

CS / MATH 4334 : Numerical Analysis

Homework Assignment 2

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Theoretical Problems

1. Sauer, ed. 3: #2c, p. 31. (Sauer, ed. 2: #2c, p. 29) Also, apply the minimum number of iterations of the Bisection Method needed to find an approximate root with an absolute error of at most $1/16$. Give this final root approximation.

(a) Prove by IVT that a root exists within some interval.

The intermediate value theorem (IVT) states if we have $f(a) < 0$, and $f(b) > 0 \rightarrow \exists c$ s.t. $f(c) = 0$. Consider an $a, b \in \mathbb{R}$. Let $a = 0.75$, $b = 1.75$. Evaluate in the function $f(a) = f(0.75) \approx -2.73 < 0$ and $f(b) = f(1.75) \approx 0.622 > 0$.
 Thus by IVT $\exists c$ s.t. $f(c) = 0$ within $(0.75, 1.75)$.

(b) Given an error of $e = 1/16$, find the number of iterations needed for the bisection method to get a root approximate.

The error for bisection is defined as $|r - x_i| = \frac{b-a}{2^{n+1}}$. This can be simplified to $|r - x_i| = 1/16 = \frac{1.75-0.75}{2^{n+1}} = \frac{1}{2^{n+1}}$. Thus, if we take $n = 3$ we obtain our desired error. Thus we need 3 iterations to obtain our desired error.

(c) Find an approximation for r by perform the function evaluations.

- 1.) $c_0 = 0.75$, $c_1 = 1.75$, $c_m = 1.25$, $f(c_m) = -1.21 < 0$
- 2.) $c_0 = 1.25$, $c_1 = 1.75$, $c_m = 1.50$, $f(c_m) = -0.34 < 0$
- 3.) $c_0 = 1.50$, $c_1 = 1.75$, $c_m = 1.625$, $f(c_m) = 0.126 > 0$

After 3 iterations, we obtain an approximate value for the root $r = 1.625$

2.) Use the Bisection Theorem (from your notes, same as Eq. 1.1, Sauer, p.30 (p.28 ed.2)) to find the minimum number of iterations n of the Bisection Method needed to achieve an approximation with absolute error less than 10^{-17} to the solution of $f(x) = 0$ lying on the initial interval $[-3, 3]$. (Assume that $f(-3)f(3) < 0$, and that the root is unique on $[-3, 3]$).

Given the bisection error theorem, $|r - x_i| = \frac{b-a}{2^{n+1}}$, we plug in our values given in the problem to obtain $|r - x_i| = \frac{3-(-3)}{2^{n+1}} = |r - x_i| = \frac{6}{2^{n+1}}$. We can rearrange to obtain:

$$6 * 10^{17} = 2^{n+1}.$$

$$\log_{10}(6 * 10^{17}) = \log_{10}(2^{n+1}).$$

$$\log_{10}(6 * 10^{17}) = (n + 1)\log_{10}(2).$$

$$\frac{\log_{10}(6*10^{17})}{\log_{10}(2)} - 1 = n$$

$n \approx 58.01 = 59$ iterations to obtain our desired error.

3. Use the Existence and Uniqueness Theorem discussed in class to show that $g(x) = e^{-x}\sin(x) + \frac{1}{2}$ has a fixed point and that it is unique (and stable) on $[\frac{1}{2}, \pi]$. Carefully show all work.

To determine existence, we evaluate all critical points and endpoints of $g(x)$. To determine uniqueness, we evaluate all critical points and endpoints of $g'(x)$.

(a) Existence

$$g(x) = e^{-x}\sin(x) + \frac{1}{2}$$

$$g'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x) \quad (\text{There is a cp at } x = \frac{\pi}{4})$$

Checking the endpoints and cps:

$$|g(\frac{1}{2})| \approx 0.79079 < 1$$

$$|g(\frac{\pi}{4})| \approx 0.82240 < 1$$

$$|g(\pi)| \approx 0.778 < 1$$

Choose $k = 0.9$. Since $k < 1$ and all cps $<$ than k , there exists a fixed point on $g(x)$.

(b) Uniqueness

$$g'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x)$$

$$g''(x) = e^{-x}\sin(x) - e^{-x}\cos(x) - e^{-x}\cos(x) - e^{-x}\sin(x)$$

$$g''(x) = -2e^{-x}\cos(x) \quad (\text{There is a cp at } x = \frac{\pi}{2})$$

Checking the endpoints and critical point:

$$|g'(\frac{1}{2})| \approx 0.24149 < 1$$

$$|g'(\frac{\pi}{2})| \approx 0.20788 < 1$$

$$|g'(\pi)| \approx 0.04321 < 1$$

Choose $k = 0.4$. Since $k < 1$ and all cps $<$ than k , there exists a unique fixed point on $g(x)$.

4. Find the set of all initial guesses for which the Fixed-Point Iteration $x_{i+1} = \frac{1}{4} - x_i^2$ converges to a stable fixed point.

By quadratic formula:

$$x_{i+1} = \frac{1}{4} - x_i^2$$

$$x = \frac{1}{4} - x^2$$

$$0 = \frac{1}{4} - x - x^2$$

$$r_1, r_2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(\frac{1}{4})}}{2(-1)} = \boxed{\frac{-1 \pm \sqrt{2}}{2}}$$

We analyze the behavior of the roots at the derivative:

$$|f'(r_1)| = |1 - \sqrt{2}| \approx 0.414 < 1 \quad \boxed{\rightarrow r_1 \text{ is stable.}}$$

$$|f'(r_2)| = |1 + \sqrt{2}| > 1 \quad \boxed{\rightarrow r_2 \text{ is NOT stable.}}$$

To determine the domain of x values in such that the initial guess x_0 converges to a fixed point, we can imagine the maximum endpoint in which the largest square encapsulating the fixed points. In this case, the range of $\boxed{x_0 = [-1.2, 1.2]}$ creates the largest square surrounding the fixed points and thus it is our domain of values.

5. Both fixed point iterations given have the same positive fixed point:

$$p_n = p_{n-1} - \frac{p_{n-1}^4 - 2}{7p_{n-1}^3}$$

$$p_n = \sqrt[3]{\frac{2}{p_{n-1}}}$$

Find the exact positive fixed point, and determine which method will converge more rapidly.

- (a) Determining the positive fixed point

$$p = \sqrt[3]{\frac{2}{p}}$$

$$p^3 = \frac{2}{p} \rightarrow p^4 = 2 \rightarrow p = \sqrt[4]{2}.$$

Therefore our $\boxed{\text{positive fixed point is } \sqrt[4]{2}}.$

- (b) Determining which method will converge faster

We analyze the derivatives to find which method has a higher multiplicity (which one converges to 0 faster).

$$f(p) = p - \frac{p^4 - 2}{7p^3}; f'(p) = 1 - \frac{4p^3 * 7p^4 - 21p^3(p^4 - 2)}{(7p^3)^2} \approx c - \frac{1}{cp^4}, \text{ where } c \text{ is an arbitrary constant.}$$

$$g(p) = \sqrt[3]{\frac{2}{p_{n-1}}}; f'(p) = c - \frac{1}{c^2}$$

Since $f(p)$ has a greater denominator value (it approaches 0 faster), we can conclude that $\boxed{\text{the first equation } f(p) \text{ will converge faster}}.$

6. Recall from class that, for a simple root r of $f(x)$, we used Taylor series centered at iterate x_i to find the error relation between the i th iterate and the $(i + 1)$ st iterate.

(a) Use Eq. (1) to find a (short) Taylor series for $f'(x)$.

$$\text{Given, } f(r) = f(x_i) + f'(x_i)(r - x_i) + \frac{1}{2}f''(c_i)(r - x_i)^2$$

$$\text{Taking the derivative, } 0 = f'(x_i) + f''(c_i)(r - x_i) \rightarrow \boxed{f'(x_i) = -f''(c_i)(r - x_i)}$$

(b) Evaluate your equation at r

$$\boxed{-f''(c_i)(r - x_i)}$$

(c) Combine your result with part (b) to find a new relation, and determine the new M and P .

$$\text{Plugging into the error function, } e_{i+1} = \frac{f''(c_i)}{2f'(x_i)}e_i^2 = \frac{f''(c_i)}{2f'(x_i)}e_i^2 = \frac{1}{2}e_i.$$

Thus we have a $\boxed{M = \frac{1}{2} \text{ and a } P = 1}.$

7. Find all values of B such that $x_{i+1} = x_i - \frac{f(x_i)}{Bf'(x_i)}$ will be a stable fixed point iteration.

Newton's method's stable fixed points is dependent on the derivative of the function. In this case, $\boxed{\text{for and } B > 0, \text{ then newton's method will converge to a stable fixed point}}.$