

Physics 210 - General Theory of Relativity

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This course was taught by Daniel Jafferis. Lectures were given at WF 3-4:30 in Jefferson 356, and the textbook was Carroll's *Spacetime and Geometry*. There were 8 undergraduates and 9 graduate students enrolled, and there were weekly problem sets and a take-home final. The teaching fellow was Ping Gao.

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1 January 25, 2017

The textbook for this class is Carroll, *Spacetime and Geometry*. If you want more practice, you can look at Schutz, and if you want a more abstract approach, you can look at Wald.

1.1 Overview

We will have 4 weeks of differential geometry, 3 weeks about the Einstein equation, and the rest 5 weeks about black hole solutions, experimental tests, gravitational waves.

The basic idea of general relativity is that spacetime has curvature, and the source of that curvature is matter (but not only). The Einstein equation looks like this:

$$G^{\mu\nu} = kT^{\mu\nu}.$$

There is the $G^{\mu\nu}$, which is supposed to represent the curvature of spacetime. The $T^{\mu\nu}$ is the stress-energy tensor. If something moves only under the influence of gravity, it will move along locally straight paths, geodesics, in spacetime. The equation that says that a path is a geodesic is given by

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0.$$

This is related to the fact that all objects fall in the same way.

1.2 Basic principles of gravity

There four main principles in gravity.

- special relativity
- equivalence principle
- general covariance
- correspondence principle with Newton's law

Newton's gravity $F = Gm_1m_2/r^2$ is not consistent with special relativity, which tells us that forces must be propagated. It is consistent with Galilean relativity, which is $t' = t$ and $x' = x + vt$. We now know that this is not right, and the right special relativistic transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \text{ where } \Lambda^\mu_\nu = \begin{pmatrix} \cosh \phi & 0 & 0 & -\sinh \phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \phi & 0 & 0 & \cosh \phi \end{pmatrix}.$$

So any theory of gravity must be consistent with this Lorentz transformation.

The equivalence principle is that the inertial mass is equal to the gravitational mass. That is, I can't distinguish between a box on Earth and an accelerating one in space. Likewise if one is in free fall in a gravitational field, then it feels the same as in an inertial frame in empty space.

The next principle is general covariance. If we take Maxwell's equations and do a Lorentz boost, it looks the same. In general relativity, there is more freedom. In curved space, there is no canonical, god-given set of coordinates. So the equation we write down must be invariant under general change of coordinates.

Under the Newtonian limit, the laws of gravity must be described by Newton's law of gravity. That is, when $Gm/rc^2 \ll 1$ we get Newton's law.

Now I'm going to derive an interesting consequence of the equivalence principle. Suppose two spaceships of distance z are accelerating with acceleration a . The first spaceship emits a light of wavelength λ and the second spaceship detects it. Then the time for the light to travel is $\delta t = z/c$ and the change of velocity in that time interval is $\delta v = az/c$. So this gives a Doppler shift of $\delta\lambda/\lambda = az/c^2$. If we believe in the equivalence principle, this situation is equivalent to the situation where there is a building of height z on Earth and one person is emitting light on the ground and one person is detecting on the roof. So there is a gravitational redshift.

But looking at the peaks of the electromagnetic waves, assuming that the space is flat, the two wavelengths must be identical. So this gravitational redshift necessarily implies that spacetime is curved.

2 January 27, 2017

Last time I gave some motivation for studying gravity and some basic principles. The main point is reconciling special relativity and Newton's law of gravity.

We are going to start with a smooth space. On top of that you are going to put various structures like vectors, forms, etc. Then you are going to describe derivatives and connections. Then we will put a metric. Finally we are going to define the curvature.

2.1 Manifolds

Definition 2.1. A **manifold** is a space that looks locally like \mathbb{R}^n , and smooth everywhere.¹

Example 2.2. One-dimensional manifolds include \mathbb{R}^1 , the closed interval, S^1 . Two intersecting segments is not a manifold because it doesn't look like \mathbb{R}^1 at the intersection.

A manifold can be fully covered by open sets that are isomorphic to open sets in \mathbb{R}^n . Here, isomorphic means that there is a one-to-one smooth map with a smooth inverse.

Example 2.3. Two-dimensional examples include more interesting ones. There is \mathbb{R}^2 , S^2 , and there is the torus, surface with two holes, surface with genus g , etc. These are orientable compact 2-manifolds. If I allow non-orientable manifolds, there are things like the Klein bottle. There is no way of embedding into flat 3-space without self-intersection.

We want to cover our manifolds with coordinates, i.e., maps $M \supseteq U \rightarrow \mathbb{R}^n$. These coordinates must be one-to-one, otherwise there are two points labeled with the same coordinate. We also want coordinates to be smooth.

Example 2.4. Let us cover S^2 with two coordinates. We first set a spherical coordinate system on S^2 . This actually isn't a coordinate because at the poles φ is not determined. So we resolve this problem using the two coordinates

$$(x_1, y_1) = (\theta \cos \varphi, \theta \sin \varphi), \quad (x_2, y_2) = ((\pi - \theta) \cos \varphi, (\pi - \theta) \sin \varphi).$$

This is actually a cheat. If you are given an abstract space and given these two coordinates, then you will only know that

$$(x_2, y_2) = \left(\frac{\pi x_1}{\sqrt{x_1^2 + y_1^2}} - x_1, \frac{\pi y_1}{\sqrt{x_1^2 + y_1^2}} - y_1 \right).$$

¹This is not a math class, I am not going to be very precise. In fact, I won't be too specific because we are ultimately trying to describe the rules of the universe, not make our own set of rules.

2.2 Functions and vectors

These are also called scalar fields. These are maps $M \rightarrow \mathbb{R}^1$. Change of coordinates take the form of $x'^\mu = x'^\mu(x)$. We will want to have

$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) \neq 0$$

so that maps are invertible.

Scalar fields transform in the following way:

$$\varphi'(x') = \varphi(x(x')).$$

This is just a relabeling of the points.

In a coordinate system we can take the derivative

$$V_\mu = \frac{\partial \varphi}{\partial x^\mu} = \partial_\mu \varphi.$$

How does this transform under coordinate change? In the x' world,

$$V'_\mu = \frac{\partial \varphi'}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu} \varphi(x(x')) = \frac{\partial \varphi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu.$$

A **covariant vector field** is a vector field that transforms under this law.

Example 2.5. Write a vector in Minkowski space time as $\vec{X} = x^\mu \vec{e}_\mu$. The Lorentz transformation is given by

$$\vec{e}'_\mu = \Lambda_\mu^\nu \vec{e}_\nu.$$

Then we have

$$V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu = \Lambda_\mu^\nu V_\nu.$$

So all vectors rotate the same way.

3 February 1, 2017

On a manifold we had a function $\varphi : M \rightarrow \mathbb{R}$ and these transform as $\varphi'(x') = \varphi(x(x'))$. On the other hand, if we try to transform a vector $V_\mu(x) = \partial_\mu \varphi$, we get an extra factor

$$V'_\mu(x') = \frac{\partial}{\partial x'^\mu} \varphi'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \varphi(x) = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu(x).$$

These are **covariant vector field**, or what mathematicians call 1-forms.

Now we may want to have something that actually points in some direction. This would be something like the tangent direction of a worldline parametrization $x^\mu(\lambda)$. This object will be

$$W^\mu = \frac{\partial}{\partial \lambda} x^\mu(\lambda).$$

In a change of coordinates, it would transform as

$$W'^\mu = \frac{\partial}{\partial \lambda} x'^\mu(\lambda) = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial \lambda}.$$

They transform in the opposite direction, and they are called **contravariant vector fields**.

Given a n -dimensional manifold M , there is a **tangent space**, which is a $2n$ -dimensional space, consisting of (x, v) where $x \in M$ is a point and v is a vector tangent to M at x . Mathematically it is called the tangent bundle.

If you have a vector and a covector, you can define

$$V_\mu W^\mu = \sum_\mu V_\mu W^\mu$$

which is a scalar. Note that if you do something bad in the construction like $\sum_\mu V_\mu W_\mu$, then this is not a scalar.

3.1 Tensors

A **tensor of rank** (p, q) is defined as an object T transforming as

$$T_{a_1 \dots a_p}{}^{b_1 \dots b_q}(x') = (\partial'_{a_1} x^{c_1} \dots)(\partial_{d_1} x'^{b_1} \dots) T_{c_1 \dots c_p}{}^{d_1 \dots d_q}.$$

General covariance says that equations of motion are tensors. This more precisely means that if an equation looks like $E^\mu{}_{\nu\rho} = 0$ then $E'^\mu{}_{\nu\rho} = 0$.

There are some operations we can perform on tensors that produce other tensors.

- For two tensors V and W of ranks (p_1, q_1) and (p_2, q_2) , we can define $V \otimes W$ by concatenating the indices. For instance,

$$V_\mu{}^\nu W_{\rho\lambda} = \Gamma_\mu{}^\nu{}_{\rho\lambda}.$$

- We can take the inner product like

$$V_\mu{}^\nu W_{\nu\lambda} = B_{\mu\lambda}.$$

- For a single tensor, we can take the contraction

$$T_{\mu\nu}{}^\rho \rightarrow T_{\mu\nu}{}^\nu = U_\mu.$$

This is not the same as $T_{\nu\mu}{}^\nu = V_\mu$.

- We define the symmetrization as

$$T_\mu{}^{\nu\rho} \rightarrow T_\mu{}^{(\nu\rho)} = T_\mu{}^{\nu\rho} + T_\mu{}^{\rho\nu}.$$

- Analogously we define the anti-symmetrization as

$$T_\mu{}^{\nu\rho} \rightarrow T_\mu{}^{[\nu\rho]} = T_\mu{}^{\nu\rho} - T_\mu{}^{\rho\nu}.$$

We know that $\partial_\mu \varphi$ is a rank 1 tensor. What about its derivative $\partial_\mu V_\nu$, which seems like a fundamental thing we need to describe the laws of the universe? It turns out $\partial_\mu V_\nu$ is not a tensor. There is a good reason for this. The derivative of a scalar field is given by

$$n^\mu \partial_\mu A = \lim_{\epsilon \rightarrow 0} \frac{A(x^\mu + \epsilon n^\mu) - A(x^\mu)}{\epsilon}.$$

For A a scalar, we know how to compare the values at different points. But if A is a vector, we don't know how to compare these two vectors at two different points. So we need a notion of moving vectors around.

One way to solve this problem is to use a connection, which tells us how to move a vector to another vector. Another way is to use another vector field to move things around.

3.2 The Lie derivative

Let us say we have a vector field $V^\mu(x)$. This gives an infinitesimal way of moving points around in a manifold.

First for scalar fields we define

$$\mathcal{L}_V \varphi = V^\mu \partial_\mu \varphi.$$

For a vector U , we are going to define

$$(\mathcal{L}_V U)^\mu = V^\nu \partial_\nu U^\mu - \partial_\nu V^\mu U^\nu.$$

If you think about it, the first term is about moving the point of a vector, and the second part is canceling out the effect of the flow generated by V . This derivative depends on the vector field, not on the single vector at a point. This transforms like

$$(\mathcal{L}_V U)'^\mu = \partial_\nu x'^\mu (\mathcal{L}_V U)^\nu + \dots = \partial_\nu x'^\mu (\mathcal{L}_V U)^\nu.$$

So this is again a vector.

4 February 3, 2017

Last time we talked about how there is no canonical notion of moving vectors around. So we defined the Lie derivative by comparing points by the flow generated by a vector field. The formula is given by

$$(\mathcal{L}_V U)^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu.$$

You can check that the Lie derivative transforms tensorially. Likewise we define the Lie derivative of the covariant vector as

$$(\mathcal{L}_V W)_\mu = V^\nu \partial_\nu W_\mu + W_\nu \partial_\mu V^\nu.$$

The Lie derivative of a (p, q) -tensor is a (p, q) -tensor and is defined similarly as

$$(\mathcal{L}_V T)^\nu{}_\mu{}^\rho = V^\lambda \partial_\lambda T^\nu{}_\mu{}^\rho + T^\nu{}_\lambda{}^\rho \partial_\mu V^\lambda - T^\lambda{}_\mu{}^\rho \partial_\lambda V^\nu - T^\nu{}_\mu{}^\lambda \partial_\lambda V^\rho.$$

There are two useful formulas that can be checked easily.

- $\mathcal{L}_U V = [U, V] = -\mathcal{L}_V U$
- $(\mathcal{L}_U \mathcal{L}_V - \mathcal{L}_V \mathcal{L}_U)T = \mathcal{L}_{[U, V]}T$

4.1 Covariant derivative

There is another notion of derivative. As said, we don't have a good notion of comparing vectors. So let us define a general form of **covariant derivative** as

$$(\nabla_\mu V)^\nu = \partial_\mu V^\nu + \Gamma_{\lambda\mu}^\nu V^\lambda$$

using an abstract matrix Γ . We are first going to want the derivative to be tensorial. To achieve this, we need Γ to be something that cancels out the non-tensorial part of $\partial_\mu V^\nu$.

So what is Γ' in x' coordinates? We want

$$\nabla'_\mu V'^\nu = (\partial'_\mu x^\alpha)(\partial_\beta x'^\nu) \nabla_\alpha V^\beta.$$

After sorting things out, we get

$$\Gamma'^\nu_{\lambda\mu} = \partial'_\lambda x^\alpha \partial'_\mu x^\beta \partial_\gamma x'^\nu \Gamma_{\alpha\beta}^\gamma - \partial'_\lambda x^\alpha \partial'_\mu x^\beta \partial_\alpha \partial_\beta x'^\nu.$$

This is the way the connection transforms. This is definitely not a tensor, and moreover it is inhomogeneous. That is, a connection being 0 in one coordinate does not imply being 0 in another coordinate. But note that for two connections Γ and $\tilde{\Gamma}$, its difference $\Gamma - \tilde{\Gamma}$ is a tensor.

For a covariant vector W_ν , its covariant is given by

$$\nabla_\mu W_\nu = \partial_\mu W_\nu - \Gamma_{\mu\nu}^\lambda W_\lambda.$$

The reason we are using the same Γ is because we want $W_\nu V^\nu$ to be a scalar and the covariant derivative to work well with respect to the Leibniz rule. Generally given a tensor of rank (p, q) , its covariant derivative

$$\nabla_\sigma T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = \partial_\sigma T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \Gamma_{\sigma\lambda}^{\mu_1} T^{\lambda \mu_2 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \dots - \Gamma_{\sigma\nu_1}^\lambda T^{\mu_1 \dots \mu_p}{}_{\lambda \nu_2 \dots \nu_q} - \dots$$

is defined to be a $(p, q+1)$ -tensor.

4.2 Parallel transport

A connection gives a way of taking the derivative. So conversely, this means that this gives a way of moving vectors around, because that is how a derivative should be defined.

Suppose that there is a path $x^\mu(\lambda)$ in the manifold and we want to transport a vector V^μ along this path. This is going to be a **parallel transport** if the derivative along the path is zero, i.e.,

$$0 = \frac{dx^\sigma}{d\lambda} \nabla_\sigma V^\mu = \frac{dV^\mu}{d\lambda} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} V^\rho.$$

This really depends on the path.

Let us see how different paths give different parallel transports. The commutator $[\nabla_\mu, \nabla_\nu]$ measures this. On a scalar, the commutator acts as

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\varphi &= \nabla_\mu(\partial_\nu\varphi) - \nabla_\nu(\partial_\mu\varphi) = \partial_\mu\partial_\nu\varphi - \Gamma_{\mu\nu}^\lambda\partial_\lambda\varphi - \partial_\nu\partial_\mu\varphi + \Gamma_{\nu\mu}^\lambda\partial_\lambda\varphi \\ &= (\Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda)\partial_\lambda\varphi. \end{aligned}$$

This anti-symmetric part $T_{\nu\mu}^\lambda = \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda$ is called the **torsion tensor** and it is a tensor because it is a difference of two connections. This can exist, but we can ignore this because it won't interfere with the general covariance principle. So we can define the **torsion-free** part

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \frac{1}{2}T_{\mu\nu}^\lambda = \frac{1}{2}(\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda).$$

5 February 8, 2017

Today we are going to derive the notion of curvature. Using the parallel transport, we can move around vectors along paths. But this moving around depends on the paths. In flat space, this does not depend on the path. So this difference is an intrinsic way of saying that the connection is curved.

Given a path parameterized by λ , a parallel transport is when

$$\frac{D}{d\lambda} V^\mu = \frac{dx^\rho}{d\lambda} \nabla_\rho V^\mu = \frac{d}{d\lambda} V^\mu + \Gamma_{\sigma\rho}^\mu \frac{dx^\rho}{d\lambda} V^\sigma = 0.$$

When we reparametrize the curve with λ' , we get

$$\frac{D}{d\lambda'} V^\mu = \frac{d}{d\lambda'} V^\mu + \Gamma_{\sigma\rho}^\mu \frac{dx^\rho}{d\lambda'} V^\sigma = \frac{d\lambda}{d\lambda'} \left(\frac{D}{d\lambda} V^\mu \right) = 0.$$

So it doesn't depend whether we pull slowly or quickly.

Parallel transports moves vectors around in the same way locally. But it has no reason to move vectors around globally. So given a loop L , we can parallel transport a vector V^μ and get V'^μ at the same point, but $V^\mu \neq V'^\mu$. This will be given by some matrix,

$$V'^\mu = \Lambda^\mu{}_\nu(L) V^\nu.$$

This matrix $\Lambda^\mu{}_\nu(L) \in \text{GL}(d)$ is called the **holonomy** around L . A **flat connection** is one such that $\Lambda = \text{id}$ always. We can require manifolds to have special holonomy group, for instance, $\{\Lambda(L)\} \subseteq \text{O}(d)$. There can be topological obstructions, but we only consider contractible L .

5.1 Geodesics

We want to now think of free fall in curved spacetime. These objects will move on locally straight lines. This will mean that acceleration is zero. But what is acceleration? The naïve definition of taking the second derivative won't work.

We know at least what velocity is: it is $\partial x^\mu / \partial \lambda$. So by acceleration being zero should mean that $\partial x^\mu / \partial \lambda$ is covariantly constant.

There is some issue here though. Normally, for a 4-vector $(\vec{x}(\lambda), t(\lambda))$, the physical velocity is not the length of the vector $(\partial \vec{x} / \partial \lambda, \partial t / \partial \lambda)$. The physical velocity is more about the direction of this vector, not the magnitude. So actually the right notion is that the direction stays the same.

So, the covariant derivative along the path of the velocity is

$$\frac{dx^\sigma}{d\lambda} \nabla_\sigma \left(\frac{dx^\mu}{d\lambda} \right) = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = f(\lambda) \frac{dx^\mu}{d\lambda}.$$

This is called the **geodesic equation**. The whole thing for $f(\lambda)$ is that reparametrization will give the same solution for different f .

Actually for a given geodesic (as a geometric path, without parametrization) there exists a parametrization λ such that $f(\lambda) = 0$. This is called the **affine**

parametrization and is unique up to linear reparametrization. An example is $\lambda = \tau$, the proper time/direction.

Note that you can always solve it, given initial position and initial velocity, because the geodesic equation is a second-order equation.

5.2 Curvature

The equivalence principle says that freely falling observers feel locally inertial. We have identified “freely falling” with “geodesic paths”. So we need to think about what locally inertial means.

The value $\Gamma_{\nu\rho}^\nu$ doesn't really matter, because one can always pick locally inertial coordinates so that $\Gamma|_{\text{pt}} = 0$. Why is this? Recall that under linear coordinate transformations, Γ is tensorial, so we need a quadratic change of coordinates. So to really change Γ , take a general quadratic transformation $x'^\mu = x^\mu + (1.2)M_{\nu\rho}^\mu x^\nu x^\rho$ so that $\partial_\mu x'^\nu|_{x=0} = \delta_\mu^\nu$. Then

$$\Gamma_{\nu\rho}^{\prime\mu}|_{x=0} = \Gamma_{\mu\rho}^\mu - \partial_\nu x'^\lambda \partial'_\rho x^\beta \partial_\lambda \partial_\beta x'^\mu = \Gamma_{\nu\rho}^\mu - M_{\nu\rho}^\mu.$$

Note that this arguments applies only to torsion-free connections. Picking $M = \Gamma|_{x=0}$, we get $\Gamma'|_{x=0} = 0$. This coordinate is called a **Riemann normal coordinate**, and is not unique.

So we need to think about what this intrinsic connect in Γ is. A good technique is to look at parallel transport. We will parallel transport a vector around a parallelogram formed by m^ν and n^μ and see what happens. The most general thing that can happen is

$$\delta V^\rho = n^\mu m^\nu R_{\mu\nu\lambda}^\rho V^\lambda.$$

Because going in the opposite direction should give the other direction, we will have

$$R_{\mu\nu\lambda}^\rho = -R_{\nu\mu\lambda}^\rho.$$

This is called the **curvature tensor**. To compute this, we write down the formula:

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu(\partial_\nu V^\rho + \Gamma_{\nu\lambda}^\rho V^\lambda) - (\mu \leftrightarrow \nu) = \dots$$

After this, you get

$$R_{\mu\nu\lambda}^\rho = \partial_\mu \Gamma_{\nu\lambda}^\rho - \partial_\nu \Gamma_{\mu\lambda}^\rho + \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\lambda}^\sigma - \Gamma_{\nu\sigma}^\rho \Gamma_{\mu\lambda}^\sigma.$$

6 February 10, 2017

From a connection we define a covariant derivative, and as a way to extract intrinsic information, we define the curvature. We define $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\mu\nu\lambda}V^\lambda$.

As an exercise, let us see what happens if we parallel transport a vector along two different paths, the two ways along a parallelogram with side ϵ_1 and ϵ_2 .

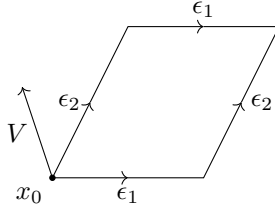


Figure 1: Parallel transporting around a parallelogram

Along the first line,

$$0 = \epsilon_1^\mu \partial_\mu V^\nu(x_0 + \lambda \epsilon_1) + \epsilon_1^\mu \Gamma_{\mu\rho}^\nu(x_0 + \lambda \epsilon_1) V^\rho(x_0 + \lambda \epsilon_1).$$

Using Taylor expansion, we can say that

$$0 = \epsilon_1^\mu \partial_\mu V^\nu(x_0) + \epsilon_1^\mu \Gamma_{\mu\rho}^\nu(x_0) V^\rho(x_0) + O(\epsilon_1^2).$$

So we can write

$$V^\mu(x_0 + \epsilon_1) = V^\mu(x_0) + \epsilon_1^\mu \Gamma_{\mu\rho}^\nu V^\rho|_{x_0} + O(\epsilon_1^2).$$

Then we can compute the next transport as

$$\begin{aligned} V_{12}^\nu &= V^\nu|_{x_0} + \epsilon_1^\mu \Gamma_{\mu\rho}^\nu V^\rho|_{x_0} + O(\epsilon_1^2) \\ &\quad + \epsilon_2^\mu \Gamma_{\mu\rho}^\nu(x_0 + \epsilon_1)(V^\rho|_{x_0} - \epsilon_1^\sigma \Gamma_{\sigma\lambda}^\rho V^\lambda|_{x_0}) + O(\epsilon_2^2) \\ &= V^\nu|_{x_0} - \epsilon_1^\mu \Gamma_{\mu\rho}^\nu V^\rho - \epsilon_2^\mu \Gamma_{\mu\rho}^\nu V^\rho + O(\epsilon_1^2 + \epsilon_2^2) \\ &\quad - \epsilon_2^\mu \epsilon_1^\sigma (\partial_\sigma \Gamma_{\mu\rho}^\nu) V^\rho + \epsilon_2^\mu \epsilon_1^\sigma \Gamma_{\mu\rho}^\nu \Gamma_{\sigma\lambda}^\rho V^\lambda + \dots \end{aligned}$$

So if we take the difference, we would get

$$V_{12}^\nu - V_{21}^\nu = \epsilon_1^\sigma \epsilon_2^\mu (\partial_\mu \Gamma_{\sigma\rho}^\nu - \partial_\sigma \Gamma_{\mu\rho}^\nu + \Gamma_{\mu\lambda}^\nu \Gamma_{\sigma\rho}^\lambda - \Gamma_{\sigma\lambda}^\nu \Gamma_{\mu\rho}^\lambda) V^\rho.$$

This thing inside the parenthesis is exactly the curvature.

What happens if the connection is flat? Suppose we have a curve of area A . Infinitesimally, the holonomy around of square of side length ϵ is $O(\epsilon^3)$. If the curve is filled with squares, then the number is A/ϵ^2 . So the holonomy is $O(\epsilon^3 \cdot A\epsilon^{-2}) = O(\epsilon)$. That is, it is zero.

There are connections in electromagnetism for the phase. The scalar field coupled to electromagnetism transforms in the following way:

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \theta(x), \quad \varphi \rightarrow \varphi'(x) = e^{iq\theta(x)} \varphi(x).$$

Then you can take the derivative as

$$D_\mu \varphi = \partial_\mu \varphi + iq A_\mu \varphi.$$

This transforms as $(D_\mu \varphi') = e^{iq\theta} D_\mu \varphi$.

6.1 Metric tensor

How do we define the geometry of the manifold? We want to specify how to measure the distance. We had some topology, vector fields and notions how to push things around, but we haven't talked about measuring length. A metric basically tells us the distance, but it will also give a connection and an integration measure.

In Euclidean geometry, an infinitesimal distance is given by $ds^2 = dx^2 + dy^2 + dz^2$. But we are going to define the **metric** generally as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

By looking random changes of coordinates, you can get non-diagonal matrices as g . That is why we allow g to be a general matrix. We don't like the mathematical definition $d(A, B)$ satisfying the triangle inequality, because this is not local. But using the metric tensor, we can measure the length of a path p as

$$d_p(A, B) = \int d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$

It is clear that length is invariant under reparametrization. Also $g_{\mu\nu} = g_{\nu\mu}$ by definition.

Example 6.1. The metric in Euclidean space looks like $g_{\mu\nu} = \delta_{\mu\nu}$. The metric in Minkowski spacetime is like

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The length ds^2 is physical, i.e., cannot depend on coordinates. So we see that $g_{\mu\nu}$ transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}.$$

So it transforms as $g \mapsto \Lambda g \Lambda^T$. You can show that with a suitable choice of Λ , we can make g be diagonal with ± 1 on the diagonal (assuming that g is not

degenerate). We call the number of +1s and -1s the **signature**. For instance, Euclidean space has signature $(3, 0)$ and Minkowski space has signature $(3, 1)$. This signature cannot change in space without going through a point where the metric is degenerate.

We will write the inverse metric $g^{\mu\nu}$, and it will have the property

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho.$$

Example 6.2. Take S^2 , the sphere in \mathbb{R}^3 given by $R^2 = x^2 + y^2 + z^2$. What is the induced 2-dimensional metric. Let us use the coordinate system (x, y) on one hemisphere. We can write

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + (d\sqrt{R^2 - x^2 - y^2})^2 \\ &= dx^2 + dy^2 + \frac{x^2 dx^2 + 2xy dx dy + y^2 dy^2}{R^2 - x^2 - y^2}, \end{aligned}$$

and so the metric tensor is

$$g = \begin{pmatrix} 1 + \frac{x^2}{R^2 - x^2 - y^2} & \frac{xy}{R^2 - x^2 - y^2} \\ \frac{xy}{R^2 - x^2 - y^2} & 1 + \frac{y^2}{R^2 - x^2 - y^2} \end{pmatrix}.$$

7 February 15, 2017

A metric is a generalization of measuring distances. It is given in general by the form $ds^2 = g_{ij}dx^i dx^j$, with $g_{ij} = g_{ji}$. If g has signature $(3,1)$, then at a point one can always choose coordinates such that g looks trivial. But you can't make it trivial everywhere.

The formula for the metric immediately gives us dot product for vectors living in the same tangent space, by

$$V \cdot W = V^\mu g_{\mu\nu} W^\nu.$$

Note that you can use the metric to move indices. For instance, we can define

$$V_\mu = g_{\mu\nu} V^\nu.$$

This is because a non-degenerate metric gives a way of identifying a vector space with its dual. In this case, we have an inverse metric $g^{\mu\nu}$ such that $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$. This gives us a way to raise indices:

$$W^\nu = g^{\nu\mu} W_\mu.$$

The inner product on vectors with lower indices are then given by

$$V \cdot W = g^{\mu\nu} V_\mu W_\nu.$$

This is the story for a metric at a point. But the interesting thing happens at the manifold.

7.1 Metric to a connection

Given a metric, we will see that there is associated a natural connection. We saw that the proper length of a path is given by

$$L = \int_{\text{path}} ds = \int \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|} d\lambda.$$

A **geodesic** is a path of extremized length. The reason I say extremized is that if the path is space-like then the path of minimal length is the special path and if the path is time-like the path of maximal length is special.

To figure out the equation, we use the calculus of derivations. Here, the whole thing is going to be invariant, under parametrization, and so we may fix a reparametrization by taking λ to be the proper distance. This gives

$$\begin{aligned} \delta L &= \int \frac{\delta(g_{\mu\nu} (dx^\mu/d\ell)(dx^\nu/d\ell))}{2\sqrt{g_{\mu\nu} (dx^\mu/d\ell)(dx^\nu/d\ell)}} d\ell \\ &= \frac{1}{2} \int d\ell \left(\delta g_{\mu\nu} \frac{dx^\mu}{d\ell} \frac{dx^\nu}{d\ell} + 2g_{\mu\nu} \frac{d\delta x^\mu}{d\ell} \frac{dx^\nu}{d\ell} \right) \\ &= \frac{1}{2} \int d\ell \left(\partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\ell} \frac{dx^\nu}{d\ell} \delta x^\sigma - 2 \frac{d}{d\ell} g_{\mu\nu} \delta x^\mu \frac{dx^\nu}{d\ell} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\ell^2} \right) \\ &= \int d\ell \left[(\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma}) \frac{dx^\mu}{d\ell} \frac{dx^\nu}{d\ell} \delta x^\sigma - 2g_{\mu\nu} \frac{d^2 x^\mu}{d\ell^2} \delta x^\nu \right]. \end{aligned}$$

After this, we find an equation of motion that looks like

$$\frac{d^2 x^\mu}{d\ell^2} + \frac{1}{2} g^{\mu\sigma} (\partial_\lambda g_{\nu\sigma} + \partial_\nu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\nu}) \frac{dx^\lambda}{d\ell} \frac{dx^\nu}{d\ell} = 0.$$

This is precisely the geodesic equation if we set

$$\Gamma_{\lambda\nu}^\mu = \frac{1}{2} g^{\mu\sigma} (\partial_\lambda g_{\nu\sigma} + \partial_\nu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\nu}).$$

This is called the **Levi-Civita connection**.

A **metric compatible connection** is one in which parallel transport preserves the length of vectors. Equivalently, this is saying that the inner product on the tangent space, $g_{\mu\nu}$ is covariantly constant. We want to pick a connection such that $\nabla_\mu g_{\alpha\beta} = 0$. Note that this implies that $\nabla_\mu g^{\alpha\beta} = 0$ and $\nabla_\mu \epsilon^{\alpha\beta\sigma\rho} = 0$. The equation we want can be written as

$$\nabla_\rho g_{\mu\sigma} = \partial_\rho g_{\mu\sigma} - \Gamma_{\rho\mu}^\lambda g_{\lambda\sigma} - \Gamma_{\rho\sigma}^\lambda g_{\mu\lambda} = 0.$$

It is not immediately clear that such a connection even exists. The trick is writing equations with mixing the indices:

$$0 = \nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu} = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - 2\Gamma_{\mu\nu}^\lambda g_{\lambda\rho}.$$

Then we find $\Gamma_{\mu\nu}^\lambda$, and this is equal to the Levi-Civita connection we have defined above. So the metric compatible connection is the same connection such that the geodesics are extremal. We are going to write $\partial_\mu g_{\nu\rho} = g_{\nu\rho,\mu}$ from now on, because we are going to see them a lot.

A **local inertial coordinate** is one in which $g_{\mu\nu}$ is the canonical (diagonal with ± 1 on the diagonal) and $\Gamma = 0$ at a point. This is possible because a quadratic change of coordinates can set Γ arbitrarily and then a linear change of coordinates can set g nicely.

Since we have a connection, we have

$$R_{\mu\nu\sigma}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda.$$

This is called the **Riemann curvature**.

8 February 17, 2017

Last time we defined the connection in terms of the metric, which is compatible with the metric:

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu}).$$

This is the same as saying that $\nabla_{\mu}g_{\alpha\beta} = 0$. Then geodesics have two interpretations: the tangent vector is parallel transported, and the proper distance is extremized.

At a point $p \in M$, one can always pick a locally inertial frame (coordinates) such that $g_{\mu\nu}|_p = \eta_{\mu\nu}$ and $\Gamma|_p = 0$. This is basically the content of the equivalence principle.

The **Riemann normal coordinate** satisfies this condition. At a point $p \in M$, we look consider the exponential map $\exp_p : T_p M \rightarrow M$. This map is defined by $\exp_p(K^{\mu}) = x^{\mu}(1)$ where $x^{\mu}(\lambda)$ is the geodesic with the initial conditions

$$x^{\mu}(0) = p, \quad \left. \frac{dx^{\mu}}{d\lambda} \right|_{\lambda=0} = K^{\mu}.$$

Because the geodesic equation is second-order, the geodesic is uniquely defined. Then the (local) inverse map gives a coordinate system.

In this coordinate system, $\hat{x}^{\mu}(\lambda) = K^{\mu}\lambda$ is always a geodesic, by definition. So this obeys $d^2\hat{x}^{\mu}/d\lambda^2 = 0$. So $\Gamma = 0$ in the Riemann normal coordinates.

8.1 Causality

In Newton's mechanics, causality is given by a single parameter t , and so there is a simple time evolution. In special relativity there is the speed limit and so the causal future looks like a cone. So given a domain in space at some fixed time, we can determine the future domain, which looks like a upside-down cone.

In general relativity, a causal curve is one which is everywhere time-like or null. Given a subset $S \subseteq M$, its **causal future** of S , denoted by $J^+(S)$, is the set of points (in spacetime) reachable by causal paths from S . This is the analogue of the picture in special relativity, and it does not have to look like a cone. A subset $S \subseteq M$ is called **achronal** if no two points in S are time-like related. A spacial slice is achronal. If I have an achronal surface S , we can define the **future domain of dependence** $D^+(S)$ as the set of points where any "inextendable causal curve" starting from that point intersect S .

This future domain might not look like a cone. For instance, there might be a singularity in the cone. Then whatever happens after the singularity, we don't know. Another mathematical possibility is that the light cone might tip to one side and make a closed time-like curve, although we believe it does not happen in physics.

We can likewise define the **past domain of dependence** $D^-(S)$, and let $D(s) = D^+(S) \cup D^-(S)$.

8.2 Symmetry of spacetime

We have some idea of what a symmetry mean: it is a transformations that leaves something invariant. A **symmetry** of spacetime is a transformation that leaves the geometry invariant.

Let us make this more precise. A symmetries of a scalar field is given as the change of coordinates that leave the field invariant. For instance, the field $f(x, y) = y^2$ is invariant under x translations, i.e., its $V = \partial/\partial x$ is 0. But if we take another coordinate $x' = x + y$ and $y' = x - y$, then we have

$$f'(x', y') = y^2 = \left(\frac{x' - y'}{2}\right)^2.$$

This now is not invariant under x' -translation. But we see that

$$0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} + \frac{\partial f}{\partial y'}.$$

So the idea is that the right way to say something about symmetries of a scalar field is to specify a vector field.

Now the geometry is specified by $g_{\mu\nu}$. In this case, the Lie derivative is the right notion, because this is the derivative along the vector flow. So a symmetry is a V^μ such that

$$\mathcal{L}_V g_{\mu\nu} = 0.$$

This is a complete coordinate independent statement.

In a coordinate system, we can write this as

$$\begin{aligned} 0 &= \mathcal{L}_V g_{\mu\nu} = V^\mu \partial_\lambda g_{\mu\nu} + \partial_\mu V^\lambda g_{\lambda\nu} + \partial_\nu V^\lambda g_{\mu\lambda} \\ &= \partial_\mu (g_{\nu\lambda} V^\lambda) + \partial_\nu (g_{\mu\lambda} V^\lambda) - V^\lambda \partial_\mu g_{\lambda\nu} - V^\lambda \partial_\nu g_{\mu\lambda} + V^\lambda \partial_\lambda g_{\mu\nu} \\ &= \partial_{(\mu} V_{\nu)} - 2\Gamma_{\mu\nu}^\lambda V^\lambda = \nabla_{(\mu} V_{\nu)}. \end{aligned}$$

This is called the **Killing equation** and such a vector V is called a **Killing vector**.

If V and W are symmetries, then $V + W$ is a symmetry, because the derivatives are linear. Also, $\mathcal{L}_{[V, W]} = [\mathcal{L}_V, \mathcal{L}_W]$ and so the commutator $[V, W]$ is also an symmetry. So it follows that the symmetries have a structure of a Lie algebra.

8.3 Symmetries of the Riemann tensor

We defined the Riemann curvature tensor as

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R_{\mu\nu}{}^\alpha{}_\beta V^\beta,$$

and derived a formula

$$R_{\mu\nu}{}^\alpha{}_\beta = \partial_\mu \Gamma_{\nu\beta}^\alpha - \partial_\nu \Gamma_{\mu\beta}^\alpha + \Gamma_{\mu\sigma}^\alpha \Gamma_{\nu\beta}^\sigma - \Gamma_{\nu\sigma}^\alpha \Gamma_{\mu\beta}^\sigma.$$

We have the identity

$$R_{\mu\nu}{}^{\alpha}{}_{\beta} = -R_{\nu\mu}{}^{\alpha}{}_{\beta}.$$

There are more relations. Because the connection is metric compatible, it commutes with lowering and raising indices. So we have

$$[\nabla_{\mu}, \nabla_{\nu}] \nabla_{\alpha} \varphi = R_{\mu\nu\alpha}{}^{\beta} \nabla_{\beta} \varphi.$$

Now we can take the cyclic permutation and add. Then we get

$$0 = \nabla_{[\mu} \nabla_{\nu} \nabla_{\alpha]} \varphi = (R_{\mu\nu\alpha}{}^{\beta} + R_{\nu\alpha\mu}{}^{\beta} + R_{\alpha\mu\nu}{}^{\beta}) \nabla_{\beta} \varphi.$$

So we get an extra identity

$$R_{\mu\nu\alpha}{}^{\beta} + R_{\nu\alpha\mu}{}^{\beta} + R_{\alpha\mu\nu}{}^{\beta} = 0.$$

9 February 22, 2017

Last time I was talking about the symmetries of the Riemann tensor. It has an obvious symmetry $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$. Then we used the identity $[\nabla_\mu, \nabla_\nu]\nabla_\rho\varphi = R_{\mu\nu\rho}{}^\sigma\nabla_\sigma\varphi$ to derive $R_{\mu\nu\rho}{}^\sigma + R_{\nu\rho\mu}{}^\sigma + R_{\rho\mu\nu}{}^\sigma = 0$.

Using the anti-symmetries and the cyclic symmetry, we can do a clever combination and get

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}.$$

So these are the full list of constraints. There is a skew-symmetry on the first two and the last two, and a symmetry between the two, and $R_{[\mu\nu\alpha\beta]} = 0$ by the cyclic identity. This shows that the number of components is

$$\frac{\frac{d(d-1)}{2}(\frac{d(d-1)}{2} + 1)}{2} - \frac{d(d-1)(d-2)(d-3)}{24} = \frac{d^2(d^2-1)}{12}.$$

So for $d = 1$, we get 0 components. For $d = 2$, there is 1 component. For $d = 3$, there are 6 components and for $d = 4$ there are 20 components. This means for instance that 1-dimensional space is always flat.

We define the **Ricci curvature** as

$$R_{\mu\nu} = R_{\mu\lambda}{}^\lambda{}_\nu = R_{\nu\mu}.$$

Then we can contract this again and define the **Ricci scalar**

$$R = R_\mu{}^\mu.$$

9.1 Bianchi identity

Another property of the Riemann tensor is the **Bianchi identity** given by

$$\nabla_{[\alpha} R_{\beta\gamma]}{}^\mu{}_\nu = 0.$$

To make things simpler, we work in the Riemann normal coordinates. Here we have $\Gamma|_p = 0$ and so the Riemann curvature only has the ∂G part and ∇ is simply ∂ . We need to check

$$\partial_{[\alpha}(\partial_{\beta}\Gamma_{\gamma]}{}^\mu{}_\nu - \partial_{\gamma}\Gamma_{\beta]}{}^\mu{}_\nu) = 0.$$

This is true because the partial derivatives commute. There is an analogous identity on the Ricci curvature:

$$\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0.$$

This is derived from the Bianchi identity.

We define the **Einstein tensor** as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

Then the Bianchi identity can be simply written as

$$\nabla^\mu G_{\mu\nu} = 0.$$

9.2 Integration

We sometimes want to integrate, like when we write down the action. We have to divide a volume into small pieces and then have to specify the volume form dV .

Let us first look at the 1-dimensional case. We have

$$1\text{-Volume} = \int_A^B ds = \int \sqrt{g_{11} dx^1 dx^1} = \int \sqrt{g_{11}} dx^1.$$

This also transforms nicely with respect to coordinate change.

What about in 2-dimensions? We want to have a volume element that gives the infinitesimal volume of a parallelogram having sides dx^1 and dx^2 . Then

$$dV = |dx^1||dx^2| \sin \theta.$$

But we know

$$|dx^1||dx^2| \cos \theta = d\vec{x}^1 \cdot d\vec{x}^2 = g_{12} dx^1 dx^2.$$

So we get

$$dV = \sqrt{g_{11}g_{22} - g_{12}^2} dx^1 dx^2 = dx^1 dx^2 \sqrt{\det g}.$$

When we change coordinates,

$$\sqrt{g'} = \sqrt{\det(\partial'_\mu x^\alpha \partial'_\nu x^\beta g_{\alpha\beta})} = \sqrt{g} \det(\partial'_\mu x^\alpha).$$

This determinant cancels out when we take the integral, as you have learned in your calculus course.

This is related to thinking of integration as a map from d -forms to \mathbb{R} . Consider the **Levi-Civita symbol**

$$\tilde{\epsilon}_{\mu_1 \dots \mu_d} = \begin{cases} +1 & \text{if } \mu_i \text{ are an even permutation of } 1, \dots, d \\ -1 & \text{if } \mu_i \text{ are an odd permutation of } 1, \dots, d \\ 0 & \text{otherwise.} \end{cases}$$

Under change of coordinates, we have

$$\tilde{\epsilon}_{\nu_1 \dots \nu_d} = \tilde{\epsilon}_{\mu_1 \dots \mu_d} \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\mu_d}}{\partial x'^{\nu_d}} \det\left(\frac{dx'^{\alpha}}{dx^{\beta}}\right).$$

So $\tilde{\epsilon}$ is not a tensor, but only a tensor density. We can make it into a tensor by multiplying a scalar density

$$\epsilon_{\mu_1 \dots \mu_d} = (\sqrt{\det g}) \tilde{\epsilon}_{\mu_1 \dots \mu_d}.$$

Then this is a true d -form since the scalar factor cancels out the contribution of coordinate change. This is called the **Levi-Civita tensor**.

9.3 The Einstein equation

In Newtonian gravity, the gravitational equation is given by

$$\nabla^2 \Phi = 4\pi G \rho.$$

The relativistic equation must be an analogue of this equation, and hence we expect the laws of gravity to be an equation on the second derivatives of the metric $g_{\mu\nu}$.

The metric $g_{\mu\nu}$ is the field variable. So I need a symmetric rank 2 tensor worth of equations. What possible equation can we write down? We can write

$$0 = a_1 g_{\mu\nu} + a_2 R_{\mu\nu} + a_3 g_{\mu\nu} R R_{\mu\nu} + a_4 R_{\mu\lambda} R^\lambda{}_\nu + \dots$$

We haven't proved this, but the only rank 2 tensors all come from the curvature and the metric and contracting. In this expression, we need to put units to make them the same units, which we call L_{pl} . So

$$0 = a_1 g_{\mu\nu} L_{\text{pl}}^{-2} + a_2 R_{\mu\nu} + a_3 g_{\mu\nu} R + a_4 R R_{\mu\nu} L_{\text{pl}}^2 + a_5 R_{\mu\nu} R^\lambda{}_\nu L_{\text{pl}}^2 + \dots$$

10 February 24, 2017

Last time we started to write down the Einstein equation. The only equations we could have written down are

$$0 = a_1 L_{\text{pl}}^{-2} g_{\mu\nu} + a_2 R_{\mu\nu} + a_3 g_{\mu\nu} R + a_4 L_{\text{pl}}^2 R R_{\mu\nu} + a_5 L_{\text{pl}}^2 R_{\mu\nu} R^\lambda{}_\nu + \dots$$

The Planck scale is around $L_{\text{pl}} = 10^{-33}\text{m}$, which is extremely small. For the gravitational field of the sun around the earth, the curvature is around

$$\text{curvature} \sim \frac{GM}{r^3} \sim 10^{-30}\text{m}^{-2}.$$

So $L_{\text{pl}}^2 R \sim 10^{-100}$. So the terms with the L_{pl} can be ignored.

This a_0 has to be very small. For some time people thought it was 0, but it is not quite zero. This is called the **cosmological constant** and is of the scale $a_0 \sim 10^{-120}$.

In vacuum, Einstein's equation is going to be

$$R_{\mu\nu} + c R g_{\mu\nu} = 0$$

for some constant c . Contracting with $g^{\mu\nu}$ gives

$$0 = g^{\mu\nu} (R_{\mu\nu} + c R g_{\mu\nu}) = R(1 + cd).$$

Then unless $c = -1/d$, we always get $R_{\mu\nu} = 0$. It turns out the Einstein equation is given by $R_{\mu\nu} - (1/2)R g_{\mu\nu} = K T_{\mu\nu}$. This is also related to the Bianchi identity of the Einstein tensor.

10.1 Diffeomorphism of redundancies

We have, in electromagnetism, A_μ giving $F_{\mu\nu} \partial_{[\mu} A_{\nu]}$, and the Maxwell equations given $\partial^\mu F_{\mu\nu} = 0$. This looks like 4 equations, but there are the solutions $A_\mu = \partial_\mu \lambda$ give $F_{\mu\nu} = 0$ and so this change gives the same physical configuration.

To get rid of this redundancy, we declare that $A_\mu + \partial_\mu \lambda = A_\mu$ are the same solution. Then we essentially have 3 variables since we can simply take $A_0 = 0$ by setting λ as we want.

In gravity we have a similar fact but it is more complicated. We have the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. But $G_{\mu\nu} = 0$ is not a fully independent set of equations because if $g_{\mu\nu} \rightarrow g'_{\mu\nu}$ by a coordinate transformation ζ_μ , then $g'_{\mu\nu}$ will also be a solution.

So we need to look at how the metric transforms under coordinate transformations (considered as an active field transformation). If we have $x'^\mu = x^\mu + \epsilon \zeta^\mu(x)$, then

$$\begin{aligned} g_{\mu\nu}(x) &= \partial_\mu x'^\alpha \partial_\nu x'^\beta g'_{\alpha\beta}(x') = (\delta_\mu^\alpha + \epsilon \partial_\mu \zeta^\alpha) (\delta_\nu^\beta + \epsilon \partial_\nu \zeta^\beta) (g'_{\alpha\beta}(x) + \epsilon + \dots) \\ &= g'_{\mu\nu}(x) + \epsilon (\zeta^\rho \partial_\rho g_{\mu\nu}(x) + \partial_\mu \zeta^\alpha g_{\alpha\nu} + \partial_\nu \zeta^\beta g_{\mu\beta}) + \dots \end{aligned}$$

So $\epsilon \delta_\zeta g_{\mu\nu} = -\mathcal{L}_\zeta g_{\mu\nu} = -\nabla_{(\mu} \zeta_{\nu)}$.

You can check that you can set the metric to $g_{00} = -1$ and $g_{0i} = 0$. This is called **temporal gauge**. After this we get only 6 variables instead of 10. This goes further because $G_{00} = 0$ and $G_{0i} = 0$ do not involve ∂_x^2 .

Another thing to note is that for the vacuum equation $G_{\mu\nu} = 0$, the Ricci curvature is zero. But the Riemann tensor might not be zero and consequently the space might still be curved. The **Weyl tensor** is the contraction free piece of the Riemann tensor, given by the formula

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(d-1)(d-2)}g_{\rho[\mu}g_{\nu]\sigma}R.$$

This has the same symmetries of the Riemann tensor but has vanishing contraction: $C_{\rho\sigma}{}^{\sigma}{}_{\nu} = 0$. So the Weyl tensors are not constrained by the Einstein equations but their derivatives are.

10.2 Newtonian limit

Newton's laws are non-relativistic, so the velocity, or the time dependence of the gravity must be small. Also we consider only weak gravitational fields.

For weak fields, we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is small. If we let

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \text{Tr}(h),$$

then the Einstein equation becomes

$$0 = G_{\mu\nu} = -\frac{1}{2}(\square h'_{\mu\nu} - h'^{\lambda}{}_{\mu, \lambda\mu} - h'^{\lambda}{}_{\nu, \lambda\mu} - h'^{\lambda\rho}{}_{,\lambda\rho} \eta_{\mu\nu}),$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$. The gauge symmetry is given by $\delta h_{\mu\nu} = \partial_{(\mu} \zeta_{\nu)}$, and then $\delta(\partial^\mu h'_{\mu\nu}) = -\square \zeta_\nu$. So we can find a ζ such that $\partial^\mu h'_{\mu\nu} = 0$. It then follows that

$$\square h'_{\mu\nu} = 0.$$

11 March 1, 2017

We wrote down the vacuum Einstein equation

$$0 = a_0 L_{\text{pl}}^{-1} g_{\mu\nu} + R_{\mu\nu} + c g_{\mu\nu} R + \dots$$

The other terms are suppressed because L_{pl} is too small. We are also going to ignore the cosmological constant in most of this class.

Somebody asked me last time of the meaning of curvature. Suppose we have an n -dimensional manifold. Consider a ball of radius ϵ . You can ask what the volume of this ball is. There is a formula for the volume of the ball in flat space. In curved space, the volume will then be

$$\frac{\text{vol}(R_\epsilon)}{\text{vol}(B_\epsilon)} = 1 - \frac{R}{6(n+2)} \epsilon^2 + O(\epsilon^4).$$

Alternatively you can look at the surface area and this obeys a similar law:

$$\frac{\text{vol}(\partial R_\epsilon)}{\text{vol}(\partial B_\epsilon)} = 1 - \frac{R}{6n} \epsilon^2 + O(\epsilon^4).$$

There is also the concept of geodesic deviation, in Section 3.10. Consider a continuous family of geodesics $\gamma_s(t)$. Then we have $x^\mu(s, t)$. Define $T^\mu = \partial x^\mu / \partial t$ and $S^\mu = \partial x^\mu / \partial s$. The **geodesic deviation** is then defined as and computed by

$$A^\mu = \frac{d^2}{dt^2} S^\mu = T^\lambda \nabla_\lambda (T^\sigma \nabla_\sigma S^\mu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma.$$

This gives another way to think about the curvature.

11.1 Matter in the Newtonian gravity

We were talking about the Newtonian limit. We looked at $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The Newtonian limit for gravity is simply $\nabla^2 \Phi = c\rho$. It turns out to be useful to define $h'_{\mu\nu} = h_{\mu\nu} - (1/2)\eta_{\mu\nu} \text{tr}(h)$. Then we have

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} \square h'_{\mu\nu},$$

by picking a gauge $\partial^\mu h'_{\mu\nu} = 0$.

Static solutions satisfy $\partial_t h'_{\mu\nu} = 0$. Then we immediately get that the Einstein equation is $\nabla^2 h'_{\mu\nu} = 0$. In particular, if $c_{\mu\nu}$ is a matter source at $r = 0$, then $h'_{\mu\nu} = c_{\mu\nu}/r$. Because we have given a gauge condition, we get $c_{i0} = 0$ and $c_{ij} = 0$. Let us call $c = c_{00}$.

If we plug this solution back, we get $h_{\mu\nu} = h'_{\mu\nu} - (1/2) \text{tr}(h') \eta_{\mu\nu}$ and so

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{c}{2r} & 0 & 0 & 0 \\ 0 & 1 + \frac{c}{2r} & 0 & 0 \\ 0 & 0 & 1 + \frac{c}{2r} & 0 \\ 0 & 0 & 0 & 1 + \frac{c}{2r} \end{pmatrix}.$$

This is a static solution by construction.

How does such a metric affect matter? Consider matter in free fall, that is slow. Then $dx^i/d\tau \ll dt/d\tau$ and the geodesic equation simplifies to

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}^\mu \left(\frac{dt}{d\tau} \right)^2 = 0.$$

For static fields, $\partial_0 g_{\mu\nu} = 0$ and so

$$\Gamma_{00}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00} = -\frac{1}{2} \partial^\mu h_{00}.$$

We can put this all together. We can say

$$\frac{d^2 t}{d\tau^2} = -\Gamma_{00}^0 \left(\frac{dt}{d\tau} \right)^2 = 0,$$

and so $t = a\tau + b$. Then

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{(dt/d\tau)^2} \frac{d^2 x^i}{d\tau^2} = \frac{1}{(dt/d\tau)^2} \frac{1}{2} \partial^i h_{00} \left(\frac{dt}{d\tau} \right)^2 = \frac{1}{2} \partial^i h_{00}.$$

This is very similar to what Newton would have said. We have $\vec{a} = \vec{F}/m = -m\vec{\nabla}\Phi/m$. So we get $h_{00} = -2\Phi$. We have $h_{00} = c^2/2r$, and this also agrees with the classical picture.

11.2 Coupling to matter

We haven't still talked about how gravity couples to matter. We want to generalize the equation $\nabla^2 \Phi = 4\pi G\rho$. We can first ask the special relativistic version of mass density.

There is the formal approach of looking at the matter Lagrangian $\mathcal{L}_{\text{matter}}$. For example, the Lagrangian $\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi$ over a scalar field gives the equation of motion $\partial_\mu \partial^\mu \varphi = 0$. The action can then be written as

$$S = \int \partial_\mu \varphi \partial^\mu \varphi \sqrt{g}.$$

So we may just define $T^{\mu\nu} = \delta\mathcal{L}/\delta g_{\mu\nu}$ as the source for the Einstein equations.

But this does not give a good physical picture. Here is another approach. For 1 particle, we can generalize energy to

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} = m \frac{dx^\mu}{d\tau}.$$

But we want to look at many particles, because we want densities. This can be described by a number density n , particles for volume, in their local rest frame. We can promote this to a vector

$$N^\mu = n \frac{dx^\mu}{d\tau}.$$

So N^0 is the number density and N^μ is the number flux. This transforms as a 4-vector because each $dx^\mu/d\tau$ transforms as a 4-vector.

Now the mass density in the rest frame is mn . So we can define

$$p^\mu N^\nu = mn \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = T^{(\mu\nu)}.$$

The interpretations of these numbers are T^{00} as energy density, T^{0i} as momentum density, T^{ij} as pressure and stress.

12 March 3, 2017

We are going to continue discussing about the right hand side of Einstein's equation, about what are the sources of $G_{\mu\nu}$. The relativistic version of energy is the four-vector $P^\mu = (E, \vec{p})$. In electrodynamics, Maxwell's equations is given by

$$\partial^\mu F_{\mu\nu} = j_\nu = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix},$$

where the right hand side is the current density. The conservation of charge $Q = \int_{\text{space}} d^3x j^0$ can be written as $\partial^\mu j_\mu = 0$.

In the gravitational case, the story is somewhat similar. We know that gravity couples to (total) energy E_{tot} . But total energy is not invariant. In special relativity, this was promoted to P_{tot}^μ . Because this has to be an integrated thing, we need another index. Then the energy and momentum is

$$E_{\text{tot}} = \int d^3x T^{00} = \int d^3x \rho, \quad \vec{P}_{\text{tot}} = \int d^3x T^{0i},$$

and other momentum flux coming from T^{ij} . The conservation law can be written as $\partial^\mu T_{\mu\nu} = 0$. This is because the equation is equivalent to

$$P^\mu = \int_R T^{\nu\mu} \epsilon_{\nu\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma$$

being conserved, for a 3-dimensional slice R .

12.1 Determining the constants

Now Einstein's equation is of the form

$$R_{\mu\nu} + cRg_{\mu\nu} = \kappa T_{\mu\nu},$$

where c and κ are the constants we need to determine. The conservation can be written as $\nabla^\mu T_{\mu\nu} = 0$ in the general case. Recall that the Bianchi identity states that

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0.$$

So c had better be $c = -1/2$.

Now we need to determine κ so that it matches the equation $\nabla^2 \Phi = 4\pi G_N \rho$. We look at the weak field limit with static sources again. In this case, the only significant limit is $T^{00} = \rho_{\text{energy}}$. If we contract the equation with $g^{\mu\nu}$, then

$$g^{\mu\nu} G_{\mu\nu} = R - \frac{1}{2} R \cdot 4 = -R = \kappa T_\mu{}^\mu = \kappa T.$$

Then we get

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} + \kappa T_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T).$$

We have $T = -\rho$ and $g_{\mu\nu} \approx \eta_{\mu\nu}$. So

$$R_{00} = \kappa\left(\rho - \frac{1}{2}\rho\right) = \frac{\kappa}{2}\rho.$$

We have

$$\begin{aligned} R_{00} &= R^i{}_{0i0} = \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{i0} + \Gamma^i_{i\lambda} \Gamma^\lambda_{00} - \Gamma^i_{0\lambda} \Gamma^\lambda_{i0} = \partial_i \Gamma^i_{00} \\ &= \partial_i \left(\frac{1}{2} g^{i\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \right) = -\frac{1}{2} \nabla^2 h_{00} = \nabla^2 \Phi. \end{aligned}$$

Here, we use the convention that Latin indices run over space and Greek indices run over all components. This gives us $\kappa = 8\pi G_N$. Therefore **Einstein's equation** is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}.$$

12.2 Examples of energy-stress tensor

There is the Klein-Gordon field in flat space given by

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0.$$

If you plug in $\varphi \sim e^{ik^\mu x_\mu}$, then you get $k^\mu k_\mu + m^2 = 0$. In curved space, we have

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi + m^2 \varphi = 0.$$

In this case, the energy-stress tensor is given by

$$T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} (\nabla \varphi)^2 - \frac{1}{2} g_{\mu\nu} m^2 \varphi^2.$$

Let me first try to justify this. In flat space,

$$\begin{aligned} T_{00} &= \dot{\varphi}^2 + \frac{1}{2} (-\dot{\varphi}^2 + \vec{\nabla} \varphi \cdot \vec{\nabla} \varphi) + \frac{1}{2} m^2 \varphi^2 \\ &= \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{1}{2} m^2 \varphi^2. \end{aligned}$$

The first two terms is the kinetic energy and the third term is the energy density. The momentum density is

$$T_{0i} = \dot{\varphi} \partial_i \varphi.$$

Now let us derive that this is indeed the right energy-stress tensor. The action for the Klein-Gordon equation is given by

$$S_{\text{KG}} = \int \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 \right) d^4 x.$$

Then the curved space generalization is

$$S_{\text{KG}} = \int \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 \right).$$

Actually there might be higher terms like $R^{\mu\nu} L_{\text{pl}}^2 \partial_\mu \varphi \partial_\nu \varphi$ but they are suppressed, by the same logic.

Now let us compute the variations. We have

$$\frac{\delta \sqrt{-g}}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} \sqrt{-g}.$$

Then

$$-\frac{\sqrt{-g}}{2} T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-g} \left(\frac{1}{2} \nabla_\mu \varphi \nabla_\nu \varphi - \frac{g_{\mu\nu}}{2} \left(\frac{m^2}{2} \varphi^2 + \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi \right) \right).$$

You can check the conservation $\nabla^\mu T_{\mu\nu} = 0$, by using the equations of motion.

Let us now look at electrodynamics. We have $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. The equation of motion is given by $\nabla^\mu F_{\mu\nu} = 0$, and the Bianchi identity $\nabla_{[\mu} F_{\nu\lambda]} = 0$ is automatically satisfied.

Recall that $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ij}^k B_k$. Using this, we see that two of the Maxwell equations are in the equation of motion and two are in the Bianchi identity.

The action is given by

$$S = -\frac{1}{4} \int \sqrt{-g} F_{\mu\nu} F_{\lambda\sigma} g^{\mu\lambda} g^{\nu\sigma}.$$

The variation with respect to the field A_μ gives $\partial^\mu F_{\mu\nu} = 0$ which is the right equation of motion. Variation with respect to the metric $g_{\mu\nu}$ is

$$T^{\mu\nu} = 2F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{2} g^{\mu\nu} F^2.$$

This is conserved because

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= 2\nabla^\mu F_{\mu\lambda} F_\nu{}^\lambda + 2F_{\mu\lambda} \nabla^\mu F_\nu{}^\lambda - g_{\mu\nu} F^{\alpha\beta} \nabla^\mu F_{\alpha\beta} \\ &= 2F_{\mu\lambda} \nabla^\mu F_\nu{}^\lambda - F^{\alpha\beta} \nabla_\nu F_{\alpha\beta} = F^{\alpha\beta} (-\nabla_\nu F_{\alpha\beta} + 2\nabla_\alpha F_{\nu\beta}) \\ &= F^{\alpha\beta} (-\nabla_\nu F_{\alpha\beta} - \nabla_\beta F_{\nu\alpha} - \nabla_\alpha F_{\beta\nu}) = -\frac{1}{2} F^{\alpha\beta} \nabla_{[\nu} F_{\alpha\beta]} = 0. \end{aligned}$$

13 March 8, 2017

In electromagnetism, the field tensor is $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ and then the stress tensor is given by

$$T^{\mu\nu} = \frac{1}{2}(F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}).$$

Then in flat space, the energy and momentum density is

$$\begin{aligned} T^{00} &= \frac{1}{2}(F^{0i}F_i^0 - \frac{1}{4}(-1)(2F^{0i}F_{0i} + F^{ij}F_{ij})) = \frac{1}{4}(E^2 + B^2), \\ T^{0i} &= \frac{1}{2}F^{0j}F^i_j = -\frac{1}{2}\vec{E} \times \vec{B}. \end{aligned}$$

13.1 Pressureless dust and ideal fluid

When we do astrophysics or cosmology, we don't describe things in terms of the standard particles. We approximate them as some kind of fluid or gas or dust.

Let us first look at pressureless dust. These are non-interacting gas of particles. In this case, it was easy to derive that the stress tensor has to be

$$T_{\mu\nu} = \rho_0 U_\mu U_\nu,$$

where $U_\mu = dx^\mu/d\tau$ is the 4-velocity. Because $U^\mu = \gamma(1, \vec{v})$, we get

$$T^{00} = \rho_0 \gamma^2 = \rho, \quad T^{0i} = \rho v^i, \quad T^{ij} = \rho v^i v^j.$$

The conservation laws must hold. Energy conservation is given by

$$\partial_0 T^{00} + \partial_i T^{i0} = \dot{\rho} + \partial_i(\rho v^i) = 0.$$

This is (almost) the conservation law for the number of particles. For momentum we have

$$\begin{aligned} \partial_0 T^{0j} + \partial_i T^{ij} &= \dot{\rho} v^j + \rho \dot{v}^j + \partial_i(\rho v^i) v^j + \rho v^i \partial_i v^j \\ &= \rho \dot{v}^j + \rho v^i \partial_i v^j = 0. \end{aligned}$$

To apply general relativity in the real world, you need more than pressureless dust. In most case, it suffices to just add pressure. We are going to consider **ideal fluid** that has only isotropic pressure p . The in the rest frame, we have

$$T^{\mu\nu} = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

To get a general form, we do a Lorentz boost and get

$$T^{\mu\nu} = (\rho_0 + p)U^\mu U^\nu + p\eta^{\mu\nu}.$$

Let me briefly explain why T^{ii} is the pressure. For R a spatial region, we have $p^i = \int_R d^3x T^{0i}$ and so

$$\dot{p}^i = \int_R \partial_0 T^{0i} = - \int_R \partial_j T^{ji} = - \int_{\partial R} T^{ji} dA_j.$$

So \dot{p}^i is the i -component of the T^{ji} flux integrated over ∂R . This is something like the force exerted on R .

Physical states give p as a function of ρ . Pressureless dust have $p = 0$. In the case of the gas of photons, we have $p = \rho/3$, although I haven't derived it. In the case of vacuum energy, we have $T_{\text{vac}}^{\mu\nu} = -\rho_{\text{vac}} g^{\mu\nu}$ and so $p = -\rho$.

Let us go back to the non-relativistic limit in flat space. Then $U^\mu = (1, v^i)$. Then the components of the stress tensor are

$$T^{00} = \rho, \quad T^{0i} = \rho v^i, \quad T^{ij} = \rho v^i v^j + p \delta^{ij}.$$

The conservation equations say

$$\begin{aligned} \partial_0 T^{00} + \partial_i T^{0i} &= \dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \\ \partial_0 T^{0i} + \partial_j T^{ji} &= \dot{\rho} v^i + \rho \dot{v}^i + \rho v^j \partial_j v^i + \partial^i p = 0. \end{aligned}$$

Then we get the Navier-Stokes equation

$$\rho(\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v}) = -\vec{\nabla} p.$$

13.2 Einstein-Hilbert action

What kind of action can you write down? In general you should write down

$$S = \int \sqrt{-g} (2\Lambda_{\text{cc}} + R + a L_{\text{pl}}^2 R^2 + \dots).$$

By the same logic, we can think that higher terms are suppressed and so we can just look at the **Einstein-Hilbert action**

$$S_{\text{EH}} = \int \sqrt{-g} R.$$

Let us compute the variation. You can actually compute this, but I am going to cheat a bit. The variation of this action is going to be something like $\delta S_{\text{EH}} = \int \sqrt{-g} E_{\mu\nu} \delta g^{\mu\nu}$ for some tensor $E_{\mu\nu}$.

The variation of $\sqrt{-g}$ can be computed in the following way. We know $\log(\det M) = \text{tr}(\log M)$ and so

$$\frac{\delta(\det M)}{\det M} = \delta \log(\det M) = \text{tr}(M^{-1} \delta M).$$

Then

$$\delta \sqrt{-g} = \frac{-g}{2\sqrt{-g}} g^{\mu\nu} \delta g_{\mu\nu} = \frac{\sqrt{-g}}{2} g^{\mu\nu} \delta g_{\mu\nu} = \frac{-\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu}.$$

We should have $E_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu}$. Consider the variation $\delta g^{\mu\nu} = \nabla^{(\mu}\epsilon^{\nu)}$. This is a diffeomorphism, and so the variation of S under this must be zero. Then we get

$$0 = \int E_{\mu\nu} \nabla^\mu \epsilon^\nu \sqrt{-g} = \int \sqrt{-g} \epsilon^\nu \nabla^\mu E_{\mu\nu}.$$

Then we get $\nabla^\mu E_{\mu\nu} = 0$ and so $E_{\mu\nu} = aG_{\mu\nu}$.

Next we look at the variation $\delta g^{\mu\nu} = \lambda g^{\mu\nu}$ for some constant λ . Then $\delta g_{\mu\nu} = -\lambda g_{\mu\nu}$ and you can compute $\delta \Gamma_{\nu\rho}^\mu = 0$. Then $\delta R_{\mu\nu} = 0$ and $\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = \lambda R$. Also $\delta \sqrt{-g} = -2\lambda \sqrt{-g}$. We then get

$$\delta S_{\text{EH}} = \delta \int \sqrt{-g} R = (-2 + 1)\lambda \int \sqrt{-g} R = \int \sqrt{-g} E_{\mu\nu} \lambda g^{\mu\nu}.$$

This implies that $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu}$.

Now if we write

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + S_{\text{matter}} + 2 \int \frac{\Lambda \sqrt{-g}}{16\pi G},$$

we get that the variation is

$$\delta S = \int \sqrt{-g} \frac{G_{\mu\nu} \delta g^{\mu\nu}}{\partial \pi G_N} + \int \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} - \frac{\Lambda}{\partial \pi G_N} \int \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}.$$

Because we want the equation $G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ at the end the action we want to write down has to satisfy

$$T_{\mu\nu} = -\frac{1}{2\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}}.$$

If we take the variation $g^{\mu\nu} = \nabla^{(\mu}\epsilon^{\nu)}$, we have

$$0 = \delta S = 2 \int \sqrt{-g} T_{\mu\nu} \nabla^\mu \epsilon^\nu = -2 \int \sqrt{-g} (\nabla^\mu T_{\mu\nu}) \epsilon^\nu.$$

So we recover the conservation law from this.

13.3 Noether's theorem

Consider a complex scalar field, with action

$$S = -\frac{1}{2} \int \sqrt{-g} (\nabla_\mu \varphi^* \nabla^\mu \varphi - m^2 \varphi^* \varphi).$$

There is a genuine symmetry $\varphi(x, t) \mapsto e^{i\lambda} \varphi(x, t)$, for a constant λ . We check that the variation of φ is zero.

Generally, we have

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi \right).$$

So if we define $j^\mu = \partial\mathcal{L}/\partial(\partial_\mu\varphi)\delta\varphi$, then the equation just says that $\partial_\mu j^\mu = 0$.

In our case, $\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi + m^2\varphi^*\varphi$ and so

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\delta\varphi^* = i(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi).$$

This is the conserved current.

14 March 10, 2017

Last time I talked about Noether's theorem. The conserved current was given by

$$j_\mu = \frac{i}{2}(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*).$$

In flat spacetime, let us see what happens if we translate a real scalar field with

$$S = \frac{1}{2} \int \partial^\mu \varphi \partial_\mu \varphi + m^2 \varphi^2.$$

The variation of the field is $\delta\varphi = \lambda^\mu \partial_\mu \varphi$, and then $\delta\mathcal{L} = \lambda^\nu \partial_\nu \mathcal{L} = \lambda^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$. If we just write $J^\mu_\nu = \delta^\mu_\nu \mathcal{L}$, the stress tensor is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta^\mu \varphi - J^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L} \\ &= \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} g^{\mu\nu} (\partial_\rho \varphi \partial^\rho \varphi + m^2 \varphi^2). \end{aligned}$$

14.1 Einstein equation as an evolution equation

In classical mechanics, we have this picture of the system at time $t = 0$ determining the system at later time. If we solve the equation, we will only get a solution that is unique up to diffeomorphism. This means that the foliation into space sections is not well-defined.

The terms g^{00} and g^{0i} have no t derivatives in \mathcal{L} . So they give constraint equations, that are like equations like $\vec{\nabla} \cdot \vec{E} = 4\pi\rho_0$.

Imagine we have a spacial region, and for each point a bunch of geodesics that start parallel. More formally, to each point is a 4-velocity $U^\mu = dx^\mu/d\tau$. Then $\theta = \nabla_\mu U^\mu$ measures whether the particles contract or expand. One can moreover define the vector

$$B^\mu{}_\nu = \nabla_\nu U^\mu, \quad B_{\mu\nu} = \frac{1}{\theta} (g_{\mu\nu} + U_\mu U_\nu) + \sigma_{(\mu\nu)} + \omega_{[\mu\nu]}.$$

Here $\sigma_{(\mu\nu)}$ is the sheaf and $\omega_{[\mu\nu]}$ is the rotation part.

At $\tau = 0$, assume that we start with parallel geodesics $U^\mu = (1, 0, 0, 0)$. We have, at time 0,

$$\begin{aligned} \frac{d\theta}{d\tau} &= 2\omega^2 - 2\sigma^2 - \frac{1}{3}(\theta)^2 - R_{\mu\nu} U^\mu U^\nu = -R_{00} \\ &= -8\pi G(T_{00} - 1/2(-T_{00} + T_{ii})) = -4\pi G(\rho + p_x + p_y + p_z). \end{aligned}$$

The $-$ sign accounts for the attraction of spacetime. Also we see that pressure makes the space shrink.

What kind of $T^{\mu\nu}$ is allowed? There is the **null energy condition** says that

$$T_{\mu\nu}l^\mu l^\nu \geq 0,$$

and this is satisfied by all reasonable matter. In the case for the perfect fluid, we have $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + \rho g_{\mu\nu}$ and so $\rho + p \geq 0$.

There is something also called the **strong energy condition** given by

$$T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}Tt_\rho t^\rho$$

for any timelike t^μ . This is just saying that gravity is attractive. In the case of a perfect fluid, it gives $\rho + 3p \geq 0$. This is not always satisfied because even our universe is expanding.

Consider a 2-dimensional surface Σ . From this surface, there is a ingoing null-ray and the outgoing null-ray starting from this surface. In flat space, they will look like shrinking and expanding 2-dimensional spheres. If both rays are contracting, then this is called a trapped surface. The **Hawking–Penrose theorems** tells us that there has to be a singularity. These singularities come in the shape of a space section, and once you get too close to it, there is no way of escaping the singularity.

The Weyl tensor is the part of the Riemann tensor that is algebraically perpendicular to the Ricci tensor. The Bianchi identity then implies

$$\nabla^\rho C_{\rho\sigma\mu\nu} = \nabla_{[\mu} R_{\nu]\rho} + \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}R = 8\pi G\left(\nabla_{[\mu}T_{\nu]\rho} + \frac{1}{3}g_{\sigma[\mu}\nabla_{\nu]}T\right).$$

This is similar to the equation $\nabla^\mu F_{\mu\nu} = J_\nu$ in electrodynamics.

14.2 Solution with a spherical symmetry

We want to find a solution for a black hole. This was done by Schwarzschild in 1916. The experience of a person near a black hole is going to be counterintuitive, and so we are going to spend some time on it.

We are assuming spherical symmetry, or more formally three Killing vector $[\cdot, \cdot]$ forming the Lie algebra $\mathfrak{su}(2) = \mathfrak{so}(3)$. We are going to work in polar coordinates. Then the symmetries tell us

$$ds^2 = -A(r, t)dt^2 + B(r, t)drdt + C(r, t)dr^2 + D(r, t)d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

We can actually do better than this. We can choose r so that $D(r, t) = r^2$. We are then going to pick a $t' = t'(t, r)$ such that $B' = 0$. This can be done by using

$$dt' = I(t, r)(Adt - (B/2)dr).$$

Since dt' must be a closed form, we need the condition $\partial_r(IA) + \partial_t(IB) = 0$. You can find a solution for I because this differential equation is linear.

15 March 22, 2017

We started by just assuming spherical symmetry. By vacuum, we are saying that there is no matter, although there can be energy in the gravitational field.

We showed that we can write down the metric in the form of

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2,$$

just by choosing a clever system.

15.1 The Schwarzschild metric

Because this is a diagonal metric, you can compute the Christoffel symbols, the Riemann tensor, and the Ricci tensor. It is going to be

$$\begin{aligned} R_{rr} &= -\left(\nu'' + \nu'^2 - \nu'\lambda' - \frac{2}{r}\lambda'\right) + e^{2(\lambda-2)}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}), \\ R_{rt} &= R_{tr} = \frac{2}{r}\dot{\lambda}, \\ R_{tt} &= \ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda} + e^{2(\nu-\lambda)}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu'\right), \\ R_{\theta\theta} &= e^{-2\lambda}(r(\lambda' - \nu') - 1) + 1, \\ R_{\varphi\varphi} &= R_{\theta\theta} \sin^2 \theta. \end{aligned}$$

Recall that $G_{\mu\nu} = 0$ is equivalent to $R_{\mu\nu} = 0$. So these all have to be zero. $R_{rt} = 0$ immediately implies $\dot{\lambda} = 0$. Then we see that

$$\dot{R}_{\theta\theta} = 0 = -e^{-2\lambda} r \lambda' \dot{\nu},$$

and hence $\dot{\nu} = 0$. That means that we can write $\nu = f(r) + g(t)$. Hence our metric takes the form of

$$ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2.$$

We can use diffeomorphism invariance, and then call $e^{2g(t)} dt^2 = dt'^2$. Then

$$ds^2 = -e^{2f(r)} dt'^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2.$$

The fact is called **Birkhoff's theorem**. If you have spherical symmetry in vacuum, it is actually static. This is more than saying that the metric is stationary, i.e., has a time-like Killing vector. Static means that there is also a spacial hypersurface that is orthogonal to the Killing vector. The metric that looks like $e^{f(r)} dt^2 + e^{g(r)} dt dr + e^{\lambda(r)} dr^2 + r^2 d\Omega^2$ is stationary but not static. Intuitively, stationary means constant velocity and static means nothing happens.

Anyways we have the equations $R_{rr} = R_{tt} = R_{\theta\theta} = 0$. Then we get

$$\lambda' + \nu' = 0, e^{-2\lambda}(r(\lambda' - \nu') - 1) + 1 = 0.$$

From the first equation, we can write $\lambda = -f + \text{const.}$ This constant can be eliminated just by rescaling t . So we can just let $\lambda = -f$. From the second equation, we get

$$0 = e^{2f}(r(-2f') - 1) + 1 = -\partial_r(re^{2f}) + 1.$$

So we get $e^{2f} = (r - R_s)/r$. Therefore

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1 - R_s/r} + r^2 d\Omega^2.$$

This is the **Schwarzschild solution** and this particular coordinate is called the **Schwarzschild coordinate**. Notice that as $r \rightarrow \infty$ the metric becomes closer to the Minkowski metric.

In the Newtonian limit,

$$g_{tt} = -1 - 2\Phi_{\text{Newton}} = -1 + \frac{2G_N M}{r},$$

and so $R_s = 2G_N M$. In the case of the sun, the Schwarzschild radius and the actual radius are $G_N M_\odot \approx 1.5\text{km}$ and $r_\odot \approx 7 \times 10^5\text{km}$. So we don't see any strong curvature around the sun.

15.2 Physics of the Schwarzschild solution

There can be naïvely two places of singularities. They are the points $r = R_s$ and $r = 0$. But we can't tell this easily, because it might just be a problem of our coordinate. We can put metrics like $dr^2 + r^2 d\varphi^2$ that look singular, but is actually flat space.

How can we tell if something is a physical singularity? If any of R , $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ or anything goes to infinity at the point, we know for sure that this is a physical singularity. Or if I have a geodesic connecting two points, but has infinite distance, that point even doesn't lie in the manifold. If you compute the contraction of the Riemann tensor, we get

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim r^{-6}.$$

So the point $r = 0$ actually is a singularity.

In the region $r < R_s$, the component dr^2 looks like time and dt^2 looks like space, and there is still spherical symmetry. This is very hard to visualize for now.

Particles follow time-like geodesics, and so it would be interesting to figure out what the geodesics look like. Because our metric is quite simple, we don't have a huge mess. The geodesic equation is given by

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}.$$

Here the non-vanishing Christoffel symbols are Γ_{tt}^r , Γ_{rr}^r , Γ_{tr}^t , $\Gamma_{r\theta}^\theta$, $\Gamma_{\theta\theta}^r$, $\Gamma_{r\varphi}^\varphi$, $\Gamma_{\varphi\varphi}^r$, $\Gamma_{\varphi\varphi}^\theta$, $\Gamma_{\theta\varphi}^\varphi$.

To simplify the solution, we want to restrict by looking at some symmetries. You should have proved in one of your problem sets that $K_\mu(dx^\mu/d\tau)$ is conserved. There are three Killing vectors, one $\partial/\partial t$ and three from the spherical symmetry. Also the velocity $1 = -g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)$ is conserved.

We may as well assume that $\theta = \pi/2$ is constant, by rotating the spherical coordinate. We have one conserved quantity coming from $K = \partial_t$,

$$K_\mu \frac{dx^\mu}{d\tau} = -\left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} = -E,$$

and another conserved quantity coming from $K = \partial_\varphi$ coming from

$$L = K_\mu \frac{dx^\mu}{d\tau} = r^2 \sin^2 \theta \frac{d\varphi}{d\tau} = r^2 \frac{d\varphi}{d\tau}.$$

16 March 24, 2017

We obtained the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1 - R_s/r} + r^2 d\Omega^2.$$

Here $R_s = 2G_N M$ where M is roughly like the total energy, not only determined by the rest mass of matter.

16.1 Geodesics in the Schwarzschild metric

We were trying to compute the orbits (or time-like or null geodesics) in this geometry. There are the Killing vectors ∂_t and those associated to spherical, including ∂_φ . To simplify our life, we can assume $\theta = \pi/2$. Then

$$E = \left(1 - \frac{R_s}{r}\right) \frac{dt}{d\tau}, \quad L = r^2 \frac{d\varphi}{d\tau}$$

are conserved, and the length is

$$-\epsilon = -\left(1 - \frac{R_s}{r}\right) \left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - R_s/r} \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\varphi}{d\tau}\right)^2.$$

Here $\epsilon = 1$ in the time-like case and $\epsilon = 0$ in the null case.

We have

$$\frac{E^2}{2} = \frac{\epsilon}{2} \left(1 - \frac{R_s}{r}\right) + \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2} \left(1 - \frac{R_s}{r}\right) \frac{L^2}{r^2}.$$

This allows us to write

$$V_{\text{eff}}(r) = \frac{\epsilon}{2} - \epsilon \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}.$$

Here the first term is a constant that we can throw out, the second term is the Newtonian gravity, the third term comes from the rotations, and the fourth term is the new term in general relativity.

Note that $r(\tau)$ changes direction when $E^2/2 = V_{\text{eff}}(r)$. The circular orbits occur when $V'(r_c) = 0$. It is instructive to have a graph of the effective potential.

The unstable orbit for light has radius $r_c = 3G_N M = \frac{3}{2}R_s$. In the case of matter, the smallest stable circular orbit is $r_c = 6G_N M$. Note that this is just for geodesics, and so if you accelerate, you could be able to get out.

16.2 Penrose diagrams

This is the most coarse feature of physics. In flat space, we know how this work. If we start at any point, we can get anything that lies in the light cone.

Let us use the coordinates

$$u = t - r, \quad v = t + r.$$

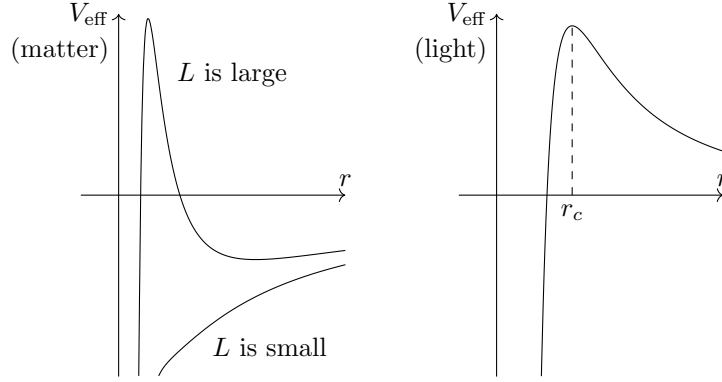


Figure 2: Graph of V_{eff} for matter and light in the Schwarzschild metric

Then the metric becomes $-dt^2 + dr^2 = -dudv$. This tells us that these coordinates u and v are null. There are different limits one can take: you can take the time-like limit i^+ , inverse time-like limit i^- , the space limit i^0 , the null limit \mathcal{I}^+ , and the inverse null limit \mathcal{I}^- . These are infinite points, but we want to choose a new coordinates that make them look finite, so that we can draw them.

Let us choose

$$\tilde{u} = \tan^{-1}(u), \quad \tilde{v} = \tan^{-1}(v).$$

The coordinates u or v being constant are equivalent to \tilde{u} or \tilde{v} being constant. So the null rays are still diagonals. This can be also seen from the metric. The metric is given by

$$ds^2 = -\frac{d\tilde{u}d\tilde{v}}{\cos^2 \tilde{u} \cos^2 \tilde{v}} + \frac{1}{4}(\tan \tilde{v} - \tan \tilde{u})^2 d\Omega^2.$$

Here the metric just look like a scaling of the original one, and so it is conformal. Hence the causal structure of Minkowski space is the same as the causal structure of this triangle. This finite conformal diagram is called the **Penrose diagram**.

A massive observer goes from i^- to i^+ . Light always start from j_- and goes to j^+ .

Now we want to do this for the Schwarzschild metric. But let us look at another example first. There is a **anti-de Sitter space**, also called AdS, given by

$$ds^2 = L^2 \left(\frac{-dt^2 + dy^2 + dx_1^2 + dx_2^2}{y^2} \right).$$

You can compute the Ricci tensor as $R_{\mu\nu} = -g_{\mu\nu}/L^2$. So it solve the vacuum Einstein equation with cosmological constant

$$\Lambda_{\text{c.c.}} = -\frac{(\lambda - 2)}{2L^2}.$$

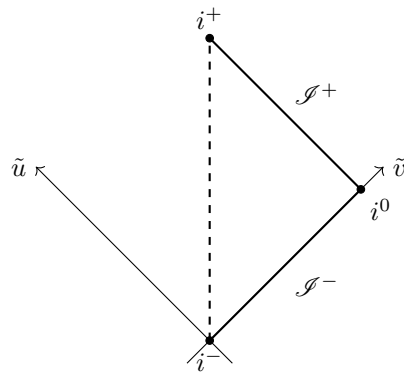


Figure 3: Penrose diagram of Minkowski space

This is a maximal symmetric space. There are translations in t, x_1, x_2 , the rotation and the two boosts. There are four more isometry. Thus there are 10 symmetries in total.

17 March 29, 2017

In some t - x diagram, the slope is the velocity if the space is flat. If we manage to pick a coordinate system without changing the angle, we can map with preserving the causal structure. For instance, if we have a metric like

$$ds^2 = e^{2w}(dx^2 + dy^2),$$

then the angles are preserved although it is highly distorting of distance.

For the flat metric, we wrote

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = -du dv + r^2 \Omega^2$$

and squeezed the space using the rescaling $\tilde{u} = \tan^{-1} u$ and $\tilde{v} = \tau^{-1} v$.

17.1 Penrose diagram of AdS_2

We started to discuss the 2-dimensional anti-de Sitter space AdS_2 given by the metric

$$ds^2 = \frac{L^2}{y^2}(-dt^2 + dy^2)$$

for $y > 0$. This is like the Lorentzian version of the Poincaré half plane. This is a maximally symmetric space, and so it has constant curvature, given by $R = -L^{-2}$. This is a good example, because there is a point at infinity that is actually of finite distance.

Let us try to draw the Penrose diagram again. Set up the null coordinates $v = t + y$ and $u = t - y$ and $\tilde{u} = \tan^{-1} u$ and $\tilde{v} = \tan^{-1} v$. If you compute the metric, we get

$$ds^2 = -4L^2 \frac{d\tilde{u}d\tilde{v}}{\sin^2(\tilde{v} - \tilde{u})}.$$

This means that if we approach the line $\tilde{v} = \pi/2$, nothing funny happens.

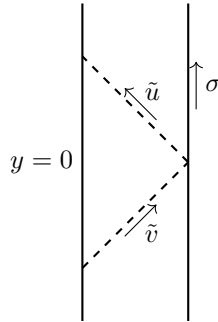


Figure 4: Penrose diagram of AdS_2

So to extend this space, we define new coordinates $\tau = \tilde{u} + \tilde{v}$ and $\sigma = \tilde{u} - \tilde{v}$. In these coordinates, the metric looks like

$$ds^2 = \frac{-d\tau^2 + d\sigma^2}{\sin^2 \sigma}.$$

Then we can extend this to $0 < \sigma < \pi$ and $-\infty < \tau < \infty$ (because we want maximal symmetry). It is clear that there is a symmetry in the time direction, but you can work out and see that there are more symmetries. In particular, $\sigma = \pi/4$ or $\sigma = \pi/2 - 0.01$ cannot be distinguished.

In this space, you can follow the line of constant y , or you can follow the line of constant σ . In both cases, you will live forever. But if you follow the line of constant y , you will live in the triangle and so you will not be able to see everything. But if you follow the line of constant σ , you will see everything.

What happens if we go along the u or v direction? We should ask whether we can reach the edge of spacetime in finite affine parameter. In 2-dimension, it is easy to find the null geodesics because as long as you stay null, it is going to be a geodesic. The line $u(\lambda)$ with $v = v_0$ is going to be a geodesic. Writing $ds^2 = -e^{2\omega} du dv$, we get

$$\frac{d^2 u}{d\lambda^2} = -\Gamma_{uu}^u \frac{du}{d\lambda} \frac{du}{d\lambda} = -\frac{d\omega}{du} \frac{du}{d\lambda} \frac{du}{d\lambda} = -2 \frac{d\omega}{d\lambda} \frac{du}{d\lambda}.$$

So $du/d\lambda = -2\omega + k$ and so

$$\lambda = \int d\lambda = e^{-k} \int_{u_0}^{\infty} du e^{2\omega(u,v)}.$$

Applying this formula to (\tilde{u}, \tilde{v}) with $e^{2\omega} = 1/\sin^2(\tilde{u} - \tilde{v})$, we get

$$\lambda = \int_{\tilde{u}_0}^{\tilde{v}_0} \frac{d\tilde{u}}{\sin^2(\tilde{u} - \tilde{v}_0)}.$$

Definition 17.1. A space is called **geodesic complete** if all geodesics are inextendable.

The things happening at the boundary is interesting because an observer can observe the light ray hitting the boundary. If we want something like energy conservation, we need to assume things like the light ray bouncing off at the boundary. These issues are subtle.

Let me talk about what happens in higher dimensions. Consider the spacetime $\mathbb{R}^{3,2}$ with the metric

$$-du^2 - dv^2 + dx^2 + dy^2 + dz^2.$$

Consider the hyperboloid defined by the equation $L^2 = -u^2 - v^2 + x^2 + y^2 + z^2$. This turns out to be the anti-de Sitter space AdS_4 . There are global coordinates

in this space given by

$$\begin{aligned} u &= L \cosh \rho \cos z, & v &= L \cosh \rho \sin z, \\ x &= L \sinh \rho \sin \theta \sin \varphi, & y &= L \sinh \rho \sin \theta \cos \varphi, & z &= L \sinh \rho \cos \theta, \\ ds^2 &= L^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2). \end{aligned}$$

Note that $\mathbb{R}^{3,2}$ has a $\text{SO}(3, 2)$ symmetry, and our hypersurface is preserved by $\text{SO}(3, 2)$. So AdS_4 is going to inherit the $\text{SO}(3, 2)$ symmetry.

If we let $1/\cos r = \cosh \rho$, then the metric is going to look like

$$ds^2 = \frac{L^2}{\cos^2 r}(-dt^2 + dr^2).$$

So we are going to have a similar Penrose diagram, with a dashed line at $r = 0$ due to the symmetry.

18 March 31, 2017

Last time we draw the Penrose diagram of 2-dimensional anti-de Sitter space, which is given by the metric

$$ds^2 = \frac{L^2}{y^2}(-dt^2 + dy^2).$$

The whole idea is the we only made coordinate changes that does not change the causal structure.

One way to write the 4-dimensional anti-de Sitter space is

$$ds^2 = \frac{-dt^2 + dw^2 + dx_1^2 + dx_2^2}{w^2}.$$

This does not cover the whole space, but there is a global coordinate given by

$$ds^2 = L^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2).$$

Another way of thinking about this space is to consider it as a hypersurface cut out by the equation $L^2 = -u^2 - v^2 + x^2 + y^2 + z^2$ in $\mathbb{R}^{3,2}$. This is going to have symmetry group $\text{SO}(3, 2)$, which is maximally symmetric. Indeed, you see that they are the same space by the coordinate transformation

$$\begin{aligned} u &= L \cosh \rho \cos \tau, & v &= L \cosh \rho \sin \tau, \\ x &= L \sinh \rho \sin \theta \sin \varphi, & y &= L \sinh \rho \sin \theta \cos \varphi, & z &= L \sinh \rho \cos \varphi. \end{aligned}$$

There is also called a **de Sitter space** given by the metric

$$dS_4 : \quad ds^2 = L^2(-dt^2 + \cosh^2 t d\Omega_3^2)$$

which comes as the hyperspace in $\mathbb{R}^{4,1}$ with symmetry group $\text{SO}(4, 1)$. This has Penrose diagram:



Figure 5: Penrose diagram of dS_4

In this space, you can never see the whole space. This space is of interest in cosmology because people think this is going to be our space after a long time.

18.1 Penrose diagram of the Schwarzschild solution

Our goal for now is not to study the anti-de Sitter or de Sitter space. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2.$$

Something strange happens at $r = R_s = 2M$ and we need to change coordinates to see what happens after $r = R_s$. What we are going to do is to zoom in near the horizon $r = R_s$.

Consider $r = 2M + \rho$ where ρ is small. If the metric is indeed nonsingular, it is going to be flat. We have

$$ds^2 \approx -\frac{\rho}{R_s} dt^2 + \frac{R_s}{\rho} d\rho^2 + R_s^2 d\Omega_2^2.$$

The last term is fine, and so we are going to focus on the first two terms.

We want to put the t, ρ plane into the conformally flat form. It is pretty clear that we want $d\rho/\rho = d(\text{sth})$ and so we take $\rho = R_s e^{u/R_s}$. Then

$$ds^2 = -e^{u/R_s} dt^2 + e^{-u/R_s} e^{2u/R_s} du^2 = e^{u/R_s} (-dt^2 + du^2).$$

Here $r \rightarrow R_s$ is $\rho \rightarrow 0$ and it is $u \rightarrow -\infty$.

We choose the null coordinates $w_{\pm} = t \pm u$ and we get

$$ds^2 = e^{w_+/2R_s - w_-/2R_s} dw_+ dw_- = (e^{w_+/2R_s} dw_+)(e^{w_-/2R_s} dw_-).$$

This is indeed just flat if we take $v_+ = e^{w_+/2R_s}$ and $v_- = -e^{-w_-/2R_s}$. The metric can be written as $ds^2 = -dv_+ dv_-$ and so the space is $\mathbb{R}^{1,1} \times S^2$ if we zoom in near $r = R_s$. Constant ρ means constant u , and this means constant $v_+ v_-$. In particular, $\rho = R_s e^{u/R_s} = -R_s v_+ v_-$. This shows these form hyperbolas in the below figure. We find new parts that were not covered in the Schwarzschild metric.

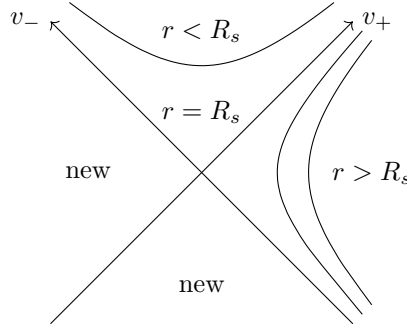


Figure 6: Schwarzschild metric near $r = R_s$

Now we know what to do. Let us look at the exact coordinates

$$ds^2 = \left(1 - \frac{R_s}{r}\right) \left(-dt^2 + \frac{dr^2}{(1 - R_s/r)^2}\right) + r^2 d\Omega_2^2.$$

We want to write $dr^* = dr/(1 - R_s/r)$. Such a coordinate can be defined as

$$r^* = \int dr^* = \int \frac{r dr}{r - R_s} = r + R_s \log\left(\frac{r}{R_s} - 1\right).$$

This r^* is also called the **tortoise coordinates**, and it takes $r^* \rightarrow -\infty$ to reach $r = R_s$.

The metric can be rewritten as

$$ds^2 = \left(1 - \frac{R_s}{r}\right)(-dt^2 + (dr^*)^2) + r^2 d\Omega^2.$$

Setting $u_{\pm} = t \pm r^*$ and $u = -2R_s e^{-u_-^*/2R_s}$ and $v = 2R_s e^{u_+^*/2R_s}$ gives

$$ds^2 = \left(1 - \frac{R_s}{r}\right) \frac{R_s^2}{uv} du dv + r^2 d\Omega^2.$$

This is called the **Kruskal metric**. We can further write

$$uv = -4R_s^2 e^{(u_+^* - u_-^*)/2R_s} = -4R_s^2 e^{r^*/R_s} = -4R_s^2 e^{r/R_s} \left(\frac{r}{R_s} - 1\right).$$

Then

$$ds^2 = -\frac{R_s}{4r} e^{-r/R_s} du dv + r^2 d\Omega^2.$$

We see that at $r = R_s$ the whole thing is smooth (and $uv = 0$). At $r \rightarrow 0$, it is singular because $R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} \rightarrow \infty$. In terms of u and v , this is $uv = 4R_s^2$. At $r \rightarrow \infty$, it goes to flat space.

Again, we can pull in ∞ by $u = \tan \tilde{u}$ and $v = \tan \tilde{v}$.

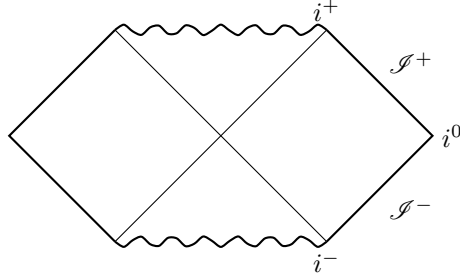


Figure 7: Penrose diagram of the Schwarzschild solution

The right side of this diagram is similar to the Penrose diagram for Minkowski space as we have seen earlier.

19 April 5, 2017

We found out that, for a different choice of coordinate, we can write the Schwarzschild metric as

$$ds^2 = -\frac{d\tilde{u}d\tilde{v}}{\cos^2 \tilde{u} \cos^2 \tilde{v}} \frac{R_s}{r} e^{-r/R_s} + r^2 d\Omega^2.$$

Here, $\tilde{u} = \tan^{-1} u$ and $\tilde{v} = \tan^{-1} v$ and

$$u = -2R_s e^{u^*/2R_s}, \quad v = 2R_s e^{v^*/2R_s},$$

$$u^* = t - r^*, \quad v^* = t + r^*, \quad r^* = r + R_s \log\left(\frac{r}{R_s} - 1\right).$$

Now I want to describe some features of this solution.

19.1 Physics of a black hole

Let us look at a spatial section of the solution. For instance, consider the locus of $\tilde{v} = -\tilde{u}$, which is the horizontal section in the middle. This corresponds to $u = -v$ and so $v^* = -u^*$. So $t = 0$ on this section. The radius of the angular sphere is going to decrease and then increase. The picture will look like this, and is called the **Einstein–Rosen bridge**. But you can't cross the bridge, and so it is called a non-transversal wormhole.

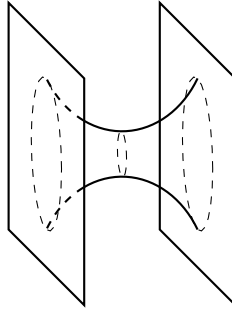


Figure 8: Einstein–Rosen bridge

When people first saw this solution, they did not understand it very well. They called this a frozen star, because to an observer far away, an astronaut falling into the horizon never actually crosses it. The astronaut just get dimmer and redder. But for the astronaut, the picture is that nothing special happens on the boundary. In fact, the geodesic maximizes the survival time and so it is better to do nothing and accept this fate.

What about the region at the bottom of the Schwarzschild solution? This is called a white hole, and things come out but cannot get it. From a thermodynamic point of view, falling into a black hole increases entropy, and the inverse process decreases entropy. That is, a white hole is a very fine tuned object and we don't expect them to form naturally.

There are two types of astrophysical black holes:

- Stellar black holes: $M \approx 10M_\odot$, $R_s \approx 30\text{km}$.
- Supermassive black holes: $M \approx 10^6\text{-}10^9M_\odot$, $R_s \approx 0.01\text{-}40(\text{Earth-Sun})$.

In fact, it is easier to make large black holes. To make a black hole with matter of density ρ , we need $R \leq R_s = 2G_N M = 2G_N \rho R^3$. So $R^2 \geq c\rho^{-1}$. That is, if we simply take any matter, and put enough of them together, they will form a black hole.

These astrophysical black holes have somewhat different Penrose diagrams, because there is matter forming the black hole. In particular, it does not have the other universe region. In fact, it might be even possible that you jump into the black hole and do not meet any matter until you reach the singularity.

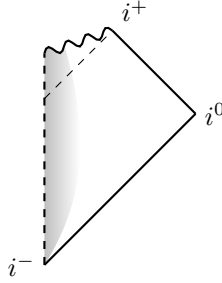


Figure 9: Penrose diagram of an astrophysical black hole

A geodesic with θ and φ fixed is given by

$$-1 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{1 - 2M/r},$$

with the equation

$$0 = \ddot{t} + 2\Gamma_{rt}^t \dot{r} \dot{t} = \ddot{t} + \dot{t} \partial_\tau \log\left(1 - \frac{2M}{r}\right).$$

Then

$$\partial_\tau \log \dot{t} = \frac{\ddot{t}}{\dot{t}} = -\partial_\tau \log\left(1 - \frac{2M}{r}\right).$$

So we can compute

$$\tau = -\frac{2}{3} \frac{r^{3/2}}{\sqrt{2M}}.$$

That is, it does not take a long time to take the horizon, and it does not take a long time to hit the singularity, from the point of view of the astronaut falling in.

The black hole interior is the region from which all causal paths hit the singularity. The horizon, which is also the boundary of the interior, is always null. The horizon is not a feature of space. Here is a simple model for black

hole formation. Consider a large spherical shell of dust, which closes up into some point. Then in the space before the shell closes in, the metric is perfectly Minkowski. But after the shell, the metric is going to be spherically symmetric and hence the Schwarzschild metric. So you might already be doomed in Minkowski space, just because of the malicious person throwing in matter at you.

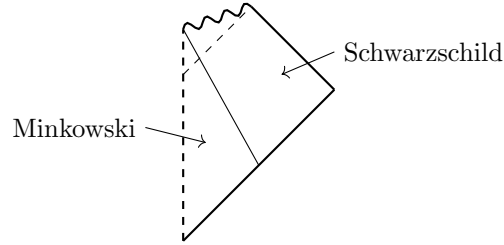


Figure 10: A collapsing shell of dust

Let us look inside a static spherical star. The metric can be written down as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2d\Omega^2.$$

In the case of a perfect fluid, we are going to have

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + \rho g_{\mu\nu},$$

where $U_\mu = (e^{\alpha(r)}, 0, 0, 0)$. The Einstein's equations are then given as

$$\begin{aligned} G_{tt} &= \frac{1}{r^2}e^{-2\beta}(2r\beta' - 1 + e^{2\beta}) = 8\pi G\rho, \\ G_{rr} &= \frac{e^{-2\beta}}{r^2}(2r\alpha' + 1 - e^{2\beta}) = 8\pi Gp, \\ G_{\theta\theta} &= e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}\alpha' - \frac{1}{r}\beta'\right) = 8\pi Gp. \end{aligned}$$

This is not enough to solve the equation, and we need some more equation coming from the state of the fluid.

20 April 7, 2017

20.1 Static spherical object

Let us consider a static spherical object, like a star or a bowling ball. Because it is static, we can write

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2.$$

The Einstein equation can be written as

$$G_{tt} = 8\pi\rho, \quad G_{rr} = 8\pi p, \quad G_{\theta\theta} = 8\pi p.$$

The mass in a sphere of radius r can be written as

$$m(r) = 4\pi \int_0^r dr \rho(r) r^2.$$

This is actually not the right definition of mass or energy, because we have not stuck in the \sqrt{g} factor in the integration. The right definition would have been

$$\bar{m}(r) = \int_0^r \sqrt{g_{\text{spatial}}} \rho(r) dr d\theta d\varphi = 4\pi \int_0^r e^{\beta(r)} r^2 \rho(r) dr.$$

But anyways, using $m(r)$ we can write

$$ds^2 = -e^{2\alpha(r)} dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2 d\Omega^2.$$

So $m(r)$ is the total mass while $\bar{m}(r)$ can be thought of the matter mass.

From the Einstein equations you will also get

$$\frac{d\alpha}{dr} = -\frac{1}{\rho + p} \frac{d\rho}{dr}, \quad \frac{dp}{dr} = -(\rho + p) \frac{(m(r) + 4\pi r^3 \rho)}{r(r - 2m(r))}.$$

There is also the equation coming from the state of matter. This additional equation will give the description of the object.

People showed that given a state, if there is enough matter, there cannot be a static solution. In the case of stars, it burns out of fuel and then it explodes and forms a

- white dwarf (supported by the Pauli exclusion principle of e^-) if $M \leq 1.4M_\odot$,
- neutron star (supported by the neutron degeneracy) if $M \leq 3M_\odot$.

20.2 Charged black holes

The charged spherical black hole is called the **Reissner–Nordström solution**. They are not astrophysical because there aren't many charged astrophysical

objects, but it is in principle interesting. There is also something in string theory related to this.

We would need to solve the Einstein equation along with the Maxwell equations:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad g^{\mu\nu} \nabla_\nu F_{\nu\sigma} = 0, \quad \nabla_{[\mu} F_{\nu\rho]} = 0.$$

The action of matter is given by

$$S_{\text{Maxwell}} = -\frac{1}{4} \int \sqrt{-g} F_{\mu\nu} F^{\mu\nu},$$

and then the stress-energy tensor is

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{Maxwell}}}{\delta g^{\mu\nu}}.$$

If you recall, the metric variation with respect to the matter action is the stress-energy tensor, with respect to the Einstein–Hilbert action is the Einstein tensor, and the field variation with respect to S_{Maxwell} gives the Maxwell's equations. We can compute the stress-energy tensor as

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2\sqrt{-g}} \left(-\frac{1}{2} \sqrt{-g} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + 2\sqrt{-g} F_{\mu\rho} F_{\nu}{}^{\rho} \right) \\ &= F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \end{aligned}$$

We can use spherical symmetry to simplify the electromagnetic tensor $F_{\mu\nu}$. The only possible nonzero components are $E_r = F_{tr}$ and $B_r = g(r, t) \sin \theta$. We also have, from the Gauss law,

$$E_r = Q/r^2,$$

because we have set our spatial sphere to have size exactly r^2 . Now we have our non-vanishing components of $F_{\mu\nu}$, that are quite simple. If we plug this in, we get the simple formula for $T_{\mu\nu}$. Then we can use the Einstein equations to compute the metric.

The process of solving the equation will be similar to obtaining the Schwarzschild solution. At the end, you will get

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{G^2 Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + G^2 Q^2/r^2} + r^2 d\Omega^2.$$

Here M is the mass and Q is the charge.

Again, by Birkhoff's theorem, ∂_t is a Killing vector and this becomes null at some fixed radius,

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

Because there are two such r , the Penrose diagram is quite different from that of the Schwarzschild solution.

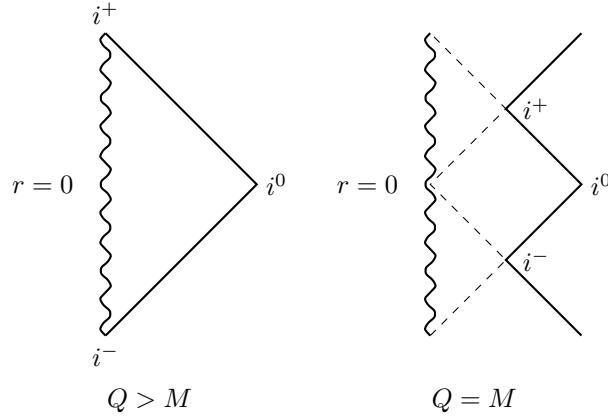


Figure 11: Penrose diagrams of charged black holes

In the case $Q > M$, there is no horizon and the singularity is time-like. The **cosmic censorship conjecture** is that this is not possible, and it is impossible to produce this from smooth and reasonable data unless it is infinitely tuned. This is proven assuming some conditions.

In the case $M = Q$, we are going to have

$$ds^2 = -\frac{(r-M)^2}{r^2} dt^2 + \frac{r^2 dr^2}{(r-M)^2} + r^2 d\Omega^2.$$

Then in this case, the horizon is at $r = M$. Also if you compute the spatial distance to the horizon, it is infinite:

$$\int_{\text{outside}}^M \frac{r dr}{r-M} = \infty.$$

But null lines can reach the horizon in finite affine parameter. If we write $dt = dv - dr/(1 - M/r)^2$, then

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega^2.$$

Then $v = \text{const}$ is a geodesic, and is parametrized by r .

You can try to maximally extend the Penrose diagram, and then you find multiple universes. But we think this is unphysical, because when you cross a horizon to get to another universe, you get an infinite blueshift and a little bit of perturbation will make this into a singularity.

21 April 12, 2017

The really interesting case (and one we can make by throwing in charge) of a charged black hole with $M > Q$. We can write the metric as

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2d\Omega^2.$$

Near the horizon r_+ , the metric looks like the Schwarzschild metric and so you don't have to accelerate to get past the horizon.

If we pick a new coordinate $v(r, t)$ as

$$dv = dt - \frac{r^2 dr}{(r-r_+)(r-r_-)},$$

then we can write the coordinates as

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dv^2 + 2dvdr + r^2d\Omega^2.$$

We can now draw the Penrose diagram.

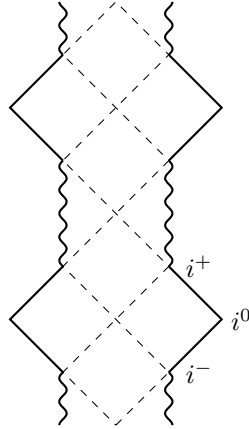


Figure 12: Penrose diagram of lightly charged black holes

Again, we don't think this is really physical because of the infinite blueshift at the boundary. If you give a little bit of perturbation, you don't get a spherical symmetry and it is not possible to solve the equation analytically. But numerical computations show that a perturbation leads to a singularity.

21.1 Komar mass

Noether's theorem says that a symmetry gives conserved currents. Without gravity, there are conservation in E and \vec{p} , and the analogue of this with gravity

is $\partial^\mu T_{\mu\nu} = 0$. In general relativity, we want everything to be covariant and so $\nabla^\mu T_{\mu\nu}^{\text{matter}} = 0$. But this means that $E_{\text{matter}} = \int d^3x T_{00}$ is not conserved, which makes sense because the metric itself contains energy. One could try to remedy this with adding $G_{\mu\nu}$ to $T_{\mu\nu}$, but the only thing that becomes a tensor is $8\pi G T_{\mu\nu} - G_{\mu\nu} = 0$. But this is stupid. So in general, there is no conserved energy.

This is the story for general spacetime. For instance, in dS_4 you might be able to generate matter. But there is some conservation in special cases.

- (1) When the spacetime is asymptotically flat or AdS. There is conserved total energy, which generates global time translation, and this is called the ADM form.
- (2) When the spacetime is stationary, there is also a conserved quantity. This is seems stupid because nothing changes anyhow, but it turns out to be useful.

In electromagnetism, the conserved current is

$$j_\mu = j_\mu^{\text{matter}} = \nabla_\nu F^{\nu\mu}.$$

Then for a 3-dimensional space-like section Σ , the conserved charge in the region is

$$Q = - \int_\Sigma d^3x \sqrt{\gamma} n_\mu \nabla_\nu F^{\nu\mu} = - \int_{\partial\Sigma} d^2\tilde{x} \sqrt{\tilde{\gamma}} n_\mu \sigma_\nu F^{\mu\nu}$$

by Stokes' theorem, where $\gamma, \tilde{\gamma}$ are the induced metric, n_μ is the normal vector to Σ , and σ_ν is the normal vector to $\partial\Sigma$ in Σ . In the static case, this is going to be $\int_{\partial\Sigma} d\vec{A} \cdot \vec{E}$.

Going back to gravity, we have the time-like Killing vector $K_\mu = \partial_t$. Then $\nabla^{(\mu} K^{\nu)} = 0$. We can define

$$j_\mu^{\text{energy}} = K^\nu T_{\mu\nu}.$$

This is indeed going to be a conserved current because

$$\nabla^\mu j_\mu = \nabla^\mu (K^\nu T_{\mu\nu}) = 0.$$

But the problem is that

$$\int_\Sigma d^3x \sqrt{\gamma} n^\mu j_\mu$$

has no nice "Stokes' theorem" to apply.

A more natural place to start is the Einstein equations. Let us define

$$j_\mu^{\text{Ricci}} = \frac{K^\nu R_{\mu\nu}}{8\pi G} = K^\nu \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right).$$

Since K is a Killing vector, we have $\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu$. Then we have

$$j_\mu^{\text{Ricci}} = \frac{1}{8\pi G} \nabla^\nu (\nabla_\mu K_\nu),$$

which is secretly some divergence. We can now apply Stokes' theorem directly and define the **Komar mass** as

$$M_{\text{Komar}} = \int_{\Sigma} d^3x \sqrt{\gamma} n_{\mu} j^{\mu}_{\text{Ricci}} = \frac{1}{8\pi G} \int_{\partial\Sigma} d^2\tilde{x} \sqrt{\tilde{\gamma}} n_{\mu} \sigma_{\nu} \nabla^{\mu} K^{\nu}.$$

So this $\nabla^{\mu} K^{\nu}$ is the analogue of the field, which obeys the Gauss law.

The same argument works for other Killing vectors. We have $L = \partial_{\varphi}$ as the Killing vector, and so we may define

$$j_{\mu}^{\text{rot}} = L^{\nu} R_{\mu\nu}, \quad J = \frac{1}{8\pi G} \int_{\partial\Sigma} d^2\tilde{x} \sqrt{\tilde{\gamma}} n^{\nu} \sigma^{\mu} \nabla_{\nu} L_{\mu}.$$

21.2 Rotating black hole

This is very relevant to astrophysics, since many stars have large angular momentum, especially compared to the charge they carry. We have only two Killing vectors ∂_t and ∂_{φ} . The **Kerr metric** is given by

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2GM r}{\rho^2}\right) dt^2 - \frac{2GM a r \sin^2 \theta}{\rho^2} (2d\varphi dt) + \frac{\rho^2}{\Delta} dr^2 \\ & + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)^2 - a^2 \Delta \sigma^2 \theta) d\varphi^2, \\ \Delta = & r^2 - 2GM r + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \end{aligned}$$

where M is the Komar energy and $a = J/M$. Note that in the limit $a \rightarrow 0$, this goes to the Schwarzschild metric.

The limit $M \rightarrow 0$ is more instructive. We should get back flat space, but it is going to be in a strange coordinates:

$$\begin{aligned} ds^2 = & -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\ & + \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} ((r^2 + a^2)^2 - a^2 (r^2 + a^2) \sin^2 \theta) d\varphi^2 \\ = & -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + \sin^2 \theta (r^2 + a^2) d\varphi^2. \end{aligned}$$

The change of coordinates not too complicated and is given by

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \varphi, \quad y = \sqrt{r^2 + a^2} \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

The constant r are given by ellipsoids, with $r = 0$ being a disc on the $z = 0$ plane of radius a .

22 April 14, 2017

22.1 Experimental tests

If you go back to Newton's time, one of the main thing people tried to do were measuring G_N . This is because this constant is really small, and still up to today this is the least well known fundamental constant, with few significant digits. Then they tested the $1/r^2$ law. Again, the precision is not great.

- (0) The zeroth experimental test in general relativity was the equivalent principle, and this was done by Galileo. He tested that acceleration due to gravity is universal.
- (1) The experiment that separated general relativity from Newtonian gravity is the precession of orbits. There will be precession of orbits due to other planets, but people know how to calculate this and there were some discrepancy with observations. Einstein computed the general relativity effect, and this made up for the discrepancy.
- (2) The next experiment was gravitational lensing (originally called bending of light). This was done by Eddington in 1919. This was done by observing stars near the sun at a total eclipse. The accuracy was only ten percent. Newtonian gravity also has a naïve effect of bending light, but the relativity effect is twice that effect.
- (3) These were old experiments. More recently, gravitational redshift was detected by Pound–Rebka in 1959, here at Jefferson. They verified that the wavelength of light that goes from the bottom of Jefferson to the top of Jefferson increases.
- (4) There is also the Shapiro delay, the retardation of light, which gives a time delay when light goes near a massive object.
- (5) There is also the precession of gyroscopes, which I won't say much.
- (6) There is detection of gravitational waves, done first by Hulse–Taylor in 1974. In a rotating binary system, the objects must emit gravitational pulses and decrease in energy. They observed that the period decreased as they rotated, just as predicted. As you all know, last year LIGO directly detected gravitational waves in 2016.

Now I would like to describe them in more detail.

- (0) The equivalence principle in general relativity means that gravitation is binding. That is, it is not some linear equation in electromagnetism. So energy gravitates and we want to detect this. If you look at the scale of gravity binding energy,

$$\frac{\text{gravity binding energy}}{\text{total energy}} \sim \frac{G_N}{c^2} \rho R^2.$$

When $R \sim 1\text{m}$, then this ratio is about 10^{-23} and we won't be able to detect this. We need bigger objects.

Fortunately, when astronauts went to the moon, they put a mirror on the surface of the moon so that we can use a laser beam to calculate the distance from the earth to the moon. Of course you need to take care of the effect due to the inhomogeneity of the objects, but using this people verified the equivalence principle in 1 part of 10^{12} .

- (2) Assume that light coming from a star passes through a massive object like the Sun, of mass M , at distance b . We want to see what the angle α of deflection is.

We are going to work in the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This is not the Newtonian limit, because in the Newtonian limit we have assumed that things move slowly. But we are dealing with light here.

For now, assume that all sources are static. Let us write

$$h_{00} = -2\Phi, \quad h_{0i} = w_i, \quad h_{ij} = 2s_{ij} - 2\psi\delta_{ij},$$

where s_{ij} is a traceless symmetric tensor. Then the metric can be written as

$$ds^2 = -(1 + 2\Phi)dt^2 + 2w_i dx^i dt + [(1 - 2\psi)\delta_{ij} + 2s_{ij}]dx^i dx^j.$$

Now let us focus on the geodesic motion. Let us write $p^\mu = dx^\mu/d\lambda$, where $\lambda = \tau/m$. Then p^0 is something like energy without the gravitational potential, and p^i is what we are interested in. We have

$$\frac{dp^i}{dt} = E[G^i + (\vec{v} \times \vec{H})^i - 2(\partial_0 h_{ij})v^j - (\partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk})v^j v^k],$$

where $G^i = -\partial_i \Phi - \partial_0 w_i$ and $H^j = \epsilon^{jkl} \partial_k w_l$ is something like a gravito-magnetic field.

We choose a transverse gauge, given by $\partial_i s^{ij} = 0$ and $\partial_i w^j = 0$. We also assume that our source is static and pressureless dust. Then $w_i = 0$ and $w_{ij} = 0$ and $\psi = \Phi$, with $\nabla^2 \psi = 4\pi G\rho$. Hence the metric is

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2).$$

If we write $dx^\mu/d\lambda = K^\mu + \ell^\mu$, then the null ray satisfies

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0.$$

In first order in ℓ^μ , we get

$$2\eta_{\mu\nu} K^\mu \ell^\nu + h_{\mu\nu} K^\mu K^\nu = 0, \quad -K^0 \ell^0 + \vec{\ell} \cdot \vec{K} = ((K^0)^2 + (\vec{K})^2)\Phi = 2\omega^2 \Phi,$$

where $\omega = K^0$.

We can work out the Christoffel symbols as

$$\Gamma_{0i}^0 = \Gamma_{00}^i = \partial_i \Phi, \quad \Gamma_{jk}^i = \delta_{jk} \partial_i \Phi - \delta_{ik} \partial_j \Phi - \delta_{ij} \partial_k \Phi.$$

Then we get

$$\frac{d\ell^0}{d\lambda} = -2\omega(\vec{K} \cdot \vec{\nabla}\Phi), \quad \frac{d\vec{\ell}}{d\lambda} = -2\omega^2 \vec{\nabla}_\perp \Phi,$$

where $\vec{\nabla}_\perp = \vec{\nabla} - \vec{K}/|\vec{K}|^2(\vec{K} \cdot \vec{\nabla})$. Then the total deflection angle can be computed as

$$\vec{\alpha} = 2 \int_{\text{path}} \vec{\nabla}_\perp \Phi ds.$$

For the potential $\Phi = -GM_\odot/r$, we would get

$$\alpha = \frac{4GM_\odot}{b}.$$

In the case of the Sun, we have $GM_\odot/c \approx 1.5\text{km}$ and $R_\odot \approx 7 \times 10^5\text{km}$. So the maximal deflection is

$$\alpha_{\text{max}} \approx 1.75\text{arcsec}.$$

- (4) The Shapiro delay comes from the change ℓ^0 . We can compute from the previous analysis that

$$\Delta t = \int \ell^0 d\lambda = -2\omega \int \Phi d\lambda = -2 \int \Phi ds.$$

That is, deflection and delay is somewhat equivalent.

- (1) We worked out the orbits of massive objects in the Schwarzschild metric. We solved the radial part:

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{2ML^2}{r^3} = \frac{1}{2}(E^2 - 1).$$

You can also analyze the angular component, and it is a bit involved so I'll tell you the answer. The precession of the orbit is

$$\Delta\varphi = \frac{6\pi G^2 M^2}{L^2} = 0.1\text{arcsec/orbit}.$$

- (3) The gravitational redshift can be also analyzed in the weak field limit. The frequency I see is related to the dot product of the four vector of the observer and the four vector of light. If you work through this analysis, the redshift is given by

$$\frac{w_2}{w_1} \approx 1 + \Phi_1 - \Phi_2.$$

- (6) We are not going to solve the gravitational wave equations. For a binary with two objects of mass $R_s = GM/c^2$, rotating around each other on an orbit of radius R , and observer at distance r will see a gravitational wave of strength

$$h \sim \frac{R_s^2}{rR} \sim 10^{-21}.$$

If you have two objects of distance L , the gravitational wave goes through this gives some change δL in distance. Then

$$\delta L \sim 10^{-18} \text{m} \left(\frac{h}{10^{-21}} \right) \frac{L}{\text{km}}.$$

So if L is like 1km then δL is like 10^{-18} m. How people detected this is using interference of light.

23 April 19, 2017

This was a guest lecture by Xi Yin.

23.1 Geometry of the Kerr metric

We're going to continue talking about the Kerr solution:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GMa r \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] d\phi^2,$$

where $\Delta = r^2 - 2GMr + a^2$ and $r^2 + a^2 \cos^2 \theta$, and also $a = J/M$.

I have this Mathematica package that computes the Ricci tensor. If you compute this out, you get zero. So this is indeed a solution to the vacuum Einstein equation. I won't tell you how to find this solution.

You can generalize it into the **Kerr–Newman solution**, of a rotating charged black hole. To get this, just replace every $2GMr$ with $2GMr - GQ^2$. In this case, the energy-stress tensor is given by

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma},$$

where the gauge field is given by

$$A_t = \frac{Qr}{\rho^2}, \quad A_\phi = -\frac{Qar \sin^2 \theta}{\rho^2}.$$

You can check that this satisfies Einstein equations.

Anyways, let's study the Kerr solution. Fix a and take the limit $M \rightarrow 0$. Then the metric becomes

$$ds^2 \rightarrow -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + \sin^2 \theta (r^2 + a^2) d\phi^2.$$

This is supposed to be a flat metric, and this is called the **Boyer–Lindquist coordinates** for \mathbb{R}^3 .

There are two Killing vectors $K = K^\mu \partial_\mu = \partial_t$ and $R = R^\mu \partial_\mu = \partial_\phi$. Surprisingly, there is a third Killing tensor $\sigma_{\mu\nu}$ (obeying the equation $\nabla_{(\rho} \sigma_{\mu\nu)} = 0$) defined by

$$\begin{aligned} \sigma_{\mu\nu} &= 2\rho^2 \ell_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}, \\ \ell &= \ell^\mu \partial_\mu = \frac{1}{\Delta} ((r^2 + a^2) \partial_t + \Delta \partial_r + a \partial_\phi) \\ n &= n^\mu \partial_\mu = \frac{1}{2\rho^2} ((r^2 + a^2) \partial_t - \Delta \partial_r + a \partial_\phi). \end{aligned}$$

What is the significance of a Killing tensor? Let $u^\mu = dx^\mu/d\tau$ be any particle along a geodesic. Then $u^\rho \nabla_\rho u^\mu = 0$ and so

$$\frac{d}{d\tau}(\sigma_{\mu\nu} u^\mu u^\nu) = u^\rho \nabla_\rho (\sigma_{\mu\nu} u^\mu u^\nu) = \nabla_\rho \sigma_{\mu\nu} u^\rho u^\mu u^\nu = 0.$$

So for a particle moving on a geodesic, there are three preserved quantities: energy given by contracting with K , angular momentum given by contracting with R , and the **Carter constant**

$$C = \sigma_{\mu\nu} u^\mu u^\nu = P_\theta^2 + \cos^2 \theta \left(a^2 (m^2 - E^2) + \left(\frac{L}{\sin \theta} \right)^2 \right).$$

So these three conserved quantities make the system integrable.

Now let us look at the causal structure. There is the Killing horizon, where $K = \partial_t$ becomes null. This is where $g_{tt} = 0$,

$$r^2 + a^2 \cos^2 \theta - 2GM r = 0.$$

This does not coincide with the event horizon, which is where $g^{rr} = 0$. This is equivalent to $\Delta = 0$ and so the locus is given by the equation

$$r^2 + a^2 - 2GM r = 0.$$

So the event horizon is contained inside the Killing horizon.

Something strange happens between the event horizon and the Killing horizon, which is called the **ergosphere**. The Killing vector becomes spacelike and so there are these matter with negative energy but can communicate.

Let us see what happens as $r \rightarrow 0$. If you think the case of $M \rightarrow 0$, the locus $r = 0$ looks like a disc. And it is not even true that all $r = 0$ is a singularity, because the singularity occurs at $\rho = 0$. This is when $r = 0$ and $\theta = \pi/2$. So the singularity looks like a ring. The Penrose diagram looks like the following picture.

Consider a particle on the plane $\theta = \pi/2$, inside the ergosphere. Suppose this is moving in the ϕ direction. Then the metric looks like

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2.$$

So the null vectors are going to have

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}} - \frac{g_{tt}}{g_{\phi\phi}} \right)}.$$

Outside the Killing horizon, one is positive and one is negative. If you emit light, one goes clockwise, and one goes counterclockwise. But once you go into the Killing horizon, both solutions are positive. So both light directions move in the same direction. Also inside the ergosphere, there is a minimal angular velocity. At the event horizon, it is going to be

$$\Omega_H = \left(\frac{d\phi}{dt} \right) \Big|_{r=r_+} = \frac{a}{r_+^2 + a^2}.$$

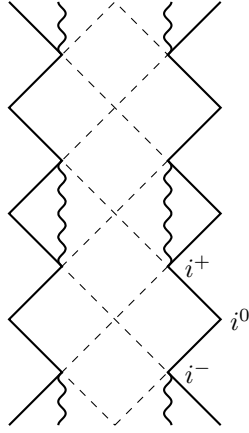


Figure 13: Penrose diagram of the Kerr solution

What about the energy and angular momentum of a particle with momentum $p^\mu = m \frac{dx^\mu}{d\tau}$? We have

$$E = -K_\mu p^\mu = -p_t = -(g_{tt}p^t + g_{t\phi}p^\phi) = -m \left(g_{tt} \frac{dt}{d\tau} + g_{t\phi} \frac{d\phi}{d\tau} \right),$$

$$L = R_\mu p^\mu = p_\phi = g_{t\phi}p^t + g_{\phi\phi}p^\phi = \dots$$

Inside the ergosphere, the Killing vector K_μ is spacelike and so a particle can have negative energy. But you can still communicate with the observer at infinity. So you can go into the ergosphere with a spaceship, create some particles with negative energy, and then come out leaving those particles. Then your total energy will increase. This was shown by Penrose to be possible, and is called a **Penrose process**. Physically, the total energy has to be conserved, and so you can think of it as extracting energy from the black hole.

24 April 21, 2017

This lecture also was given by Xi Yin. We have seen that in the Killing horizon, the light cone pointed to one angular direction. Let us write this minimal angular velocity at the outer event horizon by

$$\Omega_H = \left(\frac{d\phi}{dt} \right) \Big|_{r=r_+}.$$

24.1 Black hole thermodynamics

Inside the ergosphere, we saw that the energy $-p_{(2)}^\mu K_\mu$ of a particle can be negative, and so it is possible to extract energy from the Kerr black hole. Let us consider the Killing vector

$$\chi = K + \Omega_H R = \partial_t + \left(\frac{d\phi}{dt} \right) \Big|_{r=r_+} \frac{\partial}{\partial \phi}.$$

Then this Killing vector is null on the Killing horizon. This means that for a particle to reach the event horizon, it must have

$$-E_{(2)} + \Omega_H L_{(2)} = p_{(2)}^\mu \chi_\mu < 0.$$

That is, it must satisfy $L_{(2)} < E_{(2)}/\Omega_H$. So if we denote by δM and δL the change of the energy and the angular momentum, the inequality

$$\delta J < \frac{\delta M}{\Omega_H}$$

is always satisfied. This is why the Penrose process is not possible for the Schwarzschild metric.

There is another interesting interpretation, of increasing horizon area. We have

$$ds_{\text{Kerr}}^2 \Big|_{r=r_+} = \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (r_+^2 + a^2)^2 d\phi^2.$$

So

$$\text{Area}(\text{event horizon}) = \int \rho d\theta \wedge \frac{\sin \theta}{\rho} (r_+^2 + a^2) d\phi = 4\pi(r_+^2 + a^2).$$

Now recall that $a = J/M$ and $r_+ = GM + \sqrt{G^2 M^2 - a^2}$. We can calculate

$$\begin{aligned} \delta A &= 4\pi \left(2r_+ \left(\frac{\partial r_+}{\partial M} \delta M + \frac{\partial r_+}{\partial J} \delta J \right) + \frac{2J\delta J}{M^2} - \frac{2J^2}{M^3} \delta M \right) = \dots \\ &= 8\pi G \frac{a}{\Omega_H \sqrt{G^2 M^2 - a^2}} (\delta M - \Omega_H \delta J) > 0 \end{aligned}$$

when $\delta J < \delta M/\Omega_H$.

Let us call $\kappa = \Omega_H \sqrt{G^2 M^2 - a^2}/a$. Then

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J.$$

This κ is actually the surface gravity, satisfying $\chi^\mu \nabla_\mu \chi^\nu - \kappa \chi^\nu$. This equation is analogous to the law of thermodynamics $\delta E = T \delta S - P \delta V$. This is just an analogy, but Hawking later discovered that there is a notion of temperature of a black hole. Then we can really define

$$T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4G}.$$

24.2 Friedmann–Robertson–Walker metric

Einstein's equation is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}.$$

Let us consider this equation in the case of a perfect fluid, i.e., when there exists a four-vector U^μ such that

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}.$$

Then we would have $T^\mu{}_\nu = (p + \rho)U^\mu U_\nu + p \delta^\mu{}_\nu$, which will be a diagonal matrix in the case $U^\mu = (1, 0, 0, 0)$ and $g = \eta$.

Now we are going to look at metrics that takes the form of

$$ds^2 = -dt^2 + a(t)^2(\gamma_{ij}(u)du^i du^j).$$

In particular, we are going to make the ansatz that it is

$$\gamma_{ij}(u)du^i du^j = \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2,$$

i.e., has constant spatial curvature.

If you compute the energy-stress tensor, you get one conservation equation. We consider the three simple cases where $p = w\rho$:

- $w = 1/3$ - radiation
- $w = 0$ - dust
- $w = -1$ - cosmological constant, “dark energy”

The conservation equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a}.$$

Then $\rho(t) = \text{const}(a(t))^{-3(1+w)}$. So in the three simple cases we consider, if $w = 1/3$ then $\rho \propto a^{-4}$, if $w = 0$ then $\rho \propto a^{-3}$, if $w = -1$ then $\rho \propto a^0$.

If you look at the Einstein equations, you get

$$\frac{\ddot{a}}{a} = -4\pi G \left(w + \frac{1}{3}\right)\rho, \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}.$$

We call $H = \dot{a}/a$ the Hubble parameter.

25 April 26, 2017

25.1 Review on Penrose diagrams

Given a 2-dimensional slice of spacetime, in most cases $\theta, \varphi = \text{const}$, a Penrose diagram is a diagram having the flat metric $ds^2 = -d\tau^2 + d\sigma^2$ conformal to the physical spacetime slice $ds_{\text{real}}^2 = e^{(\dots)(\tau, \sigma)}(-d\tau^2 + d\sigma^2)$.

Let us work this out explicitly. Consider a general metric

$$ds^2 = -adt^2 + 2bdt dx + cdx^2,$$

where a, b, c are functions in x, t . We want to find null coordinates u and v . We can try $d\tilde{u} = dt + \alpha dx$ and $d\tilde{v} = dt + \beta dx$ so that $ds^2 = -ad\tilde{u}d\tilde{v}$.

But this doesn't work, because it might not be possible to find the \tilde{u} such that $d\tilde{u} = dt + \alpha dx$. So we have to put another factor and write $du = A(dt + \alpha dx)$. In order for $d(du) = 0$ as differential forms, we need to pick A such that $\partial A / \partial x = \partial(A\alpha) / \partial t$.

After conformally changing it to a flat metric, we can draw the diagram. But our coordinates might not be geodesically complete. To extend the metric to the other parts of space, we need to somehow assume some symmetry, and extend the metric analytically according to the symmetry. In this case, the formula is very suggestive of what the metric should be.

25.2 More on black hole thermodynamics

In the discussion of black hole thermodynamics, you will need to take some thing in faith. The usual first law of thermodynamics states that

$$dE = TdS - PdV.$$

There is an analogous formula for the Kerr black hole. If we throw something in a black hole, the mass becomes $M_{\text{b.h.}} \rightarrow M + dM$ and the angular momentum also becomes $J_{\text{b.h.}} \rightarrow J + dJ$. Let $A(M, J)$ be the area of the horizon. Then after some calculus one finds

$$\delta A = 8\pi G \frac{J/M}{\Omega_H \sqrt{G^2 M^2 - J^2/M^2}} (\delta M - \Omega_H \delta J).$$

It follows that

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J, \quad \kappa = \frac{\sqrt{G^2 M^2 - J^2/M^2}}{2GM(GM + \sqrt{G^2 M^2 - J^2/M^2})}.$$

This κ is called **surface gravity** and is the acceleration of an object near the horizon as observed at infinity.

Let χ^μ be the Killing vector that is timelike at ∞ . The Killing horizon is where $\chi_\mu \chi^\mu = 0$. If we look at the propagation of χ^μ along its direction, it is going to be

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa^\nu = \frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu).$$

We can normalize χ so that $\chi_\mu \chi^\mu \rightarrow -1$ as $r \rightarrow \infty$.

For a static spacetime, and a static observer, we have $U^\mu = \chi^\mu / \sqrt{-\chi_\mu \chi^\mu} = \chi^\mu / V$. This V is the redshift factor. The acceleration the static observer to their own frame is going to be

$$a^\mu = U^\sigma \nabla_\sigma U^\mu = \frac{1}{V} \nabla_\mu V = \frac{\sqrt{\nabla_\mu \chi_\nu \nabla^\mu \chi^\nu}}{2V}.$$

Then as seen from $r \rightarrow \infty$, we again have to take into account the redshift factor. Then we have $\kappa = Va = \frac{1}{2} \sqrt{\nabla_\mu \chi_\nu \nabla^\mu \chi^\nu}$.

Now M is like E . This is perfectly natural. Now $\Omega_H \delta J$ is like the work done on the black hole. Then we can identify

$$M \longleftrightarrow E, \quad \frac{A}{4G} \longleftrightarrow S, \quad \frac{\kappa}{2\pi} \longleftrightarrow T.$$

Under this identification, all the thermodynamic laws hold. The zeroth law says that for a stationary black hole, κ is constant along the horizon. The second law can be written as $\Delta A \geq 0$. This was true for Penrose processes, but this is also true for black hole mergers. The area of the merged black hole is larger than the sum of the individual black holes, although the mass can decrease by ejecting gravitational waves.

The real world involves gravity and actual entropy. So the generalized second thermodynamic law as

$$\Delta \left(\frac{A}{4G_N} + S \right) \geq 0.$$

It is good that black holes have nonzero entropy, because we cannot take a system of high entropy and then throw it into the black hole. This shouldn't violate the second law of thermodynamics.

Black holes actually have some radiation. The temperature is around κ , which is $1/4GM = 1/2R_s$ in the case of the Schwarzschild metric. This shows that the principal wavelength should be around the Schwarzschild radius. Now vacuum has these virtual particle pairs generated and collapsing. If one of these particles happen to fall into the horizon, then they can come together. Then this other particle should be the radiation. This is called the **Hawking process**, and this is similar to the quantum field theoretic prediction that if you have a strong electric field in vacuum, you can produce electron-positron pairs.

Now because of the radiation, the black hole eventually evaporates and disappears. The virtual particle pairs can be written as

$$|00\rangle + |11\rangle,$$

and after the Hawking process, we have to take the trace over inside the horizon. Then we get a density matrix. This seems to violate quantum mechanics, because the final state has to be a unitary operator times the initial state. This is called the **black hole information paradox**.

There are several possible resolutions of this paradox.

- This violates quantum mechanics: this is not very satisfying.
- The remnant remains at the scale of Planck mass. But then we would need $\exp(S_{\text{b.h.}})$ many possible microstates at a very small space.
- Quantum mechanics win and Hawking's calculation is wrong: $|\psi\rangle_{\text{final}} = U|\psi\rangle_{\text{star}}$.

There is this idea that the Einstein–Rosen bridge somehow gives a hint to this paradox.

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