Physics 253a - Quantum Field Theory I

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You need at least 10 hours a week to take this course. This course will get more difficult as we go into renormalization. Then it will get easier once we pass this and get to applications.

We will start with special relativity and quantum mechanics, put them together and see what happens. We won't start with the axioms, because they are just statements that sound reasonable but cannot be tested.

1.1 Quantum theory of radiation

When you turn on the lights, the number of particles increase. How does this happen? Max Planck in the 1900s observed that discrete energy can explain blackbody radiation. Einstein in 1916 explained spontaneous/stimulated emission, and Paul Dirac in 1927 invented quantum electrodynamics, the microscopic theory of radiation.

We have a box of size L, poke a hole and heat it up. Then light comes out. We know that the wave numbers associated with the box are $\vec{k} = \frac{2\pi}{L}\vec{n}$, and $\omega = |\vec{k}|c$. This is classical prediction. Then the number of modes $\leq n$ is proportional to n^3, and the classical equipartition theorem predicts that each mode has the same energy. Sow we would have

$$dI(\omega) \sim \omega^2 d\omega.$$

This is called the ultraviolet catastrophe. But experimentally, we have exponential decay.

Planck said that energy E is quantized, so that $E_n = \hat{h}\omega_n$. Here, $\omega_n = \frac{2\pi}{L}n$ where $n = |\vec{n}|$. Then each mode gets excited an integer number of times, E_n^{tot} is an integer times E_n. The probability of $E_n^{\text{tot}} \sim e^{-\beta E_n}$. Then

$$\langle E_n \rangle = \frac{\sum_{j=0}^{\infty} (\hbar j \omega_n) e^{-j\hbar\omega_n\beta}}{\sum_{j=0}^{\infty} e^{-j\hbar\omega_n\beta}} = \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1}.$$

Then the total energy up to ω is

$$\begin{aligned} E(\omega) &= \int_0^{\omega} d^3n \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1} = \hbar \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \int_0^{L\omega/2\pi} n^2 dn \frac{\omega_n}{e^{\hbar\omega_n\beta} - 1} \\ &= \hbar \frac{L^3}{(2\pi)^3} 4\pi \int_0^{\omega} \frac{\omega^3}{e^{\hbar\omega\beta - 1}}. \end{aligned}$$

So we get Planck's formula

$$I(\omega) = \frac{K}{2\pi^2} \frac{\omega^3}{e^{\hbar\omega\beta} - 1} \times 2.$$

The point here is that each mode gets excited an integer number of times. This is called **second quantization**. This really is just quantization, because the first quantization refers to $\vec{k} = \frac{2\pi}{L}\vec{n}$, which is just classically solving wave equations with boundary conditions.

Let us now look at a number of atoms, either in the ground state or the excited state with energy difference $E_2 - E_1 = \hbar\omega$. Let n_1, n_2 be the number of atoms with energy E_1, E_2. Also assume that there is a bath of photons of frequency ω, with intensity $I(\omega)$ and number $n_\omega = \frac{\pi^2}{\omega^3} I(\omega)$. If we look at the probability of atoms getting excited or emitting, we get

$$dn_2 = -An_2 - BI(\omega)n_2 + B'I(\omega)n_1.$$

Here, the first term is spontaneous emission, the second is stimulated emission, and the third is stimulated absorption. It's not obvious that the second term should exist, but it turns out to be nonzero. In equilibrium, we have

$$I(\omega)(B'n_1 - Bn_2) = An_2.$$

So we get

$$I(\omega) = \frac{A}{B'\frac{n_1}{n_2} - B} = \frac{A}{B'e^{\beta\hbar\omega} - B}$$

because $n_1 = e^{-\beta E_1}$ and $n_2 = e^{-\beta E_2}$.

Matching with Planck's formula, we get the relations

$$B = B', \quad A = \frac{\hbar}{\pi^2}\omega^3 B,$$

called Einstein's equations. The number B can be calculated by quantum mechanics. So we can calculate A using this relation and quantum mechanics.

This is what got to Dirac. It's great that we can compute the coefficient of spontaneous emission, but it will be good to calculate this without using thermal systems, just from fundamental laws. The second quantization really looks like the simple harmonic oscillator. So we are going to identify

$$|n\rangle = n \text{ photon state} = n\text{th excited state of the oscillator.}$$

Consider $a^\dagger$ the creation operator and a the annihilation operator so that $[a, a^\dagger] = 1$ and $N = a^\dagger a$ is the number operator with $\hat{N}|n\rangle = n|n\rangle$. We can compute

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

This turns out to be a powerful tool.

Now Fermi's golden rules says that the transition rate is $\Gamma \sim |M|^2\delta(E_f - E_i)$. If we use this, we get at the end,

$$|M_{2\to1}|^2 = |M_0|^2(n_\omega + 1), \quad |M_{1\to2}|^2 n\omega|M_0|^2.$$

So this algebra of creation and annihilation operation gives us the relation between spontaneous emission and stimulated absorbtion. Then more algebra gives

$$dn_2 = -|M_0|^2\Big(1 + \frac{\pi^2}{\hbar\omega}I(\omega)\Big)n_2 + \frac{\pi^2}{\hbar\omega^3}I(\omega)n_1.$$

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Today we are going to start the systematic development of the field. We let $c = 1$ and $\hbar = 1$.

2.1 Special relativity

There are rotations on the plane,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x_i \to R_{ij} x_j.$$

We can also rotate row vectors as

$$x^i \to x^i (R^T_{ij}),$$

and the rotations satisfy $R^T_{ij} \cdot 1_{jk} R_{kl} = 1_{il}$. This is because rotations should preserve $x^i x_i = x^2 + y^2$. In 3 dimensions, we have $x^2 + y^2 + z^2$, and in 4 dimensions, we have $t^2 - x^2 - y^2 - z^2$. So **Lorentz transformations** satisfy

$$\Lambda^T g \Lambda = g, \quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Examples include

$$\Lambda_{\theta_z} = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}, \quad \Lambda_{\beta_x} = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Four momentum is defined as

$$p^\mu = (E, p_x, p_y, p_z),$$

and it satisfies $p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$. Usually, $\vec{x}$ or x_i denotes a 3-dimensional vector, and x or x^μ denotes a 4-dimensional vector.

Tensors transform as

$$T_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}.$$

We define the **d'Alembertian** as

$$\Box = \partial_\mu^2 = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \vec{\nabla}^2 = \partial_t^2 - \Delta.$$

We say that a vector is **timelike** if $V^2 > 0$, and **spacelike** if $V^2 < 0$, and **lightlike** if $V^2 = 0$.

The proper **orthochronous** Lorentz group has $\det\Lambda = 1$ and $\Lambda_{00} > 0$. There are four components of the Lorentz group, and this is the connected component at the identity. The **Poincaré group** are Lorentz transformations plus translations.

2.2 Quantum mechanics

Remember we had normal modes in a box last time. These frequencies are quantized classically. Then Planck said that the energy should be associated to the frequency $E = j\hbar\omega$. Einstein was the one who interepreted these as particles, which we call photons, and Dirac developed this microscopic theory of $H = H_0 a^\dagger + H_0 a$.

Let us review the simple harmonic oscillator. We have a ball with a spring on it, and its equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

You can solve this, and you get

$$x(t) = \cos\Big(\sqrt{\frac{k}{m}}t\Big).$$

The classical Hamiltonian is given by

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2.$$

Then we quantize this using $[\hat{x}, \hat{p}] = i\hbar$, and define

$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} + \frac{i\hat{p}}{m\omega}\Big), \quad a^\dagger = \cdots, \quad [a, a^\dagger], \quad H = \hbar\omega(N + \tfrac{1}{2}), \quad N = a^\dagger a.$$

We found last time that

$$N|n\rangle = n|n\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

Then in the Heisenberg picture,

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0).$$

Now what can the equation of motion for the scalar field be? It should be Lorentz invariant, so the simplest possible equation is

$$\Box\phi = 0 = (\partial_t^2 - \vec{\nabla}^2)\phi = 0.$$

Take the Fourier transform, and let

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}]$$

Then the equation becomes

$$(\partial_t^2 + \vec{p}^2)a_p(t) = 0.$$

Now each component is just a classical simple harmonic oscillator. So we can quantize each separately, and then put them back together.

Electromagnetic saves are oscillators,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

This concisely encodes Maxwell's equations

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial_\mu F_{\mu\nu} = 0$$

in empty space. It's also helpful to write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This vector potential A_μ is more useful for field theory, because there are only 4 components, and also because it is invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \alpha(x),$$

called gauge invariance.

We can choose $\partial_\mu A_\mu = 0$, and this is called **Lorentz gauge**. When you do that, Maxwell's equations become

$$0 = \partial_\mu F_{\mu\nu} = \Box A_\nu.$$

So then we can make $A_\nu(x,t)$ into a set of harmonic oscillators. We write

$$A_\nu(x,t) = \int \frac{d^3p}{(2\pi)^3}(A_\nu^p(t)e^{i\vec{p}\cdot\vec{x}} + A_\nu^{p*}(t)e^{-i\vec{p}\cdot\vec{x}}), \quad (\partial_t^2 + \vec{p}^2)A_\nu^{\vec{p}} = 0.$$

Then the free electromagnetic field is equivalent to an infinite number of simple harmonic oscillators, labeled by 3 vectors $\vec{p}$ with frequencies $\omega_p = |\vec{p}|$.

Now we quantize as in quantum mechanics. Then

$$H_0 = \int \frac{d^3p}{(2\pi)^3}\omega_p(a_p^\dagger a_p + \tfrac{1}{2}).$$

The relations between these creation and annihilation operators are

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k}), \quad a_p|0\rangle = 0, \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_p}}|p\rangle.$$

What we have done is that we have constructed the Hilbert space

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p,$$

called the **Fock space**.

3 September 11, 2018

Last time we reviewed the simple harmonic oscillator. To quantize this theory, we defined $H = \omega(a^\dagger a + \frac{1}{2})$. For fields, we classically had $\Box A_\mu(x) = 0$ or $(\Box + m^2)\phi(x) = 0$. We do the Fourier transform, and we get something like

$$A(x,t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}].$$

Then the equation becomes $[\partial_t^2 + \omega_p^2]a_p(t) = 0$ and $\omega_p = \sqrt{\vec{p}^2 + m^2}$. Then we quantize and get

$$H = \int \frac{d^3p}{(2\pi)^3}[\omega_p(a_p^\dagger a_p + \frac{1}{2}].$$

3.1 Operators on the Fock space

The Fock space is then

$$\mathcal{F} = \bigoplus_p \mathcal{H}_p = \bigoplus_n \mathcal{H}_n$$

where p is the momentum and n is the number of particles. The creation and annihilation operators then behave as

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We normalize

$$a_p|0\rangle = 0, \quad |p\rangle = \sqrt{2\omega_p}a_p^\dagger|0\rangle.$$

Then we get

$$\langle p|k\rangle = \sqrt{2\omega_p}\sqrt{2\omega_k}\langle 0|a_p a_k^\dagger|0\rangle = 2\omega_p(2\pi)^3\delta^3(\vec{p} - \vec{k}).$$

We also have

$$\mathbf{1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}|p\rangle\langle p|.$$

Then you can check $|k\rangle = \mathbf{1}|k\rangle$.

Also, we define

$$A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}].$$

This is like a creation operator in position space. Indeed, we compute

$$\langle p|A(x)|0\rangle = \int d^3k\delta^3(p-k)\langle 0|0\rangle e^{-i\vec{k}\cdot\vec{x}} = e^{-\vec{p}\vec{x}}.$$

But $A(x)A(y)|0\rangle$ is not just particles at x and y.

In quantum field theory, we work with the Heisenberg picture, so we define $a_p^\dagger(t) = e^{i\omega_p t}a_p^\dagger(0)$. Then

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p(0)e^{i\vec{p}\cdot\vec{x} - i\omega_p t} + a_p^\dagger(0)e^{i\omega_p t - i\vec{p}\cdot\vec{x}}].$$

Here, you can interpret the exponent as $p^\mu x_\mu$, because $p^\mu = (\omega_p, \vec{p})$.

3.2 Classical field theory

The main object is the Hamiltonian

$$H(p,x) = \text{energy} = K + V.$$

This is not Lorentz invariant, and generates time translation. On the other hand, the Lagrantian

$$L[x,\dot{x}] = K - V$$

is not a conserved quantity, but it is Lorentz-invariant and the dynamics is determined by minimizing the action $S = \int d\epsilon L$.

For fields, we are going to have

$$L[\phi,\dot{\phi},\vec{\nabla}\phi] = L[\phi,\partial_\mu\phi], \quad H[\phi,\pi,\vec{\nabla}\phi].$$

We still talk about kinetic terms

$$K = \text{things like } \frac{1}{2}\phi\Box\phi, \quad \frac{1}{4}F_{\mu\nu}^2, \frac{1}{2}m^2\phi^2, \phi\partial_\mu A^\mu,$$

and interactions

$$V = \text{things like } A\phi^3, e\bar{\psi}A\psi, e(\partial_\mu\phi)\phi^* A_\mu.$$

Example 3.1. Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi).$$

To minimize the action, we perturb the field a little bit and look at the difference. Then

$$\delta S = \int d^4x \Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\Big] \quad = \int d^4x \Big\{\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big] + \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big]\Big\}.$$

Here, we assume $\phi(\infty) = 0$, so we get

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

This is called the **Euler–Lagrangian equations**.

Example 3.2. In the above example, we get

$$-V'(\phi) = \partial_\mu[\partial_\mu\phi] = \Box\phi.$$

3.3 Noether's theorem

Suppose $\mathcal{L}$ is invariant under some specific continuous variation. For instance, take

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^*$$

which is invariant under $\phi \to e^{i\alpha}\phi$. Then

$$0 = \frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \Big\{ \Big[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} + \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\partial \alpha} \Big\}.$$

So if the Euler–Lagrange equations are satisfied, the first term is zero so

$$\partial_\mu J^\mu = 0, \quad J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}.$$

Then if we define $Q = \int d^3x J^0$, we have $\partial_t Q = 0$. This is the statement and proof of **Noether's theorem**.

Let's think about what we get for $\phi \mapsto e^{i\alpha}\phi$. We have

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(-i\phi^*) = i\phi \partial_\mu \phi^* - i\phi^* \partial_\mu \phi.$$

We can check

$$\partial_\mu J^\mu = i\partial_\mu \phi \partial_\mu \phi^* + i\phi \Box \phi^* - i\partial_\mu \phi^* \partial_\mu \phi - i\phi^* \Box \phi = i\phi \Box \phi^* - i\phi^* \Box \phi.$$

This is zero because at the equations of motion, we have $\Box \phi = m^2 \phi$.

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Noether's theorem says that if and action has a continuous symmetry, then there exists a current J^μ with $\partial_\nu J^\mu = 0$ when the equations of motion are satisfied. In this case,

$$Q = \int d^3x J^0$$

satisfies $\partial_t Q = 0$.

Consider translation invariance. When we look at a translate of $\mathcal{L}$, we get

$$\partial_\mu(g_{\mu\nu}\mathcal{L}) = \partial_\nu \mathcal{L} = \Big[\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\Big]\frac{\delta\phi_n}{\partial\xi^\nu} + \partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\partial\xi^\nu}\Big].$$

Because the first term vanishes at equation on motion. So we have

$$\partial_\mu T_{\mu\nu}$$

where

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n - g_{\mu\nu}\mathcal{L}.$$

This is called the **energy-momentum tensor**. Here, we note that

$$\mathcal{E} = T_{00} = \sum \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\dot{\phi}_n - \mathcal{L} = \pi\dot{\phi} - \mathcal{L} = \mathcal{H}$$

is just the energy. So energy $E = \int d^3x T^{00}$ is conserved over time.

4.1 Coulomb's law

We are going to introduce an external current

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - J_\mu A^\mu.$$

(When I say current, I don't mean Noether current here.) Because $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we have

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - J_\mu A^\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\partial_\mu A_\nu \partial_\nu\text{‘}A_\mu - J_\mu A^\mu.\end{aligned}$$

Then $\partial\mathcal{L}/\partial A_\nu = -J_\nu$ and $\partial\mathcal{L}/\partial\partial_\mu A_\nu = -\partial_\mu A_\nu + \partial_\nu A_\mu = -F_{\mu\nu}$. Then the Euler–Lagrange equation is

$$\partial_\mu F_{\mu\nu} = J_\nu,$$

which is Maxwell's equations. If we go to Lorentz gauge, we get

$$\Box A_\nu = J_\nu.$$

We are going solve this by inverting the d'Alembertian $\Box$. Here, note that we have Fourier transform

$$f(x) = \int \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}, \quad \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}.$$

Then the inverse is

$$\tilde{f}(k) = \int dx f(x) e^{-ikx}.$$

We can compute

$$\Box f(x) = \int d^k k \Box \tilde{f}(x) e^{ikx} = \int d^4 k (-k^2) \tilde{f}(k) e^{ikx}.$$

So $\Box$ corresponds to $-k^2$ in Fourier space.

We want to solve the equation when there is a point charge, when $J_0 = \delta^3(x)$ and $\vec{J} = 0$. Then

$$\begin{aligned} A_0(x) &= \frac{e}{\Box}\delta^3(x) = -\frac{e}{\Delta}\delta^3(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e}{\vec{k}^2} e^{i\vec{k}\vec{x}} \\ &= \frac{e}{i4\pi^2} \int_0^\infty dk \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{e}{4\pi r}. \end{aligned}$$

This is the Coulomb potential.

4.2 Green's functions

Let's look at a complicated example,

$$\mathcal{L} = -\frac{1}{2} h \Box h + \frac{1}{3} \lambda h^3 + hJ.$$

This is a toy example for gravity, because gravitons interact with each other. Then the Euler–Lagrange equation is

$$\Box h - \lambda h^2 - J = 0.$$

We now work perturbatively in λ. For $\lambda = 0$, we know

$$h_0 = \frac{1}{\Box} J.$$

If $\lambda \neq 0$, we can write $h = h_0 + h_1$, where $h_1 = O(\lambda)$. If we plug in into the original equation, we get $\Box h_1 = \lambda h_0^2$. So we can write

$$h_1 = \frac{\lambda}{\Box}\Big(\frac{1}{\Box} J\Big)^2.$$

So we get

$$h = \frac{1}{\Box} J + \lambda \Big(\frac{1}{\Box}\Big)\Big(\frac{1}{\Box} J\Big)\Big(\frac{1}{\Box} J\Big) + \cdots.$$

We can interpret each of these in terms of Feynman diagrams. Think of each J as a source, $\frac{1}{\Box}$ as a propagation or a branch coming our from a source, and λ as an interaction between these branches. Then this is something like the Sun emitting a graviton, emitting another graviton, and they interact and become one. There are other diagrams we can draw but are not represented in the solution, and these are purely quantum mechanical effects that we will discuss. What we are doing now is classical.

If we look at the solution for $\Box_x A(x) = J(x)$ again, we have

$$A(x) = -\int d^4 y \Pi(x,y) J(y), \quad \Pi(x,y) = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2}.$$

Then you can check that $\Box_x \Pi(x,y) = -\delta^(x-y)$. We call this a **propagator** or the **Green's function**. (We have $\frac{1}{\Box} = -\Pi$.)

Let us do what we did this above in this context. Then

$$h(x) = \int d^4 y \delta^4(x-y) h(y) = -\int d^4 \pi(x,y) \Box_y h(y) = -\int d^4 y \Pi(x,y) J(y).$$

So this is the propagator of the potential from the source. We can do the same thing on the next order. We have

$$h(x) = -\int d^4 y \Pi(x,y) J(y) + \lambda \int d^4 w \int d^4 y \int d^4 z \Pi(x,w) \Pi(w,y) \Pi(w,z) J(y) J(z).$$

Then these have good physical interpretation. In quantum field theory, there will also be interactions in loops and so on.

5 September 18, 2018

This week and next week will be a bit dry. Why do we talk about cross sections in scattering? Scattering is a universal way of probing something that we can't see. We are skipping Chapter 4, which is old-fashioned perturbation theory.

5.1 Scattering

In quantum mechanics, we calculate amplitudes, $\langle f|i\rangle$, and probabilities, $|\langle f|i\rangle|^2$. In field theory, we calculate the same objects.

Let us consider the situation where two particles collide, and two or more particles come out. In the Schrödinger picture, we want to calculate

$$\langle f; t = \infty | i; t = -\infty\rangle.$$

In this Heisenberg picture, we are trying to measure $\langle f|S|i\rangle$. We are interested in this matrix S.

Classically, if we throw a beam of particles on a large particle,s we can consider the cross-section area as

$$\sigma = \frac{\#\text{particles scattered}}{\text{time} \times \text{number density of beam} \times \text{velocity of beam}}.$$

We may think this as $N = L\sigma$, where L is the luminosity.

What we want to do now is to talk about quantum mechanics. Here, σ is just a cross-section. Particles have a probability of scattering: $P = N_{\text{scatter}}/N_{\text{incident}}$. We are going to let $N_{\text{incident}} = 1$, so we are throwing one particles at a time. Then the flux is

$$\text{Flux} = \frac{|\vec{v}|}{V} = \frac{|\vec{v}_1 - \vec{v}_2|}{V}.$$

Now our formula for σ is

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}dP, \quad dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi.$$

The last factor $d\Pi$ is the density of states. On a line of size L, momenta are $p_n = \frac{2\pi}{L}n$ and so $dp = \frac{2\pi}{L}dn$. So we have

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}d^3p_j.$$

The initial and final states are given by

$$|i\rangle = |p_1\rangle|p_2\rangle, \quad |f\rangle = |p_3\rangle\cdots|p_n\rangle.$$

Because we are working in a box, we consider $|p|p\rangle = 2E_p\delta^3(0) = 2E_pV$. Then

$$|i|i\rangle = 2E_1 2E_2 V^2, \quad |f|f\rangle = \prod_{j=3}^{n}(2E_j)V.$$

Then we have

$$d\sigma = \frac{V}{T}\frac{|\langle f|S|i\rangle|^2}{|\vec{v}_1 - \vec{v}_2|\prod_j (2E_i)2E_1 2E_2 V^n}\prod_n \frac{V}{(2\pi)^3}d^3p_i.$$

We write

$$S = 1 + iT,$$

where $T = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - \cdots - p_n)M$, because momentum is conserved. Then

$$|\langle f|S|i\rangle|^2_{f\neq i} = (2\pi)^8\delta^4(\sum p)\delta^4(0)|M|^2.$$

If we plug this in, we get

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}\times |M^2|\prod_j \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^4(\sum p).$$

This second term is also called the Lorentz-invariant phase space, $d\Pi_{\text{LIPS}}$. So we can write the decay wrate as

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\Pi_{\text{LIPS}}.$$

There is no flux factor, and no $1/2E_2$.

5.2 Two-to-two scattering

Let us look at the example of a $2 \to 2$ scattering. Let us call the four particles p_1, p_2, p_3, p_4. In the center of mass frame, we have

$$|\vec{p}_1| = |\vec{p}_2| = p_i, \quad |\vec{p}_3| = |\vec{p}_4| = p_f.$$

Energy conservation is $E_1 + E_2 = E_3 + E_4 = E_{\text{CM}}$. Now we look at

$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}.$$

But this has a lot of redundancies, so we can express in terms on the direction. If we integrate over $\vec{p}_4$, we get

$$\begin{aligned} d\Pi_{\text{LIPS}} &= \frac{1}{4(2\pi)^2}\frac{dp_3}{E_3E_4}\delta(E_3 + E_4 - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int p_f^2 dp_f \frac{1}{E_3E_4}\delta(\sqrt{m_3^2 + p_f^2} + \sqrt{m_4^2 + p_f^2} - E_{\text{CM}}) \\ &= \frac{d\Omega}{16\pi^2}\int_{m_3+m_4-E_{\text{CM}}}^{\infty} dx\delta(x)\frac{p_f}{E_{\text{CM}}} = \frac{d\Omega}{16\pi^2}\frac{p_E}{E_{\text{CM}}}\theta(E_{\text{CM}} - m_3 - m_4). \end{aligned}$$

So we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_1 2E_2|\vec{v}_1 - \vec{v}_2|}\frac{1}{16\pi^2}\frac{p_f}{E_{\text{CM}}}|M|^2 = \frac{1}{64\pi^2 E_{\text{CM}}^2}\frac{p_f}{p_i}|M|^2.$$

in the center of mass frame. (This $d\Omega$ is the spherical angle $d\phi d\cos\theta$, so that $d^3p_3 = p_3^2 dp_3 d\Omega$.)

Let us look at the non-relativistic limit. Consider the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}|\tilde{V}(k)|^2.$$

Here, $\tilde{V}(k)$ is the Fourier transformation

$$\tilde{V}(k) = \int d^3x e^{-i\vec{k}\vec{x}} V(x) = \frac{e^2}{\vec{k}^2}$$

in the Coulomb potential. So we have

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}\Big(\frac{e^2}{\vec{k}^2}\Big)^2.$$

Let us how see this agrees with what we have done so far.

The free theory for the proton and the electron is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 - \phi_e^*(\Box + m_e^2)\phi_e - \phi_p^*(\Box + m_p^2)\phi_p \\ &- ieA_\mu(\phi_e^*\partial_\mu\phi_e - \phi_e\partial_\mu\phi_e^*) + ieA_\mu(\phi_p^*\partial_\mu\phi_p - \phi_p\partial_\mu\phi_p^*).\end{aligned}$$

If we take the non-relativistic limit, we have $p_\mu = (E, \vec{p}) = (\sqrt{m^2+\vec{p}^2}, \vec{p}) \approx (m, 0)$, and so $\partial_t\phi \approx im\phi$. So we redefine $\phi \to e^{im_e t}\phi$ so that the phases don't rotate. If we do that, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \phi_e\vec{\nabla}^2\phi_e + 2em_e\phi_e^*\phi_e A_0 + \phi_p\vec{\nabla}^2\phi_p - 2em_p\phi_p^*\phi_p A_0.$$

The matrix M is going to be

$$M = \frac{(2em_e)(-2em_p)}{\vec{k}^2}$$

because the coefficient of $\phi_e^*\phi_e A_0$ is $2em_e$ and this is the interaction between the electron, electron, photon, and likewise for $(-2em_p)$, and $1/|\vec{k}^2|$ is the Green's function. Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_p^2}|M|^2 = \frac{1}{64\pi^2 m_p^2}\frac{16e^4 m_e^2 m_p^2}{k^4} = \frac{1}{4\pi^2}\frac{m_e^2 e^4}{|\vec{k}^2|^2}.$$

This is the same formula we had for the Born approximation.

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We also have to need this other technical theorem. We recall that light satisfies $\Box A_\mu = 0$, and so $(\partial_t^2 + |\vec{k}|^2)A_\mu = 0$. We had our operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{i\vec{p}\cdot\vec{x}} + a_p e^{-i\vec{p}\cdot\vec{x}}).$$

Then $i\partial_t a_p = -[H, A_p] = \omega_p a_p$, so we can define even for difference time

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}}^\dagger e^{ipx} + a_{\vec{p}} e^{-ipx}).$$

6.1 LSZ reduction

We also talked about cross sections,

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|M|^2 d\Pi_{\mathrm{LIPS}},$$

where $S = 1 + iT$ and $T = (2\pi)^4\delta^4(\sum p)M$. So again, our initial state is

$$|i\rangle = |p_1p_2\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2}a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle$$

We are using $|\Omega\rangle$ because the vacuum is time-dependent. Similarly, in the final state, we can write

$$|f\rangle = |p_3\cdots p_n\rangle = \sqrt{2\omega_3}\cdots\sqrt{2\omega_n}a_{p_3}^\dagger(+\infty)\cdots a_{p_n}^\dagger(+\infty)|\Omega_{+\infty}\rangle.$$

Now the matrix element between the two things is

$$\langle f|S|i\rangle = \sqrt{2\omega_1}\cdots\sqrt{2\omega_n}\langle\Omega_\infty|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega_{-\infty}\rangle.$$

So how do we create $|p\rangle$ from $\phi(x)$? WE have

$$\langle p|\phi(x)|0\rangle = e^{ipx}, \quad \phi(x)|0\rangle = \int d^3p\frac{1}{2\omega_p}e^{i\vec{p}\cdot\vec{x}}|p\rangle = \int d^4p\delta(p^2-m^2)\theta(p^0)e^{i\vec{p}\vec{x}}|p\rangle.$$

But we have

$$-2\pi i\delta(p^2 - m^2) = \lim_{\epsilon\to 0}\Big[\frac{1}{p^2+m^2+i\epsilon} - \frac{1}{p^2-m^2-i\epsilon}\Big].$$

So we roughly have

$$\int e^{-ipx}(\Box + m^2)\phi(x)|0\rangle = |p\rangle.$$

The precise expression is

$$i\int d^4x e^{ipx}(\Box + m^2)\phi(x,t) = \sqrt{2\omega_p}(a_p(\infty) - a_p(-\infty)).$$

Let me try to derive this. If we do spatial integration by parts, we get

$$\begin{aligned}
i\int d^4x e^{ipx}(\Box + m^2)\phi(x,t) &= i\int d^4x e^{ipx}(\partial_t^2 + \omega_p^2)\phi(x,t) \\
&= \int dt\partial_t\Big[e^{i\omega_p t}\int d^3x e^{-i\vec{p}\vec{x}}(i\partial_t + \omega_p)\phi(x,t)\Big] \\
&= \int dt\partial_t\big[e^{i\omega_p t}\sqrt{2\omega_p}a_p e^{-i\omega_p t}\big] \\
&= \sqrt{2\omega_p}[a_p(\infty) - a_p(-\infty)].
\end{aligned}$$

Here, we are assuming that at $t = \pm\infty$, the field behaves like a free field, so we can compute the spatial integral simply. Now if we take the complex conjugate of both sides, we get

$$-i\int d^4x e^{-ipx}(\Box + m^2)\phi(x,t) = \sqrt{2\omega_p}(a_p^\dagger(\infty) - a_p^\dagger(-\infty)).$$

Now can compute $\langle f|i\rangle$. Here, we introduce a **time-ordering operation** T which just puts things in the correct time order. Then we have

$$\begin{aligned}
&\langle\Omega|a_{p_3}(\infty)\cdots a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle \\
&\quad = \langle\Omega|T\{[a_{p_3}(\infty) - a_{p_3}(-\infty)]\cdots[a_{p_n}(\infty) - a_{p_n}(-\infty)] \\
&\qquad [a_{p_1}^\dagger(\infty) - a_{p_1}^\dagger(-\infty)][a_{p_2}^\dagger(\infty) - a_{p_2}^\dagger(-\infty)]|\Omega\rangle,
\end{aligned}$$

because all other terms become 0. So we get

$$\begin{aligned}
\langle f|i\rangle = &\Big[i\int d^4x_1 e^{ip_1x_1}(\Box + m_1^2)\Big]\cdots\Big[-i\int d^4x_n e^{-ip_nx_n}\Big] \\
&\times\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle.
\end{aligned}$$

This is called the **LSZ reduction formula**.

6.2 Feynman propagators

The simplest example is

$$D_F(x,y) = \langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle = \lim_{\epsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{ik(x-y)}.$$

This is called the **Feynman propagator**, and satisfies $(\Box_x + m^2)D_F(x,y) = \int d^4k e^{ik(x-y)} = \delta^4(x-y)$. So this is the factor you put in when you want to talk about propagation between to interactions.

The first thing we do is to calculate without time ordering. We have

$$\begin{aligned}
\langle 0|\phi(x_1)\phi(x_2)|0\rangle &= \int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{\sqrt{2\omega_1}\sqrt{2\omega_2}}\langle 0|a_{k_1}a_{k_2}^\dagger|0\rangle e^{i(k_2x_2 - k_1x_1)} \\
&= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{ik(x_1-x_2)}.
\end{aligned}$$

If we do have time-ordering, we get

$$
\begin{aligned}
\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= \langle 0|\phi(x_1)\phi(x_2)|0\rangle\theta(t_1-t_2) + \langle 0|\phi(x_2)\phi(x_1)|0\rangle\theta(t_2-t_1) \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{ik(x_2-x_1)}\theta(\tau) + e^{ik(x_1-x_2)}\theta(-\tau)] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[e^{i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k\tau}\theta(\tau) + e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}e^{i\omega_k\tau}\theta(-\tau)] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}[e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau)].
\end{aligned}
$$

Here, we have

$$\theta(\tau) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{\omega+i\epsilon}\frac{1}{2\pi i}.$$

So we get

$$
\begin{aligned}
\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\frac{i}{\omega^2-\vec{p}^2-m^2+i\epsilon}e^{ik(x-y)} \\
&= \int d^4k \frac{i}{k^2-m^2+i\epsilon}e^{ik(x-y)}.
\end{aligned}
$$

This ϵ we are going to think of as uncertainty of energy. Feynman's ingenious idea is that you can add the two possible time orderings in one propagator, so that we don't have to think of the two cases separately.

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