{x_0\}$ and $n = 1$. The identity element $z : I^n \to X$ is given by $z(I^n) = \{x_0\}$ and the inverse can be given by

$$(-f) : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0); \quad (t_1, \dots, t_n) = f(-t_1, t_2, \dots, t_n).$$

Then $[f] + [z] = [f]$, $[f] + [-f] = [z]$, and associativity can be easily checked.

Also note that $\pi_n(X, A, x_0)$ is abelian if $n \geq 3$. Given $f : (I^n, \partial I_n, J^{n-1}) \to (X, A, x_0)$, define

$$f^r(t_1, t_2, \dots, t_n) = f(1 - t_1, 1 - t_2, t_3, \dots, t_n).$$

Note that this map preserves ∂I^n and J^{n-1}. Also if $n \geq 3$, then f and f^r are homeomorphic because $I^n \cong D^2 \times I^{n-2}$ and then you can rotate. Because $(f_1 + f_2)^r = f_2^r + f_1^r$, we get $[f_1] + [f_2] = [f_1 + f_2] = [f_2] + [f_1]$. Note that π_n is a functor from triples (X, A, x_0) to groups $(n \geq 2)$ or to pointed sets $(n = 1)$. This because $h : (X, A, x_0) \to (Y, B, x_0)$ gives a $h_* : \pi_n \to \pi_n$ by taking the composition $[f] \mapsto [h \circ f]$. You can check that the resulting map is a group homomorphism, or sends the preferred point to the preferred point.

Generally, the inclusion $A \hookrightarrow X$ gives $i : (A, x_0) \to (X, x_0)$, i.e., $i : (A, x_0, x_0) \to (X, x_0, x_0)$. There is also the map $j : (X, x_0, x_0) \to (X, A, x_0)$. So we have so far

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0).$$

While ago, we pointed out that there is a natural way to get $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$. So we can carry on.

$$\begin{aligned}
&\longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \\
&\longrightarrow \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \\
&\qquad\qquad\qquad\qquad \vdots \\
&\longrightarrow \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)
\end{aligned}$$

On this sequence, "exactness" makes sense because each set has a preferred element $e_i \in S_i$, and so we can understand ker as the inverse image of the preferred element.

Definition 33.1. Let (X, A) be a pair. We say that (X, A) satisfies the **homotopy extension property** if given any homotopy $G : I \times A \to Y$ and given any $f_0 : X \to Y$ with $f_0|_A = g_0$, there exists an extension $F : I \times X \to Y$ to all of $I \times X$ with $F|_{I \times A} = G$ and $F|_{0 \times X} = f_0$.

Lemma 33.2. *(X, A) has the homotopy extension property if and only if $Z = (I \times A) \cup (0 \times X) \subseteq I \times X$ is a retract of $I \times X$.*

34 November 21, 2016

We were looking at the homotopy extension property. For a pair (X, A), this has the homotopy extension property if and only if there is a retract $I \times X \to Z$ where $Z = I \times A \cup \{0\} \times X$. For example, $(X, A) = (I^n, \partial I^n)$ has the homotopy extension property.

More generally, if X is obtained form A by attaching a cell, then (X, A) has the homotopy extension property. This can be proved using the retraction $I \times I^n \to 0 \times I^n \cup I \times \partial I^n$. Likewise, attaching many n-cells simultaneously also gives a pair with homotopy extension property.

34.1 Exactness of the long exact sequence

Recall that we have a sequence

$$\xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A)$$
$$\xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \xrightarrow{j_*} \cdots .$$

Let us first show that $\partial \circ j_* = 0$. This is actually clear because the restriction of any thing in $\pi_n(X, A)$ that comes from $\pi_n(X)$, to A is the constant map. Now suppose that $[g] \in \ker \partial$. Then $g : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ and $g|_{I^{n-1} \times 0} : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ is homotopic to the constant map. Using the homotopy extension property on $(I^n, \partial I^n)$, we can extend this homotopy to get a homotopy from g to a map that maps the boundary to x_0. This shows that $\ker \partial \subseteq \operatorname{im} j_*$. The remaining parts can be proven similarly.

One important thing to note is that excision fails with homotopy groups. Here is an example. Let $X = S^1 \vee S^1$ and let $A = S^1$ with the base point x_0 being the point where the two circles are glued. Then

$$\pi_1(X, A) = (\mathbb{Z} * \mathbb{Z})/\mathbb{Z}$$

whereas if we let $Z = A \setminus (-\epsilon, \epsilon)$ then $\pi_1(X \setminus Z, A \setminus Z) \cong \pi_1(S^1, x_0) = \mathbb{Z}$.

Because excision fails, we can't really compute anything. So the tools we have are:

- simplicial approximation
- smooth approximation

If we have simplicial complexes $K = (V, W)$, $L = (W, T)$ and a simplicial map $\varphi : K \to L$, this gives a map $|\varphi| : |K| \to |L|$. Given finite simplicial complexes and the continuous maps $f : |K| \to |L|$, you can show that there exists a simplicial map $\varphi : K^{[n]} \to L$ such that $|\varphi|$ and f are homotopic as maps from $|K^{[n]}| = |K|$ to $|L|$.

35 November 28, 2016

Today we are going to talk about smooth approximation. If P is a smooth manifold, then Whitney's theorem says that P embeds in $\mathbb{R}^N$ smoothly, for some N.

35.1 Smooth approximation

Proposition 35.1. *Suppose—for simplicity—P is a compact manifold, and so is Q. Then any continuous map $f : P \to Q$ is homotopic to a smooth one. Furthermore, if f_0, f_1 are smooth and homotopic by a continuous homotopy $F : I \times P \to Q$, then there is also a smooth homotopy $\tilde{F}$.*

So if we denote the set of homotopy classes of maps $P \to Q$ by $[P, Q]$, then in a nutshell, $[P, Q] = [P, Q]_{\text{smooth}}$. In the proof, we are going to use the tubular neighborhood theorem, that states that if $Q \subseteq \mathbb{R}_N$ is an embedding of a compact smooth manifold, then there exists a $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ then

$$U_\epsilon = \{x \in \mathbb{R}^N : d(x, Q) \leq \epsilon\}$$

deformation retracts to Q.

Proof of proposition 35.1. We note that f can be approximated to a polynomial $\tilde{f}$ such that for all $x \in P \subseteq \mathbb{R}^{N_1}$, we have $|\tilde{f}(x) - f(x)| \leq \epsilon$. So $\tilde{f} : P \to U_\epsilon$ and $\tilde{f} \simeq f$ maps to U_ϵ. Then we can use the deformation retraction $r : U_\epsilon \to Q$ so that $r \circ \tilde{f} : P \to Q$ gives $r \circ \tilde{f} \simeq f$.

We can use the same trick for the second part, on $F : I \times P \to Q$. □

There are variants of this proposition. For instance, we can look at based maps $(P, p_0) \to (Q, q_0)$ and require that $\tilde{f}$ is smooth on $P \setminus \{p_0\}$. More generally, we can show that

$$[(P, A), (Q, B)] = [(P, A), (Q, B)]_{\text{smooth on } P \setminus A}.$$

Definition 35.2. For a smooth map $f : P \to Q$, a point $q \in Q$ is a **regular value** if for all $p \in f^{-1}(q)$, $df_p : T_pP \to T_qQ$ is onto.

35.2 Computing homotopy groups

Theorem 35.3 (Sard)**.** *Regular values are* dense *in Q. If f is proper, then regular values are open and dense in Q.*

For instance, if $\dim P < \dim Q$, then q regular means $f^{-1}(q) = \emptyset$. Then Sard says that f is not onto if $\dim P < \dim Q$.

Corollary 35.4. *If $m < n$, then $[S^m, S^n] = 0$ (i.e., every f is null-homotopic). Ditto for based maps: $\pi_m(S^n) = 0$.*

Proof. We have $[S^m, S^n] = [S^m, S^n]_{\text{smooth}}$. Also, a smooth map $f : S^m \to S^n$ is not onto. So there exists a $q \in S^n$ that is not in the image of f. So with an abuse of notation, we can write $f : S^m \to S^n \setminus q \cong \mathbb{R}^n$ and $\mathbb{R}^n$ is contractible. Thus f is homotopic to a constant map. $\square$

What if $m \geq n$? What can we say about $\pi_m(S^n)$? We can think $S^m = \mathbb{R}^m \cup \infty$ where ∞ is the base-point. Then we are looking at $f : \mathbb{R}^m \to S^n$ (assume smooth), with $f(x) \to \infty$ as $x \to \infty$.

Here is how we are going to approach this problem. Regular values are open and dense in $\mathbb{R}^n$, and so without loss of generality we may assume that 0 is a regular value (else translate the function). Then $M = f^{-1}(0)$ is a smooth manifold in $\mathbb{R}^m$ and its dimension is $\dim M = m - n$. (It is also possible that $M = \emptyset$.) So far, given f we tweaked a little bit and got a compact submanifold $M \subseteq \mathbb{R}^m$ with $\dim = m - n$. But we want extra data on M.

Definition 35.5. For $x \in M$, denote by T_xM the tangent space to M and v_xM the **normal vectors** to M at x. Then $v_xM = (T_xM)^\perp$ in $\mathbb{R}^m$, or $v_xM = \mathbb{R}^m / T_xM$. A **normal framing** of $M \subseteq \mathbb{R}^m$ is a collection of n vector fields along M with $M \ni x \mapsto v_i(x) \in v_xM$ such that $(v_1, \ldots, v_n)$ are a basis for v_xM for all x.

Our manifold M comes with a normal framing $\underline{v} = (v_1, \ldots, v_n)$ as follows. Let $e_1, \ldots, e_n \in T_0\mathbb{R}^n$ be the standard basis in $\mathbb{R}^n$. The derivative df_x maps $v_xM \to T_0\mathbb{R}^n$ isomorphically. Then let $v_i(x) = (df_x)^{-1}e_i$.

So the idea is that classifying based maps $S^m \to S^n$ (for $m \geq n$) up to homotopy is equivalent to classifying compact normally framed manifolds $(M, \underline{v})$ with $M \subseteq \mathbb{R}^m$ up to something.

Consider a smooth homotopy $F : I \times S^m \to S^n$ where f_0, f_1 for $t = 0, 1$ are based. We then get a normally framed submanifold $N \subseteq I \times \mathbb{R}^m$. This coincides at $t = 0$ and $t = 1$ with (M_0, v_0) and (M_1, v_1). This is called **framed cobordism**. Indeed, this "up to something" is "up to framed cobordism."

I am going to outline next time how the other direction works. That is, why a normally framed submanifold up to framed cobordism corresponds to homotopy classes.

36 November 30, 2016

I want to now start talking about the other direction. Suppose we have a manifold $M \subseteq \mathbb{R}^m$ with $\dim M = m-n$ with a normal framing $\underline{v} = (v_1, \dots, v_n)$. From this can we get an element of $\pi_m(S^n)$?

For a small enough $\epsilon > 0$, consider the map

$$\tau : M \times B^n \to \mathbb{R}^m; \quad (x, (y_1, \dots, y_n)) \mapsto x + \epsilon \sum_{i=1}^{n} y_i v_i.$$

The image is a tubular neighborhood U for $\epsilon \leq \epsilon_1$. Then we get a map

$$\mathbb{R}^m \supseteq U \xrightarrow{\tau^{-1}} M \times B^n \longrightarrow B^n.$$

Using this, define a map

$$f : \mathbb{R}^m \cup \{\infty\} \to B^n/S^{n-1} \cong S^n \cong \mathbb{R}^n \cup \{\infty\};$$
$$f(z) = \begin{cases} f_1(z) & \text{if } z \in \text{interior}(U) \\ \text{basepoint} & \text{if } z \notin \text{interior}(U). \end{cases}$$

You can see that these are inverse operations. So we need to classify framed manifolds.

36.1 Classifying framed manifolds

Let us look at the case when $n = 1$. That is, let us look at normally framed $m-1$ manifolds.

Consider the case $m = 2$. Because 1-manifolds are always circles, any element will be a disjoint union of circles, framed either to the inside or the outside. But any framed circle is cobordant to the empty set. This you can see though the hemisphere that is pointing either outwards or inwards. This shows that

$$\pi_2(S^1) = 0.$$

In a very similar way, you can prove that

$$\pi_m(S^1) = 0$$

for $m \geq 2$. The proof isn't very much different.

Let us go to the other extreme and look at framed 0-manifolds in $\mathbb{R}^m$. A compact 0-manifold is a finite collection of points, and a framing is a choice of an orthonormal (without loss of generality) basis at each point. The points will fall into two categories according to the orientation of the framing. Moreover, if two points are to cancel out in the cobordism, they need to have different orientations. Thus the number $d = (\#R - \#L) \in \mathbb{Z}$ is invariant under cobordism. So we obtain

$$\pi_m(S^m) = \mathbb{Z}.$$

Now let us look at $\pi_m(S^{m-1})$. This is looking at 1-dimensional manifolds. It turns out that we may just assume that these are circles that look like $S^1 \subseteq \mathbb{R}^2 \times 0 \subseteq \mathbb{R}^2 \times \mathbb{R}^{m-2}$. Let v_1 has before (the outward or inward) and let $v_2, \dots, v_{m-1}$ be the standard basis in $\mathbb{R}^{m-2}$. Let us call this the "0-framing" and denote it by $\underline{\text{v}}_0$. Then $(S^1, \underline{\text{v}}_0)$ becomes cobordant to $\emptyset$ and gives $0 \in \pi_m(S^{m-1})$.

But this gives a way to construct other framings. Given $A : S^1 \to \mathrm{GL}(m-1, \mathbb{R}$, we can define a new framing $(v_1^A, \dots, v_{m-1}^A)$ with

$$v_j^A(x) = \sum A_{ij}(x) v_i^0(x).$$

This is simply rotating the vectors around along the circle. So given any $[A] \in \pi_1(\mathrm{GL}(m-1, \mathbb{R})) = \pi_1(\mathrm{SO}(m-1))$ we get a framed manifold $(M, \underline{\text{v}})$. We haven't verified enough stuff to conclude this, but indeed,

$$\pi_3(S^2) \cong \mathbb{Z} \cong \pi_1(\mathrm{SO}(2)), \quad \pi_4(S^3) \cong \mathbb{Z}/2 \cong \pi_1(\mathrm{SO}(3)).$$

37 December 2, 2016

37.1 Fiber bundles

Definition 37.1. A **fiber bundle** is a map $p : E \to B$ such that there is a space F—called the fiber—such that for all $b \in B$, the map $p^{-1}(b)$ is homeomorphic to F, and for all $b \in B$ there is a neighborhood $U \ni b$ with $p^{-1}(U) \cong U \times F$ so that this commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"] \arrow[d, "p"] & U \times F \arrow[d, "\mathrm{pr}_U"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

Examples occur when a group acts freely on a space, e.g., a compact Lie group acting on a manifold. For instance, recall that $\mathbb{CP}^n$ can be thought as

$$\mathbb{CP}^n = \{\text{unit vectors in } \mathbb{C}^{n+1}\}/(z \sim \lambda z \text{ with } |\lambda| = 1).$$

This gives us a quotient map

$$p : S^{2n+1} \to \mathbb{CP}^n; \quad z \mapsto [z]$$

with fiber $p^{-1}([z]) = \{\lambda z : \lambda \in S^1\} \cong S^1$. Why is this locally trivial? Consider $U = \{[z] : z_0 \neq 0\} \subseteq \mathbb{CP}^n$. We want to check that $p^{-1}(U) \cong U \times S^1 \cong \mathbb{C}^n \times S^1$. The map $\mathbb{C}^n \times S^1 \to p^{-1}(U)$ can explicitly be written down as

$$((z_1, \dots, z_n), \lambda) \mapsto \frac{(\lambda, \lambda z_1, \lambda z_2, \dots, \lambda z_n)}{(1 + \sum |z_i|^2)^{1/2}}.$$

Other examples include covering spaces with n sheets. Then F are n points.

Also, there are examples where E is a group. The group $\mathrm{SO}(n)$ acts on $S^{n-1} \subseteq \mathbb{R}^n$. This gives a map

$$p : \mathrm{SO}(n) \to S^{n-1}; \quad A \mapsto Ae_1.$$

The fiber will be $\mathrm{SO}(n-1)$. Likewise, there is a map $p : U(n) \to S^{2n-1}$ with fiber $U(n-1)$. You can do this for the quaternions too.

Anyways, look at the map $p : E \to B$. Pick a point $b_0 \in B$ and let $F_0 = p^{-1}(b_0)$. Pick a base point $e_0 \in F_0$. Then $p : (E, F_0, e_0) \to (B, b_0, b_0)$. So we get a map

$$p_* : \pi_k(E, F_0, e_0) \to \pi_k(B, b_0).$$

Proposition 37.2. *This map p_* is an isomorphism.*

Recall that there is a long exact sequence

$$\pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(E, F) \longrightarrow \pi_{k-1}(F) \longrightarrow \pi_{k-1}(E) \longrightarrow \cdots.$$

Now this proposition tells us that what we have can be also written as

$$\pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \pi_{k-1}(E) \longrightarrow \cdots .$$

The proof of the proposition will tell us also what the maps are.

For example, look at a covering space $\tilde{B} \to B$ with F the discrete space. For $k \geq 1$, $\pi_k(F) = 0$. This implies that

$$\pi_k(\tilde{B}) \cong \pi_k(B)$$

for all $k \geq 2$.

Let us look at another example. Take $p : S^{2n+1} \to \mathbb{CP}^n$ with fiber $F = S^1$. We have $\pi_k(S^1) = 0$ for $k \geq 2$. So again, the long exact sequence gives

$$\pi_k(\mathbb{CP}^n) = \pi_k(S^{2n+1})$$

for $k \geq 3$. If $n = 1$, then $\pi_k(S^2) = \pi_k(S^3)$ for $k \geq 3$. In particular, $\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z}$.

Because $\pi_k(S^\infty) = 0$ for all k, we get $\pi_k(\mathbb{CP}^\infty) = 0$ for $k \geq 3$. What is $\pi_2(\mathbb{CP}^\infty)$? We can use the sequence.

$$\pi_2(S^\infty) = 0 \longrightarrow \pi_2(\mathbb{CP}^\infty) \longrightarrow \pi_2(S^1) = \mathbb{Z} \longrightarrow \pi_1(S^\infty) = 0$$

So $\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$. So all homotopy groups of $\mathbb{CP}^\infty$ except for $\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$.

The main point is that fiber bundles have the homotopy lifting property (not to be confused with the homotopy extension property). This says that given any paracompact space X and a homotopy $\{g_t\}$, $G : I \times X \to B$, and a lift of g_0, i.e., a $f_0 : X \to E$ with $p \circ f_0 = g_0$, then there exists a $F : I \times X \to E$ extending f_0 such that $p \circ F = G$.

Proof of proposition 37.2. We fisrt show that p_* is onto. Given ψ in $\pi_k(B, b_0)$, i.e., $\psi : (I^k, \partial I^k) \to (B, b_0)$, we want to find a Ψ such that $p \circ \Psi = \psi$. Now a map $(I^k, J, I^{k-1}) \to X$ can be thought as a homotopy $I \times I^{k-1} \to X$. Then the homotopy lifting property gives that such a Ψ exists.

The injectivity can be thought similarly. $\square$

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