Math 132 - Topology II: Smooth Manifolds

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This course, which is *not* a continuation of Math 131, was taught by Michael Hopkins. We met three times a week, on Mondays, Wednesdays, and Fridays, from 1:00 to 2:00 in Science Center 310. We used *Differential Topology* by Guillemin and Pollack as a textbook. There were 19 students enrolled in this course. The grading was only based on the problem sets. The course assistant for this class was Adam Al-natsheh.

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1 January 25, 2016

1.1 Outline

We will be studying something called a smooth manifold. Objects like toruses, Klein bottles, double toruses, etc., are smooth manifolds, while cones are not, because they have these corners. A natural question is "What are all the pictures of smooth surfaces that can be drawn?" The first step is to define a smooth surface, or more generally, a smooth manifold of dimension d. Then we need to define a 'picture' of a surface, which we will call an immersion $f : \Sigma \looparrowright \mathbb{R}^3$. Next, we need a notion of equivalence between pictures; if one can be smoothly deformed into another, then we call the two isotopic. Now our problem becomes:

a) Classify all smooth surfaces.
b) For each smooth surface Σ, describe the set of isotopy classes of immersions of $\Sigma \looparrowright \mathbb{R}^3$.

To approach the problem, we define the intersection form. This is a non-degenerate symmetric bilinear form on a vector space over $\mathbb{F}_2$. If something is a boundary of a sub manifold, we consider it to be zero. For two curves, we may consider the number of intersection modulo 2. This is the intersection form. It turns out that the intersection form determines the manifold, and thus the classification of immersions into the classification of bilinear forms.

Theorem 1.1. *The immersions of Σ in $\mathbb{R}^3$ is in one-to-one correspondence with the quadratic $q : V \to \mathbb{Z}/4\mathbb{Z}$ whose underlying bilinear form is the intersection form.*

The reason for the strange $\mathbb{Z}/4\mathbb{Z}$ is because a strip with 4 half-twists is the same as a strip with no twists.

2 January 27, 2016

2.1 Manifolds

Today I'm going to start by giving the official definition of a manifold. I want to consider a set $X \subset \mathbb{R}^n$ sitting inside the Euclidean space.

Definition 2.1. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a open set. A function $f : \mathcal{U} \to \mathbb{R}$ is called **smooth** if it there exists derivatives of all orders. A function $f : X \to \mathbb{R}$ is **smooth** if there is a neighborhood $X \subseteq \mathcal{U} \subseteq \mathbb{R}^n$ and a smooth function $g : \mathcal{U} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in X$.

Definition 2.2. A subset $X \subset \mathbb{R}$ is a **smooth manifold** of dimension d if for each $x \in X$ if for each $x \in X$, there is a neighborhood $x \in \mathcal{U} \subset \mathbb{R}^d$ and an injective function $f : \mathcal{U} \cap X \to \mathbb{R}^d$ such that f and $f^{-1} : f(\mathcal{U} \cap X) \to \mathcal{U} \cap X$ are both smooth.

Example 2.3. Consider the n-sphere

$$S^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i^2 = 1\}.$$

Let $n = 1$ for simplicity, and consider the function $f(x, y) = y$ on the subset of points (x, y) with $x > 0$. Then this function and its inverse is smooth, because $-1 < y < 1$. Likewise, we can do this for $x < 0$ and $y < 0$ and $y > 0$. These four open sets covers the whole space and thus S^1 is a 1-manifold sitting in $\mathbb{R}^2$.

Example 2.4. Consider the set

$$X = \{(x, y) : y = 0, x \geq 0 \text{ or } x = 0, y \geq 0\}.$$

We can define a function $f : X \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x & \text{if } x \geq 0 \\ -y & \text{if } y \geq 0. \end{cases}$$

Then the inverse $f^{-1}(t)$ is not differentiable at $t = 0$.

Example 2.5. Consider a disjoint union of S^2 and a line sitting in $\mathbb{R}^3$. We do not consider this to be a smooth manifold, because it does not have an even dimension.

2.2 Smooth structure

Definition 2.6. A **topological manifold** of dimension d is a topological space[1] X, such that each $x \in X$ has a neighborhood $\mathcal{U}$ that is homeomorphic to an open subset of $\mathbb{R}^d$.

[1] Actually, we include conditions that it should be Hausdorff and have a countable basis.

When I give a mathematical object, I should give some data and some conditions that data has to satisfy. The notion of a topological manifold only involves a topology, but we need more than that to give the condition that it is smooth.

Definition 2.7. A **chart** on X on is a pair $(\mathcal{U}, f)$ consisting of an open subset $\mathcal{U} \subset X$ and a homeomorphism $f : \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{R}^d$ with and open subset of $\mathbb{R}^d$.

Definition 2.8. Let $X \subset \mathbb{R}^n$ be a topological manifold. A chart $f : \mathcal{U} \to f(\mathcal{U}) \subset \mathbb{R}^d$ on X is **smooth** if f and f^{-1} are smooth. The manifold X is called **smooth** if X can be covered by smooth charts.

Definition 2.9. A **smooth structure** on X is a homeomorphism of X with a subspace of $\mathbb{R}^n$ for some n, which is a smooth manifold.

There are some fundamental problems in differential topology.

1) If X is a topological manifold, can it be given a smooth structure?
2) If so, if how many ways?

But clearly there are infinitely many smooth structure if there is one. This means that we need a notion of equivalence so that we don't consider the essential same structures as different.

Definition 2.10. Let $f_1 : X \to Z_1 \subset \mathbb{R}^n$ and $f_2 : X \to Z_2 \subset \mathbb{R}^m$ be two smooth structures. We consider them to be **equivalent** if there is a map $g : Z_1 \to Z_2$ such that both g and g^{-1} are smooth and the following diagram commutes.

$$\begin{tikzcd} & Z_1 \arrow[dd, "g"] \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & Z_2 \end{tikzcd}$$

Now the problems make sense. If you live in 3 dimensional space, then in fact, every topological manifold has a unique smooth structure. But around 1956, Milnor discovered the fact that this is not true for higher dimensions.

Theorem 2.11 (Milnor, 1956)**.** *S^7 has least 14 different smooth structures.*

And around 1961, Kervaire and Milnor determined the number of smooth structures on S^n where $n \neq 3, 4, 2^k-2, 2^k-3$ for $k \geq 7$. Also, in the early 1960s, Kervaire gave an example of a topological manifold with no smooth structure of dimension 10.

3 January 29, 2016

There will be a point-set topology review section next week.

Let us recall what we did last time. A smooth manifold $X \subset \mathbb{R}^n$ of dimension d is a manifold having a smooth atlas, i.e., locally diffeomorphic to $\mathbb{R}^d$.

3.1 Charts and Atlases

So for any point $x \in X$, there is diffeomorphism between an open neighborhood of x inside X, and an open set of $\mathbb{R}^d$. This diffeomorphism is called a **chart**, and is sometimes called a coordinate chart.

Example 3.1. Consider $X = \mathbb{R}^2 \setminus \{0\}$. Then we can think of a polar coordinate system and the rectangular coordinate system. Both gives a chart on a small neighborhood of X.

Suppose you have a chart $U_\alpha \to \mathbb{R}^d$ and $U_\beta \to \mathbb{R}^d$. On U_α, the chart gives a coordinate system. Also we have a coordinate system on U_β. Then in the intersection $U_\alpha \cap U_\beta$, we have two coordinate systems. One thing we can see is the the two coordinates should be smooth with respect to each other. This is how a smooth manifold is defined abstractly. An **atlas** is a covering of a manifold by charts, and a **smooth atlas** is an atlas such that the coordinate changes between the charts are smooth.

Example 3.2. Let us show that $S^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Given any $v \in S^n$, we let

$$\mathcal{U}_v = \{x \in S^n : x \cdot v > 0\}.$$

Then the projection map $\mathcal{U}_v \to v^\perp \cong \mathbb{R}^d$ gives a diffeomorphism from $\mathcal{U}_v$ to $\{y \in v^\perp : |y| < 1\}$. We could define an atlas by taking these coordinate charts for $v = \pm e_1, \ldots, \pm e_n$.

3.2 Projective space and the Grassmannian

We look at a more complex example. The **real projective space** $\mathbb{R}P^n$ is defined by

$$\mathbb{R}P^n = \text{space of lines throughout the origin in } \mathbb{R}^{n+1}.$$

Here is another way to think about it. Because any line intersects S^n in exactly two antipodal points, we have

$$\mathbb{R}P^n = S^n/(v \sim -v).$$

I want to prove that $\mathbb{R}P^n$ be given a smooth structure. Here is the problem. This $\mathbb{R}P^n$ is not defined as a subset of a Euclidean space. So we have to use the more abstract definition of a smooth manifold.

We introduce a more convenient way of representing a point in $\mathbb{R}P^n$.For an $x \in \mathbb{R}P^n$, we write it as some

$$[x_0, \dots, x_n] = [-x_0, \dots, -x_n]$$

where $x_0^2 + \cdots + x_n^2 = 1$. I'm using the square brackets to indicate that this is the equivalence class.

Cover $\mathbb{R}P^n$ by $\mathcal{U}_0, \dots, \mathcal{U}_n$, where

$$\mathcal{U}_i = \{[x_0, \dots, x_n] : x_i \neq 0\}.$$

Then every $x = [x_0, \dots, x_n] \in \mathcal{U}_i$ has a unique representation such that $x_0 > 0$. Then we get an chart on $\mathcal{U}_i$ which is

$$[x_0, \dots, x_n] \to (\tfrac{x_0}{x_i}, \tfrac{x_1}{x_i}, \dots, \tfrac{x_n}{x_i}).$$

An easy exercise is to check that the change of coordinate functions are smooth.

Definition 3.3. The **Grassmannian** $\mathrm{Gr}_k(\mathbb{R}^n)$ is defined as the set of k-dimensional vector subspaces of $\mathbb{R}^n$.

This is quite hard to embed into the Euclidean space, but it is relatively easy to describe an atlas. For a $V \in \mathrm{Gr}_k(\mathbb{R}^n)$, we let

$$\mathcal{U}_V = \{W \in \mathrm{Gr}_k(\mathbb{R}^n) : W \cap V^\perp = 0\}.$$

Then the orthogonal projection $\mathbb{R}^n \to V$ gives an isomorphism $W \to V$. Thinking W as the graph of the linear transformation $T_W : V \to V^\perp$, we see that $\mathcal{U}_V$ is diffeomorphic to $\mathrm{Hom}(V, V^\perp)$, which is simply a $k(n-k)$ dimensional vector space. This shows that

Proposition 3.4. $\mathrm{Gr}_k(\mathbb{R}^k)$ *is a smooth manifold of dimension* $k(n-k)$.

4 February 1, 2016

4.1 Multivariable calculus

So today I want to talk about derivatives in the case of smooth manifolds. Suppose that I have a smooth map $f : \mathbb{R}^k \to \mathbb{R}^\ell$, and suppose that this is given by

$$\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \mapsto \begin{bmatrix} f_1(x_1, \dots, x_k) \\ \vdots \\ f_\ell(x_1, \dots, x_k) \end{bmatrix}.$$

Then given a point $x \in \mathbb{R}^k$, we can write down all the partial derivatives

$$Df(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_\ell(x)}{\partial x_1} & \cdots & \frac{\partial f_\ell(x)}{\partial x_k} \end{bmatrix}.$$

If we have two maps

$$\mathbb{R}^k \xrightarrow{f} \mathbb{R}^\ell \xrightarrow{g} \mathbb{R}^m,$$

the chain rule says that

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x).$$

This is just a fancy way of writing down the chain rule

$$\frac{\partial(g_s \circ f)}{\partial x_t} = \sum_i \frac{\partial g_s}{\partial y_i} \frac{\partial y_i}{\partial x_t}$$

we all know.

Now suppose that we have a smooth map $f : X \to Y$ between smooth manifolds. We want to make sense of what is the derivative of f. We would like to define for $f : X \to Y$, where $\dim X = k$ and $\dim Y = \ell$, a linear transformation

$$Df : V \to W$$

such that Df becomes "the matrix of derivatives" in any local coordinate system.

Let us describe it more precisely. Given a point $x \in X$ and $f(x) \in Y$, there is a local coordinate system of a neighborhood U_X of x and U_Y of $f(x)$.

$$\begin{tikzcd} U_X \arrow[r, "f"] & U_Y \\ V_X \subset \mathbb{R}^k \arrow[u, "\Phi"] \arrow[r, "f_{\Phi\Psi}"] & V_Y \subset \mathbb{R}^\ell \arrow[u, "\Psi"'] \end{tikzcd}$$

Then we push down the map f to $f_{\Phi\Psi}$. Then we would want to do differentiate things on $f_{\Phi\Psi}$ instead. But we actually don't have a vector space, and thus we need more things.

4.2 Tangent spaces and derivative of a smooth map

We have to associate to each point $x \in X$ an abstract vector space of dimension d. Intuitively we can think of it as a space that is actually tangent to the manifold at $x \in X$.

Let $X \subset \mathbb{R}^n$ be a smooth manifold of dimension d. Then by definition, there is neighborhood $x \in \mathcal{U} \subset X$, an open set $\hat{\mathcal{U}} \subset \mathbb{R}^d$, and a smooth map $\Phi : \hat{\mathcal{U}} \to \mathcal{U}$. Then we define

Definition 4.1. The **tangent space** X at $x \in X$ is the image of $D\Phi : \mathbb{R}^d \to \mathbb{R}^n$.

Now my claim is

Proposition 4.2. *The tangent space is independent of the change of local coordinates and has dimension d.*

Proof. Suppose we have two local coordinate systems Φ and Ψ near x. We might as well assume they are defined for the same neighborhood $\mathcal{U}$ of x.

$$\begin{tikzcd} & \mathcal{U} & \\ \hat{\mathcal{U}} \arrow[ur, "\Phi"] \arrow[rr, shift left, "\Psi^{-1}\circ\Phi=F"] & & \hat{\hat{\mathcal{U}}} \arrow[ul, "\Psi"'] \arrow[ll, shift left, "\Phi^{-1}\circ\Psi=G"] \end{tikzcd}$$

Note that $G \circ F = \mathrm{Id}_{\mathbb{R}^d}$ and $F \circ G = \mathrm{Id}_{\mathbb{R}^d}$. Then the chain rule tells us that

$$DF \circ DG = \mathrm{Id}, \quad DG \circ DF = \mathrm{Id}$$

and thus

$$D\Psi^{-1} \circ DG = \mathrm{Id}\,.$$

Thus $D\Phi$ has an inverse and so $\ker D\Phi$ has image of dimension d. Also,

$$D\Phi = D\Psi(DF)$$

implies that the image of $D\Phi$ is contained in the image of $D\Psi$. Then reversing the roles, we see that the image of $D\Phi$ is the same as the image of $D\Psi$. $\square$

We can define it using the abstract definition of a smooth manifold. Suppose we have a smooth map $f : X \to Y$. The derivative of f is the linear map

$$Df : T_xX \to T_{f(x)}Y$$

defined in the following way. (T_xX is just the tangent space of X at x.) By definition we can draw

$$\begin{tikzcd} \mathcal{U} \subset X \arrow[r, "f"] & \mathcal{V} \subset Y \\ \hat{\mathcal{U}} \subset \mathbb{R}^k \arrow[u, "\Phi"] \arrow[r, "\bar{f}"] & \hat{\mathcal{V}} \subset \mathbb{R}^k \arrow[u, "\Psi"'] \end{tikzcd}$$

and then we can define Df by the diagram

$$\begin{tikzcd} T_xX \arrow[r, dotted, "Df"] & T_{f(x)}Y \\ \mathbb{R}^k \arrow[u, "D\Phi"] \arrow[r, "D\bar{f}"] & \mathbb{R}^k \arrow[u, "D\Psi"'] \end{tikzcd}$$

We can also state the chain rule.

Theorem 4.3. *If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of smooth manifolds, then*

$$D(g \circ f)|_x = D(g)|_{f(x)} \circ D(f)|_x.$$

There is an alternative definition of a tangent space.

Definition 4.4 (Version 1)**.** The **tangent space** T_xX is the equivalence classes of curves $\gamma : \mathbb{R}^1$ with $\gamma(0) = x$, where the equivalence relation is $\gamma_1 \sim \gamma_2$ if for any smooth $\mathcal{U} \to \mathbb{R}$ on a neighborhood $\mathcal{U}$ of x we have

$$\frac{d}{dt} f \circ \gamma_1|_{t=0} = \frac{d}{dt} f \circ \gamma_2|_{t=0}.$$

Definition 4.5 (Version 2)**.** We let

$$T_xX = \{\text{derivations } d : C^\infty(\mathcal{U}) \to \mathbb{R} \text{ at } x\},$$

where $\mathcal{U}$ is a neighborhood of x, and $C^\infty(\mathcal{U})$ is the set of smooth functions on $\mathcal{U}$, and a derivation is a map $C^\infty(\mathcal{U}) \to \mathbb{R}$ which satisfies

$$d(fg) = df \cdot g(x) = f(x) \cdot dg$$

for any two smooth maps.

This last definition is very nice, because it does not make up anything new. In fact, this is what is most used in algebraic geometry.

5 February 3, 2016

Now we get the idea that this is just some kind of curved linear algebra. We associated to each point a linear space and the derivative of smooth maps turned out to be linear transformations on the spaces.

5.1 Inverse function theorem

Theorem 5.1 (Inverse function theorem)**.** *Suppose that M and N are smooth manifolds of dimension d and $f : M \to N$ is a smooth map. Let $x \in M$ be a point. If $Df|_x : T_xM \to T_{f(x)}N$ is an isomorphism, then f restricts to a diffeomorphism of a neighborhood of x with a neighborhood of $f(x)$.*

Sketch of Proof. We first set up a coordinate neighborhood of x and get the following diagram, where $x \in \mathcal{U}$ and $f(x) \in \mathcal{V}$.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \mathcal{U} \arrow[u, hook] \arrow[r, "f"] & \mathcal{V} \arrow[u, hook] \\ \hat{\mathcal{U}} \subset \mathbb{R}^d \arrow[u, "\Phi"] \arrow[r, "g"] & \hat{\mathcal{V}} \subset \mathbb{R}^d \arrow[u, "\Psi"'] \end{tikzcd}$$

Note that we can shrink $\mathcal{U}$ so that the image $f(\mathcal{U})$ lies in $\mathcal{V}$. For convenience, we can assume $\Phi(0) = x$ and $\Psi(0) = f(x)$. We just have $g = \Psi^{-1} \circ f \circ \Phi$.

Because Φ and Ψ are diffeomorphisms, we see that $D\Phi_0$ and $D\Psi_0$ are isomorphisms. Also, we have

$$Dg|_0 = (D\Psi(0))^{-1} \circ Df(x) \circ D\Phi(0).$$

Then $Df(x)$ is an isomorphism if and only if $Dg(0)$ is an isomorphism.

So the theorem reduces to the inverse function theorem in $\mathbb{R}^d$: "Suppose $\hat{\mathcal{U}}, \hat{\mathcal{V}} \subset \mathbb{R}^d$ and $g : \hat{\mathcal{U}} \to \hat{\mathcal{V}}$ is smooth. If $Dg(0)$ is an isomorphism then g is a diffeomorphism in a neighborhood of zero." ☐

Locally, every diffeomorphism looks like the identity. That is, we can let $g = \mathrm{id}$. This is because given

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \hat{\mathcal{U}} \arrow[u, "\Phi"] \arrow[r, "g"] & \hat{\mathcal{V}} \arrow[u, "\Psi"] \end{tikzcd}$$

we can consider the composite map $\Psi \circ g$. Since $D(\Psi g)(0)$ is an isomorphism, we can use the inverse function theorem to get a small $\hat{\hat{\mathcal{U}}}$ of 0 on which $\Psi \circ g$ is a diffeomorphism. Therefore we can use the $\Psi \circ g$ as the coordinate chart and get the following.

$$\begin{tikzcd} M \arrow[rr, "f"] & & N \\ & \hat{\hat{\mathcal{U}}} \arrow[ul, "\Phi"] \arrow[ur, "\Psi g"'] & \end{tikzcd}$$

5.2 Immersion

Definition 5.2. A map $f : M^k \to N^\ell$[2] is an **immersion** at $x \in M$ if $Df(x)$ is an monomorphism. The map f is an **immersion** if it is an immersion at each point.

Example 5.3. Let $k \leq \ell$. The map $\mathbb{R}^k \to \mathbb{R}^\ell$ given by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_t, 0, \ldots, 0)$$

is an immersion. This is called the **standard immersion**.

Theorem 5.4. *Every immersion locally looks like a standard immersion.*

Example 5.5. The map $S^1 \to \mathbb{R}^1$ that simply sends the circle to the figure eight is a immersion.

Example 5.6. Define the torus as

$$T = \mathbb{R}^2/(x, y) \sim (x + 1, y) \sim (x, y + 1).$$

There is an smooth structure on T given by the map $T \to \mathbb{R}^4$ defined by

$$(x, y) \mapsto (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y).$$

Choose $\theta \in \mathbb{R}$ and let $f_\theta : \mathbb{R}^1 \to T$ be the map given by

$$t \mapsto (t, t\theta).$$

If θ is a irrational number, the image is dense in T, and this is a kind of pathology. We will show later that this does not happen if the manifold is compact.

[2]This just means that M is of dimension k and N is of dimension ℓ.

6 February 5, 2016[3]

Last time, we defined an immersion. The local immersion theorem states that if $f : X \to Y$ is a immersion then for any x and $f(x)$, we can set local coordinate systems so that $d\bar{f}$ looks like the standard immersion.

We can ask: is $f(X)$ always a submanifold of Y? The answer is no in general, for instance, the figure 8 is a immersion of S^1. But if we add a tiny condition the that the map is proper, it becomes a true statement.

6.1 Submersion

Definition 6.1. Let X and Y be manifolds of dimensions k and l, where $k \geq l$. A smooth map $f : X \to Y$ is called a submersion if

$$df_x : T_x X \to T_{f(x)} Y$$

is surjective for any x.

Because a submersion is simply the dual notion of an immersion, we can have an analogous theorem.

Theorem 6.2 (Local submersion theorem)**.** *Let $f : X \to Y$ be a submersion on $x \in X$. Then there is a local coordinate around x and $f(x)$ so that f looks like the canonical submersion.*

Why do we care about this? The point of both immersions and submersions is to create new manifolds from old ones. In the case of the immersion, under good enough conditions, we have seen that the image $f(X)$ is a manifold. In the case of the submersion, the preimage of a point is a manifold.

Example 6.3. We look at the map $\mathbb{R}^4 \to \mathbb{R}^2$ given by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2).$$

Then the preimage of $(0, 0)$ is the points of form $(0, 0, x_3, x_4)$, and is in fact, a $\mathbb{R}^2$ parametrized by x_3 and x_4.

Definition 6.4. Let $f : X \to Y$ be a smooth map. For a point $y \in Y$, we say that y is a **regular value** if for every $x \in f^{-1}(y)$, the derivative

$$df_x : T_x(X) \to T_y(Y)$$

is surjective. If it is not surjective, we call y a **critical value**.

Theorem 6.5 (Preimage theorem)**.** *Suppose that $f : X \to Y$ is a smooth map, with $y \in Y$ being its regular value. Then $f^{-1}(y)$ is a submanifold of X of dimension* $\dim Y - \dim X$.

[3]For some reason, Xiaolin(Danny) Shi, a graduate student, taught this day instead of Hopkins.

Example 6.6. Consider the map $f : \mathbb{R}^k \to \mathbb{R}$ given by

$$(x_1, \dots, x_k) \mapsto x_1^2 + x_2^2 + \cdots + x_k^2.$$

Then the derivative is

$$df_{(a_1,\dots,a_n)} = \begin{pmatrix} 2a_1 & 2a_2 & \cdots & 2a_k \end{pmatrix}.$$

It is not zero whenever $(a_1, \dots, a_n) \neq (0, \dots, 0)$. Thus 0 is a critical value of f, and f is locally a submersion at every other point. If we look at $f^{-1}(1)$, we immediately see that S^{n-1} is a manifold.

Let us look at one application. Let X be a smooth manifold of dimension k, and let $g_1, \dots, g_l : X \to \mathbb{R}$ be real valued smooth functions, where $k \geq l$. We want to know whether

$$g_1^{-1}(0) \cap g_2^{-1}(0) \cap \cdots \cap g_l^{-1}(0)$$

is a smooth manifold.

We can look at the product $g : X \to \mathbb{R}^l$ given by

$$x \mapsto (g_1(x), g_2(x), \dots, g_l(x)).$$

We want to know whether $g^{-1}((0, \dots, 0))$ is a smooth manifold or not. Using the preimage theorem, we can say that, if for any $x \in g^{-1}(0)$ the vectors $dg_1|_x, \dots, dg_l|_x$ are linearly independent, then $g^{-1}(0)$ is a smooth manifold.

7 February 8, 2016

Today I want to introduce arguable one of the most important ideas in the 20th century.

7.1 Transversality

Let us first look something in linear algebra. Let $\dim V = k$, and let $\dim W = l$, where $k \geq l$. Then most linear transformations $V \to W$ are surjective. And now in differential topology, the analogue of that is that $f : M^k \to N^l$ is a submersion at some point.

Likewise, if $k \leq l$ then most linear transformations are injective. In differential topology, for most cases $f : M^k \to N^l$ will be a local immersion.

Let us suppose that there is a big vector space W^n, and let $U^k \subset W$ and $V^l \subset W$ be subspaces. Then for most U and V, we will have

$$\dim U \cap V = k + l - n.$$

For instance, most of the time, two planes in space intersect in a line.

Definition 7.1. For a subspace $V \subset W$, we define the **codimension** of V as

$$\operatorname{codim} V = \dim W - \dim V.$$

We see that

$$U \cap V = \ker(U \oplus V \to W),$$

where the map is given by $(x, y) \mapsto i(x) - j(y)$ for the inclusion maps $i : U \to W$ and $j : V \to W$. In most cases, this map will be surjective and thus $\dim(U \cap V)$ will be $\dim(U \oplus V) - \dim W$. Mathematicians like to write this as the pullback.

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d, "i"] \\ V \arrow[r, "j"] & W \end{tikzcd}$$

Let us now move the stage to differential topology. Suppose we have a smooth map $f : X^l \to Y^n$ be a smooth map, and $Z^k \subset Y$ be a submanifold.

Question. *When is $f^{-1}(z)$ a smooth manifold? What is its expected dimension?*

We first look at the case when $Z = y \in Y$ is a single point. As we have seen last week, if y is a regular value of f, then $f^{-1}(y)$ is a smooth manifold of dimension $l - n$. Furthermore, we know that

$$T_x(f^{-1}(y)) = \ker df_x.$$

Now suppose we look at the general case. Because whether something is a smooth manifold is a local property we will look at a single neighborhood.

Suppose we have $x \in X$ such that $f(x) \in Z$. We can introduce a coordinate neighborhood $\mathcal{U}$ of $f(x)$ so that $\mathcal{U} \equiv \mathbb{R}^n$ and $Z \cap \mathcal{U} \equiv \mathbb{R}^k$.

$$\begin{tikzcd} X \supset f^{-1}(\mathcal{U}) \arrow[d, "f"] \arrow[dr, "h"] & \\ Y \supset \mathcal{U} \equiv \mathbb{R}^n \arrow[r, "\pi"] & \mathbb{R}^k \end{tikzcd}$$

Now $f^{-1}(Z) \cap f^{-1}(\mathcal{U}) = h^{-1}(0)$ is a smooth manifold if and only if 0 is a regular value of h. If we now move to tangent spaces, we se that this is true if and only if

$$T_{f(x)}Z \oplus T_x X \xrightarrow{(i,df)} T_{f(x)}Y$$

is surjective.

Definition 7.2. A map $f : X \to Y$ is **transverse** to $Z \subset Y$ if for all $(z, x) \in Z \times X$ with $f(x) = z$ the map

$$T_z Z \oplus T_x X \to T_{f(x)=z} Y$$

is onto. This is denoted $f \pitchfork Z$. If f is also an inclusion, then we write $X \pitchfork Z$.

Theorem 7.3. *If f is transverse to Z then $f^{-1}(Z)$ is a smooth manifold if dimension*

$$\dim Z + \dim X - \dim Y.$$

Example 7.4. The set $X = \{(x, 0) : x \in \mathbb{R}\}$ is transverse to $Z = \{(x, x^2 - 1) : x \in \mathbb{R}\}$. However, X is not transverse to $Z' = \{(x, x^2) : x \in \mathbb{R}\}$.

Example 7.5. The hyperboloid $z = xy$ and the plane $z = 1$ are transverse. But $z = xy$ and the plane $z = 0$ is not transverse.

Example 7.6. Consider two lines in $\mathbb{R}^3$. They are transverse if and only if they do not intersect at all.

In the next lecture, we will give a better interpretation for "almost all," and prove such a theorem.

Lastly I want to point out that our definition of transversality is asymmetric. One is an inclusion and the other is a smooth map.

Definition 7.7. Let $f : X \to Y$ and $g : Z \to Y$ be two smooth maps. The two maps f and g are **transverse** if for all pairs $x \in X$ and $z \in Z$ such that $f(x) = g(z)$, the map

$$T_x X \oplus T_z Z \mapsto T_{f(x)=g(z)} Y$$

is onto. In such a case, we denote $f \pitchfork g$.

Definition 7.8. If $f \pitchfork g$ then

$$W = \{(z, x) \in Z \times X : g(z) = f(x)\}$$

is a smooth submanifold of dimension

$$\dim W = \dim Z + \dim X - \dim Y.$$

Moreover, the tangent space is the pullback.

$$\begin{tikzcd} T_wW \arrow[r] \arrow[d] & T_xX \arrow[d] \\ T_zZ \arrow[r] & T_yY \end{tikzcd}$$

Proof. This reduces to the previous case:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \times X \arrow[d] \\ Y \arrow[r, hook, "\Delta"] & Y \times Y \end{tikzcd}$$

□

8 February 10, 2016

Today I'm going to make precise the notion of generic things.

8.1 Stability

Let $f : M^k \to N^l$, and consider a property P, such as 'embedding' or 'immersion' or etc.

Definition 8.1. A property P is called **stable** if for every smooth homotopy $h : M \times [0,1] \to N$ with $h(x,0) = f(x)$ there exists $\epsilon > 0$ such that for any $0 \le t < \epsilon$ the map $f_t : M \to N$ given by $x \mapsto h(x,t)$ has property P.

So intuitively it means that if you move it a little bit, it still has the property.

Example 8.2. Let $Z \subset N$. The property "$f(X)$ intersects Z" is not stable. However, the property "$f(X)$ intersects Z transversally" is stable.

These all come from facts in linear algebra. Suppose I have a linear transformation $T : \mathbb{R}^k \to \mathbb{R}^l$ of rank r, where $r \le \min(k,l)$. Suppose I also have a homotopy $h : \mathbb{R}^k \times [0,1] \to \mathbb{R}^\ell$ that is (1) continuous, (2) $h(v,0) = T(v)$, and (3) for all t, the map $v \mapsto h(v,t)$ is linear. Alternatively, we can say that we have a continuous map

$$H : [0,1] \to \mathcal{M}_{l\times k}(\mathbb{R}), \quad t \mapsto h(-,t).$$

Proposition 8.3. *If $T = H(0)$ has rank at least r, then there exists an $\epsilon > 0$ such that $H(t)$ also has rank at least r for all $0 \le t < \epsilon$.*

Proof. Since the rank of $H(0)$ is at least r, there is an $r \times r$ submatrix of t with nonzero determinant. Suppose without loss of generality, that it is the upper left $r \times r$ block. Now for any t, we denote the upper left $r \times r$ block by $A(t)$. Then we have a continuous map $\mathcal{M}_{l\times k} \to \det(A(t))$, and composing it with H, we get another continuous map $g : [0,1] \to \mathbb{R}$.

$$\begin{tikzcd} {[0,1]} \arrow[r, "H"] \arrow[dr, "g"'] & \mathcal{M}_{l\times k} \arrow[d, "\det(A(t))"] \\ & \mathbb{R} \end{tikzcd}$$

Then $g(0) \ne 0$ and thus there is a $\epsilon > 0$ such that $g(t) \ne 0$ for any $0 \le t < \epsilon$. Then $H(t)$ will have rank at least r. $\square$

In particular, the class of linear maps with trivial kernel is stable. Also, the class of surjective linear maps is also stable.

8.2 Stability of familiar properties

Theorem 8.4. *Let M, N be smooth manifolds, and assume that M is compact. The following properties of $f : M^k \to N^l$ are stable:*

a) local diffeomorphism

b) immersion

c) submersion

d) transverse to fixed $Z \subset N$

e) embedding

f) diffeomorphism

Proof. We can do all a), b), c), and d) in one swoop. A map f is called a local diffeomorphism if and only if for any $x \in M$ the map $df_x : T_xM \to T_{f(x)}N$ is an isomorphism, i.e., or rank $\geq k$. Likewise, f is a immersion if and only if the rank of df_x is $\geq k$ and f is a submersion if and only if the rank of df_x is $\geq l$. Similarly for d), we check that f is transverse to Z if and only $df : T_x(M) \to T_{f(x)}N/T_{f(x)}Z$ is surjective.

Now suppose f satisfies P for $h : M \times [0, 1] \to N$. Let $x \in M$, and from our proposition, there has to be a little neighborhood of x in M such that the rank of df_x is $\geq r$ in that neighborhood. Likewise, if we move t around a little bit, the rank will still be $\geq r$. That is, we can find, for each x an open neighborhood $U_x \times [0, \epsilon_x) \subset M \times [0, 1]$ of $(x, 0)$ such that for any (y, t) in that neighborhood, $df_t|_y$ has rank $\geq r$.

Since M is compact, M is covered by finitely many of the $U_1, \ldots, U_j$, and letting $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_j\}$, we have

$$M \times [0, \epsilon) \subset (U_1 \times [0, \epsilon_1)) \cup \cdots \cup (U_j \times [0, \epsilon_j)).$$

Then for all $y, t \in M \times [0, \epsilon)$, the derivative $df_t|_y$ has rank $\geq r$.

Now we have to do e) and f). These are both immersions that are one-to-one. So we need to show that throwing in the one-to-one property does not affect stability.

Suppose for every $\epsilon > 0$ there are points $x \neq y$ and $t < \epsilon$ such that $h(x, t) = h(y, t)$. We can find an infinte sequence x_i, y_i, t_i such that $h(x_i, t_i) = h(y_i, t_i)$, and because M is compact, there is a subsequence $x_i \to x$ and $y_i \to y$. Then by continuity, we have

$$f(x) = h(x, 0) = \lim_{i\to\infty} h(x_i, t_i) = \lim_{i\to\infty} h(y_i, t_i) = h(y, 0) = f(y).$$

Since f is one-to-one, we have $x = y$.

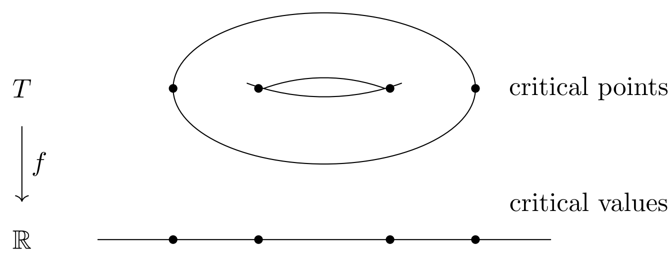
We now use the property that f is an immersion. Let $g : M \times [0, 1] \to N \times [0, 1]$ be the map given by

$$(x, t) \mapsto (h(x, t), t).$$

It is not difficult to check that $dg_{(x,0)}$ is injective and thus g is an immersion near $(x, 0)$. Then g has to be one-to-one on some neighborhood of $(x, 0)$. This contradicts the fact that there are such x_i, y_i, t_is. □

9 February 12, 2016

Consider the projection map $f : T \to \mathbb{R}$ that maps the torus embedded in 3-space to a line. Then there are four critical points and four critical values corresponding to each critical point. (Recall that critical points are $x \in M$ such that df_x is not onto, and critical values are $y \in N$ such that there exists a critical point $x \in M$ with $f(x) = y$.)



9.1 Sard's theorem

Theorem 9.1 (Sard)**.** *If $f : M \to N$ is smooth then almost all $y \in N$ are regular values. Here, almost all means that its complement is of measure zero in the Lebesgue sense.*

Definition 9.2. A subset $S \subset \mathbb{R}^k$ has **measure zero** if for every $\epsilon > 0$ there is a covering of S by countably many rectangles whose total value is less than ϵ. A **rectangle** in $\mathbb{R}^k$ is a set of the form

$$\{(x_1, \dots, x_k) : a_i \le x_i \le b_i\}$$

and it volume is defined as $\prod(b_i - a_i)$.

Example 9.3. The line $\mathbb{R}^1 \subset \mathbb{R}^2$ is of measure zero. We consider the set of rectangles

$$\{[-n, n] \times \big[-\tfrac{\epsilon}{n2^{n+2}}, \tfrac{\epsilon}{n2^{n+2}}\big]\}.$$

Then it clearly covers $\mathbb{R}^1$, and its total volume is

$$\sum \frac{\epsilon}{2^n} = \epsilon.$$

Thus it has measure zero. More generally, any proper subspace of a real vector space has measure zero.

We also need the following theorems, which we will not prove.

Theorem 9.4 (Fubini)**.** *Let $S \subset \mathbb{R}^{k+l}$, and let*

$$S_a = \{(x, y) \in S : x = a\} \subset \mathbb{R}^l$$

for $a \in \mathbb{R}^k$. If each S_a has measure zero in $\mathbb{R}^l$, then S also has measure zero in $\mathbb{R}^{k+l}$.

Theorem 9.5. *If $S \subset \mathbb{R}^k$ has measure zero and $f : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth map (in particular, a diffeomorphism) then $f(S)$ has measure zero.*

Theorem 9.6. *A countable union of sets of measure zero has measure zero.*

Proof. Let $\{S_i\}_{i=1}^{\infty}$ be a countable collection of measure zero sets. Given $\epsilon > 0$, we can choose a cover of S_n by rectangles whose total volume is less than $2^{-n}\epsilon$. then the cover of S by all these things will have total volume less than $\sum 2^{-n}\epsilon = \epsilon$. $\square$

Now the notion of measure zero in a manifold can be made sense.

Definition 9.7. A subset $S \subset M^k$ has **measure zero** if any of the following equivalent statements hold.

(1) If for every local coordinate system $\Phi : \mathcal{U} \to M$, the inverse image $\Phi^{-1}(S)$ has measure zero in $\mathbb{R}^k$.
(2) For some countable atlas $\{\Phi_i : \mathcal{U}_i \to M\}$ the sets $\Phi^{-1}(S) \subset \mathcal{U}_i \subset \mathbb{R}^k$ has measure zero.

9.2 Non-degeneracy of a critical point

Let $f : M \to \mathbb{R}$ be a smooth map. From Sard's theorem, we see that almost all values are regular values. However, if M is compact, then there has to be a maximum and a minimum, and those can't be regular values. So we can look at the next best thing that can happen.

Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is smooth and $0 \in \mathbb{R}^k$ is a critical point and

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_k} = 0.$$

Then we look at the second derivative, which is described bas the **Hessian** of f,

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{pmatrix}.$$

Definition 9.8. The critical point $0 \in \mathbb{R}^k$ is called **non-degenerate** if $\det H_f \neq 0$ at 0.

If there are no degenerate critical points, then f is called a Morse function, and there are a lot of cool stuff if we use those functions.

Definition 9.9. Suppose $f : M \to \mathbb{R}$ is smooth and $x \in M$ is a critical point. The point x is a **non-degenerate critical point** if in all coordinate systems $\Phi : \mathcal{U} \to M$ at x with $\Phi(0) = x$,

$$\det(H_{f\circ\Phi}) \neq 0$$

at 0.

Proposition 9.10. *It suffices to check this in one coordinate system.*

Proof. Suppose that we have

$$\mathcal{U} \xrightarrow{\Psi} \mathcal{U}' \xrightarrow{g} \mathbb{R}$$

with $\Psi(0) = 0$. Using the chain rule, we can check that

$$H_{g\Psi} = d\Psi^T \circ H_g \circ d\Psi.$$

Then since Ψ is a diffeomorphism, $\det H_{g\Psi} = 0$ if and only if $\det H_g = 0$. $\square$

10 February 17, 2016

We are getting to the theorem which is the culmination of chapter 1. Let us recall what we did last time. We defined the notion of measure zero, and stated but not proved Sard's theorem.

Theorem 10.1. *The set of critical values of a smooth map has measure zero.*

Suppose I have a function $f : \mathbb{R}^k \to \mathbb{R}$. A critical point $a \in \mathbb{R}^k$ is called non-degenerated if $\det H_f(a)$ is non-zero. We checked last time that if

$$\mathbb{R}^k \xrightarrow{\Psi} \mathbb{R}^k \xrightarrow{f} \mathbb{R}$$

with $\Psi(0) = 0$ and $0 \in \mathbb{R}^k$ is a critical point, then

$$H_{f\circ\Psi} = d\Psi^T \circ H_f \circ d\Psi$$

and thus critical points are independent of coordinate charts. This suggests that there is a definition of non-degeneracy that is independent of any coordinates. For now we are not going to get into more sophisticated because I think it is more important to get familiar with the tools first. That said, let us define Morse functions.

10.1 Morse functions

Definition 10.2. A **Morse function** $f : M^k \to \mathbb{R}$ is a smooth function with only non-degenerate critical points.

Example 10.3. Let us look at $k = 2$. It can be proved that H_f, in suitable coordinates, looks like one of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$x^2 + y^2 \qquad x^2 - y^2 \qquad -x^2 - y^2$$

We note that the "monkey saddle" is not a Morse function.

The miraculously cool theorem is that every manifold has a Morse function.

Theorem 10.4. *Every manifold M has a Morse function.*

In fact, almost every function is a Morse function; if you have any function, you can move it a little bit and you get a Morse function.

Theorem 10.5. *Let $M \subset \mathbb{R}^N$ be a smooth manifold. Suppose $f : M \to \mathbb{R}$ is any smooth function, Then for almost all $a_1, \ldots, a_n \in \mathbb{R}^N$,*

$$f_a = f + a_1 x_1 + \cdots + a_N x_N$$

is a Morse function.

Lemma 10.6. *Suppose $f : \mathbb{R}^k \to \mathbb{R}$ is any smooth function. Then for almost all $(a_1, \dots, a_k) \in \mathbb{R}^k$,*

$$f_a = f + a_1 x_1 + \cdots + a_k x_k$$

is a Morse function.

This is just the case when M is the whole space $\mathbb{R}^k$.

Proof. We define $g : \mathbb{R}^k \to \mathbb{R}^k$ as

$$g = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\Big).$$

Then x is a critical point of f_a if and only if $g(x) = -a$.

The derivative of g

$$dg = \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big) = H_f = H_{f_a}$$

happens to be the Hessian of f. By Sard's theorem, almost all $-a$ are regular values of g. Let us now think what this means. We see that $-a$ if a regular value if and only if for any critical point of f_a, the determinant of $dg = H_{f_a}$ is nonzero. This simply means that f_a is a Morse function. □

Proof of theorem 10.5. The manifold M is already embedded in $\mathbb{R}^N$. We see that for each point $x \in M$, there is a neighborhood for which

$$x_{\sigma(1)}, \dots, x_{\sigma(k)}$$

is a coordinate system, where $\{\sigma(1), \dots, \sigma(k)\} \subset \{1, \dots, N\}$. This means that we can cover M by countably many charts $\mathcal{U}_i \subset M \subset \mathbb{R}^N$ with the coordinate charts being $(x_1, \dots, x_N) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(k)})$.

Note that f on $\mathcal{U}_i$ is a Morse function if and only if the composite of f with the coordinate chart is a Morse function. Thus we are reduced to the following situation.

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[dr, hook] & \mathbb{R}^N \arrow[d, "(x_1,\dots,x_k)"] \\ & \mathbb{R}^k \end{tikzcd}$$

Consider the function

$$f_a = f + a_1 x_1 + \cdots + a_k x_k + a_{k+1} x_{k+1} + \cdots + a_N x_N.$$

By our lemma applied to

$$f + a_{k+1} x_{k+1} + \cdots + a_N x_N,$$

we see that for almost all $(a_1, \dots, a_k) \in \mathbb{R}^k$ the function

$$f + a_1 x_1 + \cdots + a_N x_N$$

is Morse. Then by Fubini's theorem, we see that it is Morse for almost all $(a_1, \ldots, a_N) \in \mathbb{R}^N$. Now because there are countably many charts, we have to take the union of countably many measure zero sets, but this again measure zero. $\square$

11 February 19, 2016

Today I want to finish out chapter 1.

11.1 Embedding and immersion theorems

Theorem 11.1 (Whitney embedding theorem)**.** *Every smooth n-manifold embeds in $\mathbb{R}^{2n+1}$ and immerses in $\mathbb{R}^{2n}$.*

Actually there is a harder version, which we won't prove.

Theorem 11.2. *Every smooth n-manifold embeds in $\mathbb{R}^{2n}$.*

For instance, a figure-eight 1-manifold is immersed in $\mathbb{R}^2$, but moving it a little, we get an embedding in $\mathbb{R}^3$. This generalizes into the following problem.

Immersion/Embedding problem. *Given M^n, what is the smallest m such that $M \hookrightarrow \mathbb{R}^m$, or $M \looparrowright \mathbb{R}^m$?*

Theorem 11.3 (R. Cohen)**.** *Every M^n immerses in $\mathbb{R}^{2n-\alpha(n)}$, where $\alpha(n) = e_0 + \cdots + e_k$ for $n = e_1 + e_1 2^1 + \cdots + e_k 2^k$.*

So for example, any 2-manifold is immersed in $\mathbb{R}^3$. This

Problem. *What is the smallest m such that $\mathbb{RP}^n \looparrowright \mathbb{R}^m$?*

This looks like a neat problem, but the answer is not that clean; in fact, it is still open. Every decade there is a new technique and then makes some improvement and then people lose interest and so on. There is a hall of fame for this problem.

I will first give an outline of the proof of the Whitney embedding theorem.

Proposition 11.4. *There is sufficiently large $N \gg 0$ such that $M^n \hookrightarrow \mathbb{R}^N$.*

Actually it is easy to see that it embeds into an infinite-dimensional Euclidean space. If we let

$$S = \{\text{smooth } f : M \to \mathbb{R}\}$$

then the smooth functions give a embedding $M \hookrightarrow \mathbb{R}^S$.

Proposition 11.5. *If $N > 2n+1$ there is a vector $v \neq 0 \in \mathbb{R}^N$ such that*

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, hook] & \mathbb{R}^N \arrow[d, "\pi_v"] \\ & H \cong \mathbb{R}^{N-1} \end{tikzcd}$$

where π_v is the projection to $H = v^\perp$.

11.2 Tangent bundles

Definition 11.6. Let $M \subset \mathbb{R}^N$ be a smooth k-manifold. We define the **tangent bundle** $TM \subset \mathbb{R}^N \times \mathbb{R}^N$ as

$$TM = \{(x, v) : x \in M, v \in T_x M\}.$$

We can check that TM is a smooth manifold of dimension $2k$. This is because M is locally $\mathbb{R}^k$, and thus TM is locally $T(\mathbb{R}^k) = \mathbb{R}^{2k}$. There is a canonical projection map $TM \to M$ that sends $(x, v) \mapsto x$. Then given a smooth map $f : M \to N$, the maps df_x assembles to give a smooth map $df : TM \to TN$.

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

11.3 Whitney embedding theorem

Proof of proposition 11.5. We want to find a vector v such that the projection of M inside $\mathbb{R}^N$ is a smooth manifold. First, we need to avoid vectors that are parallel to a line connecting two points in M; if we pick such a v, then the two points will be sent to the same point. Also, we need to avoid a vector that is tangent to the manifold.

So we look at the map $g : M \times M \times \mathbb{R} \to \mathbb{R}^N$ given by

$$(x, y, t) \mapsto t(f(x) - f(y))$$

and the map $h : TM \to \mathbb{R}^N$ given by

$$(x, v) \mapsto v.$$

We know that $N > 2n + 1$, and thus Sard's theorem tells us that there is a regular value $a \neq 0 \in \mathbb{R}^N$ for both g and h. This is equivalent to saying that a is not in the image of both f and g.

We now claim that the composite map $M \hookrightarrow \mathbb{R}^N \to H = a^\perp$ is an embedding. This is because a is not in the image of both g and h. $\square$

It is possible to show that there is always an immersion $M \looparrowright \mathbb{R}^{2n}$ with at worst double points. Once we have this immersion, we can always make a double point at a random point. Then we can move things around so that the two double points cancel out. This is how the hard Whitney theorem is proved.

12 February 22, 2016

Now we are moving into the material in chapter 2, which is quite important. When we look at a smooth function, the inverse image of a regular value is a submanifold. But when we look at the things below that manifold, it is something that is not a manifold but something that we want to consider.

12.1 Manifold with boundary

Let $H_n^+ \subset H_n \subset \mathbb{R}^n$ be

$$H_n = \{(x_1, \dots, x_n) : x_n \geq 0\} \quad \text{and} \quad H_n^+ = \{(x_1, \dots, x_n) : x_n > 0\}.$$

Definition 12.1. A subset $X \subset \mathbb{R}^N$ is a (smooth) **manifold with boundary** if it is locally diffeomorphic to an open subset of H^n.

On a manifold with boundary, an **interior point** is a point corresponding to a point in H_n^+, and a **boundary point** that corresponding to a point in H_n with $x_n = 0$. Actually it is not very clear that what I said even makes sense. But before addressing that issue, let me remark that every manifold is also a manifold with boundary.

Lemma 12.2. *Suppose $U, V \subset H_n$ is open and $f : U \to V$ is a diffeomorphism. Then f restrictions to a diffeomorphism of $U \cap \mathbb{R}^{n-1} \to V \cap \mathbb{R}^{n-1}$, where*

$$\mathbb{R}^{n-1} = \{(x_1, \dots, x_n) \in H_n : x_n = 0\}.$$

Proof. It suffices to show that $f(U \cap \mathbb{R}^{n-1}) \subset V \cap \mathbb{R}^{n-1}$. Suppose that $x \in U \cap \mathbb{R}^{n-1}$ but $f(x) \in H_n^+$. By the definition of a smooth map, there is a neighborhood W of x in $\mathbb{R}^n$ and a smooth $g : W \to \mathbb{R}^n$ such that f is g restricted to $H_n \cap W$.

Since f is a diffeomorphism, $df(x) = dg(x)$ is an isomorphism. So g restrictions to a diffeomorphism of a neighborhood W' of x in $\mathbb{R}^n$ with a neighborhood of $f(x)$. We might as well suppose $g(W') \subset H_n^+$. This implies that $W' \subset H_n$, and this is a contradiction. $\square$

This shows that everything what we did so far makes sense on manifolds on boundaries.

We introduce some terminology. If M is a manifold with boundary of dimension n, then we denote

$$\partial M = \{x \in M : x \text{ corresponds to a point } a \text{ in } \mathbb{R}^{n-1} \subset H_n\}.$$

This is a manifold of dimension $(n-1)$.

Example 12.3. Suppose M is a closed manifold, i.e., $\partial M = \emptyset$. If $f : M \to \mathbb{R}$ is a smooth function and $a \in \mathbb{R}$ is a regular value. Then $f^{-1}([a, \infty)) = N$ is a manifold with boundary, with $\partial N = f^{-1}(a)$.

Proof. Since $(a, \infty) \subset \mathbb{R}$ is open, $f^{-1}((a,\infty))$ is open in M, and so each point of $f^{-1}((a,\infty))$ has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n$.

Suppose $f(x) = a$. Since a is a regular value, we can choose local coordinates $x_1, \dots, x_n$ around x with $f = x_n$ gives a. Then there is a chart $\Phi : \hat{\mathcal{U}} \subset \mathbb{R}^n \to \mathcal{U}$, and then
$$(f \circ \Phi)^{-1}([a, \infty)) = \hat{\mathcal{U}} \cap H_n. \qquad \square$$

Now let me state some facts that I am sure you can even figure out by yourself. Given a manifold M^n with boundary, we can consider the tangent space $T_x M$ at a boundary point $x \in \partial M$. This is because in the local coordinates, smooth maps are those that can be extended to an open neighborhood. This gives the point at the boundary some kind of potential to "move out the manifold" a little bit. Because ∂M is a manifold with dimension $n-1$, we have a natural inclusion
$$T_x \partial M \hookrightarrow T_x M.$$

Let $f : M \to N$ be manifolds with boundary. This induces, for each $x \in M$, a map
$$df(x) : T_x M \to T_{f(x)} N$$
and moreover it satisfies the chain rule.

If M is a manifold with boundary and N is manifold with empty boundary, then the product $M \times N$ is a manifold with boundary, and $\partial(M \times N) = (\partial M) \times N$. But noted that this is not true if both M and N has nonempty boundary.

12.2 Transversality

Let's suppose I have a manifold M with boundary, and I have a Z sitting inside X both with empty boundary.

$$\begin{tikzcd} f^{-1}(Z) \arrow[r, hook] \arrow[d] & M \arrow[d, "f"] \\ Z \arrow[r, hook, "\text{codim}=k"] & X \end{tikzcd}$$

Theorem 12.4. *If $f \pitchfork Z$ and $\partial f = f|_{\partial M} \pitchfork Z$ then $f^{-1}(Z) \subset M$ is a smooth manifold with boundary of codimension k.*

Proof. Because the whole question is local, we can work locally. Locally, Z looks like $g^{-1}(0)$ for some map $g : X \to \mathbb{R}^k$. Then $f^{-1}(Z) = (gf)^{-1}(0)$.

Now M look like an open neighborhood of H_n, and so we can say that gf looks locally like a map $H_n \to \mathbb{R}^k$. If some point $z \in F^{-1}(Z)$ is an interior point of M, then by transversality, its neighborhood will look like $\mathbb{R}^{n-k}$. If $z \in F^{-1}(Z)$ is a boundary point, then one can check that its neighborhood looks like a half space. $\square$

13 February 24, 2016

Today we are going to prove the following theorem.

Theorem 13.1. *If X is a compact connected 1-manifold with boundary then X is diffeomorphic to $[0,1]$ or S^1.*

The important consequence for us will be

Theorem 13.2. *If M is a compact 1-manifold with boundary, then*

$$\#\partial M \equiv 0 \pmod 2.$$

Corollary 13.3. *Suppose X is an n-manifold with boundary ∂X. There does not exists a smooth map $f : X \to \partial X$ such that $f(x) = x$ for all $x \in \partial X$, i.e., ∂X is not a retract of X.*

Proof. Suppose $f : X \to \partial X$ is a retraction. By Sard's theorem, there is a regular value $a \in \partial X$. We see that $f^{-1}(a)$ is a 1-manifold with boundary, and

$$\partial f^{-1}(a) = f^{-1}(a) \cap \partial X = \{x \in \partial X : f(x) = a\} = \{a\}$$

since $f(x) = x$ for all $x \in \partial X$. So $\#\partial f^{-1}(a) = 1$. This contradicts the theorem. □

Corollary 13.4 ((Smooth) Brouwer fixed point theorem)**.** *If $f : D^n \to D^n$ is a smooth map, then there is an $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose not. We draw a line joining each x and $f(x)$, and let $g(x)$ be the intersection of ∂D^n and the line on the $f(x)$ side. Then we can see that g is a retraction of D^n to ∂D^n. □

13.1 Classification of 1-manifolds

A 1-manifold is something like gluing intervals together, and what can happen? You can either glue the intervals back to the starting point and get a circle, or don't and get a interval. But the problem we have to solve is that something like "not getting to the end" does not happen. Morse functions give a way to kind of cut the manifold into a finite number of parts.

Let M be a compact connected 1-manifold, and let us choose a Morse function $f : M \to \mathbb{R}$. Let S be the union of the set of critical points and ∂M. Let

$$M \setminus S = L_1 \amalg L_2 \amalg \cdots \amalg L_k$$

where each L_i are connected. We see that for each $L = L_i$, the image $f(L) \subset \mathbb{R}$ is connected and bounded. Thus $f(L) = (a, b)$ for some $a < b$.

Proposition 13.5. *The restriction $f : L \to (a, b)$ is a diffeomorphism.*

Sketch of proof. Since f has no critical points, it is a local diffeomorphism. This means that it is a covering space of (a, b), but I will explain more.

Choose a $t \in (a, b)$ and $x \in L$ with $f(x) = t$. Consider an interval $[c, d]$ containing a neighborhood of t and g such that $g(t) = x$ and $f \circ g(s) = s$ for any $s \in [c, d]$.

$$\begin{tikzcd} L \arrow[d, "f"] & \\ (a,b) & (c,d) \arrow[l, hook'] \arrow[ul, "g"'] \end{tikzcd}$$

We first note that if g exists on $[c, d]$ then such a g is unique. This is because the set

$$\{s \in [c, d] : g_0(x) = g_x(2)\}$$

is both open and closed.

Now we consider

$$\sup\{d : [c, d] \text{ such that } g \text{ exists}\} = d'.$$

If $d' < a$, then we may look at the local diffeomorphism around d' and then extend g a bit further. So $d' = a$ and gluing such gs together, we get the desired map. $\square$

We also need do some induction, so we need to understand what happens at the boundary of the L_is, that is, what happens at the critical points.

Lemma 13.6 (Morse lemma)**.** *If $f : M^1 \to \mathbb{R}$ is a Morse function and $x \in M$ is a critical pint with $f(x) = a$ then there is a coordinate system $\Phi : \hat{\mathcal{U}} \subset \mathbb{R}^1 \to \mathcal{U}$ such that $f \circ \Phi(t) = \pm t^2$.*

Sketch of proof of theorem 13.1. We know that $f : L_i \to (a_i, b_i)$ is a diffeomorphism. Using this, we get that each $\overline{L_i}$ is diffeomorphic to $[0, 1]$, and thus $\overline{L_i} \setminus L_i$ has two points. Now we need a way to order the L_i so that we can glue the different segments.

Definition 13.7. A **chain** is a sequence

$$L_{i_1}, L_{i_2}, L_{i_3}, \ldots$$

such that $\partial L_{i_j} \cap \partial L_{i_{j-1}}$ is nonempty and $i_j \neq i_{j-1}$.

Proposition 13.8. *Any maximal chain contains all the L_i.*

Proof. We show that a maximal chain is both open and closed in M. $\square$

So suppose $L_1, \ldots, L_k$ is a maximal chain. We can rearrange the Morse function so that f on L_i is replaced by $(-1)^{i-1} f + c_i$ so that they agree on the common boundary points. Let this function be F. This F is continuous, monotone, and smooth except at the p_i. We can now apply this lemma.

Lemma 13.9 (Smoothing lemma)**.** *We can change F a little to be smooth at the p_i and monotone.*

Now if $p_0 \neq p_k$, then F is diffeomorphic with a closed interval. When $p_0 = p_k$ then $e^{2\pi i c F}$ is diffeomorphic with S for some c. $\square$

14 February 26, 2016

Suppose I have a manifold M possibly with boundary, and it maps to $f : M \to X$ where X is closed. Also let $Z \subset X$ be a closed submanifold. Our goal is to show that f can always be moved a little bit so that f and ∂f are transverse to Z.

14.1 Transversality in a general family

We introduce the notion of a family. If there is a map $S \times M \to X$, then we can consider it as a set of maps f_s collected for $s \in S$.

$$\begin{tikzcd} S \times M \arrow[d, "F"] & \{s\} \times M \arrow[l, hook'] \arrow[dl, "f_s"] \\ X & \end{tikzcd}$$

Definition 14.1. A "general" family $F : S \to M \to X$ is a map which is either 1) a submersion or 2) transverse to Z with ∂F also transverse to Z.

Theorem 14.2. *If $F : S \times M \to X$ has the properties $F \pitchfork Z$ and $\partial F \pitchfork Z$, then for almost all $s \in S$, both f_s and ∂f_x is transverse to Z.*

Proof. Let us look at the pullback:

$$\begin{tikzcd} S \times M \arrow[r] & X \\ W \arrow[u, hook] \arrow[r] & Z \arrow[u, hook] \end{tikzcd}$$

Then we can look at the projection map

$$\begin{tikzcd} W \arrow[r] \arrow[dr, "g"'] & S \times M \arrow[d] \\ & S \end{tikzcd}$$

We claim that if s is a regular value of g then $f_s \pitchfork Z$.

Suppose that s is a regular value of g, and let $x \in M$. We must show that if $z \in Z$ with $f_s(x) = F(s, x) = z$ then the direct sum of the two maps

$$\begin{tikzcd} T_xM \arrow[r, "df_x"] & T_zX \\ & T_zZ \arrow[u, hook] \end{tikzcd}$$

is onto, or equivalently that

$$T_xM \to T_zX/T_zZ := V$$

is onto.

Our assumption that s is a regular value gives that

$$T_s X \oplus T_x M = T_{(s,x)} S \times M \xrightarrow{dF} V$$

is onto. Since $T_{(s,x)} W$ is just the kernel, we see that if s is regular value, then

$$\begin{tikzcd}
T_{(s,x)} W \arrow[r] \arrow[dr, two heads] & T_s S \oplus T_x M \arrow[d] \\
& T_s S
\end{tikzcd}$$

After this, some linear algebra will show that f_s is transverse to M.[4] □

Theorem 14.3 (Existence of general families)**.** *If $f : M \to X$ is a smooth map, then there is a submersion $F : \mathbb{R}^n \times M \to X$ with $F(0, x) = f(x)$. (For convenience, we assume that both M and X are compact.)*

Proof. We may assume that $X \subset \mathbb{R}^n$. By the "tubular neighborhood theorem," which we will prove next time, X is a retract of open set $U \subset \mathbb{R}^n$ containing X, i.e., a map $r : U \to X$ such that $r(x) = x$ for any $x \in X$.

Now let us construct a map $G : M \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$(x, v) \mapsto f(x) + v.$$

We look at the inverse image $G^{-1}(U)$ and because this contains M, there is an $\epsilon > 0$ such that $G^{-1}(U)$ contains $M \times B_\epsilon$. Then we can construct a new map

$$\begin{tikzcd}
M \times B_\epsilon \arrow[r, "G"] \arrow[dr, "F"'] & U \arrow[d, "r"] \\
& X
\end{tikzcd}$$

This F is clearly a submersion, because both G and r are submersions. □

[4] The full proof was given in the lecture, but I did not understand it then. In any case, some uninteresting linear algebra does finish the proof.

15 February 29, 2016

15.1 Partitions of unity

Definition 15.1. Let X be a topological space, and let $f : X \to \mathbb{R}$ be a continuous function. We define the **support** of f as

$$\operatorname{supp} f = \text{closure of } \{x : f(x) \neq 0\}$$

and

$$\operatorname{supp}^+ f = \{x : f(x) > 0\}.$$

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X.

Definition 15.2. A **partition of unity** subordinate to $\mathcal{U}$ is a sequence $\{\theta_1, \theta_2, \dots, \}$ of functions $\theta_i : X \to [0, 1] \subset \mathbb{R}$ such that

1) Each $x \in X$ has a neighborhood on which only finitely many θ_i are nonzero.
2) For each i there exists an α such that $\operatorname{supp} \theta_i \subset U_\alpha$.
3) For all $x \in X$, the sum of all θ_i is $\sum \theta_i(x) = 1$.

We note that we can replace the third condition by

3′) For all x there exists an i such that $\theta_i(x) \neq 0$.

If we have this, then $n_i = \theta_i / (\sum \theta_i)$ form a partition of unity.

Theorem 15.3. *If $X \subset M$ is a subspace of a manifold, then partitions of unity exist for any open cover of X. Moreover the θ_i can be constructed as the restriction of smooth functions on M.*

There is a half-page proof in the book, but I want to break it into digestible terms. We are not going to assume that M is embedded in the Euclidean space.

By definition of a manifold, M is Hausdorff and has a countable basis for the topology. So is, if $\{V_\alpha\}$ is an open cover, then some countable number of the V_α actually cover M.

Lemma 15.4. *Every $x \in M$ has a neighborhood with compact closure.*

Proof. Take the open ball in any coordinate neighborhood. □

Lemma 15.5. *Every compact $K \subset M$ has neighborhood $K \subset V$ with compact closure.*

Proof. For each $x \in K$ choose a neighborhood B_x with $\overline{B_x}$ compact. Then a finite number of $B_1, \dots, B_l$ cover K. Take $V = B_1 \cup \cdots \cup B_l$. Then $\overline{V} = \overline{B_1} \cup \cdots \cup \overline{B_l}$ is compact. □

Lemma 15.6. *M is a countable union of compact subspaces.*

Proof. For each x choose a neighborhood B_x with compact closure. Now $\{B_x\}$ covers M and so some countable subset $\{B_1, B_2, \dots\}$ covers M. Then we have $M = \bigcup_i \overline{B_i}$. □

Lemma 15.7. *There is a sequence*

$$K_1 \subset V_1 \subset K_2 \subset V_2 \subset K_3 \subset V_3 \subset \cdots$$

where 1) $\bigcup_i K_i = M$*, 2) each* K_i *is compact and* V_i *is open, 3)* $\overline{V_i} \subset K_{i+1}$.

Proof. Pick a family of compact sets $\{B_1, B_2, \dots\}$ such that $M = \bigcup_i B_i$. We start with $K_1 = B_1$ and find an $K_1 \subset V_1$ with compact closure. Then take $K_2 = \overline{V_1} \cup B_2$ and so on. □

Definition 15.8. A subspace $X \subset M$ is a **disconnected union** of compact sets if

$$X = \bigcup_{i=1}^{\infty} K_i$$

where each K_i is compact, and $K_i \cap K_j = \emptyset$ there are open set $K_i \subset V_i$ such that $V_i \cap V_j = \emptyset$.

Theorem 15.9. *M can be written as $M = I_1 \cup I_2 \cup I_3$ where each I_i is a disconnected union of compact sets.*

Proof. If we let

$$L_n = K_n - V_{n-2}, \quad W_n = V_n - K_{n-2},$$

we see that $L_n \subset W_n$ and $M = \cup L_n$. Also, $W_n \cap W_{n-3} = \emptyset$. Thus we can let

$$I_i = \bigcup_n L_{3n+i}$$

and get a cover of M by three disconnected unions of compact sets. □

Proof of theorem 15.3. Let us first look at the special case when $X \subset M$ is compact. We can choose for each $x \in X$ a smooth $\theta_x : X \to [0,1] \subset \mathbb{R}$ such that there exists an α with $\mathrm{supp}(\theta_x) \subset U_\alpha$. This can be done by using the bump function. Then the open sets $\mathrm{supp}^{-1}\theta_x$ cover X, and since X is compact, finitely many $\theta_1, \dots, \theta_k$ do. This shows that $\{\theta_1, \dots, \theta_k\}$ satisfies 1), 2), and 3′).

Now suppose that X is a disconnected union of compact sets K_i, where $K_i \subset V_i$ and $V_i \cap V_j = \emptyset$. For each i, there is a partition of union on K_i subordinate to $\{U_\alpha \cap V_i\}$. Let this be $\{\theta_1^i, \dots, \theta_j^i\}$. Then the set $\{\theta_j^i\}$ is a partition of unity.

Next we look at the case $X = M$. Since M is the union of three disconnected union of compact sets, we can take the union of all θs for the three unions. Thus $X = M$ has a partition of unity.

Finally, let us look at the general case $X \subset M$. Since U_α give a open cover of X, we see that there is an open $\tilde{U}_\alpha \subset M$ such that $\tilde{U}_\alpha \cap X = U_\alpha$. If we let $M' = \bigcup \tilde{U}_\alpha$, it must be a manifold. Then we can look at the partition of unity on M' and its restriction to X will give a partition of unity on X. □

16 March 2, 2016

Theorem 16.1 (Tubular neighborhood theorem)**.** *X is a retract of a neighborhood U of X.*

In the last two lectures, we have used this without proving it. Now let us prove this.

16.1 Normal bundle

Let $X \subset \mathbb{R}^m$ be a smooth manifold if dimension d. The tangent space $T_x X$ has dimension d, and does not depend on the embedding. Let the **normal space** $N_x X$ be the orthogonal complement of $T_x X$. Now this depends on the embedding, and will have dimension $m - d$.

We have defined the tangent bundle as

$$TX = \{(x, v) \in \mathbb{R}^m \times \mathbb{R}^m : v \in T_x X\}$$

and showed that TX is a manifold of dimension $2d$. Likewise, we define the **normal bundle** as

$$NX = \{(x, v) \in X \times \mathbb{R}^m : v \in N_x X\}.$$

The is actually bad notation, because NX depends on the information of the embedding, which is not present in the notation.

Proposition 16.2. *NX is a manifold if dimension $d + (m - d) = m$. In fact, $NX \to X$ is a smooth fiber bundle.*

Let us assume the proposition for a moment.

16.2 Tubular neighborhood theorem

Proof of the Tubular neighborhood theorem. We construct a map $g : N \to \mathbb{R}^m$ as $g(x, v) \mapsto x + v$. The derivative of this map at $(x, 0)$ looks like $dg_{(x,0)} : T_{(x,0)}N \to \mathbb{R}^m$, and we see that $T_{(x,0)}N = T_x X \oplus N_x X \cong \mathbb{R}^m$. Therefore $dg_{(x,0)}$ is an isomorphism.

By the inverse function theorem, g is an diffeomorphism in a neighborhood of $(x, 0)$ in N. Since X is compact, there is an ϵ such that $g|_{B_\epsilon(N)} : B_\epsilon(N) \to \mathbb{R}^m$ is a diffeomorphism with its image, where

$$B_\epsilon(N) = \{(x, v) \in N : |v| < \epsilon\}.$$

This is a local diffeomorphism, by definition. Why is it one-to-one? This, we can choose ϵ small enough so that the images of g don't overlap, because X is compact.

Then $B_\epsilon(N)$ is diffeomorphic to an open set of $\mathbb{R}^m$ containing X, and we can set $B_\epsilon(N) \to X$ to be our retraction. $\square$

More generally, let $X^d \subset Y^l \subset \mathbb{R}^m$ be manifolds. We can define

$$N_X^Y = \{(x,v) : x \in X,\ v \in T_xY,\ x \perp T_xX\}.$$

A similar argument gives an embedding $B_\epsilon N_X^Y \to Y$ as an open set. Actually there is a small technical problem. When we map $B_\epsilon(N_X^Y) \to \mathbb{R}^m$, the image might not lie in Y. So what we do is consider the tubular neighborhood U of Y, and consider a small enough ϵ so that $B_\epsilon(N_X^Y)$ lives in the U.

$$\begin{tikzcd} & Y \\ B_\epsilon(N_X^Y) \arrow[r] \arrow[ur] & U \subset \mathbb{R}^m \arrow[u, "\pi"'] \end{tikzcd}$$

Then we can project it down to Y.

Now let me justify why the normal bundle is a smooth manifold.

Proof. Locally, we can find a function $F : \mathbb{R}^m \to \mathbb{R}^{m-d}$ so that $X = F^{-1}(0)$, that is 0 is a regular value. Then T_x is the kernel of the map $dF_x : \mathbb{R}^m \to \mathbb{R}^{m-d}$. Since $N_x \cap T_x = \{0\}$, we see that the composite

$$N_x \longrightarrow \mathbb{R}^m \xrightarrow{dF} \mathbb{R}^{m-d}$$

is an isomorphism.

Now let us show that $N \to X$ is a smooth fiber bundle around each $x \in X$. If F is defined on a neighborhood W of x, then since $dF : N_x \to \mathbb{R}^{m-d}$ is an isomorphism, the map $N(W) \to W \times \mathbb{R}^{m-d}$ is a local diffeomorphism. Thus using the inverse function theorem, we can choose a neighborhood so that $N(V)$ and $V \times \mathbb{R}^{m-d}$ are diffeomorphic. $\square$

17 March 4, 2016

So far we have done transversality, partitions of unity and all this stuff, and today we are going to start applying these machinery.

Let $f : M \to X$, and let $Z \subset X$.

Question 1. *Can f be homotopic to a map that misses Z?*

Also, let $Z^{d-1} \subset X^d$.

Question 2. *Can one find a smooth $f : X \to \mathbb{R}$ for which 0 is a regular value and $f^{-1}(0) = Z$?*

These two seemingly unrelated questions are actually related very much. We can answer the second question completely, and the first question is not quite the right question to ask.

17.1 Mod 2 intersection numbers

For the first question, we can make f transverse to Z and then study $W = f^{-1}(Z) \subset M$, which is a smooth manifold.

$$\begin{array}{ccc} W & \longrightarrow & M^k \\ \big\downarrow & & \big\downarrow f \\ Z^l & \hookrightarrow & X \end{array}$$

If $\dim Z + \dim M = \dim X$, then $\dim W = 0$. In this case, we can simply count the number of points.

Definition 17.1. We define the **intersection number** $I(f, Z)$ as

$$\#\tilde{f}^{-1}(Z) \bmod 2$$

where $\tilde{f}$ is transverse to Z and is homotopic to f.

Lemma 17.2. *The intersection number $I(f, Z)$ is independent of $\tilde{f}$.*

Proof. Suppose we have two $\tilde{f}$ and $\tilde{f}'$. Then we can concatenate two homotopies between f and $\tilde{f}$, and f and $\tilde{f}$. That gives us a homotopy

$$H : M \times [-1, 1] \to X$$

where $H(\bullet, -1) = \tilde{f}$ and $H(\bullet, 1) = \tilde{f}'$. Actually this might not be smooth at $t = 0$, so we have to modify this a little bit, using a bump function, so that it slows down and almost become constant at $t = 0$. Then H becomes a smooth homotopy. Next, we can move H a little bit without changing H in a neighborhood of $M \times \{-1, 1\}$ so that H is transverse to Z.

$$\begin{array}{ccc} Y & \longrightarrow & M \times [-1, 1] \\ \big\downarrow & & \big\downarrow \tilde{H} \\ Z & \longrightarrow & X \end{array}$$

Now Y is a 1-manifold with

$$\partial Y = \partial_{-1} Y \amalg \partial_1 Y = M \times \{-1, 1\} \cap Y,$$

where $\partial_{-1} Y = \tilde{f}'^{-1}(Z)$ and $\partial_1 Y = \tilde{f}^{-1}(Z)$. By the classification of 1-manifolds we have proved, we have $\#\partial Y = 0 \pmod 2$ and thus $\#\partial_{-1} Y = \#\partial_1 Y \pmod 2$. $\square$

Now this gives you a well-defined intersection number modulo 2. This is an invariant, so if you want some thing to not happen then the easiest thing to do is check that the invariants are different. If you want something to happen, then it requires something much more sophisticated constructive methods.

There is an obvious variation.

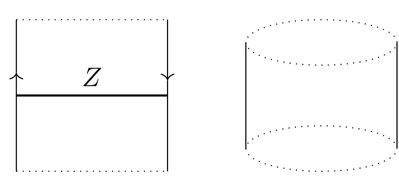
Theorem 17.3. *If $Z = \partial L$ for some submanifold L of X, then $I(f, Z) = 0$.*

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & M \arrow[d, "f"] \\
Z = \partial L \arrow[r] & X
\end{tikzcd}$$

Let us recall the question 2 I started with: when is $Z \subset X$ of codimension 1 some $f^{-1}(0)$? If $Z = f^{-1}(0)$, then clearly Z is the boundary of $f^{-1}(\{x : x \geq 0\})$. Then for any $g : S^1 \to X$, the intersection number is $I(g, Z) = 0$. This gives a necessary condition. But in fact, this is also a sufficient condition.

Theorem 17.4. *Let $Z^{n-1} \subset X^n$ be closed manifolds. Then $Z = f^{-1}(0)$ for some $f : X \to \mathbb{R}$ where 0 is a regular value of f, if and only if $I(g, Z) = 0$ for any $g : S^1 \to X$.*

In the problem set, we showed that the close Möbius band is not diffeomorphic to the closed cylinder. This was done by looking at the connected components of the boundary. But this does not work if the manifolds are open. First consider the open Möbius band.



For the circle Z on the Möbius band, we see that $I(Z, Z) = 1$, because when we move one end up, the other goes down. On the other hand, we see that any loop has intersection number 0 with itself, because we can move them up and down so that they don't meet. This proves that the Möbius band and the cylinder are not isomorphic.

18 March 7, 2016

Today I want to introduce something that is not usually taught this level of course.

18.1 Cobordism groups

Definition 18.1. Suppose M_1 and M_2 are closed n-manifolds. A **cobordism** of M_1 with M_2 is an $(n+1)$-manifold N with boundary such that $\partial N = M_1 \amalg M_2$. If there exists a cobordism between M_1 and M_2 we say that they are **cobordant**.

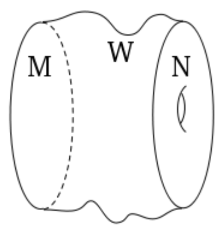


Figure 1: Cobordism between M and N.

Theorem 18.2. *"Cobordant" is an equivalence relation.*

Proof. It is obvious that it is symmetric and reflexive. For transitivity, suppose $\partial N_{12} = M_1 \amalg M_2$ and $\partial N_{23} = M_2 \amalg M_3$. Then we can glue these two manifolds together along M_2 and get a new manifold

$$N_{12} \underset{M_2}{\cup} N_{23} = (N_{12} \amalg N_{23})/(x \in M_2 \subset \partial N_{12} \sim x \in M_2 \subset N_{23})$$

with boundary $M_1 \amalg M_3$. Actually this requires a bit of machinery in the problem set.

Theorem 18.3 (Collar neighborhood theorem)**.** *If $M = \partial N$ then there exists a neighborhood U of M and a diffeomorphism $N \cong M \times [0, \infty)$.*

Once we have this we can glue N_{12} and N_{13} along the collar neighborhood, and we no longer have any problems at the boundary. $\square$

Definition 18.4. The n**-th cobordism group** MO_n is the quotient

$$\{\text{closed } n\text{-manifolds}\}/\text{cobordant}.^5$$

For addition, we make MO_n into a commutative monoid by letting $[M_1]+[M_2] = [M_1 \amalg M_2]$.

Since $[M] + [M] = 0$, we immediately see that MO_n is a $\mathbb{Z}/2$-vector space.

[5] Actually there is a set-theoretic issue here but don't worry about it.

Example 18.5. The 0-th cobordism group $MO_0 = \mathbb{Z}/2$ because the cobordism classes are determined by the number of points. The next one is $MO_1 = 0$ because the circle is a boundary of the disk. Then $MO_2 = \mathbb{Z}/2$ generated by $\mathbb{R}P^2$, and this is not obvious.

The amazing thing is that we exactly know what MO_n is.

Theorem 18.6 (Thom)**.**

$$\bigoplus_{n\geq 0} MO_n = \mathbb{Z}/2[x_j \mid j \geq 1, n \neq 2^j - 1].$$

18.2 Bordism homology

Suppose X is a topological space. A manifold over X is a manifold M together with a continuous map $f : M \to X$. We let $\partial f = f|_{\partial M} : \partial M \to X$.

Definition 18.7. Suppose $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are two closed n-manifolds over X. A **cobordism** between them is an $(n+1)$-manifold with a map $h : N \to X$ and an isomorphism $\partial h = f_1 \amalg f_2$.

Proposition 18.8. *Cobordism of maps is an equivalence relation.*

Definition 18.9. The n-th bordism homology group $MO_n(X)$ of X is the set of all closed n-manifolds over X modulo cobordism.

Again, $MO_n(X)$ is always a $\mathbb{Z}/2$-vector space.

Example 18.10. We have $MO_1(\mathbb{R}^2) = 0$. This is because any maps $f : S^1 \to \mathbb{R}^2$ extends to a map $g : D^2 \to \mathbb{R}^2$ given by

$$g(r, \theta) = rf(\theta).$$

There is actually a technical problem because this g might not be smooth at the origin. The following proposition deals with this issue.

Proposition 18.11. *If $f_1 : M_1 \to X$ and $f_2 : m_2 \to X$ are smooth and $h : N \to X$ is a cobordism between them, then there exists a smooth cobordism $\tilde{h} : N \to X$ between them.*

Example 18.12. We have $MO_1(S^2) = 0$. This is because for any $f : M^1 \to S^2$, by Sard's theorem there is a regular value $x \in S^2$. Then we can use the stereographic projection $S^2 \setminus \{x\} \to \mathbb{R}^2$ the make a map

$$\begin{array}{ccc} f : M^1 & \longrightarrow & S^2 \setminus \{x\} \\ \downarrow & & \downarrow \\ g : N^2 & \longrightarrow & \mathbb{R}^2 \end{array}$$

that shows that f cobordant to the constant map.

Example 18.13. We have $MO_1(T^2) \neq 0$. To prove this, we will produce a non-zero homomorphism $MO_1(T^2) \to \mathbb{Z}/2$. Consider a fixed loop Z around a torus, and for any $f : M^1 \to T^2$, let us look at the intersection number $I_2(f, Z) \in \mathbb{Z}/2$. From the last lecture, we see that $I(f, Z)$ depends only on the cobordism classes of f. Thus $I(f, Z)$ defines a map

$$MO_1(T^2) \to \mathbb{Z}/2.$$

This is not the zero map, and hence $MO_1(T^2) \neq 0$. In fact, we can take two loops and construct a map

$$MO_1(T^2) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

We shall prove later that this is an isomorphism.

19 March 9, 2016

Last time we talked about this bordism homology. There is a really useful tool that enables us to do a lot of interesting stuff.

19.1 Surgery

Suppose we have two 1-manifolds. Pick two points on the manifold and cut out holes (or intervals in this case) and connect them by smooth curves. Or suppose we have two points on a surface. We can cut out neighborhoods of the points and attach a cylinder connecting the two holes. Then we kind of attach a handle to it.

Definition 19.1. Let $A^k \subset M^n$. A **framing** of A in M is an open $A \subset U \subset M$ with a diffeomorphism

$$\begin{tikzcd} U \arrow[r, phantom, "\approx"] & A \times \mathbb{R}^{n-k} \\ A \arrow[u, hook] \arrow[r] & A \times \{0\} \arrow[u, hook] \end{tikzcd}$$

We remark that a framing on A exists if and only if there is a smooth $f : M \to \mathbb{R}^{n-k}$ with 0 as a regular value and $A = f^{-1}(0)$.

Definition 19.2. A **bounding manifold** of A is a manifold B with $A = \partial B$.

Let U' be the open subset of U so that

$$\begin{tikzcd} U \arrow[r, "\Phi"] & A \times \mathbb{R}^{n-k} \\ U' \arrow[u, phantom, "\cup"] \arrow[r] & A \times B^{n-k} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where B^{n-k} is the open unit ball. Now the boundary of $M \setminus U'$ is

$$\partial(M \setminus U') = \partial M \amalg (A \times S^{n-k-1}).$$

Since

$$A \times S^{n-k-1} = \partial(B \times S^{n-k-1}),$$

we can now glue $M \setminus U'$ with $B \times S^{n-k-1}$ along $A \times S^{n-k-1}$. There is always a problem when gluing; we need to somehow smooth the boundaries so that they are smooth manifolds, but this can be done. It is too technical to go over it in lecture.

We note that a surgery needs

- a submanifold $A \subset M$,
- a framing $U \approx A \times \mathbb{R}^{n-k}$ of A,
- and a bounding manifold $A = \partial B$,

and collar neighborhoods. It is possible to check that a surgery is isomorphic up to diffeomorphism. Usually we will set $A = S^{n-k}$ and $B = D^{n-k+1}$.

Example 19.3. Suppose that M_1 and M_2 are n-manifolds. Choose points $x \in M_1$ and $y \in M_2$ with coordinate neighborhoods $\Phi_x : U_x \to \mathbb{R}^n$ and $\Phi_y : U_y \to \mathbb{R}^n$. For $A = S^0 = \{-1, 1\} \hookrightarrow M_1 \amalg M_2$ that sends $-1 \mapsto x$ and $1 \mapsto y$, we can perform a surgery. Then we new manifold that is like attaching a cylinder between M_1 and M_2 at x and y. This is called the **connected sum** of M_1 and M_2, and is denoted by $M_1 \# M_2$.

Now what does surgery have to do with cobordism? When you perform a surgery, you are not changing the cobordism.

19.2 The surgery cobordism

Suppose M is closed and compact, and also suppose that B is compact. Let

$$N = M \times [0,1] \amalg B \times D^{n-k}$$

where $D^{n-k} = \{x \in \mathbb{R}^{n-k} : |x| \leq 1\}$ is the closed ball. Then we have

$$\partial N = M \times \{0\} \amalg \tilde{M}$$

where $\tilde{M}$ is the result of the surgery.

This looks hard, but it is actually just something like this. Suppose you have a 2-sphere and want to attach a handle. Then you first thicken the ball, and then attach a solid handle to the outer part. Then you get a 3-manifold with boundary the sphere and the handled sphere. This shows that the sphere is cobordant with the handled sphere, which is the torus.

We can also perform surgeries on maps. Suppose that we have a map $f : M \to X$, and $A = \partial B \subset M$ and a framing $\Phi : U \to A \times \mathbb{R}^{n-k}$. Suppose that we have an extension

$$\tilde{g} : B \times D^{n-k} \to X$$

extending $A \times D^{n-k} \to X$. Then we can map $N \to X$ by first mapping

$$M \times [0,1] \longrightarrow M \xrightarrow{f} X$$

and then mapping

$$B \times D^{n-k} \xrightarrow{\tilde{g}} X.$$

We see that these agree on $A \times D^{n-k}$ and thus gives a well-defined map. Actually we can canonically give the map $\tilde{g}$, but leaving it this way, we can make our own choices.

20 March 11, 2016

20.1 MO_1 of a torus

Let $Z \subset \Sigma$ be a connected and closed 1-manifold lying inside a 2-manifold.

Proposition 20.1. *Every $f : M \to \Sigma$ is cobordant to a map $\tilde{f}$ which is transverse to Z and with $0 \leq \#f^{-1}(Z) < 2$.*

Proof. Moving f by a homotopy if necessary, we may assume $f \pitchfork Z$. Let $z_1, \dots, z_k \in Z$ be $f(M) \cap Z$. We choose z_1 and z_2 to be neighboring points on Z, and close an embedding $[0,1] \to Z$ so that $g(0) = z_1$, $g(1) = z_2$, and $g(t) \notin Z \cap f(M)$ for $0 < t < 1$.

Choose $m_1, m_2 \in M$ such that $f(m_1) = z_1$, $f(m_2) = z_2$. Now we do surgery with $A = \{m_1, m_2\}$ and $B = [0,1]$. Then we get a map $\tilde{f}$ that does not meet M at z_1 and z_2. Then the number of intersections decrease by 2. □

There is a variation of this fact.

Proposition 20.2. *Let $Z_1, \dots, Z_n \subset \Sigma$ be connected closed 1-manifolds transverse to each other. Then any $f : M \to Z$ is cobordant to a map transverse to all Z_i with $\#f^{-1}(Z_i) < 2$.*

To prove this, you can simply make f transverse to all Z_i and $Z_i \cap Z_j$. Then $f^{-1}(Z_i \cap Z_j) = \emptyset$, and do the argument on each Z_i.

Example 20.3. Let Σ be the torus and consider two loops α and β that meets each other once. We can look at the homomorphism $MO_1(\Sigma) \to (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ given by

$$f \mapsto (I(f, \alpha), I(f, \beta)).$$

Since $\alpha \mapsto (0,1)$ and $\beta \mapsto (1,0)$, we see that this map is onto.

Proposition 20.4. *The kernel of this map is zero.*

Proof. Suppose $f : M \to \Sigma$ satisfies $I(f, \alpha) = I(f, \beta) = 0$. Then f is cobordant to a map $\tilde{f} : M \to \Sigma \setminus (\alpha \cup \beta) \cong \mathbb{R}^2$. Because we already know that $MO_1(\mathbb{R}^2) = 0$, we see that f is zero. □

So we conclude that $MO_1(\Sigma) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and moreover α and β form a basis.

21 March 21, 2016

Let Σ be a compact closed 2-manifold. We have computed the first bordism homology $MO_1(\Sigma)$ for $\Sigma = S^2$ and $\Sigma = T^2$. We remark that $MO_1(\Sigma)$ comes equipped with a symmetric bilinear form

$$MO_1(\Sigma) \times MO_1(\Sigma) \to \mathbb{Z}/2$$

that maps $(f_1, f_2) \mapsto I(f_1, f_2)$.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M_2 \arrow[d, "f_2"] \\ M_1 \arrow[r, "f_1"'] & \Sigma \end{tikzcd}$$

The pair $(MO_1(\Sigma), I)$ is called the (mod 2) intersection form of Σ.

21.1 Bilinear forms on MO_1

Suppose V is a vector space over a field k.

Definition 21.1. A **bilinear form** on V is a map $B : V \times V \to k$ satisfying $B(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 B(v, w_1) + \lambda_2 B(v, w_2)$ and $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$. A bilinear form is **symmetric** if $B(x, y) = B(y, x)$. A symmetric bilinear form is **non-degenerate** if for every $v \in V$ there is a w such that $B(v, w) \neq 0$.

Suppose V is finite dimensional, with basis $\{e_1, \ldots, e_n\}$. Then a vector corresponds to a column, and a bilinear form B corresponds to a matrix $\tilde{B}$, given by $B(x, y) = x^T \tilde{B} y$.

Example 21.2. Let $\Sigma = T^2$. Then the bilinear form on $MO_1(\Sigma)$ is

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with a certain basis. Also, if $\Sigma = T^2 \# T^2$ then the bilinear form is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that B is non-degenerate in both cases.

Theorem 21.3. *If Σ is a compact closed 2-manifold then $MO_1(\Sigma)$ is finite dimensional.*

Theorem 21.4. *The form $MO_1(\Sigma)$ is non-degenerate.*

We will prove this later and use it only.

Definition 21.5. If (V_1, B_1) and (V_2, B_2) are symmetric bilinear forms then $(V_1, B_1) \oplus (V_2, B_2)$ is defined as

$$B((v_1, w_1), (v_2, w_2)) = B_1(v_1, v_2) + B_2(w_1, w_2).$$

If B_1 and B_2 are non-degenerate, so is $B_1 \oplus B_2$.

Example 21.6. We let

$$R = \begin{cases} V = \mathbb{Z}/2 \\ B = (1) \end{cases} \qquad \text{and} \quad H = \begin{cases} V = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{cases} .$$

Lemma 21.7. *Suppose (V, B) is a non-degenerate symmetric bilinear form and $W \subset V$. Defined $B_W : W \times W \to k$ as $B_W(x, y) = B(x, y)$. If B_W is non-degenerate, then $(V, B) = (W, B_W) \oplus (W^\perp, B_{W^\perp})$, where $W^\perp$ is the orthogonal complement with respect to the bilinear form.*

21.2 Classification of finite-dimensional non-degenerate symmetric bilinear forms over $k = \mathbb{Z}/2$

First suppose there is a $v \in V$ with $B(v, v) = 1$. Then $W = (v)$ with B_W is just R. Then

$$(V, B) = R \oplus (V', B').$$

Repeating this, we find

$$(V, B) = R^{\oplus a} \oplus (V', B').$$

Here, (V', B') is even, i.e., $B(x, x) = 0$ for any x.

Now choose $0 \neq x \in V$. Since B is non-degenerate, there is y with $B(x, y) = 1$. Also, $B(y, y) = 0$. Then if we set $W = (x, y)$ then $(W, B_W) = H$. This shows that $(V, B) = H \oplus (V', B')$. Hence

$$(V, B) = R^{\oplus a} \oplus H^{\oplus b}.$$

But this is not a full classification, because some of they might be the same. In fact, $R + R + R = R + H$. Summing up, we get the following proposition.

Proposition 21.8. *Every non-degenerate symmetric bilinear form over $\mathbb{Z}/2$ is isometric to one of*

1) $H^{\oplus a}$,
2) $R \oplus H^{\oplus a}$,
3) $R \oplus R \oplus H^{\oplus a}$.

In fact, they are all different because 1) is even and the dimension is even, 2) is odd and the dimension is also odd, and 3) is odd and the dimension is even.

22 March 23, 2016

Last class I talked about symmetric bilinear forms, and now we are going to prove the non-degeneracy assertion and how surgery directly corresponds to operations on symmetric bilinear forms.

22.1 Embeddings represent every class

Theorem 22.1. *Let X be a smooth 2-manifold. Every class in $MO_1(X)$ is represented by an embedding $f : M \to X$.*

We will assume the following theorem of Whitney.

Theorem 22.2 (Whitney)**.** *Every map $M^k \to X^{2k}$ is homotopic to a* ***regular*** *immersion with at most double points. (Here a regular immersion is a map f such that for any $x \neq y \in M$ with $f(x) = f(y)$, the map $df : T_xM + T_yM \to T_{f(x)}X$ is onto.)*

Proof of theorem 22.1. Let us look at the special case $M = \mathbb{R} \amalg \mathbb{R}$, and look at the map $M \to X = \mathbb{R}^2$ that maps one line to the x-axis an the other line to the y-axis.

Now let us do surgery so that this map becomes an embedding. Then the surgery cobordism will guarantee that they are in the same cobordism class. We set $A = \{0_1, 0_2\} \subset M$ and consider the interval $B = [0, 1]$ so that $A \to \partial B$ is given by $0_1 \to 0$ and $0_2 \to 1$. We already have the framing, and we need a thickening $B \times [-1, 1] \to X$ that extends $A \to X$. We can do this by connecting the relevant points by segment. Then we can do a surgery and make it into an embedding.

Now let us go back to the general case. Using Whitney's theorem, we can assume that f is a regular immersion with at most double points. Then locally (in X) $f : M \to X$ either looks like an embedding or there are two points p and q mapping to x such that $T_pM + T_qM = T_xX$. By the standard immersion theorem, they both look like $\mathbb{R}^1 \to \mathbb{R}^2$ locally (in M) and therefore we can use surgery to remove the double point. □

In fact, we can connect things by surgery, and even show that every bordism class is represented by a embedding of a circle.

22.2 Separation theorems

Definition 22.3. A submanifold $Z \subset X$ of codimension k is **cut out** by $\varphi : X \to \mathbb{R}^k$ if 0 is a regular value of f and $f^{-1}(0) = Z$.

Question. *How can we tell when $Z \subset X$ of codimension 2 is cut out by a function $X \to \mathbb{R}$?*

First note that Z is cut out by $f : X \to \mathbb{R}$ and $\partial X = \emptyset$ then $Z = \partial f^{-1}[0, \infty)$. Then $I(Z, f) = 0$ for all $f : M^1 \to X$. But the converse is also true.

Theorem 22.4. *Z is out out by a function φ if and only if $I(Z, f) = 0$ for all $f \in MO_1(X)$.*

Proof. Suppose X is connected, and choose a point $p \in X - Z$. Given $x \in X - Z$, choose a path $\gamma : [-1, 1] \to X$ such that $\gamma(-1) = p$ and $\gamma(1) = x$ and is transverse to Z. Then set $S(x) = I(\gamma, Z)$. Then this actually does not depend of γ, because given any two paths γ and $\tilde{\gamma}$, we can glue them to get a map $f : S^1 \to X$ and

$$0 = I(f, Z) = I(\gamma, Z) + I(\tilde{\gamma}, Z).$$

This defines a function $S : X - Z \to \{0, 1\}$.

This S is locally constant, because given any $x \in X - Z$ we can let V be a connected open neighborhood of x and for every $x' \in V$ the concatenation of γ and the path joining x and x' gives a path with the same intersection number. This shows that S is constant on V. □

23 March 25, 2016

Last time I proved the theorem that if X is a smooth 2-manifold, every $\alpha \in MO_1(X)$ is represented by an embedded 1-manifold $Z \subset X$. In fact you can show in addition that if X is connected then Z can be taken to be diffeomorphic to S^1.

Suppose $Z^{k-1} \subset X^k$ be and embedded manifold.

Definition 23.1. Z is **cut out** by a function $f : X \to \mathbb{R}$ if 0 is a regular value of f and $f^{-1}(0) = Z$.

Theorem 23.2. *The manifold Z is cut out by a function f if and only if $I_Z(\alpha) = 0$ for every $\alpha \in MO_1(X)$.*

Proof. Choose a point $p \in X - Z$ and define a map $S : X - Z \to \{0, 1\}$ by choosing a $\gamma : [-1, 1] \to X$ with $\gamma(-1) = p$ and $\gamma(1) = x$ and $\gamma \pitchfork Z$. Then $S(x) = \#\gamma^{-1}(Z)$ is well-defined since $I(\alpha, Z) = 0$ for any $\alpha \in MO_1 X$. Also, this function is locally constant.

Find a countable number of diffeomorphisms $\Phi_i : U_i \to \mathbb{R}^2$ covering Z such that $Z \cap U_i$ corresponds to the x-axis. Moreover, we can assume that $S(x) = 0$ corresponds to the upper half plane $\{y > 0\}$ in each Φ_i.

For $x \in Z$, define $T_x^+ \subset T_x X$ to be the set of vectors $v \in T_x$ such that $d\Phi_i(v) \in T_{\Phi_i(x)}\mathbb{R}^2 = \mathbb{R}^2$ is in the upper half space. Note that this is independent of the choice of U_i. We can then choose a partition of unity $\{\theta_i\}$ subordinate to $\{U_i\}$ and set $f = \sum \theta_i \varphi_i$, where

$$\varphi_i : U_i \xrightarrow{\Phi_i} \mathbb{R}^2 \xrightarrow{y-\text{coord}} \mathbb{R}.$$

This f works on a neighborhood of Z, because $f(x) = 0$ for $x \in Z$, $f(x) > 0$ for $S(x) = 0$ and $f(x) < 0$ for $S(x) = 1$. Also, $df(v) > 0$ for $v \in T_x^+$, so 0 is a regular value. To make in be defined on X, we throw in the open set $X - Z$ in the cover and do the same thing. □

Now this implies that if Z is diffeomorphic to S^1, then f can be regarded as a connected sum of two manifolds. I will come back to this.

23.1 Non-degeneracy of the intersection form

Theorem 23.3. *If X is a compact closed 2-manifold, the intersection form*

$$(MO_1 X, I(-, -))$$

is non-degenerate.

Proof. We must show that if $I(\alpha, \beta) = 0$ for every β, then $\alpha = 0$. By "special position" we may assume α is represented by an embedded $Z \subset X$. Then by the separation theorem, $Z \subset X$ is cut out by a smooth $f : X \to \mathbb{R}$. The inverse image $f^{-1}([0, \infty))$ is a manifold with boundary over X with boundary equal to Z. Therefore $Z = 0$ inside $MO_1(X)$. □

Theorem 23.4 (Neighborhood theorem). *Suppose $g : S^1 \to X^2$ is an embedding such that $g(S^1) = Z \subset \operatorname{int} X$. If $I(g, g) = 0$ then Z has a neighborhood U diffeomorphic to a cylinder $Z \subset U$. If $I(g, g) = 1$, then Z has a neighborhood U diffeomorphic to the Möbius band.*

Proof. Let N be the normal bundle of Z inside X. By the ϵ-neighborhood theorem, there is a neighborhood U of Z diffeomorphic to N. We must show that either N is diffeomorphic to the cylinder or the Möbius band.

Now let

$$C = \{(x, v) \in NZ : |v| = 1\}$$

and look at the diagram.

$$\begin{tikzcd}
\{0\} \arrow[rr, "{(p,v)}"] \arrow[d] & & C \arrow[d] \\
\mathbb{R} \arrow[r, "e^{it}"'] \arrow[urr, dashed, "\gamma"] & S^1 \arrow[r, "g"'] & Z
\end{tikzcd}$$

Because C is a covering space of Z, by the path lifting property, we have a unique such γ. Then $\gamma(1) = (p, \pm v)$. In each cases, we see that the normal bundle is the cylinder and the Möbius band. $\square$

Example 23.5. Suppose $\alpha \in MO_1(X)$ with $I(\alpha, \alpha) = 1$. Represent α be an embedded circle, take the Möbius band neighborhood Z of α. If we take the the boundary W of Z, then this is diffeomorphic to a circle, and by surgery, we can make it into a connected sum of an $\mathbb{RP}^2$ and something. We will exploit this property more.

24 March 28, 2016

We have the following facts so far.

1. If X a connected surface, every $\alpha \in MO_1(X)$ is represented by an embedded circle.
2. An embedded circle $Z \subset X$ can be cut out by a smooth $f : X \to \mathbb{R}$ if and only if $I(Z, \alpha) = 0$ for all $\alpha \in MO_1(X)$.
3. An embedded circle $Z \subset X$ has a neighborhood U, diffeomorphic to $S^1 \times \mathbb{R}$ if $I(Z, Z) = 0$, and diffeomorphic to the Möbius band if $I(Z, Z) = 1$.

24.1 Classification of compact connected 2-manifolds

Let us recall the classification of non-degenerate symmetric bilinear forms (B, V) over $\mathbb{Z}/2$. If there is a $v \in V$ with $B(v, v) = 1$, then we can decompose V into $R \oplus v^\perp$, and if there isn't, we can choose v, w such that $B(v, w)$ and decompose $V = H \oplus (v, w)^\perp$.

We do the analogous thing for manifolds. We ask the question "Is there a $v \in MO_1(X)$ with $I(v, v) = 1$?" If yes, then represent v by an embedded circle $Z \subset X$. Then there is a neighborhood U of Z such that $U = \mathbb{R} \times \mathbb{R}/(x, y) \sim (x + 1, -y)$. Low let $Z \subset M \subset X$ be the set defined as

$$M = \mathbb{R} \times [-1, 1]/(x, y) \sim (x + 1, -y).$$

Then $W = \partial X$ has intersection number zero with any other α, and hence there is a function f that cut out W. One side of W is the Möbius band, and the other side is some X'.

This is actually the connected sum. Let M_1 and M_2 be two manifolds. We can choose two points each on M_1 and M_2 and take the neighborhood disks of the points and do surgery. The resulting manifold do not depend on the choice of points, because there is a diffeomorphisms of a manifold that sends any point to any point. It might depend on the choice of the framing; essentially the orientation might be different. In fact, when you do surgery on two points on S^2, the resulting manifold might become a torus of a Klein bottle depending on the orientation. But if M_1 and M_2 are not connected, then orientation does not affect the surgery. Therefore, the **connected sum** $M_1 \# M_2$ is well-defined as the result of the surgery.

Given a $M = M_1 \# M_2$, we can even disconnect them using surgery. Choose a circle on the cylinder we have attached, choose a neighborhood that looks like the cylinder, remove a cylinder inside the cylinder, and attach disks. Then we get M_1 and M_2 back.

Now given a function f that cut out W, we can do surgery on W. The Möbius band gets attached a disk, and it becomes $\mathbb{R}P^2$. That is, given a $v \in MO_1(X)$ with $I(v, v) = 1$, we can write $X = \mathbb{R}P^2 \# X'$. Moreover, it is possible to prove

$$MO_1(X_1 \# X_2) = MO_1(X_1) \oplus MO_1(X_2).$$

Then we get
$$MO_1(X) = MO_1(\mathbb{R}P^2) \oplus MO_1(X').$$

Continuing this process, we get

$$X = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \# X'$$

such that $I(\alpha, \alpha) = 0$ for any $\alpha \in MO_1(X')$. Choose $x, y \in MO_1(X')$ such that $I(x, y) = 1$, and we can represent x, y by embedded circles Z_1, Z_2 which meet only once. Because $I(Z_1, Z_1) = I(Z_2, Z_2) = 0$, both Z_1 and Z_2 thickens up to two bands intersecting at a single part. Then the union of the two bands is a punctured torus, and thus using the same argument, we can represent X' as a connected sum of the torus T and some other manifold. That is, continuing the process, we can write

$$X = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \# T \# \cdots \# T \# X',$$

where $MO_1(X') = 0$.

Theorem 24.1. *If X is a compact closed 2-manifold with $MO_1(X) = 0$ then $X = S^2$.*

This theorem, whose proof uses Morse theory, shows that X is the connected sum of $\mathbb{R}P^2$s and Ts. In fact, we have proven something stronger.

Theorem 24.2. *Suppose X is a compact connected closed 2-manifold and we have chosen an isometry*

$$R \oplus \cdots \oplus R \oplus H \oplus \cdots \oplus H \to (MO_1(X), I).$$

Then $X = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \# T \# \cdots \# T$ wit he the standard basis going to the stand basis.

Theorem 24.3. *If X_1 and X_2 are compact connected closed 2-manifolds and*

$$T : MO_1(X_1) \to MO_1(X_2)$$

is an isometry then there is a diffeomorphism $X_1 \to X_2$ inducing T by $MO_1(-)$.

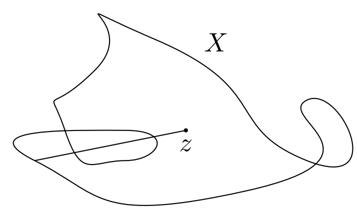
25 March 30, 2016

For the next few lectures, we are going to do winding numbers and the Jordan-Brouwer separation theorem. Classically, it states

Theorem 25.1 (Jordan curve theorem)**.** *Every simple closed curve in $\mathbb{R}^2$ divides the plane into two pieces.*

25.1 Winding numbers

Let X be a contact, connected $(n-1)$-manifold and let $f : X \to \mathbb{R}^n$ be a smooth map. The winding number "studies how f wraps X round $\mathbb{R}^n$."



Pick a point z not on X and consider the function $u(x) = (x - z)/|x - z|$. Then this is a map $u : X \to S^{n-1}$, and because they are both of dimension $(n - 1)$, mod 2 intersection theory says that the intersection number $I(u, p)$ is well-defined for a point p. This is independent of p and is called the **degree of** u and we define the **winding number** as

$$W_2(X, Z) = \deg_2(u) = I(u, p).$$

There are several things to verify.

Proposition 25.2. *Let U be a connected component of $\mathbb{R}^n - X$. If $p_0, p_2 \in U$, then $W_2(X, p_0) = W_2(X, p_1)$.*

Proof. First assume that p_0 and p_1 can be joined by a segment. Let $p_t = (1 - t)p_0 + tp_1$ and look at the homotopy $F : X \times I \to S^{n-1}$ given by

$$(x, t) \mapsto \frac{x - p_t}{|x - p_t|}.$$

Then we see that the winding number for $t = 0$ and $t = 1$ are the same.

So it suffices to show that any two points can be connected by a finite number of segments. Let

$$A = \{\text{the points that can be joined by a sequence of line segments}\}$$

and $B = U \setminus A$. Then both B and A are open, and because U is not connected, it follows that $B = \emptyset$. $\square$

Proposition 25.3. *Let $X \subset \mathbb{R}^n$ be an $(n-1)$-dimensional submanifold. Then set $\mathbb{R}^n - X$ has at most 2 connected components.*

Proof. We first show that for any $q \in X$, there exists a $B_\epsilon(q)$ such that $B_\epsilon(q) - X$ has two connected components. Let $f : X \to \mathbb{R}^n$ be the inclusion map. Because clearly f is an immersion at q, there exist charts such that the diagram commutes

$$\begin{tikzcd} U_1 \arrow[r, hook, "f"] \arrow[d, "\varphi_1"] & U_2 \arrow[d, "\varphi_2"] \\ V_1 \arrow[r, hook] & V_2 \end{tikzcd}$$

and $V_1 \to V_2$ is the inclusion map. Then the hyperplane divides the space into two parts, and therefore $B_\epsilon(q) - X$ will have two components.

Now for any $p \in \mathbb{R}^n - X$ and $q \in X$ with a neighborhood $q \in U$, there is a path joining p and U. This is because the set of points $q \in X$ such that the statement is true is a nonempty open and closed subset of X. Then if we let $U = B_\epsilon(q)$ it follows that $\mathbb{R}^n - X$ has at most two components. $\square$

26 April 4, 2016

We used surgery to do the classification of closed compact 2-manifolds.

Theorem 26.1. *If M is a compact closed connected 2-manifold then $MO_1(M)$ is finite dimensional.*

Theorem 26.2. *If (V, B) is a nondegenerate symmetric bilinear form over $\mathbb{Z}/2$, then there exists an M^2 with $(MO_1(M), I) \cong (V, B)$.*

Theorem 26.3. *If M_1, M_2 are compact closed connected 2-manifolds and there exists an isometry $MO_1(M_1) \to MO_1(M_2)$, then there is a diffeomorphism $f : M_1 \to M_2$ inducing T on MO_1.*

The first one is not proved, the second is proved, and the third is almost proved with a few loose ends. We need the following theorem.

Theorem 26.4. *If $MO_1(M) = 0$ then M is diffeomorphic to S^2.*

Example 26.5. Becasue the intersection form of $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$ is isometric to the intersection form of $\mathbb{R}P^2\#T$, we see that they are diffeomorphic.

Example 26.6. The bordism homology of the Klein bottle is $(\mathbb{Z}/2)\langle e_1, e_2\rangle$ with $B(e_1, e_1) = B(e_1, e_2) = 1$, $B(e_2, e_2) = 0$. This is isometric to $MO_1(\mathbb{R}P^2) \oplus MO_1(\mathbb{R}P^2)$ and hence the Klein bottle is diffeomorphic to $\mathbb{R}P^2\#\mathbb{R}P^2$.

There are a few more things I have to prove: the connected sum is well-defined, and sowing a disk on the boundary of a Möbius band always gives an $\mathbb{R}P^2$ no matter how you glue it.

26.1 More on Gluing

Consider

$$M \underset{f_1}{\overset{f_0}{\rightrightarrows}} N$$

diffeomorphisms with distinct boundary components. So for instance, M maps to two boundary circles of N. Let us now glue the manifolds along the boundary as

$$N' = N/(f_0(x) = f_1(x)).$$

The new manifold N' is a topological manifold, but it does not have a smooth structure.

Theorem 26.7. *The space N' can be given the structure of a smooth manifold for which the map $N \to N'$ is smooth. Moreover anay two smooth structures are diffeomorphic by a diffeomorphism which is the identity on M and the identity outside a specified neighborhood of M.*

Theorem 26.8. *Any diffeomorphism of S^1 extends to a diffeomorphism of D^2.*

This is something special to dimension 2 and somthing strange happens once we get to dimenison 6.

These two theormes imply that any two manifold obtained from the Möbius band by attaching a disk is diffeomorphic to $\mathbb{R}P^2$. Suppose there are two boundary maps $f, f' \to \partial M$ that are related by a diffeomorphism $g = f^{-1} \circ f'$. Then we can extend g to a diffeomorphism of the disk:

$$\begin{tikzcd}
\partial M & S^1 \arrow[l, "f"'] \arrow[d, hook] & S^1 \arrow[l, "g"'] \arrow[d, hook] \\
 & D^2 & D^2 \arrow[l, "\tilde{g}"']
\end{tikzcd}$$

Then it follows that the result is independent of f.

26.2 Isotopy of diffeomorphisms

Let

$$M \overset{f_0}{\underset{f_1}{\rightrightarrows}} M$$

be two diffeomorphisms.

Definition 26.9. An **isotopy** of f_0 and f_1 is a smooth map

$$h : M \times [0,1] \to M$$

such that $h(x,0) = f_0(x)$, $h(x,1) = f_1(x)$, and $h_t : M \to M$ is a diffeomorphism for each $0 \leq t \leq 1$.

Clearly isotopy is reflexive and symmetric. The only thing that might require a proof is transitivity. This can be attained by gluing the two isotopies and slowing down using a bump function at the midpoint.

Theorem 26.10. *Every diffeomorphism of S^1 is isotopic to either $(x,y) \mapsto (x,y)$ or $(x,y) \mapsto (x,-y)$.*

Proof. Consider a diffeomorphism $\phi : S^1 \to S^1$. There is a covering map

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[d] & & \mathbb{R} \arrow[d, "e^{it}"] \\
\mathbb{R}^1 \arrow[r, "e^{it}"] \arrow[urr, dotted, "f"] & S^1 \arrow[r, "\phi"] & S^1
\end{tikzcd}$$

and by the lifting property, there exists a unique f. This funciton f satisfies $f(0) = 0$ and $f'(x) \neq 0$ for every x.

If $f'(0) > 0$, because $f(0) = 0$ the intermediate value theorem and some more arguments imply that $f(2\pi) = 2\pi$ and $f(x + 2\pi k) = f(x) + 2\pi k$. We can then simply set

$$\tilde{h}(x,t) = (1-t)f(x) + tx$$

and $h(\cos\theta, \sin\theta, t) = (\cos\tilde{h}(\theta, t), \sin\tilde{h}(\theta, t))$ gives an isotopy between ϕ and the identity map.

Likewise in the $f'(0) < 0$ case, we can do the same thing and get a isotopy with $(x, y) \mapsto (x, -y)$. $\square$

Proof of theorem 26.8. We can look at the isotopy $S^1 \times [0, 1] \to S^1$ between ϕ and one of the two standard maps and map it into a neighborhood of S^1 in D^2. Then we can fill in the inner circle as the identity map. $\square$

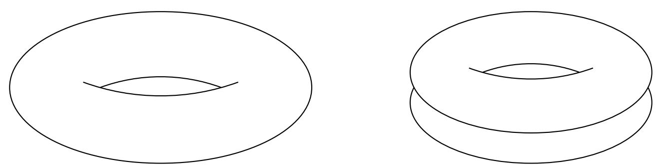
27 April 6, 2016

27.1 Immersions of surfaces

Let X be a closed compact connnected 2-manifold. Our goal is to classify all immersions $f : X \looparrowright \mathbb{R}^3$, up to isotopy and diffeomorphism.

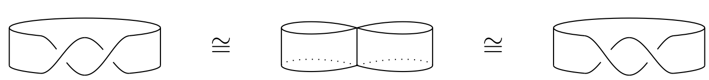
Definition 27.1. An **isotopy of immersions** $X \to Y$ is a map $h : X \times [0,1] \to Y$ such that for any t the map $h_t : X \to Y$ is an immersion.

Example 27.2. There are several immersions of the torus. One way is to map it to an ordinary torus. Another way is to first twist one direction into a figure eight and then connect the two ends. This will look like some kind of two tori attached together. When we connect the two ends, we can even twist it into a figure eight form.



Given an $\alpha \in MO_1(X)$, choose an embedded circle z on X representing α. Assume that $I(z,z) = 0$. We know that there is a neighborhood of z diffeomorphic to $S^1 \times [-1,1]$. Then $f : X \looparrowright \mathbb{R}^3$ restrics to an immersion of $S^1 \times [-1,1] \looparrowright \mathbb{R}^3$.

By moving things around, we can show that every such immersion is isotpic to an embedding that looks like a some kind of Möbius band with k full-twists. This can be given a structure of a group, by defining addition as the result of surgery. Moreover, the group is generated by "1 twist" and actually "1 twist" is isotopic to "-1 twist."



But we don't really know whether the "0 twist" and the "1 twist" are distinct. We need an invariant to check this.

27.2 Linking number

Suppose that $M_1, M_2 \hookrightarrow \mathbb{R}^3$ be embeddings of closed 1-manifolds that are disjoint. We can always write $M_1 = \partial N$. We have shown last time that it is alwasys possible to extend the map so that

$$\begin{tikzcd} M_1 \arrow[r, hook] \arrow[d] & N \arrow[dl, "f"] \\ \mathbb{R}^3 & \end{tikzcd}$$

with $f \pitchfork M_2$. We define the **linking number** of M_1 and M_2 as $L(M_1, M_2) = I(f, M_2)$.

Proposition 27.3. *The intersection number $I(f, M_2)$ is independent of the choice of N.*

Proof. Suppose we have two N' and N''. Then $N' \cup N''$ is a closed 2-manifold. Let f, f', f'' be the immersions N, N', N''. Then

$$I(f', M_2) + I(f'', M_2) = I(f, M_2) = 0$$

because M_2 is a boundary of something. □

It can also be proved that the linking number is invariant under isotopy.

Now given an embedding $f : S^1 \times [-1, 1] \hookrightarrow \mathbb{R}^3$, we can look at the $L(f|_{S^1\times\{0\}}, f|_{S^1\times\{1\}})$. In the "0 twist" embedding, this number is 0 and in the "1 twist" embedding, the linking number is 1. This shows that the two are not isotopic.

In a general setting, consider a 2-manifold X that has even intersection form $(MO_1(X), I)$. Given an immersion $f : X \looparrowright \mathbb{R}^3$, it induces a function

$$\phi : MO_1(X) \to \mathbb{Z}/2,$$

where $\phi(\alpha)$ is the isotopy class of the immersed cylinder around an embedded circle representing α.

A famous mistake of Pontryagin is assuming that ϕ is linear. But actually it is not.

Example 27.4. Consider the two circles z_1 and z_2 on the torus. Then clearly $\phi(z_1) = \phi(z_2) = 0$. But it can be checked that $\phi(z_1 + z_2) = 1$.

In fact, ϕ is a quadratic form; it satisfies the identity

$$\phi(x + y) - \phi(x) - \phi(y) = I(x, y).$$

28 April 8, 2016

Last time I drew three different immersions of the torus. You can just map it to the ordinary torus, or you can make a figure-eight torus, or a figure-eight-figure-eight torus.

Let X be a connected compact closed 2-manifold, and assume that $(MO_1(X), I)$ is even. We showed that given an immersion $X \looparrowright \mathbb{R}^3$, it incudes a quadraic function $q : MO_1(X) \to \mathbb{Z}/2$ giving a bilinear form

$$q(x+y) - q(x) - q(y) = I(x,y).$$

If we look at the quadratic forms of the torus, figure-eight torus, figure-eight-figure-eight torus, we have

$$q\begin{pmatrix} 0 & e_1 \\ e_2 & e_1+e_2 \end{pmatrix} = \begin{array}{ccc} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\ \\ \text{torus} & \text{f-e torus} & \text{f-e-f-e torus.} \end{array}$$

The theroem then tells us that the torus and the figure-eight torus are isotopic, where as the figure-eight-figure-eight torus is not.

28.1 Quadratic forms

Let V be a vector space over a field k.

Definition 28.1. A **quadratic function** on V is a function $q : V \to k$ such that

$$B(x,y) = q(x+y) - q(x) - q(y)$$

is bilinear and $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in k$.

Example 28.2. Let $V = k^n$. The map

$$q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i \le j} a_{ij} x_i x_j$$

is a quadratic form.

Definition 28.3. A quadratic function is **non-degenerate** if $B(-,-)$ is non-degenerate.

We call that q is a **quadratic refinement of** B if B is the bilinear form associated to q.

Proposition 28.4. *If q_1 and q_2 are refinemenets of B, then $q_1 - q_2$ is linear.*

Proof. It follows from that $(q_1(x+y) - q_1(x) - q_1(y)) - (q_2(x+y) - q_2(x) - q_2(y)) = B(x,y) - B(x,y) = 0$. ∎

Corollary 28.5. *The set of quadratic refinements of B is an affine space for* $\operatorname{Hom}(V,k)$*. So if $k = \mathbb{Z}/2$ and $\dim V = n$ then there are 2^n quadratic refinemenets of any fixed B.*

Definition 28.6. A **quadratic form** (over k) is a pair (V,q) consisting of a vector space V and a quadratic function $q : V \to k$.

Definition 28.7. The orthogonal sum $(V_1,q_1) \oplus (V_2,q_2)$ is the vector space $V_1 \oplus V_2$ equipped with $(q_1 \oplus q_2)(x,y) = q_1(x) + q_2(y)$.

Definition 28.8. An **isometry** $(V_1,q_1) \to (V_2,q_2)$ is a linear transformation $T : V_1 \to V_2$ such htat $q_2(Tx) = q_1(x)$ for all $x \in V_1$.

Now we want to classify quadratic forms over k up to isometry. Over $\mathbb{R}$, every q is isometic to

$$q(x_1,\dots,x_n) = x_1^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_n^2.$$

But what happens to $k = \mathbb{Z}/2$? First, because q is nondegenerate and

$$B(x,x) = 2q(x) = 0,$$

we see that $(V,B) = H^{\oplus m}$ for some m, and in particular $\dim V = 2m$.

Lemma 28.9. *Suppose (V,q) is a non-degenerate quadratic form over $\mathbb{Z}/2$, with $\dim V > 2$. Then there is a $0 \neq v \in V$ such that $q(v) = 0$.*

Proof. Because $\dim V$ is even, it is at least 4. Let $e_1,\dots,e4$ be linearly independent vectors. If $q(v) = 1$ for all $v \neq 0$, then $B(e_1,e_2) = B(e_3,e_4) = 1$ and then

$$q(e_1 + \cdots + e_4) = q(e_1+e_2) + q(e_3+e_4) + 0 = 1 + 1 + 0 + 1 + 1 + 0 = 0.$$

$\square$

Consider a $0 \neq v$ such that $q(v) = 0$. Since B is nondegenerate, there is a v such that $B(v,w) = 0$. Then $q(v+w) = q(w)+1$ and hence we can assume that one is 1, and we might as well suppose that it is w. Let $V' = \{0,v,w,v+w\}^\perp$, and let

$$H_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_- = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

be the standard 2-dimensional quadratic forms. We still have to show that they are not isometric.

Proposition 28.10. *The quadratic form (V,q) can be decomposed into $(V,q) = (V',q') \oplus H_+$.*

It folows that (V,q) is either of the form $H_+^{\oplus m}$ or $H_- \oplus H_+^{\oplus(m-1)}$.

Definition 28.11. Given a nondegenerate quadratic form (V,q), we define its **Arf invariant**

$$\operatorname{Arf}(q) = \frac{1}{\sqrt{|V|}} \sum_{v \in V} (-1)^{q(v)}.$$

Then $\operatorname{Arf}(q_1 \oplus q_2) = \operatorname{Arf}(q_1)\operatorname{Arf}(q_2)$. It follows that $\operatorname{Arf}(H_+^{\oplus m}) = 1$ and $\operatorname{Arf}(H_- \oplus H_+^{\oplus(m-1)}) = -1$.

29 April 11, 2016

Last time, we showed that if M is an oritentable closed compact connected 2-manifold, then there is a bijection

$$\left\{\begin{matrix}\text{immersions } M \looparrowright \mathbb{R}^3 \\ \text{modulo isotopy}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{quadratic refinement} \\ q : MO_1(M) \to \mathbb{Z}/2 \\ \text{of the intersection form}\end{matrix}\right\}$$

and there are 2^{2m} of these, where $2m$ is the dimension of $MO_1(M)$. When we look at immersions up to diffeomorphism, we have to take the quotient by isometry on the right hand side.

$$\left\{\begin{matrix}\text{immersions } M \looparrowright \mathbb{R}^3 \\ \text{modulo isotopy}\end{matrix}\right\}\Big/\text{diff.} \longleftrightarrow \left\{\begin{matrix}\text{quadratic refinement} \\ q : MO_1(M) \to \mathbb{Z}/2 \\ \text{of the intersection form}\end{matrix}\right\}\Big/\text{isom.}$$

Then there are 2 of them, characterized by the Arf invariant. Now let us look at the case when M is not orientable.

29.1 Quadratic forms taking values in $\mathbb{Z}/4$

If M is not orientable, then the surface M contains a Möbius band. There are two kinds of Möbius bands: the one with the right twist, and the one with the left twist. These two are different, because if you add the same Möbius by surgery then you get the 1-full-twist band, where as if you add the different Möbius bands, then you get the cylinder. So, each loop in M thickens up to an element of $\mathbb{Z}/4$.

Hence each immersion $M \looparrowright \mathbb{R}^3$ gives a quadratic $q : MO_1(M) \to \mathbb{Z}/4$ that satisfies

$$q(x + y) - q(x) - q(y) = 2I(x, y).$$

Then we have a similar theorem.

Theorem 29.1. *The set of immmersions $M \looparrowright \mathbb{R}^3$ modulo isotopy is in bijection with the set of quadratic $q : MO_1(M) \to \mathbb{Z}/4$ refining I.*

Definition 29.2. A **quadratic form** is pair (V, q) where V is a finite dimensional vector space over $\mathbb{Z}/2$ and $q : V \to \mathbb{Z}/4$ such that

$$q(x + y) - q(x) - q(y) = B(x, y)$$

is bilinear and $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in \mathbb{Z}$.

Example 29.3. There are two 1-dimensional nondegenerate quadratic forms. Let $R = \{0, e\}$ and $B(e, e) = 1$. Then there can be two

$$R_+ = (\mathbb{F}_2, q), \quad q(e) = 1,$$
$$R_- = (\mathbb{F}_2, q), \quad q(e) = -1.$$

Example 29.4. Let us look at the Klein bottle $R \oplus R$ with bases e_1 and e_2. Then there are four possibilities comming from $R_\pm \oplus R_\pm$.

Just like before, any two quadratic refinemenets of a fixed symmetric bilinear form differ by a linear function. Hence there are $2^{\dim V}$ quadratic refinements of a fixed symmetric bilinear form.

29.2 Brown-Arf invariant

Looking at the quadratic forms up to isometry, we see that $R_+ \oplus R_-$ and $R_- \oplus R_+$ are isometric (clearly), but no other are isometric. We can define an invariant.

Definition 29.5. Given a quadratic form (V, q), we define the **Brown-Arf invariant**

$$\operatorname{Arf}(q) = \frac{1}{\sqrt{|V|}} \sum_{v \in V} i^{q(v)}.$$

Again, it is not hard to check that $\operatorname{Arf}(q_1 \oplus q_2) = \operatorname{Arf}(q_1) \operatorname{Arf}(q_2)$.

Example 29.6. For R_+, its Arf invariant is

$$\frac{1}{\sqrt{2}}(i^0 + i^1) = \frac{1+i}{\sqrt{2}}.$$

Any symmetric bilinear form looks like either $R \oplus \cdots \oplus R$ or $H \oplus \cdots \oplus H$. It can be proved that the possible non-degenerate quadratic forms are

$$R_+^{\oplus k} \oplus R_-^{\oplus l} \qquad \text{or} \qquad H_+^{\oplus n} \oplus H_-^{\oplus m}.$$

(You can do the same thing, but you have to work a bit more.) Let me just state the theorem.

Theorem 29.7. *Two quadratic forms (V_1, q_1) and (V_2, q_2) are isometric if and only if (V_1, B_2) and (V_2, B_2) are isometric and the Arf invariants agree.*

We can easily calculate

$$\operatorname{Arf}(R_+) = \zeta, \quad \operatorname{Arf}(R_-) = \zeta^{-1}, \quad \operatorname{Arf}(H_+) = 1, \quad \operatorname{Arf}(H_-) = -1,$$

where $\zeta = e^{2\pi i/8}$.

Next time we will do talk some more and draw some pictures, and after that we will go back to our book.

30 April 13, 2016

We have seen last time that a quadratic form is determined up to isometry by the Arf invariant. Each surface is diffeomorphic to one of $H^{\oplus n}$ or $R \oplus H^{\oplus n}$ or $R \oplus R \oplus H^{\oplus n}$. For each case, the possible quadratic forms are

Quadratic form	Arf
$H_+^{\oplus n}$	$+1$
$H_- \oplus H_+^{\oplus(n-1)}$	-1

Quadratic form	Arf
$R_+ \oplus H_+^{\oplus n}$	ζ^1
$R_- \oplus H_+^{\oplus n}$	ζ^{-1}
$R_+ \oplus H_- \oplus H_+^{\oplus(n-1)}$	ζ^{-3}
$R_- \oplus H_- \oplus H_+^{\oplus(n-1)}$	ζ^3

Quadratic form	Arf
$R_+ \oplus R_+ \oplus H_+^{\oplus n}$	ζ^2
$R_- \oplus R_+ \oplus H_+^{\oplus n}$	ζ^0
$R_- \oplus R_- \oplus H_+^{\oplus n}$	ζ^{-2}
$R_+ \oplus R_+ \oplus H_- \oplus H^{\oplus(n-1}$	ζ^{-2}
$R_- \oplus R_+ \oplus H_- \oplus H_+^{\oplus(n-1)}$	ζ^4
$R_- \oplus R_- \oplus H_- \oplus H_+^{\oplus(n-1)}$	ζ^2

So, there are in total 2 or 4 different immersions up to isotopy and diffeomorphism.

30.1 Boy's surface and immersions of surfaces

In the case of S^2, the bordism homology has dimension 0. So it should have a unique immersion up to isotopy. This is actually pretty hard to prove, and the key part of this statement is that the sphere can be eversed. How do we do this? We can push one side of the sphere and take it back to the other side, and this will give us a cylider with some kind of carloop in the middle of each strip. This can be shown to be isotopic with a Dehn twist on the middle. Then we can untwist it to make a inverted sphere.

So, there is a unique immersion of the sphere up to isotopy. Let us now look at the dimension 1 case. The only such surface is $\mathbb{R}P^2$. The immmersion of $\mathbb{R}P^2$ in $\mathbb{R}^3$ is called Boy's surface. Hilbert assgined proving that $\mathbb{R}P^2$ cannot be immersed in 3-space as an assignment, but it was in fact false and Boy found an immersion.

The surface can be visualized as in figure 2.[6] Note that this image is chiral. This means that that there is a mirror reflection of this image, and the two immersions Boy_+ and Boy_- will be the immersions of $\mathbb{R}P^2$.

For the torus, we know there are 4 different immersions. The other immersions will be described as connected sums of these up to isotopy and diffeomorphism.

[6] Hopkins actually gave a much more detailed description of the surface, but it is impossible to draw it.

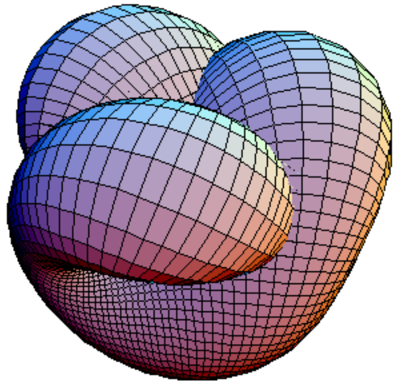


Figure 2: Boy's surface from the top view. You can see the 3-fold symmetry.

We can try to do surgery on the surface. For instance, the ordinary Klein bottle, after doing surgery on the handle part, retracts to a ball. This gives a way to easily see that the Arf invariatn of the Klein bottle is $+1$.

Theorem 30.1. *Let (V, q) be a nondegenerate $\mathbb{Z}/2$-valued quadratic form, and let $\alpha \in V$ be such that $q(\alpha) = 0$. There is a natural quadratic form (V', q') where $V' = \alpha^{\perp}/\alpha$. Then $\operatorname{Arf} q = \operatorname{Arf} q'$.*

This theorem implies that there are 8 surfaces immersed in $\mathbb{R}^3$ up to surgery. I have listed above and those will be the 8 surfaces.

31 April 15, 2016

So I want to make it easy on you by going back to be book and doing integration on manifolds.

Green's theorem says that for a region X and a loop γ as a boudary of X, we have

$$\oint_\gamma pdx + qdy = \iint_X \frac{\partial q}{\partial y} - \frac{\partial p}{\partial x} dxdy.$$

This is in the form of

$$\int_{\partial M} \text{something} = \int_M \text{something else(=some kind of derivative)}.$$

Note that integration involves

(1) an orientation,

(2) the thing you integrate, which is a differential form.

Let us think about what a differential form must look like. One observation is that it should not be a function. If we have a fuction $g(x, y)$ on $\mathbb{R}^2$ and a diffeomorphism (f_1, f_2), the interation of the pullback is different from the original integration.

$$\iint g(x, y)dxdy \neq \iint g(f_1(s, t), f_2(s, t))dsdt.$$

Instead, we have the formula

$$\iint g(x, y)dxdy = \iint g(f_1(s, t), f_2(s, t))|dF|dsdt.$$

31.1 Multilinear algebra

Let V be a vector space, and let

$$T : V \times \cdots \times V \to \mathbb{R}$$

be a multilinear form, i.e., a function linear in each factor. You can add multilinear forms, and therefore multilinear forms form a vector space. Moreover, the dimension of the vector space is $(\dim V)^n$. In our book, this is called a rank n **tensor**. In the literature, tensors are multilinear maps $V^n \times V^{*m} \to \mathbb{R}$.

An **alternating tensor** is a mutilinear map $T : V^n \to \mathbb{R}$ such that

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

More generally, for any permuation σ,

$$T(v_{\sigma_1}, \dots, v_{\sigma_n}) = (-1)^\sigma(v_1, \dots, v_n).$$

Proposition 31.1. *Suppose V has dimension n and $T : V^n \to \mathbb{R}$ is alternating. Then*
$$T(Sv_1, \dots, Sv_n) = \det(S)T(v_1, \dots, v_n)$$
for every linear $S : V \to V$.

There is another piece of algebra I need to teach you.

Let $S : V^k \to \mathbb{R}$ and $T : V^\ell \to \mathbb{R}$ be tensors of rank k, l. Then you can construct a bilinear map $S \otimes T : V^k \times V^\ell \to \mathbb{R}$ given by
$$S \otimes T(v, w) = S(v)T(w).$$

We can specifically focus on alternating tensors. Let us define the **exterior power**
$$\Lambda^k(V^*) = \{\text{alternating } T : V^k \to \mathbb{R}\}.$$
Likewise, we have a map $\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$. But the naïve guess doesn't work here. We need to somehow skew symmetrize the tensor.

Suppose $T : V^n \to \mathbb{R}$ is a rank n tensor. We will define
$$\operatorname{Alt}(T)(v_1, \dots, v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma T(v_{\sigma_1}, \dots, v_{\sigma_1}).$$
(The factor $1/n!$ arranges things so that if T were already alternating then $\operatorname{Alt} T = T$.) Given this, we can define
$$S \wedge T = \operatorname{Alt}(S \otimes T).$$

Example 31.2. Let V be a vector space with basis $e_1, \dots, e_n$. Then $\Lambda^1(V^*) = V^*$ has basis $e_1^*, \dots, e_n^*$. Next, $\Lambda^2(V^*)$ has basis $e_1^* \wedge e_2^*, \dots, e_{n-1}^* \wedge e_n^*$.

Proposition 31.3. *The space $\Lambda^k(V^*)$ has basis $\{e_{i_1}^* \wedge \dots e_{i_k}^*\}$ for $1 \le i_1 < \dots < i_k \le n$, and thus has dimension $\dim \Lambda^k(V^*) = \binom{n}{k}$.*

32 April 20, 2016

To define an integral, which is a number, we need a n-manifold M, an n-form $\omega \in \Omega^n(M)$, and an orientation of M. Then you can define $\int_M \omega$ as a number.

For instance, let $M = \mathbb{R}^1$ and $\omega = f(x)dx = e^{-x^2}dx$. You don't know what $\int_{\mathbb{R}} e^{-x^2}dx$ is, not be cause you don't know how to integrate, but because haven't told you whether you're integrating from $-\infty$ to ∞ or from ∞ to $-\infty$. In most cases, we give the orientation $1 \in T_x\mathbb{R}^1 = \mathbb{R}$.

Definition 32.1. Suppose $\omega \in \Omega^n(M)$ is an n-form. The **support** of ω is the closure of the set of all $x \in M$ for which $\omega_x \neq 0 \in \Lambda^n(T_x^*M)$.

If the support is finite, then $\int_{\mathbb{R}} \omega = \int_a^b \omega < \infty$.

32.1 Integration on manifolds

Now our goals is defining the integral. Consider a diffeomorphism $f : U \to V \subset \mathbb{R}^n$ and we have learned in our nth semester of calculus that

$$\int_V g(x_1, \dots, x_n)dx_1 \cdots dx_n = \int_U g(f(t_1, \dots, t_n))|\det df|dt_1 \cdots dt_n.$$

Now there is a quote by Grothendieck that goes like "Let mathematics be what it wants to be." We try to define things as naturally as possible. If we let

$$\omega = g(x_1, \dots, x_n)dx_1 \wedge \cdots \wedge dx_n$$

and U and V are oriented with $f : U \to V$ preserving orientation, then we can define the pull-back

$$f^*\omega = g(f(t_1, \dots, t_n)) \det(df)dt_1 \wedge \cdots \wedge dt_n.$$

Since f preserves orientation, $\det(df) = |df|$. Then we can write the change of variables law as

$$\int_V \omega = \int_U f^*\omega.$$

Here, in the notation, the orientation of both U and V are implicitly contained. This is not the best possible mathematical practice, but this is what people do.

We realize our goal when M is diffeomorphic to an open subset of $\mathbb{R}^n$, because if $\Phi : M \to V \subset \mathbb{R}^n$ is the diffeomorphism, we can simply define

$$\int_M \omega = \int_V (\Phi^{-1})^*\omega.$$

Moreover, even if $\omega \in \Omega^n$ and $\operatorname{supp}\omega$ is in a coordinate neighborhood U, then we can also define the integral because we can restrict the manifold to $\operatorname{supp}\omega$.

What if M is arbitrary? We choose a countable covering $\{U_i\}_{i=1}^{\infty}$ of M by coordinate neighborhoods $\Phi_i : U_i \to \mathbb{R}^n$. Then construct a partition of unity $\{\theta_i\}$ subordinate to the covering, and define

$$\int_M \omega = \int_M \sum_i \theta_i \omega = \sum_i \int_M \theta_i \omega.$$

Each $\int_M \theta_i \omega$ has been defined because $\operatorname{supp}(\theta_i \omega) \subset U_i$.

We need to show that the integral is well-defined.

Proposition 32.2. *$\int_M \omega$ is independent of the choice of $\{\theta_i\}$.*

Proof. Suppose $\{\tilde{\theta}_i\}$ is another partition of unity. Then we have

$$\sum_i \int_M \theta_i \omega = \sum_{i,j} \int_M \theta_i \tilde{\theta}_j \omega = \sum_j \int_M \tilde{\theta}_j \omega. \qquad \square$$

Example 32.3. Consider a curve $\gamma : [0,1] \to \mathbb{R}^2$ and consider a 1-form $pdx + qdy \in \Omega^1(\mathbb{R}^2)$. We can integrate the 1-form over the curve by

$$\oint_\gamma pdx + qdy - \int_0^1 \gamma^*(pdx + qdy).$$

Example 32.4. Let us try integrate a 2-form $\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy$ on a surface given as the image of

$$X = (X^1, X^2, X^3) : U \to \mathbb{R}^3.$$

The pull back of for instance $dx \wedge dy$ is

$$\begin{aligned} X^*(dx \wedge dy) &= (X^* dx) \wedge (X^* dy) = (X_u^1 du + X_v^1 dv) \wedge (X_u^2 du + X_v^2 dv) \\ &= (X_u^1 X_v^2 - X_v^1 X_u^2) du \wedge dv. \end{aligned}$$

If you compute everything, you get something like

$$X^* \omega = \vec{f} \cdot (X_u \times X_v).$$

This is why you were taught to integrate vector fields in you calculus course. This theory of forms encompasses all the area component, right hand rule, etc. in simply pulling back forms.

33 April 22, 2016

Today I want to introduce Stokes' theorem.

33.1 Stokes theorem

One version is the fundamental theorem of calculus

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Another version is

$$\oint_{\partial R} pdx + qdy = \iint_R (p_y - q_x)dxdy.$$

Stokes theorem just says for a manifold M^n and $\omega \in \Omega^{n-1}(M)$,

$$\int_{\partial M} d\omega = \int_{\partial M} \omega.$$

I have to explain what d is, and I have to explain what the orientation is.

Suppose M has an orientation, and let's see what happens at the boundary. We see that $T_x\partial M \oplus \mathbb{R} = T_xM$. We orient M by giving the same orientation of $\{n_{\text{out}}, e_1, \ldots, e_{n-1}\}$ to $\{e_1, \ldots, e_{n-1}\}$, where $e_1, \ldots, e_{n-1}$ form a basis for $T_x\partial M$ and n_{out} is the normal vector pointing outwards.

Now that we know the orientation for ∂M, we need a operator for taking the derivative of differential forms.

Theorem 33.1. *There is a unique linear operator $d : \Omega^{k-1}(M) \to \Omega^k(M)$ satisfying*

(1) $d(f) = df$, for any 0-form, i.e., function, f,

(2) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta$, (Leibniz rule)

(3) $d^2 = 0$,

(4) any $g : M \to N$ induces a $g^* : \Omega^k(N) \to \Omega^k(M)$ such that $dg^*\omega = g^*d\omega$ (naturality).

$$\begin{tikzcd}
\Omega^k(M) & \Omega^k(N) \arrow[l, "g^*"'] \\
\Omega^{k-1}(M) \arrow[u, "d"] & \Omega^{k-1}(N) \arrow[l, "g^*"'] \arrow[u, "d"]
\end{tikzcd}$$

Proof. Since d is linear, we may suppose that $\operatorname{supp}\omega$ is contained in a coordinate neighborhood. Using naturality, it suffices to defined d when $M = \mathbb{R}^n$. Then we can look at each component and define it explicitly. $\square$

Example 33.2. Let us consider $M = \mathbb{R}^3$. For $f \in \Omega^0$, we have

$$df = f_x dx + f_y dy + f_z dz = \operatorname{grad}(f) = \nabla f.$$

If we go from Ω^1 to Ω^2, then

$$d(f dx + g dy + h dz) = (g_x - f_y) dx \wedge dy + (h_x - f_z) dx \wedge dz + (h_y - g_z) dy \wedge dz$$

and with suitable conventions, it is the curl. Lastly, for $\Omega^2 \to \Omega^3$, it corresponds to the divergence.

Now I want to describe the steps that go into the proof of Stokes theorem. The cool thing is that once you have the formulation, the proof is easy.

Proof of Stokes theorem. I want to prove that $\int_M d\omega = \int_{\partial M} \omega$. Both sides are linear in ω, so by writing $\omega = \sum \theta_i \omega_i$, we can suppose that $\operatorname{supp} \omega$ is compact and is contained in a coordinate neighborhood U. This neighborhood can be either in the interior, or contain a part of the boundary.

Suppose first that $U \cong \mathbb{R}^n$. We might as well suppose $M = \mathbb{R}^n$. Then we can write

$$\omega = f_1 dx_2 \wedge \cdots \wedge dx_n + \cdots + f_n dx_1 \wedge \cdots \wedge dx_{n-1},$$

and again using linearity, we may suppose that $\omega = f dx_2 \wedge \cdots \wedge dx_n$. Then $d\omega = (\partial f / \partial x_1) dx_1 \wedge \cdots \wedge dx_n$. If we choose sufficiently large a such that $\operatorname{supp} \omega \subset [-a, a] \times \mathbb{R}^{n-1}$ then

$$\begin{aligned} \int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n \\ &= \int_{\mathbb{R}^{n-1}} \int_{-a}^{a} \frac{\partial f}{\partial x_1} dx_1 dx_2 \wedge \cdots \wedge dx_n \\ &= \int_{\mathbb{R}^{n-1}} f(a) - f(-a) dx_2 \wedge \cdots \wedge dx_n = 0. \end{aligned}$$

We will do the other case next time. $\square$

34 April 25, 2016

Remember that the Stokes theorem says

$$\int_M d\omega = \int_{\partial M} \omega.$$

Because the expression is linear and is diffeomorphism-invariant, we can check it only when $M = \mathbb{R}^n$ and $M = H^n$. We showed it in the case of $\mathbb{R}^n$ last time using the fundamental theorem of calculus, and now let's do it for $M = H^n$. Because we use the outward normal vector, I will unconventionally let $H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \le 0\}$.

Let our differential form $\omega \in \Omega^{n-1}(H^n)$ be

$$\omega = \sum_i f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Then its derivative is

$$d\omega = \sum \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = \Big(\sum (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \Big) dx_1 \wedge \cdots \wedge dx_n.$$

So

$$\begin{aligned} \int_{H^n} d\omega &= \int_{H^n} \sum \frac{\partial f_i}{\partial x_i} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_i}{\partial x_i} (-1)^{i-1} dx_1 \cdots dx_n. \end{aligned}$$

The first term ($i = 1$) is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} dx_1 \cdots dx_n = \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \dots, x_n) dx_2 \cdots dx_n = \int_{\partial H^n} \omega.$$

The other terms are all 0, because the ith derivative goes to zero.

This abstract apparatus reducing things to simple local problems is a theme you encounter everywhere in math.

34.1 Degree of a map

Let $f : M^n \to N^n$ be a map between compact manifolds. We have defined $\deg_2 f$ to be the number of points in $f^{-1}(y)$, where y is a regular value. This is well-defined modulo 2, because the difference of two $\# f^{-1}(y)$ and $\# f^{-1}(y')$ will be given a boundary of a 1-manifold.

If both M and N are oriented, we can do better than that. If $df_x : T_x M \to T_{f(x)} N$ is an isomorphism, we define

$$\operatorname{sign} df = \begin{cases} +1 & \text{if } df \text{ preserves orientation,} \\ -1 & \text{if } df \text{ reverses orientation.} \end{cases}$$

We the define

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x,$$

where y is a regular value. This is then a well-defined value in $\mathbb{Z}$, and if $f = \partial h$ for some h, then $\deg f = 0$. I won't build too much on this, but there is also a notion of bordism homology MSO_* of oriented manifolds.

Theorem 34.1 (Degree formula)**.** *Let $f : M^n \to N^n$ be a smooth map of compact oriented n-manifolds, and let $\omega \in \Omega^n(N)$. Then*

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

This is the starting point of the relation between topological invariants and analytic stuff. You cannot continuously deform the degree, but you can do it for ω.

Proof. First suppose that $y \in N$ is a regular value and find a neighborhood $y \in U \subset N$ be a small neighborhood such that $f^{-1}(U) = U \times S$ where U is a finite set. Consider the simple case of $\operatorname{supp} \omega \subset U$.

$$\begin{tikzcd}
f^{-1}(U) \arrow[d, "f"'] & U \times S \arrow[l, "g"] \arrow[dl] \\
U &
\end{tikzcd}$$

Then

$$\int_M f^*\omega = \int_{f^{-1}(U)} f^*\omega = \int_{U \times S} g^* f^* \omega = (\deg f) \int_N \omega.$$

Now the problem is how to reduce to the case we did above.

Proposition 34.2. *If $f : M \to N$ is the boundary of $g : W \to N$, then $\int_M f^*\omega = 0$.*

Proof. By Stokes theorem,

$$\int_M f^*\omega = \int_W dg^*\omega = \int_W g^* d\omega = \int_W g^* 0 = 0.$$

□

There is also the following fact.

Proposition 34.3. *If N is a connected manifold, and $x, y \in N$, then there is a diffeomorphism $g : N \to N$ which is isotopic to the identity, and satisfies $g(x) = y$.*

Idea of proof. You can always move the point a little bit, so the set of points you can reach is open, and the set of points you can't reach is open. So you can reach every point. □

Going back to the original theorem, we know that the theorem is true for $\operatorname{supp}\omega \subset U$ where U is a neighborhood around a regular value y. Using the proposition, for any $y' \in N$ we can choose a diffeomorphism g and an isotopy $h : N \times [0,1] \to N$ such that $g(y') = y$ and $h(z,0) = g(z)$, $h(z,1) = z$. Let $U' = g^{-1}(U)$. If $\operatorname{supp}\omega \subset U'$ then

$$\int_M f^*\omega = \int_N \omega = \int_N (g^{-1})^*\omega = \int_M f^*(g^{-1})^*\omega = (\deg f)\int_N (g^{-1})^*\omega = (\deg f)\int_N \omega.$$

Then we can cover N by finitely many U's, which we will denote by $U_1, \dots, U_k$. Using partitions of unity, we can decompose any ω into $\omega = \omega_1 + \cdots + \omega_k$ such that $\operatorname{supp}\omega_i \subset U_i$, and then

$$\int_M f^*\omega = \sum \int_M f^*\omega_i = \sum (\deg f)\int_N \omega_i = (\deg f)\int_N \omega. \qquad \square$$

35 April 27, 2016

Let $M^n \subset \mathbb{R}^{n+1}$ be a compact closed manifold. Then M is oriented, and for any $x \in M$ there is a unit normal vector n_x pointing outward. Then there is a map $g : M^n \to S^n$ that sends

$$g : x \mapsto n_x.$$

For any $x \in M$, the derivative maps $dg_x : T_xM \to T_{g(x)}S^n$. We define the **Gaussian curvature** $K(x)$ at x of M as

$$K(x) = \deg dg_x.$$

Example 35.1. Let M be the sphere or radius r. Then $g(x) = x/r$ and $K(x) = r^{-n}$. If M is a saddle point, then the curvature is negative.

So this curvature measures the shape of the surface. But this depends on the way the manifold is embedded in the Euclidean space.

Given a $M \subset \mathbb{R}^{n+1}$, there is a thing called the $\omega_{\text{vol}} \in \Omega^n(M)$ such that at $x \in M$, if $e_1, \dots, e_n$ is an oriented orthonormal basis of T_xM then

$$\omega_{\text{vol},x} = e_1^* \wedge \cdots \wedge e_n^*.$$

We are taught in calculus that

$$\int_M \omega_{\text{vol}} = \text{Volume of } M.$$

35.1 The Gauss-Bonnet theorem

Theorem 35.2 (Gauss-Bonnet)**.** *For any $M \subset \mathbb{R}^{n+1}$ and n even,*

$$\int_M g^*\omega_{\text{vol}} = \text{Vol}(S^n) \cdot \frac{1}{2}\chi(M).$$

Here, χ is something called the Euler characteristic of M, and it only depends on M as a topological space. I haven't given a definition in this course, but I'll give a definition for $\dim M = 2$ around 1:42 pm.

Actually, we already know by the degree theorem,

$$\int_M g^*\omega = \deg(g) \cdot \int_{S^n} \omega_{\text{vol}} = \deg(g) \cdot \text{Vol}(S^n).$$

The Gauss-Bonnet theorem is further saying that $\deg(g) = \frac{1}{2}\chi(M)$. Further, by the classification of 2-manifolds, we know that if M_1 and M_2 are connected oriented surfaces and $\chi(M_1) = \chi(M_2)$ then M_1 is diffeomorphic to M_2. So for instance, if we sail around the world and integrate the curvature, we can know how the earth looks like.

So let me now tell you how the Euler characteristic is defined. The first definition involves triangulating M. If we choose a triangulation and define

$$\chi(M) = \#\text{Faces} - \#\text{Edges} + \#\text{Vertices},$$

then this does not depend on the triangulation. Alternatively, we can define it as

$$\chi(M) = \dim MO_0(M) - \dim MO_1(M) + \dim MO_2(M).$$

We know by the degree formula that the integral

$$\int_M g^*(\omega_{\text{vol}})$$

depends only on the embedding $M \hookrightarrow \mathbb{R}^{n=1}$ up to isotopy. So we may assume that, for instance, if $M = S^2$ then change M into a icosahedron. The Gauss map at each vertex sweeps out a spherical polygon, and so

$$\int_M g^*(\omega_{\text{vol}}) = \sum_{\text{vertices}} \text{area of some spherical polygon.}$$

Now there is another cool theorem about the area of a spherical polygon:

Theorem 35.3. *The area of a spherical n-gon is the excess angle*

$$\sum \textit{angles} - (n-2)\pi.$$

Proof. You can play around with the fact that a "lune" of angle α has area 2α. First prove for triangles and then add them up to get the general n-gon case. □

Proof of the Gauss-Bonnet theorem. Let M be triangulated and at a vertex v, n_v triangles come together. Using the cool theorem above, we get

$$\begin{aligned}\int_M g^*(\omega_{\text{vol}}) &= \sum_{\text{vertices} v} (\sum \theta - (n_v - 2)\pi) = \sum_{\text{vertices} v} \Big(\sum (\pi - \alpha) - (n_v - 2)\pi \Big) \\ &= \sum_{\text{vertices}} (2\pi - \sum \alpha) = 2\pi V - \pi F.\end{aligned}$$

But because each face has exactly 3 edges and each edge has exactly 2 faces, $2E = 3F$. So we finally get

$$\int_M g^* \omega_{\text{vol}} = 2\pi V - \pi F = 2\pi(V - E + F) = 2\pi\chi(M) = \frac{1}{2}(4\pi)\chi(M). \qquad \square$$

There is this notion of local stuff accumulating to produce global stuff.

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