Combinatorics HW2

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Question 1 For each of the subsets of property a, b, count the number of four digit numbers whose digits are either 1, 2, 3, 4, 5.

a) The digits are distinct

b) The number is even

Proof. Start off with counting the digits that need not satisfy both a or b. By principle of multiplication, we count $5^4 = \boxed{625}$.

Now we count the digits that satisfy only condition a. Again, by principle of multiplication, $5 \cdot 4 \cdot 3 \cdot 2 = \boxed{120}$.

For the digits that only satisfy condition b, we start with choosing the last digit. If the last digit is even, the whole number is even. There are two even numbers within the set $\{1, 2, 3, 4, 5\}$. So, by principle of multiplication, we count $2 \cdot 5 \cdot 5 \cdot 5 = \boxed{250}$.

Finally, for the digits that satisfy both of the conditions, we again count from the last digit. By principle of multiplication, the answer is $2 \cdot 4 \cdot 3 \cdot 2 = \boxed{48}$. □

Question 2 How many orderings are there for a deck of cards if all the cards in the same suite are together?

Solution First, ignore the order of the cards but only consider the ordering of the suites. There are four possible suites, so there are $4! = 24$ ways of ordering. For each suite, there are 13 cards. The 13 cards can be arranged each in 13! ways. Thus, by principle of multiplication, there are a total of $\boxed{4!(13!)^4}$ ways of ordering. □

Question 4 How many distinct positive divisors does the each of the following numbers have: $3^4 \cdot 5^2 \cdot 7^6 \cdot 11, 620, 10^{10}$

Proposition Let n be a positive integer. By the Fundamental Theorem of Arithmetic, it is possible to write n as:

$$n = \prod_{i=0}^{k} {p_i}^{a_i}$$

where p_i's are prime and a_i's are positive integers. The number of positive divisiors of n is

$$\prod_{i=0}^{k} (a_i + 1) \tag{1}$$

Proof. The divisors of n must involve only the prime numbers $\{p_1, ..., p_n\}$. The power of the prime p_i can range from a_i+1. Then, we proceed with the principle of multiplication to obtain the answer. $\square$

Solution In light of the proposition, the question boils down to finding the prime factorization of the three numbers, which is given as follows.

$$3^4 \cdot 5^2 \cdot 7^6 \cdot 11, \quad 2^2 \cdot 5 \cdot 31, \quad 2^{10} \cdot 5^{10}$$

The number of their divisors are $5{\cdot}3{\cdot}7{\cdot}2 = \boxed{210}$, $3{\cdot}2{\cdot}2 = \boxed{12}$, $11{\cdot}11 = \boxed{121}$. $\square$

Question 7 In how many ways can four men and eight women be seated around a round table if there are two women between consecutive men around the table?

Solution Take any three consecutive seats of the table. We observe that there must be a men sitting in one of the three tables. Otherwise, there will be a pair of two consecutive men where there is only one women in between the men. Moreover, there can be exactly one men sitting on one of these three seats.

Designate one of the three seats to be a seat for men. Such a choice will fix the seats where the men must sit. Afterwards, we count the number of ways to permute the four men and eight women, which equals $4! \cdot 8!$.

By principle of multiplication, we deduce that the total possible seatings are $3 \cdot 4! \cdot 8!$.

We have overcounted each arrangements 12 times, for there are 12 possible rotations for each sitting. Thus, the answer is $\boxed{3 \cdot 4! \cdot 8!/12 = 3! \cdot 8!}$

$\square$

Question 8 How many ways can six women and six men could be seated if men and women are to sit in alternate seats?

Solution Apply the reasoning that we used for the previous problem. From two consecutive seats, choose one seat for the men, which decides all the seats where the men should sit. Multiply by the permutations for men and women. We conclude that the number of total possible sittings are $2 \cdot (6!)^2$.

We have overcounted each arrangements 12 times, for there are 12 possible rotations for each sitting. Thus, the answer is $\boxed{2 \cdot (6!)^2/12 = 5! \cdot 6!}$

$\square$

Question 9 In how many ways can 15 people be seated at a round table if B refuses to sit next to A? What if B only refuses to sit on A's right?

Solution We partion all possible sittings P into three parts P_1, P_2, P_3. Let P_1 refer to all the sittings where A and B do not sit next to each other. Let P_2 be the sittings where A sits to the left of B and P_3 be the sittings where B sits to the left of A. The question boils down to computing the size of P_1 and $P_1 \cup P_3$.

First observe that there is a one-to-one correspondence between P_3 and P_1. By switching the seats of A and B, it is possble to obtain a sitting in P_3 from a sitting in P_1 and vice versa. It is trivial to see that $|P| = 14!$. Divide the linear permutation of the 15 people by 15 to account for rotations.

Compute the size of the part P_3. Fix the position of A to ignore rotations. B must sit on the left of A, hence the position of B is fixed. The size of P_3 is the number of ways to permute the remaining 13 people, and hence $|P_3| = 13!$. Also, by our previous observation, $|P_2| = |P_3| = 13!$.

By the addtion principle,

$$|P| = |P_1| + |P_2| + |P_3| \quad \text{or} \quad 14! = |P_1| + 2 \cdot 13!$$

and thus

$$|P_1| = 12 \cdot 13!$$

so we have computed the sizes of all partitions.

Finally, we write the sizes of the two partitions of our interest.

$$\boxed{|P_1| = 12 \cdot 13!} \quad \text{and} \quad \boxed{|P_1 \cup P_3| = |P_1| + |P_3| = 13 \cdot 13!}$$

□

Additional Problem 1 I computed ten permutations and got nine correct. I would give myself a 90/100. I confused the permutation with a combination for some reason and divided by an extra factor of two. Other than that, the math was correct.

Also the following code will compute $P(n, k)$ in python.

```
def P(n, k):
     ans = 1
     mtp = n
     for i in range(k):
             ans = ans * mtp;
             mtp = mtp - 1;
     return ans
```

Additional Problem 2 How many positive divisors of 2310 have themselves a maximum of four positive divisors?

Solution A key observation is that 2310 factors into a product of five distint prime numbers.

$$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

Any divisor of 2310 will have some subset of these five numbers without multiplicity. Also, notice that if the number has less than two prime numbers in its factorization, it will have less than four divisors. This is by the proposition presented in Question 4 (1). So we count the number of ways to choose zero, one, or two prime numbers among the five prime numbers.

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} = 1 + 5 + 10 = \boxed{16}$$

So there are 16 such divisors. $\square$

<u>Additional Problem 3</u> Mother Bear's pizzeria offers 3 sizes of pizza, 5 protein toppings, and 5 vegetable toppings. In how many ways can I order a pizza with up to 3 protein toppings and any number of vegetable toppings?

<u>Solution</u> Apply the principle of multiplication. Choose the size, then choose the protein toppings, then the vegetable toppings. There are three ways to choose the size of the pizza. As for the protein toppings we want to compute the ways to choose three or less toppings out of five. This can be computed by adding some binomial coefficients.

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 1 + 5 + 10 + 10 = 26$$

As for the vegetable toppings, encode the choice of toppings to a binary string. For example, if we choose to add the first and the third topping, the corresponding string would be 10100. It is not hard to see a one-to-one correspondence between the binary string and the choice of toppings. There are $2^5 = 32$ binary strings of length five.

So by the principle of multiplication, we write our solution.

$$3 \cdot 26 \cdot 32 = \boxed{2496}$$

And there are a total 2496 ways to choose the pizza.

<u>Additional Problem 4</u> I got everything correct! 100/100. Also the following code will compute $\binom{n}{k}$ in python.

```
def C(n, k):
    return P(n, k) / P(k, k)
```

<u>Question 10</u> A committee of five people is to be chosen from a club that boasts a membership of 10 men and 12 women. How many ways can the committee be formed if it is to contain at least two women? How many ways if, in addition, one particular man and one particular woman who are members of the club refuse to serve together on the committee?

<u>Solution</u> From all possible choice of a 5 person comittee, we will subtract the partion with zero or one women. This will yield all ways to choose a five person comittee including at least two women.

$$\binom{22}{5} - \binom{10}{5} - \binom{10}{4}\binom{12}{1} = \boxed{23562}$$

To solve the second problem, we partition the choice of all five person committees into the choice that includes both the women and men of the problematic pair, and the ones that does not include both of the two. For the second part, the problematic women or the men can be included, but not both. In fact, it is easy to compute the size of the first part. We induct the two people into the comittee, then choose additional three people.

The problem converts to counting the number of 3-person comittees chosen from 11 women and 9 men, where the comittee must include at least one women. Again, we count by the principle of subtraction.

$$\binom{20}{3} - \binom{9}{3} = 1056$$

To obtain the size of the second part, subtract the first part from the whole set.

$$23562 - 1056 = \boxed{22506}$$

So there are 23562 ways to choose the committee without the restriction of the problematic pair, and 22506 when the pair of man and women refuses to sit in the committee together. □

Question 14 A classroom has two rows of eight seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated?

Solution Seat the students sequentially in the following order. First, seat the five students who must sit in the front row. Then, seat the four students who must sit in the back row. Finally, seat the rest of the five students in any of the remaining seven seats. By the principle of multiplication, we count the number of possible seatings. Note that we must count using permutations for each table!

$$P(8,5) \cdot P(8,4) \cdot P(7,5) = \boxed{8449792000}$$

So there are 8449792000 possible seatings. □

Question 15 At a party there are 15 men and 20 women. (a) How many ways are there to form 15 couples consisting of one man and one woman? (b) How many ways are there to form 10 couples consisting of one man and one woman?

Solution For part (a), we notice that all men must necessarily have a partner. We iterate starting from the first man and let each man to decide his partner. There are $\boxed{P(20, 15) = 20274183401472000}$ matchings possible.

For part (b), start with obtaining the list of 10 men who have a partner. There are $\binom{15}{10}$ ways to make such a choice. Using the same reasoning for part (a), there must be $P(20, 10)$ ways to assign a partner to the selected 10 men. By the principle of multiplication, the total number of couplings are as follows.

$$\boxed{\binom{15}{10} \cdot P(20,10) = 2013339046118400}$$

$\square$

Question 21 How many permutations are there of the letters of the word ADDRESSES? How many 8-permutations are there of these nine letters?

Solution Decompose the string ADDRESSES into a multiset.

$$\{A \cdot 1, D \cdot 2, E \cdot 2, R \cdot 1, S \cdot 3\}$$

The permutation of this multiset is given by the following multinomial.

$$\boxed{\binom{9}{3, 2, 2, 1, 1} = \binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{1} = 15120}$$

Consider a 8-permutation of this multiset. Notice that the one element that is not used in the permutation is uniquely determined by teh permutation. Indeed, the number of 8-permutation equals to the number of permutations of the set. There are $\boxed{15120}$ 8-permutations.

$\square$