Math 272x - Diffeomorphisms of Disks

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This course was taught by Alexander Kupers. The lectures were on Mondays, Wednesdays, and Fridays at 1–2pm. There was one final paper, and 23 students were enrolled. Lectures notes, compiled into a book, can be found on the course website. (Warning: these notes, taken in class, are incomplete, because I have stopped going to lectures at some point.)

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1 August 30, 2017

1.1 Diffeomorphisms of disks

The main object of interest is the topological group of diffeomorphisms of a disk. We have

$$D^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}, \quad \partial D^n = S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

We then define $\operatorname{Diff}_\partial(D^n)$ as the topological group of (smooth) diffeomorphisms of D^n that fix ∂D^n pointwise, in the C^∞-topology.

Question. *What is the homotopy type of* $\operatorname{Diff}_\partial(D^n)$*?*

We certainly want to compute the path components, isotopy classes. Also we might want to compute homotopy or homology groups. It turns out that this is related to algebraic K-theory.

There are several reasons for studying this object. Firstly, manifolds are interesting and so diffeomorphisms are also interesting. All manifolds are built out of disks. Also, disks capture the difference between "local" and "global" phenomena. In $\mathbb{R}^n$ you might be able to use some Eilenberg swindle to push problems away to infinity. Considering only diffeomorphisms capture the difference between "smooth" and "continuous". For instance, it is trivial to study homeomorphisms of the disk.

These objects also have interactions with other fields. As I have mentioned, surgery theory allows us to relate $\operatorname{Diff}_\partial(D^n)$ to homotopy theory and algebraic K-theory. They also have connections with cobordism categories and graph complexes.

1.2 Overview

I plan to start with low dimensions, dimensions at most 3. Define

$$\mathcal{M}_\partial(D^n) = \coprod_{[\sigma]} B\operatorname{Diff}_\partial(D^n_\sigma),$$

where $[\sigma]$ ranges over isotopy classes of smooth structures on D^n, standard near the boundary.

Theorem 1.1 (Folhlove, Radó, Perelman)**.** $\pi_0\mathcal{M}_\partial(D^n) \simeq *$.

Theorem 1.2 (Folhlove, Smale, Hatcher)**.** $\operatorname{Diff}_\partial D^n \simeq *$.

Next we go to high dimensions, when $n \ge 5$. The main tool here is the s-cobordism theorem.

Definition 1.3. If M_0 and M_1 are closed n-dimensional manifolds, then a **cobordism** between them is a compact $(n+1)$-dimensional manifold N with $\partial N \cong M_0 \amalg M_1$. If $M_0 \hookrightarrow N$ and $M_1 \hookrightarrow N$ are weak equivalences, then N is said to be an **h-cobordism**.

Theorem 1.4. *An h-cobordism N is diffeomorphism to $M_0 \times I$ if and only if $\tau(N) \in \mathrm{Wh}_1(\pi_1 M_0)$ vanishes.*

Here this Whitehead group is the quotient of $K_1(\mathbb{Z}[\pi_1(M_0)])$. These lead to the computations

$$\pi_0 \mathcal{M}_\partial(D^n) \cong \Theta_n, \quad \pi_0 \operatorname{Diff}_\partial(D^n) \cong \Theta_{n+1}.$$

These are groups of homotopy spheres and a famous paper of Kervaire and Milnor relate these to stable homotopy groups of spheres.

There is an additional structure on $\operatorname{Diff}_\partial(D^n)$. If $f_1, \dots, f_k \in \operatorname{Diff}_\partial(D^n)$ and there is an embedding $e : \amalg_k D^n \hookrightarrow D^n$, then you can use this to get a structure of an E_n-algebra. It follows from this that $B \operatorname{Diff}_\partial(D^n) \simeq \Omega^n X$ for some space X.

Theorem 1.5 (Molvet)**.** *$B \operatorname{Diff}_\partial(D^n) \simeq \Omega^n_0 \operatorname{Top}(n)/\operatorname{O}(n)$, where $\operatorname{Top}(n)$ is the homeomorphisms of $\mathbb{R}^n$ fixing the origin.*

After this we will talk about some quite recent work. Consider a category $\operatorname{Cob}(n)$ that has objects the moduli spaces of $(n+1)$-dimensional manifolds, and morphisms the moduli spaces of n-dimensional cobordism.

Theorem 1.6 (Galatius–Madsen–Tillmann–Weiss)**.** *$B \operatorname{Cob}(n) \simeq \Omega^{\infty-1} MTO(n)$.*

Theorem 1.7 (Galatius–RAndal–Williams)**.** *For the space $W_{g,1} = (\#_g S^n \times S^n) \setminus \operatorname{int}(D^{2n})$, The map $B \operatorname{Diff}_\partial(W_{g,1}) \to \Omega^{\infty-1} MT\Theta$ is an H_*-isomorphism for $* \leq (g-3)/2$.*

There is a sequence

$$\operatorname{Diff}_\partial(W_{g,1}) \to \operatorname{Emb}^{\cong}_{\frac{1}{2}\partial}(W_{g,1}) \to \mathcal{M}_\partial(D^{2n})$$

and using the information on the first two, we can get information about the last one. Here, $\operatorname{Emb}^{\cong}_{\frac{1}{2}\partial}$ is the embeddings that fix half the boundary, and we can use embedding calculus to get information.

2 September 1, 2017

We are going to do a review of differential topology. We are also going to show that $\mathrm{Diff}_\partial(D^1)$ is contractible.

2.1 C^r-manifolds and C^r-maps

Definition 2.1. A d-dimensional **topological manifold** is a second-countable Hausdorff topological space X that is locally homeomorphic to on open subset of $\mathbb{R}^d$.

Then you can make an "atlas" consisting of "charts" $X \supseteq V_i \xrightarrow{\phi_i} W_i \subseteq \mathbb{R}^d$ with transition functions $\phi_j\phi_i^{-1} : \phi_i(V_i \cap V_j) \to W_j$.

Definition 2.2. For $r \in \mathbb{N} \cup \{\infty\}$, a d-dimensional C^r**-manifold** is a second-countable Hausdorff topological space X with the data of maximal atlas of charts with C^r transition functions.

For manifolds with boundary, you replace $\mathbb{R}^d$ with $\mathbb{R}^{d-1} \times [0, \infty)$. Then there is a subset $\partial X \subseteq X$ consisting of points $p \in X$ going to $\mathbb{R}^{d-1} \times \{0\}$. This is well-defined, i.e., a point can't go to both the boundary and the interior of $\mathbb{R}^{d-1} \times \{0\}$. You can check this by computing the homology $H_*(U, U \setminus p)$. This will also show that ∂X will be a C^r-manifold of dimension $d - 1$.

There are possibly two notions of C^r-functions on $\mathbb{R}^{d-1} \times [0, \infty)$.

- extend to C^r-functions on an open neighborhood
- C^r on the interior and it and its partial derivatives extend continuously to $\mathbb{R}^{d-1} \times \{0\}$

Luckily a theorem of Whitney states that they are equivalent.

2.2 The Whitney topology

This is a topology on $C^r(M, N)$ and there are definitions in terms of a sub-basis and as a subspace of sections. There are weak Whitney and strong Whitney topologies, but they coincide for compact manifolds.

By "convergence", we mean uniform convergence of functions and the first r derivatives on compact subsets.

Definition 2.3. For $r \in \mathbb{N}$, the (weak) **Whitney topology** on $C^r(M, N)$ is the topology generated by the sub-basis consisting of

$$N^r(f, \phi, \psi, K, \epsilon) =$$

where $f : M \to N$ is a C^r-map, $\phi : V \to W \subseteq \mathbb{R}^m$ and $\psi : V' \to W' \subseteq \mathbb{R}^n$ are charts in M and N, $K \subseteq V$ is compact such that $f(K) \subseteq V'$, and $\epsilon > 0$. This set consists of $g \in C^r(M, N)$ such that $g(K) \subseteq V'$ and

$$|D^I(\psi f \phi^{-1})_k(x) - D^I(\psi g \phi^{-1})_k(x)| < \epsilon$$

for all $x \in K$, $|I| \leq r$, and $1 \leq k \leq n$.

For $r = \infty$, the topology on $C^\infty(M,N)$ is the coarsest one such that $C^\infty(M,N) \hookrightarrow C^r(M,N)$ is continuous for all $r < \infty$.

We can also define as a subspace of a section space. Let $s \leq r$. Then the set of s**-jets** of C^r functions $f : \mathbb{R}^m \to \mathbb{R}^n$ is the quotient of $C^r(\mathbb{R}^m, \mathbb{R}^n)$ by the equivalence relation

$$f \sim_s g \quad \Leftrightarrow \quad D^I f_k(0) = D^I g_k(0) \text{ for all } |I| \leq s \text{ and } 1 \leq k \leq n.$$

We can identify $J^s(\mathbb{R}^m, \mathbb{R}^n) = C^r(\mathbb{R}^m, \mathbb{R}^n)/\sim_s$ with n-tuples of real polynomials of degree at most s in m variables. This is Taylor expansion. So this is a finite-dimensional real vector space and there is a standard topology on here.

This generalizes to $C^r(M,N)$. Pick a point $p \in M$ and define an equivalence relation $f \sim_{s,p} g$ if $f(p) = g(p)$ and the first s partial derivatives coincide on charts.

Definition 2.4. The set of s**-jets** of C^r-maps $M \to N$ is defined as

$$J^r(M,N) = (M \times C^r(M,N))/\sim$$

where $(p,f) \sim (q,g)$ if $p = q$ and $f \sim_{s,p} g$.

Then there is a natural projection $\pi : J^s(M,N) \to M$ and an evaluation $\tau : J^s(M,N) \to N$. So we can combine them to get

$$(\pi, \tau) : J^s(M,N) \to M \times N.$$

Using charts, we see that this map locally looks like $\mathbb{R}^m \times \mathbb{R}^n \times J^s_0(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}^m \times \mathbb{R}^n$. We can use this to topology $J^s(M,N)$ so that $\pi : J^s(M,N) \to M \times N$ is locally trivial.

For a C^r-map $f : M \to N$ we can define

$$j^s(f) : M \xrightarrow{(\mathrm{id}, f)} M \times C^r(M,N) \xrightarrow{\text{quotient}} J^s(M,N).$$

This is a section, and so we have an embedding

$$j^s : C^r(M,N) \hookrightarrow \Gamma(M, J^s(M,N))$$

into the space of sections. Now topologize $C^r(M,N)$ by taking the subspace topology on

$$C^r(M,N) \hookrightarrow \Gamma(M, J^r(M,N)) \hookrightarrow \mathrm{Map}(M, J^r(M,N))$$

with the compact-open topology.

Proposition 2.5. *This topology is equal to the Whitney topology.*

The advantage of this is that you can use various adjunctions of the compact-open topologies to show that certain maps are continuous.

Corollary 2.6. $C^r(M,N) \hookrightarrow C^{r-1}(M,N)$ *is continuous, because the projection* $J^r(M,N) \to J^{r-1}(M,N)$ *is continuous. Likewise the composition* $C^r(M,N) \times C^r(N,P) \to C^r(M,P)$ *is continuous. The inverse map on* $\mathrm{Diff}^r_\partial(M)$ *is continuous.*

Corollary 2.7. $\mathrm{Diff}^r_\partial(M)$ *are topological groups in the Whitney topology.*

2.3 $\mathrm{Diff}^r_\partial(D^1)$

Theorem 2.8. $\mathrm{Diff}^r_\partial(D^1) \simeq *$.

Proof. This is basically a convexity argument. We are explicitly going to deformation retract to $\{\mathrm{id}\}$, which is given by

$$H : \mathrm{Diff}^r_\partial(D^1) \times [0,1] \to \mathrm{Diff}^r_\partial(D^1); \quad (f,t) \mapsto (1-t)f + t\,\mathrm{id}_{D^1}\,.$$

For continuity, check that the first r derivatives converge uniformly. Also, this actually lands in $\mathrm{Diff}^r_\partial(D^1)$. $\square$

3 September 6, 2017

Last time I defined the Whitney topology and showed that the diffeomorphism of D^1 is trivial. The argument also works for different variants, but fails for higher dimension.

3.1 Comparing diffeomorphism groups

Let M be a C^∞-manifold with boundary ∂M.

Definition 3.1. The group $\mathrm{Diff}^r_\partial(M)$ is the C^r-diffeomorphisms of M that fix ∂M pointwise. In this group, $\mathrm{Diff}^r_{\partial,D}(M)$ is the C^r-diffeomorphisms of M such that f and its first r derivatives coincide with id at ∂M. In this group, there is $\mathrm{Diff}^r_{\partial,U}(M)$ is the C^r-diffeomorphisms of M that agree with id on an open neighborhood of ∂M.

Example 3.2. I can sheer things in $\mathrm{Diff}^r_\partial(M)$. I can only make things point in the same direction in $\mathrm{Diff}^r_{\partial,U}$.

Theorem 3.3. *All the inclusions in*

$$\begin{tikzcd}
\mathrm{Diff}^1_{\partial,U}(M) \arrow[r, hook] & \mathrm{Diff}^1_{\partial,D}(M) \arrow[r, hook] & \mathrm{Diff}^1_{\partial}(M) \\
\vdots \arrow[u, hook] \arrow[r, hook] & \vdots \arrow[u, hook] \arrow[r, hook] & \vdots \arrow[u, hook] \\
\mathrm{Diff}^\infty_{\partial,U}(M) \arrow[u, hook] \arrow[r, hook] & \mathrm{Diff}^\infty_{\partial,D}(M) \arrow[u, hook] \arrow[r, hook] & \mathrm{Diff}^\infty_{\partial}(M) \arrow[u, hook]
\end{tikzcd}$$

are weak equivalences.

3.2 Collars

Definition 3.4. A **collar** of ∂M in M is an C^r-embedding $c : \partial M \times [0,1) \hookrightarrow M$ that is identity on ∂M.

We are going to show that they exist. We are going to flow along a vector field, which we should show exists.

Definition 3.5. A **partition of unity** subordinate to an open cover $\{U_i\}_{i\in I}$ of a topological space X is a collection of continuous functions $\{\eta_i : X \to [0,1]\}$ such that

(1) $\mathrm{supp}(\eta_i) \subseteq U_i$ for all i,

(2) only finitely many η_i are nonzero at a given $p \in X$,

(3) $\sum_{i\in I} \eta_i = 1$.

Partitions on unity exist on M since M is paracompact, and we can assume that η_i is C^r.

Definition 3.6. A vector field X on M is said to be **inwards pointing** if for all $p \in \partial M$, there is a chart $M \supseteq V \xrightarrow{\varphi} W \subseteq \mathbb{R}^{m-1} \times [0, \infty)$ such that in these coordinates $X(p)_m > 0$.

Note that this is a convex condition; if X and Y are inwards pointing at p, then $tX + (1-t)Y$ is inwards pointing at p for all $t \in [0, 1]$.

Lemma 3.7. *There exists a C^r-inwards pointing vector field on M.*

Proof. Locally we may take $\partial/\partial x_m$, and we can pull these back to get vector fields on V_i inwards pointing. Then we can patch these together by taking a C^r partition of unity with respect to V_i and write $X = \sum \eta_i X_i$. $\square$

Now consider the ODE for $\gamma : \mathbb{R} \to M$, given by

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)).$$

By Picard–Lindelöf, the following is well-defined.

Lemma 3.8. *There is an open neighborhood V of $\partial M \times \{0\}$ in $\partial M \times [0, \infty)$ such that there is a C^r map*

$$\Phi_X : V \to M; \quad (q, t) \mapsto \gamma_q(t).$$

Lemma 3.9. *The map Φ_X has bijective differential at $(q, 0) \in \partial M \times \{0\} \subseteq V$, for all $q \in \partial M$.*

Proof. This follows from the inwards pointing. The derivative on $T_q\partial M$ is going to be the identity, and the last diagonal entry is $X(a)_m$. $\square$

By the inverse function theorem, Φ_x is a local diffeomorphism.

Lemma 3.10. *Suppose that Y is a metric space and $f : Y \to Z$ is locally a topological embedding. If $X \subseteq Y$ is a set such that $f|_X$ is injective, then there exists a neighborhood $U \supseteq X$ such that $f|_U$ is injective.*

Using this point-set lemma, we can shrink the domain V of Φ_X to get a C^r-embedding.

Theorem 3.11. *Collars exist.*

If you were writing a differential topology book, what you should have done is to prove a relative version. These allows you to prove uniqueness results, by extending a collar on $\partial M \times \partial I$ to $\partial M \times I$ and getting an isotopy. Then you would be able to prove that the space of collars, appropriately defined, is contractible.

3.3 Application to gluing

Suppose we are given C^r-manifolds M, N, and a C^r-diffeomorphism $\varphi : \partial M \to \partial N$. Using this you can form $M \cup_\varphi N$ to get a manifold.

But you don't really know if this is a C^r-manifold, because you can have cringes at that boundary. Here existence of uniqueness of collar neighborhoods show that $M \cup_\varphi N$ has a well-defined C^r-manifolds structure.

What you do is both pick collars for them, look at $(-1, 0] \times \partial M$ and $[0, 1) \times \partial N$, and then glue them to $(-1, 1) \times \partial M$, which we know well.

3.4 The first comparison

Theorem 3.12. *The embedding* $\mathrm{Diff}^r_{\partial,U}(M) \hookrightarrow \mathrm{Diff}^r_{\partial,D}(M)$ *is a homotopy equivalence.*

Proof. There is a collar $c : \partial M \times [0, 1) \hookrightarrow M$ and then we can form a bigger manifold

$$\tilde{M} = (-1, 0] \times \partial M \cup_{\mathrm{id}} M.$$

This has a bigger collar $\tilde{c} : (-1, 1) \times \partial M \hookrightarrow \tilde{M}$.

Then $\mathrm{Diff}^r_{\partial,U}(M)$ is homeomorphic to the subgroup of $\mathrm{Diff}^r(\tilde{M})$ that are the identity on a neighborhood of $(-1, 0] \times \partial M$, and $\mathrm{Diff}^r_{\partial,D}(M)$ is homeomorphic to a subgroup of $\mathrm{Diff}^r(\tilde{M})$ that are the identity on $(-1, 0] \times \partial M$.

Now we can write out the homotopy inverse by a standard trick of sliding. Let $\eta : (-1, 1) \to (-1, 1)$ be a C^r-embedding that is the identity near 1, and sending $[0, 1)$ to $[\frac{1}{2}, 1)$. This induces a C^r-embedding $\tilde{M} \hookrightarrow \tilde{M}$

$$s_\eta(p) = \begin{cases} \tilde{c}(q, \eta(t)) & \text{if } p \in \tilde{c}(q, t) \\ p & \text{otherwise,} \end{cases}$$

and we define the homotopy inverse as

$$r(f)(p) = \begin{cases} s_\eta \circ f \circ s_\eta^{-1}(p) & \text{if } p \in \mathrm{im}(s_\eta), \\ p & \text{otherwise.} \end{cases}$$

To show that r and i are inverse up to homotopy, use the conjugation with $s_{(1-t)\eta + t \cdot \mathrm{id}}$. ∎

4 September 8, 2017

We showed in the last lecture that $\mathrm{Diff}^r_{\partial,U}(M) \hookrightarrow \mathrm{Diff}^r_{\partial,D}(M)$ are homotopy equivalences. Today we are going to show that $\mathrm{Diff}^r_{\partial,D}(M) \hookrightarrow \mathrm{Diff}^r_{\partial}(M)$ are weak equivalences.

4.1 The exponential map

This is something that takes a tangent vector and gives the geodesic.

Lemma 4.1. *M admits a C^∞ Riemannian metric.*

Proof. These locally exists, and we combine them using a C^∞ partition of unity. □

Fix a C^∞ Riemannian metric g on M, and for a moment take $\partial M = \emptyset$. Then geodesics will be extrema for the energy functional, defined for $\gamma : [a, b] \to M$ as

$$E(\gamma) = (b - a) \int_a^b \|\gamma'(t)\|^2 dt.$$

Definition 4.2. γ is a **geodesic** if for all $a' < b'$ in $[a, b]$, $\gamma|_{[a',b']}$ is a local extremum of E among all C^1 paths $[a', b'] \to M$ with the same endpoints.

Variational calculus tells us that γ is a geodesic if and only if it satisfies the Euler–Lagrange equation given by

$$\frac{d^2\gamma_i}{dt^2} = -\sum_{j,k} \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma_i}{dt}\frac{d\gamma_k}{dt}.$$

These Γ^i_{jk} are Christoffel symbols and are C^∞ functions. Picard–Lindelöf implies

Lemma 4.3. *For all $p \in M$ there exists a neighborhood $U \subseteq M$ of p and an $\epsilon > 0$ such that for all $q \in U$ and $v \in T_qM$ with $\|v\| < \epsilon$, there is a unique geodesic $\gamma : (-2, 2) \to M$ with $\gamma(0) = q$ and $\gamma'(0) = v$.*

Lemma 4.4. *There exists an open neighborhood V of the 0-section in TM and a smooth map*

$$\Gamma : U \times (-2, 2) \to M$$

such that $\Gamma|_{(q,v)\times(-2,2)}$ is the unique geodesic.

Definition 4.5. The **exponential map** $\exp : U \to M$ is the restriction of Γ to $U \times \{1\}$.

This is a C^∞ map, and if we compute the derivative at $(q, 0)$ in the 0-section, we get

$$T_{(q,0)}U \cong T_qM \oplus T_qM \to T_qM$$

which is just addition.

If $\partial M \neq \emptyset$, the above works if we pick a C^∞ collar $c : \partial M \times [0, 1) \hookrightarrow M$ and take the Riemannian metric of the form

$$g|_{\partial M} = g_\partial + dt^2.$$

4.2 Tubular neighborhoods

Let $M \subseteq N$ be a C^r submanifold with $\partial M = \emptyset$. This has a normal bundle $\nu_M = TN|_M/TM$, which can be naturally identified with $(TM)^\perp \subseteq TN|_M$ if you have a Riemannian metric.

Definition 4.6. A **tubular neighborhood** for M in N is a C^r-embedding $\Phi_M : \nu_M \hookrightarrow N$ such that

$$\begin{tikzcd} TM \oplus \nu_M \arrow[r, "D\Phi_M"] & TN|_M \arrow[d] \\ \nu_M \arrow[u, hook] \arrow[r, "\mathrm{id}"] & \nu_M \end{tikzcd}$$

commute.

Theorem 4.7. *Every C^r-submanifold $M \subseteq N$ with $\partial M = \emptyset$ has a C^r-tubular neighborhood.*

Proof. We can identify ν_M with $TM^\perp \subseteq TN|_M$ and apply the exponential map to the disk bundle $D_\epsilon \varsigma_M \subseteq \nu_M$, where ϵ is a function not a constant.

Now $\exp|_{D_\epsilon \nu_M}$ is going to have the right derivative at the 0-section. By the inverse function theorem this a local C^r-diffeomorphism and so by decreasing ϵ we may arrange it to a C^r-embedding. By fiberwise rescaling, we get $\nu_M \to D_\epsilon \nu_M \to N$. ☐

In the case $\partial M \neq \emptyset$, this same argument works if $M \subseteq N$ has "neat boundary". This in particular means that $\partial M = M \cap \partial N$ and the manifold M meats the boundary ∂N orthogonally.

4.3 The second comparison

Our goal is to show that $\mathrm{Diff}^r_{\partial,D}(M) \hookrightarrow \mathrm{Diff}^r_\partial(M)$ is a homotopy equivalence.

First pick a collar $c : \partial M \times [0, 1) \hookrightarrow M$, and then identify $T_{(q,0)}M \cong T_q\partial M \oplus \epsilon$, where ϵ is the 1-dimensional space generated by $\partial/\partial t$. To see the difference between the two groups, let's write down the derivatives in these coordinates.

A diffeomorphism $f \in \mathrm{Diff}^r_{\partial,D}$ is going to have first derivative

$$\begin{bmatrix} \mathrm{id}_{T_p\partial M} & 0 \\ 0 & \mathrm{id}_\epsilon \end{bmatrix}$$

and $g \in \mathrm{Diff}^r_\partial(M)$ is going to have first derivative

$$\begin{bmatrix} \mathrm{id}_{T_p\partial M} & \chi(q) \\ 0 & \lambda(q)\,\mathrm{id}_\epsilon \end{bmatrix}$$

where $\chi(q)$ is a C^r-vector field corresponding to sheering and $\lambda(q)$ is a function corresponding to scaling.

So we need to remove λ and χ. Pick a continuous

$$\eta : (0,\infty) \to \mathrm{Diff}^r_\partial([0,1))$$

such that $\eta(l)$ is id on $[\frac{1}{2},1)$, $(\eta(\ell))'(0) = \ell^{-1}$, and $\eta(1) = \mathrm{id}$. For a C^r-map $\lambda : \partial M \to (0,\infty)$, we define a map

$$\bar{\eta}(\lambda)(p) = \begin{cases} (q, \eta(\lambda)v) & p = (q,v) \in \partial M \times [0,1) \\ p & \text{otherwise,} \end{cases}$$

and consider

$$\bar{\eta}(\lambda(g)) \circ g \in \mathrm{Diff}^r_\partial(M).$$

This is going fix λ to 1 in the derivative at ∂M.

Now pick a C^r function $\rho : [0,1) \to [0,1]$ that is 1 near 0, 0 on $[\frac{1}{2},1)$. Given Y a C^r-vector field on ∂M, we can define

$$G(y)(p) = \begin{cases} (\Gamma_{\partial M}(q, -Y(q), \rho(t)), t) & \text{if } p = (q,t) \in \partial M \times [0,1) \\ p & \text{otherwise.} \end{cases}$$

This then fixes the $\chi(q)$ part in the first derivative on ∂M.

By suitably interpolating the η and ρ, we see that the maps are indeed homotopy inverses.

5 September 11, 2017

5.1 Whitney embedding theorem

Theorem 5.1 (Weak Whitney embedding). *Every compact C^r-manifold M of ∂M admits a C^r-embedding into $\mathbb{R}^N$.*

Proof. Take $\{V_i\}_{i=1}^k$ an open cover by domains of charts $M \supseteq V_i \xrightarrow{\varphi_i} W_i \subseteq \mathbb{R}^m$ and $\{\eta_i\}_{i=1}^k$ a partition of unity subordinate to this open cover. Define

$$\bar{\varphi}_i : M \to \mathbb{R}^{m+1}; \quad x \mapsto \begin{cases} (\eta_i(x), \eta_i(x)\varphi_i(x)) & \text{if } x \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

Then use $x \mapsto (\bar{\varphi}_1(x), \dots, \bar{\varphi}_k(x))$. □

If you be careful, you can show that for large enough N, the space of embeddings $M \hookrightarrow \mathbb{R}^N$ is contractible. Using the Whitney trick, you can show that M embeds into $\mathbb{R}^{2m}$.

There is a version for manifolds with boundary. First embed ∂M into $\mathbb{R}^N$, then extend it to $\partial M \times [0,1) \hookrightarrow \mathbb{R}^N \times [0,\infty)$ and extend by a relative version.

5.2 Convolution

Definition 5.2. For $f : \mathbb{R}^n \to \mathbb{R}$ continuous and compactly supported and $g : \mathbb{R}^n \to \mathbb{R}^m$ continuous, define the **convolution**

$$(f * g)(x) = \int_{y \in \mathbb{R}} f(x-y)g(y)dy.$$

Differentiation under the integral implies that $f * g$ is smooth if f is. So let $\eta : \mathbb{R}^n \to [0,\infty)$ that is supported in D^n and $\int_{y\in\mathbb{R}^n} \eta(y)dy = 1$. If we define $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$, then

$$\eta_\epsilon * g \to g$$

as $\epsilon \to 0$ in the C^r-topology if g was C^r.

More generally, let M, N be compact smooth manifolds with empty boundary. Take C^∞-embeddings $\varphi_M : M \hookrightarrow \mathbb{R}^{k_M}$ and $\varphi_N : N \hookrightarrow \mathbb{R}^{k_N}$ and C^∞-tubular neighborhoods

$$\Phi_M : \nu_M \hookrightarrow \mathbb{R}^{k_M}, \quad \Phi_N : \nu_N \hookrightarrow \mathbb{R}^{k_N}.$$

There are then going to be C^∞ projection maps

$$\pi_M : \mathrm{im}(\Phi_M) \to M, \quad \pi_N : \mathrm{im}(\Phi_N) \to N.$$

For $f : M \to N$ a C^r function, we can look at

$$\pi_N\biggl(\int_{y\in\mathbb{R}^{k_M}} \eta_\epsilon(y) f(\pi_M(x-y))dy\biggr).$$

This may not make sense, but this will make sense if ϵ is small enough. If ϵ is small, then $x-y$ is in the tubular neighborhood if $\eta_\epsilon(y)\neq 0$ and other things also make sense. Then

$$\eta_\epsilon *_\pi f : M \to N$$

is a well-defined map and this converges to f in the C^r-topology as $\epsilon \to 0$.

5.3 The third comparison

We now want to show

Theorem 5.3. $\mathrm{Diff}^\infty_{\partial,D}(M) \hookrightarrow \mathrm{Diff}^r_{\partial,D}(M)$ *are weak equivalences.*

Proof. Pick a collar $c : \partial M \times [0,1) \to M$ and a neat embedding $\varphi_M : M \hookrightarrow \mathbb{R}^{k_M-1}\times[0,\infty)$ and a tubular neighborhood $\Phi_M : \nu_M \to \mathbb{R}^{k_M-1}\times[0,\infty)$. This again gives a projection map $\pi_M : \mathrm{im}(\Phi_M) \to M$.

Pick a $\rho : [0,1) \to [0,1]$ that is 0 on $[0,\frac{1}{4}]$ and 1 near 1. Given a commutative diagram

$$\begin{tikzcd} S^i \arrow[r] \arrow[d] & \mathrm{Diff}^\infty_{\partial,D}(M) \arrow[d] \\ D^{i+1} \arrow[r, "G"] & \mathrm{Diff}^r_{\partial,D}(M), \end{tikzcd}$$

we need to show that we can homotope this so that there is a lift.

By sliding the collar, we can assume that all G_s for $s \in D^{i+1}$ are the identity on $\partial M \times [0,2]$. Now from $\rho : [0,1) \to [0,1]$, we can construct

$$\bar\rho : M \to [0,1]; \quad p \mapsto \begin{cases} \rho(t) & \text{if } p = (q,t) \in \partial M \times [0,1) \\ 1 & \text{otherwise.} \end{cases}$$

So I'm tuning it down near the boundary. For $s \in D^{i+1}$ define

$$\eta_\epsilon \bar{*}_\pi G_s(x) = \pi_M\left((1-\rho(x))G_s(x) + \rho(x)\int_{y\in\mathbb{R}^{k_M}} \eta_\epsilon(y) G_s(\pi_M(x-y))dy\right)$$

This, if defined, is then going to be a C^∞ function, because if $\rho(t)\neq 1$ then $H_s = \mathrm{id}$. For each $s \in D^{i+1}$ there exists a $\epsilon(s)$ such that this is well-defined and a diffeomorphism. Since $\mathrm{Diff}^r_{\partial,D}(M)$ is open in $C^r(M)$, there is a single $\epsilon_0 > 0$ that works for all $s \in D^{i+1}$.

Now our homotopy will be

$$[0,1] \ni \tau \mapsto \left(x \mapsto \pi_M\left((1-\tau\rho(x))G_s(x) + \tau\rho(x)\int_{y\in\mathbb{R}^{k_M}} \eta_\epsilon(y) G_s(\pi_M(x-y))dy\right)\right)$$

This shows that the inclusion is a weak equivalence. $\square$

5.4 Diffeomorphism groups as simplicial groups

There is of course the singular simplices $\mathrm{Sing}(\mathrm{Diff}^r_\partial(M))$ with

$$\mathrm{Sing}(\mathrm{Diff}^r_\partial(M))_k = \{\Delta^k \to \mathrm{Diff}^r_\partial(M)\}.$$

But there is also a more geometric way. Define

$$S\,\mathrm{Diff}^r_\partial(M)_k = \left\{ \begin{tikzcd} \Delta^k \times M \arrow[r] \arrow[dr] & \Delta^k \times M \arrow[d] \\ & \Delta^k \end{tikzcd} \text{ fixing } \Delta^k \times \partial M \text{ and } C^r \right\}$$

which is strictly smaller than $\mathrm{Sing}(\mathrm{Diff}^r_\partial(M))$, because we have the additional C^r condition. But you can show that the inclusion is a weak equivalence.

6 September 13, 2017

We have seen that the differentiability conditions really don't matter. So from now on, all diffeomorphisms and embeddings are C^∞. Today we are going to show that $\mathrm{Diff}_\partial(D^2) \simeq *$ by using Smale's "dynamical proof". Other proofs include

- Cerf/Gramain using "algebraic topology",
- Hatcher using "parametrized Morse theory",
- complex-analytic proof,
- curve-shortening flow.

6.1 Squares instead of disks

For Smale's proof, it's better to use squares instead of disks. To compare them, we use the following construction, If M and M' are n-dimensional manifolds without boundary, and we have an embedding $i : M \hookrightarrow M'$, then there is a continuous map

$$i_* : \mathrm{Diff}_{\partial,U}(M) \hookrightarrow \mathrm{Diff}_{\partial,U}(M'); \quad f \mapsto \left(p \mapsto \begin{cases} p & p \notin i(M) \\ i(f(i^{-1}(p))) & p \in i(M) \end{cases}\right).$$

If i is isotopic to i', then $(i)_*$ to $(i')_*$ are homotopic.

This means that if we also have an embedding $j : M' \hookrightarrow M$ such that $i \circ j$ is isotopic to id_M and $j \circ i$ is isotopic to id_M, then $(i)_*$ and $(j)_*$ are homotopy inverses.

Proposition 6.1. *$\mathrm{Diff}_{\partial,U}(I^2)$ and $\mathrm{Diff}_{\partial,U}(D^2)$ are homotopy equivalent.*

Proof. Take the scaling embedding that puts the circle in the square and square in the circle. □

6.2 Smale's theorem on $\mathrm{Diff}_\partial(D^2)$

So it suffices to prove $\mathrm{Diff}_{\partial,U}(I^2) \simeq *$. The idea is to take, for each $f \in \mathrm{Diff}_{\partial,U}(I^2)$, the vector field on I^2 given by pushing forward $\vec{e}_1$ along f:

$$\chi(f)(x,y) = Df_{f^{-1}(x,y)}(\vec{e}_1).$$

We can reconstruct f from this vector field $\chi(f)$. To see this, for convenience extend $\chi(f)$ to a vector field on $\mathbb{R}^2$ by $\vec{e}_1$. Then the flow $\Phi_{\chi(f)} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is well-defined. If we take $\{0\} \times \{y\} \in \{0\} \times I$ and flow from time t along $\chi(f)$, we get

$$\varphi(t) = \Phi_{\chi(f)}(t, 0, y) = f(t, y)$$

by looking at the differential equations they solve.

This $\chi(f)$ is not an arbitrary vector field, but has to be in

$\mathcal{S} = \{$smooth everywhere nonzero vector fields on I^2 that is $\vec{e}_1$ on $\partial I^2\}$.

On the other hand, every vector field $Y \in \mathcal{S}$ can be used to construct a diffeomorphism of I^2.

Lemma 6.2. *Let $Y \in \mathcal{S}$. For $(0, y) \in \{0\} \times I$, there exists a unique $\tau(Y, y) \in (0, \infty)$ such that $\Phi_Y(\tau(Y, y), 0, y) \in \{1\} \times I$. This time $\tau(Y, y)$ depends continuously on Y and smoothly of y.*

Proof. Uniqueness is clear, by just extending the vector field. Continuity of Y and smoothness of y follows from dependence of solutions of differential equations to initial conditions and coefficients.

The hard part is existence. The flowline γ, if it leaves I^2, has to leave through $\{1\} \times I$. So if $\tau(Y, y)$ does not exist, it has to stay inside the square. Consider the forwards limiting set

$$I^2 \supseteq \sigma^*(\gamma) = \bigcap_{\alpha>0} \overline{\gamma([\alpha, \infty))}.$$

Poincaré–Bendixson theorem says that ∞^* is one of the following:

(1) a fixed point of the flow
(2) a path between fixed points
(3) periodic orbit

Because the vector field is non-vanishing, (1) and (2) are ruled out. The periodic orbit is also ruled out by similar reasons. If there is a periodic orbit, then by something like Brouwer fixed-point theorem there has to be a fixed point. □

Now you would be tempted to write down

$$\Psi_Y : I^2 \to I^2; \quad \Psi_Y(x, y) = \Phi_Y(\tau(Y, y)x, 0, y),$$

but this is not the identity near the boundary.

So we tune this down by bump functions. Pick a smooth family of smooth embeddings $\eta_\tau : [0, 1] \to [0, \infty)$ defined for $\tau \in (0, \infty)$ such that

(i) $\eta_\tau(0) = 0$, $\eta_\tau([0, 1]) = [0, \tau]$,
(ii) η_τ has derivative 1 near 0 and 1,
(iii) $\eta_1 = \mathrm{id}$.

Now we write

$$\Psi_Y : I^2 \to I^2; \quad \Psi_Y(x, y) = \Phi_Y(\eta_{\tau(Y,y)}(y), 0, y).$$

Proposition 6.3. *Ψ_Y is a diffeomorphism of I^2 which is* id *near ∂I^2.*

Proof. It is a composition of $(x, y) \mapsto (\eta_{\tau(Y,y)}(x), 0, y)$ and $(t, y) \mapsto \Phi_Y(t, 0, y)$. □

Furthermore, we get a continuous map $\Psi : \mathcal{S} \to \mathrm{Diff}_{\partial,U}(I^2)$, and for $\chi(f)$, we get $\Psi_{\chi(f)} = f$. So to prove Smale's theorem, it suffices to prove that $\mathcal{S}$ deformation retracts onto $\chi(\mathrm{id}) = \vec{e}_1$.

Let's think of $\mathcal{S}$ as smooth maps $I^2 \to \mathbb{R}^2 \setminus \{0\}$ which equals $\vec{e}_1$ near the boundary. Let U denote the universal cover of $\mathbb{R}^2 \setminus \{0\}$. Then we can uniquely lift $Y \in S$ to $\tilde{Y} : I^2 \to U$ by specifying that $\tilde{Y}(0,0) = \vec{f}_1$. Note that U is contractible, and hence we can find a deformation retraction onto $\vec{f}_1$. This induces a deformation retraction of $\mathcal{S}$ onto $\vec{e}_1$.

Theorem 6.4 (Smale)**.** $\mathrm{Diff}_\partial(D^2) \simeq *$

6.3 Diffeomorphisms of S^2

There is certainly a map $\mathrm{O}(3) \hookrightarrow \mathrm{Diff}(S^2)$, and you have

$$\mathrm{Diff}(S^2) \to \mathrm{Fr}^{\mathrm{GL}}(TS^2) \to \mathrm{Fr}^{\mathrm{O}}(TS^2) \cong \mathrm{O}(3).$$

You can check that it follows $\mathrm{Diff}(S^2) = \mathrm{O}(3) \times G$, and this G will be contractible.

7 September 15, 2017

Today I am going to talk about the parametrized isotopy extension theorem. This is the beginning of actually proving things like $\mathrm{Diff}(S^2) \simeq \mathrm{O}(3)$ or the Gromain/Cerf's proof of Smale's theorem.

7.1 Isotopy extension

Before doing a parametrized case, let's do a non-parametrized case. Given a path $\gamma : [0,1] \to M$, is there a path of diffeomorphisms inducing it? This is intuitively clear; you assume that the space is made out of flexible material, put your finger at the point and drag along the curve.

Definition 7.1. An **isotopy of embeddings** $M \hookrightarrow N$ is a neat embedding

$$g : M \times [0,1] \hookrightarrow N \times [0,1]$$

that preserves the $[0,1]$ coordinate. An **isotopy of diffeomorphisms** is a diffeomorphism

$$f : N \times [0,1] \to N \times [0,1]$$

that preserves the $[0,1]$ coordinate.

In these cases, $g_t = g|_{M\times\{t\}} : M \hookrightarrow N$ is an embedding and $f_t = f|_{N\times\{t\}} : N \times N$ is a diffeomorphism.

Alternative definitions include $g : M \times [0,1] \to N$ being smooth and g_t is an embedding for all t. But this is slightly different. You could also say that an isotopy is a map $g : [0,1] \to \mathrm{Emb}(M,N)$, but the problem is that $M \times [0,1] \to N$ might not be smooth. So we are not going to use these definitions.

Question. *Given an isotopy of embeddings $g : M \times [0,1] \hookrightarrow N \times [0,1]$, when does there exist an isotopy of diffeomorphisms $f : N \times [0,1] \to N \times [0,1]$ such that $g_t = f_t \circ g_0$?*

The answer is yes if g is compactly supported away from ∂M, i.e., there exists a compact $K \subseteq M \setminus \partial M$ such that $g_t|_{M\setminus K} = g_0|_{M\setminus K}$.

Theorem 7.2. *For $g : M \times [0,1] \hookrightarrow N \times [0,1]$ a compactly-supported isotopy of embeddings, there exists a compactly-supported isotopy of diffeomorphisms*

$$f : N \times [0,1] \to N \times [0,1]$$

such that $f_0 = \mathrm{id}$ and $g_t = f_t \circ g_0$.

Proof. Look at the vector field $\partial/\partial t$ on $M \times [0,1]$. We can push it forward along g to get a smooth section $\chi(g)$ of $T(N \times [0,1])|_{g(M\times[0,1])}$. Then the flow along $\chi(g)$ for time t starting at $g(p,0)$ will end up at $g(p,t) = (g_t(p), t)$.

If we can extend $\chi(g)$ to a vector field $\chi(f)$ on $N \times [0,1]$ such that $\pi_*(\chi(f)) = \partial/\partial t$, then we can attempt to flow along $\chi(f)$ to define f. This will be compactly-supported and exist if $\chi(f)$ equals $\partial/\partial t$ (on $N \times [0,1]$) outside of a compact set and $\pi_*(\chi(f)) = \partial/\partial t$. To construct this, it actually suffices that $\pi_*(\chi(f))$ is a positive multiple of $\partial/\partial t$ on a compact subset. After this we can fix this by rescaling.

We will produce $\chi(f)$ locally and patch these together by a partition of unity. Take the finitely many charts covering the image of the support of g. Let these be

$$\varphi : N \times [0,1] \supseteq V_i \to W_i \subseteq \mathbb{R}^{n-1} \times [0,\infty)$$

such that $\varphi_i(g(M \times [0,1]) \cap V_i) = (\mathbb{R}^{m-1} \times [0,\infty)) \cap W_i$. To extend $\chi(g)|_{V_i}$, we can instead extend

$$(\varphi_i)_*(g_*)\Big(\frac{\partial}{\partial t}\Big)$$

can we can do this by making it constant in the remaining $(n-m)$ directions. Pull them back to V_i using φ_i to get $\chi_i(f)$ on V_i. Now $\pi_*(\chi_i(f))$ is equal to $\partial/\partial t$ on $g(M \times [0,1]) \cap C_i$. By shrinking V_i if necessary, we can guarantee that $\pi_*(\chi_i(f))$ is a positive multiple of $\partial/\partial t$.

Now take $V_0 \subseteq N \times [0,1]$ open outside $g(M \times [0,1]$, so that $V_0, V_1, \dots, V_k$ is an open cover of $N \times [0,1]$. Now take a partition of unity $\{\eta_i\}_{i=0}^k$ with respect to this cover, and

$$\chi(f) = \eta_0 \cdot \frac{\partial}{\partial t} + \sum_{i=1}^k \eta_i \chi_i(f)$$

works. $\square$

Example 7.3. Without the compact-support condition, this fails. For instance, take a knot in $\mathbb{R}^3$ with ends at infinities. Consider an isotopy that in time $[0,1]$ moves it to ∞. There can't be an isotopy of diffeomorphisms extending them, because the embeddings at $t = 0$ and $t = 1$ are not even diffeomorphic.

7.2 Parametrized isotopy extension

The proof generalizes to the following. You could either do this inductively or prove similarly.

Theorem 7.4. *Given a k-parameter family of embeddings $g : M \times \Delta^k \hookrightarrow N \times \Delta^k$ which has compact support $K \subseteq M \setminus \partial M$, there exists a compactly supported k-parameter isotopy of diffeomorphisms $f : N \times \Delta^k \to N \times \Delta^k$ such that $g_t = f_t \circ g_{(1,0,\dots,0)}$.*

Definition 7.5. The simplicial set $\mathrm{SEmb}(M,N)$ is defined as k-simplices given by k-parameter families of embeddings

$$\begin{tikzcd} M \times \Delta^k \arrow[r, hook] \arrow[dr] & N \times \Delta^k \arrow[d] \\ & \Delta^k. \end{tikzcd}$$

There is also a sub-simplicial set $\mathrm{SEmb}_{c,\partial,U}(M,N)$ consisting of compactly supported embeddings that is equal to some fixed g_0 near $\partial M \times \Delta^k$.

Corollary 7.6. *The map* $\mathrm{SDiff}_{c,\partial,U}(N) \to \mathrm{SEmb}_{c,\partial,U}(M,N)$ *is a Kan fibration.*

Theorem 7.7. *The map* $\mathrm{SEmb}_{c,\partial,U}(M,N) \hookrightarrow \mathrm{Sing}(\mathrm{Emb}_{c,\partial,U}(M,N))$ *is an equivalence.*

You can move the boundary as well as long as

(i) ∂M stays always in ∂N, or

(ii) ∂M stays always in $\mathrm{int}(N)$.

It is bad if ∂M touches ∂N at one point but lies in $\mathrm{int}(N)$ at another point.

8 September 18, 2017

Last time we proved the parametrized isotopy extension on our way to diffeomorphisms of S^2. Next time I will do Gramain/Cerf's proof of Smale's theorem. We will be using parametrized isotopy extension throughout the course.

8.1 Embeddings of $\mathbb{R}^m$

Theorem 8.1. *The inclusions*

$$\mathrm{O}(m) \hookrightarrow \mathrm{GL}_m(\mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{R}^m) \hookrightarrow \mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m)$$

are weak equivalences.

Proof. We'll start with $\mathrm{GL}_m(\mathbb{R}) \hookrightarrow \mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m)$. There will be three steps:

(i) translate to fix the origin,

(ii) make it linear near the origin,

(iii) push non-linearity out to ∞.

The reason (iii) works is because we are using the weak Whitney topology, not the strong one.

First, translation gives a deformation retract of $\mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m)$ to the subspace of embeddings fixing the origin. Explicitly, we use

$$[0,1] \ni \tau \mapsto (p \mapsto h(p) - \tau h(0)).$$

In the second step, we need to show that each commutative diagram

$$\begin{tikzcd} S^i \arrow[r] \arrow[d] & \mathrm{GL}_m(\mathbb{R}) \arrow[d] \\ D^{i+1} \arrow[r] & \mathrm{Emb}_0(\mathbb{R}^m, \mathbb{R}^m) \end{tikzcd}$$

may be homotoped through commutative diagrams such that there is a lift. Let $\eta : \mathbb{R}^m \to [0,1]$ be C^∞ which is 1 near 0 and supported in D^m. By compactness of D^{i+1}, there exists an $\epsilon_0 > 0$ such that

$$[0,1] \ni \tau \mapsto \Big(p \mapsto \Big(1 - \tau\eta\Big(\frac{p}{\epsilon_0}\Big)\Big) H_s(p) + \tau\eta\Big(\frac{p}{\epsilon_0}\Big) D_0 H_s(p)\Big).$$

This is an embedding for all $s \in D^{i+1}$ and $\tau \in [0,1]$. At $\tau = 0$, it is H_s and at $\tau = 1$, it is a linear map $D_0 H_s$ near the origin.

For the third step, for $\lambda \in (0, \infty)$ let r_λ be the diffeomorphism $p \mapsto \lambda p$. Denote the result of the second step at $\tau = 1$ by $H_s^{(1)}$. Consider

$$[0,1] \ni \tau \mapsto \begin{cases} r_{1-\tau}^{-1} \circ H_s^{(1)} \circ r_{1-\tau} & \tau < 1 \\ D_0 S_s^{(1)} & \tau = 1. \end{cases}$$

This is continuous at $\tau = 0$ is $H_s^{(1)}$ and at $\tau = 1$ is linear. Also, note that it fixes $\mathrm{GL}_m(\mathbb{R})$ pointwise.

So we get that $\mathrm{GL}_m(\mathbb{R}) \hookrightarrow \mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m)$ is a weak equivalence.

$$\begin{tikzcd} \mathrm{O}(m) \arrow[r, hook, "\simeq"] & \mathrm{GL}_m(\mathbb{R}) \arrow[r, hook, "\simeq"] \arrow[d, hook, "\simeq"] & \mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m) \\ & \mathrm{Diff}(\mathbb{R}^m) \arrow[ur, hook] & \end{tikzcd}$$

The map $\mathrm{O}(m) \hookrightarrow \mathrm{GL}_m(\mathbb{R})$ is an equivalence by Gram–Schmidt, and $\mathrm{GL}_m(\mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{R})$ being an equivalence follows from that if we started out with a diffeomorphism it would stay a diffeomorphism. ☐

Note that the map $\mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^m) \to \mathrm{GL}_m(\mathbb{R})$ given by taking the derivative is a weak homotopy inverse.

Now let us look at embeddings into a manifold M. The map taking derivative generalizes to

$$\mathrm{Emb}(\mathbb{R}^m, M) \to \mathrm{Fr}^{\mathrm{GL}}(TM); \quad f \mapsto (f(0), [D_0 f(\vec{e}_1), \ldots, D_0 f(\vec{e}_m)]).$$

Theorem 8.2. *The map* $\mathrm{Emb}(\mathbb{R}^m, M) \to \mathrm{Fr}^{\mathrm{GL}}(TM)$ *is a weak equivalence.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Emb}(\mathbb{R}^m, M) \arrow[r] \arrow[d, "\mathrm{ev}_0"] & \mathrm{Fr}^{\mathrm{GL}}(TM) \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M. \end{tikzcd}$$

Here the right vertical map is a locally trivial bundle and so a fibration, and the left vertical map is also a fibration by parametrized isotopy extension. So it suffices to prove that the map of fibers is a weak equivalence.

This map is

$$\mathrm{Emb}_0(\mathbb{R}^m, M) \to \mathrm{GL}_m(\mathbb{R}),$$

but by shrinking $\mathbb{R}^m$, we can modify any compact family in $\mathrm{Emb}_{0 \mapsto p}(\mathbb{R}^m, M)$ to land it a chart. This reduces the problem to the case $M = \mathbb{R}^m$. ☐

8.2 Connected sums

Let M and N be oriented m-dimensional manifolds, and let $\varphi_M : D^m \hookrightarrow M$ be an orientation-preserving map, and $\varphi_N : D^m \hookrightarrow N$ be an orientation-reversing map.

Definition 8.3. The **connected sum** is given by

$$M \# N = (M \setminus \mathrm{int}(\varphi_M(D^m))) \cup_{S^{m-1}} (N \setminus \mathrm{int}(\varphi_N(D^m))).$$

Note that this new manifold is oriented. We'll show that this construction is independent up to diffeomorphisms of φ_M and φ_N. (Here, we have to assume that M and N are connected.)

Lemma 8.4. *If φ'_M is isotopic to φ_M, then*

$$M\#_{(\varphi_M,\varphi_N)}N \cong M\#_{(\varphi'_M,\varphi_N)}N.$$

Proof. By isotopy extension, there is a diffeomorphism $f : M \to M$ such that $f \circ \varphi_M = \varphi'_M$. Then we can construct

$$F : M\#_{(\varphi_M,\varphi_N)}N \to M\#_{(\varphi'_M,\varphi_N)}N; \quad p \mapsto \begin{cases} f(p) & p \in M \setminus \mathrm{int}(\varphi_M(D^m)) \\ p & \text{otherwise.} \end{cases}$$

This is a diffeomorphism. □

Thus to show that connected sums are well-defined up to diffeomorphism, it suffices to show that the space of orientation-preserving embeddings is path-connected.

Lemma 8.5. *If M is path-connected and oriented, then $\mathrm{Emb}^+(D^m, M)$ is path-connected.*

Proof. We get a map

$$\mathrm{Emb}^+(D^m, M) \to \mathrm{Emb}^+(\mathbb{R}^m, M)$$

by restricting to the interior. This is a homotopy equivalence because $\mathbb{R}^m \hookrightarrow D^m$ has an inverse up to isotopy. Now we have computed $\mathrm{Emb}^+(\mathbb{R}^m, M)$, which is the oriented frame bundle $\mathrm{Fr}^{\mathrm{GL},+}(TM)$. There is then a fiber sequence

$$\mathrm{SO}(n) \to \mathrm{Fr}^{\mathrm{GL},+}(TM) \to M.$$

Because the first and last are path-connected, the total space is also path-connected. □

8.3 Diffeomorphisms of S^2

Theorem 8.6. $\mathrm{Diff}(S^2) \simeq \mathrm{O}(3)$

Proof. We have a fibration

$$\mathrm{Diff}(S^2) \to \mathrm{Emb}(D^2, S^2)$$

by isotopy extension. Now the homotopy of the base can be computed as

$$\mathrm{Emb}(D^2, S^2) \simeq \mathrm{Emb}(\mathbb{R}^2, S^2) \simeq \mathrm{Fr}^{\mathrm{GL}}(TS^2) \simeq \mathrm{Fr}^{\mathrm{O}}(TS^2) \simeq \mathrm{O}(3).$$

The fiber of the map is those diffeomorphisms of S^2 that fix D^2. Then this is $\mathrm{Diff}_{\partial,D}(D^2) \simeq *$. There is a map $\mathrm{O}(3) \hookrightarrow \mathrm{Diff}(S^2)$, and if you trace through all the maps,

$$\mathrm{O}(3) \hookrightarrow \mathrm{Diff}(S^2) \to \mathrm{Emb}(D^2, S^2) \simeq \mathrm{O}(3)$$

is the identity. □

9 September 20, 2017

Today we will finish talking about 2-dimensional stuff. We will reprove Smale's theorem $\mathrm{Diff}_\partial(D^2) \simeq *$.

9.1 Gramain's proof of $\mathrm{Diff}_\partial(D^2) \simeq *$

We will give a reformulation of Smale's theorem in terms of spaces of arcs. For the diameter γ_0, there is a action

$$\mathrm{Diff}_\partial(D^2) \xrightarrow{\text{action on } \gamma_0} \mathrm{Emb}_\partial(I, D^2).$$

This is a fibration by parametrized isotopy extension, and its fiber is going to be $\mathrm{Diff}_\partial(D^2) \times \mathrm{Diff}_\partial(D^2)$. Actually we need to be careful and sometimes replace things with fixing the neighborhood of boundary and replacing by a homotopy equivalent thing, but this is annoying and we are not going to do this.

The long exact sequence of homotopy groups based at id is going to be

$$\begin{array}{ccccccc}
\pi_2(\mathrm{Diff}_\partial(D^2))^2 & \longrightarrow & \pi_2(\mathrm{Diff}_\partial(D^2)) & \longrightarrow & \pi_2(\mathrm{Emb}_{[\gamma_0],\partial}(I, D^2)) & \longrightarrow & \\
\pi_1(\mathrm{Diff}_\partial(D^2))^2 & \longrightarrow & \pi_1(\mathrm{Diff}_\partial(D^2)) & \longrightarrow & \pi_1(\mathrm{Emb}_{[\gamma_0],\partial}(I, D^2)) & \longrightarrow & \\
\{[g], [g^{-1}]\} & \longrightarrow & \{[\mathrm{id}]\} & \longrightarrow & \{[\gamma_0]\}. & &
\end{array}$$

Note that there is a section (up to homotopy) of $\mathrm{Diff}_\partial(D^2)^2 \to \mathrm{Diff}_\partial(D^2)$ given by $g \mapsto (g, \mathrm{id})$. This shows that the maps $\pi_k(\mathrm{Diff}_\partial(D^2))^2 \to \pi_k(\mathrm{Diff}_\partial(D^2))$ is surjective, and so we have short exact sequences. Then

$$\pi_{i+1}(\mathrm{Emb}_{[\gamma_0],\partial}(I, D^2)) \cong \pi_i(\mathrm{Diff}_\partial(D^2)).$$

So Smale's theorem is equivalent to that $\mathrm{Emb}_{[\gamma_0],\partial}(I, D^2)$ is weakly contractible.

Extend the disk D^2 to $\tilde{D}^2$ on one side of γ_0, and define $T = \tilde{D} \setminus S$. Then there is a fiber sequence

$$\mathrm{Emb}_\partial(I \cup S, \tilde{D}^2 \mathrm{rel} S) \to \mathrm{Emb}_\partial(I \cup S, \tilde{D}^2) \to \mathrm{Emb}(S, \tilde{D}^2)$$

by isotopy extension. The last one is weakly equivalent to $\mathrm{O}(2)$ because it is an embedding of a disk into a manifold. Next, $\mathrm{Emb}_\partial(I \cup S, \tilde{D}^2) \simeq *$ because we can contract this lollipop to near the fixed point. So we can say that

$$\mathrm{Emb}_{[\gamma_0],\partial}(I, T) = \mathrm{Emb}_{[\gamma_0],\partial}(I \cup S, \tilde{D}^2) \simeq *.$$

Now let β_0 be the path on the handle part of T. Then clearly

$$\mathrm{Emb}_{[\gamma_0],\partial}(I, T \setminus \beta_0) \simeq \mathrm{Emb}_{[\gamma_0],\partial}(I, D^2).$$

To show that this is weakly contractible, it suffices to show that the map

$$\mathrm{Emb}_{[\gamma_0],\partial}(I, T \setminus \beta_0) \hookrightarrow \mathrm{Emb}_{[\gamma_0],\partial}(I, T) \simeq *$$

induces injections on π_*.

Consider a universal cover $\tilde{T}$ of T and identify this with $\mathbb{R} \times I$. Then $\mathrm{Emb}_{[\gamma_0],\partial}(I, T)$ can be identified with the subspace $E \subseteq \mathrm{Emb}_{[\tilde{\gamma}_0],\partial}(I, \mathbb{R} \times I)$ consisting of path that stays an embedding on T. Similarly $\mathrm{Emb}_{[\gamma_0],\partial}(I, T \setminus \beta_0)$ may be identified with $\mathrm{Emb}_{[\tilde{\gamma}_0],\partial}(I, (-1,1) \times I)$.

Now we have a diagram commuting up to homotopy on compact support:

$$\begin{tikzcd}
\mathrm{Emb}_{[\gamma_0],\partial}(I, T \setminus \beta_0) \arrow[r, hook] \arrow[dd, "\simeq"] & \mathrm{Emb}_{[\gamma_0],\partial}(I, T) \arrow[d, hook, "\cong"] \\
& E \arrow[d, hook] \\
\mathrm{Emb}_{[\tilde{\gamma}_0],\partial}(I, (-1,1) \times I) & \mathrm{Emb}_{[\tilde{\gamma}_0],\partial}(I, \mathbb{R} \times I) \arrow[l, "r"']
\end{tikzcd}$$

This shows that the first horizontal map induces injection on π_*. Therefore we get $\mathrm{Emb}_{[\gamma_0],\partial}(I, D^2) \simeq *$.

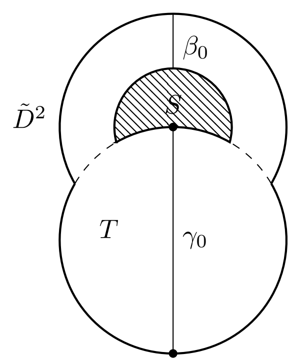

Figure 1: Extending D^2 to $\tilde{D}^2$

9.2 Space of arcs

We could replace D^2 by Σ a compact manifold. The path could go around some handles in Σ and end at the same boundary component, and we could end at a different boundary component.

Theorem 9.1. $\mathrm{Emb}_{[\gamma_0],\partial}(I, \Sigma) \simeq *$.

Proof. Let's first consider the first case, where the two paths have endpoints in different boundary components. Fill in one of a boundary component, and call this S. (We are taking Σ to be something like T.) Now we can do the same argument and use the fiber sequence

$$\mathrm{Emb}_\partial(I,\Sigma) \to \mathrm{Emb}_\partial(I\cup S,\Sigma\cup S) \to \mathrm{Emb}(S,\Sigma\cup S).$$

The middle one is contractible by the lollipop thing, and the embedding of disk is going to be given by a frame bundle $\mathrm{Fr}^{\mathrm{O}}(T(\Sigma\cup S))$. This lies in a fiber sequence

$$\mathrm{O}(2) \to \mathrm{Fr}^{\mathrm{O}}(T(\Sigma\cup S)) \to \Sigma\cup S.$$

This shows that $\mathrm{Emb}(S,\Sigma\cup S)$ has no π_i for $i>1$.

For the second case when the endpoints of the path lie in the same boundary component, we attach a handle as we've done for D^2 and form T. Take a path β_0 inside the handle. It again suffices to show that

$$\mathrm{Emb}_{[\gamma_0]}(I,T\setminus\beta_0) \to \mathrm{Emb}_{[\gamma_0],\partial}(I,T)$$

is injective on homotopy groups.

We use a cover $\tilde{T}$ of T corresponding to $\pi_1(\Sigma)\subseteq\pi_1(T)=\mathbb{Z}*\pi_1(\Sigma)$. We can use this to get a retraction of $\mathrm{Emb}_{[\gamma_0],\partial}(I,U)$ to $\mathrm{Emb}_{[\gamma_0],\partial}(I,T\setminus\beta_0)$. □

9.3 Path components of $\mathrm{Diff}_\partial(\Sigma)$

Theorem 9.2. *If Σ is an oriented path-connected compact surface with $\partial\Sigma\neq\emptyset$, then the path-components of $\mathrm{Diff}_\partial(\Sigma)$ are weakly contractible.*

Proof. It suffices to prove this for the path component $\mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma)$ of the identity. We induct over the genus g and the number of boundary components n. The case $(g,n)=(0,1)$ is D^2 and Smale's theorem.

Pick a path γ_0, and then there is a fibration

$$\mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma_{g,n}\,\mathrm{rel}\,\gamma_0) \to \mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma_{g,n}) \to \mathrm{Emb}_{[\gamma_0],\partial}(I,\Sigma_{g,n}).$$

Then this $\mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma_{g,n}\,\mathrm{rel}\,\gamma_0)$ is actually $\mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma_{g,n-1})$ or $\mathrm{Diff}^{\mathrm{id}}_\partial(\Sigma_{g-1,n+1})$. □

10 September 22, 2017

Hatcher proved $\mathrm{Diff}_\partial(D^3) \simeq *$, and we are going to prove an almost equivalent statement about embeddings.

10.1 Restating Smale's theorem

Proposition 10.1 (Restatement of Smale's theorem). *The map* $\mathrm{Emb}(D^2, \mathbb{R}^2) \to \mathrm{Emb}(S^1, \mathbb{R}^2)$ *is a weak equivalence.*

Proof. We have

$$\begin{tikzcd} \mathrm{Diff}(\mathbb{R}^2) \arrow[r, equal] \arrow[d, hook] & \mathrm{Diff}(\mathbb{R}^2) \arrow[d, hook] \\ \mathrm{Emb}(D, \mathbb{R}^2) \arrow[r] & \mathrm{Emb}(S^1, \mathbb{R}^2), \end{tikzcd}$$

where the vertical maps are fibrations by parametrized isotopy extension. The map of fibers is

$$\mathrm{Diff}(\mathbb{R}^2 \operatorname{rel} D^2) \to \mathrm{Diff}(\mathbb{R}^2 \operatorname{rel} S^1) \cong \mathrm{Diff}(\mathbb{R}^2 \operatorname{rel} D^2) \times \mathrm{Diff}_\partial(D^2).$$

Now $\mathrm{Diff}_\partial(D^2)$ is contractible by Smale's theorem, and so it is a weak equivalence. Now we can use the long exact sequence.

We still have to show that $\mathrm{Emb}(D^2, \mathbb{R}^2) \to \mathrm{Emb}(S^1, \mathbb{R}^2)$ is surjective on π_0. □

Proposition 10.2. *Every smooth embedding $\varphi : S^1 \hookrightarrow \mathbb{R}^2$ extends to a smooth embedding $D^2 \hookrightarrow \mathbb{R}^2$.*

Proof. I'm only going to give a sketch. We'll use from Morse theory that every smooth map $S^1 \to \mathbb{R}$ can be perturbed to have the following properties:

(i) the set of $p \in S^1$ such that $D_p f = 0$ is finite,

(ii) the $f(p) \in \mathbb{R}$ for p critical point are distinct,

(iii) near each critical point p we can find coordinates x on S^1 such that p corresponds to $x = 0$ and $f(x) = f(p) + \alpha x^2$ for $\alpha \neq 0$.

Let us consider

$$h : S^1 \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{\pi_2} \mathbb{R}.$$

and perturb h to $\tilde{h}$ that satisfies (i), (ii), (iii). Since embeddings are open in $C^\infty(S^1, \mathbb{R}^2)$, linear interpolation from φ to $\tilde{\varphi}$ with y-coordinate replaced by $\tilde{h}$.

By isotopy extension, φ can be extended to D^2 if and only if $\tilde{\varphi}$ can be. Now take a set $\{y_i\}_{i=1}^k \subseteq \mathbb{R}$ such that the lines $\mathbb{R} \times \{y_i\}$ separate all critical points. By linear interpolation, we can make $\mathrm{im}(\varphi)$ vertical near $\mathbb{R} \times \{y_i\}$. Cut along $\mathbb{R} \times \{y_i\}$ by innermost pairs (points in $\mathrm{im}(\tilde{\varphi}) \cap (\mathbb{R} \times \{y_i\})$ without intersection points between them).

This cuts the map $S^1 \hookrightarrow \mathbb{R}$ into many embeddings $S^1 \hookrightarrow \mathbb{R}$. Each of these individual pieces can be isotoped to have one of the five local models:

a rectangle, a parabola bounded above, parabola bounded below, and vertical reflections. Then we have formulas for disks that bound these local models, and isotopy extension gives that each of these pieces actually bound disks.

Now we need to reverse the cuts by adding or subtracting disks.

Lemma 10.3. *If M is a smooth manifold and $\varphi_0 : D^{m-1} \hookrightarrow \partial D^m$ and $\varphi_1 : D^{m-1} \hookrightarrow \partial M$ are embeddings, then $M \cong M \cup_{D^{n-1}} D^n$.*

Using this lemma, we can glue these maps together to get a extension of $\tilde{\varphi}$ to D^2. The parametrization of $\mathrm{im}(\tilde{\varphi})$ and $\tilde{\varphi}$ on ∂D^2 might differ, but $\pi_0 \operatorname{Diff}(S^1) = \mathbb{Z}/2\mathbb{Z}$ so we can line them up well. $\square$

10.2 Hatcher's theorem

Theorem 10.4 (Hatcher)**.** $\operatorname{Emb}(D^3, \mathbb{R}^3) \to \operatorname{Emb}(S^2, \mathbb{R}^3)$ *is a weak equivalence.*

Note that this is equivalent to $\operatorname{Diff}_\partial(D^3) \simeq *$ and that every embedded S^2 in $\mathbb{R}^3$ bounds a disk.

Proposition 10.5 (Alexander's theorem)**.** *Every $S^2 \hookrightarrow \mathbb{R}^3$ bounds a disk.*

Proof. First, make a height function that generic Morse. Next pick non-critical $\{z_i\}_{i=1}^k$ such that the planes $\mathbb{R}^2 \times \{z_i\}$ separate all the critical points. Cut along $\mathbb{R}^2 \times \{z_i\}$ inner most circles first. Isotope pieces to be given by one of 7 standard models using $\operatorname{Emb}(S^1, \mathbb{R}^2)$. These are all given by formulas, and we can check that they all bound D^3. Now reverse the isotopies and reassemble to D^3 using the lemma about adding and subtracting disks. Finally use $\operatorname{Diff}(S^2) \simeq \mathrm{O}(3)$ to adjust parametrization of ∂D^3. $\square$

Why can't you do this in dimension 4? In this case, the standard models will contain some handle bodies. Then the lemma about adding and subtracting disks is not going to work. It is really something special about dimension 3.

Corollary 10.6. $\operatorname{Diff}(S^3) \simeq \mathrm{O}(4)$.

Corollary 10.7 (Hatcher–Ivanov)**.** *If M is a Haken 3-manifold, then*

$$\operatorname{Diff}^{\mathrm{id}}_\partial(M) \to h \operatorname{Aut}^{\mathrm{id}}_\partial(M)$$

is a weak equivalence.

11 September 25, 2017

We have been comparing diffeomorphism groups, did some dimension 2 and a little bit of dimension 3. We are now going towards proving the s-cobordism theorem and using this to understand $\pi_i \mathcal{M}_\partial(D^n)$ for $n \geq 6$ and $i = 0, 1$.

In this lecture, all manifolds have empty boundary.

11.1 Sard's lemma

Definition 11.1. Let $f : M \to N$ be a smooth map. A point $a \in M$ is said to be a **critical point** if $D_q f : T_q M \to T_{f(q)} N$ is not surjective. Then $f(q) \in N$ is said to be a **critical value**.

If $D_q f : T_q M \to T_{f(q)} N$ is surjective, then q is said to be a **regular point**. A point $p \in N$ is a **regular value** $f^{-1}(p)$ consists of regular points, i.e., p is not a critical value.

The inverse function theorem says that $f^{-1}(M) \subseteq N$ is a smooth manifold if $p \in N$ is a regular value (and f is proper).

Theorem 11.2 (Sard's lemma)**.** *Let $U \subseteq \mathbb{R}^m$ be an open subset and let $f : U \to \mathbb{R}^n$ be a smooth map. Then the set* $\operatorname{Crit}(f)$ *of critical values has measure zero.*

Since measure zero subsets cannot contain subsets, the set of regular values is dense.

Corollary 11.3. *The set* $\operatorname{Crit}(f)$ *for $f : M \to N$ smooth has measure zero.*

Proof. Let us first do the case when $M = U_i \subseteq \mathbb{R}^m$ is open. Then cover N by a countable collection of charts. $\varphi_i : N \supseteq V_i \to W_i \subseteq \mathbb{R}^n$. Then

$$\operatorname{Crit}(f) = \bigcup_i \varphi_i^{-1}(\operatorname{Crit}(\varphi_i \circ f|_{f^{-1}(V_i)}))$$

and so is a countable union of measure zero sets. Hence it has measure 0.

For the general case, cover M by charts and do the same thing. Then the critical values are a countable union of measure 0 sets, and so is also of measure 0. ☐

Proposition 11.4. *Every compact smooth M can be embedded in $\mathbb{R}^{2m+1}$.*

Proof. Take some embedding $\varphi : M \hookrightarrow \mathbb{R}^N$. If $N \leq 2m + 1$, we are done. Otherwise, we will show how to reduce N by 1. The strategy is to consider the projection

$$\pi_{v^\perp} \circ \varphi : M \to \mathbb{R}^{N-1}$$

where $\pi_{v^\perp}$ is the projection onto the hyperplane $v^\perp$. We'll find a v such that this is still an embedding.

Two things can go wrong.

(i) The map might not be injective, or

(ii) the derivative $D(\pi_{v^\perp} \circ \varphi)$ might not be injective somewhere.

(i) happens if v lies in the image of

$$F : M \times M \setminus \Delta_M \to S^{N-1}; \quad (x, y) \mapsto \frac{\varphi(x) - \varphi(y)}{\|\varphi(x) - \varphi(y)\|},$$

and (ii) happens if v lies in the image of

$$G : S(TM) = \{w : \|w\| = 1\} \to S^{N-1}; \quad w \mapsto \frac{D\varphi(w)}{\|D\varphi(w)\|}.$$

If $N > 2m + 1$, then $\mathrm{im}(F) = \mathrm{Crit}(F)$ and $\mathrm{im}(G) = \mathrm{Crit}(G)$ because the differential can never be surjective by counting the dimension. (The codomain has dimension $\geq 2m + 1$ but the domain has dimension $2m$.) So there exists a vector $v \in S^{N-1} \setminus (\mathrm{im}(F) \cup \mathrm{im}(G))$ by Sard's lemma. $\square$

11.2 Transversality

Definition 11.5. Let $f : M \to N$ and $g : M' \to N$ be smooth maps. We say that f is **transverse** to g (or $f \pitchfork g$) if for all $p \in f(M) \cap g(M')$, $q \in f^{-1}(p)$, $q' \in g^{-1}(p)$, we have that

$$D_q f(T_q M) + D_q g(T_q M') = T_p N.$$

Example 11.6. If $m+m' < n$, then $f \pitchfork g$ if and only if the images are disjoint.

There are special cases:

(1) If g is an inclusion of a submanifold X, then we use $f \pitchfork X$ instead of $f \pitchfork g$, and this is equivalent to for all $q \in f^{-1}(f(M) \cap X)$, the map

$$T_q M \xrightarrow{D_q f} T_{f(q)} N \xrightarrow{N} (\nu_X)_q$$

is surjective. In this case, the implicit function theorem says that $f^{-1}(X)$ is a smooth submanifold. In particular, $f \pitchfork \{x\}$ is equivalent to that x is a regular value of f.

(2) If f is also an inclusion of a submanifold M, then we use $M \pitchfork X$ instead of $f \pitchfork X$. This is equivalent to $T_p M + T_p X = T_p N$ for all $p \in M \cap X$. The implicit function theorem then implies that there exists a chart V around p in which M and X look like generically intersecting affine planes.

Theorem 11.7. *Every smooth map $f : M \to N$ can be approximated arbitrarily closely by a smooth map transverse to X.*

Proof. We shall prove a strongly relative version. Consider closed sets C_{done} and D_{todo} and open set $U_{\text{done}} \supseteq C_{\text{done}}$ and $V_{\text{todo}} \supseteq D_{\text{todo}}$. Suppose f is already transverse to X on an U_{done} and we need to make it transverse to X near D_{todo} without modifying it on a neighborhood on

$$C_{\text{done}} \cup (M \setminus V_{\text{todo}}).$$

(Here U and V need not be disjoint.)

First, consider the case when $M = U \subseteq \mathbb{R}^m$ is open and $X = \{0\}$ and $N = \mathbb{R}^r$. The condition $f \pitchfork \{0\}$ is equivalent to 0 being a regular value of f. By Sard's lemma, regular values are dense in $\mathbb{R}^r$ and so we can pick a sequence of regular values $\{x_n\}$ such that $x_n \to 0$. Consider $f_k = f - x_k$. This will mess with everything, so pick a bump function

$$\eta : U \to [0,1]$$

such that η is 0 near $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$ and 1 near D_{todo} intersect a smaller neighborhood of $C_{\text{done}} \subseteq U_{\text{done}}$. Instead, take $\tilde{f}_k = f - \eta x_k$. The claim is that if x_k is small enough, then x_k is transverse to 0 both near $C_{\text{done}} \cup D_{\text{todo}}$. We will continue next time. $\square$

12 September 27, 2017

Last time I made this claim

Theorem 12.1. *Every smooth map $f : M \to N$ (with $\partial M = \partial N = \emptyset$) can be approximated by a smooth map transverse to X.*

Proof. We are going to prove the strongly relative version. Let $f : M \to N$ such that $f \pitchfork X$ on $U_{\text{done}} \supseteq C_{\text{done}}$. We are going to fix the function near D_{todo} without changing it near $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$.

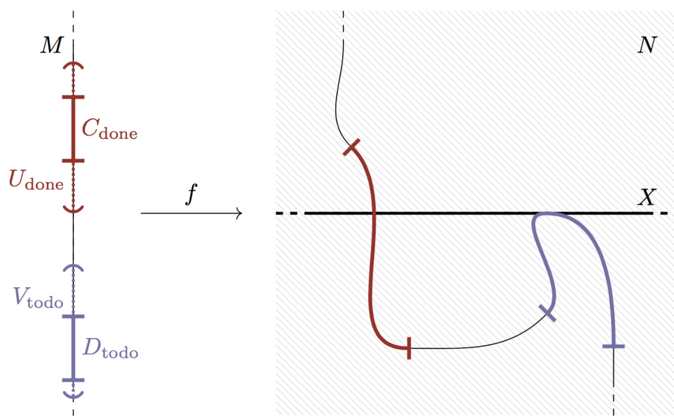

Figure 2: Figure of the strongly relative version setup

Case I. Suppose that $M \subseteq U = \mathbb{R}$ is open, $X = \{0\}$, $N = \mathbb{R}^r$ and D_{todo} is compact. The idea was to consider $f - x_k$ where x_k are regular values of f that converge to 0. Take $\eta : M \to [0, 1]$ smooth, such that $\eta = 1$ near $D_{\text{todo}} \setminus U_{\text{todo}}$, and $\eta = 0$ near $C_{\text{done}} \cup (M \setminus V_{\text{done}})$.

Let us take the function $f - x_k\eta$. This is transverse to $\{0\}$ near C_{done}, because it is f, and near $D_{\text{todo}} \setminus U_{\text{done}}$ because there it is $f - x_k$. What about $D_{\text{todo}} \cap U_{\text{done}}$? For a compact subset K, The condition of being transverse on K is an open condition, and so if k is large, the function is transverse.

Case II. Suppose $M = U \subseteq \mathbb{R}^m$ is open, D_{todo} is compact, and $N = X \times \mathbb{R}^r$. Then $f \pitchfork X$ is equivalent to $\pi \circ f \pitchfork \{0\}$. So it reduces to Case I.

Case III. Suppose $M = U \subseteq \mathbb{R}^m$ is open, D_{todo} is compact, ν_X is trivializable. Then the tubular neighborhood theorem gives an embedding $X \times \mathbb{R}^r \hookrightarrow N$. Now we substitute

$$\begin{aligned} M' &= f^{-1}(X \times \mathbb{R}^r), & f' &= f|_{M'}, \\ C'_{\text{done}} &= C_{\text{done}} \cap M', & U'_{\text{done}} &= U_{\text{done}} \cap M', \\ D'_{\text{todo}} &= f^{-1}(X \times D^r), & V'_{\text{todo}} &= V_{\text{todo}} \cap f^{-1}(X \times \operatorname{int}(2D^r)). \end{aligned}$$

Then it reduces to Case II.

Case IV. In the general case, we induct over charts. There exists a locally finite covering of X, U_α such that $\nu_X|_{U_\alpha}$ are all trivializable. Then there exists a locally finite chart of M by charts $\varphi_i : M \supseteq V_i \to W_i \subseteq \mathbb{R}^m$ such that

(i) $2D^m \subseteq W_i$,

(ii) $D_{\text{todo}} \subseteq \bigcup_i \varphi_i^{-1}(D^m)$,

(iii) for all i, there exists α such that $f(V_i) \cap X \subseteq U_\alpha$.

We can order the i and identify it with $\mathbb{N}$.

By induction we construction $f_i : M \to N$ which is transverse to X on U_i of

$$C_i = C_{\text{done}} \cup \bigcup_{j \le i} \varphi_i^{-1}(D^m).$$

For the induction step, apply Case II to

$$\begin{aligned} M' &= W_{i+1}, & f &= f_i \circ \varphi_{i+1}^{-1} \\ C'_{\text{done}} &= \varphi_{i+1}(C_i \cap V_{i+1}), & U'_{\text{done}} &= \varphi_{i+1}(U_i \cap V_{i+1}), \\ D'_{\text{todo}} &= \varphi_{i+1}(D_{\text{todo}} \cap V_{i+1}) \cap D^m, & V'_{\text{todo}} &= \varphi_{i+1}(V_{\text{todo}} \cap V_{i+1}) \cap \operatorname{int}(2D^m). \end{aligned}$$

after this, you take $\tilde{f} = \lim_{i\to\infty} f_i$, which is smooth and well-defined because near each point you change only finitely many times. Then $\tilde{f} \pitchfork X$ near $C_{\text{done}} \cup \bigcup_i \varphi_i^{-1}(D^m)$. □

By applying this inclusion of a submanifold,

Corollary 12.2. *If $M, X \subseteq N$ are compact submanifolds, then there exists an arbitrary small ambient isotopy φ_t on N such that $\varphi_1(M) \pitchfork X$.*

12.1 Jet transversality

We can reprove the previous theorem using jet transversality. Recall that $J^1(M, N)$ is given by pairs $(m, [g])$ of $m \in M$ and a 1-jet of smooth maps $g : M \to N$ near m. This 1-jet encodes the value of g and the first derivative. This is a smooth manifold of dimension $m + n + mn$. It is a locally trivial fiber bundle over $M \times N$ with fiber a finite-dimensional vector space. There is a map

$$j^1 : C^\infty(M, N) \to \Gamma(M, J^1(M, N)); \quad f \mapsto (m \mapsto (m, [f]_m)).$$

By considering local coordinates, we see that the images are smooth sections.

Let $X \subseteq M$ be a submanifold, and define $\mathcal{X} \subseteq J^1(M, N)$ as the submanifold consisting of $(m, [g])$ for $m \in X$.

Proposition 12.3. *$f \pitchfork X$ if and only if $j^1(f) \pitchfork \mathcal{X}$.*

The transversality result then also follows form the stronger theorem.

Theorem 12.4 (Jet transverality)**.** *Let $\mathcal{D} \subseteq J^r(M, N)$ be a stratified subset. Then every $f : M \to N$ can be approximated $\tilde{f}$ such that $j^r(\tilde{f}) \pitchfork \mathcal{D}$.*

Here, a subset Y of a manifold is **stratified** if it is a finite union of i-dimensional manifolds Y_i, with $Y_i \cap Y_j = \emptyset$ if $i \neq j$ and $\overline{Y}_i = \bigcup_{j \leq i} Y_j$. Transversality then is transversality to all the strata.

Morse functions are function $f : M \to \mathbb{R}$ with non-degenerate critical points, and this can be rephrased as $j^2(f) \pitchfork \mathcal{D}_{\text{Morse}}$.

Example 12.5. Let M, N be smooth manifolds and suppose $m > n$. Then $f : M \to N$ will have a dense set of regular values in N, but the rank of the derivative drops somewhere for algebraic topology reasons. How nice can we make the set where the derivative has i-dimensional cokernel?

Let $\Sigma^i \subseteq J^1(M, N)$ be the subspace of $(m, [g])$ where the cokernel of the derivative part of $[g]$ has dimension i. Then $\{\Sigma^i\}$ is a stratified subset (covering $J^1(M, N)$) and the codimension of Σ^i is $i(m - n + i)$. Jet transversality implies that we can approximate f by $\tilde{f}$ such that $\Sigma^i(f) = \{m : \dim \operatorname{coker} D = i\}$ is a codimension $i(m - n + i)$ smooth manifold.

13 September 29, 2017

We will use jet transversality and develop Morse theory.

13.1 Morse functions

Definition 13.1. **Morse functions** are smooth functions $f : M \to \mathbb{R}$ with "nondegenerate critical points".

Given a function $f : \mathbb{R}^m \to \mathbb{R}$, its Hessian $H_0(f)$ at 0 is defined in terms of the Taylor approximation

$$f(x) = f(0) + \langle \nabla_0 f, x\rangle + \frac{1}{2}\langle H_0(f)x, x\rangle + \cdots .$$

The problem is that the Hessian is not invariant under coordinate change. However, it turns out that when 0 is a critical point, it is well-defined.

For a critical point $p \in M$ of $f : M \to \mathbb{R}$, we can consider for $v, w \in T_pM$ and extend it to a vector field $\tilde{v}, \tilde{w}$ near p. Then

$$\tilde{v}(\tilde{w}(f))[p] - \tilde{w}(\tilde{v}(f))[p] = [\tilde{v}, \tilde{w}](f)[p] = 0$$

because p is a critical point. Also the left hand side is independent of the extension $\tilde{v}$ and the right hand side is independent of the extension $\tilde{w}$. Hence this expression is independent of neither and we get a well-defined function

$$\mathrm{Hess}_p(f) : T_pM \otimes T_pM \to \mathbb{R}.$$

Definition 13.2. A critical point $p \in M$ of f is **nondegenerate** if $\mathrm{Hess}_p(f)$ is nondegenerate.

Classification of symmetric bilinear forms over $\mathbb{R}$ implies that there is a basis $x_1, \ldots, x_m$ of T_pM such that $\mathrm{Hess}_p(f)$ in this basis is $-\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2$. This λ is called the **index** of the critical point.

Definition 13.3. A function $f : M \to \mathbb{R}$ is called **Morse** if all critical points are non-degenerate and f is regular near ∂M.

Existence of Morse functions can be proved using jet transversality. Consider a subset

$$\mathcal{D}_{\mathrm{notMorse}} \subseteq J^2(M, \mathbb{R})$$

of 2-jets $(m, [g])$ with $[g]$ has vanishing first derivative at m and degenerate Hessian at m. In terms of local coordinates, $J^2(\mathbb{R}^m, \mathbb{R})$ correspond to polynomials in m variables of degree ≤ 2 and no constant. So this is the subset of

$$\sum_{i,j} h_{ij} x^i x^j$$

where h_{ij} is symmetric and non-degenerate. This can be stratified by the dimension of the kernel of the Hessian. From the local description, we see that it is a stratified subset in the sense of Whitney.

One immediate observation is that f is Morse if and only if $j^2(f) \cap \mathcal{D}_{\text{notMorse}} = \emptyset$. But note that $\operatorname{codim}(\mathcal{D}_{\text{notMorse}})$ is $m+1$. This means that

$$j^2(f) \cap \mathcal{D}_{\text{notMorse}} = \emptyset \iff j^2(f) \pitchfork \mathcal{D}_{\text{notMorse}}.$$

Apply jet transversality relative to the boundary.

Theorem 13.4. *If $f : M \to \mathbb{R}$ is a smooth and regular near ∂M, then f can be approximated by $\tilde{f}$ which is Morse and $f = \tilde{f}$ near ∂M.*

Corollary 13.5. *Morse functions exist.*

13.2 Morse lemma

Definition 13.6. A critical point $p \in M$ of a smooth function $f : M \to \mathbb{R}$ is said to be a **Morse singularity** if there exists a chart around p with coordinates $(x_1, \dots, x_m)$ (p corresponds to 0) such that

$$f(x_1, \dots, x_m) = f(0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2.$$

An easy computation shows that a Morse singularity is a nondegenerate critical point. The Morse lemma shows that the converse is true.

Lemma 13.7 (Morse lemma)**.** *Every nondegenerate critical point is a Morse singularity.*

Proof. Without loss of generality, assume $f(p) = 0$. By taking a chart, we may assume $M = W \subseteq \mathbb{R}^m$ and $p = 0$. In these conditions, Taylor approximation says that there is a smooth map $Q : W \to \operatorname{Sym}(\mathbb{R}^m)$ (symmetric matrices) such that

$$f(x) = \langle Q(x)x, x\rangle$$

and $Q(0) = \operatorname{Hess}_0(f)$.

We would like to find coordinates y such that Q is independent of y. Make an ansatz $y = A(x)x$ for a smooth map $A : W \to \mathrm{GL}_m(\mathbb{R})$ with $A(0) = 1$. We need to solve the equation

$$\langle Q(x)x, x\rangle = \langle Q(0)A(x)x, A(x)x\rangle,$$

or equivalently, $A(x)^t Q(0) A(x) = Q(x)$. To do so, consider

$$\begin{aligned} G &: \operatorname{Sym}(\mathbb{R}^n) \times W \to \operatorname{Sym}(\mathbb{R}^n); \\ &\qquad (B, x) \mapsto (\mathrm{id} + \tfrac{1}{2}Q(0)^{-1}B)^t Q(0)(\mathrm{id} + \tfrac{1}{2}Q(0)^{-1}B) - Q(x). \end{aligned}$$

This map is smooth, $G(0,0) = 0$, and its derivative with respect to B at $B = 0$ is

$$(\tfrac{1}{2}Q(0)^{-1})^t Q(0) + Q(0)\tfrac{1}{2}Q(0)^{-1} = 1.$$

So by the inverse function theorem, there is a $\beta : U \to \mathrm{Sym}(\mathbb{R}^m)$ such that $G(\beta(x), x) = 0$. Thus $A(x) = \mathrm{id} + \frac{1}{2}Q(0)^{-1}\beta(x)$ solves the equation.

Now we have coordinates y such that

$$f(y) = \langle Q(0)y, y\rangle.$$

Apply some linear coordinate change to put the symmetric matrix $Q(0)$ in diagonal form with entries ± 1. $\square$

Note by the Morse lemma, non-degenerate critical points are isolated and thus if M is compact, a Morse function $f : M \to \mathbb{R}$ can only have finitely many critical points.

14 October 2, 2017

Last time we proved existence of Morse functions $f : M \to \mathbb{R}$ with non-degenerate critical points, and we showed that near a critical point, there exist coordinates such that f takes the form $f = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2$.

14.1 Generic Morse function

Definition 14.1. A Morse function $f : M \to \mathbb{R}$ is **generic** if all critical values are distinct.

Lemma 14.2. *Any Morse function can be arbitrarily approximated by generic Morse functions.*

Proof. Let $f : M \to \mathbb{R}$ be a Morse function with critical points $\{p_i\} \subseteq M$, and for each p_i pick $\eta_i : M \to \mathbb{R}$ which are compactly supported and 1 near p. Consider

$$f + \sum_i \epsilon_i \eta_i$$

for small $\epsilon_i \in \mathbb{R}$. This is a Morse function with same critical points if ϵ_i is small enough. Now take ϵ_i so that the critical points take different values. □

14.2 Understanding level sets

If we have a function $f : M \to \mathbb{R}$, then the subsets $f^{-1}(a) \subseteq M$ are called **level sets**, and is a submanifold if a is a regular value of f. We'll try to explain what $f^{-1}([a, b])$ looks like in two cases:

I. $f^{-1}([a, b])$ contains no critical points.

II. $f^{-1}([a, b])$ contains a unique critical point p, with $f(p) \neq a, b$.

These are enough to understand f generic Morse.

Proposition 14.3. *If $f : M \to \mathbb{R}$ is proper and $f^{-1}([a, b])$ contains no critical point, then $f^{-1}([a, b]) \cong f^{-1}(a) \times [a, b]$ and $f^{-1}(b) \cong f^{-1}(a)$.*

Proof. We;ll use a smooth vector field χ on $f^{-1}([a, b])$ such that $df(\chi) = 1$. (This can be done by looking at a partition of unity or a Riemannian metric.) Because f is proper, $f^{-1}([a, b])$ is compact and the flow Φ_x exists until flowlines hit the boundary. Because $df(\chi) = 1$, if we start at $p \in f^{-1}(a)$ then we hit $f^{-1}(b)$ at time $b - a$. The diffeomorphism is going to be

$$f^{-1}(a) \times [a, b] \to f^{-1}([a, b]); \quad (p, t) \mapsto \Phi_\chi(p, t - a).$$ □

Now let's look at the second case, when there is a unique critical point p in the inverse image $f^{-1}([a, b])$. Denote $c = f(p)$, and fix some nice Morse singularity coordinates $(x_1, \dots, x_n)$ around p, defined on $W \subseteq \mathbb{R}^m$. Now we can write

$$f(x) = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2.$$

Take an $\epsilon > 0$ such that $B_{\sqrt{2\epsilon}}(0) \subseteq W$ and $a < c - 2\epsilon < c + 2\epsilon < b$.

The claim is that $f^{-1}([a, c+\epsilon])$ deformation retracts onto

$$Rf^{-1}([a, c-\epsilon]) \cup C,$$

where

$$C = \Big\{(x_1, \dots, x_\lambda, 0, \dots, 0) : \sum_{i=1}^{\lambda} x_i^2 \le \epsilon\Big\}.$$

The strategy to find a manifold with boundary U, obtained from $f^{-1}([a, c-\epsilon])$ by a "handle attachment, such that the inclusions $f^{-1}([a, c-\epsilon]) \cup C \hookrightarrow U \hookrightarrow f^{-1}([a, b])$ are deformation retracts.

We modify f to F as follows. Pick a smooth $\varphi : [0, \infty) \to [0, \infty)$ such that (i) $\varphi(0) \in (\epsilon, 2\epsilon)$, (ii) $\varphi(t) = 0$ for $t \in [\epsilon, \infty)$, and (iii) $\varphi'(t) \in (-1, 0]$. Then define

$$F : M \to \mathbb{R}; \quad x \mapsto \begin{cases} f(x) - \varphi(\sum_{i=1}^{\lambda} x_i^2 + 2\sum_{i=\lambda+1}^{m} x_i^2) & x \in V \\ f(x) & \text{otherwise.} \end{cases}$$

Lemma 14.4. *F has the following properties:*

(i) $f^{-1}([a, c+\epsilon]) = F^{-1}([a, c+\epsilon])$.

(ii) F has the same critical points as f.

(iii) $F^{-1}([a, c-\epsilon]) = f^{-1}([a, c-\epsilon]) \cup (D^\lambda \times D^{n-\lambda})$, *attached along* $\partial D^\lambda \times D^{m-\lambda}$.

Proof. (i) Because $F \le f$, we have $f^{-1}([a, c+\epsilon]) \subseteq F^{-1}([a, c+\epsilon])$. Conversely, let $x \in F^{-1}([a, c+\epsilon])$ such that $\varphi(\sum_{i=1}^{\lambda} x_i^2 + 2\sum_{i=\lambda+1}^{m} x_i^2) > 0$. Let's write $x = (y, z) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$ so that this is $\|y\|^2 + 2\|z\|^2$. Then $\varphi(t) = 0$ for $t > \epsilon$ implies $\|y\| + 2\|z\| \le \epsilon$. Then

$$f(x) = c - \|y\|^2 + \|z\|^2 \le c + \frac{1}{2}\|y\|^2 + \|z\|^2 \le c + \frac{1}{2}\epsilon.$$

This proves $F^{-1}([a, c+\epsilon]) \subseteq f^{-1}([a, c+\epsilon])$.

(ii) We only need to check this if $x \in V$. We explicitly can check

$$\frac{1}{2}\nabla F(x) = (-y - \varphi'(\|y\|^2 + 2\|z\|^2)y, z - \varphi'(\|y\|^2 + 2\|z\|^2)2z).$$

Since $-1 < \varphi'(t) \le 0$, we have this equal to zero if and only if $y = 0$ and $z = 0$.

(iii) This is annoying to prove. Read Milnor's book. □

We have some conclusion. We have $f^{-1}([a, b]) \cong U \cup F^{-1}(c+\epsilon) \times [c+\epsilon, b]$. This follows from Case I, because there are no critical values in $[c+\epsilon, b]$. We then have

$$f^{-1}([a, b]) \cong (f^{-1}(a) \times [a, c-\epsilon]) \cup (D^\lambda \times D^{m-\lambda}).$$

14.3 Handle decomposition

Definition 14.5. If M is a smooth manifold with boundary, and $\varphi : \partial D^i \times D^{m-i} \hookrightarrow \partial M$ is a smooth embedding, then

$$M \cup_\varphi (D^i \times D^{m-i})$$

is a smooth manifold with boundary after smoothing corners. This is said to be obtained from M by **attaching handles**.

Definition 14.6. A **handle decomposition** of a smooth manifold M is a way of writing M as an iterative handle attachments starting at $\emptyset$.

Combining existence of generic Morse functions with the link between critical points and handle attachments, you could prove the following.

Corollary 14.7. *Every compact M admits a finite handle decomposition.*

15 October 4, 2017

Last time we showed that M has a handle decomposition.

15.1 Cobordism

Definition 15.1. Given closed m-dimensional manifolds M_0 and M_1, a **cobordism** $(W, \partial_0 W, \partial_1 W, f_0, f_1)$ from M_0 to M_1 is an $m+1$-dimensional manifold W with $\partial W = \partial_0 W \amalg \partial_1 W$ with diffeomorphisms $f_0 : M_0 \to \partial_0 W$ and $f_1 : M_1 \to \partial_1 W$.

Example 15.2. There is a cobordism between two circles and three circles. If W is a closed $(m+1)$-dimensional manifold, then this is a cobordism $\emptyset \to \emptyset$.

Handle attachments give rise to cobordism as follows. If

$$\varphi : \partial D^i \times D^{w-i} \hookrightarrow M_0 \times \{1\}$$

is a smooth embedding, where $w = m+1$, the new manifold

$$(M_0 \times I) + (\varphi) = (M_0 \times I) \cup_\varphi (D^i \times D^{w-i})$$

is a cobordism from M_0 to

$$M_1 = (M_0 \setminus \varphi(\partial D^i \times \operatorname{int}(D^{w-i}))) \cup (D^i \times \partial D^{w-i}),$$

which is the "result of **surgery** of M_0 along φ".

It turns out that all cobordisms are handle attachements. Using the relative version of existence of generic Morse functions, one proves the following.

Proposition 15.3. *Every cobordism W is diffeomorphic relative to $\partial_0 W$ to*

$$(\partial W_0 \times I) + (\varphi_1) + \cdots + (\varphi_k).$$

(Note that the order is important; it might not even make sense.) In particular, $\partial_1 W$ is diffeomorphic to $\partial_0 W$ modified by finitely many surgeries.

We'll develop tools to manipulate such handle decompositions. Today we will show that we can change φ_i up to isotopy, and that if $\operatorname{index}(\varphi_i) \geq \operatorname{index}(\varphi_{i+1})$ then we can interchange them after isotopy. Then we will show that if $\operatorname{index}(\varphi_{i+1}) = \operatorname{index}(\varphi_i) + 1$ and the attaching sphere of φ_{i+1} intersects the transverse sphere of φ_i transversally in a single point, you can cancel both.

Let's define some words. Suppose we are attaching a 1-handle $D^1 \times D^2$. This $i = 1$ is called the **index** of the handle. Inside this handle there is a **core**

$$D^1 \times \{0\},$$

and its boundary $\partial D^1 \times \{0\}$ is called the **attaching sphere**. Dually, there is a **cocore** $\{0\} \times D^2$ and a **transverse sphere** $\{0\} \times \partial D^2$.

15.2 Handle isotopy, rearrangement, and cancellation

Lemma 15.4. *If φ_1 is isotopic to φ_1', then $W+(\varphi_1)$ is diffeomorphic to $W+(\varphi_1')$ relative to $\partial_0 W$.*

Proof. Let $\varphi_i^t : \partial D^i \times D^{w-i} \times [0,1] \to \partial_1(W) \times [0,1]$ denote the isotopy of embeddings. Then there exists an isotopy of diffeomorphisms

$$f^t : \partial_1 W \times [0,1] \to \partial_1 W \times [0,1]$$

such that $f^0 = \mathrm{id}$ and $\varphi_1^t = f^t \circ \varphi_1^0$, by isotopy extension. Take a collar $c : \partial_1 W \times [0,1) \to W$ relative to $\partial_1 W$. Then the map

$$F : W + (\varphi_1) \to W + (\varphi_i'); \quad p \mapsto \begin{cases} c(f^{1-t}(q), t) & p = c(q,t) \\ p & \text{otherwise} \end{cases}$$

is a diffeomorphism. □

Lemma 15.5. *Given a cobordism $W + (\varphi_0) + (\varphi_1)$ with $i_0 = \mathrm{index}(\varphi_0) \geq i_1 = \mathrm{index}(\varphi_1)$, then we may isotope φ_1 to φ_1' with image lying in*

$$\partial_1 W \setminus \varphi_0(\partial D^i \times \mathrm{int}(D^{w-i})) \subseteq \partial_1(W + (\varphi_0)).$$

Proof. First we make the attaching sphere of φ_1 transverse to transverse sphere of φ_0, using transversality results. But the dimension of the attaching sphere of φ_1 is $i_1 - 1$ and dimension of the transverse sphere of φ_0 is $w - i_0 - 1$. But both are submanifolds of $\partial_1 W$, which has dimension $w-1$.

Next, we shrink in D^{w-i_1} to make the image of the attaching map φ_1 disjoint from the transverse sphere of φ_0. Now we can flow the attaching sphere out of the handle (φ_0). To do this, pick a vector field χ that points radially outwards in the $D^{w-i_0} \setminus \{0\}$ direction on $(D^i \setminus \{0\}) \times \partial D^{w-i_0}$.

Once we have this, the image of φ_1' lies in the complement $\partial_1 W \setminus \varphi_0(\partial D i_0 \times D^{w-i_0})$. Then we have

$$W + (\varphi_0) + (\varphi_1') \cong W + (\varphi_1') + (\varphi_0)$$

relative to $\partial_0 W$. So by handle isotopy lemma, this is diffeomorphic to $W + (\varphi_0) + (\varphi_1)$. □

We can rearrange handles inductively to get the following.

Corollary 15.6. *Every cobordism W is diffeomorphic relative $\partial_0 W$ to*

$$W = (\partial_0 W \times I) + \sum_{0\text{-}handles} (\varphi_{i_0}^0) + \cdots + \sum_{w\text{-}handles} (\varphi_{i_w}^w).$$

This is equivalent to the existence of self-index Morse functions. This means that

$$f(\partial_0 W) = -1, \quad f(\partial_1 W) = w+1,$$

and for each critical point p, $f(p)$ is its index.

Suppose we have a cobordism between 1-manifolds. If you attach a 0 sphere, which is just a separate D^2, and then you attach a 1-handle, which looks like a $D^1 \times D^1$ connecting it with the original cobordism, you just get

$$W \cup_{D_1} D_2 \cong W.$$

Lemma 15.7. *Given $W + (\varphi_0) + (\varphi_1)$ with $\operatorname{index}(\varphi_1) = \operatorname{index}(\varphi_0) + 1$ and attaching sphere of φ_1 intersecting the transverse sphere of φ_0 transversally in a single point, then $W + (\varphi_0) + (\varphi_1) \cong W$ relative to $\partial_0 W$.*

Proof. This is going to be very hard to write in symbols and very hard to read.

Firstly, what you do is use interpolation and make φ_1 standard near the intersection point. Next, by shrinking in the first D^{w-i-1}-direction and interpolating, we can make $\operatorname{im}(\varphi_1)$ standard near the intersection. Now we flow outward φ_1 along the outward vector field as before, until the intersection of the image of φ_1 with (φ_0) is standard. But now, we can use φ_1 as coordinates $(\varphi_0$ to reduce to the standard model. Now this is explicit enough to write down a diffeomorphism. $\square$

16 October 6, 2017

Today I will continue talking about handle decompositions. Before doing that we need to talk about how these are related to the topology.

16.1 Handle decompositions and the topology

Let's suppose that

$$W = (\partial_0 W \times I) + \sum_{I_0} (\phi_{i_0})^0 + \cdots + \sum_{I_w} (\phi_{i_w}^w),$$

and let $W_k \subseteq W$ be given by

$$W_k = (\partial_0 W \times I) + \sum_{I_0} (\phi_{i_0}^0) + \cdots + \sum_{I_k} (\phi_{i_k}^k).$$

Now W is an iterated pushout starting with $W_{-1} = \partial_0 W \times I$

$$\begin{array}{ccc} \coprod_{I_k} \partial D^k \times D^{w-k} & \xrightarrow{\amalg \phi_{i_k}^k} & W_{k_1} \\ \downarrow & & \\ \coprod_{I_k} D^k \times D^{w-k} & \longrightarrow & W_k. \end{array}$$

By collapsing D^{w-k} to $*$, we get a relative CW-complex X with weak homotopy equivalence $f : W \to X$ relative to $\partial_0 W$. We can do this iteratively by pushouts

$$\begin{array}{ccc} \coprod_{I_k} \partial D^k & \longrightarrow & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{I_k} D^k & \longrightarrow & X_k. \end{array}$$

Here are some properties:

- h-cells correspond bijectively to h-handles.
- The degree of attaching map $D_{i_{k+1}}^{k+1}$ with respect to $D_{i_k}^k$ is given by the intersection number of attaching sphere of $(\phi_{i_{k+1}}^{k+1})$ with the transverse of $(\phi_{i_k}^k)$.

This allows you to pass information between topology and geometry.

Here is one example. If you have a handle decomposition of W, we can read this backwards. Instead of starting at $\partial_0 W$, we can start at $\partial_1 W$ and then i-handles become $(w-i)$-handles. If W is oriented, we get we can use it to align orientations of cores and the orientations of opposite direction cocores. Then the degree of the attaching maps will be the same. So we get, if W is oriented, then

$$H_*(W, \partial_0 W) \cong H_*(X, \partial_0 W) \cong H_{w-*}(X^{\mathrm{rev}}, \partial_1 W) \cong H_{w-*}(W, \partial_1 W).$$

Definition 16.1. W is an **h-cobordism** if both maps $\partial_0 W \hookrightarrow W$ and $\partial_1 W \hookrightarrow W$ are homotopy equivalences.

16.2 Handle exchange

To trade an i-handle for an $(i+2)$-handle, we will introduce a custom-built pair of $(i+1)$-handles and $(i+2)$-handles such that the $(i+1)$-handle cancel to i-handle.

Lemma 16.2. *Given an embedding $D^{w-1} \hookrightarrow \partial_1 W$, let $\psi^i : \partial D^i \times D^{w-i} \to \partial_1(W)$ be given by restriction to a standard $\partial D^i \times D^{w-i} \subseteq D^{w-1}$, there exists a $\psi^{i+1} : \partial D^{i+1} \times D^{w-i-1} \to \partial_1(W + (\psi^i))$ such that $W \cong W + (\psi^i) + (\psi^{i+1})$.*

Proof. This can be drawn so that attaching the two is just attaching D^w. □

Let us write

$$\hat{\partial}_1(W_k) = \partial_1(W_k) \setminus \coprod_{I_{k+1}} \varphi^{k+1}_{i_{k+1}}(\partial D^{k+1} \times D^{w-k-1}).$$

Lemma 16.3 (Handle exchange). *Suppose that for an i-handle $\phi^i_{j_i}$ we have an embedding $\psi^{i+1} : \partial D^{i+1} \times D^{w-i-1} \to \hat{\partial}_1(W_i)$ such that*

(a) in $\partial_1(W_i) \supseteq \hat{\partial}_1(W_i)$, ψ^{i+1} is isotopic to an embedding whose restriction to $\partial D^{i+1} \times \{0\}$ intersects the transverse sphere of $(\phi^i_{i_j})$ once transversally,

(b) in $\partial_1(W_{i+1}) \supseteq \hat{\partial}_1(W_1)$ the map ψ^{i+1} is isotopic to an embedding which is trivial, i.e., is an restriction of an embedding $D^{w-1} \hookrightarrow \partial_1(W_{i+1})$ to the standard $\partial D^{i+1} \times D^{w-i+1}$.

Then there is an attaching map ψ^{i+2} such that

$$W = \cdots + \sum_{j \neq i_j} (\phi^i_j) + \sum_{I_{i+1}} (\phi^{i+1}_{i_{i+1}}) + (\psi^{i+2}) + \cdots.$$

Proof. Using triviality of ψ^{i+1}, we can build a ψ^{i+2} so that it cancels with ψ^{i+2}. Then

$$\begin{aligned} W &\cong \cdots + \sum_{I_i} (\phi^i_{i_i}) + \sum_{I_{i+1}} (\phi^i_{i_{i+1}}) + (\psi^{i+1}) + (\psi^{i+2}) \\ &\cong \sum_{j \neq I_j} (\phi^i_j) + \sum_{I_{i+1}} (\phi^{i+1}_{i_{i+1}}) + (\psi^{i+2}). \end{aligned}$$ □

16.3 Removing 0, 1, $w-1$, w-handles

Lemma 16.4. *If $w \geq 6$ and $\partial_0 W \hookrightarrow W$ is 1-connected, then there exists a handle decomposition with 0 or 1-handles.*

Proof. First pick a handle decomposition with handles appearing in order of increasing index. Now each 0-handle adds a new path component unless it is later reconnected by a 1-handle. This is because $\partial_0 W \hookrightarrow W$ is a bijection on π_0. Also, the attaching sphere of this 1-handle automatically intersects the transverse sphere ∂D^w transversally in a single point. So we may use cancellation to cancel all the 0-handles.

Now we exchange 1-handles for 3-handles. We want to build a $\psi^2 : \partial D^2 \times D^{w-2} \hookrightarrow \hat{\partial}_1(W_1)$ which satisfies the conditions for handle exchange with respect to the chosen 1-handle (ϕ_1^1). We build $\psi^2|_{\partial D^2}$ first. Consider the segment $S^1_+ = D^1 \times \{\vec{e}_1\}$ through the 1-handle. The endpoints are in the same component because $\partial_0 W \to W$ is a π_0-bijection. So we can connect them by $S^1_- \subseteq \hat{\partial}_1(W_0)$.

Note that

$$\hat{\partial}_1(W_0) = \partial_1(W_0) \setminus \coprod(\partial D^1 \times D^{w-1}) \cong \partial_0 W \setminus (\text{disks})$$

since there are no 0-handles. Because $w \geq 6$, we have that $\pi_1(\hat{\partial}_1(W_0)) \cong \pi_1(\partial_0 W) \twoheadrightarrow \pi_1(W)$, we can assume that $S^1 = S^1_+ \amalg S^1_-$ is null-homotopic in W by adding another loop if necessary.

This gives an embedding (after perturbation) $\psi^2|_{S_1} : S^1 \hookrightarrow \partial_1(W_1)$ that passes through the handle (ϕ_1^1) around once. Because attaching spheres of 2-handles are S^1, we can also assume that the embedding is $\psi^2|_{S_1} : S^1 \hookrightarrow \hat{\partial}_1(W_1)$. Then by a small section of the normal bundle, we get $\psi^2 : \partial D^2 \to D^{w-2} \to \hat{\partial}_1(W_1)$.

The map $\partial_1(W_2) \hookrightarrow W$ is an isomorphism of π_1, because to get from $\partial_1(W_2)$ to W, we need to attach i-cells for $i \geq 3$ and $w-2$ cells and $w-1$ cells on the backwards direction. So we're not changing π_1 in this process. This means that ψ^2 on S^1 is null-homotopic in $\hat{\partial}_1(W_1) \subseteq \partial_1(W_2)$. That is, it bounds a disc in $\partial_1(W_2)$. $\square$

17 October 11, 2017

I will prove the Whitney trick next time.

17.1 Two-index lemma

Recall that if a cobordism W is a h-cobordism if $\partial_0 W \hookrightarrow W$ and $\partial_1 W \hookrightarrow W$ are weak equivalences. Last time we showed that if $w \geq 6$ then W admits a handle decomposition without $0, 1, w-1, w$-handles.

Lemma 17.1. *Let $2 \leq q \leq w-3$, and let W be an h-cobordism of dimension $w \geq 6$. Then W admits a handle decomposition*

$$W = \partial_0(W) \times I + \sum_{I_q}(\phi^q_{i_q}) + \sum_{I_{q+1}}(\phi^{q+1}_{i_{q+1}}).$$

We'll give a proof in the case when W is simply-connected. The two main techniques will be handle sliding and the Whitney trick.

Proof. Without loss of generality we may assume an handle decomposition without $0, 1, w-1, w$-handles. It suffices to how to trade the lowest index i-handle (ϕ^i_i) for a $i+2$-handle.

We start by picking a trivial embedding $\psi^{i+1} : \partial D^{i+1} \times D^{w-i-1} \hookrightarrow \hat{\partial}_i(W^i)$. This automatically satisfies (b) of the handle exchange lemma, and we will modify it to satisfy (a). That is, the attaching sphere $\psi^{i+1}(\partial D^{i+1} \times \{0\})$ is isotopic in $\partial_1(W^i)$ to one intersecting transverse sphere of ϕ^i_i once transversally.

Since W is a h-cobordism, $H_i(W, \partial_0(W)) = 0$ and there are no i'-handles for $i' < i$. This shows that we can write

$$[\phi^i_i] = \sum_{j \in I_{i+1}} a_j d[\phi^{i+1}_j]$$

as a $\mathbb{Z}$-linear combination of attaching spheres of $(i+1)$-handles.

We will "add" this to ψ^{i+1}, and it suffices to show how to add $\pm[\phi^{i+1}_j]$ to $[\psi^{i+1}(\partial D^{i+1} \times \{0\})]$. Let $\overline{S}^{i+1}_j$ be the parallel translate of $\phi^{i+1}_i(\partial D^{i+1} \times \{0\})$ in $\hat{\partial}_1(W^i)$. Pick an embedded arc from a point in $\psi^{i+1}(\partial D^{i+1} \times \{0\})$ to a point in $\overline{S}^{i+1}_j$ and thicken it to $1 \times D^i$, with

$$\{0\} \times D^i \subseteq \psi^{i+1}(\partial D^{i+1} \times \{0\}), \quad \{1\} \times D^i \subseteq \overline{S}^{i+1}_j.$$

use this to replace $\psi^{i+1}(\partial D^{i+1} \times \{0\})$ with the embedded connected sum of $\psi^{i+1}(\partial D^{i+1} \times \{0\})$ and $\overline{S}^{i+1}_j$, lying in $\hat{\partial}_1(W^i)$. Since $\overline{S}^{i+1}_j$ bounds a $(i+1)$-disk in $\partial_1(W^i)$, we see that

$$\psi^{i+1}(\partial D^{i+1} \times \{0\}) \# \overline{S}^{i+1}_j$$

is isotopic to $\psi^{i+1}(\partial D^{i+1} \times \{0\})$, is still trivial in $\partial_1(W^{i+1})$, and can be exteded to an embedding $\tilde{\psi}^{i+1}$ of $\partial D^{i+1} \times D^{w-i-1}$.

Using this, we can find a ψ^{i+1} that is trivial in $\partial_1(W^{i+1})$, and has intersection number 1 with the transverse sphere of (ϕ_i^i). Now this can have 15 intersections with 8 positive and 7 negative signs, but the Whitney trick says that we can isotope ψ^{i+1} further such that hte number of intersection points is 1. So we can now handle exchange to replace (ϕ_i^i) with (ψ^{i+2}). □

17.2 $\mathrm{Wh}_1(\pi_1)$

Now we have

$$W = (\partial_0 W \times I) + \sum_{I_q} (\phi_j^q) + \sum_{I_{q+1}} (\phi_j^{q+1}).$$

Since W is an h-cobordism, we have $H_*(W, \partial_0(W)) = 0$. This shows that

$$\partial_q : \mathbb{Z}[I_{q+1}] \to \mathbb{Z}[I_q]$$

is an isomorphism.

From this we get an invertible matrix D with integer entries, well-defined up to

(i) multiplying rows or columns by -1,

(ii) interchanging rows or columns,

(iii) handle addition or cancellation, i.e., adding or subtracting a row and a column like

$$D \quad \longleftrightarrow \quad \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix},$$

(iv) handle slides, adding a multiple of a row to another row (or column).

If $\pi_1(W) \neq \{0\}$, then passing to the universal cover gives a matrix D with entries in the group ring $\mathbb{Z}[\pi_1]$.

Note that if $n \geq 3$, then elementary matrices are commutators,

$$e_{ik}(ab) = [e_{ij}(a), e_{jk}(b)].$$

Moreover,

$$\operatorname*{colim}_{n\to\infty} E_n(R) \subseteq (\text{commutator subgroup of } \operatorname*{colim}_{n\to\infty} \mathrm{GL}_n(R)).$$

This is because

$$\begin{pmatrix} [g,h] & \\ & \mathrm{id}_n \end{pmatrix} = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} h & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & \\ & hg \end{pmatrix},$$

$$\begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & g \\ & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & \\ -g^{-1} & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & g \\ & \mathrm{id} \end{pmatrix} \begin{pmatrix} & -\mathrm{id} \\ \mathrm{id} & \end{pmatrix}.$$

So we can take care of (ii), (iii), (iv) simultaneously, if we consider D as lying in

$$K_1(R) = \operatorname*{colim}_{n\to\infty} \mathrm{GL}_n(R)^{\mathrm{ab}} = \operatorname*{colim}_{n\to\infty} H_1(\mathrm{GL}_n(R)).$$

To take care of (1), we define

Definition 17.2. $\mathrm{Wh}_1(G) = K_1(\mathbb{Z}[G])/(\pm g)$.

Definition 17.3. For a cobordism W, the **Whitehead torsion** is defined as

$$\tau(W) = [D] \in \mathrm{Wh}_1(\pi_1(W)).$$

The conclusion is the inverse direction of the following theorem.

Theorem 17.4. *If W is an h-cobordism of dimension $w \geq 6$, then $W \cong \partial_0(W) \times I$ relative to $\partial_0(W)$ if and only if $\tau(W) = 0$.*

17.3 Poincaré conjecture

Lemma 17.5. $\mathrm{Wh}_1(\{e\}) = 0$.

Proof. Because

$$\mathrm{GL}_1(\mathbb{Z}) \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\det} \mathrm{GL}_1(\mathbb{Z})$$

is the identity, it suffices to show that $\mathrm{SL}_n(\mathbb{Z}) = E_n(\mathbb{Z})$ as $n \to \infty$. This is done by the Euclidean algorithm. $\square$

Corollary 17.6. *If M is an m-dimensional closed manifold with $M \simeq S^m$, then M is homeomorphic to S^m.*

Proof. Take a $D^m \amalg D^m$ embedded in M. The complement W is an h-cobordism and simply connected. This shows that $W \cong S^{m-1} \times I$. Then

$$M \cong D_0^m \cup_{\mathrm{id}} (S^{m-1} \times I) \cup_f D_1^m \cong D_0^m \cup_f D_1^m.$$

But every homeomorphism of S^{m-1} extends over D^m by the Alexander trick. $\square$

What we have actually proved is that there is a map

$$\begin{aligned}\pi_0(\mathrm{Diff}_\partial(D^{m-1})) &\cong \pi_0(\mathrm{Diff}^+(S^{m-1})) \\ &\twoheadrightarrow \Theta_m = \left\{\begin{matrix}\text{set of } m\text{-dimensional closed} \\ \text{manifolds homotopy equivalent to } S^m\end{matrix}\right\},\end{aligned}$$

since $\mathrm{Diff}^+(S^{m-1}) \simeq \mathrm{SO}(m) \times \mathrm{Diff}_\partial(D^{m-1})$.

18 October 13, 2017

Last time we learned that

$$\pi_0(\mathrm{Diff}_\partial(D^n)) \twoheadrightarrow \Theta_{n+1},$$

where Θ_n is the set of exotic spheres.

18.1 A Brieskorn sphere

Theorem 18.1 (Brieskorn)**.** *For $m > 1$, the manifold*

$$\Sigma = \{z \in \mathbb{C}^{2m+1} : z_0^3 + z_1^5 + z_2^2 + \cdots + a_{2m}^2 = 0,\ |z| = 1$$

is homeomorphic, but not diffeomorphic, to S^{4m-1}.

Recall that the **signature** $\sigma(m)$ of a orientable $4m$-manifold M is the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form of $H_{2m}(M;\mathbb{R})$.

Theorem 18.2 (Kervaire–Milnor)**.** *Let Σ be homeomorphic to S^{4m-1} and suppose $\Sigma = \partial W$ with W parallelizable. Then Σ is diffeomorphic to S^{2m-1} if and only if*

$$\frac{1}{8}\sigma(W) \equiv 0 \pmod{\frac{3+(-1)^{m+1}}{2} 2^{2m-2}(2^{2m-1}-1)\operatorname{Num}\Big(\frac{B_{2m}}{4m}\Big)}.$$

So we are going to show that Σ is a topological sphere and $\sigma(W) = \pm 8$. The notation we are going to use is

$$f(z) = z_0^{a_0} + \cdots + z_n^{a_n} \quad n \geq 2,$$
$$V(f) = \{z \in \mathbb{C}^{n+1} : f(z) = 0\},$$

and $\Sigma = V(f) \cap S^{2n+1}$. Also, C_{a_j} is a cyclic group of order a_j with fixed generator ω.

Because Σ is inside a sphere, Alexander duality gives a good way to understand it by understanding the complement. There is a map $\varphi : z \mapsto f(z)/|f(z)|$.

$$\begin{tikzcd}
S^{2n+1} \setminus \Sigma \arrow[r] \arrow[d, "\varphi"] & \mathbb{C}^{n+1} \setminus V(f) \arrow[d, "\tilde{\varphi}"] \\
S^1 \arrow[r, equal] & S^1
\end{tikzcd}$$

Theorem 18.3 (Milnor)**.** *These maps are smooth fiber bundles, and the top map is a fiberwise homotopy equivalence. Moreover, $\overline{F}_0 = \overline{\varphi^{-1}(1)}$ is a smooth manifold with boundary Σ, and both F_0 and Σ are connected and simply connected.*

The group $G = C_{a_0} \times \cdots \times C_{a_n}$ acts on F_0 by multiplying $e^{2\pi i/a_j}$ in the jth coordinate. Then $\pi_1(S^1, 1) = T$ acts via $T \to G$; $t \mapsto w = w_0 \times \cdots \times w_n$.

Proposition 18.4. *There is a homotopy equivalence*

$$F_0 \simeq \bigvee_{\mu(f)} S^n,$$

where $\mu(f) = \prod_j (a_j - 1)$.

Recall that
$$A * B = C(A) \times B \amalg_{A\times B} A \times C(B).$$
For instance, $(\Delta^0)^{*(n+1)} = \Delta^n$.

Proof. Let $J = C_{a_0} * \cdots * C_{a_n}$. Then there is a map

$$J \hookrightarrow \tilde{\varphi}^{-1}(1); \quad (t_j w_j^{t_j}) \mapsto (e^{2\pi i r_j/a_j} t_j^{1/r_j})$$

that is a G-equivariant deformation retract. Then F_0 is $(n-1)$-connected. Also, we can compute

$$H_*(F_0) \cong H_*(J) = H_*(\mathbb{Z}[G] \to \mathbb{Z}[G]\{\partial_j\sigma\}/(\partial_j\sigma - \omega_j\partial_j\sigma) \to \cdots).$$

Then $H_i(F_0) = 0$ for $i > n$, and

$$H_n(F_0) \cong \mathbb{Z}[G]\eta \cong \bigotimes_j \mathbb{Z}[C_{n_j}]/(1 + w_j + \cdots + \omega_j^{a_j-1}),$$

where $\eta = \prod_j (1 - \omega_j)$. □

Corollary 18.5. F_0 *is parallelizable.*

Proof. Because $\dim F_0 = 2n$ and $H_{2n}(F_0) = 0$, F_0 has no compact component. But F_0 is stably parallelizable because it is embedded in $\mathbb{C}^{n+1}$ with trivial normal bundle. □

Corollary 18.6. $\tilde{H}_i(S^{2n+1} \setminus \Sigma) \cong \begin{cases} \operatorname{coker}(1-\omega) & i = n \\ \ker(1-\omega) & i = n+1 \\ 0 & \textit{otherwise.} \end{cases}$

Proof. Use the Serre spectral sequence. Here, the coefficients aren't trivial, but we know exactly how π_1 acts. □

Let $\Delta(x) = \det(x - \omega)$ be the characteristic polynomial.

Corollary 18.7. Σ *is a topological sphere if and only if* $|\Delta(1)| = 1$.

Proof. We know that $|\Delta(1)| = 1$ if and only if $S^{2n+1} \setminus \Sigma$ has the homology of a point. Now use Alexander duality and Poincaré duality to conclude that Σ is a homology sphere. Because Σ is simply connected and we have generalized Poincaré conjecture, we get the result. □

Now we use the splitting of $H_n(F_0)$ to write

$$\Delta(x) = \prod_{0<r_j<a_j} \Big(x - \prod_j e^{2\pi i r_j/a_j}\Big).$$

Example 18.8. If $n = 2m$ and $f(z) = z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2m}^2$, then

$$\Delta(x) = \Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1.$$

So Σ is a topological sphere in this case.

18.2 Signature of the sphere

Theorem 18.9 (Pham)**.** *The intersection form on $H_n(F_0)$ is*

$$\langle g, h\rangle = \epsilon(h^{-1} g\eta)$$

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is specified as $\epsilon(1) = -\epsilon(\omega)$.

Given a linear map $\lambda : \mathbb{C}[G] \to \mathbb{C}$, define

$$\hat{\lambda} = \sum_{g\in G} \lambda(g) g \in \mathbb{C}[G].$$

Corollary 18.10. $\sigma(F_0) = \sigma_+(F_0) - \sigma_-(F_0)$*, where*

$$\sigma_+(F_0) = \#\{(\chi : G \to \mathbb{C}^\times) : \chi(\hat{\epsilon}\overline{\eta}) > 0\},$$
$$\sigma_-(F_0) = \#\{(\chi : G \to \mathbb{C}^\times) : \chi(\hat{\epsilon}\overline{\eta}) < 0\}.$$

Proof. We know that $\{\hat{\chi}\}$ is a basis for $\mathbb{C}[G]$, and we always have

$$\hat{\chi}\eta = \chi(\overline{\eta})\hat{\chi}$$

where $\overline{g} = g^{-1}$. So

$$H_n(F_0) \otimes \mathbb{C} \cong \mathbb{C}\langle \hat{x} : \hat{x}(\overline{\eta}) \neq 0\rangle.$$

By orthogonality, we have

$$\langle \hat{\chi}\eta, \hat{\chi}'\eta\rangle = \begin{cases} |G|\chi(\hat{\epsilon}\overline{\eta}) & \chi = \chi' \\ 0 & \text{otherwise.}\end{cases}$$

So we can just count these eigenvalues. □

Corollary 18.11. *If n is even, then $\sigma(F_0) = N_+ - N_-$ where*

$$N_+ = \#\{(r_0, \ldots, r_n : 0 < r_j < a_j,\ 0 < \textstyle\sum_j \frac{r_j}{a_j} \bmod 2 < 1\},$$
$$N_- = \#\{(r_0, \ldots, r_n : 0 < r_j < a_j,\ 1 < \textstyle\sum_j \frac{r_j}{a_j} \bmod 2 < 2\}.$$

Proof. We check $\hat{\epsilon}\overline{\eta} = (-1)^{n(n+1)/2}(\eta + \overline{\eta})$. So you can check

$$\chi(\hat{\epsilon}\overline{\eta}) = (-1)^{n/2}\Re(\chi(\eta)) = \cdots = |\chi(\eta)| \sin\Big(\pi \sum_j \frac{r_j}{a_j}\Big).$$

Then you can translate to the statement. □

Proof of the theorem. We have $n = 2m$, $a_0 = 3$, $a_1 = 5$, $a_j = 2$ for $j > 1$. So

$$\frac{r_0}{3} + \frac{r_1}{5} + \frac{2m-1}{2} = \frac{10r_0 + 6r_1 + 15(2m-1)}{30} = \frac{10r_0 + 6r_1}{30} + \frac{(-1)^{m+1}}{2} \pmod 2.$$

Here, $10r_0 + 6r_1 \in \{16, 26, 22, 32, 28, 38, 34, 44\}$ because $r_0 \in \{1, 2\}$ and $r_1 \in \{1, 2, 3, 4\}$. This shows that $\sigma(F_0) = 8(-1)^m$. □

19 October 16, 2017

In the last lecture I gave, we showed that

$$\pi_0(\mathrm{Diff}_\partial(D^{m-1})) \twoheadrightarrow \Theta_{n+1}$$

for $n \geq 5$. Today I am going to give the Whitney trick, and the Casson trick in dimension 4.

19.1 Whitney trick

Let N be a smooth manifold and $M, X \subseteq N$, with $m + x = n$. Also assume $M \pitchfork X$. Pick $p_0, p_1 \in M \cap X$ which are in the same path components of M and of X.

Our goal is to remove p_0 and p_1 by isotopic X. Pick embedded paths γ_M on M and γ_X on X connecting p_0 and p_1, and avoiding all other intersection points. Pick a Riemannian metric g in N such that

(i) M and X are totally geodesic near φ_M and φ_X,

(ii) $M \perp X$ at p_0 and p_1.

Pick basis $\zeta_1(0), \dots, \zeta_x(0)$ of $T_{p_0}(X) = T_{p_0}(M)^\perp$ with $\zeta_x(0) = d\gamma_x/dt$. Likewise pick a basis $\xi_1(0), \dots, \xi_m(0)$ or $T_{p_0}(M) = T_{p_0}(X)^\perp$ with $\xi_m(0) = d\gamma_m/dt$. They collectively form a basis of $T_{p_0}(N)$. Parallel transport the basis along γ_X and along γ_M to get two bases

$$\zeta_1^X(1), \dots, \zeta_x^X(1), \xi_1^X(1), \dots, \xi_m^X(1) \text{ and } \zeta_1^M(1), \dots, \zeta_x^M(1), \xi_1^M, \dots, \xi_m^M(1)$$

of $T_{p_1}(N)$. Assume that they are *oppositely oriented*. Let B^2 be the "bigon", so that we have an embedding

$$f : \partial B^2 \to N; \quad f|_{\partial_+(B^2)} = \varphi_X, \quad f|_{\partial_-(B^2)} = \varphi_M.$$

Lemma 19.1. *If $m, x < n - 2$ and $\pi_1(N) = 0$, then $\pi_1(N \setminus (M \cup X)) = 0$. If moreover $n \geq 5$, then f may be extended to an embedding $\varphi : B^2 \to N$ such that $\varphi(\mathrm{int}(B^2)) \subseteq N \setminus (M \cup X)$.*

Proof. If codimension of M and X are at least 3, then every loop or null-homotopy generically avoids M and X. Now f is null-homotopic in $N \setminus (M \cup X)$, and generically null-homotopy is embedded if $n \geq 5$. □

We want to thicken B^2 by flowing along normal vector fields. The tangent space TB^2 is $\mathrm{span}(\zeta_x^M(t), \xi_m^M(t))$ over φ_M, and $\mathrm{span}(\zeta_x^X(t), \xi_m^M(t))$ over φ_X.

Lemma 19.2. *We can extend $(\xi_1^M(t), \dots, \xi_{m-1}^M(t))$ and $(\zeta_1^X(t), \dots, \xi_{x-1}^X(t))$ to a trivialization of the normal to B^2, such that ξ^M over γ_X are orthogonal to X and ζ^X over γ_M are orthogonal to M.*

Proof. Basically we are going to extend ζ^X over γ_X to over γ_M, and then fill in B^2. Then we can do the same for ξ^M.

Pick a vector field ρ on B^2, which is orthogonal to $\partial_-(B^2)$ (corresponding to γ_M). This is possible because the two frames in $(TM)^\perp$

$$(\zeta_1^X(0), \dots, \zeta_{x-1}^X(0), \rho_0 = \tfrac{d\gamma_X}{dt}) \text{ and } (\zeta_1^X(1), \dots, \zeta_{x-1}^X(1), \rho_1 = -\tfrac{d\gamma_x}{dt})$$

have the same orientation.

So we get an extension over φ_m of $\xi_1^X, \dots, \xi_{x-1}^X$. We want to extend this to an $(x-1)$-frame in $(TB^2)^\perp$. The obstruction to extending is

$$\pi_1(\mathrm{O}(n-2)/\mathrm{O}(m-1)) = 0$$

because $m - 1 \geq 2$. So we can indeed ζ^X.

Now there is no further to extending ξ^M to trivializations of $(\mathrm{span}(\zeta^X))^\perp \subseteq (TB^2)^\perp$. Then ξ^M over γ_X are perpendicular over γ_X. $\square$

Denote these extensions by $(\zeta_1^X, \dots, \zeta_{x-1}^X, \xi_1^M, \dots, \xi_{m-1}^M)$ over B^2. We can extend them a bit outside B^2 and exponentiate to get standard coordinates around B^2. This gives a local model

$$M = D_- \times \mathbb{R}^{m-1} \times \{0\}, \quad X = D_+ \times \{0\} \times \mathbb{R}^{x-1}$$

in $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{m-1} \times \mathbb{R}^{x-1}$.

Theorem 19.3 (Whitney trick)**.** *Let M and X be submanifolds of codimension ≥ 3 in a simply-connected $n = (m+x)$-dimensional manifold N. Assume $M \pitchfork X$ and $p_0, p_1 \in M \cap N$ are points in the same path component of X and M. If they have opposite signs, then there there is a compactly supported ambient isotopy φ_t such that $M \cap \psi_1(X) = M \cap N \setminus \{p_0, p_1\}$.*

19.2 Casson trick

In dimension 4, everything in this argument goes wrong.

(i) The complement of $M \cup X$ in N might not be simply-connected.

(ii) A generic 2-disk in a 4-manifold is not embedded.

(iii) There might be framing obstructions, lying in $\pi_1(\mathrm{O}(2)/\mathrm{O}(1)) = \mathbb{Z}$.

It turns out that you can solve (i) and (iii) at the expense of (ii). The problem (iii) can be solved using some boundary twisting. Solving (i) is called the **Casson trick**.

There is something called the **finger move**. Suppose $\Sigma \subseteq N$ is a surface and there is another surface. Let γ be an arc connecting $p \in \Sigma$ and the other surface. Then we can extend Σ along the path γ and get another surface $\mathcal{F}_\gamma(\Sigma)$. This is going to have two more intersection points.

Suppose p_1 is near p_0 on Σ. Then there is an element ζ that rotates around Σ in p_0, and an element η that follows this path γ.

Lemma 19.4. $\pi_1(N \setminus \mathcal{F}_\gamma(\Sigma)) = \pi(N \setminus \Sigma)/([z, z^\eta])$.

Theorem 19.5 (Casson)**.** *Let N be simply connected, and $(D^2, \partial D^2) \looparrowright (N, \partial N)$ embedded on ∂. If there exists a $\beta \in H_2(N)$ such that $[D^2] \cdot \beta = 1$, then D^2 may be modified by finger moves such that the complement is simply-connected.*

Proof. $\pi_1(N \setminus D^2)$ is normally generated by meridians. Then $[z]$ generates $H_1(N \setminus D^2)$. This means that $\pi_1(N \setminus D^2)$ is generated by commutators. So it is normally genreated by $[z, z^w]$, which can be killed by finger moves. □

Now we can find an immersed disk with intersections. We can then find smaller disks that come from the intersections, and so they can be modified to get smaller disks. So you get the whole system of disks branching out to smaller disks, which tries to look like a Whitney disk. In the topological category, this whole thing actually contains a topological disk.

20 October 18, 2017

In the next 5 lectures, we are going to show that the map $\mathrm{Diff}_\partial(D^n) \twoheadrightarrow \Theta_{n+1}$ is surjective if $n \geq 5$ and injective if $n \geq 7$. Also, we are going to compute rational homotopy $\pi_*(\mathrm{Diff}_\partial(D^n)) \otimes \mathbb{Q}$.

20.1 Motivation for algebraic K-theory

You can think of algebraic K-theory as

- the natural home for algebraic topology invariants,
- the algebro-geometric version of topological K-theory, or
- homotopy-coherent way of doing group completioins or splitting short exact sequences.

Definition 20.1. Let R be a commutative ring. A **vector bundle** over $\operatorname{Spec} R$ is a sheaf locally isomorphic to $\mathscr{O}_{\operatorname{Spec} R}^{\oplus n}$.

Then there is a correspondence between vector bundles over $\operatorname{Spec} R$ and finitely generated projective R-module. This is given by $E \mapsto \Gamma(E)$ and $M \mapsto \tilde{M}(D_f) = M_f$. We can then group complete this. Here, we don't need to split short exact sequences, because they are already split.

Definition 20.2. $K_0(R)$ is the group completion of projective finitely generated modules with $\oplus$.

People realized that this fits with the K_1 we defined last time. For example, the following is exact.

$$K_1(\mathbb{F}_p) \to K_1(\mathbb{Z}_{(p)}) \to K_1(\mathbb{Q}) \to K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}_{(p)}) \to K_0(\mathbb{Q}).$$

Then people found out spaces whose homotopy groups are the K-groups. That is, there is a space $K(R)$ such that $K_i(R) = \pi_i K(R)$.

20.2 Group completion

Definition 20.3. A **Γ-space** is a functor $\Gamma \to \mathsf{Top}$, where Γ is the category of pointed finite sets, such that the map $X(\overline{n}) \to \prod_n X(\overline{1})$ induced by $\overline{n} = \{*, 1, \dots, n\} \to \{*, i\} = \overline{1}$ is a weak equivalence. (This implies that $X(\overline{0}) = *$.)

Here, the map

$$X(\overline{1}) \times X(\overline{1}) \xleftarrow{\simeq} X(\overline{2}) \longrightarrow X(\overline{1})$$

is sort of like multiplication, and

$$* \xleftarrow{\simeq} X(\overline{0}) \longrightarrow X(\overline{1})$$

is the unit.

Example 20.4. If M is a commutative unital monoid, then $X_M : \overline{n} \mapsto M^n$ is a Γ-space.

Example 20.5. Let R be a ring, and consider X_R that maps $\overline{n}$ to the classifying space of (the category with objects n-tuples of finitely generated projective left R-modules, and morphisms n-tuples of R-linear isomorphisms).

Now there is a functor $\Delta \to \Gamma^{\mathrm{op}}$ given by

$$[n] = \{0 \to 1 \to \cdots \to n\} \mapsto \{0 \to 1, 1 \to 2, \dots, n-1 \to n, *\} \cong \overline{n},$$

and the morphisms pulling back by looking at the gaps. We can use this to get a simplicial object

$$B_\bullet X : \Delta^{\mathrm{op}} \to \Gamma \xrightarrow{X} \mathsf{Top}.$$

This is actually something like a bar resolution.

Example 20.6. Let us look at $B_\bullet X_M$. If you look at a face map $[n] \to [n+1]$, the corresponding map will be just multiplying the two copies of M that the face map skips.

Theorem 20.7 (Segal)**.** *The geometric realization $BX = |B_\bullet X|$ is an infinite loop space.*

Here is how you do this. There is a $X(1) \times \Delta^1 \hookrightarrow BX$ that is a point on the boundary of Δ^1. So we get $X(1) \to \Omega BX$. Iterate this by noting that $S \mapsto (T \mapsto X(S \wedge T))$ is a Γ-space of Γ-spaces. This is $B^{(1)}X : S \mapsto BX^S = (T \mapsto X(S \wedge T))$. Then we get $BX \to \Omega BB^{(1)}X$, and then go on.

Definition 20.8. We define $K^{\Omega B}(R) = \Omega BX_R$, which is actually an infinite loop space. Then we define $K_i(R) = \pi_i K^{\Omega B}(R)$.

Example 20.9. Let us compute $K_0(R) = \pi_0 \Omega BX_R = \pi_1 BX_R$. But this BX_R has a cell-decomposition and we know exactly what they are. The 1-cells are objects in the category of finitely generated finitely generated projective left R-modules. Then 2-cells are isomorphisms in this category. So π_1 is generated by γ_P, and the relation is that $\gamma_P \simeq \gamma_{P'}$ if $P \cong P'$.

Let M_R be given by the classifying space of objects n-tuples of R-modules and morphisms R-linear isomorphisms. This has a natural commutative and unital structure coming from $\oplus$.

Theorem 20.10 (Segal–McDuff)**.** *If M is homotopy commutative associative unital topological monoid, then*

$$H_*(\Omega BM) \cong H_*(M)[\pi_0^{-1}].$$

Applying this to M_R, which has $BM_R \simeq BX_R$ gives

$$\begin{aligned} H_*(\Omega BX_R) &\cong H_*(M_R)[\pi_0^{-1}] \cong H_*\Bigg(\coprod_{[P]} B\,\mathrm{GL}(P)\Bigg)[\pi_0^{-1}] \\ &\cong H_*\Big(\coprod_{n\geq 0} B\,\mathrm{GL}_n(R)\Big)[\pi_0^{-1}] \cong \varinjlim H_*\Big(\coprod_{n\geq 0} B\,\mathrm{GL}_n(R)\Big). \end{aligned}$$

So we recover

$$K_1(R) = \pi_1(K^{\Omega B}(R)) = H_1(K^{\Omega B}(R)) \cong \varinjlim_n H_1(B\,\mathrm{GL}_n(R)).$$

21 October 20, 2017

Today I want to state the theorems of Igusa and Waldhausen. Last time I explained how to construct the spaces $K(R)$. Today we will spend some time constructing $K(\mathbb{S}\Omega X)$.

21.1 $S_\bullet$-construction

The input for this is a Waldhausen category.

Definition 21.1. A **Waldhausen category** $\mathcal{C}$ is a pointed category $\mathcal{C}$ with two classes of morphisms, cofibrations and weak equivalences, satisfying

(i) isomorphisms are cofibrations and weak equivalences,

(ii) $* \to A$ are cofibrations,

(iii) cofibrations and weak equivalneces are closed under composition,

(iv) if $B \hookrightarrow A$ is a cofibration and $B \to C$ is any map, the pushout $C \cup_B A$ exists and $C \to C \cup_B A$ is a cofibration,

(v) in the above situation, if $A \to A'$, $B \to B'$, $C \to C'$ are weak equivalences with $B' \hookrightarrow A'$ cofibration, then $C \cup_B A \to C' \cup_{B'} A'$ is a weak equivalence.

Let $\mathrm{Ar}(n)$ be the category of arrows in

$$0 \to 1 \to \cdots \to n$$

so that there are $n(n+1)/2$ objects.

Definition 21.2. Let $S_n\mathcal{C}$ be the subcategory of $\mathrm{Fun}(\mathrm{Ar}(n), \mathcal{C})$ such that

- $F(i \to i) = *$,
- $F(i \to j) \hookrightarrow F(i \to k)$ is a cofibration,
- the diagram

$$\begin{tikzcd} F(i \to j) \arrow[r, hook] \arrow[d] & F(i \to k) \arrow[d] \\ F(j \to j) = * \arrow[r] & F(j \to k) \end{tikzcd}$$

  is a pushout square.

Example 21.3. Let us write out what $S_2\mathcal{C}$ is. They have objects given by diagrams

$$\begin{tikzcd} * \arrow[r] & F(0,1) \arrow[r, hook] \arrow[d] & F(0 \to 2) \arrow[d] \\ & * \arrow[r] & F(1 \to 2) \arrow[d] \\ & & *. \end{tikzcd}$$

Here, you can think of $* \to F(0 \to 1) \hookrightarrow F(0 \to 2)$ is a filtered object, with $F(0 \to 1)$ and $F(1 \to 2)$ the associated graded.

Here $[n] \mapsto \mathrm{Ar}(n)$ is a cosimplicial set, so $[n] \mapsto S_n\mathcal{C}$ is a simplicial category. Consider $wS_n\mathcal{C} \subseteq S_n\mathcal{C}$ the subcategory with the same objects but morphisms natural weak equivalences. Then $[n] \mapsto wS_n\mathcal{C}$ is also a simplicial category.

Definition 21.4. $\Omega|[n] \mapsto |N_\bullet wS_n\mathcal{C}|| = K^{S_\bullet}\mathcal{C}$.

For example, take $P(R)$ the finitely generated projective left R-modules, with cofibrations inclusions with projective cokernels and weak equivalences isomorphisms.

Theorem 21.5 (Quillen–Waldhausen)**.** $K^{\Omega B}(R) \simeq K^{S_\bullet}(P(R))$.

As another example, take FinSet_* be the pointed finite sets with cofibration inclusions and weak equivalences isomorphisms. In this case, it is easy to see that

$$wS_n(\mathsf{FinSet}_*) \simeq \mathsf{FinSet}_*^n$$

because the diagram is just determined by $F(i \to i{+}1)$. So we can trace through the definitions and get

$$K^{S_\bullet}(\mathsf{FinSet}_*) \simeq \Omega B\left(\coprod_{n\geq 0} B\Sigma_n\right).$$

Theorem 21.6 (BPQS)**.** $\Omega B(\coprod_{n\geq 0} B\Sigma_n) \simeq \Omega^\infty\mathbb{S}$.

21.2 Algebraic K-theory of spaces

Let X be a space.

Definition 21.7. A **retractive finite space** over X is a triple (Y, i, r) such that $i : X \hookrightarrow Y$ is a cofibration, $r : Y \to X$ is a retraction so that $r \circ i = \mathrm{id}_X$, and (Y, X) is homotopy equivalent to a relative finite CW-complex.

Let $R^f(X)$ be the Waldhausen category with objects retractive finite spaces over X. The cofibrations will be cofibrations and weak equivalences will be weak equivalences.

Definition 21.8. The **algebraic K-theory of spaces of X** is

$$A(X) = K^{S_\bullet}(R^f(X)).$$

Example 21.9. Let us compute $\pi_0 A(*)$. Note that π_0 is given by [pointed spaces, weakly equivalent to a finite CW-complex with a retraction to $*$. There are relations $[Y] = [Y']$ if $Y \simeq Y'$ and $[Y'] = [Y] + [Y'']$ if there is a cofiber sequence $Y \hookrightarrow Y' \twoheadrightarrow Y''$.

But you can do this. We have

$$0 = [*] = [D^k] = [S^{k-1}] + [S^k]$$

because $S^{k-1} \to D^k \to S^k$. Then we see that $[S^k] = (-1)^k[S^0]$. Any finite pointed complex Y splits as

$$[Y] = \sum_{k \geq 0} \#(k\text{-cells})[S^k] = \chi(Y, *).$$

So $\pi_0 A(*) \cong \mathbb{Z}$.

There is also a group completion construction of $A(X)$ for path-connected X. Let GX be the Kan loop group, or the topological group weakly equivalent to ΩX.

Definition 21.10. $\mathrm{GL}_n(\Sigma^\infty_+ GX)$ is the homotopy colimit as $k \to \infty$ some subspace of $\prod_n \Omega^k(\bigvee_n S^k \wedge GX_+)$. If $k \geq 2$, then

$$\pi_0 \Omega^k\left(\bigvee_n S^k \wedge GX_+\right) \cong \bigoplus_n \mathbb{Z}[\pi_1(X)]^n = M_n(\mathbb{Z}[\pi_1]),$$

so we take the subspace corresponding to $\mathrm{GL}_n(\mathbb{Z}[\pi_1])$.

Theorem 21.11 (Waldhausen)**.** *If X is path-connected, then*

$$A(X) \simeq \Omega B\left(\coprod_{n \geq 0} B\,\mathrm{GL}_n(\Sigma^\infty_+ GX)\right).$$

The advantage of this is that there is a map

$$\Sigma_n \wr GX \to \mathrm{GL}_n(\Sigma^\infty_+ GX)$$

and induces the map

$$\Omega B\left(\coprod_{n \geq 0} B(\Sigma_n \wr GX)\right) \to \omega B\left(\coprod_{n \geq 0} B\,\mathrm{GL}_n(\Sigma^\infty_+ GX)\right) \simeq A(X).$$

The first thing is $\Omega^\infty \Sigma^\infty X_+ = QX_+$, so we have a map $QX_+ \to A(X)$. This map actually comes from a map of spectra $\Sigma^\infty X_+ \to \underline{A}(X)$.

Definition 21.12. We define $\underline{\mathrm{Wh}}^{\mathrm{Diff}}(X) = \mathrm{cofib}(\Sigma^\infty X_+ \to \underline{A}(X))$, and we define

$$\mathrm{Wh}^{\mathrm{Diff}}(X) = \Omega^\infty \underline{\mathrm{Wh}}^{\mathrm{Diff}}(X).$$

Theorem 21.13 (Waldhausen)**.** $A(X) \simeq QX_+ \times \mathrm{Wh}^{\mathrm{Diff}}(X)$.

21.3 Moduli spaces of h-cobordisms

The s-cobordism classified h-cobordisms starting at M with $\dim M \geq 5$. Here, we figured out that

$$\left\{\begin{matrix}\text{h-cobordisms} \\ \text{starting at } M\end{matrix}\right\} /\text{diff. rel } M \quad \cong \quad \mathrm{Wh}_1(\pi_1(M)).$$

Now we can try to figure out what the automorphisms are. At least, the automorphisms of the trivial h-cobordism $M \hookrightarrow M \times I$ are $\mathrm{Diff}(M \times I \operatorname{rel} M \times \{0\})$. This generalizes to $\partial M \neq \emptyset$.

Definition 21.14. We define $\mathcal{C}(M) = \mathrm{Diff}(M \times I \operatorname{rel} M \times \{0\} \cup \partial M \times I)$.

Then we can define the moduli space of h-cobordisms as

$$H(M) = \mathrm{Wh}_1(\pi_1(M)) \times B\mathcal{C}(M).$$

There is a map $H(M) \to H(M \times I)$ induced by homomorphisms $\mathcal{C}(M) \to \mathcal{C}(M \times I)$.

Definition 21.15. We define the stable moduli space as

$$\mathscr{H}(M) = \operatorname*{hocolim}_{h \to \infty} H(M \times I^k).$$

Theorem 21.16 (Igusa)**.** *The map $H(M) \to \mathscr{H}(M)$ is $\min(\frac{m-1}{3}, \frac{m-5}{2})$-connected.*

Theorem 21.17 (Waldhausen)**.** $\mathscr{H}(M) \simeq \Omega\,\mathrm{Wh}^{\mathrm{Diff}}(M)$.

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Last time we had this claim that there is a map

$$H(M) \to \Omega \operatorname{Wh}^{\mathrm{Diff}}(M)$$

which is approximately $\frac{m}{3}$-connected. Today we will use this to prove that $\pi_0(\mathrm{Diff}_\partial(D^n)) \to \Theta_{n+1}$ is and isomorphism if $n \geq 7$.

22.1 Computation of $\pi_0 H(X)$

Recall that for X path-connected,

$$\begin{tikzcd}
QX \simeq \Omega B\Big(\coprod_{n\geq 0} B(\Sigma_n \wr GX)\Big) \arrow[r] \arrow[d] & \Omega B\Big(\coprod_{n\geq 0} B\operatorname{GL}_n(\Sigma^\infty_+ GX)\Big) \simeq A(x) \arrow[d] \\
Q(B\pi_1)_+ \simeq \Omega B\Big(\coprod_{n\geq 0} B(\Sigma_n \wr \pi_1)\Big) \arrow[r] & \Omega B\Big(\coprod_{P} B\operatorname{GL}([P])\Big) \simeq K(\mathbb{Z}[\pi_1]).
\end{tikzcd}$$

Then we can take homotopy fibers of the vertical maps, apply homotopy groups and take cokernels of the horizontal maps.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & \pi^S_{n+1}(B\pi_1/X) \arrow[r] \arrow[d] & \pi_n \mathcal{F} \arrow[r] \arrow[d] & F_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^S_n(X_+) \arrow[r] \arrow[d] & \pi_n A(x) \arrow[r] \arrow[d] & \pi_n(\operatorname{Wh}^{\mathrm{Diff}}(X)) \arrow[r] \arrow[d] & 0 \\
 & \pi^S_n(B\pi_{1+}) \arrow[r] \arrow[d] & K_n(\mathbb{Z}[\pi_1]) \arrow[r] \arrow[d] & W_n(\pi_1) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Here we don't know if the right vertical sequence is exact.

Remember that $\pi_0 A(*) \cong \mathbb{Z}$, and this generalizes to $\pi_0 QX_+ \xrightarrow{\cong} \pi_0 A(X)$. So the first possible non-zero $\pi_n \operatorname{Wh}^{\mathrm{Diff}}(X)$ is π_1.

Lemma 22.1. *$QX_+ \to Q(B\pi_1)_+$ and $A(X)_0 \to K(\mathbb{Z}[\pi_1])_)$ are 2-connected.*

Understand and include proof

So we can include this to get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi^S_1(X_+) \arrow[r] \arrow[d, "\cong"] & \pi_1 A(X) \arrow[r] \arrow[d, "\cong"] & \pi_1 \operatorname{Wh}^{\mathrm{Diff}}(X) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi^S_1(B\pi_1)_+ \arrow[r, "(*)"] & K_1(\mathbb{Z}[\pi_1]) \arrow[r] & W_1(\pi_1) \arrow[r] & 0
\end{tikzcd}$$

So to compute $\pi_1 \mathrm{Wh}^{\mathrm{Diff}}(X)$, we only need to figure out the cokernel of $(*)$: $\pi_1^S(B\pi_1)_+ \to K_1(\mathbb{Z}[\pi_1])$.

We can compute $\pi_1^S(B\pi_1)_+$ using the Atiyah–Hirzebruch spectral sequence. Then $E^2_{p,q} = H_p(B\pi_1; \pi_q^S)$ converges to this, and so we get

$$\pi_1^S(B\pi_1)_+ \cong H_1(B\pi_1) \oplus \mathbb{Z}/2\mathbb{Z} \cong (\pi_1)^{\mathrm{ab}} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Now the map $(*)$ is just the inclusion of $[g]$ and $[-1]$, so it generates the same subgroup of $K_1(\mathbb{Z}[\pi_1])$ as $(\pm g)$. This shows that

$$\pi_0 H(X) \cong \pi_1 \mathrm{Wh}^{\mathrm{Diff}}(X) \cong W_1(\pi_1) \cong \mathrm{Wh}_1(\pi_1).$$

This is just the s-cobordism theorem again.

22.2 Computation of $\pi_1 H(X)$—the HWI sequence

Let us now compute $\pi_2 \mathrm{Wh}^{\mathrm{Diff}}(X)$.

$$\begin{tikzcd}
 & & K_3(\mathbb{Z}[\pi_1]) \arrow[r] \arrow[d] & W_3(\pi_1) \arrow[r] \arrow[d] & 0 \\
 & \pi_3^S(B\pi_1/X) \arrow[r] \arrow[d] & \pi_2\mathcal{F} \arrow[r, two heads] \arrow[d] & F_2(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_2^S(X_+) \arrow[r] \arrow[d, two heads] & \pi_2 A(X) \arrow[r, two heads] \arrow[d, two heads] & \pi_2 \mathrm{Wh}^{\mathrm{Diff}}(X) \arrow[r] \arrow[d, two heads] & 0 \\
 & \pi_2^S(B\pi_1)_+ \arrow[r] \arrow[d] & K_2(\mathbb{Z}[\pi_1]) \arrow[r] \arrow[d] & W_2(\pi_1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Lemma 22.2. $\pi_2\mathcal{F} = H_0(B\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[\pi_1])$

Proof. We have, by the group completion theorem,

$$\begin{tikzcd}
\mathrm{colim}_{n\to\infty} B\,\mathrm{GL}_n(\Sigma_+ GX) \arrow[r] \arrow[d] & A(X) \arrow[d] \\
\mathrm{colim}_{n\to\infty} B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1]) \arrow[r] & K\mathbb{Z}[\pi_1].
\end{tikzcd}$$

Vertical maps have the same relative homology in positive degress, so

$$\begin{aligned}
\pi_2\mathcal{F} &\cong \pi_3(K(\mathbb{Z}[\pi_1])) \cong H_3(K(\mathbb{Z}[\pi_1]), A(X)) \\
&\cong \operatorname*{colim}_{n\to\infty} H_3(B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1]), B\,\mathrm{GL}_n(\Sigma_+ GX)).
\end{aligned}$$

Now there is a relative Serre spectral sequence coming from $BM_n(Q_0 GX_+) \to B\,\mathrm{GL}_n(\Sigma^\infty_+) \to B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1])$ mapping to $* \to B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1]) \to B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1])$.

This spectral sequence is going to be

$$E^2_{p,q} = H_p(B\,\mathrm{GL}_n(\mathbb{Z}[\pi_1]), H_q(*, BM_n(Q_0GX_+))),$$

but the coefficient is $H_{q-1}(BM_nQ_0GX_+)$. This has first non-zero group at $q = 3$ by the Hurewicz theorem, and it is

$$H_2(BM_nQ_0GX_+) \cong \pi_2(BM_nQ_0GX_+) \cong M_nQ_0GX_1) = M_n(\pi_1(Q_0GX_+)).$$

Here, $\pi_1 Q_0GX_+ \cong \bigoplus_{\gamma\in\pi_1} \pi_2^S(X\langle 1\rangle_+)$ which are $\pi_2(X) \oplus \mathbb{Z}/2\mathbb{Z}$. So we get

$$\pi_2\mathcal{F} = H_0(BGL_n(\mathbb{Z}[\pi_1]); M_n((\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[\pi_1])),$$

and the trace map to $H_0(B\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[\pi_1])$ is an isomorphism. □

Lemma 22.3. $\pi_3^S(B\pi_1/X) \cong H_0(B\pi_1; (\pi_2)[1])$.

Proof. We have $\pi_3^S(B\pi_1, X) \cong H_3(B\pi_1, X)$ by Atiyah–Hirzebruch. We also have $H_3(B\pi_1, X) \cong \pi_3(B\pi_1, X)/\pi_1$ by Hurewicz. This is isomorphic to $\pi_2(X)/\pi_1 \cong H_0(B\pi_1; \pi_2)$. □

Once you further prove that some maps are injective, you get that the last vertical sequence is exact.

Theorem 22.4 (Hatcher–Wagoner–Igusa)**.** *The sequence*

$$K_3(\mathbb{Z}[\pi_1]) \to \frac{H_0(B\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z}))}{H_0(B\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[1])} \to \pi_2 \,\mathrm{Wh}^{\mathrm{Diff}}(X) \to \frac{K_2(\mathbb{Z}[\pi_1])}{\pi_2^S(B\pi_1)_+} \to 0$$

is exact.

Corollary 22.5. *If $\pi_1(X) = \{e\}$ with $\dim X \geq 7$, then $\pi_0\mathcal{C}(X) = \pi_1 H(M) = \pi_2 \,\mathrm{Wh}^{\mathrm{Diff}}(X) = 0$.*

Proof. The left term vanishes automatically. The right term also vanishes because

$$\pi_2^S \cong \mathbb{Z}/2\mathbb{Z} \to K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \to K_2(\mathbb{R}^{\mathrm{top}}) \cong \pi_2 ko \cong \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism. □

23 October 25, 2017

Today is the last lecture on algebraic K-theory.

23.1 Pseudo-isotopy implies isotopy

Recall we have the Hatcher–Wagoner–Igusa sequence. For M of dimension ≥ 7,

$$K_3(\mathbb{Z}[\pi_1]) \to \frac{H_0(\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[\pi_1])}{H_0(\pi_1; (\pi_2 \oplus \mathbb{Z}/2\mathbb{Z})[1])} \to \pi_1 \operatorname{Wh}^{\mathrm{Diff}}(M) \cong \pi_0 \mathcal{C}(M) \to \frac{K_2(\mathbb{Z}[\pi_1])}{\pi_2^S(B\pi_1)_+} \to 0$$

is exact. If $\pi_1(M) = \{e\}$, then we have established that $\pi_0\mathcal{C}(M) = 0$. I want to state this more concretely.

Definition 23.1. For two diffeomorphisms $f, g : M \to M$, f is **pseudo-isotopic** to g if thee is a diffeomorphism $F : M \times I \to M \times I$ such that $F|_{M\times\{0\}} = f$ and $F|_{M\times\{1\}} = g$.

This is not an isotopy because we don't have the condition that F preserves the t-coordinate.

Corollary 23.2. *If $m \geq 7$ and $\pi_1(M) = \{e\}$, then f being pseudo-isotopic to g implies f isotopic to g.*

Proof. Take an arbitrary pseudo-isotopy $F : M \times I \to M \times I$. Then $F' = (f \times \mathrm{id})^{-1} \circ F$ is a pseudo-isotopy from id to $f^{-1} \circ g$. Then F' is a concordance diffeomorphism, i.e., in $\operatorname{Diff}(M \times I, M \times \{0\})$. Because $\pi_0(\mathcal{C}(M)) = 0$, there is an isotopy F'_t from F' to $\mathrm{id}_{M\times I}$. But then $F'_t|_{M\times\{1\}}$ is an isotopy of diffeomorphisms of M from id to $f^{-1} \circ g$. Composing with f gives an isotopy from f to g. □

23.2 $\pi_0(\operatorname{Diff}_\partial(D^n))$ and Θ_{n+1}

Recall that Θ_{n+1} is the space of oriented homotopy $(n+1)$-spheres up to orientation preserving diffeomorphisms. There is an abelian monoid structure, which is to take two spheres and taking the connected sum. Here are some things we have proven.

- this connected sum is well-defined up to orientation-preserving diffeomorphism.
- it is commutative and associative.
- there is a unit $[S^{n+1}]$.

Lemma 23.3. *If $n \geq 4$, this is an abelian group.*

Proof. Let Σ be any oriented homotopy $(n+1)$-sphere. We take $\Sigma \times I$ and remove a neighborhood of $\{*\} \times I$. Let this be W. Then W is an $(n+2)$-dimensional manifold with

$$\partial W = \Sigma \# \overline{\Sigma},$$

where $\overline{\Sigma}$ is the Σ with opposite orientation. Now take $w \in W$ and remove a little dist. Then W is a h-cobordism from $\Sigma \# \overline{\Sigma}$ to S^{n+1}, so $\Sigma \# \overline{\Sigma} \cong S^{n+1}$. $\square$

Given $f \in \mathrm{Diff}^+(S^n)$, we can construct a sphere $S_f^{n+1} = D^{n+1} \cup_f D^{n+1}$. This gives a map

$$\pi_0(\mathrm{Diff}_\partial(D^n)) \xrightarrow{\cong} \pi_0(\mathrm{Diff}^+(S^n)) \longrightarrow\!\!\!\!\rightarrow \Theta_n.$$

Here, the first isomorphism follows from $\mathrm{Diff}^+(S^n) \simeq \mathrm{SO}(n+1) \times \mathrm{Diff}_\partial(D^n)$, and the second surjection follows from the h-cobordism theorem.

Lemma 23.4. $\pi_0(\mathrm{Diff}_\partial(D^n)) \twoheadrightarrow \Theta_{n+1}$ *is a group homomorphism.*

Proof. Taking the connected sum along the glued parts. Then we get

$$S_f^{n+1} \# S_g n + 1 \cong S_{f \natural g}^{n+1},$$

where $f \natural g$ is the juxtaposition, putting them next to each other. But note that $f \circ g$ is isotopic to $(f \natural 1) \circ (g \natural 1) = f \natural g$. $\square$

Now we want to show that $\pi_0(\mathrm{Diff}_\partial(D^n)) \twoheadrightarrow \Theta_{n+1}$ is an isomorphism. It suffices to show that if $S_f^{n+1} \cong S^{n+1}$ then f is isotopic to id.

Proposition 23.5. *If $n \geq 6$ and $f \in \mathrm{Diff}^+(S^n)$ and $S_f^{n+1} \cong S^{n+1}$, then f is isotopic to* id.

Proof. We have $i_1 : D^{n+1} \amalg D^{n+1} \hookrightarrow S_f^{n+1}$ and $i_2 : D^{n+1} \amalg D^{n+1} \hookrightarrow S^{n+1}$, where the first disk is mapped to a little disk around the origin in one hemisphere and the second disk is mapped to the other hemisphere.

The embeddings i_2 and $g \circ i_1$ are orientation-preserving. By isotopy extension, we can assume that the following diagram commutes.

$$\begin{tikzcd} D^{n+1} \amalg D^{n+1} \arrow[r, equal] \arrow[d, hook, "i_1"] & D^{n+1} \amalg D^{n+1} \arrow[d, hook, "i_2"] \\ S_f^{n+1} \arrow[r, "g"] & S^{n+1} \end{tikzcd}$$

The restricting the complement gives a $\overline{g} : S^n \times I \to S^n \times I$ such that $\overline{g}|_{S^n \times \{0\}} = \mathrm{id}$ and $\overline{g}|_{S^n \times \{1\}} = f^{-1}$. This shows that f^{-1} is pseudo-isotopic to id, and so f is isotopic to id. $\square$

Corollary 23.6. *If $n \geq 7$, then $\pi_0(\mathrm{Diff}_\partial(D^n)) \cong \Theta_{n+1}$.*

Cerf proved that you can actually improve this to $n \geq 5$.

Theorem 23.7 (Kervaire–Milnor)**.** *If $n \geq 2$, then Θ_{n+1} is finite.*

The strategy is to prove that each $\Sigma \in \Theta_{n+1}$ admits a stable framing. Once you've done that, send it to $p(\Sigma)$, the collection of bordism classes of Σ with a choice of stable framing. This is a subgroup of π_{n+1}^S, and get

$$0 \to bP^{n+2} \to \Theta_{n+1} \to \pi_{n+1}^S / p(S^{n+1}) \to 0.$$

Here bP^{n+1} is those Σ which admits a stable framing such that it bounds a framed manifold. Then we can just prove that bP^{n+2} is finite, but it is hard.

24 October 27, 2017

Last time we proved that $\pi_0(\mathrm{Diff}_\partial(D^n)) \cong \Theta_{n+1}$ is finite if $n \geq 5$. Today we'll use this to compute $\pi_*(\mathrm{Diff}_\partial(D^n)) \otimes \mathbb{Q}$.

24.1 Concordances rationally

Recall that that $\mathcal{C}(M) = \mathrm{Diff}(M \times I \operatorname{rel} M \times \{0\} \cup \partial M \times I)$. If we restrict to $M \times \{1\}$ we get a fiber sequence

$$\mathrm{Diff}_\partial(D^{n+1}) \to \mathcal{C}(D^n) \to \mathrm{Diff}_\partial(D^n).$$

So let us compute $\mathcal{C}(D^n)$ rationally first.

The Igusa–Waldhauset theorems give

$$\pi_*\mathcal{C}(D^n) \otimes \mathbb{Q} \cong \pi_{*+2} \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q}$$

for $* \leq \min(\frac{n-7}{3}, \frac{n-9}{2})$. This is the concordance stable range. Now there is a waldhausen spitting $\underline{A}(*) \simeq \mathbb{S} \times \underline{\mathrm{Wh}}^{\mathrm{Diff}}(*)$.

Now note that $\pi_*\mathbb{S} \otimes \mathbb{Q}$ is $\mathbb{Q}$ if $* = 0$ and 0 otherwise, and $\mathbb{S} \to \underline{A}(*)$ is an π_0-isomorphism. So we get

$$\pi_*\underline{\mathrm{Wh}}^{\mathrm{Diff}}(*) = \begin{cases} 0 & * = 0 \\ \pi_*\underline{A}(*) & \text{otherwise.} \end{cases}$$

Now

$$\prod_{n^2} Q_0 S^0 \simeq Q^* \to \mathrm{GL}_n(\mathbb{S}) \to \mathrm{GL}_n(\mathbb{Z})$$

and so $A(*) \to K(\mathbb{Z})$ is a $\mathbb{Q}$-isomorphism. The summary is that

$$\pi_i\mathcal{C}(D^n) \otimes \mathbb{Q} \cong K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 0, i = 4k+1 \text{ for } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

in the concordance stable range.

24.2 Block diffeomorphisms

Definition 24.1. $\mathrm{Diff}^b_\partial(M)$ is a simplicial group with k-simplices given by diffeomorphisms $f : M \times \Delta^k \to M \times \Delta^k$ such that $f(M \times \sigma) = M \times \sigma$ and f is id on $\partial M \times \Delta^k$.

For example $\pi_0 \mathrm{Diff}^b_\partial(M)$ is diffeomorphisms of M up to pseudo-isotopy. Here $[f] \in \pi_h \mathrm{Diff}^b_\partial(M)$ is represented by a diffeomorphism $M \times \Delta^k \to M \times k$ that is identity on $M \times \partial\Delta^k$ and $\partial M \times \Delta^k$ up to pseudo-isotopy. So it is isomorphic to $\pi_0 \mathrm{Diff}^b_\partial(M \times \Delta^k)$.

In the case $M = D^n$ with $n \geq 5$,

$$\pi_k \mathrm{Diff}^b_\partial(D^n) \cong \pi_0 \mathrm{Diff}^b_\partial(D^{n+k}) \cong \pi_0 \mathrm{Diff}_\partial(D^{n+k}) \cong \Theta_{n+k+1}$$

is finite. So rationally $\operatorname{Diff}^b_\partial(D^n)$ has contractible components and $\operatorname{Diff}_\partial(D^n) \hookrightarrow \operatorname{Diff}^b_\partial(D^n)$ is a π_0-isomorphism. So

$$\operatorname{Diff}^b_\partial(D^n)/\operatorname{Diff}_\partial(D^n) \cong \operatorname{Diff}^b_\partial(D^n)_{\mathrm{id}}/\operatorname{Diff}_\partial(D^n)_{\mathrm{id}} \simeq_{\mathbb{Q}} B\operatorname{Diff}_\partial(D^n)_{\mathrm{id}}.$$

24.3 Hatcher spectral sequence

This is a spectral sequence that computes $\pi_k(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M))$ in terms of $\pi_q(\mathcal{C}(M \times I^p))$.

Definition 24.2. Let $D^k(M)$ be the diffeomorphisms of $M \times I^k \operatorname{rel} \partial M \times I^k$ preserving projections to ∂I^k on $M \times \partial I^k$, quotient out by diffeomorphisms of $M \times I^k \operatorname{rel} \partial M \times I^k$ preserving the projection to I^k.

There is a map
$$\pi_i D^k(M) \to \pi_{i-1} D^{k+1}(M)$$
because a map from $(I^i, \partial I^i)$ to diffeomorphisms of $M \times I^k$ can be thought of as a map from $(I^{i-1}, \partial I^{i-1})$ to diffeomorphisms of $M \times I^k \times I$. There is also a map
$$\pi_0 D^k(M) \to \pi_k(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M)).$$
They give a filtration by

$$F_i \pi_k(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M)) = \operatorname{im}\begin{pmatrix} \pi_i D^{k-i}(M) \to \pi_0 D^k(M) \\ \to \pi_k(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M)) \end{pmatrix}.$$

Definition 24.3. $C^k(M)$ is the diffeomorphisms of $M \times I^k \times I \operatorname{rel} \partial M \times I^k \times I$ preserving projection to $I^k \times \{0\}$ and $\partial I^k \times I$, quotient out by diffeomorphisms of $M \times I^k \times I \operatorname{rel} \partial M \times I^k \times I$ preserving projection to $I^k \times I$.

Then we have fiber sequences $D^{k+1}(M) \to C^k(M) \to D^k(M)$ and so their homotopy groups give an exact couple

$$\begin{tikzcd}
\bigoplus \pi_* D^k(M) \arrow[rr, "k", "-1"'] & & \bigoplus \pi_* D^k(M)) \arrow[dl, "j"] \\
& \bigoplus \pi_* C^k(M)). \arrow[ul, "k"] &
\end{tikzcd}$$

This spectral sequence converges to $\pi_{p+q+1}(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M))$. The E^1 page is just $\pi_* C^k(M)$ and the d^1-differential is $\pi_* C(M \times I^k) \to \pi_* C(M \times I^{k-1})$ coming from restriction to $M \times I^k \times \{1\}$. Here, $\mathcal{C}(M \times I^k) \to C^k(M)$ is a weak equivalence.

Proposition 24.4. *There is a spectral sequence*

$$E^1_{p,q} = \pi_q(\mathcal{C}(M \times I^p)) \quad \Longrightarrow \quad \pi_{p+q+1}(\operatorname{Diff}^b_\partial(M)/\operatorname{Diff}_\partial(M))$$

with d^1 induced by restriction to $M \times I^k \times \{1\}$.

There is a "stacking operation" on $\pi_q(\mathcal{C}(M \times I^p))$ that induces a group structure on $\pi_*(\mathcal{C}(M))$. There is also the standard group structure on homotopy groups. By some Eckmann–Hilton argument, they are both commutative an are equal.

There is also a stabilization $\mathcal{C}(M \times I^{p-1}) \to \mathcal{C}(M \times I^p)$, and it is going to satisfy

$$d^1[\sigma(g)] \mapsto [g] + [\overline{g}]$$

where $\overline{g}$ is the flip of g.

Lemma 24.5. $[\sigma(g)] = -[\overline{\sigma(\overline{g})}]$.

Proof. You can verify it with a picture. □

24.4 Farrel–Hsiang theorem

If $M = D^n$, only consider homotopy in the concordance stable range $\pi_*(\mathcal{C}(D^n)) \otimes \mathbb{Q} \cong \pi_{*+2} \operatorname{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q}$.

The d^1 differential is going to be $[\sigma(g)] \mapsto [g] + [\overline{g}]$.

Lemma 24.6 (Farrel–Hsiang)**.** *Involution acts on $\pi_*(\mathcal{C}(D^n)) \times \mathbb{Q}$ as $(-1)^n$.*

So $d^1 : E_{1,q} \to E_{0,q}$ acts as 0 if n is odd and acts as 2 if n is oven. So in the E^2 of the Hatcher spectral sequence, everything vanishes of n is even and everything except for $p = 0$ vanishes if n if odd.

Theorem 24.7. *$\pi_*(B \operatorname{Diff}_\partial(D^n)) \otimes \mathbb{Q} \cong \begin{cases} K_{*+1}(\mathbb{Z}) \otimes \mathbb{Q} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$ in the concordance stable range.*

25 October 30, 2017

The goal of this week is to proof the following result:

$$\Theta_n \times B\,\mathrm{Diff}_\partial(D^n) \simeq \Omega^n\,\mathrm{Top}(n)/\,\mathrm{O}(n)$$

if $n \neq 4$. We will prove this for $n \geq 6$.

25.1 Bundles

A **principal G-bundle** is a space E with a free G-action with $\pi : E \to E/G = B$ being locally trivial. If B are nice (e.g., paracompact), these are classified by BG:

$$[B, BG] \quad \longleftrightarrow \quad \begin{Bmatrix} \text{principal } G\text{-bundles} \\ \text{over } B \end{Bmatrix}.$$

The corresponding map is given by pulling back $EG \to BG$.

Give a G-space Y we can form an associated bundle to a principal bundle $\pi : E \to B$ given by

$$\tilde{\pi} : E \times_G Y \to B,$$

and the transition functions take values in G.

If the action of G on Y is faithful, then a locally trivial Y-bundle admitting transition functions in G comes up to isomorphism from from the unique principal G-bundle.

We take $G = \mathrm{Diff}_\partial(M)$ acting faithfully on M. Then we have a correspondence

$$\begin{Bmatrix} \text{principal } \mathrm{Diff}_\partial(M)\text{-bundles} \\ \text{over } B \end{Bmatrix} \quad \longleftrightarrow \quad \begin{Bmatrix} M\text{-bundles with transition} \\ \text{functions in } \mathrm{Diff}_\partial(M) \text{ over } B \end{Bmatrix}.$$

If B is a smooth manifold, it is not a priori true that the total space of a M-bundle with transition functions in $\mathrm{Diff}_\partial(M)$ is also a smooth manifold. That is, given $f : U_i \cap U_j \to \mathrm{Diff}_\partial(M)$, it is not true that

$$(U_i \cap U_j) \times M \to (U_i \cap U_j) \times M; \quad (u, x) \mapsto (u, f(u)x)$$

is smooth. But the smoothing techniques imply that up to isomorphism we can assume transition functions have smooth adjoints. So E has a canonical smooth structure such that $\pi : E \to B$ is smooth.

Definition 25.1. A **smooth M-bundle** $\pi : E \to B$ over a smooth manifold B is a smooth map which admits bundle charts and diffeomorphisms $\pi^{-1}(U) \to U \times M$.

So if B is smooth, there is a bijection between

$$[B, B\,\mathrm{Diff}_\partial(M)] \quad \longleftrightarrow \quad \begin{Bmatrix} \text{smooth } M\text{-bundles} \\ \text{over } B \end{Bmatrix} /\text{isomorphisms}.$$

25.2 Submersions

Definition 25.2. A smooth map $\pi : E \to B$ is a **smooth submersion** if it has submersion charts. For each $e \in E$, there exists an open $U \subseteq E$ containing e and an open $V \subseteq \pi^{-1}(\pi(e))$ with diffeomorphisms

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[dr] & \pi(U) \times V \arrow[d] \\ & \pi(U). \end{tikzcd}$$

You can also generalize this definition to a submersion chart for any $K \subseteq \pi^{-1}(b)$. Then the definition is that for any open $U \subseteq E$ open containing K, there exists an open $V \subseteq \pi^{-1}(b)$ containing K with a diffeomorphism

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[dr] & \pi(U) \times V \arrow[d] \\ & \pi(U). \end{tikzcd}$$

Lemma 25.3. *Let $\pi : E \to B$ be a smooth submersion. For $K_0, K_1 \subseteq \pi^{-1}(b)$ compact admitting submersion charts, then $K_0 \cup K_1$ also admits a submersion chart.*

Corollary 25.4. *If $\pi : E \to B$ is a proper smooth submersion, then every $\pi^{-1}(b)$ admits a submersion chart.*

Corollary 25.5 (Ehresmann fibration theorem)**.** *For a proper submersion $\pi : E \to B$ and B path-connected, $E \to B$ is actually a smooth manifold bundle.*

25.3 Kirby–Siebenmann bundle theorem

In general, we can't drop the properness assumption.

Example 25.6. Consider $E = \mathbb{R} \amalg (0, \infty)$ and $B = \mathbb{R}$. This is a smooth submersion but not a manifold bundle.

Theorem 25.7 (Kirby–Siebenmann)**.** *If $\pi : E \to B$ is a smooth submersion which is a topological manifold bundle, then it is a smooth manifold bundle if* $\dim \text{fiber} \geq 6$.

Definition 25.8. $\pi : E \to B$ is a **topological manifold bundle** if it has topological bundle charts: for all $b \in B$ there exists $U \subseteq B$ containing B and homeomorphisms $\pi^{-1}(U) \to U \times \pi^{-1}(b)$.

Today we are going to look at a weak version of this. We are going to assume that E is homeomorphic to $B \times M \times \mathbb{R}$. Here is the strategy. We are going to factor

$$\begin{tikzcd} E \arrow[r, "\mathbb{Z}\text{-cover}"] \arrow[dr, "\pi"'] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\ & B \end{tikzcd}$$

and then $\tilde{\pi}$ is going to be a proper map. Then it is a proper smooth submersion and so a smooth manifold bundle. Then $\pi : E \to B$ is also going to be a smooth manifold bundle, by pulling back the smooth structure.

The input is $\pi : E \to B$ a smooth submersion and a continuous function $p : E \to \mathbb{R}$. Let

$$F_b = \pi^{-1}(b), \quad F_b(m,n) = F_b \cap p^{-1}([m,n]).$$

Definition 25.9. π satisfies the **fiberwise engulfing condition** if for all $b \in B$ and integers $m \leq n$, there is a smooth isotopy h_t of F_b such that

$$h_0 = \mathrm{id}, \quad \operatorname{int} h_1(F_b(-\infty, m)) \supseteq F_b(-\infty, n)$$

and is compactly supported in $F_b(m-1, n+1)$.

We are going to first prove a global version.

Proposition 25.10. *Suppose B is compact of dimension k. If π satisfies the fiberwise engulfing condition, then for all integers $m \leq n$ there is a smooth isotopy g_t of E over B, such that*

$$g_0 = \mathrm{id}, \quad \operatorname{int}(g_1(p^{-1}(\infty, m])) \supseteq p^{-1}(-\infty, n])$$

and is compactly supported in $p^{-1}(m-k-1, n+k+1)$.

These are special cases of $\mathcal{E}(r, c, C)$ for $r \in (0, \infty]$, $c \in (0, \infty)$, $C \subseteq B$ compact: for all $m \leq n$ such that $[m-c, n+c] \subseteq [-r+2, r-2]$ there is a smooth isotopy g_t of E over B such that

$$g_0 = \mathrm{id}, \quad \operatorname{int}(g_1(p^{-1}(-\infty, m]))) \supseteq p^{-1}(-\infty, n] \cap \pi^{-1}(C)$$

and is compactly supported in $p^{-1}(m-c, n+c)$. Here are some properties:

- $\mathcal{E}(r, c, C)$ implies $\mathcal{E}(s, d, D)$ if $r \geq s$ and $c \leq d$ and $C \supseteq D$.
- if $\mathcal{E}(r_i, c, C)$ for $r_i \to \infty$ then $\mathcal{E}(\infty, c, C)$.
- $\mathcal{E}(r, c, C)$ and $\mathcal{E}(r, d, D)$ implies $\mathcal{E}(r, c+d, C \cup D)$ because we can do the isotopy one after another. If $C \cap D = \emptyset$ we can improve it to $\mathcal{E}(r, \max(c, d), C \cup D)$.

Lemma 25.11. *Fix r, and assume the fiberwise engulfing condition. For each $b \in B$, assume that there exists an open $W_b \subseteq B$ containing b such that for all compact $C_b \subseteq W_b$, $\mathcal{E}(r, 1, C_b)$ is true.*

Proof. Since π is a smooth submersion, there is a submersion chart for $F_b(-r, r)$:

$$\begin{tikzcd} E \supseteq U_b \arrow[r] \arrow[dr] & W_b \times V_b \subseteq \pi^{-1}(b) \arrow[d] \\ & W_b. \end{tikzcd}$$

Use this to extend h_f we get for F_b to a neighborhood of $b \in W_b$. $\square$

Proof of Proposition 25.10. B has a cover by $k+1$ compact subset C_i, each of which a finite disjoint of C_{ij}. todo

Then for all i, j, we have $\mathcal{E}(r, 1, C_{ij}$ and this implies $\mathcal{E}(r, 1, C_i)$ fora ll i. Then $\mathcal{E}(r, k+1, B)$ and so $\mathcal{E}(\infty, k+1, B)$. $\square$

Corollary 25.12. *For every integers $m \leq n$, there exists an open $E_{mn} \subseteq E$ containing $p^{-1}([m, n])$ such that $\pi|_{E_{mn}} : E_{mn} \to B$ which is a smooth manifold bundle.*

26 November 1, 2017

Today we are going to finish the weak bundle theorem. Last time we proved the following.

Proposition 26.1. *For a smooth submersion $\pi : E \to B$ with compact B and continuous $p : E \to \mathbb{R}$ satisfying the fiberwise engulfing condition: all fiber $F_b = \pi^{-1}(b)$ have the property that, for all $m \le n$, there exists a smooth isotopy g_t of F_b compactly supported in $F_b(m-1, n+1) = F_b \cap p^{-1}([m-1, n+1])$ such that $g_0 = \mathrm{id}$ and*

$$\mathrm{int}(g_1(F_b(-\infty, m))) \supseteq F_b(-\infty, n).$$

Then a global version is also true: for all $m \le n$ there exists a smooth isotopy h_t compactly supported in $p^{-1}([m - \dim(B) - 1, n + \dim(B) + 1])$ such that $h_0 = \mathrm{id}$ and

$$\mathrm{int}(h_1(p^{-1}(\infty, m]))) \supseteq p^{-1}(-\infty, n]).$$

Corollary 26.2. *For all $m \le n$, there exists an open $E_{mn} \subseteq E$ containing $p^{-1}([m, n])$ such that $\pi|_{E_{mn}} : E_{mn} \to B$ is a smooth manifold bundle.*

Proof. Let $Z_{mn} = h_1(\pi^{-1}(-\infty, m])) \setminus p^{-1}(-\infty, m)$ and $E_{mn} = \bigcup_i h_1^i(Z_{mn})$. Then $E_{mn} \to B$ factors as $E_{mn} \to E_{mn}/\mathbb{Z} \to B$ and $E_{mn}/\mathbb{Z}$ is a proper smooth submersion and hence a smooth manifold bundle. So $\pi|_{E_{mn}}$ is a smooth manifold bundle. □

26.1 Verifying the fiberwise engulfing condition

Proposition 26.3. *If F_b is homeomorphic to $M \times \mathbb{R}$ where M is compact of* $\dim \ge 5$ *and $p = \pi_2$, then F_b satisfies the fiberwise engulfing condition.*

Proof. This uses the end theorem, which answer the question "when is a non-compact manifold N the interior of a compact manifold with boundary $\overline{N}$? □

Theorem 26.4. *Let N be a* Cat*-manifold (*Cat *is either* Diff *or* Top*) without boundary and* $\dim N \ge 6$*. Then N is an interior of compact* Cat*-manifold with boundary if and only if N has tame ends and for all ends ϵ a finiteness obstruction $\sigma(\epsilon) \in \tilde{K}_0(\mathbb{Z}[\pi_1^\infty(\epsilon)])$ vanishes.*

Moreover, $\overline{N}$ is unique up to h-cobordisms starting at $\partial\overline{N}$.

This tameness and obstruction is not dependent on the category Cat. Also the uniqueness part includes an important claim.

Lemma 26.5. *If $\overline{N}$ is compact and W is an h-cobordism starting at $\partial\overline{N}$, then* $\mathrm{int}(\overline{N}) \cong \mathrm{int}(\overline{N} \cup_{\partial\overline{N}} W)$.

Proof. We use the Eilenberg swindle. We have

$$\begin{aligned}\mathrm{int}(\overline{N}) &\cong \overline{N} \cup (W \cup W^{-1}) \cup (W \cup W^{-1}) \cup \cdots \\ &\cong \overline{N} \cup W \cup (W^{-1} \cup W) \cup \cdots \cong \mathrm{int}(\overline{N} \cup W).\end{aligned}$$

□

Proof of Proposition 26.3. For $m \leq n$ consider $Q = \mathrm{int}(F_b(m-1, n+1))$. This is homeomorphic to $M \times (m-1, n+1) = \mathrm{int}(M \times [m-1, n+1])$ and so by the end theorem Q has tame ends with vanishes finite obstructions. So $Q = \mathrm{int}(\overline{Q})$ and without loos of generality we may assume $\overline{Q} \cong M' \times [m-1, n+1]$ by attaching h-cobordisms.

Now note that $p^{-1}([m,n])$ is a compact subset of interior of $M' \times [m-1n+1]$. Then it is contained in $M' \times [m-1+\epsilon, n+1-\epsilon]$. (Now p is not the projection.) Use a smooth isotopy of $[m-1, n+1]$ pulling $[m-1, m+1-\epsilon]$ over $[m-1+\epsilon, n+1-\epsilon]$ with compact support in $[m-1, n+1]$ and product with $\mathrm{id}_{M'}$ to get h_t verifying fiberwise engulfing condition. ☐

26.2 Proof of Kirby–Siebenmann bundle theorem

Theorem 26.6. *If $\pi : E \to B$ is a smooth submersion that is a topological manifold bundle with fibers of dimension ≥ 6, then it is a smooth manifold bundle.*

This is true if dim $\neq 4$, but definitely false in dim $= 4$. Freedman has a counterexample with non-diffeomorphic fibers.

Proof. Without loss of generality assume $B = D^k$. Then E is homeomorphic to $N \times D^k$ over D^k. Exhaust N by compact submanifolds N_i with boundary ∂N_i and take disjoint collars $\partial N_i \times \mathbb{R} \subseteq N$.

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr] & N \times D^k \arrow[d] \\ & D^k \end{tikzcd}$$

Let $E_i = \varphi^{-1}(\partial N_i \times \mathbb{R})$. Then $\pi|_{E_i}$ is a smooth submersion whose fibers are homeomorphic to $\partial N_i \times \mathbb{R}$.

By the proposition, E_i contains an open subset $\tilde{E}_i$ containing $\varphi^{-1}(\partial N_i \times [-1,1])$ so that $\pi|_{\tilde{E}_i} : \tilde{E}_i \to B$ is a smooth manifold bundle. This is necessarily trivial and so we get diffeomorphisms $\tilde{E}_i \cong \tilde{U}_i \times B$ over B.

Consider smooth $p|_{\tilde{U}_i \times \{0\}}$ near 0 such that 0 is a regular value and set $A_i = p^{-1}(0) \subseteq \tilde{U}_i \times \{0\}$. Then $A_i \times B$ is a smooth manifold bundle which end sat a compact subset $D_i \subseteq E$. So we get a exhaustion of E by compact subsets D_i whose boundaries are smooth manifold bundles.

Apply the Ehresmann fibration theorem to $D_i \setminus \mathrm{int}(D_i)$ so that each of the $(D_i \setminus \mathrm{int}(D_i)) \setminus F_i$ are Trivializing and gluing these subbundles show that $E \to D$ finish is a smooth manifold bundle. ☐

26.3 Space of smooth structures

Definition 26.7. Let U be a topological manifold. Then $\mathrm{Sm}(U)$ is the simplicial set with k-simplices given by (E, π, φ) with

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & \Delta^k \times U \arrow[d] \\ & \Delta^k \end{tikzcd}$$

and φ is a homeomorphism, π a smooth submersion.

There is a map $\mathrm{Sing}(\mathrm{Homeo}(U)) \to \mathrm{Sm}(U)$ given a smooth structure U_Σ on U. This ma is given by sending $\Delta^k \to \mathrm{Homeo}(U)$ to

$$\begin{tikzcd} U_\Sigma \times \Delta^k \arrow[r] \arrow[dr] & U \times \Delta^k \arrow[d] \\ & \Delta^k. \end{tikzcd}$$

The bundle theorem implies that if $\dim U \geq 6$ then up to diffeomorphism over Δ^k, every k-simplicial is of this form for some Σ. So we get

$$\mathrm{Sm}(U) \simeq \coprod_{\substack{\text{concordance} \\ \text{classes } \Sigma}} \mathrm{Sing}(\mathrm{Homeo}(U))/S\,\mathrm{Diff}(U_\Sigma).$$

This term is the homotopy fiber of $B\,\mathrm{Diff}(U_\Sigma) \to B\,\mathrm{Homeo}(U)$. As an example,

$$\mathrm{Sm}(\mathbb{R}^m) \simeq \mathrm{Top}(m)/\mathrm{O}(m), \quad \mathrm{Sm}_\partial(D^n) \simeq \coprod_{\substack{\text{concordance} \\ \text{classes } \Sigma}} B\,\mathrm{Diff}_\partial(D^n_\Sigma).$$

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