Math 99r - Curves on algebraic surfaces

Taught by Ziquan Yang
Notes by Dongryul Kim

Spring 2018

This tutorial was taught by Ziquan Yang, a graduate student. The class met on Tuesdays and Thursdays from 4:30 to 6pm. The course closed followed Mumford's *Lectures on Curves on an Algebraic Surface*, and there was only a final paper on a topic related to the course.

Contents

1 January 30, 2018

The main reference will be Mumford's *Lectures on curves on an algebraic surface*. A more general reference would be Hartshorne, or Griffiths–Harris, or *Complex geometry* by D. Huybrechts, or *Foundations of algebraic geometry* by Vakil. We are going to use complex manifolds as a motivating example.

1.1 Complex manifolds

Definition 1.1. A **complex manifold** is a smooth manifold with holomorphic charts $\{(U_i, \varphi_i)\}$ given by the transition functions being biholomorphisms.

The chart $\{(U_i, \varphi_i)\}$ is supposed to tell you what it means for a local $\mathbb{C}$-valued function to be holomorphic, i.e., gives the sheaf of holomorphic functions. A map $f : M \to N$ of complex manifolds is something that pulls a holomorphic function back to a holomorphic function.

Example 1.2. For instance, consider the map $\mathbb{C}/\langle 1, i\rangle \to \mathbb{C}/\langle 2, i\rangle$ given by $x + yi \mapsto 2x + yi$. This is a smooth nice map, but is not a morphism of complex manifolds. The function $x + yi$ is holomorphic is holomorphic with respect to the complex structure of E, but its pullback $2x + yi$ is not.

Definition 1.3. A **holomorphic vector bundle** is a complex vector bundle with a holomorphic structure. It is going to be a complex vector bundle $\pi : E \to X$ where X is a complex manifold and E has a structure of a complex manifold. Locally, it is going to be given by local trivializations $\{(U_i, \psi_i)\}$ where $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ are isomorphisms such that the transition maps $U_i \cap U_j \to \mathrm{GL}(\mathbb{C}^n)$ are holomorphic.

Because of this rigidity, exact sequences of holomorphic vector bundles might not split. A real manifold can carry many different complex structures. So things like singular cohomology $H^*(-, \mathbb{Z})$ are not good enough for our purposes.This was actually studied classically by Riemann. If you integrate around paths, we get an embedding

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1)^\vee.$$

These numbers are called **periods**. This is functorial, because if you have a morphism $X \to Y$, we get

$$\begin{array}{ccc} H_1(X,\mathbb{Z}) & \longrightarrow & H^0(X,\Omega^1_X)^\vee \\ \downarrow & & \downarrow \\ H_1(Y,\mathbb{Z}) & \longrightarrow & H^0(Y,\Omega^1_Y)^\vee. \end{array}$$

This is a lattice in a complex vector space, and it is quite fine. But from a modern point of view, this is studying the Hodge structures.

The quotient $H^0(X,\Omega^1_X)^\vee/H_1(X,\mathbb{Z})$ is a complex torus, and is a variety. This is called the **Albenese variety**

$$\mathrm{Alb}(X) \cong \mathrm{Pic}^0(X) = \{\text{moduli of } \deg 0 \text{ line bundles on } X\}$$

For instance, $\dim \mathrm{Pic}^0(X) = \dim H^0(X,\Omega^1_X) = g$ is the geometric genus.

In this course, we want to study the moduli of line bundles with $c_1 = 0$ on an algebraic surface. (You can think of $c_1 = 0$ as an analogue of $\deg = 0$.) On an algebraic surface X, Severi was studying the quotient group

$$\begin{aligned} G_{\mathrm{alg}} &= \langle C - C' : C \sim_{\mathrm{alg}} C' \rangle, \\ G_{\mathrm{lin}} &= \langle C - C' : C \sim_{\mathrm{lin}} C' \rangle. \end{aligned}$$

Here, this algebraic equivalence is $C \sim C'$ there is a flat morphism $X \to S$ such that C and C' are fibers. This is a linear equivalence if $S = \mathbb{P}^1$.

Severi considered this group $\mathrm{Pic}^0 = G_a/G_l$ and showed that its dimension is $\dim G_a/G_l = \dim H^1(X,\mathscr{O}_X) = \frac{b_1}{2}$ where $b_1 = \dim_{\mathbb{Q}} H^1(X,\mathbb{Q})$. We will construct $G_{\mathrm{alg}}/G_{\mathrm{lin}}$ by purely algebraic methods.

1.2 Holomorphic vector bundles

Given a holomorphic vector bundle $E \to X$ and an open set $U \subseteq X$, let us denote

$$\mathscr{V}(U) = \{\text{holomorphic sections } s : U \to E\}.$$

This has a natural structure of an $\mathscr{O}_X(U)$-module.

Example 1.4. There is the tautological line bundle $\mathscr{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$. This is

$$E = \{([\ell], x) \in \mathbb{P}^1 \times \mathbb{C}^2 : x \in \ell\}.$$

As a sheaf $\mathscr{E}$, it will send

$$\mathscr{E}(U) = \{\text{holomorphic maps } \varphi : U \to \mathbb{C}^2 \text{ such that } \varphi([\ell]) \in \ell\}.$$

Choosing a local trivialization $\pi^{-1}(U) \to U \times \mathbb{C}$ over U is equivalent to choosing a nowhere vanishing section over U. Let me describe this local trivialization. There are homogeneous coordinates $[z_0 : z_1]$ on $\mathbb{P}^1$. The open cover is given by $U_0 = \{z_0 \neq 0\}$ with coordinates $s = \frac{z_1}{z_0}$ and $U_1 = \{z_1 \neq 0\}$ with coordinates $t = \frac{z_0}{z_1}$. The local trivialization is given by

$$U_0 \to \mathbb{C}^2; [z_0 : z_1] \mapsto (1, s)$$

and likewise $U_1 \to \mathbb{C}^2$ given by $[z_0 : z_1] \mapsto (1, t)$. If you do this, it is clear what the transition function should be. It should be given by $\times \frac{z_1}{z_0}$.

We can also ask what the global sections are. For holomorphic vector bundles, global sections tend to be very finite-dimensional. A section $U_0 \to E$

corresponds to a entire function $f : U_0 \to \mathbb{C}$, and likewise $U_1 \to E$ corresponds to $g : U_0 \to \mathbb{C}$. Then the gluing condition is

$$g(t) = \frac{1}{t} f\left(\frac{1}{t}\right).$$

You'll quickly realize that there is no such f, g except for 0.

But you can relax the condition and look for meromorphic sections. Here, $s = \frac{1}{t}$ is a meromorphic section, and it has one pole. The number of zeroes minus the number of poles is independent of the choice of section, and so it is a natural invariant of the line bundle. This is called the **degree** of the line bundle.

Given two vector bundles V and W, we can take $V \oplus W$, $V \otimes W$, and $\mathrm{Hom}(V, W)$. For the tautological line bundle $E \to \mathbb{P}^1$, we can form its dual $E^\vee = \mathrm{Hom}(\ell, \mathbb{C})$. The global sections are going to be 2-dimensional.

2 February 1, 2018

2.1 Divisors and the Picard group

Let $L \to X$ be a holomorphic line bundle over a curve X. This, with a meromorphic section will correspond to the divisors $\mathrm{Div}(X)$. When we quotient this by forgetting about sections, we get a correspondence

$$\{L \to X\} \quad \longleftrightarrow \quad \mathrm{Div}(X)/\operatorname{PDiv}(X).$$

Let see how this correspondence can be described. Let $\mathscr{K}$ be the sheaf of the meromorphic functions on X. Then for each line bundle L, we can look at the sheaf $\mathscr{L}$ corresponding to this, and then we can think about the meromorphic sections

$$H^0(X, \mathscr{L} \otimes \mathscr{K}).$$

(Think $H^0 = \Gamma$ for now.) A section $\sigma \in H^0(X, \mathscr{L} \otimes \mathscr{K})$ corresponds to an embedding $\mathscr{L} \hookrightarrow \mathscr{K}$. Given a section σ, we can define a divisor, which is an element of

$$\mathrm{Div}(X) = \bigoplus_{p \in X} \mathbb{Z}p.$$

This is given by

$$\mathrm{Div}(\sigma) = (\text{zeroes}) - (\text{poles}).$$

Given a divisor D, we can also define a line bundle. This is given by

$$U \mapsto \{f \in \mathscr{K}(U) : f + (D|_U) \geq 0\}.$$

Then adding a point is allowing a pole at p, and subtracting a point is require a zero at p. This describes the correspondence

$$\{L \to X \text{ with } \sigma \in H^0(X, \mathscr{L} \otimes \mathscr{K})/\mathbb{C}^\times\} \quad \longleftrightarrow \quad \mathrm{Div}(X).$$

Now let us see what happens when we remove the choice of a section. For any two σ_1, σ_2, they differ by a meromorphic function. That is, there exists a $f \in \mathscr{K}(X)$ such that $\sigma_1 = f\sigma_2$. Then we have $\mathrm{Div}(\sigma_1) = \mathrm{Div}(\sigma_2) + \mathrm{Div}(f)$. We define the **principal divisors** as

$$\mathrm{PDiv} = \{D : D = \mathrm{Div}(f)\}$$

for a meromorphic function f. In general, we have $\mathscr{O}_X(D_1) \cong \mathscr{O}_X(D_2)$ if $D_1 = D_2 + \mathrm{Div}(f)$ because multiplication by f gives the isomorphism.

2.2 Riemann–Roch and Riemann–Hurwitz

We can define the cotangent bundle in the following way. For each coordinate chart Z, we consider the differentials dz and then glue these bundles together.

This is well-defined because on triple intersections, we have the chain rule. This cotangent bundle is often denoted Ω^1_X. If $\dim X = n$, we denote

$$\omega_X = \bigwedge^n \Omega^1_X$$

the **canonical bundle**. If X is a curve, $\omega_X = \Omega^1_X$ and $\mathrm{Div}(\omega_X)$ (up to PDiv) is called the **canonical divisor** K.

What is the canonical bundle of $\mathbb{P}^1$? We can pick a generic differential and then compute its divisor. Consider the coordinate charts

$$s = \frac{z_1}{z_0}, \quad t = \frac{z_0}{z_1}.$$

Then $s = \frac{1}{t}$ on $U_0 \cap U_1$. If we take ds, then it is going to be $ds = -\frac{1}{t^2}dt$. So we get

$$\omega_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(-2).$$

By the way, on $\mathbb{P}^1$ we have $D \in \mathrm{PDiv}$ if and only if $\deg D = 0$. This means that the line bundle is determined uniquely by its degree. That is why we immediately said $\omega_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(-2)$.

Theorem 2.1 (Riemann–Roch)**.** *For any line bundle $\mathscr{L}$, we have*

$$\dim H^0(X, \mathscr{L}) - \dim H^0(X, \omega_X \otimes \mathscr{L}^\vee) = \deg L - g + 1.$$

Here, g is the genus of the curve X.

We're not going to prove it because it involves Serre duality. But let's plug in $\mathscr{L} = \mathscr{O}_X$. Then

$$1 - h^0(X, \omega) = 0 - g + 1$$

and so $h^0(X, \omega) = g$. For general varieties, this is called the **geometric genus**. If we put $\mathscr{L} = \omega$, we get

$$g - h^0(X, \mathscr{O}_X) = \deg \omega - g + 1$$

and then $\deg \omega = 2g - 2$. This is to be expected, from a topological perspective. In general, $\chi(M) = e(TM) = c_1(\omega^\vee)$. But $\chi(M) = 1 - 2g + 1 = 2 - 2g$. So $c_1(\omega) = 2g - 2$. A good way to remember this is that if the genus of X is high, ω_X will be positive.

Given a morphism between two curves, there is a way of relating the genus of the two curves.

Theorem 2.2 (Riemann–Hurwitz)**.** *Let $f : X \to Y$ be a morphism of two curves. Then*

$$2g_X - 2 = (\deg f)(2g_Y - 2) + \sum_i (r_i - 1)$$

where r_i is the ramification degree.

I should first define the degree $\deg f$. The easiest way is to use topology. The map $f : X \to Y$ induces a map $H^2(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z})$. This is a map $\mathbb{Z} \to \mathbb{Z}$, and it is defined to be multiplication by $\deg f$. Ramification degree is defined in the following way. Locally, if $f(p) = q$, and z is a local coordinate at q and w is a local coordinate at p, then $f^*(z)$ vanishes at p. If it is of order $n = 1$, then we say that it is **unramified**, and if $n > 1$, we say that it is **ramified** of degree n.

Proof. This should be thought of as comparing the canonical divisors of X and Y. You can show that

$$f^*\omega_Y \cong \omega_X \otimes \mathscr{O}_X\biggl(-\sum_i (r_i - 1)P_i\biggr).$$

If z is a local coordinate on Y, and w is a local coordinate on X, the bundle ω_Y locally generated by dz and so f^*dz will be generated $f^*\omega_Y$. Then this will look like $w^{r_i-1}dw$ time some invertible thing.

On the other hand, $\deg(f^*L) = \deg f \cdot \deg L$. (Or you can see this via $c_1 f^* = f^* c_1$.) $\square$

You can use this to compute the genus of a plane curve. Given an explicit curve in $\mathbb{P}^2$, you project down to $\mathbb{P}^1$. Then you can compute all the ramification data, and from this determine the genus.

2.3 Formalism with sheaves

Consider $\mathcal{C}$ the category of open subsets of a topological space X. Given two open sets $U \subseteq V$, there is one morphism $U \to Y$. A **presheaf** is a functor

$$\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}/\mathsf{Ab}/\mathsf{CRing}/\dots.$$

This means that you can restrict things on a larger open subset to a smaller open subset. A morphism of presheaves is a natural transformation of functors. This is a global notion.

A **sheaf** is a presheaf satisfying the two axioms:

- If U is covered by U_i then the section on U is determined by the sections restricted to U_i.
- If there are sections over U_i that agree on the intersections, then they come from a section on U.

A compact way to say these two together is that $\mathscr{F}(U)$ is the equalizer

$$\mathscr{F}(U) \longrightarrow \textstyle\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j).$$

3 February 6, 2018

We are going to take a functorial point of view on schemes.

3.1 Line bundles and $\mathbb{P}^n$

If you take a complex manifold M and a line bundle, and you take some sections $s_0, \dots, s_n$ such that for every point $x \in M$, there exist some $s_i(x) \neq 0$. Then you can define $M \mapsto \mathbb{P}^n$ given by

$$M \to \mathbb{P}^n; \quad x \mapsto [s_0(x), \dots, s_n(x)].$$

This is well-defined, because whatever trivialization you choose, $[s_0(x), \dots, s_n(x)]$ is going to be the same point even though the individual $s_i(x)$ doesn't make sense.

How does the converse work? Given a map $\varphi : M \to \mathbb{P}^n$, will you be able to find a line bundle L with sections $s_0, \dots, x_n$? We can just pullback

$$L \cong \varphi^* \mathscr{O}_{\mathbb{P}^n}(1), \quad s_i = \varphi^* x_i.$$

Let me now present a formalism. We define

$$\mathsf{C1} = \text{category of “morphisms to } \mathbb{P}^n\text{”}.$$

The objects are morphisms $M \to \mathbb{P}^n$, and morphism to $N \to \mathbb{P}^n$ are morphisms $M \to N$ that make the diagram commute. We also define

$$\mathsf{C2} = \text{category of “pencils”}$$

with objects $L \to M$ with $\ell_0, \dots, \ell_n \in H^0(M, L)$ such that for all $x \in M$, $\ell_i(x) \neq 0$ for some i, i.e., $\mathscr{O}_X^{n+1} \twoheadrightarrow L$. A morphism from $L \to M$ to $E \to N$ would be a morphism $M \to N$ and an isomorphism $L \cong f^* E$ such that $\ell_i \mapsto e_i$.

Proposition 3.1. *A global section $s \in H^0(M, \mathscr{L})$ corresponds to a morphism $\mathscr{O}_X \to \mathscr{L}$.*

What we just discussed give an equivalence of categories between $\mathsf{C1}$ and $\mathsf{C2}$. Explicitly, we can map

$$(M \xrightarrow{\varphi} \mathbb{P}^n) \mapsto (\varphi^* \mathscr{O}_{\mathbb{P}^n}(1), \varphi^* x_0, \dots, \varphi^* x_n).$$

In the other direction, we have

$$(L \to M, \ell_i) \mapsto (f : M \to \mathbb{P}^n;\ x \mapsto [\ell_i(x)]).$$

So given an object X in a category $\mathcal{C}$, you want to consider it as a functor $\operatorname{Hom}(-, X)$. In general, if you have a category $\mathcal{C}$, you can consider the category of presheaves

$$\widehat{\mathcal{C}} = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}).$$

There is a natural functor

$$\mathcal{C} \to \widehat{\mathcal{C}}; \quad C \mapsto h^C = \mathrm{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}.$$

This is the **Yoneda embedding**. This says that the object C is completely determined by the functor $\mathrm{Hom}(-, C)$.

Given a random contravariant functor $h : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, a natural question is whether there exists an C such that $h = h^C$. If $h = h^C$, we say that h is **represented** by C.

3.2 Schemes

These are just generalizations of commutative rings. For a commutative ring A, we give $\operatorname{Spec} A$ the set of prime ideal of A a topology with closed subsets $V(I)$ for ideals $I \subseteq A$. You can verify that $V(I)$ is precisely the image of $\operatorname{Spec} A/I \to \operatorname{Spec} A$. Given a map $\varphi : A \to B$, you get a continuous map

$$f : \operatorname{Spec} B \to \operatorname{Spec} A; \quad \mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

Open subsets will look like $\operatorname{Spec} A - V(I)$, and there is a base of distinguished open subsets consisting of

$$U_f = \{\mathfrak{p} \subseteq A : f \notin \mathfrak{p}\}.$$

The subsets U_f can be identified with $\operatorname{Spec} A_f$. Then we can define the structure sheaf as

$$\mathcal{O}_X(U_f) = A_f.$$

Then for a general open subset, we can define $\mathcal{O}_X(U)$ by gluing them.

Now we can define schemes. A **ringed space** is a topological space X with a sheaf of rings $\mathscr{F}$. Then a morphism $(X, \mathscr{F}) \to (Y, \mathscr{G})$ is given by a continuous map $X \to Y$ along with a morphism $\mathscr{G} \to f_*\mathscr{F}$ of sheaves.

A **locally ringed space** is a ringed space $(X, \mathscr{F})$ such that $\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U)$ is a local ring. A morphism between two locally ringed spaces $(X, \mathscr{F}) \to (Y, \mathscr{G})$ is a map $X \to Y$ with a morphism $\mathscr{G} \to f_*\mathscr{F}$ such that $\mathscr{G}_p \to \mathscr{F}_p$ is a local homomorphism.

Definition 3.2. An **affine scheme** is a locally ringed space that is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$.

There is an equivalence of categories

$$\mathsf{AffSch} \to \mathsf{CRing}^{\mathrm{op}}; \quad X \mapsto H^0(X, \mathscr{O}_X).$$

The real content of this is that if $A \to B$ is a ring map, and get the map $\operatorname{Spec} B \to \operatorname{Spec} A$, these consist of all morphisms as locally ringed spaces. In particular, I can take a pathological map like

$$\operatorname{Spec} B \to \{\mathfrak{p}\}, \quad \mathscr{O}_Y \to f_*\mathscr{O}_X.$$

But the thing is that if it is a local homomorphism, it can't be anything other than coming from a ring homomorphism. If $\mathfrak{p}$ is mapped to $\mathfrak{q}$, then we should have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{q}} \end{tikzcd}$$

such that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is a local homomorphism.

Definition 3.3. A **scheme** is a locally ringed space that is locally isomorphic to an affine scheme.

This is just like a manifold. In general, if you want to give a morphism $X \to Y$, you need to give the data of $f : X \to Y$ and also $f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X$. But we can also describe this using the equivalence between affine schemes and commutative rings. Given any $f : X \to Y$, we can locally find a neighborhood $f(p) \in U \cong \operatorname{Spec} B \subseteq Y$, and then find a neighborhood $p \in V \cong \operatorname{Spec} A \subseteq f^{-1}(U)$ and then just describe the ring map $A \to B$.

In practice, we work with schemes over some base scheme S. This means that the objects are morphisms $X \to S$. A scheme X can be considered as that over $\operatorname{Spec} \mathbb{Z}$. Varieties are schemes over an algebraically closed field $k = \overline{k}$ can be thought of as a scheme over k, with some special property.

4 February 8, 2018

Let us take $\overline{k} = k$. Consider the scheme $\operatorname{Spec} k[t]/(t^2)$. For a $A = k[x_1, \ldots, x_n]/I$ and a closed point $\mathfrak{m} \subseteq A$, we define the **Zariski cotangent space** as $\mathfrak{m}/\mathfrak{m}^2$, and the **tangent space** as $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$.

Proposition 4.1. *There is a natural identification*

$$\operatorname{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k[t]/(t^2), \operatorname{Spec} A) \cong \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

Proof. The map of schemes is defined by a ring map $f : A \to k[t]/(t^2)$. Given this, we can construct a map

$$\mathfrak{m}/\mathfrak{m}^2 \to k; \quad x \mapsto f(x).$$

Conversely, suppose that we have $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$. Then we get a map

$$f : A \to k[t]/(t^2); \quad x \mapsto \bar{x} + \psi(x - \bar{x})t.$$

The hard thing is to verify that this is a ring homomorphism. You can verify this. □

4.1 Fiber product of schemes

Fiber products can be discussed in general categories. This is an object $X \times_S Y$ such that for every T the following diagram can be filled in uniquely:

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

What this really means is that

$$h_{X \times_S Y} = h_X \times_{h_S} h_Y.$$

This should be interpreted as when tested against any test scheme T, we get this in Set. If S is a final object, the fiber product is the same as the product. There is a natural map $A \times_S B \to A \times B$, and this is always a monomorphism. In the case of sets, it is

$$A \times_S B = \{(a, b) \in A \times B : f(a) = f(b) \in S\}.$$

Proposition 4.2. *Fiber products preserve monomorphisms. Composition of fiber products is a fiber product.*

We would like to construct a fiber product of schemes. This is actually equivalent to proving that the functor

$$h_X \times_{h_S} h_Y$$

is representable.

Proposition 4.3. *If you can cover X by $\{X_i\}$ and each $X_i \times_S Y$ exists, then $X \times_S Y$ exists.*

Proof. The isomorphisms $U_{ij} \cong U_{ji}$ induce isomorphisms between $X \times_S U_{ij}$ and $X \times_S U_{ji}$. Then you can glue these together. $\square$

Now cover S by $\{S_i\}$. If X_i and Y_i are the preimages of S_i, then

$$X_i \times_{S_i} Y_i = X_i \times_{S_i} (S_i \times_S Y) = X_i \times_S Y.$$

So we reduce everything to the affine case. Here, you can check that $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ is just $\operatorname{Spec}(A \otimes_R B)$.

Example 4.4. This works as how you would expect. We have a pullback

$$\begin{tikzcd} \operatorname{Spec} k[x,y] \arrow[r] \arrow[d] & \operatorname{Spec} k[x] \arrow[d] \\ \operatorname{Spec} k[y] \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

That is, $\mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k$.

You can also define fibers using fiber products. If you map $\operatorname{Spec} k$ to a closed point in X and take the fiber product with $Y \to X$, you will get the fiber. Similarly, you can take intersections.

Definition 4.5. A **closed immersion** $Y \to X$ is a morphism that is, on topological spaces, a homeomorphism onto its image, and $\mathscr{O}_X \to f_* \mathscr{O}_Y$ is surjective.

Example 4.6. Consider the projection map $k[x] \to k[x,y]/(y^2 - x)$. If you take the fiber over a, you get

$$\begin{tikzcd} Y_a = \operatorname{Spec} k[y]/(y^2-a) \arrow[r] \arrow[d] & \operatorname{Spec} k[x,y]/(y^2-x) \arrow[d] \\ \operatorname{Spec} k[x]/(x-a) \arrow[r] & \operatorname{Spec} k[x]. \end{tikzcd}$$

For $a = 0$, you get $\operatorname{Spec} k[y]/(y^2)$.

If you think about $\dim_k H^0(Y_a, \mathscr{O}_{Y_a})$, this is always 2. This can be thought of as the intersection number.

5 February 13, 2018

5.1 Pullback of sheaves

Definition 5.1. For $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, we say that F and G are **adjoint functors** if for all $a \in A$ and $b \in B$,

$$\mathrm{Hom}_{\mathcal{A}}(a, G(b)) = \mathrm{Hom}_{\mathcal{B}}(F(a), b).$$

There should be compatibility conditions

$$\begin{tikzcd} \mathrm{Hom}(a', G(b)) \arrow[r] \arrow[d] & \mathrm{Hom}(F(a'), b) \arrow[d] \\ \mathrm{Hom}(a, G(b')) \arrow[r] & \mathrm{Hom}(F(a), b'). \end{tikzcd}$$

Let $f : X \to Y$ be a morphism of topological spaces. If $\mathscr{F}$ is a sheaf on X, you can **push forward** to

$$f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)).$$

Denote the left adjoint of this by f^{-1}, so that

$$\mathrm{Hom}(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}).$$

Concretely, we can write this as

$$f^{-1}\mathscr{G}(U) = \lim_{V \supseteq f(U)} \mathscr{G}(V).$$

You should verify that this is really an adjoint.

If X and Y are ringed spaces, we can talk about sheaves of modules. Let $f : X \to Y$ with $\mathscr{O}_Y \to f_*\mathscr{O}_X$. Then we can push forward an $\mathscr{O}_X$-module

$$f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$$

and consider as an $\mathscr{O}_Y$-module. In this case, we have an adjoint f^*. IN particular,

$$f^* : \mathsf{Mod}(Y) \to \mathsf{Mod}(X)$$

is a functor. This can be actually written as

$$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

5.2 Quasi-coherent sheaves

In most cases, we care about quasi-coherent sheaves. Given a A-module M, we can build out a sheaf $\tilde{M}$ on $\operatorname{Spec} A$, given by

$$\tilde{M}(D(f)) = M \otimes_A A_f.$$

Definition 5.2. A sheaf $\mathscr{F}$ (over a scheme X) is **quasi-coherent** if you can cover X by affine opens $\{X_i\}$ with $X_i = \operatorname{Spec} A_i$ such that $\mathscr{F}|_{X_i} \cong \tilde{M}_i$ for some M_i an A_i-module.

In this case, you can prove that for any $\operatorname{Spec} A \subseteq X$, we will have $\mathscr{F}|_{\operatorname{Spec} A} \cong \tilde{M}$ for some A-module M.

Another important fact is that if $X = \operatorname{Spec} A$, then there is an equivalence of categories

$$\mathsf{Mod}(X) \quad \longleftrightarrow \quad \mathsf{Mod}_A.$$

Moreover, it is an equivalence of categories, so that $0 \to M' \to M \to M'' \to 0$ is exact if and only if $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact.

If $s \in \mathscr{F}(D(f))$, then there exists an n such that $f^n s$ can be lifted to a global section. This can be used to prove that if

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

with $\mathscr{F}'$ quasi-coherent over $X = \operatorname{Spec} A$, then

$$0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to 0.$$

Let $f : X \to Y$ be a morphism with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then we have a map $B \to A$. Given a $\tilde{N}$ on Y, what is $f^*\tilde{A}$? We have

$$\operatorname{Hom}_A(N \otimes_B A, M) = \operatorname{Hom}_B(N, M_B),$$

and this shows that $f^*\tilde{N} = (N \otimes_B A)\tilde{}$.

You can prove the following:

- pullbacks of quasi-coherent sheaves are quasi-coherent
- pushforwards of quasi-coherent sheaves along quasi-compact separated morphisms are quasi-coherent
- kernels, cokernels, direct sums of quasi-coherent sheaves are quasi-coherent

A **sheaf of ideals** is a sheaf that locally looks like I an ideal of A. Then sheaves of ideals corresponds to closed subschemes. You can think of it as a closed embedding $Y \hookrightarrow X$ corresponding to

$$0 \to \mathscr{I} \to \mathscr{O}_X \to f_*\mathscr{O}_Y \to 0.$$

5.3 Sheaf of relative differentials

There are also sheaves of relative differentials. If $A \to B$ is a ring map and M is a B-module, a map $d : B \to M$ is called an **A-derivation** if it satisfies

$$d(b + b') = db + db', \quad d(bb') = bdb' + b'db, \quad da = 0.$$

The module $\Omega^1_{B/A}$ is characterized by

$$\operatorname{Der}_A(B, M) = \operatorname{Hom}_B(\Omega^1_{B/A}, M)$$

for all B-module M.

$$\begin{tikzcd} B \arrow[r, "d_M"] \arrow[d] & M \\ \Omega_{B/A} \arrow[ur, dashed] & \end{tikzcd}$$

Here is another way to construct this. Consider the kernel

$$0 \to I \to B \otimes_A B \to B \to 0$$

where the map $B \otimes_A B \to B$ is given by $b \otimes b' \mapsto bb'$. Here, I/I^2 has a B-module structure because $B \otimes_A B/I = B$. This is going to be generated by $b \otimes b' - b' \otimes b$ with

$$d : B \to I/I^2; \quad b \mapsto 1 \otimes b - b \otimes 1.$$

It can be shown that $(I/I^2, d)$ realizes $\Omega^1_{B/A}$.

For $X \to Y$ with $\mathscr{I}$ the sheaf of of ideals of

$$\Delta : X \hookrightarrow X \times_Y X,$$

we can pull back $\mathscr{I}/\mathscr{I}^2$ and get a sheaf of **relative differentials** $\Omega^1_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2)$. This is something like the $TM \cong N_\Delta(M)/M \times M$ and T^*M being equal to the conormal bundle.

If $f : X' \to X$ is a pullback of $Y' \to Y$, then

$$\Omega^1_{X'/Y'} = f^*\Omega^1_{X/Y}.$$

This is because formation of sheaves of relative differentials commutes with pullback. This is because you can locally show that

$$\Omega^1_{B'/A'} \cong \Omega^1_{B/A} \otimes_B B'.$$

Let $f : X \to Y$ be a morphism such that $f^*\mathscr{O}_Y \to \mathscr{O}_X$. For $X \to Y \to Z$, we then get

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$

You shouldn't always think of Z as a point.

Example 5.3. Consider $X = \mathbb{A}^2$ projecting to $Y = \mathbb{A}^1$ projecting to $Z = *$. Then this sequence looks like

$$k[x,y]\langle dx\rangle \to k[x,y]\langle dx, dy\rangle \to k[x,y]\langle dy\rangle \to 0.$$

Example 5.4. Consider the curve $A = k[x,y]/(y^2 - x^3)$. Then

$$\Omega^1_{A/*} = A\langle dx, dy\rangle/(2ydy - 3x^2dx).$$

At the singularity $(x,y) = (0,0)$, you see that the fiber is $M/mM = k\langle dx, dy\rangle$.

Example 5.5. Consider $k[x] \to k[x,y]/(y^2 - x)$. Then

$$\Omega^1_{A/B} = (k[x,y]/(y^2-x))\langle dx, dy\rangle/(2ydy-dx, dx) = (k[x,y]/(y^2-x))\langle dx, dy\rangle/(2ydy).$$

This is the skyscraper sheaf at $y = 0$, $i_*\mathscr{O}_{\{y=0\}}$.

We have $\omega_Y = \Omega^1_{Y/*}$, but if $X \to Y$ is a map of curves, we have

$$0 \to f^*\Omega^1_{Y/*} \to \Omega^1_{X/*} \to \Omega^1_{X/Y} \to 0.$$

Here $\mathscr{O}$(ram. divisor) is going to be $\Omega^1_{X/Y}$. This is why we were able to prove Riemann–Hurwitz.

6 February 15, 2018

6.1 Divisors

Let me talk more about divisors. For simplicity, assume that X is a smooth variety over $k = \overline{k}$. Then every local ring is a UFD, and so every local ring is a DVR.

Definition 6.1. A **prime divisor** is a closed integral subscheme of codimension 1. A **Weil divisor** is a linear combination $\sum_i n_i Y_i$ where Y_i are prime divisors.

For $f \in K(X)^\times$, we define

$$\operatorname{div}(f) = \sum_{y \text{ codim. } 1} \operatorname{val}_y(f) Y$$

where y is the generic point of Y. (Here the local ring at y is going to be a DVR.)

Definition 6.2. A divisor is **principal** if it is $\operatorname{div}(f)$ for some $f \in K(X)^\times$. Define the **class group** as

$$\operatorname{Cl}(X) = \operatorname{Div} X / \operatorname{PDiv} X.$$

It is well-known that a ring A is a UFD if and only if all prime ideals of height 1 is principal.

Proposition 6.3. *If you have $Y \subseteq X$, take $U = X - Y$. Then there exists an exact sequence*

$$\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0.$$

For instance, we have

$$\mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^n) \to \operatorname{Cl}(\mathbb{A}^n) = 0 \to 0$$

and so $\operatorname{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.

Definition 6.4. A **Cartier divisor** is an element of $\Gamma(X, \mathscr{K}^\times/\mathscr{O}_X^\times)$. It is called **principal** if it lies in the image of $\Gamma(X, \mathscr{K}^\times) \to \Gamma(X, \mathscr{K}^\times/\mathscr{O}_X^\times)$.

There is a section in Hartshorne, which is to go between Weil divisors, Cartier divisors, and invertible sheaves. From Weil divisors to Cartier divisors, you need to use a small algebraic statement that every codimension 1 integral subscheme is locally cut out by a single function.

Example 6.5. We have $\operatorname{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2$ because $\mathbb{P}^1$ minus two lines is $\mathbb{A}^2$.

6.2 Homological algbera

If you have a short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes, then you get a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to \cdots .$$

There is also the notion of homotopy of chain maps. If $f, g : A^\bullet \to B^\bullet$ the are called **homotopic** if they differ by some

$$f - g = dh + hd.$$

In this case, $f^*, g^* : H^i(A) \to H^i(B)$ are identical because $f(a) - g(a) = dh(a) + hd(a) = dh(a)$ is a boundary.

Definition 6.6. In an abelian category, an **injective object** is an object I such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

always exists.

Lemma 6.7. *Let $0 \to A \to I^\bullet$ be an injective resolution, and $0 \to B \to J^\bullet$ be any resolution. Then we can lift*

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d, dashed] & J^1 \arrow[r] \arrow[d, dashed] & J^2 \arrow[r] \arrow[d, dashed] & \cdots \\ 0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

and moreover such lift f and f' are always chain homotopic.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, and assume that $\mathcal{A}$ has enough injectives, i.e., every object is a subobject of an injective object. Now let us take an injective resolution $0 \to A \to I^\bullet$, apply the functor, and take cohomology

$$R^i F(A) = H^i(F(I^\bullet)).$$

This is well-defined because if you take any different resolutions, you can always extend the identity map and so the chain map is homotopic to the identity map.

6.3 Sheaf cohomology

Now let X be a scheme and consider the functor

$$\Gamma : \mathsf{Mod}(X) \to \mathsf{Ab}; \quad \mathscr{F} \mapsto \Gamma(X, \mathscr{F}).$$

This is left exact, and then we define as

$$H^i(X, \mathscr{F}) = R^i\Gamma(\mathscr{F}).$$

If $X \to \operatorname{Spec} A$ is relative, this lands on Mod_A.

If we don't assume the base scheme is not affine, consider $f : X \to Y$ and we can consider the **higher direct image** as $R^i f_* : \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$.

This is not very computable, and you can use Čech cohomology. Let $\mathcal{U} = \{U_i\}$ be a cover of X. For a sheaf $\mathscr{F}$, you define

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \cdots i_p})$$

and maps $d : C^p \to C^{p+1}$ as

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i}_k \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}.$$

Then **Čech cohomology** associated to this cover is just

$$\check{H}^i(\mathcal{U}, \mathscr{F}) = H^i(C^*, d).$$

There is always a natural map

$$\check{H}^i(\mathcal{U}, \mathscr{F}) \to H^i(X, \mathscr{F}),$$

and if $\mathcal{U}$ is good enough, this is an isomorphism, and it works for quasi-coherent sheaves $\mathscr{F}$.

It is a fact that a scheme X is affine if and only if $H^i(X, \mathscr{F}) = 0$ for all $\mathscr{F} \in \mathsf{Qch}(X)$. This is called Serre's affineness criterion.

7 February 20, 2018

For a functor $F : \mathsf{A} \to \mathsf{B}$, we defined $R^iF(A)$ as taking an injective resolution $0 \to A \to I^\bullet$ and taking

$$R^iF(A) = H^i(F(I^\bullet)).$$

We apply this construction to a ringed space $(X, \mathscr{O}_X)$ with an $\mathscr{O}_X$-module $\mathscr{F}$. Then we have a functor

$$\Gamma : \mathsf{Mod}_X \to \mathsf{Ab}; \quad \mathscr{F} \mapsto \Gamma(X, \mathscr{F}).$$

Then if X is a scheme over $\operatorname{Spec} A$ then we can consider it as $\Gamma : \mathsf{Mod}_X \to \mathsf{Mod}_A$. Similarly, you can push-forward $f_* : \mathsf{Mod}_X \to \mathsf{Mod}_Y$ and then we have the higher direct images R^if_*.

Normally injective resolutions are not computable, so we introduce Čech cohomology. For $\mathfrak{U}$ a covering, we define

$$C^P(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0,\dots,i_p}),$$
$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}.$$

Then we define $\check{H}(\mathfrak{U}, \mathscr{F})$ as the cohomology of this complex, and

$$\check{H}(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} \check{H}(\mathfrak{U}, \mathscr{F}).$$

Here, the limit is over refinements of covers. It is a fact that if X is affine, $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and $\mathscr{F} \in \mathsf{Qch}(X)$.

Example 7.1. Let us use $X = \mathbb{P}^1_k$ and let us compute $H^*(X, \Omega^1_X)$. We use the standard covering $U, V \cong \mathbb{A}^1$. Then

$$0 \to \Gamma(U, \Omega^1) \oplus \Gamma(V, \Omega^1) \to \Gamma(U \cap V, \Omega^1) \to 0.$$

The nonzero map is

$$d : (f(x)dx, g(y)dy) \mapsto \Big(f(x) + \frac{1}{x^2} g\Big(\frac{1}{x}\Big)\Big) dx.$$

Here, $\ker d = 0$ and $\operatorname{coker} d$ is generated by $x^{-1}dx$. So

$$H^0(X, \Omega^1_X) = 0, \quad H^1(X, \Omega^1_X) \cong k$$

for $X = \mathbb{P}^1$.

Let X be a scheme. You can easily convince yourself that

$$\check{H}^1(X, \mathscr{O}_X^\times) = \operatorname{Pic} X.$$

7.1 Chern classes

Example 7.2. Let $\underline{\mathbb{R}}$ be the constant sheaf, and let M be a manifold. Then

$$H^*(M, \underline{\mathbb{R}}) = H^*_{\mathrm{dR}}(M).$$

The reason is that

$$0 \to \underline{\mathbb{R}} \to \Omega^0_M \to \Omega^1_M \to \Omega^2_M \to \cdots$$

is an acyclic resolution, because all these sheaves Ω^i_M are flasque. So we can compute cohomology using this complex, and this is precisely de Rham cohomology.

We can also consider the holomorphic case. Let us look at the complex analytic topology here. We can consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 1$$

This is exact, because this is true on a simply connected neighborhood. This induces a map

$$c_1 : \mathrm{Pic} \cong H^1(X, \mathscr{O}_X^\times) \to H^2(X, \underline{\mathbb{Z}}) \cong H^2_{\mathrm{sing}}(X, \mathbb{Z}).$$

So this is a group homomorphism $c_1 : \mathrm{Pic}\, X \to H^2(X, \mathbb{Z})$, and is called the **first Chern class**. That is, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

If X is a smooth projective surface over $\mathbb{C}$, the group $H^2(X, \mathbb{Z})$ is equipped with a Poincaré pairing. That is, we have a pairing on $\mathrm{Pic}\, X$:

$$\langle L_1, L_2 \rangle = \langle c_1(L_1), c_2(L_2) \rangle_{\text{Poincaré}}.$$

7.2 Intersection number

If $s \in H^0(X, L \otimes K)$ such that $s_0 = M$ and $s_\infty = N$ are smooth complex manifolds, then

$$c_1(L) = [M] - [N].$$

In particular, if C and D are curves, you can convince yourself that

$$\langle \mathscr{O}_X(C), \mathscr{O}_X(D) \rangle = \#(C \cap D)$$

is sort of the intersection number. Because (co)homology class doesn't depend on homotopy, we see that the intersection number only depends on the line bundle.

Definition 7.3. Let C, D be two curves on X whose intersection is 0-dimensional. We define

$$C.D = \dim_k H^0(C \cap D, \mathscr{O}_{C \cap D})$$

where $C \cap D$ is the scheme-theoretic intersection.

If x is a closed point in $\operatorname{supp}(C) \cap \operatorname{supp}(D)$, we can define the **multiplicity** of $C \cap D$ at x as

$$\operatorname{mult}_x(C \cap D) = \dim_k \mathscr{O}_{x,X}/(f,g).$$

Here, if x is the only intersection of (f) and (g), then

$$A_{\mathfrak{m}}/(f,g) = (A/(f,g))_{\mathfrak{m}} = A/(f,g).$$

We need that $\mathfrak{m}^N \subseteq (f,g)$, but this follows from the standard fact on Artinian rings.

We know that $\mathscr{O}_X(-D) = \mathscr{I}_D$ is the ideal sheaf. Then

$$0 \to \mathscr{I}_D|_C \to \mathscr{O}_C \to i_* \mathscr{O}_{C \cap D} \to 0,$$

where $i : C \cap D \to C$ is inclusion. It is a fact that if X is projective and $\mathscr{F}$ is coherent, then $H^i(X, \mathscr{F})$ is finite-dimensional. Then we can define the **Euler characteristic** as

$$\chi(X, \mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathscr{F}).$$

Then if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is exact then $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Anyways, we get

$$\chi(C, \mathscr{O}_C) = \chi(C, \mathscr{O}_X(-D)|_C) + \chi(C \cap D, \mathscr{O}_{C \cap D}).$$

It is clear that $\chi(C \cap D, \mathscr{O}_{C \cap D}) = C.D$.

Now let me give a different formula. We have

$$0 \to \mathscr{O}_X(-C-D) \to \mathscr{O}(-C) \oplus \mathscr{O}(-D) \to \mathscr{O}_X \to \mathscr{O}_{C \cap D} \to 0.$$

This is because locally we have

$$0 \to (fg) \to (f) \oplus (g) \to A \to A/(f,g) \to 0.$$

The first part requires $(f) \cap (g) = (fg)$, and this follows from the fact that the local rings are UFD. Then we can write

$$\chi(\mathscr{O}_{C \cap D}) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-D)) - \chi(\mathscr{O}_X(-C)) + \chi(\mathscr{O}_X(-C-D)).$$

7.3 Proj construction

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, so that $S_d \cdot S_d \subseteq S_{d+e}$. We say that an ideal $\mathfrak{a} \subseteq S$ is **homogeneous** if

$$\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap S_d.$$

This means that $\mathfrak{a}$ is generated by homogeneous elements. For instance, $(x + y^2) \subseteq k[x,y]$ is not homogeneous.

On projective space $\mathbb{P}^n$, it doesn't make sense to talk about values of a polynomial f at a point, but for a homogeneous polynomial f, it makes sense to talk about the vanishing locus of f. Note that products, intersections, radicals, of homogeneous ideals are homogeneous. I previously introduced a operation Spec that takes a ring to a scheme. Now I want to introduce an operation Proj that takes a graded ring to a scheme.

For a graded ring S, we define

$$\mathrm{Proj}(S) = (X, \mathscr{O}_X),$$

where

- X as a set is the homogeneous prime ideals $\mathfrak{p} \subseteq S$ such that $\mathfrak{p} \not\supseteq S_{>0} = \bigoplus_{d>0} S_d$,
- the topology is generated by distinguished open subsets

$$X_f = \{\mathfrak{p} \not\ni f\},$$

- the functions on the distinguished open X_f is $\mathscr{O}_X(X_f) = (S_f)_0$, where $(S_f)_0$ means the degree 0 elements of S_f.

For instance, if we have $S = k[x_0, \ldots, x_n]$ then

$$\mathscr{O}_X(X_{x_0}) = (k[x_i][x_0^{-1}])_0 = k[\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}].$$

We can also say $(X_f, \mathscr{O}_{X_f}) = \mathrm{Spec}(S_f)_0$. If we assume that S is finitely generated by S_1 over S_0, then you can show that $\mathrm{Proj}(S)$ is a closed subscheme of some $\mathbb{P}^n_{S_0}$.

8 February 22, 2018

Last time we defined the Proj construction. For S a graded ring, we defined a scheme

$$\mathrm{Proj}(S) = (X, \mathscr{O}_X).$$

For $f \in S$ homogeneous, we had $X_f = \{f \notin \mathfrak{p}\}$ and

$$\mathscr{O}_X(X_f) = (S_f)_0.$$

You can further show that

$$(X_f, \mathscr{O}_X|_{X_f}) \cong \mathrm{Spec}(S_f)_0,$$

and so $\mathrm{Proj}(S)$ is a scheme. For example, if we take $S = k[x_0, \dots, x_n]$ with the normal grading, then

$$X = \mathrm{Proj}(S) = \mathbb{P}^n_k.$$

8.1 Twisted sheaves

Definition 8.1. A morphism $f : X \to Y$ is said to be **projective** if there exists a closed immersion

$$X \hookrightarrow \mathbb{P}^n_{\mathbb{Z}} \times Y$$

such that its composite with the projection map is equal to f.

If I is a homogeneous ideal in S, then there is a natural map

$$\mathrm{Proj}(S/I) \to \mathrm{Proj}(S).$$

In general, if $S \twoheadrightarrow T$ then we always get a closed immersion $\mathrm{Proj}(T) \to \mathrm{Proj}(S)$. Locally, this is going to look like $\mathrm{Spec}(Sf/I)_0 \hookrightarrow \mathrm{Spec}(S_f)_0$. Different ideals can give can give you the same subscheme. For instance, if we take $I' = \bigoplus_{d>m} I_d$, then

$$(S_f/I)_0 \cong (S_f/I')_0$$

because anything that looks like x/f^n can be thought of as xf^N/f^{m+N} for large enough N.

To classify the sheaf of ideals, we can just classify the quasi-coherent shaves. For an arbitrary graded module M, we can define a quasi-coherent sheaf

$$\tilde{M}|_{X_f} \cong \widetilde{(M_f)_0}.$$

Then this can be shown to be a quasi-coherent sheaf.

$\mathrm{Proj}(S)$ is equipped with a natural $\mathbb{Z}$-family of line bundles. Here is how you do this. For each graded module M, we can define a shifted module $M(\ell)$ such that $M(\ell)_d = M_{\ell+d}$. This is a new graded module with different grading. So we take S as a graded S-module, shift it, and take $\widetilde{S(n)}$. This is a line bundle on $\mathrm{Proj}(S)$.

Example 8.2. Take $S = k[x_0, \ldots, x_n]$ so that $\mathrm{Proj}(S) = \mathbb{P}^n_k$. We know that $\mathrm{Pic}\, \mathbb{P}^n_k = \mathbb{Z}$. In this case, $S(m)\tilde{} \cong \mathscr{O}_{\mathbb{P}^n}(m)$ for m, by comparing degree.

Unlike affine schemes, projective schemes tend to have not so many global functions. Similarly, a coherent sheaf over a projective scheme tend to have "not-so-many" global sections. But this can be remedied by twisting.

Definition 8.3. For $\mathscr{F} \in \mathsf{Qch}(\mathrm{Proj}\, S)$, we define $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{O}_S(n)$.

Theorem 8.4. *If $\mathscr{F}$ is coherent, there exists a $n \gg 0$ such that $\mathscr{F}(n)$ is globally generated.*

Proof. For $\mathbb{P}^n$, if $\mathscr{F}|_{\mathbb{A}^n_i} \cong \tilde{M_i}$, then I can take $s_{ij} \in M_i$ such that s_{ij} generate M_i. For each s_{ij}, there exists a large n such that $s_{ij} x_i^n$ can be promoted to a global section t_{ij}. Then pick N sufficiently large so that t_{ij} are global sections of $\mathscr{F}(N)$. □

This means that we can realize $\mathscr{F}$ as the quotient of a direct sum of $\mathscr{O}_X(n)$.

Definition 8.5. For $\mathscr{F}$ a coherent sheaf on $\mathrm{Proj}(S)$, we define

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathrm{Proj}(S), \mathscr{F}(n)),$$

which is a graded module over S.

Theorem 8.6. *For $\mathscr{F}$ coherent, we have $\Gamma_*(\mathscr{F})\tilde{} \cong \mathscr{F}$.*

So if we apply this to $\mathscr{I}$ a sheaf of ideals, we see that all closed subschemes are given by homogeneous ideals of S. Moreover, finite type $Y/\operatorname{Spec} A$ is projective if and only if $Y \cong \mathrm{Proj}(S)$ for some S satisfying our assumption that $S_0 = A$ and S is finitely generated by S_1 over A.

8.2 Properties of morphisms

Definition 8.7. We say that a map of schemes $X \to Y$ is **(locally) finite type** if for every $V = \operatorname{Spec} B \subseteq Y$ and every $\operatorname{Spec} A = U \subseteq f^{-1}(V)$, the induced map $B \to A$ is a ring homomorphism of finite type, that is, A is finitely generated over A.

Definition 8.8. We say that a morphism $f : X \to Y$ is **quasi-compact** if for every $V \subseteq Y$, $f^{-1}(V)$ is covered by finitely many affine opens.

If $f : X \to Y$ is quasi-compact and $\mathscr{F} \in \mathsf{Qch}(X)$, then $R^i f_* \mathscr{F} \in \mathsf{Qch}(Y)$ for all i. Higher push-forward commutes with restriction, so if $V \subseteq Y$ then

$$R^i f_* \mathscr{F}|_V \cong R^i f_*(\mathscr{F}|_{f^{-1}(V)}).$$

This gives a method to compute higher push-forwards. On each open $\operatorname{Spec} B = V \hookrightarrow Y$, we have

$$R^i f_* \mathscr{F}|_V = H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(Y)})\tilde{}.$$

Definition 8.9. A morphism $f : X \to Y$ of schemes is **affine** if there exists an affine cover $\{V_i\}$ such that each $f^{-1}(V_i)$ is affine, or equivalently, for every affine $V \subseteq Y$ the preimage $f^{-1}(V)$ is affine.

Let's prove that they are actually equivalent. This is a nice application of Serre's affineness criterion and spectral sequences. If $f : X \to Y$ is affine, then $f^* f_* \mathscr{F} \to \mathscr{F}$ is surjective for $\mathscr{F} \in \mathsf{Qch}(X)$ and $R^i f_* \mathscr{F} = 0$ for $i > 0$.

8.3 Introduction to spectral sequences

Consider a double complex

$$\begin{tikzcd}
\vdots & \vdots & \ddots \\
E^{1,0} \arrow[u] \arrow[r] & E^{1,1} \arrow[u] \arrow[r] & \cdots \\
E^{0,0} \arrow[u] \arrow[r] & E^{0,1} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

such that the square anti-commutes. Then we can define a total complex

$$\mathrm{Tot}(E^{*,*})^p = \bigoplus_{i+j=p} E^{i,j}.$$

A spectral sequence gives some filtered information on the cohomology of the filtered complex.

The E_0-page ${}_\to E_0^{*,*}$ is this double complex, with the differentials to horizontal lines. The E_1-page $E_1^{*,*}$ is the cohomology of this complex, with arrows pointing vertically. The E^2-page is the cohomology of the E_1-page with the arrows pointing left+up+up. At each page, we take the cohomology, so we are taking a subquotient. Then the things get smaller, and will converge to some

$${}_\to E_\infty^{*,*}.$$

The statement is that there exists a $H^p(\mathrm{Tot})$ such that

$${}_\to E_\infty^{0,p} \xhookrightarrow{{}_\to E_\infty^{1,p-1}} \bullet \hookrightarrow \cdots \xhookrightarrow{{}_\to E_\infty^{p,0}} H^p(\mathrm{Tot}).$$

Likewise, you can do the some thing with the vertical maps, and consider ${}_\uparrow E^{*,*}$.

The snake lemma is a special case of the spectral sequence. If we take a diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\beta"] & A'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0.
\end{tikzcd}$$

If we look at the horizontal direction, ${}_{\to}E_1^{*,*} = 0$ and so $H^*(\mathrm{Tot}) = 0$. If we look at the vertical direction, the E_1-page looks like

$$\begin{tikzcd}
\ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \\
\operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma.
\end{tikzcd}$$

If we look at the E_2-page, we should get

$$\begin{tikzcd}
? & ? & ?? \\
?? \arrow[urr] & ? & ?
\end{tikzcd}$$

and these can never cancel out. This shows that ? should all be 0, and ?? $\to$?? should be an isomorphism. So we can put them together and get an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0.$$

9 February 27, 2018

Last time I wanted to show that affineness can be checked locally. Namely, if $f : X \to Y$ with $\{Y_i\}$ an affine cover such that $f^{-1}(Y_i)$ is affine, then f if affine. We can reduce this to the case when Y is affine. I want to show that if $\mathscr{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathscr{F}) = 0$ for $i > 0$. We are going to use the Leray spectral sequence, and this is an instance of a Grothendieck spectral sequence.

Theorem 9.1 (Grothendieck spectral sequence)**.** *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be morphisms of abelian categories, and assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, with F sending F-acyclics to G-acyclics. Then there exists a spectral sequence*

$$E_2^{p,q} = R^p G \circ R^q F(A) \quad \Longrightarrow \quad R^{p+q}(G \circ F)(A).$$

Let us apply this to a special case. Let $f : X \to Y$ be a morphism of schemes over A. Then we can consider

$$\mathsf{Mod}(X) \xrightarrow{f_*} \mathsf{Mod}(Y) \xrightarrow{\Gamma} \mathsf{Mod}_A.$$

You can check that it satisfies the condition F mapping F-acyclics to G-acyclics, because it is derived pushforward is computed locally. Sow we get the **Leray spectral sequence**

$$H^p(Y, R^q f_* \mathscr{F}) \Longrightarrow H^{p+q}(X, \mathscr{F}).$$

If $f^{-1}(Y_i)$ are affine opens in X, then we get $H^p(Y, R^q f_* \mathscr{F}) = 0$ for $p > 0$ because Y is affine, and also $H^p(Y, R^q f_* \mathscr{F}) = 0$ for $q > 0$ because the map is locally affine. Therefore $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

9.1 Flat morphisms

Let me first give a motivation. We want to make sense of a "continuous family of schemes". For instance, we want a family of curves $C_t \subseteq \mathbb{P}^2$ for $t \in T$. We can take just $Z \subseteq \mathbb{P}^2 \times T$, but we want this to be something like changing continuously. But we don't want it to be smooth, because we want to allow ramification.

Definition 9.2. Let $\mathsf{F} \in \mathsf{Mod}(X)$ and $f : X \to Y$. We say that $\mathscr{F}$ is **flat** over Y if for all $x \in X$, $\mathscr{F}_x$ is flat over $\mathscr{O}_{f(x),Y}$. We say that $f : X \to Y$ is **flat** if $\mathscr{O}_X$ is flat over Y.

Flatness can be checked locally. That is, M/A is flat if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p}$. This is because computing Tor commutes with localization.

Let $f : X \to Y$ be separated finite type, $\nu : Y' \to Y$ flat, and $\mathscr{F} \in \mathsf{Qch}(X)$. Take the fiber product

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\nu} & Y. \end{array}$$

Then the claim is that there exists an isomorphism

$$\nu^* R_i f_* \mathscr{F} \xrightarrow{\cong} R^i g_* u^* \mathscr{F}.$$

First of all, there exists $\nu^* f_* \mathscr{F} \to g_* u^* \mathscr{F}$ because this is equivalent to

$$f_* \mathscr{F} \to \nu_* g_* u^* \mathscr{F} = f_* u_* u^* \mathscr{F}.$$

Hartshorne constructs this for $i > 0$ locally, but let's take this for granted. Assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Then our statement just says that

$$H^i(X, \mathscr{F}) \otimes_A A' \cong H^i(X, \mathscr{F}'),$$

where $\mathscr{F}' = u^* \mathscr{F}$. Take an affine cover $\mathfrak{U}$ of X, so that

$$H^i(X, \mathscr{F}) = h^i(C^\bullet(\mathfrak{U}, \mathscr{F})).$$

Then $\mathfrak{U}^* = \{u^{-1}(U) : U \in \mathfrak{U}\}$ is an affine cover of X', so that we can take

$$h^i(C^\bullet(\mathfrak{U}', \mathscr{F}')) = h^i(C^\bullet(\mathfrak{U}, \mathscr{F}) \otimes_A A') = H^i(X, \mathscr{F}) \otimes_A A'.$$

9.2 Serre's vanishing theorem

Let me just define this now, and explain its geometric meaning next time.

Definition 9.3. For X/k a projective scheme, we say that a line bundle $\mathscr{L}$ is **very ample** if there exists an embedding $X \hookrightarrow \mathbb{P}^n$ such that $\mathscr{L} \cong i^* \mathscr{O}_{\mathbb{P}^n}(1)$. We say that $\mathscr{L}$ is **ample** if for all $\mathscr{F} \in \mathsf{Coh}(X)$, there exists an n_0 such that for all $n > n_0$, $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is globally generated.

It can be shown that $\mathscr{L}$ is ample if and only if $\mathscr{L}^{\otimes n}$ is very ample.

Theorem 9.4 (Serre)**.** *Let X/A be projective, and let $\mathscr{O}_X(1)$ be very ample. Then for every $\mathscr{F} \in \mathsf{Coh}(X)$, for $n \gg 0$, we have $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$.*

We can first reduce to the case $X = \mathbb{P}^r_A$. This is because if $\mathscr{F}$ is coherent on X, then $i_* \mathscr{F}$ is coherent on $\mathbb{P}^r_A$ and moreover

$$H^i(X, \mathscr{F}) \cong H^i(\mathbb{P}^r, i_* \mathscr{F})$$

by because $i : X \to \mathbb{P}^r_A$ is affine.

Now for $n \gg 0$, we saw that $\mathscr{F}(n)$ is generated by global sections. Then we have a surjective

$$\bigoplus_{i=1}^N \mathscr{O}_X \to \mathscr{F}(n) \to 0,$$

and then we get some

$$0 \to \mathscr{R} \to \mathscr{E} = \bigoplus_{i=1}^N \mathscr{O}_X(-n) \to \mathscr{F} \to 0.$$

Now we begin to induct. First, the base case is for $i > r$, we have $H^i(\mathbb{P}^r, \mathscr{F}) = 0$ for all $\mathscr{F}$. The claim we want to prove is that

- $H^i(X, \mathscr{F})$ is finitely generated,
- $H^i(X, \mathscr{F}(n)) = 0$ for $n \gg 0$.

But we have a long exact sequence

$$\cdots \to H^i(X, \mathscr{E}) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{E}) \to \cdots .$$

Let X be a projective scheme over k. Let $\mathscr{O}_X(1)$ be very ample, and let $\mathscr{F}$ be a coherent sheaf. Then $\chi(\mathscr{F}(n))$ is a polynomial, called the **Hilbert polynomial**.

Theorem 9.5. *Let us assume that T is integral and Noetherian. Take $X \subseteq \mathbb{P}^n_T$ a closed subscheme. Then X is flat over T if and only if $\chi(X_t, \mathscr{O}_{X_t}(m))$ is independent of t.*

We can more generally show that if $\mathscr{F}$ is a coherent sheaf over $\mathbb{P}^n_T$, $\mathscr{F}$ is flat if and only if $\chi(X_t, \mathscr{F}_t(m))$ is independent of t. Next, we reduce to the case that $T = \operatorname{Spec} A$ where A is Noetherian local. We can do this because we can compare it with the generic point, and we can take localization.

The upshot is that checking flatness over a local ring is easy. If $(A, \mathfrak{m})$ is a local ring and M is finitely generated over A, then M is flat if and only if M is free if and only if $\dim_k M \otimes k = \dim_K M \otimes K$. We want to show that the following are equivalent:

(i) $\mathscr{F}$ is flat over A.

(ii) $H^0(X, \mathscr{F}(m))$ is finitely generated ant free over A, for $m \gg 0$.

(iii) $P_t = \chi(X_t, \mathscr{F}_t(m))$ is independent of t for $m \gg 0$.

10 March 1, 2018

I used the cohomology of line bundles on $\mathbb{P}^n_A$, but let us compute this now. We first have

$$H^0(\mathbb{P}^n_A, \mathscr{O}(m)) = A\langle \text{monomials in } x_0, \dots, x_n \text{ of degree } m\rangle.$$

Also, we are going to have

$$H^n(\mathbb{P}^n_A, \mathscr{O}(m)) = A\langle x_0^{i_0} \cdots x_n^{i_n} : i_0, \dots, i_n < 0, i_0 + \cdots + i_n = m\rangle.$$

So there is some duality, called Serre duality.

Let X be a projective scheme over k, and let $\mathscr{O}_X(1)$ be a very ample line bundle. We have a coherent sheaf $\mathscr{F} \in \mathsf{Coh}(X)$, and we have defined the Hilbert polynomial as

$$p_{\mathscr{F}}(m) = \chi(X, \mathscr{F}(m)).$$

10.1 Associated points

Let M be a finitely generated module over a Noetherian ring A.

Definition 10.1. An **associated point** $\mathfrak{p} \subseteq A$ is a point satisfying the following equivalent conditions:

(1) $\mathfrak{p}$ looks like $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M$.

(2) $\times f : M_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is never injective for all $f \in \mathfrak{p}A_{\mathfrak{p}}$.

Once we have this, we can talk about global notions.

Definition 10.2. Let X be Noetherian scheme and $\mathscr{F}$ be a coherent sheaf on X. Then $\operatorname{Ass}(\mathscr{F})$ is the set of points $x \in X$ such that $\times f : \mathscr{F}_x \to \mathscr{F}_x$ is never injective for $f \in \mathfrak{m}_{x,X} \subseteq \mathscr{O}_{x,X}$.

You can show that $\operatorname{Ass} \mathscr{F}$ is a finite set.

Lemma 10.3. *Let $\mathscr{L}$ be a line bundle, and $s \in \Gamma(X, \mathscr{L})$, then $\times s : \mathscr{F} \to \mathscr{F} \otimes \mathscr{L}$ is injective if and only if s does not vanish on any $\operatorname{Ass}(\mathscr{F})$.*

We also call $\operatorname{Ass}(\mathscr{O}_X)$ the associated points of X.

Example 10.4. Let $X = \operatorname{Spec} k[x, y]/(x^2, xy)$. This looks like $(x) \cap (x, y)^2$. There are going to be two associated points: the generic point and the point (x, y). This is because $xf = 0$ for all f, and so multiplication by f is never injective. An associated point that is not a generic point is called an **embedded point**.

Let us go back to our Hilbert polynomial. Assume that k is infinite, by base change. Choose a hyperplane $x = 0$ that does not pass through any $x \in \operatorname{Ass}(\mathscr{F})$. Then we have a map

$$\mathscr{F}(-1) \hookrightarrow \mathscr{F}; \quad s \mapsto s \otimes x.$$

We can then take

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{G} \to 0.$$

The point here is that $\dim \operatorname{supp}(\mathscr{G}) = \dim \operatorname{supp}(\mathscr{F}) - 1$ unless $\mathscr{F} = 0$, and then

$$\chi(\mathscr{G}(m)) = \chi(\mathscr{F}(m)) - \chi(\mathscr{F}(m-1)).$$

Example 10.5. For $X = \mathbb{P}^n_k$ an $\mathscr{F} = \mathscr{O}_X$, we have

$$\chi(X, \mathscr{O}_X(m)) = \sum_i (-1) i \dim H^i(X, \mathscr{O}_X(m)) = \binom{n+m}{n}.$$

We still can distill some invariants out of this. For instance,

$$p_{\mathscr{O}_X}(0) = \chi(X, \mathscr{O}_X), \quad \deg p_{\mathscr{O}_X} = \dim X.$$

Example 10.6. Consider a hypersurface $X \subseteq \mathbb{P}^n_k$ of degree d. Then there is a defining exact sequence

$$\chi(X, \mathscr{O}_X(m)) = \binom{n+m}{n} - \binom{n+m-d}{n}.$$

In particular, if $n = 2$ then we have

$$\chi(X, \mathscr{O}_X(m)) = \frac{-d^2 + 3d}{2} + md.$$

For $m = 1$, we get

$$\chi(X, \mathscr{O}_X) = \frac{-d^2+3d}{2} = 1 - \dim H^1(X, \mathscr{O}_X) = 1 - \dim H^0(X, \Omega_X).$$

So if X is smooth, we get

$$g = \frac{1}{2}(d-1)(d-2).$$

Also, note that the leading coefficient of the polynomial times $(n-1)!$ is the degree of the embedding.

10.2 Criteria for flatness

Let $\mathscr{F}$ be a coherent sheaf over $X = \mathbb{P}^n_T$, where $T = \operatorname{Spec} A$ with A Noetherian local. We want to show that the following are equivalent:

(i) $\mathscr{F}$ is flat over T,

(ii) $H^0(X, \mathscr{F}(m))$ is a free A-module of finite rank, for $m \gg 0$,

(iii) $\chi(X_t, \mathscr{F}_{X_t}(m))$ is independent of t.

Let us first show (i) $\Rightarrow$ (ii). Let $\mathfrak{U}$ be an affine cover. We know that $H^i(X, \mathscr{F}(m)) = 0$ for $m \gg 0$ and $i > 0$, so

$$0 \to H^0(X, \mathscr{F}(m)) \to C^0(\mathfrak{U}, \mathscr{F}(m)) \to C^1(\mathfrak{U}, \mathscr{F}(m)) \to \cdots$$

is exact for $m \gg 0$. But all of the C^i are flat. It is also true that $0 \to M' \to M \to M'' \to 0$ with M and M'' flat implies that M' is flat. Then we inductively get $H^0(X, \mathscr{F})$ is flat. But for a finitely generated module over a Noetherian local ring, flat is equivalent to free.

Now let us show (ii) $\Rightarrow$ (i). Take m_0 such that (ii) holds for $m \geq m_0$. Then for

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathscr{F}(m)),$$

we have $\mathscr{F} = \tilde{M}$.

For (ii) $\Rightarrow$ (iii), it suffices to show that $p_t(m) = \operatorname{rank}_A H^0(X, \mathscr{F}(m))$. Now consider a set of generators for $\mathfrak{m} \subseteq A$ and take

$$A^q \to A \to k_t \to 0.$$

Then we get

$$\mathscr{F}^q \to \mathscr{F} \to \mathscr{F}_t \to 0.$$

For $m \gg 0$, we will get an exact

$$H^0(X, \mathscr{F}(m))^q \to H^0(X, \mathscr{F}(m)) \to H^0(X, \mathscr{F}_t) \to 0,$$

and by freeness, we have

$$H^0(X, \mathscr{F}(m))^q \to H^0(X, \mathscr{F}(m)) \to H^0(X, \mathscr{F}(m)) \otimes_A k_t \to 0.$$

So we get the isomorphism we want.

11 March 20, 2018

Today we are finally going to talk about curves on surfaces.

11.1 Cartier divisors

Definition 11.1. A **Cartier divisor** is an element of $\Gamma(X, \mathscr{K}^\times/\mathscr{O}_X^\times)$. It is called **effective** if its local equations are holomorphic.

Cartier divisors are in one-to-one correspondence with Weil divisors. This is because the short exact sequence $1 \to \mathscr{O}_X^\times \to \mathscr{K}_X^\times \to \mathscr{K}_X^\times/\mathscr{O}_X^\times \to 0$ induce a connecting homomorphism

$$H^0(\mathscr{K}_X^\times/\mathscr{O}_X^\times) \xrightarrow{\delta} H^1(\mathscr{O}_X^\times) = \operatorname{Pic}(X).$$

You can pull back Cartier divisors. Assume that $g : X \to Y$, and assume that if $x \in \operatorname{Ass}(X)$ then $g(x) \notin \operatorname{supp}(D)$. Then for local equations f_i, you can check that $g^*(f_i)$ are locally not a zero divisor, because locally, the zero divisors of A are $\bigcap_{\mathfrak{p} \in \operatorname{Ass}(A)} \mathfrak{p}$. In particular, if g is flat, this condition is automatically satisfied.

Let $g : X \to X$ be a flat map, and $\mathcal{D} \subseteq X$ be a Weil divisor, and D be the corresponding Cartier divisor. The following conditions are equivalent:

(i) $\mathcal{D} \to Y$ is flat,

(ii) for all $x \in X$, a local equation F at x is not a zero divisor in the ring $\mathscr{O}_{x,X} \otimes_{\mathscr{O}_y} \kappa(y)$ for $y = g(x)$,

(iii) $\operatorname{Ass}(g^{-1}(y)) \cap \operatorname{supp}(D) = \emptyset$,

(iv) F is not in any associated prime ideal of $B/\mathfrak{p}B$, for any $\mathfrak{p} \subseteq A$ prime.

For instance, let us show that (iv) implies (i). Locally let us write $g : \operatorname{Spec} B \to \operatorname{Spec} A$, and we want to show that $B/(F)$ is flat over A. We have

$$0 \to B \xrightarrow{\times F} B \to B/(F) \to 0,$$

and want to show that $\operatorname{Tor}_A^1(B/(F), A/\mathfrak{p}) = 0$ for all $\mathfrak{p} \subseteq A$. But we have

$$0 = \operatorname{Tor}_A^1(B, A/\mathfrak{p}) \to \operatorname{Tor}_A^1(B/(F), A/\mathfrak{p}) \to B/\mathfrak{p}B \to B/\mathfrak{p}B.$$

But the map $B/\mathfrak{p}B \to B/\mathfrak{p}B$ given as multiplication by F is injective.

11.2 Curves on surfaces

Let X/k be a scheme. There are functors

$$\underline{\mathrm{Curve}}_X : \mathsf{Sch}_{/k} \to \mathsf{Set}; \quad S \mapsto \left\{ \begin{matrix} \text{relative effective} \\ \text{Cartier divisors on } X \times_k S \end{matrix} \right\}$$
$$\underline{\mathrm{Pic}}_X : \mathsf{Sch}_{/k} \to \mathsf{Set}; \quad S \mapsto \{\text{line bundles on } X \times S\} / \operatorname{Pic}(S).$$

There is an obvious natural transformation $\underline{\mathrm{Curve}}_X \to \underline{\mathrm{Pic}}_X$ given by sending D to $\mathcal{O}_{X\times S}(D)$.

For surfaces, we don't have degree, but there is still going to be a similar notion. Define

$$\mathrm{Pic}^\tau(X) = \{L \in \mathrm{Pic}(X) : L.L' = 0 \text{ for all } L' \in \mathrm{Pic}\, X\}.$$

We then define the **numerical class** as $\mathrm{Num}(X) = \mathrm{Pic}\, X/\mathrm{Pic}^\tau X$. This is also called the **Néron–Severi group**. This is finitely generated over $\mathbb{Z}$, and embeds into $H^2(X;\mathbb{Z})$.

Proposition 11.2. *Both functors can be written as*

$$\underline{\mathrm{Curve}}_X = \coprod_{\xi\in\mathrm{Num}(X)} \underline{\mathrm{Curve}}_X^\xi, \quad \underline{\mathrm{Pic}}_X = \coprod_{\xi\in\mathrm{Num}(X)} \underline{\mathrm{Pic}}_X^\xi.$$

11.3 Riemann–Roch for surfaces

Before proving this, let us talk more about curves on surfaces.

Theorem 11.3. *Let $C \subseteq X$ be a nonsingular curve. Then*

$$2g_C - 2 = C.(C+K)$$

where K is the canonical divisor.

Proof. We have $0 \to \mathscr{I}_C/\mathscr{I}_C^2 \to \Omega^1_{X/k}|_C \to \Omega^1_{C/k} \to 0$ and then

$$\omega_C \cong \omega_X|_C \otimes \mathscr{O}_X(C)|_C.$$

The degree of the left hand side is $2g_C - 2$, the degree of the right hand side is $K.C + C.C$. $\square$

With this, I can give a Riemann–Roch for surfaces. Recall that for curves, we had

$$\chi(\mathscr{L}) = \deg \mathscr{L} - g + 1.$$

The Euler characterstic is some invariant of the line bundle plus some invariant of the curve.

Theorem 11.4 (Riemann–Roch for surfaces)**.** *For $\mathscr{L}$ a line bundle over X,*

$$\chi(\mathscr{L}) = \frac{1}{2}\mathscr{L}.(\mathscr{L} - K) + \chi(\mathscr{O}_X).$$

Proof. We first reduce to the case when $\mathscr{L} = \mathscr{O}_X(C)$. Then we have $0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to i_*\mathscr{O}_C \to 0$, so

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_C \otimes \mathscr{O}_X(C) \to 0.$$

Now it suffices to compute $\chi(\mathscr{O}_C \otimes \mathscr{O}_X(C))$. But note that $i_*(\mathscr{O}_X(C)|_C) = i_*\mathscr{O}_C \otimes \mathscr{O}_X(C)$. (This is because for finite morphisms, $H^i(X,\mathscr{F}) = H^i(Y, g_*\mathscr{F})$.)

So we can compute Euler characteristic of $\mathscr{O}_X(C)|_C$ instead. But we have Riemann–Roch for curves, and we have

$$\chi(\mathscr{O}_X(C)|_C) = C^2 - \frac{C.(C+K)+2}{2} + 1 = \frac{1}{2}C.(C-K). \qquad \square$$

Lemma 11.5. *Let S be connected. For line bundle $\mathscr{L}_1, \mathscr{L}_2$ over $X \times S$, the intersection number $\mathscr{L}_{1,s}.\mathscr{L}_{2,s}$ stays constant in s.*

This implies that the numerical class stays the same, because we can take $\mathscr{L}_2$ to be the pullback of a line bundle on X.

Proof. Recall that the intersection number can be computed as

$$\mathscr{L}_1.\mathscr{L}_2 = \chi(\mathscr{O}_X) - \chi(\mathscr{L}_1^\vee) - \chi(\mathscr{L}_2^\vee) + \chi(\mathscr{L}_1^\vee \otimes \mathscr{L}_2^\vee).$$

But the coefficient of the Hilbert polynomials say constant. $\square$

In general, $\mathrm{Pic}(X \times Y) \neq \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$. Here is an example. Let E be an elliptic curve, and look at the maps

$$\mathrm{Pic}(E)^2 \to \mathrm{Pic}(E \times E) \to \mathrm{Pic}(E)^2$$

that compose to the identity. But there are other stuff in $\mathrm{Pic}(E \times E)$. For instance, take $n : E \to E$ and look at the graph $\Gamma([n]) \subseteq E \times E$. Then the image of this to $\mathrm{Pic}(E)^2$ should be $(\{0\}, \{n\text{-torsions}\})$. If we map to $\mathrm{Pic}(E\times E)$, I claim that we get a different line bundle. To see this, intersect with the diagonal. The new bundle we have is going to have n^2+1, and the intersection number of the original graph is $(n-1)^2$.

12 March 22, 2018

For $X/\mathbb{C}$ projective, we have Hodge theory.

12.1 Hodge theory

For all k, we have
$$H^k(X;\mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}$$
where $H^{i,j}$ is canonically identified with $H^j(X,\Omega^i_X)$. Then
$$H^k_{\text{sing}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^k(X,\mathbb{R}) \otimes_{\mathbb{C}} \to H^k(X,\mathbb{C})$$
are isomorphisms.

Recall that there is an exponential sequence
$$0 \to \underline{\mathbb{Z}} \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 0.$$
Then we get a long exact sequence
$$\cdots \to H^1(X,\underline{\mathbb{Z}}) \to H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X) \to \cdots.$$
This map δ i the map taking a line bundle to its numerical class. But if you take the conjugate, we see that
$$\operatorname{im}\delta \subseteq \ker(H^2(X,\mathbb{C}) \to H^{0,2} \oplus H^{2,0}) = H^{1,1}.$$
The Lefschetz $(1,1)$-theorem states that in fact we have $\operatorname{im}\delta = H^2(X,\mathbb{Z}) \cap H^{1,1}$. On the other hand, the kernel is going to be
$$\operatorname{Pic}^\tau(X) = \ker\delta = H^1(X,\mathscr{O}_X)/H^1(X,\mathbb{Z}).$$
But $H_1(X,\mathbb{Z})$ injects into $H^1(X,\mathscr{O}_X)$ and
$$\operatorname{rank} H^1(X,\mathbb{Z}) = \dim_{\mathbb{C}} H^1(X,\mathbb{C}) = \dim_{\mathbb{R}} H^{0,1}.$$
This shows that the quotient is actually going to be a torus with a complex structure.

12.2 Simple cases of the Picard scheme

Let us go back to the natural transformation of functors
$$\underline{\text{Curve}}_X \to \underline{\text{Pic}}_X.$$
Consider a point in $\underline{\text{Pic}}_X(k) = \text{Pic}(X)$, which is a line bundle $\mathscr{L}$. The set of divisors D that map to $\mathscr{L}$ can be described as
$$H^0(X,\mathscr{L})/k^\times = \mathbb{P}H^0(X,\mathscr{L}).$$

We can check that this is the correct fiber on all S-points. Consider $\mathscr{L}$ as a functor sending S to a point, with a natural transformation $\mathscr{L} \hookrightarrow \underline{\mathrm{Pic}}_X$ given by picking out $\mathscr{O}_S \boxtimes \mathscr{L}$ over $S \times X$. Then we can take the fiber

$$\underline{\mathrm{LinSys}}_L = \mathscr{L} \times_{\underline{\mathrm{Pic}}_X} \underline{\mathrm{Curve}}_X.$$

Proposition 12.1. $\underline{\mathrm{LinSys}}_L \cong h_{\mathbb{P}H^0(X,\mathscr{L})}$.

Proof. We first have

$$\underline{\mathrm{LinSys}}_L(S) = \{\mathcal{D} \subseteq X \times S\}$$

such that $\mathcal{D}$ are relative effective Cartier divisors over S, such that $\mathscr{O}_{X\times S}(\mathcal{D}) \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{K}$ for some $\mathscr{K} \in \mathrm{Pic}(S)$. Likewise, by definition, we have

$$h_{\mathbb{P}^{N-1}}(S) = \{\mathscr{K}/S \text{ with } s_1, \ldots, s_n \in H^0(K) \text{ such that } \mathscr{O}_S^n \twoheadrightarrow K\}/\cong.$$

Note that, this is the same as a section

$$s \in H^0(X \times S, \mathscr{L} \boxtimes \mathscr{K}) = H^0(X, \mathscr{L}) \otimes H^0(S, \mathscr{K})$$

that is nonvanishing, up to scalar multiplication. So then we can take the zero locus and get a divisor in $D \times S$. One nontrivial thing to check is that $\mathscr{K}_1 \cong \mathscr{K}_2$ on S if and only if $\mathscr{L} \boxtimes \mathscr{K}_1 \cong \mathscr{L} \boxtimes \mathscr{K}_2$. This can be checked by pulling back to S. $\square$

Now we can compute these. For instance, suppose we want to compute $\underline{\mathrm{Pic}}_{\mathbb{P}^2}$. We know that its k-points is $\mathbb{Z}$, but we don't know if there are other things happening. We would like to say that $\underline{\mathrm{Pic}}_{\mathbb{P}^2} = \coprod_{\mathbb{Z}} \operatorname{Spec} k$.

Theorem 12.2. *If* $H^1(X, \mathscr{O}_X) = 0$, *then* $\mathrm{Pic}(X \times S) = \mathrm{Pic}(X) \times \mathrm{Pic}(S)$.

A corollary is that

$$\underline{\mathrm{Pic}}_X = \coprod_{\xi \in \mathrm{Pic}(X)} \operatorname{Spec} k.$$

(More precisely, it is represented by that thing on the right.) To prove the theorem, we need the following fact from Lecture 7 (in Mumford's book).

Theorem 12.3. *Let* $\mathscr{F}$ *be flat over* $\mathbb{P}^n \times S$. *If for some* $s_0 \in S$, $H^{j+1}(\mathbb{P}^n_{s_0}, \mathscr{F}_{\mathbb{P}^n_{s_0}}) = 0$, *then there exists an open* U *on which*

$$g^* R^j p_* \mathscr{F} \cong R^j q_* h^* \mathscr{F}$$

for any

$$\begin{array}{ccc} \mathbb{P}^n \times T & \xrightarrow{h} & \mathbb{P}^n \times S \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array}$$

with $T \to S$ *factoring through* U *and any* $\mathscr{F}$ *on* $\mathbb{P}^n \times S$.

Proof. We only need to show that anything in $\mathrm{Pic}(X \times S)$ comes from $\mathrm{Pic}(X) \times \mathrm{Pic}(S)$. Let $\mathscr{L}$ be a line bundle on $X \times S$, and let $s \in S$. Consider the inclusion $\sigma : X \to X \times S$ with the point s. Then we can define

$$\mathscr{M} = \mathscr{L} \otimes (p_1^* \sigma^* \mathscr{L})^\vee.$$

Then $\mathscr{M}|_{X_s} \cong \mathscr{O}_{X_s}$ and $H^1(X_s, \mathscr{M}_{X_s}) = 0$.

By the theorem we stated, we can take $T = s$, and because $H^1(X_s, \mathscr{M}_{X_s})$, we obtain

$$p_{1*}\mathscr{M} \otimes \kappa(s) \cong H^0(S, \mathscr{M}_{X_s}).$$

Then because $\Gamma(U, p_{1*}\mathscr{M}) \to p_{1*}\mathscr{M} \otimes \kappa(s)$ is surjective for some affine open U, any section in $H^0(S, \mathscr{M}_{X_s})$ lifts to a section in $H^0(U \times X, \mathscr{M})$. Suppose we lift 1 to α. This gives a homomorphism

$$\Phi : p_1^* \sigma^* \mathscr{M} \to \mathscr{M}|_{U \times X}.$$

This is means that it is an isomorphism on a neighborhood of X_S. Because X is quasi-compact, we can use the tube lemma to find an open neighborhood W of s such that Φ is an isomorphism on $W \times S$.

Now we can find a covering $\{W_i\}$ of S, such that for each $W_i \times S$ we have an isomorphism $p_1^* \sigma^* \mathscr{M} \cong \mathscr{M}$ on each $W_i \times X$. Then we can glue them to get that $\mathscr{M} \cong \mathscr{O}_X \boxtimes \mathscr{K}$ for some line bundle $\mathscr{K}$ on S. $\square$

13 March 27, 2018

Last time there was this theorem about $\mathrm{Pic}(X \times Y) \cong \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$ if $H^1(X, \mathscr{O}_X) = 0$. Given $\mathscr{L}$ on $X \times S$, we want to find L and K such that X and S such that $\mathscr{L} = L \boxtimes K$.

So we defined $L = \mathscr{L}|_{X_s}$ for some s, and then defined

$$M_s = \mathscr{L} \otimes (p_1^* L_s)^\vee.$$

Using that $H^1(X, \mathscr{O}_X) = 0$, we showed that we can trivialize this on $X \times U$ for some neighborhood U of s. Then because $H^0(X \times (U \cap U'), \mathscr{O}^\times) = H^0(U \cap U', \mathscr{O}^\times)$, every transition function can be descended to a transition function on S. Moreover, because S is connected, this means that the isomorphism class of $\mathscr{L}|_{X_s}$ is locally constant in S, and thus M_s is really just M. Therefore we can patch these local sheaves together to get what we want.

So when $H^1(X, \mathscr{O}_X) = 0$, we have that $\mathrm{Pic}(X)$ is a discrete scheme. In fact, we will later identify $H^1(X, \mathscr{O}_X)$ with the tangent space of $\mathrm{Pic}(X)$ at the identity.

13.1 m-regular sheaves

Definition 13.1. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is called **m-regular** if $H^i(\mathbb{P}^n, \mathscr{F}(m-i)) = 0$ for all $i > 0$.

Serre vanishing tells us that $H^i(\mathbb{P}^n, \mathscr{F}(m))$ vanishes for $m \gg 0$, but this is telling us when this vanishes.

Proposition 13.2. *If $\mathscr{F}$ is m-regular, then*

(a) $H^0(\mathbb{P}^n, \mathscr{F}(k-1)) \otimes H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \twoheadrightarrow H^0(\mathbb{P}^n, \mathscr{F}(k))$ for all $k > m$.

(b) $H^i(\mathbb{P}^n, \mathscr{F}(k)) = 0$ for all $i > 0$ and $k + i \geq m$.

Proof. Take a hyperplane H that avoids the associated points of $\mathscr{F}$. Then we have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-H) \to \mathscr{O}_{\mathbb{P}^n} \to i_* \mathscr{O}_H \to 0,$$

and tensoring with $\mathscr{F}(k)$ gives

$$0 \to \mathscr{F}(k-1) \to \mathscr{F}(k) \to \mathscr{F}(k) \otimes i_* \mathscr{O}_H \to 0.$$

For $k = m - i$, we have

$$\to H^i(\mathbb{P}^n, \mathscr{F}(m-i)) \to H^i(\mathbb{P}^n, \mathscr{F}_H(m-i)) \to H^{i+1}(\mathbb{P}^n, \mathscr{F}(m-i-1)) \to \cdots.$$

Here, we are using induction on n. If $\mathscr{F}$ is m-regular, the first and third terms vanish, so the second term also vanishes. This shows that $\mathscr{F}_H$ is also m-regular, because we can compute cohomology wherever. So from the exact sequence

$$H^{i+1}(\mathscr{F}(m-i-1)) \to H^{i+1}(\mathscr{F}(m-i)) \to H^{i+1}(\mathscr{F}_H(m-i)),$$

the first and third vanishes by induction, and so the second vanishes. This proves (b).

For (a), we consider the diagram

$$\begin{tikzcd}
H^0(\mathscr{F}(k-1)) \otimes H^0(\mathscr{O}(1)) \arrow[r, two heads] \arrow[d] & H^0(\mathscr{F}_H(k-1)) \otimes H^0(\mathscr{O}_H(1)) \arrow[d, two heads] \\
H^0(\mathscr{F}(k)) \arrow[r] & H^0(\mathscr{F}_H(k)).
\end{tikzcd}$$

The top horizontal map is surjective, because $H^0(\mathscr{F}(k-1)) \to H^0(\mathscr{F}_H(k-1))$ is already surjective, which is because $H^1(\mathscr{F}(k-2)) = 0$. We want to show that the left vertical map is surjective. But the kernel of the bottom map is $H^0(\mathscr{F}(k-1))$, and by construction, the image of the left vertical map contains it. $\square$

Theorem 13.3. *For every n, there exists a polynomial $P_n(x_0, \dots, x_n)$ such that any ideal sheaf $\mathscr{I}$ on $\mathbb{P}_n$ is $P_n(a_0, \dots, a_n)$-regular, where a_i are defined by*

$$\chi(\mathscr{I}(m)) = \sum_{i=0}^{n} a_i \binom{m}{i}.$$

Proof. Take Z be the subscheme corresponding to $\mathscr{I}$. Take a hyperplane H that avoids the associated points of Z. Then we again have

$$0 \to \mathscr{I}(m) \to \mathscr{I}(m+1) \to (\mathscr{I} \otimes O_H)(m+1) \to 0.$$

Here, $\mathscr{I}_H$ is an ideal sheaf of H. Apply the induction hypothesis to $\mathscr{I}_H$. Then then first we have, for $m \geq m_1 - 2$,

$$0 \to H^0(\mathscr{I}(m)) \to H^0(\mathscr{I}(m{+}1)) \to H^0(\mathscr{I}_H(m{+}1)) \to H^1(\mathscr{I}(m)) \to H^1(\mathscr{I}(m{+}1)) \to 0.$$

Also, for $i \geq 2$, we get

$$0 \to H^i(\mathscr{I}(m)) \to H^i(\mathscr{I}(m+1)) \to 0$$

for $m \geq m_1 - i$.

Now, we want to show that $H^1(\mathscr{F}(m))$ stabilizes after some bounded time. From the short exact sequence, we see that either

$$H^0(\mathscr{I}(m+1)) \to H^0(\mathscr{I}_H(m+1))$$

is surjective or $\dim H^1(\mathscr{I}(m{+}1)) < \dim H^1(\mathscr{I}(m))$. So after some time, it will stabilize at 0, and from on this point, all H^0 are going to be surjective. This should happen in approximately $H^1(\mathscr{I}(m))$-time, but we know that

$$\dim H^1(\mathscr{I}(m)) = \dim H^0(\mathscr{I}(m)) - \chi(\mathscr{I}(m)) \leq \dim H^0(\mathscr{O}(m)) - \chi(\mathscr{I}(m))$$

because $H^i(\mathscr{I}(m))$ vanish for $i \geq 2$ for sufficiently large m, and $\mathscr{I}$ is an ideal sheaf. Then $\dim H^0(\mathscr{O}(m))$ is a polynomial in m, so the stabilization occurs in polynomial time. $\square$

14 March 29, 2018

Today we are finally going to construct some schemes. We introduced two functors

$$\underline{\text{Curves}}_X = \coprod_{\xi \in NS(X)} \underline{\text{Curves}}^{\xi}_X, \quad \underline{\text{Pic}}_X = \coprod_{\xi} \underline{\text{Pic}}^{\xi}_X.$$

But we can use a coarser decomposition. Let $X \subseteq \mathbb{P}^n_k$ so that we have a very ample line bundle $\mathscr{O}_X(1)$. Then each line bundle have a Hilbert polynomial, and then we can write

$$\underline{\text{Curve}}_X = \coprod_P \underline{\text{Curve}}_X.$$

This is indeed a coarser decomposition because Riemann–Roch tells us that the Hilbert polynomial $h_{\mathscr{F}}(m) = \chi(\mathscr{F}(m))$ is determined by the intersection pairing.

14.1 Grassmannians and other prerequisites

Classically, the **Grassmannian** $\mathbb{G}(n, r)$ is the space of r-dimensional spaces in $\mathbb{A}^{n+1}$. The moduli interpretation is that it sends S to the set

$$\{\mathscr{E} \text{ rank } r \text{ locally free with sections } s_0, \dots, s_n \in \Gamma(\mathscr{E}) \text{ satisfying } \mathscr{O}^{n+1} \twoheadrightarrow \mathscr{E}\}.$$

You can construct $\mathbb{G}(n, r)$ using the Plücker embedding to $\mathbb{P}^{(\cdots)}$. This is supposed to send a r-dimensional space to $v_1 \wedge \cdots \wedge v_r$.

Proposition 14.1. *Let G be a scheme over k, and let $A \subseteq B : (\mathsf{Sch}/k)^{\mathrm{op}} \to \mathsf{Set}$, with the inclusion transformation $A \to B$ factoring through h_G. Assume that for all $\alpha \in B(S)$ there exists a subscheme $Y \subseteq S$ such that for all $g : T \to S$,*

$$g^*(\alpha) \in A(T) \subseteq B(T) \quad \Leftrightarrow \quad g \text{ factors through } Y.$$

Then there exists a $G_0 \subseteq G$ such that $A \cong h_{G_0}$.

Definition 14.2. Let S be a scheme. A **stratification** of S is a set $\{S_1, \dots, S_m\}$ such that each $S_i \subseteq S$ is locally closed and $S = \coprod_i S_i$ at the level of points.

Proposition 14.3. *Let $\mathscr{F}/\mathbb{P}^n \times S$ be a coherent sheaf. Then there exists a stratification of S such that for every $T \to S$, the pullback $g_T^*\mathscr{F}$ is flat over T if and only if g factors through $\coprod_i S_i$.*

$$\begin{tikzcd} \mathbb{P}^n \times T \arrow[r, "g_T"] \arrow[d] & \mathbb{P}^n \times S \arrow[d, "p"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

Here is a trivial example. Consider the skyscraper sheaf on $\mathbb{A}^1_k$ at some point. You can make the sheaf flat on $\mathbb{A}^1_k \setminus \{0\}$ and $\{0\}$.

14.2 Embedding in the Grassmannian

Recall what we proved last time.

Theorem 14.4. *For all n, there exists a polynomial $F_n(x_0, \ldots, x_n)$ such that for every $\mathscr{I}$ on $\mathbb{P}^n$, it is $F_n(a_0, \ldots, a_n)$-regular where $\chi(\mathscr{I}(m)) = \sum a_i \binom{m}{i}$.*

So given a Hilbert polynomial P, there exists a m_0 such that all $\mathscr{I}$ with Hilbert polynomial P is m_0-regular. For curves, you can do this if you use Serre duality and

$$\dim H^1(\mathscr{L}) = \dim H^0(\omega \otimes \mathscr{L}^\vee) = 0$$

if $\deg \mathscr{L} \geq \deg \omega$.

Fix a Hilbert polynomial P and choose a m_0 sufficiently large such that

(i) If $D \subseteq X$ is any curve with Hilbert polynomial P, then $\mathscr{O}_X(-D)$ is m_0-regular. So $H^1(X, \mathscr{O}_X(-D + m_0)) = H^2(X, \mathscr{O}_X(-D + m_0)) = 0$.

(ii) $H^1(\mathscr{O}(m_0)) = 0$.

(iii) $\mathscr{O}_X(-D + m_0)$ is spanned by global sections.

Then using the long exact sequence for $0 \to \mathscr{O}_X(-D + m_0) \to \mathscr{O}_X(m_0) \to \mathscr{O}_D(m_0) \to 0$, we get that $H^1(X, \mathscr{O}_D(m_0)) = 0$.

Suppose that $\mathcal{D} \subseteq X \times S$ is a family of curves, giving Hilbert polynomial P. Then $p_*\mathscr{O}_{\mathcal{D}}(m_0)$ is locally free of rank $\chi(\mathscr{O}_X(m_0)) - P(m_0)$. By the theorem of Grothendieck, $H^1(X_s, \mathscr{O}_{\mathcal{D}_s}(m_0)) = 0$ implies that

$$p_*\mathscr{O}_{\mathcal{D}}(m_0) \otimes \kappa(s) \to H^0(X_s, \mathscr{O}_{\mathcal{D}_s}(m_0))$$

is an isomorphism. Also, if we apply the theorem, to $H^2(X, \mathscr{O}_X(-D+m_0)) = 0$, then we get that we can compute $R^1 p_* \mathscr{O}_{X \times S}(-\mathcal{D} + m_0)$ locally. This gives

$$R^1 p_* \mathscr{O}_{X \times S}(-\mathcal{D} + m_0) = 0.$$

Also, $\mathscr{F}$ globally generated means that $p^* p_* \mathscr{F} \to \mathscr{F}$ is surjective. So

$$p^* p_* \mathscr{O}_{X \times S}(-\mathcal{D} + m_0) \twoheadrightarrow \mathscr{O}_{X \times S}(-\mathcal{D} + m_0)$$

is surjective.

Fix a basis $e_0, \ldots, e_N$ of $H^0(\mathscr{O}_X(m_0))$. Then

$$0 \to p_*\mathscr{O}_{X \times S}(-\mathcal{D} + m_0) \to p_*\mathscr{O}_{X \times S}(m_0) \xrightarrow{\sigma} p_*\mathscr{O}_{\mathcal{D}}(m_0) \to 0.$$

So we have produced a locally free $p_*\mathscr{O}_{\mathcal{D}}(m_0)$ of rank r, and $(N + 1)$ sections $s_i = \Delta(1 \otimes e_i)$ of $p_*\mathscr{O}_{\mathcal{D}}(m_0)$.

15 April 3, 2018

Last time, we had to pick m sufficiently large so that $\mathcal{O}(-D + m_0)$ is spanned by global sections for all D with given Hilbert polynomial.

Proposition 15.1. *If $\mathscr{F}/\mathbb{P}^n$ is m-regular, then $\mathscr{F}(k)$ is generated by its global sections for $k \geq m$.*

Proof. It suffices to show for $k = m$. We know that we have a surjection

$$H^0(\mathbb{P}^n, \mathscr{F}(k-1)) \otimes H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \twoheadrightarrow H^0(\mathbb{P}^n, \mathscr{F}(k))$$

for $k > m$. By Serre vanishing, we know that

$$H^0(\mathbb{P}^n, \mathscr{F}(k)) \otimes \mathscr{O}_{\mathbb{P}^n} \twoheadrightarrow \mathscr{F}(k)$$

for $k \gg 0$. Then we twisted this to

$$H^0(\mathbb{P}^n, \mathscr{F}(m)) \otimes H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^{\otimes k-m} \otimes \mathscr{O}_{\mathbb{P}^n}^{(m-k)} \twoheadrightarrow \mathscr{F}(m) \otimes \mathscr{O}_{\mathbb{P}^n}(m-k).$$

But this factors through $H^0(\mathbb{P}^n, \mathscr{F}(m)) \otimes \mathscr{O}_{\mathbb{P}^n}$. ☐

15.1 Representing the curves functor

Last time, we construct a natural transformation

$$\underline{\mathrm{Curve}}_X^P(S) \to h_{\mathbb{G}(N,r)},$$

for a given Hilbert polynomial P. This curve functor is given by

$$\underline{\mathrm{Curve}}_X^P(S) = \{\mathcal{D} \subseteq X \times S \text{ relative Cartier divisor}\}$$

so that $\mathcal{D}/S$ is flat. On the other hand, the functor represented by the Grassmannian is

$$\mathrm{Hom}(S, \mathbb{G}(N,r)) = \{\text{locally free } \mathscr{E} \text{ of rank } r \text{ with } \mathscr{O}_S^{N+1} \twoheadrightarrow \mathscr{E}\}/\cong.$$

To go from $\mathcal{D}$ to $\mathscr{E}$, we choose m_0, which only depends on P, such that $p_*\mathscr{O}_{\mathcal{D}}(m_0)$ is locally free and $R^1p_*\mathscr{O}_{X\times S}(-\mathcal{D}+m_0) = 0$. We have a short exact sequence

$$0 \to \mathscr{O}_{X\times S}(-\mathcal{D} + m_0) \to \mathscr{O}_{X\times S}(m_0) \to \mathscr{O}_{\mathcal{D}}(m_0) \to 0.$$

If we push forward along p, we have vanishing of R^1p_*, so we have

$$0 \to p_*\mathscr{O}_{X\times S}(-\mathcal{D} + m_0) \to \mathscr{O}_S \otimes H^0(X, \mathscr{O}_X(m_0)) \to p_*\mathscr{O}_{\mathcal{D}}(m_0) \to 0.$$

Now we can take a basis $e_0, \ldots, e_N$ of $H^0(X, \mathscr{O}_X(m_0))$ and then consider the last map $\mathscr{O}_S^{N+1} \twoheadrightarrow \mathscr{O}_{\mathcal{D}}(m_0)$.

We are going to apply the formal lemma from last time here. Consider the diagram

$$\begin{tikzcd} \underline{\mathrm{Curve}}_X^P \arrow[r, "\Phi"] \arrow[d, hook] & h_{\mathbb{G}(n,r)} \arrow[dl, "\Psi"] \\ \underline{\mathrm{AllSubsch}}_X. & \end{tikzcd}$$

We still need to define the natural transformation $\Psi : h_{\mathbb{G}(N,r)} \to \underline{\mathrm{AllSubsch}}$. If I have a surjection,

$$0 \to \mathscr{K} \to \mathscr{O}_S \otimes H^0(X, \mathscr{O}_X(m_0)) \twoheadrightarrow \mathscr{E} \to 0$$

I can take the kernel. Then pulling back to $X \times S$ gives

$$p^*\mathscr{K} \to p^*p_*\mathscr{O}_{X\times S}(m_0) \to p^*\mathscr{E} \to 0.$$

Then if I twist by $-m_0$, I get

$$p^*\mathscr{K}(-m_0) \to p^*p_*\mathscr{O}_{X\times S} \to \mathscr{O}_{X\times S}.$$

Then I can take the image $\mathscr{I} = \mathrm{im}(p^*\mathscr{K}(-m_0)) \subseteq \mathscr{O}_{X\times S}$.

Now we need to check the formal condition. Given $g : T \to S$ and some subscheme $Z \subseteq X \times S$, we want to find a Y such that the pullback of Z to $X \times T$ is a relative Cartier divisor with Hilbert polynomial P if and only if g factors through Y. Let Z correspond to an ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_{X\times S}$. By flat stratification, there exists a $Y \subseteq S$ such that $\mathscr{I}|_Y$ is flat with Hilbert polynomial for $Z \subseteq X \times S$ is defined by

$$\chi(\mathscr{O}_{Z\times_S T}(n)) = \chi(\mathscr{O}_X(n)) - P(n).$$

The next step is to restrict Y to make sure that Z is a Cartier divisor. Note that being a Cartier divisor is open, so that if $Z_t \subseteq X_t$ is Cartier then there exists a neighborhood U such that Z is a Cartier in t. In fact, it suffices to find a neighborhood of X_t such that for each point x over t, we can find a neighborhood V such that $Z \cap V$ is cut out by one equation in V.

Consider a point $t \in T$ that is cut out by $\mathfrak{m}_t$, so that X_t is also cut out by $\mathfrak{m}_t$. For $x \in Z_t$, consider $\mathscr{O}_x$, and let $\mathscr{I}_x \subseteq \mathscr{O}_x$ to be the ideal associated to Z. The n $\mathscr{I}_x$ is cut out by one equation in $\mathscr{O}_x/\mathfrak{m}_t\mathscr{O}_x$. So we can write

$$\mathscr{I}_x + \mathfrak{m}_t\mathscr{O}_x = (f) + \mathfrak{m}_t\mathscr{O}_x$$

for some $f \in \mathscr{O}_x$. Here, we may as well assume $f \in \mathscr{I}_x$.

Now we have an exact sequence

$$0 \to \mathscr{I}_x/(f) \to \mathscr{O}_x/(f) \to \mathscr{O}_x/\mathscr{I}_x \to 0.$$

Tensoring with $k(t)$ gives

$$\mathrm{Tor}^1_{\mathscr{O}_t}(\mathscr{O}_x/\mathscr{I}_x, k(t)) \to \mathscr{I}_x/(f) \otimes k(t) \to \mathscr{O}_x/(f) + \mathfrak{m}_t\mathscr{O}_x \to \mathscr{O}_x/\mathscr{I}_x + \mathfrak{m}_t\mathscr{O}_x \to 0.$$

The first term vanishes by flatness. The last map is an isomorphism by the condition. So we have $\mathscr{I}_x/(f) \otimes k(t) = 0$. By Nakayama, we get $\mathscr{I}_x/(f) = 0$, and so Z is cut out by a single polynomial.

16 April 5, 2018

So we constructed the scheme representing the curves functor last time. Our goal was to represent $\underline{\text{Curve}}_X^P$. These are divisors $\mathcal{D} \subseteq X$ such that $P(n) = \chi(\mathcal{O}_X(-D+n))$. Then we found out that there is a m_0 such that $\mathscr{O}_X(-D+m_0)$ are spanned by global sections. Using this we constructed a functor

$$\underline{\text{Curve}}_X^P \hookrightarrow h_{\mathbb{G}(n,r)}.$$

Then we used flatness stratification and that flatness argument to show that it is representable. You can even show that this functor is going to be closed in $\mathbb{G}(n,r)$.

17 April 10, 2018

Our next goal is to construct the Picard functor. Again, we reduce it to constructing it for one numerical class. But here, for any two numerical class ξ and ξ', we have a non-canonical isomorphism

$$\underline{\mathrm{Pic}}_X^{\xi} \cong \underline{\mathrm{Pic}}_X^{\xi'}.$$

So we will construct $\underline{\mathrm{Pic}}_X^{\xi}$ for some ξ such that any L with ξ is very ample and 0-regular.

17.1 Ampleness

Recall that $\mathscr{L}/X$ is ample if for all $\mathscr{F} \in \mathsf{Coh}(X)$, there exists an n only depending on $\mathscr{F}$ such that $\mathscr{L}^{n'} \otimes \mathscr{F}$ is globally generated for all $n' \geq n$.

Theorem 17.1. *If $\mathscr{L}$ is ample, then $\mathscr{L}^n$ is very ample for $n \gg 0$.*

Proof. Let U be an affine neighborhood of P. Let $Y = X - U$ and consider $\mathscr{I}_Y \otimes \mathscr{L}^n$ globally generated for $n \gg 0$. Then we get $s \in \Gamma(\mathscr{I}_Y \otimes \mathscr{L}^n)$ such that $s \neq 0$ at P. But we can view $\mathscr{I}_Y \otimes \mathscr{L}^{\otimes n} \subseteq \mathscr{L}^{\otimes n}$ and then s vanishes on Y as a section of $\mathscr{L}^{\otimes n}$.

Now let X_s be the locus of where $s \neq 0$. This is in U, but is cut out be one equation, so it is affine. If we choose n large enough, we can cover X with X_{s_i}. Now we look at b_{ij} generators of $\Gamma(X_{s_i}, \mathscr{O}_{X_{s_i}})$. Then $c_{ij} = s_i^n b_{ij} \in \Gamma(X, \mathscr{L}^n)$ defines the closed embedding $X \to \mathbb{P}^N$. $\square$

There is a cohomological criterion for ampleness.

Theorem 17.2. *$\mathscr{L}/X$ is ample if and only if, for all $\mathscr{F} \in \mathsf{Coh}(X)$ there exist an n such that for $n' \geq n$ we have $H^i(X, \mathscr{L}^n \otimes \mathscr{F}) = 0$ for $i > 0$.*

Theorem 17.3 (Nakai–Moishezon criterion)**.** *Let X/k be a surface. Then $\mathscr{L}/X$ is ample if and only if $\mathscr{L}^2 > 0$ and $\mathscr{L}.C > 0$ for every irreducible curve C.*

Positivity comes from intersecting two effective curves.

Proof. Suppose that $\mathscr{L}$ is ample. We might as well assume that $\mathscr{L} = \mathscr{O}(D)$ for some D effective. Then we can consider D as a hyperplane section. This shows that $D^2 > 0$ and $D.C > 0$ as well. The other direction is more interesting. We use Riemann–Roch

$$h^0(\mathscr{L}) - h^1(\mathscr{L}) + h^2(\mathscr{L}) = \frac{\mathscr{L}^2 - \mathscr{L}.\mathscr{K}}{2} + \chi(\mathscr{O}_X).$$

We first show that $\mathscr{L} \otimes \mathscr{O}_D$ is ample on D. First we replace D with its reduced D_{red}, because this does not change ampleness. Then we see that $\mathscr{O}_X(D) \otimes \mathscr{O}_D|_{C_i} = \mathscr{O}_{C_i}(D \cap C_i)$ which is positive because $D \cap C_i > 0$.

Then we show that $\mathscr{L}^n$ is generated by global sections so that we have $\varphi : X \to \mathbb{P}^N$. We have then

$$\begin{aligned} 0 &\to H^0(X, \mathscr{L}^{n-1}) \to H^0(X, \mathscr{L}^n) \to H^0(X, \mathscr{L}^n \otimes \mathscr{O}_D) \\ &\to H^1(X, \mathscr{L}^{n-1}) \to H^1(X, \mathscr{L}^n) \to H^1(x, \mathscr{L}^n \otimes \mathscr{O}_D) \to \cdots . \end{aligned}$$

For $n \gg 0$, the last term is zero, so the map $H^1 \to H^1$ is surjective. Then for larger enough n, the dimension cannot decrease, so we have

$$0 \to H^0(X, \mathscr{L}^{n-1}) \to H^0(X, \mathscr{L}^n) \to H^0(X, \mathscr{L}^n \otimes \mathscr{O}_D) \to 0.$$

Since $\mathscr{L}^n \otimes \mathscr{O}_D$ is globally generated over D, we can take $s_i \in H^0(X, \mathscr{L} \otimes \mathscr{O}_D)$ that do not vanish simultaneously on D. Then we can lift them to $\tilde{s}_i \in H^0(X, \mathscr{L}^n)$. Then we can add the one section that only vanishes on D.

The claim is that this has finite fibers. If not, there exists a curve C such that $\varphi(C)$ is a point. But then $nD.C = 0$ because a point C and a general hyperplane does not intersect. So the morphism $\varphi : X \to \mathbb{P}^N$ is quasi-finite and projective, so it is finite. Now for $\varphi : X \to Y$ a finite morphism, L/Y is ample if and only if $\varphi^* L$ is ample. (This follows from the Leray spectral sequence and the cohomological criterion for ampleness.) □

Lemma 17.4. *Let H be ample and D be any divisor such that $D.H > 0$ and $D^2 > 0$. Then for $n \gg 0$, we have nD is effective.*

Proof. By Serre duality, we have

$$h^2(X, \mathscr{O}_X(nD)) = h^0(X, \mathscr{O}_X(K - nD)).$$

But the right hand side vanishes if $nD.H > K.H$. Also in Riemann–Roch, we can make the $\chi(nD)$ large enough, and then $h^0 - h^1 > 0$ implies $h^0 > 0$. □

18 April 12, 2018

For $L : V \to V$ an endomorphism, its top wedge power is 1-dimensional, so we get a determinant. We want to construct the Picard functor.

18.1 Representing the Picard functor

The strategy is that if the natural map

$$\underline{\mathrm{Curve}}_X^{\xi} \xrightarrow{\Phi} \underline{\mathrm{Pic}}_X^{\xi}$$

admits a section s, then we can realize $P(\xi)$ as a pullback functor

$$\begin{tikzcd} P(\xi) \arrow[r] \arrow[d] & C(\xi) \arrow[d, "{(1,f)}"] \\ C(\xi) \arrow[r, "\Delta"] & C(\xi) \times C(\xi). \end{tikzcd}$$

Then because $C(\xi)$ is representable, this is a fiber product of just schemes.

So we need to find a section s of Φ. Choose ξ such that L with ξ is very ample and 0-regular. Now what is a section? For each line bundle $L/X \times S$, we need a Cartier divisor $\mathcal{D} \subseteq X \times S$ such that

$$\mathscr{O}_{X \times S}(\mathcal{D}) = L \otimes p_2^* M$$

for some M. This association should be moreover functorial and well-defined in the sense that if $L' = L \otimes p_2^* M'$ then $\mathcal{D}$ is the same as for L.

Let me first explain Mumford's construction. Take $\mathscr{E} = p_{2*}L$ and $L_x = i_x^*(L)$ for the some $x \in X$ and $i_x : S \to X \times S$. Then we have a morphism $\mathscr{E} \to L_x$ induced by evaluation at x. Because $X \times S \to S$ is flat, we have that $\mathscr{E}$ is locally free. Let $r = \operatorname{rank} \mathscr{E}$ and pick $r-1$ points $x_1, \ldots, x_{r-1}$. Then we have a map

$$\mathscr{E} \to \bigoplus_{i=1}^{r-1} L_{x_i}$$

and taking the $(r-1)$th wedge power gives

$$\textstyle\bigwedge^{r-1} \mathscr{E} \to \displaystyle\bigotimes_{i=1}^{r-1} L_{x_i}.$$

Taking the dual gives

$$\left(\bigotimes_{i=1}^{r-1} L_{x_i} \right)^{\vee} \to \mathscr{E} \otimes (\textstyle\bigwedge^r \mathscr{E})^{\vee},$$

and we have

$$\mathscr{O}_S \to \mathscr{E} \otimes (\textstyle\bigwedge^r \mathscr{E})^{\vee} \otimes \displaystyle\bigotimes_{i=1}^{r-1} L_{x_i}.$$

Then we can pull this back to get

$$\mathscr{O}_{X\times S} = p_2^*\mathscr{O}_S \to p_2^*\mathscr{E} \otimes p_2^*\left(\left(\bigwedge^r \mathscr{E}\right)^\vee \otimes \bigotimes_{i=1}^{r-1} L_{x_i}\right) \to L \otimes p_2^*(\cdots).$$

This gives the desired section of the line bundle of the form $L \otimes p_2^* M$.

We need to check that if $L' = L \otimes p_2^* M'$ then we get the same divisor. To do this, we need the projection formula.

Proposition 18.1 (projection formula)**.** *On locally ringed spaces, let $f : X \to Y$ be a morphism with $\mathscr{F}$ a sheaf on X and $\mathscr{E}$ be a locally free sheaf of finite rank on Y. Then*

$$f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \cong f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} f^*\mathscr{E}).$$

If we replace L with $L \otimes p_2^* M'$, from

$$L \otimes p_2^*\left((\bigwedge^r p_{2*}L)^\vee \otimes \bigotimes_{i=1}^{r-1} L_{x_i}\right),$$

we get a bunch of $p_2^* M'$, but they cancel out. Functoriality is clear.

But what is this construction really doing? Let's just look in the case of $S = \mathrm{pt}$. Here, $\mathscr{E} = H^0(X, L)$. If $r = \dim H^0(X, L)$ and I have some points $x_1, \ldots, x_{r-1}$, I can take the kernel of

$$H^0(X, L) \to \bigoplus \mathscr{L}_{x_i} \cong k^{r-1}; \quad s \mapsto (s(x_1), \ldots, s(x_{r-1})).$$

This is going to be a 1-dimensional subspace generically, and the vanishing locus of this is going to be my curve. That is, this is the curve passing through all the $r-1$ points $x_1, \ldots, x_{r-1}$.

But the problem is that generically this will give us a unique curve, but it might be the entire surface. Can we choose $r-1$ points so that for any line bundle the curve is uniquely determined this way?

Example 18.2. Let C be a genus 2 curve, and let us look at the degree 2 case. In this case, if $\deg L = n$ then we can compute $\dim H^0(L) = 2$ for $n = 2$ and $\dim H^0(L) = n - 1$ for $n > 2$. Whatever $P_1, \ldots, P_{n-2}$ we pick, we can take Q_1, Q_2 such that $\mathscr{O}_C(Q_1 + Q_2) = K$. Then for $L = P_1 + \cdots + P_{n-2} + Q_1 + Q_2$, we will have

$$\{s \in H^0(L) : s(P_i) = 0\} = H^0(L(-P_1 - \cdots - P_{n-2})) = H^0(\mathscr{O}_C(Q_1 + Q_2))$$

a dimension 2 space.

So things don't work well.

19 April 17, 2018

Let C be a curve of genus g and let d be some large number. There is a morphism

$$C^d \to \underline{\mathrm{Pic}}^d_C; \quad (x_1, \ldots, x_d) \mapsto \mathscr{O}_C(x_1 + \cdots + x_d)$$

and then we can look at the graph $\Gamma \subseteq \underline{\mathrm{Pic}}^d_C \times C^d$. The fiber of the projection map $\Gamma \to \underline{\mathrm{Pic}}^d_C$ has fiber

$$\Gamma_L = \mathbb{P}H^0(C, L).$$

It turns out that $\dim \underline{\mathrm{Pic}}^d_C = g$, so that we have

$$h^0(L) - 1 \geq d - g.$$

We can use this to give a new perspective on the counterexample given last time. Given a curve C of genus ≥ 2, can you choose points $x_1, \ldots, x_{d-2}$ such that for all L/C of degree d, vanishing on x_is imposes exactly $d-2$ linear conditions? We can consider

$$Z = \{(L, (x_1, \ldots, x_{d-2})) : h^0(L(-\textstyle\sum x_i)) \geq 2\} \subseteq \underline{\mathrm{Pic}}^d_C \times C^{d-2}.$$

This can be thought of as a bad locus. Here, the image of the projection $Z \to \underline{\mathrm{Pic}}^d_C$ is going to be a closed subscheme, because it is the line bundles with cohomology jumps. The fibers are going to be $\dim Z_L = d - 4$, and so we have $\dim Z = d - 2$. This is equal to the dimension of C^{d-2}. So we can't conclude that such points $d_1, \ldots, d-2$ exists by just dimension counting.

For general $g \geq 2$, we need Brill–Noether theory. For C of genus g, so that $h^0(L) = d+1-g$, we want to pick $d-g$ points. So we look at $Z \subseteq \underline{\mathrm{Pic}}^d_C \times C^{d-g}$. Again, if we look at Z_L, we can use Bril–Noether to get

$$\operatorname{codim}_Z Z_L = \operatorname{codim} \underline{\mathrm{Pic}}^d_C = 2.$$

Then we have $\dim Z = d - 2$, and so the map to C^{d-g} is more likely to be surjective.

19.1 Representing the Picard functor II

Anyways, we were trying to represent $\underline{\mathrm{Pic}}^\xi_X$ for a numerical class ξ. Our strategy is the same, but now we need to do better than just taking $d-2$ points. So take G to be some parameter space, and we want a rule for getting a line bundle. Mumford adds more degree of freedom to G so that $\dim Z < \dim G$.

Choose $Nr - 1$ points and group them into $N - 1$ sets of r points and 1 set of $r - 1$ point. Let us write this

$$\mathcal{Y} = \{(x_{1,1}, \ldots, x_{1,r}), \ldots, (x_{N,1}, \ldots, x_{N,(r-1)})\}.$$

For the last $(r-1)$ points, do what we previously did, which is to take

$$\sigma_N \in H^0(X \times S, L \otimes p_2^*\{(\textstyle\bigwedge^r \mathscr{E})^{-1} \otimes [\bigotimes_{i=1}^{r-1} L_{x_{N,i}}]\}).$$

For the set of ρ points in $\mathcal{Y}$, we look at the map $\bigwedge^r \mathscr{E} \to \bigotimes_{i=1}^r L_{x_{k,i}}$ and look at the corresponding section

$$\sigma_k \in H^0(X \times S, L \otimes p_2^*\{(\bigwedge^r \mathscr{E})^{-1} \otimes [\bigotimes_{i=1}^r L_{x_{k,i}}]\}).$$

Now we consider

$$\sigma_{\mathcal{Y}} = \sigma_1 \otimes \cdots \otimes \sigma_{N-1} \otimes \sigma_N \in H^0(X \times S, L \otimes p_2^*\{(\bigwedge^r \mathscr{E})^{-N} \otimes [\bigotimes_{k,i} L_{x_{k,i}}]\}).$$

Theorem 19.1. *For suitable choices of ξ and N and $Nr-1$ point and scalars $a_{\mathcal{Y}}$, the sum $\sum a_{\mathcal{Y}}\sigma_{\mathcal{Y}}$ is flat over S if and only if $\sum a_{\mathcal{Y}}\sigma_{\mathcal{Y}}$ does not vanish identically over any $s \in S$.*

I won't prove this, but here is the basic idea. Consider a polarized surface $(X, \mathscr{O}_X(1))$. A 0-cycle $\mathfrak{U}$ is called **λ-independent** if for all $C \subseteq X$, we have

$$\deg(\mathfrak{U} \cap C) \le \lambda(\deg C)^2.$$

We also say that a 0-cycle $\mathfrak{U}$ of $\mathbb{P}^n$ is **strongly stable** if for all hyperplane H,

$$\deg(\mathfrak{U} \cap H) \le \frac{\deg \mathfrak{U}}{n+1}.$$

Then the thing we want to prove can be formulated in terms of these properties and you play with Riemann–Roch.

Next time we are going to study the tangent space of $\underline{\mathrm{Pic}}_X$. Because T_pX is $\mathrm{Hom}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$, the tangent space $T_L\underline{\mathrm{Pic}}_X(k)$ is the deformation of L over $\operatorname{Spec} k[\epsilon]/\epsilon^2$.

20 April 19, 2018

How do we differentiate? We can define the tangent space as

$$\mathrm{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X) = T_x X.$$

How does the left hand side have a structure of a k-vector space? In general, schemes don't have pushouts, but

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[rr] \arrow[d] & & \operatorname{Spec} k[\epsilon_1]/(\epsilon_1^2) \arrow[d] \\
\operatorname{Spec} k[\epsilon_2]/(\epsilon_2^2) \arrow[r] & \operatorname{Spec} k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2) &
\end{tikzcd}$$

is a pushout diagram. So if we have two tangent vectors, we get a map $\operatorname{Spec} k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2) \to X$, and then we can compose with the "diagonal" $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2)$ induced by $\epsilon_1 \mapsto \epsilon$, $\epsilon_2 \mapsto \epsilon$.

20.1 Infinitesimal structure of Curve and Pic

Here, the only thing we know about Curve_X and Pic_X are the functorial properties. So we need to abstractly talk about deformations. Let X be a surface and $D \subseteq X$ be a curve with $\mathrm{Num}(D) = \xi$. This can be considered as a point $s \in \mathrm{Curve}_X^\sigma(k)$. What is the tangent space at s? By definition, we have

$$\mathrm{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k[\epsilon]/\epsilon^2, \mathrm{Curve}_X^\xi) = \left\{ \begin{matrix} \mathcal{D} \subseteq X \times T \text{ effective relative Cartier} \\ \text{divisor with } \mathcal{D}_0 = D \end{matrix} \right\},$$

where we write $T = \operatorname{Spec} k[\epsilon]/\epsilon^2$.

The claim is that we can also describe this space as

$$H^0(D, N_{D/X}) = H^0(X, i_* N_{D/X}).$$

In particular, if $H^0(X, N_{D/X}) = 0$ then you cannot move your curve infinitesimally. This occurs when you blow up a surface at a curve and get $E.E = -1$. Then $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1)) = 0$.

The first thing to note is that X and $X_T = X \times T$ are the same as topological spaces. Let's first construct the map for relative Cartier divisors to $H^0(X, i_* N_{D/X})$. A relative Cartier divisor is represented by $\{(U_i, f_i)\}$ functions on an open cover. Then we can write

$$f_i = g_i + \epsilon h_i$$

for $g_i \in \Gamma(U_i, \mathscr{O}_X^\times)$ and $h_i \in \Gamma(U_i, \mathscr{O}_X)$, because f_i has to be invertible. Because our fiber D is fixed, g_i are already determined. We also need the compatibility conditions, f_i and f_j should differ by a unit. So we should be able to write

$$g_i + \epsilon h_i = (a_{ij} + \epsilon b_{ij})(g_j + \epsilon h_j)$$

and then $g_i = a_{ij}g_j$ and $h_i a_{ij} h_j + b_{ij} b_j$ for $a_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times)$ and $b_{ij} \in \Gamma(U_i \cap U_j), \mathscr{O}_X)$. We can write this as

$$\frac{h_i}{g_i} - \frac{h_j}{g_j} = \frac{b_{ij}}{a_{ij}}.$$

The claim now is that $\{(U_i, \frac{h_i}{g_i})\}$ defines global section in $N_{D/X}$. The normal bundle can be described as

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(D) \to i_* N_{D/X} \to 0,$$

so that $N_{D/X} = \mathscr{O}_D \otimes \mathscr{O}_X(D)$ and locally $0 \to A \to \frac{1}{f}A \to A/(f) \otimes_A \frac{1}{f}A$. Now the difference between h_i/g_i and h_j/g_j comes from $\mathscr{O}_X(U_i \cap U_j)$ and so the sections glue well.

The converse construction goes this way. If $s \in \Gamma(X, i_* N_{D/X})$, then for U_i an affine cover of X we have

$$0 \to \mathscr{O}_X(U_i) \to \mathscr{O}_X(D)(U_i) \to i_* N_{D/X}(U_i) \to 0$$

and so we can lift $s|_{U_i}$ to h_i/g_i a section of $\mathscr{O}_X(D)(U_i)$. Then we can trace back. If we write

$$\frac{h_i}{g_i} - \frac{h_i'}{g_i'} = c_i \in \mathscr{O}_X(U_i)$$

then you can show that

$$(g_i + \epsilon h_i) = (d_i + \epsilon c_i d_i)(g_i' + \epsilon h_i')$$

for d_i the unit with $g_i = d_i g_i'$. So $g_i + \epsilon h_i$ is well-defined as a section in $\Gamma(U_i, K_{X_T}^* / \mathscr{O}_{X_T}^*)$. So we get an effective Cartier divisor.

Over a local ring, Cartier divisors and flatness interact very well. Here is what I mean. Let $(A, \mathfrak{m})$ be a local ring and let X be flat over A. Consider $D \subseteq X$ a closed subscheme.

Proposition 20.1. *(1) If D is flat and D is flat and $\mathcal{D}_0$ is a Cartier divisor, then D is also a Cartier divisor. (This we proved when representing the curves functor.)*

(2) If D is a Cartier divisor, then D is flat if and only if $D_0 \neq X_0$.

Proof. We may assume that $X = \operatorname{Spec} B$ so we have $A \to B$. Let D be $B/(f)$. We want to show that $B/(f)$ is flat over A, and then it suffices to show that $\operatorname{Tor}_1^A(B/(f), A/\mathfrak{m}) = 0$ because A is a local ring. If we use $0 \to B \to B \to B/(f) \to 0$, then we get

$$0 \to \operatorname{Tor}_1^A(B/(f), A/\mathfrak{m}) \to B \otimes A/\mathfrak{m} \xrightarrow{\times f} B \otimes A/\mathfrak{m} \to \cdots.$$

Because $f \notin \mathfrak{m}B$, multiplication by f is injective. □

So for $S = \underline{\mathrm{Curve}}_X^\xi$ and $D \subseteq X$ corresponding to $s \in S$, we have constructed an isomorphism

$$\rho : T_s S \to H^0(D, N_{D/X}).$$

Once we show that $\underline{\mathrm{Curve}}_X^\xi$ is non-singular, then we can compute its dimension as the dimension of the tangent space.

21 April 24, 2018

Suppose D corresponds to $s \in \underline{\mathrm{Curve}}_X^\xi(k)$. We identified a first-order infinitesimal deformation of D inside X with the global sections of the normal bundle. This is also the same as the Zariski tangent space of the scheme $T_s\underline{\mathrm{Curve}}_X^\xi$.

21.1 Regularity of the curve functor

Today we are going to first study regularity of $\underline{\mathrm{Curve}}_X^\xi$ when ξ is sufficiently ample. This notion of regularity comes from dimension theory. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We say that the local ring is **regular** if

$$\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

This is equivalent to there being a regular sequence $(f_1, \dots, f_d)$ such that $\bar{f}_i$ span $\mathfrak{m}/\mathfrak{m}^2$.

Example 21.1. Consider $\operatorname{Spec} k[x, y]/(xy)$, localized at (x, y). This is not regular.

We want to talk about regularity of $\underline{\mathrm{Curve}}_X^\xi$, but then we need to formulate a functorial characterization of regularity. Let A be a k-algebra and $A/\mathfrak{m} = k$. Then A is regular if and only if for every $R \twoheadrightarrow S$ with R, S Artin rings over k,

$$\begin{tikzcd} & A \arrow[dl, dashed] \arrow[d] \\ R \arrow[r, two heads] & S \end{tikzcd}$$

we have $\mathrm{Hom}(A, R) \twoheadrightarrow \mathrm{Hom}(A, S)$. Here we don't worry too much because k is algebraically closed.

A curve $D \subseteq X$ is **semi-regular** if

$$H^1(\mathscr{O}_X(D)) \to H^1(\mathscr{N}_{D/X})$$

is zero.

Theorem 21.2 (Severi–Kodaira–Spencer)**.** *$\underline{\mathrm{Curve}}_X^\xi$ is regular at s if* $\operatorname{char} k = 0$ *and D is semi-regular.*

Let us denote test Artinian rings by $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$. Write $0 \to I \to A \to B \to 0$. Given a relative Cartier divisors $D_B \subseteq X \times \operatorname{Spec} B$, we want to extend it to $D_A \subseteq X \times A$ extending D_B. Denote D_B by $\{(U_i, \bar{F}_i)\}$ with $\bar{F}_i \in \Gamma(U_i, \mathscr{O}_X \otimes B)$ and $\bar{F}_i = \bar{G}_{ij}\bar{F}_j$ for $\bar{G}_{ij} \in \Gamma(U_{ij}, (\mathscr{O}_X \otimes B)^\times)$.

We note that we may assume that $\dim_k A - \dim_k B = 0$, because we can to this inductively. Let η be a element of $A \setminus B$ so that $\eta A = \ker(A \to B)$. We start with arbitrary liftings F_i and G_{ij} of $\bar{F}_i$ and $\bar{G}_{ij}$. Then

$$F_i \in \Gamma(U_i, \mathscr{O}_X \otimes A), \quad G_{ij} \in \Gamma(U_{ij}, (\mathscr{O}_X \otimes A)^\times).$$

We can do this because $A \twoheadrightarrow B$ implies $A^\times \twoheadrightarrow B^\times$.

Let us write
$$F_i - G_{ij}F_j = \eta h_{ij}$$
for $h_{ij} \in \Gamma(U_{ij}, \mathscr{O}_X \otimes A)$. Then we get
$$\eta(h_{ij} + G_{ij}h_{jk}) = \eta h_{ik} + (G_{ik} - G_{ij}G_{jk})F_k.$$
If we divide by η and then reduce modulo $\mathfrak{m}$, we get
$$h_{ij} + G_{ij}^0 h_{jk} = h_{ik} + \frac{G_{ik}^0 - G_{ij}^0 G_{jk}^0}{\eta} F_k^0 \pmod{\mathfrak{m}}.$$
If we divide by F_i^0, then we get
$$\frac{h_{ij}}{F_i^0} + \frac{h_{jk}}{F_j^0} = \frac{h_{ik}}{F_i^0} + \frac{1 - G_{ij}^0 G_{jk}^0 G_{ik}^0}{\eta}$$
in $\Gamma(U_{ijk}, \mathscr{N}_{D/X})$. This implies that $\{h_{ij}/F_i^0\}$ is a Čech 1-cocycle of $\mathscr{N}_{D/X}$. This shows that being able to choose F_i and G_{ij} such that $h_{ij} = 0$ is equivalent to this cocycle being 0 in $H^1(\mathscr{N}_{D/X})$.

Now we have $0 \to \mathscr{O}_X \to \mathscr{O}_X(D) \to \mathscr{N}_{D/X} \to 0$, and so we have
$$H^1(\mathscr{O}(D)) \xrightarrow{0} H^1(\mathscr{N}_{D/X}) \hookrightarrow H^2(\mathscr{O}_X).$$
So we an test something in H^1 being zero by looking at its image in $H^2(\mathscr{O}_X)$. Our equation tells us that the image is $\{\frac{1}{\eta}(1 - G_{ij}G_{jk}G_{kj})\}$. This is a coboundary because $\frac{1}{\eta}(1 - (G_{ij}^0)^{-1}G_{ij})$ gives a 1-cycle.

Here is one interesting thing you can see. Let ξ be an ample class. There is the natural map $\underline{\mathrm{Curve}}_X^\xi \to \underline{\mathrm{Pic}}_X^\xi$ and the fiber is just $\mathbb{P}H^0(\mathscr{O}_X(D))$. So we should have
$$\dim \mathbb{P}H^0(\mathscr{O}_X(D)) + \dim H^1(\mathscr{O}_X) = \dim H^0(\mathscr{N}_{D/X}).$$
Here, $H^1(\mathscr{O}_X)$ is the tangent space of Pic. You can also see this from
$$0 \to \frac{H^0(\mathscr{O}_X(D))}{H^0(\mathscr{O}_X)} \to H^0(\mathscr{N}_{D/X}) \to H^1(\mathscr{O}_X) \to 0.$$

Index