Statistical Modelling and Design of Experiments

Evaluation of G/G/1 Systems

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Contents

1 Presentation of the project

In this project, we will try to complete the two following objectives:

- Compare results obtained by simulation for a given G/G/1 system with the Allen-Cuneen's approximation formula. The arrival process and the service time distributions were previously given by the teacher,
- Analyze the performance of a single server queuing system (G/G/1) by simulation when service times correspond to a long-tailed distribution of probability.

Our problem is define as followed: an Erlang distribution for the inter arrival times, and a Weibull distribution for the service time, with given parameters:

- a (service) $= 0.6500$
- $K = 2$
- $E[stage] = 32$
- $E[tau] = 64$

2 Analysis of the service time

2.1 Introduction

Firstly we simulate the service time using the Weibull distribution. The main idea is to generate service time for 10,000 clients. And then compare the results we obtain with the results we expect from theory.

We know that the Weibull distribution is a long tailed distribution:

$$F_x = 1 - exp(-\frac{x}{b})^a$$

With mean and variance:

$$E[X] = b\Gamma(\frac{a+1}{a})$$

$$Var[X] = b^2(\Gamma(\frac{a+2}{a}) - \Gamma^2(\frac{a+1}{a})$$

where :

$$\Gamma(x+1) = x\Gamma(x), \Gamma(1/2) = \sqrt{\pi}$$

We then set our parameters : $a = 0.6500$ for the shape and $b = 1$ for the scale. We realize our simulation in R, it is just required to run the R code provided.

2.2 Experiments & Results

Those are the results we obtain in our simulation:

Mean:

- theory = 1.36627468447201
- experiment = 1.37056301673183

Variance:

- theory = 4.7474852709797
- experiment = 5.1572026099562

Coefficient of variation:

- theory = 1.59475433582442
- experiment = 1.65694498244225

We can observe that the theoretical and experimental results are quite the same. We have a mean service time of almost 1.4 minute, with a variability of 5 minutes. It is also important to analyze the histogram of the service time, to have a clearer idea:

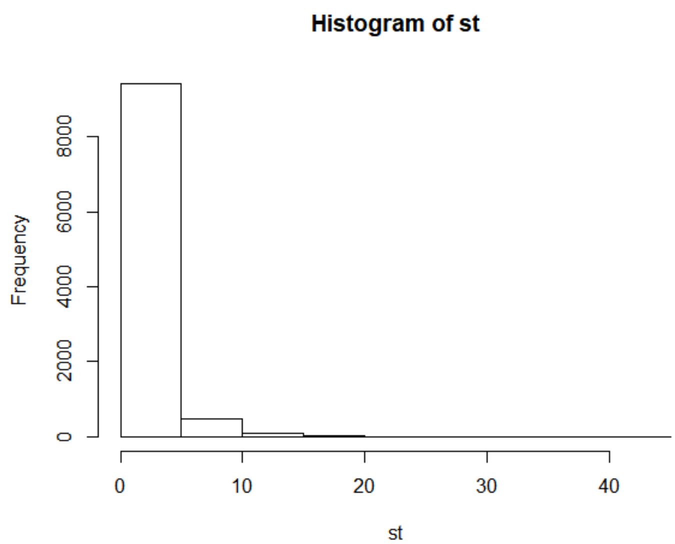


Figure 1: Histogram of the service time

We can notice that most of the service time is in the interval between 0 and 5 minutes, and the maximum we get is around 16 minutes for few samples.

3 Second Part

3.1 Introduction

In the second part we want to analyze the service time of 100,000 clients. The service distribution is still a Weibull with same shape of before (a), but now with a different scale based on the traffic factor.

$$b = \frac{\rho E[\tau]}{\Gamma(\frac{a+1}{a})}$$

We will test our service on 4 different traffic factors:

$$\rho = 0.4 \quad 0.7 \quad 0.85 \quad 0.925$$

We repeat the experiment with 10 different seeds for each traffic factor, in order to average the obtained results and get a more precise analysis.

In this second part we also consider a distribution for the inter-arrival time of each request. We use an Erlang distribution, and from this we generate 100,000 inter arrivals time, so we can have the time at which each client reaches the server with the formula:

$$t_i = \sum_{l=1}^{i-1} \tau_l$$

And then from this, we are able to compute the exit time instant from the WS, for each client i:

$$\theta_i = t_i^S + x_i$$

$$t_i^S = max\{\theta_{i-1}, t_i\}$$

Once we know the exit time, we can compute also the sojourn time in the WS and in the queue, for each client i:

$$\omega_i = \theta_i - t_i$$

$$\omega_{q,i} = t_i^S - t_i$$

From those, we get:

- The average occupancy depending on the time

$$L_{T_i} = \frac{L}{t_i - t_1}$$

- The average occupancy of the service system

$$L = \frac{\sum_{i=1}^{n} \omega_i}{t_n - t_1}$$

- The average occupancy of the queue

$$L_q = \frac{\sum_{i=1}^{n} \omega_{q_i}}{t_n - t_1}$$

- The average service time

$$W = \frac{\sum_{i=1}^{n} \omega_i}{n}$$

- The average service time of the queue

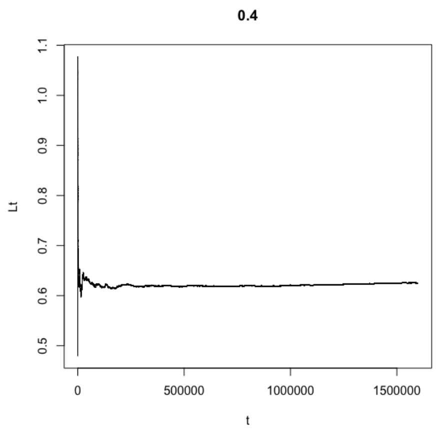
$$W_q = \frac{\sum_{i=1}^{n} \omega_{q_i}}{n}$$

At the end we obtain a confidence interval for the means of W_q and L_q and we compute the plot of L_{T_i} on time t_i for each traffic factor.

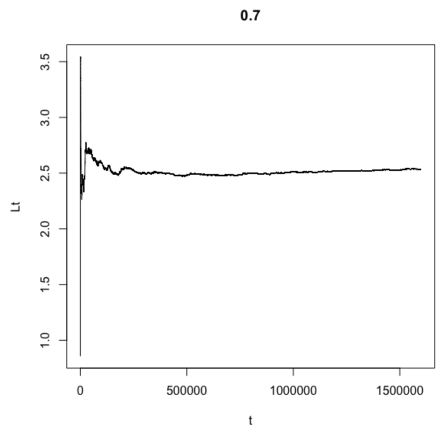
3.2 Experiment & results

In this part, we will present the results we obtained and how to interpret them.

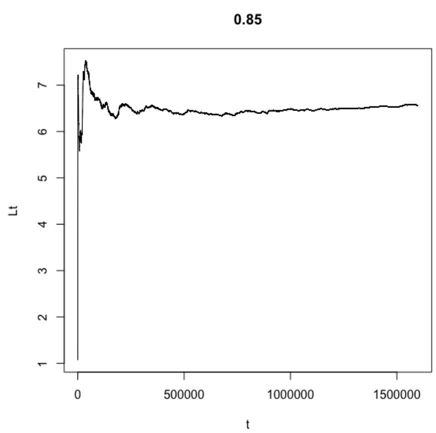
3.2.1 Plots



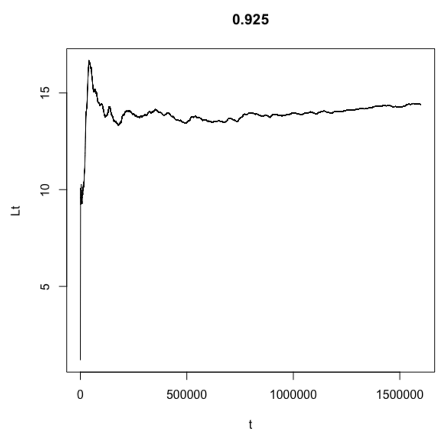
(a) $\rho = 0.4$



(b) $\rho = 0.7$



(c) $\rho = 0.85$



(d) $\rho = 0.925$

Figure 2: L_t vs. t for different values of traffic factor

Here we are plotting the variation of the average occupancy L_t with respect to the entrance time instants, for the four different traffic factors. As we can easily noticed, all the occupancy get stable after a certain amount of time. It is also important to see that there is a direct correlation between traffic factors ρ, the level of reached stability, and the time needed to reach this stability: the bigger is ρ, the bigger is the L_t of stability (for 0.4 we get stability around 0.6, for 0.925 we get something around 14), and the more is time to reach it.

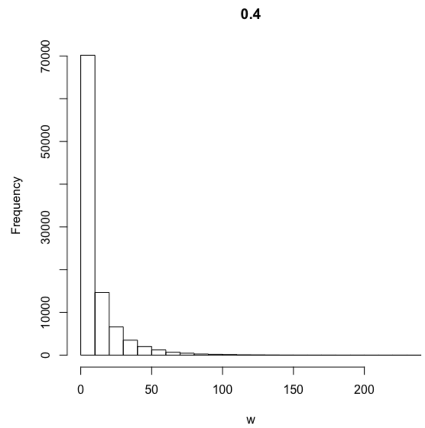
3.2.2 Values

ρ	W	L	W_q	CI	L_q	CI	Allen-Cuneen W_q	Allen-Cuneen L_q	b
0.4	4.988	0.311	1.801	[1.139,2.464]	0.112	[0.071,0.153]	2.747	0.171	4.686
0.7	20.24	1.265	14.66	[9.263,20.07]	0.916	[0.578,1.254]	16.83	1.051	8.200
0.85	53.56	3.346	46.79	[29.65,63.93]	2.923	[1.852,3.994]	49.63	3.100	9.957
0.925	119.8	7.487	112.4	[69.97,154.9]	7.027	[4.371,9.682]	117.5	7.344	10.83

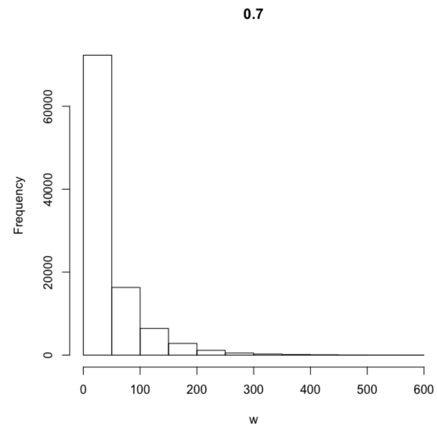
Table 1: Caption

In this table we report all the main average values we computed for all the different traffic factors. We can see that the variation of traffic factors has a direct correlation with all the other values. If the traffic factor is high then the time in the queue is increasing, as the total time in the service system and, also the occupancy is getting bigger. We got similar results even with the Allen-Cuneen approximation formulas.

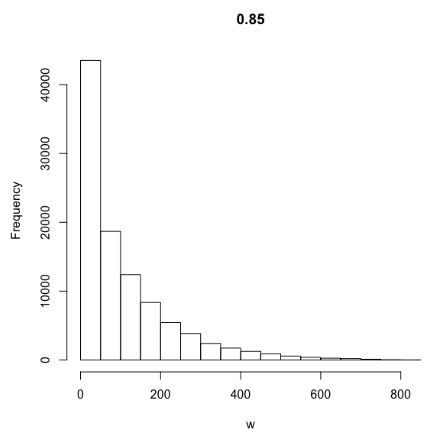
It is also interesting to take a look at the sojourn times in the queue and in the service for the different traffic factor scenarios. As expected, for the lower traffic factor ($\rho = 0.4$), the waiting time is short for most of the clients, we know we have a average of 1.801 units of time and we can see from the histogram the low variability of the results. Analyzing all the other, we noticed an increase in the average time spent in queue and therefore in the system.



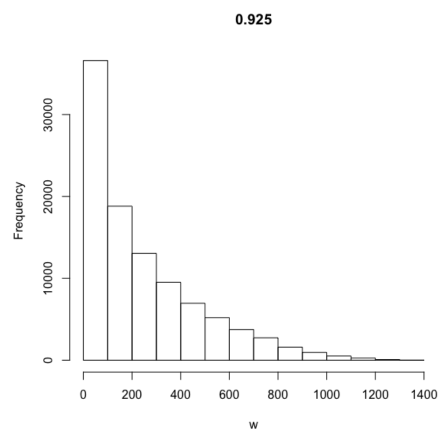
(a) $\rho = 0.4$



(b) $\rho = 0.7$

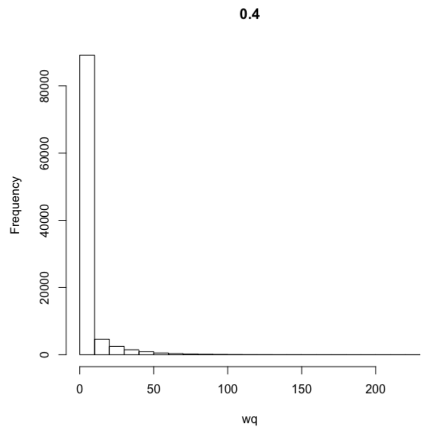


(c) $\rho = 0.85$

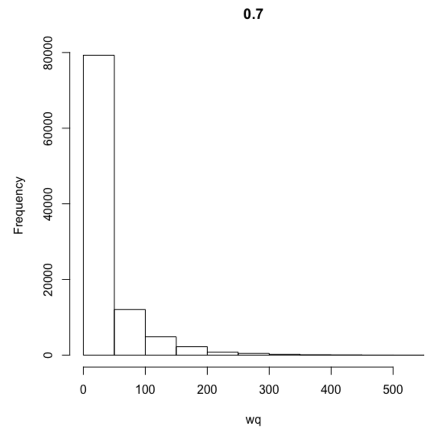


(d) $\rho = 0.925$

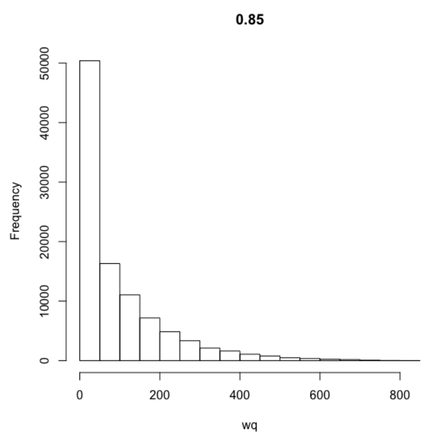
Figure 3: Histograms of sojourn times in the service system for the different traffic factors



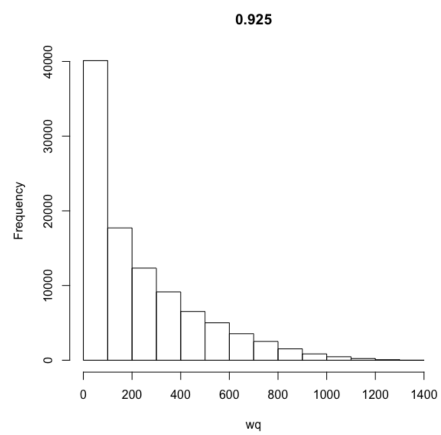
(a) $\rho = 0.4$



(b) $\rho = 0.7$



(c) $\rho = 0.85$



(d) $\rho = 0.925$

Figure 4: Histograms of sojourn times in the queue for the different traffic factors