1. Define:

$$f_n(x) := \frac{n}{1+nx^2}$$

Prove that f_n is uniformly continuous.
Is the family of function $\mathcal{F} := \{f_n(x) | n \in \mathbb{Z}_{pos}\}$ equicontinuous?

Proposition $\frac{x}{1+nx^2}$ is bounded.

Proof First, consider the function in the range $x \in \mathbb{R} \setminus [-1,1]$. Within the range, we have $|x| > 1$ and hence $x^2 > 1$. Write:

$$|\frac{x}{1+nx^2}| < |\frac{x}{nx^2}| = |\frac{1}{nx}| < 1$$

Now, consider the range $x \in [-1,1]$. Write:

$$|\frac{x}{1+nx^2}| < |\frac{x}{1}| < 1$$

This shows that our function is bounded for $x \in \mathbb{R}$ □

Claim $f_n(x)$ is uniformly continuous. Given any $\epsilon > 0$, we wish to obtain a δ_{max} where for any δ such that $|\delta| < \delta_{max}$ satisfies:

$$|f_n(x) - f_n(x+\delta)| < \epsilon$$

Or equivalently

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta)^2}| < \epsilon$$

Notice:

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta)^2}| < |\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta_{max})^2}|$$

It suffices to construct δ_{max} that satisfies:

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta_{max})^2}| < \epsilon$$

Through some algebra:

$$|\frac{n\left[1+n(x+\delta_{max})^2 - (1+nx^2)\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

$$|\frac{n\left[2nx\delta_{max} + n\delta_{max}^2\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

$$|\frac{n^2\delta_{max}\left[2x+\delta_{max}\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

Set $\delta_{max} < 1$. Also, notice that the terms in the denominators are both greater than 1. We construct a δ_{max} that satisfies a stronger condition:

$$|\frac{n^2\delta_{max}(2x+1)}{1+nx^2}| < \epsilon \quad \text{or} \quad |\delta_{max}||n^2||\frac{2x}{1+nx^2} + \frac{1}{1+nx^2}| < \epsilon$$

It is easy to see that the function $\frac{1}{1+nx^2}$ is bounded. The denominator is always greater than 1, so the function is bounded by 1. We have shown that the second summand is bounded for any real x. Again, we construct a stronger δ_{max} that satisfies:

$$|\delta_{max}|B < \epsilon$$

Where B is the maximum bound of the other terms. If $B < 0$, the statement is a tauology. Otherwise, set $\delta = \epsilon/(2B)$. This concludes the proof. $\square$

Claim The family $\mathcal{F}$ is not equicontinuous

Proof We claim that equicontinuity is violated at $x = 0$. Notice that $f_n(0) = n$. Assume for a contradiction, that $\mathcal{F}$ is equicontinuous at $x = 0$. For $\epsilon = 1$, it must be possible to obtain a δ_{max} where for all δ such that $|\delta| < \delta_{max}$, δ satisfies:

$$|f_n(0) - f_n(\delta)| < 1 \quad \text{or} \quad |n - f_n(\delta)| < 1$$

So

$$|f_n(\delta)| > n - 1$$

$\delta \neq 0$ by assumption, so as $n \to \infty$, $|f_n(\delta)| \to 1$. This is a contradiction. $\square$

2. Define:

$$f_n(x) := \begin{cases} 1 - nx & (x \in (0, 1/n)) \\ 0 & (\text{Otherwise}) \end{cases}$$

Show that the limit of this function exists and find the limit.

Solution Consider the function:

$$f(x) := \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases}$$

We claim that f_n converges to f pointwise. We must show that for any $x_0 \in \mathbb{R}$, the following equality holds:

$$\lim_{n \to \infty} f_n(x_0) = f(x_0)$$

If $x_0 \leq 0$, the problem becomes trivial. If the inequality is strict, $f_n(x_0) = 0$ regardless of the value of n. Also computing the value of $f_n(0)$, we notice that the value is identically 1, regardless of the value of n.

It remains to demonstrate the equality for $x_0 > 0$. Recall that $\lim_{n \to \infty} 1/n = 0$. Hence, it is possible to obtain a sufficiently large integer N such that for any $n > N$, we have $1/n < x_0$. By the construction of $f_n(x)$, $f_n(x_0) = 0$ for any $n > N$. This concludes the proof. $\square$

3. Define:

$$f_n(x) := \begin{cases} 1 - nx & (x \in (0, 1/n)) \\ 0 & (\text{Otherwise}) \end{cases}$$

and,

$$\mathcal{F} := \{f_n(x) | n \in \mathbb{Z}_{pos}\}$$

Is the family $\mathcal{F}$ normal?

Claim No, $\mathcal{F}$ is not normal.

Proof Assume for a contradiction, that indeed the family is normal. Then, the entire family $\mathcal{F}$ must have some subsequence of functions that converge uniformly. Let the sequence of functions $\{f_{m_1}, f_{m_2}, f_{m_3}, \dots\}$ be such a sequence of functions.

For the value $\epsilon = 1/4$, we extract some integer N such that for any $n > N$, the function achieves:

$$|f_{m_n}(x) - f(x)| < 1/4$$

For any real value x. $f(x)$ is some imaginary function that the subsequence uniformly converges to. Extract another arbitrary integer $k > N$ that satisfies the same condition. Adding the two inequalities, we obtain:

$$|f_{m_n}(x) - f(x)| + |f_{m_k}(x) - f(x)| < 1/2$$

Which implies, by the triangle inequality:

$$|f_{m_n}(x) - f_{m_k}(x)| < 1/2$$

And this is for any values of $n, k > N$. We explicitly construct a value x_0 that violates this inequality.

Take any integer n greater than N. Obtain k such that $m_k > 2m_n$. This is possible because m is a strictly increasing sequence of integers. Set $x_0 = 1/m_k$. Write:

$$|f_{m_n}(x_0) - f_{m_k}(x_0)| = |f_{m_n}(1/m_k) - f_{m_k}(1/m_k)|$$

Notice that the latter summand vanishes. Also the fraction $1/m_k$ is between zero and $1/m_n$. We proceed to write:

$$|f_{m_n}(x_0) - f_{m_k}(x_0)| = |1 - m_n/m_k| > 1/2$$

by construction. But then again, this whole absolute value must be less than 1/2, which is a contradiction. $\square$