P&M 9.26 An electromagnetic field in free space is described by the following two equations:

$$\vec{E} = \hat{y}E_0 \sin(kx + \omega t)$$
$$\vec{B} = -\hat{z}(E_0/c)\sin(kx + \omega t)$$

i) Show that the field satisfies the Maxwell's equations

Solution It is easy to observe that the divergence of both field equals to zero. The $\hat{y}$ component of the electric field has no y dependance and the $\hat{z}$ component of the magnetic field has no z dependance.

It suffices to verify the relationship regarding the divergence.

With some computation, we deduce:

$$\Delta \times \vec{E} = \hat{z}E_0 k \cos(kx + \omega t)$$
$$\Delta \times \vec{B} = \hat{y}E_0(k/c)\cos(kx + \omega t)$$

$$\frac{\partial}{\partial t}\vec{E} = \omega E_0 \cos(kx + \omega t)$$

$$\frac{\partial}{\partial t}\vec{B} = -\omega(E_0/c)\cos(kx + \omega t)$$

Recall the Faraday's Law and Ampere's Law in differential form:

$$\Delta \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad \text{and} \quad \Delta \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$$

With cancellation of the cosine terms, the two equations convert to:

$$E_0 k = \frac{E_0 \omega}{c} \quad \text{and} \quad \frac{E_0 k}{c} = \frac{\omega E_0}{c^2}$$

With more cancellation, both equations imply:

$$\boxed{c = \frac{\omega}{k}}$$

ii) Suppose $\omega = 10^{10}s^{-1}$ and $E_0 = 1kV/m$. Compute the energy density of the field per cubic meter and the rate of energy transfer per square meter.

Solution The energy density of an electromagnetic field is given as:

$$\frac{E^2\epsilon_0}{2} + \frac{B^2}{2\mu_0}$$

Where E and B denotes the magnitude of the electric and magnetic field.

The field is given as:

$$\vec{E} = \hat{y}E_0 \sin(kx + \omega t)$$

$$\vec{B} = -\hat{z}(E_0/c)\sin(kx + \omega t)$$

The energy density at a point can thus be computed by:

$$\frac{\epsilon_0 E_0^2 \sin^2(kx + \omega t)}{2} + \frac{(E_0/c)^2 \sin^2(kx + \omega t)}{2\mu_0}$$

For sufficiently large enough range of time and space, the square of the sine term averges out to 1/2. The following relationship between μ_0, ϵ_0, c comes handy:

$$\mu_0 \epsilon_0 = 1/c^2$$

The average energy density is:

$$\frac{E_0^2 \epsilon_0}{4} + \frac{1}{c^2 \mu_0}\frac{E_0^2}{4} = \boxed{\frac{E_0^2 \epsilon_0}{2} \approxeq 4.4 \cdot 10^{-6} J/m^3}$$

To compute the power flow, it is useful to use the method of Poynting vectors. The Poynting vector is a vector field defined as:

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Thus:

$$\begin{aligned}\vec{S} &= \hat{y}E_0 \sin(kx + \omega t) \times -\hat{z}(E_0/c)\sin(kx + \omega t)/\mu_0 \\ &= -\hat{x}E_0^2/(c^2\mu_0)\sin^2(kx + \omega t)\end{aligned}$$

Take a large surface perpendicular to $\hat{x}$. The power transfer along a surface can be computing by computing the surface integral of the Poyinting vector. Given that the area is sufficiently large enough, the average charge density will equal to the average of the magnitude of the Poynting vectors.

We write the average power density as:

$$\boxed{\left|\frac{\partial}{\partial t}U\right| = \frac{\epsilon_0 E_0^2}{2c} \approxeq 1300 J/(m^2 s)}$$

P&M 9.27 A sinusodial wave reflects at the surface of a medium which absorbs the half of the energy of the incident wave. Compute VSWR(Voltage Standing Wave Ratio) of the standing EM wave created by the reflection.

Solution From the Poynting vector equation, we observe that the magnitude of energy is proportional to the product of the magnitude of the electric and magnetic field. The magnetic field is a constant multiple of the electric field for traveling waves in free space. Hence, we conclude that the energy of a traveling wave is proportional to the square amplitude of the electric field.

If the medium absorbs half the energy of the incident wave, the reflection will have an amplitude reduced by a factor of $1/\sqrt{2}$. VSWR, by definition, is the maximum amplitude of the standing wave divided by the minimum amplitude. The standing wave reaches its maximum amplitude when the incident and the reflection interacts constructively. Minimum amplitude is achieved when the two waves cancel each other out. Thus, we write:

$$\boxed{VSWR = \frac{(1+1/\sqrt{2})E}{(1-1/\sqrt{2})E} = \frac{(\sqrt{2}+1)^2}{2-1} \approxeq 5.83}$$