CS / MATH 4334 : Numerical Analysis
Homework Assignment 5

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Theoretical Problems

1. Given the approximation for $f^2(x) = \dfrac{5f(x-h) - 7f(x) + f(x+h) + f(x+4h)}{11h^2}$ find the error term, along with the order of approximation. In other words, write the Taylor series for each of $5f(x-h)$, $f(x+h)$, and $f(x+4h)$, and take the appropriate linear combination of these to find a power series for the above expression.

   - $f(x-h) = f(x) - f^1(x)h + f^2(x)\frac{h^2}{2} - f^3(c_1)\frac{h^3}{6}$
   - $f(x+h) = f(x) + f^1(x)h + f^2(x)\frac{h^2}{2} + f^3(c_2)\frac{h^3}{6}$
   - $f(x+4h) = f(x) + 4f^1(x)h + 8f^2(x)h^2 - \frac{32}{3}f^3(c_3)h^3$
   - $b = 5f(x-h) - 7f(x) = -2f(x) - 5f^1(x) + \frac{5}{2}f^2(x)h^2 - \frac{5}{6}f^3(c_1)h^3$
   - $c = b + f(x+h) = -f(x) + 4f^1(x)h + 3f^2(x)h^2 - \frac{1}{6}f^3(c_1)h^3 + \frac{1}{6}f^3(c_2)h^3$
   - $d = c + f(x+4h) = 11f^2(x)h^2 - \frac{32}{3}f^3(c_3)h^3 - \frac{1}{6}f^3(c_1)h^3 + \frac{1}{6}f^3(c_2)h^3$
   - Simplifying, we get...
     $f^2(x) + [\frac{32}{33}f^3(c_3) + \frac{1}{66}f^3(c_2) - \frac{1}{66}f^3(c_1)]h$
   - $\boxed{\text{Our order of approximation is 1 and our error is } \frac{32}{33}f^3(c_3) + \frac{1}{66}f^3(c_2) - \frac{1}{66}f^3(c_1)}$

2. Suppose $f^2(x)$ is approximated by a variation on the three-point centered difference formula with the following constraints (given in hw). Find the value of the step size h which minimizes the upper bound of $E(f,h)$.

   - Plugging in our errors, we get $\dfrac{1}{4h^2}[y_- + \epsilon_- - 2(y_0 + \epsilon_0) + y_+ + \epsilon_+] =$
     $\dfrac{y_- - 2y_0 + y_+}{4h^2} + \dfrac{\epsilon_- - 2\epsilon_0 + \epsilon_+}{4h^2} - \dfrac{f^4(c)}{6}h^2.$
   - Since $\mid E_{total} \mid = \mid E_r - E_t \mid$, $\mid E_{total} \mid = \mid \dfrac{\epsilon_- - 2\epsilon_0 + \epsilon_+}{4h^2} - \dfrac{f^4(c)}{6}h^2 \mid \leq \dfrac{\epsilon}{h^2} + \dfrac{f^2(c)}{6}h^2$
   - $\dfrac{d}{dh}(\dfrac{\epsilon}{h^2} + \dfrac{f^2(c)}{6}h^2) = \dfrac{-2\epsilon}{h^3} + \dfrac{f^2(c)}{3}h^2 = 0 \to h = \boxed{\sqrt[4]{\dfrac{6\epsilon}{f^2(c)}} \text{ is our step size}}$

3. Apply Richardson's Extrapolation once, starting with the three-point centered difference formula given in Theoretical Problem #2, to find a higher-order formula to approximate $f^2(x)$. This new formula is of what order?

   - Richardson's Extrapolation gives us $\dfrac{2^2F_2(h/2) - F_2(h)}{2^2 - 1}$
   - Expanding, we obtain:
     $2^2[\dfrac{f(x-h) - 2f(x) + f(x+h)}{h^2} - \dfrac{f(x-2h) - 2f(x) + f(x+2h)}{4h^2}] * 3^{-1} =$
     $\dfrac{4f(x-h) - 8f(x) + 4f(x+h) - f(x-2h) - 2f(x) + f(x+2h)}{3h^2}$
   - $\boxed{\text{Our order is } O(h^3)}$ since our original order is $O(h^2)$, Richardson's Extrapolation adds one to the order.

4. Integrate Newton's divided-difference interpolating polynomial to prove the formula (5.19).

   - Given the points (-h,1), (0,0), (h,0), we can use newtdd to find our coefs, 1, $\dfrac{-1}{h}$, and $\dfrac{1}{2h^2}$. Thus, $P_2(x) = 1 - \frac{1}{h}(x+h) + \frac{1}{2h^2}(x+h)(x-0)$
   - $\int_{-h}^{h} P_2(x)dx = |_{-h}^{h}\, x - \dfrac{1}{2h}x^2 - x + \dfrac{1}{6h^2}x^3 + \dfrac{1}{4h}x^2 = \boxed{\dfrac{1}{3}h, \text{ which is what we expected}}$

5. Consider the quadrature rule given in the hw. Find c_1, c_2, c_3 such that the quadrature rule integrates the functions $f(x) = 1, x, x^2$ exactly. (Note that the points are not evenly spaced.) What is the degree of precision of this quadrature formula?

   - We can form a linear system out of our function, as seen below:
     $2 = c_1 + c_2 + c_3$
     $0 = -c_1 + \frac{1}{2}c_2 + c_3$
     $\frac{2}{3} = c_1 + \frac{1}{4}c_2 + c_3$
   - we can form a matrix as seen below:

     $$\begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2} & 1 \\ 1 & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix}$$

   - Solving the system, we get a solution vector of

     $$\boxed{\begin{bmatrix} \dfrac{5}{9} \\ \dfrac{16}{9} \\ \dfrac{-1}{3} \end{bmatrix}. \text{ Since our third derivative is 0, our rule is of degree 3.}}$$

6. Recall that the error in a quadrature rule based on an interpolant $P(x)$ is written as given in the assignment. Find an upper bound for $\mid E(x) \mid$ in terms of h.

   - Our bounds are given by:
     $x_0 = x_0 + ht \rightarrow t = 0$
     $x_2 = x_0 + ht \rightarrow x_2 - x_0\ h \rightarrow t = 2$
     $dx = hdt$
   - Simplifying, we obtain the following integral:

     $\frac{1}{6}M_3 \int_0^2 (ht)(h(t-1))(h(t-2))dt \rightarrow \frac{1}{6}M_3h^4\ [\frac{1}{4}t^4 - t^3 + \frac{1}{2}h^2]\ |_0^2$

   - We get that our integral evaluates to $\boxed{\dfrac{-1}{3}h^4M_3}$, which is our upper bound in terms of h.

7. If $\int_0^1 e^{x^2}dx$ is approximated with the Composite Simpson's Rule, determine the minimum number of panels m needed for the upper bound of the absolute value of the error term to be less than any positive real number E.

- Simpsons is given by: $\frac{h}{3}[y_0 + y_{2m} + 4\Sigma_{i=1}^{m} y_{2i-1} + 2\Sigma_{i=1}^{m-1} y_{2i}]$
- Simpsons error is given as $\dfrac{(b-a)h^4}{180} f^4(c)$
- $\dfrac{(b-a)h^4}{180} f^4(c) \rightarrow \dfrac{(1)h^4}{180} f^4(c)$
- Simplifying $f^4(c) = 2(8e^{x^2}x^4 + 12e^{x^2}x^2 + 6e^{x^2})$ has a max of $76e$ on the interval. Thus we obtain $\dfrac{h^4}{180} 76e \le E$
- Simplifying, we obtain that $\boxed{h \le \sqrt[4]{\dfrac{180}{76e} E}}$

8. Apply the Composite Simpsons rule with m=4 panels to the integral $\int_0^{2\pi} xsin(x)dx$ Compute the absolute error between the exact integral and the approximation.

- Our 'nodes' are $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$.
- Simpsons is given by: $\frac{h}{3}[y_0 + y_{2m} + 4\Sigma_{i=1}^{m} y_{2i-1} + 2\Sigma_{i=1}^{m-1} y_{2i}]$. Our h is $(b-a)/2m$ $= \pi/4$.
- Applying Simpsons, we get $\frac{\pi}{12}[sin(0) + sin(2\pi) + 4 * [sin(\frac{\pi}{4}) + sin(\frac{3\pi}{4}) + sin(\frac{5\pi}{4}) + sin(\frac{7\pi}{4})] + 2[sin(\frac{\pi}{2}) + sin(\pi) + sin(\frac{3\pi}{2})]]$ = -6.28319. Our actual is 6.29751.
- The abserr is given by $\mid x - x_a \mid = \mid -6.28319 - 6.29751 \mid = \boxed{0.01432}$

9. Apply Romberg Integration to find R_{44} for the integral $\int_0^1 x^2dx$.

- Using the formula in the book, we can find the values of R.
$R_{11} = (b-a)\dfrac{f(a)+f(b)}{2}$
$R_{j1} = \dfrac{1}{2}R_{j-1,1} + h_j\Sigma_{i=1}^{2^{2j-2}} f(a + (2i-1)h_j$
$R_{jk} = \dfrac{4^{k-1}R_{j,k-1} - R_{j-1,k-1}}{4^{k-1}-1}$
- Thus we solve for the following R's below:
$R_{11} = 1/2$
$R_{21} = 3/8$
$R_{22} = 1/3$
$R_{31} = 11/32$
$R_{32} = 1/3$
$R_{33} = 1/3$
$R_{41} = 43/128$
$R_{42} = 1/3$

$R_{43} = 1/3$
$R_{44} = 1/3$