# Title

Benevolent Tomato

**Theorem 1.** *The characteristic polynomial $ch(z)$ always has a real root.*

**Theorem 2.** *If $f \geq 1/N$, then $ch(z)$ has a unique positive, real root that has a magnitude strictly greater than any of the other complex roots.*

*Proof.* Consider the polynomial

$$h(z) := (z-1)\mathrm{ch}_N(z) = z^{N+1} - (f+1)z^N + f \tag{0.1}$$

which has a simpler algebraic expression. We split the polynomial $h(x)$ into two summands, and invoke Rouche's Theorem ([**?**] p91). Let $C_{1+\epsilon}$ be a circular contour centered at the origin with radius $1+\epsilon$ for arbitrarily small $\epsilon$. Write

$$h(z) = (z^{N+1} + f) + (1+f)z^N \tag{0.2}$$

and Taylor expand the two summands at $z = 1$.

$$(z^{N+1} + f) = 1 + f + (N+1)\epsilon \tag{0.3}$$

$$z^N(1+f) = (1+N\epsilon)(1+f) = 1 + f + N(1+f)\epsilon \tag{0.4}$$

By assumption, $f \geq 1/N$, which implies $(N+1) \leq N(1+f)$. The modullus of the two terms along the contour can be compared as follows.

$$\left|z^{N+1} + f\right| \leq \left|(1+f)z^N\right| \tag{0.5}$$

By Rouche's theorem, $h(z)$ has the same number of roots as the term that has a larger modullus in the countour $C_{1+\epsilon}$, which is the summand $(1-f)z^N$. It is trivial to see that this summand has $N$ roots inside the countour, and by fundamental theorem of algebra, $h(z)$ has $N+1$ roots.

We know that $\mathrm{ch}_N(z)$ is positive somewhere in the interval $[1, \infty)$. We consider the following:

$$\mathrm{ch}_N(1) = 1 - fN \leq 0. \tag{0.6}$$

By the Intermediate Value Theorem, we conclude that the one root outside the unit circle is a positive real value. □

**Theorem 3.** *If $f < 1/N$, then all the roots of $ch_N(z)$ have a modullus strictly less than* 1.

*Proof.* It suffices to show that

$$\widetilde{h}(z) = h(1/z)z^{N+1} = fz^{N+1} - (f+1)z + 1 \tag{0.7}$$

has exactly one root within the unit circle which comes from multiplying $(z-1)$. Again, consider the countour $C_{1+\epsilon}$ and split $\widetilde{h}(z)$ into two summands.

$$\widetilde{h}(z) = (fz^{N+1} + 1) - (f+1)z \tag{0.8}$$

Taylor expand the two summands at $z = 1$, and notice that under the condition $f < 1/N$, the second summand has a larger modullus along the contour $C_{1+\epsilon}$.

$$
\begin{aligned}
fz^{N+1} + 1 &= f(1 + (N+1)\epsilon) + 1 && (0.9)\\
(f+1)z &= (f+1)(1+\epsilon) && (0.10)
\end{aligned}
$$

Clearly, the second summand has one root inside the contour $C_{1+\epsilon}$, which originates from $(z-1)$. By Rouche's theorem, $\widetilde{h}(z)$ has exactly one root inside the unit circle, i.e. $z = 1$, and all other roots have a modullus greater than 1. Consequently, $\mathrm{ch}_N(z) = h(z)/(z+1)$ has all of its roots strictly inside the unit circle. ☐

**Theorem 4** (Bounds for the dominant eigenvalue)**.** *Given that $f \geq 1/N$, the dominant eigenvalue of $L_f$ of order $N$ is given by*

$$1 + f - \frac{1}{N} \leq \lambda_{\max} < 1 + f. \tag{0.11}$$

*Proof.* The upper bound is trivial:

$$\mathrm{ch}_N(1+f) = f > 0. \tag{0.12}$$

We have $\mathrm{ch}_N(0) = -f < 0$, and thus by the Intermediate Value Theorem the maximum root is bounded.

To obtain the lower bound, we write $f = 1/N + \epsilon$ for some $\epsilon \geq 0$. With some algebra listed below , we compute $\mathrm{ch}_N(z)$ at the claimed lower bound. If we show that this value is less than zero, the dominating root must be greater than the purported lower bound. We find

$$\mathrm{ch}_N\left(1 + f - \frac{1}{N}\right) = -\left(1 + f - \frac{1}{N}\right)^N \left[\frac{1}{fN-1}\right] + \frac{fN}{fN-1}. \tag{0.13}$$

We wish to bound this value by zero. It suffices to show

$$fN - \left(1 + f - \frac{1}{N}\right)^N \leq 0, \tag{0.14}$$

which, using the $\epsilon$ substitution, converts to

$$1 + N\epsilon - (1+\epsilon)^N \geq 0. \tag{0.15}$$

Expanding the power term by the binomial theorem, we see that inequality indeed holds. ☐