RMT cheetsheet

Benevolent Tomato

1 Preliminaries

Definition *Wigner Matricies*

These info are taken from feier's pdf

Consider two probability distributions, Y, Z with zero mean. Let Z to have a variance of 1. Assume all the moments of these two distributions to be finite.

These distributions form a random matrix ensenble. To construct a matrix from these two distributions, take the distribution Y N times and fill the diagonal entries. Then, fill up the upper diagonal by drawing from Z, $N(N-1)/2$ times.

Definition *ESD and DSDs*

Consider a Wigner Matrix ensemble $\mathcal{M}_n$. The n eigenvalues of any matrix in the ensenble form a **spectral density**. Let $M_n \in \mathcal{M}_n$ to have a ESD of $f(x)$. The ESD is written as follows.

$$f(x) = \frac{1}{n} \sum_{\lambda \in \mathrm{Spec}(M_n)} \delta(x - \frac{\lambda}{\sqrt{n}})$$

The average of the ESDs provide a deterministic spectral density(DSD) of the entire ensenble.

Theorem *Markov's Inequality*

This is a nice tool that bounds the probability of a random variable being "too large" by the expected value. **This only works when the random variable is positive!**

$$\frac{\langle X \rangle}{a} \geq \mathcal{P}(x \geq a)$$

Theorem *Chebyshev's Inequality*

We use this inequality to bound the random variable to the mean. Unilike the Markov's inequality, the probability is bounded by the standard deviation.

$$\frac{1}{k^2} \geq \mathcal{P}(|X - \mu| > k\sigma)$$

Remark *Strategy of using probability inequalities*

Take the ESD. Approximate it in some compact region, and compute the probability that the ESD is bounded to the desired distribution. Bound the probability using markov. Then, use a delta-epsilon argument to bound the probability.

Concept *Stieltjes Transform*

Define the Stieltjes Transform by the following equation.

$$s_n(z) := \int_{x\in\mathbb{R}} \frac{f(x)}{x-z}dx = \int_{x\in\mathbb{R}} \frac{1}{x-z}d\mu$$

Where μ is the measure corresponding to the ESD of some random matrix.

Stieltjes transforms are nice, for they can be written as an expression of the trace.

$$s_n(z) = \mathrm{tr}(M_n/\sqrt{n} - zI)^{-1}$$

Proof. We deduce some properties about traces. Consider a n-by-n matrix A and its spectrum $\mathrm{Spec}(A)$. By the eigenvalue-trace lemma, we know

$$\sum_{\lambda\in\mathrm{Spec}(A)} \lambda = \mathrm{tr}(A)$$

We easily deduce

$$\sum_{\lambda\in\mathrm{Spec}(A)} \frac{1}{\lambda} = \mathrm{tr}(A^{-1}) \qquad \text{and} \qquad \mathrm{Spec}(A - zI) = \{\lambda_i - z\}$$

and infer

$$\mathrm{Spec}[(M_n/\sqrt{n} - zI)^{-1}] = \left\{\frac{1}{\lambda_i/\sqrt{n} - z}\right\}$$

which implies

$$\mathrm{tr}[(M_n/\sqrt{n} - zI)^{-1}] = \sum_{\lambda\in\mathrm{Spec}(M_n)} \frac{1}{\lambda/\sqrt{n} - z}$$

. Notice that the sum can be expressed as an integral using the ESD of M_n. Let $f(x)$ to be the ESD, and we can write the following.

$$\mathrm{tr}[(M_n/\sqrt{n} - zI)^{-1}] = \int_{x\in\mathbb{R}} \frac{f(x)}{x-z}dx := s_n(z)$$

. $\square$

Theorem *Moment of the GOE*

The odd moment of the GOE ensemble vanishes. The $2k$th moment of the GOE ensenble matches the kth Catalan number.

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

Proof. The theorem holds by five observations.

1. Using the eigenvalue trace lemma, we can express the trace of A^{2k} as circuits/cycles of length 2k. By the nature of expected values, all cycles that have an unmatched edge can be ignored. This indicates that all the odd moments vanish.

2. All cycles that have more than three kinds of the same edges, regardless of direction, can be ignored. Consider the expansion

$$a_{i1i2}a_{i2i3}\dots a_{iki1}$$

   and simplify them to the form

$$a_{i1i2}^3 a_{i2i3}^2 \dots a_{iki1}^2$$

   . These contribution dies out, since they have a total of degree freedom k, hence a contribution of N^k.

3. All cycles that have arrows pointing in the same direction dies out. All such cycles can be bijected to a partition of $[k]$. The degree of freedom is also k, and this dies out.

4. So all circuits must be comprised of sets of edges that travel in the different direction. Count this by the nmber of Dych words and the Catalan numbers appear!

5. Note that the double variance in the center entries are countered in step 3. All pairs a_{ii}^2 can be considered as two edges that connect to each other in different directions.

□

2 Free Probability

These ideas are taken from Mingo's book.

2.1 Cumulants, Gaussian, Wick's Formula

A **Characteristic Function** is a power series that has moments multiplied by the power of i. Basically, it is like the moment generating function evaluated at the imaginary line. We are mainly concerned with PDFs which have a 0th moment of 1. Hence, we write the Characteristic Function as follows.

$$\varphi(t) \;=\; 1 + \alpha_1 \frac{it}{1!} + \alpha_2 \frac{(it)^2}{2!} + \alpha_3 \frac{(it)^3}{3!} \cdots$$

[1]

Take the logarithm of the characteristic function to obtain cumulants.

$$\log(\varphi(t)) \;=\; k_1 \frac{it}{1!} + k_2 \frac{(it)^2}{2!} + k_3 \frac{(it)^3}{3!} \cdots$$

[2] Use Taylor series expansion to explicitly compute the cumulatns in terms of the moments.

Also, remarkably, the characteristic function of the gaussian is

$$\varphi(t) \;=\; \exp(i\mu t - \frac{\sigma^2 t^2}{2})$$

and thus the cumulant vanishes after k_2.

Moreover, there is a nice combinatorial way to extract the moments from the cumulants. Refer to Mingo.

Even Moments of the gaussian distribution have a nice combinatorial interpretation.

$$\alpha_{2k} = (2k-1)!!$$

Prove using by-parts integration.

Gaussian Vectors are defined as tuples of random variables that are not necessarily free. We write the vector as $(X_1, \ldots, X_N)$. For this vector to be Gaussian, the tuple must have a density that resembles the gaussian. For example,

$$\mathbb{E}(X_{i1}, \ldots, X_{ik}) = \int_{\mathbb{R}^n} t_{i1} \ldots t_{ik} \frac{\exp(-\langle C^{-1}t, t\rangle/2)dt}{\sqrt{2\pi}^{N/2} \det(C)^{1/2}}$$

Where C is the covariance matrix.

Using independance of the random variables, it is possible to derive **Wick's Formula** for Gaussian Vectors. The key idea is to collect the repetitive random variables and use the combinatorial interpretation for the even moments. With some more group theory, we can derive the **Genus Expansion**. [3]

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[1] αi denotes the ith moment

[2] $\log = \ln$ here

[3] Be aware that tr and Tr refer to diffrerent quantities. $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$

We discuss Wick's Formula and the Genus Expansion in greater detail. Here is the formulation of Wick's Formula.

Theorem 1 (Wick's Formula)**.** *Let X be a gaussian vector. Then*

$$\mathbb{E}(X_{i1} \cdots X_{in}) \;=\; \sum_{\pi \in \mathcal{P}_2[n]} \mathbb{E}_\pi(X_{i1} \cdots X_{in})$$

Where $\mathcal{P}_2[n]$ denotes all the partitions with block size 2.

A theorem by Biane comes in handy.

Theorem 2. *Let σ, π be permutations in which the subgroup of permutations generated by the two generators descirbe a transitive group action. That means,*

$$\langle \sigma, \pi \rangle = \langle \delta \rangle$$

for some permutation δ. Then, the following holds.

$$\#(\sigma) + \#(\sigma \circ \pi) + \#(\pi) \;=\; N + 2(1-g)$$

where g is the minimum genus which imbeds the graph.

This is a topology/graph theory result. For us, we just need to know that $g = 0$ when the graph is planar. That is, when the partitions are non-crossing.