CH-232-A

Answers to ICS 2020 Problem Sheet #4

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Problem 4.1: Prefix Order Relation

a) $The\ givens\ are:$

$$\prec \subseteq \textstyle\sum^* x \sum^*,$$
$$p \preceq w,$$
$$p, w \in \textstyle\sum^*,$$

To show that the relation $\prec$ $is\ a\ partial\ order, then\ it\ must\ be\ reflective, antisymmertic\ and\ transitive.$

i. $Antisymmetric\ Test$

$\forall a, b \in \sum^* . \big((a, b) \in \prec \wedge (b, a) \in \prec\big) \Rightarrow a = b$
$since\ p \preceq w, then\ this\ means\ they\ are\ antisymmetric, reflective\ and\ transitive.$
$\therefore pRw \wedge wRp \Rightarrow p = w$
$This\ will\ lead\ us\ to\ conclue\ the\ relation\ is\ in\ antisymmetric\ because\ every\ domain\ in \sum^*, can\ be\ equal\ to\ its\ codomain\ such\ that\ if\ (a, b) \in \prec\ and\ (b, a) \in \prec.$

ii. $Reflexiveness$

$$\forall a \in \textstyle\sum^* . \big((a, a) \in \prec\big)$$

$This\ is\ true\ since\ there\ could\ be\ "a\ prefix\ word"\ and\ that\ would\ lead\ the\ relation\ to\ have\ some\ elements\ whose\ domain\ and\ codomain\ to\ be\ the\ same.$

iii. Transitivity

$$\forall a, b, c \in \textstyle\sum^* . \big((a, b) \in \prec \wedge (b, c) \in \prec\big) \Rightarrow (a, c) \in \prec$$

$This\ is\ also\ true\ since, p \preceq w, then\ this\ means\ it's\ transitive.$
$For\ Example: if\ p \Rightarrow w, and\ w \Rightarrow p, then\ p \Rightarrow p\ and\ yes\ this\ could\ exist\ in\ the\ relation.$

$\therefore It\ is\ partially\ orderd.$

b) $The\ givens\ are:$

$$\prec \subset \sum^* x \sum^*$$

$$p \prec w$$

$$p, w \in \sum^*$$

$$p \neq w$$

To show that $\prec$ is a partial order we need to prove that $\prec$ is irreflexive, asymmetric, and transitive on Σ^*.

i. Irreflexive

$$\forall a \in \sum^* . \left((a,a) \notin \prec\right)$$

$Since\ p \prec w\ \&\ p \neq w, this\ would\ lead\ us\ to\ conclude\ that\ there\ will\ not\ exist\ any\ word\ in\ the\ relation\ with\ the\ same\ domain\ and\ codomain. P\ is\ a\ proper\ prefix\ means\ that\ p\ is\ always\ different\ from\ w\ and\ this\ implies\ that\ the\ relation\ can\ not\ have\ same\ domain\ and\ codomain. Thus, it\ is\ irreflexive.$

ii. $Asymmetric$

$$\forall a, b \in \sum^* .\ (a,b) \in R \Rightarrow (b,a) \notin R$$

$Let\ a = p, and\ b = w, and\ w = pxq, but\ w \neq p.$
$If\ a\ is\ a\ prefix\ of\ b, then\ this\ does\ not\ mean\ b\ could\ be\ a\ prefix\ of\ a\ since\ the\ defintion\ states\ that, both\ values\ can\ not\ be\ the\ same.\ So\ it\ is\ Asymmetric.$

iii. $\forall a, b, c \in \sum^* . \left((a,b) \in \prec \wedge (b,c) \in \prec\right) \Rightarrow (a,c) \in \prec$

$This\ is\ false. Having\ word\ pair\ (a,b)\ and\ (b,c)\ in\ the\ relation\ does\ not\ mean\ you\ does\ not\ mean\ neccessary\ the\ pair\ (a,c)\ exists\ in\ the\ relation.$

$\therefore It\ is\ not\ strict\ partial\ orderd.$

C)

Problem 4.2: Function Composition

$$\boldsymbol{Given}: \boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{B}\ \&\ \boldsymbol{g}: \boldsymbol{B} \rightarrow \boldsymbol{C}$$

a) $g\ o\ f: A \rightarrow C$

$$(g\ o\ f)(x) = g\big(f(x)\big)$$

$if\ g\ o\ f\ is\ bijective\ then\ this\ means, it\ is\ injective\ and\ subjective.$

$\boldsymbol{To\ show\ that\ f\ is\ injective}$:

$let\ x1, x2\ \in A,$
$f(x1) = f(x2), \quad then$
$(g\ o\ f)(x1) = g\big(f(x1)\big) = g\big(f(x2)\big) = (g\ o\ f)(x2)$
$this\ implies, x1 = x2.$
$i.e.\ Since\ g\ o\ f\ is\ bijective.$
$\therefore f\ is\ injective.$
$\boldsymbol{To\ show\ that\ g\ is\ subjective}$:
$let\ z\ \in C,$
$x\ \in A, \quad then$
$(g\ o\ f)(x) = g\big(f(x)\big) = z$
$this\ implies: y = f(x)\ \in B, \quad then\ g(y) = z$
$\therefore g\ is\ subjective.$

b) $Example\ of: g\ o\ f\ is\ not\ bijective, f\ is\ injective\ and\ g\ is\ surjective.$
$f: A \to B\ \&\ g: B \to C, \quad g\ o\ f\ = g(f(x))$

$take\ g: \{0,1,2\} \to \{0,1\}$
$by\ 0\ \to 0, \quad 1 \to 1, \quad 2 \to 1, \quad then\ g\ is\ surjective.$

$take\ f: \{0,1\} \to \{0,1,2,3\}$
$by\ 0\ \to 0, \quad 1 \to 1, \quad then\ f\ is\ injective.$

$by\ (g\ o\ f)(x) = g\big(f(x)\big) \to g\ o\ f: A\ \to C.$
$\therefore\ g\ o\ f: \{0,1,2\} \to \{0,1,2,3\}\ is\ not\ bijective\ as\ it\ is\ not\ surjective$

c) $Let\ f:\ \mathbb{R}\ \to\ \mathbb{R}^2\ by\ f(x) = (x, 0),\ \ and\ g: \mathbb{R}^2\ \to\ \mathbb{R}\ by\ g(x\ , y) = x,\ then$
$g\ o\ f: \mathbb{R}\ \to\ \mathbb{R}\ \ is\ bijective\ but, f\ is\ not\ surjective\ and\ g\ is\ not\ injective.$