Math 222 - Lie Groups and Lie Algebras

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This course was taught by Fabian Haiden, at MWF 10-11am in Science Center 310. The textbook was *An Introduction to Lie Groups and Lie Algebras* by A. Kirillov. There were 6 undergraduates and 10 graduate students enrolled. There were weekly problem sets and a final presentation. The course assistant was Yusheng Luo.

Contents

1 January 23, 2017

1.1 Logistics

A famous mathematician said that you can do nothing without Lie groups and Lie algebras. The goal of this course is to learn and use this language. Office hours are Tuesdays 5–6 pm and Fridays 2–3 pm. Prerequisites include:

- (multi-)linear algebra
- abstract algebra: language of group theory
- calculus on manifolds
- topology: fundamental groups and covering spaces

The textbook for this course is *An introduction to Lie groups and Lie algebras* by Kirillov. Another good reference is *Notes on Lie groups* by Richard Borcherds. For those who are taking this course for a grade, there will be weekly homeworks due Mondays. There will be a take-home midterm and a final paper (which will be due the end of reading period).

1.2 Lie groups

Definition 1.1. A **Lie group** is a C^∞ manifold G with a group structure:

- a composition $G \times G \to G; (g, h) \mapsto gh$ that is smooth and associative,
- an identity element $1 \in G$,
- for every $g \in G$ an inverse $g^{-1} \in G$ so that the inverse map $G \to G; g \mapsto g^{-1}$ is smooth.

Example 1.2. The space $\mathrm{GL}(n, \mathbb{R})$ of invertible $n \times n$ real matrices is a Lie group. There is also $\mathrm{GL}(n, \mathbb{C})$, the invertible complex manifolds.

For finite or finitely generated groups, there are ways to write it down. For example,

$$\pi_1(\text{genus 2 surface}) = \langle a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle.$$

But Lie groups are uncountable, unless it is 0-dimensional. So we use "infinitesimal generators" instead. For example, the vector field

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

generates the 1-parameter group of diffeomorphisms. This gives a morphism

$$\mathbb{R} \to \mathrm{GL}(2, \mathbb{R}); \quad t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

1.3 Lie algebras

For a fixed $g \in G$, denote by $Lg : G \to G$ the map $h \mapsto gh$. The infinitesimal generators of a Lie group G is its Lie algebra.

Definition 1.3. The **Lie algebra** of G is defined as

$$\mathfrak{g} = \mathrm{Lie}(G) = T_1 G.$$

For example, the Lie algebra of $G = \mathrm{GL}(n, \mathbb{R})$ is $\mathrm{Lie}(G) = \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}(n, \mathbb{R})$. The 1-parameter family that is generated by $A \in \mathrm{Mat}(n, \mathbb{R})$ is

$$t \mapsto \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

If we look at the composition, something interesting happens. Heuristically,

$$(1 + \epsilon A)(1 + \epsilon B) = 1 + \epsilon(A + B) + O(\epsilon^2)$$

and so composition is commutative to first order. At the second order, the commutator is

$$\begin{aligned}
&(1 + \epsilon A)(1 + \epsilon B)(1 + \epsilon A)^{-1}(1 + \epsilon B)^{-1}) \\
&\qquad = (1 + \epsilon A)(1 + \epsilon B)(1 - \epsilon A + \epsilon^2 A^2)(1 - \epsilon B + \epsilon^2 B^2) \\
&\qquad = \cdots = 1 + \epsilon^2 (AB - BA) + O(\epsilon^3).
\end{aligned}$$

We write $[A, B] = AB - BA$. This is only for $\mathrm{GL}(n, \mathbb{R})$, but in general, the Lie algebra $\mathfrak{g}$ carries a product map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Definition 1.4. A **Lie algebra** is a vector space $\mathfrak{g}$ with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ written as $(a, b) \mapsto [a, b]$ such that

(1) $[a, b] = -[b, a]$ (antisymmetry),
(2) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ (Jacobi identity).

The Jacobi identity is Leibniz's rule in disguise, if you write it this way:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

Just to make the analogy, note that

$$\frac{\partial}{\partial t}(fg) = \Big(\frac{\partial}{\partial t} f\Big) g + f \Big(\frac{\partial}{\partial t} g\Big).$$

There are two things that $\mathrm{Lie}(G)$ does not see. The first is other connected components, and the second is passing the covering spaces/quotients by discrete groups. For example, there is a covering map $(\mathbb{R}, +) \to S^1 \subseteq \mathbb{C}^*$ and so the Lie algebra of S^1 is $(\mathbb{R}, +)$. This might not be surprising, but the surprising thing is that this is all. There is an equivalence of categories:

$$\begin{gathered}
\{\text{connected, simply connected Lie groups}\} \\
\updownarrow \\
\{\text{finite dimensional Lie algebras}\}
\end{gathered}$$

That is, every Lie algebra comes from a Lie group, and group homomorphisms $G \to H$ comes from algebra homomorphisms $\mathfrak{g} \to \mathfrak{h}$.

2 January 25, 2017

Today I want to talk about some motivations and a review of smooth manifolds.

Consider the Laplacian $\Delta_{S^2} : C^\infty(S^2) \to C^\infty(S^2)$ on the 2-sphere. This is defined as

$$\Delta_{S^2} f = (\Delta_{\mathbb{R}^3} \bar{f})|_{S^2},$$

where $\bar{f}$ is the extension such that it is constant on the rays from the origin. A physically and mathematically interesting problem is to determine the eigenvalues and eigenfunctions of this differential operator. This is a second-order differential equation $\Delta_{S^2} f = \lambda f$ but it has non-constant coefficients; it does not have translational symmetry. So we can't just apply Fourier theory to solve it. But it has a SO(3) symmetry. Compare this with the same problem on S^1, which can be solved via Fourier theory.

What is Fourier theory? This is basically saying that locally compact *abelian* groups G correspond to its characters $\chi : G \to S^1 = \mathrm{O}(1)$. If G is non-abelian, then any character χ is trivial on the commutator, i.e., it only sees $G/[G,G]$. So we instead consider linear representations $G \to \mathrm{GL}(n, \mathbb{C})$. This gives rise to the problem of understanding the representations of a given Lie group.

One line of attack is to look at representations of the Lie algebra $\mathfrak{g}$. These are the maps $\mathfrak{g} \to \mathrm{Mat}(n, \mathbb{C})$ that preserves bracket. This problem is solvable if $\mathfrak{g}$ is "semisimple", which we will talk about later.

It turns out that the eigenfunctions of Δ_{S^2} are the restrictions of polynomials on $\mathbb{R}^3$.

2.1 Smooth manifolds

Intuitively manifolds are pieces of $\mathbb{R}^n$ glued together.

Definition 2.1. A **smooth manifold** is a Hausdorff, second countable topological space M with an **atlas** (collection of **charts**) $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ so that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^∞ where it is defined.

Example 2.2. $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{Mat}(n, \mathbb{R})^2$ is a manifold because it is open. This can be covered by a single chart. The circle S^1 is a smooth manifold that can be covered by two charts.

The Lie group

$$\begin{aligned} \mathrm{SU}(2) &= \{M \in \mathrm{GL}(2, \mathbb{C}) : M\bar{M}^T = 1, \det M = 1\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &= \{x \in \mathbb{R}^4 : x_1^2 + \cdots + x_4^2 = 1\} = S^3 \end{aligned}$$

is a smooth manifold. This raises the question, what spheres can be given a Lie group structure? It turns out S^1 and S^3 are the only ones.

Definition 2.3. A map $f : M \to N$ is C^∞ or **smooth** if for any charts

$$V_\alpha \xleftarrow{\varphi_\alpha} U_\alpha \subseteq M \xrightarrow{f} N \supseteq U_\beta \xrightarrow{\psi_\beta} V_\beta$$

the composition $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is a smooth map where defined. A **diffeomorphism** is a smooth map with smooth inverse.

The multiplication map $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is a smooth map because the entries are polynomials. The inverse $\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is also smooth because there is a formula for the inverse. Alternatively you can use the inverse function theorem.

Definition 2.4. A **map between Lie groups** is a C^∞ group homomorphisms.

2.2 Fiber bundles

Definition 2.5. A **fiber bundle** with fiber F is a map $p : E \to M$ such that, locally in M, looks like $F \times M \to M$. More formally, for any $x \in M$ there is a open neighborhood $U \ni x$ such that there is a diffeomorphism $\varphi : p^{-1}(U) \to F \times U$ so that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"] & F \times U \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd}$$

It is customary to write this as a sequence

$$F \longrightarrow E \xrightarrow{p} M,$$

even though there is no literal map $F \to E$.

Example 2.6 (The **Hopf fibration**)**.** The group $\mathrm{SU}(2)$ acts on $\mathbb{C}^2$ and so it acts on $\mathbb{C}P^1 \cong S^2$. This action is transitive. Now fix a point $p \in \mathbb{C}P^1$. Then we get a map

$$\mathrm{SU}(2) \to \mathbb{C}P^2; \quad A \mapsto Ap.$$

The stabilizer of this action are the rotations $\{\left(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{smallmatrix}\right)\} \cong \mathrm{U}(1)$. So this gives a fibration $S^1 \to S^3 \to S^2$.

A special case of a fiber bundle is vector bundle.

Definition 2.7. A **vector bundle** is a fiber bundle with fiber $F = \mathbb{R}^n$. Then fibers have vector space structures and trivializations are fiber-wise linear.

2.3 Tangent vectors

There is a mathematician's way of defining a tangent vector and a physicist's way.

Definition 2.8. A **tangent vector** at $p \in M$ is a derivation $X : C^\infty(M) \to \mathbb{R}$ that satisfies the following conditions:

- if $f = g$ in a neighborhood of p then $X(f) = X(g)$
- $X(\mu f + \lambda g) = \mu X(f) + \lambda X(g)$
- $X(fg) = X(f)g(p) + f(p)X(g)$

Definition 2.9. In coordinates $x_1, \ldots, x_n$, a **tangent vector** is something of the form

$$a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

that in different coordinates transforms via the chain rule.

The space of tangent vectors at p is called the **tangent space** and is denoted by T_pM. There can be given a vector bundle structure on $\bigcup_{p \in M} T_pM$ and this is called the **tangent bundle** and is denoted TM.

3 January 27, 2017

Today is going to be mostly about topology, fundamental groups and covering spaces.

3.1 Submanifolds

There are two notions: immersed manifolds and embedded manifolds.

Definition 3.1. An **immersion** is a smooth map $i : M \to N$ such that Di is injective at every point.

For example, the map $\mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2$ given by $x \mapsto (x, ax)$ is an immersion. If a is irrational, then the image is dense.

Definition 3.2. An **embedding** is an immersion that is an homeomorphism onto its image. In this case, the image is called a **submanifold**.

Definition 3.3. A **closed Lie subgroup** is a subgroup which is an embedded submanifold.

A subgroup which is also an embedded manifold is always closed. This is an exercise. Note that embedded manifolds are not necessarily closed: a open interval in the plane is not closed.

Theorem 3.4. *If G is a Lie group and $H \subseteq G$ is a closed subgroup, then it is a closed Lie subgroup.*

3.2 Fundamental groups and covering spaces

Definition 3.5. For a topological space with basepoint (X, x_0), its **fundamental group** is defined as

$$\pi_1(X, x_0) = \{\text{paths } x_0 \to x_0\}/\text{homotopy},$$

with multiplication being concatenation.

The fundamental group of any contractible space is trivial. $\pi_1(S^1) = \mathbb{Z}$). For $n \geq 2$, $\pi_1(S^n) = 1$ and $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$.

A good way to compute π_1 is to use the cell complex structure on X. The generators will be all the 1-cells and relations will come from the 2-cells.

The interesting thing about Lie groups is that they have abelian fundamental groups. How do I show that $\pi_1(G, 1)$ is abelian? For $\alpha, \beta \in \pi_1(G, 1)$ we want to show that $\alpha \circ \beta$ is homotopic to $\beta \circ \alpha$. Consider a map

$$I^2 \to G; \quad (s, t) \mapsto \alpha(s)\beta(t).$$

Then this square gives a homotopy between $\alpha\beta$ and $\beta\alpha$. This is not very specific to Lie groups. If we have some kind of multiplication, i.e., for H-spaces, this argument works.

Definition 3.6. A topological space is **simply connected** if it is path connected and has trivial fundamental group.

Definition 3.7. A **covering** of X is a fiber bundle $F \to E \to X$ such that F is discrete (0-dimensional).

Example 3.8. Let

$$\mathrm{Sp}(1) = \{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\} \cong S^3 = \mathrm{SU}(2).$$

This acts on the subspace $\mathbb{R}^3 = \{bi + cj + dk\} = \mathrm{Lie}(\mathfrak{sp}(1))$ by conjugation. $\mathrm{Sp}(1)$ acts by isometries, and so we get a map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$. It is an exercise to show that this map is onto and has kernel $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. So

$$\mathrm{SO}(3) = \mathrm{Sp}(1)/\{\pm 1\} = \mathrm{SU}(2)/\{\pm 1\} \cong \mathbb{R}P^3.$$

This gives another description $\mathrm{SU}(2) = \mathrm{Spin}(3)$. Also we see that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$. There is a Dirac belt trick that demonstrates this fact.

Let X be a "nice" topological space (for example cell complexes or manifolds). Also assume that it is path connected. Any covering map $p : Y \to X$ gives an injective group homomorphism

$$p_* : \pi_1(Y) \to \pi_1(X).$$

This gives a correspondence

$$\left\{\begin{matrix}\text{connected}\\ \text{coverings of } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups of } \pi_1(X)\end{matrix}\right\}.$$

The **universal covering** $\tilde{X}$ of X is the one corresponding to the trivial subgroup $\{1\} \subseteq \pi_1(X)$. Explicitly it is constructed as

$$\tilde{X} = \{(x, \alpha) : x \in X, \alpha \text{ a homotopy class of paths } x_0 \to x\}.$$

Note that $\pi_1(X, x_0)$ acts on $\tilde{X}$. So for a subgroup $N \subseteq \pi_1(X)$ we get a covering $\tilde{X}/N \to X$.

There is a lifting property. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. The it uniquely lifts to the universal covers:

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[r, dashed, "\tilde{f}"] \arrow[d] & (\tilde{Y}, \tilde{y}_0) \arrow[d] \\
(X, x_0) \arrow[r, "f"] & (Y, y_0)
\end{tikzcd}$$

4 January 30, 2017

The first problem set is problems 2.1, 2.2, 2.3, 2.5, 2.11 from the textbook.

4.1 Universal covers and connected components of Lie groups

Theorem 4.1. *Let G be a connected Lie group. Then the universal cover $\tilde{G}$ has a canonical structure of a Lie group, such that $p : \tilde{G} \to G$ is a morphism of Lie groups.*

Proof. There is going to be a basepoint $\tilde{1} \in \tilde{G}$ that projects to $1 \in G$. To get multiplication, we consider the lifting

$$\begin{tikzcd} \widetilde{G \times G} \cong \tilde{G} \times \tilde{G} \arrow[r, dashed] \arrow[d] & \tilde{G} \arrow[d] \\ G \times G \arrow[r] & G \end{tikzcd}$$

which gives multiplication. To get associativity, you can use uniqueness. Likewise lift inversion to get inversion. These maps are C^∞ because $\tilde{G}$ is locally homeomorphic to G and so you can use local charts. □

Example 4.2. The covering $\mathbb{R} \to S^1$ gives a group structure on $\mathbb{R}$. There is a natural Lie group structure on $\mathrm{Spin}(n) = \widetilde{\mathrm{SO}(n)}$.

For the universal covering $p : \tilde{G} \to G$, its kernel $\ker p$ is a discrete central subgroup. (One of the exercises is going to prove that this is central.) This gives an exact sequence

$$1 \longrightarrow \ker(p) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Theorem 4.3. *Let G be a Lie group and G^0 be the connected component of $1 \in G$. Then $G^0 \subseteq G$ is a normal subgroup, and G/G^0 is discrete. So we get another exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G/G^0 \longrightarrow 1.$$

Proof. This is because $1 \cdot 1 = 1$ and so $G^0 \times G^0 \to G^0$ by connectedness. Normality also follows similarly. □

We have stated the following theorem last time.

Theorem 4.4. *A closed Lie subgroup $H \subseteq G$ is a closed subgroup.*

Corollary 4.5. *(1) If G is connected and $U \subseteq G$ is a neighborhood of $1 \in G$, then U generates G.*
(2) If $f : G \to H$ is a map of Lie groups, $f_* : T_1G \to T_1H$ is surjective, and H is connected, then f is surjective.

Proof. (1) Let $H \subseteq G$ be the subgroup generated by U. Then $H \subseteq G$ is open because $gU \subseteq H$ for all $g \in H$. Then it is automatically a closed Lie subgroup (i.e., embedded submanifold) and hence closed. Then H is closed and open and G is connected, so $H = G$.

(2) Because $f_* : T_1G \to T_1H$ is surjective, implicit function theorem tells us that $\mathrm{im}(f)$ contains a neighborhood of $1 \in H$. So by (1), $\mathrm{im}(f) = H$. $\square$

4.2 Fiber bundle from a map of Lie groups

Theorem 4.6. *If G is a Lie group and $H \subseteq G$ is a closed Lie subgroup, then G/H has a canonical structure of a smooth manifold, such that we get a fiber bundle*

$$H \longrightarrow G \xrightarrow{p} G/H.$$

Moreover $T_H(G/H) = T_1G/T_1H$. If H is a normal subgroup, then G/H is going to be a Lie group.

Proof. It suffices to choose a local section, because we can then construct a local trivialization by multiplying H. Choose a submanifold $M \subseteq G$ with transverse intersection to $g \cdot H$ at g: $T_gG = T_g(gH) \oplus T_gM$. Then we get a map $M \times H \to G$ which is a diffeomorphism in a neighborhood of $\{g\} \times H$. Set U to be the neighborhood of $g \in M$ such that $U \times H \to G$ is a embedding. We can then use $\tilde{U} = p(U) = p(UH)$ as charts. These are also local trivializations.

$$\begin{tikzcd} UH \arrow[d, "p"'] & U \times H \arrow[l] \arrow[dl] \\ \tilde{U} & \end{tikzcd}$$

You can check that the charts are compatible and thus the smooth structure on G/H is independent of the choice of g and M. It is also clear that $\ker(p_* : T_gG \to T_{\tilde{g}}(G/H)) = T_gH$. $\square$

Corollary 4.7. *(1) If $H \subseteq G$ is a connected subgroup then $\pi_0(G/H) = \pi_0(G)$.*
(2) If $H \to G \to G/H$ is a fiber bundle, and G and H are connected, then we get an exact sequence of group

$$\pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow 1.$$

5 February 1, 2017

Lie groups appear as symmetries of spaces.

5.1 Group actions

Definition 5.1. An **action** of a Lie group G on a manifold M is a group homomorphism $\phi : G \to \mathrm{Diff}(M)$ such that the induced map $G \times M \to M$ is smooth.

Example 5.2. The group $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$. This is even a linear representation. The group $\mathrm{SO}(n, \mathbb{R})$ acts on $S^{n-1} \subseteq \mathbb{R}^n$, and the complex version $\mathrm{SU}(n)$ acts on S^{2n-1}.

Definition 5.3. A **representation** of a Lie group G on a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. In the case of $\dim V < \infty$, we also require $G \times V \to V$ be smooth.

Definition 5.4. A **morphism** between representations $(V, \rho_V) \to (W, \rho_W)$ is a linear map $f : V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "f"] & V \arrow[d, "f"] \\ W \arrow[r, "\rho_W(g)"] & W \end{tikzcd}$$

commutes for all $g \in G$.

Definition 5.5. If G acts on M, then the **orbit** of m is the set $G \cdot m = \{gm : g \in G\}$. The **stabilizer** of m is defined as

$$G_m = \mathrm{Stab}_G(m) = \{g \in G : gm = m\}.$$

Note that if m and m' are in the same orbit, then $\mathrm{Stab}(m)$ and $\mathrm{Stab}(m')$ are conjugates.

Theorem 5.6. *(1)* $\mathrm{Stab}_G(m) \subseteq G$ *is a closed Lie subgroup.*
(2) We get an injective immersion $G/\mathrm{Stab}_G(m) \hookrightarrow M$*.*

We will prove this later using the technology of Lie algebras.

Definition 5.7. A **G-homogeneous space** is a manifold M with a transitive action of G.

If we fix a point $m \in M$ then we can identify $M = G/\mathrm{Stab}_G(m)$. This will be a diffeomorphism. We can also say that $G \to M$ is a fiber bundle with fiber $\mathrm{Stab}_G(m)$.

Example 5.8. We have $\mathrm{SO}(n,\mathbb{R})$ acting on S^{n-1}. So we have a fiber bundle

$$\mathrm{SO}(n-1,\mathbb{R}) \to \mathrm{SO}(n,\mathbb{R}) \to S^{n-1}.$$

Likewise there is an action $\mathrm{SU}(n,\mathbb{R})$ on S^{2n-1} and so

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}.$$

Example 5.9. A **flag** in $\mathbb{R}^n$ is a sequence of subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{R}^n$$

with $\dim V_k = k$. Let $\mathcal{F}_n(\mathbb{R})$ be the set of all flags in $\mathbb{R}^n$. Then $\mathrm{GL}(n,\mathbb{R})$ acts transitively, and the stabilizer of the standard flag is $B(n,\mathbb{R})$, the space of invertible upper triangular matrices.

The quotient space M/G can be very pathological, even non-Hausdorff. For example, let $G = \mathrm{GL}(n,\mathbb{C})$ act on $\mathrm{Mat}(n,\mathbb{C})$ by conjugation. Then M/G is the set of Jordan normal forms. There are some fields devoted to these pathological quotients like geometric invariance theory or non-commutative geometry.

Any Lie group G acts on itself in 3 canonical way:

$$\begin{array}{lrl} \text{Left action} & L_g : G \to G, & h \mapsto gh, \\ \text{Right action} & R_g : G \to G, & h \mapsto hg^{-1}, \\ \text{Adjoint action} & \mathrm{Ad}_g : G \to G, & h \mapsto ghg^{-1}. \end{array}$$

The adjoint action always fixed $1 \in G$. So we get an induced action $(\mathrm{Ad}_g)_* : T_1G \to T_1G$. This already gives a representation of G on T_1G. But it doesn't need to be faithful. In fact, there are Lie groups that cannot be embedded into $\mathrm{GL}(n,\mathbb{R})$. For example, $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$ does not have a faithful finite-dimensional representation.

6 February 3, 2017

6.1 Classical groups

We will look at some examples of groups.

- The **general linear group** $\mathrm{GL}(n, \mathbb{K})$, the group of invertible linear transformations, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.
- The **special linear group** $\mathrm{SL}(n, \mathbb{K})$, the transformations that preserve the volume form.
- The **orthogonal group** $\mathrm{O}(n, \mathbb{K})$, the transformations preserving the usual bilinear inner product.
- The **special orthogonal group** $\mathrm{SO}(n, \mathbb{K}) = \mathrm{O}(n, \mathbb{K}) \cap \mathrm{SL}(n, \mathbb{K})$, transformations preserving the volume and the inner product.
- More generally the groups $\mathrm{O}(p, q)$ and $\mathrm{SO}(p, q)$ that preserve the non-degenerate quadratic form of signature (p, q) (over $\mathbb{R}$).
- The **symplectic group** $\mathrm{Sp}(n, \mathbb{K}) = \{A \in \mathrm{GL}(2n, \mathbb{K}) : \omega(X, Y) = \omega(A(X), A(Y))\}$, transformations preserving the symplectic form, a non-degenerate skew-symmetric form.
- The **unitary group** $\mathrm{U}(n)$, transformations preserving the hermitian inner product (over $\mathbb{C}$).
- The **special unitary group** $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$, transformations preserving the hermitian inner product and have determinant 1.
- The **compact symplectic group** $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n, \mathbb{C})$.

One surprising coincidence is that $\mathrm{SO}^+(1, 3) = \mathrm{PSL}(2, \mathbb{C})$. This is useful in physics.

G	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{C})$	$\mathrm{Sp}(n,\mathbb{C})$
$\mathfrak{g}$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathrm{tr} = 0$	$X + X^t = 0$	$X + X^t = 0$	$JX + X^t J = 0$
$\dim_{\mathbb{C}}$	n^2	$n^2 - 1$	$n(n-1)/2$	$n(n-1)/2$	$n(2n+1)$
π_0	1	1	$\mathbb{Z}/2$	1	1
π_1	$\mathbb{Z}$	1	$\mathbb{Z}/2$ $(n \geq 3)$	$\mathbb{Z}/2$ $(n \geq 3)$	1
G	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{O}(n,\mathbb{R})$	$\mathrm{SO}(n,\mathbb{R})$	$\mathrm{Sp}(n)$
$\mathfrak{g}$	$X + X^* = 0$	$X + X^* = 0$ $\mathrm{tr}\, X = 0$	$X = \overline{X}$ $X^t + X = 0$	$X = \overline{X}$ $X^t + X = 0$	$X + X^* = 0$ $JX + X^t J = 0$
G	$\mathrm{Sp}(n,\mathbb{R})$		$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{SL}(n,\mathbb{R})$	

Table 1: Complex groups and their maximal compact subgroups

6.2 The exponential map

For a complex matrix $A \in \mathrm{Mat}(n, \mathbb{C})$, we define the **exponential map** $\exp : \mathfrak{gl}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ as

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

This converges absolutely and is defined holomorphically. The map has a partial inverse near the identity given by

$$\log(1 + A) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^n}{n}.$$

This converges only for A close to 0.

Theorem 6.1. *(1)* $\exp(\log A) = A$ *and* $\log(\exp A) = A$ *when they are defined.*
(2) $\exp(0) = 1$ *and* $\exp_*(0) = \mathrm{id}_{\mathfrak{gl}(n,\mathbb{K})}$.
(3) $XY = YX$ *implies* $e^X e^Y = e^{X+Y}$ *and* $\log(XY) = \log X + \log Y$.
(4) For $X \in \mathfrak{gl}(n, \mathbb{K})$ *the map* $\mathbb{K} \to \mathrm{GL}(n, \mathbb{K})$ *given by* $t \mapsto e^{tX}$ *is a homomorphism of Lie groups.*
(5) $e^{AXA^{-1}} = Ae^X A^{-1}$ *and* $e^{A^t} = (e^A)^t$.

Proof. (1) You can check this at the level of power series. This follows from the 1-variable case.

(2) Clear.

(3) Again check this for the power series in 2 variables.

(4) This follows from (3).

(5) Plug in the definition of as the power series. Same for the transpose. □

These maps give local charts for the classical groups we have defined above.

Theorem 6.2. *If* $G \subseteq \mathrm{GL}(n, \mathbb{K})$ *is a classical group, then there is a subspace* $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$*, a neighborhood* U *of* $0 \in \mathfrak{gl}(n, \mathbb{K})$*, a neighborhood* V *of* $1 \in \mathrm{GL}(n, \mathbb{K})$ *such that* exp *and* log *give diffeomorphisms on* $G \cap V$ *and* $\mathfrak{g} \cap U$.

Proof. We have to go through that list and identify what the Lie algebra is.

For $\mathrm{GL}(n, \mathbb{K})$ we have $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$ and the theorem follows from the inverse function theorem.

For $\mathrm{SL}(n, \mathbb{K})$, we have $A = e^X \in \mathrm{SL}(n, \mathbb{K})$ near 1 and want to figure out what X is. We have $1 = \det A = \det e^X = \exp(\operatorname{tr} X)$. So $\operatorname{tr} X = 0$. This shows that $\mathfrak{g}$ is the space of traceless matrices.

For $\mathrm{O}(n, \mathbb{K})$, we want $A^t A = 1$ so $e^{X+X^t} = 1$. This shows that $\mathfrak{g}$ is the skew-symmetric matrices.

The group $\mathrm{SO}(n, \mathbb{K})$ is a connected component so it has the same $\mathfrak{g}$.

The group $\mathrm{U}(n)$ has the algebra $\mathfrak{u}(n) = \{X + X^* = 0\}$ and $\mathfrak{su}(n)$ has another traceless condition.

The group $\mathrm{Sp}(n, \mathbb{K})$ has algebra $\mathfrak{sp}(n, \mathbb{K}) = \{JX + X^t J = 0\}$ and for the compact symplectic group you put the two conditions together. □

Corollary 6.3. *Each classical group G is a Lie group with $T_1G = \mathfrak{g}$. So $\dim G = \dim \mathfrak{g}$.*

Proof. We get a chart on G near 1. Then using left multiplication by g we get a chart on G near g. The reason $T_1G = \mathfrak{g}$ is because $\exp(0) = 1$ and $\exp_*(0) = \mathrm{id}$. $\square$

7 February 6, 2017

The second problem set due next Monday is: 2.7, 2.13, 2.14, 2.15, 3.1.

7.1 The exponential map

We want to define $\exp : \mathfrak{g} \to G$ for general Lie groups, not only for matrix groups.

Proposition 7.1. *Let G be a Lie group and $x \in \mathfrak{g} = T_1 G$. Then there is a unique map of Lie groups $\gamma_x : \mathbb{K} \to G$ such that $\dot{\gamma}_x(0) = x$. This is called the **one-parameter subgroup** corresponding to x.*

Proof. We want $\gamma(s+t) = \gamma(s)\gamma(t)$. That is, we want $\dot{\gamma}(t) = \gamma(t)\dot{\gamma}(0) = \dot{\gamma}(0)\gamma(t)$. Then γ_x solves the initial value problem $\dot{\gamma} = \gamma x$ and $\gamma(0) = 1$. (In other words, γ is the trajectory of left-invariant vector fields.) By general theory of uniqueness and existence of solutions to ODEs, we see that γ_x is unique on some small t. Then extend the solution to the whole space via the homomorphism property. □

Definition 7.2. Define the **exponential map** $\exp : \mathfrak{g} \to G$ by $\exp(x) = \gamma_x(1)$.

From uniqueness, we have $\gamma_x(t) = \gamma_{tx}(1) = \exp(tx)$. For $G = \mathrm{GL}(n, \mathbb{K})$, exp coincides with the matrix exponential.

Theorem 7.3. *(1)* $\exp(0) = 1$, $\exp_*(0) = \mathrm{id}_g$.
(2) exp *is* C^∞ *(analytic in the case of complex groups), is a diffeomorphism from a neighborhood of* $0 \in \mathfrak{g}$ *to a neighborhood of* $1 \in G$.
(3) $\exp((s+t)x) = \exp(tx)\exp(sx)$ *for* $s, t \in \mathbb{R}$.
(4) exp *is natural: if* $\phi : G \to H$ *induces* $\phi_* : \mathfrak{g} \to \mathfrak{h}$*, then*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi_*"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

commutes.
(5) If $g \in G$ *and* $x \in \mathfrak{g}$ *then* $g\exp(x)g^{-1} = \exp(\mathrm{Ad}(g)x)$*, a special case of (4).*

Proof. (1) Clear from the definition.
(2) C^∞ follows from the smooth dependence of solutions to ODEs.
(3) This also follows from the definition.
(4) Uniqueness of the exp map shows that the two things are the same. $\phi(\exp(tx))$ is a one-parameter subgroup with derivative at 0 being $\phi_* x$. □

The map $\exp : \mathfrak{g} \to G$ has image contained in $G^0 \subseteq G$. A natural question is whether $\operatorname{im}\exp = G^0$. In general it is not. (One of the homework gives you an explicit counterexample.) We have $\operatorname{im}(\exp) = G^0$ if

(1) $G = \mathrm{GL}(n, \mathbb{C})$,

(2) G is compact, (in this case, you can give a Riemannian metric and the theory of exp in Riemannian manifolds show that exp is surjective.)

(3) G is nilpotent.

Proposition 7.4. *If G, H are Lie groups with G connected, then $\phi : G \to H$ is uniquely determined by $\phi_* : \mathfrak{g} \to \mathfrak{h}$.*

Proof. The image of exp contains a neighborhood of G and this neighborhood generates G. ☐

7.2 Defining the Lie bracket

Our next goal is to define $[-,-]$ on $\mathfrak{g}$ in general. For matrix groups, this is going to be the usual matrix commutator.

Definition 7.5. Define a map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ locally near 0 given by $\exp(x)\exp(y) = \exp(\mu(x, y))$ for x, y in some small neighborhood of 0.

Lemma 7.6. *The Taylor series for μ is given by*

$$\mu(x, y) = x + y + \lambda(x, y) + \textit{terms of order } \geq 3,$$

with λ a bilinear skew-symmetric map.

Proof. Let us write

$$\mu(x, y) = \alpha_1(x) + \alpha_2(y) + Q_1(x) + Q_2(y) + \lambda(x, y) + \cdots,$$

where α are linear and Q are quadratic. Computing $\mu(x, 0)$ yields $\alpha_1(x) = x$ and $Q_1 = 0$. Computing $\mu(0, y)$ yields $\alpha_2(y) = y$ and $Q_2 = 0$. Because $\exp(x)\exp(x) = \exp(2x)$, we have $\lambda(x, x) = 0$. This means that λ is skew-symmetric. ☐

Definition 7.7. Define the **Lie bracket** as $[x, y] = 2\lambda(x, y)$.

8 February 8, 2017

There was a function $\mu(x, y)$ defined by

$$\exp(x)\exp(y) = \exp(\mu(x, y))$$

for x, y small, and found out that

$$\mu(x, y) = x + y + \frac{1}{2}[x, y] + \text{higher order}.$$

where $[x, y]$ is bilinear and skew-symmetric.

Proposition 8.1. *(1) If $\phi : G \to H$ induces $\phi_* : \mathfrak{g} \to \mathfrak{h}$, then $\phi_*[x, y] = [\phi_* x, \phi_* y]$.*
(2) In particular, $\mathrm{Ad}(g)[x, y] = [\mathrm{Ad}(g)x, \mathrm{Ad}(g)y]$.
(3) $\exp(x)\exp(y)\exp(x)^{-1}\exp(y)^{-1} = \exp([x, y] + \cdots)$.

Because of (3), we can think the Lie bracket as the commutator.

Proof. (1) $[-, -]$ is defined in terms of exp, and exp is natural.
(3) This follows from direct computation. □

Corollary 8.2. *If G is commutative, then $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.*

8.1 The Jacobi identity

We have an adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. So taking the derivative we get $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Lemma 8.3. *(1)* $\mathrm{ad}(x)y = [x, y]$.
(2) $\mathrm{Ad}(\exp x) = \exp(\mathrm{ad}\, x)$.

Proof. (1) By definition, $\mathrm{Ad}(g)y = (d/dt)|_{t=0}\, g\exp(ty)g^{-1}$. So

$$\begin{aligned}\mathrm{ad}(x)y &= \frac{d}{ds}\frac{d}{dt}\Big|_{t=s=0} \exp(sx)\exp(ty)\exp(-sx) \\ &= \frac{d}{dx}\frac{d}{dt}\Big|_{t=s=0} \exp(t)\exp(st[x, y] + \cdots) = [x, y].\end{aligned}$$

(2) is clear. □

Theorem 8.4. *If G is a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$, then the bracket $[-, -]$ satisfies the Jacobi identity:*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

i.e., $\mathfrak{g}$ is a Lie algebra.

Proof. Note that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ preserves $[-, -]$. So

$$\mathrm{ad}[x, y] = [\mathrm{ad}\, x, \mathrm{ad}\, y] = \mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g}).$$

Acting on z, we get $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$. □

So we get a functor from the category of Lie groups to the category of Lie algebras.

8.2 Sub-Lie algebras

Definition 8.5. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a **sub-Lie algebra** if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an **ideal** if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

If $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra with induced bracket.

Theorem 8.6. *(1) Let $H \subseteq G$ is a Lie subgroup. Then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra. (2) If $H \subseteq G$ is a closed normal Lie subgroup, then $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, with* $\mathrm{Lie}(G/H) = \mathfrak{g}/\mathfrak{h}$. *Conversely, if $H \subseteq G$ is a closed Lie subgroup, and G and H are connected, then $\mathfrak{h} \subseteq \mathfrak{g}$ being an ideal implies $H \subseteq G$ normal.*

Proof. (1) This just follows from the naturality of the Lie bracket.

(2) A subgroup $H \subseteq G$ is a closed normal subgroup if and only if $\mathrm{Ad}(g)$ preserves H for any $g \in G$. This is equivalent to $\mathrm{Ad}(g)$ preserving $\mathfrak{h}$ for any $\mathfrak{g}$, because H is connected. This is equivalent to $\mathrm{ad}(x)$ preserving $\mathfrak{h}$ for any $x \in \mathfrak{g}$. This just means $\mathfrak{h}$ is and ideal. $\square$

8.3 Lie algebra of vector fields

This doesn't really fit into what we have done, but it is interesting. For a manifold M, take

$$G = \mathrm{Diff}(M) = \{\text{group of diffeomorphisms}\}.$$

Then we can think its Lie algebra as

$$\mathfrak{g} = \{\text{vector fields on } M\} = \Gamma(M, TM).$$

Then a 1-parameter subgroup corresponding to $X \in \mathfrak{g}$ is the vector flow generated by the field. This is the exponential map. If M is non-compact, the flow is only defined in a neighborhood of $M \times \{0\}$.

We can also define the Lie bracket as

$$[X, Y](f) = Y(X(f)) - X(Y(f))$$

like we do in differential geometry. (There is this sign issue, but Kirillov convinces you why the sign should be changed.)

9 February 10, 2017

For vector fields X, Y on M, we have the Lie bracket

$$[X,Y]f = Y(X(f)) - X(Y(f)).$$

In coordinates, this can be written as

$$\left[\sum_i f_i \frac{\partial}{\partial x_i}, \sum_i g_i \frac{\partial}{\partial x_i}\right] = \sum_{i,j} \Big(g_i \frac{\partial f_j}{\partial x_i} - f_i \frac{\partial g_j}{\partial x_i} \Big) \frac{\partial}{\partial x_j}.$$

Equivalently, $\Phi_X^t \circ \Phi_Y^s \circ \Phi_X^{-t} \circ \Phi_Y^{-s} = \Phi_{[X,Y]}^{st} +$ higher order terms.

Theorem 9.1. *For an action $\rho : G \to \mathrm{Diff}(M)$ of an Lie group on a manifold, there is a map $\rho_* : \mathfrak{g} \to \mathrm{Vect}(M)$ of Lie algebras.*

Proof. This map can be defined using the diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, dashed, "\rho_*"] \arrow[d, "\exp"] & \mathrm{Vect}(M) \arrow[d, "\exp"] \\
G \arrow[r, "\rho"] & \mathrm{Diff}(M)
\end{tikzcd}$$

This is compatible with the Lie bracket because ρ preserves the commutator in groups, and the Lie bracket is determined using the commutator. ☐

Example 9.2. $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$. So any $A \in \mathfrak{gl}(n, \mathbb{R})$ gives a vector field $\rho_*(A)$ given by $A : \mathbb{R}^n \to T_p\mathbb{R}^n = \mathbb{R}^n$.

Example 9.3. Let G act on G by left multiplication. Then we get a map $\rho_* : \mathfrak{g} \to \mathrm{Vect}(G)$ given by sending a vector to the associated right-invariant vector field. So we get an isomorphism

$$\mathfrak{g} \longrightarrow \{\text{right-invariant vector fields}\}$$

of Lie algebras.

9.1 Stabilizer of a Lie group action

Theorem 9.4. *Let a Lie group G act on M, and let $m \in M$ be a point.*
(1) $\mathrm{Stab}_G(m) \subseteq G$ is a closed Lie subgroup.
(2) The map $G/\mathrm{Stab}_G(m) \hookrightarrow M$ is an immersion.

Proof. (1) Define

$$\mathfrak{h} = \{x \in \mathfrak{g} : \rho_x(x)(m) = 0\} = \ker(\mathfrak{g} \to \mathrm{Vect}(M) \to T_mM).$$

The formula for $[-,-]$ shows that $\mathfrak{h}$ is a Lie algebra.

To show that $\mathrm{Stab}_G(m)$ is a closed Lie subgroup it suffices to find a neighborhood $U \subseteq G$ of 1 such that $\mathrm{Stab}_G(m) \cap U$ is a smooth manifold. This is because you can use multiplication to move charts around.

Take a vector space $\mathfrak{u}$ such that $\mathfrak{h} \oplus \mathfrak{u} = \mathfrak{g}$. By definition, $\rho_* : \mathfrak{u} \to T_m M$ is now an isomorphism of vector spaces. Then the map

$$\mathfrak{u} \to M; \quad x \mapsto (\rho \exp(x))(m)$$

is injective locally at the origin by the implicit function theorem, because $\rho \exp(x) \in \mathrm{Stab}_G(m)$ is equivalent to $x \in \mathfrak{h}$ locally.

Any $g \in G$ near 1 can be written uniquely as $g = \exp(y)\exp(x)$ for $x \in \mathfrak{h}$ and $y \in \mathfrak{u}$ by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$. Then $gm = \exp(y)m$ and so $g \in \mathrm{Stab}_G(m)$ if and only if $y = 0$ locally. This shows that $\exp : \mathfrak{h} \to \mathrm{Stab}_G(m)$ is a bijection near $0 \in \mathfrak{h}$.

(2) We already know $T_1(G/\mathrm{Stab}_G(m)) = \mathfrak{u}$ and this injects into $T_m M$. So we get an immersion. $\square$

Corollary 9.5. *If $f : G \to H$ is a map of Lie groups,*
(1) $\ker(f) \subseteq G$ is a closed Lie subgroup.
(2) $\mathrm{im}(f) \subseteq H$ is an immersed subgroup.

Proof. We let G act on H by $g(h) = f(g)h$. $\square$

Example 9.6. Let V be a finite dimensional vector space and $B : V \times V \to \mathbb{K}$ be a nondegenerate bilinear form. We have an action $\mathrm{GL}(V)$ on $\mathrm{Hom}(V \otimes V, \mathbb{K})$. Then the stabilizer of B, $\mathrm{O}(V, B)$, is a closed subgroup of $\mathrm{GL}(V)$. The Lie algebra is

$$\mathfrak{o}(V, B) = \{A \in \mathfrak{gl}(V) : B(Av, w) + B(v, Aw) = 0\}.$$

10 February 13, 2017

Example 10.1. Let A be a finite dimensional (possibly nonassociative) algebra over $\mathbb{K}$. So there is a multiplication $\mu \in \operatorname{Hom}(A \otimes A, A)$. The group $\mathrm{GL}(A)$ acts on A, so it acts on the space $\operatorname{Hom}(A \otimes A, A)$. So $M/\mathrm{GL}(A)$ classifies the algebra structures on A up to isomorphism. The stabilizer of μ will be

$$\mathrm{Stab}_{\mathrm{GL}(A)}(\mu) = \{\phi \in \mathrm{GL}(A) : \phi(ab) = \phi(a)\phi(b)\} = \mathrm{Aut}(A).$$

Its Lie algebra is

$$\{D \in \mathfrak{gl}(A) : D(ab) = D(a)b + aD(b)\} = \mathrm{Der}(A).$$

We can do this for $A = \mathfrak{g}$ a Lie algebra.

Definition 10.2. The **center** of a Lie algebra $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

Theorem 10.3. *If a Lie group G is connected, then $Z(G) \subseteq G$ is a closed subgroup and $Z(\mathfrak{g}) = \mathrm{Lie}(Z(G))$.*

The proof is similar.

10.1 Baker–Campbell–Hausdorff formula

We know that $e^{X+Y} = e^X e^Y$ for matrices X, Y if $[X, Y] = 0$.

Theorem 10.4. *Let G be a Lie group and $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$. Then $\exp(X)\exp(Y) = \exp(X + Y)$.*

Theorem 10.5 (Baker–Campbell–Hausdorff formula)**.** *Let G be a Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. For sufficiently small $x, y \in \mathfrak{g}$,*

$$\begin{aligned}\log(\exp(x)\exp(y)) &= x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots \\ &= \sum_{n=1}^{\infty} \mu_n(x, y),\end{aligned}$$

where μ_n is a Lie polynomial over $\mathbb{Q}$, homogeneous of degree n in x, y.

The useful part of this theorem is that there exists a formula. Then the Lie bracket locally determines multiplication near the identity. Or in special cases, all except for finite terms disappear and so it becomes a useful formula.

We are not going to prove this. There are several proofs. The first one uses an analytic approach. You first prove

$$\log(e^x e^y) = x + \int_0^1 \psi(e^{\mathrm{ad}(x)} e^{t\,\mathrm{ad}(y)})dt, \qquad \text{where } \psi(z) = \frac{z \log z}{z - 1}.$$

Alternatively, you can look at the completion of the free nonassociative algebra $\mathbb{K}\langle\langle x, y\rangle\rangle$ with the coproduct map $\Delta : A \to A \otimes A$ given by $\Delta(ab) = \Delta(a)\Delta(b)$ and $\Delta(x) = 1 \otimes x + x \otimes 1$. You can define the formal exp and log,

$$\{ax + by + \cdots\} \underset{\log}{\overset{\exp}{\rightleftarrows}} \{1 + zx + by + \cdots\},$$

which is actually a bijection. The free Lie algebra generated by x and y turns out to be the set of primitive elements, $\{\Delta(a) = 1 \otimes a + a \otimes 1\}$, and the image of this set under exp is the set of group-like elements, $\{\Delta(a) = a \otimes a\}$. So for x, y primitive, $\exp(x)$ and $\exp(y)$ are group-like. Then $\exp(x)\exp(y)$ is group-like, so $\log(\exp(x)\exp(y))$ is primitive.

Corollary 10.6. *Suppose G is connected. Then the product on G is uniquely determined by* $\mathrm{Lie}(G)$*.*

10.2 Recovering the Lie group from its algebra

So far, we have:

(1) We have a functor $\{\text{Lie groups}/\mathbb{K}\} \to \{\text{Lie algebras}/\mathbb{K}\}$. If G is connected, then $\mathrm{Hom}(G, H) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ is injective.

(2) We can recover multiplication on a connected G from $\mathfrak{g}$.

Theorem 10.7. *(1) If G is a Lie group, then there is a one-to-one correspondence*

$$\left\{\begin{matrix} \textit{connected} \\ \textit{subgroups } H \subseteq G \end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} \textit{subalgebras} \\ \mathfrak{h} \subseteq \mathfrak{g} \end{matrix}\right\}.$$

(2) If G is simply connected, then

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}), \quad \phi \mapsto \phi_*$$

is a bijection.
(3) If $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{K}$, then there is a corresponding Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$*.*

11 February 15, 2017

The third problem set is 3.5, 3.6, 3.7, 3.9, 3.13 due Wednesday, February 22.

Theorem 11.1. *(1) If G is any Lie group over $\mathbb{K}$, there is a correspondence*

$$\left\{\begin{matrix} H \subseteq G \\ \textit{connected} \end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} \mathfrak{h} \subseteq \mathfrak{g} \\ \textit{subalgebras} \end{matrix}\right\}.$$

(2) If G, H are Lie groups, and G is simply connected, then

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}); \quad \phi \mapsto \phi_*$$

is a bijection.
(3) If $\mathfrak{g}$ is a finite dimensional Lie algebra, then there is a Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$.

Proof. (3) By Ado's theorem, which we haven't proved, we have $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$ for some n. Then (1) shows that $\mathfrak{g}$ corresponds to some $G \subseteq \mathrm{GL}(n, \mathbb{K})$ with $\mathrm{Lie}(G) = \mathfrak{g}$.

(2) Injectivity is clear, because we can use the fact that exp and ϕ commute, and G is generated by a neighborhood of 1.

Now suppose G_1, G_2 be Lie groups and $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a morphism of Lie algebras. We want to construct a map $\phi : G_1 \to G_2$. Take the Lie group $G = G_1 \times G_2$ and let

$$\mathfrak{h} = (\text{graph of } f) = \{(x, f(x)) : x \in \mathfrak{g}_1\} \subseteq \mathfrak{g}_1 \times \mathfrak{g}_2 = \mathfrak{g}.$$

Then (1) tells us that there is a corresponding connected subgroup $H \subseteq G_1 \times G_2$. The composite of $H \hookrightarrow G_1 \times G_2 \to G_1$ induces an isomorphism of Lie algebras, and so it must be a covering map. But G_1 is simply connected, and thus this is an isomorphism. Now let us look at

$$G_1 \to H \hookrightarrow G_1 \times G_2 \to G_2.$$

This gives the map, inducing $\phi_* = f$.

(1) In the case $\dim \mathfrak{h} = 1$, we can produce $H = \exp(\mathfrak{h}) \subseteq G$ easily. But if $\dim \mathfrak{h} > 1$, then $\exp(\mathfrak{h})$ is not a subgroup in general. Consider the subbundle $E \subseteq TG$ with $E_g = \mathfrak{h}g \subseteq T_g G$. (A subbundle of TM is also called a **distribution**.) Then because $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, for any vector fields $X, Y \in \mathrm{Vect}(G)$ with $X(p), Y(p) \in E_p$ for all p, we have $[X, Y](p) \in E_p$. You can check this by Leibniz's rule. (This condition is called **integrability**.) Frobenius' theorem then says that there is a maximal connected submanifold $H \subseteq G$ such that $T_p H = E_p$ for all $p \in H$.

By maximality, we see that for any $x \in \mathfrak{h}$, the curve e^{tx} is always in H. Then $\exp(\mathfrak{h}) \subseteq H$. Since $1 \in H^{-1}$, maximality tells us that $H^{-1} = H$. Then for any $h \in H$, $1 \in Hh$ so we get $Hh = H$. Hence H is a subgroup. $\square$

Corollary 11.2. *If $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{K}$, then there is a unique simply connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$. *If G' is any connected Lie group with* $\mathrm{Lie}(G') = \mathfrak{g}$, *then $G' = G/Z$ for some discrete central subgroup $Z \subseteq G$.*

11.1 Real and complex forms

If $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, then there is a complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ with the bracket extended in the obvious way.

Example 11.3. If you complexify $\mathfrak{sl}(2,\mathbb{R})$, you get $\mathfrak{sl}(2,\mathbb{C})$. If you complexify $\mathfrak{su}(2)$, you also get $\mathfrak{sl}(2,\mathbb{C})$, because $\mathfrak{su}(2)$ is the traceless anti-hermitian matrices. But we see that $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are not isomorphic as Lie algebras. You can see that by looking at the corresponding Lie groups. Alternatively, you can look at the Killing form $(x, y) \mapsto \mathrm{tr}(xy)$, which has different signature for the two algebras.

Definition 11.4. We see that $\mathfrak{g}$ is a **real form** of $\mathfrak{g}_\mathbb{C}$. If G is a Lie group over $\mathbb{C}$, a (connected) real subgroup $H \subseteq G$ is called a **real form** of G if $\mathfrak{h} \subseteq \mathfrak{g}$ is a real form.

Example 11.5. $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{C})$ is a real form. But $\mathrm{SL}(2,\mathbb{R})$ has a $\mathbb{Z}$-to-1 universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. But $\mathrm{SL}(2,\mathbb{C})$ is already simply connected. So $\mathrm{SL}(\tilde{2},\mathbb{R})$ is not a real form of anything.

In general, you can define the complexification $G_\mathbb{C}$ using the universal property

$$\begin{tikzcd} G \arrow[r] \arrow[d, "f"] & G_\mathbb{C} \arrow[dl, dashed] \\ H, & \end{tikzcd}$$

where H is a Lie group over $\mathbb{C}$. Note that $G \to G_\mathbb{C}$ may not be injective.

12 February 17, 2017

12.1 Representations

Definition 12.1. A **representation** is a morphism $\rho : G \to \mathrm{GL}(V)$ or Lie groups. A **representation** of a Lie algebra is a morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras.

We are mostly going to work with V finite dimensional over $\mathbb{C}$. This is because the case V over $\mathbb{R}$ is generally more complicated.

Theorem 12.2. *(1) A representation $\rho : G \to \mathrm{GL}(V)$ induces a representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$. This is a functor $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathfrak{g})$ because G-equivariant Lie group morphisms induce G-equivariant Lie algebra morphisms.*
(2) If G is simply connected, the functor $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathfrak{g})$ is an equivalence of categories.

If G is only connected, we have $G = \tilde{G}/Z$ where Z is a discrete central subgroup. Then a representation of G gives a representation of $\tilde{G}$ by

$$\begin{tikzcd} \tilde{G} \arrow[d, "\pi"] \arrow[dr, "\rho_{\tilde{G}}"] & \\ G \arrow[r, "\rho_G"] & \mathrm{GL}(V). \end{tikzcd}$$

Then representations of G are representations of $\tilde{G}$ that is trivial on Z.

Theorem 12.3. *If $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, then there is a equivalence of categories $\mathsf{Rep}(\mathfrak{g}) \cong \mathsf{Rep}(\mathfrak{g}_{\mathbb{C}})$.*

This is just the universal property of complexification. So for instance

$$\mathsf{Rep}(\mathrm{SU}(2)) \cong \mathsf{Rep}(\mathfrak{su}(2)) \cong \mathsf{Rep}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathsf{Rep}(\mathrm{SL}(2,\mathbb{C})).$$

Example 12.4. There is the trivial representation $\rho : G \to \mathrm{GL}(\mathbb{C})$ given by $g \mapsto 1$, and for the Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C})$ given by $x \mapsto 0$. This is sometimes denoted by just $\mathbb{C}$.

Example 12.5. There is the adjoint representation given by $\rho : G \to \mathrm{GL}(\mathfrak{g})$ with $g \mapsto \mathrm{Ad}(g)$ and the corresponding $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $x \mapsto \mathrm{ad}(x) = [x, -]$.

Definition 12.6. A **subrepresentation** of a representation $\rho : G \to \mathrm{GL}(V)$ is a subspace W such that $\rho(G)W \subseteq W$. Likewise for the Lie algebra.

Likewise one can define quotient representations. If G is connected, subrepresentations of G on V correspond to subrepresentations of $\mathfrak{g}$ on V.

You can define $V \oplus W$, $V \otimes W$, V^*, $\mathrm{Hom}(V, W) = V^* \otimes W$ of representations V and W. When taking the dual, you have to be careful and define

$$(g \cdot v)(w) = v(g^{-1}w), \quad (x \cdot v)(w) = -v(x \cdot w)$$

for $g \in G$, $x \in \mathfrak{g}$, $v \in V^*$, and $w \in V$. So the representation for $\mathrm{Hom}(V, W)$ will be

$$gA = \rho_W(g) A (\rho_V(g))^{-1}.$$

Now for a representation, we define the **invariant vectors** as

$$V^G = \{v \in V : gv = v \text{ for all } g \in G\} \subseteq V,$$
$$V^{\mathfrak{g}} = \{v \in V : xv = 0 \text{ for all } x \in \mathfrak{g}\} \subseteq V.$$

Then we see that

$$\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W), \quad V^G = \mathrm{Hom}_G(\mathbb{C}, V).$$

12.2 Irreducibility

Definition 12.7. A representation V is **irreducible** or **simple** if $V \neq 0$ and $0, V$ are the only subrepresentations of V.

Example 12.8. We have $\mathrm{SL}(n, \mathbb{C})$ acting on $\mathbb{C}^n$, and this is irreducible because the action is almost transitive. If $\dim V = 1$, then it is automatically irreducible.

These are the elementary building blocks in some sense. If V which is not irreducible, there is a non-trivial representation $W \subseteq V$ and so we get a exact sequence

$$0 \to W \to V \to V/W \to 0$$

in the category of representations. So every representation is an extension of irreducible ones.

Definition 12.9. A representation V is **completely reducible** or **semisimple** if $V = \bigoplus_i n_i V_i$ with $n_i \geq 1$ multiplicities and V_i pairwise non-isomorphic irreducible representations.

Example 12.10. Take $G = \mathbb{R}$ so that $\mathfrak{g} = \mathbb{R}$. A representation on V is determined by $\rho(1) = A$, where $A \in \mathfrak{gl}(V)$ is an arbitrary linear map. Then the representation of G will be given by $t \mapsto e^{tA} \in \mathrm{GL}(V)$.

If $v \in V$ is an eigenvector of V, then $\mathbb{C}v \subseteq V$ is a subrepresentation. So only the 1-dimensional representations are irreducible. This means that the representation is completely reducible if and only if A is diagonalizable. So not every representation of $\mathbb{R}$ is semisimple. This is because $\mathbb{R}$ is not compact.

Lemma 12.11. *If V is a representation of G and $A \in \mathrm{End}_G(V)$, then every eigenspace $V_\lambda \subseteq V$ of A is a subrepresentation. If A is diagonalizable, then $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ as a representation of G.*

Proof. If $v \in V_\lambda$, then $A(g \cdot v) = g \cdot Av = g \cdot \lambda v = \lambda(g \cdot v) = g \cdot v \in V_\lambda$. ☐

13 February 22, 2017

The fourth problem set is 3.16, 3.17, 3.18, 4.2, 4.6 due February 27.

Lemma 13.1. *Let V be a representation and $A \in \mathrm{End}_G(V)$. Any eigenspace V_λ of A is a subrepresentation and if A is diagonalizable then $V = \bigoplus_\lambda V_\lambda$.*

Corollary 13.2. *If $Z \subseteq G$ is a central subgroup and $\rho(Z)$ is diagonal, then $V = \bigoplus_\lambda V_\lambda$ where V_λ are the eigenspaces of $\rho(Z)$.*

Example 13.3. Consider the action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. Then the map $v \otimes w \mapsto w \otimes v$ is map of G-representations. The eigenspaces are $V_1 = \mathrm{Sym}^2 \mathbb{C}^n$ and $V_{-1} = \bigwedge^2 \mathbb{C}^n$. In this case, it turns out that they are both irreducible representations.

13.1 Schur's lemma

Lemma 13.4 (Schur's lemma)**.** *Let V and W be irreducible representations of G. Then*

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Proof. If $f \in \mathrm{Hom}_G(V, W)$, then $\ker(f) \subseteq V$ and $\mathrm{im}(f) \subseteq W$ are subrepresentations. Since V and W are irreducible, either $f = 0$ or f is an isomorphism.

Now assume that V and W are indeed isomorphic. For any $f \in \mathrm{End}_G(V)$, f has an eigenvalue $\lambda \in \mathbb{C}$. This means that $f - \lambda\,\mathrm{id}_V \in \mathrm{End}_G(V)$. Then $f - \lambda\,\mathrm{id}_V$ is not invertible, and hence has to be 0. So $f = \lambda\,\mathrm{id}_V$. □

Example 13.5. The action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ is irreducible. So by Schur, $Z(\mathrm{GL}(n, \mathbb{C})) = \mathbb{C}^*$ and $Z(\mathfrak{gl}(n, \mathbb{C})) = \mathbb{C}$. This is because $g \in Z(\mathrm{GL}(n, \mathbb{C}))$ implies that $\rho(g) : \mathbb{C}^n \to \mathbb{C}^n$ is in $\mathrm{End}_G(\mathbb{C}^n)$. You can do the same for $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$.

Corollary 13.6. *If $V = \bigoplus_i n_i V_i$ and $W = \bigoplus_i m_i V_i$ then*

$$\mathrm{Hom}_G(V, W) = \bigoplus_i \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{m_i}).$$

Theorem 13.7. *If G is abelian, then any irreducible representation is 1-dimensional.*

Proof. If $\rho : G \to V$ then $\rho(g) \in \mathrm{End}_G(V)$ for any g. By Schur's lemma, we get $\rho(g) \in \lambda(g) \cdot 1_V$ for some $\lambda(g) \in \mathbb{C}$. Then every subspace $W \subseteq V$ is a subrepresentation. So $\dim V = 1$. □

Example 13.8. The group $S^1 = \mathrm{U}(1)$ has irreducible representations $z \mapsto z^k$ for some $k \in \mathbb{Z}$.

13.2 Unitary representations

Definition 13.9. A representation $\rho : G \to \mathrm{GL}(V)$ is **unitary** if there is a inner product on V such that $\rho : G \to \mathrm{U}(V)$. Likewise a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is **unitary** if there is a inner product such that $\rho : \mathfrak{g} \to \mathfrak{u}(V)$.

Example 13.10. For $G = \mathbb{R}$, the map $t \mapsto e^{tA}$ for $A : V \to V$ is unitary if and only if A is diagonalizable with eigenvalues in $i\mathbb{R}$.

Theorem 13.11. *Unitary representations are completely reducible (semisimple).*

Proof. Let V be a unitary representation and we have an invariant inner product. We want to prove that if $W \subseteq V$ is a subrepresentation then

$$0 \to W \to V \to V/W \to 0$$

splits. Take $W^\perp \subseteq V$, which is a subrepresentation. Then $W^\perp \cong V/W$. Now by induction we can decompose V into irreducible representations. ☐

Theorem 13.12. *Any representation of a finite group is unitary.*

Proof. We start with any hermitian inner product $B(v, w)$. Produce an invariant inner product by averaging:

$$\langle v, w\rangle = \frac{1}{|G|} \sum_{g \in G} B(gv, gw).$$

☐

Corollary 13.13. *Any representation of of a finite group is completely reducible.*

This extends to compact groups. Take any compact Lie group G over $\mathbb{R}$. A volume element dV (of positive density) on G if **left** (resp. **right**) **invariant**

$$\int_G f(hg)dV(g) = \int_G f(g)dV(g), \quad \text{resp.} \int_G f(gh)dV(g) = \int_G f(g)dV(g).$$

Example 13.14. On $\mathbb{R}_{>0}$ the invariant measure is dx/x. On $\mathrm{GL}(n, \mathbb{R})$ the measure $|\det X|^{-1} dX$ is invariant.

14 February 24, 2017

We're getting into the theory of compact Lie groups. First we are going to show that any representation of a compact Lie group over $\mathbb{R}$ is completely reducible. There is then the question of computing the irreducible representations. We are also going to talk about harmonic analysis on G, which is the analogue of doing Fourier analysis. This is decomposing $L^2(G)$, which is naturally a representation of G into irreducible representations.

14.1 Integration on compact Lie groups

We started talking about integration with a volume element dV on G. This can be left-invariant or right-invariant. As in the case of left-invariant vector fields, we have one-to-one correspondences

$$\begin{Bmatrix}\text{left-invariant} \\ \text{densities}\end{Bmatrix} \quad \longleftrightarrow \quad |{\textstyle\bigwedge}^{\dim G}\mathfrak{g}^*| \quad \longleftrightarrow \quad \begin{Bmatrix}\text{right-invariant} \\ \text{densities}\end{Bmatrix}.$$

Fix a right-invariant volume form dV. Then $g \cdot dV$ is still right-invariant. By uniqueness of the volume-element, we have

$$g \cdot dV = \phi(g) dV$$

for some $\phi(g) \in \mathbb{R}_{>0}$. So this gives a map of Lie groups $\phi : G \to \mathbb{R}_{>0}$.

Definition 14.1. The Lie group G is called **unimodular** if $\phi \equiv 1$. This is equivalent to saying that the left-invariant densities are the right-invariant densities.

Any compact group is going to be unimodular, because the image of ϕ must be a compact subgroup of $\mathbb{R}_{>0}$.

Example 14.2. The group $\mathrm{GL}(n, \mathbb{R})$ is unimodular, because the volume element $dV = |\det X|^{-n} dX$ is a both right-invariant and left-invariant volume element.

Example 14.3. The group of affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ is not unimodular.

Theorem 14.4. *Any representation of a compact group is unitary and therefore completely reducible.*

Proof. Take any inner product $B(v, w)$ on V. For a representation $\rho : G \to \mathrm{GL}(V)$, we define

$$\langle v, w \rangle = \int_G B(gv, gw) dV.$$

This is again an inner product. $\square$

14.2 Characters

Definition 14.5. Given a representation $\rho : G \to \mathrm{GL}(V)$, we define its **character** as the function

$$\chi_V : G \to \mathbb{C}; \quad g \mapsto \operatorname{tr} \rho(g).$$

Theorem 14.6. *(1) If $V = \mathbb{C}$ is the trivial representation, then $\chi_V = 1$.*
(2) $\chi_{V \oplus W} = \chi_V + \chi_W$.
(3) $\chi_{V \otimes W} = \chi_V \chi_W$.
(4) $\chi_V(ghg^{-1}) = \chi_V(h)$.
(5) If V is unitary, $\chi_{V^*} = \bar{\chi}_V$.

So if $V = \bigoplus_i n_i V_i$ then $\chi_V = \sum_i n_i \chi_{V_i}$.

Theorem 14.7. *If G is compact, and V and W are irreducible representations, then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Here the inner product is defined as

$$\langle f_1, f_2 \rangle = \int_G f_1 \bar{f}_2 dV.$$

We are going to prove a stronger statement later.

Corollary 14.8. *(1) V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*
(2) If $V = \bigoplus_i n_i V_i$ then $n_i = \langle \chi_{V_i}, \chi_V \rangle$.

Given a representation $\rho : G \to \mathrm{GL}(V)$ with V having a basis, its entries considered as matrices are functions $\rho_{i,j} : G \to \mathbb{C}$.

Theorem 14.9. *(1) If $V \not\cong W$ are irreducible representations, then*

$$\langle \rho^V_{i,j}, \rho^W_{k,l} \rangle = 0.$$

(2) If V is an irreducible representation and the basis on V is orthonormal, then

$$\langle \rho^V_{i,j}, \rho^V_{k,l} \rangle = \frac{1}{\dim V} \delta_{ik} \delta_{jl}.$$

Lemma 14.10. *(1) If $V \not\cong W$ are irreducible and $T : V \to W$ is any linear map, then*

$$\int_G g T g^{-1} dg = 0.$$

(2) If V is irreducible and $T : V \to V$, then

$$\int_G g T g^{-1} dg = \frac{\operatorname{tr}(T)}{\dim V} \operatorname{id}_V.$$

Proof. (1) The integrating can be thought of as a projection $\mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W)$. But we have $\mathrm{Hom}_G(V, W) = 0$ by Schur's lemma.

(2) This is because the projection preserves trace. $\square$

15 February 27, 2017

There is another problem set 5 due next week: 4.4, 4.7, 4.11, 4.13. Today I want to at least state the Peter–Weyl theorem and prove one part.

15.1 Peter–Weyl theorem

Theorem 15.1. *(1) If $V \cong W$ are non-isomorphic unitary irreducibles with basis, then $\langle \rho_{ij}^V, \rho_{kl}^W \rangle = 0$.*
(2) For a unitary irreducible V with basis,

$$\langle \rho_{ij}^V, \rho_{kl}^V \rangle = \frac{1}{\dim V} \delta_{ik}\delta_{jl}.$$

Proof. Let E_{jk} be the matrix $V \to W$ with 1 in the jth row kth column and 0 everywhere else. Using the projection lemma, we get

$$0 = \int_G \rho^W(g) E_{jk} \rho^V(g)^{-1} dg = \sum_{i,l} E_{il} \int_G \rho_{ij}^W \overline{\rho_{lk}^V} dg.$$

So $\langle \rho_{ij}^W, \rho_{lk}^V \rangle = 0$, for any i, j, k, l.

The second part is analogous. We have

$$\frac{\delta_{jl}}{\dim V} = \frac{\operatorname{tr}(E_{jl})}{\dim V} = \int_G \rho^V(g) E_{jl} \rho^V(g) dg = \sum_{i,k} E_{ik} \int_G \rho_{ij}^V \overline{\rho_{kl}^V} dg.$$

Compare the entries. □

There is a basis-invariant formulation. Let V be an inner product space. Then we can put an inner product on $\operatorname{End}(V)$ given by

$$\langle A, B \rangle = \frac{1}{\dim V} \operatorname{tr}(AB^*).$$

For a unitary representation $\rho : G \to \mathrm{GL}(V)$ we can define a map

$$\operatorname{End}(V) \to C^\infty(G, \mathbb{C}); \quad A \mapsto \operatorname{tr}(\rho(g)A)$$

that preserves the inner product.

Define, in the case when G is compact,

$$\hat{G} = \text{set of isomorphism classes of irreducible representaions.}$$

Then we can put what we did together to get an isometric embedding

$$\bigoplus_{V \in \hat{G}} \operatorname{End}(V) \to C^\infty(G).$$

Theorem 15.2 (Peter–Weyl theorem)**.** *This map induces an isomorphism of Hilbert spaces*

$$\widehat{\bigoplus_{V\in\hat{G}}} \operatorname{End}(V) \to L^2(G,\mathbb{C}).$$

In other words, we have $L^2(G,\mathbb{C}) \cong \widehat{\bigoplus}_{V\in\hat{G}}(\dim V)V$.

Corollary 15.3. *We have* $\widehat{\bigoplus}_{V\in\hat{G}}\mathbb{C}1_V \cong L^2(G,\mathbb{C})^G$, *with* χ_V *being the basis of* $L^2(G,\mathbb{C})^G$.

15.2 Representation theory of $\mathfrak{sl}(2,\mathbb{C})$

Theorem 15.4. *Any representation of* $\mathfrak{sl}(2,\mathbb{C})$ *is completely reducible.*

Proof. Because $\mathrm{SU}(2,\mathbb{C})$ is simply connected,

$$\mathsf{Rep}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathsf{Rep}(\mathfrak{su}(2,\mathbb{C})) \cong \mathsf{Rep}(\mathrm{SU}(2)).$$

Since $\mathrm{SU}(2)$ is compact, all representations are unitary. So everything is completely reducible. $\square$

Let us explicitly write down the basis for $\mathfrak{sl}(2,\mathbb{C})$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then there are the Lie bracket relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

Consider the eigenspaces of h:

$$V_\lambda = \{v \in V : hv = \lambda v\}.$$

Lemma 15.5. $eV_\lambda \subseteq V_{\lambda+2}$, $fV_\lambda \subseteq V_{\lambda-2}$.

Proof. This is a direct computation:

$$h(ev) = [h,e]v + ehv = 2ev + e\lambda v = (2+\lambda)ev.$$

The same computation works for f. $\square$

Theorem 15.6. $\rho(h)$ *is diagonalizable, and so* $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$.

Proof. We can assume that V is irreducible. We have $V' = \bigoplus_\lambda V_\lambda \subseteq V$, and this is a subrepresentation by the previous lemma. But $\rho(h)$ has eigenvalues and so $V' \neq 0$. Then $V' = V$. $\square$

Theorem 15.7. *Any irreducible representation is of the following form: there is a basis* $v^0, \dots, v^n$ *and*

$$hv^i = (n-2i)v^i, \quad ev^i = (n+1-i)v^{i-1}, \quad fv^i = (i+1)v^{i+1}.$$

16 March 1, 2017

We were classifying representations of $\mathfrak{sl}(2, \mathbb{C})$. The interesting thing was that there is an eigenvalue decomposition $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$.

For the irreducible representation V, take the nonzero $0 \neq v \in V_\lambda$ such that $\Re(\lambda)$ is maximal. We then define

$$v_k = \frac{f^k}{k!} v \in V_{\lambda - 2k}.$$

These are linearly independent as long as they are nonzero. This shows that $v_{n+1} = 0$ but $v_n \neq 0$.

Lemma 16.1. *$fv_k = (k+1)v_{k+1}$, $hv_k = (\lambda - 2k)v_k$, $ev_0 = 0$, $ev_k = (\lambda - k + 1)v_{k-1}$.*

Proof. We have $ev_0 = 0$ since $ev_0 \in V^{\lambda+2}$ and $\Re(\lambda)$ is maximal. You can check $ev_k = (\lambda - k + 1)v_{k-1}$ for $k > 0$ by induction. □

Corollary 16.2. *The vectors $v = v_0, \dots, v_n$ is a basis of V.*

Proof. V is irreducible. □

We further have $ev_{n+1} = 0 = (\lambda - n)v_n$ and so $\lambda = n$. An alternative way of seeing this is $\operatorname{tr} \rho(h) = \operatorname{tr}[\rho(e), \rho(f)] = 0$. So we completely understand the structure of the representation of $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 16.3. *Irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are classified by the highest weight $\lambda = 0, 1, 2, \dots$.*

Theorem 16.4. *Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$.*
(1) There is a weight decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$ by eigenspaces of $\rho(h)$.
(2) $e^n : V_{-n} \to V_n$ and $f^n : V_n \to V_{-n}$ are isomorphisms.

16.1 Spherical Laplacian

The spherical Laplacian is given by

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2}\Delta_{S^2} + \Delta_{\text{radial}} = \frac{1}{r^2}\Delta_{s^2} + \partial_r^2 + \frac{2}{r}\partial_r,$$

and more explicitly,

$$\Delta_{S^2} = J_x^2 + J_y^2 + J_z^2, \quad J_x = y\partial_z - z\partial_y, \quad \dots.$$

Now there are two actions by Δ_{S^2} and $\mathrm{SO}(3, \mathbb{R})$ on $C^\infty(S^2)$.

Lemma 16.5. *Δ_{S^2} commutes with the action of $\mathrm{SO}(3, \mathbb{R})$.*

Proof. We know that both $\Delta_{\mathbb{R}^3}$ and Δ_{radial} are rotation invariant. □

Algebraically, this can be interpreted as $\Delta_{S^2} \in U\mathfrak{so}(3,\mathbb{R})$ being a central element. This is called a "Casimir" element.

Consider

$$P_n = \left\{ \begin{matrix} \text{functions } S^2 \to \mathbb{C} \text{ which can be written as} \\ \text{polynomials in } x, y, z \text{ of degree} \leq n \end{matrix} \right\}.$$

Now it is clear that ∇_{S^2} maps P_n to P_n. Because $\bigcup P_n \subseteq C^\infty(S^2)$ is dense, it suffices to diagonalize Δ_{S^2} on P_n. Here, we note that the complexification of $\mathfrak{so}(3,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{C})$ with the identification being

$$-\frac{i}{2}(e+f) \leftrightarrow J_x, \quad \frac{1}{2}(f-e) \leftrightarrow J_y, \quad -\frac{ih}{2} \leftrightarrow J_z.$$

We are now going to decompose $P_n = \bigoplus_i n_i V_i$ into irreducible representations. Let us write

$$u = x + iy = \rho e^{i\varphi}, \quad v = x - iy = \rho e^{-i\varphi}.$$

Then $uv = x^2 + y^2 = 1 - z^2 = \rho^2$. We see that $z^p, z^p u^k, z^p v^k$ for $p + k \leq n$ span P_n, because any polynomial in x, y, z can be written as a polynomial in u, v, z. Let us write

$$f_{p,k} = z^p \sqrt{1-z^2}^{|k|} e^{i\varphi k}.$$

These form a basis, by the uniqueness of the Fourier series.

Now we note that $J_z = \partial/\partial\varphi$ and so $J_z f_{p,k} = ik f_{p,k}$. This means that the vectors of weight $2k$ in P_n have basis

$$f_{0,k}, f_{1,k}, \dots, f_{n-|k|,k}.$$

17 March 3, 2017

We are going to be looking at finite dimensional Lie algebras over $\mathbb{C}$ now. One major class is the semisimple ones, meaning $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and nonabelian simple ones are contained here. The other class is the solvable ones, extension of abelians, and an important subclass is nilpotent algebras, satisfying $[x_1, [\dots, [x_{n-1}, x_n]]] = 0$ for large enough n. A special case is abelian, $[\mathfrak{g}, \mathfrak{g}] = 0$. You can classify semisimple Lie algebras, but solvable Lie algebras are hard to classify.

17.1 Solvable and nilpotent Lie algebras

Definition 17.1. A $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **subalgebra** if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and is called an **ideal** if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

If $I, J \subseteq \mathfrak{g}$ are ideals, then $I \cap J$ and $[I, J]$ and $I + J$ are ideals. There is the ideal of commutators, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$. Also, if $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ is an Lie algebra. Then $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is the largest abelian quotient.

Example 17.2. We have $[\mathfrak{gl}(n), \mathfrak{gl}(n)] = [\mathfrak{sl}(n), \mathfrak{sl}(n)] = \mathfrak{sl}(n)$. One direction is obvious. For the other direction, you can use elementary matrices.

You can take the commutator ideal repeatedly:

$$D^0\mathfrak{g} = \mathfrak{g}, \quad D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \quad D^2\mathfrak{g} = [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], \dots, \quad D^{n+1}\mathfrak{g} = [D^n\mathfrak{g}, D^n\mathfrak{g}].$$

This is called the **derived series**. Then $D^{n+1}\mathfrak{g} \subseteq D^n\mathfrak{g}$ are ideals and $D^n\mathfrak{g}/D^{n+1}\mathfrak{g}$ are abelian. So $\mathfrak{g}$ being **solvable** just means $D^n\mathfrak{g} = 0$ for some n.

Proposition 17.3. *A Lie algebra $\mathfrak{g}$ is solvable if and only if there is a filtration of subalgebras*

$$0 = \mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_n = \mathfrak{g}$$

such that $\mathfrak{a}_{i+1} \subseteq \mathfrak{a}_i$ is an ideal and $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is abelian.

Proof. We have $\mathfrak{a}_{n-1} \supseteq [\mathfrak{g}, \mathfrak{g}]$ and by induction, $\mathfrak{a}_{n-k} \supseteq D^k\mathfrak{g}$. So having such a filtration implies solvable. The other direction is obvious. □

There is also another filtration

$$D_0\mathfrak{g} = \mathfrak{g}, \quad D_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \quad D_2\mathfrak{g} = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \quad \dots, \quad D_n\mathfrak{g} = [\mathfrak{g}, D_{n-1}\mathfrak{g}],$$

called the **lower central series**. We say that $\mathfrak{g}$ is **nilpotent** if $D_n\mathfrak{g} = 0$ for some n. This just means that $(n+1)$-fold products in $\mathfrak{g}$ vanish.

Example 17.4. Let $\mathfrak{b} \subseteq \mathfrak{gl}(n, \mathbb{C})$ be the Lie algebra of upper triangular matrices and $\mathfrak{n} \subseteq \mathfrak{b}$ be the algebra of strictly upper triangular matrices. Then $\mathfrak{b}$ is solvable but not nilpotent and $\mathfrak{n}$ is nilpotent.

Here is why. Geometrically we can look at $\mathfrak{b}$ and $\mathfrak{n}$ in terms of

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

with $\dim V_i = i$ as $\mathfrak{b} = \mathfrak{a}_0$ and $\mathfrak{n} = \mathfrak{a}_1$, where

$$\mathfrak{a}_k = \{x \in \mathfrak{gl}(n) : xV_i \subseteq V_{i-k}\}.$$

Because $[\mathfrak{a}_k, \mathfrak{a}_l] \subseteq \mathfrak{a}_{k+l}$, we see that $\mathfrak{n} = \mathfrak{a}_1$ is nilpotent. We also have $D^1\mathfrak{b} = [\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{a}_1$ since the diagonal cancel out and so $\mathfrak{b} = \mathfrak{a}_0$ is solvable. You can check that the lower central series for $\mathfrak{b}$ stabilizes at $\mathfrak{n}$ again using elementary matrices.

Theorem 17.5. *(1) If $\mathfrak{g}$ is solvable (resp. nilpotent), then its complexification $\mathfrak{g}_{\mathbb{C}}$ is again solvable (resp. nilpotent).*
(2) If $\mathfrak{g}$ is solvable (resp. nilpotent), then so is any subalgebra or quotient.
(3) If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is solvable.
(4) If $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ is an extension and $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable, then $\mathfrak{g}$ is solvable.

Proof. Clear. □

Theorem 17.6 (Lie)**.** *If $\mathfrak{g}$ is solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ then there is a basis of V such that all $\rho(x)$ are upper triangular.*

Proposition 17.7. *If $\mathfrak{g}$ is solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ then there is a $v \in V$ which is a common eigenvector of all $\rho(x)$.*

We will discuss the proof next time.

18 March 6, 2017

The sixth problem set is 4.5, 4.14, 5.3, 5.4 due March 20. We started talking about the general structure theory of Lie algebras.

18.1 Representations of solvable algebras

Theorem 18.1 (Lie)**.** *If $\mathfrak{g}$ is solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then there is a basis of V such that all $\rho(x)$ are upper triangular. (This works for any algebraic closed field with characteristic zero.)*

Proposition 18.2. *If $\mathfrak{g}$ is solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then there is a $v \in V$, $v \neq 0$ which is a common eigenvector of all $\rho(x)$.*

Proof. We induct on $\dim \mathfrak{g}$. The case $\dim \mathfrak{g} = 0$ is trivial. Assume $\dim \mathfrak{g} > 0$. Then $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ must be a proper ideal. Choose a 1-codimensional subspace $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}^1 \subseteq \mathfrak{g}$, which is automatically going to be an ideal.

Now write $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathbb{C}x$ as a vector space. Induction tells us that there exists a $v \in V$ such that $hv = \lambda(h)v$ for all $h \in \mathfrak{g}^1$. Consider the space

$$W = \operatorname{span}\{v, xv, x^2v, \dots\} \subseteq V.$$

Then $\mathfrak{g}^1$ acts on W as

$$hx^k v = \lambda(h) x^k v + \sum_{\rho < k} a_{kl} x^k v,$$

because the commutators are in $\mathfrak{g}^1$.

Choose the smallest n such that $x^{n+1} \in \operatorname{span}\{v, \dots, x^n v\}$. Then $v, \dots, x^n v$ form a basis of W. We know that

$$\operatorname{tr}_W(\rho(h)) = (n+1)\lambda(h).$$

Then

$$0 = \operatorname{tr}_W[\rho(x), \rho(h)] = \operatorname{tr}_W \rho([x, h]) = (n+1)\lambda([x, h]).$$

That is, $\lambda([x, h]) = 0$.

This in particular shows that $hx^k v = \lambda(h)x^k v$. That is, any vector in W is a common eigenvector of all $\rho(h)$ for $h \in \mathfrak{g}^1$. Since $\rho(x)$ acts on W, it has an eigenvector in W that is the common eigenvector of $\mathfrak{g}$. □

Proof of Theorem 18.1. We induct on $\dim V$. Pick a common eigenvector v, and pick a basis of $V/\mathbb{C}v$ and lift it to V and add constant multiples of v. □

Corollary 18.3. *(1) If $\mathfrak{g}$ is solvable and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is irreducible, then* $\dim V = 1$.
(2) If $\mathfrak{g}$ is solvable, then is a series

$$0 \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = \mathfrak{g}$$

of ideals in $\mathfrak{g}$ such that $\dim(I_k/I_{k-1}) = 1$.
(3) If $\mathfrak{g}$ is solvable then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. (1) is clear. For (2) just apply the theorem to the adjoint representation. For (3), the reverse direction is clear. The forward direction follows from the adjoint representation. There is a basis such that every matrix is upper triangular, and then $\mathrm{ad}([\mathfrak{g},\mathfrak{g}])$ are strictly upper triangular. Then for sufficiently large n,

$$\mathrm{ad}([x_1,\dots,[x_{n-1},x_n]]) = 0$$

for every $x_1,\dots,x_n \in [\mathfrak{g},\mathfrak{g}]$. By definition of the adjoint operator,

$$[y,[x_1,\dots,[x_{n-1},x_n]]] = 0$$

for every $y, x_1,\dots,x_n \in [\mathfrak{g},\mathfrak{g}]$. □

Theorem 18.4. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent operators. Then there is a basis of V in which all $\mathfrak{g}$ are strictly upper triangular.*

Proof. The idea is the same. It is in the textbook. □

Theorem 18.5 (Engel)**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.*

Proof. The forward direction is clear. For the other direction, by the theorem there exists a series of ideals

$$0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $[x,\mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$. This just implies that the lower central series converges to zero. □

Definition 18.6. A Lie algebra $\mathfrak{g}$ is **simple** if it is not abelian and has no nontrivial ideals. A Lie algebra $\mathfrak{g}$ is **semisimple** if it contains no solvable ideals (except $0 \subseteq \mathfrak{g}$).

Lemma 18.7. *If $\mathfrak{g}$ is simple then it is semisimple.*

Proof. If suffices to show that if $\mathfrak{g}$ is simple, then it is not solvable. This is because if it is solvable then $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{g}$ and so $[\mathfrak{g},\mathfrak{g}] = 0$. □

Example 18.8. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then the adjoint representation is irreducible, and so $\mathfrak{sl}(2,\mathbb{C})$ is simple.

19 March 8, 2017

19.1 Radical of a Lie algebra

Proposition 19.1. *The set*

$$\mathrm{rad}(\mathfrak{g}) = \bigcup_{\substack{I \subseteq \mathfrak{g} \\ solvable}} I$$

is a solvable ideal.

Proof. If I and J are solvable, then we have an exact sequence

$$0 \to I \to I + J \to J/(I \cap J) \to 0.$$

Then $I + J$ is solvable. $\square$

Theorem 19.2. *(1) $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.*
(2) If $\mathfrak{b} \subseteq \mathfrak{g}$ is solvable and $\mathfrak{g}/\mathfrak{b}$ is semisimple, then $\mathfrak{b} = \mathrm{rad}(\mathfrak{g})$.

Proof. For any $I \subseteq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ solvable, we have an exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \tilde{I} = \rho^{-1}(I) \to I \to 0.$$

Then $\tilde{I}$ is solvable and so $\tilde{I} = \mathrm{rad}(\mathfrak{g})$. Hence $I = 0$. $\square$

So every Lie algebra can be uniquely split into a solvable one and a semisimple one.

Theorem 19.3 (Levi)**.** *The exact sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to 0$$

always splits. In other words, there is a subalgebra $\mathfrak{g}_{ss}$ so that $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$ and $\mathfrak{g}_{ss} \cong \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$.

Here is an exact sequence that does not split:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \twoheadrightarrow \mathbb{C}^2.$$

So this is really a deep statement.

Definition 19.4. A Lie algebra is **reductive** if $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ (or equivalently, if $\mathfrak{g}/Z(\mathfrak{g})$ is semisimple).

Theorem 19.5. *Let V be an irreducible representation of $\mathfrak{g}$. Then any $x \in \mathrm{rad}(\mathfrak{g})$ acts by scalar, i.e., $xv = \lambda(x)v$ for some $\lambda : \mathrm{rad}(\mathfrak{g}) \to \mathbb{C}$. In particular, $x \in [\mathrm{rad}(\mathfrak{g}), \mathfrak{g}]$ acts trivially.*

Proof. Since $\mathrm{rad}(\mathfrak{g})$ is solvable, there is some $0 \neq v \in V$ that is a common eigenvector of all $x \in \mathrm{rad}(\mathfrak{g})$ with $xv = \lambda(x)v$. Define

$$V_\lambda = \{x \in V : xw = \lambda(x)w \text{ for all } x \in \mathrm{rad}(\mathfrak{g})\}.$$

Then you can check that $\mathfrak{g}V_\lambda \subseteq V_\lambda$, and because $v \in V_\lambda$, we get $V = V_\lambda$. $\square$

19.2 Classical groups are reductive

Definition 19.6. A bilinear form B on $\mathfrak{g}$ is called **invariant** if

$$B([x,y],z) + B(y,[x,z]) = 0$$

for all $x, y, z \in \mathfrak{g}$.

Lemma 19.7. *If B is an invariant form on $\mathfrak{g}$, and $I \subseteq \mathfrak{g}$ is an ideal, then*

$$I^\perp = \{x \in \mathfrak{g} : B(x, I) = 0\}$$

is an ideal. (B can be degenerate.) In particular, $\mathfrak{g}^\perp = \ker(B) \subseteq \mathfrak{g}$ is an ideal.

Example 19.8. On $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, the bilinear form $B(x,y) = \operatorname{tr}(xy)$ is a symmetric invariant bilinear forms. On the space of symmetric matrices, B is positive-definite, and on the space of anti-symmetric matrices, B is negative-definite.

More generally, if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is any representation, then

$$B_V(x,y) = \operatorname{tr}(\rho(x)\rho(y))$$

is a symmetric invariant bilinear form.

Theorem 19.9. *If there exists a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that B_V is non-degenerate, then $\mathfrak{g}$ is reductive.*

Proof. We need to show that $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] = 0$. There exists an irreducible subrepresentation $W \subseteq V$. For any $x \in [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$, we have $xW = 0$. Then $x \in \ker B_W$, and we have $B_V = B_W + B_{V/W}$. Inductively we obtain $x \in \ker B_V$. Because B_V is non-degenerate, $x = 0$. □

Theorem 19.10. *All classical Lie algebras are reductive, and the semisimple ones are*

$$\mathfrak{sl}(n, \mathbb{K})\ (n \geq 2), \quad \mathfrak{so}(n, \mathbb{K})\ (n \geq 3), \quad \mathfrak{su}(n)\ (n \geq 2), \quad \mathfrak{sp}(n)\ (n \geq 1).$$

Proof. We find a representation with B_V nondegenerate. In all this cases, let V be the defining representation $\mathbb{K}^n$ (or $\mathbb{K}^{2n}$). Then $B_V = \operatorname{tr}(x,y)$, and you can check that this is nondegenerate. □

20 March 10, 2017

20.1 Cartan's criterion

To every Lie algebra is associated a **Killing form** given by

$$K(x,y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)).$$

Example 20.1. Consider $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ with basis e, h, f. Then

$$\operatorname{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \operatorname{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Then the Killing form can be computed as

$$K(e,f) = K(f,e) = 4, \quad K(h,h) = 8, \quad K(h,e) = K(h,f) = 0.$$

So you can check that $K(x,y) = 4\operatorname{tr}(xy)$. Generally, in $\mathfrak{sl}(n,\mathbb{K})$, the Killing form is $K(x,y) = 2n\operatorname{tr}(xy)$. In $\mathfrak{gl}(n,\mathbb{K})$, the Killing form is $K(x,y) = 2n\operatorname{tr}(xy) - 2\operatorname{tr}(x)\operatorname{tr}(y)$.

Theorem 20.2 (Cartan's criterion)**.**
(1) $\mathfrak{g}$ is solvable if and only if $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$.
(2) $\mathfrak{g}$ is semisimple if and only if K is non-degenerate.

Theorem 20.3. *Suppose $V/\mathbb{C}$ is a finite dimensional vector space and $A : V \to V$ be a linear map.*
(1) A can be uniquely written as $A = A_s + A_n$, where $[A_s, A_n] = 0$ and A_s is semisimple and A_n is nilpotent.
(2) For $\operatorname{ad}(A) \in \operatorname{End}(\operatorname{End}(V))$, we have $(\operatorname{ad} A)_s = \operatorname{ad}(A_s)$ and $\operatorname{ad} A_s = P(\operatorname{ad} A)$ for some $P \in t\mathbb{C}[t]$.
(3) For $\overline{A}_s$ the semisimple operator with the same eigenspaces as A_s and complex conjugate eigenvalues, $\operatorname{ad}(\overline{A}_s) = Q(\operatorname{ad} A)$ for $Q \in t\mathbb{C}[t]$.

Proof. This is just Jordan normal form. □

Theorem 20.4. *If $\mathfrak{g} \in \mathfrak{gl}(V)$ is a Lie subalgebra such that $x \in \mathfrak{g}$ and $y \in [\mathfrak{g},\mathfrak{g}]$ implies $\operatorname{tr}(xy) = 0$, then $\mathfrak{g}$ is solvable.*

Proof. Consider any $x \in [\mathfrak{g},\mathfrak{g}]$ and look at its Jordan decomposition $x = x_s + x_n$. We want to show that $x_s = 0$. We have

$$\operatorname{tr}(x\bar{x}_s) = \sum \lambda_i \bar{\lambda}_i = \sum |\lambda_i|^2,$$

where λ_i are the eigenvalues. But $x \in [\mathfrak{g},\mathfrak{g}]$ and so $x = \sum_i [y_i, z_i]$. Then

$$\operatorname{tr}(x\bar{x}_s) = \operatorname{tr}\Big(\sum [y_i, z_i]\bar{x}_s\Big) = -\sum \operatorname{tr}(y_i[\bar{x}_s, z_i]).$$

But $[\bar{x}_s, z_i] = \operatorname{ad}(\bar{x}_s)z_i) = Q(\operatorname{ad}(x))z_i \in [\mathfrak{g},\mathfrak{g}]$. Then the right hand side is zero and so $\operatorname{tr}(x\bar{x}_s) = 0$. This shows that all λ_i are zero and so x is nilpotent for all $x \in [\mathfrak{g},\mathfrak{g}]$. Then $\mathfrak{g}$ is solvable. □

Proof of Theorem 20.2. (1) Suppose that $\mathfrak{g}$ is solvable. Because $\mathfrak{g}$ is solvable, using Lie's theorem we can find a basis of $\mathfrak{g}$ such that all $\mathrm{ad}(x)$ is upper triangular for all $x \in \mathfrak{g}$. Then for any $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$, $\mathrm{ad}(x)$ is upper triangular and $\mathrm{ad}(y)$ is strictly upper triangular. This implies that $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$.

For the other direction, $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$ and so by the lemma, $\mathfrak{ad}(g)$ is solvable. Since there is an exact sequence

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \mathrm{ad}(\mathfrak{g}) \to 0,$$

the Lie algebra $\mathfrak{g}$ is also solvable.

(2) Suppose that $\mathfrak{g}$ is semisimple. Consider the ideal $I = \ker(K) \subseteq \mathfrak{g}$. The Killing form on I is 0 and so (1) shows that I is solvable. The $\mathfrak{g}$ being semisimple implies that K is non-degenerate.

Suppose that K is non-degenerate. Then $\mathfrak{g}$ is reductive by what we have done last time. Since $Z(\mathfrak{g}) \subseteq \ker(K) = 0$, the radical is $\mathrm{rad}(\mathfrak{g}) = 0$. That is, $\mathfrak{g}$ is semisimple. $\square$

Corollary 20.5. *For any Lie algebra, $\mathfrak{g}/\mathbb{R}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semisimple.*

Theorem 20.6. *For semisimple $\mathfrak{g}$ and an ideal $I \subseteq \mathfrak{g}$, there is an ideal $I' \subseteq \mathfrak{g}$ such that $\mathfrak{g} = I \oplus I'$.*

Proof. Look at the orthogonal complement with respect to the Killing form. $\square$

Corollary 20.7. *Any semisimple $\mathfrak{g}$ is a direct sum of simple Lie algebras: $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$.*

Corollary 20.8. *If $\mathfrak{g}$ is semisimple then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Proposition 20.9. *If $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ is semisimple, any ideal $I \subseteq \mathfrak{g}$ takes the form of $I = \bigoplus_{i \in J} \mathfrak{g}_j$ for some $J \subseteq \{1, \dots, n\}$.*

Proof. Assume $I \subseteq \bigoplus_{i=1}^n \mathfrak{g}_i$ is some ideal. Look at the projection $p_n : \mathfrak{g} \to \mathfrak{g}_n$. Because $\mathfrak{g}_n$ is simple, either $p_n(I) = 0$ or $p_n(I) = \mathfrak{g}_n$. In the case $p_n(I) = 0$, we have $I \subseteq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ and can use the induction hypothesis. If $p_n(I) = \mathfrak{g}_n$, then $[\mathfrak{g}_n, I] = [\mathfrak{g}_n, p_n(I)] = \mathfrak{g}_n$. Because I is an ideal, $\mathfrak{g}_n \subseteq I$. Then $I = I' \oplus \mathfrak{g}_n$ for some ideal $I' \subseteq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. $\square$

Corollary 20.10. *If $\mathfrak{g}$ is semisimple, and $I \subseteq \mathfrak{g}$ is an ideal, then I and $\mathfrak{g}/I$ are semisimple.*

21 March 20, 2017

The automorphism group $\mathrm{Aut}(\mathfrak{g})$ has a natural Lie group structure, and its Lie algebra is

$$\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g}) = \{(D : \mathfrak{g} \to \mathfrak{g}) : D([x,y]) = [Dx, y] + [x, Dy]\}.$$

Proposition 21.1. *If G is a connected Lie group, and $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple, then $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$ and $\mathrm{Aut}(\mathfrak{g})/\mathrm{Ad}(G)$ is discrete.*

Proof. We note that there is a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$, and $\ker \mathrm{ad} = Z(\mathfrak{g}) = 0$ because $\mathfrak{g}$ is semisimple.

We can extend the Killing form to $\mathrm{Der}(\mathfrak{g})$ by simply setting $K(\delta_1, \delta_2) = \mathrm{tr}(\delta_1 \delta_2)$. We can check that $[\delta, \mathrm{ad}(x)] = \mathrm{ad}(\delta(x))$ for any $\delta \in \mathrm{Der}(\mathfrak{g})$ and $x \in \mathfrak{g}$. Now K is nondegenerate and so there is a splitting $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ as Lie algebras. why?

For any $\delta \in \mathfrak{g}^\perp$ and $x \in \mathfrak{g}$,

$$\mathrm{ad}(\delta(x)) = [\delta, \mathrm{ad}(x)] = 0.$$

This means that $\delta(x) = 0$ and so $\delta = 0$. Hence $\mathfrak{g}^\perp = 0$ and $\mathfrak{g} = \mathrm{Der}(\mathfrak{g})$. ☐

Theorem 21.2. *(1) Let G be a compact Lie group over $\mathbb{R}$. Then $\mathfrak{g} = \mathrm{Lie}(G)$ is reductive and its Killing form is negative semidefinite.*
(2) Let $\mathfrak{g}$ be semisimple over $\mathbb{R}$ and suppose that the Killing form is negative definite. Then $\mathfrak{g} = \mathrm{Lie}(G)$ for some compact connected G.

Proof. (1) Because G is compact, any representation $\rho : G \to U(V)$ is unitary for some suitable metric on V. On $\mathfrak{u}(n)$, the form $\mathrm{tr}(xy) = -\mathrm{tr}(xy^*)$ is negative definite. Applying this to the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{u}(\mathfrak{g})$ shows that K is negative semidefinite. Also $\ker K = Z(\mathfrak{g})$.

(2) Let G be a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. The bilinear form $B(x,y) = -K(x,y)$ is positive definite, symmetric, and $\mathrm{Ad}(G)$-invariant. This shows that $\mathrm{Ad} : G \to \mathrm{SO}(\mathfrak{g})$. Note that $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathfrak{g})$ is a connected component, and $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$ is a closed Lie subgroup. Thus $\mathrm{Ad}(G) \subseteq \mathrm{SO}(\mathfrak{g})$ is a closed Lie subgroup, and hence compact. Also $\mathrm{Lie}(\mathrm{Ad}(G)) = \mathrm{Lie}(G/Z(G)) = \mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$. ☐

Note that if $\mathfrak{g}$ is semisimple over $\mathbb{R}$ and K is negative definite, then any connected G with $\mathrm{Lie}(G) = \mathfrak{g}$ is compact. The same argument does not work over $\mathbb{C}$.

21.1 Compact real form

Theorem 21.3. *Leg $\mathfrak{g}$ be semisimple over $\mathbb{C}$. Then there is a compact Lie group K over $\mathbb{R}$ with $\mathfrak{k}_\mathbb{C} \cong \mathfrak{g}$.*

This $\mathfrak{k}$ is called the **compact real form** of $\mathfrak{g}$, and it is unique up to conjugation. If $\mathfrak{g} = \mathrm{Lie}(G)$, then we can choose $K \subseteq G$. Here are some examples:

$\mathfrak{g}$	K
$\mathfrak{sl}(n,\mathbb{C})$	$\mathrm{SU}(n)$
$\mathfrak{o}(n,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{R})$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathrm{Sp}(n)$

Table 2: Simple Lie algebras over $\mathbb{C}$ and their compact real forms

Corollary 21.4. *Any representation over a semisimple Lie algebra over $\mathbb{K}$ is completely reducible(=semisimple).*

Proof. If $\mathbb{K} = \mathbb{C}$, then we can take a compact real form so that $\mathsf{Rep}(K) \cong \mathsf{Rep}(\mathfrak{k}) \cong \mathsf{Rep}(\mathfrak{g})$. If $\mathbb{K} = \mathbb{R}$, then it follows from the theorem and the remark after it. $\square$

This corollary is true for any semisimple Lie algebra over a field of characteristic 0. There is an algebraic proof based on homological algebra.

For $\mathfrak{g}$ a Lie algebra over $\mathbb{C}$, we say that $x \in \mathfrak{g}$ is **semisimple** (resp. **nilpotent**) if $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is.

Theorem 21.5. *If $\mathfrak{g}$ is semisimple over $\mathbb{C}$ and $x \in \mathfrak{g}$, then there exists a unique Jordan decomposition $x = x_s + x_n$ so that $x_s \in \mathfrak{g}$ is semisimple and $x_n \in \mathfrak{g}$ is nilpotent, and $[x_s, x_n] = 0$. Moreover $[x, y] = 0$ implies $[x_s, y] = 0$.*

Uniqueness is clear sine $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is a Jordan decomposition in $\mathfrak{gl}(\mathfrak{g})$. The basic idea for the proof is that there is a Jordan decomposition $\mathrm{ad}(x) = \mathrm{ad}(x)_s + \mathrm{ad}(x)_n$ and $\mathrm{ad}(x)_s$ is a derivation which must be inner. Then it comes from some x_s.

22 March 22, 2017

When we studied representations of $\mathfrak{sl}(2,\mathbb{C})$, we looked at eigenspaces of the element $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. This is called a Cartan subalgebra, and we will see that it always exists for a semisimple Lie algebra. In terms of groups, this corresponds to a maximal torus.

22.1 Jordan decomposition in semisimple algebras

The main tool is the Jordan decomposition for semisimple algebras.

Theorem 22.1. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$. For each $x \in \mathfrak{g}$, there exists a unique Jordan decomposition into $x = x_s + x_n$, where $\operatorname{ad}(x_s)$ is semisimple and $\operatorname{ad}(x_n)$ is nilpotent, and $[x_s, x_n] = 0$.*

Proof. Let us first decompose $\operatorname{ad}(x) = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$ in $\mathfrak{gl}(\mathfrak{g})$. Consider the decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda$$

into generalized eigenspaces of $\operatorname{ad} x$. This is a graded structure, i.e., $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$, because

$$(\operatorname{ad}(x) - (\lambda+\mu))^n [y,z] = \sum_{k=0}^{n} \binom{n}{k} [(\operatorname{ad} x - \lambda)^k y, (\operatorname{ad} x - \mu)^{n-k} z]$$

by induction. The consequence is that $(\operatorname{ad} x)_s$ is a derivation, because if $y \in \mathfrak{g}_\lambda$ and $z \in \mathfrak{g}_\mu$ then

$$(\operatorname{ad} x)_s [y,z] = (\lambda+\mu)[y,z] = [(\operatorname{ad} x)_s y, z] + [y, (\operatorname{ad} x)_s z].$$

Since $\mathfrak{g}$ is semisimple, there exists a $x_s \in \mathfrak{g}$ such that $\operatorname{ad} x_s = (\operatorname{ad} x)_s$. Also $(\operatorname{ad} x)y = 0$ implies $(\operatorname{ad} x_s)y = (\operatorname{ad} x)_s y = 0$. $\square$

Definition 22.2. $\mathfrak{h} \subseteq \mathfrak{g}$ is a **toral subalgebra** if $\mathfrak{h}$ is commutative and consists of semisimple elements.

Theorem 22.3. *If $\mathfrak{g}$ is semisimple over $\mathbb{C}$, $\mathfrak{h} \subseteq \mathfrak{g}$ is toral, and $(-,-)$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, then*
(1) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x\}$,
(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$,
(3) if $\alpha \neq -\beta$, then $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$,
(4) $(-,-)$ restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.

Proof. (1) Note that $\operatorname{ad}(h)$ is diagonalizable for every $h \in \mathfrak{h}$ and also for any $h_1, h_2 \in \mathfrak{h}$, we have $[\operatorname{ad}(h_1), \operatorname{ad}(h_2)] = 0$. So they can be simultaneously diagonalized.

(2) If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, we have

$$\operatorname{ad}(h)[x,y] = [\operatorname{ad}(h)x, y] + [x, \operatorname{ad}(h)y] = (\alpha+\beta)(h)[x,y].$$

(3) By invariance, for $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$,

$$0 = ([h,x],y) + (x,[h,y]) = (\alpha+\beta)(h)(x,y)$$

for all h. So $(x,y) = 0$ unless $\alpha + \beta = 0$.

(4) This follows from (3) since the pairing is non-degenerate. □

Lemma 22.4. *Let $\mathfrak{g}, \mathfrak{h}$ and $(-,-)$ be as above. Then*
(1) $(-,-)$ is non-degenerate on $\mathfrak{g}_0$,
(2) $x \in \mathfrak{g}_0$ implies $x_s, x_n \in \mathfrak{g}_0$,
(3) $\mathfrak{g}_0$ is reductive.

Proof. (1) is clear. For (2), if $x \in \mathfrak{g}_0$ then $[x, \mathfrak{h}] = 0$. Then $[x_s, \mathfrak{h}] = 0$ an so $x_s \in \mathfrak{g}_0$. For (3), note $\mathfrak{g}$ is a representation of $\mathfrak{g}_0$. Because K is non-degenerate on $\mathfrak{g}$, by (1) it is also non-degenerate on $\mathfrak{g}_0$. Then by Theorem 19.9 $\mathfrak{g}_0$ is reductive. □

22.2 Cartan subalgebra

Definition 22.5. Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$. We say that $\mathfrak{h} \subseteq \mathfrak{g}$ is a **Cartan subalgebra** if it is toral with

$$C(\mathfrak{h}) = \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\} = \mathfrak{h}.$$

Example 22.6. In $\mathfrak{sl}(n, \mathbb{C})$, the subalgebra

$$\mathfrak{h} = \{\text{traceless diagonal matrices}\}$$

is a Cartan subalgebra.

Theorem 22.7. *If $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal toral algebra, then it is Cartan.*

Proof. We have a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$. We claim that $\mathfrak{g}_0$ is a toral subalgebra. Since $\mathfrak{h} \subseteq \mathfrak{g}_0$, maximality will imply $\mathfrak{h} = \mathfrak{g}_0$ and then $\mathfrak{h} = C(\mathfrak{h}) = \mathfrak{g}_0$.

Consider any $x \in \mathfrak{g}_0$. If $(\operatorname{ad} x)|_{\mathfrak{g}_0}$ is not nilpotent, then $0 \neq \operatorname{ad} x_s|_{\mathfrak{g}_0}$ and so $x_s \notin \mathfrak{h}$. On the other hand, $[\mathfrak{h}, x_s] = 0$ so $\mathfrak{h} \oplus \mathbb{C}x_s$ is a bigger toral subalgebra. This shows that $(\operatorname{ad} x)|_{\mathfrak{g}_0}$ is nilpotent for every $x \in \mathfrak{g}_0$. By Engel's theorem, $\mathfrak{g}_0$ is nilpotent, and it is also reductive. Therefore $\mathfrak{g}_0$ is abelian.

Now we want to show that any $x \in \mathfrak{g}_0$ is semisimple. Using the Jordan decomposition, we may show that $x_n = 0$. For any $y \in \mathfrak{g}_0$, we see that $\operatorname{ad}(x_n)\operatorname{ad}(y)$ is nilpotent because $\mathfrak{g}_0$ is commutative. So $\operatorname{tr}(\operatorname{ad}(x_n)\operatorname{ad}(y)) = 0$, and because the Killing form restricted to $\mathfrak{g}_0$ is non-degenerate, we get $x_n = 0$. Therefore $x = x_s$ is semisimple. □

23 March 24, 2017

Last time we defined Cartan subalgebras. If $\mathfrak{g}$ is semisimple over $\mathbb{C}$, a subalgabra $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra if $\mathfrak{h} = C(\mathfrak{h})$ and every $x \in \mathfrak{h}$ is semisimple.

23.1 Root decomposition

This is what we proved last time.

Theorem 23.1. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra. Let $(-,-)$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$.*

(1) There is a ***root decomposition***

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x\}$ is the root space and $R = \{\alpha \in \mathfrak{h}^* : \alpha \neq 0\}$ is the set of roots.

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(3) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)$ if $\alpha + \beta \neq 0$.

(4) $(-,-)$ restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.

Theorem 23.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with each $\mathfrak{g}_i$ simple.*

(1) If $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ are Cartan subalgebras, then $\bigoplus \mathfrak{h}_i = \mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra with root system $R = \amalg R_i$.

(2) Conversely, if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra then there exist Cartan subalgebras $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ such that $\mathfrak{h} = \bigoplus \mathfrak{h}_i$.

Proof. (1) is clear.

(2) Define $\mathfrak{h}_i = p_i(\mathfrak{h}) \subseteq \mathfrak{g}_i$, where $p_i : \mathfrak{g} \to \mathfrak{g}_i$ is the projection. For $x \in \mathfrak{g}_i$, we have $[h, x] = [p_i(h), x]$ and so $\mathfrak{h}_i \subseteq \mathfrak{g}_i$ is a Cartan subalgebra. Then $\mathfrak{h} \subseteq \bigoplus \mathfrak{h}_i$ where $\bigoplus \mathfrak{h}_i$ is a toral algebra. By maximality of $\mathfrak{h}$, we get $\mathfrak{h} = \bigoplus \mathfrak{h}_i$. $\square$

Example 23.3. Consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with the Cartan subalgebra $\mathfrak{h} = \{X : X \text{ diagaonal}, \operatorname{tr} X = 0\}$. We can compute

$$[h, E_{i,j}] = (h_i - h_j)E_{i,j}$$

for any $h \in \mathfrak{h}$. Consider the maps $e_i : \mathfrak{h} \to \mathbb{C}; h \mapsto h_i$. The formula tells us that the root decomposition is

$$R = \{e_i - e_j : i \neq j\}, \quad \mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{i,j}.$$

Using this decomposition, we can compute the Killing from as

$$K(h, h') = \operatorname{tr}(\operatorname{ad}(h)\operatorname{ad}(h')) = \sum_{i \neq j} (h_i - h_j)(h'_i - h'_j) = 2n \sum h_i h'_i = 2n \operatorname{tr}(hh')$$

at least on $\mathfrak{h} \subseteq \mathfrak{g}$.

Generally, let $(-,-)$ be a invariant, symmetric, nondegenerate bilinear form on $\mathfrak{g}$. For a root $\alpha \in R$, we can find an element $H_\alpha \in \mathfrak{h}$ such that

$$(\alpha, \beta) = \alpha(H_\beta) = (H_\alpha, H_\beta).$$

Lemma 23.4. *For $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$,*

$$[e, f] = (e, f)H_\alpha.$$

Proof. For any $h \in \mathfrak{h}$,

$$([e, f], h) = (e, [f, h]) = -(e, [h, f]) = \alpha(h)(e, f) = (H_\alpha, h)(e, f).$$

Then $[e, f] = (e, f)H_\alpha$ by non-degeneracy of $(-,-)$ on $\mathfrak{h}$. $\square$

Lemma 23.5. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$ with $\mathfrak{h}$ a Cartan subalgebra.*

(1) If $\alpha \in R$ then $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

(2) Suppose $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ satisfy $(e, f) = 2/(\alpha, \alpha)$. Define $h_\alpha = 2H_\alpha/(\alpha, \alpha)$. Then $\alpha(h_\alpha) = 2$ and e, f, h_α span a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

(3) The element h_α is independent of $(-,-)$.

Proof. (1) Suppose $(\alpha, \alpha) = 0$. Then $\alpha(H_\alpha) = 0$. There exist $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) \neq 0$. Define

$$h = [e, f] = (e, f)H_\alpha,$$

and let $\mathfrak{a}$ be the subalgebra generated by e, f, h. Then $[h, e] = \alpha(h)e = 0$ and $[h, f] = \alpha(h)f = 0$. Hence $\mathfrak{a}$ is solvable with $[\mathfrak{a}, \mathfrak{a}]$ containing h. This shows that $\mathrm{ad}(h)$ is nilpotent, and by definition of a toral algebra, $\mathrm{ad}(h)$ is also semisimple. Then $\mathrm{ad}(h) = 0$ and so $h = 0$ and so $H_\alpha = 0$ and so $\alpha = 0$.

(2) and (3) are exercises. $\square$

Lemma 23.6. *Let α be a root and $\mathfrak{sl}(2, \mathbb{C})_\alpha$ be the subalgebra spanned by $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, h_α. Then*

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{0 \neq k \in \mathbb{Z}} \mathfrak{g}_{k\alpha}$$

is an irreducible representation.

24 March 27, 2017

Tomorrow I will post the midterm, consisting of eight problems.

Last time we had for each $\alpha \in R$, a copy $\mathfrak{sl}(2,\mathbb{C})_\alpha \subseteq \mathfrak{g}$ spanned by $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$.

Lemma 24.1. *Let $\mathfrak{g}$ be a semisimple algebra over $\mathbb{C}$. Then*

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{0\neq k\in\mathbb{Z}} \mathfrak{g}_{k\alpha}$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.

Proof. We have $[e, \mathfrak{g}_{k\alpha}] \subseteq \mathfrak{g}_{(k+1)\alpha}$ and $[e, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}h_\alpha$. Similar statements hold for $f \in \mathfrak{g}_{-\alpha}$.

We have $(\alpha, h_\alpha) = 2$ and so there is a weight decomposition, with $V[2k] = \mathfrak{g}_{2k}$, $V[2k+1] = 0$ and $\dim V[0] = 1$. This shows that V is irreducible. □

24.1 Symmetries of the roots

Theorem 24.2. *Suppose $\mathfrak{g}$ is semisimple over $\mathbb{C}$, and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ be the root decomposition and let $(-,-)$ be a symmetric invariant form on $\mathfrak{g}$.*

(1) R spans $\mathfrak{h}^*$ over $\mathbb{C}$ and $\{h_\alpha : \alpha \in R\}$ spans $\mathfrak{h}$.

(2) For any $\alpha \in R$, $\dim \mathfrak{g}_\alpha = 1$.

(3) For $\alpha, \beta \in R$,

$$\beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

(4) For any $\alpha, \beta \in R$ the reflection

$$s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha.$$

is in R. In particular, $s_\alpha(\alpha) = -\alpha \in R$.

(5) For $\alpha \in R$, its scalar multiple $\lambda\alpha$ for $\lambda \in \mathbb{C}$ is in R if and only if $\lambda = \pm 1$.

(6) If α and $\beta \neq \pm\alpha$ are roots, then

$$V = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.

(7) If $\alpha, \beta, \alpha+\beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. (1) Suppose there exists a $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in R$. This means that $\operatorname{ad}(h) = 0$ by the root decomposition. This means that $h \in Z(\mathfrak{g}) = 0$ and hence $h = 0$.

(2) This follows from the previous lemma. If V is reducible, all eigenspaces have dimension at most 1.

(3) Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2, \mathbb{C}_\alpha$. The weight of $\mathfrak{g}_\beta$ is $\beta(h_\alpha) \in \mathbb{Z}$.

(4) Assume $n = \beta(h_\alpha) \geq 0$ is the weight. If $v \in \mathfrak{g}_\beta$ has n then $f_\alpha^n v \neq 0 \in \mathfrak{g}_{\beta - n\alpha}$. Then $0 \neq \mathfrak{g}_{\beta - n\alpha} = \mathfrak{g}_{s_\alpha(\beta)}$. That is, $s_\alpha(\beta) \in R$.

(5) Suppose $\alpha, \lambda\alpha \in R$. By (3), $2c \in \mathbb{Z}$ and so $c \in \frac{1}{2}\mathbb{Z}$. Also we have $1/c \in \frac{1}{2}\mathbb{Z}$. This shows that $c \in \{\pm 1/2, \pm 1, \pm 2\}$. Then we only need to exclude the case $c = 2$.

Consider the irreducible representation

$$V = \mathbb{C}h_\alpha \oplus \bigoplus_{0 \neq k \in \mathbb{Z}} \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}.$$

Note that $V[2] = \mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ and $\mathrm{ad}(e_\alpha) : V[2] \to V[4]$ is 0. This shows that $V[4] = V[6] = \cdots = 0$. In particular $\mathfrak{g}_{2\alpha} = 0$ and so $2\alpha \notin R$.

(6) Because $\dim \mathfrak{g}_{\beta + k\alpha} = 1$ for $\beta + k\alpha \in R$, it is irreducible.

(7) This follows from the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. The map $\mathrm{ad}(e_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$ has to be nonzero. □

Theorem 24.3. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra.*

(1) Let $\mathfrak{h}_\mathbb{R} \subseteq \mathfrak{h}$ be the vector space over $\mathbb{R}$ generated by h_α for $\alpha \in R$. Then $\mathfrak{h}^* = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$, and K is real-valued and positive definite on $\mathfrak{h}_\mathbb{R}$.

(2) Let $\mathfrak{h}_\mathbb{R}^* \subseteq \mathfrak{h}^*$ be generated over $\mathbb{R}$ by R. Then $\mathfrak{h}^* = \mathfrak{h}_\mathbb{R}^* \oplus i\mathfrak{h}_\mathbb{R}^*$.

Proof. We know that

$$K(h_\alpha, h_\beta) = \mathrm{tr}(\mathrm{ad}(h_\alpha)\,\mathrm{ad}(h_\beta)) = \sum_{\gamma \in R} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z}.$$

So the restriction of K to $\mathfrak{h}_\mathbb{R}$ is real-valued and also positive definite. Then $i\mathfrak{h}_\mathbb{R}$ is negative definite. So they have trivial intersection but $\mathfrak{h}$ is generated by $\mathfrak{h}_\mathbb{R}$ and $i\mathfrak{h}_\mathbb{R}$. Thus $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$. □

If $\mathfrak{k} \subseteq \mathfrak{g}$ is a compact real form, then $\mathfrak{k} \cap \mathfrak{h} = i\mathfrak{h}_\mathbb{R}$. For example, if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{su}(n)$, then they are both imaginary diagonal traceless matrices.

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and let $x \in \mathfrak{sl}(2, \mathbb{C})$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then the eigenvalues of $\mathrm{ad}(x)$ are $\lambda_i - \lambda_j$, and $[x, y] = 0$ implies that y is also diagonal with respect to the eigenbasis of x. Then the centralizer $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. This is true for general semisimple Lie algebras over $\mathbb{C}$.

25 March 29, 2017

Definition 25.1. Define the **nullity** of $x \in \mathfrak{g}$ as

$$n(x) = \dim(\ker(\mathrm{ad}(x)^n)), \quad n \gg 0.$$

Define the **rank** of $\mathfrak{g}$ as

$$\mathrm{rank}(\mathfrak{g}) = \min_{x \in \mathfrak{g}} n(x).$$

Definition 25.2. The **regular elements** are

$$\mathfrak{g}^{\mathrm{reg}} = \{x \in \mathfrak{g} : n(x) = \mathrm{rank}(\mathfrak{g})\} \subseteq \mathfrak{g}.$$

This set is open and dense. If we are over $\mathbb{C}$, it further follows that it is connected, because the complement will have complex codimension at least 1.

Proposition 25.3. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$.*

(1) For $x \in \mathfrak{g}^{\mathrm{reg}}$, $\mathrm{ad}(x)$ is semisimple and $C(x) = \{y \in \mathfrak{g} : [x, y] = 0\}$ is a Cartan subalgebra.

(2) For $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra and

$$x \in \mathfrak{h} \cap \mathfrak{g}^{\mathrm{reg}} = \{x \in \mathfrak{h} : \alpha(x) \neq 0 \text{ for all } \alpha \in R\}$$

then $C(x) = \mathfrak{h}$.

(3) Any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$ are conjugate under an action of G by the adjoint action.

25.1 Abstract root system

Definition 25.4. An **abstract root system** is a pair (E, R) with E and Euclidean vector space and $R \subseteq E \setminus \{0\}$ such that

(R1) R spans E,

(R2) for $\alpha, \beta \in R$,

$$n_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z},$$

(R3) if α is a root, then

$$s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

sends R to R. In particular, $s_\alpha(\alpha) = -\alpha \in R$.

Definition 25.5. A root system is **reduced** if $\alpha, c\alpha \in R$ implies $c = \pm 1$.

Example 25.6. Consider the $(n-1)$-dimensional root system

$$E = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \textstyle\sum \lambda_i = 0\}, \quad R = \{e_i - e_j : i \neq j\}.$$

One can check that this satisfies all the axioms. This also follows from the fact that it is the root system of $\mathrm{SL}(n, \mathbb{C})$. This root system is called **type** A_{n-1}.

An **isomorphism** of root system is given by a linear isomorphism $\phi : E_1 \to E_2$ such that

- $\phi(R_1) = R_2$,
- $n_{\phi(\alpha)\phi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R$.

Any $\alpha \in R$ gives an automorphism $s_\alpha \in \mathrm{Aut}(E, R)$. So this generates some group.

Definition 25.7. The **Weyl group** is the subgroup $W \subseteq \mathrm{O}(E)$ is the subgroup generated by s_α for $\alpha \in R$.

In the case of A_{n-1}, that the Weyl group is S_n.

Lemma 25.8. *The Weyl group W is a finite group.*

Proof. W injects in to the permutation group S_R. $\square$

There can be automorphisms that are not in W. For instance, in A_2 there is the automorphism $\alpha \mapsto -\alpha$ that is not in W.

25.2 Classification of rank 2 reduced root systems

Theorem 25.9. *Let $\alpha, \beta \in R$ be linearly independent with $|\alpha| \geq |\beta|$. Then there are the following possibilities:*

$\angle(\alpha, \beta) = \phi$	$\lvert\alpha\rvert/\lvert\beta\rvert$	$n_{\alpha\beta}$	$n_{\beta\alpha}$
$\pi/2$	?	0	0
$\pi/3$	1	1	1
$2\pi/3$	1	-1	-1
$\pi/4$	$\sqrt{2}$	2	1
$3\pi/4$	$\sqrt{2}$	-2	-1
$\pi/6$	$\sqrt{3}$	3	1
$5\pi/6$	$\sqrt{3}$	-3	-1

Table 3: Possible configurations of two roots

Proof. We have $(\alpha, \beta) = |\alpha||\beta| \cos\phi$ and so $n_{\alpha\beta} = 2|\alpha|/|\beta| \cos\phi$. Then $n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\phi$ is an integer. So $n_{\alpha\beta} n_{\beta\alpha} \in (0, 1, 2, 3)$. Now do a case-by-case analysis. $\square$

For rank 1, there is the unique reduced root system A_1.

For rank 2, there the root systems $A_1 \amalg A_1 = D_2$ coming from $\mathfrak{so}(4, \mathbb{C})$ or $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$, A_2 coming from $\mathfrak{so}(3, \mathbb{C})$, $B_2 = C_2$ coming from $\mathfrak{so}(5, \mathbb{C})$ or $\mathfrak{sp}(2, \mathbb{C})$, and G_2.

26 March 31, 2017

Last time I drew the four root systems and claimed that they are all the rank 2 reduced root systems.

Theorem 26.1. *Any rank 2 reduced root system is isomorphic to $A_1 \amalg A_1$, A_2, B_2, or G_2.*

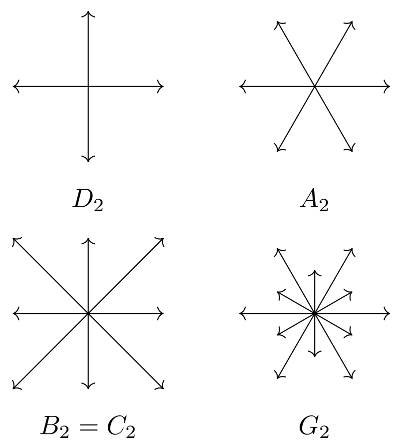


Figure 1: Root systems of rank 2

Proof. Pick $\alpha, \beta \in R$ such that $\phi = \angle(\alpha, \beta) < \pi$ is maximal and $|\alpha| \geq |\beta|$. Then $\phi \geq \pi/2$ since otherwise $\angle(-\alpha, \beta) > \alpha(\alpha, \beta)$. There are cases $\phi = \pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$.

In the case $\phi = 2\pi/3$ we have $|\alpha| = |\beta|$ and reflect them around to get all of A_2. There cannot be more roots because of maximality of ϕ and reducedness.

In the case $\phi = 3\pi/4$ we have $|\alpha| = \sqrt{2}|\beta|$. Doing the same thing gives B_2, and likewise for $\phi = 5\pi/6$ which gives G_2 and $\phi = \pi/2$ which gives D_2. $\square$

Lemma 26.2. *If R is a reduced root system, then $(\alpha, \beta) < 0$ implies $\alpha + \beta \in R$.*

Proof. It suffices to check in the rank 2 case, because the intersection of a root system with a subspace is root system. Just look at all the possible cases. $\square$

26.1 Positive and simple roots

Choose a $v \in E$ such that $(v, \alpha) \neq 0$ for all $\alpha \in R$. We define

$$R_\pm = \{\alpha \in R : \pm(\alpha, v) > 0\}$$

so that $R = R_+ \cup R_-$. The roots R_+ and R_- are called **positive** and **negative roots**, and this is called the **polarization** of R.

Definition 26.3. A root $\alpha \in R^+$ is **simple** if it is not a sum of positive roots.

Lemma 26.4. *Every $\alpha \in R^+$ is a sum of simple roots.*

Proof. Break the roots into smaller positive roots if it is not simple. This process is going to end because the inner product is decreasing and there is only a finite number of roots. □

Lemma 26.5. *If $\alpha \neq \beta$ are simple roots, then $(\alpha, \beta) \leq 0$.*

Proof. If $(\alpha, \beta) > 0$ then $(-\alpha, \beta) < 0$ and so $\beta - \alpha \in R$. Either $\beta - \alpha \in R_+$ or $\alpha - \beta \in R_+$. In any case, this contradicts that both α and β are simple. □

Theorem 26.6. *Simple roots form a basis of E.*

Proof. We only need to further show linear independence, which follows from $(\alpha, \beta) \leq 0$. □

Corollary 26.7. *Every $\alpha \in R$ can be written uniquely as*

$$\alpha = \sum_{i=1}^{\text{rank}} n_i \alpha_i$$

with $n_i \in \mathbb{Z}$. Also $\alpha \in R_\pm$ if and only if $\pm n_i \geq 0$ for all i.

We define the **height** to be

$$h^+(\alpha) = \sum n_i.$$

Example 26.8. Take A_{n-1}. We have $R = \{e_i - e_j : i \neq j\}$ and choose $R_+ = \{e_i - e_j : i < j\}$. Then the simple roots are $\alpha_i = e_i - e_{i+1}$. The height is $\text{ht}(e_i - e_j) = j - i$.

26.2 Lattices from root system

We define the **root lattice** as

$$\begin{aligned} Q &= \text{abelian group generated by } R \\ &= \text{abelian group generated by simple roots} \\ &= \bigoplus_i \mathbb{Z} e_i \cong \mathbb{Z}^{\text{rank}}. \end{aligned}$$

Also there are the **coroots** given by

$$\alpha \in R \quad \rightsquigarrow \quad \alpha^\vee \in E^* : \alpha^\vee(\beta) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

Similarly we can define the **coroot lattice** as

$$Q^\vee = \text{abelian group generated by coroots} = \bigoplus_{\mathbb{Z}} \alpha_i^\vee$$

The coroot lattice, is not the dual lattice of the root lattice. In fact, we define the **weight lattice** as

$$P = \text{dual to } Q^\vee = \{\lambda \in E : \alpha^\vee(\lambda) \in \mathbb{Z} \text{ for all } \alpha^\vee \in Q^\vee\}.$$

There are the **fundamental weights** $w_i \in P$ such that

$$\alpha_j^\vee(w_i) = \delta_{ij},$$

where α_j are simple roots. Then $P = \bigoplus_i \mathbb{Z}w_i$. Since $\beta^\vee(\alpha) = n_{\alpha\beta} \in \mathbb{Z}$ for $\alpha, \beta \in R$, we have $Q \subseteq P$. Then P/Q is a finite abelian group.

Example 26.9. Let us look at the case of A_1. We have $Q = \mathbb{Z}\alpha$ and $P = \mathbb{Z}(\alpha/2)$.

27 April 3, 2017

Given an abstract root system (E, R) a reduced root system, we can break the symmetry and define positive and negative roots $R = R_+ \amalg R_-$. Then we can define simple roots $S = \{\alpha_1, \dots, \alpha_r\}$ that form a basis.

27.1 Weyl chamber

Recall that we can polarize with respect to a vector

$$v \in E \setminus \bigcup_{\alpha \in R} L_\alpha, \text{ where } L_\alpha = \{\lambda \in E : (\lambda, \alpha) = 0\}.$$

A **Weyl chamber** is a connected component of $E \setminus \bigcup_\alpha L_\alpha$.

Example 27.1. The root system A_2 has 6 Weyl chambers, each of the form of a cone with angle $\pi/3$.

Lemma 27.2. *Let C be a Weyl chamber.*

(1) Its closure $\overline{C}$ is an unbounded convex cone.

(2) $\partial\overline{C}$ is a union of codimension 1 faces, which are called the ***walls*** *of C.*

Lemma 27.3. *There is a one-to-one correspondence between polarizations and Weyl chambers.*

Theorem 27.4. *The Weyl group W acts transitively on the set of Weyl chambers.*

Proof. Two Weyl chambers C and C' are adjacent if $\overline{C}$ and $\overline{C}'$ intersect in a codimension 1 face. Let them intersect in L_α. Then it can be checked that $C' = s_\alpha(C)$.

In a general situation, given any two chambers C and C', we can always find a sequence $C = C_0, C_1, \dots, C_n = C'$ so that C_i and C_{i+1} are adjacent by choosing a generic enough path. □

Each Weyl chamber has $r = \dim E$ faces, because it is given by r inequalities coming form the simple roots.

Corollary 27.5. *Any two polarization $R = R_+ \amalg R_- = R'_+ \amalg R'_-$ are related by an element of the Weyl group. In other words, there exists a $w \in W$ such that $w(R_+) = R'_+$.*

We would want to recover R by recording only the simple roots S.

Theorem 27.6. *Let R be a reduced root system with polarization $R = R_+ \amalg R_-$. Let $S = \{\alpha_1, \dots, \alpha_r\}$ be the simple roots, and let $s_i = s_{\alpha_i}$ be the reflections.*

(1) $s_1, \dots, s_r$ generate W.

(2) $W \cdot S = R$.

Proof. (1) It suffices to show that if G is the subgroup generated by $s_1, \ldots, s_r$, then G acts transitively on the set of Weyl chambers. Let C^+ be the positive Weyl chamber corresponding to the polarization $R = R_+ \amalg R_-$.

If C' is adjacent to C^+, then $C = s_i C^+$ for some $1 \leq i \leq r$. If $C = wC^+$ for some G and C' is adjacent to G, then $w^{-1}C'$ is adjacent to $w^{-1}C = C^+$. Then $w^{-1}C' = s_i C^+$ for some s_i and so $C' = ws_i C^+$. This shows that G acts transitively on the set of Weyl chambers and so $G = W$.

(2) For $\alpha \in R$ choose a chamber C with L_α as a wall. We know that $C = wC^+$ for some $w \in W$. So $L_\alpha = wL_{\alpha_i}$ for some $\alpha_i \in S$ and so $\alpha = \pm w\alpha_i$. The sign does not matter because $s_\alpha(\alpha) = -\alpha$. □

Example 27.7. Consider the A_{n-1} with roots $R = \{e_i - e_j : i \neq j\}$. The simple roots are $\{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ and $W = S_n$ are generated by these transposition. The positive Weyl chamber corresponding to this polarization is

$$C^+ = \{(\lambda_1, \ldots, \lambda_n) \in E : \lambda_1 > \lambda_2 > \cdots > \lambda_n\}.$$

28 April 5, 2017

The eighth problem set due April 10 is 6.3, 6.4, 6.5, 6.7.

So far, given a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we looked at the Weyl group and saw that this is generated by the simple reflections.

28.1 Cayley graph of the Weyl group

Given a generator of the group, we can construct the Cayley graph. For any $w \in W$, let us define its **length** as

$$\rho(w) = \min\{n : w = s_{i_1} \cdots s_{i_n}\}.$$

Theorem 28.1. *$\rho(w)$ is the number of root hyperplanes L_α between C^+ and $w(C^+)$.*

Proof. If $w = s_{i_1} \cdots s_{i_n}$ then there is a path from C^+ to $w(C^+)$, which crosses n hyperplanes. So there are at most n hyperplanes between C^+ and $w(C^+)$. For the other side of the inequality, if there are n hyperplanes, then take a path that crosses exactly n hyperplanes and follow it. □

Let us draw the Cayley graph of A_2. The Weyl group is S_3 and the elements can be identified with the Weyl chambers.

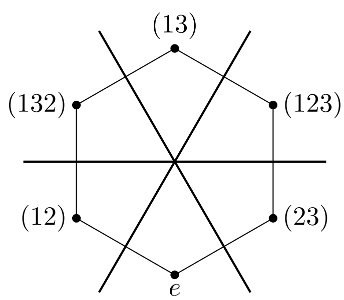


Figure 2: Cayley graph of the Weyl group for A_2

Let R be a reduced root system with polarization, and give an order on the set of simple roots.

Definition 28.2. The **Cartan matrix** is a $r \times r$ matrix A with entries

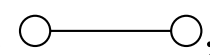
$$a_{ij} = n_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

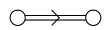
We know that $a_{ii} = 2$ and $a_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$. This matrix A does not depend on polarization, up to permutation of rows and columns. Also from this matrix we can recover the whole root system up to isomorphism.

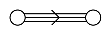
28.2 Dynkin diagram

Let us construct a graph with

- simple roots as vertices,
- for edges, connect α_i with α_j if

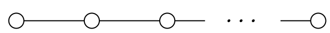
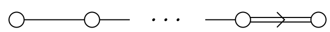
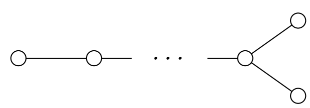
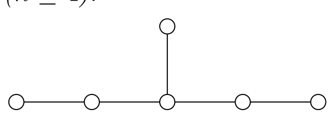
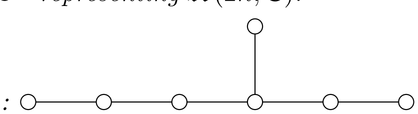
  (a) if $\angle(\alpha_i, \alpha_j) = \frac{2\pi}{3}$ then ,

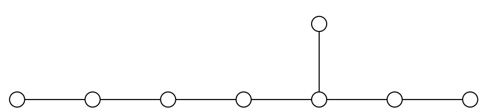
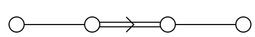
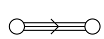
  (b) if $\angle(\alpha_i, \alpha_j) = \frac{3\pi}{4}$ then  with the arrow pointing to the shorter root,

  (c) if $\angle(\alpha_i, \alpha_j) = \frac{5\pi}{6}$ then  with the arrow pointing to the shorter root.

Given two root systems (E_1, R_1) and (E_2, R_2), we can form the direct sum $(E_1 \oplus E_2, R_1 \amalg R_2)$. We say that a root system (E, R) is **irreducible** if there is no non-trivial partition $R = R_1 \amalg R_2$ such that $R_1 \perp R_2$. This is equivalent to the Dynkin diagram being connected.

Theorem 28.3. *This is the full list of Dynkin diagrams of irreducible reduced root systems.*

- A_n *($n \geq 0$):*  *representing* $\mathfrak{sl}(n+1, \mathbb{C})$.
- B_n *($n \geq 2$):*  *representing* $\mathfrak{so}(2n+1, \mathbb{C})$.
- C_n *($n \geq 3$):*  *representing* $\mathfrak{sp}(n, \mathbb{C})$.
- D_n *($n \geq 4$):*  *representing* $\mathfrak{so}(2n, \mathbb{C})$.
- E_6*:*  E_7*:* 

  E_8*:* 
- F_4*:* 
- G_2*:* 

There are some coincidences,

$$A_1 = B_1 = C_1$$

coming from $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{so}(3, \mathbb{C}) = \mathfrak{sp}(1, \mathbb{C})$. Also we have

$$A_3 = D_3$$

coming from $\mathfrak{sl}(4, \mathbb{C}) = \mathfrak{so}(6, \mathbb{C})$.

The classification is not hard; you will eventually figure out if you are stranded on an island and want to prove this. For now let us consider the simply laced algebras, i.e., those with no multiple edges. In this case, all roots have the same length.

29 April 7, 2017

Last time I put this famous list of Dynkin diagrams on the board. I will sketch a proof of this classification.

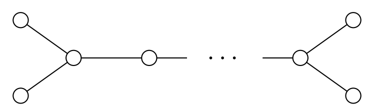
29.1 Classification of simply laced Dynkin diagrams

Let D be a connected, simply laced (no multiple edges) Dynkin diagram of a roots system. This implies that all roots have the same length. So we can normalize everything so that $(\alpha_i, \alpha_i) = 2$. Note that the Cartan matrix

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j)$$

is a positive definite matrix. This is the essential property.

If we take any (full) subgraph of the Dynkin diagram, it also has to be positive definite. The sum of the roots on the cycle is a kernel, and so it is a contradiction. Next every vertex is connected to at most 3 edges. This is because if one vertex is connected to four, the sum of the four vertices minus 2 times the center vertex will be in the kernel. Similarly we can exclude



by again finding something in the kernel.

Now this shows that there is either no branching points, which is A_n, or there is a single trivalent branching point. Let the branching point be α, and the three branches be

$$\beta_1 - \beta_2 - \cdots - \beta_{p-1} - \alpha, \quad \gamma_1 - \gamma_2 - \cdots - \gamma_{q-1} - \alpha, \quad \delta_1 - \delta_2 - \cdots - \delta_{r-1} - \alpha,$$

where $p, q, r \geq 2$. Take the weighted sum

$$\beta = \sum_{n=1}^{p-1} n\beta_n, \quad \gamma = \sum_{n=1}^{q-1} n\beta_n, \quad \delta = \sum_{n=1}^{r-1} n\gamma_n.$$

Then β, γ, δ are pairwise orthogonal and $\alpha, \beta, \gamma, \delta$ are linearly independent. By the Pythagorean theorem,

$$\Big(\alpha, \frac{\beta}{|\beta|}\Big)^2 + \Big(\alpha, \frac{\gamma}{|\gamma|}\Big)^2 + \Big(\alpha, \frac{\delta}{|\delta|}\Big)^2 < |\alpha|^2 = 2.$$

We can compute $(\beta, \beta) = p(p-1)$ and $(\alpha, \beta) = -(p-1)$. Therefore

$$\frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r} < 2.$$

This is equivalent to $1/p + 1/q + 1/r > 1$. So either $p = q = 2$ or $p = 2, q = 3$ and $3 \leq r \leq 5$. The first case is D_{n+2} and the second case is E_6, E_7, E_8.

29.2 Recovering the Lie algebra

Theorem 29.1. *Suppose $\mathfrak{g}$ is semisimple over $\mathbb{C}$, and $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra with root system $R \subseteq \mathfrak{h}^*$. Let $R = R_+ \amalg R_-$ be a polarization, with $S = (\alpha_1, \dots, \alpha_r)$ the set of simple roots.*

(1) Subspaces

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

are subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as vector spaces.

(2) Choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $(e_i, f_i) = 2/(\alpha_i, \alpha_i)$. Let $h_i = 2H_{\alpha_i}/(\alpha_i, \alpha_i)$ where H_α is dual to α. Then $e_1, \dots, e_r$ generate $\mathfrak{n}_+$ and $f_1, \dots, f_r$ generate $\mathfrak{n}_-$ as a Lie algebra. Also $h_1, \dots, h_r$ is a basis of h. Thus all of these generate $\mathfrak{g}$ as a Lie algebra.

*(3) The **Serre relations** hold:*

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$
$$[e_i, f_j] = \delta_{ij} h_i, \quad (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0.$$

The Serre relations turn out to be all the relations among e_i, f_i, h_i. That is, if you have an arbitrary Dynkin diagram, take the free Lie algebra and quotient by these relations, then you get a finite dimensional semisimple Lie algebra.

Proof. (1) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and also the sum of positive roots is positive.

(2) This is quite clear. Every positive root is a sum of some simple roots. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if α and β are positive roots.

(3) Maybe $[e_i, f_j] = \delta_{ij} h_i$ requires some explanation. If $i = j$ then this is some lemma we proved before. If $i \neq j$ then $[e_i, f_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$, but $\alpha_i - \alpha_j$ is not a root since it is neither positive nor negative.

The other two relations follow form the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. Consider

$$\bigoplus_{k \geq 0} \mathfrak{g}_{\alpha_j + k\alpha_i},$$

which is a representation of $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$. Then $\operatorname{ad}(f_i) e_j = 0$. This means that the lowest weight is a_{ij}. But the highest weight is the minus the lowest weight and hence $-a_{ij}$. So $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$. ☐

30 April 10, 2017

This is going to be the last week of lectures. The ninth problem set is Kirillov 7.1, 7.3, 7.11, 7.13 due April 17.

We had these Serre relations on the simple roots.

Theorem 30.1. *Let R be a reduced system with polarization $R = R_+ \amalg R_-$ and let $S = \{\alpha_1, \dots, \alpha_r\}$ be the simple roots. Let $\mathfrak{g}(R)$ be the Lie algebra over $\mathbb{C}$ with generators e_f, f_i, h_i for $i = 1, \dots, r$ and Serre relations. Then $\mathfrak{g}(R)$ is finite dimensional and semisimple with root system R.*

This gives a correspondence

$$\left\{\begin{matrix}\text{semisimple Lie algebras}/\mathbb{C} \\ \text{up to isomorphism}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{reduced root systems} \\ \text{up to isomorphism}\end{matrix}\right\}.$$

We are now going to study representations.

30.1 Representations of semisimple Lie algebras

If $\mathfrak{g}$ is semisimple over $\mathbb{C}$, any representation V of $\mathfrak{g}$ is completely reducible:

$$V = \bigoplus_i n_i V_i,$$

where V_i are irreducible and $n_i > 0$. So we want to classify irreducible representations of $\mathfrak{g}$ and compute n_i for a given V.

Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.

Definition 30.2. The **weight space** of weight $\lambda \in \mathfrak{h}^*$ is

$$V[\lambda] = \{v \in V : hv = \lambda(h)v, h \in \mathfrak{h}\}.$$

An element $\lambda \in \mathfrak{h}^*$ is a **weight** of V if $V[\lambda] \neq 0$. We also write

$$P(V) = \text{set of weights of } V.$$

Since $V[\lambda] \cap V[\mu] = 0$ if $\lambda \neq \mu$, the representation V being finite implies $P(V)$ is finite.

Theorem 30.3. *If V is a finite dimensional representation of $\mathfrak{g}$, then*

(1) $V = \bigoplus_{\lambda \in P(V)} V[\lambda]$,

(2) weights are integral, i.e., $P(V) \subseteq P$.

Proof. (1) V restricts to a representation of $\mathfrak{sl}(2, \mathbb{C})_{\alpha_i}$, which is spanned by e_i, f_i, h_i. Then h_i acts diagonally on V (by representation theory of $\mathfrak{sl}(2, \mathbb{C})$). Also by definition of Cartan subalgebras, $[h_i, h_j] = 0$. So they can be simultaneously diagonalized and so V splits into weight spaces.

(2) It suffices to show that h_α has eigenvalues in $\mathbb{Z}$. This follows from the statement for $\mathfrak{sl}(2, \mathbb{C})_\alpha$ acting on V. $\square$

Note that if $x \in \mathfrak{g}_\alpha$ then $xV[\lambda] \subseteq V[\lambda + \alpha]$. So V is some kind of graded module over a graded algebra.

We now want to record the numbers $\dim V[\lambda]$ for $\lambda \in P(V)$. We define the **character**

$$\operatorname{ch}(V) = \sum_{\lambda \in P} (\dim V[\lambda]) e^\lambda \in \mathbb{C}[P],$$

where e^λ is just a formal symbol. The algebra $\mathbb{C}[P]$ has an interpretation as a functions on an algebraic torus

$$T = \mathfrak{h}/2\pi i Q^\vee \cong (\mathbb{C}^*)^r$$

by the correspondence

$$e^\lambda \in \mathbb{C}[P] \quad \leftrightarrow \quad e^{\lambda(t)} \in \mathbb{C}.$$

If G is a simply connected group corresponding to $\mathfrak{g}$, and V is both a representation of $\mathfrak{g}$ and G, then we can interpret

$$\operatorname{ch}(V)(t) = \operatorname{tr}_V(\exp(t)).$$

We only had a function defined on the torus, but it can naturally be extended to the entire group.

Example 30.4. Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then $P = \mathbb{Z}\frac{\alpha}{2}$ where α is a root. Then $\mathbb{C}[P] = \mathbb{C}[x^{\pm 1}]$ is the Laurent polynomials in a single variable. The character is given by

$$\operatorname{ch}(V_n) = x^{-n} + x^{-n+2} + \cdots + x^{n-2} + x^n = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}.$$

In terms of groups, we had a way to compute the multiplicity of an irreducible representation by taking the inner product with respect to the invariant measure. There is going to be an analogue of this fact.

Lemma 30.5. *We have* $\operatorname{ch}(\mathbb{C}) = 1$, $\operatorname{ch}(V_1 \oplus V_2) = \operatorname{ch}(V_1) + \operatorname{ch}(V_2)$, $\operatorname{ch}(V_1 \otimes V_2) = \operatorname{ch}(V_1)\operatorname{ch}(V_2)$, *and* $\operatorname{ch}(V^*) = \overline{\operatorname{ch}(V)}$ *where we interpret* $\overline{e^\lambda} = e^{-\lambda}$.

Theorem 30.6. *Let V be a finite dimensional representation of $\mathfrak{g}$. Then* $\operatorname{ch}(V)$ *is invariant under the Weyl group W. In other words,* $\dim V[\lambda] = \dim V[w\lambda]$.

31 April 12, 2017

We were studying representations of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, there is a splitting

$$V = \bigoplus_{\lambda \in P(V)} V[\lambda]$$

into eigenspaces of $\mathfrak{h}$. Then we can define the character

$$\mathrm{ch}(V) = \sum_{\lambda \in P(V)} (\dim V[\lambda]) e^{\lambda}.$$

Theorem 31.1. *Let V be finite dimensional. Then $\mathrm{ch}(V)$ is invariant under the Weyl group W.*

Proof. It suffices to show that $\mathrm{ch}(V)$ is invariant under reflections $s_\alpha(\lambda) = \lambda - \alpha^\vee(\lambda)\alpha$. Here since $\lambda \in P(V)$ we have $n = \alpha^\vee(\lambda) \in \mathbb{Z}$.

V restricts to a representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$. Then

$$f_i^n : V[\lambda] \to V[\lambda - n\alpha], \quad e_i^n : V[\lambda - n\alpha] \to V[\lambda]$$

are isomorphisms. That is, $\dim V[\lambda] = \dim V[\lambda - n\alpha]$. □

In fact, $\mathrm{ch}(V)$ for V irreducible generate $\mathbb{C}[P]^W$. Recall that once we choose a polarization, we get a splitting

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha.$$

31.1 Highest weight representations

Definition 31.2. A vector $0 \neq v \in V[\lambda]$ is called a **highest weight vector** if $\mathfrak{n}_+ v = 0$. A representation V is called a **highest weight representation** if it is generated by a highest weight vector $v \in V[\lambda]$.

Theorem 31.3. *If V is an irreducible finite dimensional representation of $\mathfrak{g}$ then V is a highest weight representation.*

Proof. Choose $\lambda \in P(V)$ such that $\lambda + \alpha \notin P(V)$ for any $\alpha \in R_+$. (Take an $h \in \mathfrak{h}$ so that $\alpha(h) > 0$ for all $\alpha \in R_+$, and then take $\lambda \in P(V)$ with $\lambda(h)$ maximal.) If we choose $0 \neq v \in V[\lambda]$, then $\mathfrak{n}_+ v = 0$. Then v generates a nonzero subrepresentation of V, and this has to be V by irreducibility. □

In any height weight representation V with highest weight $v \in V[\lambda]$, we would have some relations:

$$hv = \lambda(h)v \text{ for } h \in \mathfrak{h}, \quad ev = 0 \text{ if } e \in \mathfrak{n}_+.$$

The **Verma module** M_λ is the universal module with these relations. Its construction uses the universal enveloping algebra of $\mathfrak{g}$.

If $\rho : \mathfrak{g} \to \operatorname{End}(V)$ is a representation, it image $\{\rho(\mathfrak{g})\} \subseteq \operatorname{End}(V)$ is going to be a sub Lie algebra, but not in general an algebra. It generates some algebra, and we want this to be a quotient of some **universal enveloping algebra**. So we define

$$U\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \Big/ ([x, y] - (xy - yx)).$$

It is the free unital algebra generated by $\mathfrak{g}$ with relations $[x, y] = xy - yx$.

Theorem 31.4 (Poincaré–Birkhoff–Witt theorem)**.** *As vector spaces, $U\mathfrak{g} \cong S\mathfrak{g}$ where $S\mathfrak{g}$ is the symmetric algebra.*

Using this, we can construct the Verma module as

$$M_\lambda = U_\mathfrak{g}/(I_\lambda = (\mathfrak{n}_+, h - \lambda(h) : h \in \mathfrak{h})).$$

Example 31.5. For example, when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the Verma module has a $\mathbb{C}$ at each weight $\lambda, \lambda - 2, \lambda - 4, \ldots$. This is going to irreducible unless λ is a nonnegative integer.

Lemma 31.6. *If V is a highest weight representation with highest weight λ, $V \cong M_\lambda/W$ for some subrepresentation $W \subseteq M_\lambda$.*

Theorem 31.7. *If $\lambda \in \mathfrak{h}^*$ and M_λ is the corresponding Verma module with highest weight vector v_λ, then*

1. *every $v \in M_\lambda$ can be uniquely written as $v = uv_\lambda$ with $u \in U\mathfrak{n}_-$. So $M_\lambda = U\mathfrak{n}_-$ as vector spaces.*
2. *there is a weight decomposition $M_\lambda = \bigoplus_\mu M_\lambda[\mu]$ with*

$$P(M_\lambda) = \lambda - Q_+, \quad Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i.$$

32 April 14, 2017

There was this Verma module M_λ, the universal highest weight representation. Every highest weight representation is a quotient of M_λ, and every irreducible representation is a highest weight representation.

32.1 Classification of irreducible representations

Lemma 32.1. *There exists a unique maximal proper submodule of M_λ.*

Proof. Take the sum of all proper submodules $W \subseteq M_\lambda$ with $W[\lambda] = 0$. □

Define $L_\lambda = M_\lambda/\text{maximal proper submodule}$. By construction, L_λ is irreducible, and any irreducible heights weight representation is isomorphic to some L_λ, in particular, finite dimensional irreducible representations.

For which $\lambda \in \mathfrak{h}^*$ is L_λ finite dimensional? It has to be integral, and it also has to be on the positive Weyl chamber because it is a highest weight. We define the **dominant integral weights** as

$$P_+ = \{\lambda \in P : \alpha^\vee(\lambda) \geq 0 \text{ for } \alpha \in R_+\} = P \cap \overline{C}_+ = P/W.$$

Theorem 32.2. $\dim L_\lambda < \infty$ *if and only if* $\lambda \in P_+$.

Proof. If $\dim L_\lambda < \infty$ then by the $\mathfrak{sl}(2,\mathbb{C})$ case, $\lambda \in P$. It is clear that $\lambda \in R_+$.

For the other direction, recall that in the case of $\mathfrak{sl}(2,\mathbb{C})$ we had an exact sequence

$$0 \to M_{-n-2} \to M_n \twoheadrightarrow L_n \to 0.$$

In the general case, we define the action

$$\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha, \quad w_\bullet \lambda = w(\lambda + \rho) - \rho \text{ for } w \in W.$$

Then we can show that

$$L_\lambda = M_\lambda \Big/ \bigoplus_{i=1}^{r} M_{s_{i\bullet}\lambda}.$$

It follows that L_λ is finite dimensional. □

Corollary 32.3. *Finite dimensional irreducible representations $\mathfrak{g}$ are classified by P_+.*

There is the **Bernstein–Gelfand–Gelfand resolution**, which says that if $\lambda \in P_+$ then there is a long exact sequence

$$0 \leftarrow L_\lambda \leftarrow M_\lambda \leftarrow \bigoplus_{i=1}^{r} M_{s_{i\bullet}\lambda} \leftarrow \bigoplus_{w \in W, \ell(w)=2} M_{w_\bullet \lambda} \leftarrow \cdots \leftarrow M_{w_{0\bullet}\lambda} \leftarrow 0,$$

where w_0 is the unique element of W with largest length, which sends the positive Weyl chamber to the negative Weyl chamber.

This is secretly some topological invariant.

32.2 Weyl character formula

The problem now is to find the characters $\mathrm{ch}(L_\lambda)$ for $\lambda \in P_+$. The strategy is to compute the characters of Verma modules $\mathrm{ch}(M_\lambda)$, and then use the Bernstein–Gelfand–Gelfand resolution. One problem is that $\mathrm{ch}(M_\lambda)$ is not in $\mathbb{C}[P]$, but you can use its some kind of completion:

$$\widehat{\mathbb{C}[P]} = \left\{ f = \sum_{\lambda \in P} c_\lambda e^\lambda : \mathrm{supp}(f) \subseteq \bigcup_{\text{finite}} (\lambda_i - P_+) \right\}.$$

Then we have

$$\mathrm{ch}(M_\lambda) = e^\lambda \,\mathrm{ch}(U\mathfrak{n}_-).$$

Here, the Poincaré–Birkhoff–Witt theorem says that $\prod_{\alpha \in R_+} f_\alpha^{n_\alpha}$ form a basis of $U\mathfrak{n}_-$. So

$$\mathrm{ch}(U\mathfrak{n}_-) = \sum_{\mu \in Q_+} e^{-\mu} P(\mu) = \prod_{\alpha \in R_+} (1 - e^{-\alpha} + e^{-2\alpha} + \cdots) = \prod_{\alpha \in R_+} \frac{1}{1 - e^{-\alpha}}.$$

Thus

$$\mathrm{ch}(M_\lambda) = e^\lambda \prod_{\alpha \in R_+} \frac{1}{1 - e^{-\alpha}}.$$

Now we use the Bernstein–Gelfand–Gelfand resolution to compute

$$\mathrm{ch}(L_\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \,\mathrm{ch}(M_{w_\bullet \lambda}) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w_\bullet \lambda}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

This is called the first form of the **Weyl character formula**. Both the numerator and the denominator are roughly alternating under a reflection.

33 April 17, 2017

33.1 Representations of $\mathfrak{sl}(n, \mathbb{C})$

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with the root system

$$R = \{e_i - e_j : i \neq j\} \subseteq \mathfrak{h}^* = \mathbb{C}^n/\mathbb{C}(1, \dots, 1).$$

We choose the polarization

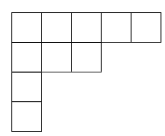
$$R_+ = \{e_i - e_j : i < j\}.$$

Then the integral weights are

$$\begin{aligned} P &= \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^* : \lambda_i - \lambda_j \in \mathbb{Z}\}, \\ P_+ &= \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^* : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0 \text{ are integers}\}. \end{aligned}$$

If you thing about this, these are partitions. So you can represent them using **Young diagrams**, with λ_i boxes on the ith row.

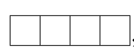
$$\lambda = (5, 3, 1, 1, 0) \quad \leftrightarrow$$



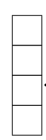
There is a tautological action of $\mathfrak{sl}(n, \mathbb{C})$ on $\mathbb{C}^n$. This representation has highest weight e_1, which is $(1, 0, \dots, 0)$. This has the diagram:

If you take its symmetric power $V = \mathrm{Sym}^k(\mathbb{C}^n)$, its highest weight is going to be $(k, 0, \dots, 0)$. Its exterior power $V = \bigwedge^k(\mathbb{C}^n)$ is going to have highest weight $(1, \dots, 1, 0, \dots 0)$. So their diagrams are

$\mathrm{Sym}^k(\mathbb{C}^n) \quad \leftrightarrow$



, $\bigwedge^k(\mathbb{C}^n) \quad \leftrightarrow$



.

The ring $\mathbb{C}[P]$ can be described as

$$\mathbb{C}[P] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(x_1 \cdots x_n = 1).$$

For $\lambda \in P_+$, the Weyl character formula tells us that

$$\mathrm{ch}(L_\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{\alpha/2})}.$$

In the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we have $\rho = (n-1, n-2, \ldots, 1, 0)$ and so

$$\operatorname{ch}(L_\lambda) = \frac{\sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{\sigma(1)}^{\lambda_1 + n - 1} \cdots x_{\sigma(n)}^{\lambda_n}}{\prod_{i<j} (x_i - x_j)}$$

For $V = \mathbb{C}$, we have $\operatorname{ch}(V) = 1$, and indeed we have the Vandermonde formula

$$\prod_{i<j} (x_i - x_j) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{\sigma(1)}^{n-1} \cdots x_{\sigma(n)}^{0}.$$

33.2 The McKay correspondence

This was a presentation by Wyatt Mackey. There is a special collection of groups called spin groups. There is a double cover

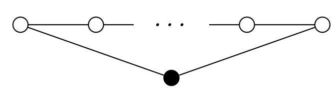
$$1 \to \mathbb{Z}/2 \to \operatorname{Spin}(n) \to \operatorname{SO}(n, \mathbb{R}) \to 1.$$

Finite subgroups of $\operatorname{SU}(2) = \operatorname{Spin}(3)$ correspond to lifts from $\operatorname{SO}(3, \mathbb{R})$.

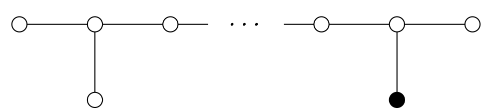
Let G be a finite group and fix a representation ρ. Let $\{\rho_i\} = \operatorname{Ir}(G)$ be the irreducible representations of G. Form a (weighted directed) graph with vertices $\{\rho_i\}$ labeled with $\dim \rho_i$, and edges $\rho_i \to \rho_j$ with multiplicity m_{ij} if ρ_j appears m_{ij} times in $\rho \otimes \rho_i$. This looks like a contrived thing to consider.

Theorem 33.1 (McKay)**.** *For a finite $G \subseteq \operatorname{SU}(2)$, the McKay graph with respect to ρ the canonical representation given by $G \hookrightarrow \operatorname{SU}(2) \hookrightarrow \operatorname{GL}(2, \mathbb{C})$, is of the following form, where the black dot represents the trivial representation:*

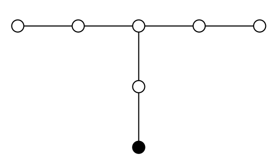
- *G is cyclic:*



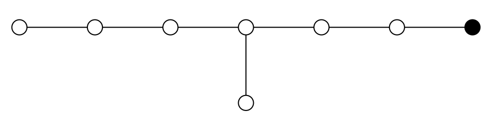
- *G is the spin cover of a dihedral group:*



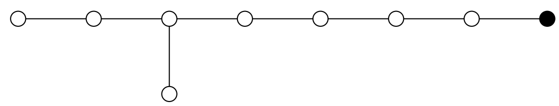
- *G is the spin cover of the tetrahedral group:*



- *G is the spin cover of the octahedral group:*



- *G is the spin cover of the icosahedral group:*



If you remove the trivial representations in each case, you get the Dynkin diagrams. It is really surprising because we are only looking at finite subgroups of SU(2).

Theorem 33.2. *These are the graphs of the minimal resolutions of $\mathbb{C}^2/G$.*

Here is a way to construct this graph. We have G-Hilb($\mathbb{A}^2$) the resolution and take E to be the exceptional locus. This Hilbert scheme is constructed in the following way. Consider the functor

$$\mathcal{H}_{X,n} : \mathsf{Schemes} \to \mathsf{Sets}; \quad S \mapsto \begin{Bmatrix} \text{subschemes of } S \times X \\ \text{flat finite of degree } n \end{Bmatrix}.$$

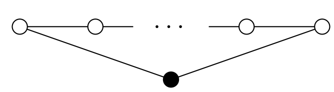
This is represented by $X^{[n]}$. Now given an action of G on X, we have X^{reg} the points on which G acts freely. Then G-Hilb(X) is the irreducible component of $X^{[n]}$ which contains an orbit of a point $x \in X^{\text{reg}}$.

Anyways, let us define

$$E(i) = \{J \in E : [M_J : S_i] \neq 0\}.$$

The socles decompose into simple modules, over $\Pi(Q)$. Then we connect i and j if $E(i) \cap E(j)$ is nonempty.

Let us look at the example of the cyclic group $G \cong \mathbb{Z}/n\mathbb{Z}$. The irreducible representations are $\rho_i(1) = e^{2\pi ik/n}$. Then the graph $\Gamma(G, \rho_1)$ is going to be



since $\mathrm{SU}(2) = \rho_1 \oplus \rho_{-1}$.

Definition 33.3. A **quiver** is a 1-dimensional set $Q = (Q_0, Q_1)$. A **representation of a quiver** is a map $F : Q \to \mathscr{C}$ to a category $\mathscr{C}$, viewed as a simplicial set.

Definition 33.4. The **path algebra** KQ of a quiver is its completion to a category.

Set $Q_0 = \{0, \dots, m\}$, and fix a dimension vector $\underline{d} = (d_0, \dots, d_m)$. We define

$$\mathrm{GL}(\underline{d}) = \prod \mathrm{GL}(d_i), \quad \mathsf{Rep}(Q, \underline{d}) = \prod_{a \in Q_1} \mathrm{Maps}(d_{h(a)}, d_{t(a)}).$$

This can be thought of as an affine scheme. For an action of G on $X = \operatorname{Spec} A$, we can set

$$A^G(\chi) = \bigoplus A^G(\chi_n) = \bigoplus \{\phi \in A : g^*(\phi) = \chi(g)^n \phi\}.$$

This gives the construction of McKay quivers, which makes the correspondence less mysterious.

34 April 19, 2017

34.1 Spin groups and spin representations

This was a talk by Gregory Parker. Recall we have a short exact sequence

$$1 \to \mathbb{Z}/2 \xrightarrow{i} \mathrm{SU}(2) \xrightarrow{\pi} \mathrm{SO}(3) \to 1,$$

where the projection π is

$$z + wj \in \mathrm{SU}(2) \subseteq \mathbb{H} \quad \mapsto \quad (\mathbb{R}^3 = \{bj + ck + dk\} \subseteq \mathbb{H} \to \mathbb{R}^3; \quad v \mapsto qvq^{-1}).$$

Given an irreducible representation $\mathrm{SU}(2) \to V$ with weight k, this factors through $\mathrm{SO}(3)$ if and only if k is even.

A lot of motivation comes from physics. For a spin-1/2 particle, there is the operator J_z, and these look like they generate rotation in 3-space. But in fact, e^{tJ_z} is in $\mathrm{SU}(2)$ not in $\mathrm{SO}(3)$. So for instance, you have to rotate an electron twice to make it look the same. There are also Weyl fermions,with an $\mathrm{SO}(3,1)$ action on $\mathbb{C}^2$.

Definition 34.1. For (V, q) a vector space with inner product over $\mathbb{R}$ or $\mathbb{C}$, we define the **tensor algebra** as

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k}.$$

We can also define $\bigwedge^\bullet(V) = T(V)/(v \otimes w + w \otimes v)$.

Definition 34.2. The **Clifford algebra** $\mathrm{Cl}(V, q)$ is defined as

$$T(V)/(v \otimes w + w \otimes v + 2q(v, w)1).$$

Notice that there is no well-defined grading here.

Proposition 34.3. *As a vector space,* $\bigwedge^\bullet(V) \cong \mathrm{Cl}(V, q)$. *Thus* $\dim \mathrm{Cl}(V, q) = 2^{\dim V}$ *and* $e_{i_1} \cdots e_{i_n}$ *are a basis.*

Example 34.4. Take $\mathbb{R}$ with the normal inner product $\cdot$. The Clifford relations says $2e_1 \cdot e_1 = -2$ and so there is an isomorphism $\mathrm{Cl}(\mathbb{R}, \cdot) \cong \mathbb{C}$.

Example 34.5. Take $\mathbb{C}^2$ with the Hermitian inner product. This Clifford algebra turns out to be $\mathrm{Mat}_2(\mathbb{C})$ with the isomorphism

$$1 \mapsto \mathrm{id}, \quad e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_1 e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 34.6 (Periodicity of $\mathrm{Cl}(\mathbb{C}^n)$)**.** *There is an isomorphism* $\mathrm{Cl}(\mathbb{C}^{n+2}) \cong \mathrm{Cl}(\mathbb{C}^n) \otimes \mathrm{Cl}(\mathbb{C}^2)$.

In particular,

$$\mathrm{Cl}(\mathbb{C}^2) = \mathrm{Mat}_2(\mathbb{C}), \quad \mathrm{Cl}(\mathbb{C}^3) = \mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}), \quad \mathrm{Cl}(\mathbb{C}^4) = \mathrm{Mat}_4(\mathbb{C}), \dots.$$

Take the Lie group $\mathrm{Cl}^\times(V,q) \subseteq \mathrm{Cl}(V,q)$ of units. Let us call this Lie algebra $\mathfrak{cl}^\times(V,q)$. There is an adjoint action

$$\mathrm{Ad} : \mathrm{Cl}^\times(V,q) \to \mathrm{GL}(\mathrm{Cl}(V,q)); \quad v \mapsto v(-)v^{-1}.$$

Proposition 34.7. *The action of Ad_v is (for $q(v,v) \neq 0$)*

$$-\mathrm{Ad}_v(w) = w - \frac{2q(v,w)}{q(v,v)} v.$$

Proof. We can use $v^{-1} = -v/q(v,v)$ and the Clifford relations. □

To get rid of the minus sign, define the twisted adjoint as

$$\widetilde{\mathrm{Ad}}_v = \alpha(v) w v^{-1}, \quad \alpha(v) = \begin{cases} -1 & \text{if } v = v_1 \cdots v_k \text{ with } k \text{ odd}, \\ 1 & \text{if } v = v_1 \cdots v_k \text{ with } k \text{ even}. \end{cases}$$

Define the **Clifford group** as

$$\Gamma(V,q) = \{ v \in \mathrm{Cl}^\times(V,q) : \widetilde{\mathrm{Ad}}_v(V) \subseteq V \}$$

Proposition 34.8. *There is a natural map $\widetilde{\mathrm{Ad}} : \Gamma(V) \to \mathrm{GL}(V)$.*

(i) The image falls in $\mathrm{O}(V,q)$.

(ii) It is surjective with kernel $\mathbb{K}^\times$.

$$1 \to \mathbb{K}^\times \to \Gamma(V,q) \to \mathrm{O}(V,q) \to 1$$

Proof. (ii) Calculate the center.

(i) We check

$$q(\widetilde{\mathrm{Ad}}_v x, \widetilde{\mathrm{Ad}}_v y) = \cdots = q(x,y)$$

using the Clifford relations. To show that it is surjective, show that all reflections are in the image. □

Define a norm map $\mathrm{Cl}(V,q) \to \mathbb{K}^\times$ as

$$\|v_1 \cdots v_k\| = \|v_1\| \cdots \|v_k\|.$$

Now define

$$\mathrm{Spin}(n) = \{ v_1 \cdots v_k \in \Gamma(V) : \|v\| = \pm 1, k \text{ even} \}$$

so that

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q) \to 1.$$

We can likewise define

$$\mathrm{Pin}(n) = \{ v_1 \cdots v_k \in \Gamma(V) : \|v\| = \pm 1 \}$$

so that

$$1 \to \mathbb{Z}/2 \to \mathrm{Pin}(V,q) \to \mathrm{O}(V,q) \to 1.$$

34.2 Symmetric spaces

This was a presentation by Elizabeth Himwich. The goal is to classify symmetric spaces, and the idea is that we can think of them as Lie groups, and classify the associated algebras.

On a Riemannian manifold there is an affine connection ∇_X that satisfies $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$ and $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$. There is a torsion free metric compatible connection, and we can also define curvature.

For any two close points $p, q \in M$, there is a geodesic $t \mapsto \gamma(t)$ such that $\gamma(0) = p$ and $\gamma(1) = q$, and let $\gamma(-1) = q'$. This map $s_p : q \mapsto q'$ is called the **involutive isometry**.

The isometries of M form a group under composition, and it has the compact-open topology.

Theorem 34.9. *Let $I(M)$ be the isometry group of M.*

(1) $I(M)$ is a locally compact Lie group.

(2) For $p \in M$, the subgroup K fixing p is compact.

Theorem 34.10. *Consider M a symmetric space. Let $p_0 \in M$ and $G = I_0(M)$ with $K \subseteq G$ the subgroup fixing p_0.*

(1) G/K is diffeomorphic to M.

(2) $\phi : g \mapsto s_\bullet g s_\bullet$ is an involution isometry.

(3) If we define $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(K)$ with

$$\mathfrak{t} = \{x \in \mathfrak{g} : (d\sigma)_e x = x\}, \quad \mathfrak{p} = \{x \in \mathfrak{g} : (d\sigma)_e x = -x\},$$

then $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$.

For a pair $(\mathfrak{l}, s)$ of an Lie algebra $\mathfrak{l}$ and s and involution, The set $\mathfrak{t}$ of fixed points is a compact subalgebra of $\mathfrak{g}$.

There are three types:

- compact type: $\mathfrak{l}$ is compact and semisimple
- noncompact type: $\mathfrak{l}$ is semisimple
- Euclidean

There is a duality between the compact and noncompact types.

It turns out this reduces to finding all simple Lie algebras over $\mathbb{C}$, and finding all the involutions, and finding the centers.

35 April 21, 2017

35.1 Formal group laws

This was a talk by James Hotchkiss. Over $\mathbb{R}$ or $\mathbb{C}$, a Lie group gives a Lie algebra, and the Baker–Campbell–Hausdorff formula gives access to "higher order data". Our goal is to elaborate this over more general fields or geometries.

Definition 35.1. A **formal group law** of dimension n over any ring R is an n-tuple

$$F(X,Y) \in (R[[X_1,\dots,X_n,Y_1,\dots,Y_n]])^n$$

such that $F(X,0) = F(0,X) = X$ and $F(F(X,Y),Z) = F(X,F(Y,Z))$.

This is actually a group object in the category of formal schemes.

Example 35.2. Given a Lie group G, choose local coordinates such that $0 = 1_G$, and choose x, y near 1_G, and expand the coordinates in x and y.

We can also consider analytic groups. Let k be complete with respect to an absolute value $|\cdot|$. This is further called an ultrametric if $|x-y| \leq \sup\{|x|,|y|\}$. Over such a field, you can do pretty much the same thing.

Given a formal group law, you can construct a group in certain cases. Suppose k is complete with valuation ring $A \subseteq k$, with a maximal ideal $\mathfrak{m} \subseteq A$. Suppose we have a formal group law F of dimension n. Then you can actually give a group structure on $(\mathfrak{m}_A)^n$.

If R is a ring and F is a formal group law over R, then there is a well-defined group structure on the nilpotents $\mathfrak{N}(R)$. This gives a functor $\mathsf{Alg}_r \to \mathsf{Grp}$.

Let us see how to classify formal group laws. A 1-dimensional homomorphism is an element $\alpha \in R[[x]]$ such that

$$\alpha(F(X,Y)) = G(\alpha(X), \alpha(Y)).$$

It is further called a strict isomorphism if $\alpha = x + \cdots$.

Example 35.3. Let k be a field, and define $\hat{G}_a(X,Y) = X+Y$, and $\hat{G}_m(X,Y) = X+Y+XY$. If α is an homomorphism, then

$$\alpha(X)^3 = \alpha(\hat{G}_m(X, \hat{G}_m(X,X))) = \hat{G}_a(\alpha(X), \hat{G}_a(\alpha(X),\alpha(X))) = 3\alpha(X).$$

So if the field have positive characteristic, these two are not isomorphic.

Example 35.4. Consider $R = \mathbb{F}_p[c]/(c^2)$. Then $F(X,Y) = X+Y+XY^p$ is a 1-dimensional formal group law but is not commutative.

Theorem 35.5. *If $\mathfrak{N}(R) = 0$, then every 1-dimensional formal group law over R is commutative.*

35.2 Groups of Lie type

This was a talk by Andrew Gordon. If you were like me two weeks ago, all the simple groups you have encountered are $\mathbb{Z}/p\mathbb{Z}$ and alternative groups. There is an important theorem proved not too long ago, which is the classification of finite simple groups. There are $\mathbb{Z}/p\mathbb{Z}$, A_n, 26 sporadic groups, and groups of Lie type.

If $\mathfrak{g}$ is a simple Lie algebra, we have proved that there is an isomorphism $\mathrm{Der}(\mathfrak{g}) \cong \mathfrak{g}$. This gives a group

$$G_{\mathrm{ad}} \cong \langle \exp(th_\alpha), \exp(g_\beta) \rangle_{\alpha\in\Pi,\beta\in R},$$

which has Lie algebra $\mathfrak{g}$.

We now want to do this over an arbitrary field, which is going to be hard because we don't have exp: what is $1/k!$ if the characteristic is positive? Note that $\mathrm{Ad}(h_\alpha)g_\beta = \langle \alpha, \beta \rangle g_\beta$, and so we can define the analogue of $\exp(th_\alpha)$ as

$$h_\alpha(z)g_\beta = z^{\langle \alpha,\beta\rangle} g_\beta.$$

If we have $[g_\alpha, g_\beta] = \pm(r+1)g_{\alpha+\beta}$, then we also have

$$\exp(t \,\mathrm{Ad}\, g_\alpha)g_\beta = g_\beta + (r+1)g_{\alpha+\beta} + \frac{(r+1)(r+2)}{2} g_{\alpha+2\beta} + \cdots .$$

Here all $(r+1)\cdots(r+n)/n!$ are integers if r is an integer.

There is an abstraction called a **B-N pair**. These are subgroups $B, N \subseteq G$ such that:

(1) B, N span G,

(2) $B \cap N = H$ is a normal subgroup of N,

(3) $N/H = W$ is a group that is generated by elements of order 2,

(4) W does not normalize B,

(5) for $s, w \in W$ with s one of the order 2 generator of W, $sBw \subseteq BswB \cup BwB$.

In the case of the above construction, we would want H to be generated by the Cartan algebra.

Theorem 35.6. *If G has a B-N pair, $G = [G, G]$, and B contains no normal group of G, then G is simple.*

36 April 24, 2017

36.1 Virosoro algebra

This was a presentation by Adam Ball. Our plan is to define Virosoro algebras, Verma modules, and the Kac determinant. The **Viroroso algebra** is defined as

$$\mathrm{Vir} = \mathbb{C}C \oplus \bigoplus_{n\in\mathbb{Z}} L_n,$$

with the commutation relations $[C, L_n] = 0$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}.$$

There is further a conjugation $*$ on Vir as $L_n^* = L_{-n}$ and $C^* = C$. Then it can be checked that $[L_m^*, L_n^*] = [L_n, L_m]^*$.

This shares many properties with semisimple Lie algebras. This algebra comes in looking at projective representations of the Witt algebra. We can look at the universal enveloping algebra $U = U(\mathrm{Vir})$ and also define $\mathrm{Vir}_\pm = \bigoplus_{n\in\mathbb{Z}^+} L_{\pm n}$. Then we write $U_\pm = U(\mathrm{Vir}_\pm)$.

For physics applications, we are only interested in irreducible lowest weight representations. A representation (ρ, V) is going to be unitary if V has a positive definite hermitian form and $\rho(A^*) = \rho(A)^*$. Let $V(c,h)$ be the Verma module generated by v_0 with $L_0 v_0 = hv_0$, $Cv_0 = cv_0$, and $U_+V(c,h) = 0$. Because we want our representation to be unitary, c, h are real.

The weight spaces of the Verma module V also has to be orthogonal. For a generaic $A, B \in U_-$, we would have

$$(Av_0, Bv_0) = (B^*Av_0, v_0) = (\text{coefficient of 1 in } D)$$

if we assume $(v_0, v_0) = 1$. So the inner product is unique. This is further well-defined, because if $Av_0 = Bv_0$ then $(A-B)v_0 = 0$ and so $A - B \in U_+$, and

$$(Dv_0, Av_0) - (Dv_0, Bv_0) = ((A^* - B^*)Dv_0, v_0) = (0, v_0) = 0.$$

Now we have a unique well-defined hermitian form on $V(c,h)$. Let K be the kernel of the inner product. If $k \in K$, then $(V, k) = 0$ and this means that $(A^*V, k) = 0$. Let us write $T(c,h) = V(c,h)/K$. You can show that T is actually irreducible, and so K is the maximal subalgebra.

For the representation to be unitary, it also has to be positive definite. This gives more restriction on c and h. In fact, if $0 < c < 1$ there is only a discrete family irreducible unitary representations.

36.2 Universal enveloping algebras

This was a presentation by Shang Liu. We have briefly talked about universal enveloping algebras in class.

Definition 36.1. For $\mathfrak{g}$ a Lie algebra over $\mathbb{K}$, we define

$$U\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}/(x\otimes y - y\otimes x - [x,y]).$$

Theorem 36.2. *For A an algebra and $\rho : \mathfrak{g} \to A$ a representation in the sense that*

$$\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x,y]),$$

there exists a unique algebra homomorphism $\rho' : U\mathfrak{g} \to A$ such that ρ factors through $U\mathfrak{g}$.

So if $A = \mathrm{End}(V)$, then representations of $\mathfrak{g}$ by V is the same as $U\mathfrak{g}$-module structures on V. There is no natural grading, but there is a filtration

$$U_p\mathfrak{g} = \mathrm{Span}\{\sigma(x_1)\cdots\sigma(x_k)\}_{k\leq p}.$$

Then

$$\mathbb{K} = U_0\mathfrak{g} \subseteq U_1\mathfrak{g} \subseteq \cdots, \quad U\mathfrak{g} = \bigcup_{p\geq 0} U_p\mathfrak{g}.$$

Proposition 36.3. *(a) This is a filtered algebra, i.e., $a \in U_p\mathfrak{g}$ and $b \in U_q\mathfrak{g}$ implies $ab \in U_{p+q}\mathfrak{g}$. Moreover, $ab - ba \in U_{p+q-1}\mathfrak{g}$.*
(b) If $v_1, \dots, v_n$ is a basis of $\mathfrak{g}$ then

$$\{\sigma(v_1)^{k_1}\cdots\sigma(v_n)^{k_n} : \textstyle\sum k_i \leq p\}$$

generate $U_p\mathfrak{g}$.

Proof. This is basically because if $x_1, \dots, x_p \in \mathfrak{g}$ and π is a permutation of $\{1, \dots, p\}$ then

$$\sigma(x_1)\cdots\sigma(x_p) - \sigma(x_{\pi(1)})\cdots\sigma(x_{\pi(p)}) \in U_{p-1}\mathfrak{g}.$$

(a) immediately follows from this and (b) also can be proved. $\square$

Theorem 36.4 (Poincaré–Birkhoff–Witt)**.** *If $v_1, \dots, v_n$ is a basis of $\mathfrak{g}$, then the set $\sigma(v_1)^{k_1}\cdots\sigma(v_n)^{k_n}$ with $\sum k_i \leq p$ is a basis of $U_p\mathfrak{g}$ (assuming characteristic zero).*

The linear independence is hard to show. The idea is that if we have a representation/module $U_\mathfrak{g} \to \mathrm{End}(V)$, it suffices to show that the linear transformations are linearly independent. Consider the polynomial algebra $P = \mathbb{K}[z_1, \dots, z_n]$ with $n = \dim\mathfrak{g}$. Let P_i be the space of polynomials of degree at most i. Denote $Z_I = z_{i_1}\cdots z_{i_p}$ for $I = (i_1, \dots, i_p)$. By $i \leq I$ we will mean $i \leq i_k$ for all k.

Lemma 36.5. *Let $v_1, \dots, v_n$ be a basis of $\mathfrak{g}$. For any $p \geq 0$ there is a unique map $f_p : \mathfrak{g}\otimes P_p \to P$ such that*

(A) $f_p(v_i \otimes Z_I) = z_i z_I$ if $i \leq I$ and $z_I \in P_p$,

(B) $f_p(v_i \otimes Z_I) - z_i Z_I \in P_q$, *for* $z_I \in P_q$ *and* $q \leq p$,

(C) $f_p(v_i \otimes f_p(v_j \otimes Z_J)) = f_p(v_j \otimes f_p(v_i \otimes Z_J)) + f_p([v_i, v_j] \otimes Z_J)$ *for* $Z_J \in P_{p-1}$.

Moreover, $f_p|_{\mathfrak{g} \otimes P_{p-1}} f_{p-1}$.

We can show that $\phi(\sigma(x_{i_1}) \cdots \sigma(x_{i_p}))1 = x_{i_1} \cdots x_{i_p}$. Since these are linearly independent in the polynomial algebra, the elements $\sigma(x_{i_1}) \cdots \sigma(x_{i_p})$ are linearly independent in $U\mathfrak{g}$.

Corollary 36.6. *(1) The map* $\sigma : \mathfrak{g} \to U\mathfrak{g}$ *is injective.*
(2) If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *then* $U\mathfrak{g}_1 \otimes U\mathfrak{g}_2 \to U_{\mathfrak{g}}$ *is an isomorphism of vector spaces.*

37 April 26, 2017

37.1 Affine Lie algebras

This was a talk by Or Eisenberg. A generalized Cartan matrix satisfies

(1) $a_{ii} = 2$,

(2) $a_{ij} = 0$ implies $a_{ji} = 0$,

(3) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i \neq j$.

There are the Chevalley generators $\{h_i, e_i^+, e_i^-\}_{1 \leq i \leq n}$ with the Serre relations

$$[h_i, h_j] = 0, \quad [h_i, e_j^{\pm}] = \pm a_{ji} e_j^{\pm}, \quad [e_j^{\pm}, e_i^{\mp}] = \pm \delta_{ij} h_i, \quad (\mathrm{ad}_{e_i^{\pm}}^{\gamma - a_{ij}}) e_j^{\pm} = 0.$$

For a general matrix, it might be that the roots are not linearly independent and has nonzero kernel. There is a trick of making them linearly independent by extending the rows.

todo

Given a Lie algebra $\mathfrak{g}$, you can define its **loop algebra** as

$$\mathscr{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}, \quad [t^n \otimes g, t^m \otimes h] = t^{n+m} \otimes [g, h].$$

37.2 Peter–Weyl theorem

This was a talk by Aaron Slipper. The whole theory was started by the study of the regular representation of a finite group. For a finite group G, the regular representation is defined as

$$\phi : G \to \mathrm{End}(\mathbb{C}^{|G|}); \quad \phi(g)(e_{g'}) = e_{gg'}.$$

It turns out that this decomposes as $\bigoplus V_i^{\dim V_i}$ over irreducible representations. For a Lie group G, the regular representation is

$$G \to \mathrm{End}(L^2(G)); \quad g \mapsto (f(x) \mapsto f(g^{-1}x)).$$

Lemma 37.1. *For $f \in L^2(G)$, the map $G \to L^2(G)$; $g \mapsto f(g^{-1}x)$ is continuous.*

Proof. Find a continuous function c such that $\|c - f\|_2 < \epsilon$. The function c is defined on a compact set, and so it is uniformly continuous. Then you can approximate this. □

The convolution is defined as

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy.$$

Lemma 37.2. *Let G be a compact group and $f \in L^2(G)$. For all $\epsilon > 0$, there exists a finite collection of Borel sets $E_i \subseteq G$ that E_i disjointly cover G and $y_i \in E_i$ such that $\|f(y^{-1}x) - f(y_i^{-1}x)\|_2 < \epsilon$ for all $y \in E_i$.*

Proof. By the previous lemma, there exists an neighborhood of $1 \in G$ such that $\|f(gx) - f(x)\|_2 < \epsilon$ for all $g \in U$. Then translates of U has a finite cover. $\square$

Lemma 37.3. *Let $f \in L^1(G)$, $h \in L^2(G)$, and*

$$F(x) = \int_G f(y)h(y^{-1}x)dy.$$

Then F is the limit of a sequence of functions, each of which is a finite linear combination of left translates of h.

Theorem 37.4 (Peter–Weyl)**.** *The (Hilbert) linear span of the matrix coefficients of G is all of $L^2(G)$.*

Proof. If $h(x) = (\phi(x)u, v)$ is a matrix coefficient, then $\overline{h(x^{-1})} = (\phi(x)v, u)$ and $h(gx) = (\phi(x)u, \phi(g^{-1})v)$ and $h(xg) = (\phi(x)\phi(g)u, v)$ are also matrix coefficients.

Now let U be the span of matrix coefficients of G. Suppose that $U^\perp \neq 0$, and let $F_1(x)$ be a continuous function in $U^\perp$ with $0 \neq F_1(1) \in \mathbb{R}$. Define

$$F_2(x) = \int_G F_1(yxy^{-1})dy.$$

Because U is closed under the operations, $U^\perp$ is also closed. So $F_2 \in U^\perp$. Then $F(x) = F_2(x) + \overline{F_2(x^{-1})}$ is a conjugation-invariant, unitary function. Consider the function $K(x, y) = F(x^{-1}y)$. Then the Hilbert operator

$$Tf = \int_G K(x, y)f(y)dy$$

is going to be compact. Now because of the spectral theory, there exists a nonzero eigenvalue λ such that V_λ is finite dimensional.

We claim that V_λ is invariant under action of G. Then this decomposes into irreducible representations W_λ. You can show that then the inner product of F with some linear combination of matrix coefficients is positive, but F is orthogonal to U. This is a contradiction. $\square$

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