CH-232-A

Answers to ICS 2020 Problem Sheet #6

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1. The two elementary Boolean functions → (implication) and ¬ (negation) combined are sufficient to express all the possible Boolean functions. But individually, they are not universal function and they can't be used to sufficiently express the rest of the Boolean functions like that of NOT-AND and NOT-OR.
   i.e. NOT-AND and NOT-OR are universal functions since they can be used to derive all elementary Boolean functions.

   But combined:

   Let $S = \{\rightarrow, \neg\}$,

   For ¬, it is obvious why it is expressible – It is a member of S and it itself is an elemental Boolean function. The remaining → can express the terms of ∧ and ∨ with ¬.
   Such that:

   $x \vee y = \neg x \rightarrow y$ $\quad$ *De Morgans law*

   $x \wedge y = \neg(x \rightarrow \neg y)$ $\quad$ …

   Thus, $\{\rightarrow, \neg\}$ are sufficient to express all the possible Boolean functions but not individually.

2.
   a) $\varphi(A,B) = (\neg A \vee \neg B) \wedge (\neg A \vee B) \wedge (A \vee \neg B)$

   $\varphi(A,B) = (A' + B')(A' + B)(A + B')$ *(writing it in a simplified form)*

   $(A'A' + A'B + A'B' + B'B)(A + B')$ *(distribution)*

   $(A' + A'(B + B') + 0)(A + B')$ *(idempotent law)*

   $(A' + A'(1) + 0)(A + B')$ *(complement)*

   $(A')(A + B')$ *(idempotent law)*

   $(A'A)(A'B')$ *(distribution)*

   $(0 + A'B')$ *(complement)*

   $A'B'$

   $\therefore \neg A \wedge \neg B$ *(re-writing)*

   b) $\varphi(A,B,C) = (A \wedge \neg B) \vee (A \wedge \neg B \wedge C)$

   $\varphi(A,B,C) = (AB') + (AB'C)$ *(writing it in a simplified form)*

   $(AB')(1 + C)$ *(distribution)*

   $(AB')(1)$ *(domination)*

   AB'

   $\therefore A \wedge \neg B$ *(re-writing)*

   c) $\varphi(A,B,C,D) = \big(A \vee \neg(B \wedge A)\big) \wedge (C \vee (D \vee C))$

   $\varphi(A,B,C,D) = (A + (BA)').(C + (D + C))$ *(writing it in a simplified form)*

   $(A + (BA)')(C + D)$ *(idempotent law)*

   $(A + B' + A')(C + D)$ *(De Morgan's law)*

$(1 + B')(C + D)$ *(Complement law)*

$(1)(C + D)$ *(Domination)*

$C + D$ *(Identity law)*

$\therefore C \lor D$ *(re-writing)*

d) $\varphi(A, B, C) = (\neg(A \land B) \lor \neg C) \land (\neg A \lor B \lor \neg C)$

$\varphi(A, B, C) = ((AB)' + C')(A' + B + C')$ *(writing it in a simplified form)*

$(A' + B' + C')(A' + B + C')$ *(De Morgan's law)*

$By\ using\ distritutions, Let\ us\ first\ distribute\ A'\ with\ (A' + B + C')\ and\ simplify\ it$

$A'A' + A'B + C'A'$

$A' + A'B + C'A'$ *(Idempotent law)*

$A'(1 + B) + A'C'$ *(distribution)*

$A' + A'C'$ *(dominance law)*

$A'(1 + C')$ *(distribution law)*

$\therefore A'$

$Secondly\ distribute\ B'\ with\ (A' + B + C')\ and\ simplify\ it$

$B'A' + B'B + B'C'$

$\therefore B'A' + B'C'$ *(complement law)*

$Thirdly\ distribute\ B'\ with\ (A' + B + C')\ and\ simplify\ it$

$C'A' + C'B + C'C'$

$C'A' + C'B + C'$ *(idempotent law)*

$C'A' + C'(B + 1)$

$C'A' + C'$ *(dominance law)*

$C'(A + 1)$ *(distribution)*

$\therefore C'$

$Combinin\ the\ three\ distributions\ we\ get\ A' + B'A' + B'C' + C'$

$A'(1 + B') + C'(B' + 1)$ *(dominance law)*

$A' + C'$

$\therefore \neg A' \lor \neg C$ *(re-writing)*

e) $\varphi(A, B) = (A \lor B) \land (\neg A \lor B) \land (A \lor \neg B) \land (\neg A \lor \neg B)$

$\varphi(A, B) = (A + B)(A' + B)(A + B')(A' + B')$ *(writing it in a simplified form)*

$(AA' + AB + BA' + BB)(AA' + AB' + B'A' + B'B')$

$(0 + AB + BA' + BB)(0 + B'A + A'B' + B'B')$ *(complement law)*

$(AB + BA' + B)(B'A + A'B' + B')$ *(idempotent law)*

$(B(A + A') + B)(B'(A + A') + B')$

$(B(1) + B)(B'(1) + B')$ *(complement law)*

$(B)(B')$ *(complement law)*

$\therefore 0$

3. $\varphi(P, Q, R, S) = (\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$

a)

	P	Q	R	S	$\neg P$	$\neg Q$	$\neg R$	$\neg S$	$\neg P \vee Q$	$\neg Q \vee R$	$\neg R \vee S$	$\neg S \vee P$	$(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$
1	0	0	0	0	1	1	1	1	1	1	1	0	1
2	0	0	0	1	1	1	1	0	1	1	1	1	0
3	0	0	1	0	1	1	0	1	1	1	0	0	0
4	0	0	1	1	1	1	0	0	1	1	1	1	0
5	0	1	0	0	1	0	1	1	1	0	1	0	0
6	0	1	0	1	1	0	1	0	1	0	1	1	0
7	0	1	1	0	1	0	0	1	1	1	0	0	0
8	0	1	1	1	1	0	0	0	1	1	1	1	0
9	1	0	0	0	0	1	1	1	0	1	1	1	0
10	1	0	0	1	0	1	1	0	0	1	1	1	0
11	1	0	1	0	0	1	0	1	0	1	0	1	0
12	1	0	1	1	0	1	0	0	0	1	1	1	0
13	1	1	0	0	0	0	1	1	1	0	1	1	0
14	1	1	0	1	0	0	1	0	1	0	1	1	0
15	1	1	1	0	0	0	0	1	1	1	0	1	0
16	1	1	1	1	0	0	0	0	1	1	1	1	1

$\therefore$ *Only two conditions satisfy the* φ.
i.e. *when* $\varphi(P, Q, R, S) = (0, 0, 0, 0)$
and when $\varphi(P, Q, R, S) = (1, 1, 1, 1)$

b) From the truth table we see that the conditions that satisfy φ are when $\varphi(P, Q, R, S) = (0, 0, 0, 0)$ *and* $\varphi(P, Q, R, S) = (1, 1, 1, 1)$.
This means, when $\varphi(P, Q, R, S)$ is 0 and $\varphi(P, Q, R, S)$ is negated.

P	Q	R	S	$\neg P$	$\neg Q$	$\neg R$	$\neg S$	φ
0	0	0	0	1	1	1	1	1
1	1	1	1	0	0	0	0	1

$(\neg P \wedge \neg Q \wedge \neg R \wedge \neg S)$ *OR* $(P \wedge Q \wedge R \wedge S)$

$\therefore$ *DNF of* $X = (\neg P \wedge \neg Q \wedge \neg R \wedge \neg S) \vee (P \wedge Q \wedge R \wedge S)$