Math 278x - Categorical Logic

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This course was taught by Jacob Lurie, on Mondays, Wednesdays, and Fridays at 3-4pm. I stopped going to lectures, and thus the notes are incomplete. A complete set of lecture notes can be found on the instructor's website, `http://math.harvard.edu/~lurie/278x.html`.

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This course is on first-order logic.

Definition 1.1. A **linearly ordered set** is a set X with a relation $R \subseteq X^2$ satisfying

(A1) $R(x, x)$ for all x,

(A2) $R(x, y)$ and $R(y, z)$ implies $R(x, z)$,

(A3) $R(x, y)$ and $R(y, x)$ implies $x = y$.

(A4) $R(x, y)$ or $R(y, x)$ for all x, y.

This is an example of first-order theory.

1.1 First-order logic

Definition 1.2. A **language** is a set of predicate symbols $\{P_i\}$ with arities $n_i \geq 0$. A **theory** is just a language and a set of first-order sentences. These are the axioms of the theory.

Definition 1.3. Let T be a theory. A **model** of T is a set M, with for each predicate P_i, a subset $M^{P_i} \subseteq M^{n_i}$, such that for every $\varphi \in T$, φ is true in M. (This is written as $M \vDash \varphi$.)

There are theories that are not first-order. For instance, torsion groups are not first-order. Finite fields are not first-order.

Theorem 1.4 (Completeness)**.** *Let T be a theory and φ be a sentence. The following are equivalent:*

(1) There is a proof of φ from T. (Written as $T \vdash \varphi$.)

(2) φ is true in every model of T. (Written as $T \vDash \varphi$.)

This gives relation between syntax and semantics. But to define this rigorously, we need to consider formulas. These are going to be sentences, such as $\varphi(x) = \forall y R(x, y)$. For φ a formula, let us define

$$M^\varphi = \{(a_1, \dots, a_n) \in M^n : M \vDash \varphi(a_1, \dots, a_n)\}.$$

Theorem 1.5 (Completeness)**.** *Let T be a theory and $\varphi(x_1, \dots, x_n)$ and $\varphi'(x_1, \dots, x_n)$ be formulas. The following are equivalent:*

(1) There is a proof (from T) that $\varphi \Leftrightarrow \varphi'$.

(2) For every model M, we have $M^\varphi = M^{\varphi'}$.

Question. *Say that for each $M \vDash T$, we have subset $\tilde{M} \subseteq M$. When does there exist $\varphi(x)$ such that $\tilde{M} = M^\varphi$ for all M?*

Definition 1.6. Let M and M' be models of T. We say that a function $f : M \to M'$ is an **elementary embedding** if for every $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$,

$$M \vDash \varphi(a_1, \ldots, a_n) \Leftrightarrow M \vDash \varphi(f(a_1), \ldots, f(a_n)).$$

If $f : M \to M'$, then we should have $\tilde{M} = f^{-1}(\tilde{M}')$. This is a necessary condition, but not sufficient.

Example 1.7. Let T be the theory of fields. Inside every F, consider

$$F \supseteq \tilde{F} = \{\text{roots of unity in } F\}.$$

But the claim is that there is no single formula defining $\tilde{F}$. One way to show this is to use the theory of ultrafilers. Let U be a non-principal ultrafilter on $\mathbb{Z}_{>0}$. We want to construct a field

$$F = \left(\prod_{n>0} \mathbb{C}\right)/U.$$

But for any $\varphi(x)$, and any sequence $\{z_n\}$, we have $F \vDash \varphi(\{z_n\})$ if and only if $\{n : \mathbb{C} \vDash \varphi(z_n)\} \in U$. Let's suppose that $\varphi(x)$ was satisfied by exactly the roots of unity. Consider

$$z = (e^{2\pi i}, e^{2\pi i/2}, e^{2\pi i/3}, \ldots).$$

Here, $z^k = 1$ is not true for any k, but it should be true that $M \vDash \varphi(z)$

Theorem 1.8 (Makkai definablity)**.** *Assume that $\tilde{M} = f^{-1}(\tilde{M}')$ for any elementary embedding $f : M \to M'$, and is compatible with ultraproducts, i.e.,*

$$\tilde{M} = \left(\prod_{i} \tilde{M}_i\right)/U.$$

This should be considered as a companion for Gödel's completeness theorem. This gives the existence result, and Gödel's completeness gives uniqueness.

Theorem 1.9 (Makkai)**.** *Let T be a first-order theory. Then T can be recovered (up to equivalence) from $\mathsf{Mod}(T)$, the category with objects models and morphisms elementary embeddings. But this $\mathsf{Mod}(T)$ is a "category with ultraproducts".*

We need to define a lot of stuff to make this rigorous. Of course, if every axiom of T can be proven from T' and vice versa, we would want to call them equivalent. But this is more subtle.

Example 1.10. Consider $\mathbb{P}^2(K)$ for a field K. There are points and lines in $\mathbb{P}^2(K)$, and incidence relations. An abstract **projective plane** is a set of points P and a set of lines L. The axioms will be:

(A1) Every pair of distinct points lies on a unique line.

(A2) Every pair of distinct lines intersects on a unique point.

Actually K can be recovered from $\mathbb{P}^2(K)$. But does every projective plane come from a field? This requires Papuus' theorem.

(A3) Pappus' theorem

So you can go back and forth between these theories, although a priori they look different.

So for each theory T, we are going to define a category $\mathrm{Syn}(T)$, called the syntactic category of T. Then we can define

Definition 1.11. T and T' are equivalent if $\mathrm{Syn}(T) \simeq \mathrm{Syn}(T')$ as categories.

Theorem 1.12 (Makkai)**.** *Let T be a first-order theory. Then* $\mathrm{Syn}(T) \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{ultra}}(\mathsf{Mod}(T), \mathsf{Set})$ *is an equivalence. Conversely,* $\mathsf{Mod}(T)$ *is a subcategory of* $\mathsf{Fun}(\mathsf{Syn}(T), \mathsf{Set})$*.*

This syntactic category $\mathrm{Syn}(T)$ is a pretopos. But to make a one-to-one dictionary between classical first-order logic, we will need to give up the law of excluded middle. This is because of a pragmatic reason.

Example 1.13. Consider T the first-order theory of groups. The morphisms of $\mathsf{Mod}(T)$ are elementary embeddings. But there is a notion of group homomorphism, which is strictly weaker than being an elementary embedding. For a group homomorphism, we have $z = xy$ implies $f(z) = f(x)f(y)$, but we don't conclude $z \neq xy$ implies $f(z) \neq f(x)f(y)$. This is what we mean by giving up the law of excluded middle.

2 January 24, 2018

We'll begin with review of first-order logic.

2.1 First-order logic

Let L be a language with predicate symbols $\{P_i\}$ and an infinite set V of free variables.

Definition 2.1. A **formula** with free variables in $V_0 \subseteq V$ consist of

(1) $x = y$ for $x, y \in V_0$,

(2) $P_i(x_1, \dots, x_{n_i})$ for $x_1, x_2, \dots \in V_0$,

(3) $\varphi \vee \psi$ for φ and ψ already defined,

(4) $\neg\varphi$ for φ already defined,

(5) $\exists x \psi$ for ψ a formula with free variables in $V_0 \amalg \{x\}$.

Recall that an L-structure is a set M with $M[P_i] \subseteq M^{n_i}$. Given any formula φ with variables in $V_0 = \{x_1, x_2, \dots, x_n\}$ and $c_1, \dots, c_n \in M$, we define $M \vDash \varphi(c_1, \dots, c_n)$ inductively as the following:

(1) for $\varphi =$ "$x_i = x_j$", $M \vDash \varphi(\vec{c})$ if and only if $c_i = c_j$.

(2) for $\varphi =$ "$P_i(x_{j_1}, \dots, x_{j_{n_i}})$", $M \vDash \varphi(\vec{c})$ if and only if $\vec{c} \in M[P_i]$,

(3) ...

With these primitive notions, we can build others like

(1) $\varphi \wedge \psi$ means $\neg(\neg\varphi \vee \neg\psi)$

(2) $\varphi \Rightarrow \psi$ means $\neg\varphi \vee \psi$

(3) $\exists \vec{x} \varphi(\vec{x}, \vec{y})$ means $\exists x_1 \exists x_2 \cdots \exists x_n \varphi(\vec{x}, \vec{y})$

(4) $\forall \vec{x} \varphi(\vec{x}, \vec{y})$ means $\neg \exists \vec{x} \neg \varphi(\vec{x}, \vec{y})$.

A **sentence** is a formula with no free variables, and a **theory** T of a language is a set of sentences. A **model** is a L-structure M such that $M \vDash \varphi$ for all $\varphi \in T$. If M is a model and φ is a formula, we write

$$M[\varphi] = \{(c_1, \dots, c_n) : M \vDash \varphi(c_1, \dots, c_n)\}.$$

Suppose we have a map $V_0 = \{x_1, \dots, x_n\} \to V_1$ with $x_i \mapsto y_i$. Given a formula $\varphi(x_1, \dots, x_n)$, we would like to substitute every x_i with y_i. But there is a caveat, when one of the variables we want to use as free is already bound. But life is too short to worry about such problems.

2.2 The weak syntactic category

Given a first order theory T, we define a category $\mathrm{Syn}_0(T)$, called the **weak syntactic category** of T. The objects are formulas $\varphi(\vec{x})$ in the language of T. (I'm going to write $[\varphi(\vec{x})]$ for the object.) Let $[\varphi(\vec{x})]$ and $[\psi(\vec{y})]$ be objects of $\mathrm{Syn}_0(T)$. A morphism $f : [\varphi] \to [\psi]$ is a collection of maps

$$f_M : M[\varphi] \to M[\psi]$$

for each model M, such that there exists a formula $\theta(\vec{x}, \vec{y})$ such that for all M, $M[\theta]$ is the graph in $M[\varphi] \times M[\psi]$. We need to check that this is a category.

Proposition 2.2. *If $f : [\varphi] \to [\varphi']$ and $g : [\gamma'] \to [\gamma'']$ are morphisms in $\mathrm{Syn}_0(T)$, so is $g \circ f$.*

Proof. Suppose that $M[\theta(\vec{x}, \vec{y})] = \Gamma(f_M)$ and $M[\theta'(\vec{y}, \vec{z})] = \Gamma(g_M)$. Then the first-order formula

$$\rho(\vec{x}, \vec{z}) = \exists \vec{y} \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{y}, \vec{z})$$

cuts out the graph of $g \circ f$. $\square$

Here is another way to describe this category. The morphisms from $[\varphi(\vec{x})]$ to $[\varphi(\vec{y})]$ are formulas $\theta(\vec{x}, \vec{y})$ such that

$$T \vDash \forall \vec{x}, \vec{y}, (\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})) \wedge (\forall \vec{x}[\varphi(\vec{x}) \Rightarrow \exists! \vec{y} \theta(\vec{x}, \vec{y})]),$$

modulo the equivalence relation

$$T \vDash \forall \vec{x}, \vec{y} \theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y}).$$

If we really want to be syntactic, we should replace it with $\vdash$, but then we would be using Gödel's completeness theorem, which we are supposed to prove. So we leave it as $\vDash$.

Let $X = [\varphi(\vec{x})] \in \mathrm{Syn}_0(T)$. For each M, we have a subset $M[X] = M[\varphi] \subseteq M^n$. A morphism $f : X \to Y$ induces a map $M[X] \to M[Y]$ by definition. That is, $X \mapsto M[X]$ is a functor $\mathrm{Syn}_0(T) \to \mathsf{Set}$.

Proposition 2.3. *Let M and N be models of T. The following are equivalent:*

(1) an elementary embedding $f_0 : M \to N$

(2) natural transformations $M[-] \to N[-]$

Proof. First, if f_0 is elementary, for any φ and $c_1, \ldots, c_n \in M$, we know that

$$M \vDash \varphi(c_1, \ldots, c_n) \quad \Leftrightarrow \quad N \vDash \varphi(f_0(c_1), \ldots, f_0(c_n)).$$

So $f : M^n \to N^n$ restricts to a map $f_\varphi : M[\varphi] \to N[\varphi]$. The claim is that this data gives a natural transformation. We need to check that the diagram

$$\begin{array}{ccc} M[\psi] & \xrightarrow{f_\psi} & N[\psi] \\ \downarrow{\scriptstyle g_M} & & \downarrow{\scriptstyle f_N} \\ M[\varphi] & \xrightarrow{f_\varphi} & N[\varphi] \end{array}$$

commutes. This follows from the fact that f is elementary used on formula θ.

For the other direction, we first use on $\varphi(x) = (x = x)$. Then we get a map $f_0 : M \to N$. We need to show that f_0 is an elementary embedding, which just means that f_0 takes $M[\varphi]$ to $N[\varphi]$. This follows from functoriality. □

So we have shown that the functor

$$\mathsf{Mod}(T) \hookrightarrow \mathsf{Fun}(\mathrm{Syn}_0(T), \mathsf{Set})$$

is fully faithful. There is a natural question, which we will postpone this question until later.

Question. *What is the image? Given $F : \mathrm{Syn}_0(T) \to \mathsf{Set}$, when does it arrise from a model?*

In the abstract projective plane, we had lines and points.

Definition 2.4. A **typed language** is a set Typ of types, predicate symbols $\{P_i\}$ with arity $(t_1, \ldots, t_n)$. Now every variable has a type (e.g., $\{p, q, r\}$ for points and $\{\ell, \ell', \ell''\}$ for lines). Formulas are as before, except $x = y$ is only allowed when x and y have the same type and $P_i(x_1, \ldots, x_n)$ is allowed only when the type of x_j is t_j. We also add the formula $\perp$, which is always false. A model is a collection of sets $M[t]$ for every $t \in Typ$, and $M[P_i] \subseteq M[t_1] \times \cdots \times M[t_n]$.

If there is only one type, everything reduces to the previous definition. If there are only finitely many types, this is pointless. When there are no types at all, this is **propositional logic**. Here, the formulas are $\perp$, P_i, and their disjunctions and negations. In particular, every formula is a sentence.

What does $\mathrm{Syn}_0(T)$ look like for propositional logic? We have

$$\mathrm{Hom}_{\mathrm{Syn}_0(T)}([\varphi], [\psi]) = \begin{cases} * & T \vDash \varphi \Rightarrow \psi, \\ \emptyset & \text{otherwise.} \end{cases}$$

So this is a poset of sentences up to provable equivalences.

3 January 26, 2018

Our goal is to find a criterion for when a functor $\mathrm{Syn}_0(T) \to \mathsf{Set}$ comes from a model.

3.1 Fiber products

Proposition 3.1. *The category* $\mathrm{Syn}_0(T)$ *have fiber products, and* $M[-] : \mathrm{Syn}_0(T) \to \mathsf{Set}$ *preserves them, where* $M \vDash T$*.*

Proof. First we show that there are fiber products. For $f : X \to Z$ given by $\theta(\vec{x}, \vec{z})$ and $g : Y \to Z$ given by $\theta'(\vec{y}, \vec{z})$, the fiber product is going to be $\exists \vec{z}(\theta(\vec{x}', \vec{z}') \wedge \theta'(\vec{y}, \vec{z}))$. The projection maps are just going to be $\vec{x}' = \vec{x}$ and $\vec{y}' = \vec{y}$. To check that this is a fiber product, we first see that for every model, this will give a fiber product. If $\delta(\vec{w}, \vec{x})$ defines $W \to X$ and $\delta(\vec{w}, \vec{y})$ defines $W \to Y$, then $W \to P$ should be defined by $\delta(\vec{w}, \vec{x}') \wedge \delta(\vec{w}, \vec{y}')$. $\square$

Corollary 3.2. *The corollary* $\mathrm{Syn}_0(T)$ *has finite limits, preserved by* $M[-]$ *for each* $M \vDash T$*.*

Proof. It suffices to find an initial object. Take a sentence φ such that $T \vDash \varphi$, for instance, $\forall x(x = x)$. If we denote $[\varphi] = 1$, then $M[1] = \{*\}$ and so there is just unique map from $M[\varphi] \to M[1]$ for each φ. This map is a morphism $X \to 1$ in the syntactic category. $\square$

3.2 Factorization of morphisms

Lemma 3.3. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. The following are equivalent:*

(1) f *is an isomorphism.*

(2) Each $f_M : M[X] \to M[Y]$ *is bijective.*

Proof. (1) to (2) is obvious. For the other direction, just switch the variables. $\square$

Corollary 3.4. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. The following are equivalent:*

(1) f *is a monomorphism.*

(2) $f_M : M[X] \to M[Y]$ *for all* $M \vDash T$*.*

Proof. Apply the lemma to $X \to X \times_Y X$. $\square$

Example 3.5. Let $\varphi(\vec{x})$ and $\varphi_0(\vec{x})$ be formulas. If $T \vDash (\forall \vec{x} \varphi_0(\vec{x}) \Rightarrow \varphi(x))$, then $M[\varphi_0] \subseteq M[\varphi]$. This defines a map $[\varphi_0] \to [\varphi]$, which is a monomorphism. Let's call this a "special" monomorphism.

Proposition 3.6. *Let $f : X \to Y$ be any morphism in the syntactic category. Then f has a canonical factorization as $X \xrightarrow{g} Y \xrightarrow{h} Z$ where g is an epimorphism and h is a special monomorphism. Moreover, for any $M \vDash T$, we would have $M[X] \twoheadrightarrow M[Y] \hookrightarrow M[Z]$.*

Proof. Let us write $X = [\varphi(\vec{x})]$ and $Z = [\psi(\vec{z})]$ and f given by $\theta(\vec{x}, \vec{z})$. Now define $Y = [\exists y \theta(\vec{x}, \vec{z})]$, with g given by $\theta(\vec{x}, \vec{z})$ and h given by the identity map. We haven't proven that g is an epimorphism, but let's hold this for now. □

The statement that g is an epimorphism does not characterize the factorization.

Definition 3.7. Let $\mathcal{C}$ be a category with fiber products, and consider

$$X \times_Y X \underset{\pi'}{\overset{\pi}{\rightrightarrows}} X \xrightarrow{\ g\ } T.$$

We say that g is an **effective epimorphism** if g is the coequalizer.

Effective epimorphisms are always epimorphisms, but the converse is not true. The converse is true in Set. But in CRing, effective epimorphisms are surjections. Localizations are epimorphisms, but usually not surjective.

Proof. To show that g is an epimorphism, we show that g is an effective epimorphism. Assume that a map $U : [X] \to [W]$ is given by $\beta(\vec{x}, \vec{w})$, and on the level of models, we should have a map $[Y] \to [W]$. The formula that cuts out this is going to be $\exists \vec{x}(\theta(\vec{x}, \vec{z}) \wedge \beta(\vec{x}, \vec{w}))$. □

In fact, the fact that g is an effective epimorphism determines this factorization uniquely.

Proposition 3.8. *Let $\mathcal{C}$ be any category with fiber products, and $f : X \to Z$ be any morphism. Suppose f factors as $X \xrightarrow{g} Y \xrightarrow{h} Z$ with g an effective epimorphism and h a monomorphism. Then this factorization is unique.*

Proof. This is just a general categorical fact. □

Corollary 3.9. *Let $X = [\varphi(\vec{x})]$. Then every monomorphism $f : X_0 \hookrightarrow X$ is isomorphic to a special monomorphism.*

Proof. There is a factorization $X_0 \twoheadrightarrow Y \hookrightarrow X$ with $Y \hookrightarrow X$ a special monomorphism, and there is also the factorization $X_0 \cong X_0 \hookrightarrow X$. □

Definition 3.10. Let $\mathcal{C}$ be any category and $X \in \mathcal{C}$. Consider poset of **subobjects**

$$\mathrm{Sub}(X) = \{\text{monomorphisms } X_0 \hookrightarrow X\}/\text{isomorphism}.$$

Proposition 3.11. *Le $X = [\varphi(\vec{x})]$. Then*

(1) every object of $\mathrm{Sub}(X)$ *has the form $[\varphi_0(\vec{x})]$ where $T \vDash (\varphi_0 \Rightarrow \varphi)$.*

(2) *Given any two formale* $\varphi_0(\vec{x})$ *and* $\varphi(\vec{x})$, $[\varphi_0(\vec{x})] \subseteq [\varphi(\vec{x})]$ *if and only if* $T \vDash (\varphi_0 \Rightarrow \varphi)$.

(3) $[\varphi_0(\vec{x})] = [\varphi(\vec{x})]$ *in* $\mathrm{Sub}(X)$ *if an only if* $T \vDash (\varphi_0 \Leftrightarrow \varphi)$.

Corollary 3.12. *For* $X \in \mathrm{Syn}_0(T)$, $\mathrm{Sub}(X)$ *is a Boolean algebra.*

Proof. The least upper bound is given by $\varphi_0 \vee \varphi_1$, the greatest lower bound is given by $\varphi_0 \wedge \varphi_1$, and the complement is given by $\neg\varphi_0 \wedge \varphi$. □

Moreover, for any model $M \vDash T$ the map $\mathrm{Sub}(X) \to \mathrm{Sub}(M[X])$ is homomorphism of Boolean algebras.

Proposition 3.13. *Let* $F : \mathrm{Syn}_0(T) \to \mathsf{Set}$ *be a functor. Then* F *comes from a model* $M \vDash T$ *if and only if*

(a) F *preserves finite limits,*

(b) F *carries effective epimorphisms to surjections, and*

(c) *for any* $X \in \mathrm{Syn}_0(T)$, $F : \mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ *preserves least upper bounds of finite subsets.*

4 January 29, 2018

Let T be a theory, and let $\mathcal{C} = \mathrm{Syn}_0(T)$. We proved that

(A1) $\mathcal{C}$ has finite limits.

(A2) any $f : X \to Z$ factors as $f = h \circ g$ with h monic and g effective epic.

(A3) for $X \in \mathcal{C}$, $\mathrm{Sub}(X)$ as least elements and joints $X_0 \vee X_1$.

Given a morphism $f : X \to Z$ in $\mathrm{Syn}_0(T)$, how do we check if f is an effective epimorphism? This is true if and only if each $f_M : M[X] \to M[Y]$ is a surjection. This is because we have uniqueness of factorization. We have can always factorize $f : X \to Z$ into $M[X] \twoheadrightarrow M[Y] \hookrightarrow M[Z]$, and then $f : X \to Z$ is an effective epimorphism if and only if $Y \to Z$ is an isomorphism.

We actually showed that $\mathrm{Sub}(X)$ is a Boolean algebra, and we can deduce this from $X_0 \times_X X_1$ being the meet $X_0 \wedge X_1$.

4.1 Criteria for representability

Theorem 4.1. *Let $F : \mathrm{Syn}_0(T) \to \mathsf{Set}$ be a functor. This comes from a model if and only if*

(1) F preserves finite limits,

(2) F preserves effective epimorphisms,

(3) each $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a map of upper semi-lattices.

Just as the previous discussion, (3) is equivalent to $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ being a Boolean algebra homomorphism.

The interesting direction is the converse. Assume that F satisfies (1), (2), and (3). Pick a variable e, and consider $E = [e = e] \in \mathrm{Syn}_0(T)$. Then for any model N, we have $N[E] = N$. So let us define

$$M = F(E).$$

If F came from a model, this is the right thing to do. Now let us assume we have a finite set $V_0 = \{x_1, \dots, x_n\}$ of variables. Then

$$E^{V_0} = [(x_1 = x_1) \wedge \cdots \wedge (x_n = x_n)]$$

is always true, so $N[E^{V_0}] = N^{V_0} = \prod_{x \in V_0} N$. This give bijection $E^{V_0} \cong \prod_{x \in V_0} E$ in the syntactic category. Because F preserves finite limits, we get

$$F(E^{V_0}) = \prod_{x \in V_0} F(E) \cong M^{V_0}.$$

For each $\varphi(\vec{x})$, $[\varphi(\vec{x})]$ is a subobject of E^{V_0}. So we have

$$F[\varphi(\vec{x})] \subseteq M^{V_0}$$

because monomorphisms are preserved.

Given any map $V_0 \to V_1$ with $x_i \mapsto y_i$, we have a morphism

$$\begin{tikzcd} [\varphi(\vec{y})] \arrow[r, hook] \arrow[d] & E^{V_1} \arrow[d] \\ [\varphi(\vec{x})] \arrow[r, hook] & E^{V_0} \end{tikzcd}$$

which is a pullback square in $\mathrm{Syn}_0(T)$. Because F preserves finite limits, we have $F[\varphi(\vec{y})] = F[\varphi(\vec{x})] \times_{M^{V_0}} M^{V_1}$. Now for each predicate symbol P_i of arity n, choose $x_1, \dots, x_n$. We define

$$M[P_i] = F(P_i(x_1, \dots, x_n)) \subseteq M^n.$$

This lets us define $M[\varphi(\vec{x})] \subseteq M^{V_0}$ for each $\varphi(\vec{x})$ with variables in V_0.

Now the claim is that for any formula $\varphi(\vec{x})$, we have $F[\varphi(\vec{x})] = M[\varphi(\vec{x})]$ as subsets of M^{V_0}. This should be some kind of induction on the formation of $\varphi(\vec{x})$.

(i) If φ is $x = y$. You can do this.

(ii) If φ is $P_i(y_1, \dots, y_n)$, this is tautological (unless some of the y_is have the same variables, in which case it follows from the compatibility.)

(iii) Assume $\varphi = \varphi_0 \vee \varphi_1$. If we assume that $M[\varphi_0] = F[\varphi_0]$ and $M[\varphi_1] = F[\varphi_1]$, then $M[\varphi_0 \vee \varphi_1] = M[\varphi_1] \cup M[\varphi_2] = F[\varphi_0] \cup F[\varphi_1]$, but F preserves joins.

(iv) If $\varphi = \neg\psi$, then it follows from the fact that $F : \mathrm{Sub}(E^{V_0}) \to \mathrm{Sub}(M^{V_0})$ is a Boolean algebra homomorphism.

(v) If $\varphi(\vec{x}) = \exists y \psi(\vec{x}, y)$, then $[\varphi(\vec{x}, y)] \subseteq E^{V_0 \cup \{y\}}$. Here, $[\psi(\vec{x}, y)] \to [\varphi(\vec{x})]$ is an effective epimorphism, so F takes it to a surjective morphism. So $M[\psi(\vec{x}, y)] \to M[\varphi(\vec{x})]$ is a surjection, and it is just going to be the projection.

Corollary 4.2. *M is a model of T.*

Proof. Let φ be a sentence. Then

$$F[\varphi] = \begin{cases} \{*\} & M \vDash \varphi, \\ \emptyset & \text{otherwise.} \end{cases}$$

So if φ is a true sentence, it is a final object, and F should map it to a final object. So φ should be true in T. $\square$

4.2 Categories that are weak syntactic

Now I want to ask the following question.

Question. *What is special about this category? Given a category $\mathcal{C}$, when is it of the form* $\mathrm{Syn}_0(T)$ *for some typed first-order theory T?*

We first have the three properties (A1), (A2), and (A3). This can be thought of some structure on the category $\mathrm{Syn}_0(T)$. Here are some more compatibility conditions.

(A4) Effective epimorphisms are stable under pullback.

This is a true statement in any $\mathrm{Syn}_0(T)$. This can be checked against any model, and pullback of surjective maps in Set is a surjection. There is another way to formulate this. Let's say that $\mathcal{C}$ is a category with pullbacks, and consider a morphism $g : Y' \to Y$. Then there is a morphism $\mathrm{Sub}(Y) \to \mathrm{Sub}(Y')$ given by $Y_0 \mapsto Y_0 \times_Y Y'$. Then the compatibility between (A1) and (A3) is

(A5) For any $g : Y' \to Y$, the map $g^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(Y')$ is a homomorphism of upper semi-lattices.

Again, this is true in $\mathrm{Syn}_0(T)$ because we can check against any model and this is true for sets.

Proposition 4.3. *Assume (A1) and (A2). Then (A4) is equivalent to for any pullback square from $f : X \to Y$ to $f' : X' \to Y'$, we have* $\mathrm{im}(f') = g^{-1}\,\mathrm{im}(f)$.

Proof. Exercise. □

Definition 4.4. A **weak pretopos** is a category $\mathcal{C}$ satisfying these axioms (A1)–(A5).

This terminology is not standard.

Example 4.5. We know that $\mathrm{Syn}_0(T)$ is a weak pretopos. Also, Set is a weak pretopos.

Example 4.6. Let P be a poset. (A1) says that P has a largest element 1 and meets $p \wedge q$. (A2) and (A4) are automatic because effective epimorphisms are identities and all morphisms are monomorphisms. (A3) states that P has a smallest element 0 and joins $p \vee q$. (A5) states that $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. So P is a weak pretopoi if and only if P is a distributive lattice.

Definition 4.7. A weak pretopos $\mathcal{C}$ is **Boolean** if for every $X \in \mathcal{C}$, $\mathrm{Sub}(X)$ is a Boolean algebra.

Theorem 4.8. *Let $\mathcal{C}$ be a small category. The following are equivalent:*

(1) $\mathcal{C} \cong \mathrm{Syn}_0(T)$ of some typed theory T.

(2) $\mathcal{C}$ is a Boolean weak pretopos.

If you want T to be untyped, you need to add an extra axiom

(A6) There exists $X \in \mathcal{C}$ such that for all $Y \in \mathcal{C}$, there exist monomorphisms $Y \hookrightarrow X^n$ for some $n \gg 0$.

This object is $E = [e = e]$ that we have used in the proof of the previous theorem.

5 January 31, 2018

We started with a first-order theory, and we extracted some of the properties:

(A1) $\mathcal{C}$ has finite limits.

(A2) Every morphism $f : X \to Y$ has an image.

(A3) For $X \in \mathcal{C}$, $\mathrm{Sub}(X)$ is an upper semilattice.

(A4) Pullbacks of effective epimorphisms are effective epimorphisms.

(A5) For $f : X \to Y$, $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ preserves least elements and joins.

I gave the name "weak pretopos" to this.

5.1 Weak pretopoi

What are morphisms between weak pretopoi? This should be a functor $F : \mathcal{C} \to \mathcal{C}'$ such that

(1) F preserves finite limits,

(2) F preserves effective epimorphisms,

(3) for all $X \in \mathcal{C}$, the induced $F : \mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices. (Equivalently, it is a homomorphism of lattices.)

Definition 5.1. Let $\mathcal{C}$ be a weak pretopos. A **model** of $\mathcal{C}$ is a morphism of pretopoi $\mathcal{C} \to \mathsf{Set}$. If $\mathcal{C} = \mathrm{Syn}_0(T)$, then $\mathsf{Mod}(\mathcal{C}) = \mathsf{Mod}(T)$ in the classical sense.

Example 5.2. Let P and P' be distributive lattices. Morphisms $P \to P'$ as weak pretopoi is the same as lattice homomorphisms $P \to P'$.

Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$ be an object. You can make a new category $\mathcal{C}_{/X}$ consisting of objects $(U, f : U \to X)$ and morphisms $(U, f) \to (V, g)$ given by a map $h : U \to V$ making the diagram commute.

Proposition 5.3. *If $\mathcal{C}$ is a weak pretopos, so is $\mathcal{C}/X$.*

Proof. Fiber products can be formed in $\mathcal{C}$. Images also can be formed in $\mathcal{C}$. Unions of subobjects are also fomred in $\mathcal{C}$. Compatibility can be checked in $\mathcal{C}$. $\square$

But the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ is not going to be a morphism of weak pretopoi in most cases.

Proposition 5.4. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$ a weak pretopos. Then we get a functor $f^* : \mathcal{C}_Y \to \mathcal{C}_X$ given by $U \mapsto U \times_Y X$. Then f^* a morphism of weak pretopoi.*

This is the right adjoint of the functor given by composition by f.

5.2 Constructing a theory

Let $\mathcal{C}$ be a small weak pretopos. Our goal is to construct an associated first-order theory T such that models of $\mathcal{C}$ are models of $\mathcal{C}$.

First we need to describe the language. Types are going to be objects of $\mathcal{C}$. (This is a structure that associates to each type X a set $M[X]$.) For each morphism $f : X \to Y$, we are going to consider a predicate P_f with arity (X, Y). So we would have $M[P_f] \subseteq M[X] \times M[Y]$. Here are the axioms we will have:

- (for each $f : X \to Y$) $\forall x \exists! y P_f(x, y)$ (This will give $M[P_f]$ the interpretation of a graph of a function.)
- (for $\mathrm{id} : X \to X$) $\forall x P_{\mathrm{id}}(x, x)$ (This will be f_{id} the identity map on $M[X]$.)
- (for $X \xrightarrow{f} Y \xrightarrow{g} Z$) $\forall x, y, z[P_f(x, y) \wedge P_g(y, z) \Rightarrow P_{g\circ f}(x, z)$ (This makes sure that composition have as compositions.)

So far, models are simply functors $M : \mathcal{C} \to \mathsf{Set}$. But we don't want all functors.

- (for E the final obejct) $\exists! e[e = e]$ (M preserves final objects.)
- (for $f : X \to Y$ an effective epimorphism) $\forall y \exists x P_f(x, y)$ (M preserves effective epimorphisms.)
- ...

The defining feature of this theory is that models of $T(\mathcal{C})$ are "the same" as models of $\mathcal{C}$. There literally is a bijection between these models. But $\mathsf{Mod}(T(\mathcal{C}))$ is usually not equivalent to $\mathsf{Mod}(\mathcal{C})$. Because $\mathsf{Mod}(T(\mathcal{C})) = \mathsf{Mod}(\mathrm{Syn}_0(T(\mathcal{C})))$, maybe we should compare $\mathcal{C}$ and $\mathrm{Syn}_0(T(\mathcal{C}))$.

There is always a functor

$$\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C})); \quad X \mapsto [x = x].$$

On the level of maps, this is going to be given by $f : X \to Y$ mapped to $P_f(x, y)$.

Proposition 5.5. *λ is a morphism of weak pretopoi.*

Proof. We need to show $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$, but this is one of the axioms. We also need to show that $\lambda(g \circ f) = \lambda(g) \circ \lambda(f)$, and it is also one of the axioms. So λ is a functor. Now we need to show more stuff like if f is an effective epimorphism then $\lambda(f)$ is an effective epimorphism. But this is also an axiom, and everything will be an axiom. $\square$

So we can compose the morphisms of weak pretopoi and get

$$\mathsf{Mod}(T(\mathcal{C})) \cong \mathsf{Mod}(\mathrm{Syn}_0(T(\mathcal{C}))) \xrightarrow{-\circ\lambda} \mathsf{Mod}(\mathcal{C}).$$

This is bijective on objects, but not necessarily an equivalence.

Theorem 5.6. *A **weak pretopos** if and only if $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ is an equivalence.*

One direction is obvious, because Syn_0 is Boolean. If $\mathcal{C}$ is not Boolean, you can think of $\mathrm{Syn}_0(T(\mathcal{C}))$ as a "Booleanization" of $\mathcal{C}$. Let me kind of almost prove that. Assume $f : \mathcal{C} \to \mathcal{D}$ is a morphism of pretopoi, and construct the following:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[d, "\lambda_{\mathcal{C}}"] & & \mathcal{D} \arrow[d, "\lambda_{\mathcal{D}}"] \\ \mathrm{Syn}_0(T(\mathcal{C})) \arrow[rr, "f^{\mathrm{Bool}}"] & & \mathrm{Syn}_0(T(\mathcal{D})). \end{tikzcd}$$

If $\mathcal{D}$ is Boolean, λ_D is an equivalence so we and construct $\mathrm{Syn}(T(\mathcal{C})) \to \mathcal{D}$. What you need to show is that this factorization is unique.

Lemma 5.7. *Let $\mathcal{C}$ be Boolean and let $X \in \mathcal{C}$ factor as a product $X \cong X_1 \times \cdots \times X_n$. If $\varphi(x_1, \dots, x_n)$ is a formula in $T(\mathcal{C})$, and interpret*

$$[\varphi(x_1, \dots, x_n)] \subseteq \lambda(X_1) \times \cdots \times \lambda(X_n) = \lambda(X).$$

Then there exists a $Y \in \mathrm{Sub}(X)$ such that $\lambda(Y) = [\varphi(x_1, \dots, x_n)]$ in $\mathrm{Sub}(\lambda(X))$.

Proof. We induct on φ.

(i) If φ is $x_i = x_j$, then we can take $Y = X \times_{X_i \times X_j} X_i$.

(ii) If φ is $P_f(x_i, x_j)$ then we can take $Y = X \times_{X_i \times X_j} X_i$ where $X_i \to X_i \times X_j$ is given by $\mathrm{id} \times f$.

(iii) If φ is $\varphi_0 \vee \varphi_1$, with φ_0 and φ_1 represented by Y_0 and Y_1, we can take $Y = Y_0 \vee Y_1$ in $\mathrm{Sub}(X)$.

(iv) If $\varphi(\vec{x})$ is $\neg\psi(\vec{x})$, we get $[\psi(\vec{x})] = \lambda(Z)$ for some $Z \subseteq X$. If $\mathcal{C}$ is Boolean, it has a complement Y in $\mathrm{Sub}(X)$. But if there is a morphism of upper semilattices that happen to Boolean, it is a Boolean algebra homomorphism.

(v) If $\varphi(\vec{x}) = \exists w \psi(\vec{x}, w)$, then we take Y to be the image of $[\psi] \to [\varphi]$.

$\square$

Example 5.8. We will see that there is a canonical (weak) pretopos $\mathcal{C}$ such that $\mathsf{Mod}(\mathcal{C})$ is groups with group homomorphisms. But this is not Boolean, and if you Booleanize it you will get $\mathsf{Mod}(\mathrm{Syn}_0(T(\mathcal{C})))$ will be groups with elementary embeddings.

Example 5.9. Let X be a quasi-compact quasi-separated scheme. A set $K \subseteq X$ is **constructible** if it belongs to the Boolean algebra generated by the quasi-compact open sets. You can then make a new topological space X^c with basis given by the constructible sets in X. This is an example of a Stone space, i.e., compact Hausdorff totally disconnected. If P is the lattice of quasi-compact open sets in X, then $\mathrm{Syn}_0(T(P))$ is the lattice corresponding to X^c.

6 February 2, 2018

I was informed that what I have been calling a weak pretopos is actually called a **coherent category**.

6.1 Versions of the completeness theorem

Theorem 6.1 (Gödel's completeness theorem, version 0)**.** *If T is consistent, then it has a model.*

Theorem 6.2 (Deligne's completeness theorem, version 0)**.** *If $\mathcal{C}$ is a small coherent category and $\mathcal{C}$ is consistent, then $\mathcal{C}$ has a model.*

Here, we are saying that $\mathcal{C}$ is **consistent** if $\mathrm{Sub}(1)$ has more than one element. That is, 0 and 1 are different in $\mathrm{Sub}(1)$. As an exercise, you can show that $\mathcal{C}$ is consistent if it is not equivalent to $\{*\}$.

Theorem 6.3 (Gödel's completeness theorem, version 1)**.** *If φ is a sentence of T that is true in every model, then φ is provable from T. Equivalently, if $T \not\vdash \varphi$, then there is a model $M \vDash \varphi$.*

Theorem 6.4 (Deligne's completeness theorem, version 1)**.** *If $\mathcal{C}$ is small and coherent, if $U \in \mathrm{Sub}(1)$ and $U \neq 1$ (in $\mathrm{Sub}(1)$), then there exists a model $M : \mathcal{C} \to \mathsf{Set}$ such that $M[U] = \emptyset$.*

We will take these to be the official statements. We easily see that version 0 and 1 are equivalent for Gödel, but it is harder to see this for Deligne.

Theorem 6.5 (Gödel's completeness theorem, version 2)**.** *If $\varphi(\vec{x})$ and $\psi(\vec{x})$ are two formula,s and $T \not\vdash \varphi \Rightarrow \psi$, then there exists an M such that $M[\varphi] \not\subseteq M[\psi]$.*

Theorem 6.6 (Deligne's completeness theorem, version 2)**.** *If $\mathcal{C}$ is small and coherent, and $U \subseteq X$ is a subjective with $U \neq X$, then there exists a model $M : \mathcal{C} \to \mathsf{Set}$ such that $M[U] \subsetneq M[X]$.*

It is easy to see that version 2 implies version 1. I claim that version 1 implies version 2. Let $U \subsetneq X$ be a subobject. Then we can look at the slice category $\mathcal{C}_{/X}$ with objects $Y \in \mathcal{C}$ along with maps $Y \to X$. The final object is X, and now $U \subsetneq X$ is a subject of the final object. Then there exists a model $N : \mathcal{C}_{/X} \to \mathsf{Set}$ such that $N[U] = \emptyset$. There also exists a morphism $\mathcal{C} \to \mathcal{C}_{/X}$ given by $Y \mapsto Y \times X$. We compose them to get $M : \mathcal{C} \to \mathsf{Set}$. Now my claim is that

$$M[U] \subsetneq M[X]$$

which is equivalent to $N[U \times X] \subsetneq N[X \times X]$. But we have a pullback

$$\begin{tikzcd}
N[U \times X] \arrow[r, hook] & N[X \times X] \\
N[U] \arrow[u] \arrow[r, hook] & N[X] \arrow[u]
\end{tikzcd}$$

where $N[U]$ is empty and $N[X]$ has one element. So it cannot be a bijection.

Proposition 6.7. *The following data are equivalent:*

(1) models of $\mathcal{C}_{/X}$

(2) models M of $\mathcal{C}$ with a chosen element of $M[X]$.

You can think of this as adding a constant.

Theorem 6.8 (Deligne's completeness theorem, version 3)**.** *Let $\mathcal{C}$ be a small coherent category and $f : X \to Z$ be a morphism in $\mathcal{C}$. If f is not an isomorphism, then there exists a model $M : \mathcal{C} \to \mathsf{Set}$ such that $M[X] \to M[Z]$ is not bijective.*

Let's now prove that version 2 implies version 3. Assume that f induces a bijection in every model. We know that f can be factorized as

$$M[X] \xrightarrow{g} M[Y] \xrightarrow{h} M[Z]$$

where g is an effective epimorphism and h is a monomorphism. Then g and h is bijective in every model, and version 2 implies that h is an isomorphism. But what about g? The definition of an effective epimorphism is that

$$X \times_Y X \rightrightarrows X \xrightarrow{g} Y$$

is a coequalizer. Here we can take X as a subobject $X \hookrightarrow X \times_Y X$, and the fact that $M[X] \to M[Y]$ is bijective implies that $M[X] = M[X \times_Y X]$. Then $X \cong X \times_Y X$ as subobjects of $X \times X$. Then Y can be computed by the coequalizer of $X \rightrightarrows X$, which is X.

6.2 Boolean coherent categories are syntactic

Corollary 6.9. *Let $\lambda : \mathcal{C} \to \mathcal{D}$ be a morphism of coherent categories. Assume that*

(1) λ is essentially surjective, (for all $D \in \mathcal{D}$ there exists an isomorphism $D \cong \lambda(C)$)

(2) for each $C \in \mathcal{C}$, λ induces a surjection $\mathrm{Sub}(C) \twoheadrightarrow \mathrm{Sub}(\lambda(C))$,

(3) every model of $\mathcal{C}$ extends to a model of $\mathcal{D}$.

Then λ is an equivalence.

Example 6.10. Last time we had a morphism $\mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$. If $\mathcal{C}$ is Boolean, then (2) is satisfied. This is what we finished with last time. Then (1) is satisfied and (3) is obvious. This will finish the proof of the theorem we stated last time.

Now let us prove the corollary. We want to know that for $X, Y \in \mathcal{C}$, that

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\lambda(X), \lambda(Y))$$

is bijective. First, we show that it is injective. Suppose f and g are mapped to the same morphism. If we look at the equalizer, we get

$$\lambda(\mathrm{Eq}(f,g)) = \mathrm{Eq}(\lambda(f), \lambda(g)) = \lambda(X)$$

as subobjects of $\lambda(X)$. If $f \neq g$, then $\mathrm{Eq}(f,g) \subsetneq X$, and then Deligne's completeness theorem version 2 shows that there exists a model M such that $M[\mathrm{Eq}(f,g)] \subsetneq M[X]$. That is, $f, g : M[X] \to M[Y]$ are not equal. But every model of $\mathcal{C}$ should come from a model of $\mathcal{D}$, and this gives a contradiction.

Now we show that it is surjective. Given any $\bar{f} : \lambda(X) \to \lambda(Y)$, we want some $f : X \to Y$ to satisfies $\bar{f} = \lambda(f)$. We encode this as the graph

$$\Gamma(\bar{f}) \subseteq \lambda(X) \times \lambda(Y) \cong \lambda(X \times Y).$$

Then (2) tells us that there exists a $U \subseteq X \times Y$ such that $\Gamma(\bar{f}) = \lambda(U)$ as subobjects of $\lambda(X \times Y)$. Now we want to show that $\pi : U \to X$ is an isomorphism. It suffices to show that $\pi : M[U] \to M[X]$ is an isomorphism for all M a model of $\mathcal{C}$. This can be checked for models of $\mathcal{D}$, and then $\lambda(U) \to \lambda(X)$ is an isomorphism by definition.

Recall that for a theory T, we defined the syntactic category to be objects $X = [\varphi(\vec{x})]$ and $Y = [\psi(\vec{x})]$ and morphisms given by $\theta(\vec{x}, \vec{y})$ such that

$$T \vDash \forall \vec{x}, \vec{y}(\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})) \wedge \forall \vec{x}(\varphi(\vec{x}) \Rightarrow \exists! \vec{y} \theta(\vec{x}, \vec{y})).$$

But this is semantic. If we really want a syntactic definition, we should replace $T \vDash$ by $T \vdash$. Suppose we have a reasonable definition of proof, and use this to define $\mathrm{Syn}'_0(T)$. Can we show that the functor

$$\lambda : \mathrm{Syn}'_0(T) \to \mathrm{Syn}_0(T)$$

is an equivalence of categories? Suppose that we can justify all arguments made so far, i.e., $\mathrm{Syn}'_0(T)$ is a category, and λ is a morphism of coherent categories. If we can justify that models of $\mathrm{Syn}'_0(T)$ are models of $\mathrm{Syn}_0(T)$, then we can apply our corollary to show that λ is a equivalence of categories. A corollary is Gödel's completeness theorem.

7 February 5, 2018

We have been talking about coherent categories. Most categories you are familiar with, of abelian groups or of commutative rings, don't have (finite) unions of subobjects.

We saw that you can produce semantics from syntax, i.e., $\mathrm{Syn}_0(T)$ to $\mathsf{Mod}(T)$. But we want to go in the other direction.

Example 7.1. Consider a theory T with no predicates and $\exists! x(x = x)$. Compare this with the theory T' with a single predicate P with axioms $\exists! y[P(y)]$ and $(\exists! z)[\neg P(z)]$. These have only one model, so are semantically equivalent, in a boring way. But $\mathrm{Syn}_0(T)$ and $\mathrm{Syn}_0(T')$ are not equivalent. The category $\mathrm{Syn}_0(T)$ is equivalent to $\{0 < 1\}$ and the category $\mathrm{Syn}_0(T')$ is equivalent to finite sets.

So to produce a syntax, we need choose what to pick. We will choose $\mathrm{Syn}_0(T')$, and this has the property that we can take disjoint union.

7.1 Disjoint unions

Proposition 7.2. *Let $\mathcal{C}$ be coherent. Consider $x \in \mathcal{C}$ and consider subobjects $X_1, X_2, \ldots, X_n \subseteq X$ which are disjoint, i.e., $X_i \wedge X_j$ is the least subobject. Then $X_1 \vee \cdots \vee X_n \subseteq X$ is a coproduct of $X_1, \ldots, X_n$.*

Proof. Given $f_i : X_i \to Y$, we want to a unique $X_1 \vee \cdots \vee X_n \to Y$. X does not play a role, so assume that $X = X_1 \vee \cdots \vee X_n \to Y$. We encode they by the graph. We can take $\Gamma(f_i) \subseteq X_i \times Y \subseteq X \times Y$ and then take

$$Z = \Gamma(f_1) \vee \cdots \vee \Gamma(f_n) \subseteq X \times Y.$$

Now the claim is that Z is the graph of some morphism. This means that $h : Z \hookrightarrow X \times Y \to X$ is an isomorphism. My first claim is that h is a monomorphism, which is that $Z \hookrightarrow Z \times_X Z$ is an isomorphism. Applying (A5) twice, we can describe

$$Z \times_X Z = \bigvee_{i,j} \Gamma(f_i) \times_X \Gamma(f_j) \subseteq X \times Y \times Y.$$

If $i \neq j$, then $\Gamma(f_i) \times_X \Gamma(f_j) = \emptyset$ because $\Gamma(f_i) \times_X \Gamma(f_j) \to X$ factors through $X_i \wedge X_j$. (Here, we're using (A5) again.) So

$$Z \times_X Z = \bigvee_i \Gamma(f_i) \times_X \Gamma(f_i) \subseteq Z.$$

This shows that h is a monomorphism.

Going back to the picture, we see that $Z \subseteq X \times Y \to X$ is a monomorphism. So we can take Z as a subobject of X, and it should contain each X_i. This shows that it is Z and so h is an isomorphism. $\square$

Example 7.3. Let us take $n = 0$. Then this shows that the least element $\emptyset \in \mathrm{Sub}(X)$ is actually the initial object in $\mathcal{C}$.

Definition 7.4. Let $\mathcal{C}$ be a category which has fiber products, and consider $X, Y \in \mathcal{C}$ such that $X \amalg Y$ exists. We say that the coproduct is **disjoint** if

(1) $X \hookrightarrow X \amalg Y$ and $Y \hookrightarrow Y$ are monomorphisms and

(2) $X \times_{X \amalg Y} Y$ is the initial object in $\mathcal{C}$.

Proposition 7.5. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) $\mathcal{C}$ is coherent and has disjoint coproducts.

(2) $\mathcal{C}$ satisfies (A1), (A2), (A4), and

(A3′) $\mathcal{C}$ has finite coproducts and they are disjoint,

(A5′) The formation of coproducts commutes with pullback.

Proof. Let's first check (A5′). Given $f : X \to Y$ and $Y_1, \ldots, Y_n$, we want to show that

$$\coprod (X \times_Y Y_i) \to X \times_Y \coprod Y_i$$

is an isomorphism. Y really does't matter, so we may replace Y by $\coprod_i Y_i$. But now, it suffices to show that $X \times_Y Y_i$ are disjoint subobjects of X with union being X. But this is (A5).

Now let $\mathcal{C}$ be a category satisfying (2). Let $X \in \mathcal{C}$ and $X_1, \ldots, X_n \subseteq X$. We want to form $X_1 \vee \cdots \vee X_n \subseteq X$, and to get this, we look at $X_1 \amalg X_2 \amalg \cdots \amalg X_n$. Then we have a map $X_1 \amalg \cdots \amalg X_n \to X$, with image U. This should contain all X_i, and cannot have more, and so is $X_1 \vee \cdots \vee X_n$. □

So we this is a strengthening of the original axioms that allows us to take disjoint unions.

7.2 Equivalence relations

Let me give a loose analogy with algebraic geometry. Let T be theory and consider $[\varphi(x_1, \ldots, x_n)]$. Loosely speaking, this is like cutting out an affine variety. But we would like to construct projective varieties as well, and this is quotienting $\mathbb{A}^{n+1} \setminus \{0\}$.

Definition 7.6. Let $\mathcal{C}$ be any category with finite limits, and let $X \in \mathcal{C}$. An **equivalence relation** on X is a subobject $R \subseteq X \times X$ such that for all $Y \in \mathcal{C}$,

$$\mathrm{Hom}(Y, R) \subseteq \mathrm{Hom}(X, R) \times \mathrm{Hom}(X, R)$$

is an equivalence relation.

For instance, $X \times_Y X \subseteq X \times X$ is an equivalence relation. In fact, an effective epimorphism can be thought of as a coequalizer of a equivalence relation, in the category of sets.

Definition 7.7. We say that an equivalence relation $R \subseteq X \times X$ is **effective** if $R = X \times_Y X$ for some $f : X \to Y$.

Now I am going to replace axiom (A2) with

(A2′) Every equivalence relation is effective.

Proposition 7.8. *Let $\mathcal{C}$ be a category satisfying (A1), (A4), and (A2′). Then it also satisfies (A2).*

Lemma 7.9. *Let $\mathcal{C}$ satisfy (A1) and (A4). Suppose we have a pullback square*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X. \end{tikzcd}$$

If f is an effective epimorphism and f' is an isomorphism, then f is an isomorphism. That is, we can test isomorphisms locally.

Proof. Because g and g' are effective epimorphisms, we have

$$\begin{tikzcd} Y' \times_{X'} Y' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \times_X Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r, "g"] & X. \end{tikzcd}$$

If f' is an isomorphism, then the left vertical map is an isomorphism. Then f is induced by the coequalizer, and so an isomorphism. □

Proof of Proposition. We need to construct an image of $f : X \to Z$. We can look at $X \times_Z X$ and look at the coequalizer $X \xrightarrow{g} Y$. Now we have a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, and need to show that h is a monomorphism. We have pullback squares

$$\begin{tikzcd} X \times_Y X \arrow[r, "\cong"] \arrow[d] & X \times_Z X \arrow[r] \arrow[d] & X \times X \arrow[d, "g \times g"] \\ Y \arrow[r] & Y \times_Z Y \arrow[r] & Y \times Y. \end{tikzcd}$$

Because g is an effective epimorphism, $X \times_Z X \to Y \times_Z Y$ is an effective epimorphism. Then $Y \to Y \times_Z Y$ is an isomorphism. □

Definition 7.10. A **pretopos** is a category $\mathcal{C}$ such that

(A1) $\mathcal{C}$ has finite limits,

(A2′) every equivalence relation is effective,

(A3) $\mathcal{C}$ has disjoint coproducts,

(A4′) pullbacks of effective epimorphisms are effective epimorphism,

(A5′) formation of corproducts commute with pullbacks.

Example 7.11. Set and $\mathsf{Set}^{\mathrm{fin}}$ are pretopoi.

8 February 7, 2018

So pretopoi are special coherent categories.

Definition 8.1. A first-order theory T **eliminates imaginaries** if $\mathrm{Syn}_0(T)$ is a pretopos.

In fact, given a first-order theory, you can construct a first-order theory T^{eq} that eliminates imaginaries. Here is how you do this. Given $X = [\varphi(x)]$ and an equivalence relation $R \subseteq X \times X$ given by $[\psi(x, x')]$, you might worry that X/R does not exist. But you can add a new type t (interpreted as X/R) and a new predicate $P(x, y)$ with arity $t_0, t)$ (where t_0 is the type of x) and axioms

$$\begin{aligned}
&\forall \vec{x}, \vec{y}(P(x, y) \Rightarrow \varphi(x), \quad \forall x \varphi(x) \Rightarrow \exists! y P(x, y), \\
&\forall y \exists x P(x, y), \quad \forall x, x', y, y'(P(x, y) \wedge P(x', y') \Rightarrow (y = y' \Leftrightarrow \psi(x, x'))).
\end{aligned}$$

In fact, this new theory has the same models, because the axioms tell you how to do this.

Theorem 8.2. *Let $\mathcal{C}$ be a small coherent category. Then there exists a pretopos $\mathcal{C}^{\mathrm{eq}}$ and a functor of coherent categories $\lambda : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ with the following property: for any pretopos $\mathcal{D}$,*

$$\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{eq}}, \mathcal{D}) \xrightarrow{\circ \lambda} \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$$

*is an isomorphism. We call this the **pretopos completion** of $\mathcal{C}$.*

Taking $\mathcal{D} = \mathsf{Set}$, we see that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{eq}}$ have the same models.

Proposition 8.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and $l : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

(1) λ is a morphism of coherent categories.

(2) λ is a morphism of coherent categories and preserves finite coproducts.

We will also see that $\mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ is fully faithful. Let me give you some intuition for what this should be. For $X, Y \in \mathcal{C}$, we should be allowed to form $X \amalg Y$. Maybe the coproduct does not exist, or maybe there is a coproduct but it is not disjoint.

So here is a question we need to answer. What is map $Z \to X \amalg Y$ in $\mathcal{C}^{\mathrm{eq}}$? It is clear that both $\mathrm{Hom}_{\mathcal{C}}(Z, X)$ and $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$ lies in $\mathrm{Hom}_{\mathcal{C}}(Z, X \amalg Y)$. But there should be more. Suppose $Z = Z_0 \vee Z_1$ where $Z_0 \wedge Z_1 = \emptyset$. We have seen that in this case, $Z = Z_0 \amalg Z_1$. Then given $Z_0 \to X$ and $Z_1 \to Y$, we can map

$$f : Z = Z_0 \amalg Z_1 \to X \amalg Y.$$

This is a correct description, but we will eventually find $\mathcal{C}^{\mathrm{eq}}$ as a full subcategory of a much larger category $\mathsf{Shv}(\mathcal{C})$.

8.1 Sheaves

Definition 8.4. Let $\mathcal{C}$ be any category with fiber products. A **Grothendieck topology** on $\mathcal{C}$ is a specification of a collection of maps $\{f_i : U_i \to X\}_{i \in I}$ that we will call **covering families** satisfying

(T1) If $\{f_i : U_i \to X\}$ is a covering and $Y \to X$ is any map, then $\{U_i \times_X Y \to Y\}$ is a covering.

(T2) If $\{U_i \to X\}$ is a covering and $\{V_{ij} \to U_i\}$ is a covering, then $\{V_{ij} \to U_i \to X\}$ is a covering.

(T3) If $\{f_i : U_i \to X\}$ such that some f_i admits a section, then it is a covering.

Proposition 8.5. *Any enlargement of a covering is a covering.*

Example 8.6. Let X be a topological space, and let U be the collection of open subsets of X. We declare that $\{U_i \to U\}$ is a covering if $U = \bigcup_i U_i$.

This was motivated by algebraic geometry, where maps are not necessarily inclusions.

Example 8.7. Let X be a scheme, and let $\mathcal{C}$ be the category of schemes U with an étale map $U \to X$. We can take $\{U_i \to U\}$ is a covering if $\coprod U_i \to U$ is surjective.

Proposition 8.8. *Let $\mathcal{C}$ be a coherent category. Then $\mathcal{C}$ has a Grothendieck topology where we declare that $\{f_i : U_i \to X\}_{i \in I}$ is a covering if there is some finite subset $I_0 \subseteq I$ such that $\vee_{i \in I_0} \operatorname{im}(f_i) = X$ (in $\operatorname{Sub}(X)$).*

Proof. (T1) follows since taking images and joins commutes with pullback. For (T2) you can replace all coverings with finite coverings. If the maps $V_{ij} \to X$ factors through some $X' \subseteq X$, then all $V_{ij} \to U_i$ factors through $X' \times_X U_i$, and so $X' \times_X U_i \cong U_i$. Then $X' \cong X$. You can check (T3). $\square$

Definition 8.9. Let $\mathcal{C}$ be a category. A **presheaf** of sets is just a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. If $\mathcal{C}$ has a Grothendieck topology, we say that $\mathscr{F}$ is a **sheaf** if for every covering $\{U_i \to X\}$, the diagram

$$\mathscr{F}(X) \longrightarrow \textstyle\prod_i \mathscr{F}(U_i) \rightrightarrows \textstyle\prod_{i,j} \mathscr{F}(U_i \times_X U_j)$$

is an equalizer. The collection of sheaves forms a category $\mathsf{Shv}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$.

Example 8.10. If X is a topological space, U is the collection open subsets, then $\mathsf{Shv}(X)$ is the sheaves on X.

Let me introduce a bit of notation. If $\mathcal{C}$ is a category and $Y \in \mathcal{C}$, we denote $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ given by

$$h_Y(X) = \operatorname{Hom}_{\mathcal{C}}(X, Y).$$

Proposition 8.11. *Let $\mathcal{C}$ be a coherent category, and let $Y \in \mathcal{C}$. Then $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf.*

Proof. We want to check that

$$\mathrm{Hom}(X, Y) \longrightarrow \prod_i \mathrm{Hom}(U_i, Y) \rightrightarrows \prod_{i,j} \mathrm{Hom}(U_i \times_X U_j, Y)$$

is an equalizer. We first denote $X_i = \mathrm{im}(U_i \to X)$ so that $U_i \twoheadrightarrow X_i \hookrightarrow X$. Then by what we did last time, $f_i : U_i \to Y$ uniquely factors as $\bar{f}_i : X_i \to Y$.

To get $X \to Y$, we encode the objects in the graph $\Gamma(\bar{f}_i)$ in $\mathrm{Sub}(X \times Y)$. Denote $Z = \bigvee \Gamma(\bar{f}_i)$. Then it suffices to show that $Z \to X$ is an isomorphism. To show that it is a monomorphism, we look at

$$\bigvee_i \Gamma(\bar{f}_i, \bar{f}_i) \cong Z \to Z \times_X Z = \bigvee_{i,j} \Gamma(\bar{f}_i, \bar{f}_j).$$

In the previous lecture, the non-diagonal terms were just empty. But here, we claim that $\Gamma(\bar{f}_i, \bar{f}_j) \subseteq \Gamma(\bar{f}_i, \bar{f}_i)$. This follows from the fact that $\bar{f}_i$ and $\bar{f}_j$ agree on $U_i \times_X U_j$. $\square$

Note that

$$\mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}); \quad Y \mapsto h_Y$$

is always fully faithful, and it factors through $\mathsf{Shv}(\mathcal{C})$ if $\mathcal{C}$ is coherent. We will study categories that look like $\mathsf{Shv}(\mathcal{C})$, and if $\mathcal{C}$ is coherent, we are going to find

$$\mathcal{C} \subseteq \mathcal{C}^{\mathrm{op}} \subseteq \mathsf{Shv}(\mathcal{C}).$$

Definition 8.12. A **topos** is a category of the form $\mathsf{Shv}(\mathcal{C})$ where $\mathcal{C}$ is a small category with a Grothendieck topology.

Example 8.13. If X is a topological space, $\mathsf{Shv}(X)$ is a topos. If $\mathcal{C}$ is a coherent category, $\mathsf{Chv}(\mathcal{C})$ is a topos. In particular, if T is a first-order theory, $\mathsf{Shv}(\mathrm{Syn}_0(T))$ is a topos, called the **classifying topos** of T.

This $\mathsf{Shv}(\mathrm{Syn}_0(T))$ is the "space of all models" of T. In general, for $\mathcal{X}$ a topos, the category $\mathcal{C}$ is not part of the data. For instance, $\mathsf{Shv}(\mathcal{C})$ and $\mathsf{Shv}(\mathcal{C}^{\mathrm{eq}})$ is going to be the same topos.

9 February 9, 2018

Let's recall what we were talking about. A category $\mathcal{X}$ is a topos if there is an equivalence $\mathcal{X} \simeq \mathsf{Shv}(\mathcal{C})$ for $\mathcal{C}$ is small, has limits, with a Grothendieck topology.

9.1 Topoi are pretopoi

Theorem 9.1. *Every topos $\mathcal{X}$ is a pretopos.*

Let's assume that $\mathcal{X} = \mathsf{Shv}(\mathcal{C})$.

Theorem 9.2. *The inclusion $\mathsf{Shv}(\mathcal{C}) \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ has an effect adjoint L, and L preserves finite limits.*

It want to show why the second theorem implies the first. Assume Theorem 9.2.

Lemma 9.3. *$\mathcal{X}$ has arbitrary limits.*

Proof. We can compute limits in $\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$. Then the limit is also going to be a sheaf, because limits commute with limits. □

Lemma 9.4. *$\mathcal{X}$ as all colimits.*

Proof. Note that this not just the colimit computed in $\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, which is not necessarily a sheaf. Nevertheless, you can still form the direct limit, and then apply the sheafification L to it. This will do it. □

Lemma 9.5. *Coproducts in $\mathcal{X}$ are disjoint.*

Proof. The product of $\mathscr{F}$ and $\mathscr{G}$ should be $L(\mathscr{F} \amalg \mathscr{G})$. Because L preserves finite limits, we can compute

$$L\mathscr{F} \times_{L(\mathscr{F}\amalg\mathscr{G})} L\mathscr{G} \cong L(\mathscr{F} \times_{\mathscr{F}\amalg\mathscr{G}} \mathscr{G}) \cong L(\emptyset).$$

But $L(\emptyset)$ is the initial object because L preserves finite limits. □

Lemma 9.6. *Every equivalence relation in $\mathcal{X}$ is effective.*

Proof. Let $\mathscr{F}$ be a sheaf and consider an equivalence relation $\mathscr{R} \subseteq \mathscr{F} \times \mathscr{F}$. Now if we define

$$\mathscr{E}(C) = \mathscr{F}(C)/\mathscr{R}(C),$$

then $\mathscr{E}(C)$ is a presheaf. This can be described as $\mathscr{R} \cong \mathscr{F} \times_{\mathscr{E}} \mathscr{F}$. But then,

$$\mathscr{F} \times_{L\mathscr{E}} \mathscr{F} \cong L(\mathscr{F} \times_{\mathscr{E}} \mathscr{F}) \cong L(\mathscr{R}) \cong \mathscr{R}.$$

This is precisely what it means for the equivalence relation to be effective. □

Lemma 9.7. *Colimits in $\mathcal{X}$ are universal, i.e., commute with fiber products.*

Proof. Suppose I give you $\mathscr{F} \to \mathscr{G}$ and a collections of sheaves $\{\mathscr{G}_\alpha\}$ over $\mathscr{G}$. What we want to show is that

$$L(\varinjlim(\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha)) \cong \mathscr{F} \times_{\mathscr{G}} L(\varinjlim \mathscr{G}_\alpha).$$

But we have an isomorphism $\varinjlim_\alpha(\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) \cong \mathscr{F} \times_{\mathscr{G}} \varinjlim \mathscr{G}_\alpha$, and then we can apply L. □

This immediately implies

Lemma 9.8 (A5)**.** *Finite coproducts in $\mathcal{X}$ are preserved by pullback.*

Lemma 9.9. *Pullbacks of effective epimorphisms in $\mathcal{X}$ are effective epimorphisms.*

Proof. Suppose I have a pullback of an effective epimorphism $\mathscr{F} \to \mathscr{G}$ to $\mathscr{F}' \to \mathscr{G}'$. Then the pullback of $\mathscr{R} = \mathscr{F} \times_{\mathscr{G}} \to \mathscr{F}$ is $\mathscr{R}' = \mathscr{F}' \times_{\mathscr{G}'} \mathscr{F}'$. By definition, $\mathscr{G}$ is the coequalizer, which is a colimit, and it should pullback to the coequalizer. This shows that $\mathscr{G}'$ is the coequalizer and $\mathscr{F}' \to \mathscr{G}'$ is an effective epimorphism. □

We didn't use much here, except Theorem 9.2.

Proposition 9.10. *Let $\mathcal{C}$ be a pretopos, and $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory, where $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint L. If L preserves finite limits, then $\mathcal{C}_0$ is a pretopos.*

Proof. Exercise. □

9.2 Constructing sheafification

Now let us see how to prove Theorem 9.2.

Theorem 9.11. *There exists a sheafification functor $L : \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Shv}(\mathcal{C})$ and L is left exact.*

Recall that $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf if, for every covering $\{U_i \to X\}_{i \in I}$, we can write

$$\mathscr{F}(X) \longrightarrow \textstyle\prod_i \mathscr{F}(U_i) \rightrightarrows \textstyle\prod_{i,j} \mathscr{F}(U_i \times_X U_j).$$

But it is convenient to understand this in a different way. You can think of it as

$$\mathscr{F}(X) \to \varprojlim_{V \to X \text{ factoring through some } U_i} \mathscr{F}(V)$$

being an isomorphism.

Definition 9.12. A **sieve** on an object $X \in \mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ such that for any $U \to V$ in $\mathcal{C}_{/X}$, if $V \in \mathcal{C}^{(0)}_{/X}$ then $U \in \mathcal{C}^{(0)}_{/X}$.

Definition 9.13. A sieve $\mathcal{C}_{/X}^{(0)}$ is a **covering** if it contains a covering $\{f_i : U_i \to X\}$.

Example 9.14. For any collection of maps $\{U_i \to X\}$, the sieve $\mathcal{C}_{/X}^{(0)}$ consisting of maps $V \to X$ that factors through some U_i is a covering sieve.

Now we can say that $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf if and only if, for every covering sieve $\mathcal{C}_{/X}^{(0)} \subseteq \mathcal{C}_{/X}$,

$$\mathscr{F}(X) = \varinjlim_{U \in \mathcal{C}_{/X}^{(0)}} \mathscr{F}(U).$$

The good thing is that the space of covering sieves is partially ordered by reverse inclusion. For $\mathscr{F}$ any presheaf, we can construct $\mathscr{F}^\dagger : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ by

$$\mathscr{F}^\dagger(X) = \varinjlim_{\mathcal{C}_{/X}^{(0)} \text{ cov.}} \varprojlim_{U \in \mathcal{C}_{/X}^{(0)}} \mathscr{F}(U).$$

Note that there is a canonical map $\mathscr{F} \to \mathscr{F}^\dagger$. Also, note that this is naturally equivalent to the original sheaf if $\mathscr{F}$ is a sheaf to begin with.

Proposition 9.15. *For any presheaf $\mathscr{F}$, the composite*

$$\mathscr{F} \to \mathscr{F}^\dagger \to \mathscr{F}^{\dagger\dagger}$$

exhibits $\mathscr{F}^{\dagger\dagger}$ as a sheafification of $\mathscr{F}$.

Let me explain why we need the double dagger.

Definition 9.16. A presheaf $\mathscr{F}$ is **separated** if every covering $\{U_i \to X\}$, the map

$$\mathscr{F}(X) \to \prod_i \mathscr{F}(U_i)$$

is injective.

Lemma 9.17. *For any $\mathscr{F}$, $\mathscr{F}^\dagger$ is separated.*

Lemma 9.18. *For any separated $\mathscr{F}$, $\mathscr{F}^\dagger$ is a sheaf.*

Proof. We want to check that $\mathscr{F}^\dagger$ is a sheaf. Suppose we are given a covering $\{U_i \to X\}$, and let us assume we have sections $s_i \in \mathscr{F}^\dagger(U_i)$ that agree on overlaps. Each section s_i comes up in a covering $V_{ij} \to U_i$. Then s_i is given by specifying sections $t_{ij} \in \mathscr{F}(V_{ij})$. We can say that $V_{ij} \to X$ is a covering. We would like to say that $\{t_{ij}\}$ is in $\mathscr{F}^\dagger(X)$.

We need to check the gluing conditions in order to say this. That is, we need to check that t_{ij} and $t_{i'j'}$ agree on $V_{ij} \times_X V_{i'j'}$. Note that s_i and $s_{i'}$ agree as sections of $\mathscr{F}^\dagger(U_i \times_X u_j)$, and hence there exists a covering $W_k \to U_i \times_X U_k$, the representatives agree. Then you need to use the fact that $\mathscr{F}$ is separated. $\square$

10 February 12, 2018

Last time we showed that every topos $\mathcal{X}$ is a pretopos, in particular, with infinite coproducts. What I want to talk about today, is a sort of a converse. But let me introduce a semi-temporary definition.

Definition 10.1. Say $\mathcal{X}$ is a pretopos with infinite coproducts. We say that a collection $\{f_i : U_i \to X\}$ is a **covering** if $\coprod U_i \to X$ is an effective epimorphism.

10.1 Giraud's theorem

Theorem 10.2 (Giraud). *Let $\mathcal{X}$ be a category. The following are equivalent:*

(1) $\mathcal{X}$ is a topos. (This means that $\mathcal{X} \simeq \mathsf{Shv}(\mathcal{C})$ where $\mathcal{C}$ is a small category with finite limits and a Grothendieck topology.)

(2) There exists a fully faithful embedding $i : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ such that i has a left adjoint preserving finite limits, where $\mathcal{C}$ is a small category.

*(3) $\mathcal{X}$ satisfies **Giraud's axioms***

(G1) $\mathcal{X}$ has finite limits

(G2) every equivalence relation is effective

(G3) $\mathcal{X}$ has arbitrary coproducts and they are disjoint

(G4) effective epimorphisms are closed under pullback

(G5) pullbacks of coproducts are coproducts

(G6) there exists a set of objects $\mathcal{U}$ of $\mathcal{X}$ such that every $X \in \mathcal{X}$ has a covering $\{U_i \to \mathcal{X}\}$ where each $U_i \in \mathcal{U}$.

We have proven (1) implies (2), and also (2) implies (3) except for (G6). For this, we note that there is a covering

$$\coprod_{X \in \mathcal{C}, \eta \in \mathscr{F}(X)} L h_X \twoheadrightarrow \mathscr{F}.$$

The interesting direction is (3) implies (1).

Using (G6) we can choose a small full subcategory $\mathcal{C} \subseteq \mathcal{X}$ such that $\mathcal{C}$ contains generators of $\mathcal{X}$, and $\mathcal{C}$ is closed under finite limits in $\mathcal{X}$. (Maybe take the skeleton if you're worried about the difference of small and essentially small.)

Theorem 10.3. *(a) Say that a collection of maps $\{U_i \to X\}$ in $\mathcal{C}$ is a covering if it is so in $\mathcal{X}$. This defines a Grothendieck topology on $\mathcal{C}$.*

(b) For $X \in \mathcal{X}$, define $h_X : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. Then each h_X is a sheaf.

(c) The construction $X \mapsto h_X$ is a an equivalence $\mathcal{X} \simeq \mathsf{Shv}(\mathcal{C})$.

Proof. (a) If $\{U_i \to X\}$ is a covering, and $Y \to X$ is a morphism, we need to show that $\{U_i \times_X Y \to Y\}$ is also a covering. This follows from (G4) and (G5). We also need to check that it is transitive, that if $\{V_{i,j} \to U_i\}$ and $\{U_i \to X\}$ is a covering then $\{V_{i,j} \to X\}$ is a covering. Here, we can imitate the proof for checking that the coverings on a coherent category gives a Grothendieck topology.

(b) For each $\{U_i \to X\}$ a covering, we need to check that

$$\operatorname{Hom}_{\mathcal{X}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{X}}(\textstyle\coprod_i U_i, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{X}}(\textstyle\coprod_{i,j} U_i \times_X U_j, Y)$$

is an equalizer. But this follows from that

$$(\textstyle\coprod_i U_i) \times_X (\textstyle\coprod_j U_j) \rightrightarrows \textstyle\coprod_i U_i \longrightarrow X$$

is a coequalizer. Then we have the isomorphism $\coprod_{i,j} U_i \times_X U_j \cong (\coprod_i U_i) \times_X (\coprod_j U_j)$ by (G5).

(c) Now consider the functor $h : \mathcal{X} \to \mathsf{Shv}(\mathcal{C})$. First note that h preserves inverse limits (in particular, monomorphisms). First let us prove that h is fully faithful. We want to show that

$$\theta_X : \operatorname{Hom}_{\mathcal{X}}(X,Y) \to \operatorname{Hom}_{\mathsf{Shv}(\mathcal{C})}(h_X, h_Y)$$

is bijective. Fix Y, and say that X is "good" of θ_X is bijective.

(i) Any $X \in \mathcal{C}$ is good, by Yoneda.

(ii) If $\{U_i \to X\}$ is a covering, U_i are good, and $U_i \times_X U_j$ are good, then X is good. This is because both

$$\textstyle\coprod_{i,j} U_i \times_X U_j \rightrightarrows \textstyle\coprod_i U_i \longrightarrow X$$

$$\textstyle\coprod_{i,j} h_{U_i} \times_{h_X} h_{U_j} \rightrightarrows \textstyle\coprod_i h_{U_i} \longrightarrow h_X$$

are coequalizers by the following lemma.

(iii) If $X \in \mathcal{X}$ is a subobject of some $\overline{X} \in \mathcal{C}$, then X is good. If $\{U_i \to X\}$ is a covering with $U_i \in \mathcal{C}$, then $U_i \times_X U_j \cong U_i \times_{\overline{X}} U_j$ is in $\mathcal{C}$ because $\mathcal{C}$ is closed under finite limits. Then we can apply (ii).

(iv) All $X \in \mathcal{X}$ are good. For $\{U_i \to X\}$ with $U_i \in \mathcal{C}$, we have $U_i \times_X U_j \subseteq U_i \in U_j \in \mathcal{C}$. So $U_i \times_X U_j$ is good, and by (ii), X is good.

This can't be the end, because we have not used a lot of axioms. We need to show that $h : \mathcal{X} \to \mathsf{Shv}(\mathcal{C})$ is essentially surjective.

(i) h preserves initial objects. If $\{U_i \to X\}$ is a covering in $\mathcal{X}$ then $\{h_{U_i} \to h_X\}$ is a covering in $\mathsf{Shv}(\mathcal{C})$.

(ii) h preserves coproducts. Take $\{X_i\}_{i\in I}$ and $X = \coprod_i X_i$. Then $\{h_{X_i} \to h_X\}$ is a covering by the lemma, and so $\coprod_i h_{X_i} \to h_X$ is an effective epimorphism. We need to show that this is a monomorphism. That is, we want to show that

$$\textstyle\coprod_i h_{X_i} \to (\coprod_i h_{X_i}) \times_{h_X} (\coprod_j h_{X_j}) \cong \coprod_{i,j} h_{X_i \times_X X_j}.$$

This follows from (G3), because $X_i \times_X X_j$ is an initial object so $h_{X_i\times_X X_j}$ is an initial object as well.

Now fix a sheaf $\mathscr{F} \in \mathsf{Shv}(\mathcal{C})$. We want $\mathscr{F} \simeq h_X$ for some $X \in \mathcal{X}$. We know that there exists a covering $\{h_{C_i} \to \mathscr{F}\}$ with $C_i \in \mathcal{C}$. (We had an explicit form.) If we take $U = \coprod_i C_i$, then $h_U = \coprod_i h_{C_i}$. Then $h_U \twoheadrightarrow \mathscr{F}$.

(a) Suppose $\mathscr{F} \subseteq h_{\overline{X}}$ for some $\overline{X} \in \mathcal{X}$. Then we have $h_U \twoheadrightarrow \mathscr{F} \subseteq h_{\overline{X}}$, and it comes from $U \to \overline{X}$. Decompose this into $U \twoheadrightarrow X \hookrightarrow \overline{X}$. Then we get

$$h_U \twoheadrightarrow h_X \hookrightarrow h_{\overline{X}},$$

and it is a fact that a decomposition is unique. So $h_X \simeq \mathscr{F}$.

(b) Let $\mathscr{F}$ be arbitrary. We have an effective epimorphism $h_U \twoheadrightarrow \mathscr{F}$, and so

$$h_R = h_U \times_{\mathscr{F}} h_U \rightrightarrows h_U \twoheadrightarrow \mathscr{F}$$

where h_R is representable because it is inside $h_{U\times U}$. Now you can show that R is an equivalence relation, and then write its quotient of X by (G2). Then $h_X \simeq \mathscr{F}$.

This finishes the proof. □

Lemma 10.4. *Let $\{U_i \to X\}$ be a covering of $\mathcal{X}$. Then $\{h_{U_i} \to h_X\}$ is a covering in $\mathsf{Shv}(\mathcal{C})$.*

Proof. We need to show that $\coprod_i h_{U_i} \to h_X$ is an effective epimorphism. Let us evaluate at a point $\eta \in h_X(C)$. Then we can cover $U_i \times_X C$ by $V_{i,j} \in \mathcal{C}$, and then we can map $\eta \in h_X(C)$ to $h_X(V_{i,j})$, and then pull back to $h_{U_i}(V_{i,j})$. □

11 February 16, 2018

Let $\mathcal{X}$ be a topos. We will say that a collection of maps $\{f_i : U_i \to X\}$ is a **covering** if the induced

$$\coprod U_i \to X$$

is an **effective epimorphism**.

11.1 Quasi-compactness

Definition 11.1. An object $X \in \mathcal{X}$ is **quasi-compact** if every covering $\{U_i \to X\}_{i \in I}$ has a finite subcover $\{U_i \to X\}_{i \in I_0}$ for $I_0 \subseteq I$ finite.

Note that $\{f_i : U_i \to X\}$ is a covering if and only if $\{\mathrm{im}(f_i) \to X\}$ is a covering.

Proposition 11.2. *Let $f : X \to Y$ be an effective epimorphism. If X is quasi-compact, so is Y.*

Proof. If $\{U_i \to Y\}$ is a covering, we have $\{U_i \times_Y X \to X\}$ a covering. Then there is a finite subcover, and then $\{U_i \times_Y X \to Y\}$, is a covering, with each of them factoring through U_i. □

Proposition 11.3. *Let $\{X_i\}_{i \in I}$ be a collection of objects with coproduct $X \in \mathcal{X}$. If I is finite, then X is quasi-compact if and only if all X are quasi-compact.*

Proof. Exercise. □

For instance, the initial object is quasi-compact.

Definition 11.4. Let $\mathcal{X}$ be a topos. An object $X \in \mathcal{X}$ is **quasi-separated** if for any quasi-compact $U, V \in \mathcal{X}$ and any $U \to X \leftarrow V$, their fiber product $U \times_X V$ is quasi-compact.

Sometimes objects are quasi-separated for stupid reasons. In order for this to be a useful notion, we need enough quasi-compact objects.

Example 11.5. Let $\mathfrak{X} \in \mathsf{Shv}(\mathbb{R}^n)$. Then $\mathscr{F} \in \mathcal{X}$ is quasi-compact if and only if $\mathscr{F} \cong \emptyset$. So everything is quasi-separated.

Definition 11.6. A topos $\mathcal{X}$ is **coherent** if there is a collection on objects $\mathcal{U}$ such that

(1) every $U \in \mathcal{U}$ is quasi-compact and quasi-separated,

(2) $\mathcal{U}$ generates X (every X has a covering by objects of $\mathcal{U}$),

(3) $\mathcal{U}$ is closed under finite products.

In particular, this implies that the collection of quasi-compact objects of $\mathcal{X}$ are closed under finite products.

Definition 11.7. Let $\mathcal{C}$ be a small category with finite limits. We say that a Grothendieck topology on $\mathcal{C}$ is **finitary** if every covering $\{U_i \to X\}_{i\in I}$ has a finite subcover.

Proposition 11.8. *If $\mathcal{C}$ has a finitary Grothendieck topology, then $\mathsf{Shv}(\mathcal{C})$ is coherent.*

Proof. We need to construct a class with good finiteness property. Fr $C \in \mathcal{C}$, denote $h_C : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ be $h_C(D) = \mathrm{Hom}(D, C)$ and $L : \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Shv}(\mathcal{C})$. I claim that $\mathcal{U} = \{Lh_C\}_{C\in\mathcal{C}}$ works.

First note that $C \mapsto Lh_C$ preserves finite limits because L preserves all limits. We want for $C \in \mathcal{C}$, Lh_C is quasi-compact. Let $\{\mathscr{F}_i \to Lh_C\}$ be a covering. First note that $\mathrm{id} : C \to C$ determines an element $s \in Lh_C(C)$. This is a covering means that there exists a covering locally, so that for some $\{C_j \to C\}$ can be $s_j \in Lh_C(C_j)$ can be lifted to $\tilde{s}_j \in \mathscr{F}_{ij}(C_j)$ for some $i_j \in I$. But our Grothendieck topology is finitary, so that we may assume that $\{C_j \to C\}$ is finite. Then $\{\mathscr{F}_i \to Lh_C\}_{i=i_j}$ is a finite covering.

Now we show that Lh_C is quasi-separated. Let $\mathscr{F}, \mathscr{G}$ be quasi-compact, and we want to show that $\mathscr{F} \times_{Lh_C} \mathscr{G}$ is quasi-compact. There exists a covering $\{Lh_{D_i} \to \mathscr{F}\}$ which we assume to be finite. Here,

$$\mathrm{Hom}(Lh_{D_i}, Lh_C) = \mathrm{Hom}(h_{D_i}, Lh_C) = Lh_C(D_i).$$

This is a section, so each D_i admits a finite cover such that each of $Lh_{D_i} \to \mathscr{F} \to Lh_C$ comes from a map $D_i \to C$. So we may assume that $\mathscr{F} = h_D$, with $Lh_D \to Lh_C$ coming form a map $D \to C$. Likewise, we can replace $\mathscr{G}$ with Lh_E. Then we have that the fiber product is $Lh_{D\times_C E}$, which is quasi-compact. □

Proposition 11.9. *Every coherent topos $\mathcal{X}$ arises in this way, for a canonically chosen $\mathcal{C}$.*

Definition 11.10. Let $\mathcal{X}$ be a coherent topos. We say that $X \in \mathcal{X}$ is **coherent** if it is quasi-compact and quasi-separated.

Inside $\mathcal{X}$ there is a full subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ of coherent objects.

Lemma 11.11. *Let $\mathcal{X}$ be a coherent topos. An object $X \in \mathcal{X}$ is quasi-separated if and only if for any quasi-compact Y and any $f, g : Y \to X$, their equalizer $\mathrm{Eq}(f, g)$ is quasi-compact.*

Proof. We first want to show that X is quasi-separated under this condition. If $U \to X$ and $V \to X$ with U, V quasi-compact, we can write

$$U \times_X V = \mathrm{Eq}(U \times V \to X).$$

For the converse, assume that X is quasi-separated. For $f, g : Y \to X$ with Y quasi-compact, we want to show that $\mathrm{Eq}(f, g)$ is quasi-compact. We are going to use the full strength of the coherent condition. Take a finite covering $\{U_i \to Y\}$, and then take the fiber product. Then $\mathrm{Eq}(U_i \to X)$ cover $\mathrm{Eq}(Y \to X)$, and it suffices to show that Eq($U_i \to X$ is quasi-compact. Taht is, we may assume $Y \in \mathcal{U}$. Now we observe that

$$\begin{tikzcd} \mathrm{Eq}(Y \to X) \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Y \times_X Y \arrow[r, hook] & Y \times Y \end{tikzcd}$$

is a pullback. Here $Y \times_X Y$ and Y are quasi-compact, and $Y \times Y$ is quasi-separated because it is in $\mathcal{U}$. $\square$

Corollary 11.12. *Let $\mathcal{X}$ be coherent and $X, Y \in \mathcal{X}$ be quasi-separated. Then $X \times Y$ is quasi-separated.*

Proof. We want to know that $\mathrm{Eq}(Z \to X \times Y)$ is quasi-compact. But

$$\mathrm{Eq}(Z \to X \times Y) = \mathrm{Eq}(\mathrm{Eq}(Z \to X) \to Y)$$

is quasi-compact. $\square$

Corollary 11.13. *Let $\mathcal{X}$ be a coherent topos. The coherent objects of $\mathcal{X}$ are closed under finite limits.*

Proof. We want to know that if $X \to Y \leftarrow Z$ and X, Y, Z are coherent then $X \times_Y Z$ is coherent. It is quasi-compact because Y is quasi-separated. Also $X \times Z$ is quasi-separated, and it is not hard to show that subobjects of quasi-separated are quasi-separated. $\square$

Lemma 11.14. *Let $\mathcal{X}$ be a topos, and consider $\mathcal{X}_{\text{q.c.}} \subseteq \mathcal{X}$ the full subcategory of quasi-compact objects. $\mathcal{X}_{\text{q.c.}}$ is essentially small.*

Proof. We know that there is a set $\mathcal{U}$ of generators for $\mathcal{X}$. This means that I can find, for every quasi-compact X, a finite covering $\{U_i \to X\}$. Then there is only a set's worth of choices of this U_i, and a set's worth of choices for the equivalence relations. $\square$

Theorem 11.15. *Let $\mathcal{X}$ be a coherent topos.*

(a) $\mathcal{X}_{\text{coh}}$ has a finatary Grothendieck topology, where $\{U_i \to X\}$ is a covering if it is a covering in $\mathcal{X}$.

(b) The construction $X \mapsto h_X$ induces an equivalence $\mathcal{X} \hookrightarrow \mathsf{Shv}(\mathcal{X}_{\text{coh}})$.

Proof. We checked that $\mathcal{X}_{\text{coh}}$ is a finitary Grothendieck topology. Then (b) is what we proved in Giraud's theorem. $\square$

12 February 21, 2018

Last time we construct from a category $\mathcal{C}$ with finite limits and finitary Grothendieck topology, a topos $\mathsf{Shv}(\mathcal{C})$. We want to see what is lost in this construction.

Definition 12.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi. A **geometric morphism** from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f^* : \mathcal{Y} \to \mathcal{X}$ such that

(1) f^* preserves finite limits,

(2) f^* preserves effective epimorphisms,

(3) f^* preserves coproducts.

Example 12.2. If $f : X \to Y$ is continuous, it induces a functor $f^* : \mathsf{Shv}(Y) \to \mathsf{Shv}(X)$, and it is an example of a geometric morphism.

12.1 Characterizing geometric morphisms

Theorem 12.3. *Let $f^* : \mathcal{Y} \to \mathcal{X}$ be a functor preserving finite limits. Then the following are equivalent:*

(1) f^* is a geometric morphism.

(2) f^* preserves all colimits.

(3) f^* has a right adjoint.

The equivalence between (2) and (3) is the adjoint functor theorem. From (2) to (1), we only need to show that it preserves effective epimorphisms, but this is because coequalizers are send to coequalizers. So the main content of this theorem is (1) implies (2).

We want to show that f^* preserves coequalizers. Let $g, g' : U \to Y$. Then the coequalizer can be thought of as quotienting out by an equivalence relation $R = Y \times_{\mathrm{coeq}} Y \subseteq Y \times Y$. From U, this equivalence relation can be built in the following way. First, without loss of generality we can assume $U \subseteq Y \times Y$ (by taking the image $U \to Y \times Y$). This may not be symmetric, so we replace U by $U \vee U^{\mathrm{op}}$. Then for each $n \geq 0$, we form the n-fold fiber product

$$U \times_Y U \times_Y \cdots \times_Y U \subseteq Y^{n+1} \to Y \times Y.$$

Take the image $U_n \subseteq Y \times Y$ of this map. Now, we can take $R = \bigvee_{n \geq 0} U_n$. Note that all of these operations can be constructed using images, fiber products, and joins.

Definition 12.4. For $\mathcal{X}, \mathcal{Y}$ topoi, we denote $\mathsf{Fun}^*(\mathcal{Y}, \mathcal{X}) \subseteq \mathsf{Fun}^*(\mathcal{Y}, \mathcal{X})$ the full subcategory consisting of geometric morphisms.

How do we compute this in practice? Let $\mathcal{C}$ be a small category with finite limits and a Grothendieck topology. Then we have

$$j : \mathcal{C} \xrightarrow{h} \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \xrightarrow{L} \mathsf{Shv}(\mathcal{C}).$$

Theorem 12.5. *Let $\mathcal{X}$ be any topos. Then composition with j determines a fully faithful*
$$\mathsf{Fun}(\mathsf{Shv}(\mathcal{C}), \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
such that the essential image is those $f : \mathcal{C} \to \mathcal{X}$ such that

(1) f preserves finite limits,

(2) f preserves coverings.

Example 12.6. Let $\mathcal{C}$ be a category with finite limits. Then $\mathcal{C}$ has a topology where $\{f_i : C_i \to C\}$ is a covering of f where todo

We have the following general statement.

Theorem 12.7. *Let $\mathcal{X}$ be any category with small colimits. Then composition with h determines an equivalence*

$$\mathsf{LFun}(\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

where LFun *is the colimit-preserving functors.*

Let me only give a construction in the reverse direction. If $f : \mathcal{C} \to \mathcal{X}$ is any functor, define $\mathscr{F}_X : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ with $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$. Then $X \mapsto \mathscr{F}_X$ has a left adjoint.

Now let $\mathcal{X}$ be a topos, and let $\mathcal{C}$ be a category with finite limits. In this case, there is a subcategory

$$\mathsf{Fun}^*(\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \mathcal{X}) \subseteq \mathsf{LFun}(\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \mathcal{X}).$$

This should correspond to some full subcategory, and my claim is that it corresponds to
$$\mathsf{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{X}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathcal{X}),$$
the functors preserving finite limits. This means that if $F : \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathcal{X}$ preserves colimits, then F preserves finite limits if and only if $f = F \circ h$ preserves finite limits.

Assume that f preserves finite limits. We are going to do something similar to the proof of Giraud's theorem. Let $\mathcal{D} \subseteq \mathcal{X}$ be a small full subcategory containing a set of generators, images of f, and closed under finite limits. We can write this functor $\mathcal{C} \to \mathcal{X}$ as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "h"] & \mathcal{D} \arrow[r, hook] \arrow[d, "h'"] & \mathcal{X} \arrow[d, "\simeq"] \\ \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \arrow[r, "f'"] & \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathsf{Set}) \arrow[r, "L"] & \mathsf{Shv}(\mathcal{D}). \end{tikzcd}$$

By the universal property we have mentioned earlier, we have $F \simeq L \circ F'$. Because L preserves finite limits, it suffices to show that F' preserves finite limits.

Note that F' is given by the left Kan extension along $f^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$. This just means that

$$F'(\mathscr{F})(D) = \varinjlim_{D \to f(C)} \mathscr{F}(C).$$

Then for any object $D \in \mathcal{D}$, we want that the construction

$$\mathscr{F} \mapsto \varinjlim_{D \to f(C)} \mathscr{F}(C)$$

preserves finite limits. This is because the diagram is filtered.

So this was when $\mathcal{C}$ did not have a topology. Now let $\mathcal{C}$ be a category with finite limits and a Grothendieck topology. We have

$$\begin{tikzcd} \mathsf{Fun}^*(\mathsf{Shv}(\mathscr{C}), \mathcal{X}) \arrow[r, "\circ j"] \arrow[d, "\circ L"] & \mathsf{Fun}(\mathcal{C}, \mathcal{X}) \\ \mathsf{Fun}^*(\mathsf{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set}), \mathcal{X}). \arrow[ur, "\circ h"'] & \end{tikzcd}$$

We have shown that the diagonal map is fully faithful, with essential image being the left exact functors.

Let $f : \mathcal{C} \to \mathcal{X}$ be left exact. Then f extends to a geometric morphism $F : \mathsf{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set}) \to \mathcal{X}$. But we want to find a dotted arrow in

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set}) \arrow[r, "F"] \arrow[d, "L"] & \mathcal{X} \\ \mathsf{Shv}(\mathcal{C}). \arrow[ur, dashed] & \end{tikzcd}$$

So when does this exist? This is easier to think in the right adjoints. Recall that $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$. Then we are asking when there exists

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set}) & \mathcal{X} \arrow[l, "X \mapsto \mathscr{F}_X"'] \arrow[dl, dashed] \\ \mathsf{Shv}(\mathcal{C}). \arrow[u, hook] & \end{tikzcd}$$

So when is $\mathscr{F}_X$ a sheaf on $\mathcal{C}$ for all $X \in \mathcal{X}$?

We can just check the sheaf axioms. For every covering $\{C_i \to C\}$, we need

$$\mathrm{Hom}_{\mathcal{X}}(f(C), X) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(\textstyle\coprod f(C_i), X) \rightrightarrows \mathrm{Hom}_{\mathcal{X}}(\textstyle\coprod_{i,j} C_i \times_C C_j, X)$$

to be an equalizer for all X. This just means that

$$\textstyle\coprod f(C_i \times_C C_j) \rightrightarrows \coprod f(C_i) \longrightarrow f(C).$$

But because we are in a topos, we have that the left term is

$$\coprod_{i,j} f(C_i \times_C C_j) \cong \coprod_{i,j} f(C_i) \times_{f(C)} f(C_j) \cong (\textstyle\coprod f(C_i)) \times_{f(C)} (\coprod_j f(C_j)).$$

So that being a coequalizer is just that $\{f(C_i) \to f(C)\}$ is a covering.

Next time, we will apply these ideas in some examples, like $\mathcal{C}$ a coherent category.

Example 12.8. Consider $\mathcal{C} = \{*\}$. Then $\mathsf{Shv}(\mathcal{C}) = \mathsf{Set}$. So what we proved is that

$$\mathsf{Fun}^*(\mathsf{Set}, \mathcal{X}) \cong \mathsf{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{X}) \cong \{*\}$$

because left exact functors should take the final object to the final object. So in the (2-)category of topoi, $\mathsf{Set} = \mathsf{Shv}(*)$ is a final object.

Definition 12.9. Let $\mathcal{X}$ be a topos. A **point** of X is a geometric morphism from Set to $\mathcal{X}$, that is, objects of $\mathsf{Fun}^*(\mathcal{X}, \mathsf{Set})$.

13 February 23, 2018

We introduced two lectures ago the notion of a coherent topos. Recall that $\mathcal{X}$ is coherent if it has "enough" quasi-compact quasi-separated objects. In this case, $\mathcal{X}_{\mathrm{coh}}$ is closed under finite limits.

Lemma 13.1. *If $\mathcal{X}$ is coherent, then $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under finite coproducts.*

Proof. Let $X = \coprod_{i \in I} X_i$, with I finite and X_i coherent. Because each X_i is quasi-compact, X is quasi-compact. For $U \to X \leftarrow V$ where U and V quasi-compact, we want to show that $U \times_X V$ is quasi-compact. We can then write $U_i \to X_i \leftarrow V_i$, and then $U_i \times_{X_i} V_i$ are all quasi-compact. Now we know that $U \times_X V = \coprod_i U_i \times_{X_i} V_i$. □

Lemma 13.2. *Let $\mathcal{X}$ be coherent, and let $U \to X$ be an effective epimorphism. Let $R = U \times_X U$. If U is coherent and R is quasi-compact, then X is coherent.*

Proof. X is quasi-compact because it takes an effective epimorphism from a quasi-compact object. Now we want to prove that X is quasi-separated, and we use the other characterization of quasi-separated objects in the context of coherent topoi. We need to show that if Y is quasi-compact and $f, g : Y \to X$ then $\mathrm{Eq}(f, g) \subseteq Y$ is quasi-compact. We first change Y by taking the pullback

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d, two heads] & U \times U \arrow[d, two heads] \\ Y \arrow[r, "(f,g)"] & X \times X. \end{tikzcd}$$

Then we have $\mathrm{Eq}(\tilde{f}, \tilde{g}) \twoheadrightarrow \mathrm{Eq}(f, g)$. So we may assume that f and g factor through U. Then we get

$$\mathrm{Eq}(f, g) = Y \times_{U \times U} R.$$

Y and R are quasi-compact, and $U \times U$ is coherent. □

So we have proven the following.

Proposition 13.3. *Let $\mathcal{X}$ be a coherent topos. Then $\mathcal{X}_{\mathrm{coh}}$ is a pretopos. (That is, it is closed under finite limits, finite coproducts, quotients by equivalence relations.)*

Proposition 13.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be coherent categories, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

(1) f is a morphism of coherent categories (i.e., f preserves finite limits, effective epimorphisms, finite joins)

(2) f preserves finite limits and also sends coverings to coverings.

Proof. Exercise. □

Proposition 13.5. *Let $\mathcal{C}$ be a coherent category, and let $\mathcal{X}$ be a topos. Then*

$$\begin{tikzcd} \mathsf{Fun}^*(\mathsf{Shv}(\mathcal{C}), \mathcal{X}) \arrow[r, hook] \arrow[dr, "\simeq"'] & \mathsf{Fun}(\mathcal{C}, \mathcal{X}) \\ & \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{X}). \arrow[u, hook] \end{tikzcd}$$

Proposition 13.6. *Let $\mathcal{C}$ be a pretopos. Then $h : \mathcal{C} \to \mathsf{Shv}(\mathcal{C})_{\mathrm{coh}}$ is an equivalence.*

Proof. Because it is a Yoneda embedding, it is fully faithful. So the content is that it is essentially surjective. First lat X be in $\mathcal{C}$ and assume that $\mathscr{F} \subseteq h_X$. If $\mathscr{F}$ is coherent (equivalently quasi-compact), then we want to show that $\mathscr{F} \simeq h_{X_0}$ for some $X_0 \subseteq X$. Let $\{h_{U_i} \to \mathscr{F}\}_{i \in I}$ be a covering with I finite. Then $h_{U_i} \to \mathscr{F} \subseteq h_X$ and so we get $f_i : U_i \to X$. Then define $X_0 = \bigvee_i \mathrm{im}(f_i) \subseteq X$. Then $h_{X_0} = \mathscr{F}$.

Now we want to show that any coherent $\mathscr{F} \in \mathsf{Shv}(\mathcal{C})$ has the form h_X for some $X \in \mathcal{C}$. The functor $h : \mathcal{C} \to \mathsf{Shv}(\mathcal{C})$ is a morphism of coherent categories because it preserves coverings. This implies h preserves finite coproducts. Now the strategy is the same. We have a finite covering $\{h_{U_i} \to \mathscr{F}\}$ and then $h_U \simeq \coprod h_{U_i} \twoheadrightarrow \mathscr{F}$ where $U = \coprod U_i$. Sow we have $h_U \times_{\mathscr{F}} h_U \to h_U \to \mathscr{F}$, and by the first part, we have $h_U \times_{\mathscr{F}} h_U \simeq h_R$. So we have

$$h_R \rightrightarrows h_U \longrightarrow \mathscr{F}.$$

Because $\mathcal{C}$ is a pretopos, R is effective. Then we can quotient U by R and get $R \to U \twoheadrightarrow X$. Then $h_X \simeq \mathscr{F}$. $\square$

So the upshot is that there are mutually inverse constructions

$$\left\{\begin{matrix}\text{coherent} \\ \text{topoi}\end{matrix}\right\} \underset{\mathcal{C} \mapsto \mathsf{Shv}(\mathcal{C})}{\overset{\mathcal{X} \mapsto \mathcal{X}_{\mathrm{coh}}}{\rightleftarrows}} \left\{\begin{matrix}\text{essentially} \\ \text{small pretopoi}\end{matrix}\right\}.$$

For $\mathcal{C}$ a coherent category and $\mathcal{D}$ a pretopos, we get

$$\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathsf{Shv}(\mathcal{D})) \simeq \mathsf{Fun}^*(\mathsf{Shv}(\mathcal{C}), \mathsf{Shv}(\mathcal{D})).$$

We can define

$$\mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{eq}} = \mathsf{Shv}(\mathcal{C})_{\mathrm{coh}}.$$

This is going to be the pretopos completion of $\mathcal{C}$. We have just proved that this is an equivalence for pretopoi $\mathcal{C}$.

Proposition 13.7. *$\mathcal{C}_{\mathrm{eq}}$ is a "pretopos-completion" of $\mathcal{C}$, i.e., for any pretopos $\mathcal{D}$,*

$$\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{D}) \to \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Proof. Without loss of generality, assume that $\mathcal{D}$ is small, because we can make it larger and larger. Then $\mathcal{D} \simeq \mathcal{X}_{\mathrm{coh}}$ for some $\mathcal{X}$ a coherent topos. We have

$$\begin{tikzcd}
\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{X}_{\mathrm{coh}}) \arrow[r] \arrow[d, hook] & \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{X}_{\mathrm{coh}}) \arrow[d, hook] \\
\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{X}) \arrow[r] & \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{X}).
\end{tikzcd}$$

But landing in $\mathcal{X}_{\mathrm{coh}}$ can be tested on $\mathcal{C}$, because we can built everything in $\mathcal{C}_{\mathrm{eq}}$ can be built from $\mathcal{C}$ by finite limits, coproducts, and equivalence relations. So this is a pullback square, and

$$\mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{X}) \simeq \mathsf{Fun}^*(\mathsf{Shv}(\mathcal{C}_{\mathrm{eq}}), \mathcal{X}) \simeq \mathsf{Fun}^*(\mathsf{Shv}(\mathcal{C}), \mathcal{X}) \simeq \mathsf{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{X}). \quad \square$$

Let's look at an example.

Definition 13.8. Let $\mathcal{C}$ be a category with finite limits. A **group object** of $\mathcal{C}$ is an object $G \in \mathcal{C}$ with multiplication $m : G \times G \to G$ such that for each $C \in \mathcal{C}$,

$$\mathrm{Hom}(C, G) \times \mathrm{Hom}(C, G) \xrightarrow{m} \mathrm{Hom}(C, G)$$

is a group structure on $\mathrm{Hom}(C, G)$.

Proposition 13.9. *Let G be a group object $\mathcal{C}$, and let Γ be a finitely presented group. Then the functor*

$$C \mapsto \mathrm{Hom}_{\mathsf{Group}}(\Gamma, \mathrm{Hom}_{\mathcal{C}}(C, G))$$

is represented by an object $G^\Gamma \in \mathcal{C}$.

This construction gives a functor $G \mapsto \{G^\Gamma\}$ and this we get a functor

$$\mathsf{Group}(\mathcal{C}) \to \mathsf{Fun}(\mathsf{Group}^{\mathrm{op}}_{\mathrm{f.p.}}, \mathcal{C}).$$

Proposition 13.10. *This functor is fully faithful, and the image is the left exact functor.*

Definition 13.11. $\mathcal{X}_{\mathrm{group}} = \mathsf{Fun}(\mathsf{Group}_{\mathrm{f.p.}}, \mathsf{Set})$ is the classifying topos of groups.

This is because if $\mathcal{Y}$ is any topos,

$$\mathsf{Fun}^*(\mathcal{X}_{\mathrm{group}}, \mathcal{Y}) \cong \mathsf{Fun}^{\mathrm{lex}}(\mathsf{Group}^{\mathrm{op}}_{\mathrm{f.p.}}, \mathcal{Y}) \simeq \mathsf{Group}(\mathcal{Y}).$$

Note that $\mathcal{X}_{\mathrm{group}}$ is coherent. So $\mathcal{X}_{\mathrm{group}} = \mathsf{Shv}(\mathcal{X}_{\mathrm{group,coh}})$ and so its models will be

$$\mathsf{Mod}(\mathcal{X}_{\mathrm{group,coh}}) \simeq \mathsf{Group}.$$

Contemplate about this pretopos.

14 February 26, 2018

What I want to do today and Wednesday is to talk about a large class of topoi that come from topology.

14.1 Locales

Definition 14.1. A **locale** is a poset $\mathcal{U}$ such that

(1) every subset $S \subseteq \mathcal{U}$ has a least upper bound $\bigvee_{U \in S} U$ (this implies that every T has a greatest lower bound given by $\bigvee\{U \text{ a lower bound of } T\}$)

(2) $U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I}(U \wedge V_i)$.

Example 14.2. Let X be a topological space. Then $\mathcal{U}(X)$ the collection of open subsets $U \subseteq X$ is a locale. It has joins given by taking unions, and meets given by the interior of the intersections. Then (2) is satisfied, because $U \cap \bigcup_{i \in I} V_i = \bigcup_{i \in I}(U \cap V_i)$.

Proposition 14.3. *Let $\mathcal{U}$ be a poset satisfying (1). Then $\mathcal{U}$ is a local if and only if it is a Heyting algebra: that is, for $U, V \in \mathcal{U}$, there exists another object $(U \to V)$ such that $W \subseteq (U \to V)$ if and only if $W \cap U \subseteq V$. (This is also equivalent to that it is cartesian closed.)*

Proof. Exercise. □

Example 14.4. Any complete Boolean algebra is a locale. This is because $(U \to V) = V \bigvee U^c$.

Example 14.5. Let $\mathcal{X}$ be a topos, and let $X \in \mathcal{X}$ be an object. Then $\mathrm{Sub}(X)$ is a locale.

Proof. For $\{U_i \subseteq X\}_{i \in I}$ a set of subobjects of X, we have

$$\bigvee U_i = \mathrm{im}(\textstyle\coprod U_i \to X).$$

Then we have

$$\begin{aligned} V \wedge \bigvee U_i = V \times_X U_i &= V \times_X \mathrm{im}(\textstyle\coprod U_i \to X) = \mathrm{im}((\textstyle\coprod U_i \times_X V) \to V) \\ &= \mathrm{im}(\textstyle\coprod (U_i \times_X V) \to V) = \bigvee_i (V \wedge U_i). \end{aligned}$$ □

So any topos gives you lots and lots of locales.

Definition 14.6. Let $\mathcal{X}$ be a topos and let $1 = 1_{\mathcal{X}}$ be a final object. We define $\mathrm{Sub}(1)$ as the **underlying locale** of $\mathcal{X}$.

Note that $\mathrm{Sub}(1) \subseteq \mathcal{X}$ is a full subcategory.

Definition 14.7. A topos $\mathcal{X}$ is **localic** if $\mathrm{Sub}(1)$ generates $\mathcal{X}$, that is, every object $X \in \mathcal{X}$ has a covering $\{U_i \to X\}$ with $U_i \hookrightarrow 1$.

Proposition 14.8. *Let $\mathcal{U}$ be any locale. Then $\mathcal{U}$ has a topology whose coverings are families $\{U_i \leq X\}$ such that $X = \bigvee U_i$.*

Proof. Exercise. Note that for X a topological space and $\mathcal{U} = \mathcal{U}(X)$, we recover the usual Grothendieck topology. □

So for any locale $\mathcal{U}$, we can associate to it a topos $\mathsf{Shv}(\mathcal{U})$.

Proposition 14.9. *Let $\mathcal{X}$ be a localic topos. Then*

$$h : \mathcal{X} \to \mathsf{Shv}(\mathrm{Sub}(1))$$

is an equivalence.

Proof. This follows from the proof of Giraud's theorem. Note that Sub(1) is closed under finite limits. □

Proposition 14.10. *Let $\mathcal{U}$ be a locale. Then $\mathcal{U} \simeq \mathrm{Sub}(1_{\mathsf{Shv}(\mathcal{U})})$. More precisely,*

$$h : U \to \mathsf{Fun}(\mathcal{U}^{\mathrm{op}}, \mathsf{Set})$$

as essential image consisting of sheaves which are subobjects of the final object.

Proof. First, let $U \in \mathcal{U}$. Then

$$h_U(V) = \begin{cases} \{*\} & V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\{V_i \to V\}$ be a covering. We want

$$h_U(V) \to \prod h_U(V_i) \rightrightarrows \prod h_U(V_i \wedge V_j)$$

to be an equalizer. That is, $V \leq U$ if and only if $V_i \leq U$. This shows that it is a sheaf, and that it is subobject of 1.

To complete the proof, suppose that $\mathscr{F}$ is any sheaf on $\mathcal{U}$ with $\mathscr{F} \subseteq 1$. Then $\mathscr{F}(U)$ is either empty or a singleton for each U. We want $\mathscr{F}$ to be representable. So we define

$$U = \bigvee_{\mathscr{F}(V) \neq \emptyset} V.$$

The claim is that $\mathscr{F} \simeq h_U$, which means that $\mathscr{F}(V) \neq \emptyset$ if and only if $V \leq U$. For the reverse direction, invoke the fact that V is a sheaf. It suffices to know that $\mathscr{F}(U) \neq \emptyset$. We have $\{V \to U\}_{\mathscr{F}(V) \neq \emptyset}$ a covering of U, and then we can recover $\mathscr{F}(U)$ as a one-element set. □

The upshot is that the following are equivalent:

$$\left\{ \begin{matrix} \text{localic} \\ \text{topos } \mathcal{X} \end{matrix} \right\} \underset{\mathcal{U} \mapsto \mathsf{Shv}(\mathcal{U})}{\overset{\mathcal{X} \mapsto \mathrm{Sub}(1_{\mathcal{X}})}{\rightleftarrows}} \{\text{locale } \mathcal{U}\}$$

I want to sharpen this a bit.

Definition 14.11. Let $\mathcal{U}$ and $\mathcal{V}$ be locales. A **morphisms of locales** is a map $f^* : \mathcal{V} \to \mathcal{V}$ such that f^* preserves ordering, infinite joins, finite meets. Let me write $\mathsf{Fun}^*(\mathcal{V},\mathcal{U}) \subseteq \mathsf{Fun}(\mathcal{V},\mathcal{U})$ for the full subcategory morphisms of locales.

Let $\mathcal{U}$ be a locale, and $\mathcal{X}$ be a topos. What can we say about $\mathsf{Fun}^*(\mathsf{Shv}(\mathcal{U}), \mathcal{X})$? We have shown that

$$\mathsf{Fun}^*(\mathsf{Shv}(\mathcal{U}), \mathcal{X}) \xrightarrow{\circ h} \mathsf{Fun}(\mathcal{U}, \mathcal{X})$$

is fully faithful, with image consisting of functors preserving finite limits and coverings. If $f : \mathcal{U} \to \mathcal{X}$ preserves finite limits, then $f(1) \cong 1$. Then $f(\mathcal{U}) \subseteq \mathsf{Sub}(1_{\mathcal{X}})$. Then the condition of preserving finite limits coverings is the same as preserving finite meets and joins. Thus the image is equivalent to $\mathsf{Fun}^*(\mathcal{U}, \mathrm{Sub}(1_{\mathcal{X}}))$.

Proposition 14.12. *Geometric morphisms of topoi from $\mathcal{X}$ to $\mathsf{Shv}(\mathcal{U})$ are equivalent to morphisms of locales $\mathrm{Sub}(1_{\mathcal{X}})$ to $\mathcal{U}$.*

So we have a pair of adjoint functors

$$\left\{\text{topoi } \mathcal{X}\right\} \underset{\mathcal{U}\mapsto \mathsf{Shv}(\mathcal{U})}{\overset{\mathcal{X}\mapsto \mathrm{Sub}(1_{\mathcal{X}})}{\rightleftarrows}} \left\{\text{locales } \mathcal{U}\right\}.$$

Question. *What is the relationship between locales and topological spaces?*

I already explained the construction $X \mapsto \mathcal{U}(X)$. This is a functor. That is, if $f : X \to Y$ is continuous, then we get a map

$$f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$$

of locales. But this functor

$$\mathsf{Top} \to \{\text{locales}\}; \quad X \mapsto \mathcal{U}(X)$$

is not fully faithful.

Example 14.13. Let X have the trivial topology: $\mathcal{U}(X) = \{\emptyset, X\}$. This does not depend on X at all.

It is also not essentially surjective. Many locales do not come from topological spaces. We are going to prove the following next time.

Theorem 14.14. *The construction $X \mapsto \mathcal{U}(X)$ is induces an equivalence*

$$\left\{\begin{matrix}\textit{sobor topological}\\ \textit{spaces}\end{matrix}\right\} \xrightarrow{\simeq} \left\{\begin{matrix}\textit{spatial}\\ \textit{locales}\end{matrix}\right\}.$$

15 February 28, 2018

Last time we defined a locale, which is a poset behaving like the open sets of a topological space. We had a construction

$$\{\text{topological spaces}\} \to \{\text{locales}\}; \quad X \mapsto \mathcal{U}(X).$$

15.1 Points of a locale

Definition 15.1. Let $\mathcal{U}$ be a locale. A **point** x of $\mathcal{U}$ is:

(A) a morphism of locales $\{0,1\} \to \mathcal{U}$, which is a morphism $x^* : U \to \{0 < 1\}$ that preserves joins and finite meets.

(B) a subset $\mathcal{U}_x \subseteq \mathcal{U}$ (the elements $U_x = \{U \in \mathcal{U} : x^* U = 1\}$) such that

  (a) if $U \leq V$ and $U \in U_x$ then $V \in U_x$,

  (b) if $\bigvee U_i \in \mathcal{U}_x$, then some $U_i \in \mathcal{U}_x$,

  (c) $\mathcal{U}_x$ is closed under finite limits.

Let us write $\mathrm{Pt}(\mathcal{U})$ as the set of points of $\mathcal{U}$.

Note that (a) and (b) imply that there is a $U(x) = \bigvee_{U \notin \mathcal{U}_x} U$ such that $\mathcal{U}_x = \{U \not\leq U(x)\}$. (c) translates to the assertion that $U(x)$ is prime: $U(x)$ is not the largest element of $\mathcal{U}$ and $U(x) = V \wedge W$ implies $U(x) = V$ or $U(x) = W$. Then a point of $\mathcal{U}$ just a prime element.

For any $U \in \mathcal{U}$, associate the set of points

$$\tilde{U} = \{x : U \in \mathcal{U}_x\} \subseteq \mathrm{Pt}(\mathcal{U}).$$

That is, we have a map

$$\mathcal{U} \longrightarrow \{\text{subsets of } \mathrm{Pt}(\mathcal{U})\}.$$

The conditions tell us that

(a) this is order-preserving, i.e., $U \leq V$ implies $\tilde{U} \subseteq \tilde{V}$,

(b) $U \mapsto \tilde{U}$ preserves joins,

(c) $U \mapsto \tilde{U}$ preserves finite meets.

So we they form a topology, and we can set $\mathrm{Pt}(\mathcal{U})$ to be a topological space. Then we have a construction

$$\mathcal{U} \longrightarrow \mathcal{U}(\mathrm{Pt}(\mathcal{U})).$$

Let X be a topological space, and let $\mathcal{U}$ be any locale. Then a continuous map $f : X \to \mathrm{Pt}(\mathcal{U})$ is taking $x \in X$ to $\mathcal{U}_{f(x)}$. This is a binary relation on $X \times \mathcal{U}$ meaning "$f(x) \in U$". This relation satisfies

(a) "$f(x) \in U$" implies "$f(x) \in V$" if $U \leq V$,

(b) "$f(x) \in \bigvee U_i$" implies "$f(x) \in U_i$" for some i,

(c) "$f(x) \in U_i$" implies "$f(x) \in \bigwedge U_i$" where the meet is finite

(d) for each $U \in \mathcal{U}$, the set $\{x \in X : \text{“}f(x) \in U\text{”}\}$ is open.

You can also think of this as taking $\mathcal{U}$ to subsets of X. Then this is just a map from $\mathcal{U}(X)$ to $\mathcal{U}$ of locales. That is, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{cont}}(X, \mathrm{Pt}(\mathcal{U})) \cong \mathrm{Hom}_{\mathrm{Locales}}(\mathcal{U}(X), \mathcal{U}) = \mathsf{Fun}^*(\mathcal{U}(X), \mathcal{U}).$$

So we have a pair of adjoint functors

$$\begin{Bmatrix} \text{topological} \\ \text{spaces} \end{Bmatrix} \underset{\mathcal{U} \mapsto \mathrm{Pt}(\mathcal{U})}{\overset{X \mapsto \mathcal{U}(X)}{\rightleftarrows}} \{\text{locales}\}.$$

What happens if we do this construction in succession? The points in $\mathrm{Pt}(\mathcal{U}(X))$ is the open subsets $U \subseteq X$ which are **prime**, that is, $U \neq X$ and $U = V \cap W$ implies $U = V$ or $U = W$. Equivalently, if we write $K = X - U$, then K is **irreducible**, that is, $K \neq \emptyset$ and $K = K_0 \cup K_1$ implies $K = K_0$ or $K = K_1$.

15.2 Sober spaces and spatial locales

Definition 15.2. A topological space X is called **sober** if $X \to \mathrm{Pt}(\mathcal{U}(X))$ is bijective, i.e., every irreducible set in X has a unique "generic point".

If this is satisfied, then $X \to \mathrm{Pt}(\mathcal{U}(X))$ is a homeomorphism.

Example 15.3. Any Hausdorff space is sober. Any scheme is sober.

Proposition 15.4. *For any locale $\mathcal{U}$, the points $\mathrm{Pt}(\mathcal{U})$ is sober.*

Proof. We can show that

$$\mathrm{Pt}(\mathcal{U}) \xrightarrow{f_{\mathrm{Pt}(\mathcal{U})}} \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U}))) \xrightarrow{\mathrm{Pt}(g : \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U})} \mathrm{Pt}(\mathcal{U})$$

is the identity. We want to show that $f_{\mathrm{Pt}(\mathcal{U})}$ is bijective. So it suffices to show that $\mathrm{Pt}(g)$ is injective. But note that $g^* : \mathcal{U} \to \mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is surjective, so the points should be injective as sets. You can also prove this directly. $\square$

Definition 15.5. A locale $\mathcal{U}$ is called **spatial** if $g : \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U}$ is and isomorphism of locales.

We know that $\mathcal{U} \to \mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is necessarily surjective. Note that $\mathcal{U}$ is spatial if and only if $\mathcal{U}$ has "enough points", i.e., if $U, V \in \mathcal{U}$ are distinct, then there exist a point $x \in \mathrm{Pt}(\mathcal{U})$ such that $U \in \mathcal{U}_x$ and $V \notin \mathcal{U}_x$ or vice versa. (This just means U and V are different in $\mathcal{U}(\mathrm{Pt}(\mathcal{U}))$.)

Proposition 15.6. *A locale $\mathcal{U}$ is spatial if and only if $\mathcal{U} \cong \mathcal{U}(X)$ for some topological space X.*

Proof. Only if is clear. For if, we note that for $U, V \in \mathcal{U}(X)$ these are open sets and $f : X \to \mathrm{Pt}(\mathcal{U}(X))$. $\square$ todo

Corollary 15.7. *The adjunction gives an equivalence*

$$\left\{\begin{matrix}\textit{sober}\\ \textit{topological spaces}\end{matrix}\right\} \simeq \{\textit{spatial locales}\}.$$

The inclusion

$$\{\textit{sober spaces}\} \hookrightarrow \{\textit{spaces}\}$$

has a left adjoint $X \mapsto \mathrm{Pt}(\mathcal{U}(X))$. The inclusion

$$\{\textit{spatial locales}\} \hookrightarrow \{\textit{locales}\}$$

has a right adjoint $\mathcal{U} \mapsto \mathcal{U}(\mathrm{Pt}(\mathcal{U})) = \mathcal{U}/\simeq$ with $U \simeq V$ if $\tilde{U} = \tilde{V}$.

Example 15.8. If X has the trivial topology and nonempty, then this is $X \mapsto \{*\}$.

Example 15.9. This doesn't have to be surjective. If X is a variety over $\mathbb{C}$ with the Zariski topology (in the classical sense). Then $\mathrm{Pt}(\mathcal{U}(X))$ is the underlying space of X as a scheme.

In practice, failure to be sober is a pathology. So X to $\mathrm{Pt}(\mathcal{U}(X))$ is an improvement. For locales, $\mathcal{U} \mapsto \mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ often loses interesting information. There are many interesting examples of locales with no points.

Example 15.10 (Deligne)**.** Let $\mathcal{U}$ be the set of equivalence classes of measurable sets $X \subseteq [0,1]$, with $X \sim Y$ if $\mu((X - Y) \cup (Y - X)) = 0$. This is a poset, and in fact a Boolean algebra. It is even a complete: every $\{X_i\}_{i \in I}$ has a join $\bigvee X_i$. (This is generally not given by the union unless I is countable.) For any collection $\{X_i\}_{i\in I}$, we can find a countable subcollection with the same join, because the measure of the union of the segment can only increase in a countable way. So $\mathcal{U}$ is a locale. But $\mathcal{U}$ has no points. If $U \subseteq [0,1]$ is prime, then $[0,1] - U$ does not have measure zero and cannot decompose into two smaller sets. But this is impossible.

16 March 2, 2018

So we finished the first part of the class. Let me give a outline of this class. We first translated a first-order theory into a Boolean pretopos. By passing to a sheaf, we showed that these are equivalent to coherent topoi, and they are contained in topoi. In topoi, there are localic topoi, which are equivalent to locales. Inside tehre are spatial locales, which were equivalent to sober topological spaces.

Question. *Let $\mathcal{X}$ be a topos. How far can $\mathcal{X}$ be from localic topoi?*

Theorem 16.1 (Joyale–Tierney)**.** *Let $\mathcal{X}$ be a topos. Then there is a geometric morphism $\pi : \mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is localic and*

$$\operatorname{colim}(\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows\!\!\to \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows \mathcal{U}) \xrightarrow{\cong} \mathcal{X}.$$

Moreover, everything in this diagram (except $\mathcal{X}$) will be localic.

I haven't explained what these are, but topoi are going to be 2-categories, and this is supposed to be quotienting $\mathcal{U}$. Let me tell you the end of the proof, which is constructing $\mathcal{U}$ from $\mathcal{X}$.

16.1 Classifying topoi

Definition 16.2. Let $\mathsf{Set}_{\text{fin}}$ be the category of finite sets. Now define

$$\mathcal{X}_{\text{Set}} = \mathsf{Fun}(\mathsf{Set}_{\text{fin}}, \mathsf{Set}) \simeq \mathsf{Fun}^{\text{fil.colim}}(\mathsf{Set}, \mathsf{Set}).$$

This is going to be a topos, and we are going to call this the **classifying topos of sets**. There is a canonical object $X_0 \in \mathcal{X}_{\text{Set}}$, given by the inclusion functor. In other words, it is corepresented by $*$.

Proposition 16.3. *Let $\mathcal{X}$ be any topos. Evaluation on $X_0 \in \mathcal{X}_{\text{Set}}$ induces and equivalence*

$$\mathsf{Fun}(\mathcal{X}_{\text{Set}}, \mathcal{X}) \simeq \mathcal{X}.$$

Proof. Note that $\mathcal{X}_{\text{Set}}$ are presheaves on $\mathsf{Set}_{\text{fin}}^{\text{op}}$, and have finite limits. So we have

$$\mathsf{Fun}^*(\mathcal{X}_{\text{Set}}, \mathcal{X}) \simeq \mathsf{Fun}^{\text{lex}}(\mathsf{Set}_{\text{fin}}^{\text{op}}, \mathcal{X}).$$

But note that if S is a finite set, we have $S = \coprod_{s \in S}\{s\}$ in $\mathsf{Set}_{\text{fin}}$ so we have $S = \prod_{s \in S}\{s\}$ in $\mathsf{Set}_{\text{fin}}^{\text{op}}$. So we should have $f(S) = f(*)^S$. $\square$

It follows that $\mathcal{X}_{\text{set}}$ is not localic. If it were localic, given any $X \in \mathcal{X}$ we can write down a geometric morphism $\pi : \mathcal{X} \to \mathcal{X}_{\text{Set}}$ such that $\pi^* X_0 \cong X$, and then we would get that X is covered by subobjects of 1. So this is the furthest possible topos from being localic.

Proposition 16.4. *There is a Grothendieck topology on $\mathsf{Set}_{\text{fin}}^{\text{op}}$ where a collection of maps $\{S \to T_i\}_{i \in I}$ is a covering if some $S \to T_i$ is injective.*

Definition 16.5. Let us define $\mathcal{X}_{\mathrm{Set}\neq\emptyset} = \mathsf{Shv}(\mathsf{Set}_{\mathrm{fin}}^{\mathrm{op}}) \subseteq \mathcal{X}_{\mathrm{Set}}$. This is the **classifying topos of nonempty sets**.

Proposition 16.6. *$F : \mathsf{Set}_{\mathrm{fin}} \to \mathsf{Set}$ is a sheaf if and only if $F(\emptyset) \to F(*) \rightrightarrows F(**)$ is an equalizer.*

You can also check that $\mathcal{X}_{\mathsf{Set}\neq\emptyset} \cong \mathsf{Fun}(\mathsf{Set}_{\mathrm{fin}}^{\neq\emptyset}, \mathsf{Set})$. Note that $X_0 \in \mathcal{X}_{\mathsf{Set}\neq\emptyset}$.

Proposition 16.7. *Let $\mathcal{X}$ be any topos. Then*

$$\mathsf{Fun}^*(\mathcal{X}_{\mathsf{Set}\neq\emptyset}, \mathcal{X}) \to \mathcal{X}$$

given by evaluation on X_0 is fully faithful, and the essential image consists of $X \in \mathcal{X}$ such that $X \to 1$ is an effective epimorphism.

Proof. There is a fully faithful embedding of the left hand side to $\mathsf{Fun}(\mathsf{Set}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X})$ with image the functors preserving coverings and finite limits. If $f(*) = X$, then we have $f(S) \cong X^S$. When does this preserve coverings? We need every injection $S \hookrightarrow T$ to induce an effective epimorphism $X^T \to X^S$. This is automatic if $S \neq \emptyset$. If $S = \emptyset$, then $X^T \to 1$ should be an effective epimorphism. $\square$

So $\mathcal{X}_{\mathsf{Set}\neq\emptyset}$ and this is again some free thing generated by X_0. This is also very very far from localic, but let us look at morphisms from a localic topos to $\mathcal{X}$.

Example 16.8. Consider $\mathrm{Equiv}(\mathbb{Z})$ the set of all equivalence relations on $\mathbb{Z}$. There is a topology generated by

$$U_{i,j} = \{E : iEj\}.$$

On this topological space, we have a sheaf $\mathscr{F}$ informally described as

$$\mathscr{F}_E = \mathbb{Z}/E.$$

More formally, we are taking the quotient

$$\underline{\mathbb{Z} \times \mathbb{Z}} \supseteq \mathscr{R} \rightrightarrows \underline{\mathbb{Z}} \twoheadrightarrow \mathscr{F}.$$

Here, $\mathscr{F} \to 1$ is an effective epimorphism. Now we have a geometric morphism

$$\varphi : \mathsf{Shv}(\mathrm{Equiv}(\mathbb{Z})) \to \mathcal{X}_{\mathrm{Set}\neq\emptyset}$$

such that $\mathscr{F} \simeq \varphi^* X_0$.

More generally, let $\mathcal{X}$ be any topos, and let $X \in \mathcal{X}$ be a "nonempty" object. (Again, this means that the map to 1 is an effective epimorphism.) Then there exists a geometric morphism $\psi : \mathcal{X} \to \mathcal{X}_{\mathrm{Set}\neq\emptyset}$ such that $X \simeq \psi^* X_0$. In the 2-category of topoi, you can take fiber products.

$$\begin{tikzcd}
\mathrm{Enum}(X) \arrow[r, "\psi'"] \arrow[d, "\varphi'"] & \mathsf{Shv}(\mathrm{Equiv}(\mathbb{Z})) \arrow[d, "\varphi"] \\
\mathcal{X} \arrow[r, "\psi"] & \mathcal{X}_{\mathrm{Set}\neq\emptyset}.
\end{tikzcd}$$

Now we have

$$\psi'^*\underline{\mathbb{Z}} \twoheadrightarrow \psi'^*\mathscr{F} \simeq \varphi'^*X.$$

So $\mathcal{X}$ is covered by countable copies. So here is how the proof of the theorem of Joyale–Tierney will work. Let $\mathcal{X}$ be any topos, and choose a set of generators $\{X_i\}$. Take $X = \coprod_i X_i$. Without loss of generality, assume that X is "nonempty". Then we have a covering

$$\mathcal{U} = \mathrm{Enum}(X) \to \mathcal{X}.$$

Example 16.9. Let $\mathcal{X} = \mathsf{Set}$ and X some set. Consider

$$\begin{tikzcd} \mathrm{Enum}(X) \arrow[r] \arrow[d] & \mathsf{Shv}(\mathrm{Equiv}(\mathbb{Z})) \arrow[d] \\ \mathsf{Set} \arrow[r] & \mathcal{X}_{\mathrm{Set}^{\neq\emptyset}}. \end{tikzcd}$$

Let us take points here. Then we should get

$$\begin{tikzcd} \mathrm{Surjection}(\mathbb{Z} \to X) \arrow[r] \arrow[d] & \mathrm{Equiv}(\mathbb{Z}) \arrow[d] \\ * \arrow[r, "X"] & \mathsf{Set}^{\neq\emptyset}. \end{tikzcd}$$

If X was uncountable, the this the points of $\mathrm{Enum}(X)$ is empty, but $\mathrm{Enum}(X) \to \mathsf{Set}$ is still going to be an open surjection.

17 March 5, 2018

We were trying to prove Joyal and Tierney's theorem of resolving topoi.

$$\begin{tikzcd} \mathrm{Enum}(X) \arrow[r] \arrow[d] & \mathsf{Shv}(\mathrm{Equiv}(\mathbb{Z})) \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}_{\mathrm{Set}^{\neq\emptyset}} \end{tikzcd}$$

17.1 Open maps

Recall that that a map $f : X \to Y$ of topological spaces is open if $U \subseteq X$ implies $f(U) \subseteq Y$ open. In this case, the map $\mathcal{U}(X) \to \mathcal{U}(Y)$ has a left adjoint $U \mapsto f(U)$.

Definition 17.1. Let $\mathcal{U}$ and $\mathcal{V}$ be locales. We say that $f : \mathcal{U} \to \mathcal{V}$ is a morphism of locales is **open** if

(1) f^* has a left adjoint $f_!$,

(2) for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$, $f_!(U \wedge f^*V) = f_!(U) \wedge V$.

Example 17.2. Let $f : X \to Y$ be a map of topological spaces. If f is open then the induced map $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open. This is because $U \mapsto f(U)$ gives the adjoint and $f(U \cap f^{-1}(V)) = f(U) \cap V$.

The converse can fail even for sober spaces.

Proposition 17.3. *Let $f : X \to Y$ is a continuous map and assume that each $y \in Y$ is closed. Then the following are equivalent:*

(1) f is open.

(2) $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open.

(3) $f^* : \mathcal{U}(Y) \to \mathcal{U}(X)$ has a left adjoint.

Proof. We only need to show (3) $\Rightarrow$ (1). Then for each $U \subseteq X$, we have $f_!(U) \subseteq Y$. This should be the smallest open subset V of Y such that $U \subseteq f^{-1}(V)$, or $f(U) \subseteq V$. Tautologically, we have $f_!(U) \supseteq f(U)$. Suppose that $y \in f_!(U)$. Then $f_!(U)$ is not contained in $Y - \{y\}$ which is open. This shows that $f(U)$ is not contained in $Y - \{y\}$ and $y \in f(U)$. So $f_!(U) = f(U)$ and we get (1). $\square$

Example 17.4. Consider $Y = \{0, \eta\}$ with $X = \{0\}$. There exists an $f_!$ given by $f_!(X) = Y$, but it is not an open morphism of locales because $f_!(X \cap f^{-1}(\{\eta\})) = \emptyset$ while $f_!(X) \cap \{\eta\} = \{\eta\}$.

Definition 17.5. A map of locales $f : \mathcal{U} \to \mathcal{V}$ is an **open surjection** if f is open and $f_!(1_U) = 1_V$.

Note that $f_!(1_U) = 1_V$ implies more generally that $f_!(f^*V) = V$ for any $V \in \mathcal{V}$ by the projection formula.

Proposition 17.6. *Let $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ be maps of locales. If f and g are open, then $g \circ f$ is open. (The corresponding statement for open surjection also holds.)*

Example 17.7. Let $\mathcal{X}$ be a topos, and let $f : X \to Y$ be a map in $\mathcal{X}$. Then we have a morphism

$$\mathrm{Sub}(X) \to \mathrm{Sub}(Y); \quad V \mapsto X \times_Y V,$$

and this is an open map of locales. This is because we have $f_!(U) = \mathrm{im}(U \to Y)$ and

$$\begin{aligned} \mathrm{im}(U \to Y) \wedge V &= \mathrm{im}(U \to Y) \times_Y V \\ &= \mathrm{im}(Y \times_Y V \to Y) = \mathrm{im}(U \times_X (X \times_Y V) \to Y) \end{aligned}$$

giving the projection formula. This is an open surjection if and only if f is an effective epimorphism.

Now let me give the general definition.

Definition 17.8. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then for each $Y \in \mathcal{Y}$, we get a map of locales $\mathrm{Sub}(f^*Y) \to \mathrm{Sub}(Y)$. We say that f is **open** if each f_Y is open, and it is an **open surjection** if each f_Y is.

Proposition 17.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. Suppose I give you $\mathcal{Y}_0 \subseteq \mathcal{Y}$ a set of generators. Then f is open if and only if f_Y is open for each $Y \in \mathcal{Y}_0$, and similarly for open surjections.*

Proof. One direction is obvious. Assume that f_Y is open for $Y \in \mathcal{Y}_0$. Choose any $Y \in \mathcal{Y}$, and we want to show that f_Y is open. Choose a covering $\{Y_i \to Y\}_{i \in I}$ with $Y_i \in \mathcal{Y}_0$. We have a morphism of locales $\mathrm{Sub}(Y) \to \mathrm{Sub}(f^*Y)$ and first want to show that this has a left adjoint. When is $U \subseteq f^*V$ (in $\mathrm{Sub}(f^*Y)$)? This this true if and only if $U \times_{f^*Y} f^*Y_i \subseteq f^*V \times_{f^*Y} f^*Y_i = f^*(V \times_Y Y_i)$ for all i. Because $\mathrm{Sub}(f^*Y_i) \to \mathrm{Sub}(Y)$ is an open surjection, this is equivalent to

$$f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \subseteq V \times_Y Y_i$$

for all i. The conclusion is that given $U \subseteq f^*Y$,

$$f_{Y!}(U) = \bigvee_{i \in I} \mathrm{im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y)$$

is the smallest $V \subseteq Y$ such that $U \subseteq f^*V$. This gives the left adjoint $f_{Y!}$ for $f_Y^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(f^*Y)$. To complete the proof, we need to check the projection formula for $f_{Y!}$, and this is an exercise. □

Proposition 17.10. *Let $\mathcal{X}$ be a topos, and let $\mathcal{U}$ be a locale. Let $f : \mathcal{X} \to \mathsf{Shv}(\mathcal{U})$ be a geometric morphism, which corresponds to $g : \mathrm{Sub}(1_{\mathcal{X}}) \to \mathcal{U}$. Then f is open (as a map of topoi) if and only if g is open (as a map of locales).*

Proof. The only if direction is clear. For the if direction, assume that g is open. We want that for any $Y \in \mathsf{Shv}(\mathcal{U})$ the map $f_Y : \mathrm{Sub}(f^*Y) \to \mathrm{Sub}(Y)$ is open. But we can check this on representable sheaves h_U for $U \in \mathcal{U}$. So we check that

$$f_Y : \mathrm{Sub}(f^* h_U) \to \mathrm{Sub}(h_U)$$

is open. Then we are actually proving that

$$\{W \subseteq 1_{\mathcal{X}} : W \leq g^* U\} \to \{V \in \mathcal{U} : V \leq U\}$$

is open. But we can construct the left adjoint of f_Y^* by just restricting $g_!$ to the subset, and also the projection formula follows immediately. □

Corollary 17.11. *Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales. Then f is open (open surjective) if and only if $\mathsf{Shv}(\mathcal{U}) \to \mathsf{Shv}(\mathcal{V})$ is open (open surjective) as a map of topoi.*

Corollary 17.12. *Let $\mathcal{X}$ be any topos. Then the unit map $u : \mathcal{X} \to \mathsf{Shv}(\mathrm{Sub}(1_{\mathcal{X}}))$ is an open surjection.*

Proof. This is because $\mathrm{Sub}(1_{\mathcal{X}}) \to \mathrm{Sub}(1_{\mathcal{X}})$ is an open surjection. □

Let me advertise what we will do next time. We will study the map

$$\mathsf{Shv}(\mathrm{Equiv}(\mathbb{Z})) \to \mathcal{X}_{\mathrm{Set}^{\neq\emptyset}}$$

is an open surjection.

Proposition 17.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, and let $Y \in \mathcal{Y}$ be an object. Given a map $\alpha : X \to f^*Y$, we get a morphism*

$$\mathrm{Sub}(X) \to \mathrm{Sub}(f^*Y) \to \mathrm{Sub}(Y)$$

of locales.

(1) If f_Y is open, then so is $u_X : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is open.

(2) If f^*Y has a covering $\{X_i \to f^*Y\}$ and each U_{X_i} is open, so is f_Y.

Proof. For (2), we can find this smallest open object by

$$f_{Y!}(U) = \bigvee_{i \in I} u_{x_i!}(U \times_{f^*Y} X_i).$$

The projection formula follows. □

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