Combinatorics HW5

Daniel Son

Counting Derangements A derangement is a permutation of the set where no elements are fixed. We define D_n to be the number of derangements of the cannonical set $[n]$. By the inclusion-exclusion principle, we derive

$$D_n = n!\left(1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

By the alternating series test, we conclude

$$D_n = \left\{\frac{n!}{e}\right\}$$

Posets and Convolutions

Let $(X, \leq)$ be a finite poset. We consider a class of functions that map pairs of the poset X to the reals. Let $f, g : X \times X \to \mathbb{R}$. Define a discrete convolution of the two posets as follows.

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$$

We define three important functions, each corresponding to the identity, the ordering, and the inverse of the ordering. They are called the Kronecker Delta, Zeta, and the Mobius Function.

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

It is trivial to find out that the delta function is the convolutional identity.

Before writing out the Mobius function, we introduce a constructive method to obtain the convolutional inverse of an arbitrary function f. We requre $f(y, y)$ to be nonzero.

Let g be the left inverse of f. We easily observe that for nondistinct paris, g must be the reciprocal of f.

$$g(y, y) = \frac{1}{f(y, y)} \qquad \forall y \in X$$

For distinct pairs, the convolution of f, g must yield zero. If $x > y$, then the convolution is automatically zero. That is, assuming $x < y$,

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y) = 0$$

Break down the sum.

$$f(x, x) \cdot g(x, y) + \sum_{x < z \leq y} f(x, z) \cdot g(z, y) = 0$$

Sove for $g(x,y)$.

$$g(x,y) = -\frac{1}{f(x,x)} \sum_{x<z\leq y} f(x,z)\cdot g(z,y)$$

It is not hard to see that convolutions are associative. This leads us to conclude that the left inverse equals to right inverse.

$$f_l * f * f_r = \delta * f_r = \delta * f_l \quad \text{or} \quad f_r = f_l$$

Finally, we present the Mobius Function. The mobius function is defined as the inverse of the zeta function. plug in $f \mapsto \zeta$.

$$\mu(x,y) = \begin{cases} 1 & \text{if x = y} \\ -\sum_{x<z\leq y} \mu(z,y) & \text{otherwise} \end{cases} \quad \text{then} \quad \mu * \zeta = \delta$$

Proof of Mobius Inversion

Proof. Let ζ be the zeta function of $(X,\leq)$. Using the properties of ζ and μ previously discussed, we calculate as follows for x an arbitrary element in X:

$$\begin{aligned}
\sum_{\{y:y\leq x\}} G(y)\mu(y,x) &= \sum_{\{y:y\leq x\}} \sum_{\{z:z\leq y\}} F(z)\mu(y,x) \\
&= \sum_{\{y:y\leq x\}} \mu(y,x) \sum_{\{z:z\in X\}} \zeta(z,y)F(z) \\
&= \sum_{\{z:z\in X\}} \sum_{\{y:y\leq x\}} \zeta(z,y)\mu(y,x)F(z) \\
&= \sum_{\{z:z\in X\}} \left(\sum_{\{y:z\leq y\leq x\}} \zeta(z,y)\mu(y,x) \right) F(z) \\
&= \sum_{\{z:z\in X\}} \delta(z,x)F(z) \\
&= F(x).
\end{aligned}$$

□

Tips for Mobius Inversion

It is necessary that the cumulative function G is of simple form. If is is not clear what G is, then take the compliment of G's argument with respect to the universal set.

For example, it is horrendous to compute:

$$G(n) = \sum_{i|n} \phi(i)$$

However, consider

$$G(n) = \sum_{i|n} \phi(n/i)$$

Each divisor i uniquely maps to another divisor n/i. If a number ξ is coprime with n/i, $gcd(\xi \cdot i, n) = i$. More precisely, $(\xi, n/i) = 1$ iff $(\xi \cdot i, n) = i$. $\phi(n/i)$ counts the number of such ξ, and this corresponds to the numbers that have a gcd i with n. Each number in $[n]$ must have some gcd that divides n. Thus, $G(n)$ counts all numbers between $1, n$.

Classic Mobius Inversion

Memorize this sum:

$$\sum_{i|n} \mu(n/i) i = \phi(i)$$