Math 213a - Complex Analysis

Taught by Wilfried Schmid
Notes by Dongryul Kim

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This course was taught by Wilfried Schmid. We met on Tuesdays and Thursdays from 2:30pm to 4:00pm in Science Center 216. We did not use any textbook, and there were 13 students enrolled. There was a take-home final and also an oral exam. The course assistance was Yusheng Luo, a graduate student.

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There will be either a take-home and an oral exam, or a sit-down final. There are no textbooks, but Ahlfors and Greene-Krantz is will be a reference.

We are going to do a rapid review of complex analysis. Then we are going to talk about Mittag-Leffler and Weierstrass theorems and refinements, and then normal families and Riemann mapping theorem with refinements. Next we are going to do learn the Picard theorems, and then special functions like the Γ funcitons.

We will typically look at an open set $U \subseteq \mathbb{C}$ and a function $f : U \to \mathbb{C}$. Some conventions we will use is if $z \in U$ then we write $z = x + iy$ and for such functions $f(z)$, we write $f(z) = u(x, y) + iv(x, y)$ with $u, v : U \to \mathbb{R}$ real.

1.1 Holomorphic functions

Theorem 1.1. *The following conditions of $f(z) = u + iv$ are equivalent:*

(1) for each $z \in U$, f is differentiable in the complex sense, i.e.,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

(2) for each $z \in U$, $f'(z)$ exists and $z \mapsto f'(z)$ is continuous.

(3) the map $(x, y) \mapsto (u(x, y), v(x, y))$ is a C^1 map which satisfies the Cauchy-Riemann equations $\partial u/\partial x = \partial y/\partial v$ and $\partial u/\partial y = -\partial v/\partial x$.

(4) the function f is continuous and $\int_{\partial D} f(z)dz = 0$ for every Green's domain $D \subseteq M$, i.e., D that is relatively compact in U and ∂D is a disjoint union of piecewise C^1 simple closed curves.

(5) the function f is continuous and $\int_{\partial R} f(z)dz = 0$ for every rectangle R whose closure is contained in U.

(6) the function f is continuous, and for every $z_0 \in U$, there exsits an open neighborhood $V \subseteq U$ and $F : V \to \mathbb{C}$ such that $f|_V = F'$ with F having a derivative at every $z \in V$.

(7) for every $z_0 \in U$, there exists an open neighborhood V in U and a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ that converges absolutely on X with value equal to f.

One remark is that if $f, g : U \to \mathbb{C}$ are differentiable in the complex sense at $z \in U$, then $f + g$, fg, cg for any $c \in \mathbb{C}$, and f/g if $g(z) \neq 0$, are all differentiable in the complex sense and the usual rules of differentiation apply. Also, (2) is Cauchy's definition of a "holomorphic function" where (7) is Weierstrass' definition of a holomorphic function. One funny anecdote is that when John Tate taught this course, he used only (7) to develop the whole theory. The reason was because if you are working for instance in $\mathbb{Q}_p$, then you have to do everything in terms of power series.

The implication (2) $\Rightarrow$ (4) is Cauchy's theorem, and (1) $\Rightarrow$ (4) is Goursat's theorem. Statement (3) can be formulated as the Jacobian matrix of the C^1 map $(x, y) \mapsto (u(x, y), v(x, y))$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}$$

is either 0 or a positive multiple of a special orthogonal matrix, i.e., the C^1 map is conformal.

Proof of (2) $\Rightarrow$ (3). If $f'(z)$ exists at z, then so do

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h)-f(z)}{h}, \quad \lim_{\substack{k\to 0\\ k\in\mathbb{R}}} \frac{f(z+ik)-f(z)}{ik}$$

and both equal $f'(z)$. The expression for the first one is

$$\lim_{h\to 0 h\in\mathbb{R}} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{v(x+h,y)-v(x,y)}{h}$$

and the second one is

$$\lim_{k\to 0 k\in\mathbb{R}} \frac{u(x,y+k)-u(x,y)}{ik} + i \lim_{\substack{k\to 0\\ k\in\mathbb{R}}} \frac{v(x,y+k)-v(x,y)}{ik}.$$

Comparing the real and imaginary parts, you get the result. □

Proof of (3) $\Rightarrow$ (4). We apply Green's theorem to

$$f(z)dz = (u+iv)(dx+idy) = (udx - vdy) + i(vdx + udy).$$

Then

$$\begin{aligned} \int_{\partial D} f(z)dz &= \int_{\partial D} (udx - vdy) + i\int_{\partial D} (vdx + udy) \\ &= \int_D \Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)dxdy + i\int_D \Big(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\Big)dxdy = 0. \end{aligned}$$ □

Proof of (1) $\Rightarrow$ (5). Suppose f satisfies (1), but $\int_{\partial R} f(z)dz \neq 0$ for some R whose closure is in U. We may suppose that $\int_{\partial R} f(z)dz = 1$. Divide R into 4 equal sized rectangles. Then there must exists at least one of the four, call it R_1 such that $|\int_{\partial R_1} f(z)dz| \geq 1/4$. Continue inductively and we get a sequence of subrectangles $R_1, R_2, \ldots$ with $|\int_{\partial R_n} f(z)dz| \geq 4^{-n}$, $\operatorname{diam} R_n = 2^{-n} \operatorname{diam} R$, and $\operatorname{length} R_n = 2^{-n} \operatorname{length} R$.

Now the intersections of all the rectangles R_n will be a single point $z_0 \in U$ because it is a nested sequence of compact sets. Becasue f is differentiable at z_0, given any $\epsilon > 0$ there is a N such that $n \geq N$ implies

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$$

for $z \in \partial R_n$. Then

$$\begin{aligned}
4^{-n} &\leq \left| \int_{\partial R} f(z) dz \right| = \left| \int_{\partial R_n} (f(z) - f(z_0)) dz \right| \\
&\leq \left| \int_{\partial R_n} f'(z_0)(z - z_0) dz \right| + \epsilon \int_{\partial R_n} |z - z_0| dz \\
&= \epsilon \int_{\partial R_n} |z - z_0| dz \leq \epsilon \operatorname{length} R_n \operatorname{diam} R_n.
\end{aligned}$$

Because this is true for every ϵ, we get a contradiction. $\square$

2 September 9, 2016

2.1 Continuing the proof

Let us recall where we ended last time. We are working in the settting of $f : U \to \mathbb{C}$ with $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. So far, we have (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (2) $\Rightarrow$ (1) $\Rightarrow$ (5).

Proof of (5) $\Rightarrow$ (6). May as we suppose that $U = \Delta(z_0, r)$, wher $\Delta(z_0, r)$ denotes the open disc centered at z_0 with radius r.

For $z \in \Delta(z_0, r)$ define the two paths in $\Delta(z_0, r)$ from z_0 to z, given as in the following diagram.

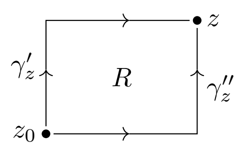

Figure 1: The paths γ'_z and γ''_z

By (5), we see that

$$\int_{\gamma'_z} f(z)dz = \int_{\gamma''_z} f(z)dz$$

since $\gamma''_z - \gamma'_z = \partial R$. Let us define

$$F(z) = \int_{\gamma'_z} f(z)dz = \int_{\gamma''_z} f(z)dz.$$

On horizontal lines, $dz = dx$ and on vertical lines, $dz = idy$. This implies

$$\frac{\partial F}{\partial x} f, \quad \frac{\partial F}{\partial y} = if$$

by the fundamental theorem of calculus. Write $F(z) = \tilde{u}(x, y) + i\tilde{v}(x, y)$. Since f is continuous, the functions $\tilde{u}$ and $\tilde{v}$ are both C^1. Also

$$\frac{\partial \tilde{u}}{\partial x} = \Re f, \quad \frac{\partial \tilde{v}}{\partial x} = \Im f, \quad \frac{\partial \tilde{u}}{\partial y} = -\Im f, \quad \frac{\partial \tilde{v}}{\partial y} = \Re f$$

and so $\tilde{u}$ and $\tilde{v}$ satisfies the Cauchy-Riemann equations, which is condition (3). But we need that F satisfies (2), i.e., is differentiable. But why is (3) $\Rightarrow$ (2)?

Recall that the Cauchy-Riemann equations are equivalent to the Jacobian of the map $(x, y) \mapsto (\tilde{u}, \tilde{v})$ is multiplication by a complex number when we make the identification $\mathbb{R}^2 \cong \mathbb{C}$. Then the derivative of the map, thought of as a map from $\mathbb{R}^2 \cong \mathbb{C}$ to itself, is multiplication by a complex number. So F' exists at every point. So $F' = f$. □

The proof of (2) $\Rightarrow$ (7) depends on the following lemma:

Lemma 2.1. *Suppose f satisfies (2), and let $D \subseteq U$ be a Green's domain. For any $z \in D$,*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

Proof. Let us choose $\eta > 0$ that less less than the distance of z and ∂D. (By convention, the distance between z and $\emptyset$ is $+\infty$.) Define $D_\eta = D - \operatorname{clos}\Delta(z,\eta)$. This is a Green's domain in $U - \{z\}$, and its boundary is

$$\partial D_\eta = \partial D - \{|\zeta - z| = \eta\}.$$

Since (2) $\Rightarrow$ (4), and since $\zeta \mapsto f(\zeta)/(\zeta - z)$ satisfies (2) on $U - \{z\}$, we have

$$\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = \int_{|\zeta - z| = \eta}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

By (2), given $\epsilon > 0$, if η is sufficiently small and $|\zeta - z| = \eta$ then

$$|f(\zeta) - f(z) - (\zeta - z)f'(z)| \leq \epsilon\eta.$$

So

$$\begin{aligned}\epsilon\int_{|\zeta-z|=\eta}|d\zeta| &\geq \left|\int_{|\zeta-z|=y}\frac{f(\zeta)}{\zeta - z}d\zeta - f(z)\int_{|\zeta-z|=\eta}\frac{d\zeta}{\zeta - z} - f'(z)\int_{|\zeta-z|=\eta}d\zeta\right| \\ &= \left|\int_{|\zeta-z|=\eta}\frac{f(\zeta)d\zeta}{\zeta - z} - 2\pi i f(z) - 0\right|.\end{aligned}$$

As $\epsilon \to 0$, we get the lemma. □

Proof of (2) $\Rightarrow$ (7). Choose a $z_0 \in U$. Let R between distance between z_0 and ∂D. Pick a $0 < r < R$.

Suppose that $|z - z_0| < r$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Let $D = \Delta(z_0, r)$. For $z \in \Delta(z_0, r)$ and $\zeta \in \partial D$, we have the geometric series

$$\frac{1}{\zeta - z} = \sum_{k=0}^{\infty}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}$$

which converges absolutely and uniformly[1] in ζ and z proved $|z - z_0| \leq r - \delta$ for some small $\delta > 0$.

[1]This, by convention, means that it absolutely converges and the absolute convergence is uniform. It is literally not what it says, but people usually mean this.

Then the lemma tells us that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}}$$

$$= \sum_{k=0}^{\infty}(z-z_0)^k\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)d\zeta}{(\zeta-z)^{k+1}} = \sum_{k=0}^{\infty}a_k(z-z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{k+1}}.$$

This convergence is uniform and absolute in z. □

We note that (7) is a local statement, so U can be replaced, for example, by a disc centered at z_0, of radius at most the distance between z_0 and ∂U. Also, what we have actually proved is stronger than (7). We have proved that if (2) is true, then for any $z_0 \in U$, the function $f(z)$ can be expressed as $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ on $\Delta(z_0, r)$ given that r is less than the distance of z_0 and ∂U, and that the convergence is absolute and uniform.

Consider a formal power sereis $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ for $z_0, a_k \in \mathbb{C}$ so that z_0 is the center of the power series.

Lemma 2.2. *There exists a uniquely determined R, $0 \le R \le \infty$, such that*

(1) for any r with $0 < r < R$and $|z-z_0| \le r$, the series converges uniformly and absolutely.

(2) if $|z-z_0| > R$ then the series diverge.

*This R is called the **radius of convergence**. Suppose for $|z-z_0| < R$,*

$$s(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

Then $s(z)$ is differentiable at z and

$$s'(z) = \sum_{k=0}^{\infty} ka_k(z-z_0)^k$$

absolutely and uniformly. This new function s' has the same radius of convergence with $s(z)$.

If we have this lemma, we are done.

Proof of (7) ⇒ (2). Assume the lemma. Then on $\Delta(z_0, r)$ with $0 < r < R$, $s(z)$ is differentiable. Its derivative $s'(z)$ is also differentiable and hence continuous. □

Proof of (6) ⇒ (7). Locally, we can say $f = F'$ and hence F satisfies (2). Then F can be expressed locally as a power series with positive radius of convergence, and by the lemma, same is true for $f = F'$. □

3 September 11, 2016

3.1 Finishing the proof

So we need to prove this lemma to finish the proof of the theorem. Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a formal power series, with $z_0, a_k \in \mathbb{C}$.

Lemma 3.1. *There exists a uniquely determined $0 \le R \le \infty$, such that*

(1) for $z \in \mathbb{C}$, $|z-z_0| < R$, the series converges absolutely and the convergence is uniform and absolute on any compact subset of $\Delta(z_0, R)$

(2) the series diverges for $z \in \mathbb{C}$, $|z-z_0| > R$.

This R is called the radius of convergence of the power series. For $z \in \Delta(z_0, R)$, let $s(z)$ denote the limit. Then $s(z)$ is differentiable in the complex sense and $s'(z) = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1}$ with this series having the same radius of convergence.

Proof. We define

$$R = \sup\{|z-z_0| : \text{the series converges at } z\}.$$

Now suppose $0 < r < R$. Then there exists z_1 with $r_1 = |z_1 - z_0|$ such that the series converges absolutely at z_1 and $r < r_1 \le R$. Also convergence implies boundedness of the terms: there exists an M such that $|a_k r_1^k| \le M$, i.e., $|a_k| \le M r_1^{-k}$. For $|z-z_0| \le r$, we have

$$|a_k(z-z_0)^k| \le |a_k| r^k \le M \frac{r^k}{r_1^k}.$$

Therefore the series converges absolutely and uniformly.

Now we want to say the same thing for convergence of the series $\sum k a_k (z-z_0)^{k-1}$. Choose r_2 with $r < r_2 < r_1$. Because the series converges also for $|z-z_0| = 0$, we can easily check that the series converges absolutely and uniformly on $\Delta(z_0, r)$.

Let us define for $z \in \Delta(z_0, r)$, and let

$$s(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

Consider $h \in \mathbb{C}$ where $0 < |h| < r - |z-z_0|$. The difference quotient can be computed as

$$\begin{aligned}
&\left|\frac{s(z+h)-s(z)}{h}-\sum_{k=1}^{\infty}ka_k(z-z_0)^{k-1}\right| = \left|\sum_{k=2}^{\infty}a_k\sum_{l=2}^{k}\binom{k}{l}(z-z_0)^{k-l}h^{l-1}\right| \\
&\leq \sum_{k=2}^{\infty}|a_k|\sum_{l=2}^{k}\binom{k}{l}|z-z_0|^{k-l}|h|^l \leq M\sum_{k=2}^{\infty}\sum_{l=2}^{k}\binom{k}{l}\frac{r^{k-l}}{r_1^k}|h|^l \\
&= M\sum_{k=2}^{\infty}\left(\frac{(r+|h|)^k-r^k}{|h|r_1^k}-k\frac{r^{k-1}}{r_1^k}\right) \leq M\sum_{k=2}^{\infty}\frac{k(r+|h|)^{k-1}-kr^{k-1}}{r_1^k} \\
&= Mr_1^{-1}\left(\frac{1}{(1-(r+|h|)/r_1)^2}-\frac{1}{(1-r/r_1)^2}\right).
\end{aligned}$$

Letting $h\to 0$, this sum goes to zero. This shows that $\sum a_k(z-z_0)^k$ is differentiable and its derivative is $\sum ka_k(z-z_0)^{k-1}$. □

As I proved last time, this lemma implies the theorem.

Definition 3.2. A function $f : U \to \mathbb{C}$ is said to be **holomorphic** or **complex analytic** if it satisfies any one (equivalently, all) of the statements (1)–(7) in the theorem.

3.2 A bunch of corollaries

There are some consequences.

Corollary 3.3 (Cauchy integral formula)**.** *Suppose f is holomorphic on U with $z\in U$. If $D\subseteq U$ is a Green's domain with $z\in D$, then*

$$f(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

Corollary 3.4. *Let $f : U\to\mathbb{C}$ be holomorphic and $z_0\in U$. Let R be the distance of z_0 and ∂U ($R=+\infty$ if $U=\mathbb{C}$). Then on $\Delta(z_0,R)$,*

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k$$

with appropriately chose a_k, convergence is absolute, and uniform and absolute on a compact subset of $\Delta(z_0,R)$. This power series has radius of convergence at least R. Moreover, formally differentiated series has the same radius of convergence as the original series, and converges to $f'(z)$.

The argument for (2) ⇒ (7) contains the formula for a_k:

$$a_k=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta$$

where $0<r<R$.

Corollary 3.5. *If $f : U \to \mathbb{C}$ is holomorphic, then f has derivatives of all orders (in the complex sense). Moreover, if $z \in U$ and $D \subseteq U$ is a Green's domain containing z, then*

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta.$$

Proof. You can do this by differentiating the Cauchy integral formaula (but needs to be justified). Alternatively, use the previous corollary. $\square$

Corollary 3.6 (Morera's theorem)**.** *Suppose $f : U \to \mathbb{C}$ is continuous and for any rectifiable closed curve γ in U the integral $\int_\gamma f(z)dz = 0$. Then f is holomorphic.*

The inverse is not always true. For instance, the integral of $1/z$ over the unit circle is $2\pi i$. If the curve is cohomologous to zero, then the integral should be zero.

We define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Note that with this notation, $\partial/\partial z$ sends $z \mapsto 1$, $\bar{z} \mapsto 0$, and likewise $\partial z/\partial\bar{z} = 0$ and $\partial\bar{z}/\partial\bar{z} = 1$.

Corollary 3.7. *Suppose $f : U \to \mathbb{C}$ is C^1. Then f is holomorphic if and only if $\partial f/\partial\bar{z} = 0$.*

Corollary 3.8. *Suppose $f : U \to \mathbb{C}$ is holomorphic and let $f(z) = u(x, y) + iv(x, y)$. Then u and v are harmonic.*

Proof. We have

$$0 = \frac{\partial^2 f}{\partial z\partial\bar{z}} = \frac{1}{4}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)f.$$

$\square$

Corollary 3.9. *Suppose $f : U \to \mathbb{C}$ is holomorphic and $z_0 \in U$, U is holomorphic and $f \not\equiv 0$. Then there exists a uniquely determined integer $n \geq 0$ and a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$.*

Proof. First suppose $f \not\equiv 0$ on any neighborhood of z_0. Then for z near z_0, the function f can be presented as

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$

The a_k cannot all be zero.

Let n be the least integer such that $a_n \neq 0$. Then

$$f(z) = (z - z_0)^n\sum_{k=n}^{\infty} a_k(z - z_0)^{k-n}$$

and we can let the sum on the right hand size be $g(z)$.

This proves that $\{z \in U : f \not\equiv 0$ in any neighborhood $\}$ is open in U. Likewise the complement is open in U. Because U is connected and $f \not\equiv 0$ on U, it must be that $f \not\equiv 0$ on some neighborhood of z_0. $\square$

4 September 13, 2016

4.1 More and more corollaries

We proved this corollary last time.

Corollary 4.1. *Suppose $f : U \to \mathbb{C}$ is holomorphic and $z \in U$, with $f \not\equiv 0$. Then there exists an integer $n \geq 0$ and a holomorphic $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$.*

If $n \geq 1$ says that f has a zero at z_0 (or "vanishes at z_0"), and n is the **order** of zero.

Corollary 4.2. *Under the same hypotheses, $S = \{z \in U : f(z) = 0\}$ is discrete in U.*

Corollary 4.3. *Suppose $f_n : U \to \mathbb{C}$ is holomorphic with $n = 1, 2, \ldots$. If $f_n \to f$ uniformly on a compact sets (or equivalently, locally uniformly) then f is holomorphic.*

Proof. Let R be a closed rectangle with $R \subseteq U$. Then $\operatorname{int}(R)$ is a Green's domain in U, so $\int_{\partial R} f_n dz = 0$. Then $\int_{\partial R} f dz = 0$. This implies that f is holomorphic. □

Corollary 4.4 ("Open mapping theorem"). *Suppose $f : U \to \mathbb{C}$ is holomorphic, and U is connected, with f non-constant. Then $f(U)$ is open.*

Proof. Suppose $z_0 \in U$, and let $f(z) - f(z_0) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$ and $n \geq 1$. Since $g(z_0) \neq 0$, the function $g(z)^{1/n}$ has a locally defined holomorphic function, because $\log g(z)$ is locally well-defined. Define $h(z) = (z - z_0) g(z)^{1/n}$. Then $h(z_0) = 0$ and $h'(z_0) = g(z_0)^{1/n} \neq 0$ and satisfies

$$f(z) = f(z_0) + h(z)^k.$$

Since $h(z_0) = 0$ and $h'(z_0) \neq 0$, the map h, viewed as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, has a nonzero differential at $z_0 = x_0 + iy_0$, namely multiplication by $h'(z)$, hence the differential is invertible. By the inverse function theorem, $h(z)$ has a local inverse from an open neighborhood of $h(z_0) = 0$ onto an open neighborhood of z_0. That is, the inverse of z_0 under h covers a full neighborhood of 0. Then map $w \mapsto w^n$ is surjective on the neighborhood of zero. Because $f(z) = f(z_0) + h(z)^n$, the image of f includes a neighborhood of $f(z_0)$. □

Corollary 4.5 ("Inverse function theorem"). *Suppose $f : U \to \mathbb{C}$ is holomorphic and 1-to-1. Then by the open mapping theorem $f(U)$ is open, and by the hypothesis, $f^{-1} : f(U) \to U$ is well-defined set-theoretically. Then f^{-1} is holomorphic.*

Proof. As in the proof of the previous corollary, for $z_0 \in U$ I can write $f(z) = f(z_0) + h(z)^n$ where $h(z) = (z - z_0) g(z)^{1/n}$. Also note that we may as well suppose that U is connected. Then f is not constant.

So $f(z) = f(z_0) + h(z)^n$. Arguing as in the previous corollary, $z \mapsto f(z)$ is locally n-to-1 near z_0. So we must have $n = 1$. Then $f(z) = f(z_0) + (z - z_0)g(z)$ with $g(z_0) \neq 0$. Viewed as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, this f has a invertible differential at z_0. So by the inverse function theorem of multivariable calculus, f has a C^1 inverse locally near z_0. Since $f : U \to \mathbb{C}$ is globally 1-to-1, this proves that $f^{-1} : f(U) \to U$ is at least C^1. At any $f(z_0) \in f(U)$, the differential of f^{-1} (thought of as $\mathbb{R}^2 \to \mathbb{R}^2$) is the inverse of the differential of f at z_0, i.e., the inverse of multiplication by $f'(z_0)$ (thought of as $\mathbb{R}^2 \to \mathbb{R}^2$). But that is multiplication by $1/f'(z_0)$ and so it satisfies to Cauchy-Riemann equations. That is f^{-1} is holomorphic. □

Corollary 4.6 ("Maximum principle")**.** *Suppose $f : U \to \mathbb{C}$ is holomorphic and U connected, with f non-constant. Then the function $z \mapsto |f(z)|$ cannot assume a maximum at any point of U.*

Proof. This follows from the open mapping theorem. □

Let $\Delta = \Delta(0, 1)$.

Corollary 4.7 (Schwarz lemma)**.** *Suppose $f : \Delta \to \mathbb{C}$ is holomorphic, $f(0) = 0$, $|f(z)| \leq 1$ on Δ. Then for $z \in \Delta$, $|f(z)| \leq |z|$, and $|f'(0)| \leq 1$. Both of these inequalities are strict (except for $|f(0)| = 0$) unless $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. The function $f(z)/z$ is holomorphic on Δ since $f(z)$ has a zero at 0. Define $g_r(z) = f(rz)/rz$ for $0 < r < 1$. This is holomorphic on an open neighborhood of closure of Δ. Also $|g_r(z)| \leq 1/r$ on $\partial\Delta$. This immplies that $|g_r(z)| \leq 1/r$ for all $z \in \Delta$ by the maximum principle. By continuity, $|f(z)/z| \leq 1$ for all $z \in \Delta$. Either $f(z)/z$ is constant, in which case the statement is OK, or $f(z)/z$ is not. Then by the maximal principle, $|f(z)/z| < 1$ and $f'(0) = \lim_{f(z)/z}$ and hence $|f'(0)| < 1$ again by maximal principle. □

This is a very important statement, especially in the context of complex manifolds.

4.2 Laurent series

Suppose $0 \leq r_1 < r_2 \leq \infty$ and $z_0 \in \mathbb{C}$. Let

$$A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$$

Theorem 4.8 (Laurent)**.** *Suppose $f : A \to \mathbb{C}$ is holomorphic. Then for $z \in A$,*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

converges locally uniformly and absolutely on A. More precisely, $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges uniformly and absolutely on compact subsets of $\Delta(z_0, r_2)$, and

$\sum_{k=-\infty}^{-1} a_k(z-z_0)^k$ *converges uniformly and absolutely on compact subsets of* $\mathbb{C} - \operatorname{clos}\Delta(z_0, r_1)$. *Finally,*

$$a_k = \frac{1}{2\pi i}\int_{|\zeta - z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta$$

for any r *with* $r_1 < r < r_2$.

Proof. Suppose ρ_1, ρ_2 are chosen so that $r_1 < \rho_1 < \rho_2 < r_2$. Let $\tilde{A} = \{z \in \mathbb{C} : \rho_1 < |z - z_0| < \rho_2\}$. This is a Green's domain in A. The map $\zeta \mapsto f(\zeta)/(\zeta - z_0)^{k+1}$ is holomorphic on A, so

$$0 = \int_{\partial\tilde{A}} \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta = \int_{|\zeta-z_0|=\rho_2} \frac{f(\zeta)}{(\zeta-z_2)^{k+1}}d\zeta - \int_{|\zeta-z_0|=\rho_1}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta.$$

This shows that the formula that was stated is independent of r.

For $z \in \tilde{A}$, by Cauchy

$$f(z) = \frac{1}{2\pi i}\int_{\partial\tilde{A}}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho_2}\frac{f(\zeta)}{\zeta-z_0}d\zeta - \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho_1}\frac{f(\zeta)}{\zeta-z_0}d\zeta.$$

Now

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-z_0)-(z-z_0)} = \begin{cases} \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}} & \text{for } |\zeta - z_0| = \rho_2 \text{ and } |z-z_0| < \rho_2, \\ -\sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}} & \text{for } |\zeta-z_0| = \rho_1 \text{ and } |z - z_0| > \rho_1. \end{cases}$$

So

$$f(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int_{|\zeta-z_0|=\rho_2}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta\right)(z-z_0)^k + \frac{1}{2\pi i}\sum_{k=-\infty}^{-1}\left(\int_{|\zeta-z_0|=\rho_1}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta\right)(z-z_0)^k.$$

This converges as stated in the theorem. □

As a special case, $r_1 = 0$ and $r_2 = r$. I will write $\Delta^*(z_0, r) = \{z \in \Delta(z_0, r) : z \neq z_0\}$ to denote the punctured disc. Suppose $f(z)$ is holomorphic on $\Delta^*(z_0, r)$. Then there exists a_k for $-\infty < k < \infty$ such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

converges uniformly and absolutely on compact subsets of $\Delta^*(z_0, r)$.

We call

$$a_{-1} = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} f(\zeta)d\zeta$$

is the **residue** of f at z_0, and is denoted by $\operatorname{Res} f|_{z_0}$. Also, the series $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ is called the **principal part** of f at z_0.

The function f has a **removable singularity** at z_0 if $a_k = 0$ for all $k \leq -1$. We say that f has a **pole of order** n if $a_{=}0$ for $k < -n$ and $a_{-n} \neq 0$. We say that f has a **essential singularity** at z_0 if $a_k \neq 0$ for infinitely many $k < -1$.

Suppose $U \subseteq \mathbb{C}$ is open, and $S \subset U$ is a discrete subset with $f : U - S \to \mathbb{C}$ holomorphic. Then for any $z_0 \in S$, the function $f : \Delta^*(z_0, r) \to \mathbb{C}$ is also holomorphic for some small $r = r(z_0) > 0$. So one has the notion of removable singularity, pole, essential singularity, residue, in the context. One says that "f is holomorphic on U except for isolated singularities" at the points of S.

5 September 15, 2016

Suppose $U \subseteq \mathbb{C}$ is open and $S \subseteq U$ be a discrete subset. One calls a holomorphic function $f : U - S \to \mathbb{C}$ **holomorphic on U with isolated singularities at S**.

5.1 Isolated singularities

If $z_0 \in S$ then there exists a $\eta > 0$ such that $\Delta(z_0, \eta) \cap S = \{z_0\}$. So $f : \Delta^*(z_0, \eta) \to \mathbb{C}$ is holomorphic, and hence we have notions of residue, pole, isolated singularities. We say that f is **meromophic** on U if all the singularities are poles.

Theorem 5.1. *For $z_0 \in S$,*

(1) f has a removable singularity at z_0 if and only if $|f(z)|$ is bounded near z_0.

(2) f has a pole at z_0 if and only if $\lim_{z\to z_0} |f(z)| = +\infty$.

(3) f has an essential singularity at z_0 if and only if for any δ with $0 < \delta \ll 1$, $f(\Delta^*(z_0, \delta))$ is dense in $\mathbb{C}$.

The (3) is called the Casorati-Weierstrass theorem, and in fact, $f(\Delta^*(z_0, \delta))$ is all of $\mathbb{C}$ possibly except for one point. This stronger result is called Big Picard's theorem, and is much much harder to prove.

Proof. (1) The forward direction is obvious. Now suppose $|f(z)|$ is bounded near z_0. As before, we know that $f : \Delta^*(z_0, \eta) \to \mathbb{C}$ is holomorphic. For $k \leq -1$ and $0 < r < \eta$, we have

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right| \leq \max_{|\zeta - z_0| = r} |f(\zeta)| r^{-k}.$$

Because the right hand side goes to zero as $r \to 0$ if $k \leq -1$, we see that $a_k = 0$ for all $k \leq -1$.

(2) For the forward direction, we can write $f(z) = (z - z_0)^{-n} g(z)$ for some $n \geq 1$ and g holomorphic with $g(z_0) \neq 0$. So $|f(z)| \to \infty$ as $z \to z_0$. For the other direction, suppose $|f(z)| \to +\infty$ as $z \to z_0$. Then after making η smaller if necessary, $f(z) \neq 0$ on $\Delta^*(z_0, \eta)$. Then $z \mapsto 1/f(z)$ has an isolated singularity at z_0, and it is locally bounded near z_0. So $1/f(z)$ has a removable singularity at z_0. Then $1/f(z) = (z - z_0)^n g(z)$ for some holomorphic g with $g(z_0) \neq 0$. Then f has a pole of order n at z_0.

(3) The backwards direction follows from (1) and (2). Now suppose that f has an essential singularity. Suppose $0 < \delta < \eta$ and $f(\Delta^*(z_0, \delta))$ is not dense in $\mathbb{C}$. Then there exists $\omega_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $f(\Delta^*(z_0, \delta)) \cap \Delta(w_0, \epsilon) = \emptyset$. Then $|f(z) - \omega_0| \geq \epsilon$ for $z \in \Delta^*(z_0, \delta)$ and so $1/(f(z) - \omega_0)$ is holomorphic on $\Delta^*(z_0, \delta)$, bounded by ϵ^{-1}. Then $g(z) = 1/(f(z) - \omega_0)$ is holomorphic even at z_0, because it is a removable singularity. Then f at z_0 is either a removable singularity or a pole. $\square$

5.2 Residue theorem and its applications

Let $U \subseteq \mathbb{C}$ be and open set, and $S \subseteq U$ discrete. A function f is holomorphic on U excpt for isolated singularities at points of S. Let $D \subseteq U$ be a Green's domain and $\partial D \cap S \neq \emptyset$.

Theorem 5.2 (Residue theorem)**.** *In this situation,*

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{z\in D\cap S} \operatorname{Res} f(\zeta)|_z.$$

(Note that $D \cap S$ is finite.)

Although it is an important statement, the proof is a triviality.

Proof. Enumerate $S \cap D = \{z_1, \dots, z_N\}$. For each j with $1 \le j \le N$, we can choose $r_j > 0$ such that $\operatorname{clos}\Delta(z_j, r_j) \subseteq D$, and $\operatorname{clos}(\Delta(z_j, r_j)) \cap \operatorname{clos}(\Delta(z_i, r_i)) = \emptyset$ for $i \neq j$. Consider $\tilde{D} = D - \bigcup \operatorname{clos}(\Delta(z_j, r_j))$. This is a Green's domain in $U - S$ and then

$$0 = \int_{\partial\tilde{D}} f(z)dz = \int_{\partial D} f(z)dz - \sum_{j=1}^{N} \int_{|z-z_j|=r_j} f(z)dz = \int_{\partial D} f(z)dz - 2\pi i \sum_{j=1}^{N} \operatorname{Res} f|_{z_j}.$$

□

The residue theorem is useful in computing certain definite integrals. I'll give you three examples.

Example 5.3. Let $R(x, y)$ be a rational function without poles on the unit circle in $\mathbb{R}^2$. Then

$$\begin{aligned}\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta &= \frac{1}{i}\int_{|z|=1} R\Big(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\Big)\frac{dz}{z} \\ &= 2\pi \sum_{|\zeta|<1} \operatorname{Res}\Big(R\Big(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2}\Big)z^{-1}\Big)\Big|_{\zeta}.\end{aligned}$$

Example 5.4. Let $R(x)$ be a rational function with no poles on $\mathbb{R}$, and assume that $|R(x)| = O(x^{-2})$ as $x \to \pm\infty$. For $a \in \mathbb{R}$, we want to compute $\int_{-\infty}^{\infty} e^{iax}R(x)dx$. We may assume $a \ge 0$ because otherwise we can take the complex conjugate. Then e^{iaz} is bounded on $\Im z > 0$. For $r \gg 0$, let $D_r = \{z \in \mathbb{C} : |z| < r, \Im z > 0\}$. This is a Green's domain on $\mathbb{C}$. If r is large enough, $R(z)$ has no poles outside $\Delta(0, r)$. Then by the Residue Theorem,

$$\begin{aligned}2\pi i \sum_{\Im z>0} \operatorname{Res}(e^{ia\zeta}R(\zeta))|_z &= \int_{\partial D_n} f(z)e^{iaz}dz \\ &= \int_{-r}^{r} e^{iax}R(x)dx + \int_{|z|=r, \Im z\ge 0} e^{iaz}R(z)dz.\end{aligned}$$

Because

$$\left|\int_{|z|=r,\Im z\geq 0} e^{iaz}R(z)dz\right| \leq \pi r \sup_{|z|=r}|zR(z)| = O(rR(r)) \leq \text{const}\cdot r^{-1}.$$

So the second integral goes to zero as $r\to\infty$, and therefore

$$\int_{-\infty}^{\infty} e^{iax}R(x)dx = -2\pi i \sum_{\Im z>0} \operatorname{Res}(e^{ia\zeta}R(\zeta))|_z.$$

Example 5.5. Let $R(x)$ be a rational function without poles on $\mathbb{R}_{\geq 0}$. Let $s\notin\mathbb{N}$, $s>0$ be a real number and suppose that $|R(x)| = O(x^{-s})$ as $x\to+\infty$. Because s is not an integer, the integral

$$\int_0^\infty x^{s-1}R(x)dx$$

converges absolutely. We want to evaluate this domain. Choose $0<\delta\ll 1$, $1\ll r<\infty$, and $0<\eta\ll 1$ so that all poles of R lie in $D(r,\delta,\eta) = \{z : \delta < |z| < r, 0 < \arg z < 2\pi-\eta\}$. This is a Green's domain in $\mathbb{C} - \{re^{-\eta i/2} : 0\leq r<\infty\}$ and contains all poles of $R(z)$. Because $\log z$ has a wel-defined branch around D, the function z^{1-s} also has a well-defined holomorphic determination with agrees with the normal definition on $\mathbb{R}_{>0}$. Then

$$\begin{aligned}2\pi i\sum_z \operatorname{Res}(\zeta^{s-1}R(\zeta))|_z &= \int_{\partial D} z^{s-1}R(z)dz\\ &= \int_\delta^r x^{s-1}R(x)dx - e^{i(2\pi-\eta)s}\int_\delta^r x^{s-1}R(e^{i(2\pi-\eta)}x)dx\\ &+ \int_0^{2\pi-\eta} re^{i\theta(s-1)}R(re^{i\theta})rde^{i\theta} - \int_0^{2\pi-\eta}\delta e^{i\theta(s-1)}R(\delta e^{i\theta})\delta de^{i\theta}.\end{aligned}$$

First letting $\eta\to 0$, we get

$$\begin{aligned}2\pi i\sum_z \operatorname{Res}(\zeta^{s-1}R(\zeta))|_z &= (1-e^{2\pi is})\int_\delta^r x^{s-1}R(x)dx\\ &+ \int_0^{2\pi} r^{s-1}e^{i\theta(s-1)}R(re^{i\theta})rde^{i\theta} - \int_0^{2\pi}\delta^{s-1}e^{i\theta(s-1)}R(\delta e^{i\theta})\delta de^{i\theta}.\end{aligned}$$

Letting $r\to\infty$ and $\delta\to 0$, we get

$$2\pi i\sum_z \operatorname{Res}(\zeta^{s-1}R(\zeta))|_z = (1-e^{2\pi is})\int_0^\infty x^{s-1}R(x)dx.$$

Although this method does not work for s complex, if we get an holomorphic answer, then the answer for s complex should also be the same.

6 September 20, 2016

Recall that for an open set $U \subseteq \mathbb{C}$, a function f being meromorphic on U means having isolated singularities, all of which are poles (or removable).

Definition 6.1. For a $z_0 \in U$, the **order** of f at z_0 is

(i) $n \geq 1$ if f has a zero of order n,

(ii) 0 if f is holomorphic on some neighborhood of z_0 with $f(z_0) \neq 0$,

(iii) $-n$ with $n \geq 1$ if f has a pole of order n at z_0.

Locally this means that $f(z) = (z - z_0)^n g(z)$ with $g(z)$ holomorphic on a neighborhood of z_0 and $g(z_0) \neq 0$. Then the logarithmic derivative is

$$\frac{f'}{f}(z) = \frac{n}{z - z_0} + \frac{g'}{g}(z).$$

Because g'/g is holomorphic near z_0, we see that $\mathrm{Res}(f'/f)|_{z_0}$ is the order of f at z_0.

6.1 The argument principle and total order

Theorem 6.2 (Argument principle)**.** *Let $D \subset U$ be a Green's domain such that f has no zeros or poles on ∂D. Then*

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f} dz$$

is the total order of f in D, i.e., the total number of zeros in D minus the total number of poles, both counted with appropriate multiplicity.

Proof. This follows from the Residue theorem. $\square$

Why is this called the argument principle? We can write $f(z) = r(z)e^{i\theta(z)}$ for some C^∞ functions r and θ, except at zeros and poles. On ∂D this is well-defined with $r > 0$, and $(f'/f)dz = dr/r + id\theta$ so

$$\int_{\partial D} \frac{f'}{f} dz = 0 + i \int_{\partial D} d\theta = i(\text{total variation of } \theta \text{ along } \partial D).$$

Corollary 6.3 (Rouche's theorem)**.** *Let $U \subseteq \mathbb{C}$ be open and connected, with f, g meromorphic functions on U. Let $D \subset U$ be a Green's domain such that f and g has no zeros nor poles on ∂D. Suppose $|f(z) - g(z)| < |g(z)|$ for $z \in \partial D$. Then the total order of f and g in D coincide.*

Proof. On ∂D, $|(f/g)(z) - 1| < 1$. This means that the image of ∂D under f/g in $\mathbb{C}$ lies in $\Delta(1,1)$, which is in $\mathbb{C} - \mathbb{R}_{\geq 0}$. This means that $\log(f/g)$ has a well-defined branch on ∂D. Then

$$\frac{1}{2\pi i} \int_{\partial D} \Big(\frac{f'}{f} - \frac{g'}{g} \Big) dz = \int_{\partial D} d \log \frac{f}{g} = 0,$$

which is the difference between the total order of f and total order of g. $\square$

Corollary 6.4 (Fundamental theorem of calculus)**.** *Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$. Then $P(z)$ has exactly n zeros in $\mathbb{C}$, counted with multiplicity.*

Proof. Suppose $R > 0$ and very big. Then on $\partial\Delta(0, R)$,

$$|P(z) - a_n z^n| \leq \sum_{k=0}^{n-1} |a_k| R^k < |a_k| R^n$$

if R is very large. Apply Rouche's theorem on $\Delta(0, R)$ and find that $P(z)$ has as many zeros in $\Delta(0, R)$ as z^n. □

Corollary 6.5 (Open mapping theorem)**.** *Suppose $U \subseteq \mathbb{C}$ is open, connected, and $f : U \to \mathbb{C}$ is holomorphic. Let $z_0 \in U$ and $w_0 = f(z_0)$. For every sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that for $w \in \Delta(w_0, \delta)$, $f(z) - w$ has exactly many zeros (counted with multiplicity) in $\Delta(z_0, \epsilon)$ as the order of zero of $f(z) - w_0$ at z_0.*

Proof. We know that there exists $\eta > 0$ such that $\Delta(z_0, \eta)$ is contained in U, and $f(z) - w_0 \neq 0$ on $\Delta^*(z_0, \eta)$. For $0 < \epsilon < \eta$, the disc $\Delta(z_0, \epsilon) \subset U$ is a Green's domain, and $f(z) - w_0 \neq 0$ on $\partial\Delta(z_0, \epsilon)$. Let $\delta = \min_{|z - z_0| = \epsilon} |f(z) - w_0| > 0$. Now suppose $w \in \Delta(w_0, \delta)$. Then for $z \in \partial\Delta(z_0, \epsilon)$, $w \in \Delta(w_0, \delta)$, we have $|w - w_0| < \delta \leq |f(z) - w_0|$. Then

$$|(f(z) - w) - (f(z) - w_0)| < |f(z) - w_0|.$$

Now apply Rouche, with $f(z) - w$ playing the role of f and $f(z) - w_0$ the role of g. This implies that the total number of $f(z) - w_0$ on $\Delta(z_0, \epsilon)$ is equal to the order of zero of $f(z) - w_0$ at z_0. □

6.2 Projective 1-space

I should say something about what is coming up. We are going to define the projective 1-space and define holomorphic/meromorphic functions on this space. Then we are going to talk about automorphisms of the unit disc. Although this may seem unmotivated, this is important because if you have a Riemannian surface, its universal cover is either the projective 1-space, $\mathbb{C}$, or the unit disc. In particular, the proof of the Picard theorems will depend on this.

The group $\mathbb{C}^* = \mathbb{C} - \{0\}$ acts on $\mathbb{C}^2 - \{0\}$ by scalar multiplication.

Definition 6.6. We define $\mathbb{P}^1$ (or simply $\mathbb{P}^1$ as $\mathbb{C}^* \setminus (C^* - \{0\})$ which is the set of lines through the origin in $\mathbb{C}^2$.

The group $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{C}^2 - \{0\}$, and this commutes with the $\mathbb{C}^*$ action, and hence induces an action on $\mathbb{P}$. Topologize $\mathbb{P}$ by giving it the quotient topology induced by $\mathbb{C}^2 - \{0\} \to \mathbb{P}$. Define $\varphi_0 : \mathbb{C} \to \mathbb{P}$, $\varphi_\infty : \mathbb{C} \to \mathbb{P}$ by $\varphi_0(z) =$ line through $\binom{z}{1}$ and $\varphi_\infty(z) =$ line through $\binom{1}{z}$. Then

(i) $\varphi_0, \varphi_\infty$ are continuous and injective,

(ii) $\operatorname{im}\varphi_0 \cup \operatorname{im}\varphi_\infty = \mathbb{P}$,

(iii) $\varphi_\infty^{-1}\varphi_0(\mathbb{C}) = \mathbb{C}^*$, $\varphi_0^{-1}\varphi_\infty(\mathbb{C}) = \mathbb{C}^*$,

(iv) $\varphi_0^{-1}\varphi_\infty(\mathbb{C}^*) = \mathbb{C}^* = \varphi_\infty^{-1}\varphi_0(\mathbb{C}^*)$,

(v) for any $z \in \mathbb{C}^*$, $\varphi_0^{-1}\varphi_\infty(z) = 1/z$, $\varphi_\infty^{-1}\varphi_0(z) = 1/z$,

(vi) via φ_0, we can identify $\mathbb{C} \cup \{\infty\} \to \mathbb{P}^1$.

The proof is an exercise.

Suppose $U \subseteq \mathbb{P}$ is open. Then define the notion of a holomorphic/meromorphic/function with isolated singularities as those f for which $f \circ \varphi_0$ and $f \circ \varphi_\infty$ have those property on $\mathbb{C}$. Equivalently, this is when

(i) f has these properties on $U \cap \mathbb{C}$ and

(ii) $z \mapsto f(1/z)$ has these properties on some neighborhood of 0, when $\infty \in U$.

Recall the notion of residue of a function at isolated singularity z_0:

$$\frac{1}{2\pi i}\int_{|z-z_0|=\delta} f(z)dz.$$

So I can, and should, think the residue as associated to the holomorphic 1-form $f(z)$ with isolated singularities.

Then a special case of the residue theorem states:

Theorem 6.7. *Suppose fdz is a holomorphic 1-form with isolated singularities on $\mathbb{P}$. Then*

$$\sum_{z_0 \in \mathbb{P}} \operatorname{Res}(fdz)|_{z_0} = 0.$$

Proof. Choose $r > 0$ so big that fdz has no singularity outside $\operatorname{clos}\Delta(0,r)$ except possibly at ∞. Then

$$\frac{1}{2\pi i}\int_{\partial\Delta(0,r)} f(z)dz = \sum_{z_0 \in \Delta(0,r)} \operatorname{Res}(fdz)|_{z_0}.$$

So we need to show that this integral also equals $-\operatorname{Res} fdz|_\infty$. This is because a small circle around infinity is a large circle, with orientation reserved by $z \mapsto 1/z$. $\square$

Corollary 6.8. *Suppose fdz is a holomorphic 1-form with isolated singularities on $\mathbb{P}$. Then the total number of zeros minus the total number of poles of f'/fdz is equal to zero.*

Suppose f is a meromorphic function on $\mathbb{P}$. Then we can choose a polynomial $p(z)$ which as exactly the same zeros as f on $\mathbb{C}$. Likewise, we can choose a polynomial $q(z)$ that has exactly the same poles as f on $\mathbb{C}$. Then the function

$$h = f\frac{q}{p}$$

has no zeros or poles on $\mathbb{C}$, but it may have zero or pole at ∞. So either h or $1/h$ has no poles on $\mathbb{P}$. So $h : \mathbb{P} \to \mathbb{C}$ is holomorphic, possibly zero at ∞. Thus h is constant by the maximal principle. Thus we have proved:

Theorem 6.9. *Any meromorphic function defined on $\mathbb{P}$ is rational, i.e., is a quotient of polynomials.*

Essentially the same argument proves

Theorem 6.10 (Liouville's theorem)**.** *Any bounded entire function (i.e., holomorphic function on $\mathbb{C}$) is constant.*

Proof. Think of it as a function on $\mathbb{P}$ with possibly a singularity at ∞. That singularity is removable, and so we can use maximal principle. $\square$

7 September 22, 2016

Theorem 7.1. *If $f(z)dz$ is a meromorphic 1-form on $\mathbb{P}$, then $\sum_{\zeta\in\mathbb{P}} \operatorname{Res}(f(z)dz)|_\zeta = 0$.*

Now suppose f is a meromorphic function on $\mathbb{P}$. Then df/f is a meromorphic 1-form and thus has sum of residues zero. What about $\operatorname{Res}(df/f)|_\infty$? Let $w = 1/z$. Then $w = 0$ corresponds to $z = \infty$, and so

$$\operatorname{Res} \frac{df}{f}\Big|_\infty = \operatorname{Res} \frac{\partial f/\partial w}{f} dw\Big|_{w=0} = \text{order of } f \text{ at } \infty.$$

So we conclude that $\sum_{\zeta\in\mathbb{P}} \operatorname{ord} f|_\zeta = 0$. Indeed, df/f is invariant of local coordinates and its residue is the order of f.

7.1 Automorphisms of $\mathbb{P}$

We have define the notion of meromorphic/holomorphic functions on open $U \subseteq \mathbb{P}$. What about the notion of a holomorphic map $F : U \to \mathbb{P}$? First of all, we require continuity. Suppose $z_0 \in U$. There are two cases:

(i) $\lim_{z\to z_0} F(z) \in \mathbb{C}$ - then F holomorphic near z_0 as map into $\mathbb{P}$ is equivalent to F holomorphic near z_0 as a map into $\mathbb{C}$.

(ii) $\lim_{z\to z_0} F(z) = \infty$ - then F holomorphic near z_0 is equivalent to $1/F$ being holomorphic near z_0 as a map to $\mathbb{C}$.

So we have a notion of an automorphism $F : \mathbb{P} \to \mathbb{P}$, which means that it is continuous, bijective, and holomorphic in both directions.

Recall that $GL(2,\mathbb{C})$ acts on $\mathbb{C}^2$, and the actions commutes with scalar multiplication. So $GL(2,\mathbb{C})$ acts as a group of automorphisms on $\mathbb{P}$. The center of this action is $\mathbb{C}^*\mathbf{1}$.

We have identified $\mathbb{P} \cong \mathbb{C} \cup \{\infty\}$, with z corresponding to the line $\binom{z}{1}$. Then a matrix g act as

$$gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \frac{az+b}{cz+d}.$$

These are call the **linear fractional transformations**. This makes sense even when $cz + d = 0$, because we can interpret it as ∞.

We denote

$$SL(2,\mathbb{C}) = \{g \in GL(2,\mathbb{C}) : \deg g = 1\},$$
$$PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\text{center} = GL(2,\mathbb{C})/\mathbb{C}^* = SL(2,\mathbb{C})/\{\pm 1\}.$$

Theorem 7.2. *The action of $GL(2,\mathbb{C})$ on $\mathbb{P}$ induces an isomorphism $SL(2,\mathbb{C})/\{\pm1\} \cong \operatorname{Aut}(\mathbb{P})$. Furthermore, $SL(2,\mathbb{C})$ acts on $\mathbb{P}^1$ in a triply transitive manner.*

Proof. Suppose $g \in SL(2,\mathbb{C})$ acts trivially: $(az+b)/(cz+d) = z$. Then setting $z = 0$ and $z = \infty$, we see that $b = c = 0$. So the map $SL(2,\mathbb{C})/\{\pm 1\} \to \mathrm{Aut}(\mathbb{P})$ is injective.

Now suppose $\mathbb{P} \to \mathbb{P}$ is an automorphism. Then $z \mapsto F(z)$, $z \mapsto F(1/z)$, $z \mapsto 1/F(z)$, and $z \mapsto 1/F(1/z)$ are holomorphic where defined. That is, F can be interpreted as a meromorphic function. Since $F : \mathbb{P}^1 \to \mathbb{P}^1$ is bijective, it must have a single zero and a single pole. That is, $F(z) = az + b/cz + d$. This shows that $SL(2,\mathbb{C})/\{\pm 1\} \to \mathrm{Aut}(\mathbb{P})$ is surjective.

Why is this action triply transitive? It suffices to show that if $z_0, z_1, z_\infty \in \mathbb{P}$ are distinct, then there exists a $g \in SL(2,\mathbb{C})$ such that $gz_0 = 0, gz_1 = 1, gz_\infty = \infty$. Define the linear fractional transformation g as

$$z \mapsto \frac{z - z_0}{z - z_\infty}\frac{z_1 - z_\infty}{z_1 - z_0}.$$

By construction, this sends $z_0 \to 0$, $z_\infty \to \infty$, and $z_1 \mapsto 1$. $\square$

7.2 Conjugacy classes of matrices

To compute the conjugacy classes of $SL(2,\mathbb{C})$, we can use the Jordan canonical form. Any $g \in SL(2,\mathbb{C})$ is conjugate to one of the following:

$$\pm 1, \quad \begin{pmatrix} \lambda & 0 \\ 0, \lambda^{-1} & \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The first one is trivial, and the second one which is $z \mapsto \lambda^2 z$ has two fixed points 0 and ∞. The third one has only one fixed point ∞, and it is the unipotent one.

Define

$$SU(2) = \left\{\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\right\},$$
$$SU(1,1) = \left\{\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\right\}.$$

These are all contained in $SL(2,\mathbb{C})$. Also $SU(2)$ is the group of matrices which preserves the standard inner product on $\mathbb{C}^2$, of determinant 1, and $SU(1,1)$ is the group of matrices which preserves the standard indefinite hermitian form on $\mathbb{C}^2$, of determinant 1.

Theorem 7.3. *The action of $SU(2)$ on $\mathbb{P}$ is transitive and induces $SU(2)/\{\pm 1\} \hookrightarrow \mathrm{Aut}(\mathbb{P})$. Any finite subgroup $\Gamma \subset SL(2,\mathbb{C})$ is $SL(2,\mathbb{C})$ conjugate to a subgroup of $SU(2)$.*

Proof. Suppose $\Gamma \subset SL(2,\mathbb{C})$ is a finite subgroup. Define an inner product $(,)_\Gamma$ in terms of the standard one on $\mathbb{C}^2$ as

$$(v_1, v_2)_\Gamma = \frac{1}{\#\Gamma}\sum_{\gamma \in \Gamma}(\gamma v_1, \gamma v_2).$$

Then $(\gamma v_1, \gamma v_2)_\Gamma = (v_1, v_2)_\Gamma$, so this new $(,)_\Gamma$ is Γ-invariant. Any two inner products on $\mathbb{C}^2$ are $GL(2,\mathbb{C})$ conjugate since there is an orthonormal basis, up to $GL(2,\mathbb{C})$ conjugacy, the two are the same. So under some $GL(2,\mathbb{C})$ conjugate, and hence $SL(2,\mathbb{C})$ conjugate, Γ preserves $(,)$ i.e. is a subgroup of $SU(2)$. ☐

Constructing this new invariant from an old one is called **Weyl's unitary trick**.

Proposition 7.4. *Any $g \in SU(2)$ is conjugate to*

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}$$

for some $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof. g will have an eigenvector v_1 with eigenvalue absolute value 1. Then take a unit vector that is perpendicular to v_1. Then v_2 will be an eigenvector, corresponding to the reciprocal of the previous eigenvalue. ☐

7.3 Automorphisms of Δ

From now one, we will write

$$\Delta = \Delta(0,1), \quad H = \{z \in \mathbb{C} : \Im z > 0\}, \quad c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in SU(2).$$

This c is called the **Cayley transform**.

Lemma 7.5. *The map c is an isomorphism from H to Δ.*

Proof. For $x \in \mathbb{R} \cup \{\infty\}$,

$$c \cdot x = \frac{x - i}{-ix + 1} = i \frac{x - i}{x + i} \in \partial\Delta.$$

So c maps $\mathbb{R} \cup \{\backslash\}$ to $\partial\Delta$ isomorphically. So c must map H onto one of the connected components of $\mathbb{P} - \partial\Delta$. But because $c \cdot i = 0$, c maps H to Δ. ☐

Theorem 7.6. *(a) The action of $SU(1,1)$ on $\mathbb{P}$ preserves Δ, is transitive on Δ, and induces an isomorphism $SU(1,1)/\{\pm 1\} \to \mathrm{Aut}(\Delta)$.*

There are more statements, but let me just state and prove this one.

Proof. $SU(1,1)$ contains the two subgroups

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

The former acts by rotation, and the latter maps $0 \mapsto \tanh t$. So $SU(1,1)$ acts transitively on Δ.

Suppose $F : \Delta \to \Delta$ is an automorphism. By Schwartz's lemma, we see that $|F(z)| \leq 1$ for all $z \in \Delta$ implies $|F'(0)| \leq 1$. By symmetry, between F and F^{-1}, we see that $|1/F'(0)| \leq 1$ and so $|F'(0)| = 1$. This implies that $F(z) = e^{2i\theta} z$. Therefore

$$F(z) = e^{2i\theta} z = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}.$$

$\square$

8 September 27, 2016

8.1 Automorphisms and conjugacy classes

There is a bijection

$$c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} : H \to \Delta.$$

Theorem 8.1. *(a) The action of* $\mathrm{SU}(1,1)$ *on* $\mathbb{P}$ *preserves* $\Delta \subseteq \mathbb{P}$*. This action is transitive on* Δ*, and induces* $SU(1,1)/\{\pm 1\} \cong \mathrm{Aut}(\Delta)$*.*

(b) Similarly, $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\} \cong \mathrm{Aut}(H)$*.*

(c) The isotropy groups of $\mathrm{SU}(1,1)$ *acting on* Δ *and those of* $\mathrm{SL}(2,\mathbb{R})$ *acting on* H *are connected and compact.*

(d) $c^{-1}\,\mathrm{SU}(1,1)c = \mathrm{SL}(2,\mathbb{R})$*.*

(e) Every $g \in \mathrm{SU}(1,1)$ *is conjugate in* $\mathrm{SU}(1,1)$ *to exactly one of the following:*

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (center)}, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ for } -\pi < \theta < 0 \text{ or } 0 < \theta < \pi \text{ (elliptic)},$$

$$\pm \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \text{ for } 0 < t < \infty \text{ (hyperbolic)}, \quad \pm \begin{pmatrix} 1 \pm \frac{i}{2} & \pm\frac{1}{2} \\ \pm\frac{1}{2} & 1 \mp \frac{i}{2} \end{pmatrix} \text{ (unipotent)}.$$

(f) Every $g \in \mathrm{SL}(2,\mathbb{R})$ *is conjugate, in* $\mathrm{SL}(2,\mathbb{R})$*, to exactly one of:*

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ for } -\pi < \theta < 0 \text{ or } 0 < \theta < \pi,$$

$$\pm \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ for } 0 < t < \infty, \quad \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. (a) Suppose

$$g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in \mathrm{SU}(1,1),$$

and let $e^{i\theta} \in \partial\Delta$. Then

$$g \cdot e^{i\theta} = \frac{\alpha e^{i\theta} + \beta}{\bar\beta e^{i\theta} + \bar\alpha} = e^{-i\theta} \frac{\alpha e^{i\theta} + \beta}{\bar\alpha e^{-\theta} + \bar\beta} \in \partial\Delta.$$

So $g : \partial\Delta \to \partial\Delta$. So either g maps Δ to Δ or $\mathbb{P} - \mathrm{clos}(\Delta)$. But $g \cdot 0 = \beta/\bar\alpha$ with $|\alpha|^2 - |\beta|^2 = 1$, and so $g \cdot 0 \in \Delta$. This shows that $g : \Delta \to \Delta$, and the rest of (a) was proved last time.

(b) The action of $SL(2,\mathbb{R})$ fixes $\mathbb{R} \cup \{\infty\}$. Hence any $g \in \mathrm{SL}(2,\mathbb{R})$ maps H to either H or $\mathbb{P} - \mathrm{clos}(H)$. But

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : i \mapsto \frac{ai + b}{ci + d} = \frac{(ad - bc)i}{c^2 + d^2} + \frac{bd + ac}{c^2 + d^2}$$

which is in H. So $g : H \to H$ is a homeomorphism, and we get $\mathrm{SL}(2,\mathbb{R})/\pm 1 \hookrightarrow \mathrm{Aut}(H)$. Since H and Δ are isomorphic, every automorphism of H comes from some $g \in \mathrm{SL}(2,\mathbb{C})$. Suppose $g \in \mathrm{SL}(2,\mathbb{C})$ maps $H \to H$. Then $g : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$. $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ in a triply transitive manner. So there exists $g_1 \in \mathrm{SL}(2,\mathbb{R})$ such that $g \cdot 0 = g_1 \cdot 0$, $g \cdot 1 = g_1 \cdot 1$, and $g \cdot \infty = g_1 \cdot \infty$. Then $g_1^{-1} \circ g$ fixes 0, 1, ∞ individually. Therefore $g_1^{-1} \circ g = \pm 1$, so $g \in \mathrm{SL}(2,\mathbb{R})$. Hence $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\} \cong \mathrm{Aut}(H)$. The fact that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on H follows from the corresponding fact about $\mathrm{SU}(1,1)$ action on Δ, since $H \cong \Delta$.

(d) This follows from (a), (b), and the existence of $c : H \cong \Delta$.

(c) It suffices to prove this for the $\mathrm{SU}(1,1)$-action on Δ, by (d). Since $\mathrm{SU}(1,1)$ acts transitively on Δ, it suffices to show that the isotropy subgroup of $\mathrm{SU}(1,1)$ at 0 is

$$\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

This is clear because

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

maps 0 to 0, then $\beta/\bar{\alpha} = 0$ or $\beta = 0$. Then $|\beta|^2 = 1$ because $|\beta|^2 - |\alpha|^2 = 1$.

(e) + (f) The Jordan canonical form applies to real 2×2 matrices which have real eigenvalues. So any $g \in \mathrm{SL}(2,\mathbb{R})$ which as real eigenvalues λ, λ^{-1} is $\mathrm{GL}(2,\mathbb{R})/\text{center} \cong \mathrm{SL}^{\pm}(2,\mathbb{R})/\{\pm 1\}$. Suppose $g \in \mathrm{SL}(2,\mathbb{R})$ has eigenvalues $\lambda, \lambda^{-1} \in \mathbb{R}$. Then g is $\mathrm{SL}^{\pm}(2,\mathbb{R})$-conjugate to exactly one of

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ for } 0 < t < \infty.$$

Now there is a small problem of whether the elements of $\mathrm{SL}(2,\mathbb{R})$ will be, not $\mathrm{SL}^{\pm}(2,\mathbb{R})$-conjugate but $\mathrm{SL}(2,\mathbb{R})$-conjugate to one of these. Because $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ fixes both $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$, things that are $\mathrm{SL}^{\pm}(2,\mathbb{R})$-conjugate to them are also $\mathrm{SL}(2,\mathbb{R})$-conjugate to them. However, conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ sends $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. So the question is whether they are $\mathrm{SL}(2,\mathbb{R})$-conjugate or not. It turns out that they are not.

Now let us consider the case when $g \in \mathrm{SL}(2,\mathbb{R})$ has eigenvalues λ, μ, not both real. Then $\{\lambda, \mu\} = \{\bar{\lambda}, \bar{\mu}\}$ and $\lambda\mu = 1$, so $\lambda = e^{i\theta}$ for some $\theta \notin \pi\mathbb{Z}$. Since the eigenvalues are distinct, there are two linearly independent eigenvectors in $\mathbb{C}^2$. The must be each other's complex conjugate. Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on pairs of linearly independent vectors in $\mathbb{R}^2$, up to $\mathrm{SL}(2,\mathbb{R})$-conjugacy we may assume the eigenvectors are $\binom{\pm i}{1}$ (as points in $\mathbb{P}$). Since $c : H \to \Delta$ maps i to 0, asking which $g \in \mathrm{SL}(2,\mathbb{R})$ fix i is the same as asking which $g \in \mathrm{SU}(1,1)$ fix 0. They are precisely those of the form $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$, and so the isotropy subgroup of $\mathrm{SL}(2,\mathbb{R})$ at i is

$$\left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

The question remains whether $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ SU$(1,1)$-conjugate to $\left(\begin{smallmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{smallmatrix}\right)$. It turns out that they are not SU$(1,1)$-conjugate to each other, and so $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ is not SL$(2,\mathbb{R})$-conjugate to its inverse. This proves (f), and an easy computation translates (f) to (e). $\square$

Note that

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} z = e^{2i\theta} z, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z = z+1, \quad \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} z = e^{2t} z.$$

Elliptic elements of SU$(1,1)$, respectively SL$(2,\mathbb{R})$ have exactly one fixed point in Δ, respectively H, and the other fixed point in $\mathbb{P}-\mathrm{clos}(\Delta)$, respectively $\mathbb{P}-\mathrm{clos}(H)$. Hyperbolic elements elements of SU$(2,\mathbb{R})$, respectively SL$(2,\mathbb{R})$ has two fixed points, located at the boundary $\mathbb{R}\cup\{\infty\}$, respectivly $\partial\Delta$. Unipotent elements have one fixed point on the boundary.

Theorem 8.2.

$$\mathrm{Aut}(\mathbb{C}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

with the latter acting by $z \mapsto az+b$.

Proof. I leave it as an exercise. What needs to be proved is that any continuous function on $\mathbb{C}$ to $\mathbb{P}$. $\square$

9 September 29, 2016

Suppose X is a Hausdorff space, and Γ a group acting on X by homeomorphisms. We say that Γ **acts without fixed points** if for any $x_0 \in X$ and $\gamma \in \Gamma$, $\gamma x_0 = x_0$ implies that γ acts on X as the identity map. (It is more common to require $\gamma = e$ in this case, but we are using this definition because we want to allow centers.) We also say that Γ **acts properly discontinuously**, if for every $x_0 \in X$ there exists an open neighborhood U_0 of x_0 such that $\{\gamma \in \Gamma : \gamma U \cap U \neq \emptyset\}$ is finite. It is easy to see that Γ acting property discontinuously is equivalent to (i) $\{\gamma \in \Gamma : \gamma x_0 = x_0\}$ is finite and (ii) there exists an open subset U_0 such that $\gamma U_0 \cap U_0 = \emptyset$ if $\gamma x_0 \neq x_0$, and $\gamma \in \Gamma$ and $\gamma x_0 = x_0$ implies $\gamma U_0 = U_0$.

As an exercise, prove the following fact, which is tricky. Suppose X is locally compact and Hausdorff, and Γ acts properly continuously. Then the map $X \to \Gamma \setminus X$ induces a Hausdorff topology on $\Gamma \setminus X$. If Γ acts also without fixed points, then $X \to \Gamma \setminus X$ is a covering map.

9.1 Riemann surfaces

Now suppose that X is a connected **Riemann surface**, meaning that it is a 1-dimensional complex manifold. Let $\tilde{X}$ be its universal cover. Then $\tilde{X}$ clearly is a Riemann surface as well, and $\Gamma = \pi_1(X)$ acts on $\tilde{X}$ by biholomorphisms with a natural identification $X = \Gamma \setminus \tilde{X}$.

Theorem 9.1 (Köbe, 1907)**.** *Up to isomorphism, the only connected, simply connected Riemann surfaces are $\mathbb{P}$, $\mathbb{C}$, and $H \cong \Delta$.*

The notion of the Riemann surface actually was established after 1907, but once it was established, it was clear that Köbe had proved this theorem.

The only Riemann surface with covering space $\mathbb{P}$ is $\mathbb{P}$, because any automorphism of $\mathbb{P}$ has a fixed point.

To understand those connected Riemann surfaces whose universal covering space is $\mathbb{C}$, we need to know the group of automorphisms of $\mathbb{C}$ which act properly discontinuously without fixed points. $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b\}$, and so the only such subgroups are the additive group $\mathbb{C}$ which are free of rank 1 or 2. Therefore these Riemann surfaces are, up to isomorphism,

$$\mathbb{C}, \quad \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*, \quad \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \text{ with } \tau \in H.$$

The third one is an elliptic curve, and they contain a lot of arithmetic information.

Now let us look at the Riemann surfaces whose universal covering is H. They are all of the form $\Gamma \setminus H$ where $\Gamma \subseteq \mathrm{SL}(2, \mathbb{R})$ is a discrete subgroup, containing -1, and not containing elliptic elements (this implies Γ acts properly discontinuously, without fixed points).

9.2 Mittag-Leffler theorem

Let $U \subseteq \mathbb{C}$ be open.

Lemma 9.2. *Let $\{f_m\}$ be a sequence of holomorphic functions on U which converges locally uniformly on U to a function f. Then f is holomorphic, and $f_n' \to f'$ also locally uniformly.*

Proof. We already saw that f is holomorphic, by using the characteristic of integrating over an rectangle. Now suppose $z_0 \in U$, and choose $r > 0$ so that $\operatorname{clos}(\Delta(z_0, r)) \subseteq U$. Then $\Delta(z_0, r)$ is a Green's domain, and for $z \in \Delta(z_0, r)$,

$$f_n'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta) d\zeta}{(\zeta - z)^2}$$

and similarly for f. Because $f_n \to f$ uniformly on $\partial\Delta(z_0, r)$, and $1/(\zeta - z)^2$ is bounded for $|z - z_0| < r/2$, we see that $f_n' \to f'$ uniformly. $\square$

Let $\{f_n\}$ be a sequence of meromorphic functions on U.

Definition 9.3. The sequence $\{f_n\}$ is **locally unipolar** if each $z_0 \in U$ has an open neighborhood V_0 such that, for an appropriately chosen integer $n_0 = n_0(z_0, V_0)$, $n, m \geq n_0$ implies that $f_m - f_n$ is holomorphic on V_0.

Definition 9.4. The sequence $\{f_n\}$ is **unipolar** on U, if for each $z_0 \in U$ there exists n_0, V_0 as in the previous definition, and in addition $f_n - f_{n_0}$ converges locally uniformly on V_0. In this situation, define $\lim_{n\to\infty} f_n = f$ so that with V_0, n_0 as abouve, $\lim_{n\to\infty}(f_n - f_{n_0}) = f - f_{n_0}$ on V_0.

This definition of the limit makes sense, because (i) if $n_1 > n_0$ then $(f_n - f_{n_0}) = (f_n - f_{n_1}) + (f_{n_1} - f_{n_0})$ and (ii) restricting the domain does not affect the limit. As a direct consequence of the definition, $f_n \to f$ locally uniformly on $U - \{\text{poles of } f\}$. Hence $f_n' \to f'$ locally uniformly on this open set.

Suppose $\{z_k\}_{k\geq 0}$ is a set of distinct points in $\mathbb{C}$ without points of accumulation. Equivalently, $z_k \to \infty$ in $\mathbb{P}$. Suppose further, for each k we are given a potential principal part at z_k, $P_k(z) = \sum_{\ell=1}^{N_k} a_{k,l}(z - z_0)^{-l}$. Then $P_k(z)$ is holomorphic on $\mathbb{C} - \{z_k\}$, and hence on $\Delta(0, |z_k|)$. Now choose $r_k \geq 0$ and $\epsilon_k > 0$ such that $0 < r_k < |z_k|$ unless $z_k = 0$ in which case $r_k = 0$, and $r_k \to +\infty$ as $k \to +\infty$, and $\sum \epsilon_k < \infty$. Then we can choose Q_k for each k, a polynomial obtained by taking enough terms of the Taylor expansion of P_k at 0 so that $|P_k(z) - Q_k(z)| < \epsilon_k$ on $\Delta(0, |z_k|)$.

Let

$$s_n = \sum_{k=0}^{n} (P_k(z) - Q_k(z)).$$

This function has poles at $z_1, \ldots, z_n$ with principal part P_j at z_j. Also

$$|s_m(z) - s_n(z)| \leq \sum_{k=m}^{n} |P_k(z) - Q_k(z)| \leq \sum_{k=m}^{n} \epsilon_k$$

for $m > n$. So we can conclude that $\{s_n(z)\}$ is unipolar, and converges uniformly locally on $\mathbb{C}$. This proves the following theorem.

Theorem 9.5 (Mittag-Leffler)**.** *Let $\{z_k\}$ be a sequence of distinct points in $\mathbb{C}$, with $z_k \to \infty$ and $P_k(z)$ a sequence of potential principal parts at z_k. Then there exists a meromorphic function f on $\mathbb{C}$ which has principal part P_k at z_k, $k \geq 0$, and no other poles. This f can be expressed as*

$$f(z) = \sum_{k=0}^{\infty}(P_k(z) - Q_k(z)) + g(z),$$

with $g(z)$ entire.

For example, $f(z) = \pi^2/\sin^2 \pi z$ is meromorphic on $\mathbb{C}$ and has poles at $k \in \mathbb{Z}$ with principal part $(z-k)^{-2}$ at $z = k$. Note that for $|z| < |k|/2$, $|(z-k)^{-2}| < 4k^{-2}$, and we can choose $Q_k = 0$ and $\epsilon_k = 4k^{-2}$, $r_k = |k|/2$. So the conclusion is

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k=\infty}^{\infty} \frac{1}{(z-k)^2} + g(z).$$

What is $g(z)$? Note that $|\sin \pi z| = \frac{1}{2}|e^{i\pi x} - e^{-i\pi x + \pi y}|$ and so $|1/\sin \pi z| \to 0$ as $|y| \to \infty$ uniformly on $|x| \leq 1/2$. Also on $|x| \leq 1/2$ and $k > 0$, $|(z-k)^{-2}| \leq 4k^{-2}$, and so

$$\left|\sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}\right| \leq \left|\frac{1}{z^2}\right| + \sum_{k=1}^{\infty} \frac{\infty}{k^2},$$

and hence for all x because it is periodic of period 1. The conclusion is that g is periodic of period 1, and bounded on $\{|\Re(z)| \leq 1/2\}$ and hence globally bounded. Therefore g is a constant, and thus

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k=-\infty}^{\infty} \frac{1}{(k-z)^2} + c$$

for some c constant.

10 October 4, 2016

I had shown that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + c$$

for some constant c. We had also shown that $\pi^2/\sin^2 \pi z \to 0$ as $|y| \to +\infty$ uniformly in x. We now claim that $c = 0$, and it suffices to show that $\sum_{n=-\infty}^{\infty} \frac{1}{(iy-n)^2} \to 0$ as $|y| \to \infty$. This is because

$$\begin{aligned}
\Big|\sum_{n=-\infty}^{\infty} \frac{1}{(iy-n)^2}\Big| &\le \sum_{n=-\infty}^{\infty} \frac{1}{n^2+y^2} = \frac{1}{y^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+y^2} \\
&\le \frac{1}{y^2} + 2\sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dx}{x^2+y^2} = \frac{1}{y^2} + 2\int_0^{\infty} \frac{1}{n^2+y^2} \\
&= \frac{1}{y^2} + 2|y|^{-1} \int_0^{\infty} \frac{dx}{x^2+1} = O(|y|^{-1}).
\end{aligned}$$

This shows that $c = 0$.

Recall that $(d/dx)\cot x = -1/\sin^2 x$ on the real axis, so it is true also for complex z. Hence

$$\frac{d}{dz}\pi \cot \pi z = -\frac{\pi^2}{\sin^2 \pi z}.$$

Likewise $1/(z-n)^2 = -(d/dz)(1/(z-n))$. Then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = -\frac{d}{dz}\Big(\frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z-n} + \frac{1}{z+n}\Big)\Big)$$

because the sum in the right hand side converges, locally uniformly. This implies that

$$\pi \cot \pi z = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2-n^2} + c$$

and since both $\cot \pi z$ and the right hand size are odd functions of z, we have $c = 0$.

10.1 Mittag-Leffler on an open set

Suppose $U \subseteq \mathbb{C}$ is open and $\{z_k\}$ a sequence in U without points of accumulation in U. Let $z_k \neq z_l$ if $k \neq l$, and for each k consider a potential principal part $P_k(z) = \sum_{l=1}^{N_k} a_{k,l}(z-z_k)^{-l}$ at z_k.

Theorem 10.1 (Mittag-Leffler)**.** *There exists a meromorphic function f on U with principal part P_k at z_k and no other poles. It is determined up to addition of a holomorphic function on U.*

Proof. We may assume $U \neq \mathbb{C}$. Initially assume that ∂U is compact (but U need not compact) and $\{z_k\}$ is bounded. Let $\delta_k = \mathrm{dist}(\{z_k\}, \partial U) > 0$. I want to claim that $\delta_k \to 0$ as $k \to \infty$. If not, we can go to a subsequence of $\{z_k\}$ such that $\delta_k \geq \delta > 0$ for all k. Since $\{z_k\}$ is bounded, there must be a point of accumulation z_∞, which can be either on U or ∂U. But the latter is precluded by $\mathrm{dist}(\{z_k, \partial U\}) \geq \delta > 0$. This proves the claim.

Now for each k, choose $w_k \in \partial U$ such that $|z_k - w_k| = \delta_k$. Consider the Laurent series of $P_k(z)$ centered at w_k. Consider the Laurent series of $P_k(z)$ centered at w_k:

$$\frac{1}{z - z_k} = \frac{1}{(z - w_k) - (z_k - w_k)} = \frac{1}{z - w_k} \sum_{n=0}^{\infty} \Big(\frac{z_n - w_k}{z - w_k}\Big)^n = \sum_{n=1}^{\infty} \frac{(z_k - w_k)^{n-1}}{(z - w_k)^n}.$$

Given $\epsilon > 0$, we can choose M_k large enough so that

$$\Big|\frac{1}{z - z_k} - \sum_{n=1}^{M_k} \frac{(z_k - w_k)^{n-1}}{(z - w_k)^n}\Big| < \epsilon \text{ for } |z - w_k| \geq 2\delta_k.$$

Because we have estimated $1/(z - z_k)$ by a polynomial in $1/(z - w_k)$, we can get the same type of estimate for

$$\Big|\frac{1}{(z - z_k)^l} - \Big(\sum_{n=1}^{M_k} \frac{(z_k - w_k)^{n-1}}{(z - w_k)^n}\Big)^l\Big|$$

uniformly on $\{|z - w_k| \geq 2\delta_k\}$. So for each k there exists a polynomial without constant terms Q_k in $1/(z - w_k)$ such that $|z - w_k| \geq 2\delta_k$ implies $|P_k(z) - Q_k(z)| < 2^{-k}$.[2]

Q_k has no poles in U and P_k has a pole at z_k only, and $|z_k - w_k| = \delta_k \to 0$, with $w_k \in \partial U$. This implies that the partial sums of $\sum_{k=1}^{\infty} (P_k(z) - Q_k(z))$ is locally equipolar on U, and locally uniformly convergent. This prove that

$$f(z) = \sum_{k=1}^{\infty} (P_k(z) - Q_k(z))$$

has the require property.

Note that in this construction, both $P_k(z)$ and $Q_k(z)$ are holomorphic on a neighborhood of ∞, vanishing at ∞, and $\sum_{n=1}^{\infty} (P_k(z) - Q_n(z))$ converges uniformly on some neighborhood of ∞, where it has the value 0.

Now let us look at the general case. By translating U if necessary, we can arrange $0 \in U$ and $z_k \neq 0$ for all k. Since $\{z_k\}$ has no point of accumulation in U, $\{0\} \notin \mathrm{clos}(\{z_k\})$. Now define $\tilde{U} = \{1/z : z \in U\}$, and define $\tilde{z}_k = 1/z_k$. We know that $\{\tilde{z}_k\}$ are a bounded sequence and $\tilde{z}_k \neq \tilde{z}_l$ for $k \neq l$, and also $\partial \tilde{U}$ is compact because $0 \in U$. So the previous argument applies to $\tilde{U}$, $\{\tilde{z}_k\}$: there exists a meromorphic function $\tilde{f}(z)$ on $\tilde{U}$ with poles at $\tilde{z}_k$, with specifiable principal part $\tilde{P}_k$ at $\tilde{z}_k$, no other poles, holomorphic at ∞.

[2]This is known as the "pole moving lemma".

Now define $\tilde{f}(z) = f(1/z)$. Then f will be meromorphic on U and holomorphic at a neighborhood of 0, with poles at z_k and principal part $\tilde{P}_k(1/z)$ at z_k. How do I have to choose $\tilde{P}_k$ so that f has principal part P_k at z_k? We need

$$\tilde{P}_k\left(\frac{1}{z}\right) = \sum_{k=1}^{\tilde{N}_k} \tilde{a}_{k,l}\left(\frac{1}{z} - \frac{1}{z_k}\right)^{-l} \equiv P_k(z) = \sum_{k=1}^{N_k} a_{k,l}(z - z_k)^{-l}$$

where $\equiv$ shall mean equivality modulo a function which is holomorphic on U. We have

$$\begin{aligned}
\tilde{P}_k\left(\frac{1}{z}\right) &= \sum_{l=1}^{\tilde{N}_k}(-1)^l \tilde{a}_{k,l} z^l z_k^l (z - z_k)^{-l} = \sum_{l=1}^{\tilde{N}_k}(-1)^l z_k^l ((z - z_k) + z_k)^l (z - z_k)^{-l} \\
&= \sum_{l=1}^{\tilde{N}_k}(-1)^l \tilde{a}_{k,l} \sum_{r=0}^{l} \binom{l}{r} z_k^{2l-r}(z - z_k)^{r-l} \\
&\equiv \sum_{l=1}^{\tilde{N}_k}\sum_{r=0}^{l-1}(-1)^l \tilde{a}_{k,l} \binom{l}{r} z_k^{2l-r}(z - z_k)^{r-l}
\end{aligned}$$

where $r - l < 0$ in all cases. We want this to be equal to P_k. We can make it so by downward induction. $\square$

10.2 Infinite products

We want to construct holomorphic functions, with zeros at specified locations, of specified order, with no other zeros. To do this directly, I need infinite products.

Definition 10.2. Let $\{c_n\}$ be a sequence of complex numbers. The infinite product $\prod_{k=0}^{\infty} c_k$ exists if for $N \gg 0$ the limit $\lim_{n\to\infty} \prod_{k=N}^{n} c_k$ exists and is nonzero. In that case, $\lim_{n\to\infty} \prod_{k=1}^{n} c_k$ also exists.

The definition implies that $c_k \neq 0$ for all but finitely many k. One can observe that if $\prod_{k=1}^{\infty} c_k$ converges then $c_k \to 1$ as $k \to \infty$. This is because for $n > N$

$$\prod_{k=N}^{n+1} c_k - \prod_{k=N}^{n} c_k = (c_{n+1} - 1) \prod_{k=N}^{n} c_k$$

and as $n \to \infty$ we get $\lim_{n\to\infty} c_n = 1$.

11 October 6, 2016

Last time I introduced the convergence of a infinite product. This time I will use a different notation.

Definition 11.1. The infinite product $\prod_{k=1}^{\infty}(1+a_k)$ converges if for some N, $\lim_{n\to\infty}\prod_{k=N}^{\infty}(1+a_k)$ exists and is nonzero.

If so,

$$\prod_{k=1}^{N-1}(1+a_k)\lim_{n\to\infty}\prod_{k=N}^{n}(1+a_k)$$

is independent of N. Also $a_k \neq -1$ for all but finitely many k. In the following, define a "branch" of $\log z$, $z \neq 0$ requiring $-\pi < \Im \log z \leq \pi$.

Lemma 11.2. *The product* $\prod_{k=1}^{\infty}(1+a_k)$ *converges if and only if*

(i) $a_k \neq -1$ *for all but finitely many* k*, and*

(ii) $\sum_{k=N}^{\infty}\log(1+a_k)$ *exists for* $N \gg 0$.

Proof. We may as well suppose $a_k \neq -1$. If $\sum_{k=1}^{\infty}\log(1+a_k)$ converges, then

$$\lim_{n\to\infty}\prod_{k=1}^{\infty} = \lim_{n\to\infty}\exp\sum_{k=1}^{n}\log(1+a_k).$$

Because exp is continuous, we get $\Leftarrow$.

For $\Rightarrow$, assume $a_k \neq -1$ for all k, and $\prod_{k=1}^{\infty}(1+a_k) = P$ exists. We can assume that $P \notin i\mathbb{R}_{\geq 0}$. Then if we exponentiate

$$\sum_{k=1}^{n}\log(1+a_k) - \log P$$

and take the limit as $n \to \infty$, we get 1. Then

$$\left|\sum_{k=1}^{n}\log(1+a_k) - \log P\right|$$

becomes arbitrarily small in $\mathbb{R}/2\pi i\mathbb{Z}$ as $n \to \infty$. We know that $\prod_{k=1}^{\infty}(1+a_k)$ converges and so $1 + a_k \to 1$ as $k \to \infty$. So we cannot have jumps by integral multiples of $2\pi i$ for an large. The conclusion is that there exists $m \in \mathbb{Z}$ such that

$$\left|\sum_{k=1}^{n}\log(1+a_k) - \log P - 2\pi i m\right| \to 0$$

as $n \to \infty$. Then $\sum\log(1+a_k)$ converges. □

11.1 Weierstrass problem and product theorem

Definition 11.3. The infinite product $\prod_{k=1}^{\infty}(1+a_k)$ converges **absolutely** if $\sum_{k=N}^{\infty}|\log(1+a_k)|$ converges for $N \gg 0$.

We have

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = z\Big(1 - \frac{z}{2} + \frac{z^2}{3} - \cdots\Big)$$

and that $1 - z/2 + \cdots$ is a holomorphic function on $\Delta(0,1)$ with value 1 at $z = 0$. So $|z|/2 < |\log(1+z)| < 2|z|$ for $|z| < \delta$ for $\delta > 0$. Therefore $\prod_{k=1}^{\infty}(1+a_k)$ converges absolutely if and only if $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Let $U \subseteq \mathbb{C}$ be an open set, and let $\{f_n\}$ be a sequence of holomorphic functions on U such that $\sum_{n=1}^{\infty} f_n$ converges locally uniformly and absolutely on U. (Again, this means that the convergence is absolute and the absolute convergence is locally uniform.) In that situation, $\prod_{k=1}^{\infty}(1+f_k)$ converges locally uniformly and absolutely on U. Let $F(z)$ be the limit. Then for $z_0 \in U$, $F(z_0) = 0$ if and only if $1 + f_k(z_0) = 0$ for some k. If U is connected and $f_k \not\equiv -1$ for all k, then

$$\operatorname{ord} F|_{z_0} = \sum_{k=1}^{\infty} \operatorname{ord}(1+f_k)|_{z_0}$$

and the latter sum is finite.

Let $\{z_k\}$ be a sequence in $\mathbb{C}$, and $z_k \neq z_l$ for $k \neq l$, without points of accumulation in $\mathbb{C}$ (or equivalently, $|z_k| \to \infty$ as $k \to \infty$). Suppose we are given $n_k \in \mathbb{Z}_{>0}$ for each k.

Weierstrass problem. *Construct an entire function $W(z)$ which has a zero of order exactly n_k at z_k, with no other zeros.*

Of course you can ask this for any open subset of $\mathbb{C}$, but then the problem has a different problem.

What should we multiply to get this? We would like to multiply stuff like $(1 - z/z_k)^{n_k}$ to get the right orders, but this gets big as $z \to \infty$. So we also multiply $e^{P_k(z)}$ for some polynomial $P_k(z)$. And there is another problem of $z_k = 0$.

As a slight modification of notation, always include $z_0 = 0$ and $z_k \neq 0$ for all $k > 0$, but allow $n_0 = 0$. We can choose r_k, ϵ_k for $k > 0$ such that $0 < r_k < |z_k|$, $r_k \to \infty$, and $\sum_{k=0}^{\infty} \epsilon_k < \infty$.

Now fix k and consider any $|z| \leq r_k$. Then $|z/z_k| \leq r_k/|z_k| < 1$. The series

$$-\sum_{l=1}^{\infty} \frac{1}{l}\Big(\frac{z}{z_k}\Big)^l = \log\Big(1 - \frac{z}{z_k}\Big)$$

converges absolutely and uniformly on $|z| \leq r_k$.

Lemma 11.4. *We can choose $m_k > 0$ so that, for $|z| \le r_k$,*

$$\left|1 - \left(1 - \frac{z}{z_k}\right)^{n_k} e^{n_k \sum_{l=1}^{\infty} \frac{1}{l}(\frac{z}{z_k})^l}\right| < \epsilon_k.$$

Proof. We have

$$\left|\log\left(1 - \frac{z}{z_k}\right) + \sum_{l=1}^{m_k} \frac{1}{l}\left(\frac{z}{z_k}\right)^l\right| = \Big|\sum_{k=m_k+1}^{\infty} \frac{1}{l}\left(\frac{z}{z_k}\right)^l\Big| \le \frac{1}{m_k} \sum_{l=m_k+1} \left|\frac{z}{z_k}\right|^l$$
$$\le \frac{(r_k/|z_k|)^{m_k+1}}{m_k} \frac{1}{1 - r_k/|z_k|}.$$

Multiply by m_k and exponentiate, and conclude that we can make that thing in the statement as close to 1 as I please for $|z| \le r_k$ by choosing m_k large enough. ☐

Now using this lemma, we can choose the polynomials $P_k(z)$ nicely so that the function converges locally uniformly and absolutely. If $n_0 > 0$ then we can simply multiply z^{n_0}. This proves:

Theorem 11.5 (Weierstrass product theorem)**.** *There exists an entire function $F(z)$ which has a zero of order exactly n_k at z_k, and no other zeros. The most general such function can be expressed as $F(z) = e^{g(z)}W(z)$ with $g(z)$ entire and*

$$W(z) = z^{n_0} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)^{n_k} e^{n_k \sum_{l=1}^{m_k} \frac{1}{l}(\frac{z}{z_n})^l}$$

with an appropriate choice of m_k, which makes the product locally uniformly and absolutely convergent.

Suppose it is possible to choose m_k uniformly. Then the least possible choice of $m = m_k$ is called the **genus** of the Weierstrass product. In this situation, $W(z)$ is uniquely determined. If in addition, $g(z)$ is a polynomial (necessarily unique modulo a multiple of $2\pi i$) then the larger of m and $\deg g$ is called the genus of $F(z)$. It is not obvious from what I have said, but if there are more zeros, this forces the function to grow more rapidly.

Example 11.6. Consider $F(z) = \sin \pi z$, which has first order zeros at $z = k \in \mathbb{Z}$. Does the product $z \prod_{k \in \mathbb{Z}\setminus\{0\}}(1 - z/k)$ converge locally uniformly and absolutely? This would be equivalent to saying that $\sum z/k$ converges uniformly absolutely. The answer is not, because $\sum 1/k = \infty$. What about

$$z \prod_{k \in \mathbb{Z}\setminus\{0\}} (1 - z/k)e^{z/k}?$$

Because $\log(1 - z/k) + z/k$ is $O(k^{-2})$, this converges absolutely. So

$$\sin \pi z = e^{g(z)} z \prod_{k \in \mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{k}\right)e^{z/k} = e^{g(z)} z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

This needs to be justified, but for now I am going to work formally. Take the logarithmic derivative. Then

$$\begin{aligned}\frac{\pi\cos\pi z}{\sin\pi z} &= g'(z) + \frac{1}{z} + \sum_{k=1}^{\infty}\Big(-\frac{1}{k}\frac{1}{1-z/k} + \frac{1}{k}\frac{1}{k}\frac{1}{1+\frac{z}{k}}\Big)\\ &= g'(z) + \frac{1}{z} + \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2} = g'(z) + \pi\cot\pi z.\end{aligned}$$

So $g'(z)=0$, and so $e^{g(z)} = c \neq 0$. That is,

$$\sin\pi z = cz\prod_{k=1}^{\infty}\Big(1-\frac{z^2}{k^2}\Big).$$

Differentiating and setting $z=0$ yields $\pi = c$. In conclusion,

$$\sin\pi z = \pi z\prod_{k=1}^{\infty}\Big(1-\frac{z^2}{k^2}\Big).$$

What about the logarithmic derivative? Assume that $\sum f_k$ converges locally uniformly and absolutely. Then

$$\begin{aligned}\Big(\prod_{k=1}^{\infty}(1+f_k)\Big)^{-1}\frac{d}{dz}\prod_{k=1}^{\infty}(1+f_k) &= \lim_{n\to\infty}\Big(\prod_{k=1}^{n}(1+f_k)\Big)^{-1}\frac{d}{dz}\lim_{n\to\infty}\prod_{k=1}^{\infty}(1+f_k)\\ &= \lim_{n\to\infty}\Big(\prod_{k=1}^{n}(1+f_k)^{-1}\frac{d}{dz}\prod_{k=1}^{n}(1+f_k)\Big) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f_k}{1+f_k}.\end{aligned}$$

So logarithmic derivatives of Weierstrass product can be computed formally term by term.

12 October 11, 2016

12.1 The Gamma function

The **Gamma function** is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dz.$$

Note that $t^{z-1} = t^{\Re z - 1}$ for $t \geq 0$ and so this converges absolutely for $\Re z > 0$. So the above formula defines a holomorphic function on $\{\Re z > 0\}$, because the integral of it over any rectangle will be zero.

Doing integration by parts, we get

$$\Gamma(z) = \frac{1}{z} \int_0^\infty e^{-t} d(t^z) = \frac{1}{z} \Big((e^{-t} t^z)|_{t=0}^\infty + \int_0^\infty e^{-t} t^z dt \Big) = \frac{1}{z} \Gamma(z+1).$$

We can use this identity to analytically continue $\Gamma(z)$ to $\{\Re z > -1\}$, but with a first order pole at $z = 0$, with $\operatorname{Res} \Gamma(z)|_{z=0} = \Gamma(1)$. In a similar way, we have

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z(z+1)}\Gamma(z+2) = \cdots = \frac{1}{z(z+1)\cdots(z+k)}\Gamma(z+k+1). \tag{$*$}$$

So inductively we can continue $\Gamma(z)$ to all of $\mathbb{C}$ with first order poles at $z = -k$, $k \in \mathbb{Z} \geq 0$.

Proposition 12.1. *For $k \geq 1$, $\Gamma(k) = (k-1)!$ and for $k \geq 0$ the residue $\operatorname{Res} \Gamma(z)|_{z=-k} = (-1)^k / k!$.*

Proof. Evaluate $(*)$ at $z = 1$ and find $\Gamma(k+z) = (k+1)!\Gamma(z) = (k+1)!$. Because we have

$$\Gamma(z) = \frac{1}{(z+k)-k} \frac{1}{(z+k)-(k-1)} \cdots \frac{1}{(z+k)-0} \Gamma(z+k+1)$$

and evaluating near $z = -k$, we get

$$\Gamma(z) = \frac{(-1)^k}{k!} \frac{1}{z+k} (1 + (z+k)(\text{holomorphic function}))$$

and so the residue at $z = -k$ is $\operatorname{Res} \Gamma(z)|_{z=-k} = (-1)^k / k!$. $\square$

Why are people interested in this function? One of the big tools in number theory is L-functions, and these satisfy functional equations that are expressed in terms of the Gamma function.

Recall that

$$\sin \pi z = \pi z \prod_{k \in \mathbb{Z} \setminus \{0\}} \Big(1 - \frac{z}{k}\Big) e^{z/k}$$

and this converges locally uniformly and absolutely. Define

$$G(z) = \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) e^{-z/k},$$

which is an entire function, with first order zeros at $z = -k$ for $k \in \mathbb{Z}_{\geq 1}$. Then $G(z-1)$ is entire and has the same zeros as $G(z)$ plus one other at $z = 0$. So

$$G(z-1) = z e^{g(z)} G(z)$$

for some entire function $g(z)$.

Lemma 12.2. *$g(z) = \gamma$ is a constant, called **Euler's constant**, and*

$$\gamma = \lim_{n\to\infty} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big) \sim 0.577\cdots.$$

Note that

$$\gamma = \int_1^{\infty} \Big(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\Big) dx.$$

Proof. We know that $G(z-1) = z e^{g(z)} G(z)$ and so

$$\prod_{k=1}^{\infty} \Big(1 + \frac{z-1}{k}\Big) e^{-(z-1)/k} = z e^{g(z)} \prod_{k=1}^{\infty} (1 + \frac{z}{k}) e^{-z/k}.$$

Now let us take logarithmic derivatives. We get

$$\sum_{k=1}^{\infty} \Big(\frac{1}{z+k-1} - \frac{1}{k}\Big) = \frac{1}{z} + g'(z) + \sum_{k=1}^{\infty} \Big(\frac{1}{z+k} - \frac{1}{k}\Big).$$

Then

$$g'(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} \Big(\frac{1}{z+k-1} - \frac{1}{z+k}\Big) = 0.$$

This shows that $g = \gamma$ is a constant.

Now let us evaluate at $z = 1$ and take the logarithm. Then

$$\begin{aligned} 0 &= \gamma + \sum_{k=1}^{\infty} \Big(\log\Big(1 + \frac{1}{k}\Big) - \frac{1}{k}\Big) = \gamma - \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) + \log 1\Big) \\ &= \gamma - \lim_{n\to\infty} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big). \end{aligned}$$

□

So we have

$$G(z) = \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) e^{-z/k}, \quad G(z-1) = z e^{\gamma} G(z).$$

Define $H(z) = e^{\gamma z} G(z)$ so that $H(z)$ is entire and has zeros at $z = k$ for $k \in \mathbb{Z}_{\geq 1}$ with no other zeros. We have

$$H(z-1) = e^{\gamma z - \gamma} G(z-1) = z e^{\gamma z} G(z) = z H(z).$$

What do we know about $1/(zH(z))$? It is meromorphic, has first order poles at $z = -k$ for $k \in \mathbb{Z}_{\geq 0}$, and no other poles. Then

$$\frac{1}{(z+1)H(z+1)} = \frac{1}{H(z)} = \frac{z}{zH(z)}.$$

So $1/(zH(z))$ has exactly the same polar locations as $\Gamma(z)$ and behaves the same way under $z \mapsto z+1$.

12.2 Gamma as an infinite product

Theorem 12.3.

$$\Gamma(z) = \frac{1}{zH(z)} = \frac{1}{z} e^{-\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k}.$$

This requires a lot of work to prove, so we first state some corollaries.

Corollary 12.4. *$\Gamma(z)$ has no zeros.*

Corollary 12.5. $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$.

Proof. We have

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \Gamma(z)(-z)\Gamma(-z) \\
&= \frac{1}{z} e^{-\gamma z} \left(\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right) (-z) \frac{1}{-z} e^{\gamma z} \left(\prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} e^{-z/k} \right) \\
&= \frac{1}{z} \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{k}\right)^{-1} e^{z/k} = \frac{\pi}{\sin \pi z}
\end{aligned}$$

□

Define

$$f_n(t) = \begin{cases} (1 - t/n)^n & 0 \leq t \leq n, \\ 0 & t > n. \end{cases}$$

Lemma 12.6. *The functions f_n satisfy the following:*

(1) $0 \leq f_n(t) \leq e^{-t}$ *for all t and $n > 0$.*

(2) $\lim_{n \to \infty} f_n(t) = e^{-t}$.

Proof. For (1), we may assume $0 < t < n$. We need to show $\log f_n(t) \leq -t$, i.e., $n \log(1 - t/n) + t \leq 0$. This is true becuase

$$n \log\left(1 - \frac{t}{n}\right) = -n \sum_{k=1}^{\infty} \frac{t^k}{kn^k} + t = -\sum_{k=2}^{\infty} \frac{t^k}{kn^{k-1}} = O(n^{-1})$$

This shows both (1) and (2). □

Proof of Theorem 12.3. We have

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^k t^{z-1} dt,$$

by the dominated convergence theorem. This is the crucial step, and I will continue next time. $\square$

13 October 13, 2016

Let us recall that we have defined

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for $\Re z > 0$ and $\Gamma(z+1) = z\Gamma(z)$. We had shown

$$\Gamma(x) = \lim_{n\to\infty} \int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{x-1} dt$$

for $x > 0$. We have

$$\begin{aligned}
\int_0^n \Big(1-\frac{t}{n}\Big)^n t^{x-1} dt &= \frac{1}{x}\int_0^n \Big(1-\frac{t}{n}\Big)^n d(t^n) = \frac{n}{n}\frac{1}{x}\int_0^n \Big(1-\frac{t}{n}\Big)^{n-1} t^x dt \\
&= \frac{n}{n}\frac{1}{x}\frac{n-1}{n}\frac{1}{x+1}\int_0^n \Big(1-\frac{t}{n}\Big)^{n-2} t^{x+1} dt \\
&= \cdots = \frac{n}{n}\frac{1}{x}\frac{n-1}{n}\frac{1}{x+1}\cdots\frac{1}{n}\frac{1}{x+n-1}\int_0^n t^{x+n-1} dt \\
&= \frac{n}{n}\frac{1}{x}\frac{n-1}{n}\frac{1}{x+1}\cdots\frac{1}{n}\frac{1}{x+n-1}\frac{1}{x+n} n^{x+n} = \frac{n! n^x}{x(x+1)\cdots(x+n)} \\
&= \frac{1}{x} n^x n! \prod_{k=1}^n \frac{1}{x+k} = \frac{1}{x} e^{x\log n} \prod_{k=1}^n \Big(1+\frac{x}{k}\Big)^{-1} \\
&= \frac{1}{x} e^{x(\log n - 1 - 1/2 - \cdots - 1/n)} \prod_{k=1}^n (1+\frac{x}{k})^{-1} e^{x/k}.
\end{aligned}$$

Taking $\lim_{n\to\infty}$, we get

$$\Gamma(x) = \frac{1}{x} e^{-\gamma x} \prod_{k=1}^\infty \Big(1+\frac{x}{k}\Big)^{-1} e^{x/k}.$$

So I have shown this formula for $\Re x > 0$ (and hence for all z for which both sides make sense).

Suppose f is a continuous function on $\mathbb{R}_{>0}$ satisfying appropriate growth conditions at 0 and ∞. Then the function

$$M(f,s) = \int_0^\infty f(t) t^{s-1} dt$$

is called the **Mellin transform** of $f(t)$ for $s \in \mathbb{C}$ where this makes sense. Then Γ is just the Mellin transform of e^{-z}.

13.1 Legendre duplication formula

Theorem 13.1 (Legenedre duplication formula)**.** *For any $z \in \mathbb{C}$,*

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big).$$

Proof. The derivative of the logarithmic derivative of Γ is

$$\begin{aligned}\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{d}{dz}\biggl(-\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}\Bigl(\frac{1}{z+k} - \frac{1}{k}\Bigr)\biggr) \\ &= \frac{1}{z^2} + \frac{k=1}{\infty}\frac{1}{(z+k)^2} = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}.\end{aligned}$$

Then the derivative of the logarithmic derivative of $\Gamma(2z)$ is

$$2\frac{d}{dz}\frac{\Gamma'(2z)}{\Gamma(2z)} = 4\sum_{k=0}^{\infty}\frac{1}{(2z+k)^2} = \sum_{k=0}^{\infty}\frac{1}{(z+k/2)^2}.$$

Likewise we have

$$\frac{d}{dz}\biggl(\frac{\frac{d}{dz}\Gamma(z)\Gamma(z+1/2)}{\Gamma(z)\Gamma(z+1/2)}\biggr) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2} + \sum_{k=0}^{\infty}\frac{1}{(z+k+1/2)^2} = \sum_{k=0}^{\infty}\frac{1}{(z+k/2)^2}.$$

This implies that

$$\Gamma(z)\Gamma\Bigl(z+\frac{1}{2}\Bigr) = e^{az+b}\Gamma(2z).$$

Now we know $\operatorname{Res}\Gamma(z)|_{z=0} = 1$ and $\Gamma(1) = 1$. Also from $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ we see that $\Gamma(1/2) = \sqrt{\pi}$. Taking the residues of both sides at $z = 0$, we get

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = \operatorname{Res}\Bigl(\Gamma(z)\Gamma\Bigl(z+\frac{1}{2}\Bigr)\Bigr)\Big|_{z=0} = \frac{1}{2}e^b.$$

The value at $z = 1/2$ give us

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = e^{a/2+b}.$$

From this we get $e^a = 1/4$ and $e^b = 2\sqrt{\pi}$. $\square$

13.2 Dirichlet series

Suppose $\{a_n\}_{n\geq 1}$ is a sequence of complex numbers such that $|a_n| = O(n^c)$ for some $c \in \mathbb{R}$.

Definition 13.2. The series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the **Dirichlet series** with coefficients a_n, where $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$. The series converges locally uniformly and absolutely for $\sigma > c+1$.

Let us define

$$\sigma_0 = \inf\{\sigma \in \mathbb{R} : \textstyle\sum_{n=1}^{\infty}|a_n|n^{-\sigma} < \infty\}.$$

This is the **abscissa** of the absolute convergence of the Dirichlet series–it converges uniformly and absolutaly on $\{s \in \mathbb{C} : \Re s \geq \sigma_0 + \epsilon\}$ for any $\epsilon > 0$.

The convergence behavior of the Dirichlet series is much more complicated than that of the power series. That is, it can have uniform convergence to the left of σ_0.

Now suppose that (i) $a_1 = 1$ and (ii) $a_{mn} = a_m a_n$ whenever $(m, n) = 1$. Then formally at least,

$$\sum_n a_n n^{-s} = \prod_p \Big(\sum_{k=0}^{\infty} a_{p_k} p^{-ks} \Big) = \prod_p \Big(1 + \sum_{k=1}^{\infty} a_{p^k} p^{-ks} \Big).$$

Because

$$\sum_p \sum_{k=1}^{\infty} |a_k| p^{-k\sigma} \leq \sum_{n=1}^{\infty} |a_n| n^{-\sigma} < \infty,$$

we conclude that

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \Big(1 + \sum_{k=1}^{\infty} a_{p^k} p^{-ks} \Big)$$

and both the sum and the product converges uniformly and absolutely for $\Re s > \sigma_0 + \epsilon$.

Example 13.3. The **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

has abscissa of absolute convergece $\sigma_0 = 1$, and so it is holomorphic (at least) on $\{s \in \mathbb{C} : \Re s > 1\}$. This can also be written as

$$\zeta(s) = \prod_p \Big(1 + \sum_{k=1}^{\infty} p^{-ks} \Big) = \prod_p (1 - p^{-s})^{-1}.$$

In particular, $\zeta(s) \neq 0$ if $\Re s > 1$.

Theorem 13.4 (Riemann)**.** *$\zeta(s)$ extends to all of $\mathbb{C}$ as a meromorphic function which has only one pole of order 1 at $s = 1$ with residue $\operatorname{Res} \zeta(s)|_{s=1} = 1$. Moreover,*

$$\zeta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s).$$

Proof. We have

$$n^{-s} \Gamma(s) = \int_0^{\infty} e^{-nt} t^{s-1} dt$$

and so for $t \geq 0$ and $\Re s > 1$,

$$\zeta(s)\Gamma(s) = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kt} t^{s-1} dt = \lim_{n\to\infty} \int_0^{\infty} e^{-t} \frac{1 - e^{-nt}}{1 - e^{-t}} t^{s-1} dt.$$

By dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^{\infty} e^{-t} \frac{e^{-nt}}{1 - e^{-t}} t^{s-1} dt = 0$$

locally uniformly in s for $\Re s > 1$. This shows that

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt. \qquad \square$$

14 October 18, 2016

We had seen that

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

for $\Re s > 1$.

14.1 Functional equation for ζ

To compute this integral, for $0 < r_1 < r_2$ consider a curve C_{r_1,r_2} that starts from r_2, goes around the circle of radius r_2 counterclockwise to r_2, goes to r_1 along the real axis, goes around the circle of radius r_1 clockwise, and then goes to r_2 along the real axis. The two integrals along the real axis does not cancel because t^{s-1} changes by a factor of $e^{2\pi i(s-1)} = e^{2\pi is}$. Thus if $r_1, r_2 \notin 2\pi\mathbb{Z}$,

$$\int_{C_{r_1,r_2}} \frac{t^{s-1}dt}{e^t - 1} = (1 - e^{2\pi is}) \int_{r_1}^{r_2} \frac{t^{s-1}dt}{e^t - 1} + \int_{t=r_2e^{i\theta}} \frac{t^{s-1}dt}{e^t - 1} - \int_{t=r_1e^{i\theta}} \frac{t^{s-1}dt}{e^t - 1}.$$

On the other hand, by the residue theorem,

$$\int_{C_{r_1,r_2}} \frac{t^{s-1}dt}{e^t - 1} = 2\pi i \sum_{r_1 \le 2\pi|k| \le r_2} \mathrm{Res}\Big(\frac{t^{s-1}}{e^t - 1}\Big)\Big|_{s=2\pi ik}.$$

First suppose $0 < r_1 < r_2 < 2\pi$, $\Re s > 1$, and then let $r_1 \to 0$. Then we get

$$0 = (1 - e^{2\pi is}) \int_0^{r_2} \frac{t^{s-1}dt}{e^t - 1} + \int_{t=r_2e^{i\theta}} \frac{t^{s-1}dt}{e^t - 1}.$$

For $0 < r < 2\pi$, define

$$I_r(s) = (e^{2\pi is} - 1) \int_r^\infty \frac{t^{s-1}dt}{e^t - 1} + \int_{t=re^{i\theta}} \frac{t^{s-1}dt}{e^t - 1}.$$

This is an entire function of s! But on $\Re s > 1$, we can write

$$\begin{aligned} I_r(s) &= (e^{2\pi is} - 1) \int_0^\infty \frac{t^{s-1}dt}{e^t - 1} + (1 - e^{2\pi is}) \int_0^r \frac{t^{s-1}dt}{e^t - 1} + \int_{t=re^{i\theta}} \frac{t^{s-1}dt}{e^t - 1} \\ &= (e^{2\pi is} - 1) \int_0^\infty \frac{t^{s-1}dt}{e^t - 1}. \end{aligned}$$

This is independent of r provided $0 < r < 2\pi$ and $\Re s > 1$. So let us define $I(s) = I_r(s)$. The conclusion is that

(1) $I(s)$ is an entire function.

(2) for $\Re s > 1$, $(e^{2\pi is} - 1)\zeta(s)\Gamma(s) = I(s)$.

From this we immediately see that $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$.

Where are the poles of $\zeta(s)$? Potentially they are at $s \in \mathbb{Z}$, but we know that it has no poles for $\Re s > 1$. Also $\Gamma(s)$ has first order poles at $s \in \mathbb{Z}_{\leq 0}$, which cancels the zeros of $e^{2\pi i s} - 1$. So the only possible poles occur at $s = 1$. We can quickly calculate the residue. We have

$$\begin{aligned}\operatorname{Res} \zeta(s)|_{s=1} &= \frac{I(1)}{\Gamma(1)} \operatorname{Res} \frac{1}{e^{2\pi i s} - 1}\Big|_{s=1} = \frac{1}{2\pi i} I(1) \\ &= \frac{1}{2\pi i} \int_{t=re^{i\theta}} \frac{dt}{e^t - 1} = \frac{2\pi i}{2\pi i} = 1.\end{aligned}$$

Now the identity

$$(e^{2\pi i 2} - 1)\zeta(s)\Gamma(s) = (e^{2\pi i s} - 1) \int_r^\infty \frac{t^{s-1} dt}{e^t - 1} + \int_{t=re^{i\theta}} \frac{t^{s-1} dt}{e^t - 1}$$

holds for any s. Now suppose $0 < r < 2\pi$ as before and $r_2 > r$, $r_2 \notin 2\pi\mathbb{Z}$. Then

$$\int_{C_{r,r_2}} \frac{t^{s-1} dt}{e^t - 1} = 2\pi i \sum_{0 < 2\pi|k| < r_2} \operatorname{Res} \frac{t^{s-1}}{e^t - 1}\Big|_{s=2\pi i k}.$$

For $0 < 2\pi k < r_2$,

$$\begin{aligned}\operatorname{Res} \frac{t^{s-1}}{e^t - 1}\Big|_{t=2\pi i k} &= (t^{s-1}|_{t=2\pi i k}) \operatorname{Res}\Big(\frac{1}{e^t - 1}\Big)\Big|_{t=2pik} = e^{i\pi(s-1)/2}(2\pi k)^{s-1}, \\ \operatorname{Res} \frac{t^{s-1}}{e^t - 1}\Big|_{t=-2\pi i k} &= (t^{s-1}|_{t=-2\pi i k}) \operatorname{Res}\Big(\frac{1}{e^t - 1}\Big)\Big|_{t=-2pik} = e^{3i\pi(s-1)/2}(2\pi k)^{s-1}.\end{aligned}$$

So

$$\int_{C_{r,r_2}} \frac{t^{s-1} dt}{e^t - 1} = 2\pi i \sum_{0 < 2\pi k < r_2} (2\pi)^{s-1} k^{s-1} (e^{i\pi(s-1)/2} + e^{3i\pi(s-1)/2}).$$

As $r_2 \to \infty$, we get

$$\lim_{r_2 \to \infty} \int_{C_{r,r_2}} \frac{t^{s-1} dt}{e^t - 1} = 2\pi i (2\pi)^{s-1} (e^{i\pi(s-1)/2} + e^{3i\pi(s-1)/2}) \zeta(1-s)$$

provided that $1 - \Re s > 1$, i.e., $\Re s < 0$. So for $\Re s < 0$,

$$\begin{aligned}\lim_{r \to \infty} \frac{t^{s-1} dt}{e^t - 1} + \lim_{n \to \infty} \int_{t=(2n+1)\pi e^{i\theta}} \frac{t^{s-1} dt}{e^t - 1} - \int_{t=re^{i\theta}} \frac{t^{s-1} dt}{e^t - 1} \\ = 2\pi i (2\pi)^{s-1} (e^{i\pi(s-1)/2} + e^{3i\pi(s-1)/2}) \zeta(1-s).\end{aligned}$$

Lemma 14.1. *For $|t| = (2n-1)\pi$ for $n \in \mathbb{Z}_{>0}$, $|e^t - 1|$ is bounded away from 0 uniformly in t.*

If this is true, then

$$\lim_{n\to\infty}\int_{t=(2n+1)\pi e^{i\theta}}\frac{t^{s-1}dt}{e^t-1}=0$$

given that $\Re s<0$.

Proof. Because e^t lies on the circle of radius $e^{\Re t}$ centered at the origin, $|e^t-1|\geq |e^{\Re t}-1|$. So for $|\Re t|\geq\delta$ with $0<\delta\ll 1$, we have

$$|e^t-1|\geq |e^{\Re t}-1|\geq\min\{e^\delta-1,1-e^{-\delta}\}.$$

So for $|\Re t|\geq\delta$, the claim is good. For $|\Re t|<\delta$, $t=\Re t+i\Im t$, so $\Im t\sim(2n-1)\pi$. So $e^t-1\sim -2$. $\square$

15 October 20, 2016

We have an entire function, defined for but independent of $0 < r < 2\pi$,

$$I(s) = (e^{2\pi i s} - 1)\int_r^\infty \frac{t^{s-1}dt}{e^t - 1} + \int_{t=re^{i\theta}} \frac{t^{s-1}dt}{e^t-1}$$

and $I(s) = (e^{2\pi i s} - 1)\zeta(s)\Gamma(s)$ for $\Re s > 1$. This showed that $\zeta(s)$ is a meromorphic function with a single pole of order 1 at $s = 1$.

We also have

$$\begin{aligned} 2\pi i(2\pi)^{s-1}(e^{i\pi(s-1)/2} + e^{3i\pi(s-1)/2})\zeta(1-s) &= \lim_{n\to\infty}\int_{C_{r,(2n-1)\pi}} \frac{t^{s-1}dt}{e^t-1} \\ &= -I(s) = -(e^{2\pi i s} - 1)\zeta(s)\Gamma(s). \end{aligned}$$

Therefore

$$(2\pi)^s(e^{i\pi s/2} - e^{3i\pi s/2})\zeta(1-s) = -(e^{2\pi i s} - 1)\zeta(s)\Gamma(s),$$

and therefore

$$(2\pi)^s \sin\frac{\pi s}{2}\zeta(1-s) = \sin \pi s \zeta(s)\Gamma(s).$$

So

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

This finishes the proof of Theorem 13.4.

Suppose $\Re s < 0$. Then $\Re(1-s) > 1$ and we have $\zeta(1-s)$ as a Euler product. This shows that $\zeta(1-s) \neq 0$ for $\Re s < 0$. Still for $\Re s < 0$, $\Gamma(s)\cos(\pi s/2)$ has a pole if and only if s is an integer and $\cos(\pi s/2) \neq 0$. This shows that $\zeta(s)$ first order zeros at the strictly negative even integers, and has no other zeros for $\Re s < 0$. These are the **trivial zeros** of $\zeta(s)$.

Γ has a first order pole of residue 1 at $s = 1$, and $\cos 0 = 1$, and $\zeta(1-s)$ has a first order pole at $s = 0$ with residue -1. This shows that

$$\zeta(0) = -1/2.$$

Recall the Legendre duplication formula $\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)$. Then $\sqrt{\pi}\Gamma(s) = 2^{s-1}\Gamma(s/2)\Gamma((s+1)/2)$ and so we get

$$\sqrt{\pi}\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)2^{s-1}\Gamma(s/2)\Gamma((s+1)/2)\zeta(s).$$

Because $\Gamma((s+1)/2)\Gamma((1-s)/2) = \pi/\cos(\pi s/2)$, we get

$$\sqrt{\pi}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) = \pi^{1-s}\Gamma\Big(\frac{s}{2}\Big)\zeta(s).$$

To make it more symmetric, we can write it as

$$\pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s).$$

Define

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Then

$$\xi(s) = \xi(1-s).$$

Is this an accident? In the view of algebraic number theory, this $\zeta(s)$ encodes the information of all primes, and the other part $\pi^{-s/2}\Gamma(s/2)$ can be thought as attached to the completion $\mathbb{R}$ of $\mathbb{Q}$. Then $\xi(s)$ is the completed zeta function.

15.1 Hadamard's theorem

Let $f(z)$ be an entire function, written as

$$f(z) = z^{n_0}e^{g(z)}\prod_k \Big(1-\frac{z}{z_k}\Big)e^{\sum_{l=1}^{m_k}\frac{1}{l}(\frac{z}{z_k})^l},$$

setting $n_k = 1$ for $k > 0$ but allowing repetitions of z_k. Recall if it is possible to choose $m_k = m$ independent of k and if $g(z)$ is a polynomial, then the genus of f is the larger of the minimal choice of m and the degree of $g(z)$. Otherwise, the genus of f is ∞.

For $r > 0$, define $M(r) = \max\{|f(z)| : |z| = r\}$. We know that if f is not constant, $M(r)$ is strictly increasing (by the maximal principle) and $\lim_{n\to\infty} = +\infty$ by Liouville's theorem. Then

$$\lambda(f) = \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}$$

is well-defined unless f is a constant—from now on we suppose f is non-constant. This $\lambda(f)$ is called the **order** (of growth) of f

Theorem 15.1 (Hadamard). *If g is a non-constant entire function then either both or neither of "the genus of f" and $\lambda(f)$ are finite. In the finite case, genus of $f \le \lambda(f) \le$ genus of $f+1$.*

What this means is that if a function has many zeros, this forces the function to grow rapidly. This theorem will take some time to prove.

Observe that $\lambda = \lambda(f)$, in the finite case, means that $M(r) \le e^{r^{\lambda+\epsilon}}$ for every $\epsilon > 0$, and for no $\epsilon < 0$. This implies, in particular, if f_1, f_2, f_1f_2 are non-constant, then $\lambda(f_1f_2) \le \max(\lambda(f_1), \lambda(f_2))$. Also if $g(z)$ is a non-constant polynomial, then $\lambda(e^{g(z)}) = \deg g$.

Proof of $\lambda \leq genus + 1$. Define

$$E_m(w) = (1-w)e^{\sum_{l=1}^m \frac{1}{l}w^l}.$$

Then if suffices to show that

$$\log|E_m(w)| \leq C(m)|w|^{m+1}$$

for an absolute constant $C(m)$, because g will be a product of functions of the form $E_m(w)$ and $e^{g(z)}$.

We prove this claim by induction on m. For $m = 0$, we have

$$\log|E_0(w)| = \log|1-w| \leq \log(1+|w|) \leq |w|.$$

For $m \geq 1$,

$$\begin{aligned}\log|E_m(w)| &\leq \log|E_{m-1}(w)| + \frac{1}{m}\Re w^m \leq C(m-1)|w|^m + \frac{1}{m}|w|^m \\ &\leq \frac{C(m-1)+1/m}{|w|}|w|^{m+1}.\end{aligned}$$

So we are done when w is bounded away from 0. If $|w| < 1$, then

$$\log E_m(w) = -\sum_{l=m+1}^{\infty} \frac{1}{l}m^l = \frac{1}{m+1}w^{m+1}\varphi(w)$$

for some holomorphic $\varphi(w)$. For, say $|w| \leq 1/2$, we have $\log|E_m(w)| \leq c/(m+1)|w|^{m+1}$. $\square$

16 October 25, 2016

Recall Hadamard's theorem: if f is an entire function, then

$$\text{genus of } f \leq \text{order of } f \leq \text{genus of } f + 1.$$

We have proved the second inequality. The proof of the first inequality depends on the Poisson-Jensen formula.

Before stating and proving the formula, we remark that if $\theta_j \in \mathbb{R}/2\pi$, then $\int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_j}|d\theta$ converges absolutely. To see this, we may suppose $\theta_j = 0$. Then $\log|e^{i\theta} - 1| \approx \log|\theta|$ as $\theta \to 0$. This is integrable.

Suppose that f is holomorphic on a neighborhood of $\partial\Delta$. Then $\int \log|f(e^{i\theta})|d\theta$ converges absolutely. To see this, write $f(z) = g(z)\prod_{j=1}^n (z - e^{i\theta_j})$ whre g is holomorphic and nonzero on a neighborhood of $\partial\Delta$. Then

$$\int_0^{2\pi} \log|f(e^{i\theta})|d\theta = \int_0^{2\pi} \log|g(e^{i\theta})|d\theta + \sum_{j=1}^n \int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_j}|d\theta.$$

This converges absolutely.

16.1 Poisson-Jensen formula

Theorem 16.1 (Poisson-Jensen formula)**.** *Suppose $f(z)$ is holomorphic on a neighborhood of* $\mathrm{cl}(\Delta)$*. Let $z_1, \ldots, z_n$ be the zeros of f on* $\mathrm{cl}(\Delta)$*, enumerated with multiplicity. Then for $z \in \Delta$, $z \neq z_1, \ldots, z_n$,*

$$\log|f(z)| = \sum_{j=1}^N \log\left|\frac{z - z_j}{1 - \bar{z}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) \log|f(e^{i\theta})|d\theta.$$

Corollary 16.2 (Jensen's formula)**.** *Under the same hypotheses,*

$$\log|f(0)| = \sum \log|z_j| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta.$$

We are first going to prove Jensen's formula.

Proof. **Step 1.** Suppose $f(z) \neq 0$ for $z \in \mathrm{cl}(\Delta)$. Then $f(z) = e^{g(z)}$ for some holomorphic $g(z)$ defined on a (possibly smaller) neighborhood of $\mathrm{cl}(\Delta)$. Then

$$\log|f(0)| = \Re g(0) = \Re \frac{1}{2\pi}\frac{g(e^{i\theta})}{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta.$$

Step 2. Suppose f has zeros $z_1, \ldots, z_n$ in Δ and no zeros on $\partial\Delta$. Then $f(z) = \prod_{j=1}^N (z - z_j)/(1 - \bar{z}_j z)h(z)$, where h is holomorphic on a neighborhood of cl(Δ) and nonzero on all of cl(Δ). Then

$$\begin{aligned}\log|f(0)| = \log|h(0)| + \sum_{j=1}^{N}\log|z_j| &= \sum_{j=1}^{N}\log|z_j| + \frac{1}{2\pi}\int_0^{2\pi}\log|h(e^{i\theta})|d\theta \\ &= \sum_{j=1}^{N}\log|z_j| + \frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta.\end{aligned}$$

Step 3. Let us now look at the general case. Let $z_1, \dots, z_r$ be the zeros of f on Δ and let $e^{i\theta_j}$ for $1 \le j \le s$ be the zeros of f on $\partial\Delta$. Then we can write $f(z) = g(z)\prod_{j=1}^{s}(z - e^{i\theta_j})$ where g is holomorphic on a neighborhood of cl(Δ) but $g(z) \neq 0$ on $\partial\Delta$. Then

$$\log|f(0)| = \log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|g(e^{i\theta})|d\theta + \sum_{j=1}^{r}\log|z_j|.$$

It then suffices to show that $\int_0^{2\pi}\log|e^{i\theta} - e^{i\theta_j}|d\theta = 0$, which is true. $\square$

Proof of Poisson-Jensen formula. Suppose $z \in \Delta$ with $z \neq z_j$, for $1 \le j \le N$. Define $L = L_z$ by

$$L_z(\zeta) = \frac{\zeta + z}{1 + \zeta\bar{z}}.$$

Then $L_z(0) = z$ and $|L(e^{i\theta})| = 1$ and so $L : \Delta \cong \Delta$. Let M_z be the inverse, which is given by

$$M_z(\zeta) = \frac{\zeta - z}{1 - \zeta\bar{z}}.$$

Now $\log|f(z)| = \log|f \circ L(0)|$. The zeros of $f \circ L$ on cl(Δ) are (with multiplicity) the $M(z_j)$. So

$$\begin{aligned}\log|f(z)| &= \sum_{j=1}^{N}\log|M(z_j)| + \frac{1}{2\pi}\int_0^{2\pi}\log|f \circ L(e^{i\theta})|d\theta \\ &= \sum_{j=1}^{N}\log\left|\frac{z - z_j}{1 - \bar{z}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi}\log|f \circ L(e^{i\theta})|d\theta.\end{aligned}$$

The final equality

$$\int_0^{2\pi}\log|f \circ L(e^{i\theta})|d\theta = \int_0^{2\pi}\Re\Big(\frac{e^{i\theta} + z}{e^{i\theta} - z}\Big)\log|f(e^{i\theta})|d\theta$$

can be shown by changing variables. $\square$

17 October 27, 2016

17.1 Proof of Hadamard's theorem

Apply Jensen's formula in the following way. Let f be an entire function. We need to show that $g \leq \lambda$, where g is the genus of f and λ is the order of f. Multiplying f by a polynomial does not affect either λ or γ. So I may suppose $f(0) \neq 1$. Also, I can assume f has infinite many zeros, since it is easy if f has finitely many zeros. Enumerate them as

$$0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots$$

with $n_k = 1$, and define $N(r) = \#\{z_k : |z_k| \leq r\}$ and $M(r) = \max\{|f(z)| : |z| = r\}$. For $r > 1$, define $f_r(z) = f(rz)$. Using Jensen's formula on f_r, we get

$$0 = \sum_{|z_k| \leq r} \log\left|\frac{z_k}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| d\theta,$$

and thus

$$-\sum_{|z_k| \leq r} \log\left|\frac{z_k}{r}\right| \leq \log M(r).$$

Applying this inequality to $2r$, we get

$$-\sum_{|z_k| \leq r} \log\left|\frac{z_k}{2r}\right| \leq -\sum_{|z_k| \leq 2r} \log\left|\frac{z_k}{2r}\right| \leq \log M(2r).$$

Recall that for any $\epsilon > 0$, if $r \gg 0$ then $\log M(r) \leq r^{\lambda+\epsilon}$. So $\log 2N(r) \leq (2r)^{\lambda+\epsilon}$. Because $k \leq N(|z_k| + 1)$, we get

$$\log 2k \leq (2(|z_k| + 1))^{\lambda+\epsilon},$$

for $r \gg 0$. Then for any $\epsilon > 0$, there exists $k \gg 0$ such that $k \leq |z_k|^{\lambda+\epsilon}$ because the choice of ϵ allows to absorb 2, $\log 2$, and 1.

Define $m = \lfloor \lambda \rfloor$. Then $\lambda < m+1$. Choose $\epsilon > 0$ so that $\lambda + \epsilon < m+1$. Then for $k \gg$, we have $|z_k|^{-m-1} \leq k^{-(m+1)/(\lambda+\epsilon)}$ for $k \gg 1$. Thus $\sum_{k=1}^{\infty} |z_k|^{-m-1} < \infty$. It follows that

$$P(z) = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{z_k}\Big) e^{\sum_{k=1}^{m}(\frac{z}{z_k})^l}$$

converges uniformly and absolutely on compact sets. Therefore the genus of $P(z) \leq m \leq \lambda$.

We can now write $f = e^h P(z)$ for some entire function h. It suffices to show that h is a polynomial of degree at most m. Let us apply Poisson-Jensen to

$f_r(z) = f(rz)$. For a fixed z and $|z| < r$,

$$\begin{aligned}\log|f(z)| &= \sum_{k=1}^{N} \log\Big|\frac{r^{-1}z - r^{-1}z_k}{1 - r^{-2}z\bar{z}_k}\Big| + \frac{1}{2\pi}\int_0^{2\pi} \Re\frac{e^{i\theta} + r^{-1}z}{e^{i\theta} - r^{-1}z}\log|f(re^{i\theta})|d\theta \\ &= \sum_{k=1}^{N} \log\Big|\frac{rz - rz_k}{r^2 - z\bar{z}_k}\Big| + \frac{1}{2\pi}\int_0^{2\pi} \Re\frac{re^{i\theta} + z}{re^{i\theta} - z}\log|f(re^{i\theta})|d\theta.\end{aligned}$$

Note that if F is a holomorphic function, then

$$\frac{F'}{F} = \frac{\partial}{\partial z}(\log F) = \frac{\partial}{\partial z}(\log F + \log \bar{F}) = 2\frac{\partial}{\partial z}\log|F(z)|.$$

Thus

$$\begin{aligned}\frac{f'}{f} &= 2\frac{\partial}{\partial z}\log|f(z)| \\ &= 2\frac{\partial}{\partial z}\sum_{k=1}^{N} \log\Big|\frac{rz - rz_k}{r^2 - z\bar{z}_k}\Big| + 2\frac{\partial}{\partial z}\int_0^{2\pi} \Re\frac{re^{i\theta} + z}{re^{i\theta} - z}\log|f(re^{i\theta})|d\theta \\ &= \sum_{k=1}^{N}\Big(\frac{1}{z - z_k} + \frac{\bar{z}_k}{r^2 - z\bar{z}_k}\Big) + \frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{i\theta}}{(re^{i\theta} - z)^2}\log|f(re^{i\theta})|d\theta.\end{aligned}$$

Because $f = e^h P(z)$,

$$\frac{f'}{f} = h' + \sum_{k=1}^{\infty}\Big(\frac{1}{z - z_k} + \text{polynomial of degree } \leq m - 1\Big),$$

with the sum converging uniformly and absolutely. Then

$$\frac{\partial^m}{\partial z^m}\frac{f'}{f} = h^{(m+1)} + (-1)^m m! \sum_{k=1}^{\infty} \frac{1}{(z - z_k)^{m+1}}.$$

Let me finish next time.

18 November 1, 2016

Because

$$\frac{f'}{f}(z) = \sum_{|z_k|\le r}\Big\{\frac{1}{z-z_k} + \frac{\bar{z}_k}{r^2 - z\bar{z}_k}\Big\} + \frac{(m+1)!}{2\pi}\int_0^{2\pi}\frac{2r}{(re^{i\theta}-z)^2}\log|f(re^{i\theta})|d\theta,$$

differentiating m times we get

$$\frac{d^m}{dz^m}\frac{f'}{f}(z) = (-1)^m m!\sum_{|z_k|\le r}\frac{1}{(z-z_k)^{m+1}} + m!\sum_{|z_k|\le r}\frac{\bar{z}_k^{m+1}}{(r^2-z\bar{z}_k)^{m+1}}$$
$$+\frac{(m+1)!}{2\pi}\int_0^{2\pi}\frac{2r}{(re^{i\theta}-z)^{m+2}}\log|f(re^{i\theta})|d\theta.$$

If suffices to show that the two last terms tned to 0 as $r\to\infty$, since then $h^{(m+1)}(z)\equiv 0$.

Recall that $m=\lfloor\lambda\rfloor$. We have

$$\Big|\sum_{|z_k|\le r}\frac{\bar{z}_k^{m+1}}{(r^2-z\bar{z}_k)^{m+1}}\Big| \le \sum_{|z_k|\le r}\Big|\frac{1}{(\frac{r^2}{\bar{z}_k}-z)^{m+1}}\Big| \le N(r)\frac{1}{(r-|z|)^{m+1}}.$$

Since $N(r)\le r^{\lambda+\epsilon}$ and we can set $\lambda+\epsilon<m+1$, we get that $\sum_{|z_k|\le r}(r^2/\bar{z}_k - z)^{-m-1}$ tends to 0 as $r\to\infty$. For the next term,

$$\Big|\int_0^{2\pi}\frac{2r}{(re^{i\theta}-z)^{m+2}}\log|f(re^{i\theta})|d\theta\Big| \le Cr^{-m-1}\log M(r)$$

for $r\gg 1$ and again choosing $\epsilon>0$ such that $\lambda+\epsilon<m+1$ we see that it tends to 0 as $r\to\infty$. Therefore $h^{(m+1)}(z)\equiv 0$ and so h is a polynomial of degree at most M.

We have defined $m=\lfloor\lambda\rfloor$ and from this had deduced the uniform and absolute convergence on compact sets of

$$\prod_{k=1}^{\infty}\Big(1-\frac{z}{z_k}\Big)e^{\sum_{l=1}^{m}(\frac{z}{z_k})^l}.$$

This shows that the genus of the canonical Weierstrass product for f is at most m. If so, in the previous argument replace m in the definition of the Weierstrass product by $m_1\le m$ with m_1 the genus of the Weierstrass product for f. This finishes the proof of Hadamard's theorem.

18.1 Poisson's formula

Suppose $u(x,y)$ is a harmonic function defined on some neighborhood of $\operatorname{clos}(\Delta)$. Recall that this means that u is C^2 and

$$\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)u(x,y)=0.$$

For $0 < r < 1$ define

$$P_r(\theta) = \Re\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}}\right\} = \frac{1}{2}\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}} + \frac{1+re^{-i\theta}}{1-re^{-i\theta}}\right\} = \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

Then Poisson's formula states that for $z = x + iy = re^{i\theta} \in \Delta$,

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(e^{i\varphi})d\phi.$$

Lemma 18.1. *There exists a holomorphic function $f(z)$ defined on a neighborhood of* $\mathrm{clos}(\Delta)$ *such that $\Re f(x+iy) = u(x,y)$.*

Proof. Consider the 1-form

$$\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx.$$

Therefore there exists a real valued C^2 function $v(x,y)$ on a neighborhood of $\mathrm{clos}(\Delta)$ such that $dv = (\partial u/\partial x)dy - (\partial u/\partial y)dx$. Then by the Cauchy-Riemann equations $f(z) = u(x,y) + iv(x,y)$ is holomorphic. $\square$

Corollary 18.2. *Real valued harmonic functions on an open set in $\mathbb{R}^2$ do not have a maximum unless they are constant.*

Proof. If not, by scaling and translation we may suppose u is defined on $\mathrm{clos}(\Delta)$ and the maximum occurs in Δ. Write $u = \Re f$ as in the lemma and define $F(z) = e^{u(z)}$. Then $|F(z)| = e^{u(x,y)}$ cannot have a maximum. This shows that u cannot have a maximum, unless u is a constant. $\square$

As before, suppose u is harmonic on a neighborhood of $\mathrm{clos}(\Delta)$, and $u = \Re f$ with f holomorphic. Define $F(z) = e^{f(z)}$. Write $z = x + iy = re^{i\theta} \in \Delta$. By Poisson-Jensen applied to F and $r = 1$,

$$\begin{aligned} u(x,y) = \log|F(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right\}u(e^{i\varphi})d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(e^{i\varphi})d\varphi. \end{aligned}$$

This argument, applied to $u_r(x,y) = u(rx,ry)$, implies Poisson's formula for harmonic functions $u(x,y)$ defined on Δ, provided it extends continuously to $\mathrm{clos}(\Delta)$ by letting $r \searrow 1$.

More generally, this applies if u is harmonic on Δ which have L^p boundary values on $\partial\Delta$ ($p \geq 1$) or even distribution boundary values, or even hyperfunction boundary values.

18.2 Normal family

Theorem 18.3 (Riemann mapping theorem)**.** *Suppose $U \subseteq \mathbb{C}$ is open, connected, and simply connected with $U \neq \mathbb{C}$. Then U is biholomorphic to Δ.*

Suppose $U \subseteq \mathbb{C}$ is open connected, and let $\mathcal{F}$ be a family (i.e., set) of holomorphic functions on U.

Definition 18.4. $\mathcal{F}$ is a **normal family** if every sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence which converges locally uniformly. (Note that in that case the limit is holomorphic.)

Lemma 18.5. *The following two conditions are equivalent:*

(a) $\mathcal{F}$ is locally uniformly bounded.

(b) $\mathcal{F}$ is a normal family.

This is a special case of Ascoli's theorem.

Proof. Let us first prove (b) $\Rightarrow$ (a). Suppose $\mathcal{F}$ satisfies (b) but not (a). Then there exists $z_0 \in U$ such that $\mathcal{F}$ is not uniformly bounded on any neighborhood of z_0. Hence for $n \in \mathbb{N}$ there exist $z_n \in U$ and $f_n \in \mathcal{F}$ such that $|f_n(z_n)| \geq n$ and $z_n \to z_0$. Then $\{f_n\}$ cannot have a subsequence which converges uniformly on any neighborhood of z_0.

For (a) $\Rightarrow$ (b), suppose $\mathcal{F}$ satisfies (a). We claim that $\mathcal{F}$ is locally uniformly equicontinuous. Suppose $z_0 \in U$. Choose $r > 0$ and $M > 0$ such that $z \in \Delta(z_0, r)$ implies $z \in U$ and $|f(z)| \leq F$ for any $f \in \mathcal{F}$, with $\operatorname{clos} \Delta(z_0, r) \subseteq U$. Now suppose $0 < \rho < r$ and $z_1, z_2 \in \Delta(z_0, \rho)$. Then

$$|f(z_1) - f(z_2)| \leq \frac{1}{2\pi} \int_{\partial\Delta(z_0,r)} \left| \frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2} \right| Mrd\theta \leq \frac{\mathrm{Const}_{r,\rho}}{2\pi} M|z_2 - z_1|,$$

by Cauchy's integral formula. Then $\mathcal{F}$ is uniformly equicontinuous on $\Delta(z_0, \rho)$. $\square$

19 November 3, 2016

Let $U \subseteq \mathbb{C}$ be open, and $\mathcal{F}$ be a family of holomorphic functions on U. We have shown that if $\mathcal{F}$ is normal then $\mathcal{F}$ is locally uniformly bounded, and that if $\mathcal{F}$ is locally uniformly bounded, then $\mathcal{F}$ is locally uniformly equicontinuous. We need to show that if $\mathcal{F}$ is locally uniformly bounded, then it is normal.

Choose a countable dense subset $\{s_k\} \subseteq U$. Given a sequence of functions in $\mathcal{F}$, choose a subsequence that converge at s_1 (which exists since $\{|f(s_k)| : f \in \mathcal{F}\}$ is bounded). Then choose a subsequence of the first subsequence which converges also at s_2, and continue. Take the diagonal subsequence and call it $\{f_n\}$. By construction, for each k, $f_n(s_k)$ converges. Let some arbitrary $z_0 \in U$ be given, and choose $r > 0$ such that $\operatorname{clos}(\Delta(z_0, r)) \subseteq U$. We shall show that $\{f_n\}$ converges uniformly on $\Delta(z_0, r)$, since z_0 was arbitrary, this will imply that $\{f_n\}$ converges locally uniformly.

Let $\epsilon > 0$ be given. Then by local uniform equicontinuity, there exists a $\delta > 0$ such that $z_1, z_2 \in \operatorname{clos}(\Delta(z_0, r))$ and $|z_1 - z_2| < \delta$ implies $|f(z_1) - f(z_0)| < \epsilon/3$ for all $f \in \mathcal{F}$. The disks $\Delta(s_k, \delta)$ cover $\operatorname{clos}(\Delta(z_0, r))$. So by compactness, finitely many $\Delta_{1 \le j \le m}\Delta(s_{k_j}, \delta)$ cover $\operatorname{clos}(\Delta(z_0, r))$. We know that $\{f_n(s_{k_j})\}$ converges. So for these finitely many s_{k_j}, we can choose N such that $m, n \ge N$ and $1 \le j \le m$ implies $|f_m(s_{k_j}) - f_n(s_{k_j})| < \epsilon/3$. Now suppose $z \in \Delta(z_0, r)$. Then there exists j such that $|z - s_{k_j}| < \delta$. Then for $m, n \ge N$,

$$|f_m(z) - f_n(z)| \le |f_m(z) - f_m(s_{k_j})| + |f_m(s_{k_j}) - f_n(z)| + |f_n(s_{k_j}) - f_n(z)| < \epsilon.$$

So I have proved Lemma 18.5.

19.1 Riemann mapping theorem

Riemann mapping theorem. *Let $U \subseteq \mathbb{C}$ be open, connected, simply connected, and $U \neq \mathbb{C}$. Then $U \cong \Delta$.*

Proof. After translating U if necessary, we may suppose $0 \notin U$. Then $\log z$ has a well-defined holomorphic determination on U because U is simply connected. Therefore $z \mapsto \sqrt{z} = e^{\log z/2}$ also has a well-defined holomorphic determination on U. By the open mapping theorem, $U_1 =$ image of U under $z \mapsto z^{1/2}$ is open and $U \cong U_1$ since $w \mapsto w^2$ is the inverse. If $w \in U_1$ then $-w \notin U_1$ because $w^2 \in U$ has a "unique" square root. Becuase U_1 is open, there exist $a, \delta > 0$ such that $U_1 \cap \Delta(a, \delta) = \emptyset$. Let U_1 be the image of U_1 under $z \mapsto (z-a)^{-1}$. Then $U \cong U_1 \cong U_2$, and U_2 is a bounded set. Translating and scaling U_2 allows us to find $0 \in U_3 \subseteq \Delta$ such that $U_3 \cong U_2$. So after all this, I can assume without loss of generality that $0 \in U$ and $U \subseteq \Delta$. Of course, U is open, connected, and simply connected.

Define

$$\mathcal{F} = \{f : U \to \Delta : f(0) = 0, f \text{ is one-to-one }, f'(0) \in \mathbb{R}_{>0}\}.$$

Clearly $\mathcal{F} \neq \emptyset$ because the identity is an element, and by the lemma, $\mathcal{F}$ is a normal family. Let $M = \sup\{f'(0) : f \in \mathcal{F}\} > 0$. Choose a sequence in $\mathcal{F}$

such that the derivative at 0 converges to M, and go to the subsequence that converges locally uniformly. The conclusion is that there exists a $\{f_n\} \subseteq \mathcal{F}$ such that $f_n'(0) \to M$ and $f_n \to f$ locally uniformly. Clearly $f(0) = 0$ and $f'(0) = \lim_{n\to\infty} f_n'(0) = M$. So we need to show that f is one-to-one and $f : U \to \Delta$.

Suppose that $z_0 \in U$ and $w_0 = f(z_0)$. Because f is non-constant, there exists a $\delta > 0$ such that $f(z) - w_0 \neq 0$ on $\operatorname{clos}(\Delta(z_0, \delta)) \setminus \{z_0\}$, because the zeros are isolated. Let $\epsilon = \min_{\zeta\in\partial\Delta(z_0,\delta)} |f(z) - w_0|$, and let $f_n \to f$ be uniform on $\partial\Delta(z_0, \delta)$. Choose N such that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon$ on $\partial\Delta(z_0, \delta)$. Then

$$|(f_n(z) - w_0) - (f(z) - w_0)| < \epsilon \leq |f(z) - w_0|$$

for $z \in \partial\Delta(z_0, \delta)$ and $n \geq N$. By Rouché, there exists a $z_n \in \Delta(z_0, \delta)$ such that $f_n(z_n) = w_0$ for $n \geq N$. Suppose there exists $\tilde{z}_0 \in U$ such that $\tilde{z}_0 \neq z_0$ and $f(\tilde{z}_0) = w_0 = f(z_0)$. Then apply the previous argument with small enough δ for both z_0 and $\tilde{z}_0$, and conclude that there exist $z_n \in \Delta(z_0, \delta)$, $\tilde{z}_n \in \Delta(\tilde{z}_0, \delta)$ such that $f_n(z_n) = w_n = f_n(\tilde{z}_n)$. Since δ is small enough, $\Delta(z_0, \delta) \cap \Delta(\tilde{z}_0, \delta) \neq 0$ and so $\tilde{z}_n \neq z_n$. This contradicts the injectivity of f_n. Therefore $f \in \mathcal{F}$.

So we have constructed $f \in \mathcal{F}$ such that $f'(0) = M \geq h'(0)$ for every $h \in \mathcal{F}$. If $U = \Delta$, then there is nothing to prove. Suppose $a \in \Delta \setminus U$. Define $S(z) = (z - a)/(1 - \bar{a}z)$. Then $S(a) = 0$ and $|S(e^{i\theta})| = 1$ and so $\Sigma : \Delta \cong \Delta$. Then $S \circ f : U \to \Delta$ is one-to-one and $S \circ f(z) \neq 0$ for all $z \in u$. Arguing as before, we conclude that $S \circ f$ has a well-defined holomorphic square root, i.e., $h : U \to \Delta$ such that $S \circ f = h^2$. Write σ for the squaring map: $\sigma(z) = z^2$. Then $\sigma \circ h = S \circ f$ and so $f = S^{-1} \circ \sigma \circ h$. Since f and S are one-to-one, so is h.

Define $T : \Delta \cong \Delta$, $T(z) = \alpha(z - b)/(1 - \bar{b}z)$, where $b = h(0)$ and α is chosen so that $\frac{d}{dz}T \circ h(0) \in \mathbb{R}_{>0}$. (Note that the derivative cnanot be 0 because h and S are one-to-one.) We have

$$f = S^{-1} \circ \sigma \circ T^{-1}(T \circ h).$$

We know that $T \circ h \in \mathcal{F}$ because $T \circ h(0) = 0$. So $\frac{d}{dz}T \circ h|_{z=0} \leq f'(0) = M$. On the other hand, $S^{-1} \circ \sigma \circ T^{-1} : \Delta \to \Delta$ and $0 \mapsto 0$. Also it is clearly not a multiple of z. So by Schwartz's lemma, $|d/dz(S^{-1} \circ \sigma \circ T^{-1})|_0| < 1$. This cannot be possible, and thus f is a biholomorphism between U and Δ. $\square$

20 November 8, 2016

Let $C \subseteq \mathbb{R}^2$ be a Jordan curve, i.e., the image of $\partial\Delta = \mathbb{R}/2\pi\mathbb{Z}$ under an injective continuous map. Then the inverse map is continuous because images of closed sets are images of compact sets and hence closed. Therefore the original maps is a homeomorphism. The Jordan curve theorem then states that if $C \subseteq \mathbb{R}^2$ is a Jordan curve, then $\mathbb{R}^2 \setminus C$ has two connected components, one of which is unbounded, and C is the boundary of each of these. Assume this.

20.1 Carathéodory's theorem

Let $U \subseteq \mathbb{C}$ be open, connected, simply connected, and bounded. Suppose also that ∂U is a Jordan curve. Let $F : \Delta \cong U$ be the essential unique biholomorphism guaranteed by the Riemann Mapping Theorem.

Theorem 20.1 (Carathéodory)**.** *In this construction, F extends to a continuous map $\bar{F} : \operatorname{clos}\Delta \to \operatorname{clos}U$, whose restriction to $\partial\Delta$ is a homeomorphism onto the Jordan curve ∂U.*

For $0 < r < 1$, let $C_r = \Delta \cap \partial\Delta(1, r)$. Let $\gamma_r = F(C_r)$. Now let us parameterize C_r as $z = 1 - re^{i\theta}$ for $-\theta_r < \theta < \theta_r$. Explicitly, $\theta_r = \cos^{-1}(r/2)$. Composing this parametrization with F, we get a parametrization of γ_r which is real analytic. Therefore γ_r has a well-defined length $\ell(\gamma_r)$ which may be $+\infty$.

Lemma 20.2. *The function $r \mapsto \ell(\gamma_r)$ is Lebesgue integrable and $\int_0^1 \ell(\gamma_r)^2/r dr < \infty$.*

Proof. The function $r \mapsto \ell(\gamma_r)$ is the pointwise limit of the length of the F-image of the portion of C_r parametrized by $-(1 - 1/2n)\theta_r < \theta < (1 - 1/2n)\theta_r$ which is continuous as a function of r. So $r \mapsto \ell(\gamma_r)$ is measurable.

F is conformal with scaling factor $|F'(z)|$ at z. So

$$\ell(\gamma_r) = \int_{-\theta_r}^{\theta_r} |F'(1 - re^{i\theta})| r d\theta$$

where $+\infty$ is allowed as value. Then

$$\begin{aligned}
\int_0^1 \frac{\ell(\gamma_r)^2}{r} dr &= \int_0^1 r\left(\int_{-\theta_r}^{\theta_r} |F'(1 - re^{i\theta})| d\theta\right)^2 dr \\
&\leq \int_0^1 \left(\int_{-\theta_r}^{\theta_r} |F'(1 - re^{i\theta})|^2 \int_{-\theta_r}^{\theta_r} 1 d\theta\right) dr \\
&\leq \pi \int_0^1 \int_{-\theta_r}^{\theta_r} |F'(1 - re^{i\theta})|^2 r d\theta dr \leq \pi \cdot (\text{area of } U) < \infty.
\end{aligned}$$

□

Note that $\int_0^\delta dr/r = +\infty$ for every $\delta > 0$. Thus $\ell(\gamma_r)$ cannot be bounded away from 0 on any interval $(0, \delta)$ with $\delta > 0$.

Corollary 20.3. *There exists a sequence* $1 > r_1 > r_2 > \cdots > 0$ *such that* $\infty > \ell(\gamma_{r_1}) > \ell(\gamma_{r_2}) > \cdots$ *with* $r_n \to 0$ *and* $\ell(\gamma_{r_n}) \to 0$.

Lemma 20.4. *If* $\ell(\gamma_r) < \infty$ *for some* r*, then* F *extends continuously to* $\overline{C_r} = \text{clos}(C_r)$.

Proof. If $\theta_n \nearrow \theta_r$ then $F(1 - re^{i\theta_n})$ converges in $\mathbb{C}$ and the limit is independent of a particular choice of θ_n. The same is true for $\theta_n \searrow -\theta_r$. $\square$

With r_n chosen as before, let

$$a_n = \lim_{\theta \searrow -\theta_r} F(1 - r_n e^{i\theta}), \quad b_n = \lim_{\theta \nearrow \theta_r} F(1 - r_n e^{i\theta})$$

with γ_n for γ_{r_n}. Observe that $F : \Delta \to U$ is a homeomorphism, and thus F extends to a homeomorphism of 1-point compactification. Therefore $a_n \in b_n \in \partial U$. If we know the statement of the theorem, a_n would be different from b_n. But we don't know this yet.

Lemma 20.5. $|a_n - b_n| \to 0$ *as* $n \to \infty$*. For* $n \gg 1$*, either* $a_n = b_n$ *or* $a_n \neq b_n$ *and* $\partial U - \{a_n, b_n\}$ *has two connected components, exactly one of which, call it* δ_n*, satisfies* $\text{diam}(\delta_n) < \text{diam}(\partial U)$.

Proof. Let us parametrize ∂U as $\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto w(\theta)$. Then both $\theta \mapsto w(\theta)$ and its inverse are continuous, hence uniformly continuous. Note that $|a_n - b_n| \leq \ell(\gamma_n) \to 0$. Thus as $n \to \infty$, the points $a_n, b_n \in \partial U$ get infinitely close. Hence $w^{-1}(a_n) - w^{-1}(b_n) \in \mathbb{R}/2\pi\mathbb{Z}$ tends to 0 as $n \to \infty$. By uniform continuity of ω, δ_n, which is the image under w of "$w^{-1}(a_n) - w^{-1}(b_n)$" will have diameter tending to zero as $n \to \infty$. $\square$

There are two possibilities: $a_n = b_n$ or $a_n \neq b_n$. Define $\bar{\delta}_n = \{a_n\} = \{a_n, b_n\}$ if $a_n = b_n$ and define $\bar{\delta}_n = \delta \cup \{a_n, b_n\}$ if $a_n \neq b_n$. Also let $\bar{\gamma}_n = \gamma_n \cup \{a_n, b_n\}$.

We know that γ_n and δ_n are one-to-one images of open intervals of under continuous maps F, w. Also $\gamma_n \subseteq U$ and $\bar{\delta}_n \subseteq \partial U$ with $\{a_n, b_n\}$ the intersection of the closures of γ_n and δ_n. The conclusion is that

$$\gamma_n \cup \bar{\delta}_n = \begin{cases} \bar{\gamma}_n & \text{if } a_n = b_n \\ \bar{\gamma}_n \cup \delta_n & \end{cases}$$

is a Jordan curve. Let U_n denote the interior o $\gamma_n \cup \bar{\delta}_n$. Note that $\text{diam}\, U_n = \text{diam}\, \partial U_n \leq \text{diam}\, \delta_n + \ell(\gamma_n)$. This goes to 0 as has been shown already.

Lemma 20.6. *For* $n \gg 1$, $U_n = F(\Delta \cap \Delta(1, r_n))$.

Proof. $\bar{\delta}_n \subseteq \partial U$ and $\gamma_n \subseteq U$. Thus $\gamma_n \cup \bar{\delta}_n$ is idsjoint from $\mathbb{C} \setminus \text{clos}(U)$. Thus $\mathbb{C} \setminus \text{clos}(U)$ is in the exterior of $\gamma_n \cup \bar{\delta}_n$. Now suppose $w_0 \in \partial U \setminus \bar{\delta}_n$. Now suppose $w_0 \in \partial U \setminus \bar{\delta}_n$. Then w_0 has an open neighborhood in $\mathbb{C}$ which is disjoint from $\bar{\delta}_n$ and $\bar{\gamma}_n$. Thus the exterior of $\gamma_n \cup \bar{\delta}_n$ contains both points in U and points in $\mathbb{C} \setminus \text{clos}(U)$. That means the exterior of $\gamma_n \cup \bar{\delta}_n$ contains $\mathbb{C} \setminus \text{clos}(U)$ and $\partial U \setminus \bar{\delta}_n$, some points in U.

$\Delta \setminus C_{r_n}$ has two connected components. Thus its F-image also has two connected components, both of which are disjoint from $\gamma_n \cup \bar{\delta}_n$ because $\gamma_n = F(C_{r_n})$ and $\bar{\delta}_n \subseteq \partial U$. So each of these can be contained either in the interior of exterior of $\gamma_n \cup \bar{\delta}_n$. Then by exclusion one of the components must be the interior of $\gamma_n \cup \bar{\delta}_n$ and the other contained in the exterior.

As $n \to \infty$, we have $\bigcup(\Delta \setminus \operatorname{clos} \Delta(1, r_n)) = \Delta$ and so $\bigcup F(\Delta \setminus \operatorname{clos} \Delta(1, r_n)) = U$. But $\operatorname{diam}(\gamma_n \cup \bar{\delta}_n) = \operatorname{diam} U_n \to 0$. Therefore we conclude that $F(\Delta \cap \Delta(1, r_n)) = \operatorname{int}(\gamma \cup \bar{\delta}_n)$ for $n \gg 1$. $\square$

Note that $\operatorname{clos} U_n = U_n \cup \gamma_n \cup \bar{\delta}_n$ is compact, and decreases as n increases. Thus $\bigcap_{n \gg 1} \operatorname{clos} U_n \neq \emptyset$ by the finite intersection property. We know that $\operatorname{diam} \operatorname{clos} U_n \to 0$.

Corollary 20.7. *The set* $\bigcap_{r>0} \operatorname{clos}(F(\Delta \cap \Delta(1, r)))$ *consists of a single point.*

21 November 10, 2016

Let us recall where we are. $U \subseteq \mathbb{C}$ is an open, connected, simply connected, bounded domain with ∂U a Jordan curve. Let $F : \Delta \to U$ be a biholomorphism. We have shown that there exists a sequence r_n with $0 < r_n < 1$ such that $r_n \to 0$ as $n \to \infty$ and $\operatorname{diam} F(\Delta \cap \Delta(1, r_n)) = \operatorname{diam}(\operatorname{clos}(F(\Delta \cap (1, r_n)))) \to 0$ as $n \to \infty$. Then $\bigcap_n \operatorname{clos}(F(\Delta \cap \Delta(1, r_n)))$ consists of a single point whith lies on ∂U.

In this argument, we can replace 1 by any $\alpha \in \partial\Delta$. Let $\cap_{r>0} \operatorname{clos} F(\Delta \cap \Delta(\alpha, r)) = \{a(\alpha)\}$ so that $a(\alpha) \in U$. Define $\bar{F} : \operatorname{clos} \Delta \to \operatorname{clos} U$ such that

$$\overline{F}(z) = \begin{cases} F(z) & \text{if } z \in \Delta, \\ a(z) & \text{if } z \in \partial\Delta. \end{cases}$$

Lemma 21.1. *$\overline{F}$ is continuous.*

Proof. For $z \in \Delta$, this is clear. Suppose $\alpha \in \partial\Delta$. We know that $(\operatorname{clos} \delta) \cap \Delta(\alpha, r)$ is a neighborhood for the topology of $\operatorname{clos} \Delta$ at α. We also know that

$$\operatorname{diam} \operatorname{clos} F(\Delta \cap \Delta(\alpha, r)) \to 0.$$

So it suffices to show that $\overline{F}(\operatorname{clos} \Delta \cap \Delta(\alpha, r/2)) \subseteq \operatorname{clos} F(\Delta \cap \Delta(\alpha, r))$. It is obvious that $\overline{F}(\Delta \cap \Delta(\alpha, r/2)) \subseteq \operatorname{clos} F(\Delta \cap \Delta(\alpha, r))$. Thus it suffices to show:

$$\overline{F}(\partial\Delta \cap \Delta(\alpha, r/2)) \subseteq \operatorname{clos} F(\Delta \cap \Delta(\alpha, r)).$$

Suppose $\beta \in \partial\Delta(\alpha, r/2)$. Then $\Delta \cap \Delta(\beta, r/2) \subseteq \Delta(\alpha, r)$. We know that $a(\beta) \in \operatorname{clos} F(\Delta \cap \Delta(\beta, r/2)) \subseteq \operatorname{clos} F(\Delta \cap \Delta(\alpha, 1))$. This shows that $\overline{F}$ is continuous. $\square$

Corollary 21.2. *$\overline{F} : \operatorname{clos} \Delta \to \operatorname{clos} U$ is surjective.*

Proof. By continuity, $\overline{F}(\operatorname{clos} \Delta)$ is compact, and is contained in $\operatorname{clos} U$ but contains U. So $\overline{F}(\operatorname{clos} \Delta) = \operatorname{clos} U$. $\square$

By construction, $\overline{F}(\partial U) \subseteq \partial U$ and $\overline{F}(\Delta) = F(\Delta) = U$.

Corollary 21.3. *$\overline{F}(\partial\Delta) = \partial U$.*

To complete the proof of Carathéodory's theorem, it suffices to show:

Lemma 21.4. *$\overline{F} : \partial\Delta \to \partial U$ is one-to-one.*

Proof. Suppose for some $\alpha, \beta \in \partial\Delta$, $\alpha \neq \beta$ but $\overline{F}(\alpha) = \overline{F}(\beta)$. Define

$$\ell_\alpha = F(\{t\alpha : 0 \le t < 1\}), \quad \ell_\beta = F(\{t\beta : 0 \le t < 1\}).$$

Then $\overline{\ell_\alpha} = \operatorname{clos}(\ell_\alpha) = \ell_\alpha \cup \{a(\alpha)\}$ and $\overline{\ell_\beta} = \operatorname{clos}(\ell_\beta) = \ell_\beta \cup \{a(\beta)\}$. Define $\gamma = \overline{\ell_\alpha} \cup \beta = \ell_\alpha \cup \overline{\ell_\beta}$. This is a Jordan curve in $\operatorname{clos} U$ whose intersection with

∂U consists of a single point $\{a(\alpha)\} = \{a(\beta)\}$. Let U_γ denote the interior of this Jordan curve.

We know that $\mathbb{C} \setminus \operatorname{clos} U$ is open and disjoint from γ. Thus $\mathbb{C} \setminus \operatorname{clos} U \subseteq$ exterior of γ. Any point p in $\partial U \setminus \{a(\alpha)\}$ is disjoint from γ, and every neighborhood of p contains points from $\mathbb{C} \setminus \operatorname{clos} U$ as well as points in U. In particular, $\partial U \setminus \{a(\alpha)\} \subseteq$ exterior of γ. We know that $\Delta \setminus (\{t\alpha : 0 \le t < 1\} \cup \{t\beta : 0 \le t < 1\})$ has two connected components. By exclusion, the F-imeage of one of these must be U_γ and the F-image of the other must lie in the exterior of γ. So the $\overline{F}$-iamge of the one of the two components of $\partial \setminus \{\alpha, \beta\}$ must get mapped to $a(\alpha) = a(\beta)$ because: suppose δ is a point in this component of $\partial\Delta \setminus \{\alpha, \beta\}$ that goes in the U_γ direction. Then $\{t\delta : 0 \le t < 1\}$ is a curve in Δ which dense to $a(\delta)$ as $t \to 1$. But $a(\delta) \in \operatorname{clos} U_\gamma \cap \partial U = \{a(\alpha)\}$.

This shows that if $\overline{F} : \partial\Delta \to \partial U$ fails to be one-to-one, then it is constant on some open nonempty $/\theta$-interval I of $\{e^{i\theta}\}$. We are free to tranlate U and may suppose without loss of generality that $\overline{F}$-image of this open interval is 0. Then $\overline{F}(z)$ for $z \in \Delta$ cannot be zero. By Jensen's formula, for $0 < r < 1$,

$$\log|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|F(re^{i\theta})| d\theta.$$

By uniform continuity of $\overline{F}$, for $\theta \in I$, $F(re^{i\theta}) \to 0$ uniformly in θ. Then it follows that the right hand side of the interval is $-\infty$ as $r \to 1$. Then $F(0) = 0$. $\square$

This finishes the proof of Carathéorody.

21.1 Riemann surfaces again

Recall that the only connected, simply connected Riemann surfaces are $\mathbb{P}$, $\mathbb{C}$, H. The surface $\mathbb{P}$ doesn't cover any other surface, because every automorphism has a fixed point. The next one $\mathbb{C}$ is properly covers only $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ or $\mathbb{C}/L$ which is topologically a 2-torus. Aside from the "few" cases covered by $\mathbb{C}$, and apart from $\mathbb{P}$, connected Riemanns surfaces arise as covered by H: the surface $\Gamma\ H$ where $\Gamma = \pi_1(\Gamma\ H$ is determined as a subgroup of $\operatorname{Aut}(H) \cong \mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$ up to inner automorphism. Then $\Gamma \subseteq \mathrm{SL}(2, \mathbb{R})$ with $\{\pm 1\} \subseteq \Gamma$ arises as a fundamental group of a Riemann surface if and only if

(1) $\Gamma \subseteq \mathrm{SL}(2, \mathbb{R})$ is discrete,

(2) Γ contains no elliptic elements.

For $N \in \mathbb{N}$, define the **principal congruence subgroup of level** N as

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) : \gamma \cong 1_{2\times 2} \bmod N\}.$$

Note that $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ is a well-defined finite group (but $\mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ is not unless N is prime). Then $\Gamma(N)$ can be thought as a kernel of $\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$. So $\Gamma(N) \subseteq \mathrm{SL}(2, \mathbb{Z})$ is a normal subgroup of finite index.

Lemma 21.5. *For $N \geq 2$, $\Gamma(N)$ contains no elliptic elements. (But for $N = 2$ only, $-1 \in \Gamma(N)$.)*

Proof. Suppose $\gamma \in \Gamma(N)$ is elliptic. Then γ must have eigenvalues $e^{i\theta}$ for $\theta \notin \pi\mathbb{Z}$. So $\operatorname{tr}\gamma = 2\cos\theta$ for $\theta \in \pi\mathbb{Z}$, and in particular, $\operatorname{tr}\gamma \in (-2, 2)$. But $\operatorname{tr}\gamma \cong 2 \pmod{N}$. So the only possibilities are $N \leq 3$ and $\operatorname{tr}\gamma \in \{-1, 0, 1\}$.

For $N = 3$, $\operatorname{tr}\gamma = -1$ so $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Write $a = 1 + 3k$ and $d = -2 - 3k$. Then

$$1 = ad - bc = -2 - 9k - 9k^2 - bc \equiv -2 \pmod{9}.$$

This is impossible.

The remaining case is $N = 2$. In this case, write $a = 1 + 2k$ and $d = -1 - 2k$. Then

$$1 = ad - bc = -1 - 4k - 4k^2 - bc \equiv -1 \pmod{4}$$

and we again get a contradiction. □

Next time I will prove the following theorem:

Theorem 21.6. *The set $\mathbb{C} \setminus \{0, 1\}$ has H as a universal cover with fundamental group $\Gamma(2)/\{\pm 1\}$.*

Corollary 21.7 (Little Picard theorem)**.** *Any non-constant entire function omits at most one value.*

Proof. Assume that $f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. (We can assume this because this is general up to translation, rotation, and scaling.) We have a universal covering of $\mathbb{C} \setminus \{0, 1\}$ by H. Then by topological monodromy theorem, we can lift this to get $\tilde{f}$.

$$\begin{tikzcd} & H \arrow[d, "\text{hol.}"] \\ \mathbb{C} \arrow[ur, "\tilde{f}"] \arrow[r, "f", "\text{hol.}"'] & \mathbb{C} \setminus \{0, 1\} \end{tikzcd}$$

Compose $\tilde{f}$ with $H \cong \Delta$. Then we get a bounded entire function, which must be constant. Then f is constant. □

22 November 15, 2016

Let $\Gamma \subseteq \mathrm{SL}(2,\mathbb{R})$ be a subgroup.

Definition 22.1. Γ acts **property discontinuously** on H if every $z_0 \in H$ has a neighborhood U such that $U \cap \gamma U \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.

You will show in the homework that:

Proposition 22.2. $\Gamma \subseteq \mathrm{SL}(2,\mathbb{R})$ *is discrete if and only if* Γ *acts properly discontinuously on* H. *Also* $H \to \Gamma\backslash H$ *is a covering map if and only if* Γ *acts properly discontinuously without fixed points.*

For simplicity, say "Γ acts without fixed points" if $\gamma z = z$ for $\gamma \in \Gamma$ implies $\gamma = \pm 1$.

Theorem 22.3. $\Gamma(2)\backslash H \cong \mathbb{C}\backslash\{0,1\}$, *and this is the universal cover of* $\mathbb{C}\backslash\{0,1\}$ *with fundamental group* $\Gamma(2)$.

Definition 22.4. A **fundamental domain** for the action of Γ on H is a closed subset $F \subseteq H$ such that

(a) ∂F consists of a finite number of smooth curves,

(b) $\{\gamma \in \Gamma : \gamma F \cap F \neq \emptyset\}$ is finite,

(c) $\bigcup_{\gamma\in\Gamma} \gamma F = H$.

(d) $z_1, z_2 \in F$, $z_1 \neq z_2$, $z_2 = \gamma z_1$ for $\gamma \in \Gamma$ implies $z_1, z_2 \in \partial F$.

22.1 A fundamental domain for $\Gamma(2)$

Let us write

$$F = \{z \in \mathbb{C} : |\Re z| \leq 1, |z - 1/2| \geq 1/2, |z + 1/2| \geq 1/2\}.$$

Proposition 22.5. F *is a fundamental domain for the action of* $\Gamma(2)$ *on* H. *If* $z_1, z_2 \in \partial F$ *are distinct but* $\Gamma(2)$*-related, then* $z_2 = -\bar{z}_1$.

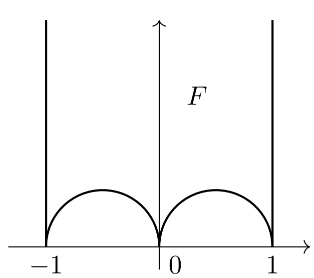

Figure 2: The fundamental domain F

Proof. Define

$$t = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $s, t \in \Gamma(2)$ and although $w \in \Gamma(2)$, it normalizes $\Gamma(2)$, i.e., $w\Gamma(2)w^{-1} = \Gamma(2)$. Also $w \equiv w^{-1} \bmod \mathbf{1}_{2\times 2}$ and so w, w^{-1} have the same action on H.

Suppose $z, tz \in F$. Since $tz = z + 2$, it follows that $\Re(z) = -1$ and $\Re(tz) = +1$. Then $tz = -\bar{z}$. Because $w\{|z - 1/2| = 1/2\}$ is the w-image of the circle of radius $1/2$, centered at $1/2$, so must be a circle or straight line. Because it $w(0) = \infty$ and $w(1) = -1$, it is a straight line through -1 and perpendicular to $\mathbb{R}$. So

$$w\{|z - 1/2| = 1/2\} = \{\Re z = -1\} \quad \text{and similarly} \quad w\{|z + 1/2| = 1/2\} = \{\Re z = 1\}.$$

Now

$$\begin{aligned} s\{z \in H : |z - 1/2| = 1/2\} &= wtw^{-1}\{z \in H : |z - 1/2| = 1/2\} \\ &= wt\{z \in H : \Re z = -1\} = w\{z \in H : \Re z = 1\} \\ &= \{z \in H : |z + 1/2| = 1/2\}. \end{aligned}$$

Also

$$s\{1/2 = 1/2e^{i\theta} : 0 < \theta < \pi\} = \{-1/2 - 1/2e^{-i\theta} : 0 < \theta < \pi\}$$

and so if s maps $z \in \partial\Delta(1/2, 1/2) \cap H$ to $\partial\Delta(-1/2, 1/2) \cap H$, then $sz = -\bar{z}$.

To complete the proof, we proceed in the following steps:

Step 1. Suppose $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ with $z \in H$. Then $\Im\gamma z \leq \max\{\Im z, 1/\Im z\}$. To see this, write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Then

$$\Im\gamma z = \Im\frac{az + b}{cz + d} = \Im\frac{adz + bc\bar{z}}{|cz + d|^2} = \frac{\Im z}{|cz + d|^2}.$$

If $c = 0$, then $d = \pm 1$ and we may assume $d = 1$ without loss of generality. Then $\operatorname{im}\gamma z = \Im z$. If $c \neq 0$, then

$$\Im\gamma z = \frac{\Im z}{|cz + d|^2} \leq \frac{\Im z}{|c|^2(\Im z)^2} \leq \frac{1}{\Im z}.$$

Step 2. For $z \in H$, the supremum $\sup\{\Im\gamma z : \gamma \in \Gamma(2)\}$ is assumed, i.e., there is a maximal value.

Suppose $\{\gamma_n\} \subseteq \Gamma(2)$ is a sequence such that $\Im(\gamma_n z) \to \sup\{\Im\gamma z : \gamma \in \Gamma(2)\}$. Because $\Im(t^k\gamma_n z) = \Im\gamma_n z$, we may assume that $|\Re\gamma_n z| \leq 1$. By Step 1 and a compactness argument, we may suppose—by going to a subsequence—that $\gamma_n z \to z_\infty$ in H with $|\Re z_\infty| \leq 1$. By the homework problem, $\Gamma(2)$ acts property discontinuously without fixed points, so $H \to \Gamma(2)\backslash H$ is a covering map. So z_∞ has a neighborhood U in H such that $\gamma U \cap U = \emptyset$. But then because $\gamma_n z \to z_\infty$, there exist $\gamma_n z, \gamma_m z \in U$ and then $(\gamma_m\gamma_n^{-1})U \cap U \neq \emptyset$.

Step 3. $\bigcup_{\gamma\in\Gamma(2)} \gamma F = H$.
We know that any $\Gamma(2)$-orbit in H contains some point z such that $\Im\gamma z \leq \Im z$ for all $\gamma \in \Gamma(2)$, and also $|\Re z| \leq 1$. Then $\Im(sz) \leq \Im z$ and so $|2z-1| \geq 1$. Likewise, applying s gives $|2z+1| \geq 1$ and so $z \in F$.

Step 4. Suppose $z, \gamma z \in G$, with $\gamma \neq \pm 1$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$. Then Γ is one of the following modulo multiplication by $\pm\mathbf{1}_{2\times 2}$: s, s^{-1}, t, t^{-1}, and the identification by γ is one of those we have described above.
Let us first look at the case $c = 0$. Then $a = d = \pm 1$ and we may assume without loss of generality that $a = d = 1$. Hence $\gamma = \left(\begin{smallmatrix} 1 & 2k \\ 0 & 1 \end{smallmatrix}\right)$ for some $k \in \mathbb{Z}$. Because $\gamma z = z + 2k$, we have either $\Re z = -1$ and $k = 1$ or $\Re z = 1$ and $k = -1$.
Suppose now that $c \neq 0$ and $c = \pm 2d$. Then we may suppose $d = 1$ and $c = \pm 2$ and $a = 1 \pm 2b$.

$$\gamma z = \frac{az+b}{cz+d} = \frac{az+b}{d(1\pm 2z)} = \frac{(1\pm 2b)z+b}{\pm 2z+1} = \frac{b(2z\pm 1)+z}{2z\pm 1} = \frac{z}{1\pm 2z} + 2k$$

for $k = b/2 \in \mathbb{Z}$. So $\gamma z = s^{\mp 1}z + 2k$. By assumption, $z, \gamma z \in F$, and $z \in F$ implies $s^{\pm z} \in \{z \in H : |z - 1/2| \leq 1/2\} \cup \{z \in H : |z + 1/2| \leq 1/2\}$. The conclusion is that z and γz must both be in the boundary. Finally, if $c \neq 0$ and $c \neq 2d$, then we claim that $|cz+d| > 1$. Assume this for now. Then

$$\Im\gamma z = \frac{\Im z}{|cz+d|^2} < \Im z.$$

If γ^{-1} lands in one of the previous two conditions, then we are done by symmetry. So we may assume that both γ and γ^{-1} is in this case. Then $\Im z < \Im\gamma z$ and we get a contradiction. Thus it suffices to prove our claim $|cz+d| > 1$. Suppose that $|cz+d| \leq 1$. Then $|z \pm d/|c|| \leq 1/|c|$. We know that c is a nonzero even integer. If $c = \pm 2$, then $|z \pm d/2| \leq 1/2$. Since $c \neq \pm 2d$, we have $|d| > 1$, but this gives a contradiction. If $c = 2k$ with $|k| > 1$, in which case $|z \pm d/|c|| \leq 1/|c|$ is a semicircle that is disjoint from $F \in \mathbb{C}$.

This completes the proof of the proposition, and in particular, $\{\gamma \in \Gamma(2) : \gamma F \cap F \neq \emptyset\}$ is finite. $\square$

23 November 17, 2016

Last time, we have proved:

Proposition 23.1. *F is a fundamental domain for the action of $\Gamma(2)$ on H. If $z, \gamma z \in F$ and $z \neq \gamma z$, then $z, \gamma z \in \partial F$ and $\gamma z = -\bar{z}$.*

23.1 Universal covering space for the thrice-punctured sphere

Theorem 23.2. *There exists a holomorphic covering map $\Phi : H \to \mathbb{C} \setminus \{0, 1\}$ with covering group $\Gamma(2)$. Moreover, Φ can be chosen so that $\lim_{z\to 0} \Phi(z) = 0$, $\lim_{z\to 1} \Phi(z) = 1$, and $\lim_{z\to\infty} \Phi(z) = \infty$.*

We have already seen that this implies the "Little Picard theorem".

Proof. Define $F^+ = F \cap \{\Re z \geq 0\}$ and $F^- = F \cap \{\Re z \leq 0\}$. Then $F = F^+ \cup F^-$ and $F^+ \cap F^- = i\mathbb{R}_{>0}$. Also F^- is the image of F^+ under $z \mapsto -\bar{z}$.

Define $\varphi^+ :$ (clos. of $F^+ \subseteq \mathbb{P}$) $\to$ (clos. of $H \subseteq \mathbb{P}$) as a homeomorphism and holomorphic on interiors—such a map exists by Riemann–Carathéodory—such that $\varphi^+(0) = 0$, $\varphi^+(1) = 1$, and $\varphi^+(\infty) = \infty$, which can be done because $\mathrm{Aut}(H)$ acts on $\partial H = \mathbb{R} \cup \{\infty\}$ in a triply transitive manner. Then the only way they can fit with the right orientation is

$$\varphi^+(i\mathbb{R}_{>0}) = \mathbb{R}_{<0}, \quad \varphi^+(1 + i\mathbb{R}_{>0}) = \mathbb{R}_{>1},$$
$$\varphi^+(F^+ \cap |z - 1/2| = 1/2) = \{x \in \mathbb{R} : 0 < x < 1\}.$$

Now define $\varphi :$ (clos. of $F \subseteq \mathbb{P}$) $\to$ (clos. of $H \subseteq \mathbb{P}$) as

$$\varphi(z) = \begin{cases} \varphi^+(z) & z \in \text{clos. of } F^+ \subseteq \mathbb{P} \\ \overline{\varphi^+(-\bar{z})} & z \in \text{clos. of } F^- \subseteq \mathbb{P}. \end{cases}$$

We know that φ is well-defined and holomorphic on (the interior of) F possible on $i\mathbb{R}_{>0}$.

We claim that φ is holomorphic even on points of $i\mathbb{R}_{>0}$. The easiest way to do it is to integrate over rectangles. It suffices to show that $\int_{\partial R} \varphi dz = 0$ for every rectangle R in the interior of F. Suppose R passes thorough the imaginary axis. Define $R_n^+ = R \cap \{\Re z > 1/n\}$ and $R_n^- = R \cap \{\Re z < -1/n\}$. Since φ is continuous, it is uniformly continuous on a compact set. Thus

$$\int_{\partial R} \varphi dz = \int_{\partial R^+} \varphi dz + \int_{\partial R^-} \varphi dz = \lim_{n\to\infty} \int_{\partial R_n^+} \varphi dz + \lim_{n\to\infty} \int_{\partial R_n^-} \varphi dz = 0.$$

So φ is holomorphic on the F.

By construction, $\varphi : F \to \mathbb{C} \setminus \{0, 1\}$ is surjective because we already have things not on the real axis, and also we have $\mathbb{R}_{<0}$, $(0, 1)$, and $\mathbb{R}_{>1}$. Moreover it is injective except possible on the boundary of F in H, where it is 2-to-1, but the two inverse images of any point are related by the action $z \mapsto -\bar{z}$, i.e., by

$\Gamma(2)$. So $\varphi : F \to \mathbb{C} \setminus \{0,1\}$ is surjective, one-to-one module the identifications on ∂F effected by the action of $\Gamma(2)$.

Define $\Phi : H \to \mathbb{C} \setminus \{0,1\}$ by

$$\Phi(z) = \{\varphi(\gamma^{-1}z) : \gamma \in \Gamma(2), z \in F\}.$$

This is well-defined, because if $\gamma_1 z = \gamma_2 z$ with $\gamma_2 \neq \pm\gamma_1$ for $z \in F$, then $\gamma_2^{-1}\gamma_1$ must preserve z, which must lie on ∂F, and $\gamma_2^{-1}\gamma_1 z = -\bar{z}_1$. By construction, Φ is continuous (as composition of continuous function which agree on overlaps), $\Gamma(2)$-invariant, and 1-to-1 modulo the action on $\Gamma(2)$. Also Φ is holomorphic except possibly on points in H which are either $\Gamma(2)$-conjugate to points in $\{i\mathbb{R}_{>0} \pm 1\}$ and points in $w\{i\mathbb{R}_{>0} \pm 1\}$. These two cases can be treated exactly like the earlier case on $i\mathbb{R}_{>0}$. □

This "showing holomorphy using continuity" is called the **reflection principle**. If f is continuous and holomorphic except for on a line, then it is holomorphic on the line.

23.2 Invariant form of Schwartz lemma

Theorem 23.3 (Big Picard theorem)**.** *In any neighborhood of an essential singularity, a holomorphic function assumes every value, with at most one exception, infinitely often.*

For example, $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. Big Picard implies little Picard. This is because any entire function that is not a polynomial can be thought of having an essential singularity at $z = \infty$.

Let H and Δ have invariant metrics

$$\frac{dx^2 + dy^2}{(1-|z|^2)^2} \text{ on } \Delta, \quad \frac{dx^2+dy^2}{y^2} \text{ on } H.$$

Let d_Δ and d_H be the corresponding distance functions. They are $\mathrm{Aut}(\Delta)$(resp. $\mathrm{Aut}(H)$)-invariant.

Theorem 23.4 (Invariant form of Schwarz lemma)**.** *Suppose $F : H \to H$ is holomorphic and $z_1, z_2 \in H$ with $z_2 \neq z_1$. Then $d_H(F(z_1), F(z_2)) \leq d_H(z_1, z_2)$ and the inequality is strict unless F is a biholomorphic. Equivalently, the same thing holds for $F : \Delta \to \Delta$.*

Proof. Because d_h is $\mathrm{Aut}(H)$-invariant, it suffices to prove that if $F : H \to H$ is holomorphic and $F(0) = 0$ and $z \in \Delta$, then $d_\Delta(F(z), 0) \leq d_\Delta(z, 0)$. By radial symmetry, we have $d_\Delta(z,0) = d_\Delta(|z|, 0)$ and $d_\Delta(F(z), 0) = d_\Delta(|F(z)|, 0)$.

We know that $d(||z||, 0)$, for $z \in \Delta$, is strictly increasing in terms of $|z|$. Then what we must prove is $|F(z)| \leq |z|$ and that this is inequality is strict unless F is a biholomorphism. This is precisely the Schwartz lemma. □

Theorem 23.5 (Ahlfors)**.** *Suppose M is a hermitian manifold, whose holomorphic sectional curvatures are negative and uniformly bounded away from 0. Let $F : \Delta \to M$ be holomorphic. Then there exists a constant c_{curv} such that $d_M(F(z_1), F(z_2)) \leq c_{\mathrm{curv}} d_\Delta(z_1, z_2)$.*

24 November 29, 2016

It's been two weeks since last class, so let remind you where we stood. We proved the theorem:

Theorem 24.1. *There exists a holomorphic covering $H \to \mathbb{C} \setminus \{0, 1\}$ with covering group $\Gamma(2)$.*

We further constructed a surjective holomorphic map $H \to \mathbb{C} \setminus \{0, 1\}$ which drops to a one-to-one map $\Gamma(2)\backslash H \to \mathbb{C} \setminus \{0, 1\}$. We did not exactly show that $H \to \mathbb{C} \setminus \{0, 1\}$ is a covering, but a homework problem implies this. We also proved the invariant form of Schwartz lemma.

24.1 Big Picard theorem

The next business is to prove the big Picard theorem.

Theorem 24.2 (Big Picard theorem)**.** *In any neighborhood of an essential singularity a holomorphic function takes on any value infinitely often, with at most one exception.*

It suffices to show that if $f : \Delta^* \to \mathbb{C} \setminus \{0, 1\}$ is holomorphic then f has a removable singularity or a pole at the origin. The reason this suffices is because we can shrink the disc as much as we like.

Consider the map $p : H \to \Delta^*$ with $p(z) = e^{2\pi i z}$. This is a covering with covering group generated by $z \mapsto z + 1$. Suppose $f : \Delta^* \to \mathbb{C} \setminus \{0, 1\}$. Then we get a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\tilde{f}} & H \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \Delta^* & \xrightarrow{f} & \mathbb{C} \setminus \{0,1\} \end{array}$$

by lifting the map f to $\tilde{f}$. Suppose $z_0 \in H$. Then $p(z_0 + 1) = p(z_0)$ so $\tilde{f}(z_0 + 1) = \gamma \tilde{f}(z_0)$ for some $\gamma \in \Gamma(2)$. But γ can be chosen independently of z_0, at least locally, because $\Gamma(2)$ is discrete, and hence globally by holomorphy. Thus there exists a $\gamma \in \Gamma(2)$ such that $\tilde{f}(z + 1) = \gamma \tilde{f}(z)$ for all $z \in H$.

Before finding out what γ is, let us think how unique γ is. We may replace $\tilde{f}$ by $z \mapsto \gamma_1 \tilde{f}(z)$ for any fixed $\gamma_1 \in \Gamma(2)$. This has the effect of replacing γ by $\gamma_1 \gamma \gamma_1^{-1}$. More than that, $\Gamma(2)$ is normalized by $\mathrm{SL}(2, \mathbb{Z})$. So the finite group $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$ acts on $\mathbb{C} \setminus \{0, 1\}$.[3] Replacing $\tilde{f}$ by $z \mapsto \gamma_1 \tilde{f}(z)$ has the effect of replacing f by $A \circ f$ where $A : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{0, 1\}$ is induced by γ_1. So the conclusion is that in proving the theorem, we may replace γ by $\gamma_1 \gamma \gamma_1^{-1}$ for $\gamma_1 \in \mathrm{SL}(2, \mathbb{Z})$.

[3] We don't need to know that this action is, but this finite group is S_3 and it acts on $\mathbb{P} \setminus \{0, 1, \infty\}$ by permuting these three points in a fractional linear manner.

Recall that $H \cong G/K$ with $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$, which is the isotropy subgroup of $\mathrm{SL}(2,\mathbb{R})$ at $i \in H$. So we can choose $g_n \in \mathrm{SL}(2,\mathbb{R})$ for $n \in \mathbb{Z}_{>0}$ such that $\tilde{f}(in) = g_n \cdot i$. Then by the Schwartz lemma,

$$d_H(\tilde{f}(in+1), \tilde{f}(in)) \leq d_H(in+1, in) = \int_0^1 \frac{dt}{n} = \frac{1}{n}$$

while on the other hand,

$$d(\tilde{f}(in+1), \tilde{f}(in)) = d_H(\gamma\tilde{f}(in), \tilde{f}(in)) = d_H(\gamma g_n \cdot i, g_n \cdot i) = d_H(g_n^{-1}\gamma g_n i, i).$$

The conclusion is hat $d_H(g_n^{-1}\gamma g_n i, i) \leq 1/n$. So as $n \to \infty$, the element $g_n^{-1}\gamma g_n \in \mathrm{SL}(2,\mathbb{R})$ lies in any open neighborhood of $\mathrm{SO}(2)$. The coefficients of the characteristic polynomial of any $g \in \mathrm{SL}(2,\mathbb{R})$ are continuous functions of g, and constant on conjugacy classes. This shows that γ has eigenvalues $e^{\pm i\theta}$. But $\Gamma(2)$ contains no elliptic elements. Therefore γ has eigenvalues $e^{i\theta}$ with $\theta \in \pi\mathbb{Z}$. So γ has eigenvalues both 1 or both -1. Because we are allowed to change γ by the center, we may suppose without loss of generality that γ is unipotent or the identity.

Case (a) $\gamma = 1$.
Then $\tilde{f}(z+1) = \gamma\tilde{f}(z) = \tilde{f}(z)$. That is, we can think $\tilde{f}$ has a holomorphic map from Δ^* to H, or equivalently, can lift f to a map $\pi^{-1} \circ f : \Delta^* \to H$. Comparing this with $H \cong \Delta$, we see that $\pi^{-1} \circ f$ has a removable singularity.

Case (b) $\gamma \neq 1$, γ is unipotent.
We now that γ has a unique fixed point. Since $\Gamma(2) \subseteq \mathrm{GL}(2,\mathbb{Q})$, the fixed point must lie in $\mathbb{Q} \cup \{\infty\}$. The group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$. This is because $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends $\gamma(\infty) = \frac{a}{c}$. Then by the previous remark, we may suppose $\gamma\infty = \infty$. Then

$$\gamma = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

for $k \in \mathbb{Z} \setminus \{0\}$.

Let us define a new map $\tilde{h} : H \to \mathbb{C}$ as $\tilde{h}(z) = \tilde{f}(z) - 2kz$. Then

$$\begin{aligned} \tilde{h}(z+1) &= \tilde{f}(z+1) - 2k(z+1) = \gamma\tilde{f}(z) - 2k(z+1) \\ &= \tilde{f}(z) + 2k - 2k(z+1) = \tilde{h}(z). \end{aligned}$$

The fact that $\tilde{h}$ is invariant under the fundamental group implies that there is a map $h : \Delta^* \to \mathbb{C}$ that makes

$$\begin{tikzcd} H \arrow[r, "\tilde{h}"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\ \Delta^* \arrow[r, "h"] & \mathbb{C} \end{tikzcd}$$

commute.

Proposition 24.3. *For $y \gg 1$ and $0 \le x \le 1$, the function $|\tilde{h}(z)|$ is bounded by a polynomial in y where $z = x + iy$.*

Assume this for now. We have $\tilde{h}(z) = h(t)$ where $t = e^{2\pi i z}$. Then $|t| = e^{-2\pi y}$ and so $y = (1/2\pi)\log|t|^{-1}$. Because $e^{2\pi i z}$ for $0 \le x \le 1$ covers all of Δ^*, the function $|h(t)|$ is bounded by a polynomial in $\log|t|$ for $|t| \ll 1$, by the proposition. Then h has a removable singularity at 0. This gives

$$\begin{aligned}\tilde{f}(z) &= \tilde{h}(z) + 2kz = h(e^{2\pi i z}) + 2kz \\ &= (h(0) + a_1 e^{2\pi i z} + a_2 e^{4\pi i z} + \cdots) + 2kz \in H.\end{aligned}$$

So $0 < \Im \tilde{f}(z) = \Im h(0) + 2ky + O(e^{-2\pi y})$ for $y \gg 1$. It follows that $k > 0$. Also as $\tilde{f}(z) = h(0) + 2kz + O(e^{-2\pi y})$. It follows that $\pi^{-1} \circ f(e^{2\pi i z}) = h(0) + 2kz + O(e^{-2\pi y})$ for $0 \le x \le 1$. Then f has a pole of order k at 0.

Proof of proposition 24.3. Because $\tilde{h}(z) = \tilde{f}(z) - 2kz$, it suffices to show for $0 \le x \le 1$, $|\tilde{f}(z)|$ is bounded by a polynomial in y. We have

$$\begin{aligned}d_H(\tilde{f}(z), i) &\le d_H(\tilde{f}(z), \tilde{f}(i)) + \text{const} \le d_H(z, i) + \text{const} \\ &\le d_H(x + iy, iy) + d_H(iy, i) + \text{cosnt} \le \frac{1}{y} + \log y + \text{const} \le \log y^2\end{aligned}$$

for $y \gg 1$. So $d_H(\tilde{f}(z), i) \le d_H(iy^2, i)$. Recall that $c : H \to \Delta$ with $c(z) = (z - i)/(z + i)$ relates d_H to d_Δ. Then $d_\Delta(c(\tilde{f}(z)), ci) \le d_\Delta(c(iy^2), c(i))$. Write $\tilde{f}(z) = w \in \Delta$. Then

$$d_\Delta\Big(\frac{w - i}{w + i}, 0\Big) \le d_\Delta\Big(\frac{iy^2 - i}{iy^2 + i}, 0\Big)$$

for $y \gg 1$ and $|t| \ll 1$. It follows that $|(w - i)/(w + i)| \le |(y^2 - 1)/(y^2 + 1)|$.

Because $y \gg 1$, we have

$$\Big|\frac{y^2 - 1}{y^2 + 1}\Big| = \Big|\frac{1 - y^{-2}}{1 + y^{-2}}\Big| = |1 - 2y^2 + \cdots| \le 1 - y^{-2}.$$

Likewise

$$\Big|\frac{w - i}{w + i}\Big| = \Big|\frac{1 - i/w}{1 + i/w}\Big| = \Big|1 - \frac{2i}{w} + \cdots\Big| \ge 1 - \frac{4}{|w|}$$

for $|w| \gg 1$. It follows that $|w| \le 4y^2$. Therefore $|\tilde{f}(z)| \le 4y^2$ for $0 \le x \le 1$ and $y \gg 1$. □

25 December 1, 2016

25.1 Compact Riemann surfaces coming from $\mathbb{C}$

Recall that connected Riemann surfaces with universal cover $\mathbb{C}$ are, up to isomorphism, $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{C}/L$ where $L \subseteq \mathbb{C}$ is a lattice, i.e., a $\mathbb{Z}$-module spanned in an $\mathbb{R}$-basis of $\mathbb{C}$. The first two spaces are not very interesting.

Let $L \subseteq \mathbb{C}$ be a lattice. What can be said about holomorphic functions on $\mathbb{C}/L$? Because it is compact, it takes a maximal value at some point. Then we see that it is constant. But consider meromorphic functions on $\mathbb{C}/L$. These are called **elliptic functions**. Suppose f is an elliptic function, and $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ where $\{\omega_1, \omega_2\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.

Theorem 25.1. *Let $\zeta_1, \ldots, \zeta_r \in \mathbb{C}/L$ be an enumeration of the poles of f. Then*

$$\sum \operatorname{Res} f|_{\xi_i} = 0.$$

Proof. For $a \in \mathbb{C}$, define $F_a = \{a + t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 \le 1\}$. This is a fundamental domain for the action of L on $\mathbb{C}$. We can choose a so that none of the poles lie on ∂F_a. Then

$$\sum_{\zeta \in \mathbb{C}\setminus L} \operatorname{Res} f|_\zeta = \sum_{\eta \in F_a} \operatorname{Res} f|_\eta = \frac{1}{2\pi i}\int_{\partial F_a} f(z)dz.$$

But because f is periodic, the integral along the opposite sides cancel out. □

Let f be an elliptic function. Let $\zeta_1, \ldots, \zeta_r$ be an enumeration of zeros and poles of f in $\mathbb{C}/L$. Let n_i be the order of f at ζ_i (so $n > 0$ if f has a zero and $n < 0$ if f has a pole).

Corollary 25.2. $\sum n_i = 0$.

Proof. Apply the theorem to f'/f. □

Corollary 25.3. *Suppose f is a non-constant elliptic function. Then the total number of poles in $\mathbb{C}/L$, counted with multiplicity is at least 2.*

Proof. Suppose f has a first order pole at $\zeta \in \mathbb{C}/L$ and no other poles. Then $\operatorname{Res} f|_\zeta = 0$ and so there are no poles. It follows that f is constant. □

Theorem 25.4. $\mathbb{C}/L^1 \cong \mathbb{C}/L^2$ *if and only if* $L_2 = \alpha L_1$ *with* $\alpha \in \mathbb{C}^*$.

Proof. Suppose $\mathbb{C}/L_1 \cong \mathbb{C}/L_2$. Consider the following diagram.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/L_1 \arrow[r, "\cong"] & \mathbb{C}/L_2 \end{tikzcd}$$

Then $f(z+w_1) = f(z) + \varphi_1$ for $\omega_1 \in L_1$ and $\varphi_1 \in L_2$. That means that f' is L_1-periodic and so f' is constant. This shows that $f(z) = \alpha z + b$ with $\alpha \in \mathbb{C}^*$. It follows that $L_1 = \alpha L_2$. $\square$

Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice. After rescaling, we may suppose that the ordered $\mathbb{R}$-basis is positively oriented, i.e., going from ω_1 to ω_2 involves rotation by an angle θ with $0 < \theta < \pi$. So we can write $\mathbb{C}/L \cong \mathbb{C}/L_\tau$ where $\tau = \omega_2/\omega_1$ and $L_\tau = \mathbb{Z} \oplus \mathbb{Z}/\tau$. By construction $1, \tau$ is a positively oriented $\mathbb{Z}$-basis and so $\tau \in H$. Thus $\mathbb{C}/L \cong \mathbb{C}/L_\tau$ for some $\tau \in H$.

A natural question is: with $\tau_1, \tau_2 \in H$, when are $\mathbb{C}/L_{\tau_1} \cong \mathbb{C}/L_{\tau_2}$? By the previous argument, they are isomorphic if and only if there is an $\alpha \in \mathbb{C}^*$ such that $\mathbb{Z} \oplus \mathbb{Z}_{\tau_2} = \alpha(\mathbb{Z} \oplus \mathbb{Z}_{\tau_1})$. This is when $1, \tau_2$ and $\alpha, \alpha\tau_1$ are related by some matrix in $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\alpha\tau_1 = a\tau_2 + b, \quad \alpha = c\tau_2 + d.$$

This is possible if and only if τ_1 and τ_2 are related by the action of some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ on H. This proves:

Theorem 25.5. *The set of connected, compact Riemann surfaces with universal cover $\mathbb{C}$ modulo isomorphism is naturally isomorphic to $\mathrm{SL}(2,\mathbb{Z})\backslash H$, with τ corresponding to $\mathbb{C}/L_\tau$.*

Here, this $\mathrm{SL}(2,\mathbb{Z}) \cong \mathbb{C}$ when unramified. Historically, this is the first answer to what we now call a "moduli problem".

25.2 Elliptic curves

Suppose now L is a lattice, where $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\omega_2/\omega_1 \in H$. The answer was given by Weierstrass. Fix L as above, and define the **Weierstrass $\wp$-function** as

$$\wp(z) = \sum_{\omega \in L\setminus\{0\}} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big) + \frac{1}{z^2}.$$

This function is a function depending on L.

Theorem 25.6. *The series converges locally uniformly on $\mathbb{C}/L$ to a meromorphic function on $\mathbb{C}$, which is L-periodic, hence an elliptic function on $\mathbb{C}/L$ with second order pole at $0 \in \mathbb{C}/L$ and no other poles in $\mathbb{C}/L$.*

Note that $\mathbb{C}/L$ has a natural structure of an abelian group. Then we can say that $\wp(z)$ is an even function. $\wp'$ is also an elliptic function, with pole of order 3 at $0 \in \mathbb{C}/L$, no other poles, and as a function on $\mathbb{C}$, odd.

Theorem 25.7. *The field of elliptic functions on $\mathbb{C}/L$ is generated over $\mathbb{C}$ by $\wp$ and $\wp'$. Moreover,*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

with some $g_2, g_3 \in \mathbb{C}$.

Define $F : \mathbb{C}/L \to \mathbb{P}^2$ where $F(z) = [1, \wp(z), \wp'(z)]$.

Theorem 25.8. *This gives an isomorphism between* $\mathbb{C}/L$ *and a smooth curve in* $\mathbb{P}^2$*—smooth closed* 1*-dimensional complex submanifold.*

So what is the image? Because $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, the image is the closure in $\mathbb{P}^2$ of the curve $y^2 = 4x^3 - g_2x - g_3$. Let us look at what happens at $z = 0$. We have

$$F(z) = \left[1, \frac{1}{z^2} + \cdots, -\frac{2}{z^3} + \cdots\right] = [z^3, z + \cdots, -2 + \cdots].$$

So $z = 0$ corresponds to $[0, 0, 1]$, which is the point at infinity. Therefore $F(\mathbb{C}/L)$ intersects the "line at ∞" in $\mathbb{P}^2$ at the point $[0, 0, 1]$ and the order of the intersection is 3. By general nonsense in algebraic geometry, $F(\mathbb{C}/L)$ intersects any line (i.e., copy of $\mathbb{P}^1$) in $\mathbb{P}^2$ with multiplicity 3.

Recall that $\mathbb{C}/L$ has a structure of an abelian group. Can you see that in terms of the curve $F(\mathbb{C}/L) \subseteq \mathbb{P}^2$? The additional law, in terms of this curve, is that $a + b + c = 0$ if there exists a line π such that $\pi \cap F(\mathbb{C}/L) = \{a, b, c\}$ with multiplicity.

If I had another lecture, I could have proved all of this. But the conclusion is that all elliptic curves (i.e., compact Riemann surfaces, covered by $\mathbb{C}$) are of the form: completion in $\mathbb{P}^2$ of the affine curve $y^2 = 4x^3 - g_2x - g_3$ with addition law algebraic.

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