Math 25a - Honors Linear Algebra and Real Analysis I

Dongryul Kim

Fall 2015

Contents

1 September 2, 2014

1.1 Introduction

Math 25a is abstract and rigorous linear algebra.

- Math 23 - similar material, less abstract, more examples
- Math 25 - more abstractly, less examples
- Math 55 - even more abstract, also treat algebra

This class is going to consume your life, i.e. you will use 30 hours a week out of class for this course. Take it only if you are fascinated by the material covered in the course. We will start with set theory and other basic stuff, and then speed up.

How are you supposed to study:

- Be present each class.
- Take careful notes of what is written, and as much as possible of what is said; If you miss class, borrow and rewrite notes.
- After class, read through your notes, line-by-line, and understand *definitions, theorems, proofs*.
- Be honest with yourself.
- Try to solve the homework problems *alone*.
- Try to explain the material to someone else (in the class).
- Seek help from CAs and me.

It is a "tall" subject, so you cannot understand anything without knowing the whole thing.

Some mistakes:

- Believe you understand, because you have some vague pricier, repeat words.
- Ask CAs for help with homework.

Formally saying, you can collaborate with your peers, but you must write down the solutions by yourself.

There are three book recommended for this course. But none are required.

- Axler, *Linear algebra done right*, 3rd edition
- Lax, *Linear algebra and appplications*, 2nd edition
- Hoffman-kunze, *Linear algebra*, 2nd edition

Grading policy:

Homework	30%
Midterm	30%
Final	40%

1.2 Basic set theory

Definition 1.1 (Cantor)**.** *A **set** is a collection of definite and distinct objects of our perception or our mind.*

Example 1.2.

$$S = \{\diamondsuit, \heartsuit, \clubsuit, \spadesuit\}$$

$$C = \textit{Standard deck of cards}$$

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\emptyset = \{\}$$

Given a set A, we write $a \in A$, if a is an object belonging to A, and $a \notin A$ if a is not an object belonging to A.

Definition 1.3. *If A and B are sets, then we write $A \subset B$, and say A is a **subset** of B, if*

$$a \in A \Rightarrow a \in B.$$

Example 1.4. *We have $\{\heartsuit, \diamondsuit\} \subset S$, $S \subset S$, and $\emptyset \subset S$.*

Definition 1.5. *Two sets A, B are equal, $A = B$, if $A \subset B$ and $B \subset A$.*

Definition 1.6. *$A \subsetneq B$ if $A \subset B$ and $A \neq B$. We way A is a **proper subset** of B.*

Sometimes authors use $\subseteq$ for subsets and $\subset$ for proper subsets, but in this class, we are going to use $\subset$ and $\subseteq$ interchangeably just for subsets.

1.3 Set operations

Let A, B be sets. Define

(union)	$A \cup B = \{a : a \in A \text{ or } a \in B\}$
(intersection)	$A \cap B = \{a : a \in A \text{ and } a \in B\}$
(difference)	$A \setminus B = \{a : a \in A \text{ or } a \notin B\}$
(symmetric difference)	$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

We will use these basic operations throughout the course.

2 September 4, 2015

2.1 More set theory

Last week we introduced sets and set operations. Recall the definitions.

$$A \cup B = \{a \mid a \in A \text{ or } a \in B\}$$
$$A \cap B = \{a \mid a \in A \text{ and } a \in B\}$$
$$A \setminus B = \{a \mid a \in A \text{ and } a \notin B\}$$

Proposition 2.1. *Let A, B, and C be sets. Then*

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. For (i), we first prove $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. This is true because

$$\begin{aligned} a \in A \cap (B \cup C) &\Leftrightarrow a \in A \text{ and } a \in B \cup C \\ &\Leftrightarrow a \in A \text{ and } (a \in B \text{ or } a \in C) \\ &\Leftrightarrow (a \in A \text{ and } a \in B) \text{ or } (a \in A \text{ and } a \in C) \\ &\Leftrightarrow a \in A \cap B \text{ or } a \in A \cap C \\ &\Leftrightarrow a \in (A \cap B) \cup (A \cap C). \end{aligned}$$

But since each step is '$\Leftrightarrow$', we actually get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(ii) is left as an exercise. □

Proposition 2.2. *Let A, B, and C be sets. Then*

(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. Exercise. □

Definition 2.3. *If I is any set and $\{A_i\}_{i \in I}$ is a collection of sets, then*

$$\bigcup_{i \in I} A_i = \{a \mid a \in A_i \textit{ for at least one } i\},$$

$$\bigcap_{i \in I} A_i = \{a \mid a \in A_i \textit{ for all } i \in I\}.$$

Definition 2.4. *Given two sets A, B, their Cartesian product is defined as*

$$A \times B = \{(a, b) \mid a \in A \textit{ and } b \in B\}.$$

Example 2.5. *Let $S = \{\diamondsuit, \heartsuit, \clubsuit, \spadesuit\}$ and C = standard deck of cards. Then*

$$C = S \times \{2, 3, 4, \ldots, 10, J, Q, K, A\}.$$

2.2 Relations

Definition 2.6. *A* ***relation*** *between two sets A, B is any subset $\mathcal{R} \subset A \times B$.*

We call this a relation, because given $a \in A, b \in B$, we think of a being relate to b if $(a, b) \in \mathcal{R}$. We also write $a\mathcal{R}b$.

Example 2.7. *Let $A = B = \{1, 2, 3\}$. Some relation can be $\mathcal{R} = \{(1, 2), (1, 3), (2, 3)\}$. This is the $<$ relation.*

The relation $\mathcal{R} = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ is the $\leq$ relation.

And the relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ is the $=$ relation.

The most important relations (for now) are the equivalence relations.

Definition 2.8. *A relation $\mathcal{R} \subset A \times A$ is an* ***equivalence relation****, if*

(reflexive)	*$a\mathcal{R}a$ for all $a \in A$,*
(symmetric)	*$a\mathcal{R}b$ if and only if $b\mathcal{R}a$ for all $a, b \in A$,*
(transitive)	*$a\mathcal{R}b$ and $b\mathcal{R}c$ imply $a\mathcal{R}c$ for all $a, b, c \in A$.*

Definition 2.9. *If $\mathcal{R} \subset A \times A$ is an equivalence relation, define for $a \in A$ the* ***equivalence class***

$$[a]_{\mathcal{R}} = \{b \in A \mid b\mathcal{R}a\}.$$

Proposition 2.10. *Let $\mathcal{R}$ be a equivalence relation on A. Then for $a, b \in A$,*

1. *If $b \in [a]_{\mathcal{R}}$, then $[b]_{\mathcal{R}} = [a]_{\mathcal{R}}$.*
2. *Either $[a]_{\mathcal{R}}$ or $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} = \emptyset$.*

Proof. We first show $[a]_{\mathcal{R}} \subset [b]_{\mathcal{R}}$. If $c \in [a]_{\mathcal{R}}$ then $c\mathcal{R}a$. But $b\mathcal{R}a$, and hence $a\mathcal{R}b$ by symmetry. So $c\mathcal{R}b$ by transitivity and $c \in [b]_{\mathcal{R}}$. To show $[b]_{\mathcal{R}} \subset [a]_{\mathcal{R}}$, we use the above argument with $a\mathcal{R}b$.

For the second part, we show both statements cannot hold simultaneously by contradiction. Given $a, b \in A$, we get $\emptyset = [a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} = [a]_{\mathcal{R}}$, but $a \in [a]_{\mathcal{R}}$. Therefore we arrive at a contradiction.

Let us then assume $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset$. There exists a $c \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{R}}$, i.e., $c \in [a]_{\mathcal{R}}$ and $c \in [b]_{\mathcal{R}}$. Then $c\mathcal{R}b$ and $c\mathcal{R}b$, so $a\mathcal{R}c$ and $c\mathcal{R}b$, so $a\mathcal{R}b$, so by the first part, $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$. □

Two equivalence classes are either equal or disjoint. We sometimes denote $A \cup B = A \dot{\cup} B$ if A and B are disjoint. So with an abuse of notation, we can write

$$A = \dot{\bigcup}_{a \in A} [a]_{\mathcal{R}}.$$

Definition 2.11. *If $\mathcal{R}$ is an equivalence relation on A, define the* ***quotient*** *of A by $\mathcal{R}$ as*

$$A/\mathcal{R} = \{[a]_{\mathcal{R}} \mid a \in A\}.$$

This is actually what we do all the time without noticing. For instance, when we think of colors, we don't think of the humongous set of all object with a certain color. Rather, we think it as an abstract thing.

Example 2.12. *Let $A \in \mathbb{N}$ and define $\mathcal{R}$ as $a\mathcal{R}b$ if and only if $a+b$ is even. Then*

$$A/\mathcal{R} = \{\{2,4,6,\dots\},\{1,3,5,\dots\}\}.$$

Index