PHYS 202 HW4

Daniel Son

Q1 Two masses m_1, m_2 are connected by a spring with a spring constant k.
a) Write out a general solution for the position of the two masses.
b) Find a specific solution for $x_1(0) = x_2(0) = 0$ and $v_1(0) = v$
c) Sketch $x_1(t)$ and $x_2(t)$.

Solution

Each of the masses must have a linear term which describes the constant velocity of the center of mass. Also, each mass must have a oscillating term. By Newton's 2nd law, we write

$$\ddot{x_1} = -k(x_1 - x_2) \quad \text{and} \quad \ddot{x_2} = -k(x_2 - x_1)$$

Also, in light of our previous observation, we guess the solution to be in the form of

$$x_1 = Re[Ae^{i\omega t} + Ct]x_2 = Re[Be^{i\omega t} + Ct]$$

It is possible to include a constant term, but it is possible to redefine x_1, x_2 to be zero at the defined place of axis, so we discard the extra variables for simplicity.

Complexify the equation and plug them into the coupled differential eqation. We obtain the following matrix equation.

$$k\begin{bmatrix} -1/m_1 & 1/m_1 \\ 1/m_2 & -1/m_2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = -\omega^2\begin{bmatrix} A \\ B \end{bmatrix} \quad \text{or} \quad k\begin{bmatrix} 1/m_1 & -1/m_1 \\ -1/m_2 & 1/m_2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \omega^2\begin{bmatrix} A \\ B \end{bmatrix}$$

The eigensystem of the 2x2 matrix narrows down the candidates of the angular frequency and the amplitude A, B. According to mathematica, the eigensystem is as follows.

```
In[6]:= Coeff2 = {{k / m, -k / m}, {-k / M, k / M}};
        Eigensystem[Coeff2] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{k\,(m+M)}{m\,M} \\ \{1,\ 1\} & \left\{-\frac{M}{m},\ 1\right\} \end{pmatrix}$$

Thus, we obtain the two possible solutions.

$$(\omega, A, B) = (0, G, G) \quad \text{or} \quad (\sqrt{\frac{k(m+M)}{mM}}, -MG/m, G)$$

Where G is an arbitrary constant. The two restrictions does not provide constraints on C. Also, we defined the initial position to be zero at $t = 0$. Thus, we compute the imaginary part of the complexified equation instead of the real part. We write the general solution as follows.

$$\boxed{(x_1, x_2) = (Ct, Ct) \quad \text{or} \quad (-\frac{MG}{m}\sin(\sqrt{\frac{k(m+M)}{mM}}t) + Ct, G\sin(\sqrt{\frac{k(m+M)}{mM}}t) + Ct)}$$

We simplify the equation by using the definition of reduced mass.

$$\mu := \frac{mM}{m+M}$$

$$\boxed{(x_1, x_2) = (Ct, Ct) \quad \text{or} \quad (-\frac{MG}{m}\sin(\sqrt{k/\mu}t) + Ct, G\sin(\sqrt{k/\mu}t) + Ct)}$$

The center of mass mode cannot satisfy the given initial condition To compute the particular solution, compute the time derivative of $x1$ and apply the condition $\dot{x_1}(0) = v, \dot{x_2}(0) = 0$.

$$\dot{x_1}(0) = -\frac{MG}{m}\sqrt{\frac{k}{m}} + C = v \quad \text{and} \quad \dot{x_2}(0) = G\sqrt{k/\mu} + C = 0$$

By subtracting each equation from one another, we derive an expression for G and C.

$$G = -\frac{v}{M}\sqrt{\frac{\mu^3}{k}} \quad \text{and} \quad C = v\frac{\mu}{M}$$

Plugging in, we obtain

$$\boxed{(x_1, x_2) = \left(\frac{v}{m}\sqrt{\mu^3/k}\sin(\sqrt{k/\mu}t) + \frac{v\mu}{M}t, -\frac{v}{M}\sqrt{\mu^3/k}\sin(\sqrt{k/\mu}t) + \frac{v\mu}{M}t\right)}$$

Q2 Find the normal modes of a two-body oscillator that is vertically attatched to a ceiliing.

Solution Let x_1, x_2 be the displacement from the equilibrium positon of the two masses. Gravity can be ignored by considering the equilibrium position. Writing out the force eqution in matrix form

$$m\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Assuming a solution in the form of $x = e^{i\omega t}$, ω^2 will be the eigenvalues of the matrix

$$\frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

. The eigenvalues will correspond to the amplitude of the two displacements.

$$\begin{pmatrix} \frac{(3+\sqrt{5})\,k}{2\,m} & \frac{(3-\sqrt{5})\,k}{2\,m} \\ \left\{\frac{1}{2}(-1-\sqrt{5}),\ 1\right\} & \left\{\frac{1}{2}(-1+\sqrt{5}),\ 1\right\} \end{pmatrix}$$

Thus, the two frequencies of the normal modes are

$$\sqrt{\frac{(3+\sqrt{5})k}{2m}},\ \sqrt{\frac{(3-\sqrt{5})k}{2m}}$$

And the first frequency is the higher frequency and the second frequcncy is the lower frequency. For the higher frequency, the two masses move in the opposite direction, and the top mass moves at a greater amplitude. For the lower frequency, the two masses move in the same direction, and the top mass moves with less magnitude.

□

Q3 Asymmetric strings

a) Set up equations and find the matrix whose eigenvalues can be related to normal mode frequencies

Solution

Let x_1, x_2 denote the equilibrium position of the two masses. By Newton's 2nd law

$$m\ddot{x}_1 = -k_1x_1 - k_C(x_1 - x_2) \quad \text{and} \quad m\ddot{x}_2 = -k_2x_2 - k_C(x_2 - x_1)$$

Dividing my masses

$$\ddot{x}_1 = -\omega_1^2x_1 - \omega_C^2(x_1 - x_2) \quad \text{and} \quad \ddot{x}_2 = -\omega_2^2x_2 - \omega_C^2(x_2 - x_1)$$

Complexify the equations, and try solutions in the form of $(x_1, x_2) = (Ae^{-i\omega t}, Be^{i\omega t})$. By plugging in the equation and dividing by the exponential term, we obtain

$$\begin{bmatrix} -\omega_C^2 - \omega_1^2 & \omega_C^2 \\ \omega_C^2 & -\omega_C^2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = -\omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

Multiply both sides by negative one.

$$\begin{bmatrix} \omega_C^2 + \omega_1^2 & -\omega_C^2 \\ -\omega_C^2 & \omega_C^2 + \omega_2^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

Collect ω_1^2 from the LHS, and divide by it both sides.

$$\begin{bmatrix} \beta + 1 & -\beta \\ -\beta & \beta + \alpha \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = (\omega/\omega_1)^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

The eigensystem of the coefficient matrix above will yield the value of $(\omega/\omega_1)^2$ and its corresponding eigenvectors. The following mathematica code shows the result.

In[23]:=

```
Coeff = {{β + 1, -β}, {-β, β + α}};
Clear[α, β];
Eigensystem[Coeff] // MatrixForm
Map[
 Sqrt,
 Eigenvalues[Coeff]
]
```

Out[25]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(1+\alpha+2\beta-\sqrt{1-2\alpha+\alpha^2+4\beta^2}\right) & \frac{1}{2}\left(1+\alpha+2\beta+\sqrt{1-2\alpha+\alpha^2+4\beta^2}\right) \\ \left\{-\frac{1-\alpha-\sqrt{1-2\alpha+\alpha^2+4\beta^2}}{2\beta}, 1\right\} & \left\{-\frac{1-\alpha+\sqrt{1-2\alpha+\alpha^2+4\beta^2}}{2\beta}, 1\right\} \end{pmatrix}$$

Out[26]= $\left\{\frac{\sqrt{1+\alpha+2\beta-\sqrt{1-2\alpha+\alpha^2+4\beta^2}}}{\sqrt{2}}, \frac{\sqrt{1+\alpha+2\beta+\sqrt{1-2\alpha+\alpha^2+4\beta^2}}}{\sqrt{2}}\right\}$

Plugging in $\alpha = 1$, the eigensystem simplies as follows.

In[27]:=

```
α = 1;
Eigensystem[Coeff] // MatrixForm
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 1 & 1+2\beta \\ \{1, 1\} & \{-1, 1\} \end{pmatrix}$$

So for the $\alpha = 1$ case, we confirm that $\omega/\omega_1 = 1$ results in the center of mass mode, and $\omega/\omega_1 = \sqrt{1+2\beta}$ results in the breathing mode. $\alpha = 1$ means that the string constant of the two strings connected to the wall are identical, and the eigenvectors concur with the physical phenomena.

Now, apply the taylor approximation on both of the resonant frequencies. Call them ω_s, ω_f respectively. For real values x, y where $y \ll 1$, we have an approximation

$$\sqrt{x+y} \approxeq \sqrt{x} + \frac{y}{2\sqrt{x}}$$

.

Applying this approximation,

$$\xi := \sqrt{(1-a)^2 + 4\beta^2} \approxeq (1-\alpha) + 2\frac{\beta^2}{1-\alpha} \approxeq (1-\alpha)$$

and also,

$$\omega_s = \frac{\sqrt{1+\alpha+2\beta-\xi}}{\sqrt{2}} \approxeq \sqrt{\frac{1+\alpha+2\beta-(1-\alpha)}{2}} = \sqrt{\alpha+\beta} \approxeq \sqrt{\alpha} + \frac{\beta}{2\sqrt{\alpha}}$$

$$\omega_f = \frac{\sqrt{1+\alpha+2\beta+\xi}}{\sqrt{2}} \approxeq \sqrt{\frac{1+\alpha+2\beta+(1-\alpha)}{2}} = \sqrt{1+\beta} \approxeq 1+\frac{\beta}{2}$$

To sum up

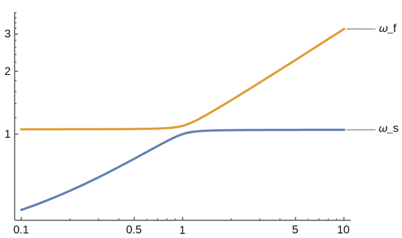
$$\boxed{(\omega_s, \omega_f) = (\sqrt{\alpha}+\frac{\beta}{2\sqrt{\alpha}}, 1+\frac{\beta}{2})}$$

Remark If $\alpha > 1$, then the signs of ξ swap. The abve equation is correct for $\alpha \ll 1$, but for $\alpha \gg 1$, ω_s and ω_f must be swapped.

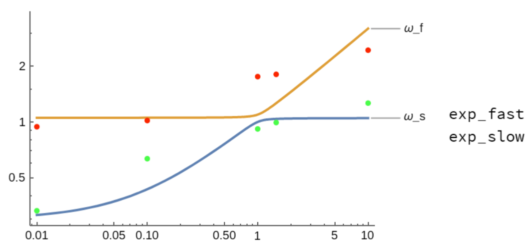
In[30]:=
```
Clear[α, β];
β = .1;
```

In[32]:=
```
LogLogPlot[{Sqrt[1 + α + 2 β - Sqrt[1 - 2 α + α^2 + 4 β^2]]/Sqrt[2], Sqrt[1 + α + 2 β + Sqrt[1 - 2 α + α^2 + 4 β^2]]/Sqrt[2]}, {α, .1, 10}
 ,
 PlotLabels → {"ω_s", "ω_f"}
]
```

Out[32]=

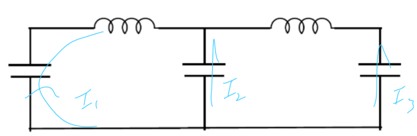


```
C2 = {.1, .68, 1, 10, 100}^-1;  (* in nanoFarads *)
fs = {201, 158, 146, 101, 53} * (2 π * 10^(-3));  (* kHz *)
ff = {387, 287, 279, 162, 150} * (2 π * 10^(-3));  (* kHz *)

Cfs = Partition[Riffle[C2, fs], 2];

Cff = Partition[Riffle[C2, ff], 2];

Show[
 LogLogPlot[{Sqrt[1 + α + 2 β - Sqrt[1 - 2 α + α^2 + 4 β^2]]/Sqrt[2], Sqrt[1 + α + 2 β + Sqrt[1 - 2 α + α^2 + 4 β^2]]/Sqrt[2]}, {α, .01, 10}
  ,
  PlotLabels → {"ω_s", "ω_f"}
 ]
 ,

 ListLogLogPlot[Cff, PlotStyle → Red, PlotLegends → "exp_fast"],
 ListLogLogPlot[Cfs, PlotStyle → Green, PlotLegends → "exp_slow"]
]
```

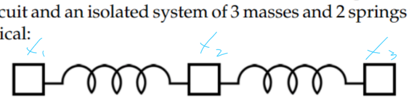


4) Lab Circuit

Consider again the coupled LC circuit we've studied in lab on several occasions:



Last week you showed this circuit to be equivalent to a 2-mass, 3-spring system, which is how we thought about it for Lab #3. But, this week it is more useful to recognize an alternative equivalence between this same circuit and an isolated system of 3 masses and 2 springs with the masses matched and the springs identical:



For the LC circuit let's define the charges on the upper plates of the capacitors to be q_1, q_2, and q_3, respectively. For the mechanical system define the displacements of the three masses to be x_1, x_2, and x_3. Show that the differential equations which govern the time evolution of the charges q_i are identical to those which describe the motion of the mechanical system, identifying

$$q \leftrightarrow x$$
$$L \leftrightarrow m$$
$$1/C \leftrightarrow k.$$

Solution 1 Using impedences

Use Newton's second law to write out the equations that govern the movement of the mechanical oscillator.

$$m\ddot{x}_1 = -k(x_1 - x_2)$$
$$m\ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3)$$
$$m\ddot{x}_3 = -k(x_3 - x_2)$$

For convinience, manipulate the second equation in the form

$$\ddot{x}_2 = -\ddot{x}_1 - \ddot{x}_3$$

As for the RLC circuit, define the circuit direction to point from bottom to the top, and call the currents I_1, I_2, I_3 from left to right. The voltage through each of the three paths must be conserved by the loop rule. Complexifying the currents and applying the complex version of Ohm's law, we write

$$\tilde{I}_1(\frac{1}{i\omega C} + i\omega L) = \tilde{I}_2\frac{1}{i\omega C} = \tilde{I}_3(\frac{1}{i\omega C} + i\omega L) \tag{1}$$

The solution of the currents will be in the form $\tilde{I} = Ae^{i\omega t}$. We deduce

$$i\omega\tilde{I} = \frac{d}{dt}\tilde{I} = \ddot{\tilde{Q}} \quad \text{and} \quad \frac{\tilde{I}}{i\omega} = \tilde{Q}$$

Where $\ddot{Q}$ is the complexified charge of any capacitor. The script is omitted to imply that the rule applies to all capacitors in the circuit.

In light of this identity, rewrite (1) in terms of charge.

$$\frac{1}{C}\tilde{Q}_1 + L\ddot{\tilde{Q}}_1 = \frac{1}{C}\tilde{Q}_2 = \frac{1}{C}\tilde{Q}_3 + L\ddot{\tilde{Q}}_3$$

Call each side of the equation A, B, C. Rewriting $A = B, C = B$, we obtain

$$L\ddot{\tilde{Q}}_1 = \frac{1}{C}(\tilde{Q}_2 - \tilde{Q}_1) \tag{a}$$

$$L\ddot{\tilde{Q}}_3 = \frac{1}{C}(\tilde{Q}_3 - \tilde{Q}_1) \tag{b}$$

Finally, by applying the junction rule on the top center junction,

$$\dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 = 0$$

and differentiating by time and with some manipulation

$$\ddot{Q}_2 = -\ddot{Q}_1 - \ddot{Q}_3 \tag{c}$$

Indeed, the isomorphism $(k, m) \mapsto (1/C, L)$ verifies that the equations a, b, c are indeed isomorphic to the equations of the mechanical system.

□

Q5 Describe the normal modes of the CO2 molecule.

Solution We start with deriving the normal mode frequencies. Consider the molecules as three masses located at equilibrium position x_1, x_2, x_3. Their respective masses will be m_O, m_C, m_O respectively. For convinience, define the vector $\vec{x} := (x_1, x_2, x_3)$. By Newton's 2nd law,

$$M\frac{d^2}{dt^2}\vec{x} = -K\vec{x} \tag{1}$$

Where the matricies M, K is

$$M := diag\{m_O, m_C, m_O\}, K := -k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The inverse of a diagonal matrix can also be written as a diagonal matrix where each entry is the reciprocal of the original entry. From (1), multiply both sides by M^{-1} to the left.

$$\frac{d^2}{dt^2}\vec{x} = -M^{-1}K\vec{x} = diag\{1/m_O, 1/m_C, 1/m_O\}(-k)K\vec{x}$$

Multiplying by the diagonal matrix on the left results in each column being multiplied by the corresponding entry. Also, pull out the scalar $1/m_O$.

$$\boxed{\frac{d^2}{dx^2}\vec{x} = -\frac{k}{m_O}diag\{1, 4/3, 1\}K\vec{x} = -\frac{k}{m_O}\begin{bmatrix} 1 & -1 & 0 \\ -4/3 & 8/3 & -4/3 \\ 0 & -1 & 1 \end{bmatrix}\vec{x}}$$

For convinience, we define

$$C := \begin{bmatrix} 3 & -3 & 0 \\ -4 & 8 & -4 \\ 0 & -3 & 3 \end{bmatrix}$$

We wish to compute the eigenvalues and the eigevectors that correlate to $M^{-1}K = -\frac{k}{3m_O}C$. We compute these quantities for C and make adjustments.

The characteristic polynomial of C is

$$|C - \lambda I| = \begin{vmatrix} 3-\lambda & -3 & 0 \\ -4 & 8-\lambda & -4 \\ 0 & -3 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda-8)\lambda$$

The spectrum i.e. the set of eigenvalues of C is $0, 3, 8$. We derive the corresponding eigenvectors by computing the nullspace of $C - \lambda I$.

With some algebra, we conclude that the eigensystem of C to be as follows.

$$\begin{aligned} \lambda = 0 &\mapsto (1,1,1) \\ \lambda = 3 &\mapsto (1,0,-1) \\ \lambda = 8 &\mapsto (3,-8,3) \end{aligned}$$

Thus, the eigensystem of the matrix $M^{-1}K = -\frac{k}{3m_O}C$ is

$$\lambda = 0 \mapsto (1, 1, 1)$$
$$\lambda = -\frac{k}{m_O} \mapsto (1, 0, -1)$$
$$\lambda = -\frac{8k}{3m_O} \mapsto (3, -8, 3)$$

From top to bottom, each mode corresponds to the center of mass mode, breathing mode, and the egyptian mode. $\square$