Notes on higher moments of the anticommutator

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1 Higher moments of a regular GOE

Let A be a square matrix of order N. We all know the following shorthand to compute trace.

$$\operatorname{tr}(A^k) \;=\; \sum_s \prod_{i=0}^{n-1} a_{si,s(i+1)}$$

Where $s_0, \cdots, s_{n-1}$ ranges over all finite sequences of length k drawn from $[1, N]$. Also, for convinience, we let $s_n = s_0$.

Now, let A to be drawn from a GOE. We also know that the following formula for the moment of a N-by-N matrix ensenble.

$$\mu_k \;=\; \lim_{n\to\infty} \frac{\langle \operatorname{tr}(A^k)\rangle}{N^{k/2+1}}$$

[1]

Consider the sequence $\{s\}$ as a set of verticies and the pair of indicies s_i, s_{i+1} that occur in the trace expansion as edges. Each summand in the trace expansion corresponds to a closed walk. We have established the following.

Theorem 1 (Graph theoretical computation of moments)**.** *Let S_k be the set of all closed walks of length k over N verticies. Then, the moment can be computed as follows.*

$$\mu_k \;=\; \lim_{n\to\infty} \frac{\langle \sum_{s\in S_k} \prod_{i=0}^{n-1} a_{si,s(i+1)}\rangle}{N^{k/2+1}} \tag{1}$$

By using the nature of expected values, it is not hard to derive the following Corollary.

Corollary 1 (Trees)**.** *Let T_{2k} be the set of all traversals over a tree with $k+1$ verticies. Then, the moment of the experimental density of a GOE can be computed as follows.*

$$\mu_k \;=\; \begin{cases} \langle \sum_{s\in T_k} \prod_{i=0}^{n-1} a_{si,s(i+1)}\rangle & 2|k \\ 0 & 2\nmid k \end{cases} \tag{2}$$

2 Anticommutator of two GOEs

Let A, B be two matrices drawn from two GOEs that are asymptotically free. We wish to compute the moments of $AB + BA$. We wish to compute

$$\mu_k \;=\; \lim_{n\to\infty} \frac{\langle \operatorname{tr}[(AB+BA)^k]\rangle}{N^{k+1}} \;=\; \lim_{n\to\infty} \sum_{P\in\mathcal{P}_k} \frac{\langle \operatorname{tr}(P)\rangle}{N^{k+1}} \tag{3}$$

[1]denotes the expected value. That is, for a random variable X, $\langle X\rangle = \mathbb{E}[X]$

The set $\mathcal{P}_k$ is the set of **Product Words** of length $2k$, that is, all strings of length $2k$ that are combinations of AB and BA. For example,

$$\mathcal{P}_2 = \{ABAB, ABBA, BAAB, BABA\}$$

Definition 1 (Characteristic of product words)**.** *Consider the product word $P \in \mathcal{P}_n$. The characteristic of the product word is denoted by $\chi(P)$, and it is the number of integers $0 \leq i < 2n$ such that $P_i = P_{i+1}$. For example, if $P = ABBA$*

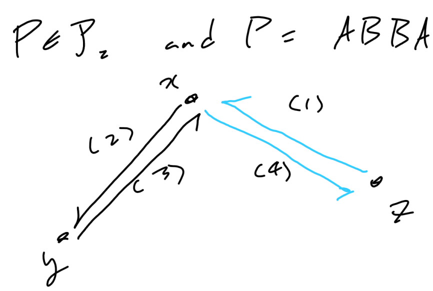
$$\chi(P) \;=\; 2$$

since $P_2 = P_3$ and $P_4 = P_1$.

We wish to analyze the anticommutator trace in (3) expansion using graph theory. In light of Theorem 1, we construct a colored a graph for each product word. Before we move on, however, we provide some definitions to simplify our analysis.

Definition 2 (Cliff Edges)**.** *Consider T_{2k}, a tree traversal over a tree with $k+1$ verticies. We define a cliff edge to be an edge which the traversal passes through and returns right after the passing. Formally, it is the edge corresponding to T_iT_{i+1} where $T_{i+2} = T_i$.*

It is possible to relate each summand in (3) to a colored traversal of a tree. Below is an example.



We have a simple tree with three verticies, and the traversal consists of four directed edges. Each edge that corresponds to the matrix A is colored blue, and the ones that correspond to B black. We refer to the colored traversals corrresponding to the product words as **Matched Colored Traversals(MAT)** For all the matrix entries have mean zero, we notice that each edge must be repeated twice. Using a degree of freedom argument, we verify that the two pair of edges must come from opposite directions[2]. Moreover, by the mean zero property of each matrix entry, each pair of directed edges must have a same color. We present the following observations.

[2]e.g. the summand can contain two occurences A_{12} and A_{21} but not A_{12} and A_{12}. This is without applying symmetry condition of the GOE.

Proposition 1. *In a graph traversal, there necessarily exists a cliff edge. Moreover in a MAT, there exists two cliff edges with different colors.*

Theorem 2. *Let $P \in \mathcal{P}_k$ be a product word that has a characteristic less than k. Then*

$$\lim_{n\to\infty} \frac{\langle tr(P)\rangle}{N^{k+1}} \;=\; 0$$

Proof. We induct on k. For $k = 1$, the theorem follows trivially. First, we write

$$\lim_{n\to\infty} \frac{\langle \mathrm{tr}(P)\rangle}{N^{k+1}} \;=\; \sum_{s\in T_k} \langle \prod_{i=0}^{n-1} P_i(s_i, s_{i+1})\rangle$$

where P_i is the ith character of the product word, and can either be A or B. The parantheses that follows denote the entry of the matrix. For example, if $P_i = A$ then $P_i(s_i, s_{i+1}) = A_{s_i,s_{i+1}}$.

Consider each summand which correspond to a colored tree traversal. In the tree traversal T_k, we know from proposition 1 that there must exist a cliff edge. If the traversal has two directed edges with differing color for a cliff edge, the entire term vanishes.

Otherwise, by the proposition, we exclude the two cliff edges that have different colors to obtain a new traversal $\bar{s} \in T_{k-1}$. By the inductive hypothesis, the summand vanishes. □

Corollary 2. *The odd moments of the anticommutator product of GOEs vanish. Moreover, the even moments are dominated by two terms with the maximum characteristics. In symbols,*

$$\mu_{2k+1} = 0 \quad \text{and} \quad \mu_{2k} = 2\langle \prod_{i=0}^{n-1} L_i^{(k)}(s_i, s_{i+1})\rangle \tag{4}$$

Where $L^{(k)}$ is an alternating combination of AA and BB. For example, $L^{(4)} = AABBAABB$[3]

3 The Challenge

We have narrowed down the moment computation from counting traces of 2^k product words to counting a single matrix product that has a simpler form. We wish to solve the following combinatorial problem.

Question 1 (The number of AABB-MATs)**.** *Consider T_{2k} to be a MAT of any tree with $k+1$ verticies that correspond to the word $L^{(k)}$. How many traversals T_{2k} exist?*

[3]Note that $L^{(k)} \notin \mathcal{P}_k$. $L^{(k)}$ is derived from the cyclicity of trace.