Midterm II-part I

Daniel Son

1. Prove or disprove: there is an entire analytic function with real part $x - xy$. If there is such an analytic function, find all such functions. Also, find the series expansion of the function of z around the origin.

Solution Let function $f(z) = f(x + iy)$ be such a function that satisfies the condition. Analytic functions are necessarily holomorphic and vice versa. Hence, it is possible to apply the Cauchy-Riemann Equations in this context. Define:

$$u := Re(f(x + iy)) \quad \text{and} \quad v := Im(f(x + iy))$$

It is given that $u = x - xy$. We compute:

$$u_x = 1 - y \quad \text{and} \quad u_y = -x$$

By the Cauchy-Riemann Equations, we deduce:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$v_x = -u_y = x \quad \text{and} \quad v_y = u_x = 1 - y$$

The function $v(x, y)$ must be expressed as the following:

$$v(x, y) = x^2/2 + C(y) = y - y^2/2 + D(x)$$

Where C, D are functions that map real values to real values that depend solely on y and x respectively. The two expressions of $v(x, y)$ must equate each other. Write:

$$C(y) - y + y^2/2 = D(x) - x^2/2$$

Recognize that the LHS is independent of x and the RHS independent of y. Thus, we conclude that both expressions equal to a constant, say C.

$$D(x) = x^2/2 + C \quad \text{and} \quad v(x, y) = x^2/2 + y - y^2/2 + C$$

Compute the complex derivative of f by differentiating it over the real axis. The holomorphicity of f guarantees that the derivative is unique. Write:

$$\frac{d}{dz} f(z) = \frac{\partial}{\partial x} u(x, y) + \frac{\partial}{\partial x} v(x, y) i$$

$$= (1 - y) + xi = 1 - iz$$

Taking the antiderivative, we conclude, for some complex constant C',

$$f(z) = z - iz^2/2 + C'$$

The real part of f does not contain a constant. Hence, we narrow down $C' = Ci$ where C is a real value.

We have shown that a function f that satisfies $Re(f) = x - xy$ must be in the form of:

$$f(x) = Ci + z - iz^2/2 \qquad (C \in \mathbb{R})$$

Indeed all such functions must be holomorphic, for f is a complex polynomial of order two. Moreover, by some algebra, we notice that such functions always have a real part $x - xy$. We conclude that the functions of the form above are all the analytic entire functions that have a real part of $x - xy$. The function is already written as its series expansion about the origin.

□

2. Compute four integrals.

i) Compute:

$$I := \int_{-\infty}^{\infty} \frac{dx}{cosh(x)} = \int_{-\infty}^{\infty} \frac{2dx}{e^x + e^{-x}}$$

<u>Soultion</u> The integrand is an even function. Hence we write:

$$I = 4\int_{0}^{\infty} \frac{dx}{e^x + e^{-x}} \qquad \text{and} \qquad I/4 = \int_{0}^{\infty} \frac{e^x dx}{e^{2x} + 1}$$

Apply the u-subsitution, $u = e^x$:

$$I/4 = \int_{-\infty}^{\infty} \frac{du}{u^2 + 1} = arctan(u)\Big|_{-\infty}^{\infty} = \pi$$

Hence:

$$I = 4\pi$$

□

ii) Let ζ be any real number and $a > 0$. Evaluate:

$$I := \int_{-\infty}^{\infty} \frac{e^{-2\pi\zeta x}}{x^2 + a^2} dx$$

<u>Solution</u> Define a holomorphic function $f(z)$ as follows:

$$f(z) = \frac{e^{-2\pi\zeta z}}{z^2 - a^2}$$

The numerator and the denominator are known to be holomorphic. Thus the function is holomorphic everywhere other than the poles which are located at $z = \pm a$. Draw a semicircular contour centered at the origin that occupies quadrant I and IV. Call this contour γ, and denote the radius as R.

Take the contour integral of $f(z)$ over γ. Let the straight segment of the contour be called S, and the circular region C.

We claim that the integral over the circular region vanishes. That is, a as $R \to \infty$, $\oint_C f = 0$

Notice:

$$\left| \oint_C f \right| = \left| \int_{z \in C} \frac{e^{-2\pi\zeta z}}{z^2 + a^2} dz \right| \leq \int_{z \in C} \frac{max|e^{-2\pi\zeta z}|}{R^2 - a^2} dz$$

Note that the modulus of an exponent is the exponent of the modulus of the argument. That is:

$$|e^{-z}| = e^{Re(-2\pi\zeta z)}$$

And for $z \in C$, the quality is bounded under 1. Thus:

$$\left| \oint_C f \right| \leq \frac{2\pi R}{R^2 - a^2}$$

And the upper bound converges to zero as R approaches infinity. This shows that the circular region converges to zero. ✓

By the residue theorem:

$$\oint_C f + \oint_S f = 2\pi i Res_f(a)$$

The first summand of the LHS vanishes. The second summand can be computed with some algebra. We write:

$$\oint_S f = \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix} \cdot (-i) dx}{(xi)^2 - a^2} = i \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix} dx}{x^2 + a^2} dx = iI$$

The residue can be computed with ease:

$$Res_f(a) = \lim_{z \to a} \frac{e^{-2\pi\zeta z}(z-a)}{z^2 - a^2} = \lim_{z \to a} \frac{e^{-2\pi\zeta z}}{z + a} = \frac{e^{-2\pi\zeta a}}{2a}$$

Combining the results, we write:

$$iI = 2\pi i \frac{e^{-2\pi\zeta a}}{2a} \quad \text{or} \quad \boxed{I = \frac{\pi e^{-2\pi\zeta a}}{a}}$$

iii) Compute:

$$\frac{I}{2\pi i} = \frac{1}{2\pi i}\oint_{|z|=2} \frac{zdz}{z^2-1}$$

<u>**Solution**</u> The function

$$f(z) = \frac{z}{z^2-1}$$

is holomorphic outisde the two poles $z = \pm 1$. By the residue theorem, the integral I equals to the sum of the residues multiplied by $2\pi i$. Our answer is the following sum:

$$Res_f(1) + Res_f(-1)$$

Write:

$$Res_f(1) = \lim_{z\to 1} \frac{z(z-1)}{z^2-1} = z/(z+1)\Big|_{z=1} = 1/2$$

$$Res_f(-1) = \lim_{z\to -1} \frac{z(z+1)}{z^2-1} = z/(z-1)\Big|_{z=-1} = 1/2$$

Thus:

$$\boxed{\frac{I}{2\pi i} = 1}$$

iv) Compute:

$$I := \int_0^\infty \frac{x^{-1/2}}{x+1} dx$$

Solution We use two identities about the beta function. Recall the definiton:

$$B(n,m) := \int_0^1 x^n (1-x)^m dx$$

And the two identities:

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \quad \text{and} \quad B(n,m) = \int_0^\infty \frac{u^{n-1}}{(1+u)^{n+m}} du$$

By the seond condition, the integral simplies to:

$$I = B(1/2, 1/2)$$

And by the first identity:

$$B(1/2, 1/2) = \Gamma(1/2)^2 / \Gamma(1) = \pi$$

We conclude:

$$I = \pi$$

□

3. Consider the following infinite products:

$$I_1(a) := \prod_{n=1}^{\infty}(1 + a_n) \quad \text{and} \quad I_2(b) := \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}(1 + b_{mn})$$

a) State the definition of convergence of $I_1(a)$. Give an example of a product that converges to a finite, nonzero number, and an example that diverges.

<u>Definition</u> We define the partial product S_N as follows:

$$S_N := \sum_{n=1}^{N}(1 + a_n)$$

If the partial product converges as $N \to \infty$, then the infinite product $I_1(a)$ is defined to converge.

Consider the case where $a_n = 0$ identically. Trivially, $S_N = 1$ regardless of N. The infinite series converges to 1.

Now, let $a_n = 1/n$. By induction, it is possible to show $S_N = N + 1$. For the base case, $S_1 = 1 + a_1 = 2$. For the inductive case:

$$S_{N+1} = \prod_{n=1}^{N+1}(1 + \frac{1}{n}) = \frac{N+2}{N+1}S_N = N + 2$$

which proves the claim. Ergo, S_{N+1} diverges to infinity.

b) State the definition for the convergence of the infinite product $I_2(b)$.

<u>Definition</u> It would be nice if the nested products all converge. That is: $I_1(b_k)$ converges for any k. The sequence b_k denotes the sequence:

$$b_{k1}, b_{k2}, b_{k3}, \ldots, b_{kn}, \ldots$$

Given that $I_1(b_k)$ converges for any k, we define $I_2(b)$ to converge if

$$I_1(I_1(b_k) - 1)$$

converges. In words, if each row of $b+1$ converges, we write out all the products associated with row. Then, take the infinite product of the results. If any of the rows has a divergent infinite product, we define that the double product is undefined.

c) Does there exist a bounded sequence $\{b_{mn}\}$ such that each $b_{mn} \neq -1$ and:

$$\prod_{m=1}^{\infty}\left[\prod_{n=1}^{\infty}(1+b_{mn})\right] \neq \prod_{n=1}^{\infty}\left[\prod_{m=1}^{\infty}(1+b_{mn})\right]$$

Either prove no such sequence exists, or find one where the two products are not equal.

<u>Solution</u> There exists a sequence where the left product is undefined but the right product converges to zero. Consider the following sequence:

$$b_{mn} = \frac{(-1)^m}{(m+1)^2}$$

Note that b_{mn} is independent with regards to n. Hence the product

$$\prod_{n=1}^{\infty}(1+b_{mn})$$

diverges for even m. Each term is constantly greater than 1. By the definition of nested infinite products in part b), we conclude that the left product is undefined.

However, the right product converges to zero. It suffices to show:

$$0 < \prod_{m=1}^{\infty}(1+b_{mn}) < 1$$

Before proving the inequality, we first demonstrate that the product indeed converges. Recall(from the textbook) that if the infinite sum $\sum^{\infty} a_k$ converges absolutely, then so does $\prod^{\infty}(1+a_k)$. To show convergence of the infinite product, we show the convergence of the following sum:

$$\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}$$

This sum converges by the p-series test. $p = 2 > 1$ so the sequence converges, and hence the product converges too. Evidently, the product is greater than zero.

It remains to indeed show that the infinite product converges to a value less than 1. Now that the convergence of the product is guaranteed, we group the product into pairs. That is:

$$\prod_{m=1}^{\infty}\left(1+\frac{(-1)^m}{m+1}\right) = \left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots$$

$$= \prod_{m=1}^{\infty}\left(1-\frac{1}{2m-1}\right)\left(1+\frac{1}{2m}\right) \leq \prod_{m=1}^{\infty}\left(1-\frac{1}{2m-1}\right)\left(1+\frac{1}{2m-1}\right) = \prod_{m=1}^{\infty}\left(1-\frac{1}{(2m-1)^2}\right) < 1$$

We conclude that the nested product on the right converges to zero. $\square$

d) What was the two favorite and the two least favorite topics?

My favorite topics were contour integrals and fourier analysis. I believe that I will use the technique of contour integrals multiple times in the future taking more math and physics courses. To be honest, I did not understand too much about the fourier analysis, but decomposing functions into complex exponential bases seems like an exciting subject.

My least favorite topics were the prime number theorem and conformal mapping. The prime number theorem was too much for me to digest in a course of two weeks. The conformal mappings seemed theoretically interesting, but I wish we would have covered more applications around the topic.

4.

Let X_1, X_2 be Cauchy distributions with parameter $a, b > 0$. Assuming that the two random variables are independent, prove that the sum, that is $X := X_1 + X_2$, is also a Cauchy distribution. Also, compute the parameter of X

<u>Solution</u> The parameter of X is $a + b$. It suffices to prove to following:

$$\int_{x=-\infty}^{\infty} \left(\frac{1}{\pi a}\right)\left(\frac{1}{\pi b}\right)\frac{1}{1+(x/a)^2}\frac{1}{1+((x-t)/b)^2}dx = \frac{1}{\pi(a+b)}\frac{1}{1+[t/(a+b)]^2}$$

Take out the constants from the LHS and rewrite the integral as:

$$\frac{1}{\pi^2 ab}\int_{x=-\infty}^{\infty}\frac{1}{1+(x/a)^2}\frac{1}{1+((x-t)/b)^2}dx$$

If the integrand is considered as a complex valued function, we notice that the function is memomorphic and has four poles at

$$z = \pm ai, t \pm bi$$

We take a semicircular contour centered at the origin with radius R. The contour sits on the first and the second quadrant. The function vanishes in the order of $1/z^4$, hence the circular region of the integral vanishes. (a rigorous justification is added as an appendix). Denote the integrand as $f(z)$. By the residue theorem we write:

$$\oint f(z)dz = 2\pi i(Res_f(ai) + Res_f(t+bi))$$

Also, rewrite the integrand as:

$$\frac{1}{(z/a+i)(z/a-i)[(z-t)/b+i][(z-t)/b-i]}$$

We infer the residues at $z = ai, t + bi$ with ease. Simply remove the terms that vanish at the poles. Ergo:

$$Res(ai) = \frac{1}{2i[(ai-t)/b+i][(ai-t)/b-i]}$$

$$Res(t+bi) = \frac{1}{2i[(t+bi)/a+i][(t+bi)/a-i]}$$

And with multiple lines of algebra, we conclude:

$$2\pi i(Res(ai) + Res(t+bi)) = \frac{\pi\frac{ab}{a+b}}{1+\frac{t^2}{(a+b)^2}}$$

please trust me...

Combining the results, we conclude:

$$\frac{1}{\pi^2 ab}\int_{x=-\infty}^{\infty}\frac{1}{1+(x/a)^2}\frac{1}{1+((x-t)/b)^2}dx = \frac{1}{\pi^2 ab}2\pi i(Res(ai)+Res(t+bi))$$

$$= \frac{1}{\pi^2 ab}\frac{\pi\frac{ab}{a+b}}{1+\frac{t^2}{(a+b)^2}} = \frac{1}{\pi(a+b)}\frac{1}{1+\frac{t^2}{(a+b)^2}}$$

□

Appendix The circular region of the integral indeed vanishes.

Proof Consider the same complex function:

$$f(z) := \frac{1}{1+(z/a)^2}\frac{1}{1+((z-t)/b)^2}$$

Let C denote the circular region of the semicircular contour centered at the origin. Say C has a radius of R. We notice:

$$max_{z\in C}|f(x)| = \frac{1}{(R/a)^2-1}\frac{1}{(R-t)^2/b^2-1}$$

Also:

$$\left|\oint_{z\in C} f(z)dz\right| \leq 2\pi R\cdot max_{z\in C}|f(x)|$$

$$= \frac{2\pi R}{((R/a)^2-1)((R-t)^2/b^2-1)}$$

As $R\to\infty$, this value converges to zero. Hence, the circular region of the integral vanishes. □