Math 231br - Advanced Algebraic Topology

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This course was taught by Eric Peterson. The class met at MWF 1-2 in Science Center 221. The textbook for reference was *Algebraic Topology—Homotopy and Homology* by Switzer. There were 8 undergraduates and 11 graduate students enrolled. The grading was based on 1/3 bi-weekly problem sets, 1/3 a midterm paper, and 1/3 a final paper. The course assistant was Xiaolin (Danny) Shi. There are also lecture notes posted on the official website: `http://math.harvard.edu/~ecp/teaching/Spring2017/231b/`.

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1.1 Introduction

The goal of this course is to give an introduction to the homotopy theory of spaces.

- Decomposition of spaces (co/fiber sequences)
- Invariants of theses sorts of decompositions ($\pi_*, H_*, H^*, \ldots$) and their properties (Whithead, Hurewicz, ...)
- Representability theorems (Brown, Adams, ...) and the stable category ($\mathbb{S}, MO, H\mathbb{Z}, KO, \ldots$)
- Computations (characteristic classes, Bott periodicity, π_* of, say, spheres)

Suppose you have some favorite (simply connected) space X and you want to compute $\pi_*(X)$. The place to start is the following theorem.

Theorem 1.1 (Hurewicz)**.** *If* $\pi_{*,n}X = 0$ *then* $H_n(X;\mathbb{Z}) \cong \pi_n X$.

There is a decomposition of X into

$$X[n+1,\infty) \longrightarrow X \longrightarrow K(\pi_n X, n),$$

which is a fiber sequence. The homotopy groups of $K(\pi_n X, n)$ are well known:

$$\pi_* K(\pi_n X, n) = \begin{cases} \pi_n X & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

There is also a long exact sequence coming from the fiber sequence. So

$$\pi_* X[n+1,\infty) = \begin{cases} \pi_* X & \text{if } * > n \\ 0 & \text{otherwise.} \end{cases}$$

Now we need to know $H_* X[n+1,\infty)$. There is this machine called the Serre spectral sequence, and this computes $H_* X[n+1,\infty)$ out of $H_* X$ and $H_* K(\pi_n X, n)$.

The textbook we are going to use is Switzer's *Algebraic Topology*. A more reasonable book is Spanier's *Algebraic Topology*. I am going to assign two papers.

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2.1 Some facts about the category of spaces

I am going to prove some technical lemmas and some basic constructions.

Suppose $X = \bigcup_j A_j$ is a decomposition into a locally finite collection of closed subsets A_j. Then a continuous function $f : X \to T$ is the same data as continuous functions $f_j : A_j \to T$ that agree on the overlap:

$$f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}.$$

This is called **locality**, or **sheaf condition**.

Definition 2.1. A **sheaf** is an assignment from subsets (usually taken to be open) to arbitrary sets ($\mathcal{F}(A)$) together with restriction maps $\mathrm{res}^A_B : F(A) \to F(B)$ for $B \subseteq A$ such that:

- $\mathrm{res}^B_C \circ \mathrm{res}^A_B = \mathrm{res}^A_C$
- for a "cover" $\{A_j\}$ of X, the following sequence is an equalizer

$$\mathcal{F}(X) \longrightarrow \prod_j \mathcal{F}(A_j) \underset{\mathrm{res}^{A_j}_{A_j \cap A_k}}{\overset{\mathrm{res}^{A_j}_{A_k \cap A_j}}{\rightrightarrows}} \prod_{k,l} \mathcal{F}(A_k \cap A_l)$$

Definition 2.2. An **equalizer** E of sets $f, g : S \to T$ is the subset of S such that f and g restrict to the same functions.

Lemma 2.3. *Suppose that $h : F \to S$ is some function with the property $f \circ h = g \circ h$. Then there is unique factorization $F \to E \to S$.*

Example 2.4. For some fixed space T, the assignment $\mathcal{F}(A) = \{\text{cont. func. } A \to T\}$ is a sheaf.

Let us look at another property. For two spaces X and Y, there is a space $X \times Y$ with the property

$$\{\text{cont. func. } T \to X \times Y\} = \{(f, g) : \text{cont. } (f : T \to X), (g : T \to Y)\}.$$

In the category of spaces, we are going to write $\mathsf{Spaces}(A, B)$ for the set of continuous functions $A \to B$. Then

$$\mathsf{Spaces}(T, X \times Y) = \mathsf{Spaces}(T, X) \times \mathsf{Spaces}(T, Y).$$

Given an equivalence relation R on a space X, the space of equivalence relations X/R also forms a space with a continuous map $X \to X/R$.

Lemma 2.5. *If $f : X \to Y$ is a continuous function with the property that xRx' then $f(x) = f(x')$, then there is a factorization*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & T \\ X/R \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where all arrows are continuous.

As a special case, for $A \subseteq X$ take the maximal relation on A and extend by the identity relation on X. The space formed by this relation is called X/A. (By convention, $X/\emptyset = X \amalg \{*\}$.) This also have a categorical interpretation. If $A \to X \to Y$ is constant, then there is a factorization.

$$\begin{tikzcd} A \arrow[r, hook, "i"] & X \arrow[r, "f"] \arrow[d] & Y \\ & X/A \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

Lemma 2.6. *Suppose R is a relation on X and S is a relation on Y, then there is a map*

$$\frac{X \times Y}{R \times S} \to \frac{X}{R} \times \frac{Y}{S}$$

which is always a continuous bijection. If Y is locally compact and S is the identity relation, then this map is a homeomorphism.

Example 2.7. Take $Y = [0,1]$ and $S = \mathrm{id}$, then $(X/\alpha) \times I \cong (X \times I)/(\alpha \times \mathrm{id})$.

For X and Y spaces, Y^X is the space of continuous functions $X \to Y$ (with the compact-open topology).

Lemma 2.8. *If X is locally compact, then the evaluation map $Y^X \times X \to Y$ is continuous. If X and Z are additionally Hausdorff, then $Y^{Z \times X} \cong (Y^Z)^X$.*

Corollary 2.9. *There is a natural bijection $\mathsf{Spaces}(X \times Z, Y) \cong \mathsf{Spaces}(X, Y^Z)$ with respect to X and Y. For example, given $g : Y \to Y'$ the diagram commutes.*

$$\begin{tikzcd} \mathsf{Spaces}(X \times Z, Y) \arrow[r] \arrow[d, "g_*"] & \mathsf{Spaces}(X, Y^Z) \arrow[d, "g_*"] \\ \mathsf{Spaces}(X \times Z, Y') \arrow[r] & \mathsf{Spaces}(X, (Y')^Z) \end{tikzcd}$$

Often we will care about "pairs of spaces" ($A \subseteq X$) or spaces with a choice of basepoint ($\{x_0\} \subseteq X$). Maps of such things are continuous functions $f : X \to Y$ with the property $f(A) \subseteq B$.

Lemma 2.10. *The product of two such pairs is*

$$(X, A) \times (Y, B) = (X \times Y, (X \times B) \cup (A \times Y)).$$

You can also build function spaces $(Y, B)^{(X,A)}$ and there is a natural bijection

$$\mathsf{Pairs}((X, A) \times (Z, C), (Y, B)) \cong \mathsf{Pairs}((X, A), ((Y, B)^{(Z,C)}, Y^C)).$$

Example 2.11. With a basepoint, the product of spaces is given by

$$(X, \{x_0\}) \times (Y, \{y_0\}) = (X \times Y, (X \times \{y_0\}) \cup (\{x_0\} \times Y)) = (X \times Y, X \vee Y).$$

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Note that pointed spaces are pairs. That is, there is a functor $\mathsf{Spaces}_* \to \mathsf{Pairs}$. In fact, there is a functor in the other direction $\mathsf{Pairs} \to \mathsf{Spaces}_*$ given by $(X, A) \mapsto (X/A, *)$.

As I've said last time, the product of two pairs (X, x_0) and (Y, y_0) is $(X \times Y, X \vee Y)$. But then we can collapse it down to

$$(X \times Y, X \vee Y) \mapsto \Big(\frac{X \times Y}{X \vee Y}, *\Big).$$

This is some kind of a product. So we get a bijection

$$\mathsf{Spaces}(X, Y^Z) \cong \mathsf{Spaces}(X \wedge Z, Y).$$

This is called the **smash product**.

3.1 The fundamental group

Recall that the path space of X is $X^I = X^{[0,1]}$ and the set $\pi_0(X)$ is the set of connected components of X^I. Let us write $[Y, X]$ for the set of homotopy classes of maps $Y \to X$. Then $\pi_0(X) = [*, X]$, or in Spaces_*,

$$\pi_0(X, x_0) = [S^0, X].$$

As a remark, $[Y, X]$ can be expressed as $\pi_0(X^Y)$.

Also recall that $\pi_1(X, x_0)$ is the collection of based loops in X, taken up to homotopy. Then

$$\pi_1(X, x_0) = [S^1, X] = [S^0 \wedge S^1, X] = [S^0, X^{S^1}] = \pi_0(X^{S^1}).$$

This X^{S^1} is called the **loop space** and is denoted $\Omega X = X^{S^1}$.

Note that $\pi_1(X)$ is also a group. Expressing π_1 as above, what's special about $\pi_0\Omega(-)$ or $[S^1, -]$ that makes these functors group-valued?

Definition 3.1. A **group** is a set G together with maps $\mu : G \times G \to G$, $\eta : * \to G$, $\chi : G \to G$ satisfying

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \cong G \times * \arrow[d, "1\times\eta"] \arrow[dr, "\mu"] & \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[dr] & G \times G \arrow[r, "\chi\times 1"] & G \times G \arrow[d, "\mu"] & G \times G \arrow[l, "1\times\chi"'] & G \arrow[l, "\Delta"'] \arrow[dl] \\
& * \arrow[r, "\eta"] & G & * \arrow[l, "\eta"'] &
\end{tikzcd}$$

all commute.

Lemma 3.2. *If G is a group, then $\mathsf{Sets}(X, G)$ forms a group for any other set X.*

Proof. The multiplication is given by

$$\mu : \mathsf{Sets}(X, G) \times \mathsf{Sets}(X, G) \cong \mathsf{Sets}(X, G \times G) \xrightarrow{\mu} \mathsf{Sets}(X, G).$$

The other maps can be defined and check that the diagrams commute. This is actually automatic, because you can take every diagram in the definition and the apply $\mathsf{Sets}(X, -)$. □

So why are $\pi_0\Omega(-)$ or $[S^1, -]$ valued in groups? The answer is that ΩX is a group object in the homotopy category of Spaces_*. So if you know this, then $[S^0, \Omega X]$ is automatically a group. (In fact, $[\varphi, \Omega X]$ is automatically a group.)

There is a second answer. It is that S^1 forms a cogroup inside the homotopy category of Spaces_*. You at least know how to define this thing.

Definition 3.3. A **cogroup** is a group with all the arrows turned around and all the products converted to coproducts. Namely, there are maps $\mu' : C \to C \vee C$ satisfying all the things we wanted. (Note that the initial object and the terminal object are the same, so we don't have to replace the point.)

Example 3.4. The S^1 with the multiplication map $S^1 \to S^1 \vee S^1$ the pinching map is a cogroup.

This explanation is more satisfying because we are not really using anything about X.

Note that $S^1 \wedge (-)$ is also a cogroup inside the homotopy category of Spaces_*. (This is because of the distributive law $Z \wedge (X \vee Y) = (Z \wedge X) \vee (Z \wedge Y)$.) This functor is also quite important that it gets its own name. We call $\Sigma(-) = S^1 \wedge (-)$ the **(reduced) suspension**.

Lemma 3.5. *The natural map $[\Sigma X, Y] = [X, \Omega Y]$ is an isomorphism of groups.*

Proof. Take $f, g : \Sigma X \to Y$ and $f', g' : X \to \Omega Y$. Then their multiplication is defined by the two rows of the following diagram.

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\Delta'} Y$$

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f' \times g'} \Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y$$

You can check that the two maps are homotopic. □

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4.1 Higher homotopy groups

Definition 4.1. We define the n**th homotopy group** as

$$\pi_n(X, x_0) = [\Sigma^n S^0, X] = \cdots = [S^0, \Omega^n X].$$

In the intermediate objects, there are two group structures coming from Σ on the left and Ω on the right. Are they the same?

Lemma 4.2 (Eckmann–Hilton). *Let S be a set with two product, $*$ and $\circ$, that share a unit and $(x * x') \circ (y * y') = (x \circ y) * (x' \circ y')$. Then $* = \circ$ and both are commutative and associative.*

Proof. We have

$$\begin{aligned} x \circ y &= (x * e) \circ (e * y) = (x \circ e) * (e \circ y) = x * y \\ &= (e * x) \circ (y * e) = (e \circ y) * (x \circ e) = y * x. \end{aligned}$$

Associativity can be checked. □

Corollary 4.3. $[\Sigma^{n-1} S^0, \Omega X]$ *has only one multiplication and it is commutative. In fact, if K is a H-cogroup and L is a H-group, then $[K, L]$ has only one group structure.*

Proof. We need to check that for $f, g, f', g' \in [K, L]$ the following diagram commutes.

$$\begin{tikzcd}
 & K \times K \arrow[r, "\mu' \times \mu'"] & (K \vee K) \times (K \vee K) \arrow[r, "(f \vee f') \times (g \vee g')"] & (L \vee L) \times (L \vee L) \arrow[r, "\mathrm{fold} \times \mathrm{fold}"] & L \times L \arrow[dr, "\mu"] & \\
K \arrow[ur, "\Delta"] \arrow[r] \arrow[dr, "\mu'"'] & \Delta(K) \arrow[u, hook] \arrow[r] & \begin{array}{c}(K \times *) \times (K \times *) \cup \\ (* \times K) \times (* \times K)\end{array} \arrow[u, hook] \arrow[r] \arrow[d, "1 \times T \times 1"] & \begin{array}{c}(L \times *) \times (L \times *) \cup \\ (* \times L) \times (* \times L)\end{array} \arrow[u, hook] \arrow[d, "1 \times T \times 1"] \arrow[ur] & & L \\
 & K \vee K \arrow[r, "\Delta \vee \Delta"] & (K \times K) \vee (K \times K) \arrow[r] & (L \times L) \vee (L \times L) \arrow[r, "\mu \times \mu"] & L \vee L \arrow[ur, "\mathrm{fold}"'] &
\end{tikzcd}$$

You can check that each small piece commutes. □

Note that there is a homeomorphism $S^{n+1} = \Sigma S^n$. So we can just write $\pi_n X = [S^n, X]$.

4.2 Exact sequences

Recall that an **exact sequence** of group is a pair of maps $N \xrightarrow{f} G \xrightarrow{g} K$ with $g^{-1}(e) = \operatorname{im} f$. A natural question to ask if you were fascinated by the previous lectures is, when does a sequence of spaces $A \to B \to C$ induce an exact sequence of sets

$$[-, A] \to [-, B] \to [-, C] \quad \text{or} \quad [A, -] \leftarrow [B, -] \leftarrow [C, -]?$$

Note that they are pointed sets, and so there will be a notion of exactness. If the first sequence is exact, we call $A \to B \to C$ a **exact sequence** and if the second is exact, we call it a **coexact sequence**.

Lemma 4.4. *Any map of $f : X \to Y$ extends to a coexact sequence $X \to Y \to Z$.*

Proof. We construct Z. Define

$$X = Y \cup_f CX = \frac{Y \amalg CX}{f(x) \sim (x, 1)},$$

where $CX = X \wedge I$ is the cone on X. Note that a null-homotopy of a map $X \to T$ is the same thing as a map $CX \to Y$. So a null-homotopy of a map $f^*\gamma$ for $\gamma \in [Y, T]$ is a map $CX \to T$. Then by the sheaf condition there exists a map $Y \cup_f CX \to T$. The other side is easier. $\square$

You can iterate this process:

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{f} (Y \cup_f CX) \cup_i CY \longrightarrow \cdots.$$

These look nasty, but they're not.

Lemma 4.5. $(Y \cup_f CX) \to ((Y \cup_f CX) \cup_i CY)/CY$ *is a homotopy equivalence.*

Lemma 4.6. *For $A \subseteq X$ a subspace $(X \cup_i CA)/CA \cong X/A$ is a homeomorphism.*

As a consequence,

$$(Y \cup_f CX) \cup_i CY \simeq ((Y \cup_f CX) \cup_i CY)/CY \simeq (Y \cup_f CX)/Y \cong \Sigma X.$$

The 5th term becomes ΣY. So there is an infinitely long coexact sequence

$$X \xrightarrow{f} Y \to Y \cup_f CX \to \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \to \Sigma(Y \cup_f CX) \to \Sigma^2 X \to \cdots.$$

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Let us write $Y \cup_f CX = Cf$. We have

$$X \xrightarrow{f} Y \to Cf \to \Sigma X \to \Sigma Y \to \cdots$$

and so we get

$$[X,T] \xleftarrow{f^*} [Y,T] \leftarrow [Cf,T] \leftarrow [\Sigma X,T] \leftarrow [\Sigma Y,T] \leftarrow [\Sigma Cf,T] \leftarrow \cdots$$

Lemma 5.1. *There is an action of* $[\Sigma X, T]$ *on* $[Cf, T]$.

Proof. You can collapse the part in the middle of CX in Cf we get a map $Cf \to \Sigma X \vee Cf$. □

Lemma 5.2. *There is a factorization* $[Cf,T] \to [Cf,T]/[\Sigma X,T] \to [Y,T]$.

Proof. It is clear from the construction that the action of ΣX doesn't change the map on the Y part. □

5.1 Exact sequences

When we constructed coexact sequences we needed that a null-homotopy of $f : X \to Y$ is the same as a map $H : CX \to Y$. This is a map $X \to Y^I$ satisfying some conditions. Guided by this, we can construct an exact sequence in the following way.

For a map $f : X \to Y$ define

$$P_f = \{(x,\gamma) \in X \times (Y^I) : \gamma(1) = f(x)\}.$$

Proposition 5.3. $P_f \to X \to Y$ *is an exact sequence of* Spaces_*.

Iterating this yields a long exact sequence of pointed spaces,

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to \Omega P_f \to \Omega X \to \Omega Y \to P_f \to X \to Y.$$

Applying π_0 gives

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 P_f \to \pi_1 X \to \pi_1 Y \to \pi_0 P_f \to \pi_0 X \to \pi_0 Y,$$

which is is a long exact sequence of homotopy groups associated to any map $f : X \to Y$.

What is this P_f object, and how can we calculate anything about its π_*? This P_f is really big to be satisfied.

For now, restrict attention to inclusions $i : A \hookrightarrow X$ rather than $f : X \to Y$. Then $(X, A, \{x_0\})$ is a pair of pointed spaces. Then we can write $P_i = (X, A, \{x_0\})^{(I, \partial I, 0)}$ and use the exponential adjunction:

$$\pi_{n-1} P_i = [(S^{n-1}, *), (P_i, \gamma_0)] = [(D^n, S^{n-1}), (X, A)] = \pi_n(X, A).$$

We define the **relative homotopy group** $\pi_n(X, A)$ in this way.

Corollary 5.4. *There is an long exact sequence*

$$\cdots \to \pi_2(A) \to \pi_2(X,A) \to \pi_1(A) \to \pi_1(X) \to \pi_1(X,A) \to \pi_0(A) \to \pi_0(X).$$

Definition 5.5. A pair (X, A) is called n**-connected** when $\pi_{\leq n}(X,A) = 0$. An inclusion $i : A \hookrightarrow X$ is a **weak equivalence** if it is ∞-connected.

You can convert any map into an inclusion as follows:

$$\begin{tikzcd}
 & M_f = \dfrac{Y \times I \cup X}{(y,0) \sim f(y), * \times I} \\
Y \arrow[ur, hook, "i"] \arrow[r, "f"] & X \arrow[u, hook]
\end{tikzcd}$$

Then $X \to M_f$ is a homotopy equivalence so we have $P_f = P_i$. This M_f is called the **reduced mapping cylinder**.

Consider a pair of inclusions $B \subseteq A \subseteq X$. Then there are three pairs and so we have:

$$\begin{tikzcd}
\pi_{n+1}(X,A) \arrow[rr] \arrow[dr] & & \pi_n(A,B) \arrow[rr] \arrow[dr] & & \pi_{n-1}B \arrow[rr] \arrow[dr] & & \pi_{n-1}X \\
 & \pi_n A \arrow[ur] \arrow[dr] & & \pi_n(X,B) \arrow[ur] \arrow[dr] & & \pi_{n-1}A \arrow[ur] \arrow[dr] & \\
\pi_n B \arrow[rr] \arrow[ur] & & \pi_n X \arrow[rr] \arrow[ur] & & \pi_n(X,A) \arrow[rr] \arrow[ur] & & \pi_{n-1}(A,B)
\end{tikzcd}$$

Proposition 5.6. *The sequence in the middle of the diagram only consisting of relative homotopy groups is exact.*

This smells like homology.

Proof. You can check this by diagram chasing. $\square$

Consider the special case of a retraction $r : X \to A$ so that $r \circ i \simeq \mathrm{id}_A$. Then $r_* \circ i_* = \mathrm{id}_{\pi_* A}$ implies that i_* is an inclusion. Then

$$\pi_{n+1}(X,A) \xrightarrow{0} \pi_n A \hookrightarrow \pi_n X \to \pi_n(X,A) \xrightarrow{0} \pi_{n-1}(A) \to \cdots.$$

So if $n \geq 2$ then we get a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_n A \arrow[r, shift left, "i_*"] & \pi_n X \arrow[l, shift left, "r_*"] \arrow[r] & \pi_n(X,A) \arrow[r] & 0.
\end{tikzcd}$$

That is, $\pi_n X \cong \pi_n A \oplus \pi_n(X,A)$.

6 February 3, 2017

6.1 The action of the fundamental group

You might recall there is this action of $\pi_1 X$ on $\pi_n X$. At least you might recall that a path $\gamma : I \to X$ induces an isomorphism $\Gamma : \pi_1(X, \gamma(0)) \leftarrow \pi_1(X, \gamma(1))$.

Definition 6.1. A group G is said to act **compatibly** an another group A, if

(i) G acts on A by $\alpha : G \times A \to A$,

(ii) the multiplication map $A \times A \to A$ is G-equivariant, i.e., $g(a_1 a_2) = (ga_1)(ga_2)$.

Example 6.2. G acts compatibly on itself by conjugation: $(g_1 g_2)^g = g_1^g g_2^g$.

If A is abelian, then a compatible G-action is identical information to a $\mathbb{Z}[G]$-module structure on A.

Theorem 6.3. *S^1 coacts compatibly on S^n for all $n \geq 1$.*

If K coacts compatibly on L, then $[K, T]$ acts compatibly on $[L, T]$.

Corollary 6.4. *$\pi_1(X)$ act compatibly on $\pi_n X$ for all $n \geq 1$.*

Proof of Theorem 6.3. We compose the following maps:

$$\alpha' : S^n \xrightarrow{\text{collapse}} \frac{S_1^n \amalg S_2^n}{*_1 \sim (-*)_2} \xrightarrow{\text{collapse}} \frac{S_1^n \amalg I_2}{*_1 \sim 1_2} \xrightarrow{\text{collapse } \partial I} S^n \vee S^1.$$

This is the candidate for our coaction. We have to check coassociativity, counitality, cocompatibility. The first two things are not interesting. Cocompatibility can be checked in the following diagram.

$$\begin{tikzcd}
S^n \arrow[r, "\mu'"] \arrow[d, "\alpha'"] & S^n \vee S^n \arrow[r, "\alpha' \vee \alpha'"] & S^n \vee S^1 \vee S^n \vee S^1 \arrow[d, "\text{fold}"] \\
S^n \vee S^1 \arrow[rr, "\mu' \vee \mathrm{id}"] & & S^n \vee S^n \vee S^1
\end{tikzcd}$$

You can check that this commutes up to homotopy. $\square$

There is a relative version of this story. The group $\pi_1 A$ acts compatibly on $\pi_n(X, A)$ for $n \geq 2$. Because we constructed the coaction of S^1 on S^n, the resulting action of $\pi_1 X$ on $\pi_n X$ is natural. A map $f : (X, A) \to (Y, B)$ induces:

$$\begin{tikzcd}
\pi_n X \times \pi_n X \arrow[r] \arrow[d] & \pi_n X \arrow[d] \\
\pi_n Y \times \pi_n Y \arrow[r] & \pi_n Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1 A \times \pi_n(X, A) \arrow[r] \arrow[d] & \pi_n(X, A) \arrow[d] \\
\pi_1 B \times \pi_n(Y, B) \arrow[r] & \pi_n(Y, B)
\end{tikzcd}$$

$$\begin{tikzcd} \pi_1 A \times \pi_n(X, A) \arrow[r] \arrow[d] & \pi_n(X, A) \arrow[d] \\ \pi_1 A \times \pi_{n-1} A \arrow[r] & \pi_{n-1} A \end{tikzcd}$$

If $f : X \to Y$ is an homotopy equivalence, then $f_* : \pi_{*\geq 2} X \to \pi_{*\geq 2} Y$ is an isomorphism of $\mathbb{Z}[\pi, X]$-modules. This is considerably stronger, and it is the first piece of structure of "π-algebras".

There is also an "unbased" version of this that encompasses the change of basepoint map Γ. You just need a strong enough version of relative maps.

7 February 6, 2017

We were talking about coexact and exact sequences. We had a special object $P_f \xrightarrow{j} X \xrightarrow{f} Y$, where

$$P_f = \{(\gamma, x) \in Y^I \times X : \gamma(1) = f(x)\}.$$

Then we have

$$P_j = \{(\alpha, (\gamma, x)) \in X^I \times P_f : \alpha(1) = x, \gamma(1) = f(x)\}.$$

This is homotopy equivalent to ΩY, because we can map the path α and post-compose with γ.

Proposition 7.1. *For two spaces X and Y, $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$.*

We can state this as $\pi_n Y \to \pi_n(X \times Y) \to \pi_n X$ being exact.

7.1 Fibrations

Definition 7.2. A map $p : E \to B$ has the **homotopy lifting property** with respect to X where given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "{(-,0)}"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

there exists a lift $\tilde{H}$. The map p is called a **fibration** if it has the homotopy lifting property for all spaces X, and it is called a **weak fibration** if it has the homotopy lifting property for sphere.

Example 7.3. The projection $X \times Y \to X$ is a fibration. Just don't mess with the Y coordinate when lifting the homotopy.

Example 7.4. The map $PX \to P$ also is a fibration.

Given a fibration $p : E \to B$, set $F = p^{-1}(b_0)$ to be its **fiber**. We want to show that there is a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1} B \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots.$$

Lemma 7.5. *Consider $p : E \to B$, $B' \subseteq B$, and $E' = p^{-1}(B')$. If p has the homotopy lifting property for $X \times I$, then $p' : (E, E')^{(I, \partial I)} \to (B, B')^{(I, \partial I)}$ has the homotopy lifting property for X.*

Proof. We want to lift the homotopy.

$$\begin{tikzcd} T \arrow[r, "f'"] \arrow[d] & (E, E')^{(I,\partial I)} \arrow[d, "p'"] \\ T \times I \arrow[r, "H'"] & (B, B')^{(I,\partial I)} \end{tikzcd}$$

This has the same data as

$$\begin{tikzcd} T \times (I \vee I) \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ T \times I \times I \arrow[r, "H"] & B \end{tikzcd}$$

where $H(T \times I \times \partial I) \subseteq B'$, $H(t_0 \times 0 \times I) \subseteq B'$, $f(T \times (0 \times 0)) \subseteq E'$, and $f(t_0 \times (I \vee I)) \subseteq E'$. Reparametrizing gives the right diagram. □

Corollary 7.6. *If $p : E \to B$ is a weak fibration, then $\pi_n(E, E') \to \pi_n(B, B')$ is an isomorphism for $n \geq 1$.*

Proof. We are going to just think about $n = 1$. This is because of the previous lemma. Given $w : (I, \partial I) \to (B, B')$, draw it as

$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & E \arrow[d] \\ I \arrow[r, "w"] \arrow[ur, dashed, "\tilde{w}"] & B \end{tikzcd}$$

so $\tilde{w} : (I, \partial I) \to (E, E')$. This gives surjectivity. If $w_1, w_2 : (I, \partial I) \to (E, E')$ and there is a homotopy $H : p \circ w_1 \sim p \circ w_2$, form

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ I \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

Now use the lemma. □

Corollary 7.7. $\cdots \to \pi_{n+1} B \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots$ *is a long exact sequence.*

Proof. Let $B' = \{b_0\}$ and $E' = F$. □

Recall that $PX \to X$ is a fibration with $F = \Omega X$. And we have a long exact sequence

$$\cdots \to \pi_{n+1} X \to \pi_n \Omega X \to \pi_n PX \to \pi_n X \to \pi_{n-1} \Omega X \to \cdots .$$

But PX is contractible. So we get $\pi_{n+1} X \cong \pi_n \Omega X$, which we already knew.

The fibers of a fibration (as the basepoint int B varies inside a path component) are all homotopy equivalent. The maps are going to come from the lifting of

$$\begin{tikzcd} F_0 \times * \arrow[r, hook] \arrow[d] & E \arrow[d, "p"] \\ F_0 \times I \arrow[r, "\gamma \circ \text{collapse}"] & B. \end{tikzcd}$$

Note that maps with homotopy equivalent fibers are not all fibrations. Look at the example

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} x - 1 & x \geq 1 \\ 0 & -1 \leq x \leq 1 \\ x + 1 & x \leq -1. \end{cases}$$

8 February 8, 2017

Last time we proved this theorem that if you have a fibration then you have a long exact sequence of homotopy groups.

8.1 Fiber bundles and examples

Definition 8.1. A **fiber bundle** consists of data $p : E \to B$ and F such that there exists an open cover $\{U_\alpha\}$ of B with $p^{-1}(U_\alpha) \cong U_\alpha \times F$ that commutes with projection and p to U_α.

Lemma 8.2. *Every fiber bundle is a weak fibration, i.e., has the homotopy lifting property for S^n.*

Proof. Given a map $w : S^n \to B$, its image is covered by a finite sub-collection. You can subdivide S^n into smaller subspace, each of which lies in one of U_α. In any one piece, the homotopy lifting problem is solvable. Glue. □

If B is compact, then this is actually a fibration.

Lemma 8.3. *If $H \subseteq G$ is a closed subgroup of a topological group G, and $H \in G/H$ has an open neighborhood U with a section $s : U \to G$, then $p : G \to G/H$ is a fiber bundle with fiber H.*

Proof. This section s lets us write points in $p^{-1}(U)$ as H-displacement $s(p(g))g^{-1} \in H$. All the points in G/H acquire a similar ϕ by left-translation by $g \in G$. □

Example 8.4. Take $\mathrm{O}(n) \hookrightarrow \mathrm{O}(n+k)$. This gives a quotient $\mathrm{O}(n+k)/\mathrm{O}(n)$. The set $\mathrm{O}(n+k)$ is space of orthonormal $(n+k)$-frames in $\mathbb{R}^{n+k}$. So $\mathrm{O}(n+k)/\mathrm{O}(n)$ is the space of orthonormal k-frames in $\mathbb{R}^{n+k}$. This is called the **Stiefel manifold**.

To check that this is a fiber bundle, we need an open neighborhood of $(e_{n+1}, \dots, e_{n+k})$ in $\mathrm{O}(n+k)/\mathrm{O}(n)$ with a section. For n other vectors $v_1, \dots, v_n$, select an open subset determined by the condition $0 \neq \det(v_1, \dots, v_n, e_{n+1}, \dots, e_{n+k})$. Make it an orthonormal frame by Gram–Schmidt.

Example 8.5. There is a further quotient you can do. Take $\mathrm{O}(n+k)/\mathrm{O}(n) \times \mathrm{O}(k)$. This receives a map from the Stiefel manifold. This things is called the **Grassmannian** and is the space of k-dimensional subspaces of $\mathbb{R}^{n+k}$.

Example 8.6. There is $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n) \to \{\pm 1\}$. You can do all theses over $\mathbb{R}$ and $\mathbb{C}$.

Now these together with the long exact sequence of homotopy groups, you can compute a bunch of stuff. For instance,

$$\mathrm{SU}(n) \to \mathrm{U}(n) \to \mathrm{U}(1) \cong S^1$$

gives isomorphisms $\pi_{>2}(\mathrm{SU}(n)) \cong \pi_{>2}(\mathrm{U}(n))$.

Lemma 8.7. *We say that a group G acts **properly discontinuously** on a space X if*

(a) for every x there exists a $x \in U_x$ such that $gU_x \cap U_x \neq \emptyset$ implies $g = e$,

(b) for any two x, y in different orbits, there exist $x \in U_x$ and $y \in U_y$ with $gU_x \cap U_y = \emptyset$.

In this case, $X \to X/G$ is a fiber bundle with fiber G.

Example 8.8. The action of $\mathbb{Z}$ on $\mathbb{R}$ is properly discontinuous. So $\mathbb{Z} \to * \to S^1$ is a fiber bundle.

Here are some remarks.

(i) If $\pi_0 X = 0$, then

$$0 \to \pi_1 X \to \pi_1 X/G \to \pi_0 G \to 0$$

is a short exact sequence of groups. If $\pi_1 X = 0$, then $\pi_1 X/G \cong \pi_0 G$. Hence you put a g-action on $\pi_n X/G$ for all $n \geq 1$. In fact, $\pi_n X \to \pi_n X/G$ respects the action.

(ii) A covering is **regular** if $p_*(\pi_1 E) \subseteq \pi_1 B$ is normal. All regular covers arise in this way.

(iii) If G is finite, X is Hausdorff, and G has no fixed points, then the action is proper discontinuous.

9 February 10, 2017

9.1 CW-complexes

A **CW-structure** on a space X is a presentation of X by iteratively attaching n-cells.

Definition 9.1. Given a space Y and a pointed map $g : \bigvee_\alpha S^{n-1} \to Y$, the space $Y \cup_g \vee_\alpha CS^{n-1}$ is called Y **with n-cells attached**.

Definition 9.2. A **CW-complex** is a sequence of pointed space $\{X^n\}$ (we call X^n the n-skeleton) such that

(i) $X^{-\infty} = \cdots = X^{-1} = \{x_0\}$,

(ii) X^n is formed from X^{n-1} by attaching n-cells.

Sometimes, we will refer to the union $\bigcup_{n=-\infty}^{n=\infty} X^n$ as "the CW-complex", taken with the weak topology. There is a relative version of this definition: start with a CW-complex A and set $X^{-\infty} = \cdots = X^{-1} = A$.

This is not a transitive notion: (X, A) and (A, B) being relative CW-complexes might not give a CW-pair (X, B). Also we remark that attaching a 0-cell is adding a point.

Example 9.3. The n-cell attached to a $-\infty$-cell presents S^n. You can also present S^n by attaching one 0-cell, attaching two 1-cells, two 2-cells, . . . , two n-cells also presents S^n.

Example 9.4. Using the second presentation of S^n, we can define S^∞ as the colimit of the skeleta $(S^\infty)^n = S^n$. Likewise we can take $\mathbb{R}P^n = \mathbb{A}^n \cup \mathbb{R}P^{n-1}$ and so we can take the colimit $\mathbb{R}P^\infty$. You can do the same thing with $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

Lemma 9.5. *A map $f : X \to Y$ is continuous if and only if, for a CW-presentation of X,*

$$\bigvee_n \bigvee_{\alpha \in A_n} CS^{n-1} \xrightarrow{g} X \xrightarrow{f} Y$$

is continuous.

Lemma 9.6. *If X is a CW-complex and Y is a finite CW-complex, then $X \times Y$ is also a CW-complex, using $D^n \times D^m \cong D^{n+m}$.*

Sometimes the weak topology on the product CW-complex and the product topology might be different. In this case, we use the first one.

Lemma 9.7. *If A is a subcomplex of a CW-complex X, then X/A is naturally a CW-complex.*

Corollary 9.8. *Homotopy of maps of CW-complexes can also be constructed inductively.*

Corollary 9.9. *Smashing with a finite complex, like S^1, send CW-structures to CW-structures.*

Let us take a smash of a gluing diagram

$$\begin{tikzcd} \bigvee_\alpha S^{n-1} \arrow[r, "g"] \arrow[d, hook] & Y^{n-1} \arrow[d] \\ \bigvee_\alpha D^n \arrow[r] & Y^n \end{tikzcd}$$

with a circle S^1. Then we get the following diagram.

$$\begin{tikzcd} \bigvee_\alpha S^n \arrow[r, "\Sigma g"] \arrow[d] & \Sigma(Y^{n-1}) = (\Sigma Y)^n \arrow[d] \\ \bigvee_\alpha CS^n \arrow[r] & (\Sigma Y)^{n+1} \end{tikzcd}$$

Example 9.10. The map $S^n \to Y$ has the same data as a map $* \to Y$ and a map $D^n \to Y$ mapping the boundary to the point.

Example 9.11. Likewise a map $\omega : S^n \to Y$ is the same data of a map $S^{n-1} \to Y$ and two null-homotopies of this map. So ω is a difference of two null-homotopies of the same class $\varphi : S^{n-1} \to Y$.

Example 9.12. The real projective plane $\mathbb{R}P^2$ has the CW-structure with the gluing map given by

$$\begin{tikzcd} S^1 \arrow[r, "\times 2"] \arrow[d, hook] & S^1 = \mathbb{R}P^1 \arrow[d] \\ D^2 \arrow[r] & \mathbb{R}P^2. \end{tikzcd}$$

So maps $\mathbb{R}P^2 \to Y$ is the same data as a map $\omega : S^1 \to Y$ with a null-homotopy of 2ω.

Lemma 9.13. *Let $X = A \cup_g D^n$, let (K, L) a finite simplicial pair, and $f : (K, L) \to (X, A)$. There exists a subdivision (K', L') and a map $f' : (K', L') \to (X, A)$ such that*

(i) $f|_{f^{-1}(A)} = f'|_{f^{-1}(A)}$ *and* $f \simeq f'$ *relative to* $f^{-1}(A)$,

(ii) for $\sigma \in K'$, *if* $f'(\sigma)$ *meets* $\mathrm{int}(D^n)$, *then it contain:* $f(\sigma) \subseteq \mathrm{int}(D^n)$ *and* $f'|_\sigma$ *is a linear map.*

10 February 13, 2017

Given a space X and a map $g : S^{n-1} \to X$, we can attach on n-cell as $X \cup_g CS^{n-1}$.

Now consider only a homotopy class $[g] \in \pi_{n-1}(X)$. Suppose $g_1 \simeq g_2$ are two maps in this class. Then $X \cup_{g_1} CS^{n-1} \simeq X \cup_{g_2} CS^{n-1}$ are homotopy equivalent. This is because you can use the base part of the cone to run the homotopy. So the space

$$X \cup_g CS^{n-1}$$

is well-defined up to homotopy equivalence.

10.1 The homotopy theory of CW-complexes

Our goal is now to get familiar with the behavior of $X \cup_g CS^{n-1}$ as a homotopy type.

Lemma 10.1. *For (X, A) a relative CW-complex, the pair $(X, (X, A)^n)$ is n-connected, i.e., $\pi_{\leq n}(X, (X, A)^n) = 0$.*

Corollary 10.2. *The inclusion $X^n \hookrightarrow X$ is n-connected.*

The proof of both of these is difficult, in the sense that it uses the simplicial approximation lemma.

Corollary 10.3. *$\pi_{<n} S^n = 0$.*

There is also a converse to the statement.

Lemma 10.4. *If a relative CW-complex (X, A) is n-connected, then there exists a relative CW-complex (X', A'), homotopic to (X, A), with $(X', A')^n = A'$.*

Corollary 10.5. *Suppose X is a n-connected CW-complex and Y is a m-connected CW-complex. Then $X \wedge Y$ is $(n + m + 1)$-connected.*

Proof. First pick a model of X and a model of Y such that there are no cells of X of dimension below n and no cells of Y below m. (You might worry about A' not being a point after picking a model, but if $A' \leq X'$ is a contractible subcomplex, then $X'/A' \simeq X'$.)

The cells in $X \times Y$ take the form $* \times *$, $e^i_\alpha \times *$, $* \times e^j_\beta$, $e^i_\alpha \times e^j_\beta$. We then quotient out by the first three types, so $X \wedge Y$ is populated by cells of type $e^i_\alpha \times e^j_\beta$. These have dimension $i + j \geq n + m + 2$. Hence $X \wedge Y$ is $(n + m + 1)$-connected. □

Theorem 10.6 (Homotopy excision)**.** *Let $A, B \leq X$ be subcomplexes of a CW-complex X with $X = A \cup B$, and $(A, A \cap B)$ be n-connected, and $(B, A \cap B)$ be m-connected. Then $\pi_*(A, A \cap B) \to \pi_*(X, B)$ is an isomorphism for $* < n + m$ and is surjective at $* = n + m$. (Note that this is always an isomorphism for homology.)*

Corollary 10.7. *$\pi_*(A \vee B) \cong \pi_* A \oplus \pi_* B$ for $* < n+m$, where A is n-connected and B is m-connected.*

Proof. After picking appropriate models, $A \vee B = (A \times B)^{n+m+1}$. Also $\pi_*(A \times B) = \pi_*(A) \oplus \pi_*(B)$ always. □

Corollary 10.8. *If (X, B) is n-connected and B is m-connected, then $\pi_*(X, B) \to \pi_*(X/B, *)$ is an isomorphism for $1 < * \leq n + m$.*

Proof. Apply the theorem to

$$\begin{tikzcd}
(X, B) \arrow[r, hook] \arrow[d] & (X \cup_B CB, CB) \arrow[d] \\
(X/B, *) \arrow[r, "\cong"] & (X \cup_B CB/CB, *).
\end{tikzcd}$$

The cone is contractible, so the right vertical map is an isomorphism. □

Corollary 10.9 (Freudenthal suspension theorem)**.** *Suppose X is n-connected for $n \geq 1$. Then $\pi_* X \to \pi_{*+1} \Sigma X$ is an isomorphism for $* \leq 2n$ and onto for $* = 2n + 1$.*

Proof. We have $\pi_{*+1}(CX, X) \to \pi_{*+1}(CX/X)$. The second space is ΣX and the first one is $\pi_* X$. □

We can start computing homotopy groups. We first know that $\pi_{<n} S^n = 0$. Also, using the fibration $\mathbb{Z} \to \mathbb{R} \to S^1$ we see that $\pi_n(S^1) = \pi_{n-1}(\mathbb{Z})$. Our S^n is $(n-1)$-connected, so we have isomorphisms $\pi_m(S^n) \cong \pi_{m+1}(S^{n+1})$ for $m \leq 2n - 2$.

π_8	0	0	?	?	?	?	?
π_7	0	0	?	?	?	?	?
π_6	0	0	?	?	?	?	$\mathbb{Z}$
π_5	0	0	?	?	?	$\mathbb{Z}$	0
π_4	0	0	?	?	$\mathbb{Z}$	0	0
π_3	0	0	?	$\mathbb{Z}$	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0
π_1	0	$\mathbb{Z}$	0	0	0	0	0
	S^0	S^1	S^2	S^3	S^4	S^5	S^6

Figure 1: Homotopy groups of spheres

We have a fibration $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$. So as $n \to \infty$, we get a fibration

$$S^1 \to S^\infty \to \mathbb{C}P^\infty.$$

This shows that $\pi_{*+1} \mathbb{C}P^\infty = \pi_* S^1$. Note that the 3-skeleton of $\mathbb{C}P^\infty$ is $\mathbb{C}P^1 \cong S^2$. So

$$\pi_3(\mathbb{C}P^\infty, \mathbb{C}P^1) = 0 \to \pi_2(\mathbb{C}P^1) \to \pi_2(\mathbb{C}P^\infty) = \mathbb{Z} \to \pi_2(\mathbb{C}P^\infty, \mathbb{C}P^1) = 0$$

is an exact sequence. That is, $\pi_2(S^2) = \mathbb{Z}$.

11 February 15, 2017

Eric is on vacation, so I (Jun-Hou Fung) am going to tell you about model categories.

11.1 Introduction to model categories

Suppose $\mathscr{C}$ is a category. Suppose $\mathscr{W} \subseteq \mathrm{Mor}(\mathscr{C})$. We want to consider objects connected with $\mathscr{W}$ the same. For instance, let $\mathscr{C}$ be Spaces and $\mathscr{W}$ be weak equivalences. So we want to define something like $\mathscr{C}[\mathscr{W}^{-1}]$. We can let the objects of $\mathscr{C}[\mathscr{W}^{-1}]$ be the same as $\mathscr{C}$, and let the morphisms be

$$\mathscr{C}[\mathscr{W}^{-1}](X, Y) = \{X \to * \xleftarrow{\simeq} * \xrightarrow{\simeq} * \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} * \xleftarrow{\simeq} Y\}.$$

This is called the **Gabriel–Zisman localizatoin**, but this is not a locally small category. So by equipping $(\mathscr{C}, \mathscr{W})$ with a model structure, we will get a better behaviored category $\mathrm{Ho}(\mathscr{C})$.

Proposition 11.1. *There exists a simplicial category $L^H\mathscr{C}$ called a "hammock localization" that is initial amongst all simplicial categories with W inverted.*

The 0-simplices will be zigzags as above, 1-simplices will be hammocks, and n-simplices will be wider hammocks.

Suppose we have a functor $F : \mathscr{C} \to \mathscr{D}$. We would then want a functor $\mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$. The naïve thing to do is to is Kan extensions, but there are problems. Model categories are also supposed to tell you which functors you can derive, and how to do it.

Suppose you don't care. There are stuff in homological algebra, and they give very similar situations. Model categories are supposed to give a theory of homotopy theories. Then you can start to think about the homotopy theory of homotopy theories. There also is something called a $(\infty, 1)$-category, which is an ∞-category of ∞-categories.

11.2 Model categories

Definition 11.2. We say that i has the **left-lifting property** with respect to p or that p has the **right-lifting property** with respect to i if

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

there is a map. We the write $i \in \mathrm{llp}(p)$ and $p \in \mathrm{rlp}(i)$.

Definition 11.3. A **model category** $\mathscr{C}$ is a category with three distinguished wide subcategories

- $\mathscr{W}$ weak equivalences ($\xrightarrow{\sim}$)

- fib fibrations ($\twoheadrightarrow$)
- cof cofibrations ($\hookrightarrow$)

such that

(MC1) $\mathscr{C}$ is bicomplete, i.e., has small limits and colimits,

(MC2) $\mathscr{W}$ has the 2-out-of-3 property, i.e., if two of $f, g, g \circ f$ are in $\mathscr{W}$ then the other one is too,

(MC3) $\mathscr{W}, \mathsf{fib}, \mathsf{cof}$ are closed under retracts, i.e., if g has a property and

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

has identity horizontal maps then f has the property,

(MC4) $\mathsf{cof} \subseteq \mathrm{llp}(\mathscr{W} \cap \mathsf{fib})$ and $\mathsf{fib} \subseteq \mathrm{rlp}(\mathscr{W} \cap \mathsf{cof})$,

(MC5) there exist a functorial factorization f as $f = pi$ where $i \in \mathsf{cof}$ and $p \in \mathscr{W} \cap \mathsf{fib}$, and also a factorization f as $f = pi$ with $i \in \mathscr{W} \cap \mathsf{cof}$ and $p \in \mathsf{fib}$.

Note that the MC axioms are dual. So we only need to prove half the theorems.

Definition 11.4. Let $\mathscr{C}$ have an initial object ϕ and a terminal object $*$. Say A is **cofibrant** if $\phi \to A$ is a cofibration, and **fibrant** if $A \to *$ is a fibration.

Example 11.5. Consider the category of spaces with $\mathscr{W}$ weak homotopy equivalences, fib Serre (weak) fibrations, and cof retracts of cell complexes.

Example 11.6. We can take the same thing with $\mathscr{W}$ homotopy equivalneces, fib Hurewicz fibrations, and cof closed Hurewicz cofibrations.

Example 11.7. Fix a ring R, and look at the category Ch_R of chain complexes. Let $\mathscr{W}$ be the quasi-isomorphisms, fib be the degree-wise epimorphisms in positive degrees, and cof be monomorphisms in all degrees with projective cokernels. This is called the projective model structure.

Example 11.8. We can give a similar model, with the same $\mathscr{W}$, but with fib being the degree-wise epimorphisms in all degrees with injective kernel, and cof degree-wise monomorphisms in positive degrees.

Example 11.9. Other examples include simplicial sets.

Proposition 11.10. *$\mathsf{cof} = \mathrm{llp}(W \cap \mathsf{fib})$, $\mathscr{W} \cap \mathsf{cof} = \mathrm{llp}(\mathsf{fib})$, $\mathsf{fib} = \mathrm{rlp}(\mathscr{W} \cap \mathsf{cof})$, $\mathscr{W} \cap \mathsf{fib} = \mathrm{rlp}(\mathsf{cof})$.*

Proof. I am just going to prove the first one. It suffices to prove that $\mathrm{llp}(\mathscr{W} \cap \mathsf{fib}) \subseteq \mathsf{cof}$. We use the factorization property. Suppose that $f \in \mathrm{llp}(\mathscr{W} \cap \mathsf{fib})$. Factorize it to $f = pi$ with $p \in \mathsf{fib} \cap \mathscr{W}$ and $i \in \mathsf{cof}$. Then there is a lifting

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "f"'] & L' \arrow[d, "p", "\wr"'] \\ L \arrow[r, "\mathrm{id}"] \arrow[ur, "h"] & L \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & K \arrow[r, "\mathrm{id}"] \arrow[d, "i"] & K \arrow[d, "f"] \\ L \arrow[r, "h"] & L' \arrow[r, "p"] & L. \end{tikzcd}$$

We then have a retract and then $f \in \mathsf{cof}$. $\square$

Proposition 11.11. cof *and* $\mathscr{W} \cap \mathsf{cof}$ *are stable under cobase change, and* fib *and* $\mathscr{W} \cap \mathsf{fib}$ *are stable under base change.*

Proof. Easy exercise. $\square$

11.3 Homotopy relation

Definition 11.12. Fix an $A \in \mathrm{ob}(\mathscr{C})$. A (good) cylinder object for A is a factorization of the fold map $\nabla : A \amalg A \to A$

$$A \amalg A \overset{i_0 \amalg i_1}{\hookrightarrow} \mathrm{cyl}(A) \xrightarrow{\sim} A.$$

Two maps $f, g : A \to X$ are **left homotopic** ($f \overset{l}{\sim} g$) if there exists a cylinder object and a map $H : \mathrm{cyl}(A) \to X$.

$$\begin{tikzcd} A \arrow[d] \arrow[drr, "f"] & & \\ A \amalg A \arrow[r] & \mathrm{cyl}(A) \arrow[r, dashed, "H"] & X \\ A \arrow[u] \arrow[urr, "g"'] & & \end{tikzcd}$$

Lemma 11.13. *If A is cofibrant, and i_0 and i_1 are cofibrations.*

Proposition 11.14. *If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $\mathscr{C}(A, X)$.*

12 February 17, 2017

Last time we defined a cylinder object, and this gave the notion of a left homotopy.

Proposition 12.1. *Left homotopy $\overset{l}{\sim}$ is an equivalence relation.*

Proof. We only need to check transitivity. Let $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$ be given by cylinders $\mathrm{cyl}(A)$ and $\mathrm{cyl}(A)'$. Then we can glue them by taking the pushout

$$\begin{tikzcd} & & A \arrow[d, hook, "i_0"] \\ & A \arrow[r, hook, "\sim"] \arrow[d, hook, "\wr"] & \mathrm{cyl}(A) \arrow[d, hook, "\wr"] \\ A \arrow[r, hook, "i_1'"] & \mathrm{cyl}(A)' \arrow[r, hook, "\sim"] & C \end{tikzcd}$$

Then use the universal property and some of the axioms. □

12.1 The homotopy category of a model category

Definition 12.2. $\pi^l(A, X)$ is the set of equivalence classes $\mathscr{C}(A, X)/\overset{l}{\sim}$.

Lemma 12.3. *Let X be fibrant and $f \overset{l}{\sim} g : A \to X$. Then if $h : A' \to A$, then $fh \overset{l}{\sim} gh$.*

Proof. There exists a homotopy $H : \mathrm{cyl}(A) \to X$ witnessing $f \overset{l}{\sim} g$. We can factorize $\mathrm{cyl}(A) \to A$ into $\mathrm{cyl}(A) \overset{\sim}{\hookrightarrow} \mathrm{cyl}(A)' \overset{\sim}{\twoheadrightarrow} A$. Then we can use the homotopy lifting property and get

$$\begin{tikzcd} \mathrm{cyl}(A) \arrow[r] \arrow[d, hook, "\wr"] & X \arrow[d, two heads] \\ \mathrm{cyl}(A)' \arrow[r] \arrow[ur, dashed, "\tilde{H}"] & *. \end{tikzcd}$$

Then you can construct a lift $k : \mathrm{cyl}(A') \to \mathrm{cyl}(A)'$ and then

$$\begin{tikzcd} A' \arrow[r, "h"] \arrow[d] & A \arrow[d] \arrow[dr, "f"] & \\ \mathrm{cyl}(A') \arrow[r, "k"] & \mathrm{cyl}(A) \arrow[r, "H"] & X \\ A' \arrow[r, "h"] \arrow[u] & A \arrow[u] \arrow[ur, "g"] & \end{tikzcd}$$

gives the desired homotopy. □

We can now define the dual notion of a cylinder.

Definition 12.4. A **(good) path object** is a factorization

$$X \xrightarrow{\sim} \mathrm{cocyl}(X) \overset{p}{\twoheadrightarrow} X \times X.$$

$f \overset{r}{\sim} g : A \to X$ are right homotopic if they factor through a common $A \to \mathrm{cocyl}(X)$.

Lemma 12.5. *Let $f, g : A \to X$.*
(a) If A is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$.
(b) If X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.

Proof. You do more liftings. □

If A is cofibrant and X is fibrant, then $\overset{l}{\sim}$ and $\overset{r}{\sim}$ are equivalent and so we can define

$$\pi(A, X) = \pi^l(A, X) = \pi^r(A, X).$$

Here is a theorem in classical topology.

Theorem 12.6 (Whitehead)**.** *Let $f : A \to X$ be a map between bifibrant (fibrant and cofibrant) objects. Then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose $f : A \xrightarrow{\sim} X$ be a weak equivalence. Then we can factor f as $A \overset{\sim}{\hookrightarrow} C \overset{\sim}{\twoheadrightarrow} X$. Then C is bifibrant. Now we use the homotopy lifting and get a map $s : X \to C$ such that $ps = \mathrm{id}_X$. Using the following lemma, we see that $sp \overset{l}{\sim} \mathrm{id}_C$. Likewise we can find a homotopy inverse for q. Then we can compose the two inverses to find a homotopy inverse for f.

Now we show that if f is a homotopy equivalence, then it is a weak equivalence. We know that f has a homotopy inverse $g : X \to A$ and so we have a homotopy $H : \mathrm{cyl}(X) \to X$. Then lift the map to $k : \mathrm{cyl}(X) \to C$. □

Lemma 12.7. *If A is cofibrant, and $C \to X$ is an acyclic fibration, then $\pi_* : \pi^l(A, C) \to \pi^l(A, X)$ is an bijection.*

For any map X, we define the **cofibrant replacement** QX for X as $\emptyset \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$. Likewise we define the **fibrant replacement** RX for X as $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$.

Definition 12.8. The **homotopy category** $\mathrm{Ho}(\mathscr{C})$ is the category with objects same as $\mathscr{C}$ and morphisms

$$\mathrm{Ho}(\mathscr{C})(X, Y) = \pi(RQX, RQY).$$

It turns out that RQX is bifibrant. Also there is a functor $\gamma : \mathrm{Ho}(\mathscr{C})$

Theorem 12.9. *The functor γ is a localization of $\mathscr{C}$ with respect to $\mathscr{W}$, i.e.,*

(1) γ takes $\mathscr{W}$ to isomorphisms,

(2) it is the initial category such that satisfies (1).

So if we have defined $\mathrm{Ho}(\mathscr{C})(X, Y) = \pi(QRX, QRY)$ it would have been an equivalent category.

12.2 Derived functors

Definition 12.10. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between model categories. The **total left derived functor** $\mathbb{L}F : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ is the terminal functor with

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\gamma_C"] & \mathscr{D} \arrow[d, "\gamma_D"] \\ \mathrm{Ho}(\mathscr{C}) \arrow[r, "\mathbb{L}F"] & \mathrm{Ho}(\mathscr{D}). \end{tikzcd}$$

Definition 12.11. F is

- a **left Quillen functor** if it is a left adjoint and preserves cof and $\mathscr{W} \cap \mathsf{cof}$.
- a **right Quillen functor** if it is a right adjoint and preserves fib and $\mathscr{W} \cap \mathsf{fib}$.

Lemma 12.12 (Ken Brown's lemma)**.** *Let F be a functor between model categories. If F takes acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

This is tricky to prove. But the upshot is that a left Quillen functor preserves weak equivalence between cofibrant objects. So you can construct the left derived functor as

$$\mathbb{L}F = F \circ Q.$$

In the category of chain complexes, this is the same as looking at the projective resolution and applying F.

So the homotopy category, derived adjunctions, and derived natural transformations give the data of a pseudo-2-functor between $\mathsf{ModelCat} \xrightarrow{\mathrm{Ho}} \mathsf{Cat}_{\mathrm{ad}}$.

13 February 22, 2017

We were looking at the question of what cell complexes there are. If there are 0 cells, then it is a point. If there are one cells, it is just S^k. If there are two cells, these are classified by $S^n \cup_\alpha D^m$ for each $\alpha \in \pi_{m-1} S^n$.

There was the connected region, where the homotopy groups are zero, and the stable region, where the homotopy groups are stable along the rays. To compute $\pi_2(S^2)$, we used the fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$. The rest of the semester we are going to fill the chart Figure 1.

There is a Hopf fibration $S^1 \to S^3 \to S^2$ and so there is a long exact sequence

$$\cdots \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \cdots .$$

Then $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. This is pretty much what we can do so far.

13.1 The homotopy theory of CW-complexes II

Lemma 13.1. *For all n-equivalence $B \hookrightarrow Y$,*

$$\begin{tikzcd} B \arrow[r, hook] & Y \\ S^{n-1} \arrow[u, "\omega|_{\partial D^n}"] \arrow[r, hook] & D^n \arrow[u, "\omega"'] \arrow[ul, dashed] \end{tikzcd}$$

there is a dashed map with a homotopy on the top right triangle that fixes S^{n-1}.

Lemma 13.2. *If $Z \to Y$ is an n-equivalence and (X, A) is a relative CW-complex with $\dim(X, A) \le n$, there exists an arrow*

$$\begin{tikzcd} Z \arrow[r, "f"] & Y \\ A \arrow[u, "g"] \arrow[r, hook] & X \arrow[u, "h"'] \arrow[ul, dashed] \end{tikzcd}$$

with a homotopy on the top right triangle.

Corollary 13.3. *For $f : Z \to Y$ an n-equivalence, for all complexes X of $\dim X \le n$, the map $[X, Z] \to [X, Y]$ is onto. If $\dim X < n$, it is a bijection, because you can lift homotopies.*

Corollary 13.4 (Whitehead)**.** *A weak homotopy equivalence between two CW-complexes is a homotopy equivalence.*

Proof. You just use some kind of Yoneda trick. □

Corollary 13.5. *All maps $f : (X, A) \to (Y, B)$ of CW-pairs are homotopic (relative to A) to cellular maps, and homotopies between cellular maps admit cellular replacements.*

Recall that if X is n-connected and Y is m-connected, then $(X \times Y)^{n+m+1} = X \vee Y$. So for $1 < * < n+m$, the map $\pi_*(X \vee Y) \to \pi_*(X \times Y)$ is an isomorphism. So for $n \ge 2$, $\pi_n(\bigvee_\alpha S^n_\alpha) = \bigoplus \pi_n S^n_\alpha$.

13.2 Eilenberg–MacLane space

Lemma 13.6. *For an abelian group A and $n \geq 2$ there is a space $K(A, n)$ with the property*

$$\pi_* K(A,n) = \begin{cases} A & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Pick a presentation $\mathbb{Z}^l \hookrightarrow \mathbb{Z}^k \twoheadrightarrow A$. Replace the product by bouquet of spheres:

$$(S^n)^{\vee l} \xrightarrow{j} (S^n)^{\vee k} \to (S^n)^{\vee k} \cup_j C(S^n)^{\vee l} = X_n.$$

We claim that $\pi_{*<n} X_n$ is correct and $\pi_n X_n \supseteq A$. This is because a coexact sequence gives the exact sequence in homotopy up to around $2n$.

To form X_{m+1} from X_m, select all the homotopy classes in $\pi_m X_m$ that doesn't belong. Then elements determine a map in from a bouquet of spheres. Take the mapping cone, X_{m+1}, take the union over. ☐

Lemma 13.7. *For X with $\pi_{*<n} X = 0$ and Y with $\pi_{*>n} Y = 0$, $[X, Y] \cong [\pi_n X, \pi_n Y]$.*

14 February 24, 2017

Lemma 14.1. *There exist space $K(A,n)$ with $\pi_n K(A,n) = A$ and other homotopy groups* 0*.*

Lemma 14.2. *If X is $(n-1)$-connected and Y has $\pi_{*>n} Y = 0$, then $[X,Y] \cong [\pi_n X, \pi_n Y]$.*

Corollary 14.3. *The space $K(A,n)$ is unique up to homotopy.*

Corollary 14.4. $\Omega K(A,n) \simeq K(A,n-1)$.

The collection is these kind of spaces is super interesting. This $K(A,n-1)$ is the loop space of a space, and so $K(A,n)$ can be thought of as a "delooping".

14.1 Brown representability

Recall that $\mathsf{Spaces}_*(-,T)$ describes a sheaf, i.e., the pasting lemma hold.

Theorem 14.5 (Brown)**.** *Take a functor $F : \mathsf{hSpaces}_*^{\mathrm{op}} \to \mathsf{Sets}_*$, such that*

(i) Wedge axiom: $F(\bigvee_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$,

(ii) Sheaf condition, Mayer–Vietoris axiom: if $X = A_1 \cup A_2$ (with compatible CW-structure) and $f_1 \in F(A_1)$ and $f_2 \in F(A_2)$ such that $f_1|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2}$, then there exists a $f \in F(X)$ such that $f_1 = f|_{A_1}$ and $f_2 = f|_{A_2}$.

Then there exists a complex Y and an element $u \in F(Y)$ such that $[X,Y] \to F(X); f \mapsto f^* u$ is a natural bijection.

As a consequence of Yoneda, for a natural transformation $\tau : F \to G$ between two such functors there exists a unique $t : Y_F \to Y_G$ such that $t_* : [X, Y_F] \to [X, Y_G]$ is compatible with the isomorphism.

Definition 14.6. An element $u \in F(Y)$ is n**-universal** if $[S^q, Y] \to F(S^q); w \mapsto w^* u$ is onto for all $q = n$ and an isomorphism for $q < n$.

Proof of Brown representability part 1. Suppose that $u_n \in F(Y)$ is n-universal. We want to correct u_n and Y so that they are an $(n+1)$-universal pair, fixing the isomorphism at dimension n and onto at dimension $n+1$.

Consider the set

$$A = \{\alpha \in \pi_n Y : \alpha^* u_n = 0\}, \quad L = F(S^{n+1}).$$

Take the mapping cone

$$\bigvee_\alpha S^n_\alpha \xrightarrow{\bigvee_\alpha \vee *} Y \vee \bigvee_{\lambda \in L} S^{n+1}_\lambda \to Y'.$$

Applying F, we get

$$\prod_\alpha F(S^n_\alpha) \leftarrow F(Y) \times \prod_\lambda F(S^{n+1}_\lambda) \leftarrow F(Y').$$

This sequence is exact, because of the sheaf axiom for F. Now (u_n, λ) maps to 0 in $\prod_\alpha F(S^n_\alpha)$ by definition. So this lift to an element $u_{n+1} \in F(Y')$. This modification works because when we attached the $(n+1)$-cells we killed exactly what needed to be killed in $\pi_n(Y)$ and added all the things in $\pi_{n+1}(Y)$. $\square$

Lemma 14.7. *Suppose Y is a space with universal element $u \in F(Y)$, (X, A) is a CW-pair, $v \in F(X)$, and $g : A \to Y$ is a cellular map such that $g^*u = v|_A$. Then there exists a cellular $X \to Y$ extending g and classifying v.*

Proof. We construct the space

$$T = X \amalg (A \times I) \amalg Y/((a \in X \sim (a, 0) \in A \times I), (g(a) \in Y \sim (a, 1) \in a \times I)).$$

Let T_1 be the part of X and the half of $A \times I$ and T_2 be the part of Y and the half of $A \times I$ so that $T_1 \cup T_2 = T$. Then there exists a $u \in F(T_1)$ since T_1 retracts to A and $v \in F(T_2)$ likewise. They agree because $g^*u = v|_A$. So we get $\omega \in F(T)$.

We can extend T to a CW-pair (Y', T) with universal element u' restricting to ω on T. Then there is a weak equivalence $Y \to Y'$. Then Whitehead's theorem tells that j is a homotopy equivalence. Now the composite $X \hookrightarrow Y' \to Y$ does the job. $\square$

Proof of Brown representability part 2. To get surjectivity (of $F(X) \leftarrow [X, Y]$) take $A = *$ in the lemma. To get injectivity, set $X' = X \times I$ and $A' = X \times \partial I$ with the map $A' \to Y$ being $\tilde{g}_1$ on one and $\tilde{g}_2$ on the other. $\square$

If you apply $\tilde{H}^n(-, A)$ to this theorem and follow the proof, you see that this will give exactly the space $K(A, n)$.

15 February 27, 2017

I did not attend class, and I made these notes after class. Thanks to Reuben Stern who provided me with the copy of his notes.

15.1 Spectra

Recall that Brown representability tells us that a functor $\mathsf{hSpaces}_*^{\mathrm{op}} \to \mathsf{Set}_*$ satisfying the wedge axiom and the sheaf condition are representable. In particular, the (reduced) cohomology functors $\tilde{h}^n$ satisfies these conditions by the Mayer–Vietoris sequence. We also have a natural isomorphism $\tilde{h}^{n+1}(\Sigma X) \cong \tilde{h}(X)$. This shows that there is a natural sequence of spaces $\underline{h}_n$ such that

$$\begin{tikzcd}
\tilde{h}^n(X) \cong \mathsf{hSpaces}_*(X, \underline{h}_n) \arrow[d, "\cong"] \arrow[dr] & \\
\tilde{h}^{n+1}(\Sigma X) \cong \mathsf{hSpaces}_*(\Sigma X, \underline{h}_{n+1}) \arrow[r, "\cong"] & \mathsf{hSpaces}_*(X, \Omega \underline{h}_{n+1}).
\end{tikzcd}$$

So Yoneda gives us a natural map $\Sigma \underline{h}_n \to \underline{h}_{n+1}$. We can read this backwards and think that these sequence of spaces is the cohomology theory $\tilde{h}$. Our goal is to use this presentation to blend homotopy theory into the category of cohomology theories.

More precisely, we want a category in which

- cohomology theories live inside this category as objects,
- Spaces_* maps to this category so that $[X, h]$ computes $h^0(X)$,
- this functor from Spaces_* is compatible with all the connectivity-stabilized theorems.

Definition 15.1. A **spectrum** is a sequence $\{E_n\}_{n \in \mathbb{Z}}$ of CW-complexes together with inclusions (as subcomplexes) $\Sigma E_n \hookrightarrow E_{n+1}$.

Note that using the mapping cylinder, you can always make maps into inclusions.

Example 15.2. To each $X \in \mathsf{Spaces}_*$ is associated the **suspension spectrum** $\Sigma^\infty X$ given by $\{\Sigma^n X\}_{n=0}^\infty$. In particular, we write

$$\mathbb{S}^0 = \Sigma^\infty S^0.$$

Example 15.3. There is the **Eilenberg–MacLane spectrum** given by

$$HA = \{K(A, n)\}_{n=0}^\infty.$$

The naïve notion of a morphism between spectra will be maps compatible with the embedding $\Sigma E_n \hookrightarrow E_{n+1}$. Using this definition, we will compute $[\mathbb{S}^0, \mathbb{S}^0]$ to be $[S^0, S^0] = \{\pm 1\}$. But this is not the answer we want.

Definition 15.4. A **subspectrum** (F_n) is a collection of subcomplexes $F_n \subseteq E_n$ with compatible linking maps, i.e., $\Sigma F_n \subseteq F_{n+1}$. A **cofinal subspectrum** is one for which any cell $e^m \subseteq E_n$ eventually appears in some F_{n+l}, i.e., $\Sigma^l e^m \subseteq F_{n+l}$ for some large enough l.

For example, $(*, \ldots, *, S^n, S^{n+1}, \ldots)$ is a cofinal subspectrum of $\mathbb{S}^0$.

Definition 15.5. A **morphism** between spectra $E \to G$ is a choice of a cofinal spectrum $F \subseteq E$ and a compatible sequence of maps $F \to G$. Two spectra are called **equivalent** if they agree on a mutually cofinal spectrum.

We easily see that for cofinal $F \hookrightarrow E$, the morphism is equivalent to the identity morphism. Also, note that the intersection of two cofinal subspectra is a cofinal subspectrum.

Definition 15.6. Two maps $f, h : E \to G$ are **homotopic** if there is a cofinal subspectrum F'' and a witness of level-wise homotopy

$$F'' \wedge I^+ \to G,$$

where I^+ is the interval with a disjoint basepoint.

Lemma 15.7. *Spectra has wedge sums and cofiber sequences/mapping cones.*

Corollary 15.8. *Let us define* $\pi_n E = [\mathbb{S}^n, E] = \lim_{k\to\infty} \pi_{n+k} E_k$. *If a map* $E \to G$ *induces an isomorphism on* π_*, *then it is an isomorphism.*

The negative indices for E_n really is useful.

Lemma 15.9. *The inclusion* $(\Sigma E_n)_n \to (E_{n+1})_n$ *is a cofinal subspectrum and thus an isomorphism.*

Corollary 15.10. *There is an inverse functor* $(\Sigma^{-1} E_n)_n = (E_{n-1})_n$.

Corollary 15.11. *For any two spectra* X *and* Y, *the set* $[X, Y]$ *can be given a structure of an abelian group via the isomorphism with* $[\Sigma^2\Sigma^{-2}X, \Sigma^2\Sigma^{-2}Y]$.

16 March 1, 2017

16.1 Cohomology theory from spectra

Last time we built this category $\mathsf{Spectra}$. There were some notion of homotopy between spectra, and there was a functor constructed a functor $\Sigma^\infty : \mathsf{Spaces} \to \mathsf{Spectra}$.

$$\begin{tikzcd} \mathsf{Spaces} \arrow[r] \arrow[d] & \mathsf{Spectra} \arrow[d] \\ \mathsf{hSpaces} \arrow[r, "\Sigma^\infty"] & \mathsf{hSpectra} \end{tikzcd}$$

There was also a Brown representability $\mathsf{CohomThy} \to \mathsf{hSpectra}$ given by

$$\tilde{E}^0(X) = \mathsf{hSpectra}(\Sigma^\infty X, E).$$

Definition 16.1. Given a spectrum E, we define its associated co/homology functor by

$$\tilde{E}^n(X) = \mathsf{hSpectra}(\Sigma^\infty X, \Sigma^n E), \quad \tilde{E}_n(X) = \pi_n(E \wedge X).$$

Proposition 16.2. *These are in fact co/homology functors.*

Lemma 16.3. *We have* $[\Sigma^\infty X, \Sigma^\infty Y] = \lim_{m\to\infty}[\Sigma^m X, \Sigma^m Y]$.

If E is presented by (E_n), then we have $\varinjlim \Sigma^{-n}\Sigma^\infty E_n$. In other words, E is an ind-system of desuspensions of the suspension spectra. I won't define an ind-system, but maps between ind-systems are generally computed as

$$\varprojlim_\alpha \varinjlim_\beta [X_\alpha, Y_\beta],$$

which looks like our business with cofinal subspectra.

Proof of Proposition 16.2. We check the axioms. We have

$$\begin{aligned}\tilde{E}^{n+1}(\Sigma X) &= \mathsf{hSpectra}(\Sigma^\infty \Sigma X, \Sigma^{n+1}E) = \mathsf{hSpectra}(\Sigma\Sigma^\infty, \Sigma^{n+1}E) \\ &= \mathsf{hSpectra}(\Sigma^\infty X, \Sigma^n E) = \tilde{E}^n(X).\end{aligned}$$

Likewise

$$\tilde{E}_n(X) = [\mathbb{S}^n, \mathbb{E} \wedge X] = [\mathbb{S}^{n+1}, E \wedge \Sigma X] = \tilde{E}_{n+1}(\Sigma X).$$

Suppose we have a cofiber sequence $A \xrightarrow{i} X \to X \cup_i CA$. Then the sequence

$$\Sigma^\infty \hookrightarrow \Sigma^\infty X \to \Sigma^\infty(X \cup_i CA) = \Sigma^\infty X \cup_i C\Sigma^\infty A$$

is again coexact. Smashing with E gives

$$E \wedge \Sigma^\infty A = E \wedge A \to E \wedge \Sigma^\infty X = E \wedge X \to E \wedge (\Sigma^\infty X \cup_i C\Sigma^\infty A).$$

Finally, we have to show that wedge axiom. We have

$$\begin{aligned}\tilde{E}^n(\textstyle\bigvee_\alpha X_\alpha) &= \mathsf{hSpectra}(\Sigma^\infty \textstyle\bigvee_\alpha X_\alpha, \Sigma^n E) = \mathsf{hSpectra}(\textstyle\bigvee_\alpha \Sigma^\infty X_\alpha, \Sigma^n E)\\ &= \prod_\alpha \mathsf{hSpectra}(\Sigma^\infty X_\alpha, \Sigma^n E) = \prod_\alpha \tilde{E}^n(X_\alpha).\end{aligned}$$

For homology,

$$\begin{aligned}\tilde{E}_n(\textstyle\bigvee_\alpha X_\alpha) &= \pi_n(E \wedge \textstyle\bigvee_\alpha X_\alpha) = [\mathbb{S}^n, E \wedge \textstyle\bigvee_\alpha X_\alpha]\\ &= \left[\mathbb{S}^n, \varinjlim_{S \subseteq A \text{ finite}}, E \wedge \textstyle\bigvee_\alpha X_\alpha\right] = \varinjlim_{S \text{ finite}} [\mathbb{S}^n, E \wedge \textstyle\bigvee_\alpha X_\alpha]\\ &= \lim_S \prod_{\alpha \in S} E_n(X_\alpha) = \bigoplus E_n(X_\alpha).\end{aligned}$$

This works because S^n is compact and so the image lies in a finite number of cells. $\square$

So now we have written down two maps:

$$\mathsf{CohomThy} \rightleftarrows \mathsf{hSpectra}.$$

This can be checked to be an equivalence. One side, from spectra to cohomology to spectra, is essentially because of Whitehead's theorem. The answer to the other side is spectral sequences.

The cohomology of a union of spaces does not agree with the limit of the cohomologies. Instead, there is a Milnor sequence: see homework 3.

The map $\mathsf{CohomThy} \to \mathsf{hSpectra}$ is secretly defined on objects at the level of $\mathsf{Spectra}$ without passing through the homotopy category. These spectra satisfy the condition that $\Sigma \underline{h}_n \to \underline{h}_{n+1}$ is adjoint to a weak equivalence $\underline{h}_n \to \Omega \underline{h}_{n+1}$. These are called **$\Omega$-spectra**, and they satisfy that

$$\mathsf{hSpectra}(\Sigma^\infty X, h) = \varinjlim [\Sigma^n X, \underline{h}_n]$$

is a constant system.

In general, one can convert a spectrum into an Ω-spectrum as follows. Look at

$$\mathsf{hSpectra}(\Sigma^\infty X, \Sigma^\infty Y) = \varinjlim [\Sigma^n X, \Sigma^n Y] = \varinjlim [X, \Omega^n \Sigma^n Y] = [X, \varinjlim \Omega^n \Sigma^n Y]$$

and define $y_0 = \varinjlim \Omega^n \Sigma^n Y$. If we define $y_1 = \varinjlim \Omega^n \Sigma^{n+1} Y$, then we have $\Omega y_1 = y_0$. In general, applying this recipe to the spaces in a spectrum recovers a sequence of spaces in an equivalent Ω-spectrum: $\underline{E}_0 = \varinjlim \Omega^n E_n$.

The functor $(-) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ preserves an exact sequence of abelian groups. This means that

$$X \mapsto E^0(X) \mapsto E^0(X) \otimes \mathbb{Z}_{(p)}$$

also send cofiber sequences to long exact sequences, and hence is represented by a spectrum $E_{(p)}$. Likewise $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ also does this, and so

$$X \mapsto E_0(X) \mapsto \mathrm{Hom}(E_0 X, \mathbb{Q}/\mathbb{Z})$$

also is represented by a spectrum $\mathbb{I}_{\mathbb{Q}/\mathbb{Z}}(E)$.

17 March 3, 2017

Recall that $S^n \to K(\mathbb{Z}, n)$ is an n-equivalence. This show that $\mathbb{S}^{\to} H\mathbb{Z}$ is a 0-equivalence. This means that there is an $F \to \mathbb{S}^0 \to H\mathbb{Z}$ with $\pi_{*\leq 0} F = 0$.

Theorem 17.1 (Hurewicz). *If X is $(n-1)$-connected (= n-connective), then $\pi_n X \cong H_n(X, \mathbb{Z})$.*

Proof. Apply $\wedge X$ to the fiber sequence to get

$$\pi_* F \wedge X \to \pi_* \mathbb{S}^0 \wedge X \to \pi_* H\mathbb{Z} \wedge X \to \pi_{*-1} F \wedge X \to \cdots .$$

We just need to know that $F_{*\leq 0}(X) = 0$. □

The tool to compute anything about F is spectral sequences.

17.1 Spectral sequences

The idea is that $A \hookrightarrow X \to X \cup_A CA$ induces a long exact sequence of homology groups. Most of our spaces come as gluing sequences, like

$$\begin{tikzcd}
X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X \\
A_1 \arrow[u] & A_2 \arrow[u] & & A_n \arrow[u] & &
\end{tikzcd}$$

I want a tool to compute compute $H_* X$. Ideally it wouldn't depend visibly on the homology of each X_n.

We apply homology to this whole picture. This gives us lots of going around maps:

$$\begin{tikzcd}
H_* X_1 \arrow[r] & H_* X_2 \arrow[r] \arrow[dl, "{[-1]}"] & H_* X_3 \arrow[r] \arrow[dl, "{[-1]}"] & \cdots \arrow[r] & H_* X \\
H_* A_1 \arrow[u] & H_* A_2 \arrow[u] & H_* A_3 \arrow[u] & &
\end{tikzcd}$$

We have $X = \bigcup_{n=1}^{\infty} X_n$ and so $H_*(X) = \lim_{n\to\infty} H_*(X_n)$. Hence for any $\sigma \in H_*(X)$ there is some n for which $\sigma_n \in H_*(X_n)$ and σ is the image. But this lives in $H_* X_n$ and we don't want this, we can map it down to $a_{n-1} \in H_* A_{n-1}$. In fact, the minimal n for which such a class σ_n appears has nonzero image a_{n-1}, because of the long exact sequence.

We now want to sort out such classes in $H_* A_{*-1}$ from those classes without lifts to $H_*(X)$.

$$\begin{tikzcd}
\cdots \arrow[r] & H_* X_{n-2} \arrow[r] & H_* X_{n-1} \arrow[r] \arrow[dl] & H_* X_n \arrow[r] \arrow[dl] & H_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
& H_* A_{n-2} \arrow[u] & H_* A_{n-1} \arrow[u] & H_* A_n \arrow[u] & H_* A_{n+1} \arrow[u] &
\end{tikzcd}$$

Suppose I start with a class $a_n \in H_*A_n$. Does there exists a $\sigma_{n+1} \in H_*X_{n+1}$ that maps down to a_n? The natural thing to do is to map it to $\sigma_n \in H_*X_n$ and then ask whether this is zero. But I don't like H_*X_n and so we can push it down to $a_{n-1} \in H_*A_{n-1}$. If $a_{n-1} \neq 0$, then $\sigma_n \neq 0$ and so there is no lift.

If $a_{n-1} = 0$, then we have still have to ask if $\sigma_n = 0$. If it is zero, then we have a lift to $\sigma_{n-1} \in H_*X_{n-1}$. Then we can push it down to $a_{n-2} \in H_*A_{n-2}$ and repeat the same thing until you get to $H_*(*)$.

Definition 17.2. We define $d_1 : H_*A_n \to H_*X_n \to H_*A_{n-1}$. We define the map d_r as the map that sends a_n to a_{n-r}.

It really makes sense to call this a differential because $d_1^2 = 0$. These maps d_r is well-defined modulo homology with respect to d_{r-1}. The homology with respect to d_{r-1} is written $E^r_{*,n}$, with $E^1_{*,n} = H_*A_n$. We can check that each $E^r_{*,n}$ is a subquotient of $E^{r-1}_{*,n}$.

Theorem 17.3. *The group H_*X admits a filtration such that its nth filtration quotient is $E^\infty_{*,n}$. To be more precise, we have a filtration*

$$\cdots \subseteq \operatorname{im} H_*X_n \subseteq \operatorname{im} H_*X_{n+1} \subseteq \cdots \subseteq H_*X,$$

and we have

$$E^\infty_{*,n} \cong \frac{\operatorname{im} H_*X_n}{\operatorname{im} H_*X_{n-1}}.$$

This is basically saying that to be in H_*X, you need to survive the process. There is some tedious algebra involved.

Things get complicated when you work with a descending filtration. Relatedly, if you use cohomology instead of homology. Cohomology is related to the Milnor sequence. Nonvanishing of the terms of the Milnor sequences can mess up.

If you filter the space by skeleta, then $A_n = \bigvee_\alpha S^n_\alpha$. Then we have

$$h_*(A_n) = h_*(\textstyle\bigvee_\alpha S^n_\alpha) = \displaystyle\bigoplus_\alpha h_{*\pm n}.$$

If you work things out,

$$d_1 : E^1_{*,n} \to E^1_{*,n}$$

is given by $d_1^{\text{cell}} : C_n^{\text{cell}}(X, h_*) \to C_{n-1}^{\text{cell}}(X, h_*)$, and then $E^n_{*,*} = h_*(X)$.

18 March 6, 2017

Last time we set up spectral sequences. From $E^2_{*,*} = H^{\text{cell}}_*(X; h_*)$ gives $h_*(X)$. This is called the **Atiyah–Hirzebruch spectral sequence**.

Corollary 18.1. *Cellular homology computes homology.*

Definition 18.2. A map of spectral sequences is a map between two diagrams

$$\begin{tikzcd} H_*X_1 \arrow[r] & H_*X_2 \arrow[r] & \cdots \\ H_*A_1 \arrow[u] & H_*A_2. \arrow[u] & \end{tikzcd}$$

Lemma 18.3. *If there is some page E^r on which a map of spectra sequences induces an isomorphism, then the map $H_*X \to H_*X'$ is an isomorphism.*

For instance, suppose that $H_*A \to H_*A'$ is an isomorphism. We have that $H_*X_0 \to H_*X_0'$ is an isomorphism, so the five lemma shows that $H_*X_1 \to H_*X_1'$ is an isomorphism. Then $H_*A_2 \to H_*A_2'$ an isomorphism and the five lemma implies that H_*X_2 is and isomorphism. This is just for the case $r = 1$, but you can do something similar.

Corollary 18.4. *If $\tau : E_*(-) \to F_*(-)$ is a natural transformation of homology theories and $E_*(S^n) \cong F_*(S^n)$ is an isomorphism for all S^n, then τ is a natural isomorphism.*

18.1 Obstruction theory

The goal is to understand this situation. Given a diagram, when can we find interesting maps?

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] \arrow[dr, dashed] & B \\ X \arrow[r, dashed] \arrow[ur, dashed] & Y \arrow[u] \end{tikzcd}$$

Today we are going to summarize a bunch of these results in terms of a spectral sequence computing $\pi_0 Y^X$.

Lemma 18.5. *If $\pi_{*<n}Y = 0$ and $\pi_{*>n}Z = 0$, then $[Y, Z] \cong \operatorname{Hom}(\pi_n Y, \pi_n Z)$.*

Corollary 18.6. *Set $Z = K(\pi_n Y, n)$, then we have a canonical element*

$$p_n \in [Y, K(\pi_n Y, n)] = \operatorname{Hom}(\pi_n Y, \pi_n Y) \ni \text{id}$$

that induces an isomorphism on π_n.

The fiber of π_n, with exact sequence $\mathrm{fib} \to Y \to K(\pi_n Y, n)$ then gives a long exact sequence

$$\begin{array}{ccccccc}
? & \longrightarrow & \pi_{n+1}Y & \longrightarrow & 0 & \longrightarrow & \\
? & \longrightarrow & \pi_n Y & \longrightarrow & \pi_n Y & \longrightarrow & \\
? & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow &
\end{array}$$

This shows that

$$\pi_*\mathrm{fib} = \begin{cases} \pi_* Y & \text{if } * > n, \\ 0 & \text{otherwise.} \end{cases}$$

We are going to call $\mathrm{fib} = Y(n, \infty)$. Because $\pi_{*<(n+1)}Y(n,\infty) = 0$, we can iterate this.

$$\begin{tikzcd}
Y \arrow[d] & Y(n,\infty) \arrow[l] \arrow[d] & Y(n+1,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
K(\pi_n Y, n) & K(\pi_{n+1}Y, n+1) & K(\pi_{n+2}Y, n+2) &
\end{tikzcd}$$

We can then apply $F(X, -)$ and then π_*. This is a spectral sequence, with

$$E^1_{*,n} = \pi_* F(X, K(\pi_n Y, n)) = H^{n-*}(X; \pi_n Y)$$

that computes $\pi_* F(X, Y)$.

Example 18.7. if $Y = \underline{h}_0$ is the ∞-loopspace underlying an Ω-spectrum, this spectral sequence agrees with the Atiyah–Hirzebruch spectral sequence.

The behavior of this spectral sequence is complicated in low degrees: the sequences entries may just be sets or groups. These are called **fringed spectral sequences**. There is a theory of this, you can get out of trouble by setting $X = \Sigma^2 X'$ or $Y = \Omega^2 Y'$.

If $\pi_{*<n}X = 0$ and $\pi_{*>n}Y = 0$, the this spectral sequence has a range of zero. The only remaining group on degree $* = 0$ is $H^n(X; \pi_n Y)$. Then

$$[X, Y] = \pi_0 F(X, Y) = H^n(X; \pi_n Y) = \mathrm{Hom}(H_n X, \pi_n Y) = \mathrm{Hom}(\pi_n X, \pi_n Y)$$

by the universal coefficients theorem and the Hurewicz theorem.

What information do we need in order to compute a nontrivial example of this spectral sequence? Let us return to our diagram. We can look at the fiber of maps like $Y(n, \infty) \to Y$.

$$\begin{tikzcd}
Y \arrow[dd] & & Y(n,\infty) \arrow[ll] \arrow[dd] & & Y(n+1,\infty) \arrow[ll] \arrow[dd] \\
& K(\pi_n Y, n-1) \arrow[ur] \arrow[dr, "d_1"] & & K(\pi_{n+1}Y, n) \arrow[ur] \arrow[dr, "d_1"] & \\
K(\pi_n Y, n) & & K(\pi_{n+1}Y, n+1) & & K(\pi_{n+2}Y, n+2)
\end{tikzcd}$$

Here, Brown representability identifies the map "d_1" from $K(\pi_n Y, n-1)$ to $K(\pi_{n+1}Y, n+1)$ with a natural transformation

$$H^{n-*}(X; \pi_n Y) \to H^{(n+1)-*+1}(X; \pi_{n+1} Y).$$

These are called **cohomology operations**. The goal is to understand this.

19 March 8, 2017

In an ideal world, I would give you a spectral sequence that is not too hard, tangible, well-motivated, and not too easy. But such an example does not exist and I am going to give an example that is not well-motivated.

19.1 A complicated example

For some reason, we are interested in computing the cohomology of a certain complex:

$$C_n = \{n\text{-variate power series beginning with } 1 + \cdots\} \text{ under multiplication.}$$

We define the differential maps as

$$\delta^1(f) = \frac{f(x)f(y)}{f(x+y)}, \quad \delta^2(g) = \frac{g(x,y)g(t,x+y)}{g(t+x,y)g(t,x)},$$

and generally a similar-looking alternating product.

Here is one strategy to compute the cohomology. You first find the cocycles. If you guess, something like $f(x) = 1 + x$, then it is not a cocycle. But we have

$$\delta^1(f)(x,y) = \frac{(1+x)(1+y)}{1+x+y} = 1 + xy + \cdots .$$

So this is not a cocycle. There are many power series, and so this is a bad strategy.

To make life easier, we can do the following:

(1) Work over $\mathbb{F}_2$ instead of $\mathbb{Z}$ (Throwing away unrecoverably).

(2) Work modulo terms of higher degree.

If we work modulo degree 2, then $f = 1 + x$ is indeed a cocycle. Note that if we do this, products are converted into sums: $(1+f)(1+g) = 1 + f + g$. This is now a tractable problem. For instance, for $n = 1$ we want

$$1 = \delta^1(f) = \frac{(1+f_+(x))(1+f_+(y))}{1+f_+(x+y)} = 1 + f_+(x) + f_+(y) - f_+(x+y)$$

module products. In $\mathbb{F}_2$, the polynomials $f_+(x) = x^{2^j}$ are those. For $n = 2$, there are $g_+(x,y) = x^{2^j}y^{2^j}$ but there are also other stuff like $\delta^1(x^7) = x^6y + \cdots + xy^6$.

Theorem 19.1. *The cohomology of this additive complex is $\mathbb{F}_2[a_i : i \geq 0]$ where a_i is represented by x^{2i}.*

Now our goal is to reconstruct the multiplicative version of this, from this answer. I have the entire cochain complex C^* and inside there is a cochain complex with no linear terms C_2^*, and inside there is a complex with no linear or quadratic terms C_3^*, and so on.

$$\begin{tikzcd}
C^* \arrow[d] & C_2^* \arrow[l, hook'] \arrow[d] & C_3^* \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] \\
\left\{\begin{matrix}\text{hom. poly.}\\ \text{of deg. 1}\end{matrix}\right\} & \left\{\begin{matrix}\text{hom. poly.}\\ \text{of deg. 2}\end{matrix}\right\} & \cdots &
\end{tikzcd}$$

Taking the cohomology of this system of cochain complexes, we get a spectral sequences that computes the multiplicative cohomology from the addition cohomology $E^1_{*,*}$. (The first star is the cohomological degree and the second star is the homogeneous degree.)

The first page look like:

$$\begin{array}{c|cccc}
7 & & & & a_0a_1a_2 \\
6 & & & a_1a_2 & a_1^3, a_0^2a_2 \\
5 & & & a_0a_2 & a_0a_1^2 \\
4 & & a_2 & a_1^2 & a_0^2a_1 \\
3 & & & a_0a_1 & a_0^3 \\
2 & & a_1 & a_0^2 & \\
1 & & a_0 & & \\
0 & 1 & & & \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

Figure 2: First page $E^1_{*,*}$

Now how do you compute all the differentials in this spectral sequence? First select a class in the additive cohomology, represented by some polynomial f_+. Then pick a preimage for this, like $1 + f_+$. Apply the multiplicative differential δ to this choice of lift, and finally project it back to the associated graded.

Let us do this. For instance, take $a_0 = [x]$. Lift this to $1 + x$. Take the boundary and then we get $1 + xy + \cdots$. Projecting it down give a_0^2. If you have some faith, you would believe that $d_{2^j}(a_j) = a_j^2$. This always tell you that all the a_i are eventually killed. That is, $H^1(C^*) = 0$.

Let us look at what happens to a_0a_1. We have

$$\delta(1 + xy^2) = 1 + txy^4 + t^2x^2y^2 + \cdots,$$

and so $d_3(a_0a_1) = a_0^2a_2 + a_1^3$. So you can compute everything. Still it is very hard to actually compute it because everything participates in the spectral sequence.

Most spectral sequences don't work like this, because they are not explicit.

20 March 10, 2017

We want to talk about the multiplicative structure of spectral sequences, but we don't even know the multiplication of cohomology.

20.1 Smash products

Our goal is to understand what it means for a cohomology theory to take values in rings. Ideally this will involve giving the representing spectrum with some structure.

Definition 20.1. A **ring** is a (commutative, unital) monoid in AbGrp under the $\otimes$-monoidal structure. In other words, there are maps $\mu : R \otimes_{\mathbb{Z}} R \to R$ and $\eta : \mathbb{Z} \to R$ satisfying $R = R \otimes_{\mathbb{Z}} \mathbb{Z} \to R \otimes_{\mathbb{Z}} R \to R$ is the identity.

A **monoidal structure** on $\mathscr{C}$ is the is the data of a functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ with various natural transformations encoding associativity and maybe unitality or commutativity. The smash product is also the monoidal product in the homotopy category of pointed spaces.

We want a monoidal structure on $\mathsf{Spectra}$, so that we can talk about rings (or, monoids in this monoidal structure), compatible with

$$\mathsf{AbGrps} \xrightarrow{H} \mathsf{Spectra} \xleftarrow{\Sigma^\infty} \mathsf{Spaces}_*.$$

For a space X and a spectrum E, we have previous defined

$$(E \wedge X)_n = E_n \wedge X.$$

Lemma 20.2. *You can take this as a definition of $E \wedge \Sigma^\infty X$.*

There is something prove here, because there are two ways to compute $\Sigma^\infty X \wedge \Sigma^\infty Y$. Actually this is clear.

Now a generic spectrum is an ind-system of desuspensions of suspension spectra: a spectrum $E = (E_n)$ can be thought as

$$E = \{\cdots \to \Sigma^{-n}\Sigma^\infty E_n \to \Sigma^{-(n+1)}\Sigma^\infty E_{n+1} \to \cdots\}.$$

If I know how to smash together two suspension spectra, this suggests a definition for the smash product of two arbitrary spectra:

$$\begin{array}{ccccccc}
\Sigma^\infty E_0 \wedge F_0 & \longrightarrow & \Sigma^{-1}\Sigma^\infty E_1 \wedge F_0 & \longrightarrow & \Sigma^{-2}\Sigma^\infty E_2 \wedge F_0 & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\Sigma^{-1}\Sigma^\infty E_0 \wedge F_1 & \longrightarrow & \Sigma^{-2}\Sigma^\infty E_1 \wedge F_1 & \longrightarrow & \Sigma^{-3}\Sigma^\infty E_2 \wedge F_1 & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\Sigma^{-2}\Sigma^\infty E_0 \wedge F_2 & \longrightarrow & \Sigma^{-3}\Sigma^\infty E_1 \wedge F_2 & \longrightarrow & \Sigma^{-4}\Sigma^\infty E_2 \wedge F_2 & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

You can try to pick a cofinal subsystem and use this as the smash product. But this is a bad idea since associativity of the smash product fails. A better way is to avoid making a choice by setting $(E \wedge F)_n$ to be the "mapping cylinder" of the degree $\leq n$ part of the diagram. For instance, we take

$$(E \wedge F)_1 = (E_0 \wedge F_1) \cup_{I \times (E_0 \wedge F_0)} (E_1 \wedge F_0).$$

Proposition 20.3. *This gives* $\mathsf{Spectra}$ *an associative monoidal structure that passes to the homotopy category, where it is unital and symmetric.*

Note that the general definition of the smash product doesn't agree with our initial definition, although they are weak equivalent.

Corollary 20.4. *The* ***cup product****, which is a map* $K(R, n) \times K(R, m) \to K(R, n+m)$*, induces a map*

$$HR \wedge HR \xrightarrow{\mu} HR.$$

The sphere spectrum $\mathbb{S}$ is the unit object for $\wedge$, hence it is too a ring: $\mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}$. Any cohomology theory valued in rings belongs to a ring spectrum. To construct a multiplication from a ring spectrum, we look at, for $\omega \in E^n(X)$ and $\psi \in E^m(X)$,

$$\Sigma^\infty X \xrightarrow{\Delta} \Sigma^\infty X \wedge \Sigma^\infty X \xrightarrow{\omega \wedge \psi} \Sigma^{-n} E \wedge \Sigma^{-m} E \xrightarrow{\mu} \Sigma^{-(n+m)} E.$$

Definition 20.5. A **Kronecker pairing** of two classes $\omega \in E^n(X)$ and $\sigma \in E_m(X)$ is:

$$\mathbb{S}^m \xrightarrow{\sigma} E \wedge \Sigma^\infty X \xrightarrow{\mathrm{id} \wedge \omega} E \wedge \Sigma^{-n} E \xrightarrow{\mu} \Sigma^{-n} E.$$

This gives $\langle \omega, \sigma \rangle \in \pi_{n+m} E$.

Lemma 20.6. *Suppose that* $f : X \to Y$ *is a map of spaces. Then*

$$\langle f^* \omega, \sigma \rangle = \langle \omega, f_* \sigma \rangle.$$

Proof. We look at the diagram

$$\begin{tikzcd} \mathbb{S}^m \arrow[r, "\sigma"] \arrow[dr, "f_*\sigma"'] & E \wedge \Sigma^\infty E \arrow[d, "f"] \arrow[dr, "f^*\omega"] & & \\ & E \wedge \Sigma^\infty Y \arrow[r, "\omega"] & E \wedge \Sigma^{-n} E \arrow[r, "\mu"] & \Sigma^{-n} E. \end{tikzcd}$$

Follow the arrows. ☐

Defining function objects is now easy to define. We want the adjunction

$$[X, F(E_1, E_2)] = [X \wedge E_1, E_2].$$

Now the right hand side is a functor satisfying Brown representability. We don't get an explicit description but that is fine.

21 March 20, 2017

If you have a cofiber/coexact sequence, you get a long exact sequence in homology. If you have a fiber/exact sequence, you get a long exact sequence in homotopy. (Note that π_* of a cofiber sequence is a long exact sequence through a range, but not everywhere.) The Serre spectral sequence is supposed to give you, from a fiber sequence a spectral sequence in homology.

21.1 Serre spectral sequence

Fix a fibration $F \to E \to B$ over a CW-complex B. The associated-graded of the skeletal filtration on B consists of bouquets of spheres. Pull back the fibration along the skeletal inclusions.

$$\begin{tikzcd} F \arrow[r] & E \arrow[r] & B \\ F \arrow[r] \arrow[u, equal] & E^{(k)} \arrow[r] \arrow[u, hook] & B^{(k)} \arrow[u, hook] \end{tikzcd}$$

This gives a filtration $E^{(k)}$ of E.

Proposition 21.1. *The associated-graded of $E^{(k)}$ looks like*

$$E^{(k)}/E^{(k-1)} = F \wedge (B^{(k)}/B^{(k-1)}) = F \wedge \bigvee_\alpha S^k_\alpha.$$

This gives a spectral sequence converging to $\tilde{h}_*(E)$, beginning with

$$E^1_{*,k} = \tilde{h}_*\Big(F \times \bigvee_\alpha S^k_\alpha\Big) = \tilde{h}_*\Big(\bigvee_\alpha \Sigma^k F\Big) = \bigoplus_\alpha \tilde{h}_{*-k}(F) = C^{\mathrm{cell}}_k(B; \tilde{h}_* F).$$

The last identification is non-canonical and non-natural.

Theorem 21.2 (Serre)**.** *If the action of $\pi_1 B$ on $h_* F$ is trivial (e.g., if $\pi_1 B = 0$), then d_1 is the cellular differential, and hence*

$$E^2_{p,q} = H_p(B; h_q F),$$

and it computes $h_{p+q}(E)$.

Proof. We are not going to prove it because it takes some actual work. □

If h_* is a field, then $H_*(B; h_* F) = H_*(B; h_*) \otimes h_* F$. It is remarkable how much this automates.

Example 21.3. There is a fibration $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$. We are going to compute the integral homology of $\mathbb{C}P^{n-1}$ (for $n = 5$). If either $p = 0$ or $q = 0$, then

$$E^2_{p,q} = H_p(B; H_q(F; \mathbb{Z})) = \begin{cases} H_q(F; \mathbb{Z}) & p = 0, \\ H_p(B; \mathbb{Z}) & q = 0. \end{cases}$$

(p,q)	0	1	2	3	4	5	6	7	8	9
3	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	$\mathbb{Z}$	?	?	?
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	$\mathbb{Z}$	?	?	?

Figure 3: E^2 page of the spectral sequence

We also have $E^2_{p,1} = E^2_{p,0} = H_p(B;\mathbb{Z})$. The differentials will be $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

Now because $H_1(S^{2n-1}) = 0$, the $\mathbb{Z}$ in the position $(p,q) = (0,1)$ cannot survive the sequence. The whole spectral sequence lies in the first quadrant, hence the only differential that has a change of being nonzero is d_2. So $d^2 : E^2_{2,0} \to E^2_{0,1}$ has to be surjective. It also has to be injective, lest $H_2 S^{2n-1} \neq 0$. So $d^2 : E^2_{2,0} \to E^2_{0,1}$ is an isomorphism. Then $E^2_{2,0} = E^2_{2,1} = H_2(B;\mathbb{Z})$ and so there as to be another $\mathbb{Z}$.

You can run the same argument over. Because $H_3(S^{2n-1}) = H_4(S^{2n-1}) = 0$, we have $E^2_{4,0} = \mathbb{Z} = E^2_{4,1}$. You can do this until we get to $E^2_{8,0} = \mathbb{Z} = E^2_{8,1}$. Then you can't run the argument further because $H^9(\mathbb{C}P^4) \neq 0$.

We can do the same thing for odd p. We have $E^2_{1,0} = 0\ E^2_{1,1}$, and we can keep climbing up by 2. Then you get up to $E^2_{7,0} = E^2_{7,1} = 0$. Finally, $\dim \mathbb{C}P^4 = 8$ and so all higher homology vanish. This gives a complete description of $H_p(\mathbb{C}P^4)$, which is the bottom row.

The exactly same argument applies to the spectral sequence for $S^1 \to * \to \mathbb{C}P^\infty$.

You could also apply cohomology and get a spectral sequence

$$E_2^{p,q} = H^p(B; h^q(F))$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

The construction of the spectral sequence is natural, and this converges to the map in co/homology of total spaces.

22 March 22, 2017

Last time I defined the Serre spectral sequence. Today I want to run some more examples. I also want to show you why this is also theoretically useful.

22.1 More about the Serre spectral sequence

Theorem 22.1 (Freudenthal suspension theorem)**.** *For X an n-connected space with $n \geq 1$, the map $X \to \Omega\Sigma X$ is a $(2n)$-equivalence.*

Proof. There is a fibration $\Omega\Sigma X \to P\Sigma X \to \Sigma X$, and build the Serre spectral sequence.

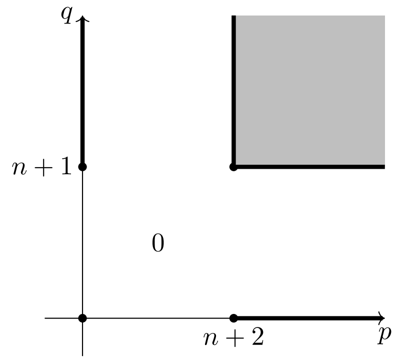

Figure 4: Serre spectral sequence of $\Omega\Sigma X \to P\Sigma X \to \Sigma X$

This has to converge to the homology of $P\Sigma X \simeq *$. So the differential connecting the p-axis to the q-axis must be isomorphisms for the range $p \leq 2n + (\text{const})$.

Now consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"] & CX \arrow[r] \arrow[d, "J"] & \Sigma X \arrow[d, equal] \\ F = \Omega\Sigma X \arrow[r] & E = P\Sigma X \arrow[r] & B = \Sigma X. \end{tikzcd}$$

The top row is a cofiber sequence and the bottom row is a fiber sequence. This induces

$$\begin{tikzcd} \tilde{H}_r X & H_{r+1}(CX, X) \arrow[l, "\cong"', "\partial"] \arrow[r, "\cong"] \arrow[d, "{(J,j)}"] & \tilde{H}_{r+1}\Sigma X \arrow[d, equal] \\ \tilde{H}_r F \arrow[from=u, "j"] & H_{r+1}(E, F) \arrow[l, "\partial"'] \arrow[r] & \tilde{H}_{r+1}(B). \end{tikzcd}$$

Now if you stare at the diagram for a long time, you will be able to convince yourself that $\tilde{H}_{r+1}(B) \to \tilde{H}_r F$ is just the differential d_r. This implies that j is also an isomorphism. $\square$

Lemma 22.2. *Given a diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, equal] & B & \\ F \arrow[r, "i"] & E \arrow[r] & C(i) \arrow[u, "c"] \arrow[r, "\delta"] & \Sigma F, \end{tikzcd}$$

a class $x \in H_n(B)$ has $d_n(x)$ defined (meaning $d_{<n}x = 0$) and $d_n(x) = y \in H_{n-1}(F)$ exactly when there exists a class $\tau(x) \in H_nC(i)$ such that $\delta_*\tau(x) = y$ and $c_*\tau(x) = x$.

Proof. This is left up to you. ☐

The cohomological Serre spectral sequence is superior to the homologycal Serre spectral sequence.

Lemma 22.3. *The Serre filtration $(E^{(k)})$ is multiplicative: $E^{(k)} \times E'^{(k')} \subseteq (E \times E')^{(k+k')}$.*

Corollary 22.4. *There is a map of spectral sequences*

$$\mathrm{SSS}(E/B) \otimes \mathrm{SSS}(E'/B') \to \mathrm{SSS}(E \times E'/B \times B').$$

Corollary 22.5. *If you pick $E = E'$, then $\Delta : E \to E \times E$ induces a product structure on $\mathrm{SSS}(E/B)_*$ satisfying $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \cdot d_r(y)$.*

Let us again look at the case of $S^1 \to * \to \mathbb{C}P^\infty$.

Lemma 22.6. *If h^*F is a free h_*-module, then all the classes of the Serre spectral sequence appear as products of classes on the edges.*

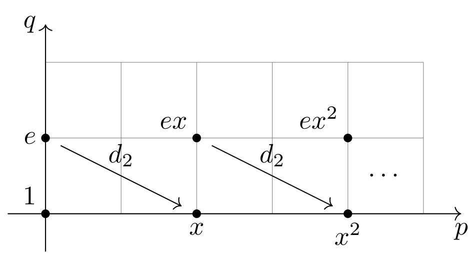

Figure 5: Serre spectral sequence of $S^1 \to * \to \mathbb{C}P^\infty$

Let $e \in H^1S^1$ be the generator, and let us call $d_2(e) = x$. Then the generator of the $E^2_{2,1}$ has to be ex. The generator of $E^2_{4,0}$ has to be $d_2(ex) = d_2(e)x - ed_2(x) = x^2$. Then the generator of $E^2_{4,1}$ has to be ex^2 and we can do the same thing. This shows that

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \quad x \in H^2(\mathbb{C}P^\infty).$$

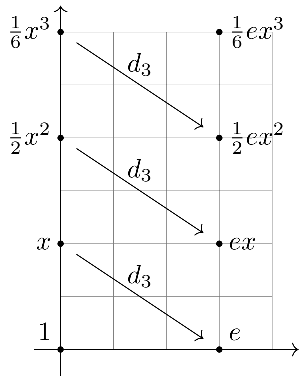

Figure 6: Serre spectral sequence of $\Omega S^3 \to * \to S^3$

Let us look at another example of

$$\Omega S^3 \to * \to S^3.$$

We can doing the same thing, but here we have $d_3(x^2) = d_3(x)x + xd_3(x) = 2ex$. The generator of $E^2_{0,4}$ is going to be $\frac{1}{2}x^2$. If you do the same thing, we get $E^2_{0,6}$ has generator $\frac{1}{6}x^3$. The conclusion is

$$H^*(\Omega S^3) = \Gamma[x] = \mathbb{Z}\langle 1, x^{(1)}, x^{(2)}, \ldots\rangle \text{ with } x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)}.$$

Likewise we can try to do the same thing for

$$\Omega S^2 \to * \to S^2.$$

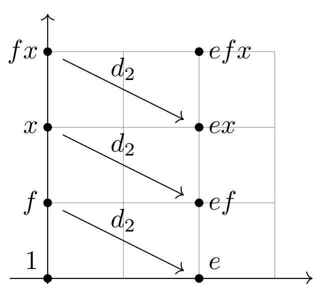

Figure 7: Serre spectral sequence of $\Omega S^2 \to * \to S^2$

In this case, we have $d_2(f^2) = d_2(f)f - fd_2(f) = 0$ and so we just have set a new generator $x \in E^2_{0,2}$. After more computations, the conclusion is

$$H^*\Omega S^2 \cong \Lambda[f] \otimes \Gamma[x] = (\mathbb{Z}[f]/(f^2)) \otimes \Gamma[x].$$

23 March 24, 2017

23.1 More spectral sequences

We were thinking about $\mathbb{C}^\times \to S^\infty \to \mathbb{C}P^\infty$. We have

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r, equal] \arrow[d] & \mathbb{C}^\times \arrow[d] \\
\mathbb{R}P^\infty \arrow[r] \arrow[d] & \mathbb{C}^\infty \setminus 0 \arrow[d] \\
\mathbb{C}P^\infty \arrow[r, "\cdot 2"] & \mathbb{C}P^\infty.
\end{tikzcd}$$

The bottom square is a pullback square because the long exact sequence $\pi_*(K(\mathbb{Z}/2, 1)) \to \pi_*(K(\mathbb{Z}, 2)) \to \pi_*(K(\mathbb{Z}, 2))$ is a long exact sequence. So we get a map of fibrations. Then we get a map of spectral sequences.

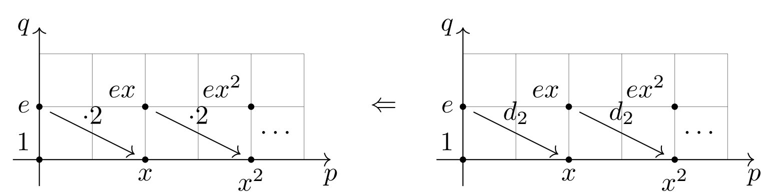

Figure 8: Map of Serre spectral sequences

You can check that the pullback of $(\cdot 2) : \mathbb{C}P^2 \to \mathbb{C}P^2$ sends x to x. Then it automatically follows that d_2 on the left is multiplication by 2. Then the product formula tells you what all the other maps are: they are multiplication by 2. So the E_3 page have only stuff on the bottom row. This shows that

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}/2[x].$$

We can also compute the cohomology ring in $\mathbb{F}_2$. In this case, the d_2 is just zero and nothing vanishes on the E_3 page. Then

$$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2\langle 1, e, x, ex, x^2, \dots \rangle$$

as modules. You would expect $e^2 = 0$ because in the spectral sequence, but this is not true. If you go back to our first discussion of the spectral sequence, we only have access to $H^* X_n / X_{n-1}$. When you take two elements $\alpha \in H^* X_m / X_{m-1}$ and $\beta \in H^* X_n / X_{n-1}$, and multiply them, what you are doing is pushing them down to $H^* X_m$ and $H^* X_n$ and multiplying them. This should give something in $H^* X_{m+n}$. But you don't know if this has a lift in $H^* X_{m+n} / X_{m+n-1}$. To see this in the spectral sequence, you need to push it down until right before you get zero. That is why e^2 can appear as x. These are called **multiplicative extensions**. There is nothing you can do about it.

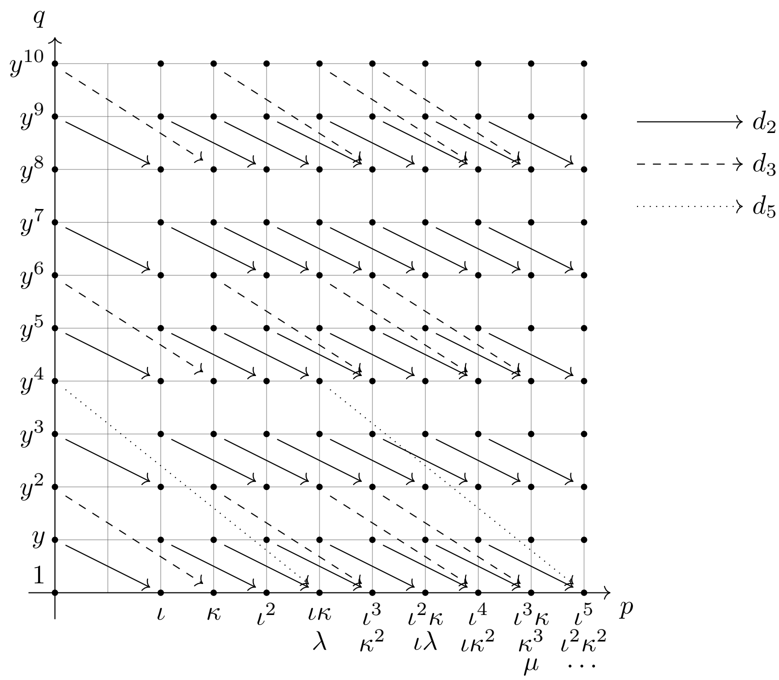

Figure 9: Serre spectral sequence of $K(\mathbb{Z}/2,1) \to * \to K(\mathbb{Z}/2,2)$

Let us look $K(\mathbb{Z}/2,1) = \Omega K(\mathbb{Z}/2,2) \to * \to K(\mathbb{Z}/2,2)$ and compute the $\mathbb{F}_2$-cohomology of $K(\mathbb{Z}/2,2)$.

We first see that to cancel out y, there has to be an isomorphism on $E_2^{0,1}$. So there should be a $\mathbb{F}_2$ at $E_2^{2,0}$. Then there are dots on all $E_2^{2,q}$. You can do the same thing and conclude that there are classes ι^p of $E_2^{2p,0}$. You see that $d_2(y^2) = 2y\iota = 0$ but $d_2(y^3) = 3y^2\iota = y^2\iota$. So the d_2 maps come in these alternating rows.

But you see that the classes y^2 is not killed by a d_2. The only way is for the d_3 map to do this job. This shows that there is a class $\kappa \in E_2^{3,0}$. Then there are classes $\kappa \in E_2^{3,0}$. This induces classes $y^n\kappa \in E_2^{3,n}$ and likewise generates a bunch of d_3 maps. Also the class κ also has to give d_2 maps according to the product formula. After we figure out what is going on in d_2 and d_3, we note that y^4 has yet to be canceled out. The only thing that can happen is d_5 killing it. So we get a class $\lambda \in E_2^{5,0}$.

After doing all of this, you see an inductive pattern. The conclusion is that

$$H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2[\iota,\kappa,\lambda,\mu,\ldots],$$

where $\iota,\kappa,\lambda,\mu,\ldots$ live in degree $2,3,5,9,\ldots$.

24 March 27, 2017

24.1 G-bundles

Definition 24.1. A **(principal)** G**-bundle** is a particular sort of fiber bundle $p : E \to B$ with fiber G, where G acts on E in such a way that for $U \subseteq B$, the identification $\varphi_U : p^{-1}(U) \cong G \times U$ is equivariant.

If G acts on some other space F, then from a G-bundle, one can form a fiber bundle $E \times_G F$ over B with fiber F, by

$$E \times F/(eg, f) \sim (e, fg).$$

This is sometimes called the **Borel construction** or the associated fiber bundle.

Example 24.2. The group $U(n)$ acts on $\mathbb{C}^n$. So from a $U(n)$-bundle you get a vector bundle. (In fact, this induction is a bijection.)

Lemma 24.3. *The assignment*

$$X \mapsto \{\textit{isomorphism classes of } G\textit{-bundles overt } X\}$$

satisfies the wedge axiom and the Mayer–Vietoris axiom, (where X is a finite complex).

Corollary 24.4. *There is a homotopy type BG representing this functor.*

I would like to know about this BG, and in particular its cohomology. Here is why. Suppose I have a G-bundle $E \to B$. This is the same data of a homotopy class of $B \to BG$. If I have my favorite element H^*BG, then we get an element $\xi^*\omega \in H^*B$ that is natural. These are called **characteristic classes** of G-bundles.

Lemma 24.5. *Let $p : E \to B$ be a G-bundle where E is n-connected. Then the classifying map $\xi : B \to BG$ is an n-equivalence. Equivalently, the natural transformation $[-, B] \to [-, BG]$ is an isomorphism on complexes of dimension $\leq n$.*

Corollary 24.6. *The universal bundle classified by* $\mathrm{id} : BG \to BG$ *has contractible total space, often denoted by EG.*

Corollary 24.7. *$\pi_{*+1}BG \cong \pi_*G$, with the indicated shift.*

There is a model of $K(A, n)$ which is an actual topological group, so that $BK(A, n) = K(A, n + 1)$.

For the last few lectures, we have been using spectral sequences on $G \to * \to BG$. So if we know about H^*G, we might be able to get a handle on H^*G.

Let us try to compute $H^*BU(n)$. We first have a fibration $U(n-1) \to U(n) \to S^{2n-1}$. Using this fibration, we compute

$$H^*U(2) = \Lambda[e_1, e_2].$$

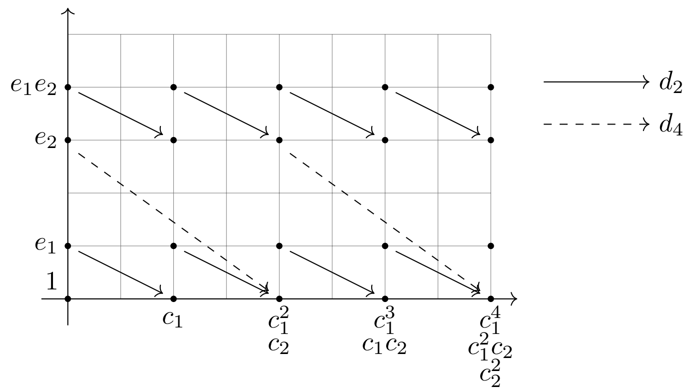

Figure 10: Serre spectral sequence of $U(2) \to * \to BU(2)$

From the fibration $U(2) \to * \to BU(2)$, we get the following spectral sequence.

This is the same thing we did last time. The conclusion is

$$H^*BU(2) \cong \mathbb{Z}[c_1, c_2].$$

This pattern is general. We learn that

$$H^*U(n) = \Lambda[e_1, \dots, e_n], \quad H^*BU(n) \cong \mathbb{Z}[c_1, \dots, c_n].$$

These classes c_j are called **Chern classes**.

24.2 The bar complex

For G a finite group, there is an especially useful model for BG, called the **bar complex**.

Definition 24.8. For a category $\mathscr{C}$, define its **nerve** to be the simplicial set with:

- 0-simplices: objects in $\mathscr{C}$,
- 1-simplices: morphisms $f : x \to y$ in $\mathscr{C}$,
- 2-simplices: the data of

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"'] & & z, \end{tikzcd}$$

- 3-simplices: the data of ....

Functors give rise to maps of simplicial sets. This is kind of an inspiration for natural transformations giving homotopies and co/limits giving cones.

Example 24.9. Consider the category $G//G$ with objects G and morphisms $h : g \to gh$. There is also the category $*//G$ with one object $*$ and morphisms $h : * \to *$.

Lemma 24.10. *$G//G$ is contractible, i.e., any map $* \to G//G$ is fully faithful and essentially surjective.*

The G-action on $G//G$ is discrete.

Corollary 24.11. *$N(G//G) \to N(*//G)$ models $EG \to BG$.*

In fact, there are equivalences of categories

$$\begin{tikzcd} G//G \arrow[r] \arrow[d] & *//G \arrow[d] \\ \left\{\begin{matrix} G\text{-torsors with} \\ \text{marked point} \end{matrix}\right\} \arrow[r] & \{G\text{-torsors}\}. \end{tikzcd}$$

Recall that $\xi : X \to BG$ is the data of G-bundles. This data is the same thing as the data of

$$\operatorname{Sing} X \to N(*//G),$$

where $\operatorname{Sing} X$ is the simplicial set of all $\Delta^n \to X$. But $N(*//G)$ has no interesting ≥ 2-simplices in the sense that the edges determine everything. So this is just the data of

$$\Pi_1 X \to N(*//G) \simeq \{G\text{-torsors}\}.$$

This line of though is called the **Grothendieck construction**.

25 March 29, 2017

25.1 Properties of Chern classes

Theorem 25.1. *For each $U(n)$-bundle ξ over a CW-complex X, there are unique elements $c_j(\xi) \in H^{2j}(X)$ satisfying*

(0) they depend only on the isomorphism class of ξ,

(1) for a map $f : Y \to X$, $c_j(f^*\xi) = f^*c_j(\xi)$,

(2) $c_0(\xi) = 1$

(3) for γ the tautological bundle over $\mathbb{C}P^\infty$, $c_1(\gamma) = x \in \mathbb{Z}[x] = H^*\mathbb{C}P^\infty$

(4) for ξ a $U(n)$-bundle and ζ a $U(m)$-bundle both over X,

$$\chi_k(\xi \oplus \zeta) = \sum_{i+j=k} c_i(\xi)c_j(\zeta).$$

Lemma 25.2. *For ξ a $\mathbb{C}^n$-vector bundle on X, there exists a space $f : Y \to X$ such that*

(a) $f^* : H^*X \hookrightarrow H^*Y$ is injective,

(b) $f^*\xi = \xi' \oplus \eta$ where η is a line bundle.

This lemma allows us to pull back the bundle and compute the Chern class on the pullback bundle, which splits into a line bundle and a bundle of smaller rank.

Proof. Set $Y = \mathbb{P}(\xi)$, the fiberwise projectivization of ξ, to get a bundle $\mathbb{C}P^{n-1} \to \mathbb{P}(\xi) \to X$. Now pull back ξ along f to form

$$f^*\xi = \{(l, v) : l \in \mathbb{P}(\xi), v \in \xi|_{f(l)}\},$$

where l is a line in one of the fibers of ξ. Inside this, there is a subbundle of those pairs (l, v) where $v \in l$, which is a line bundle η over $\mathbb{P}(\xi)$. This comes with an orthogonal complement $\eta^\perp$.

Now let us show (a). We have

$$\begin{tikzcd}
\mathbb{C}^n \arrow[d, hook] & \mathbb{C}^n \setminus \{0\} \arrow[l, hook'] \arrow[r] \arrow[d, hook] & \mathbb{C}P^{n-1} \arrow[r] \arrow[d, hook] & BU(1) \arrow[d, equal] \\
\xi \arrow[d] & \xi \setminus \{0\} \arrow[l, hook'] \arrow[d] & \mathbb{P}(\xi) \arrow[l] \arrow[r] \arrow[d, "f"] & BU(1) \arrow[d] \\
X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r] & *.
\end{tikzcd}$$

We have a map of fiber bundles from $\mathbb{C}P^{n-1} \to \mathbb{P}(\xi) \to X$ to $BU(1) \to BU(1) \to *$. We know that the map $H^*BU(1) \to H^*\mathbb{C}P^{n-1}$ looks like $\mathbb{Z}[x] \to$

$\mathbb{Z}[x]/(x^n)$. Because the spectral sequence for $BU(1) \to BU(1) \to *$ collapses, this shows that in the spectral sequence for $\mathbb{C}P^{n-1} \to \mathbb{P}(\xi) \to X$, we have

$$d_r(x^j) = d_r(F^* x^j) = F^* d_r(x^j) = F^*(0) = 0.$$

So the spectral sequence for $\mathbb{C}P^{n-1} \to \mathbb{P}(\xi) \to X$ also collapses.

Now look at the map from $\mathbb{C}P^{n-1} \to \mathbb{P}(\xi) \to X$ to $* \to X \to X$, induced by f. This shows that $H^*X \hookrightarrow H^*\mathbb{P}(\xi)$ is injective. $\square$

This moreover shows that

$$H^*\mathbb{P}(\xi) \cong \mathbb{Z}\langle 1, x, \ldots, x^{n-1}\rangle \otimes H^*(X;\mathbb{Z}),$$

with the relation $x \cdot x^{n-1} = b_1 x^{n-1} - b_2 x^{n-2} + \cdots + (-1)^{n-1} b_n$ for some classes $b_j \in H^{2j}$.

Proposition 25.3. *The classes b_j satisfy the condition of the theorem, and they agree with the classes $x_j \in \mathbb{Z}[x_1, \ldots, x_n] \cong H^*BU(n)$.*

Proof. From the theorem, naturality and normalization are immediate. For the sum formula, consider $\mathbb{P}(\xi), \mathbb{P}(\zeta) \hookrightarrow \mathbb{P}(\xi, \zeta)$. You can check that $\mathbb{P}(\xi \oplus \zeta) \setminus \mathbb{P}(\xi)$ deformation retracts onto $\mathbb{P}(\zeta)$.

Define the classes

$$\begin{aligned} \alpha &= x^n - b_1(\xi)x^{n-1} + \cdots + (-1)^n b_n(\xi) \in H^*\mathbb{P}(\xi \oplus \zeta), \\ \beta &= x^m - b_1(\zeta)x^{m-1} + \cdots + (-1)^m b_m(\zeta) \in H^*\mathbb{P}(\xi \oplus \zeta). \end{aligned}$$

Note that $\alpha|_{\mathbb{P}(\xi)} = 0$ and $\beta|_{\mathbb{P}(\zeta)} = 0$. Then by Mayer–Vietoris, we have $\alpha \cdot \beta = 0 \in H^*\mathbb{P}(\xi \oplus \zeta)$. But $\alpha \cdot \beta$ is a degree $n+m$ polynomial in x, and so it has to be $x^{n+m} - b_1(\xi \oplus \zeta)x^{n+m-1} + \cdots$. This proves the sum formula. $\square$

Now let us shows that $c_j(\xi)$ is unique. Any vector bundle has some Y over which it decomposes as a sum of line bundles, $\xi = \bigoplus \eta_j$. Then

$$\begin{aligned} \sum_{j=0}^n c_j(\xi)x^{n-j} &= c(\xi) = c(\textstyle\bigoplus_j \eta_j) = \displaystyle\prod_j c(\eta_j) \\ &= \prod_j (x - c_1(\eta_j)) = \prod_j (x - c_1(l_j^*\gamma)) = \prod_j (x - l_j^* c_1(\gamma)), \end{aligned}$$

where l_j are the maps $Y \to BU(1)$ classifying η_j.

Let us now show that these agree with $x_j \in \mathbb{Z}[x_1, \ldots, x_n] \cong H^*BU(n)$. There is a map $\xi_n : (BU(1))^n \to BU(n)$ classifying n-dimensional vector bundles with a choice of splitting into n lines. There is a fibration $S^{2n-1} \to BU(n-1) \to BU(n)$ obtained by identifying $\Omega BU(n) = U(n)$.

If you run the Serre spectral sequence on this fibration, what you get is

$$\begin{tikzcd} H^{*+2n-1}BU(n-1) \arrow[r, "0"] & H^*BU(n) \arrow[r, "\cdot x_n"] \arrow[d, "\xi_n"] & H^{*+2n}BU(n) \arrow[r] \arrow[d, "\xi_n"] & H^{*+2n}BU(n-1) \arrow[r, "0"] \arrow[d, "\xi_{n-1}"] & H^{*+1}BU(n) \\ & H^*BU(1)^n \arrow[r] & H^{*+2n}BU(1)^n \arrow[r] & H^{*+2n}BU(1)^{n-1} & \end{tikzcd}$$

These are going to land inside the symmetric polynomials, and ξ_n are all ring morphisms.

26 March 31, 2017

The plan today is to use this technology to compute the homology of Eilenberg–MacLane spaces.

26.1 The Steenrod algebra

There is a Federer spectral sequence given by

$$d_1 : H^n(-;\pi_n Y) \to H^{n+2}(-;\pi_{n+1}Y)$$

associated to the Postnikov tower of Y. What do these look like? In general, such a functor is going go be a map $\omega : K(\pi_n Y, n) \to K(\pi_{n+1}Y, n+2)$. This is in therm the same as an element $\omega \in H^{n+2}(K(\pi_n Y, n); \pi_{n+1}Y)$. We know have a fibration

$$K(A,n) \to * \to B(K(A,n)) \cong K(A,n+1).$$

If you remember the bar construction, you have a filtration on BG, and the associated graded is going to be $(G \wedge S^1)^{\wedge n}$.

$$\begin{tikzcd} \cdots & BG^{(n)} \arrow[l, hook'] \arrow[d] & BG^{(n-1)} \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] \\ & (G \wedge S^1)^{\wedge n} & (G \wedge S^1)^{\wedge (n-1} & \end{tikzcd}$$

Corollary 26.1. *There is a spectral sequence $E^1_{*,*} = \tilde{h}_*((G \wedge S^1)^{\wedge n})$ giving $h_* BG$.*

If h has Künneth isomorphisms, then $\tilde{h}_*((G \wedge S^1)^{\wedge n}) \cong (\tilde{h}_{*-1}G)^{\otimes n}$. The differential d_1 then looks like

$$g_1 \otimes \cdots \otimes g_n \mapsto \sum_j (-1)^j g_1 \otimes \cdots \otimes g_{j-1}g_j \otimes \cdots \otimes g_n.$$

If you know homological algebra, this chain complex computes

$$E^2_{*,*} = \operatorname{Tor}^{h_*G}_{*,*}(h_*, h_*).$$

Let us call $R = h_*G$ and $k = h_*$.

Let me give you some examples of some Tor algebras. This is due to Tate. Tor can always be computed by picking a free R-resolution of h_*, tensoring the resolution with $- \otimes_R h_*$, and then taking homology. You can do this in such a way that the resolution itself is a commutative differential graded algebra.

Example 26.2. Let us consider $R = k[e]/e^2$. We are going to resolve this by a free R-module resolution. The second claim says that we can be clever and give it a structure of a commutative differential graded algebra. This can be done by setting $y = \frac{1}{2}x^2$, $z = \frac{1}{6}x^3$, ... Then our free resolution can be written as

$$C_* = \Gamma_R[x].$$

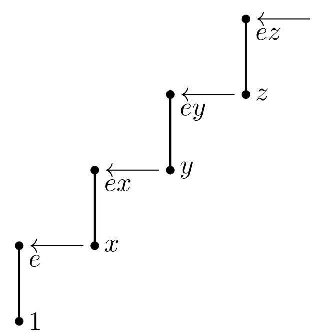

Figure 11: Free resolution of k

Now to compute Tor, we tensor it with k. This leaves no differentials and so

$$\mathrm{Tor} = H_*(C_* \otimes k) = \Gamma_k[x].$$

Example 26.3. Let us look at $R = k[x]$. In this case, we get a free resolution that has just two things. By the same line of argument, we are going to have

$$\mathrm{Tor} = \Lambda_k[y].$$

I am now going to actually compute something in topology. Set $G = \mathbb{Z}/2$ and take the ordinary mod 2 homology. We have a spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{H_*(\mathbb{Z}/2;\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \quad \Rightarrow \quad H_*(B\mathbb{Z}/2; \mathbb{F}_2).$$

The ring is going to be

$$R = H_*(\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}_2\langle [0], [1] \rangle, \quad [i] \cdot [j] = [i + j]$$

This is isomorphic to $\mathbb{F}_2[a]/a^2$, with $a = [1]-1$. Then we have already computed $E^2_{*,*} = \Gamma[x]$. We already know that $H_*(B\mathbb{Z}/2; \mathbb{F}_2) = H_*(\mathbb{R}P^\infty; \mathbb{F}_2)$. Then we see that there is no differentials in the spectral sequence.

As we'll see, this is the start of a general pattern: the bar spectral sequence computing $H_*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ for each n collapses at E^2. So the calculation of theses homology groups reduces to computing the E^2 Tor terms.

Over $\mathbb{F}_2$, you can recognize

$$\Gamma_{\mathbb{F}_2}[x] = \Lambda[x^{(1)}, x^{(2)}, x^{(4)}, \ldots].$$

Tor also has a kind of Künneth isomorphism,

$$\mathrm{Tor}^{\Gamma[x]} = \mathrm{Tor}^{\bigotimes \Lambda[x^{2^j}]} \cong \bigotimes_j \mathrm{Tor}^{\Lambda[x^{2^j}]} \cong \bigoplus_j \Gamma[y_j \text{ associated to } x^{2^j}].$$

27 April 10, 2017

27.1 The Steenrod algebra II

We were looking at natural transformations between cohomology functors. We need to input the Serre spectral sequence and the transgression theorem.

Note that

$$\mathbb{F}_2[\iota_1] \cong H^*(\mathbb{R}P^\infty, \mathbb{F}_2) \cong [\mathbb{R}P^\infty, K(\mathbb{F}_2, *)] \cong (H^1(-;\mathbb{F}_2) \to H^*(-;\mathbb{F}_2)).$$

For example, when $* = 1$, there are only two things in $(\mathbb{F}_2[\iota_1])_1 = \{0, 1\}$. So there are two such natural transformations, and they are 0 and id. This is true for higher $*$. In general, there are two natural transformation $H^1(-;\mathbb{F}_2) \to H^n(-;\mathbb{F}_2)$, which is 0 and something more mysterious. Our subgoal is to identify in more explicit terms the other nonzero natural transformations for $* \neq 1$.

Previously we computed the cohomology $H^*(K(\mathbb{F}_2, 2); \mathbb{F}_2)$. (See Figure 9.)

Lemma 27.1. *For a Serre fibration $F \to E \to B$ and classes $x \in H^{n+1}B$ and $y \in H^n F$ satisfying $p^*x = q^*y$*

$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[r] \arrow[d] & B \\ & (E,F) \arrow[ur, "p"] \arrow[r, "q"] & \Sigma F \end{tikzcd}$$

are exactly the same as transgressive differentials $d_n(y) = x$.

Consider ι_1^2, which participates in the transgressive differential $d_3(\iota_1^2) = \iota_2$. The condition $p^*x = q^*y$ is natural against cohomology operations. Expand ι_1^2 as

$$\begin{tikzcd} & X \arrow[d, "\omega"] \arrow[dr, "\omega^2"] & \\ \mathbb{R}P^\infty \arrow[r, "\mathrm{id}=\iota_1"] & \mathbb{R}P^\infty \arrow[r, "\mathrm{Sq}"] & K(\mathbb{F}_2, 2), \end{tikzcd}$$

where the map on the right is ω^2 becomes of the naturality of the ring structure on H^*. Then from $p^*\iota_2 = q^*\iota_1$, we get

$$p^*(\mathrm{Sq}\,\iota_2) = \mathrm{Sq}(p^*\iota_2) = \mathrm{Sq}(q^*\iota_1) = q^*(\mathrm{Sq}\,\iota_1) = q^*\iota_1^2.$$

This is a proof of Kudo's transgression theorem, which says that the transgressive differentials commute with cohomology operations. So we defined Sq on the first cohomology, but it also takes things in the second cohomology and spits out a third cohomology.

Any map $K(\mathbb{F}_2, 1) \to K(\mathbb{F}_2, 2)$ induces a map

$$\begin{tikzcd} \Sigma K(\mathbb{F}_2, 1) \arrow[r] \arrow[d] & \Sigma K(\mathbb{F}_2, 2) \arrow[d] \\ K(\mathbb{F}_2, 2) \arrow[r, "\mathrm{Sq}^1"] & K(\mathbb{F}_2, 3). \end{tikzcd}$$

You can define the squares similarly so that $\iota^8 = \operatorname{Sq}^4 \operatorname{Sq}^2 \operatorname{Sq}^1 \iota_1$ and so on. Then we have

$$H^*(K(\mathbb{F}_2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[\iota_2, \operatorname{Sq}^1 \iota_2, \operatorname{Sq}^2 \operatorname{Sq}^1 \iota_2, \ldots].$$

In fact, this process is exhaustive.

Theorem 27.2 (Serre)**.** *The cohomology of $K(\mathbb{F}_2; q)$ is*

$$H^*(K(\mathbb{F}_2, q); \mathbb{F}_2) \cong \mathbb{F}_2[\operatorname{Sq}^I \iota_q : I_j \geq 2I_{j+1}, \textstyle\sum(I_j - 2I_{j+1}) < q].$$

So here are some properties of the squares:

(0) $\operatorname{Sq}^0(x) = x$

(1) $\operatorname{Sq}^{|x|}(x) = x^2$

(2) $\operatorname{Sq}^n(x + y) = \operatorname{Sq}^n x + \operatorname{Sq}^n y$

(3) $\operatorname{Sq}^{>|x|}(x) = 0$

(4) $\operatorname{Sq}^n(xy) = \displaystyle\sum_{i+j=n} \operatorname{Sq}^i x \operatorname{Sq}^j y$

(5) "The Adem relations" $\operatorname{Sq}^{2n-1} \operatorname{Sq}^n = 0$, and $d(\operatorname{Sq}^n) = \operatorname{Sq}^{n-1}$ extends to a derivation.

Here is a way to think about the Adem relations. For instance,

$$\begin{aligned} 0 = d^3(0) = d^3(\operatorname{Sq}^5 \operatorname{Sq}^3) &= d(d^2(\operatorname{Sq}^5 \operatorname{Sq}^3)) = d(\operatorname{Sq}^3 \operatorname{Sq}^3 + \operatorname{Sq}^5 \operatorname{Sq}^1) \\ &= \operatorname{Sq}^2 \operatorname{Sq}^3 + \operatorname{Sq}^3 \operatorname{Sq}^2 + \operatorname{Sq}^5 \operatorname{Sq}^0 + \operatorname{Sq}^4 \operatorname{Sq}^1 = \operatorname{Sq}^2 \operatorname{Sq}^3 + \operatorname{Sq}^5 + \operatorname{Sq}^4 \operatorname{Sq}^1. \end{aligned}$$

The conclusion is that a lot of Sq^n are redundant, i.e., they are in the algebra generated by the other squares.

Lemma 27.3. Sq^{2^j} *are the indecomposables.*

Proof. This uses the bar spectral sequence. $\square$

Theorem 27.4. *These exhaust of collections of cohomology operations $H^*(-; \mathbb{F}_2) \to H^{*+n}(-; \mathbb{F}_2)$ which are compatible with suspension.*

28 April 12, 2017

28.1 A Serre spectral sequence trick

The trick I want to tell you is how to kill a particular class in the cohomology of a space. Suppose that $\omega \in H^n(X;\mathbb{Z})$. This is the same thing as a classifying map $\omega : K(\mathbb{Z},n)$. Sitting over $K(\mathbb{Z},n)$ is a fibration $K(\mathbb{Z},n-1) \to * \to K(\mathbb{Z},n)$.

$$\begin{tikzcd}
K(\mathbb{Z},n-1) \arrow[r, equal] \arrow[d] & K(\mathbb{Z},n-1) \arrow[d] \\
P_\omega \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r, "\omega"] & K(\mathbb{Z},n)
\end{tikzcd}$$

This gives a map between spectral sequences. The cohomology of $H^*(K(\mathbb{Z},n-1);\mathbb{F}_2)$ starts with ι_{n-1} and $\mathrm{Sq}^2\,\iota_{n-1}$, and the differentials must kill $\iota_n, \mathrm{Sq}^2\,\iota_n \in H^*(K(\mathbb{Z},n);\mathbb{F}_2)$. On the spectral sequence for $K(\mathbb{Z},n-1) \to P_\omega \to X$, we see that the classes ι_{n-1} on the right has to map to ι_{n-1} on the left, and ι_n on the right has to maps to ω on the left. So the differential maps ι_{n-1} to ω, in the spectral sequence coming from $K(\mathbb{Z},n-1) \to P_\omega \to X$.

The consequence of this construction is that H^*P_ω doesn't have ω in it. The trade is that, if $\mathrm{Sq}^2\,\omega = 0$ then $\mathrm{Sq}^2\,\iota_{n-1}$ has no target and so survives to contribute to higher cohomology. Additionally, π_*P_ω is mostly equal to π_*X except around degree n.

The idea for Friday is to use this to calculate homotopy groups of spaces, by picking ω to be a generator of $H^n(X)$ for X an $(n-1)$-connected space. We will be precise about this next time.

We're computing using the mod 2 Serre spectral sequence. The idea is to use the Hurewicz map $\pi_n X \xrightarrow{\cong} H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{F}_2)$. There is a discrepancy here: the mod 2 co/homology of P_ω does not give full information of its integral co/homology.

28.2 Serre classes

The idea with Serre classes is to build of version of algebraic topology that works "at a prime". The starting point for this is to build a version of algebra that works at a prime.

Definition 28.1. A **class** of abelian groups is a collection $\mathcal{C}$ such that

(i) if $0 \to A'' \to A \to A' \to 0$ is a short exact sequence, then $A \in \mathcal{C}$ if and only if $A'', A' \in \mathcal{C}$,

(ii) for $A \in \mathcal{C}$ and B an abelian group, $A \otimes B \in \mathcal{C}$,

(iii) for $A \in \mathcal{C}$, $H_{*>0}(BA;\mathbb{Z}) \in \mathcal{C}$.

Example 28.2. The class $\mathcal{C}_p$ consists of abelian torsion groups with finite exponent coprime to p.

Here is the big idea. Algebra works "up to $\mathcal{C}$". Mod 2 co/homology only looks at things related to 2. So $\mathcal{C}_p$ contains everything we don't care about. We define **$\mathcal{C}$-monomorphisms** as maps having kernel in $\mathcal{C}$, **$\mathcal{C}$-epimorphisms** as maps with cokernel in $\mathcal{C}$, and **$\mathcal{C}$-isomorphisms** as maps with both in $\mathcal{C}$. Two groups are **$\mathcal{C}$-isomorphic** if they are connected by a zig-zag of $\mathcal{C}$-isomorphisms.

You can check that most of algebra works here. In particular, all of homological algebra can be done "up to $\mathcal{C}$", e.g., the 5-lemma, the snake lemma, or the notion of exactness. Exactness here will be that the kernel and the cokernel almost agree, up to some $\mathcal{C}$.

The biggest idea is that homotopy theory can also be done "up to $\mathcal{C}$".

Definition 28.3. A map $f : X \to Y$ is a **weak $\mathcal{C}$-equivalence** if $\pi_* f : \pi_* X \to \pi_* Y$ is a $\mathcal{C}$-isomorphism for all $*$.

We can go back to January and start proving theorems all over.

Theorem 28.4 (Hurewicz)**.** *For X a simply connected space, if $\pi_{<n} X \in \mathcal{C}$ then $H_{<n} \in \mathcal{C}$ and $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.*

Theorem 28.5 (Whitehead)**.** *For $f : X \to Y$ a map of simply connected space which is an isomorphism on π_2, f induces a $\mathcal{C}$-isomorphism on $\pi_{<n}$ and a $\mathcal{C}$-epimorphism on π_n if and only if f induces a $\mathcal{C}$-isomorphism on $H_{<n}$ and a $\mathcal{C}$-epimorphism on H_n.*

Theorem 28.6 (Approximation)**.** *Let X, Y be simply connected spaces and let $f : Y \to X$ have π_2 onto. The following are equivalent:*

(i) $H^{<n}(X; \mathbb{Z}/p) \to H^{<n}(Y; \mathbb{Z}/p)$ *is an isomorphism and* $H^n(X; \mathbb{Z}/p) \to H^n(Y; \mathbb{Z}/p)$ *is an epimorphism.*

(ii) The same thing with homology mod p.

(iii) $H_{\leq n}(X, Y; \mathbb{Z}/p) = 0$.

(iv) $H_{\leq n}(X, Y; \mathbb{Z}) \in \mathcal{C}_p$.

(v) $\pi_{\leq n}(X, Y) \in \mathcal{C}_p$.

(vi) $\pi_{<n} Y \to \pi_{<n} X$ *is a* $\mathcal{C}_p$*-isomorphism and* $\pi_n Y \to \pi_n X$ *is a* $\mathcal{C}_p$*-epimorphism.*

(vii) $\pi_{<n} X$ *and* $\pi_{<n} Y$ *have isomorphic p-components.*

Proof. (i) and (ii) are related by duality. (ii) and (iii) are related by the long exact sequence of a pair. (iii) and (iv) are related by the universal coefficients theorem. (iv) and (v) are related by the mod $\mathcal{C}$ Hurewicz theorem. (v) and (vi) are related by the long exact sequence. (vi) and (vii) is not obvious. □

Lemma 28.7. *For $f : A \to B$ a $\mathcal{C}_p$-isomorphism of finitely generated abelian groups. Then A and B have isomorphic p-components.*

Proof. Let ${}_pA$ denote the subgroup of prime-to-p elements torsion elements in A, so we want $A/{}_pA \cong B/{}_pB$. We have

$$\begin{tikzcd}
0 \arrow[r] & {}_pA \arrow[r, "f"] \arrow[d] & A \arrow[r] \arrow[d] & A/{}_pA \arrow[r] \arrow[d, "\hat{f}"] & 0 \\
0 \arrow[r] & {}_pB \arrow[r] & B \arrow[r] & B/{}_pB \arrow[r] & 0.
\end{tikzcd}$$

Because f is a $\mathcal{C}_p$-isomorphism, $\hat{f}$ is also a $\mathcal{C}_p$-isomorphism using the $\mathcal{C}_p$-five lemma. Using the fundamental theorem of finitely generated abelian groups, it follows that $\hat{f}$ is a monomorphism and $\hat{f}$ induces an isomorphism of torsion groups with its image a subgroup of maximal rank. $\square$

29 April 14, 2017

29.1 Serre's method

We want to compute the homotopy group of spheres. The big idea is to build a given space from Eilenberg–MacLane spaces in such a way that $X \to \{Y_i\}$ is a cohomology isomorphism. If you build stuff out of Eilenberg–MacLane spaces, their homotopy groups are going to be simple.

Recall that

$$\begin{aligned} H^*(K(\mathbb{Z}/2, q); \mathbb{F}_2) &\cong \mathbb{F}_2[\mathrm{Sq}^I \iota_q : I_j \geq 2I_{j+1}, 2I_1 - I_+ < q], \\ H^*(K(\mathbb{Z}, q); \mathbb{F}_2) &\cong \mathbb{F}_2[\mathrm{Sq}^I \iota_q : I_j \geq 2I_{j+1} : 2I_1 - I_+ < q, I_{\mathrm{last}} \neq 1]. \end{aligned}$$

Set $n \gg 1$ and consider S^n. We first look at the first approximation

$$S^n \to K(\mathbb{Z}, n)$$

which is a isomorphism of π_n. The cohomology of the left hand side and the right hand side do not agree.

Our complain is that $\mathrm{Sq}^2 \iota_n \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{F}_2)$ is not present in $H^{n+2}(S^n; \mathbb{F}_2)$.

$$\begin{tikzcd} K(\mathbb{Z}/2, n+1) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}/2, n+1) \arrow[d] \\ P_{\mathrm{Sq}^2 \iota_n} \arrow[r] \arrow[d] & * \arrow[d] \\ K(\mathbb{Z}, n) \arrow[r, "\mathrm{Sq}^2 \iota_n"] & K(\mathbb{Z}/2, n+2) \end{tikzcd}$$

Then the left fibration is killing $\mathrm{Sq}^2 \iota_n$ and so the lift $S^n P_{\mathrm{Sq}^2 \iota_n}$ won't have our initial complaint. Then we will have our next complaint. But we need to compute the cohomology of $P_{\mathrm{Sq}^2 \iota_n}$ first. We use the Serre spectral sequence.

k	0	1	2	3	4	5	6	7
$H^{n+k}K(\mathbb{Z}, n)$	$\boxed{\iota_n}$		Sq^2	$\mathrm{Sq}^{1,2}$	$\boxed{\mathrm{Sq}^4}$	Sq^5	$\boxed{\mathrm{Sq}^6}$, $\mathrm{Sq}^{4,2}$	$\boxed{\mathrm{Sq}^7}$, $\mathrm{Sq}^{5,2}$
$H^{n+k}K(\mathbb{Z}/2, n+1)$		ι_{n+1}	Sq^1	$\boxed{\mathrm{Sq}^2}$	$\boxed{\mathrm{Sq}^{1,2}}$, $\mathrm{Sq}^{2,1}$	$\boxed{\mathrm{Sq}^{3,1}}$, Sq^4	$\boxed{\mathrm{Sq}^5, \mathrm{Sq}^{4,1}}$	Sq^6, $\boxed{\mathrm{Sq}^{5,1}}$, $\boxed{\mathrm{Sq}^{4,2}}$

Table 1: Computing the cohomology of $P_{\mathrm{Sq}^2 \iota_n}$

Here, the differentials go like $\iota_{n+1} \mapsto \mathrm{Sq}^2$, $\mathrm{Sq}^1 \mapsto \mathrm{Sq}^{1,2}$, $\mathrm{Sq}^{2,1} \mapsto \mathrm{Sq}^5$, $\mathrm{Sq}^4 \mapsto \mathrm{Sq}^{4,2}$, $\mathrm{Sq}^5 \mapsto \mathrm{Sq}^{5,2}$, $\mathrm{Sq}^{4,1} \mapsto \mathrm{Sq}^{5,2}$, $\ldots$. Then we have an approximation $S^n[0, n+1]$. We can now do this process again.

$$\begin{tikzcd} S^n \arrow[d] \arrow[dr] \arrow[drr] & & \\ K(\mathbb{Z}, n) \arrow[d] & S^n[0, n+1] \arrow[l] \arrow[d] & S^n[0, n+2] \arrow[l] \\ K(\mathbb{Z}/2, n+2) & K(\mathbb{Z}/2, n+3) & \end{tikzcd}$$

But we need to first compute the cohomology of $S^n[0, n+2]$.

k	0	1	2	3	4	5	6	7
$H^{n+k} S^n[0, n+1]$	$\boxed{\iota_n}$	0	0	$\mathrm{Sq}^2\, \iota_{n+1}$	$\boxed{\mathrm{Sq}^4\, \iota_n}$, $\mathrm{Sq}^3\, \iota_{n+1}$	$\mathrm{Sq}^{3,1}\, \iota_{n+1}$	$\boxed{\mathrm{Sq}^6\, \iota_n}$, $(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1})\iota_{n+1}$	$\boxed{\mathrm{Sq}^7\, \iota_n}$, $\mathrm{Sq}^{5,1}\, \iota_{n+1}$, $\mathrm{Sq}^{4,2}\, \iota_{n+1}$
$H^{n+k} K(\mathbb{Z}/2, n+2)$	0	0	ι_{n+2}	Sq^1	Sq^2	$\boxed{\mathrm{Sq}^3}$, $\mathrm{Sq}^{2,1}$	$\mathrm{Sq}^{3,1}$, Sq^4	$\boxed{\mathrm{Sq}^5, \mathrm{Sq}^{4,1}}$.

Table 2: Computing the cohomology of $S^n[0, n+2]$

Now we do the next approximation. Here, you run into a problem. There is a class ι_{n+3} and this knocks out $\mathrm{Sq}^4\, \iota_n$, but there is a class $\mathrm{Sq}^1\, \iota_{n+3}$ that transgress to zero, i.e., lives. Then there is still a class in H^{n+4}.

In this specific situation, what you can use is $K(\mathbb{Z}/8, n+4)$ instead of $K(\mathbb{Z}/2, n+4)$:

$$
\begin{tikzcd}
S^n \arrow[d] \arrow[dr] & \\
S^n[0, n+2] \arrow[r] \arrow[d, "\mathrm{Sq}^4\, \iota_n"] & S^n[0, n+3] \\
K(\mathbb{Z}/8, n+4) &
\end{tikzcd}
$$

Here is what you get for computing cohomology.

k	0	1	2	3	4	5	6	7
$H^{n+k} S^n[0, n+2]$	$\boxed{\iota_n}$				$\mathrm{Sq}^4\, \iota_n$	$\mathrm{Sq}^3\, \iota_{n+2}$	$\mathrm{Sq}^6\, \iota_n$	$\mathrm{Sq}^7\, \iota_n$, $(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1})\iota_{n+2}$
$H^{n+k} K(\mathbb{Z}/8, n+3)$				ι_{n+3}	$\beta_3 \iota_{n+3}$	$\mathrm{Sq}^2\, \iota_{n+3}$	$\mathrm{Sq}^3\, \iota_{n+3}$, $\mathrm{Sq}^2\, \beta_3 \iota_{n+3}$	$\boxed{\mathrm{Sq}^4\, \iota_{n+3}}$, $\mathrm{Sq}^3\, \beta_3 \iota_{n+3}$

Table 3: Computing the cohomology of $S^n[0, n+3]$

Here, β_3 is the thing that relates cohomology classes in the universal coefficients theorem sequence. This plays well with the Serre spectral sequence. After this, we get a good approximation and get $S^n[0, n+1]$. So using this approximation, you will read off the homotopy groups.

k	0	1	2	3	4	5	6	7
$\pi_{n+k} S^n$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\cdots$

Table 4: 2-torsion parts of stable homotopy groups

Note that we are working with $H^*(-; \mathbb{F}_2)$, so Serre classes from last time tells us that we're only getting $(\pi_* S^n)_2$. The $n \gg 0$ condition was important, because $\Sigma(H^* K(\mathbb{Z}/2, n)) \cong H^*(K(\mathbb{Z}/2, n+1))$ is not true for small n.

Also, I might have given you the impression that you can compute all the 2-torsion parts of the homotopy groups if you work sufficiently hard, but this is not true. There is an indeterminacy at $k = 14$ that cannot be resolved.

30 April 17, 2017

Today is going to be quite fast. You can go over a whole semester over this.

30.1 The Adams spectral sequence

We computed $\pi_{*+n}S^n$ for $n \gg 0$ and $* < 2n$. We needed to understand some things, namely $H^*(S^n[n, n+k]; \mathbb{F}_2)$ as a module over the Steenrod algebra. The Adams spectral sequence is supposed to be a repackaging of this process. The idea is to strip homotopy out of a space iteratively using the Hurewicz homomorphism.

Lemma 30.1. *Take X to be $(n-1)$-connected with $\pi_* X$ finite exponent and 2-torsion. Let $X' = \mathrm{fib}(\mathbb{S} \wedge X \to H\mathbb{F}_2 \wedge X)$. Then $\pi_n X' \leq \pi_n X$.*

$$\begin{tikzcd} \mathbb{S} \wedge X \arrow[d, "Hurewicz"] & X' = \overline{H\mathbb{F}_2} \wedge X \arrow[l] \\ H\mathbb{F}_2 \wedge X & \end{tikzcd}$$

Proof. We have a long exact sequence

$$\cdots \longrightarrow \pi^s_{n+1}X \longrightarrow H_{n+1}(X;\mathbb{F}_2) \longrightarrow$$
$$\pi^s_n X' \longrightarrow \pi^X_n \longrightarrow H_n(X;\mathbb{F}_2) \longrightarrow$$
$$\pi^s_{n-1}X' \longrightarrow \pi^s_{n-1}(X) \longrightarrow \cdots.$$

The Hurewicz theorem tell us that $\pi^s_* X = 0$ for $* \leq n$, and also that $\pi^s_* \twoheadrightarrow H_* X \twoheadrightarrow H_*(X;\mathbb{F}_2)$. This shows the theorem. □

Now we have a filtration

$$\begin{tikzcd} X \arrow[d] & \overline{H\mathbb{F}_2} \wedge X \arrow[l] \arrow[d] & \overline{H\mathbb{F}_2}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\ H\mathbb{F}_2 \wedge X & H\mathbb{F}_2 \wedge \overline{H\mathbb{F}_2} \wedge X & H\mathbb{F}_2 \wedge \overline{H\mathbb{F}_2}^{\wedge 2} \wedge X & \end{tikzcd}$$

So we get a spectral sequence

$$\begin{aligned} E^1_{*,*} &= \pi_* H\mathbb{F}_2 \wedge \overline{H\mathbb{F}_2}^{\wedge *} \wedge X \cong \tilde{H}_*(\overline{H\mathbb{F}_2}^{\wedge *} \wedge X; \mathbb{F}_2) \\ &\cong H_*(X;\mathbb{F}_2) \otimes \bigoplus_* H_*(\overline{H\mathbb{F}_2}; \mathbb{F}_2) \end{aligned}$$

that computes $\pi_* X$.

Lemma 30.2. $E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathrm{Steenrod}}(H^*(X;\mathbb{F}_2);\mathbb{F}_2)$.

Sketch of Proof. The resolution at left is a kind of normalized bar construction. This is then a matter of naming the derived functor that is being calculated. $\square$

In our case, $X = \mathbb{S}$. But this is still hard to compute. As a warm-up, consider the small subalgebra of Steenrod algebra:

$$\mathcal{A}(1)^* = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \subseteq \text{full Steenrod algebra}.$$

We note that this algebra actually terminates, and $\mathrm{Sq}^{2,2} = \mathrm{Sq}^{1,2,1}$. The resolution looks like this:

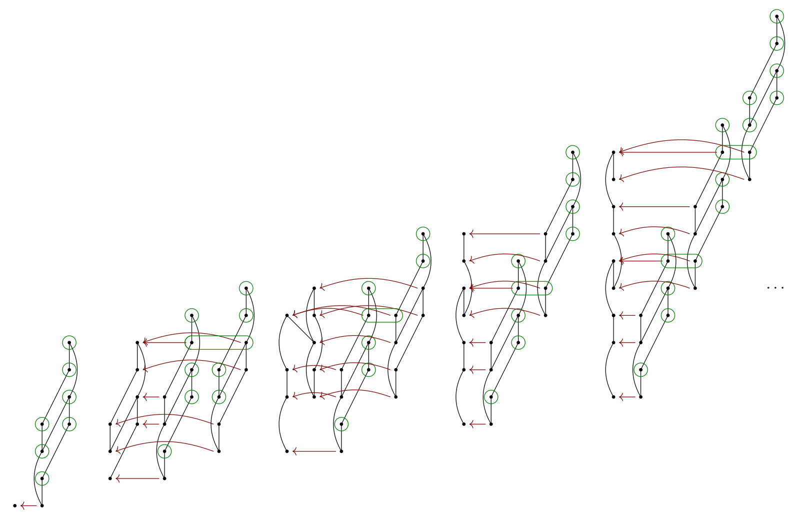

Figure 12: Free resolution of $\mathbb{F}_2$ over $\mathcal{A}(1)^*$

If you notice, we get a repeating pattern and we start over again. Now we apply $\mathrm{Hom}_{\mathcal{A}(1)}(-, \mathbb{F}_2)$ and compute the cohomology. Then we get the following picture.

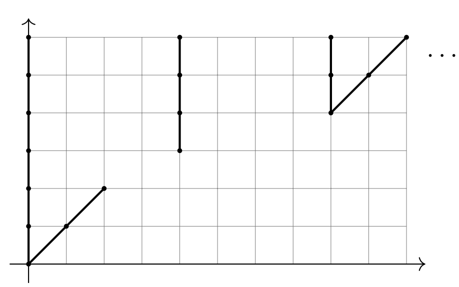

Figure 13: Adams spectral sequence over $\mathcal{A}(1)^*$

This spectral sequence is actually computing the homotopy groups of the real K-theory KO. If you can justify this, it would be a proof of Bott periodicity.

31 April 19, 2017

Today we are going to take a detour and come back to fun spectral sequences next time.

31.1 Hopf invariants and the EHP sequences

Theorem 31.1 (James)**.** *For X a path connected space,*

$$\Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{j=1}^{\infty} X^{\wedge j}.$$

The left hand side is the free loopspace construction, and the right hand side is the free monoid construction.

Sketch of Proof. Calculate the homology of both sides over $\mathbb{F}_p$ and over $\mathbb{Q}$. $\square$

This splitting is highly nontrivial and we will use it to construct some interesting maps. There is a projection

$$\Sigma\Omega\Sigma X \to \Sigma \bigvee_{j=1}^{\infty} X^{\wedge j} \to \Sigma(X^{\wedge 2})$$

and induces a map

$$h : \Omega\Sigma X \to \Omega\Sigma X^{\wedge 2}.$$

This map is called the **Hopf invariant** of X.

This map is important for the "Hopf invariant 1 problem", the vector fields on spheres problem, and the H-space structure on spheres problem. The last two are related in the following way. If there is a multiplication $\mu : S^n \times S^n \to S^n$, then we can construct a $H(\mu) : S^{2n+1} \to S^{n+1}$ by

$$\begin{tikzcd} CS^n \times S^n \arrow[rr] & & CS^n \\ & S^n \times S^n \arrow[ul] \arrow[dl] \arrow[r, "\mu"] & S^n \arrow[u] \arrow[d] \\ S^n \times CS^n \arrow[rr] & & CS^n. \end{tikzcd}$$

The main scaffolding of this problem is to compare $H(\mu)$ and h.

For us, we want to understand the map h in the case $X = S^n$, as well as its homotopy fiber P_h, the thing participating in a fiber sequence $P_h \to \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1}$.

Recall that we have already calculated the cohomology of the loopspace:

$$H^*\Omega S^{2n+1} \cong \Gamma[x_{2n}], \quad H^*\Omega S^{2n} \cong \Lambda[e_{2n-1}] \otimes \Gamma[x_{4n-2}].$$

Now we look at the Serre spectral sequence. We want to show that the picture is the right picture and there are no differentials. This claim will follow

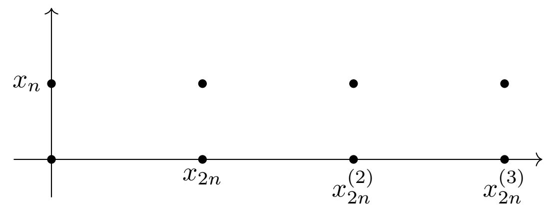

Figure 14: Serre spectral sequence for $P_h \to \Omega S^{n+1} \to \Omega S^{2n+1}$

by showing that the edge homomorphism $H^*\Omega S^{2n+1} \to H^*\Omega S^{n+1}$ is injective.

In the case when n is odd, we are looking at a map

$$H^*\Omega S^{2n+1} \cong \Gamma[x_{2n}] \to H^*\Omega S^{n+1} \cong \Lambda[e] \otimes \Gamma[y_{2n}],$$

and if $x_{2n} \mapsto y_{2n}$ then we are done. This morphism h began life as

$$\Sigma\Omega\Sigma S^n \xrightarrow{\cong} \Sigma \bigvee_{j=1}^{\infty} S^{jn} \to \Sigma S^{2n},$$

where these are isomorphisms in degree $2n+1$. Then $x_{2n} \mapsto y_{2n}$ because of some transgression mumbo-jumbo. Now the only space with $H^*(X;\mathbb{Z}) = \mathbb{Z}\{1, x_n\}$ is the n-sphere. So $P_h \simeq S^n$.

In the case when n is even, you can show that

$$\Gamma[x_{2n}] \to \Gamma[y_n]; \quad x_{2n} \mapsto \frac{1}{2} y_n^2.$$

The algebra structure on both sides then show that

$$x_{2n}^{(k)} \mapsto \frac{(2k)!}{2^k k!} y_n^{(2k)}.$$

The coefficient is a unit 2-locally, and thus in Serre's $\mathcal{C}_{(2)}$ theory, there is an weak equivalence $P_h \simeq S^n$.

The composite $S^n \to \Omega\Sigma S^n \to \Omega\Sigma S^{2n}$ is zero. So there is a lift:

$$\begin{tikzcd}
 & P_h \arrow[d] \\
S^n \arrow[ur, "\simeq"] \arrow[r, "e"] & \Omega S^{n+1} \arrow[d, "h"] \\
 & \Omega\Sigma S^{2n}
\end{tikzcd}$$

The map e is a cohomology isomorphism in degree n and so the lifted map is also a cohomology isomorphism in degree n. In particular, it is a weak equivalence.

Anyways, we have a fiber sequence

$$\cdots \to \Omega^2\Sigma S^{2n} \xrightarrow{p} S^n \xrightarrow{e} \Omega\Sigma S^n \xrightarrow{h} \Omega\Sigma S^{2n} \to \cdots.$$

This is called the **EHP sequence** and the letter stand for Einhängung, Hopf, and Whitehead products.

32 April 21, 2017

32.1 Calculations in the EHP spectral sequence

Last time we had the EHP sequence

$$\begin{tikzcd}
\Omega^0 S^0 \arrow[r] & \cdots \arrow[r] & \Omega^{n-1} S^{n-1} \arrow[r] \arrow[d] & \Omega^n S^n \arrow[r, "e"] \arrow[d, "h"] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d, "h"] & \cdots \arrow[r] & \Omega^\infty \Sigma^\infty S^0 \\
 & & \cdots & \Omega^n S^{2n-1} & \Omega^{n+1} S^{2n+1} & &
\end{tikzcd}$$

We are going to apply π_* to this. Then we get a spectral sequence computing

$$E^1_{s,t} = \pi_s \Omega^t S^{2t-1} = \pi_{s+t} S^{2t-1} \quad \Longrightarrow \quad \pi_s \Omega^\infty \Sigma^\infty S^0 = \pi_s \mathbb{S}^0.$$

This is the EHP spectral sequence. This is populated by unstable homotopy groups of sphere, converging to $\pi_* \mathbb{S}^0$.

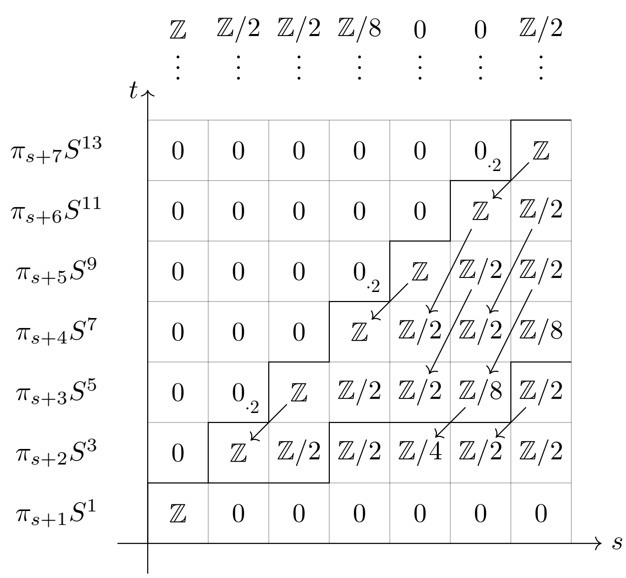

Figure 15: E^1 page of the EHP spectral sequence

There are some stuff we know about the input. We have Freudenthal's theorem, the connectivity estimates, and $S^1 \simeq K(\mathbb{Z}, 1)$. Also we have some information about $\pi_* \mathbb{S}$, from Serre's method.

Because we know that $\pi_2 \mathbb{S}$ is $\mathbb{Z}/2$, the differential d_2 must be $\cdot 2$. But we can't get anywhere more. So let us figure out a differential on the main diagonal.

There is a diagram

$$\begin{tikzcd}
\Omega^n S^n \arrow[r] & \Omega^{n+1} S^{n+1} \arrow[r] & \Omega^{n+1} S^{2n+1} \\
\mathrm{O}(n) \arrow[u] \arrow[r, hook] & \mathrm{O}(n+1) \arrow[u] \arrow[r] & S^n \arrow[u] \\
\mathbb{R}P^{n-1} \arrow[u] \arrow[r, hook] & \mathbb{R}P^n \arrow[u] \arrow[r] & S^n. \arrow[u, equal]
\end{tikzcd}$$

Here the map $\mathrm{O}(n)$ is given by rotating around some fixed axis. The middle row is also a fiber sequence, but the bottom row is a cofiber sequence. Still we have

extracting d_1 as $S^{n-1} \to S^{n-1}$

Proposition 32.1. *The map "d_1" is the cellular differential associated to $\mathbb{R}P^\infty$, i.e., $C_n \xrightarrow{(-1)^n+1} C_{n-1}$. So the differentials on the main diagonal alternate between 0 and 2.*

But we need more ideas to learn about other differentials. We can truncate the spectral sequence:

$$\begin{tikzcd}
* \arrow[r] & \Omega S^2 \arrow[r, hook] \arrow[d, equal] & \Omega^2 S^2 \arrow[r, hook] \arrow[d] & \Omega^3 S^3 \arrow[r, equal] \arrow[d] & \Omega^3 S^3 \arrow[r, equal] \arrow[d] & \Omega^3 S^3 \arrow[r, equal] \arrow[d] & \cdots \\
 & \Omega^1 S^2 & \Omega^2 S^3 & \Omega^3 S^5 & * & * &
\end{tikzcd}$$

So using these techniques, you can get more information about the homotopy groups. On different truncations, some stuff should be killed by a differential but not because the source isn't there.

In fact, you can show using this method that all homotopy groups except for these diagonal ones are 2-torsion. Then you can show that

$$\pi_* S^{2n+1} \cong \mathbb{Q}[x_{2n+1}], \quad \pi_* S^{2n} \otimes \mathbb{Q} \cong \mathbb{Q}[x_{2n}] \oplus \mathbb{Q}[x_{4n-1}].$$

33 April 24, 2017

33.1 The May spectral sequence

We have looked that the Adams spectral sequence, and it seemed really arduous. The idea is to filter whatever algebra is under consideration by word length, in particular the Hopf algebra. This has the effect of trivializing the (co)multiplication; the product of words of length n is $2n$. The trade is that, up to filtration, this now looks like a tensor product of exterior algebras, and their Ext groups are easy to compute.

This gives a spectral sequence with E_1-page isomorphic to a polynomial algebra converging to the Ext you want. We are going to have some hypotheses: the Hopf algebra must be connected, graded, and of finite type. This essentially prohibits the presence of multiplicative inverses.

Theorem 33.1. *In the case of a Steenrod algebra, this gives a spectral sequence of signature*

$$E^1_{*,*,*} = \mathbb{F}_2[h_{ij} : i \geq 1, j \geq 0] \quad \Longrightarrow \quad \textit{Adams } E_2\textit{-page for the sphere.}$$

Lemma 33.2. *d_1 in this spectral sequence is completely determined:*

$$d_1(h_{ij}) = \sum_{k=1}^{i-1} h_{kj} h_{(i-k)(j+k)},$$

and the spectral sequence is one of algebras, i.e., it satisfies the Leibniz rule.

But higher differentials are much more mysterious. This machinery also applies to $\mathcal{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$, which has E^1-page $\mathbb{F}_2[h_{10}, h_{11}, h_{20}]$, where $h_{10} = \mathrm{Sq}^1$, $h_{11} = \mathrm{Sq}^2$, and h_{20} is the relation between the two.

I can just start writing down the spectral sequence, but it is going to be really messy. Recall that what we should be getting out of this is that messy drawing. To simplify this, we note that this spectral sequence is $(E^1)^2$-linear. That is, every square class is a cycle and so $d(xs^2) = d(x)s^2$ by Leibniz. Then we can consider this whole thing as an $(E^1)^2$-module:

$$E^1 = (E^1)^2\{1, h_{10}, h_{11}, h_{20}, h_{10}h_{11}, h_{10}h_{20}, h_{11}h_{20}, h_{10}h_{11}h_{20}\}.$$

Using this basis, we can write d_1 as a big matrix

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{10}^2 h_{11}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_{11}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_{10}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we can compute the E^2 page as

$$E^2 = \frac{Z}{B} = (E^1)^2\{1, h_{10}, h_{11}, h_{10}h_{11}\}/(1(h_{10}^2h_{11}^2), h_{10}(h_{11}^2), h_{11}(h_{10}^2), h_{10}h_{11}(1)).$$

Missing figure

the E^2 page

To get further, you need to know how to calculate the d_2 differentials.

33.2 Nakamura's squaring operations

The idea is the Steenrod algebra acts on "all kinds" of things—anything mod 2 where "square" even makes sense.

Lemma 33.3. *There exists an operation on the May spectral sequence for $\mathcal{A}$ (or $\mathcal{A}(1)$) such that:*

(1) $\mathrm{Sq}^0(h_{ij}) = h_{i(j+1)}$,

(2) $\mathrm{Sq}^1(h_{ij}) = h_{ij}^2$,

(3) they are linear,

(4) $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$,

(5) (Kudo): $d(\mathrm{Sq}^n x) = \mathrm{Sq}^n d(x)$.

Now let us compute the differential of h_{20}^2. Then

$$\begin{aligned} d_2(h_{20}^2) &= d_2(\mathrm{Sq}^1 h_{20}) = \mathrm{Sq}^1 d_2(h_{20}) = \mathrm{Sq}^1(h_{10}h_{11}) \\ &= \mathrm{Sq}^1 h_{10}\,\mathrm{Sq}^0 h_{11} + \mathrm{Sq}^0 h_{10}\,\mathrm{Sq}^1 h_{11} = h_{10}^2h_{12} + h_{11}h_{11}^2 = h_{11}^3. \end{aligned}$$

So we get more differentials, and after knocking them out, we get the following E^3 page.

Missing figure

this is E^3: this looks like the wedges and columns picture

The interpretation is that the kth homotopy group is the kth column. On the first column, h_0-multiplication indicates nontrivial extension. So $\pi_0 = \mathbb{Z}_2^\wedge$ is the 2-adics. Also $\pi_1 = \pi_2 = \mathbb{Z}/2$ and $\pi_3 = 0$ and so on. As a ring, we get

$$\mathbb{Z}_2^\wedge[\eta, u, \beta]/(2\eta = 0, \eta^3 = 0, u^2 = 4\beta, u\eta = 0).$$

This is the homotopy of BO, real Bott-periodicity.

34 April 26, 2017

34.1 Oh, the places you'll go

Today is the last class, and so I'm going to talk about the things you can study. This course is 231br and the r means that you can take this course over and over, and oftentimes the course will do one of these topics.

- Vector fields on spheres problem

  How many mutually orthogonal nowhere vanishing vector fields can you find on S^n? This is related to groups structures on S^n, or parallelizability. This was done in the 1970s, and most of the stuff you've learned was done in the 1960s. There is also K-theory and vector bundles, and power operations, which is a topic of ongoing research.

- Bordism homology

  This is probing a manifold with other manifolds, form a chain complex using maps into X from n-manifolds with boundary. This ties into serious geometry, especially intersection theory. You can also do this where the probe manifolds have some tangential structure. If the tangent bundles are all trivialized, you get stable homotopy $\pi_*(-)$.

- Algebraic K-theory

  This is a general machine for extracting a spectrum from any kind of remotely geometric context: manifolds, spaces, modules over a ring, etc. Each time, it captures exceptionally interesting information about the category. From our perspective, computations are super hard and deep.

- Homological stability

  This showed up when we were talking about the bar spectral sequence or the Serre class. Often, things (like symmetry groups Σ_n) come in families (like $\{\Sigma_n\}_n$) and as n ranges they display some kind of stable phenomenon (like $\tilde{H}_* B\Sigma_n \to \tilde{H}_{*+1} B\Sigma_{n+1}$ is an isomorphism through degree $\approx 2n$). All of these have interesting geometric consequences that you can learn about.

- Equivariant homotopy theory

  Geometric things have symmetries. You can rebuild homotopy theory so that it tracks the symmetry group while keeping a homotopical flavor. This forces geometry back into the picture in an interesting way, which also admits accessible computation.

- Interactions with algebraic geometry

  There are two major contact points I want to discuss: motivic homotopy theory and étale homotopy theory. Both are methods for extracting a space(-like object) from a scheme in a way that algebro-topological invariants re-encode the invariants of the original object. You might be able to go and read about étale homotopy theory now, but you could study about motivic homotopy theory for the next half decade. But motivic homotopy theory is bidirectional. For instance, Isaksen computed $\pi_*\mathbb{S}$ for $60 \leq * \leq 80$.

- Goodwillie calculus

  These are "polynomial" approximations to homotopical functors. This is applied to classical functors to get various other classical things. The slogan is that it is a mechanical way to approximate "unstable" homotopy theory by "stable" homotopy theory.

- Spectral algebraic geometry

  This is trying to do algebraic geometry with ring spectra. It ties together all of homotopy theory with all of algebraic geometry and is still very much in flux. It is a hot topic of research.

- Chromatic homotopy theory

  Spectra and stable homotopy theory is known to be hard. What we would like is an algebraic model of it and a functor sending spectra to objects in the model in a computable way. Cohomology somewhat establishes this, but there are distinct spaces with the same cohomology. Chromatic homotopy theory gives a certain arithmetic-geometric such model. This is the best approximation we know to stable homotopy theory, and it occupied as 1980-2000, though there are plenty unresolved features.

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