Combinatorics HW4

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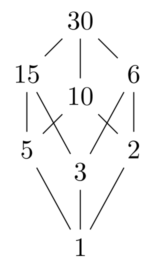
Section 4.6: 36, 37, 43, 50
Section 5.7: 3, 4, 7, 9, 15, 23, 24

Additional Problem 1. Draw the Hasse diagram for the divides rela- tion on the positive divisors of 30. Then explain in your own words the relationship between that diagram, and the diagram in Figure 4.7 from the textbook.

Solution



The Hasse diagram of the divisors of 30 are isomorphic to the Hasse diagram of the subsets of $\{1, 2, 3\}$. We can describe the isomorphism as follows. Let define function f to as $f(1) = 2, f(2) = 3, f(3) = 5$. The isomorphism ϕ that maps subsets to numbers is

$$\phi(S) = \prod_{k \in S} f(k)$$

where $S \subseteq \{1, 2, 3\}$. For example, $\phi(\{1, 3\}) = f(1) \cdot f(3) = 2 \times 5 = 10$. By the fundamental theorem of arithmetic, it is easy to see that each divisor correspond to a unique subset via the inverse of ϕ. $\square$

Sec4.6Q36 Let X be a set of n elements. How many different relations on X are there? How many of these relations are reflexive? Symmetric? Antisymmetric? Reflexive and symmetric? Reflexive and anti-symmetric?

Solution

We distinguish any ordered pairs of two integers into distinct and nondistinct pairs. The former refers to the pairs which have distinct entries. That is (a, b) where $a \neq b$. The latter refers to (a, a).

Relations can be considered as a set of ordered pairs. If the relation is symmetric, the choice of a nondistinct pair of the relation forces the relation to include the corresponding pair. That is, if $(a, b) \in R$ for where $a \neq b$, then $(b, a) \in R$. The choice of nondistinct pairs does not affect the symmetry of the relation. For a relation to be reflexive, it must include all the nondistinct pairs. For a relation to be antisymmetric it must not include any of the nondistinct pairs.

From the observations made above, we construct relations that satisfy the given conditions. If the relation is defined from the cannonical set $[n]$ to $[n]$, there exists n nondistinct pairs and $\binom{n}{2}$ couples of distinct pairs.

It is trivial that there exists $\boxed{2^{n^2}}$ relations overall.

To construct all the symmetric relations, we either choose or leave each nondistinct pair or a distinct couple of pairs. There are a total of $n+\binom{n}{2}$ such objects. Thus, the number of symmetric relations are $\boxed{2^{n(n+1)/2}}$.

As for the reflexive relations, we choose all the nondistinct pairs and choose or leave the distinct pairs. There are $n(n+1)$ distinct pairs so we conclude that there are $\boxed{2^{n(n-1)}}$ reflexive relations.

For antisymmetric relations, we can either choose or leave all the nondistinct pairs. For the distinct pair couples, we are allowed to choose one of the two pairs, or include both of them from the relation. Thus, there are three possible choices for each distinct couple. We count $\boxed{3^{\binom{n}{2}}2^n}$.

For reflexive and symmetric relations, we choose or leave the distinct couples and choose all the nondistinct pairs. There are $\boxed{2^{n(n-1)/2}}$ such relations.

For reflexive and antisymmetric relations, we choose all the nondistinct pairs and choose between the three options for each distinct couple. We count $\boxed{3^{\binom{n}{2}}}$.

Sec4.6Q37 Let R', R'' be partial orders on a set X. Define the intersection R such that $xRy \leftrightarrow (xR'y) \wedge (xR''y)$. Prove that R is a partial order.

Proof. We demonstrate symmetry, reflexivity, and transitivity of the relation R. For R', R'' are both partial orders, they must be reflexive. Hence, $xR'x$ and $xR''x$ for any $x \in X$. By the definition of the intersect R, xRx. R is reflexive.

To demonstrate symmetry, assume xRy. We deduce $xR'y$ and $xR''y$ from definition. R' and R'' are symmetric, so $yR'x$ and $yR''x$. Thus, yRx and R is symmetric.

By the same logic, we demonstrate transitivity. Assume xRy and yRz. $xR'y$ and $yR'z$ from the definition of R and we infer $xR'z$ from the transitivity of R'. Likewise, $xR''z$. Thus, xRz which concludes the proof. $\square$

Sec4.6Q43 Let $X = a, b, c, d, e, f$ and let the relation R on X be defined by $aRb, bRc, cRd, aRe, eRf, fRd$. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.

Solution Drawing a Hasse diagram, we identify that element a is maximum and d the minimum. All elements are connected either upwards or downwards of a and d. By transitivity, we infer all the orders, and set xRx for all $x \in X$.

As for the number of all linear extensions, we will count all the linear arrangement of the elements. Arrange the elements from the smallest to largest The position of d, a are fixed to be in the beginning and in the end of the queue. Any arrangement where c comes before b and f comes before e will work. Observe that by reseving the positions of b, c, a linear order is created. There are

$\boxed{\binom{4}{2} = 6}$ ways to do this.

Sec4.6Q50 Consider the partially ordered set of subsets of the set $X = \{a, b, c\}$ of three elements. How many linear extensions are there?

Solution

Again, refer to the Hasse diagram from the book. We fix the entire set and the emptyset to be the maximum and the minimum of the linear order. Considering arranging the 8 subsets in a line, we have deduced the necessary position of the two elements.

The nonempty proper subsets occupy the six remaining slots. We first determine the order of the three elements $\{a\}, \{b\}, \{c\}$. Afterwards, we insert the subsets with two elements in between to come up with a valid ordering. Notice that from the permutation (a, b, c) (abusing notation so that $a = \{a\}$ and so on), the subsets $\{a, b\}, \{a, c\}$ must come before $\{a\}$. The set $\{b, c\}$ can either come before $\{b\}$ or $\{a\}$. For the former case, the three subsets of size 2 can be arranged in any of the six possible orderings amongst them. For the latter case, $\{a, b\}, \{a, c\}$ has two possible arrangements. By the principle of addition and multiplication, we deduce that the total number of linear extensions are

$$6 \cdot (2 + 6) = \boxed{48}$$

Sec5.7q3 Compute the first few diagonal sums of the Pascal's triangle. Determine the relationship between them.

Solution

The nth diagonal can be considely written as the following sum.

$$a_n := \sum_{i=0}^{n-i \geq i} \binom{n-i}{i}$$

Where $n = 0, 1, 2, ...$

The following python code generates the diagonal sum.

```
def P(n, k):
    ans = 1
    for i in range(k):
        ans = ans * (n-i)
    return ans

def C(n, k):
    return P(n, k)/P(k, k)

def nthDiag(N):
    i = 0
    ans = 0
```

```
while (N - i >= i):
    ans = ans + C(N - i, i)
    i = i + 1

return ans
```

According to python, the first few of the sums are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

and we recognize that this is the fibbonacci sqeuence. That is, denoting the nth diagnoal sum a_n,

$$a_{n+2} = a_{n+1} + a_n$$

Sec5.7q4 Expand the following two products by the binomial theorem.

Solution

$$(x+y)^5 = \sum_{i=0}^{i=5} \binom{5}{i} x^i y^{5-i} = y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + x^5$$

$$(x+y)^6 = \sum_{i=0}^{i=6} \binom{6}{i} x^i y^{6-i} = y^6 + 6xy^5 + 15x^2y^4 + 20x^3y^3 + 15x^4y^2 + 6x^5y + x^6$$

Sec5.7q7 Prove that the weighted sum of the binomials by the geometric series is in the form of $(r+1)^k$.

Solution We start with considering a specific sum. Write

$$\sum_{i=0}^{i=n} \binom{n}{i} 2^i = \sum_{i=0}^{i=n} \binom{n}{i} 2^i 1^{n-i}$$

By plugging in $(x, y) = (2, 1)$ to the binomial theorem, we recognize that the sum is exactly 3^n.

$$\sum_{i=0}^{i=n} \binom{n}{i} x^i y^{n-i} = (x+y)^n \quad \text{and} \quad \sum_{i=0}^{i=n} \binom{n}{i} 2^i 1^{n-i} = (2+1)^n = 3^n$$

Likewise, for the sum

$$\sum_{i=0}^{i=n} \binom{n}{i} r^i$$

plug in $(x, y) = (r, 1)$ to show that the sum is $(r+1)^n$ $\square$

Sec5.7q9 Evaluate the sum

$$\sum_{i=0}^{i=n}(-1)^i\binom{n}{i}10^k$$

Solution The sum equals

$$\sum_{i=0}^{i=n}\binom{n}{i}(-10)^k1^{n-k} = (-10+1)^n = \boxed{(-9)^n}$$

Sec5.7q15 Prove the equality for every integer $n > 1$.

$$\binom{n}{1} - 2\binom{n}{2} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0$$

Proof. We prove the identity by differentiating the binomial identity. For any positive integer n,

$$(1-x)^n = \sum_{i=0}^{i=n}\binom{n}{i}(-x)^i = 1 + \sum_{i=1}^{i=n}\binom{n}{i}-x^i$$

Take the derivative with respect to x both sides.

$$\frac{d}{dx}(1-x)^n = \sum_{i=1}^{i=n}\binom{n}{i}(-1)^i\frac{d}{dx}x^i$$

$$n(1-x)^{n-1} = \sum_{i=1}^{i=n}\binom{n}{i}(-1)^i ix^{i-1}$$

Set $x = 1$ and multiply -1 both sides.

$$0\cdot(-1) = \sum_{i=1}^{i=n}(-1)^{i+1}i\binom{n}{i} = \binom{n}{1} - 2\binom{n}{2} + \cdots + (-1)^{n-1}n\binom{n}{n}$$

$\square$