# Math 253y - Symplectic Manifolds and Lagrangian Submanifolds

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Better notes are available on the course webpage `http://www.math.harvard.edu/~auroux/253y18/`.

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## Contents

# 1 September 4, 2018

The main goal of this class is to learn symplectic manifolds, Lagrangian submanifolds, pseudo-holomorphic curves, Floer homology, Fukaya categories, etc. This is not a beginning course on symplectic geometry. For people who do need grade, I will try to have homeworks.

## 1.1 Overview I

Let us start with reviewing basic symplectic geometry.

**Definition 1.1.** A **symplectic manifold** $(M^{2n}, \omega)$ is an even-dimensional manifold with $\omega \in \Omega^2(M)$ that is closed, i.e., $d\omega = 0$, and non-degenerate, i.e., $\omega : TM \cong T^*M$. This non-degenerate condition is equivalent to $\omega^n \neq 0$.

**Example 1.2.** Oriented surfaces with an area form, are symplectic manifolds. In higher dimension, there is $M = \mathbb{R}^{2n}$ with the canonical symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. You can also take the cotangent bundle $M = T^*N$ of an arbitrary manifold $N$. Here, there is

$$\omega = d\lambda, \quad \lambda = pdq$$

where $q$ are coordinates on the base and $p$ are coordinates on the fiber. This $\lambda$ is called the **Liouville form**. Complex projective space $M = \mathbb{C}P^n$ has a Fubini–Kähler form, and so do smooth complex projective varieties.

**Theorem 1.3** (Darboux). *For all $p \in M$, there exists a neighborhood $p$ and local coordinates $(x_i, y_i)$ such that $\omega = \sum_i dx_i \wedge dy_i$.*

So there are no local invariants. On the other hand, there is an obvious global invariant $[\omega] \in H^2(M, \mathbb{R})$.

**Theorem 1.4** (Moser). *If $M$ is compact closed, and $(\omega_t)_{t\in[0,1]}$ are a continuous symplectic family with $[\omega_t] \in H^2(M, \mathbb{R})$ independent of $t$, then there exists an isotopy $\varphi_t \in \mathrm{Diff}(M)$ such that $\varphi_t^*\omega_t = \omega_0$. In particular, $\varphi_1 : (M, \omega_0) \cong (M, \omega_1)$.*

In this field, the group of symplectomorphisms is very large. Inside $\mathrm{Symp}(M, \omega)$, there is a subgroup

$$\mathrm{Ham}(M, \omega) = \text{flows of (time-dependent) Hamiltonian vector fields.}$$

Given any $H \in C^\infty(M.\omega)$, there exists a unique vector field $X_H$ such that $\omega(X_H, -) = -dH$. This vector field preserves the symplectic form, because

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H} d\omega = 0$$

by the Cartan formula.

Lagrangian submanifolds are your savior if you think there are no interesting thing to do, due to Darboux's theorem.

**Definition 1.5.** A **Lagrangian submanifold** is $L^n \subseteq M^{2n}$ such that $\omega|_L = 0$.

**Example 1.6.** In $(\mathbb{R}^{2n}_{x_i,y_i}, \omega_0)$, the space $\mathbb{R}^n_{x_i}$ is a Lagrangian submanifold. On a surface, any simple closed curve is a Lagrangian submanifold. In $T^*N$, there is the zero section, and there are also the cotangent fibers. In $\mathbb{R}^{2n}$, the product $\prod S^1(r_i)$ is Lagrangian. More generally, the $T^n$-orbits in a toric symplectic manifold is Lagrangian.

Observe that $(TL)^{\perp\omega} = TL$. So we get an isomorphism

$$NL = (TM|_L)/TL \to T^*L; \quad [v] \mapsto \omega(v, -)|_{TL}.$$

**Theorem 1.7** (Weinstein)**.** *A neighborhood of $L$ in $M$ is symplectomorphic to a neighborhood of of the zero section in $T^*L$.*

So deformations of $L \subseteq M$ corresponds to sections of $T^*L$. But for $\alpha \in \Omega^1(L)$, its graph is Lagrangian if and only if $\alpha$ is closed. Moreover, the deformation is an Hamiltonian isotopy if and only if $\alpha$ is exact. This leads to the notion of a flux, lying in $H^1(L, \mathbb{R})$. For $L = S^1$, this is equal to the area swept, and it vanishes if and only if the isotopy is Hamiltonian.

What kinds of Lagrangian submanifolds $L$ exist in a given symplectic manifold $(M, \omega)$? For example, on a oriented surface, Lagrangians are simple closed curves. To consider Hamiltonian isotopies, you need to keep track of the area swept.

**Conjecture 1.8** (Arnold, nearby)**.** *Let $N$ be a closed manifold. In its cotangent bundle, look at closed exact $L \subseteq T^*N$. Then $L$ is Hamiltonian isotopic to the zero section.*

What is exact? Recall that $\omega = d\lambda$. Then $L$ is Lagrangian if and only if $\lambda|_L$ is closed. We say that $L$ is exact Lagrangian if and only if $\lambda|_L = df$ is exact. If we know that $L$ is a section, it is easy. The conjecture is proved for $T^*S^1$, which is easy, and for $T^*S^2$ (2013) and $T^*T^2$ (2016). On the other hand, the homology was known for a bit longer.

**Theorem 1.9** (Abonzaid–Kragh, 2016)**.** *Let $L \subseteq T^*N$ be a closed exact Lagrangian, and consider the projection $\pi_L : L \to N$. Then $\pi_L$ is a (simple) homotopy equivalence.*

What about in $\mathbb{R}^4$? Let us look at closed compact Lagrangians $L \subseteq \mathbb{R}^4$. If $L$ is orientable, then $L \cong T^2$. (This is because the normal bundle is isomorphic to the cotangent bundle, and then you do some computations.) If $L$ is not orientable, we should have $\chi(L) < 0$ and divisible by 4. (The Klein bottle was excluded by Nemirovski in 2006.) So these problems are surprisingly hard. All known Lagrangian tori in $\mathbb{R}^4$ are Hamiltonian isotopic to

- a product torus $S^1(r_1) \times S^1(r_2)$,
- Chekanov (v1990) torus $T_{\mathrm{Ch}}(r)$.

But we don't know if these are all.

The manifold $\mathbb{R}^{2n}$ is a cotangent bundle, so we can talk about exact Lagrangians. It is a theorem of Gromov that there is no closed exact Lagrangian in $\mathbb{R}^{2n}$. If $L$ is exact, then any disk bound by $L$ has zero area. But Gromov showed that $L \subseteq \mathbb{R}^{2n}$ must bound holomorphic discs, and these have positive area. The next best are monotone Lagrangians. These are such that the symplectic area of a disc bound by $L$ is positively proportional to its Maslov index.

In $\mathbb{C}P^2$, which is a toric manifold, we know about product tori. Monotone ones are $\{(x : y : 1) : |x| = |y| = 1\}$. There is also the Chekanov monotone torus, which bounds more holomorphic discs. In 2014, R. Vianna showed that there are infinitely many types of monotone Lagrangian tori.

In $\mathbb{R}^6$, there is a result of Fukaya that states that monotone closed Lagrangians must be diffeomorphic to $S^1 \times \Sigma_g$. On the other hand, there is a construction due to Ekholm–Evashberg–Murphy–Smith 2013 that any $N\#(S^1 \times S_2)$ has a Lagrangian embedding into $\mathbb{R}^6$.

**Theorem 1.10.** *There exist infintely many different families of monotone Lagrangian $T^3 \subseteq \mathbb{R}^6$.*

All known ones are Lagrangian isotopic to product tori. In $\mathbb{R}^8$, there exist knotted monotone Lagrangian tori $T^4$. There is a bunch of complicated results like this.

# 2 September 6, 2018

Last time I started with and overview of all the things that will appear in the class.

## 2.1 Overview II

The latter part of the course will use $J$-holomorphic curves to study Lagrangians. These are key tools to study Lagrangians in modern symplectic geometry. In general, there is no reason for a symplectic manifold to carry of complex structure. But they carry almost-complex structures $J : TM \to TM$ with $J^2 = -1$ compatible with the symplectic structure. Here, compatibility means that $\omega(-, J-)$ is a Riemannian metric. The choice is contractible, so it is not too important.

**Definition 2.1.** A ***J*-holomorphic curve** is a smooth map from a Riemann surface

$$u : (\Sigma, j) \to (M, J)$$

that satisfy $\overline{\partial}_J u = 0$, i.e., $J \circ du = du \circ j$.

The Riemann surface $\Sigma$ can have boundaries, and then we sill require that the boundary of $\Sigma$ maps to a given Lagrangian in $M$. The space of maps will have finite expected dimension, because $\bar{\partial}$ is Fredholm. Using an index theorem, we can compute the dimension of the moduli space $\mathcal{M}(\Sigma, J, [u])$, where $[u]$ is the homology. Then Gromov compactness says that this moduli space has a suitable compactification. The area of the $J$-holomorphic curve with respect to the metric $g$ is equal to the symplectic area, which is $\langle [\omega], [u] \rangle$.

Once we have this notion, we can define Lagrangian Floer (co)homology, invented by Floer. Given two Lagrangians $L_1, L_2$, we will define a chain complex and define

$$HF(L_1, L_2) = H^*(CF(L_1, L_2), \partial).$$

The chain complex is going to be the vector space generated by $L_1 \cap L_2$. We might ask what the coefficient of $q$ in $\partial p$ is. This is going to be a weighted count of $J$-holomorphic

$$u : \mathbb{R} \times [0, 1] \to M$$

such that the ends goes to $q$ and $p$. What Floer showed is that if $L_i$ do not bound any holomorphic discs (for instance, the exact case) then $\partial^2 = 0$ and the cohomology is invariant under Hamiltonian isotopies and of $J$. Moreover, $HF(L, L) \cong H^*(L)$. There is no grading by default, but when they exist, it is going to be a graded isomorphic.

**Corollary 2.2.** *If $L$ does not bound discs, $\#(L, \psi(L)) \geq \dim H^*(L)$ where $\psi \in \mathrm{Ham}(M, \omega)$ and $\psi(L) \pitchfork L$.*

**Example 2.3.** Consider the cylinder and $L_1$ a circle. Suppose we push it around with a Hamiltonian isotopy and get $L_2$, with two intersections $p, q$. Then we have

$$\partial p = q - q = 0,$$

and then $HF(L_1, L_2) = CF = H^*(S^1)$.

**Example 2.4.** Consider the same $L_1$, but now let $L_2$ a boundary of a small disc passing $L_1$. Let the intersection by $p, q$. In this case,

$$\partial p = (\cdots)q, \quad \partial q = (\cdots)p,$$

and so $\partial^2 \neq 0$.

Near the end of the course, we will talk about other disc-counting invariants, e.g., distinguishing exotic monotone Lagrangians by counting holomorphic discs. Then we will also talk about Fukaya categories. This is a way to package all the Lagrangians with intersections in one category. The objects are (nice) Lagrangian submanifolds, with extra data, with morphisms given by Floer complexes and differentials. Composition

$$CF(L_2, L_3) \otimes CF(L_1, L_2) \to CF(L_1, L_3)$$

is given by counting holomorphic discs bound by $L_1, L_2, L_3$. This is really an $A_\infty$-category. The reason this language is useful is because the category can be generated by some Lagrangians we are familiar with.

## 2.2 Symplectic manifolds

A reference for this is *Lectures on symplectic geometry* by A. Cannas da Silva, and *Introduction to symplectic topology* by McDuff–Salamon. Recall that a **symplectic manifold** is a manifold $(M^{2n}, \omega)$ equipped with a real 2-form $\omega$ that is closed and non-degenerate. This also gives a map $\omega_x : T_xM \to T_x^*M$. Also, $\omega^{\wedge n}$ is nonzero, so we get a top exterior form.

**Example 2.5.** Here are some examples:

- An oriented surface $M$ with an area form.
- Euclidean space $M = \mathbb{R}^{2n}$ with $\omega = \sum_i dx_i \wedge dy_i$. Actually, every nondegenerate skew-symmetric bilinear form on a vector space always looks like this. There are other symplectic structures on $\mathbb{R}^{2n}$ but these are because something interesting happens at infinity.
- Take $M = T^*N$, and $\omega = d\lambda$. In local coordinates, take $(q_1, \dots, q_n)$ coordinates on $N$, and take $p_1, \dots, p_n)$ dual coordinates on the fiber. Then consider $\lambda = \sum_{i=1}^n p_i dq_i$, so that $\omega = \sum dp_i \wedge dq_i$. This $\lambda$ is independent on coordinates, and you can check this. Or you can intrinsically define $\lambda$ as

  $$\lambda_{(x,\xi)}(v) = \langle \xi, d\pi(v) \rangle$$

  where $\pi : T^*N \to N$ is the projection.

- Products of symplectic manifolds are $M_1 \times M_2$ with symplectic form $\omega_1 \oplus \omega_2 = \pi_1^*\omega_1 + \pi_2^*\omega_2$.
- Symplectic submanifolds $V \subseteq W$ are submanifolds with $\omega|_{TV}$ is nondegenerate.

Try to think about for which $n$ the sphere $S^{2n}$ has a symplectic form.

A **Lagrangian submanifold** is $L^n \subseteq (M^{2n}, \omega)$ such that $\omega|_L = 0$. In this case,

$$T_x L^{\perp\omega} = \{u \in T_x M : \omega(u, v) = 0 \text{ for all } v \in T_x L\} = T_x L$$

for dimension reasons.

**Example 2.6.** Again, here are examples.

- The zero section in $T^*N$.
- More generally, consider graphs of 1-forms $\alpha \in \omega^1(M, \mathbb{R})$. First, we note that
$$\operatorname{graph}(\alpha) \subseteq T^*N \xrightarrow{\pi} N$$
is a diffeomorphism. Then the restriction of the Liouville form is tautologically
$$\lambda|_{\operatorname{graph}(\alpha)} = \alpha \quad \text{(or rather, } \pi^*\alpha).$$
So $\operatorname{graph}(\alpha)$ is Lagrangian if and only if $d\lambda|_{\operatorname{graph}(a)} = 0$ if and only if $d\alpha = 0$. (It is called an exact Lagrangian if and only if $\alpha$ is exact.)
- The conormal bundle to a smooth submanifold $V^k \subseteq N^n$ is defined as
$$N^*V = \{(x, \xi) : x \in V, \xi \in T_x^*N, \xi|_{T_x N} = 0\} \subseteq T^*N.$$
This is going to be a rank $n - k$ subbundle of $T^*N|_V$. Then $\lambda|_{N^*V} = 0$ because $d\pi(v) \in T_x V$ implies $\lambda(v) = \langle \xi, d\pi(v)\rangle = 0$. So $N^*V$ are exact Lagrangians.
- Let $\varphi \in \operatorname{Diff}(M)$ be a diffeomorphism. Consider
$$\operatorname{graph}(\varphi) = \{(x, \varphi(x)) \in M^- \times M\}.$$
Here, $M^-$ is the symplectic manifold with $-\omega$ instead of $\omega$. Then $\varphi \in \operatorname{Symp}(M, \omega)$ if and only if $\operatorname{graph}(\varphi)$ is a Lagrangian in $M^- \times M$. This is because $i^*\hat{\omega} = -\omega + \varphi^*\omega$.

# 3 September 11, 2018

Today we will do some basic symplectic geometry.

## 3.1 Hamiltonian vector fields

Remember for $(M, \omega)$ a symplectic manifold and $H \in C^\infty(M, \mathbb{R})$, there exists a unique vector field $X_H$ such that

$$\iota_{X_H} \omega = -dH.$$

Recall that given a time-dependent vector field $V_t$, the corresponding flow $\varphi_t$ generated by this is a family of diffeomorphisms

$$\varphi_0(p) = p, \quad \frac{d}{dt}(\varphi_t(p)) = v_t(\varphi_t(p)).$$

Then it is a general fact that

$$\frac{d}{dt}(\varphi_t^* \alpha) = \varphi_t^*(L_{V_t} \alpha).$$

There is also Cartan's formula

$$L_v \alpha = d\iota_v \alpha + \iota_v d\alpha.$$

So given a Hamiltonian $H_t$, we get a flow of $X_{H_t}$ and it satisfies

$$\varphi_t^* \omega = \omega$$

because

$$\frac{d}{dt}(\varphi_t^* \omega) = \varphi_t^*(L_{X_{H_t}} \omega) = \varphi_t^*(d\iota_{X_{H_t}} \omega + \iota_{X_{H_t}} d\omega = \varphi_t^*(-dH_t + 0) = 0.$$

**Definition 3.1.** We define the group of **Hamiltonian diffeomorphisms** $\mathrm{Ham}(M, \omega)$. (This is a group, because if we concatenate, we can reparametrize so that the flow is smooth at the boundary.)

Also note that $dH(X_H) = -\omega(X_H, X_H) = 0$. So the flow of $H$ preserves the level sets of $H$.

**Example 3.2.** Here are some examples:

- Take $\mathbb{R}^2$ with $\omega_0 = dx \wedge dy = rdr \wedge d\wedge$. Let us take the Hamiltonian $H = \frac{1}{2} r^2$. Then the Hamiltonian vector field is

  $$X_H = \frac{\partial}{\partial \theta},$$

  which is rotation.

- On $S^2$, consider the standard area form $\omega_0 = d\theta \wedge dz$. If we take the Hamiltonian $H = z$, then
$$X_H = -\frac{\partial}{\partial\theta}.$$
These are examples of Hamiltonian $S^1$-actions.
- Take $M = T^*N$ with coordinates $(q, p)$. Let us first consider $H = H(q)$, a Hamiltonian that factors through the projection $\pi : T^*N \to N$. It turns out that
$$X_H(q,p) = (0, -dH_{(q)}) \in T_{(q,p)}(T^*N).$$
On the other hand, we can give a Riemannian metric on $N$ so that $TN \cong T^*N$. Consider $H = \frac{1}{2}|p|^2$. Then
$$X_H = \text{geodesic flow}.$$
If you couple these together $H = \frac{1}{2}|p|^2 + V(q)$, we will get the dynamics of a particle with potential $V$.

Hamiltonian vector fields are those with $\iota_X\omega = -dH$ is exact. A **symplectic vector field** is that with $\iota_X\omega$ is closed. The flow still preserves $\omega$. Given a symplectic isotopy $(\varphi_t)$ generated by a symplectic vector field $V_t$, we get the identity component in Symp, and actually $\pi_0\,\mathrm{Symp}$ is also very interesting.

To look at the difference between symplectic flows and Hamiltonian flows, we define
$$\mathrm{Flux}(\varphi_t) = \int_0^1 [-\iota_{V_t}\omega]dt \in H^1(M, \mathbb{R}).$$
Any Hamiltonian isotopy has flux zero. For small enough isotopy, you can also show that a symplectic isotopy with zero flux can be made into a Hamiltonian isotopy.

The flux has a concrete interpretation. Given $\gamma : S^1 \to M$, the image $\varphi_t(\gamma)$ sweeps out a cylinder. Write $\Gamma(s,t) = \gamma_t(\gamma(s))$, and we compute
$$\begin{aligned}\int_\Gamma \omega = \int \Gamma^*\omega &= \int_0^1 \int_{S^1} \omega\Big(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\Big) ds dt \\ &= \int_0^1 \int_{S^1} -\iota_{V_t}\omega\Big(\frac{\partial\Gamma}{\partial x}\Big) ds dt = \int_0^1 \langle[-\iota_{V_t}\omega], [\varphi_t(\gamma)]\rangle dt = \langle \mathrm{Flux}, [\gamma]\rangle.\end{aligned}$$

## 3.2 Moser's theorem

All these theorems use what is called Moser's trick.

**Theorem 3.3** (Moser's theorem)**.** *Let $M$ be a compact closed manifold, and let us say I have $(\omega_t)_{t\in[0,1]}$ a smooth family of symplectic forms, such that $[\omega_t] \in H^2(M, \mathbb{R})$ is independent of $t$. Then there exists an isotopy $\varphi_t \in \mathrm{Diff}(M)$ such that $\varphi_t^*\omega_t = \omega_0$. Hence $\varphi_1$ gives $(M, \omega_0) \cong (M, \omega_1)$.*

*Proof.* If we look at $\frac{d\omega_t}{dt}$, this is going to be exact because they all lie in the same cohomology class. So there exist 1-forms $\alpha_t$ such that

$$d\alpha_t = \frac{d\omega_t}{dt}.$$

Then there exists a smooth family $\alpha_t$. (This requires an explicit version of Poincaré's lemma, and some partition of unity argument.) We now know that there exists a vector field $X_t$ such that

$$\iota_{X_t}\omega_t = -\alpha_t.$$

Let us now take $\varphi_t$ the flow generated by $X_t$. Then

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*\Big(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\Big) = \varphi_t^*\Big(d\iota_{X_t}\omega_t + \frac{d\omega_t}{dt}\Big) = 0$$

because $\iota_{X_t}\omega_t = -\alpha_t$ and $\frac{d\omega_t}{dt} = d\alpha_t$. □

This is only for compact closed manifolds, but if don't assume this, you need to make assumptions at infinity. It is also not true that any two symplectic forms in the same cohomology class are isotopic in the same $[\omega]$. (This is different from the Kähler world, where you can just linearly interpolate.) For instance, McDuff has the following example. On $S^2 \times S^2 \times T^2$, we can take

$$\omega_0 = \pi_1^*\omega_{X^2} + \pi_2^*\omega_{S^2} + ds \wedge dt, \quad \omega_1 = \psi^*\omega_0.$$

Here, $\psi(z, w, s, t) = (z, R_{z,t}(w), s, t)$ where $R_{z,t}$ is the rotation by axis $z$ with angle $t$. They are in the same cohomology class, but there does not exist a $\omega_t$ connecting them in this class. You can find the proof in McDuff's book on $J$-holomorphic curves (in something like section 9.7).

## 3.3 Darboux's theorem

**Theorem 3.4** (Darboux's theorem)**.** *For any $p \in (M, \omega)$, there exist local coordinates near $p$ in which $\omega = \sum dx_i \wedge dy_i$.*

We need the following linear algebra fact.

**Lemma 3.5.** *We have $(T_pM, \omega_p) \cong (\mathbb{R}^{2n}, \omega_0)$ as a symplectic vector space.*

*Proof.* You build a standard basis $e_i, f_j$ such that $\omega(e_i, f_i) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. You first choose $e_1$ and then find $f_1$ such that $\omega(e_1, f_1) = 1$. Then look at the orthogonal to the span of $e_1$ and $f_1$ and continue by induction. □

Using the standard basis on $T_pM$, we find local coordinates

$$\mathbb{R}^{2n} \supset U \xrightarrow{f} M$$

such that $\omega_1 = f^*\omega$ agrees with $\omega_0$ at the origin. Define

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Since non-degeneracy is an open condition, and $\omega_0 = \omega_1$ at 0, these are all symplectic on a neighborhood of 0. Shrink the domain $U$ if necessary. Note that $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ is closed, hence exact. So define

$$d\alpha = d\omega_1 - d\omega.$$

We can assume that $\alpha = 0$ at the origin. But then, the first-order terms in $\alpha$ become constant forms in $d\alpha$, and this is zero. So we can discard these terms and assume that $\alpha$ vanishes to order 2 at the origin.

Let $v_t$ be the vector field such that $\iota_{v_t}\omega_t = -\alpha$. Then $v_t(0) = 0$. Also, let $\varphi_t$ be the flow of $v_t$, well-defined and staying inside $U$ in a small neighborhood of 0. If we let Moser do its thing, we get

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*\Big(d\iota_{v_t}\omega_t + \frac{d\omega_t}{dt}\Big) = \varphi_t(-d\alpha + d\alpha) = 0.$$

Then we find that

$$(f \circ \varphi_1)^*\omega = \varphi_1^*(f^*\omega) = \varphi_1^*\omega_1 = \omega_0.$$

# 4 September 13, 2018

Homework 1 is under construction. We have talked about Moser and Darboux next time.

## 4.1 Lagrangian neighborhood theorem

**Proposition 4.1.** *If $L \subseteq M$ is a Lagrangian submanifold, then $NL \cong T^*L$.*

*Proof.* We have $TM|_L \cong T^*M|_L \twoheadrightarrow T^*L$, and the kernel is $TL$. So $TM/TL \cong NL \cong T^*L$. □

**Theorem 4.2** (Lagrangian neighborhood theorem, Weinstein)**.** *If $L \subseteq M$ is a Lagrangian, then there exist neighborhoods $U$ of $L$ in $(M, \omega)$ and $U_0$ of the zero section in $(T^*L, \omega_0)$, and a symplectomorphism*

$$\varphi : (U_0, \omega_0) \xrightarrow{\sim} (U, \omega)$$

*that maps $L$ as you think.*

*Proof.* We first pick a complement to $TL$, i.e., a subbundle $N \subseteq TM|_L$ such that $TM|_L = TN \oplus N$. Here, we can ensure $N$ is a Lagrangian subbundle, for instance, by picking an $\omega$-compatible metric. Now we can use the exponential map to build

$$\psi : T^*L \cong N \supseteq U_0 \to U \subseteq M$$

such that (i) $\psi$ along the zero section is the inclusion, (ii) the pullback of the symplectic form $\psi^*\omega = \omega_1$ coincides with $\omega_0$ at the zero section.

We are now in position to use Moser. Consider

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

These are going to be exact symplectic forms on a neighborhood of the zero section. So we can take

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\alpha$$

for some 1-form $\alpha$. In fact, we can choose $\alpha$ so that it vanishes at every point of the zero section (even to order 2). Let $v_t$ to be the vector field corresponding to it, so that $\iota_{v_t}\omega_t = -\alpha$. Let $\varphi_t$ be the flow generated by it. Then we have the customary calculation

$$\frac{d}{dt}(\varphi_t^*\omega_t) = 0.$$

Our final answer is then going to be $\varphi = \psi \circ \varphi_1$ (defined over some neighborhood of the zero section). □

There also exist neighborhood theorems for symplectic submanifolds or isotropic manifolds. But they are not nice.

## 4.2 Hamiltonian group actions

A lot of construction comes with Hamiltonian group actions. Let us say we have a Lie group $G$. If $G$ acts on $M$, it induces a map of Lie algebras

$$T_eG = \mathfrak{g} \to \mathfrak{X}(M) = (\text{vector fields on } M); \quad \xi \mapsto X_\xi = \frac{d}{dt}\Big|_{t=0}(\exp(t\xi)x).$$

This is a Lie algebra homomorphism, that is, $X_{[\xi,\eta]} = [X_\xi, X_\eta]$.

Let us now look at actions which preserve $\omega$,

$$G \to \mathrm{Symp}(M, \omega).$$

For instance, a symplectic $S^1$-action are symplectic vector fields whose flow are $2\pi$-periodic. We say that an $S^1$-action is Hamiltonian if the vector field $X_{\partial/\partial\theta}$ is Hamiltonian. More generally, if I have an action of a torus $T^k$, we will need each $S^1$ factor to act in a Hamiltonian way. In this case, there exist $k$ functions $H_1, \dots, H_k \in C^\infty(M, \mathbb{R})$ such that $X_{\partial/\partial\theta_i} = X_{H_i}$. This is the notion of a moment map. This is a package that contains all the Hamiltonians for $X_g$, where $g \in \mathfrak{g}$.

**Definition 4.3.** We say that a $G$-action is **Hamiltonian** if there exists a **moment map** $\mu : M \to \mathfrak{g}^*$ with the following properties:

(1) For all $\xi \in \mathfrak{g}$, let $H_\xi = \langle \mu, \xi \rangle : M \to \mathbb{R}$. Then $X_\xi$ is the Hamiltonian vector field generated by $H_\xi$.

(2) The moment map $\mu$ is $G$-equivariant, i.e.,

$$\langle \mu(g \cdot x), \mathrm{Ad}_g(\xi) \rangle = \langle \mu(x), \xi \rangle.$$

Note that if $G$ is abelian, this is just $\mu(gx) = \mu(x)$. For $G = T^k$, we can call

$$\mu = (H_1, \dots, H_k) : M \to \mathfrak{g}^* \cong \mathbb{R}^k.$$

These Hamiltonians need to satisfy the $2\pi$-periodicity condition, and also they should commute with each other. Also (2) says that $\mu$ is invariant under the $G$-action, so the flow of $X_{H_i}$ preserves not just $H_i$ but also all $H_j$. Actually condition (2) comes for free in this case. Because the Lie bracket is zero, we have

$$X_{\{H_i,H_j\}} = [X_{H_i}, X_{H_j}] = 0,$$

where $\{f, g\} = dg(X_f) = \omega(X_f, X_g)$ is the **Poisson bracket**. So $\{H_i, H_j\}$ is constant, and in fact 0 because you can integrate along the $S^1$ corresponding to $X_{H_i}$. So $dH_j(X_{H_i}) = 0$, i.e., $H_j$ is invariant under the $i$th action.

So for torus actions, we found that

- $\omega(X_{H_i}, X_{H_j}) = \{H_i, H_j\} = 0$, orbits are isotropic, (so you can't have an effective Hamiltonian $T^n$-action on an symplectic manifold with dimension less than $2n$)

- the level sets of $\mu = (H_1, \ldots, H_k) : M \to \mathfrak{g}^* \cong \mathbb{R}^k$ are foliated by orbits, and moreover coisotropic (this means $(TN)^{\perp\omega} \subseteq TN$), because orthogonal complement to the orbit is the tangent space.

**Example 4.4.** There is a standard $T^n$-action on $(\mathbb{C}^n, \omega)$, given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n).$$

This has moment map

$$\mu = \Big(\frac{1}{2}|z_1|^2, \ldots, \frac{1}{2}|z_n|^2\Big).$$

Note that the image of this map is the positive orthant of $\mathbb{R}^n$.

**Definition 4.5.** A **toric manifold** is a symplectic manifold $(M^{2n}, \omega)$ with a Hamiltonian $T^n$-action. Then the regular levels of the moment map $\mu : M \to \mathbb{R}^n$ are Lagrangian.

Regular levels correspond to where the action has discrete stabilzer. So by dimension reasons, regular levels are disjoint unions of $T^n$-orbits. In fact, we can show connectedness assuming that $M$ is something like compact and closed. So each nonempty fiber of $\mu$ is a single orbit.

**Theorem 4.6** (Atiyah, Guillemin–Sternberg 1982)**.** *Let $(M, \omega)$ be a compact connected symplectic manifold with a Hamiltonian action of a torus. Denote the moment map $\mu : M \to \mathbb{R}^k$. Then the level sets of $\mu$ are connected, and the image of $\mu$ is a convex polytope (which is the convex hull of the image under $\mu$ of the fixed points).*

# 5 September 18, 2018

Last time we were talking about Hamiltonian actions. For now we are going to focus on Hamiltonian torus actions. If $G$ acts on $(M, \omega)$, you can package them into

$$\mu : M \to \mathfrak{g}^*,$$

so that for any $\xi \in \mathfrak{g}$, the Hamiltonian flow corresponding to $\langle \mu, \xi \rangle : M \to \mathbb{R}$ is $X_\xi$ the vector flow generating the action. Also, there was an equivariance condition.

In the $T^k$-action, we get

$$\mu = (H_1, \ldots, H_k) : M \to \mathbb{R}^k$$

packaging the $k$ Hamiltonians. These actions should commute, $\{H_i, H_j\} = 0$. The orbits are isotropic, and the regular levels are coisotropic.

## 5.1 Atiyah–Guillemin–Sternberg convexity theorem

**Theorem 5.1** (Atiyah, Guillemin–Sternberg)**.** *If $(M, \omega)$ is a compact symplectic manifold with a Hamiltonian $T^k$ action with moment map $\mu : M \to \mathbb{R}^k$, then the level sets of $\mu$ are connected and the image $\mu(M) \subseteq \mathbb{R}^k$ is a convex polytope. Moreover, this is the convex hull of the image of the fixed points under $\mu$, and the faces has rational slope.*

You can see this if you have a toric Kähler manifold, but it is not clear that every symplectic toric manifold comes from this case. Connectedness is essentially saying that there are no index 1 saddle points.

Let us first look at the local picture. Consider $p \in M$ a fixed point for $T^k$ (or a subtorus $T^l \subseteq T^k$). This is the same thing as saying that $p$ is a critical point of $\mu$ (or a linear projection of $\mu$ onto $\mathbb{R}^l$). Then there is an action of $T^k$ on the symplectic vector space $(T_pM, \omega_p)$. Then by averaging, there exists an invariant metric on $T_pM$, compatible with the invariant complex structure. (This is complicated than it looks, but it is quite standard.) Then we can look at the generators of the linearized action.

So the generators of the linearized action are going to be $k$ commuting matrices in $\mathfrak{sp}(2n, \mathbb{R})$, and in fact $\mathfrak{u}(n)$. Because these are anti-hermitian commuting matrices, they can be simultaneously diagonalized. That is, there is a block decomposition so that each $\frac{\partial}{\partial \theta_j}$ acts by

$$\bigoplus_{i=1}^{n} \begin{pmatrix} 0 & -\lambda_i^j \\ \lambda_i^j & 0 \end{pmatrix},$$

where $\lambda_i^j$ are integers since the action should be periodic. Let us package them as $\vec{\lambda}_i = (\lambda_i^j)_{j=1,\ldots,k} \in \mathbb{Z}^k$. (This is sometimes called the weights of the action.)

That tells you that the moment map is something like

$$\mu = \mu(p) + \sum_{i=1}^{n} \frac{1}{2}|z_i|^2 \vec{\lambda}_i + \cdots .$$

Now there is a linearization theorem the tells us that the action is conjugate to this on a neighborhood. Look at the direction $\bigcap_{j=1}^{k} \ker(M_j)$ where the action is degenerate. Then the theorem says that this is really $T_p$ of the fixed point set. The idea is that if it is not a fixed point, you can't go around and come back in time $2\pi$. Or you can prove this by using the exponential chart for an invariant metric and showing that geodesics makes geodesics.

So we get the moment map really looks like the quadratic equation above. Locally, the image is then a convex cone spanned by $\vec{\lambda}_i \in \mathbb{Z}^k$ inside $\mathbb{R}^k$.

*Proof.* For connectedness, observe that any linear projection $\langle \mu, \xi \rangle$ is a Morse–Bott function of even index. This prevents any index 1 or coindex 1 saddles, and this shows connectedness. So you keep slicing the moment map until you get the inverse image of a point.

Global convexity follows from local convexity and connectedness. This is not particularly illuminating. □

In the toric case, where $T^n$ acts on $M^{2n}$, what we get is that the level sets of $\mu$ are single orbits.

**Example 5.2.** If we look at $\mathbb{C}^2$ with a $T^2$ action, the moment map is

$$\mu = \Big(\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2\Big).$$

The image is the first orthant in $\mathbb{R}^2$. (By the way, the theorem doesn't really apply here.) The level sets are $S^1(r_1) \times S^1(r_2) \subseteq \mathbb{C}^2$, which are tori if $r_1, r_2 > 0$, and $S^1(r_1) \times \{0\}$ if it is degenerate.

**Example 5.3.** Consider $\mathbb{C}P^1 = S^2$, with the round volume form and the action

$$(z_0 : z_1) \mapsto (z_0 : e^{i\theta} z_1).$$

In this case, the moment map is

$$\mu = \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

In this case, the image is and interval.

**Example 5.4.** Take $\mathbb{C}P^2$ with action

$$(z_0 : z_1 : z_2) \mapsto (z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2).$$

If you work out, the image of the moment map is going to be a triangle, with vertices $(0 : 0 : 1)$ and $(0 : 1 : 0)$ and $(1 : 0 : 0)$.

## 5.2 Delzant's theorem

**Theorem 5.5** (Delzant)**.** *A compact toric symplectic $(M^{2n}, \omega)$ up to $T^n$-equivariant symplectomorphism is in one-to-one correspondence with Delzant polytopes $\Delta = \mathrm{im}(\mu) \subseteq \mathbb{R}^n$ up to translation.*

In general, there is an action of $\mathrm{GL}(n, \mathbb{Z})$ on both sides. On the toric manifold side, it acts on the torus, and so $\mathfrak{t}^*$. On the polytope side, it is this action on $\mathfrak{t}^* \cong \mathbb{R}^n$. But I need to tell you what a Delzant polytope is.

**Definition 5.6.** A **Delzant polytope** is a convex polytope that is

- simple, $n$ edges meet at each vertex, i.e., locally combinatorially looks like $(\mathbb{R}_{\geq 0})^n$,
- rational, the normals to the facets can be chosen in $\mathbb{Z}^n$, i.e., the edges are parallel to vectors in $\mathbb{Z}^n$,
- smooth, the primitive integer normals (or edges) at each vertex form a basis of $\mathbb{Z}^n$.

Without the smoothness condition, you get toric orbifolds.

For instance, the triangle with vertices $(0,0), (0,1), (1,0)$ is a Delzant polytope, but $(0,0), (0,1), (2,0)$ is not because the smoothness condition fails at $(0,1)$.

You can only keep the combinatorial data, with the slopes of all the edges. If you only see this, you only recover the complex structure (as a complex algebraic variety or a complex manifold). The length of the edges are giving the data of the cohomology class. In fact, if you take any edge in the Delzant polytope, and integrate $\omega$ along that $S^2$, you get the length of the edge.

One direction of Delzant's theorem can be done by looking at the image of the moment map. The converse direction is interesting. One possibility is to look at local patches and try to glue them together. This works, but it is slightly painful.

A faster and counterintuitive way is to consider the Delzant polytope as the intersection

$$\Delta = (\mathbb{R}_{\geq 0})^N \cap (\text{affine } n\text{-plane in } \mathbb{R}^N \text{ with rational slope}).$$

You can take $N$ to be the number of facets, with the $i$th coordinate the distance to the $i$th facet, measured in an appropriate way. Then you will be able to do this. This is useful because we can build $M$ as a symplectic reduction of a Hamiltonian $T^{N-n}$-action on $(\mathbb{C}^N, \omega_0)$.

**Theorem 5.7** (Marsden–Wienstein)**.** *Let there be a Hamiltonian action of a compact Lie group $G$ on $(M, \omega)$, with moment map $\mu : M \to \mathfrak{g}^*$. Assume $G$ acts freely on $\mu^{-1}(0)$. (By equivariance, $\mu^{-1}(0)$ is preserved by $G$.) Then*

$$M_{\mathrm{red}} = \mu^{-1}(0)/G$$

*is a smooth manifold, and carries a natural symplectic form* $\omega_{\mathrm{red}}$ *with the following property: if*

$$M \supseteq \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/G = M_{\mathrm{red}},$$

*then* $\pi^*\omega_{\mathrm{red}} = \omega|_{\mu^{-1}(0)}$.

# 6 September 20, 2018

The symplectic reduction was the following theorem.

## 6.1 Symplectic reduction

**Theorem 6.1.** *Let $G$ be a compact Lie group with a Hamiltonian action on $(M, \omega)$, corresponding to the moment map $\mu : M \to \mathfrak{g}^*$. Assume that $G$ acts freely on $\mu^{-1}(0)$. (By equivariance $G$ preserves $\mu^{-1}(0)$.) Then $M_{\text{red}} = \mu^{-1}(0)/G$ is a smooth manifold, and carries a natural symplectic form $\omega_{\text{red}}$ such that $\pi^*\omega_{\text{red}} = \omega|_{\mu^{-1}(0)}$.*

*Proof.* Consider a basis $\xi_1, \dots, \xi_k$ of the Lie algebra, and $X_1, \dots, X_k$ the corresponding vector fields. Let us call $H_1, \dots, H_k$ be the components of $\mu$, defined by $H_i = \langle \mu, \xi_i \rangle$. The condition that $G$ acts freely means that $X_1, \dots, X_k$ are linearly independent for all $x \in \mu^{-1}(0)$. So $dH_1, \dots, dH_k$ are linearly independent. This shows that $d\mu$ is surjective at every $x \in \mu^{-1}(0)$. Then 0 is a regular value and $\mu^{-1}(0)$ is smooth. Now $G$ compact acts freely, and some basic fact tells that $\mu^{-1}(0)/G$is a smooth manifold.

We now show that $\mu^{-1}(0)$ is coisotropic. At $x \in \mu^{-1}(0)$, we have

$$\begin{aligned} T_x\mu^{-1}(0) &= \ker d\mu = \ker dH_1 \cap \dots \cap \ker dH_k \\ &= X_1^{\perp\omega} \cap \dots \cap X_k^{\perp\omega} = \operatorname{span}(X_1, \dots, X_k)^{\perp\omega} = T_x(G \cdot x)^{\perp\omega}. \end{aligned}$$

But $T_x(G \cdot x)^{\perp\omega}$ is contained in $T_x\mu^{-1}(0)$. Now it follows from a general fact of coisotropic submanifolds. If we look at any coisotropic submanifold $N \subseteq M$, we can look at $(TN)^{\perp\omega} \subseteq TN$. As a consequence of $\omega$ being closed, this $(TN)^{\perp\omega}$ is closed under the Lie bracket. This you use the formula for $d\omega(X, Y, Z)$ versus Lie derivatives. Then the Frobenius integrability theorem says that we get a foliation, called the **isotropic foliation**.

In our case, the leaves are the $G$-orbits. This is fibered as $N = \mu^{-1}(0) \xrightarrow{\pi} M_{\text{red}} = \mu^{-1}(0)/G$. But how do we find $\omega_{\text{red}}$ such that $\pi^*\omega_{\text{red}} = \omega|_N$? At each point $x \in N$ we look at $\pi(x)$ and see that $\omega|_x$ is pulled back from some $\omega|_{\pi(x)}$. Then we use $G$-equivariance to translate to other points. $\square$

**Example 6.2.** Look at the diagonal $S^1$ action on $(\mathbb{C}^n, \omega_0)$, with

$$e^{i\theta}(z_1, \dots, z_n) = (e^{i\theta}z_1, \dots, e^{i\theta}z_n), \quad \mu = \frac{1}{2}\sum |z_i|^2.$$

If we look at $\mu^{-1}(\frac{1}{2})$, it is the unit circle $S^{2n-1} \subseteq \mathbb{C}^n$. The degenerate direction is $iz$, and if I look at the quotient, we get

$$S^{2n-1}/S^1 \cong \mathbb{C}P^{n-1}.$$

Just to check with what we did last time, we had a $T^n$-action on $\mathbb{C}^n$, with moment map $\vec{\mu} = (\frac{1}{2}|z_1|^2, \dots, \frac{1}{2}|z_n|^2)$. The diagonal action is $\mu_{\text{diag}} = \sum_i \mu_i$, and so the toric reduction is going to correspond to the simplex

$$\{\textstyle\sum_i \mu_i = \frac{1}{2}\} \cap (\mathbb{R}_{\geq 0})^n.$$

This even gives a formula for the standard Kähler form on $\mathbb{C}P^{n-1}$ and its moment map. The moment map is

$$\mu((z_1 : \cdots : z_n)) = \left( \frac{|z_1|^2}{|z_1|^2 + \cdots + |z_n|^2}, \ldots, \frac{|z_{n-1}|^2}{|z_1|^2 + \cdots + |z_n|^2} \right).$$

**Example 6.3.** Let us do a non-abelian example. Consider the Grassmannian $G(k,n)$ of complex $k$-planes in $\mathbb{C}^n$, which is the $k \times n$ matrices of full rank modulo $\mathrm{GL}(k,\mathbb{C})$. So we look at $\mathbb{C}^{k\times n}$ with the standard symplectic form $\omega_0$ and look at the $\mathrm{U}(k)$ action by left multiplication. If you think about this, you will see that there is a relatively simple formula,

$$\mu(M) = \frac{i}{2} M M^* \in \mathfrak{u}(k) \cong \mathfrak{u}(k)^*.$$

Then we can take $\mu^{-1}(\frac{i}{2} I)$, and then we get

$$M_{\text{red}} = \{MM^* = I\} / \mathrm{U}(k) = G(n,k).$$

You can compute $\dim_{\mathbb{C}} M_{\text{red}} = k(n-k)$.

**Example 6.4.** Let's talk about a fun example, of polygon spaces. Start with $\mathrm{SO}(3)$ acting on $S^2(r) \subseteq \mathbb{R}^3$, with the standard area form. This action is Hamiltonian. There is standard basis of $\mathrm{so}(3)^*$ given by rotation by the standard axes. With that understood, the moment map of this action

$$\mu : S^2(r) \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$$

is just the inclusion map.

**Example 6.5.** But now let's think about the diagonal action of $\mathrm{SO}(3)$ on $S^2(r_1) \times \cdots \times S^2(r_n)$. The moment map for that will be

$$\mu(v_1, \ldots, v_n) = v_1 + \cdots + v_n \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*.$$

So if we look at the quotient, we get

$$\mu^{-1}(0)/\mathrm{SO}(3) = \{(v_1, \ldots, v_n) : |v_i| = r_i, \textstyle\sum_i v_i = 0\}.$$

This is the space of $n$-gons in $\mathbb{R}^3$, modulo translations and rotations. This may be singular, and to ensure smoothness, we may need $r_i$ generic. (We are afraid about polygons being contained in a line, so that the $\mathrm{SO}(3)$-action is not free.) Then you will get that $\mu^{-1}(0)/\mathrm{SO}(3)$ is a symplectic manifold of dimension $\dim_{\mathbb{R}} = 2n - 6$.

The triangle space ($n = 3$) is a point. For $n = 4$, there is a "bending" $S^1$-action that rotates a point along the diagonal. This is a Hamiltonian with moment map the length of the diagonal. There is not much choice for a 2-dimensional symplectic manifold with a Hamiltonian $S^1$-action. It is going to be $S^2$ with the moment polytope

$$\ell \in [|r_1 - r_2|, r_1 + r_2] \cap [|r_3 - r_4|, r_3 + r_4].$$

For $n = 5$, the bending action will not work always, because maybe the diagonal is degenerate. But for generic lengths, this makes the polygon space a toric symplectic manifold. If the lengths are generic and close to 1, the polytope is a heptagon, if you graph the triangle inequalities in the $(\ell_1, \ell_2)$-space, where $\ell_i$ are the two diagonals. This is $S^2 \times S^2$ blown up at 3 points. (Blowing up corresponds to chopping up a corner of your polytope.)

We will now talk about constructions of Lagrangians. One example is, of course, levels of moment maps. There is a generalization of this. Assume we are in the situation of a symplectic reduction.

**Proposition 6.6.** *Let $L_{\mathrm{red}} \subseteq (M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be a Lagrangian. Then $\pi^{-1}(L_{\mathrm{red}}) \subseteq \mu^{-1}(c) \subseteq M$ is a Lagrangian in $(M, \omega)$. (Here, $c$ is a central element in $\mathfrak{g}^*$.) Moreover, every $G$-invariant Lagrangian in $(M, \omega)$ is contained in a (central) level of $\mu$.*

# 7 September 25, 2018

We will start from symplectic reduction. Let $G$ on $(M,\omega)$ be a Hamiltonian action with $\mu : M \to \mathfrak{g}^*$. For $c \in Z(\mathfrak{g}^*)$ a regular value of $\mu$ (with free $G$-action), we define

$$\mu^{-1}(c) \xrightarrow{\pi} M_{\text{red}} = \mu^{-1}(c)/G.$$

There is $\omega_{\text{red}}$ on $M_{\text{red}}$ such that $\pi^*\omega_{\text{red}} = \omega|_{\mu^{-1}(c)}$.

## 7.1 Lagrangians from symplectic reductions

**Proposition 7.1.** *Let $L_{\text{red}} \subseteq (M_{\text{red}}, \omega_{\text{red}})$ be a Lagrangian submanifold of the reduced space. Then $\pi^{-1}(L_{\text{red}})$ is a Lagrangian in $(M,\omega)$.*

*Proof.* We have that $\pi : \mu^{-1}(c) \to M_{\text{red}}$ is a locally trivial fibration with fiber $G$. So $\pi^{-1}(L_{\text{red}})$ is a smooth submanifold of dimension

$$\frac{1}{2}\dim M_{\text{red}} + \dim G = \frac{1}{2}\dim M.$$

Also, $\omega_{\text{red}}$ vanishes on $L_{\text{red}}$, so $\pi^*\omega_{\text{red}}$ vanishes on $\pi^{-1}(L_{\text{red}}) = L$. That is, $\omega|_L = 0$. □

Alternatively, we can say that graph($\pi$) gives a Lagrangian correspondence in $M_{\text{red}}^- \times M$. Then we can compose this with the Lagrangian in $M_{\text{red}}$ and get a Lagrangian of $M$. There is also a converse.

**Proposition 7.2.** *If $L \subseteq (M,\omega)$ is a $G$-invariant Lagrangian, then $L$ is contained in a level $\mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

*Proof.* We need to check that $\mu$ is constant on $L$, i.e., $d\mu|_L = 0$. (Then $c$ is central by equivariance.) Fix $\xi \in \mathfrak{g}$ and $v \in TL$, and let us look at $d(\langle \mu, \xi\rangle)(v)$. This is

$$d(\langle \mu, \xi\rangle)(v) = \omega(v, X_\xi).$$

Here, $X_\xi$ generates the $\xi$ action, so is tangent to $L$ because $L$ is $G$-invariant. So this is 0 because $\omega|_L = 0$. □

**Example 7.3.** Consider a toric manifold $(M^{2n},\omega)$ with a $T^n$ action. If $c$ is in the interior of moment polytope, then $M_{\text{red}} = \{\text{pt}\}$. Then it is its own Lagrangian submanifold, and you get that the orbits are Lagrangian.

**Example 7.4.** Here is a slightly more interesting interesting. Take $S^2 \times S^2$ with the equal areas. We look at the diagonal action of SO(3), $g(v_1, v_2) = (gv_1, gv_2)$. The moment map is

$$\mu(v_1, v_2) = v_1 + v_2.$$

The antidiagonal $\mu^{-1}(0) = \{(v, -v)\}$ is smooth and Lagrangian, even though SO(3) does not act freely.

**Example 7.5.** Consider $\mathbb{C}^2 \times (\mathbb{C}P^1)^-$, and the subset

$$\{((z_1, z_2), (z_1 : z_2)) : |z_1|^2 + |z_2|^2 = 1\} \cong S^3.$$

(This is the graph of the Hopf map.) This is a Lagrangian sphere, because it is the Lagrangian correspondence underlying the reduction of $\mathbb{C}^2$ by the diagonal $S^1$ action. We can also think it as the following. The circle $S^1$ acts on $M = \mathbb{C}^2 \times (\mathbb{C}P^1)^-$ by a diagonal action on $\mathbb{C}^2$ and trivial on $\mathbb{C}P^1$. Inside there, we know that

$$\Delta \subseteq M_{\text{red}} = (\mathbb{C}P^1) \times (\mathbb{C}P^1)^-$$

is a Lagrangian. The inverse image of this is our sphere $\pi^{-1}(\Delta) = S^3$.

Here is a more fun way of interpreting this. The group SU(2) acts on $\mathbb{C}^2$. This is a Hamiltonian action, with moment map

$$\mu : (z_1, z_2) \mapsto \Big(\frac{1}{2}(|z_1|^2 - |z_2|^2), \Im(z_1\bar{z}_2), \Re(z_1, \bar{z}_2)\Big).$$

The levels are the diagonal $S^1$-orbits. We also see that SU(2) acts on $(\mathbb{C}P^1, \omega_0)$ because it commutes with the diagonal $S^1$-action. The moment map is the same formula divided by $|z_1|^2 + |z_2|^2$. Having done that, I can look at the SU(2) on $M$ diagonally. This is going to be Hamiltonian with moment map

$$\mu = \mu_{\mathbb{C}^2} - \mu_{\mathbb{C}P^1} : M \to \mathfrak{su}(2)^* \cong \mathbb{R}^3.$$

Now we observe that $\mu^{-1}(0)$ is the Lagrangian we were looking at.

**Example 7.6.** Consider the $S^1$-action on $(\mathbb{C}^2, \omega_0)$ by $e^{i\theta}(x, y) = (e^{i\theta}x, e^{-i\theta}y)$. This has moment map

$$\mu = \frac{1}{2}(|x|^2 - |y|^2).$$

For $c \neq 0$, we have $M_{\text{red}} \cong \mathbb{C}$, because its moment polytope is just a half-line. Or you can see this directly as

$$\pi : \mathbb{C}^2 \to \mathbb{C}; \quad (x, y) \to xy.$$

The fiber over $a$ looks like a hyperboloid, because this is $xy = a$. For $a = 0$, we get a double cone. The moment map $\mu$ is like the height of these, and so we see that each $\mu^{-1}(c) \cap \pi^{-1}(a)$ is a $S^1$-orbit, for $(a, c) \neq (0, 0)$. So we get

$$\mu^{-1}(c)/S^1 \cong \mathbb{C}$$

where $\mathbb{C}$ has some area form. Given a simple closed curve $\gamma \subseteq \mathbb{C}$, we get a Lagrangian torus

$$\pi^{-1}(\gamma) \cap \mu^{-1}(c).$$

The **Chekanov torus** is the case $\mathbb{R}^4 = \mathbb{C}^2$ with $c = 0$ and $\gamma$ not enclosing the origin. (This is the only known example of a Lagrangian torus in $\mathbb{R}^4$ that is not Hamiltonian isotopic to the product torus.) This also works in $S^2 \times S^2$ or $\mathbb{C}P^2$.

## 7.2 Symplectic fibrations

Let us now talk about **symplectic fibrations** over surfaces. In general, this is

$$\pi : (M, \omega) \to \Sigma$$

with symplectic fibers $F_p = \pi^{-1}(p)$ symplectic submanifolds. Another way of saying it is that $\omega|_{\ker d\pi}$ is non-degenerate. We want it to be a submersion, possibly except at some set of critical points.

We are going to stay away from the critical points for now. There is the **horizontal distribution**

$$\mathcal{H} = (\ker d\pi)^{\perp\omega},$$

and $d\pi$ identifies $d\pi_q : \mathcal{H}_q \cong T_q\Sigma$. Then we can look at the **horizontal lift** of $v \in T\Sigma$ to $v^\# \in \mathcal{H}$ such that $d\pi(v^\#) = v$.

If the fibers are compact, we can do parallel transport along paths in $\Sigma$. They induce symplectomorphisms between smooth fibers. To see this, take $v$ a vector field on $\Sigma$ and $v^\#$ its horizontal lift. We then compute

$$\mathcal{L}_{v^\#}\omega = d\iota_{v^\#}\omega$$

by the Cartan formula. This is not zero, but if restrict to the fiber, then

$$d\iota_{v^\#}\omega|_{F_p} = d(\iota_{v^\#}\omega|_F) = 0.$$

**Proposition 7.7.** *Given a Lagrangian $\ell \subseteq F_{p_0}$ and an arc $\gamma \subseteq \Sigma$ through $p_0$, let $L$ be the image of $\ell$ under the parallel transport over $\gamma$. Then $L$ is Lagrangian in $M$. Conversely, every Lagrangian in $M$ which fibers over an arc in $\Sigma$ is of this form.*

In the above example, parallel transport takes $S^1$-orbits at $\mu = c$ to $S^1$-orbits at $\mu = c$. One thing you can do is, instead $T^2 \subseteq \mathbb{C}^2$, we can use this to build $S^1 \times S^{n-1} \subseteq \mathbb{C}^n$, or for odd $n$, non-orientable $S^{n-1}$ bundle over $S^1$. We can look at

$$\mathbb{C}^n \to \mathbb{C}; \quad (z_1, \dots, z_n) \mapsto z_1^2 + \cdots + z_n^2.$$

This is the as before, because $z_1^2 + z_2^2 = (z_1 + iz_2)(z_1 - iz_2)$. In $\sum z_i^2 = c$, we can look at the real part $S^{n-1}$. Then we can parallel transport around a loop, it closes up, and then we get the Lagrangian we want.

# 8 September 27, 2018

So we were discussing construction of Lagrangians. We just discussed the case of the base having two dimensions. We have

$$\pi : (M, \omega) \to \Sigma^2$$

with symplectic fibers, possibly with isolated critical points. Using the orthogonal complement, we can lift a vector on the base to a vector on the total space. Then parallel transport along a path is a symplectomorphism, and a Lagrangian in one fiber makes a Lagrangian in the total space.

## 8.1 Symplectic Lefschetz fibrations

**Example 8.1.** Take

$$\pi : \mathbb{C}^n \to \mathbb{C}; \quad (z_1, \dots, z_n) \mapsto \sum z_i^2.$$

This is singular away from the origin. The fibers $Q_a = \pi^{-1}(a)$ are actually symplectomorphic to $T^*S^{n-1}$, so they contain the "core sphere"

$$S = Q_a \cap (e^{i\theta/2}\mathbb{R})^n.$$

where $a = re^{i\theta}$. These happen to be preserved under parallel transport. This implies that parallel transport along closed curves close up. By parallel transporting $S_a$ over a circle, we can build Lagrangians $S^1 \times S^{n-1} \subseteq \mathbb{C}^n$, if the loop does not go around the origin. If you go around the origin for odd $n$, you get a non-orientable $S^{n-1}$-bundle over $S^1$.

This is an example of a symplectic **Lefschetz fibration**. This means that the fibration locally looks like this.

**Theorem 8.2** (Donaldson, 1998)**.** *Every compact symplectic manifold carries a symplectic Lefschetz fibration over $S^2$ outside a (real) codimension 4 locus. Moreover, this extends if you blow up the base locus.*

The intuition is that if you have a Kähler manifold and a sufficiently ample line bundle, you can look at the ratio of two sections to get a map to $\mathbb{C}P^1$, except where you get 0/0. This is of complex codimension 2. This theorem is telling you that you can do this in the symplectic world.

**Example 8.3.** On a 4-dimensional symplectic manifold, the Lefschetz fibration looks like, after blowing up, the fibers are surfaces generically and on some isolated points, the fibers are surfaces with loops contracted.

We can also use the singular fibers. You can think of parallel transporting to the singular fiber, which can be done because we have an explicit formula. If we take $S = Q_a \cap (e^{i\theta/2}\mathbb{R})^n$ and let $a \to 0$, we get this singular point. So a singular fiber is obtained by a singular fiber by collapsing a Lagrangian sphere, called the **vanishing cycle**. Monodromy around the singular fiber is the Dehn twist.

We can try to do this on both sides. Suppose that we have a path on the base between two singular points. If vanishing cycles coming from the two singular fibers happen to match, we can look at the parallel transport of the vanishing cycles on the entire interval. This gives a Lagrangian sphere.

**Theorem 8.4** (Donaldson, Seidel, ...)**.** *Every Lagrangian sphere is of this form in a sufficiently complicated Lefschetz fibration.*

You can also imagine doing this along a path between many singular points.

**Theorem 8.5** (A.–Muñoz–Presas, 2006)**.** *Let $L \hookrightarrow (M, \omega)$ be a compact symplectic manifold. Then there exists a symplectic Lefschetz fibration $f$ on the blowup of $M$, a Lagrangian $\tilde{L}$ Hamiltonian isotopic to $L$, and $h : L \to \mathbb{R}$ a Morse function, such that $\tilde{L}$ fibers over an arc and $f|_{\tilde{L}} \to \mathrm{arc} \subset S^2$ looks like the Morse function $h$.*

## 8.2 Generating functions

Let's go back to something that is standard. This is a way to build exact Lagrangians in $(T^*N, \omega = d\lambda)$. This is something more general than the graph of an exact form.

We take a smooth function

$$F : N \times \mathbb{R}^k \to \mathbb{R}$$

and use coordinates $(x, u)$. Assume that $x$ is non-degenerate in the $u$ direction. (This roughly means that fixing $x$ gives a Morse function on $u$, but we want there to be interesting things such as canceling out of critical points.) Then we define

$$L_F = \{(x, \xi) \in T^*N : \exists u \in \mathbb{R}^k, \tfrac{\partial F}{\partial u}(x, u) = 0, \tfrac{\partial F}{\partial x}(x, u) = \xi\}.$$

Why is there any reason to think that this is a Lagrangian? This fits into the language of coisotropics. If we look at

$$\mathrm{graph}(dF) \subseteq T^*N \times T^*\mathbb{R}^k,$$

this is Lagrangian, and we are intersecting with $\{v = 0\}$, which is coisotropic. In fact, this is fibered coisotorpic with isotropic fibers $\{\mathrm{pt}\} \times \mathbb{R}^k \times 0$ and quotient $T^*N$. The nondegeneracy condition is that the intersection is transverse. Then we are projecting the intersection to $T^*N$. The claim is that this is always an immersion, and it is an embedding if it is injective. (Injectivity is that the existence of $u \in \mathbb{R}^k$ is unique.) This is exact because

$$\lambda|_{L_F} = \xi dx|_{L_F} = dF|_{L_F}.$$

**Example 8.6.** Let's look at $N = \mathbb{R}$ and $\mathbb{R}^k = \mathbb{R}$. If we take

$$F(x, u) = f(x) + u^2,$$

this is just the graph of $df$. If we take

$$F(x, u) = \frac{1}{4}u^4 + \frac{a}{2}u^2 + ux,$$

we get the sideways graph

$$L_F = \{(x, \xi) : \xi^2 + a\xi + x = 0\}.$$

This can generate very general Lagrangians.

**Theorem 8.7** (Laudenbach–Sikorav, 1985)**.** *If $L \subseteq T^*N$ is Hamiltonian isotopic to the zero section, then $L$ has a generating function which is quadratic at $\infty$ in the $u$-direction.*

This was used to prove a special case of Arnold's conjecture for Lagrangians intersections in $T^*N$. This says that if $N$ is closed and $L \subseteq T^*N$ is Hamiltonian isotopic to the zero section $0_N$, and $L \pitchfork 0_N$, then $\#(L \cap 0_N) \geq \dim H^*(N)$.

For the graph graph$(df)$, the transversality condition is that $f$ is Morse. Then this is an instance of the Morse inequality. For $L_F$, the intersection is the critical points of $F$ on $N \times \mathbb{R}^k$. This implies Arnold's conjecture for $C^0$ Hamiltonian isotopies. Understanding the general case motivated Floer to introduce Lagrangian Floer homology.

## 8.3 Gromov's h-principle

There is something mysterious called Gromov's h-principle. This is for Lagrangian immersions (or isotropic embeddings), not for Lagrangian embeddings.

**Definition 8.8.** A **formal Lagrangian immersion** of $L$ into $(M, \omega)$ is a pair $(f, F)$ where

$$\begin{tikzcd} TL \arrow[r, "F"] \arrow[d] & TM \arrow[d] \\ L \arrow[r, "f"] & TM, \end{tikzcd}$$

(this is equivalent to $F$ being a bundle map $TL \to f^*TM$) such that

(1) $F$ is injective on every fiber of $TL$,

(2) for each $p \in L$, we have $F(T_pL) \subseteq T_{f(p)}M$ is a Lagrangian subspace,

(3) $[f^*\omega] = 0$ in $H^2(L, \mathbb{R})$.

**Theorem 8.9** (Gromov, Lees)**.** *Every formal Lagrangian immersion is homotopic among formal Lagrangians immersions to an actual Lagrangian immersion $(\tilde{f}, d\tilde{f})$.*

# 9 October 2, 2018

At the end of last time, we stated Gromov's h-principle for Lagrangian immersions (or isotropic embeddings).

**Definition 9.1.** A **formal Lagrangian immersion** of some manifold $L$ into $(M, \omega)$ is a pair $(f, F)$ where $f : L \to M$ and $F : TL \to f^*TM$ such that

(1) $F$ is injective on each fiber,

(2) at every $p \in L$ the immersion $F(T_pL) \subseteq T_{f(p)}M$ gives a Lagrangian subspace,

(3) we have $[f^*\omega] = 0$ in $H^2(L, \mathbb{R})$ (this is clearly an obstruction).

**Theorem 9.2** (Gromov, Lees)**.** *Every formal Lagrangian immersion is homotopic among formal Lagrangian immersions to a Lagrangian immersion.*

*Proof.* You can look at Elashberg–Mishachev. □

Let's look at what this says.

**Example 9.3.** If $M = (\mathbb{C}^n, \omega_0)$, then $f^*TM$ is just trivial and (3) always holds. Then the theorem implies that there exists a Lagrangian immersion $L \looparrowright (\mathbb{C}^n, \omega_0)$ if and only if $TL \otimes \mathbb{C}$ is trivial.

**Example 9.4.** This implies that there exists a Lagrangian immersion $S^n \looparrowright (\mathbb{C}^n, \omega_0)$. But this doesn't tell you how to build one. We can actually find this with just one transverse double point. Consider

$$(z_1, \dots, z_n) \mapsto z_1^2 + \cdots + z_n^2$$

which is a Lefschetz fibration. Each fiber contains a Lagrangian sphere

$$\sum z_i^2 = c, \quad \arg(z_i^2) = \arg(c)$$

which is an $S^{n-1}$. Last time we stated that if we go along a loop not passing the origin, we get an $S^{n-1} \times S^1$. Instead, if we go pass the origin, we get an immersion of the Lagrangian sphere. In fact, in dim $= 3$ every closed oriented 3-manifold has a Lagrangian immersion in $\mathbb{C}^3$ with just one double point. On the other hand, there is no h-principle for embeddings.

**Theorem 9.5** (Gromov)**.** *If $H_1(L) = 0$, then there is no Lagrangian embedding $L \to \mathbb{C}^n$. (Actually, every Lagrangian in $\mathbb{C}^n$ must bound a holomorphic disc with positive symplectic area.)*

## 9.1 Modifications of Lagrangians

There is this **stabilization** due to Audin–Lalonde–Polterovich. Suppose $i : L \looparrowright (M, \omega)$ is a Lagrangian immersion. Then there exists a Lagrangian embedding of $L \times S^1 \to M \times \mathbb{R}^2$ close to $i \times \text{incl.}$ (This was in the homework.) You can turn $i$ into an isotropic embedding

$$i : L \looparrowright M \times \{0\} \hookrightarrow M \times \mathbb{R}^2.$$

Then we can look at the normal bundle $(T^*L \times \mathbb{R}^2, 0 \oplus \omega_0)$. Then we take small $S^1$ in the $\mathbb{R}^2$ factor.

**Example 9.6.** There exist Lagrangian embeddings $T^4 \hookrightarrow (\mathbb{R}^8, \omega_0)$ not Lagrangian isotopic. The trick is to look at different trivializations of $T(T^3) \otimes \mathbb{C}$ and then use the h-principle, and then to this stabilization. This is in D. Rizell–Evans I think.

But it is more interesting to talk about how you can do this without increasing the dimension. This is called **Lagrangian surgery** or Lagrangian connected sum, due to Polterovich. The model is

$$A = \mathbb{R}^n_{x_1,\dots,x_n} \times \{0\}, \quad B = \{0\} \times \mathbb{R}^n_{y_1,\dots,y_n}$$

in $\mathbb{R}^{2n}$. Now we want to smooth this intersection point and stay Lagrangian.

In dimension 1, it is clear what we are doing. In higher dimension, consider the function

$$f_\epsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad f_\epsilon(x) = \epsilon \log(\|x\|)$$

for some $\epsilon > 0$. If we look at the graph of the differential of this map,

$$L_\epsilon = \operatorname{graph}(df_\epsilon) = \left\{ \left( x_1, \dots, x_n, y_1 = \frac{\epsilon x_1}{\|x\|^2}, \dots, y_n = \frac{\epsilon x_n}{\|x\|^2} \right) \right\}.$$

So actually this is completely symmetric in $x, y$. Topologically, we built $\mathbb{R}^n - \{0\} \cong \mathbb{R}^n \times S^{n-1}$ with a neck of size $\sqrt{\epsilon}$. It is also asymptotic to $A \cup B$ at infinity.

To make it a global construction, we should make it actually equal to $A \cup B$ at infinity. So we use a cut-off function so that the modification is compactly supported. The fasted way to do is to replace $f_\epsilon$ by something such that the graph of $f'$ connects the $\|x\|$-axis and $f'$-axis smoothly.

Now we talk about what happens when $A$ and $B$ embedded Lagrangians in $(M, \omega)$, with $A \cap B = \{p\}$ transversely. Then we can define $B \#_p A$, which is the outcome of this construction.

**Proposition 9.7.** *This is a Lagrangian embedding of the connected sum $A \# B$, and up to Hamiltonian–isotopy, independent of the choices of local charts, profile of $f_\epsilon$, and parameter $\epsilon > 0$.*

*Proof.* You can convince yourself that to do this construction, if you do something on one local chart, you need to do the inverse on the other local chart. If you reflect one local chart, you have to reflect the other one to keep the symplectic form standard on $\mathbb{C}^n$. Then you should check that changing $f_\epsilon$ or $\epsilon > 0$ doesn't sweep any area. $\square$

The significance is that in the Fukaya category, $B\#_p A$ is the mapping cone for $B \xrightarrow{p} A$. That is, there exists a $L$ such that there is a long exact sequence in Floer cohomology

$$\cdots \to HF(L, B) \xrightarrow{p} HF(L, A) \to HF(L, B\#_p A) \to \cdots .$$

This was proved by Fukaya–Oh–Ohta–Ono, in "Chapter 10" of the book. In the case when $B$ is a sphere, this was proved by Seidel. In that case, this is a symplectic Dhen twist.

The reason I insisted on $\epsilon > 0$ is that $A\#_p B$ is not Hailtonian isotopic (in general not even topologically isotopic) to $B\#_p A$. This is because in the local model, switching $x$ and $y$ corresponds to switching the sign of $\epsilon$. (My convention is that $B\#_p A$ is the mapping cone.)

In $\dim_2$ only, they are actually Lagrangian isotopic via hyperKähler geometry. Here, define $u = x_1 + ix_2$ and $v = y_1 - iy_2$ (rather than $x_1 + iy_1$ and $x_2 + iy_2$), and then my Lagrangian is

$$L_\epsilon = \{uv = \epsilon\}.$$

Now I can take any $\epsilon$ and move it in $\mathbb{C}^*$ around the origin.

If two Lagrangians intersect transversely in more than one point, or there is a self-intersection, we can perform surgeries at all the intersections. At each intersection, we can choose the direction of the surgery (the sign of $\epsilon_i$) and the neck sizes $\epsilon_i$ at each point. Here, the outcome may be a non-orientable Lagrangian.

**Example 9.8.** Let us start form the Lagrangian torus $T^2 \subseteq \mathbb{R}^4$, and then deform it to create self-intersections. Then there is an even number of intersections, for both signs. If we do surgery on these, we get Lagrangian non-orientable surfaces with $\chi < 0$ and $4 \mid \chi$.

The Hamiltonian isotopy class does depend on the neck size. For instance, let's do a self-sum. The Hamiltonian isotopy class is detected by a symplectic area of a disc bounding it. If we take the disc around a neck, this disc has zero area, so we don't get anything. But because we're doing a self-sum, there is a curve that passes the neck, goes around the manifold, and comes back. This does sweep area depending on $\epsilon$.

**Example 9.9.** In $\mathbb{C}^n$, we looked at an immersed $S^n$, with a transverse double point. In this case, the two surgeries we can do give the two different perturbation of the curve. Then we get Lagrangian $S^1 \times S^{n-1}$.

# 10 October 4, 2018

We are going to talk about Lagrangian fibrations.

## 10.1 Lagrangian fibrations

Let's assume that for some reason, we happen to know a map

$$\pi : (M^{2n}, \omega) \to B^n$$

with Lagrangian fibers over regular values, and assume that $\pi$ is proper, i.e., fibers are compact. Let us set the notation $F_b = \pi^{-1}(b)$. There is a notion of flux, and this gives local charts near any regular value $b_0$. This gives a canonical map

$$\phi : b_0 \in U_{b_0} \to H^1(F_{b_0}, \mathbb{R})$$

where $U_{b_0}$ is a neighborhood in $B$. Abstractly, this is how this works. First identify a neighborhood of $F_{b_0}$ with a piece of $T^*F_{b_0}$. Then for $b$ close to $b_0$, the fiber $F_b$ is going to be a graph of a closed 1-form. Then define

$$\phi(b) = [\alpha_b] \in H^1(F_{b_0}, \mathbb{R}).$$

Why is this canonical? Explicitly, given a loop $\gamma \in H_1(F_{b_0})$, you can sweep a cylinder

$$\Gamma : S^1 \times [0,1] \to M$$

that projects to the path $b_0 \to b$. Then the claim is that

$$\langle \phi(b), \gamma \rangle = \int_\Gamma \omega.$$

By Stokes's theorem and the fact that the fibers are Lagrangians, this does not depend on the choices of $\gamma$ and $\Gamma$. So this gives a concrete description of $\phi(b)$. You can see that these are the same things.

**Proposition 10.1.** *The map $\phi(b)$ is a local embedding.*

*Proof.* For injectivity, note that if $\phi(b) = \phi(b')$, then $\alpha_b - \alpha_{b'}$ is exact, so it has to be $df$ for some function $f$. But then, $f$ has critical points because $F_{b_0}$ is compact, hence this contradicts the fact that the fibers are disjoint.

This works similarly on tangent spaces. For $v \in T_{b_0}B$, pick $v^\#$ any vector field on $F_{b_0}$ a lift under $\pi$. Then

$$d\phi(v) = [-\iota_{v^\#}\omega] \in H^1(F_{b_0}, \mathbb{R}).$$

If it's a closed 1-form, it has to have a zero. But the vector field $v^\#$ vanishes nowhere on $F_{b_0}$. $\square$

What this implies is that the dimension of $H^1(F_{b_0}, \mathbb{R})$ has to be at least $n$. Moreover, the tangent bundle to the fiber $TF$ must be trivial, because the lift of a basis of $T_{b_0}B$ gives frames in $NF_{b_0} \cong T^*F_{b_0}$.

The case of interest for us is $F \cong T^n$. Then there are local charts

$$\phi : U_{b_0} \to H^1(F_{b_0}, \mathbb{R}) \cong \mathbb{R}^n,$$

which is going to be a local diffeomorphism. This gives an **integer affine structure** on the regular values $B^{\text{reg}} \subseteq B$. This structure is given by local charts in $\mathbb{R}^n$, with gluing maps given in the semi-direct product

$$\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n.$$

This is because change if basis of $H_1(F, \mathbb{Z})$ changes the identification of $H^1(F, \mathbb{R})$ by an element of $\mathrm{GL}(n, \mathbb{Z})$, and change if reference fiber amounts to translation by $\phi(b)$. Equivalently, an integer affine structure is the data of lattices

$$TB^{\mathbb{Z}} \subseteq TB$$

corresponding to integer tangent vectors, closed under Lie brackets.

We can build $M$ from $TB$, by the following process. First take $T^*B$ which is a symplectic manifold, and then take the fiberwise quotient by the dual lattice $T^*B^{\mathbb{Z}}$. The symplectic form descends to give a symplectic manifold

$$(T^*B/T^*B^{\mathbb{Z}}, \omega_0).$$

**Example 10.2.** Let $(M^{2n}, \omega)$ be a toric symplectic manifold, and let $\mu = (\mu_1, \dots, \mu_n)$ be the moment map $\mu : M \to \Delta \subseteq \mathbb{R}^n$, with $\Delta$ the moment polytope. The level sets are Lagrangian tori, so we get a Lagrangian fibration. The affine coordinates on $B = \Delta$ is exactly (up to $2\pi$) the standard affine structure on $\mathbb{R}^n \supseteq \Delta$. To see this, we look at each $S^1$-action separately. Let us look at the flux, of a cylinder bounding two $S^1$-orbits. This cylinder can be made by taking a path $\gamma$ and looking at the $S^1$-orbit of $\gamma$. If $X_i$ is the vector field from the $i$th $S^1$-action, then we get

$$\omega(\dot{\gamma}, X_i) = d\mu_i(\dot{\gamma})$$

by definition. Using this and invariance, we get that that $\int_\Gamma \omega = (2\pi)(\Delta\gamma)$.

## 10.2 Completely integrable systems

Having a global Hamiltonian torus action is very special. There are many Lagrangian torus fibrations where you don't have a global action.

**Definition 10.3.** A **completely integrable system** is map $f = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ such that

$$\{f_i, f_j\} = df_j(X_{f_i}) = \omega(X_{f_i}, X_{f_j}) = 0$$

for all $i, j$, and $df_1, \dots, df_n$ are linearly independent over regular values.

This implies that $X_i = X_{f_i}$ satisfy the condition that $df_i(X_j) = 0$. Also, linearly independent implies that they span the tangent space to the fiber of $f$. Then $\omega(X_i, X_j) = 0$ implies that the regular fibers are Lagrangian. So $f$ is a Lagrangian fibration over regular values, but there is no reason for them to close up. Conversely, given a Lagrangian fibration $\pi : M \to B$, over a neighborhood of $b_0 \in B$, pick coordinates $(x_1, \dots, x_n)$ on $B$, and then $f_i = x_i \circ \pi$ for a completely integrable system, locally.

A priori, the flows of $X_{f_i}$ in a locally integrable system has no reason to be periodic. But it seems that choosing the canonical affine coordinates given by flux is a good idea. If we do this, over the regular parts, periodicity is automatic. This is called the **action-angle coordinates**.

Given a locally integrable system $f = (f_1, \dots, f_n)$ near a regular value $b_0 = 0$, if the fiber $F_0 = f^{-1}(b_0)$ is compact an closed, we have seen that it is Lagrangian. Along $F_0$, the 1-forms $df_1, \dots, f_n$ are linearly independent, an they give a trivialization/pointwise basis on

$$N^* F_0 = ((\text{annihilator of } TF_0) \subseteq T^*M|_{F_0}) \cong TF_0.$$

I didn't do anything so far. Identify a neighborhood of $F_0$ with a piece of $(T^*F_0, \omega_0)$, with $F_0$ being the zero section. Realize

$$f^{-1}((\lambda_1, \dots, \lambda_n)) = \left(\text{graph of } \alpha = \sum \lambda_i \alpha_i + O(\lambda^2)\right),$$

for some closed 1-forms $\alpha_i$, pointwise linearly independent. (This is the same thing as $\alpha_i = \iota_{v^\#}\omega$ for $v_i^\#$ the lift of the $i$th basis vector in $T_{b_0}\mathbb{R}^n$.) These $\alpha_i$ form a basis of $T^*F_0$, and it is dual to $X_i$, because

$$\alpha_j(X_i) = \omega_0(\alpha_i, X_i) = df_i(\alpha_i) = \delta_{ij},$$

where $\omega_0$ is the canonical symplectic form on $T^*F_0 \cong M$ and $\alpha_i$ is thought of as in $T^*F_0 \subseteq T(T^*F_0)$. (This might be confusing, but $X_i$ are just the vectors tangent to the fibers, and $\alpha_i$ point out the fibers.)

Now we want to get some coordinates. Because $\alpha_i$ are closed 1-forms, on the universal cover the lift of $\alpha_j$ is exact:

$$\theta_j : \tilde{F}_0 \to \mathbb{R}, \quad d\theta_j = \alpha_j.$$

Then flows of $S_i$ lifts, with $d\theta_j(X_i) = \delta_{ij}$. That is, $X_i = \frac{\partial}{\partial \theta_i}$. The other thing is, that because $X_i$ and $d\theta_j$ are pointwise linearly independent,

$$\theta : (\theta_1, \dots, \theta_n) : \tilde{F}_0 \to \mathbb{R}^n$$

is a local diffeomorphism. We also know that we can flow by $X_i$, so in fact, $\tilde{F}_0 \cong \mathbb{R}^n$. So $F_0$ has to be a quotient of $\mathbb{R}^n$ by a subgroup of translations, the group of **periods** of $(\alpha_1, \dots, \alpha_n)$. This is the group of things that we get by integrating $\alpha_i$ along a loop.

**Theorem 10.4** (Arnold–Liouville)**.** *Compact regular fibers of completely integrable systems are Lagrangian tori.*

Now what you do is to do a change of basis so that the periods are $\mathbb{Z}^n$ instead of some crazy lattice in $\mathbb{R}^n$. If we pick $f_1, \dots, f_n$ the flux coordinates, then $[\alpha_i]$ will be a basis of $H^1(F_0, \mathbb{Z})$. And then the periods will be $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Then $\theta_j$ will be the usual angular coordinates on $F_0 = T^n = (\mathbb{R}/\mathbb{Z})^n$. Then $X_j = \partial/\partial\theta_j$ are periodic. So locally we have a Hamiltonian $T^n$-action.

**Corollary 10.5.** *Locally, $\pi : M \to B$ in flux coordinates is the same thing as a Hamiltonian $T^n$-action, with the moment map.*

The angular coordinates $\theta_j$ on $F_0$ (and any regular fiber) are only defined up to additive constants. But if you do this globally, you see that there are choices that are better than others. We normalize these so that $\theta = 0$ defines a Lagrangian section of $f$. By invariance, it follows that $\theta = \text{const}$ are also Lagrangian sections. Then locally we can write the symplectic form as

$$\omega = \sum df_i \wedge d\theta_i.$$

So locally, $\pi : M \to B$ looks (equivariantly symplectomorphic) to $T^*B/T^*B^{\mathbb{Z}} \to B$. But globally, even over $B^{\text{reg}}$, there may be obstruction to the existence of Lagrangian sections.

# 11 October 9, 2018

Last time we looked at the relation between Lagrangian fibrations and locally completely integrable systems. So supposing that

$$\pi : (M^{2n}, \omega) \to B^n$$

is proper with regular fibers Lagrangian, then Arnold–Liouville says that compact smooth fibers are tori. Also, there were canonical local coordinates on $B^{\text{reg}} \subseteq B$ modelled on $H^1(F, \mathbb{R})$. We also saw that $B^{\text{reg}}$ has an integer affine structure ($\mathbb{R}^n$-charts glued by $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$). We also had action-angle coordinates that locally made this look like

$$\omega = \sum d\mu_i \wedge d\theta_i,$$

so $M$ locally looks like $T^*B/(T^*B)^{\mathbb{Z}}$.

## 11.1 Lagrangian fibrations with singular points

This is beautiful/boring depending on your point of view. But there are nontrivial things happening around the singular points. If you look at a monodromy around a singular point, this gives some bending in the integral affine structure.

**Example 11.1.** Take $\mathbb{C}P^2\{(x : y : z)\}$ (or $\mathbb{C}^2$), as a toric manifold with the moment map.

$$\mu = \left( \frac{|x|^2}{|x|^2 + |y|^2 + |z|^2}, \frac{|y|^2}{|x|^2 + |y|^2 + |z|^2} \right).$$

This has a $T^2$-action, and the fibers are product tori. Regular fibers are then $\{(x : y : 1)\}$ with $|x|$ and $|y|$ fixed.

Let's look at another example on $(\mathbb{C}P^2, \omega_0)$, with Poisson-commuting

$$f = \left( \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2 + |z|^2}, \frac{|sz - cz^2|^2}{|x|^2 + |y|^2 + |z|^2} \right).$$

One of the moment map rotates $(x : y : z) \mapsto (e^{i\theta}x : e^{-i\theta}y : z)$. The other one is not normalized. So $f_1$ is the moment map for the $S^1$-action, and $f_2$ is on the reduced space for the $S^1$-action, $\mathbb{C}P^1 = \{[xy : z^2]\}$. If we look at regular values of $f$, these are going to be tori, with the $S^1$-action. What do I mean by not normalized? If you take a $S^1$-orbit and then look at the area swept, this is going to be just $|x|^2 - |y|^2 = f_1$. But if you take a loop in the other direction, we don't have a canonical choice, and moreover, the area is not just $|xy - c|^2$ but the symplectic area in the symplectic reduction. So we should be using this area as the coordinate.

This has one interior singularity at the origin on $f^{-1}(0, |c|)$. Now let's look at the monodromy around this. By this, I really mean changing the radius $r$ of the circle in the reduction and the value $f_1$, so that it goes around the loop. If we look at what happens, we get the map

$$\gamma_1 \mapsto \gamma_1, \quad \gamma_2 \mapsto \gamma_2 + \gamma_1,$$

where $\gamma_1$ is the $S^1$-action and $\gamma_2$ is the other loop. We can't draw its moment polytope globally, but we can draw this locally, introducing a branched cut. The two sides are actually straight.

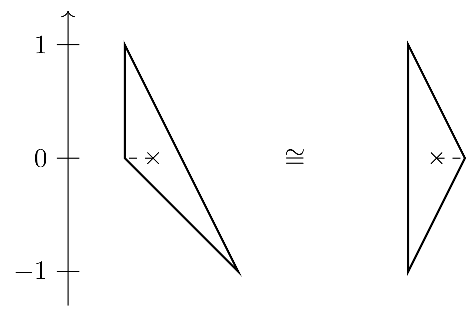


Figure 1: Integral affine structure on the base

As $c$ varies, here is what happens.

- For $c = 0$, this is the toric case, and there is a **nodal trade** operation. There is Symington's work on almost-toric symplectic 4-manifolds.
- As $c$ changes, the singular point slides in the $x$-axis. This is called a **nodal slide**.
- For $c = \frac{1}{2}$, the node hits the side. You can try to diagonalize the complex quadratic form $xy - cz^2$ with respect to the Hermitian quadratic form. Here, we can see that $|xy| \leq \frac{1}{2}(|x|^2 + |y|^2)$, and so we always have

  $$\frac{|xy - \frac{1}{2}z^2|}{|x|^2 + |y|^2 + |z|^2} \leq \frac{1}{2}.$$

  These two $x$ and $y$ have to line up to get equality, and this is when $y = -\bar{x}$ after normalizing $z = 1$. This condition implies $\mu_1 = 0$. So

  $$f^{-1}(0, \tfrac{1}{2}) \cong \mathbb{R}P^2,$$

  which is the fixed-point set of the complex conjugation $(x : y : z) \mapsto (-\bar{y} : -\bar{x} : \bar{z})$. Then you get something like Figure 2, which is a toric orbifold.

Here are other things you can do. You can take $\mathbb{C}P^2$, and then do a nodal trade at each vertex. Then you get a triangle with some branch cuts near each vertex. Then you do a nodal slide to get other strange-looking shapes. For instance, you can get right triangles with slope 4, of slope 45/4, or so on. In

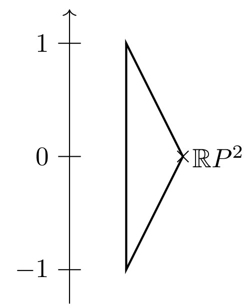


Figure 2: Integral affine structure on the base, for $c = \frac{1}{2}$

fact, you can classify shapes by the determinants of primitive vectors along at each vertex. The set of shapes we can reach are all $(a^2, b^2, c^2)$ such that

$$a^2 + b^2 + c^2 = 3abc.$$

This is called the "Markov triples", and we can obtain them by the "mutation" operation

$$(a, b, c) \leftrightarrow (a, b, c' = 3ab - c).$$

These are achieved by doing nodal slides and redrawing.

Now you can look at

$$T_{a\cdot b^2,c^2} = \text{fiber at (affine) barycenter of the triangle.}$$

The one $T_{1,1,1}$ is just the standard Clifford torus in $\mathbb{C}P^2$. Then $T_{1,1,4}$ is the monotone Chekanov torus, and so on. The claim is that $T_{a^2,b^2,c^2}$ are monotone Lagrangian tori, and they are distinguished by their enumerative geometry, namely number of families of lowers area holomorphic discs they bound (R. Vianna, 2014).

Intuitively, as we do a nodal slide, the node passes through the barycenter. As this happens, the torus becomes a singular sphere at that point. This is called **mutation** of Lagrangian. There are many ways to think about this. You can think of this as modifying the Lagrangian along a Lagrangian disc with boundaries on $L$. You can also think of this as transition between the two Lagrangian surgeries on an immersed $S^2$. More recent work of Pasceleff–Tonkonog gives an "wall-crossing formula" for the enumerative geometry of mutation. Under mirror-symmetry, this amounts to understanding "cluster charts" $(\mathbb{C}^*)^2 \hookrightarrow$ mirror.

# 12 October 11, 2018

So let us now talk about almost-complex structures.

## 12.1 Almost-complex structures

**Definition 12.1.** A **complex structure** on a real vector space $V$ is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = -1$, so that $V$ becomes a complex vector space with $iv = Jv$. We say that this is **compatible with the symplectic structure** $\Omega$ if

$$G(u, v) = \Omega(u, Jv)$$

defines a positive-definite and symmetric bilinear form.

Compatibility is equivalent to

- symmetric, $\Omega(Jv, Jv) = \Omega(u, v)$,
- positive definite, or **tameness**, $\Omega(u, Ju) > 0$ for all $u \neq 0$.

**Proposition 12.2.** *Any compatible $(V, \Omega, J)$ is always isomorphic to the standard one $(\mathbb{R}^{2n}, \Omega_0, J_0)$. Moreover, given any Lagrangian subspace $L \subseteq V$, we can arrange the isomorphism so that*

$$L \cong \mathbb{R}^n_{x_1,\dots,x_n}.$$

*Proof.* Pick $e_1 \in L$ nonzero, and rescale it so that $\Omega(e_1, Je_1) = 1$. Then look at the symplectic orthogonal to $\mathrm{span}(e_1, f_1 = Je_1)$, and then $(W, \Omega|_W, J|_W)$ is compatible and $L \cap W$ is Lagrangian in $W$. Now we can induct. $\square$

**Proposition 12.3.** *For $(V, \Omega)$ a symplectic vector space and given any inner product $\langle -, - \rangle$, we can build out of it a canonical compatible complex structure $J$. (The inner product $\langle -, - \rangle$ to start out with need not be compatible with $\Omega$!)*

*Proof.* We have two isomorphisms

$$V \cong V^*, \quad u \mapsto \Omega(u, -), \quad u \mapsto \langle u, - \rangle.$$

So there exists a unique invertible $A : V \to V$ such that $\Omega(u, v) = \langle Au, v \rangle$, and $A$ is skew-symmetric with respect to the inner product, because

$$\langle Au, v \rangle = \Omega(u, v) = -\Omega(v, u) = -\langle Av, u \rangle = -\langle u, Av \rangle.$$

Now we can look at $AA^* = -A^2$, which is symmetric positive definite. Hence it has a square root $\sqrt{AA^*}$. Then we define

$$J = (\sqrt{AA^*})^{-1} A.$$

(This $A = (\sqrt{AA^*})J$ is called the polar decomposition of $A$.) Here, $A$ commutes with $AA^* = -A^2$, so $A$ also commutes with $\sqrt{AA^*}$, so $A$ also commutes with $J$. We then check

- $J^* = A^*(\sqrt{AA^*})^{-1} = \sqrt{(AA^*)^{-1}}A^* = -J$,
- $J^*J = A^*(AA^*)^{-1}A = 1$.

So we know that $J$ is an almost-complex structure. Also,

$$\Omega(u, Jv) = \langle Au, Jv \rangle = \langle -JAu, v \rangle = \langle \sqrt{AA^*}u, v \rangle$$

shows that this is symmetric and positive-definite. □

If our inner product was already compatible, we would have gotten $A = 1$, and so you have done nothing. What this buys us is, given a symplectic manifold, we can take an arbitrary Riemannian metric, and use it to get an almost-complex structure on the entire manifold.

**Definition 12.4.** An **almost-complex structure** on a manifold $M$ is $J \in \mathrm{End}(TM)$ sch that $J^2 = -1$. We say that it is compatible/tame with respect to $\omega$ if it is at every point of $M$.

Then you can also get $\omega(-, J-) = g$ a Riemannian metric on the manifold.

**Corollary 12.5.** *Every symplectic manifold admits a compatible almost-complex structure; the space $\mathcal{J}(M, \omega)$ of these is path-connected (in fact, contractible).*

Path-connectedness is good, because if we construct something and it doesn't jump when we change the almost-complex structure, it means that the construction does not depend on the choice.

*Proof.* Pick any Riemannian metric $g_0$. Then using the polar decomposition, we get a compatible $J$. Conversely, given two $J_0$ and $J_1$ the almost complex structures, let $g_0, g_1$ be the corresponding metrics. Define $g_t = tg_1 + (1-t)g_0$ and then apply polar decomposition to get a family $\tilde{J}_t$ of almost-complex structures. Here, $\tilde{J}_0 = J_0$ and $\tilde{J}_1 = J_1$. □

Note that we didn't use $d\omega = 0$ in this construction. So it works just as well on any symplectic vector bundle. The upshot is that the classification of symplectic vector bundles over a manifold is equivalent to the classification of complex vector bundles. We can conversely ask, given a almost-complex manifold $(M, J)$, the space of $J$-compatible symplectic forms on $M$. This can be empty, but if it is nonempty, it is contractible. This is because we can interpolate $\omega = t\omega_0 + (1-t)\omega_1$ and tameness $\omega(u, Ju) > 0$ gives non-degeneracy. But many almost-complex manifolds have no symplectic structures; $S^6$ caries an almost-complex structure coming from imaginary octonions, but there is no symplectic form since we would have $[\omega] + 0 \in H^2(S^6; \mathbb{R})$ but $\int \omega^3 = 0$. Another example is $S^1 \times S^3$, which is a complex manifold since it is $(\mathbb{C}^2 \setminus 0)/\mathbb{Z}$.

Note that any two of $(J, g, \omega)$ determine the third one. So preserving two of these preserve all three. If we apply this to $(V, J, \Omega, G) \cong (\mathbb{R}^{2n}, J_0, \Omega_0, G_0)$, we get

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n).$$

Also, if we look at the compatible complex structures on $(\mathbb{R}^{2n}, \Omega_0)$, this is

$$\mathcal{J}(2n) = \mathrm{Sp}(2n)/\,\mathrm{U}(n).$$

That is, $\mathrm{Sp}(2n)$ retracts onto $\mathrm{U}(n)$. Similarly, the space of Lagrangian $n$-planes in $(\mathbb{R}^{2n}, \Omega_0)$, this is

$$\mathrm{LGr}(n) = \mathrm{U}(n)/\,\mathrm{O}(n),$$

because $\mathrm{U}(n)$ acts transitively on $(\mathbb{R}^{2n}, \Omega_0, J_0, L)$. A consequence is that $\pi_1 \mathrm{LGr}(n) = \mathbb{Z}$.

## 12.2 Types

If $J^2 = -1$, we can extend this linearly to $TM \otimes \mathbb{C}$. Then $J^2$ is diagonalizable, with eigenvalues $+i$ with eigenspace $TM^{1,0}$ and eigenvalue $-i$ with eigenspace $TM^{0,1}$. We can define projection maps

$$\pi^{1,0} : TM \to TM^{1,0}; \quad v \mapsto \frac{1}{2}(v - iJv) = v^{1,0},$$
$$\pi^{0,1} : TM \to TM^{0,1}; \quad v \mapsto \frac{1}{2}(v + iJv) = v^{0,1},$$

Each of these are $\mathbb{R}$-bundle isomorphisms. But if we look at the complex structures, we get

$$(Jv)^{1,0} = iv^{1,0}, \quad (Jv)^{0,1} = -iv^{0,1}.$$

So as complex vector bundles,

$$(TM, J) \cong (TM^{1,0}, i) \cong \overline{TM^{0,1}}.$$

On the cotangent bundle, we can similarly define

$$T^*M^{1,0} = \{\eta \in T^*M \otimes \mathbb{C} : \eta(Jv) = i\eta(v)\},$$
$$T^*M^{0,1} = \{\eta \in T^*M \otimes \mathbb{C} : \eta(Jv) = -i\eta(v)\},$$

There is a similar formula

$$\eta^{1,0} = \frac{1}{2}(\eta - i\eta \circ J) = \frac{1}{2}(\eta + iJ^*\eta), \quad J^* : T^*M \to J^*M; \quad \eta \mapsto -\eta J.$$

**Example 12.6.** On a complex manifold (where there exist charts with $J$ coming from $\mathbb{C}^n$), in local coordinates we have

$$z_j = x_j + iy_j, \quad J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J^* dx_j = dy_j.$$

Then we have

$$TM^{1,0} = \mathrm{span}\left(\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\right), \quad T^*M^{1,0} = \mathrm{span}(dz_j = dx_j + iy_j).$$

On differential forms, we similarly decompose

$$\bigwedge^k T^*M \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M, \quad \bigwedge^{p,q} = (\bigwedge^p T^*M^{1,0}) \otimes (\bigwedge^q T^*M^{0,1}).$$

A $(p,q)$-form pairs non-trivially only when fed $p$ vectors in $TM^{1,0}$ and $q$ vectors in $TM^{0,1}$.

Given $f : M \to \mathbb{C}$, we can define

$$df = (df)^{1,0} + (df)^{0,1} = \partial f + \bar{\partial} f.$$

On a complex manifold, we can define

$$d = \partial + \bar{\partial} : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

But not quite on an almost-complex manifold. So here is where we need to pay attention.

## 12.3 Integrability of almost-complex structures

**Definition 12.7.** An almost-complex structure is **integrable** if

$$[TM^{1,0}, TM^{1,0}] \subseteq TM^{1,0}.$$

Any complex structure is necessarily integrable. But you don't expect on this to be true for any almost-complex structure, because it is a pointwise condition. So define the **Nijenhuis tensor**

$$N(u,v) = [Ju, Jv] - J[u, Jv] - J[Ju, v] - [u, v].$$

A more practical way of thinking about this is,

$$N(u,v) = -8\Re([u^{1,0}, v^{0,1}]^{0,1}).$$

So we really have that $N = 0$ if and only if $J$ is integrable.

**Proposition 12.8.** *$N$ is a tensor, i.e., does not depend on the derivatives of $u$ and $v$, and it is skew-symmetric and complex anti-linear. So we can view it as*

$$N : \bigwedge^2 T^{1,0} \to T^{0,1}, \text{ or } N \in \Omega^{2,0} \otimes T^{0,1}.$$

So if $\dim_{\mathbb{C}} M = 1$, we always have $N \equiv 0$ by dimension reason. So every almost-complex structure is integrable.

**Theorem 12.9** (Newlander–Nirenberg)**.** *An almost-complex structure $J$ is a complex structure if and only if it is integrable.*

There is a dual interpretation of this. If $\alpha \in \Omega^{0,1}$ then we can take

$$d\alpha = (d\alpha)^{2,0} + (d\alpha)^{1,1} + (d\alpha)^{0,2}.$$

This wrong type $d\alpha^{2,0}$ can be expressed in terms of the Nijenhuis tensor as

$$d\alpha^{2,0}(u,v) = d\alpha(u^{1,0}, v^{1,0}) = -\alpha([u^{1,0}, v^{1,0}]) = N^*\alpha.$$

In particular, this term $d\alpha^{2,0}$ is not a differential but some tensor.

Here is a funny consequence. Consider $f : M \to \mathbb{C}$ and let us look at the $(0,2)$ part of $d^2 f = 0$. Then

$$\bar\partial(\bar\partial f) + d^{0,2}(\partial, f), \quad \bar\partial(\bar\partial f) = -\bar N^*(\partial f).$$

So if $f$ is holomorphic, we get $\bar N^*(\partial f) = 0$. If $N$ is nondegenerate, this implies that $\partial f = 0$ so that $f$ is constant.

# 13 October 16, 2018

Last time we were talking about almost complex structures.

## 13.1 Some Kähler geometry

**Definition 13.1.** A **Kähler manifold** $(M, \omega, J)$ is by definition a symplectic manifold with a compatible almost structure which is integrable.

On a Kähler manifold, we have $d = \partial + \bar{\partial}$, with $\bar{\partial}^2 = 0$. Then we can we can do Hodge theory, and harmonic forms split by types, thus cohomology also splits as

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}.$$

If we add the assumption that $[\omega] \in H^2(M, \mathbb{Z})$, then $M$ embeds into some $\mathbb{C}P^N$ and so we can do algebraic geometry.

Given a complex manifold, how would you give a Kähler structure? Let $(M, J)$ be a complex manifold (or an almost-complex manifold).

**Definition 13.2.** A (real-valued) function $f : M \to \mathbb{R}$ is **strictly plurisubharmonic** if

$$dd^c f(v, Jv) > 0$$

for all $v \neq 0$, where $d^c f = -df \circ J = J^* df$.

**Example 13.3.** Consider $\mathbb{C}$ and the function

$$f(z) = \frac{1}{4}|z|^2 = \frac{1}{4}r^2.$$

Then we have

$$df = \frac{1}{2} r dr, \quad d^c f = \frac{1}{2} r^2 d\theta, \quad dd^c f = r dr \wedge dd\theta = \omega_0.$$

So this is strictly plurisubharmonic.

On a complex manifold, we have

$$d = \partial + \bar{\partial}, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \quad \partial^2 = \bar{\partial}^2 = 0.$$

So we have

$$d^c f = J^*(\partial f + \bar{\partial} f) = i(\partial f - \bar{\partial} f),$$
$$dd^c f = i(\partial + \bar{\partial})(\partial - \bar{\partial}) f = -2i\partial\bar{\partial} f.$$

This shows that this is always of type $(1, 1)$.

What this shows is that if $f$ is strictly plurisubharmonic and $J$ is integrable, then $dd^c f$ is an (exact) $J$-compatible symplectic form, which is a Kähler form. Similarly, if $J$ is only almost-complex, we still get an exact $J$-tame symplectic form. Conversely, every Kähler form and be locally (on subsets where $[\omega] = 0$) be expressed as $dd^c$ of a function $f$, called the **Kähler potential**.

This is one reason people love strictly plurisubharmonic functions. Another reason is because we get maximum principles. Let $u : (\Sigma^2, j) \to (M, J)$ be a $J$-holomorphic curve. (This means that $du \circ j = J \circ du$.) If $f : M \to \mathbb{R}$ is a function, then

$$d^c(f \circ u) = j^*(u^* df) = u^*(J^* df) = u^* d^c f.$$

This shows that $dd^c(f \circ u) = u^*(dd^c f)$. So if $f$ is strictly plurisubharmonic and $du \neq 0$, then this is a positive area form. But in complex dimension 1, with $z = dx + iy$, we find that

$$dd^c h = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) dx \wedge dy.$$

Hence $\Delta(f \circ u) \geq 0$ (and $> 0$ where $du \neq 0$). This gives the **maximum principle** for $f \circ u$ : there is no local maximum.

Here is a consequence. Let $(M, J)$ be an almost-complex manifold, and let $h : M \to \mathbb{R}$ that is proper (i.e., goes to $\infty$ at $\infty$) and strictly plurisubharmonic near $\infty$. Then $h$ has no local maximum outside a compact set along $J$-holomorphic curves. This shows that non-constant closed $J$-holomorphic curves stays inside a compact subset. This is the sort of argument that gives well-definedness of holomorphic curve theory and Lagrangian Floer homology on non-compact symplectic manifolds which are convex at $\infty$.

If I have a toric symplectic manifold, it will quickly become a toric Kähler manifold. Then they admit local Kähler potentials which are $T^n$-invariant (by averaging over the action). In this case, $f$ is a function only of the norms of the moment map (or norms of coordinates in the context of toric varieties). It is easier to start with a toric complex manifold, which is a partial compactification of $(\mathbb{C}^*)^n$, and then put a Kähler potential that is $T^n$-invariant. But how do I know if my Kähler potential is strictly plurisubharmonic? There is a simple formula: if $z_j = \exp(\rho_j + i\theta_j)$ and $f = f(\rho_1, \ldots, \rho_n)$ then

$$dd^c f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \rho_i \partial \rho_j} d\rho_i \wedge d\theta_j = \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial \rho_j}\right) \wedge d\theta_j.$$

So $f(\rho_1, \ldots, \rho_n)$ being strictly plurisubharmonic just means that $f$ is a strictly convex function of $\rho_j = \log|z_j|$. We also get the moment map for free. Looking at the coefficient of $d\theta_j$, we get that the moment map has to be given by

$$\mu = (\mu_1, \ldots, \mu_n) = \left(\frac{\partial f}{\partial \rho_1}, \ldots, \frac{\partial f}{\partial \rho_n}\right).$$

In general, we have a collection of local potentials for affine charts of the toric variety, and they differ only by linear functions of the $\rho$s.

**Example 13.4.** If I look at $(\mathbb{C}^n, \omega_0)$, then we have $\omega_0 = dd^c(\frac{1}{4}\|z\|^2)$. On $\mathbb{C}P^n$, we can write this directly by

$$\omega_{\text{std}} = dd^c\left(\frac{1}{4}\log(|z_0|^2 + \cdots + |z_n|^2)\right).$$

This is cheating, because the form is not globally exact. You can calculate this on $\mathbb{C}^{n+1} \setminus \{0\}$ and find it is scaling invariant, so it comes from $\mathbb{C}P^n$. Another better way to think about this is that on each affine chart $\{z_0 \neq 0\}$, say $z_0 = 1$, we can take make sense of this potential up to constant. If I do a change of charts and compare the two expressions, they differ by

$$dd^c \log|z_j|^2 = 0.$$

(This works because this is a sum of a holomorphic and an anti-holomorphic function.)

## 13.2 $J$-holomorphic curves

Let's return to the definition.

**Definition 13.5.** A map $u : (\Sigma, j) \to (M, J)$ is **$J$-holomorphic** if $du \circ j = J \circ du$. Or, we can define

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) \in \Omega^{0,1}(\Sigma, u^*TM)$$

and say that $\bar{\partial}_J u = 0$. Here, $\Omega^{0,1}(\Sigma, u^*TM)$ means that it is a section of $T^*\Sigma^{0,1} \otimes_{\mathbb{C}} u^*TM$.

The image of $u$ is then a (possibly singular) almost complex submanifold of $M$. For generic $J$, when $\dim_{\mathbb{C}} u(\Sigma) > 1$, this is an over-determined problem so we expect no $J$-holomorphic submanifolds other than curves. The definition makes sense in almost-complex geometry, but soon we will need to assume that $M$ is symplectic and $J$ is compatible with $\omega$ (or at least tame). This is because we want to do Gromov compactness and energy estimates. In this setting, $J$-holomorphic submanifolds are always symplectic submanifolds.

For a non-constant $J$-holomorphic curve with $\Sigma$ a closed Riemann surface, we have

$$\int_\Sigma u^*\omega = \langle [\omega], [u(\Sigma)] \rangle > 0$$

because $u^*\omega$ is pointwise positive at non-critical points. A consequence is that $\omega$ can't be exact, and $[u(\Sigma)]$ also can't be zero. Similarly, if $\Sigma$ has boundary on a Lagrangian $L$, the paring should be positive in relative $H^2$ and $H_2$.

In fact, we have more if we are in the compatible case. We have

$$\int_\Sigma u^*\omega = \langle [\omega], [u(\Sigma)] \rangle$$

is also equal to the are of $u(\Sigma)$ for the compatible metric $g(-,-) = \omega(-, J-)$. This is because the area for the metric $g$ is just equal to the symplectic area.

We will want to study the **moduli space** of $J$-holomorphic curves in $(M, \omega, J)$, with domain $(\Sigma, j, (z_1, \dots, z_n))$ representing a given (relative) homology class, where $\Sigma$ is a Riemann surface possibly with boundary and $z_1, \dots, z_n$ are marked points on $\Sigma$. We might want to fix the marked points, or we might want to fix the genus of $\Sigma$ and vary $j$. This set is going to too large, so we are going to look at these up to reparametrizations. This means that we are going to identify

$$\begin{tikzcd} (\Sigma, j, z) \arrow[r, "u"] \arrow[d, "\phi"] & M \\ (\Sigma', j', z') \arrow[ur, "u'"] & \end{tikzcd}$$

that are given by composition with a biholomorphism $\phi$ taking $z$ to $z'$.

# 14 October 18, 2018

Let us return to $J$-holomorphic curves. Given a Riemann surface $(\Sigma, j)$ and an almost-complex manifold $(M, J)$, a $J$-holomorphic map is a map $u : (\Sigma, j) \to (M, J)$ such that

$$\bar{\partial}_J u = (du)^{0,1} = \frac{1}{2}(du + J du j) = 0$$

in $\Omega^{0,1}(\Sigma, u^*TM)$. We will let $(\Sigma, j)$ vary in some set of domain curves, and quotient by reparametrization. For now, let us fix a closed $(\Sigma, j)$.

## 14.1 First-order variation of curves

Given one map $u : \Sigma \to M$ that is $J$-holomorphic, the tangent space at $u$ to the space of maps $\Sigma \to M$ (of some regularity, which I am not telling you now) are going to be vector fields along $u$. We can think of

$$v \in C^\infty(\Sigma, u^*TM), \quad (\text{or } W^{k,p}(\Sigma, u^*TM) \text{ which is a Banach space}).$$

For Sobolev spaces, we typically take $k = 1$ and $p > 2$. This is because we need $k - \frac{2}{p} > 0$ so that $W^{k,p} \hookrightarrow C^0$.

Having fixed a regularity, we want to think about the first-order variation of the $J$-holomorphic equation. This is

$$D_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, T^*\Sigma^{0,1} \otimes u^*TM),$$

and it is going to be given by

$$D_u(v) = \frac{d}{dt}\bigg|_0 \bar{\partial}_J(\exp_u(tv)).$$

But here there is a problem. When we pull back $TM$ using $\exp_u(tv)$, this is a different bundle from when we pull back along $u$. So we need a connection on $M$, so that we can identify the bundles. Then we can write

$$D_u(v) = \nabla_{\frac{d}{dt}|_{t=0}} \bar{\partial}_J(\exp_u(tv)).$$

A convenient choice is the Levi-Civita connection for some metric on $M$.

But here, we can think of this $D_u$ as a map of Banach bundles over the space $\{\text{maps}\}$, and we are looking at a connection on this bundle. But at the zero section, the zero section gives a horizontal distribution, so the connection really doesn't matter. We are going to look at general things, so let me use the connection and just write down the formula.

This $D_u(v)$ is given by

$$D_u(v) = \frac{1}{2}(\nabla v + J \circ \nabla v \circ j + \nabla_v J \circ du \circ j).$$

Here, the $\nabla$ is the pullback to $u^*TM \to \Sigma$ of the Levi-Civita connection on $TM$ for some $J$-compatible $g$. The last term comes from $J$ changing along the change of the map $u$. This is usually called a **real Cauchy–Riemann operator**. It is called real because this is not $\mathbb{C}$-linear. What this means is that in local trivializations of the complex vector bundle $u^*TM$, it looks like

$$D(v) = \bar{\partial} v + (Av)^{0,1}$$

for some order 0 real matrix $A$.

The advantage is that we can now use elliptic theory. This operator $D$ is an elliptic differential operator of order 1. This gives estimates like

$$\|v\|_{W^{k,p}} \leq C(\|v\|_{W^{k-1,p}} + \|Dv\|_{W^{k-1,p}})$$

So if $Dv = 0$, then the $W^{k,p}$-norm is bounded by the $W^{k-1,p}$-norm, and it follows that it is finite-dimensional. That is, the kernel of $D$ is finite-dimensional and is contained in $C^\infty$ (if $J$ is).

**Theorem 14.1.** *The operator $D_u$ is a Fredholm operator, i.e., has finite dimensional kernel and cokernel).*

So we can look at the index

$$\mathrm{ind}_{\mathbb{R}}(D_u) = \dim \ker - \dim \mathrm{coker},$$

and it is given by Riemann–Roch. It is

$$\mathrm{ind}_{\mathbb{R}}(D_u) = n\chi(\Sigma) + 2\langle c_1(u^*TM), [\Sigma]\rangle = n(2-2g) + 2\langle c_1(TM), [u(\Sigma)]\rangle.$$

This is because, first, we can deform the $\bar{\partial}$ operator and $A$ to the holomorphic $\bar{\partial}$ and $A = 0$ and not change the index. Then we are in the usual setting of Riemann surfaces, and we get $n$ because we are looking at the rank $n$ vector bundle on $\Sigma$. The factor of 2 comes from looking at real dimension instead of complex dimension.

The cokernel can be thought of as the kernel of the formal adjoint $D^*$. This is going to be

$$D^* : W^{k,p}(\Sigma, \textstyle\bigwedge^{0,1} u^*TM) \to W^{k-1,p}(\Sigma, u^*TM).$$

But via complex conjugation, we can thing of this as a Cauchy–Riemann operator on $T^*\Sigma \otimes \overline{u^*TM}$. This will be useful, because we can test injectivity of Cauchy–Riemann operators pretty easily, and injectivity of $D^*$ is like surjectivity of $D$. Assume that $D_u$ is surjective for all $J$-holomorphic curves $u : \Sigma \to M$. (Then we say that $J$ is **regular**.) Then the space

$$\tilde{\mathcal{M}} = \{(u : \Sigma \to M) : \bar{\partial}_J u = 0, [u(\Sigma)] = A\}$$

is going to be a smooth manifold of dimension equal to the index. Moreover, we will have $T_u\tilde{\mathcal{M}} = \ker D_u$. This is because surjectivity gives something like transversality. The intuition is correct, but you will need the Banach implicit function theorem, which uses some fixed-point type argument.

## 14.2 The Maslov index

Recall the first Chern class. Take a complex vector bundle $E \to \Sigma$ of rank $n$ on a closed oriented surface $\Sigma$. Let us cut open a disc $\Sigma^0 = \Sigma \setminus \text{disc}$, and this retracts onto $\bigvee S^1$. Over this, $E$ is automatically trivial.

So $E$ is trivial over both $\Sigma^0$ and $D^2$, and so the change of the trivialization along the separating $S^1$ is given by

$$S^1 \to \mathrm{GL}(n, \mathbb{C}).$$

But $\pi_1(\mathrm{GL}(n,\mathbb{C}) \cong \mathbb{Z}$ with isomorphism given by $\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^*$. So rank $n$ complex vector bundles over $\Sigma$ are classified by the degree $\mathbb{Z}$. We then define the **first Chern class** by

$$\deg(E) = \langle c_1(E), [\Sigma] \rangle.$$

Let's also talk about when $\Sigma$ is a Riemann surface with boundary. Here, we will need to impose a totally real boundary condition. That is, we are going to require that

$$u : (\Sigma, \partial\Sigma) \to (M, L)$$

to have $u(\partial\Sigma)$ to lie in some Lagrangian $L$. In this case, $u^*TL \subseteq (u^*TM, J)$ is going to be a totally real subbundle over $\partial\Sigma$. The holomorphic equation is still given by the same formula, but the domain is different. The linearized operator is

$$D_u : W^{k,p}(\Sigma, u^*TM, u^*TL) \to W^{k-1,p}(\Sigma, \textstyle\bigwedge^{0,1} \otimes u^*TM),$$

where the domain is the space of vector fields in $W^{k,p}(\Sigma, u^*TM)$ such that the restriction to $\partial\Sigma$ is in $u^*TL$. The range doesn't change because we mix up derivatives. But the formal adjoint still takes the same form, because we want to do it with respect to the $L^2$-pairing.

This operator $D_u$ is again Fredholm, and but what is the new formula for the index? We write

$$\mathrm{ind}_{\mathbb{R}}(D_u) = n\chi(\Sigma) + \mu(u^*TM, u^*TL).$$

This $\mu(u^*TM, u^*TL)$ is called the **Maslov index**. Here, note that $u^*TM \to \Sigma$, where $\Sigma$ has boundary, is trivial as a complex line bundle. But we need to impose the condition that $u^*TL \subseteq u^*TM$ over $\partial\Sigma = S^1$. So this gives an element of

$$\pi_1 \mathrm{LGr}(n) = \pi_1(\mathrm{U}(n)/\mathrm{O}(n)) = \mathbb{Z},$$

where LGr is the Lagrangian Grassmannian. We can see this isomorphism by looking at

$$\det{}^2 : \mathrm{U}(n)/\mathrm{O}(n) \to S^1.$$

This suggests a relation between the Maslov index and twice of the Chern class.

# 15 October 23, 2018

We were looking at $J$-holomorphic curves, maps $u : (\Sigma, j) \to (M, J)$ from a Riemann surface to an almost complex manifold. The equation was given by

$$\bar{\partial}_J u = \frac{1}{2}(du + J du j) = 0.$$

The linearized operator was given by

$$D_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \textstyle\bigwedge^{0,1} \otimes u^*TM),$$
$$D_u(v) = \frac{1}{2}(\nabla_v + J\nabla_v j + (\nabla_v J) du \circ j).$$

This is Cauchy–Riemann, Fredholm, and its index is $n\chi(\Sigma) + 2c_1(TM)u_*[\Sigma]$. If $\Sigma$ has a boundary, then we have

$$\operatorname{ind}_{\mathbb{R}}(D_u) = n\chi(\Sigma) + \mu(u^*TM, u^*TL).$$

## 15.1 The Maslov index is twice the Chern class

This $\mu$ is the Maslov index I have started talking about last class. If $\Sigma$ is a Riemann surface with boundary, then the bundle $u^*TM \to \Sigma$ is trivial as a complex vector bundle. So with respect to a choice of trivialization, $u^*TL$ gives a loop in

$$LGr(n) = \{\text{Lagrangian planes in } (\mathbb{R}^{2n}, \omega_0)\} = U(n)/O(n).$$

Then we can show that $\pi_1 LGr(n) \cong \mathbb{Z}$, and this is induced by the map $\det^2 : U(n)/O(n) \to S^1$.

Here are other ways of thinking about this. The Maslov index of a loop of Lagrangian planes in $\mathbb{C}^n$ is also the signed count (with multiplicities) of the times when $L(t)$ fails to be transverse to a given Lagrangian plane $L_0$. In $\mathbb{C}^n$, if you fix some $L_0$, then the condition of being not transverse to $L_0$ is a codimension 1 phenomenon. So you can count this number and get the Maslov index.

**Example 15.1.** Consider the disc $u : D^2 \to (\mathbb{R}^2, S^1)$ given by the identity map. Then the index is

$$\operatorname{ind}(D_u) = n\chi(D^2) + \mu = 1 + \mu = 3.$$

But how do I know that $\mu = 2$? Note that the bundle is already trivialized. Then our convention is counterclockwise is positive, so we get two counts.

Here is another way of thinking about the Maslov index. If $u^*TL$ is an orientable subbundle of $u^*TM|_{\partial M}$, then $\mu$ is twice the relative first Chern class. (At least, you should expect to get an even number because the oriented Lagrangian Grassmannian is a double cover of the unoriented Grassmannian, so it corresponds to the even things.) In this oriented case, we can trivialize

$u^*TM$ over $\partial\Sigma$. Then we can take the complex bundle $u^*TM \otimes \mathbb{C}$ and identify $u^*TM \cong u^*TL \otimes \mathbb{C}$ over $\partial\Sigma$. Then we can look at the relative $c_1$ with respect to this trivialization, and it turns out that $\mu = 2c_1$.

To understand this index formula $\operatorname{ind}(D_u) = n\chi(\Sigma) + \mu$, we build a double surface

$$\hat{\Sigma} = (\Sigma, j) \cup_{\partial\Sigma} (\bar{\Sigma}, -j)$$

and glue the bundle $E = u^*TM$ to $\bar{E} = \overline{u^*TM} \to \overline{\Sigma}$ using reflection about $F = u^*TL$ over $\partial\Sigma$. Then we get a complex bundle $\hat{E} \to \hat{\Sigma}$ with $c_1(\hat{E}) = 2\mu(E, F)$. (Here, we don't even need orientability.) Up to 0th order terms, we have $D = \bar{\partial}$, and this commutes with complex conjugation involution $I$, because

$$\nabla_v + (-J)\nabla_v(-j) = \nabla_v + J\nabla_v j.$$

So the kernel and the cokernel both $\pm 1$ eigenspaces for $I$. We can now show that the $+1$ eigenspace is the solutions over $(E, F) \to (\Sigma, \partial\Sigma)$, and the $-1$ eigenspace is the solutions over $(E, JF) \to (\Sigma, \partial\Sigma)$. This shows that

$$\operatorname{ind}(\hat{D}_{\hat{E}\to\hat{\Sigma}}) = \operatorname{ind}(D_{(E,F)\to\Sigma}) + \operatorname{ind}(D_{(E,JF)\to\Sigma}).$$

The two indices agree, because they have homotopic boundary conditions. So

$$\operatorname{ind}(D) = \frac{1}{2}\operatorname{ind}(\hat{D}) = \frac{1}{2}(n\chi(\hat{\Sigma}) + 2c_1(\hat{E})) = n\chi(\Sigma) + \mu(E, F).$$

## 15.2 Reparametrization and extra data

Let us again take the example of the disc $D^2$ on $\mathbb{R}^2$. We have seen that

$$\operatorname{ind}(D_u) = 1 + \mu = 3,$$

and $T_u\tilde{\mathcal{M}} = \ker D_u$ has dimension 3. But here, this 3-dimension just corresponds to reparametrization. We have

$$\operatorname{Aut}(D^2) = \mathrm{PSL}(2, \mathbb{R}),$$

and this has real dimension 3. This acts on $\tilde{M}$, and what we will care about is

$$\mathcal{M} = \tilde{\mathcal{M}} / \operatorname{Aut}.$$

and indeed you can see that $\mathcal{M} = \{\mathrm{pt}\}$ as a (stupid incidence) of the Riemann mapping theorem.

The domains we are going to take are $(\Sigma, j, z)$, a Riemann surfaces (with given topological type) and a complex structure $j$ and marked point $z$. We let the complex structure $j$ vary, and also we make the points $z$ move around in the surface. Then we can ask how many additional data we are giving, and also how many automorphisms they have.

First suppose that $\Sigma$ is closed.

- If $\Sigma$ is $(S^2, j)$, then $\dim_{\mathbb{C}} \operatorname{Aut} = 3$. If we start adding marked points, we get $\dim_{\mathbb{C}} \operatorname{Aut} = 2$ for one point, $\dim_{\mathbb{C}} \operatorname{Aut} = 1$ for two points, and $\operatorname{Aut} = \{1\}$ for three points. For more points, there is a moduli and we get $\dim_{\mathbb{C}} \mathcal{M}_{0,k} = k - 3$. We call this **stable** if there is no nontrivial automorphism, so when $k \geq 3$.
- Now we let us look at $T^2$, which we can think of as $T^2 \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \{\operatorname{Im} > 0\}/\operatorname{PSL}_2(\mathbb{Z})$. So with no marked points, we have $\dim_{\mathbb{C}} \mathcal{M}_{1,0} = 1$, but there is an automorphism $\dim_{\mathbb{C}} \operatorname{Aut} = 1$ because we have translation. If we add one point, this solves the translation issue, so $\dim_{\mathbb{C}} \mathcal{M}_{1,1} = 1$ with no automorphism. Then for $k$ points, $\dim_{\mathbb{C}} \mathcal{M}_{1,k} = k$.
- For genus $g \geq 2$, the automorphism is always discrete and then we always have $\dim_{\mathbb{C}} \mathcal{M}_{k,g} = 3g - 3 + k$ (as an orbifold).

Now we can define

$$\mathcal{M}_{g,k}(M, J, A) = \left\{ (u : (\Sigma, j, z) \to M) : \begin{array}{c} \Sigma_g = g, k \text{ marked points,} \\ \bar{\partial}_J u = 0, u_*[\Sigma] = A \end{array} \right\} / \sim,$$

for $A \in H_2(M, \mathbb{Z})$. Then we say that $u \sim u'$ if there is a biholomorphism between $(\Sigma, j, z)$ and $(\Sigma, j', z')$ that makes $u$ into $u'$.

The first order deformations also come from deformation of $j$ and moving points around in $z$. For the deformation of $j$, we get

$$T_j \mathcal{J}_\Sigma = \{j' \in \operatorname{End}(T\Sigma) : jj' + j'j = 0\} = \Omega^{0,1}(\Sigma, T\Sigma),$$

living in some finite-dimensional subspace which is a local slice for the $\operatorname{Diff}(\Sigma)$-action. Then the linearized operator is given by

$$D_{(u,j)}(v, j') = D_u(v) + \frac{1}{2} J du j' \in \Omega^{0,1}(\Sigma, u^*TM),$$

with the additional term coming from the $Jduj$ term. This is again Fredholm. If $J$ is **regular**, i.e., if $D_{(u,j)}$ is onto, then $\mathcal{M}_{g,k}(M, J, A)$ is smooth and its dimension is

$$d = n(2 - 2g) + 2c_1(TM)A + 2(3g - 3 + k).$$

Even if we are in the unstable range, this is what we get after quotienting out by automorphisms. So we get

$$d = (2n - 6)(1 - g) + 2c_1(TM)A + 2k.$$

Similarly, if $\Sigma$ has boundary, and we look at $k$ interior marked points and $l$ marked points on the boundary, then

$$d = (n - 3)\chi(\Sigma) + \mu(A) + 2k + l.$$

For instance, for $D^2$ with $l$ marked points on $S^1$, we get $\dim_{\mathbb{R}} = l - 3$, and it is contractible.

Consider the case $\Sigma = S^2$ and $J$ integrable. (So we are looking at rational curves inside some complex manifold.) Then $D$ is just the $\bar{\partial}$ operator on $u^*TM$, which is a holomorphic vector bundle over $\mathbb{P}^1$. Grothendieck proved this theorem that this implies that

$$u^*TM = \bigoplus_i \mathscr{O}(d_i)$$

and so we get

$$\ker \bar{\partial} = \bigoplus_i H^0(\mathbb{P}^1, \mathscr{O}(d_i)), \quad \operatorname{coker} \bar{\partial} = \bigoplus_i H^1(\mathbb{P}^1, \mathscr{O}(d_i)).$$

In this case, $\bar{\partial}$ is surjective if and only if $d_i \geq -1$. So $u$ is regular if and only if all $d_i$ are $d_i \geq -1$.

# 16 October 25, 2018

We were discussing regularity for $J$-holomorphic curves. When is it true that for every $u : (\Sigma, j) \to (M, J)$ (in a given moduli space), the linearized operator $D_u$ is onto?

## 16.1 Regularity

Here is the **automatic regularity** example I gave last time. If $\Sigma = S^2$ and $J$ is integrable, then $D_u$ is just the $\bar{\partial}$-operator on the holomorphic line bundle $u^*TM \to S^2 = \mathbb{C}P^1$. Then we have a splitting

$$u^*TM = \bigoplus_i \mathscr{O}(d_i),$$

so we just have

$$\operatorname{coker} D_u = \bigoplus_i H^1_{\bar{\partial}}(\mathbb{P}^1, \mathscr{O}(d_i)) = 0$$

if and only if $d_i \geq -1$ for all $i$. In practice, the hard thing is knowing exactly this splitting. In fact, even in the non-integrable case, if somehow $u^*TM = \bigoplus$ line bundle in such a way that $D_u$ splits with respect to this (or just upper triangular), we can get the surjective criterion.

Let us look at a more general case. Let $D$ be a Cauchy–Riemann operator on a complex line bundle $L \to \Sigma$. If $\deg(L) < 0$, then automatically $D$ is injective. This is because if $s \neq 0$ and $Ds = 0$, then the zeros of $s$ are isolated and have positive multiplicity. This is the Carleman similarity principle. If $Ds = 0$ where we write $Ds = \bar{\partial}s + A^{0,1}s$, then

$$\bar{\partial}s = -A^{0,1}s,$$

and then the leading order in the Taylor expansion should look like a holomorphic function.

Now recall that $D^*$ is also a Cauchy–Riemann operator on $T^*\Sigma \otimes L^*$. Therefore if $\deg(L) > 2g - 2$, then $D^*$ is injective, so $D$ is surjective.

**Example 16.1.** Now consider $u : S^2 \to M^4$ an embedded $J$-holomorphic sphere. Then we get a subbundle

$$TS^2 \subseteq u^*TM,$$

and by formal arguments, you can see that this is preserved by $D_u$. Moreover, this is $\mathscr{O}(2)$. This shows that the tangent part of $D_u$ is onto. (Actually you can absorb this direction into the deformation of the complex structures.) So regularity holds if and only if the projected $D$ to the normal bundle is onto. This holds if and only if the degree of the normal bundle is at least $-1$. Because we are in a 4-manifold, we can look at the self-intersection number, and this means that if we write $A = [u(S^2)] \in H_2(M)$, then $A \cdot A = -1$. This is just that expected dimension is nonnegative. We are later going to see that generic $J$ are regular, but these types of arguments are useful when you want to really compute something.

We can do a similar thing for discs. If we have

$$u : (D^2, \partial D^2) \to (M, L)$$

and if $u^*TM$ splits into $\bigoplus$ complex line bundles with on $\partial D^2$ the bundle $u^*TL$ splits into $\bigoplus$ real line bundles, and moreover $D_u$ splits or is upper triangular, then we get a similar surjectivity criterion. (For instance, when $M$ is a product of surfaces and also $L$ is a product of Lagrangians in the surfaces.) Here, $D_u$ is onto if and only if each factor is onto, and this hold if

$$\mu(L_i, \Lambda_i) \geq -1$$

for all $i$.

**Example 16.2.** Similarly, we look at embedded $D^2 \hookrightarrow (M^4, L)$. Then we get an embedding

$$du : (TD^2, TS^1) \hookrightarrow (u^*TM, u^*TL),$$

with the quotient being a line bundle. So we want the quotient to have $\mu \geq -1$. So we can say that this is regular if and only if

$$\mu \geq 1 \quad \Leftrightarrow \quad \dim \geq 0,$$

where this $\mu$ is the total Maslov index.

## 16.2 Generic regularity

Now let us look at a generic $J$.

**Definition 16.3.** We say that $u : \Sigma \to M$ (a $J$-holomorphic curve) is **somewhere injective** if there exists a $z \in \Sigma$ such that

$$du(z) \neq 0, \quad u^{-1}(u(z)) = \{z\}.$$

For such a $z$, we call it an **injective point**.

**Proposition 16.4.** *If $\Sigma$ is closed, either $u$ factors through a (branched) covering map*

$$u : \Sigma \xrightarrow{\pi} \Sigma_1 \xrightarrow{u_1} M$$

*or $u$ is injective at all but finitely many points.*

In the latter case, we say that $u$ is **simple**. This is not true for the case with boundary, because we can have an embedding of the disc on the plane with overlaps. In this case, the disc is somewhere injective, but it is not injective at infinitely many points. Define

$$\mathcal{M}^* \subseteq \mathcal{M}$$

be the subset of somewhere injective $J$-holomorphic curves $u : \Sigma \to M$.

**Theorem 16.5.** *There exists a Baire dense set $\mathcal{J}^{\mathrm{reg}} \subseteq \mathcal{J}(M, \omega)$ such that if $J \in \mathcal{J}^{\mathrm{reg}}$ and $u$ is a somewhat injective $J$-holomorphic curve then $D_u$ is onto. (That is, $\mathcal{M}^*$ is smooth of expected dimension.)*

Here, we are giving $\mathcal{J}(M, \omega)$ the $C^\ell$-topology for some sufficiently large $\ell$. To prove this theorem, let us look at the universal moduli space

$$\begin{aligned} \tilde{\mathcal{M}}^* &= \{(\Sigma, j, u : \Sigma \to M, J) : J \in \mathcal{J}(M, \omega), \bar{\partial}_J u = 0, u_*[\Sigma] = A, u \text{ some. inj.}\}/\sim \\ &= \coprod_{J \in \mathcal{J}} \mathcal{M}^*(J, A). \end{aligned}$$

There is a natural projection $\mathrm{pr} : \tilde{\mathcal{M}}^* \to \mathcal{J}$. If we look at the first-order deformation of the even bigger $\bar{\partial}$ operator, then

$$\tilde{D}_{(u,j,J)}(v, j', J') = D_u(v) + \frac{1}{2} J du j' + \frac{1}{2} J' du j,$$

where $v \in W^{k,p}(\Sigma, u^*TM)$, and $j \in T_j \mathcal{J}_\Sigma$, and

$$J' \in T_J \mathcal{J} = \{J' \in C^l(M, \mathrm{End}(TM)) : JJ' + J'J = 0, \omega(J'-, -) = -\omega(-, J'-)\}.$$

(We assume something like $\ell > k$.)

**Proposition 16.6.** *If $u$ is somewhere injective, then $\tilde{D}$ is always onto.*

The moral reason is that $D_u(v) + \frac{1}{2} J du j'$ was Fredholm, and then $\frac{1}{2} J'(u) du j$ is huge. If we have this, then $\tilde{M}^*$ is a smooth Banach manifold, and $\mathrm{pr} : \tilde{\mathcal{M}}^* \to \mathfrak{J}$ is a Fredholm projection. There is the Sard–Smale theorem that shows that the regular values are Baire dense, and then we can define $\mathcal{J}^{\mathrm{reg}}$ to be this set.

*Proof.* We have that $\mathrm{Im}\, \tilde{D}_{(u,j,J)} \supseteq \mathrm{Im}\, D_u$, and $\mathrm{Im}\, D_u$ is closed and finite codimension. So it is enough to show that the orthogonal complement to the big image $\mathrm{Im}\, \tilde{D}_{(u,j,J)}$ is zero. Assume $\eta \in L^q(\Sigma, \bigwedge^{0,1} \otimes u^*TM)$ satisfies the condition that

$$\eta \in \ker D_u^* = (\mathrm{Im}\, D_u)^\perp$$

and $\eta \perp \{J'(u) du j, J' \in T_J \mathcal{J}\}$.

Now we want $\eta \equiv 0$, and assume not. Because $\eta \in \ker D_u^*$ and $D_u^*$ is a Cauchy–Riemann operator, we see that $\eta$ is $C^\ell$ and has isolated zeros. But there is this other condition that $\eta \perp \{J'(u) du j\}$, and we claim that this shows that for all $z \in \Sigma$ an injective point, we have $\eta(z) = 0$. This will show that $\eta = 0$ everywhere. To show this claim, we note that there exists a $J'_{u(z)}$ mapping the complex line $\mathrm{Im}(du(z)) \subseteq T_{u(z)} M$ to any complex line in $T_{u(z)} M$. This is because we can arrange this locally, and then use a cut-off function to make this to a global construction. Then this tells us that $\eta$ could not be orthogonal to all $J'(u) du j$. □

So we get that

$$\tilde{\mathcal{M}}^* = \coprod_J \mathcal{M}^*(A, J)$$

is a Banach manifold, and the projection $\pi : \tilde{\mathcal{M}}^* \to \mathcal{J}$ is a Fredholm map of index $d = \operatorname{ind} D_{(u,j)}$. By the Sard–Smale theorem, the set

$$\mathcal{J}^{\mathrm{reg}} = \{\text{regular values of } \pi\}$$

is Baire dense. (You can even intersect this over genus of $\Sigma$ and the homology classes, because they are going to be countably many things.)

Now we can check transversality. If $J \in \mathcal{J}^{\mathrm{reg}}$ and $(u, j) \in \mathcal{M}^*(A, J)$ then

$$d\pi : \ker \tilde{D} = T_{(u,j,J)} \tilde{\mathcal{M}}^* \to T_J \mathcal{J}$$

is onto. But $\tilde{D}_{(u,j,J)} : (v, j', J') \mapsto D_{(u,j)}(v, j') + \frac{1}{2} J' du j$ is onto. So it follows that $D_{(u,j)}$ is onto.

A similar argument shows that for all $J_0, J_1 \in \mathcal{J}^{\mathrm{reg}}$, there exists a path $(J_t)$ (in fact, any generic path would do) such that

$$\coprod_{t \in [0,1]} \mathcal{M}^*(A, J_t) = \pi^*(\{J_t\})$$

is smooth of expected dimension, $\dim = d + 1$.

# 17 November 1, 2018

Today we are going to prove Gromov compactness.

## 17.1 Gromov compactness

**Theorem 17.1** (Gromov). *Let $u_n : (\Xi_n, j_n, z) \to (M, \omega)$ be a sequence of $J$-holomorphic curves, with $J \in \mathcal{J}(M, \omega)$. If*

$$E(u_n) = \int_{\Sigma_n} u_n^* \omega = \langle [\omega], u_{n*}[\Sigma_n] \rangle$$

*is uniformly bounded, there is a subsequence converging to a stable map $u_\infty : \Sigma_\infty \to M$.*

Here, $\Sigma_\infty$ is the nodal configuration of Riemann surfaces, where there are marked points and modal points that are all distinct in the domain.

Before being precise about the statement, let us look at the geometric consequences. The phenomena that can happen are

- possible degenerations of the domain, and collision of special points,
- **bubbling** of spheres (and discs when $\partial\Sigma \neq \emptyset$).

What do we mean by bubbling?

**Example 17.2.** Define

$$u_n : S^2 = \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1; \quad (x_0 : x_1) \mapsto (x_0 : x_1), (nx_1, x_0).$$

So on affine charts, this is $x \mapsto (x, 1/nx)$. Away from $x = 0$, this converges (uniformly) to $(x, 0)$. But not so fast. If you draw this in $\mathbb{P}^1 \times \mathbb{P}^1$, the curve looks like parabolas approaching the two axes. So if we say that the limit is $(x, 0)$, we are missing a lot of things. Also, there is no fixed parametrization of the domain. If we reparametrize $\tilde{x} = nx$, then the map looks like

$$u_n \circ \varphi_n : \tilde{x} \mapsto (\tfrac{1}{n}\tilde{x}, \frac{1}{\tilde{x}}).$$

So what we find is that this converges uniformly (away from $\infty$) to $(0, 1/\tilde{x})$. So the actual domain of the limit curve is $\mathbb{C}P^1 \vee \mathbb{C}P^1$, with $x = 0$ glued to $\tilde{x} = \infty$.

Here is the ideas. To handle domain degenerations, we are going to look at stable compactifications

$$\overline{\mathcal{M}}_{g,k} = \{\text{stable genus } g \text{ curves with } k \text{ marked points}\}.$$

of $\mathcal{M}_{g,k} = \{(\Sigma, j, z)\}$. This gives a candidate limit domain for $\lim_{n\to\infty}(\Sigma_n, j_n, z_n)$. Now that doesn't handle bubbling, because all our domains were $\mathbb{C}P^1$. To detect this, we need to identify bubbling regions of the domain. These are the points where $\sup|du_n| \to \infty$, where we fix a parametrization. Outside these regions, there is theorem of Arzela–Ascoli that tells us that there exists a convergent subsequence. In the bubbling regions, we rescale the domain, by setting $v_n(z) = u_n(z_n^0 + \epsilon_n z)$. Here, we have to choose $\epsilon_n \to 0$ in a suitable way. For instance, we could take $\epsilon_n = (\sup_{\text{neighbor}}|du_n|)^{-1}$ and then $\sup|dv_n| = 1$, where $v_n$ is defined on larger and larger balls going to small discs in $\Sigma_n$. So we can again use some Arzela–Ascoli to get $v_\infty : \mathbb{C} \to M$ of finite energy. Then the removable singularity theorem says that $v_\infty$ extends to a $J$-holomorphic $v_\infty : \mathbb{C}P^1 \to M$.

But this is not the end of the story, because this might miss intermediate bubbles. One way to proceed is to look at the energy content by changing $\epsilon_n$. If we pick a sequence $\epsilon_n$ that approaches zero more gently, you are still going to have bubbling, but then we are not going to see anything because it is something between bubbles. So we can change $\epsilon_n$ around and see how much energy we get. So we can "catch" all the bubble components by consider different rescalings.

This process ends, because of the energy bounds. There is an energy estimate that says, for a nonconstant $J$-holomorphic curve, we have

$$E(u) = \int u^*\omega \geq \hbar > 0.$$

So the total number of components is bounded. This comes from **Gromov's monotonicity lemma**. This says that if I have a holomorphic curve in $\mathbb{C}^n$ passing through the origin 0, the intersection with the ball $B(r)$ has area at least $\pi r^2$.

What happens with the case with boundary? If the points $z_n^\circ$ where bubbling happens are at (or close) to the boundary $\partial\Sigma$, then the limit map looks like $v_\infty : \mathbb{H} \to M$. The removable singularity theorem again applies, and this shows that we can extend to $\mathbb{H} \cup \{\infty\} \cong D^2 \to (M, L)$.

## 17.2 Stable curves

Suppose I have a sphere bubble on a disc, and the bubbling point is trying to escape to the boundary. What happens in this case is that we get a "ghost" component that looks like a disc bubble and is the constant map, and then a sphere bubbling on the ghost map. This is what stable limits will give us.

**Definition 17.3.** A **stable curve** is a (finite) union of Riemann surfaces with all distinct marked points, modulo the identification of pairs of marked points, called "nodes", (identifications can occur in the same Riemann surface) such that each component is stable, i.e., the automorphism group is discrete. Leftover non-node points become marked points.

The closed case is given by the **Deligne–Mumford moduli space**

$$\overline{\mathcal{M}}_{g,k} = \{\text{stable connected curves of genus } g \text{ with } k \text{ marked points}\}/\text{iso}.$$

For example, $\overline{\mathcal{M}}_{0,k}$ is just a tree of spheres, each with at least 3 special points (marked points plus nodes). We have $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{\text{pt}\}$ as the simplest example. The next case is

$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{C}P^1.$$

How do we see that? First we can take on $\mathbb{C}P^1$, the special points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$. Then the fourth marked point can be $z_4 \in \mathbb{C}P^1 - \{0, 1, \infty\} = \mathcal{M}_{0,4}$, in the uncompactified space. If we take the limiting behavior $z_4 \to z_i$, we are going to get $\mathbb{C}P^1 \vee \mathbb{C}P^1$ with $z_i, z_4$ on one sphere and the other two points on the other sphere. You can think of this as reparametrizing the sphere so that $z_i$ and $z_4$ are separate, and then the other two points come together. For $\overline{\mathcal{M}}_{0,5}$, it will have $\dim 2$ stratum with one $\mathbb{C}P^1$ with 5 points, and then $\dim 1$ strata with $\mathbb{C}P^1 \vee \mathbb{C}P^1$, and then $\dim 0$ strata. It turns out that this is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 3 points. In general, $\overline{\mathcal{M}}_{g,k}$ will be an algebraic stack (compact complex orbifold) of dimension $\dim_{\mathbb{C}} = 3g - 3 + k$ with codimension of strata equal to the number of nodes.

Let us now look at the disc case. These are going to be **Stasheff associahedron**. We have

$$\mathcal{A}_k = \overline{\mathcal{M}}_{0,k}^{\text{discs}} = \{\text{stable discs with } k \text{ boundary marked points in order}\}/\sim .$$

If I have three marked points in the correct order, we have $\mathcal{A}_3 = \{\text{pt}\}$. Then we have

$$\mathcal{A}_4 = [0, 1].$$

This is because if I fix the marked points $1, 2, 3$, then 4 slide between 3 and 1, and then once it hits 3 or 1 it becomes a disc bubble. The next one

$$\mathcal{A}_5 = \text{pentagon}.$$

This is because each edge corresponds to two edge points coming together. The next one $\mathcal{A}_6$ is a polytope with faces squares and pentagons.

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