Math 253y - Symplectic Manifolds and Lagrangian Submanifolds

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Strictly better notes are available on the course webpage `http://www.math.harvard.edu/~auroux/253y18/`.

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Contents

1 September 4, 2018

The main goal of this class is to learn symplectic manifolds, Lagrangian submanifolds, pseudo-holomorphic curves, Floer homology, Fukaya categories, etc. This is not a beginning course on symplectic geometry. For people who do need grade, I will try to have homeworks.

1.1 Overview I

Let us start with reviewing basic symplectic geometry.

Definition 1.1. A **symplectic manifold** (M^{2n}, ω) is an even-dimensional manifold with $\omega \in \Omega^2(M)$ that is closed, i.e., $d\omega = 0$, and non-degenerate, i.e., $\omega : TM \cong T^*M$. This non-degenerate condition is equivalent to $\omega^n \neq 0$.

Example 1.2. Oriented surfaces with an area form, are symplectic manifolds. In higher dimension, there is $M = \mathbb{R}^{2n}$ with the canonical symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. You can also take the cotangent bundle $M = T^*N$ of an arbitrary manifold N. Here, there is

$$\omega = d\lambda, \quad \lambda = pdq$$

where q are coordinates on the base and p are coordinates on the fiber. This λ is called the **Liouville form**. Complex projective space $M = \mathbb{C}P^n$ has a Fubini–Kähler form, and so do smooth complex projective varieties.

Theorem 1.3 (Darboux)**.** *For all $p \in M$, there exists a neighborhood p and local coordinates (x_i, y_i) such that $\omega = \sum_i dx_i \wedge dy_i$.*

So there are no local invariants. On the other hand, there is an obvious global invariant $[\omega] \in H^2(M, \mathbb{R})$.

Theorem 1.4 (Moser)**.** *If M is compact closed, and $(\omega_t)_{t\in[0,1]}$ are a continuous symplectic family with $[\omega_t] \in H^2(M, \mathbb{R})$ independent of t, then there exists an isotopy $\varphi_t \in \mathrm{Diff}(M)$ such that $\varphi_t^*\omega_t = \omega_0$. In particular, $\varphi_1 : (M, \omega_0) \cong (M, \omega_1)$.*

In this field, the group of symplectomorphisms is very large. Inside $\mathrm{Symp}(M, \omega)$, there is a subgroup

$$\mathrm{Ham}(M, \omega) = \text{flows of (time-dependent) Hamiltonian vector fields.}$$

Given any $H \in C^\infty(M.\omega)$, there exists a unique vector field X_H such that $\omega(X_H, -) = -dH$. This vector field preserves the symplectic form, because

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H}d\omega = 0$$

by the Cartan formula.

Lagrangian submanifolds are your savior if you think there are no interesting thing to do, due to Darboux's theorem.

Definition 1.5. A **Lagrangian submanifold** is $L^n \subseteq M^{2n}$ such that $\omega|_L = 0$.

Example 1.6. In $(\mathbb{R}^{2n}_{x_i,y_i}, \omega_0)$, the space $\mathbb{R}^n_{x_i}$ is a Lagrangian submanifold. On a surface, any simple closed curve is a Lagrangian submanifold. In T^*N, there is the zero section, and there are also the cotangent fibers. In $\mathbb{R}^{2n}$, the product $\prod S^1(r_i)$ is Lagrangian. More generally, the T^n-orbits in a toric symplectic manifold is Lagrangian.

Observe that $(TL)^{\perp\omega} = TL$. So we get an isomorphism

$$NL = (TM|_L)/TL \to T^*L; \quad [v] \mapsto \omega(v, -)|_{TL}.$$

Theorem 1.7 (Weinstein)**.** *A neighborhood of L in M is symplectomorphic to a neighborhood of of the zero section in T^*L.*

So deformations of $L \subseteq M$ corresponds to sections of T^*L. But for $\alpha \in \Omega^1(L)$, its graph is Lagrangian if and only if α is closed. Moreover, the deformation is an Hamiltonian isotopy if and only if α is exact. This leads to the notion of a flux, lying in $H^1(L, \mathbb{R})$. For $L = S^1$, this is equal to the area swept, and it vanishes if and only if the isotopy is Hamiltonian.

What kinds of Lagrangian submanifolds L exist in a given symplectic manifold (M, ω)? For example, on a oriented surface, Lagrangians are simple closed curves. To consider Hamiltonian isotopies, you need to keep track of the area swept.

Conjecture 1.8 (Arnold, nearby)**.** *Let N be a closed manifold. In its cotangent bundle, look at closed exact $L \subseteq T^*N$. Then L is Hamiltonian isotopic to the zero section.*

What is exact? Recall that $\omega = d\lambda$. Then L is Lagrangian if and only if $\lambda|_L$ is closed. We say that L is exact Lagrangian if and only if $\lambda|_L = df$ is exact. If we know that L is a section, it is easy. The conjecture is proved for T^*S^1, which is easy, and for T^*S^2 (2013) and T^*T^2 (2016). On the other hand, the homology was known for a bit longer.

Theorem 1.9 (Abonzaid–Kragh, 2016)**.** *Let $L \subseteq T^*N$ be a closed exact Lagrangian, and consider the projection $\pi_L : L \to N$. Then π_L is a (simple) homotopy equivalence.*

What about in $\mathbb{R}^4$? Let us look at closed compact Lagrangians $L \subseteq \mathbb{R}^4$. If L is orientable, then $L \cong T^2$. (This is because the normal bundle is isomorphic to the cotangent bundle, and then you do some computations.) If L is not orientable, we should have $\chi(L) < 0$ and divisible by 4. (The Klein bottle was excluded by Nemirovski in 2006.) So these problems are surprisingly hard. All known Lagrangian tori in $\mathbb{R}^4$ are Hamiltonian isotopic to

- a product torus $S^1(r_1) \times S^1(r_2)$,
- Chekanov (v1990) torus $T_{\mathrm{Ch}}(r)$.

But we don't know if these are all.

The manifold $\mathbb{R}^{2n}$ is a cotangent bundle, so we can talk about exact Lagrangians. It is a theorem of Gromov that there is no closed exact Lagrangian in $\mathbb{R}^{2n}$. If L is exact, then any disk bound by L has zero area. But Gromov showed that $L \subseteq \mathbb{R}^{2n}$ must bound holomorphic discs, and these have positive area. The next best are monotone Lagrangians. These are such that the symplectic area of a disc bound by L is positively proportional to its Maslov index.

In $\mathbb{C}P^2$, which is a toric manifold, we know about product tori. Monotone ones are $\{(x : y : 1) : |x| = |y| = 1\}$. There is also the Chekanov monotone torus, which bounds more holomorphic discs. In 2014, R. Vianna showed that there are infinitely many types of monotone Lagrangian tori.

In $\mathbb{R}^6$, there is a result of Fukaya that states that monotone closed Lagrangians must be diffeomorphic to $S^1 \times \Sigma_g$. On the other hand, there is a construction due to Ekholm–Evashberg–Murphy–Smith 2013 that any $N\#(S^1 \times S_2)$ has a Lagrangian embedding into $\mathbb{R}^6$.

Theorem 1.10. *There exist infintely many different families of monotone Lagrangian $T^3 \subseteq \mathbb{R}^6$.*

All known ones are Lagrangian isotopic to product tori. In $\mathbb{R}^8$, there exist knotted monotone Lagrangian tori T^4. There is a bunch of complicated results like this.

2 September 6, 2018

Last time I started with and overview of all the things that will appear in the class.

2.1 Overview II

The latter part of the course will use J-holomorphic curves to study Lagrangians. These are key tools to study Lagrangians in modern symplectic geometry. In general, there is no reason for a symplectic manifold to carry of complex structure. But they carry almost-complex structures $J : TM \to TM$ with $J^2 = -1$ compatible with the symplectic structure. Here, compatibility means that $\omega(-, J-)$ is a Riemannian metric. The choice is contractible, so it is not too important.

Definition 2.1. A **J-holomorphic curve** is a smooth map from a Riemann surface

$$u : (\Sigma, j) \to (M, J)$$

that satisfy $\overline{\partial}_J u = 0$, i.e., $J \circ du = du \circ j$.

The Riemann surface Σ can have boundaries, and then we sill require that the boundary of Σ maps to a given Lagrangian in M. The space of maps will have finite expected dimension, because $\bar{\partial}$ is Fredholm. Using an index theorem, we can compute the dimension of the moduli space $\mathcal{M}(\Sigma, J, [u])$, where $[u]$ is the homology. Then Gromov compactness says that this moduli space has a suitable compactification. The area of the J-holomorphic curve with respect to the metric g is equal to the symplectic area, which is $\langle [\omega], [u] \rangle$.

Once we have this notion, we can define Lagrangian Floer (co)homology, invented by Floer. Given two Lagrangians L_1, L_2, we will define a chain complex and define

$$HF(L_1, L_2) = H^*(CF(L_1, L_2), \partial).$$

The chain complex is going to be the vector space generated by $L_1 \cap L_2$. We might ask what the coefficient of q in ∂p is. This is going to be a weighted count of J-holomorphic

$$u : \mathbb{R} \times [0, 1] \to M$$

such that the ends goes to q and p. What Floer showed is that if L_i do not bound any holomorphic discs (for instance, the exact case) then $\partial^2 = 0$ and the cohomology is invariant under Hamiltonian isotopies and of J. Moreover, $HF(L, L) \cong H^*(L)$. There is no grading by default, but when they exist, it is going to be a graded isomorphic.

Corollary 2.2. *If L does not bound discs, $\#(L, \psi(L)) \geq \dim H^*(L)$ where $\psi \in \mathrm{Ham}(M, \omega)$ and $\psi(L) \pitchfork L$.*

Example 2.3. Consider the cylinder and L_1 a circle. Suppose we push it around with a Hamiltonian isotopy and get L_2, with two intersections p, q. Then we have

$$\partial p = q - q = 0,$$

and then $HF(L_1, L_2) = CF = H^*(S^1)$.

Example 2.4. Consider the same L_1, but now let L_2 a boundary of a small disc passing L_1. Let the intersection by p, q. In this case,

$$\partial p = (\cdots)q, \quad \partial q = (\cdots)p,$$

and so $\partial^2 \neq 0$.

Near the end of the course, we will talk about other disc-counting invariants, e.g., distinguishing exotic monotone Lagrangians by counting holomorphic discs. Then we will also talk about Fukaya categories. This is a way to package all the Lagrangians with intersections in one category. The objects are (nice) Lagrangian submanifolds, with extra data, with morphisms given by Floer complexes and differentials. Composition

$$CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_1, L_3)$$

is given by counting holomorphic discs bound by L_1, L_2, L_3. This is really an A_∞-category. The reason this language is useful is because the category can be generated by some Lagrangians we are familiar with.

2.2 Symplectic manifolds

A reference for this is *Lectures on symplectic geometry* by A. Cannas da Silva, and *Introduction to symplectic topology* by McDuff–Salamon. Recall that a **symplectic manifold** is a manifold (M^{2n}, ω) equipped with a real 2-form ω that is closed and non-degenerate. This also gives a map $\omega_x : T_xM \to T_x^*M$. Also, $\omega^{\wedge n}$ is nonzero, so we get a top exterior form.

Example 2.5. Here are some examples:

- An oriented surface M with an area form.
- Euclidean space $M = \mathbb{R}^{2n}$ with $\omega = \sum_i dx_i \wedge dy_i$. Actually, every nondegenerate skew-symmetric bilinear form on a vector space always looks like this. There are other symplectic structures on $\mathbb{R}^{2n}$ but these are because something interesting happens at infinity.
- Take $M = T^*N$, and $\omega = d\lambda$. In local coordinates, take $(q_1, \dots, q_n)$ coordinates on N, and take $p_1, \dots, p_n)$ dual coordinates on the fiber. Then consider $\lambda = \sum_{i=1}^n p_i dq_i$, so that $\omega = \sum dp_i \wedge dq_i$. This λ is independent on coordinates, and you can check this. Or you can intrinsically define λ as

  $$\lambda_{(x,\xi)}(v) = \langle \xi, d\pi(v) \rangle$$

  where $\pi : T^*N \to N$ is the projection.

- Products of symplectic manifolds are $M_1 \times M_2$ with symplectic form $\omega_1 \oplus \omega_2 = \pi_1^*\omega_1 + \pi_2^*\omega_2$.
- Symplectic submanifolds $V \subseteq W$ are submanifolds with $\omega|_{TV}$ is nondegenerate.

Try to think about for which n the sphere S^{2n} has a symplectic form.

A **Lagrangian submanifold** is $L^n \subseteq (M^{2n}, \omega)$ such that $\omega|_L = 0$. In this case,

$$T_x L^{\perp\omega} = \{u \in T_x M : \omega(u, v) = 0 \text{ for all } v \in T_x L\} = T_x L$$

for dimension reasons.

Example 2.6. Again, here are examples.

- The zero section in T^*N.
- More generally, consider graphs of 1-forms $\alpha \in \omega^1(M, \mathbb{R})$. First, we note that
$$\operatorname{graph}(\alpha) \subseteq T^*N \xrightarrow{\pi} N$$
is a diffeomorphism. Then the restriction of the Liouville form is tautologically
$$\lambda|_{\operatorname{graph}(\alpha)} = \alpha \quad \text{(or rather, } \pi^*\alpha).$$
So $\operatorname{graph}(\alpha)$ is Lagrangian if and only if $d\lambda|_{\operatorname{graph}(\alpha)} = 0$ if and only if $d\alpha = 0$. (It is called an exact Lagrangian if and only if α is exact.)
- The conormal bundle to a smooth submanifold $V^k \subseteq N^n$ is defined as
$$N^*V = \{(x, \xi) : x \in V, \xi \in T_x^*N, \xi|_{T_x N} = 0\} \subseteq T^*N.$$
This is going to be a rank $n - k$ subbundle of $T^*N|_V$. Then $\lambda|_{N^*V} = 0$ because $d\pi(v) \in T_x V$ implies $\lambda(v) = \langle \xi, d\pi(v) \rangle = 0$. So N^*V are exact Lagrangians.
- Let $\varphi \in \operatorname{Diff}(M)$ be a diffeomorphism. Consider
$$\operatorname{graph}(\varphi) = \{(x, \varphi(x)) \in M^- \times M\}.$$
Here, M^- is the symplectic manifold with $-\omega$ instead of ω. Then $\varphi \in \operatorname{Symp}(M, \omega)$ if and only if $\operatorname{graph}(\varphi)$ is a Lagrangian in $M^- \times M$. This is because $i^*\hat{\omega} = -\omega + \varphi^*\omega$.

3 September 11, 2018

Today we will do some basic symplectic geometry.

3.1 Hamiltonian vector fields

Remember for (M, ω) a symplectic manifold and $H \in C^\infty(M, \mathbb{R})$, there exists a unique vector field X_H such that

$$\iota_{X_H}\omega = -dH.$$

Recall that given a time-dependent vector field V_t, the corresponding flow φ_t generated by this is a family of diffeomorphisms

$$\varphi_0(p) = p, \quad \frac{d}{dt}(\varphi_t(p)) = v_t(\varphi_t(p)).$$

Then it is a general fact that

$$\frac{d}{dt}(\varphi_t^*\alpha) = \varphi_t^*(L_{V_t}\alpha).$$

There is also Cartan's formula

$$L_v\alpha = d\iota_v\alpha + \iota_v d\alpha.$$

So given a Hamiltonian H_t, we get a flow of X_{H_t} and it satisfies

$$\varphi_t^*\omega = \omega$$

because

$$\frac{d}{dt}(\varphi_t^*\omega) = \varphi_t^*(L_{X_{H_t}}\omega) = \varphi_t^*(d\iota_{X_{H_t}}\omega + \iota_{X_{H_t}}d\omega = \varphi_t^*(-dH_t + 0) = 0.$$

Definition 3.1. We define the group of **Hamiltonian diffeomorphisms** $\mathrm{Ham}(M, \omega)$. (This is a group, because if we concatenate, we can reparametrize so that the flow is smooth at the boundary.)

Also note that $dH(X_H) = -\omega(X_H, X_H) = 0$. So the flow of H preserves the level sets of H.

Example 3.2. Here are some examples:

- Take $\mathbb{R}^2$ with $\omega_0 = dx \wedge dy = rdr \wedge d\wedge$. Let us take the Hamiltonian $H = \frac{1}{2}r^2$. Then the Hamiltonian vector field is

  $$X_H = \frac{\partial}{\partial\theta},$$

  which is rotation.

- On S^2, consider the standard area form $\omega_0 = d\theta \wedge dz$. If we take the Hamiltonian $H = z$, then
$$X_H = -\frac{\partial}{\partial \theta}.$$
These are examples of Hamiltonian S^1-actions.
- Take $M = T^*N$ with coordinates (q, p). Let us first consider $H = H(q)$, a Hamiltonian that factors through the projection $\pi : T^*N \to N$. It turns out that
$$X_H(q, p) = (0, -dH_{(q)}) \in T_{(q,p)}(T^*N).$$
On the other hand, we can give a Riemannian metric on N so that $TN \cong T^*N$. Consider $H = \frac{1}{2}|p|^2$. Then
$$X_H = \text{geodesic flow}.$$
If you couple these together $H = \frac{1}{2}|p|^2 + V(q)$, we will get the dynamics of a particle with potential V.

Hamiltonian vector fields are those with $\iota_X \omega = -dH$ is exact. A **symplectic vector field** is that with $\iota_X \omega$ is closed. The flow still preserves ω. Given a symplectic isotopy (φ_t) generated by a symplectic vector field V_t, we get the identity component in Symp, and actually $\pi_0 \operatorname{Symp}$ is also very interesting.

To look at the difference between symplectic flows and Hamiltonian flows, we define
$$\operatorname{Flux}(\varphi_t) = \int_0^1 [-\iota_{V_t} \omega] dt \in H^1(M, \mathbb{R}).$$
Any Hamiltonian isotopy has flux zero. For small enough isotopy, you can also show that a symplectic isotopy with zero flux can be made into a Hamiltonian isotopy.

The flux has a concrete interpretation. Given $\gamma : S^1 \to M$, the image $\varphi_t(\gamma)$ sweeps out a cylinder. Write $\Gamma(s, t) = \gamma_t(\gamma(s))$, and we compute
$$\begin{aligned}\int_\Gamma \omega = \int \Gamma^* \omega &= \int_0^1 \int_{S^1} \omega\Big(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\Big) ds dt \\ &= \int_0^1 \int_{S^1} -\iota_{V_t} \omega \Big(\frac{\partial \Gamma}{\partial x}\Big) ds dt = \int_0^1 \langle [-\iota_{V_t} \omega], [\varphi_t(\gamma)] \rangle dt = \langle \operatorname{Flux}, [\gamma] \rangle.\end{aligned}$$

3.2 Moser's theorem

All these theorems use what is called Moser's trick.

Theorem 3.3 (Moser's theorem)**.** *Let M be a compact closed manifold, and let us say I have $(\omega_t)_{t \in [0,1]}$ a smooth family of symplectic forms, such that $[\omega_t] \in H^2(M, \mathbb{R})$ is independent of t. Then there exists an isotopy $\varphi_t \in \operatorname{Diff}(M)$ such that $\varphi_t^* \omega_t = \omega_0$. Hence φ_1 gives $(M, \omega_0) \cong (M, \omega_1)$.*

Proof. If we look at $\frac{d\omega_t}{dt}$, this is going to be exact because they all lie in the same cohomology class. So there exist 1-forms α_t such that

$$d\alpha_t = \frac{d\omega_t}{dt}.$$

Then there exists a smooth family α_t. (This requires an explicit version of Poincaré's lemma, and some partition of unity argument.) We now know that there exists a vector field X_t such that

$$\iota_{X_t}\omega_t = -\alpha_t.$$

Let us now take φ_t the flow generated by X_t. Then

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*\Big(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\Big) = \varphi_t^*\Big(d\iota_{X_t}\omega_t + \frac{d\omega_t}{dt}\Big) = 0$$

because $\iota_{X_t}\omega_t = -\alpha_t$ and $\frac{d\omega_t}{dt} = d\alpha_t$. □

This is only for compact closed manifolds, but if don't assume this, you need to make assumptions at infinity. It is also not true that any two symplectic forms in the same cohomology class are isotopic in the same $[\omega]$. (This is different from the Kähler world, where you can just linearly interpolate.) For instance, McDuff has the following example. On $S^2 \times S^2 \times T^2$, we can take

$$\omega_0 = \pi_1^*\omega_{X^2} + \pi_2^*\omega_{S^2} + ds \wedge dt, \quad \omega_1 = \psi^*\omega_0.$$

Here, $\psi(z, w, s, t) = (z, R_{z,t}(w), s, t)$ where $R_{z,t}$ is the rotation by axis z with angle t. They are in the same cohomology class, but there does not exist a ω_t connecting them in this class. You can find the proof in McDuff's book on J-holomorphic curves (in something like section 9.7).

3.3 Darboux's theorem

Theorem 3.4 (Darboux's theorem)**.** *For any $p \in (M, \omega)$, there exist local coordinates near p in which $\omega = \sum dx_i \wedge dy_i$.*

We need the following linear algebra fact.

Lemma 3.5. *We have $(T_pM, \omega_p) \cong (\mathbb{R}^{2n}, \omega_0)$ as a symplectic vector space.*

Proof. You build a standard basis e_i, f_j such that $\omega(e_i, f_i) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. You first choose e_1 and then find f_1 such that $\omega(e_1, f_1) = 1$. Then look at the orthogonal to the span of e_1 and f_1 and continue by induction. □

Using the standard basis on T_pM, we find local coordinates

$$\mathbb{R}^{2n} \supset U \xrightarrow{f} M$$

such that $\omega_1 = f^*\omega$ agrees with ω_0 at the origin. Define

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Since non-degeneracy is an open condition, and $\omega_0 = \omega_1$ at 0, these are all symplectic on a neighborhood of 0. Shrink the domain U if necessary. Note that $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ is closed, hence exact. So define

$$d\alpha = d\omega_1 - d\omega.$$

We can assume that $\alpha = 0$ at the origin. But then, the first-order terms in α become constant forms in $d\alpha$, and this is zero. So we can discard these terms and assume that α vanishes to order 2 at the origin.

Let v_t be the vector field such that $\iota_{v_t}\omega_t = -\alpha$. Then $v_t(0) = 0$. Also, let φ_t be the flow of v_t, well-defined and staying inside U in a small neighborhood of 0. If we let Moser do its thing, we get

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*\Big(d\iota_{v_t}\omega_t + \frac{d\omega_t}{dt}\Big) = \varphi_t(-d\alpha + d\alpha) = 0.$$

Then we find that

$$(f \circ \varphi_1)^*\omega = \varphi_1^*(f^*\omega) = \varphi_1^*\omega_1 = \omega_0.$$

4 September 13, 2018

Homework 1 is under construction. We have talked about Moser and Darboux next time.

4.1 Lagrangian neighborhood theorem

Proposition 4.1. *If $L \subseteq M$ is a Lagrangian submanifold, then $NL \cong T^*L$.*

Proof. We have $TM|_L \cong T^*M|_L \twoheadrightarrow T^*L$, and the kernel is TL. So $TM/TL \cong NL \cong T^*L$. ☐

Theorem 4.2 (Lagrangian neighborhood theorem, Weinstein)**.** *If $L \subseteq M$ is a Lagrangian, then there exist neighborhoods U of L in (M, ω) and U_0 of the zero section in (T^*L, ω_0), and a symplectomorphism*

$$\varphi : (U_0, \omega_0) \xrightarrow{\sim} (U, \omega)$$

that maps L as you think.

Proof. We first pick a complement to TL, i.e., a subbundle $N \subseteq TM|_L$ such that $TM|_L = TN \oplus N$. Here, we can ensure N is a Lagrangian subbundle, for instance, by picking an ω-compatible metric. Now we can use the exponential map to build

$$\psi : T^*L \cong N \supseteq U_0 \to U \subseteq M$$

such that (i) ψ along the zero section is the inclusion, (ii) the pullback of the symplectic form $\psi^*\omega = \omega_1$ coincides with ω_0 at the zero section.

We are now in position to use Moser. Consider

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

These are going to be exact symplectic forms on a neighborhood of the zero section. So we can take

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\alpha$$

for some 1-form α. In fact, we can choose α so that it vanishes at every point of the zero section (even to order 2). Let v_t to be the vector field corresponding to it, so that $\iota_{v_t}\omega_t = -\alpha$. Let φ_t be the flow generated by it. Then we have the customary calculation

$$\frac{d}{dt}(\varphi_t^*\omega_t) = 0.$$

Our final answer is then going to be $\varphi = \psi \circ \varphi_1$ (defined over some neighborhood of the zero section). ☐

There also exist neighborhood theorems for symplectic submanifolds or isotropic manifolds. But they are not nice.

4.2 Hamiltonian group actions

A lot of construction comes with Hamiltonian group actions. Let us say we have a Lie group G. If G acts on M, it induces a map of Lie algebras

$$T_eG = \mathfrak{g} \to \mathfrak{X}(M) = (\text{vector fields on } M); \quad \xi \mapsto X_\xi = \frac{d}{dt}\Big|_{t=0} (\exp(t\xi)x).$$

This is a Lie algebra homomorphism, that is, $X_{[\xi,\eta]} = [X_\xi, X_\eta]$.

Let us now look at actions which preserve ω,

$$G \to \mathrm{Symp}(M, \omega).$$

For instance, a symplectic S^1-action are symplectic vector fields whose flow are 2π-periodic. We say that an S^1-action is Hamiltonian if the vector field $X_{\partial/\partial\theta}$ is Hamiltonian. More generally, if I have an action of a torus T^k, we will need each S^1 factor to act in a Hamiltonian way. In this case, there exist k functions $H_1, \dots, H_k \in C^\infty(M, \mathbb{R})$ such that $X_{\partial/\partial\theta_i} = X_{H_i}$. This is the notion of a moment map. This is a package that contains all the Hamiltonians for X_g, where $g \in \mathfrak{g}$.

Definition 4.3. We say that a G-action is **Hamiltonian** if there exists a **moment map** $\mu : M \to \mathfrak{g}^*$ with the following properties:

(1) For all $\xi \in \mathfrak{g}$, let $H_\xi = \langle \mu, \xi \rangle : M \to \mathbb{R}$. Then X_ξ is the Hamiltonian vector field generated by H_ξ.

(2) The moment map μ is G-equivariant, i.e.,

$$\langle \mu(g \cdot x), \mathrm{Ad}_g(\xi) \rangle = \langle \mu(x), \xi \rangle.$$

Note that if G is abelian, this is just $\mu(gx) = \mu(x)$. For $G = T^k$, we can call

$$\mu = (H_1, \dots, H_k) : M \to \mathfrak{g}^* \cong \mathbb{R}^k.$$

These Hamiltonians need to satisfy the 2π-periodicity condition, and also they should commute with each other. Also (2) says that μ is invariant under the G-action, so the flow of X_{H_i} preserves not just H_i but also all H_j. Actually condition (2) comes for free in this case. Because the Lie bracket is zero, we have

$$X_{\{H_i, H_j\}} = [X_{H_i}, X_{H_j}] = 0,$$

where $\{f, g\} = dg(X_f) = \omega(X_f, X_g)$ is the **Poisson bracket**. So $\{H_i, H_j\}$ is constant, and in fact 0 because you can integrate along the S^1 corresponding to X_{H_i}. So $dH_j(X_{H_i}) = 0$, i.e., H_j is invariant under the ith action.

So for torus actions, we found that

- $\omega(X_{H_i}, X_{H_j}) = \{H_i, H_j\} = 0$, orbits are isotropic, (so you can't have an effective Hamiltonian T^n-action on an symplectic manifold with dimension less than $2n$)

- the level sets of $\mu = (H_1, \dots, H_k) : M \to \mathfrak{g}^* \cong \mathbb{R}^k$ are foliated by orbits, and moreover coisotropic (this means $(TN)^{\perp\omega} \subseteq TN$), because orthogonal complement to the orbit is the tangent space.

Example 4.4. There is a standard T^n-action on $(\mathbb{C}^n, \omega)$, given by

$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_1, \dots, z_n) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n).$$

This has moment map

$$\mu = \Big(\frac{1}{2}|z_1|^2, \dots, \frac{1}{2}|z_n|^2\Big).$$

Note that the image of this map is the positive orthant of $\mathbb{R}^n$.

Definition 4.5. A **toric manifold** is a symplectic manifold (M^{2n}, ω) with a Hamiltonian T^n-action. Then the regular levels of the moment map $\mu : M \to \mathbb{R}^n$ are Lagrangian.

Regular levels correspond to where the action has discrete stabilzer. So by dimension reasons, regular levels are disjoint unions of T^n-orbits. In fact, we can show connectedness assuming that M is something like compact and closed. So each nonempty fiber of μ is a single orbit.

Theorem 4.6 (Atiyah, Guillemin–Sternberg 1982)**.** *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian action of a torus. Denote the moment map $\mu : M \to \mathbb{R}^k$. Then the level sets of μ are connected, and the image of μ is a convex polytope (which is the convex hull of the image under μ of the fixed points).*

5 September 18, 2018

Last time we were talking about Hamiltonian actions. For now we are going to focus on Hamiltonian torus actions. If G acts on (M, ω), you can package them into

$$\mu : M \to \mathfrak{g}^*,$$

so that for any $\xi \in \mathfrak{g}$, the Hamiltonian flow corresponding to $\langle \mu, \xi \rangle : M \to \mathbb{R}$ is X_ξ the vector flow generating the action. Also, there was an equivariance condition.

In the T^k-action, we get

$$\mu = (H_1, \dots, H_k) : M \to \mathbb{R}^k$$

packaging the k Hamiltonians. These actions should commute, $\{H_i, H_j\} = 0$. The orbits are isotropic, and the regular levels are coisotropic.

5.1 Atiyah–Guillemin–Sternberg convexity theorem

Theorem 5.1 (Atiyah, Guillemin–Sternberg)**.** *If (M, ω) is a compact symplectic manifold with a Hamiltonian T^k action with moment map $\mu : M \to \mathbb{R}^k$, then the level sets of μ are connected and the image $\mu(M) \subseteq \mathbb{R}^k$ is a convex polytope. Moreover, this is the convex hull of the image of the fixed points under μ, and the faces has rational slope.*

You can see this if you have a toric Kähler manifold, but it is not clear that every symplectic toric manifold comes from this case. Connectedness is essentially saying that there are no index 1 saddle points.

Let us first look at the local picture. Consider $p \in M$ a fixed point for T^k (or a subtorus $T^l \subseteq T^k$). This is the same thing as saying that p is a critical point of μ (or a linear projection of μ onto $\mathbb{R}^l$). Then there is an action of T^k on the symplectic vector space (T_pM, ω_p). Then by averaging, there exists an invariant metric on T_pM, compatible with the invariant complex structure. (This is complicated than it looks, but it is quite standard.) Then we can look at the generators of the linearized action.

So the generators of the linearized action are going to be k commuting matrices in $\mathfrak{sp}(2n, \mathbb{R})$, and in fact $\mathfrak{u}(n)$. Because these are anti-hermitian commuting matrices, they can be simultaneously diagonalized. That is, there is a block decomposition so that each $\frac{\partial}{\partial \theta_j}$ acts by

$$\bigoplus_{i=1}^n \begin{pmatrix} 0 & -\lambda_i^j \\ \lambda_i^j & 0 \end{pmatrix},$$

where λ_i^j are integers since the action should be periodic. Let us package them as $\vec{\lambda}_i = (\lambda_i^j)_{j=1,\dots,k} \in \mathbb{Z}^k$. (This is sometimes called the weights of the action.)

That tells you that the moment map is something like

$$\mu = \mu(p) + \sum_{i=1}^{n} \frac{1}{2}|z_i|^2 \vec{\lambda}_i + \cdots .$$

Now there is a linearization theorem the tells us that the action is conjugate to this on a neighborhood. Look at the direction $\bigcap_{j=1}^{k} \ker(M_j)$ where the action is degenerate. Then the theorem says that this is really T_p of the fixed point set. The idea is that if it is not a fixed point, you can't go around and come back in time 2π. Or you can prove this by using the exponential chart for an invariant metric and showing that geodesics makes geodesics.

So we get the moment map really looks like the quadratic equation above. Locally, the image is then a convex cone spanned by $\vec{\lambda}_i \in \mathbb{Z}^k$ inside $\mathbb{R}^k$.

Proof. For connectedness, observe that any linear projection $\langle \mu, \xi \rangle$ is a Morse–Bott function of even index. This prevents any index 1 or coindex 1 saddles, and this shows connectedness. So you keep slicing the moment map until you get the inverse image of a point.

Global convexity follows from local convexity and connectedness. This is not particularly illuminating. $\square$

In the toric case, where T^n acts on M^{2n}, what we get is that the level sets of μ are single orbits.

Example 5.2. If we look at $\mathbb{C}^2$ with a T^2 action, the moment map is

$$\mu = \Big(\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2\Big).$$

The image is the first orthant in $\mathbb{R}^2$. (By the way, the theorem doesn't really apply here.) The level sets are $S^1(r_1) \times S^1(r_2) \subseteq \mathbb{C}^2$, which are tori if $r_1, r_2 > 0$, and $S^1(r_1) \times \{0\}$ if it is degenerate.

Example 5.3. Consider $\mathbb{C}P^1 = S^2$, with the round volume form and the action

$$(z_0 : z_1) \mapsto (z_0 : e^{i\theta} z_1).$$

In this case, the moment map is

$$\mu = \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

In this case, the image is and interval.

Example 5.4. Take $\mathbb{C}P^2$ with action

$$(z_0 : z_1 : z_2) \mapsto (z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2).$$

If you work out, the image of the moment map is going to be a triangle, with vertices $(0 : 0 : 1)$ and $(0 : 1 : 0)$ and $(1 : 0 : 0)$.

5.2 Delzant's theorem

Theorem 5.5 (Delzant)**.** *A compact toric symplectic (M^{2n}, ω) up to T^n-equivariant symplectomorphism is in one-to-one correspondence with Delzant polytopes $\Delta = \operatorname{im}(\mu) \subseteq \mathbb{R}^n$ up to translation.*

In general, there is an action of $\mathrm{GL}(n, \mathbb{Z})$ on both sides. On the toric manifold side, it acts on the torus, and so $\mathfrak{t}^*$. On the polytope side, it is this action on $\mathfrak{t}^* \cong \mathbb{R}^n$. But I need to tell you what a Delzant polytope is.

Definition 5.6. A **Delzant polytope** is a convex polytope that is

- simple, n edges meet at each vertex, i.e., locally combinatorially looks like $(\mathbb{R}_{\geq 0})^n$,
- rational, the normals to the facets can be chosen in $\mathbb{Z}^n$, i.e., the edges are parallel to vectors in $\mathbb{Z}^n$,
- smooth, the primitive integer normals (or edges) at each vertex form a basis of $\mathbb{Z}^n$.

Without the smoothness condition, you get toric orbifolds.

For instance, the triangle with vertices $(0, 0), (0, 1), (1, 0)$ is a Delzant polytope, but $(0, 0), (0, 1), (2, 0)$ is not because the smoothness condition fails at $(0, 1)$.

You can only keep the combinatorial data, with the slopes of all the edges. If you only see this, you only recover the complex structure (as a complex algebraic variety or a complex manifold). The length of the edges are giving the data of the cohomology class. In fact, if you take any edge in the Delzant polytope, and integrate ω along that S^2, you get the length of the edge.

One direction of Delzant's theorem can be done by looking at the image of the moment map. The converse direction is interesting. One possibility is to look at local patches and try to glue them together. This works, but it is slightly painful.

A faster and counterintuitive way is to consider the Delzant polytope as the intersection

$$\Delta = (\mathbb{R}_{\geq 0})^N \cap (\text{affine } n\text{-plane in } \mathbb{R}^N \text{ with rational slope}).$$

You can take N to be the number of facets, with the ith coordinate the distance to the ith facet, measured in an appropriate way. Then you will be able to do this. This is useful because we can build M as a symplectic reduction of a Hamiltonian T^{N-n}-action on $(\mathbb{C}^N, \omega_0)$.

Theorem 5.7 (Marsden–Wienstein)**.** *Let there be a Hamiltonian action of a compact Lie group G on (M, ω), with moment map $\mu : M \to \mathfrak{g}^*$. Assume G acts freely on $\mu^{-1}(0)$. (By equivariance, $\mu^{-1}(0)$ is preserved by G.) Then*

$$M_{\text{red}} = \mu^{-1}(0)/G$$

is a smooth manifold, and carries a natural symplectic form ω_{red} *with the following property: if*

$$M \supseteq \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/G = M_{\mathrm{red}},$$

then $\pi^*\omega_{\mathrm{red}} = \omega|_{\mu^{-1}(0)}$.

6 September 20, 2018

The symplectic reduction was the following theorem.

6.1 Symplectic reduction

Theorem 6.1. *Let G be a compact Lie group with a Hamiltonian action on (M, ω), corresponding to the moment map $\mu : M \to \mathfrak{g}^*$. Assume that G acts freely on $\mu^{-1}(0)$. (By equivariance G preserves $\mu^{-1}(0)$.) Then $M_{\text{red}} = \mu^{-1}(0)/G$ is a smooth manifold, and carries a natural symplectic form ω_{red} such that $\pi^*\omega_{\text{red}} = \omega|_{\mu^{-1}(0)}$.*

Proof. Consider a basis $\xi_1, \dots, \xi_k$ of the Lie algebra, and $X_1, \dots, X_k$ the corresponding vector fields. Let us call $H_1, \dots, H_k$ be the components of μ, defined by $H_i = \langle \mu, \xi_i \rangle$. The condition that G acts freely means that $X_1, \dots, X_k$ are linearly independent for all $x \in \mu^{-1}(0)$. So $dH_1, \dots, dH_k$ are linearly independent. This shows that $d\mu$ is surjective at every $x \in \mu^{-1}(0)$. Then 0 is a regular value and $\mu^{-1}(0)$ is smooth. Now G compact acts freely, and some basic fact tells that $\mu^{-1}(0)/G$is a smooth manifold.

We now show that $\mu^{-1}(0)$ is coisotropic. At $x \in \mu^{-1}(0)$, we have

$$\begin{aligned} T_x\mu^{-1}(0) &= \ker d\mu = \ker dH_1 \cap \cdots \cap \ker dH_k \\ &= X_1^{\perp\omega} \cap \cdots \cap X_k^{\perp\omega} = \operatorname{span}(X_1, \dots, X_k)^{\perp\omega} = T_x(G \cdot x)^{\perp\omega}. \end{aligned}$$

But $T_x(G \cdot x)^{\perp\omega}$ is contained in $T_x\mu^{-1}(0)$. Now it follows from a general fact of coisotropic submanifolds. If we look at any coisotropic submanifold $N \subseteq M$, we can look at $(TN)^{\perp\omega} \subseteq TN$. As a consequence of ω being closed, this $(TN)^{\perp\omega}$ is closed under the Lie bracket. This you use the formula for $d\omega(X, Y, Z)$ versus Lie derivatives. Then the Frobenius integrability theorem says that we get a foliation, called the **isotropic foliation**.

In our case, the leaves are the G-orbits. This is fibered as $N = \mu^{-1}(0) \xrightarrow{\pi} M_{\text{red}} = \mu^{-1}(0)/G$. But how do we find ω_{red} such that $\pi^*\omega_{\text{red}} = \omega|_N$? At each point $x \in N$ we look at $\pi(x)$ and see that $\omega|_x$ is pulled back from some $\omega|_{\pi(x)}$. Then we use G-equivariance to translate to other points. $\square$

Example 6.2. Look at the diagonal S^1 action on $(\mathbb{C}^n, \omega_0)$, with

$$e^{i\theta}(z_1, \dots, z_n) = (e^{i\theta}z_1, \dots, e^{i\theta}z_n), \quad \mu = \frac{1}{2}\sum |z_i|^2.$$

If we look at $\mu^{-1}(\frac{1}{2})$, it is the unit circle $S^{2n-1} \subseteq \mathbb{C}^n$. The degenerate direction is iz, and if I look at the quotient, we get

$$S^{2n-1}/S^1 \cong \mathbb{C}P^{n-1}.$$

Just to check with what we did last time, we had a T^n-action on $\mathbb{C}^n$, with moment map $\vec{\mu} = (\frac{1}{2}|z_1|^2, \dots, \frac{1}{2}|z_n|^2)$. The diagonal action is $\mu_{\text{diag}} = \sum_i \mu_i$, and so the toric reduction is going to correspond to the simplex

$$\{\textstyle\sum_i \mu_i = \frac{1}{2}\} \cap (\mathbb{R}_{\geq 0})^n.$$

This even gives a formula for the standard Kähler form on $\mathbb{C}P^{n-1}$ and its moment map. The moment map is

$$\mu((z_1 : \cdots : z_n)) = \left(\frac{|z_1|^2}{|z_1|^2 + \cdots + |z_n|^2}, \ldots, \frac{|z_{n-1}|^2}{|z_1|^2 + \cdots + |z_n|^2}\right).$$

Example 6.3. Let us do a non-abelian example. Consider the Grassmannian $G(k,n)$ of complex k-planes in $\mathbb{C}^n$, which is the $k \times n$ matrices of full rank modulo $\mathrm{GL}(k,\mathbb{C})$. So we look at $\mathbb{C}^{k\times n}$ with the standard symplectic form ω_0 and look at the $\mathrm{U}(k)$ action by left multiplication. If you think about this, you will see that there is a relatively simple formula,

$$\mu(M) = \frac{i}{2}MM^* \in \mathfrak{u}(k) \cong \mathfrak{u}(k)^*.$$

Then we can take $\mu^{-1}(\frac{i}{2}I)$, and then we get

$$M_{\text{red}} = \{MM^* = I\}/\,\mathrm{U}(k) = G(n,k).$$

You can compute $\dim_{\mathbb{C}} M_{\text{red}} = k(n-k)$.

Example 6.4. Let's talk about a fun example, of polygon spaces. Start with $\mathrm{SO}(3)$ acting on $S^2(r) \subseteq \mathbb{R}^3$, with the standard area form. This action is Hamiltonian. There is standard basis of $\mathrm{so}(3)^*$ given by rotation by the standard axes. With that understood, the moment map of this action

$$\mu : S^2(r) \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$$

is just the inclusion map.

Example 6.5. But now let's think about the diagonal action of $\mathrm{SO}(3)$ on $S^2(r_1) \times \cdots \times S^2(r_n)$. The moment map for that will be

$$\mu(v_1, \ldots, v_n) = v_1 + \cdots + v_n \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*.$$

So if we look at the quotient, we get

$$\mu^{-1}(0)/\,\mathrm{SO}(3) = \{(v_1, \ldots, v_n) : |v_i| = r_i, \textstyle\sum_i v_i = 0\}.$$

This is the space of n-gons in $\mathbb{R}^3$, modulo translations and rotations. This may be singular, and to ensure smoothness, we may need r_i generic. (We are afraid about polygons being contained in a line, so that the $\mathrm{SO}(3)$-action is not free.) Then you will get that $\mu^{-1}(0)/\,\mathrm{SO}(3)$ is a symplectic manifold of dimension $\dim_{\mathbb{R}} = 2n - 6$.

The triangle space ($n = 3$) is a point. For $n = 4$, there is a "bending" S^1-action that rotates a point along the diagonal. This is a Hamiltonian with moment map the length of the diagonal. There is not much choice for a 2-dimensional symplectic manifold with a Hamiltonian S^1-action. It is going to be S^2 with the moment polytope

$$\ell \in [|r_1 - r_2|, r_1 + r_2] \cap [|r_3 - r_4|, r_3 + r_4].$$

For $n = 5$, the bending action will not work always, because maybe the diagonal is degenerate. But for generic lengths, this makes the polygon space a toric symplectic manifold. If the lengths are generic and close to 1, the polytope is a heptagon, if you graph the triangle inequalities in the (ℓ_1, ℓ_2)-space, where ℓ_i are the two diagonals. This is $S^2 \times S^2$ blown up at 3 points. (Blowing up corresponds to chopping up a corner of your polytope.)

We will now talk about constructions of Lagrangians. One example is, of course, levels of moment maps. There is a generalization of this. Assume we are in the situation of a symplectic reduction.

Proposition 6.6. *Let $L_{\mathrm{red}} \subseteq (M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be a Lagrangian. Then $\pi^{-1}(L_{\mathrm{red}}) \subseteq \mu^{-1}(c) \subseteq M$ is a Lagrangian in (M, ω). (Here, c is a central element in $\mathfrak{g}^*$.) Moreover, every G-invariant Lagrangian in (M, ω) is contained in a (central) level of μ.*

Index