Physics 151 - Mechanics

Taught by Arthur Jaffe
Notes by Dongryul Kim

Fall 2016

This course was taught by Arthur Jaffe. We met on Tuesdays and Thursdays from 11:30am to 1:00pm in Jefferson 356. We did not use a particular textbook, and there were 17 students enrolled. Grading was based on two in-class midterms and six assignments. The teaching follow was David Kolchmeyer.

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1 September 1, 2016

This class is about classical mechanics, but this is actually going to be an introduction to theoretical physics. There is a correspondence between conservation laws and symmetry, which was discovered by Noether, and this has connections with mechanics, quantum mechanics, and pure mathematics.

Let us look at a classical problem. Suppose we have a point particle μ at $\vec{r}$ moving under a central force $\vec{F} = -k/r^2$. I'm going to write $\vec{r} = r\vec{n}$ where r is the magnitude and $\vec{n}$ is the unit vector. The question is: What is the orbit? The standard answer is to use Newton's Law and solve the corresponding differential equation. What I am going to do today is to answer the same equation with conservation laws.

As you all know, the answer is that the orbit is an ellipse. An ellipse is the locus of points whose sum of the distance from the two foci is constant.

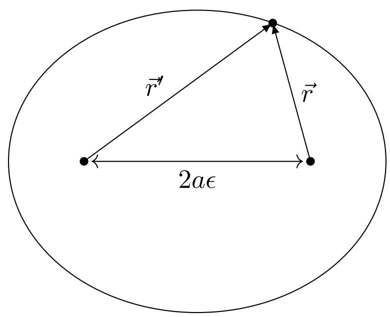

Figure 1: Figure of an ellipse

The ellipse has a semi-major axis whose length we will denote $2a$. The distance between the two foci will be denoted $2a\epsilon$ where ϵ is the **eccentricity** of the ellipse. Because we have $\vec{r}' = \vec{r} + 2a\epsilon\vec{e}_1$, we have

$$\vec{r}'^2 = r^2 + 4a^2\epsilon^2 + 4a\epsilon r\cos\theta = (2a - r)^2$$

and therefore the equation of the ellipse is given by

$$r(1 + \epsilon\cos\theta) = a(1 - \epsilon^2).$$

This is where we are heading to.

1.1 Three conservation laws

The first one is energy conservation. The energy E is given by

$$E = \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{k}{r}$$

where the dot denotes time differentiation. To show that the energy is conserved, we differentiate E to get

$$\dot{E} = \frac{dE}{dt} = \dot{\vec{r}} \cdot (\mu\ddot{\vec{r}} - \vec{F}) = 0$$

by Newton's law.

Item number two is angular momentum around the origin. The angular momentum $\vec{L}$ is given by $\vec{r} \times \vec{p}$ so

$$\frac{d\vec{L}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \vec{F} = 0 + 0 = 0.$$

From this we can conclude that the motion lies on the plane perpendicular to $\vec{L}$.

The third one is the highlight of this lecture. Energy and angular momentum are general conservation laws that come over everywhere, but this is specific to the $1/r^2$ potential. We define the **Lenz vector** $\vec{\epsilon}$ as

$$\vec{\epsilon} = \frac{\vec{p} \times \vec{L}}{\mu k} - \vec{n}.$$

This is a dimensionless vector. But why is this conserved? The only things that change in this formula are $\vec{p}$ and $\vec{n}$. We know well what the derivative of $\vec{p}$ is. The derivative of $\vec{n}$ is given as

$$\dot{\vec{n}} = -\frac{1}{r^2}\vec{n} \times (\vec{r} \times \dot{\vec{r}}).$$

If this formula is correct, then

$$\dot{\vec{\epsilon}} = \frac{\vec{F} \times \vec{L}}{\mu k} - \dot{\vec{n}} = -\frac{k\mu}{r^2 \mu k}\vec{n} \times (\vec{r} \times \dot{\vec{r}}) - \dot{\vec{n}} = 0$$

and hence $\vec{\epsilon}$ is conserved.

Now once we have this, we immediately get the orbit. The quantity

$$\vec{\epsilon} \cdot \vec{r} = \frac{L^2}{\mu k} - r = \epsilon r \cos\theta$$

is conserved. This is precisely the equation of the ellipse. In fact, you even see why I named it ϵ; the magnitude of the Lenz vector is precisely the eccentricity of the orbit.

2 September 6, 2016

There is going to be sections on Mondays at 6:30 at Lyman 330. There is going to be a dinner tonight in Lowell House Small Dining Room at 5:50.

2.1 More on elliptical orbits

We looked at the orbit of the planet last time. There were three quantities that are conserved.

(1) Energy - $E = \frac{1}{2}\mu\dot{\vec{r}}^2 - k/r$

(2) Angular momentum - $\vec{L} = \vec{r} \times \vec{p}$

(3) The Lenz vector - $\vec{\epsilon} = \vec{p} \times \vec{L}/\mu k - \vec{n}$

Actually I cheated a bit last time. When we derived the formula $r(1 + \epsilon\cos\theta) = L^2/\mu k$, I set θ to be the angle between $\vec{r}$ and $\vec{\epsilon}$. But in the formula, we let θ to be the angle between $\vec{r}$ and $\vec{e}_1$. So we need to check that $\vec{\epsilon}$ and $\vec{e}_1$ have the same direction.[1] Because $\vec{\epsilon}$ is constant, we can simply evaluate ϵ at a few points. On the point on the right, we see that both $\vec{p} \times \vec{L}$ and $\vec{n}$ points in the $+e_1$ direction. On the point on the top, we see that $\vec{p} \times \vec{L}$ points in the $+e_2$ direction and $\vec{n}$ points in the slightly $-e_1$ direction. So $\vec{\epsilon}$ will point in the $+e_1$ direction.

What about the magnitude of ϵ? We have

$$\begin{aligned}\epsilon^2 = \vec{\epsilon} \cdot \vec{\epsilon} &= 1 + \Big(\frac{\vec{p} \times \vec{L}}{\mu k}\Big)^2 - \frac{2\vec{r} \cdot (p \times \vec{L})}{\mu k r} = 1 + \frac{p^2 L^2}{\mu^2 k^2} - \frac{2L^2}{\mu k r} \\ &= 1 + \frac{2L^2}{\mu k^2}\Big(\frac{p^2}{2\mu} - \frac{k}{r}\Big) = 1 + \frac{2L^2 E}{\mu k^2}.\end{aligned}$$

One remark is that you can get the energy levels of the hydrogen atom from this formula. Also since $\epsilon = 0$ for the circle, you can get the energy of the circular orbit.

2.2 Kepler's laws

What are Kepler's laws?

1. Elliptical Motion
2. $\vec{r}$ sweeps out equal area in equal time. This comes from the equation
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu}.$$
3. The square of the period τ^2 is proportionate to a^3. This is actually not true, but approximately true if $m \ll M$.

[1] I personally don't understand this. Can't we set the coordinate system in the first place so that $\vec{\epsilon}$ aligns with $\vec{e}_1$.

2.3 Hyperbolic orbits

Can we take this further? What about unbound orbits?

There are two different cases. The central force can be either attractive or repulsive. In both cases, the locus is an hyperbola. This is the locus of points with $r' - r = 2a$. On a hyperbola, people usually uses the left focus to be the origin. The eccentricity ϵ is greater than 1.

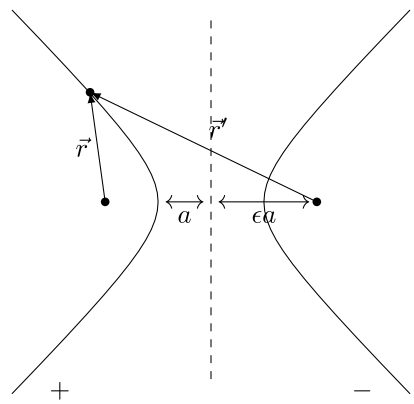

Figure 2: Figure of an hyperbola

The equation is again given by

$$r(1 + \epsilon \cos\theta) = a(\epsilon^2 - 1).$$

The other side is given by

$$r(-1 + \epsilon \cos\theta) = a(\epsilon^2 - 1).$$

There is no doubt that the Lenz vector is conserved, because the problem is just the same. Where does it point? In the $+$ branch, that is, when the force is attractive, the Lenz vector $\vec{\epsilon}$ points to the right, because we are continuously increasing ϵ in the same problem. On the other hand, in the $-$ branch, when the force is repulsive, $\vec{\epsilon}$ points to the left.

Because the angular momentum is conserved, we can add the rotational energy and consider

$$E - \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu k} - \frac{k}{r}.$$

The sum of the last two terms is called the **effective potential** $V_{\text{eff}}(r)$.

3 September 8, 2016

3.1 The Rutherford scattering

Today I want to talk about the Rutherford scattering. This is the case of an unbound, repulsive potential. The formula for the hyperbola is given by

$$r(-1+\epsilon\theta) = -\frac{L^2}{\mu k}.$$

We see that there is a maximal value of θ, and I am going to write it $\theta_{\max}$. From the equation, we see that $\cos\theta_{\max} = 1/\epsilon$. Then

$$\tan\theta_{\max} = \sqrt{\epsilon^2 - 1} = \sqrt{\frac{2EL^2}{\mu k^2}}.$$

In the Rutherford experiment, the scattering angle is given by $\Theta = \pi - 2\theta_{\max}$. Hence we can say that

$$\cot\frac{\Theta}{2} = \sqrt{\frac{2EL^2}{\mu k^2}}.$$

But we need to parameterize in terms of the **impact parameter**, which is the distance between the asymptote and the origin. Because the energy and the angular momentum is conserved, we can compute it at large distance. Then we see that

$$\sqrt{2\mu E} = p_{\text{in}}, \qquad L = sp_{\text{in}} = s\sqrt{2\mu E}.$$

Then we can write

$$\cot\frac{\Theta}{2} = \sqrt{\frac{2EL^2}{\mu k^2}} = \frac{2sE}{|k|}, \qquad s = \frac{|k|}{2E}\cos\frac{\Theta}{2}.$$

To analyze the situation in 3-dimensions, we use spherical polar coordinates (r, Θ, ϕ). We write Θ to make it match our scattering angle. Here, Θ is the angle between $\vec{r}$ and the $+z$-axis. The volume form and the area form on the unit sphere are given by

$$dV = r^2 dr \sin\Theta d\Theta d\phi, \qquad d\Omega = \sin\Theta d\Theta d\phi.$$

To see how much of the incoming beam go to the sphere, we divide $sdsd\phi$ by $d\Omega$. Then the **differential scattering cross section** is given by

$$\frac{d\sigma}{d\Omega} = \left|\frac{sdsd\phi}{\sin\Theta d\Theta d\phi}\right| = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4(\Theta/2)}$$

if you plug it in the formula. This is the Rutherford formula.

There is actually a big problem. These are atomic particles and thus they can't be really described by classical mechanics. But if you actually compute it for the quantum mechanical picture, you get the same answer.

4 September 13, 2016

Today we are going back to Newtonian mechanics. I want to start by giving examples.

4.1 Newton's equation in coordinates

In Cartesian coordinates, $\vec{r} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$. Newton's equation states that $\vec{F} = m\vec{a}$ where $\vec{a} = \ddot{\vec{r}}$ and $\vec{F} = -\nabla V$. If we write it in coordinates,

$$m\ddot{x} = F_x = -\frac{\partial F}{\partial x}.$$

We might want to label $x_1, x_2, \dots, x_N$ where N is 3 times the number of particles.

If we are working in planar coordinates (r, θ) replacing (x, y), so that $\vec{r} = r\vec{n}$ with $\vec{l} \perp \vec{n}$, then we can go back and forth in coordinates as

$$\begin{aligned} r &= \sqrt{x^2 + y^2} & x &= r\cos\theta \\ \tan\theta &= \frac{y}{x} & y &= r\sin\theta. \end{aligned}$$

Because we know how to take the derivative in Cartesian coordinates. We have

$$\begin{aligned} \dot{\vec{r}} &= \dot{r}\vec{n} + r\dot{\theta}\vec{l}, \\ \ddot{\vec{r}} &= \ddot{r}\vec{n} + \dot{r}\dot{\theta}\vec{l} + \dot{r}\dot{\theta}\vec{l} + r\ddot{\theta}\vec{l} - r\dot{\theta}^2\vec{n} = (\ddot{r} - r\dot{\theta}^2)\vec{n} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{l}. \end{aligned}$$

Each of these components of a name: $\ddot{r}$ is the radial acceleration, $-r\dot{\theta}^2$ is the centripetal acceleration, $2\dot{r}\dot{\theta}$ is the Coriolis acceleration, and $r\ddot{\theta}$ is the angular acceleration.

Now that we know the acceleration in the components, let us compute the components of the force. We can write

$$\nabla V = \frac{\partial V}{\partial x}\vec{e}_1 + \frac{\partial V}{\partial t}\vec{e}_2 = (\nabla V)_r\vec{n} + (\nabla V)_\theta\vec{l}.$$

The $(\nabla V)_r$ can be computed as

$$(\nabla V)_r = (\nabla V)\cdot\vec{n} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial V}{\partial r}.$$

You have to be careful because each derivative has one variable that changes and one variable that is fixed. Likewise,

$$(\nabla V)_\theta = (\nabla V)\cdot\vec{l} = -\frac{\partial V}{\partial x}\sin\theta + \frac{\partial V}{\partial y}\cos\theta = \frac{1}{r}\frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{1}{r}\frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} = \frac{1}{r}\frac{\partial V}{\partial \theta}.$$

So the Newton equations can be written as

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}, \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\frac{1}{r}\frac{\partial V}{\partial \theta}.$$

This looks very different from what we had in Cartesian coordinates. In fact they are exactly the same equation when we express in terms of the Lagrangian. This is some kind of covariance, or symmetry.

4.2 The Lagrangian

The kinetic energy is given as

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2,$$

and let $V = V(x, y)$ be an arbitrary potential.

Newton's equation gives

$$\begin{aligned} -\left(\frac{\partial V}{\partial x}\right)_{y,\dot{x},\dot{y}} &= m\ddot{x} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)_{\dot{y},x,y}, \\ -\left(\frac{\partial V}{\partial y}\right)_{x,\dot{x},\dot{y}} &= m\ddot{y} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right)_{\dot{x},x,y}. \end{aligned}$$

To make this is simpler, we define the **Lagrangian** as $\mathcal{L} = T - V$. Then we can rewrite the equation as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}.$$

This is called the Lagrangian's equation, and this form is *independent of coordinate system*.

Let us look at $\mathcal{L}$ in terms of plane polar coordinates. We have

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta).$$

Then the Lagrangian equations are

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}, \qquad \frac{d}{dt}(mr^2\dot{\theta}) = -\frac{\partial V}{\partial \theta}.$$

This way we didn't need to go through all the algebra. This is the philosophy of the Lagrangian equation.

4.3 Tensors

Let $V(x)$ with $x = (x_1, \ldots, x_N)$. Set up a new coordinate $x(q)$ with $q = (q_1, \ldots, q_N)$. To translate a vector in x coordinates q, we can apply the chain rule and write

$$\frac{\partial}{\partial q_i}V(x(q)) = \sum_{j=1}^{N} \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial q_i}.$$

This transformation looks like $\tilde{\vec{W}} = J\vec{W}$, and things that satisfy this relation are called **covariant vectors**. There are vectors that transform in the other direction, like

$$dq_i = \sum_j \frac{\partial q_i}{\partial x_j}dx_j.$$

These are called **contravariant vectors**.

The Jacobian is defined as

$$J_{ij} = \left(\frac{\partial x_j}{\partial q_i}\right)_{q_1,\ldots,\cancel{q_i},\ldots,q_N}.$$

This must be invertible, because the coordinate transform must be invertible in the first place. The inverse will be given by

$$\tilde{J}_{ij} = \left(\frac{\partial q_j}{\partial x_i}\right)_{x_1,\ldots,\cancel{x_i},\ldots,x_N}.$$

Then $\tilde{J}J = I = \tilde{J}J$.

5 September 15, 2016

Suppose there are functions of coordinates $q = (q_1, \dots, q_N)$ and $q = q(q')$. Let V_i be functions of coordinates q, nad V_i' be functions of q' with $V_i' = V_i(q(q'))$. We are going to say that V_i is a **covariant vector** if these satisfy the relation

$$V_i' = \sum_{j=1}^{N} \frac{\partial q_j'}{q_i} V_j.$$

5.1 Coordinate independence of the Lagrangian

The Lagrangian is given as $\mathcal{L} = T - V$ where $T = T(x, \dot{x}, t)$ and $V = V(x, t)$. We are assuming that the potential is only dependent on the position, i.e., this is not like the system of a charged particle in a magnetic field. Let q be another coordinates with $x = x(q)$. We are going to derive

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{from} \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0.$$

To make simple, let us take out the time-dependence of $\mathcal{L}$. Consider

$$\frac{\partial \mathcal{L}(x(q), \dot{x}(q, \dot{q}))}{\partial q_i} = \sum_{j=1}^{N} \frac{\partial \mathcal{L}}{\partial x_j}\frac{\partial x_j}{\partial q_i} + \sum_{j=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial q_i}.$$

This doesn't work by itself, and so let us compute the other term. Before this, we note that

$$\dot{x}_i(q, \dot{q}) = \frac{d}{dt} x_i(q) = \sum_{j=1}^{N} \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} = \sum_{j=1}^{N} \Big(\frac{\partial x_i}{\partial q_j}\Big)\dot{q}_j,$$

so it follows that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}.$$

This is called the **cancellation of dots**. The other term in Lagrange's equation is

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} &= \frac{d}{dt}\sum_{j=1}^{N}\left(\frac{\partial \mathcal{L}}{\partial x_j}\frac{\partial x_j}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial \dot{q}_i}\right) = \frac{d}{dt}\sum_{j=1}^{N}\frac{\partial L}{\partial \dot{x}_j}\frac{\partial x_j}{\partial q_i} \\ &= \sum_{j=1}^{N}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right)\frac{\partial x_j}{\partial q_i} + \sum_{j=1}^{N}\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\left(\frac{d}{dt}\frac{\partial x_j}{\partial q_i}\right).\end{aligned}$$

Then the difference is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{j=1}^{N}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} - \frac{\partial \mathcal{L}}{\partial x_j}\right)\frac{\partial x_j}{\partial q_i} + \sum_{j=1}^{N}\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\left(\frac{d}{dt}\frac{\partial x_j}{\partial q_i} - \frac{\partial \dot{x}_j}{\partial q_i}\right).$$

The second term is zero because

$$\frac{\partial \dot{x}_j}{\partial q_i} = \sum_{l=1}^{N} \frac{\partial^2 x_j}{\partial q_l \partial q_i} \dot{q}_l, \qquad \frac{d}{dt}\frac{\partial x_j}{\partial q_i} = \sum_{l=1}^{N} \frac{\partial^2 x_j}{\partial q_l \partial q_i} \dot{q}_l.$$

So what we are left with is

$$\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} \right) = \sum_{j=1}^{N} \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} - \frac{\partial \mathcal{L}}{\partial x_j} \right) \left(\frac{\partial x_j}{\partial q_i} \right).$$

If we write the Lagrange vector V^{Lag} and Newton vector V^{New} as

$$V_i^{\text{Lag}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}, \qquad V_i^{\text{New}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i},$$

then what we have just proved can be written as

$$V^{\text{Lag}} = J V^{\text{New}}.$$

Now if Newton's Equations are correct, then $V^{\text{New}} = 0$. It follows from the formula that $V^{\text{Lag}} = 0$.

6 September 20, 2016

Let me start with a bit of review. For the Lagrangian $\mathcal{L}(q, \dot{q}, t)$, we have

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0.$$

In this equation, $p_i = \partial \mathcal{L}/\partial \dot{q}_i$ is called the **generalized momentum** and $F_i = \partial \mathcal{L}/\partial q_i$ is called the **generalized force**. For instance, if

$$T = \frac{1}{2}\sum m_i \dot{q}_i^2, \quad V = V(q)$$

then $p_i = m_i \dot{q}_i$ and if

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \quad V = V(r, \theta)$$

then $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$. This p_θ is the angular momentum L. If we look only at the dimension, then

$$[p_i][q_i] = [p_i][\dot{q}_i]T = [\mathcal{L}]T = [\text{Action}].$$

6.1 The Lagrangian in fields

Take a charged particle in coordinate $\vec{x}(t)$ with charge e. The electric and magnetic field can be written as

$$\vec{B}(\vec{x}, t) = \nabla \times \vec{A}(\vec{x}, t), \qquad \vec{E} = -\nabla\Phi(\vec{x}, t) - \frac{\partial \vec{A}}{\partial t}$$

where Φ and $\vec{A}$ are scalar and vector potentials. We can write down the Lagrangian of this charge as

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 - e\Phi(\vec{x}, t) + e\sum_{j=1}^{3} A_j(\vec{x}, t)\dot{x}_j(t).$$

What are the Lagrange equations for this? After a bit of algebra, you will get

$$m\ddot{\vec{x}} = e(\vec{E} + \dot{\vec{x}} \times \vec{B}).$$

This is the Lorentz force. That Lagrangian is the minimal coupling.

6.2 Lagrange's equation with constraints

Sometimes it is useful to introduce redundant coordinates. For instance, consider a particle on a inclined surface. In this case, we have $q_1 = x\cos\alpha$ and $q_2 = -x\sin\alpha$. So to be on the plane, we need a constraint

$$f(q_1, q_2) = q_1 \tan\alpha + q_2 = 0.$$

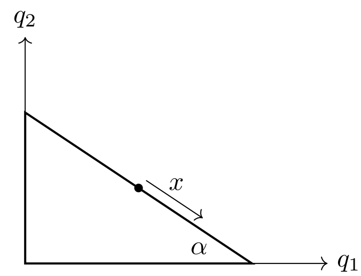

Figure 3: Particle on an inclined surface

The formula for coordinate change we had last time is, for $\mathcal{L}(x(q), \dot{x}(q, \dot{q}))$, something like

$$(V)^{\text{Lagrange}} = J(V)^{\text{Newton}}.$$

But we can't immediately do this because we have more than one way to express x by q.

Suppose we have coordinates $q = (q_1, \ldots, q_N)$ and $x = (x_1, \ldots, x_n)$ with $N > n$, then we would have

$$V^x = \tilde{J}V^q \text{ which looks like } \begin{pmatrix} \square \\ \square \end{pmatrix} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}.$$

If we denote the constraints as f_i, then

$$0 = \frac{\partial f}{\partial x_i} = \sum_{j=1}^{N} \frac{\partial q_j}{\partial x_i} \frac{\partial f}{\partial q_j}.$$

That is, V_j^q is in the kernel of J. We want few equations as possible, so we look at $N - n = k$ independent constraints functions $f^{(1)}, \ldots, f^{(k)}$. The Lagrange equation will give

$$0 = JV^q$$

and then because V^q is in the kernel, we have

$$V_i^q = \sum_{l=1}^{k} \lambda_l \frac{\partial f^{(l)}}{\partial q_i}$$

where λ is the **Lagrange multiplier** which is also an unknown. This, along with the constraint functions, will have $k + N$ equations and $k + N$ unknowns.

Let us look at a cylinder rolling down without slipping. The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}Ma^2\dot{\theta}^2 + Mgx\sin\alpha,$$

and the constraint is $f = x - a\theta$. Then the equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \lambda \frac{\partial f}{\partial x}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}, \qquad f = 0.$$

These are three equations in three unknowns x, θ, λ. Explicitly,

$$M\ddot{x} - Mg\sin\alpha = \lambda, \qquad \frac{1}{2}Ma^2\ddot{\theta} = -\lambda a, \qquad x = a\theta.$$

Then you can get $(3/2) \ddots x = g\sin\alpha$. In this case, $\lambda = -(1/3)Mg\sin\alpha$, and this you can interpret as the force of the constraint.

To sum up, the Lagrange's equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f^{(j)}}{\partial q_i}.$$

These type of constraints are called **holonomic constraints**.

7 September 22, 2016

7.1 Calculus of variations

If we have a function f, then the maxima and minima are attained at the points where f' is zero. If f has more that one component, i.e., take a vector, there are many choices of directions in which we can take the derivative. For a direction η, the **directional derivative** is

$$(D_{\vec{\eta}} f)(\vec{x}) = \lim_{\epsilon \to 0} \frac{f(\vec{x} + \epsilon \vec{\eta}) - f(\vec{x})}{\epsilon} = (\nabla f) \cdot \vec{\eta}.$$

A more complicated situation is where $\vec{x}$ is not a point in Euclidean space but a function. In mechanics, this comes in as f being the Lagrangian. Let x be $q(t) = (q_1(t), \dots, q_N(t))$ for $t_A \le t \le t_B$. We define the **action** as

$$S_{\text{action}} = \int_{t_A}^{t_B} \mathcal{L}(q(t), \dot{q}(t), t) dt = S(q).$$

This S now plays the role of f with q playing the role of x. To analyze this situation, we must ask what it means to differentiate a function over a function.

Let us define the derivative of $S(q)$. As in the case of a function over vectors, the most natural thing is to look at the directional derivatives. For another function $\eta = (\eta_1(t), \dots, \eta_N(t))$, the directional derivative of $S(q)$ will be

$$(D_\eta S)(q) = \lim_{\epsilon \to 0} \frac{S(q + \epsilon \eta) - S(q)}{\epsilon}.$$

Because we have a definition of S, we can straightforwardly compute it as

$$\begin{aligned}
(D_\eta S)(q) &= \lim_{\epsilon \to 0} \int_{t_A}^{t_B} \frac{\mathcal{L}(q(t) + \epsilon \eta(t), \dot{q}(t) + \epsilon \dot{\eta}(t), t) - \mathcal{L}(q(t), \dot{q}(t), t)}{\epsilon} dt \\
&= \int_{t_A}^{t_B} \sum_{j=1}^{N} \biggl(\frac{\partial \mathcal{L}}{\partial q_j} \eta_j(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{\eta}_j(t) \biggr) dt.
\end{aligned}$$

Let us additionally require the endpoints are fixed, i.e., $\eta(t_A) = \eta(t_B) = 0$. If this is true, then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{d}{dt} \Bigl(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \eta_j(t) \Bigr) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \eta_j(t).$$

So we can continue the expression using this identity as

$$\begin{aligned}
(D_\eta)(q) &= \int_{t_A}^{t_B} \sum_{j=1}^{N} \biggl(\frac{\partial \mathcal{L}}{\partial q_j} \eta_j(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \dot{\eta}_j(t) \biggr) dt \\
&= \int_{t_A}^{t_B} \sum_{j=1}^{N} \biggl(\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \biggr) \eta_j(t) dt + \int_{t_A}^{t_B} \sum_{j=1}^{N} \frac{d}{dt} \biggl(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \eta_j(t) \biggr) dt \\
&= \int_{t_A}^{t_B} \sum_{j=1}^{N} \biggl(\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \biggr) \eta_j(t) dt = \langle V^{\text{Lag}}, \eta \rangle.
\end{aligned}$$

Now Hamilton's principle says that $(D_\eta S)(Q)$ for all η if and only if $V^{\text{Lag}}(Q) = 0$. The backward direction is trivial. For the forward direction, let $\eta_j(t)$ to be the same as $\partial\mathcal{L}/\partial q_j - (d/dt)(\partial\mathcal{L}/\partial\dot{q}_j)$. Then the integral becomes the integral of some nonnegative functions, and so it must be zero.

8 September 27, 2016

The principle of least action says that the Lagrangian equations come from minimizing action $S(q)$ with fixed endpoints for q.

8.1 Least action principle for the oscillator

For example, consider the harmonic oscillator in one dimensions, whose equation is given by $\ddot{q}(t) = -\omega^2 q(t)$ for $t_1 \le t \le t_2$. The solution to this equation is

$$Q(t) = Q(t_1)\cos(\omega(t-t_1)) + \frac{\dot{Q}(t_1)}{\omega}\sin(\omega(t-t_1)).$$

The action of q is

$$S(q) = \frac{1}{2}\int_{t_1}^{t_2} (\dot{q}(t)^2 - \omega^2 q(t)^2)dt.$$

In terms of the position at the two endpoints, the solution can be written as

$$\begin{aligned} Q(t) = Q(t_1)\Big(&\cos(\omega(t-t_1)) - \frac{\cos(\omega(t_2-t_1))}{\sin(\omega(t_2-t_1))}\sin(\omega(t-t_1))\Big) \\ &+ Q(t_2)\frac{\sin(\omega(t-t_1))}{\sin(\omega(t_2-t_1))}. \end{aligned}$$

If $\sin(\omega(t_2-t_1)) = 0$, then some problem occurs, and you can even see this physically; the two endpoints does not determine the solution.

Now let $Q(t)$ be the solution to the Euler-Lagrange equations and $\eta(t)$ be a variation with vanishing endpoints at t_1 and t_2. Let $q = Q + \eta$. Then

$$\begin{aligned} S(q) &= S(Q) + S(\eta) + \int_{t_1}^{t_2} (\dot{Q}\dot{\eta} - \omega^2 Q\eta)dt \\ &= S(Q) + S(\eta) + \int_{t_1}^{t_2} (-\ddot{Q}\eta - \omega^2 Q\eta)dt = S(Q) + S(\eta). \end{aligned}$$

This means that the principle of least action for the harmonic oscillator is a question whether $S(\eta) \ge 0$. We have

$$S(\eta) = \frac{1}{2}\int_{t_1}^{t_2} (\dot{\eta}(t)^2 - \omega^2\eta(t)^2)dt$$

If we find any η that makes $S(\eta) < 0$, then the principle of least action is false. To make $\dot{\eta}$ small and η big, we consider the function $\eta(t) = \sin(\pi(t-t_1)/(t_2-t_1))$. Then

$$S(t) = \frac{1}{2}\int_{t_1}^{t_2} (\dot{\eta}(t)^2 - \omega^2\eta(t)^2)dt = \frac{1}{2}\Big(\Big(\frac{\pi}{t_2-t_1}\Big)^2 - \omega^2\Big)\frac{t_2-t_1}{2}.$$

So $S(\eta) > 0$ if $t_2 - t_1 < \pi/\omega$, for my specific choice of η. On the other hand if $t_2 - t_1 > \pi/\omega$ then the action is not minimized. This shows that if $t_2 - t_1 > \pi/\omega$ the principle of least action is false.

What if we integrate $S(\eta)$ by parts? Then

$$S(\eta) = \frac{1}{2}\int_{t_1}^{t_2} \eta(t)\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\eta(t)dt = \frac{1}{2}\Big\langle \eta, \Big(\frac{d^2}{dt^2} - \omega^2\Big)\eta\Big\rangle.$$

So this is in fact an eigenvalue problem.

Let us write down the normalized eigenvectors. These will be

$$f^{(n)}(t) = \sqrt{\frac{2}{t_2 - t_1}}\sin\Big(\frac{\pi(t - t_1)n}{t_2 - t_1}\Big) \text{ with } \lambda_n = \Big(\frac{\pi}{t_2 - t - 1}\Big)^2 n^2 - \omega^2.$$

Moreover, if $n_1 \neq n_2$ then $\langle f^{(n_1)}, f^{(n_2)}\rangle = \delta_{n_1 n_2}$.

In fact $f^{(n)}(t)$ are a basis and everything can be expanded in terms of these $f^{(n)}$. If

$$\eta(t) = \sum_{j=1}^{\infty} c_j f^{(j)}(t)$$

then

$$S(\eta) = \sum_{j=1}^{\infty} c_j^2\Big(\frac{\pi^2}{(t_2 - t_1)^2} j^2 - \omega^2\Big).$$

Then whether $S(\eta) \geq 0$ for all η reduces to the question of whether $\pi^2/(t_2 - t_1)^2 j^2 > \omega^2$ for all j. That is, it is related to the smallest eigenvalue of the transformation.

9 September 29, 2016

9.1 The Hamiltonian

The quantity

$$H = \sum_{j=1}^{N} p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t)$$

is called the **Hamiltonian** and plays an important role.

So far, we have look at the case when the Lagrangian takes the form of

$$\mathcal{L} = T - V, \quad T = \frac{1}{2} \sum_{i,j=1}^{N} t_{ij}(q) \dot{q}_i \dot{q}_j, \quad V = V(q),$$

where $t_{ij} = t_{ji}$ are the coefficients. The p_is are defined as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{j=1}^{N} t_{ij}(q) \dot{q}_j.$$

In this case,

$$\sum_{i=1}^{N} \dot{q}_i p_i = \sum_{i,j=1}^{N} \dot{q}_i t_{ij}(q) \dot{q}_j = 2T.$$

So $H = T + V$. This is why the Hamiltonian is generally identified with energy.

Let us calculate the time derivative of H along a trajectory satisfying Lagrange's equation. First

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \sum_{i=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} = \sum_{i=1}^{N} (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} \\ &= \frac{d}{dt} \sum_{i=1}^{N} p_i \dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}. \end{aligned}$$

So

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

if we go along the trajectory that satisfies the Lagrangian equation. So H is constant if $\mathcal{L}$ is time-independent.

The Lagrangian for the relativistic particle is given by

$$\mathcal{L} = -mc^2 \sqrt{1 - \left(\frac{v}{c} \right)}.$$

Note that for small v, this is $-mc^2 + 1/2mv^2 + O(v^4)$. The momentum can be computed as

$$p_i = \frac{\partial \mathcal{L}}{\partial v_i} = \frac{mv_i}{\sqrt{1-\beta^2}} \text{ where } \beta = \frac{v}{c}.$$

The Hamiltonian is then

$$H = \vec{p} \cdot \vec{v} - \mathcal{L} = \sqrt{p^2c^2 + m^2c^4}$$

after some algebra.

9.2 Symmetry and Noether's theorem

What is a symmetry? We are going to consider a family of transformations q_ϵ, and consider the Lagrangian $\mathcal{L}(q_\epsilon, \dot{q}_\epsilon, t)$. For instance, Let x_ϵ be the rotation by ϵ about axis $\vec{n}$ of $\dot{x}$. We are then going to ask whether $\mathcal{L}$ stays the same under the rotation.

For example, take

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}(k_1x_1^2 + k_2x_2^2 + k_3x_3^2).$$

The rotation is a symmetry for the kinetic energy, and if the potential energy is also conserved under coordinate change, then this rotation is a symmetry of $\mathcal{L}$. Noether's theorem says that every symmetry is associated to a conservation of a quantity.

Theorem 9.1 (Noether's theorem 1)**.** *If $\mathcal{L}(q, \dot{q}, t) = \mathcal{L}(q_\epsilon, \dot{q}_\epsilon, t)$ for a family q_ϵ differentiable in ϵ near $\epsilon = 0$, and identity $\epsilon = 0$, then the quantity*

$$Q = \sum_{i=1}^{N} p_i \frac{dq_{\epsilon,i}}{d\epsilon}\bigg|_{\epsilon=0}$$

is conserved.

Proof. We have

$$0 = \frac{d}{d\epsilon}\mathcal{L}(q_\epsilon, \dot{q}_\epsilon, t) = \sum_{i=1}^{N}\left(\frac{\partial \mathcal{L}}{\partial q_{\epsilon,i}}\frac{dq_{\epsilon,i}}{d\epsilon} + \frac{\partial \mathcal{L}}{\partial \dot{q}_{\epsilon,i}}\frac{d\dot{q}_{\epsilon,i}}{d\epsilon}\right) = \sum_{i=1}^{N}\left(\dot{p}_{\epsilon,i}\frac{dq_{\epsilon,i}}{d\epsilon} + p_{\epsilon,i}\frac{d\dot{q}_{\epsilon,i}}{d\epsilon}\right).$$

This is just $dQ/dt = 0$. □

Consider the system with two particles $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$. The kinetic energy is given by

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2.$$

Consider new coordinates $\vec{x}_\epsilon^{(1)} = \vec{x}^{(1)} + \epsilon\vec{a}$ and $\vec{x}_\epsilon^{(2)} = \vec{x}^{(2)} + \epsilon\vec{a}$. Then $\vec{R}_\epsilon = \vec{R} + \epsilon\vec{a}$ and $\vec{r}_\epsilon = \vec{r}$. So the kinetic energy

$$T(\vec{R}_\epsilon, \dot{\vec{R}}_\epsilon, \vec{r}_\epsilon, \dot{\vec{r}}_\epsilon) = \frac{1}{2}M\dot{\vec{R}}_\epsilon^2 + \frac{1}{2}\mu\dot{\vec{r}}_\epsilon^2$$

is the same. So this is a symmetry. In this case,

$$Q = \vec{P} \cdot \vec{a} = (\vec{p}^{(1)} + \vec{p}^{(2)}) \cdot \vec{a}$$

is conserved.

10 October 4, 2016

Recall that Noether's theorem 1 states that if $q \to q_\epsilon$ is a family of translations such that $\mathcal{L}_\epsilon(q, \dot{q}, t) = \mathcal{L}(q_\epsilon, \dot{q}_\epsilon, t)$ is equal to $\mathcal{L}$, then the charge

$$Q = \sum_{i=1}^{N} p_i \frac{dq_{\epsilon_i}}{d\epsilon}\Big|_{\epsilon=0}$$

is conserved on a trajectory satisfying Lagrange's equations.

There is a generalized version of this theorem.

Theorem 10.1 (Noether's theorem 2)**.** *Suppose there is a function $G(q, \dot{q}, t)$ such that $(d/d\epsilon)\mathcal{L}_\epsilon|_{\epsilon=0} = dG/dt$. Then the charge*

$$Q = \sum_{i=1}^{N} p_i \frac{dq_{\epsilon_i}}{d\epsilon}\Big|_{\epsilon=0} - G$$

is conserved along a trajectory satisfying Lagrange's equations.

Proof. To see this, look at

$$\frac{d}{d\epsilon}\mathcal{L}_\epsilon = \sum_{i=1}^{N}\Big(\frac{\partial \mathcal{L}}{\partial q_{\epsilon_i}}\frac{dq_{\epsilon_i}}{d\epsilon} + \frac{\partial \mathcal{L}}{\partial \dot{q}_{\epsilon_i}}\frac{d\dot{q}_{\epsilon_i}}{d\epsilon}\Big) = \frac{d}{dt}\Big(\sum_{i=1}^{N} p_{\epsilon_i}\frac{dq_{\epsilon_i}}{d\epsilon}\Big).$$

Then we immediately get the theorem. $\square$

For example, take $q_{\epsilon_i}(t) = q_i(t+\epsilon)$. Then

$$\frac{d}{d\epsilon}\mathcal{L}_\epsilon(q(t), \dot{q}(t), t)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\mathcal{L}(q(t+\epsilon), \dot{q}(t+\epsilon), t)\Big|_{\epsilon=0} = \frac{d}{dt}\mathcal{L}(q, \dot{q}, t) - \frac{\partial \mathcal{L}}{\partial t}.$$

In the case of $\partial\mathcal{L}/\partial t = 0$, we can choose $G = \mathcal{L}$. In that case,

$$Q = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} = H.$$

is conserved.

10.1 Legendre transformation

We know how to get H to $\mathcal{L}$. It turns out that it is also possible to obtain $\mathcal{L}$ from H, and these transformations are in fact the same transformation. The Lagrangian $\mathcal{L}$ is expressed in terms of the varibles q and $\dot{q}$, and the Hamiltonian is expressed in terms of q and p.

Example 10.2. Consider the example

$$\mathcal{L} = \frac{1}{2}\langle \dot{q}, M\dot{q}\rangle - V(q)$$

for some symmetric matrix M. Then $p = M\dot{q}$ and

$$H(q, p) = \frac{1}{2}\langle \dot{q}, M\dot{q}\rangle + V(q) = \frac{1}{2}\langle p, M^{-1}p\rangle + V(q).$$

Example 10.3. Consider a particle in a magnetic field. The Lagrangian and momentum is given by

$$\mathcal{L} = \frac{1}{2}mv^2 + e\vec{A}\cdot\vec{v} - V(x), \quad \vec{p} = m\vec{v} + e\vec{A}.$$

So

$$H = \vec{p}\cdot\vec{v} - \mathcal{L} = \frac{1}{2m}|\vec{p} - e\vec{A}|^2 + V(x)$$

Because there is a way of writing p_i in terms of $\mathcal{L}$ and $\dot{q}_i$, there also has to be a way of writing $\dot{q}_i$ in terms of H and p_i. We have

$$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\Big(\sum_{j=1}^{N} p_j\dot{q}_j - \mathcal{L}\Big) = \dot{q}_i + \sum_{j=1}^{N}\Big(p_j\frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial\mathcal{L}}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial p_i}\Big) = \dot{q}_i.$$

This is half of Hamilton's equations. We also have

$$\frac{\partial H}{\partial q_i} = \sum_{j=1}^{N}\Big(p_i\frac{\partial\dot{q}_j}{\partial q_i} - \frac{\partial\mathcal{L}}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial q_i}\Big) - \frac{\partial\mathcal{L}}{\partial q_i} = -\dot{p}_i.$$

The **Hamilton's equations** are

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$

We got off a bit to the side track, but let us get back to the Legendre transform.

Example 10.4. Take a 1-dimensional system with $\mathcal{L} = (1/2)mv^2 - V(x)$. Then $H = pv - \mathcal{L}$ where $p = \partial\mathcal{L}/\partial v = mv$. The fact that $p = \partial\mathcal{L}/\partial v$ implies that v is the maximal point of the function $pv - \mathcal{L}(x,v)$ as a function over v. Thus

$$H(x,p) = pv - \mathcal{L} = \max_v(pv - \mathcal{L}).$$

Likewise, because $\mathcal{L} = pv - H$, we have an analogous equation for $\mathcal{L}$ in terms of H.

Consider $\mathcal{L}(v) = v^\alpha/\alpha$ for $v > 0$ and $\alpha > 1$. Then

$$H(p) = \max_v(pv - \frac{1}{\beta}p^\beta \text{ for } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

As a consequence,

$$pv \le \frac{1}{\alpha}v^\alpha + \frac{1}{\beta}p^\beta$$

for any p and v. The equality holds when the maximum is attained. This is **Young's inequality**.

11 October 13, 2016

What I want to start on today is Hamilton's equations.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{\partial H}{\partial q_i}.$$

Now we can think the whole thing as a $2N$ vector

$$\xi = \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix}$$

that is in the **phase space**. Then Hamilton's equations can be written as

$$\dot{\xi}(t) = \Gamma \nabla \xi H(\xi), \quad \Gamma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

This is just a short-hand notation for Hamilton's equations.

11.1 Oscillations

Consider a general quadratic Lagrangian

$$\mathcal{L} = \frac{1}{2}\langle \dot{q}, M\dot{q} \rangle - \frac{1}{2}\langle q, Mq \rangle,$$

where are M and K are positive definite symmetric real matrices. These can be diagonalized with positive eigenvalues, because they are hermitian and thus normal.[2]

The Lagrange equations say that

$$M\ddot{q} + Kq = 0.$$

To solve this equation, we let $Q = M^{1/2}\ddot{q}$ and we get $M^{1/2}\ddot{Q} = -KM^{-1/2}Q$. Then

$$\ddot{Q} = -M^{-1/2}KM^{-1/2}Q.$$

Then $M^{-1/2}KM^{-1/2}$ turns out to be a real symmetric matrix with positive eigenvalues, and so we can write $M^{-1/2}KM^{-1/2} = \Omega^2$. Then our equation is $\ddot{Q} = -\Omega^2 Q$. Then we can solve it as

$$Q(t) = \cos(\Omega t)Q(0) + \Omega^{-1}\sin(\Omega t)\dot{Q}(0).$$

So the final solution is

$$q(t) = M^{-1/2}\cos(\Omega t)M^{1/2}q(0) + M^{-1/2}\Omega^{-1}\sin(\Omega t)M^{1/2}\dot{q}(0).$$

[2]These are the matrices that satisfy $MM^\dagger = M^\dagger M$.

12 October 18, 2016

Last time we look at the oscillator whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\langle \dot{Q}, \dot{Q}\rangle - \frac{1}{2}\langle Q, \Omega^2 Q\rangle$$

where $Q = M^{1/2}q$ and $\Omega^2 = M^{-1/2}KM^{-1/2}$. Then the solution is given by

$$Q(t) = \cos(\Omega t)Q(0) - \Omega^{-1}\sin(\Omega t)\dot{Q}(0).$$

Suppose $f^{(j)}$ is a normalized eigenvectors of Ω, and let ω_j be the frequencies. Write

$$Q(0) = \sum_{j=1}^{N} \alpha_j f^{(j)}, \quad \dot{Q}(0) = \sum_{j=1}^{N} \beta_j \omega_j f^{(j)}.$$

Let us compute the Hamiltonian $H = \frac{1}{2}\langle \dot{Q}, \dot{Q}\rangle + \frac{1}{2}\langle Q, \Omega^2 Q\rangle$. We have

$$\frac{1}{2}\langle Q, \Omega^2 Q\rangle = \frac{1}{2}\left\langle \sum_{j=1}^{N} \alpha_j f^{(j)}, \Omega^2 \sum_{k=1}^{N} \alpha_k f^{(k)}\right\rangle = \frac{1}{2}\sum_{j=1}^{N} \alpha_j^2 \omega_j^2,$$

$$\frac{1}{2}\langle \dot{Q}, \dot{Q}\rangle = \frac{1}{2}\sum_{j,k=1}^{N} \langle \beta_j \omega_j f^{(j)}, \beta_k \omega_k f^{(k)}\rangle = \frac{1}{2}\sum_{j=1}^{N} \beta_j^2 \omega_j^2.$$

This shows that

$$H = \frac{1}{2}\sum_{j=1}^{N} (\alpha_j^2 + \beta_j^2)\omega_j^2.$$

That is, energy is additive over normal modes!

12.1 Hamilton equations for the oscillator

Recall that Hamilton's equations are

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}.$$

The Hamiltonian is given by

$$H = \frac{1}{2}\langle p, M^{-1}p\rangle + \frac{1}{2}\langle q, Kq\rangle = \frac{1}{2}\left\langle \xi, \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix} \xi \right\rangle, \quad \text{where } \xi = \begin{pmatrix} q \\ p \end{pmatrix}.$$

The Hamilton equations are given by

$$\dot{\xi} = \begin{pmatrix} 0 & M^{-1} \\ -K & 0 \end{pmatrix} \xi(t) = T\xi(t).$$

So the solution is given by

$$\xi(t) = e^{tT}\xi(0).$$

This solution must agree with the solution we have obtained before. Let us check this. It's easier if we also do the same scale transformation. If we have $Q = M^{1/2}q$, then

$$P = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = \dot{Q} = M^{1/2}\dot{q} = M^{-1/2}p.$$

So

$$\Xi = \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} M^{1/2} & 0 \\ 0 & M^{-1/2} \end{pmatrix} \xi.$$

In this new coordinates, the Hamilton equations become

$$\frac{d\Xi}{dt} = \begin{pmatrix} 0 & I \\ -\Omega^2 & 0 \end{pmatrix} \Xi = X\Xi.$$

Then

$$\Xi(t) = e^{tX}\Xi(0) = \begin{pmatrix} \cos(\Omega t) & \Omega^{-1}\sin(\Omega t) \\ -\Omega\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \Xi(0),$$

and thus

$$\xi(t) = \begin{pmatrix} M^{-1/2} & 0 \\ 0 & M^{1/2} \end{pmatrix} \begin{pmatrix} \cos\Omega t & \Omega^{-1}\sin\Omega t \\ -\Omega\sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} M^{1/2} & 0 \\ 0 & M^{-1/2} \end{pmatrix} \xi(0).$$

13 October 20, 2016

Today we are going to talk about Hamilton's equation. It is given by

$$\frac{d\xi}{dt} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla_\xi H.$$

13.1 Poisson brackets

We can write this as

$$\frac{d\xi_i}{dt} = \left(\sum_{j,k=1}^{2N} \Gamma_{lj} \frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial \xi_l} \right) \xi_i.$$

This is a first order differential equation and in mathematics is called a vector field.

The coefficients appear a lot in physics that is given a special name, the **Poisson bracket**. It is defined as

$$[A(\xi), B(\xi)]_\xi = \sum_{i,j=1}^{2N} \frac{\partial A}{\partial \xi_i} \Gamma_{ij} \frac{\partial B}{\partial \xi_j} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

So $[A, B]_\xi = -[B, A]_\xi$, i.e., the Poisson bracket is skew-symmetric. Using this notation, this can be written as

$$\frac{d\xi_i}{dt} = -[H, \xi_i]_\xi.$$

We also write $[A, B]_\xi = D_A B$, and this D_A is called the **Lie derivative**. Then we can write $d\xi(t)/dt = -D_H \xi(t)$. Let us look at some examples. Clearly

$$[q_i, q_j]_\xi = 0, \quad [p_i, p_j]_\xi = 0, \quad [q_i, q_j] = \delta_{ij}.$$

These are called the **fundamental Poisson brackets**. This can be also written as $[\xi_i, \xi_j]_\xi = \Gamma_{ij}$.

Let us do a change of coordinates and write

$$\begin{pmatrix} q \\ p \end{pmatrix} = \xi \quad \to \quad \begin{pmatrix} Q \\ P \end{pmatrix} = \Xi.$$

This change of coordinates is said to be **canonical** if

$$[\Xi_i, \Xi_j] = \Gamma_{ij}.$$

For example, the scaling transformation $Q_i = (M^{1/2} q)_i$ and $P_j = (M^{-1/2} p)_j$ is canonical.

The solutions to Hamilton's equation are actually canonical transformations. We have

$$\xi(t) = \xi(0) - \int_0^t dt_1 D_H \xi(t_1).$$

Using perturbation theory, we can write this as

$$\begin{aligned}\xi(t) &= \xi(0) - \int_0^t dt_1 D_H\left(\xi(0) - \int_0^t dt_2 D_H \xi(t_2)\right) = \cdots \\ &= \sum_{j=0}^{n} (-1)^j \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j D_H \cdots D_H \xi(0) + R_{n+1}.\end{aligned}$$

This is called Dyson's formula.

14 October 25, 2016

14.1 Examples of canonical transformations

Let us look at a general Hamilton's equations

$$\frac{d\xi(\lambda)}{d\lambda} = -(D_A\xi)(\lambda),$$

where A is a function on phase space and λ is a parameter that may or may not be time. Let us make a table.

A	D_A	$e^{-\lambda D_A}$	$\Xi_i = e^{-\lambda D_A}\xi_i$	$[\Xi_i, \Xi_k]_\xi$
q_j	$\frac{\partial}{\partial p_j}$	$e^{-\lambda\frac{\partial}{\partial p_j}}$	$\begin{cases}\xi_i & \text{if } i \neq j+n \\ \xi_i - \lambda & \text{if } i = j+n\end{cases}$	$[\Xi_i, \Xi_j]_\xi = [\xi_i, \xi_j]_\xi$
p_j	$-\frac{\partial}{\partial q_j}$	$e^{\lambda\frac{\partial}{\partial q_j}}$	$\begin{cases}\xi_i & \text{if } i \neq j \\ \xi_i + \lambda & \text{if } i = j\end{cases}$	$[\Xi_i, \Xi_j]_\xi = [\xi_i, \xi_j]_\xi$
qp	$p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$	$e^{-\lambda D_A}$	$q \mapsto e^\lambda q,\ p \mapsto e^{-\lambda}p$	$[Q,Q]_\xi = [P,P]_\xi = 0,$ $[Q,P]_\xi = [q,p]_\xi$

Table 1: Examples of Hamilton equations

If we give different weights $\omega_1, \dots, \omega_N$ and let $A = \sum_{j=1}^N \omega_j q_j p_j$, then $Q_j = e^{\lambda\omega_j}q_j$ and $P_j = e^{-\lambda\omega_j}p_j$ and so $[Q_i, P_j]_\xi = [q_i, p_j]_\xi$. So it is still a canonical transformation.

If ωs are all equal and

$$V(q) = \sum_{|i|=k} q_1^{i_1} q_2^{i_2} \cdots q_N^{i_N} c_{i_1,\dots,i_N}$$

is a homogeneous polynomial of degree k, then

$$e^{-\lambda D_A}V(q) = e^{\lambda(\sum_{j=1}^N i_j)\omega}V(q) = e^{\lambda k\omega}V(q).$$

Let us consider a 2-dimensional coordinate and represent it by $q = q_1 + iq_2$. Then

$$T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) = \frac{1}{2}m\dot{\bar{q}}\dot{q},$$

and then $p = \frac{1}{2}m\dot{\bar{q}}$ and $\bar{p} = \frac{1}{2}m\dot{q}$. So

$$T = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} = \frac{2p\bar{p}}{m}.$$

In this case,

$$\begin{aligned}
&[q,p]_\xi = [q_1 + iq_2, \tfrac{1}{2}(p_1 - ip_2)]_\xi = 1,\\
&[q,\bar{p}]_\xi = 0.
\end{aligned}$$

So $q, \bar{q}, p, \bar{p}$ is also a canonical transformation.

We will see later that all solutions to Hamilton's equations are canonical transformations. For now, consider the case

$$H = \frac{1}{2}\langle P, P\rangle + \frac{1}{2}\langle Q, \Omega^2 Q\rangle$$

so that

$$D_H = \sum_{i=N}^{N}\Big((\Omega^2 Q)_i \frac{\partial}{\partial P_i} - P_i \frac{\partial}{\partial Q_i}\Big) = \langle Q, \Omega^2 \tfrac{\partial}{\partial P}\rangle - \langle P, \tfrac{\partial}{\partial Q}\rangle.$$

15 October 27, 2016

In the homework assignment, you have shown that

$$[L_i, L_j] = \sum_k \epsilon_{ijk} L_k$$

for the angular momentum L of a single particle. The third component is $L_3 = q_1 p_2 - q_2 p_1$ and the Lie derivative is given by

$$D_{L_3} = -\Big(q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1}\Big) - \Big(p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1}\Big).$$

Thus

$$e^{-\theta D_{L_3}} = e^{\theta(q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1}) + \theta(p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1})}.$$

Note that $q_1(\partial/\partial q_2) - q_2(\partial/\partial q_1) = \partial/\partial\theta$. So D_{L_3} is generating a rotation by the third axis. Likewise, if $\vec{n}$ is a fixed unit vector, then $e^{\theta D_{\vec{L}\cdot\vec{n}}}$ is a canonical transformation that is the rotation in phase space by θ about $\vec{n}$.

15.1 Symmetry in elliptical orbits

Suppose we have three functions V_1, V_2, V_3 on phase space that satisfies

$$[L_i, V_j]_\xi = \sum_{k=1}^{3} \epsilon_{ijk} V_k.$$

$V_j = L_j$ will be one example, and $V_j = q_j$ will also be an example. This is another way to define a **vector** in phase space.

In the potential $V = -k/r$, the vector

$$\vec{\epsilon} = \frac{\vec{p} \times \vec{L}}{mk} - \vec{n}$$

played a special role. This also satisfies $[L_i, \epsilon_j] = \sum_{k=1}^{3} \epsilon_{ijk}\epsilon_k$. If $E < 0$ (in the case of elliptical orbits) define

$$\vec{K} = \Big(-\frac{mk^2}{2H}\Big)^{1/2} \vec{\epsilon}.$$

Then K has the same dimension as $\vec{L}$ and

$$H = -\frac{mk^2}{2(L^2 + K^2)}$$

by the formula $\epsilon^2 = 1 + 2EL^2/mk^2$. Then it turns out that

$$[L_i, L_j]_\xi = [K_i, K_j]_\xi = \sum_k \epsilon_{ijk} L_k, \quad [L_i, K_j]_\xi = \sum_k \epsilon_{ijk} K_k.$$

If we let $\vec{M} = (\vec{L} + \vec{K})/2$ and $\vec{N} = (\vec{L} - \vec{K})/2$, then the above relations can be written as

$$[M_i, M_j]_\xi = \sum_k \epsilon_{ijk} M_k, \quad [N_i, N_j]_\xi = \sum_k \epsilon_{ijk} N_k, \quad [M_i, N_j]_\xi = 0.$$

These relations come from the rotational symmetry of the 4-space. In 3-space, there are three rotations coming from,

$$D_{L_1} = q_2 \frac{\partial}{\partial q_3} - q_3 \frac{\partial}{\partial q_2}, \quad \ldots$$

but in 4-space, there are three more infinitesimal rotations, that can be written as

$$D_{K_1} = q_4 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_4}, \quad D_{K_2} = q_4 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_4}, \quad \ldots.$$

What about unbound orbits? In this case, there is a hyperbolic symmetry.

16 November 1, 2016

Let $H = H(\xi)$ be the Hamiltonian and $A(\xi)$ be a function on phase space. Then Hamilton's equations tells us that

$$\frac{dA}{dt} = -D_H A = -[H, A]_\xi = -\sum_{i,j=1}^{2N} \frac{\partial H}{\partial \xi_i} \Gamma_{ij} \frac{\partial A}{\partial \xi_j}.$$

Then

$$A(\xi(t)) = e^{-tD_H} A(\xi(0))$$

is the flow on phase space. If $[A, H]_\xi = 0$ then A is conserved by H. If we have a conserved quantity A, then $e^{-\lambda D_A} H = H$, i.e., the flow generated by a conserved quantity does not change H. Then $e^{-\lambda D_A}$ is a **symmetry** of H.

Example 16.1. Consider the case of Kepler:

$$H = \frac{p^2}{2\mu} - \frac{k}{r}.$$

Because $[H, L_1]_\xi = 0$, $[H, L_2]_\xi = 0$, it follows from $[L_1, L_2]_\xi = L_3$ that $[H, L_3]_\xi = 0$. So all angular momentum components are conserved. In this case, we further have symmetries K_i from the rotation of 4-space, SO(4).

Example 16.2. Consider the harmonic oscillator $H = p^2/2\mu + kr^2$. The rotations does not change the solution, and this can be written as

$$e^{\theta D_{\vec{L}\cdot\vec{n}}} e^{-tD_H} \xi = e^{-tD_H} e^{-\theta D_{\vec{L}\cdot\vec{n}}} \xi.$$

Then $D_H D_{\vec{L}\cdot\vec{n}} - D_{\vec{L}\cdot\vec{n}} D_H = D_{[H,\vec{L}\cdot\vec{n}]} = 0$ if $\vec{L}$ is conserved. This means that $[H, \vec{L}\cdot\vec{n}] = 0$.

Example 16.3. In the positive energy case of the Kepler problem, we have $\vec{K} = (mk^2/2H)^{1/2}\vec{\epsilon}$ and

$$[L_i, L_j] = \sum \epsilon_{ijk} L_k, \quad [K_i, K_j] = -\sum \epsilon_{ijk} L_k, \quad [L_i, K_j] = \sum \epsilon_{ijk} K_k$$

generates the Lorentz group $\mathrm{SO}(3, 1)$.

There is a very nice representation of $\mathrm{SO}(3, 1)$ by 2×2 matrices of determinant 1. Note that for any complex matrix A with $\det A = 1$, we can decompose $A = HU$ for a hermitian matrix H and a unitary matrix U. To see this, just set $H = (AA^\dagger)^{1/2}$ and $U = H^{-1}A$.

Also there is a correspondence between hermitian matrices and points in 4-space as

$$x = (t, \vec{x}) \quad \leftrightarrow \quad tI + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \hat{x},$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the map $\hat{x} \mapsto A\hat{x}A^\dagger$ gives an action 2×2 matrices on 4-space. We can recover each component by taking $x_\mu = \mathrm{Tr}(\hat{x}\sigma_\mu)/2$. If A is a unitary matrix, then this rotates x_1, x_2, x_3 and leave x_0 unchanged. If A is a hermitian matrix, then this gives a boost.

17 November 3, 2016

Consider the classical oscillator given by $\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$. There is a field analogue of this. The Klein-Gordon wave equation is given by $(\Box + m^2)q(t, \vec{x}) = 0$, where

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}.$$

17.1 Lagrange equations for fields

What does the Lagrangian $\mathcal{L}$ look like? Denote $x = (t, \vec{x})$ and assume

$$\mathcal{L} = \int L d\vec{x}, \quad S = \int \mathcal{L} dt = \int L dx$$

where $dx = dt d\vec{x}$. To avoid confusion, let us write $\varphi = q$ and

$$(\partial\varphi)_\mu = \frac{\partial \varphi}{\partial x_\mu} \text{ for } \mu = 1, 2, 3, \quad \text{and} \quad (\partial\varphi)_0 = \frac{\partial\varphi}{\partial t}.$$

Then the Lagrangian density function is given in the form $L = L(\varphi, \partial\varphi)$.

Let us use Hamilton's principle. This means that for any directional derivative of the action S is 0. Consider a variation $\varphi \mapsto \varphi(x) + \epsilon\eta(x)$, such that η is supported in some set B in space-time. If you compute, it turns out that $(D_\eta S)(\varphi) = 0$ for all φ with fixed boundary conditions at a fixed φ, then φ satisfies **Lagrange's equations**

$$\sum_{j=1}^{3} \Big(\frac{\partial}{\partial x_\mu} \frac{\partial L}{\partial(\partial\varphi)_\mu} \Big) = \frac{\partial L}{\partial \varphi}.$$

Let us show this. We have

$$\begin{aligned} D_\eta = S_B(\varphi) = D_\eta \int_B L dx &= \frac{d}{d\epsilon} \int L(\varphi + \epsilon\eta, \partial\varphi + \epsilon d\eta) dx \Big|_{\epsilon=0} \\ &= \int \left(\frac{\partial L}{\partial \varphi}\eta + \sum_\mu \frac{\partial L}{\partial(\partial\varphi)_\mu} (\partial\eta)_\mu \right) dx \Big|_{\epsilon=0} \\ &= \int_B \left(\frac{\partial L}{\partial\varphi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} \frac{\partial L}{\partial(\partial\varphi)_\mu} \right) \eta dx + \sum_\mu \int_B \frac{\partial}{\partial x_\mu} \left(\frac{\partial L}{(\partial\varphi)_\mu} \eta \right) dx \\ &= \int_B \left(\frac{\partial L}{\partial\varphi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x_\mu} \frac{\partial L}{\partial(\partial\varphi)_\mu} \right) \eta dx. \end{aligned}$$

So the integrand must be zero.

Example 17.1. Suppose L is given by

$$L(\varphi, \partial\varphi) = \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x_0}\Big)^2 - \frac{1}{2}\sum_{i=1}^{3}\Big(\frac{\partial\varphi}{\partial x_i}\Big)^2 - \frac{1}{2}m^2\varphi^2.$$

Then the Lagrange equations can be computed as

$$\frac{\partial^2\varphi}{\partial x_0^2} - \sum_{i=1}^{3}\frac{\partial^2\varphi}{\partial x_i^2} + m^2\varphi = (\Box + m^2)\varphi = 0.$$

If we had several components, then we can add the Lagrangians for each component.

17.2 Noether's theorem again

In this case, there is the notion of conserved "current". There is a charge density ρ associated to each flow, and so for a region D

$$\frac{d}{dt}Q(\mathcal{D}) = \frac{d}{dt}\int_{\mathcal{D}}\rho d\vec{x} = -\int_{\partial\mathcal{D}}\vec{J}\cdot d\vec{\rho}.$$

If we let $J = (\rho, \vec{J})$, then we can write this as

$$\sum_{\mu=0}^{3}\frac{\partial J_\mu}{\partial x_\mu} = 0.$$

Recall that the classical Noether's theorem states that if

$$Q = p\frac{dq}{d\epsilon}\Big|_{\epsilon=0} - G$$

is conserved where $d\mathcal{L}/d\epsilon|_{\epsilon=0} = dG/dt$.

The field analogue of this is

$$J_\mu = \pi_\mu\frac{d\varphi}{d\epsilon}\Big|_{\epsilon=0} - G_\mu$$

is conserved where $dL/d\epsilon|_{\epsilon=0} = \sum_{\mu=0}^{3} dG_\mu/dx_\mu$.

18 November 8, 2016

Last time we had the notion of a conserved current J_μ satisfying $\sum_{\mu=0}^3 \partial J_\mu/\partial x_\mu = 0$. Then the charge $Q = \int J_0(x)d\vec{x}$ does not depend on time x_0. Now Noether's theorem states that if $\mathcal{L} = \int L d\vec{x}$ and $L_\epsilon(\varphi, \partial\varphi) = L(\varphi(\epsilon), \partial\varphi(\epsilon))$ and

$$\frac{dL_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = \sum_{\mu=0}^{3} \frac{\partial G_\mu}{\partial x_\mu},$$

then

$$J_\mu = \pi_\mu \frac{d\varphi(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} - G_\mu$$

is a conserved current.

Proof of Noether's theorem. We have

$$\begin{aligned}
\frac{d}{d\epsilon}L_\epsilon &= \frac{\partial L_\epsilon}{\partial\varphi(\epsilon)}\frac{d\varphi(\epsilon)}{d\epsilon} + \sum_{\nu=0}^{3}\frac{\partial L_\epsilon}{\partial(\partial_\mu\varphi(\epsilon))}\frac{d}{d\epsilon}\partial_\mu\varphi(\epsilon) \\
&= \frac{\partial L}{\partial\varphi(\epsilon)}\frac{d\varphi(\epsilon)}{d\epsilon} + \sum_{\nu=0}^{3}\left(\frac{\partial}{\partial x_\nu}\left(\frac{\partial L_\epsilon}{\partial\partial_\nu\varphi(\epsilon)}\frac{d\varphi(\epsilon)}{d\epsilon}\right) - \left(\frac{\partial}{\partial x_\nu}\frac{\partial L_\epsilon}{\partial\partial_\nu\varphi(\epsilon)}\right)\frac{d\varphi(\epsilon)}{d\epsilon}\right) \\
&= \sum_{\nu=0}^{3}\frac{\partial}{\partial x_\nu}\left(\frac{\partial L_\epsilon}{\partial\partial_\nu\varphi(\epsilon)}\frac{d\varphi(\epsilon)}{d\epsilon}\right).
\end{aligned}$$

So if $dL_\epsilon/d\epsilon = \sum_{\nu=0}^3 \partial G_\nu/\partial x_\nu$, then

$$J_\nu = \pi_\nu \frac{d\varphi(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} - G_\nu; \quad \pi_\mu(x,\epsilon) = \frac{\partial L_\epsilon}{\partial\partial_\nu\varphi(\epsilon)}.$$

is a conserved current. □

18.1 The energy-momentum density

Consider the family $\varphi_\epsilon(x) = \varphi(x + \epsilon e_\nu)$ where e_ν is a unit vector in direction ν. If this works out for all 4 directions, we get 4 conserved vectors $J_\mu^{(\nu)} = T_{\mu\nu}$, which is called the **energy-momentum density**. In the case $\nu = 0$, we get the energy density $T_{00}(x) = J_0^{(0)}$ and

$$H = \int T_{00}(x)d\vec{x}$$

is the energy. For $\nu = 1, 2, 3$, $T_{0\nu}(x)$ is the momentum density and

$$P_\nu = \int T_{0\nu}(x)d\vec{x}.$$

In this case, it is easy to find functions G. If we take $G_\mu = L\delta_{\mu\nu}$, then

$$\frac{dL}{dx_\nu} = \sum_{\mu=0}^{3} \frac{\partial G_\mu}{\partial x_\mu}$$

is trivially verified. Then we get the formula

$$T_{\mu\nu}(x) = \pi_\mu(x)\frac{\partial\varphi(x)}{\partial x_\nu} - \delta_{\mu\nu}L$$

for the energy-momentum density tensor.

Let us look at the Klein-Gordon equation, given by

$$L = \frac{1}{2}\Big(\frac{\partial\varphi}{c\partial t}\Big)^2 - \frac{1}{2}\sum_{j=1}^{3}\Big(\frac{\partial\varphi}{\partial x_j}\Big)^2 - V(\varphi(x)).$$

Then $\pi_0 = \partial\varphi/\partial x_0$ and $\pi_j = -\partial\varphi/\partial x_j$ for $j = 1, 2, 3$. So

$$H(x) = T_{00}(x) = \frac{1}{2}\sum_{j=0}^{3}\Big(\frac{\partial\varphi}{\partial x_j}\Big)^2 + V(\varphi(x)).$$

Likewise

$$P_\nu(x) = T_{0\nu}(x) = \frac{\partial\varphi}{\partial x_0}\frac{\partial\varphi}{\partial x_\nu}.$$

This is the simplest example. There are also space-time symmetry, i.e., rotation in space and boots, and also the symmetry in the target space, i.e., rotation among the φ_i, where the field has many components.

19 November 10, 2016

Let us look at the 2-dimensional wave equation

$$\mathcal{L} = \int L d\vec{x}, \quad L = \frac{1}{2}\Big(\frac{\partial\varphi}{c\partial t}\Big)^2 - \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x}\Big)^2 - V(\varphi(x,t))$$

in space-time (t, x). If $V = 0$, we get the normal wave equation $\Box\varphi = 0$ and if $V = m^2\varphi^2/2$ then we get the Klein-Gordon equation $(\Box + m^2)\varphi = 0$. The static phase wave

$$\varphi = e^{i\omega t + ikx}$$

is a solution for $\Box\varphi = 0$ if $\omega^2 = k^2c^2$, and is a solution for the Klein-Gordon equation if $\omega^2 = (k^2 + m^2)c^2$. In this case, the energy density can be computed as

$$\begin{aligned} H(x) &= \frac{1}{2}\Big(\frac{\partial\varphi}{c\partial t}\Big)^2 + \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \frac{1}{2}m^2\varphi^2 \\ &= \frac{1}{2}\Big(\frac{\omega^2}{c^2} + k^2\Big)\sin^2(\omega t + kx) + \frac{1}{2}m^2\cos^2(\omega t + kx) \\ &= k^2\sin^2(\omega t + kx) + \frac{1}{2}m^2. \end{aligned}$$

We also have the momentum density

$$P(x) = \frac{\partial\varphi}{c\partial t}\frac{\partial\varphi}{\partial x} = \frac{\omega k}{c}\sin^2(\omega t + kx).$$

Because the energy density oscillates and is positive, it has ∞ total energy. To make it finite, we may look at wave packets instead.

19.1 Solitons

Now let us look at an example of a non-linear equation. Let $V(\varphi) = (\varphi^2 - 1)^2/2$. Then the wave equation is

$$\Box\varphi + 2\varphi(\varphi^2 - 1) = 0.$$

There is a solution that has a single peak, and in many ways, this behaves like a particle. This is called a **soliton**.

Let us look at the simplest case: the static soliton of the form $\varphi(t, x) = \varphi(x)$. Then the equation is

$$-\frac{d^2\varphi(x)}{(dx)^2} + V'(\varphi(x)) = 0,$$

where x is the spatial variable. This looks like the normal Newton's equations but with potential being $-V$. So $-\varphi'^2 + V(\varphi)$ is conserved. Assuming this is 0, we get a very simple formula for the energy density:

$$H(x) = \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + V(\varphi(x)) = \Big(\frac{\partial\varphi}{\partial x}\Big)^2 = 2V(\varphi(x)).$$

Also $P = 0$.

Let us go back to the example $V(\varphi) = \frac{1}{2}(\varphi^2 - 1)^2$. We have $\varphi'^2 = 2V(\varphi) = (\varphi^2 - 1)^2$. So $\varphi' = 1 - \varphi^2$. The solution to this equation is $\varphi(x) = \tanh x$. In this case, the total energy is finite, and

$$H = \int H(x)dx = \int_{-\infty}^{\infty} \frac{1}{\cosh^4 x} dx = \frac{4}{3}.$$

The energy is concentrated near 0.

We can also make this soliton move. We can take a Lorentz boost and apply it to the static soliton. More explicitly, we take

$$\begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

and let $\varphi_\beta(x,t) = \varphi'(x')$. This new soliton will move with velocity v, for $\beta = v/c = \tanh\psi$. We can further check that

$$\begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} 4/3 \\ 0 \end{pmatrix} = \begin{pmatrix} H_\beta \\ P_\beta \end{pmatrix}.$$

Note that instead of solving $\varphi' = 1-\varphi^2$, we could have solved $\varphi' = -(1-\varphi^2)$. The solution to this equation gives a anti-soliton, where the calling something a soliton or an anti-soliton is quite arbitrary.

More generally, let us consider the potential

$$V(\varphi) = \frac{\lambda^2}{8}\Big(\varphi^2 - \frac{m^2}{\lambda^2}\Big)^2,$$

where $m, \lambda > 0$. Then the static soliton is

$$\varphi(x) = \frac{m}{\lambda} \tanh\Big(\frac{m}{2}(x-a)\Big)$$

with energy $H = 2m^3/3\lambda^2$.

There are other examples. If a solution is localized in space-time, it is called an instanton because it exists for an instance. There is also the Sine-Gordan equation given by $V(\varphi) = 1+\cos\varphi$. Then there are solitons that tunnels through one peak, and it is actually given by arctan. Again, applying a boost gives a moving solution.

20 November 15, 2016

Today we are going to look at target space symmetries.

20.1 Target space symmetry

Consider a vector field $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ and the Lagrangian

$$L = \frac{1}{2}\sum_{j=1}^{3}\Big(\frac{\partial \varphi_i}{\partial x_0}\Big)^2 - \frac{1}{2}\sum_{i,j=1}^{3}\Big(\frac{\partial \varphi_j}{\partial x_i}\Big) - V(\vec{\varphi})^2.$$

The rotation

$$\vec{\varphi}_\epsilon = R_\epsilon \vec{\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{pmatrix} \vec{\varphi}$$

preserves the Lagrangian. So Noether's theorem gives the conserved current

$$J_\mu = \sum_{j=1}^{3} \pi_{\mu j} \frac{\partial \varphi_j}{\partial \epsilon}\Big|_{\epsilon=0}.$$

For example,

$$J_0(x) = \sum_{j=1}^{3} \frac{\partial \varphi_j}{\partial t}\Big(\frac{dR_\epsilon}{d\epsilon}\Big|_{\epsilon=0}\Big)_{jk}\varphi_k = -\frac{\partial \varphi_2}{\partial t}\varphi_3 + \frac{\partial \varphi_3}{\partial t}\varphi_2.$$

Then the conserved charge for a target space symmetry of L can be computed as integrating $Q = J_0$ on a region in space.

Recall that in the proof of Noether's theorem, we get

$$\begin{aligned} \frac{dL_\epsilon}{d\epsilon} &= \sum_i \Big(\frac{\partial L}{\partial \varphi_i}\frac{d\varphi_i}{d\epsilon} + \sum_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)}\frac{d\partial_\mu \varphi_i}{d\epsilon}\Big) \\ &= \sum_\mu \frac{\partial}{\partial x_\mu}\Big(\sum_i \frac{\partial L}{\partial(\partial_\mu \varphi_i)}\frac{d\varphi_i}{d\epsilon}\Big). \end{aligned}$$

Then for $\varphi_\epsilon = \varphi + \epsilon\eta$ and $S_\epsilon = \int L_\epsilon d\vec{x}dt$,

$$\begin{aligned} \frac{dS_\epsilon}{d\epsilon} &= (\text{Lag. eqn. part}) + \int_{\mathcal{D}} \sum_\mu \frac{\partial}{\partial x_\mu}\Big(\sum_i \frac{\partial L}{\partial \partial_\mu \varphi_i}\eta_i\Big) d\vec{x}dt \\ &= (\text{Lag. eqn. part}) + \int_{\partial \mathcal{D}} \sum_i \Big(\frac{\partial L}{\partial \partial_\mu \varphi_i}\eta_i\Big) d\sigma^\mu. \end{aligned}$$

But from this argument, we can get more. Instead of letting $\varphi_\epsilon = \varphi + \epsilon\eta$, let $\eta = (\partial \varphi_\epsilon / \partial \epsilon)|_{\epsilon=0}$. Consider $\mathcal{D} = (\text{some region in space}) \times (\text{some time interval})$.

Then there are two types of boundary in $\partial\mathcal{D}$: time boundary and surface boundary. The time boundary gives surface terms

$$\int_{\text{end time}} \sum_i \frac{\partial\varphi_i}{\partial t}\frac{d\varphi_{\epsilon i}}{d\epsilon}\Big|_{\epsilon=0} d\vec{x} - \int_{\text{start time}} \sum_i \frac{\partial\varphi_i}{\partial t}\frac{d\varphi_{\epsilon i}}{d\epsilon}\Big|_{\epsilon=0} d\vec{x}.$$

The spatial boundary something, but the important thing is that for a finite energy solution on ∞ space, we can set the spatial component go to infinity so that there is no spatial boundary. For instance, look at the soliton, which has finite energy and is time independent. Suppose $S_\epsilon(\varphi) = S(\varphi_\epsilon)$ where $\varphi_{\epsilon=1}$ is a soliton solution. Then we can take $\vec{x}$ to be all space and then

$$\frac{dS_\epsilon}{d\epsilon}\Big|_{\epsilon=1} = 0$$

for all variations.

20.2 Topological conservation law

For the soliton, there is another conservation law, that does not arise from Noether's theorem. In the 1-dimensional soliton, the solution is like a tunneling event between two ground states at ± 1. Then the quantity

$$\text{topological charge} = \int_{-\infty}^{\infty} \int \frac{\partial\varphi}{\partial x} dx$$

is conserved. This is a very simple case, but the Maxwell's equations and also Yang-Mills theory. I wanted to point out that there are symmetries that have other interpretations.

21 November 17, 2016

This class was a review session for the in-class midterm, by the TF.

21.1 Review for the exam—lots of examples

Let us look at the 2-dimensional system

$$\mathcal{L} = \frac{1}{2}(\dot{q}_1^2 - \dot{q}_2^2) - V(q_1, q_2); \quad V(q_1, q_2) = (q_1^2 - q_2^2)^2.$$

Then $p_1 = \dot{q}_1$ and $p_2 = -\dot{q}_2$. So the Hamiltonian is given by

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - \mathcal{L} = \frac{1}{2}(p_1^2 - p_2^2) + V(q_1, q_2).$$

Then the transformation

$$\begin{pmatrix} q_1(\epsilon) \\ q_2(\epsilon) \end{pmatrix} = \begin{pmatrix} \cosh\epsilon & \sinh\epsilon \\ \sinh\epsilon & \cosh\epsilon \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

is a symmetry.

We now apply Noether's theorem. Because $(dq_1(\epsilon)/d\epsilon)|_{\epsilon=0} = q_2$ and $(dq_2(\epsilon)/d\epsilon)|_{\epsilon=0} = q_1$, we get

$$G = p_1 q_2 + q_2 p_1$$

as the conserved quantity. Let's compute some Poisson brackets for practice. We have

$$[q_1, G] = q_2, \quad [q_2, G] = q_1, \quad [p_1, G] = -p_2, \quad [p_2, G] = -p_1.$$

Let's now compute the canonical flow generated by this function. There is a nice trick to compute this. Recall that such a transformation is canonical. Because G is a conserved quantity,

$$\begin{aligned} \frac{d}{d\lambda}\Xi(\lambda) &= -D_G q_1(\lambda) = [q_1(\lambda), G]_{q,p} = [q_1(\lambda), G(\lambda)]_{q,p} \\ &= [q_1(\lambda), G(\lambda)]_{q(\lambda),p(\lambda)} = q_2(\lambda). \end{aligned}$$

Then

$$\frac{d}{d\lambda}\Xi(\lambda) = \begin{pmatrix} q_2(\lambda) \\ q_1(\lambda) \\ -p_2(\lambda) \\ -p_1(\lambda) \end{pmatrix}.$$

This is a simple differential equation, and so can solve it easily.

Let's now move on and talk about classical field theory. Consider the Lagrangian given by

$$L = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2.$$

The equation of motion will be

$$\partial_t^2\varphi - \partial_x^2\varphi = 0.$$

There is the boost symmetry, but there is also a scaling symmetry given by

$$\varphi_\epsilon(t,x) = \varphi(\epsilon t, \epsilon x).$$

Then

$$S_\epsilon = \int \epsilon^2 \left(\frac{1}{2}(\partial_t\varphi)^2(\epsilon t, \epsilon x) - (\partial_x\varphi)^2(\epsilon t, \epsilon x) \right) dt dx = S.$$

22 November 29, 2016

Today and Thursday, I am going to talk about the connection between classical mechanics and quantum theory. The first point of view is looking at the **Schrödinger equation**, given by

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t).$$

Another way of looking at this is through the **Heisenberg equation**, which looks like

$$\hbar\frac{d}{dt}A(t) = i(HA(t) - A(t)H) = i[H, A(t)].$$

The connection between the two ways were noticed mainly by Dirac. The solutions to the two equations are given by

$$\psi(t) = e^{-itH/\hbar}\psi(0), \quad A(t) = e^{itH/\hbar}A(0)e^{-itH/\hbar}.$$

Today we are going to concentrate on the Schrödinger picture.

22.1 The Feynman formula

We have the solution to the Schrödinger equation. We can write it as

$$\psi(t,x) = (e^{-itH}\psi)(0,x) = \int \mathfrak{K}_t(x,x')\psi(0,x')dx',$$

where $\mathfrak{K}_t$ is the **Green's function**.

Let us look at the simplest example of the freely moving particle in 1-dimension. In this case, $H = p^2/2m$. Consider the Fourier transform

$$\psi(0,x) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}\tilde{\psi}(0,p)dp,$$
$$\tilde{\psi}(0,p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(0,x)dx.$$

Then the formula for ψ is

$$\begin{aligned}\psi(t,x) &= \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}e^{-\frac{itp^2}{2m\hbar}}\tilde{\psi}(p)dp \\ &= \frac{1}{2\pi\hbar}\int\left(\int dp\, e^{ip(x-x')/\hbar - i\frac{tp^2}{2m\hbar}}\right)\psi(0,x')dx'.\end{aligned}$$

So the Green's function we want is

$$\mathfrak{K}_t(x,x') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar - \frac{itp^2}{2m\hbar}}.$$

Actually this doesn't converge, so we have to put a convergence factor and set

$$\mathfrak{K}_t(x,x') = \lim_{\epsilon\to 0} \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar - \frac{itp^2}{2m\hbar}} e^{-\epsilon p^2}.$$

If you work it out, then we get the answer

$$\mathfrak{K}_t(x,x') = \sqrt{\frac{m}{2\pi i\hbar t}} e^{i\frac{m(x-x')^2}{2\hbar t}}.$$

The action of a free particle satisfying the Newton-Lagrange equation is given by

$$S_{\text{free,Newton}} = \frac{m}{2}\int_0^t \dot{x}^2 ds = \frac{m}{2}\frac{(x-x')^2}{t},$$

where x is the position at time t and x' is position at time 0. So we can write

$$\mathfrak{K}_t(x,x') = \sqrt{\frac{m}{2\pi i\hbar t}} e^{iS_{\text{free,Newton}}/\hbar}.$$

Now Feynman thought a lot about this equation and tried to see what happens if we take an arbitrary path and average over all paths. And then we would get

$$\mathfrak{K}_t(x,x') = \int e^{iS_{\text{free}(x)}/\hbar}\mathcal{D}x.$$

The mathematical problem here is what exactly $\mathcal{D}(x)$ is. But you can formulate it, and this was Feynman's thesis.

Why is it? Note that $e^{-it_1H/\hbar}e^{-it_2H/\hbar} = e^{-i(t_1+t_2)H/\hbar}$ and so

$$\int \mathfrak{K}_{t_1}(x,x_1)\mathfrak{K}_{t_2}(x_1,x')dx_1 = \mathfrak{K}_{t_1+t_2}(x,x').$$

So if we let $x_0 = x$, and x_k be x at time kt/n, then we get

$$\begin{aligned}
\mathfrak{K}_t(x,x') &= \int dx_1\cdots dx_{n-1}\, \mathfrak{K}_{t/n}(x,x_1)\mathfrak{K}_{t/n}(x_1,x_2)\cdots\mathfrak{K}_{t/n}(x_{n-1},x') \\
&= \left(\frac{m}{2\pi i\hbar t/n}\right)^{n/2} \int e^{i\frac{m}{2\hbar}\frac{t}{n}\sum_{j=0}^{n-1}\frac{(x_j-x_{j+1})^2}{(t-n)^2}} dx_1\cdots dx_{n-1} \\
&\approx \left(\frac{m}{2\pi i\hbar t/n}\right)^{n/2} \int e^{i\frac{m}{2\hbar}\int_0^t \dot{x}^2 ds} dx_1\cdots dx_{n-1}.
\end{aligned}$$

Here we get a constant, that tends to infinity as $n\to\infty$, and some $dx_1\cdots dx_{n-1}$, that also doesn't make sense as $n\to\infty$. But at least in some sense, we have the formula

$$\mathfrak{K}_t(x,x') \approx \int e^{iS_{\text{free}}/\hbar}\mathcal{D}x.$$

This is the **Feynman formula**.

What if you have a particle that is not free? The answer is that you can just replace S_{free} by a general S. This is actually the same as first letting the particle move freely for one time increment, then giving a jolt by the potential, then letting the particle move freely again, giving a second jolt, and alternating this process. So if $H = H_{\text{free}} = V$, then

$$\lim_{n\to\infty}(e^{A/n}e^{B/n})^n = e^{A+B}.$$

This is why we can simply replace the free action with the general action.

23 December 1, 2016

Last time we looked at the Schrödinger equtation and got

$$\mathfrak{K}_t(x,x') = \int e^{iS(x(t))/\hbar}\mathcal{D}x.$$

In the case of the free particle, we further had

$$\mathfrak{K}_t(x,x') = \int e^{iS(x(t))/\hbar}\mathcal{D}x = e^{iS_{\text{classical}}/\hbar}.$$

23.1 The Heisenberg equation

The **Heisenberg equation** is given by

$$e^{itH/\hbar}Ae^{-itH/\hbar} = A(t)$$

for an observable A.

There is a connection between this equation and the Poisson brackets we saw in classical mechanics. Recall that

$$[x_i,p_j]_\xi = \sum_k \Big(\frac{\partial x_i}{\partial x_k}\frac{\partial p_i}{\partial p_k} - \frac{\partial x_i}{\partial p_k}\frac{\partial p_j}{\partial x_k}\Big) = \delta_{ij}.$$

In quantum theory, we have the relation (where we put the hat to distinguish from the classical counterpart)

$$[\hat{x}_i,\hat{p}_j] = \hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i = \hat{x}_i\Big(\frac{\hbar}{i}\frac{\partial}{\partial\hat{x}_j}\Big) - \Big(\frac{\hbar}{i}\frac{\partial}{\partial\hat{x}_j}\Big)\hat{x}_i = \frac{\hbar}{i}\delta_{ij}.$$

So in some sense, $[\hat{x}_i,\hat{p}_j] = [x_i,p_j]_\xi i\hbar$. This suggests the classical limit identity is given by

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[\hat{A},\hat{B}] = [A,B]_\xi.$$

Given a function A, in classical mechanics, we get a flow

$$e^{-\lambda D_A}B = B - \lambda[A,B]_\xi + \frac{\lambda^2}{2}[A,[A,B]_\xi]_\xi + \cdots.$$

This corresponds to the unitary operator $e^{i\lambda\hat{A}/\hbar}$ in quantum mechanics. If we expand $e^{i\lambda\hat{A}/\hbar}\hat{B}e^{-i\lambda\hat{A}/\hbar}$, we get

$$\begin{aligned} e^{i\lambda\hat{A}/\hbar}\hat{B}e^{-i\lambda\hat{A}/\hbar} &= f(\lambda) = \sum_{k=0}^{N} f^{(k)}(0)\frac{\lambda^k}{k!} + O(\lambda^{N+1}) \\ &= \hat{B} + \frac{i\lambda}{\hbar}[\hat{A},\hat{B}] + \Big(\frac{i\lambda}{\hbar}\Big)^2\frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \cdots. \end{aligned}$$

So as $\hbar \to 0$, this goes to

$$B - \lambda[A,B]_\xi + \frac{\lambda^2}{2}[A,[A,B]_\xi]_\xi + \cdots .$$

In the classical setting, $e^{-\lambda D_A}$ is a canonical transformation, and in the quantum setting, $e^{i\lambda\hat{A}/\hbar}$ is a unitary transformation.

We also had another important set of Poisson brackets, which are the ones between the angular momenta. Continuing the analogue, we have

$$[\hat{L}_i,\hat{L}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{L}_k, \quad [L_i,L_j]_\xi = \sum_k \epsilon_{ijk}L_k.$$

Still, we cannot talk about it in more generality. For instance, recall that x_1p_1 in classical mechanics generated the scale transformation. But here we run into an ambiguity. Is the right analogue $\hat{x}_1\hat{p}_1$ or $\hat{p}_1\hat{x}_1$? If $e^{i\lambda\hat{A}/\hbar}$ is to be a unitary operator, the operator $\hat{A}$ has to be hermitian. But the adjoint of $\hat{x}_1\hat{p}_1$ is $\hat{p}_1\hat{x}_1$ and vice versa. So what we do is to take

$$xp \quad \to \quad \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}).$$

Let me show you something more disturbing. What about $A = px^3$? Will $\hat{A} = \frac{1}{2}(\hat{p}\hat{x}^3 + \hat{x}^3\hat{p})$ do? Let us compute the eigenvalues of $\hat{A}$. If you solve the equation $\hat{A}\psi(x) = \lambda\psi(x)$, you get

$$\psi(x) = \tilde{c}\frac{1}{x^{3/2}}e^{-i\lambda 2x^2}.$$

This is square-integrable if and only if λ is in the lower half plane. So every λ in the lower half plane is an eigenvalue of $\hat{A}$! This is because $\hat{A}$ is not self-adjoint.

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