Math 232a - Introduction to Algebraic Geometry I

Notes by Dongryul Kim

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There are going to be weekly problem sets due Fridays, a midterm, and a take-home final. The textbook is going to be Mumford, *Algebraic Geometry I: Complex Projective Varieties.*

Algebraic geometry is the study of the geometry of sets of solutions to systems of polynomial equations. Special cases include linear algebra (when polynomials are all linear) and Galois theory (when there is one variable).

Consider an algebraic curve, like $x^2 + y^2 = 1$ over the rational numbers. Finding solutions $x, y \in \mathbb{Q}$ amounts to finding Pythagorean triples. These rational points can be found using the stereographic projection to the y-axis. If t is rational, then the second intersection point of $x^2 + y^2 = 1$ and $y = t(x + 1)$ is going to be a rational point.

To which equations or curves can we generalize this trick? We need at least one solution P, so equations like $x^n + y^n = 1$ for $n > 2$ don't work. Next, a generic line through P has to intersect the curve in exactly one other point. This second condition will follow if the curve is of degree 2 (or some singular higher degree curve). This is a special case of

Theorem 1.1 (Bezout)**.** *Curves C_1 and C_2 with $\deg C_1 = m$ and $\deg C_2 = n$ intersect at most mn points.*

Actually, they intersect in exactly mn points if counted correctly. This is sort of a philosophy, and we need to look at virtual fundamental classes to make this true.

1.1 Complex projective space

Still there are obvious counterexamples. To get rid of these, we solve over $\mathbb{C}$. For instance, we look at $\{x^2 + y^2 = 1\} \subseteq \mathbb{C}^2$. Topologically, this is a cylinder and is non-compact. This means that there are some missing solutions. Compactifying is solving in $\mathbb{P}^2$. Consider

$$\mathbb{C}P^2 = \{(X, Y, Z) : \text{not all zero}\}/(X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z).$$

This contains a copy of $\mathbb{C}^2$ as $Z = 1$. In this space, we are solving the homogeneous equation $X^2 + Y^2 = Z^2$. There are two additional solutions $(1, \pm i, 0)$. If we add these in, we will get a set homeomorphic to S^2.

Look at the map $\mathbb{C}^2 \to \mathbb{C}P^1$ given by $(x, y) \to x/y$. The map is undefined at $(0, 0)$, because it tries to take every value. The procedure to fix this is to "blow-up" at the origin. We replace the origin in $\mathbb{C}^2$ by a copy of $\mathbb{C}P^1$, which we call E. This is a set-theoretic description, and we can glue them so that the line L through $(0, 0)$ intersects the blowup in $L \in E$.

The new space is not $\mathbb{C}P^2$, because the curves cut out by $X = 0$ and $Y = 0$ don't intersect. In fact, this is $\mathbb{C}P^1 \times \mathbb{C}P^1$.