CS / MATH 4334 : Numerical Analysis
Homework Assignment 2

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Theoretical Problems

1. Sauer, ed. 3: #2c, p. 31. (Sauer, ed. 2: #2c, p. 29) Also, apply the minimum number of iterations of the Bisection Method needed to find an approximate root with an absolute error of at most 1/16. Give this final root approximation.

   (a) Prove by IVT that a root exists within some interval.
   The intermediate value theorem (IVT) states if we have $f(a) < 0$, and $f(b) > 0$ $\rightarrow \exists c$ s.t. $f(c) = 0$. Consider an a,b $\in \mathbb{R}$. Let a = 0.75, b = 1.75. Evaluate in the function $f(a) = f(0.75) \approx -2.73 < 0$ and $f(b) = f(1.75) \approx 0.622 > 0$. $\boxed{\text{Thus by IVT } \exists c \text{ s.t. } f(c) = 0 \text{ within } (0.75, 1.75)}$.

   (b) Given an error of $e = 1/16$, find the number of iterations needed for the bisection method to get a root approximate.
   The error for bisection is defined as $|r - x_i| = \frac{b-a}{2^{n+1}}$. This can be simplified to $|r - x_i| = 1/16 = \frac{1.75-0.75}{2^{n+1}} = \frac{1}{2^{n+1}}$. Thus, if we take $n = 3$ we obtain our desired error. $\boxed{\text{Thus we need 3 iterations to obtain our desired error}}$.

   (c) Find an approximation for r by perform the function evaluations.
   1.) $c_0 = 0.75$, $c_1 = 1.75$, $c_m = 1.25$, $f(c_m) = -1.21 < 0$
   2.) $c_0 = 1.25$, $c_1 = 1.75$, $c_m = 1.50$, $f(c_m) = -0.34 < 0$
   3.) $c_0 = 1.50$, $c_1 = 1.75$, $c_m = 1.625$, $f(c_m) = 0.126 > 0$
   After 3 iterations, we obtain an approximate value for the root $\boxed{r = 1.625}$

2.) Use the Bisection Theorem (from your notes, same as Eq. 1.1, Sauer, p.30 (p.28 ed.2)) to find the minimum number of iterations n of the Bisection Method needed to achieve an approximation with absolute error less than 10^{-17} to the solution of f(x) = 0 lying on the initial interval [-3,3]. (Assume that $f(-3)f(3) < 0$, and that the root is unique on [-3,3]).

   Given the bisection error theorem, $|r - x_i| = \frac{b-a}{2^{n+1}}$, we plug in our values given in the problem to obtain $|r - x_i| = \frac{3-(-3)}{2^{n+1}} = |r - x_i| = \frac{6}{2^{n+1}}$. We can rearrange to obtain:

   $6 * 10^{17} = 2^{n+1}$.

   $log_{10}(6 * 10^{17}) = log_{10}(2^{n+1})$.

   $log_{10}(6 * 10^{17}) = (n + 1)log_{10}(2)$.

   $\frac{log_{10}(6*10^{17}}{log_{10}(2)}$ - $1 = n$

   $\boxed{n \approx 58.01 = 59 \text{ iterations to obtain our desired error.}}$

3. Use the Existence and Uniqueness Theorem discussed in class to show that g(x) = $e^{-x}sin(x) + \frac{1}{2}$ has a fixed point and that is is unique (and stable) on $[\frac{1}{2}, \pi]$. Carefully show all work.

To determine existence, we evaluate all critical points and endpoints of $g(x)$. To determine uniqueness, we evaluate all critical points and endpoints of $g^{'}(x)$.

(a) Existence

$$g(x) = e^{-x}sin(x) + \frac{1}{2}$$
$$g^{'}(x) = -e^{-x}sin(x) + e^{-x}cos(x) \text{ (There is a cp at x = } \frac{\pi}{4} \text{)}$$

Checking the endpoints and cps:

$$\mid g(\tfrac{1}{2}) \mid \approx 0.79079 < 1$$
$$\mid g(\tfrac{\pi}{4}) \mid \approx 0.82240 < 1$$
$$\mid g(\pi) \mid \approx 0.778 < 1$$

$\boxed{\text{Choose } k = 0.9. \text{ Since } k < 1 \text{ and all cps} < \text{than } k, \text{ there exists a fixed point on } g(x).}$

(b) Uniqueness

$$g^{'}(x) = -e^{-x}sin(x) + e^{-x}cos(x)$$
$$g^{''}(x) = e^{-x}sin(x) - e^{-x}cos(x) - e^{-x}cos(x) - e^{-x}sin(x)$$
$$g^{''}(x) = -2e^{-x}cos(x) \text{ (There is a cp at x = } \frac{\pi}{2} \text{)}$$

Checking the endpoints and critical point:

$$\mid g^{'}(\tfrac{1}{2}) \mid \approx 0.24149 < 1$$
$$\mid g^{'}(\tfrac{\pi}{2}) \mid \approx 0.20788 < 1$$
$$\mid g^{'}(\pi) \mid \approx 0.04321 < 1$$

$\boxed{\text{Choose } k = 0.4. \text{ Since } k < 1 \text{ and all cps} < \text{than } k, \text{ there exists a unique fixed point on } g(x).}$

4. Find the set of all initial guesses for which the Fixed-Point Iteration $x_{i+1} = \frac{1}{4} - x_i^2$ converges to a stable fixed point.

By quadratic formula:

$$x_{i+1} = \frac{1}{4} - x_i^2$$
$$x = \frac{1}{4} - x^2$$
$$0 = \frac{1}{4} - x - x^2$$

$$r_1, r_2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(\frac{1}{4})}}{2(-1)} = \boxed{\frac{-1 \pm \sqrt{2}}{2}}$$

We analyze the behavior of the roots at the derivative:

$$\mid f'(r_1) \mid = \mid 1 - \sqrt{2} \mid \approx 0.414 < 1 \; \boxed{\rightarrow r_1 \text{ is stable.}}$$

$$\mid f'(r_2) \mid = \mid 1 + \sqrt{2} \mid > 1 \; \boxed{\rightarrow r_2 \text{ is NOT stable.}}$$

To determine the domain of x values in such that the initial guess x_0 converges to a fixed point, we can imagine the maximum endpoint in which the largest square encapsulating the fixed points. In this case, the range of $\boxed{x_0 = [\text{-}1.2,\ 1.2]}$ creates the largest square surrounding the fixed points and thus it is our domain of values.

5. Both fixed point iterations given have the same positive fixed point:

$$p_n = p_{n-1} - \frac{p_{n-1}^4 - 2}{7p_{n-1}^3}$$

$$p_n = \sqrt[3]{\frac{2}{p_{n-1}}}$$

Find the exact positive fixed point, and determine which method will converge more rapidly.

(a) Determining the positive fixed point

$$p = \sqrt[3]{\frac{2}{p}}$$

$$p^3 = \frac{2}{p} \rightarrow p^4 = 2 \rightarrow p = \sqrt[4]{2}.$$

Therefore our $\boxed{\text{positive fixed point is } \sqrt[4]{2}}$.

(b) Determining which method will converge faster

We analyze the derivatives to find which method has a higher multiplicity (which one converges to 0 faster).

$f(p) = p - \dfrac{p^4 - 2}{7p^3}$; $f'(p) = 1 - \dfrac{4p^3 * 7p^4 - 21p^3(p^4 - 2)}{(7p^3)^2} \approx c$ - $\dfrac{1}{cp^4}$, where c is an arbitrary constant.

$$g(p) = \sqrt[3]{\frac{2}{p_{n-1}}} \; ; \; f'(p) = c - \frac{1}{c^2}$$

Since $f(p)$ has a greater denominator value (it approaches 0 faster), we can conclude that $\boxed{\text{the first equation } f(p) \text{ will converge faster}}$.

6. Recall from class that, for a simple root r of $f(x)$, we used Taylor series centered at iterate x_i to find the error relation between the ith iterate and the $(i+1)$st iterate.

   (a) Use Eq. (1) to find a (short) Taylor series for $f^{'}(x)$.

   Given, f(r) $= f(x_i) + f^{'}(x_i)(r - x_i) + \frac{1}{2}f^{''}(c_i)(r - x_i)^2$

   Taking the derivative, $0 = f^{'}(x_i) + f^{''}(c_i)(r - x_i) \rightarrow \boxed{f^{'}(x_i) = -f^{''}(c_i)(x - x_i)}$

   (b) Evaluate your equation at r

   $\boxed{-f^{''}(c_i)(r - x_i)}$

   (c) Combine your result with part (b)to find a new relation, and determine the new M and P.

   Plugging into the error function, $e_{i+1} = \frac{f^{''}(c_i)}{2f^{'}(x_i)}e_i^2 = \frac{f^{''}(c_i)}{2f^{'}(x_i)}e_i^2 = \frac{1}{2}e_i$.

   Thus we have a $\boxed{M = \frac{1}{2} \text{ and a } P = 1}$.

7. Find all values of B such that $x_{i+1} = x_i - \frac{f(x_i)}{Bf^{'}(x_i)}$ will be a stable fixed point iteration.

   Newtons method's stable fixed points is dependent on the derivative of the function. In this case, $\boxed{\text{for and } B > 0\text{, then newton's method will converge to a stable fixed point}}$.