PHYS 411T Final Exam

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1 Minis

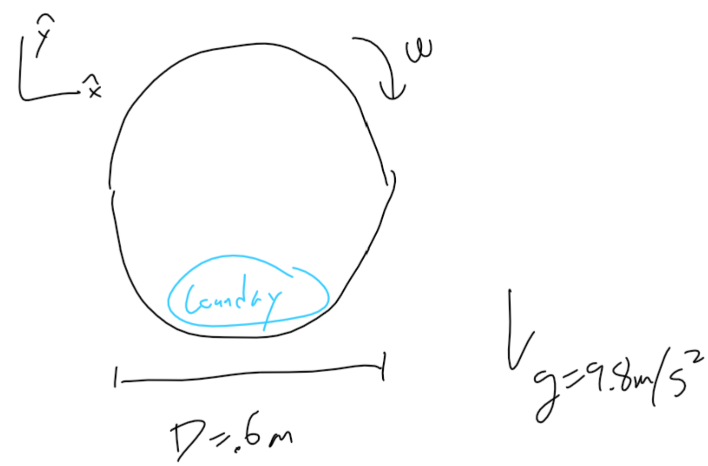

Figure 1: A simplified laundry basket.

Part A. Suppose the laundry basket starts rotating at some high angular frequency and slows down gradually. This way, we can ignore the effects of friction and assume that the clothes also move in the same angular velocity as the basket. We wish to find the angular velocity where the laundry loses contact with the surface. This occurs when the gravity exceeds the centrepetal acceleration of the clothes. The threshold angular velocity ω must satisfy the following.

$$\vec{F}_{\text{cent}} = \frac{mv^2}{r} = \frac{m(r\omega)^2}{r} = mg \tag{1.1}$$

$$\boxed{\omega = \sqrt{\frac{g}{r}} = 5.715 rad/s} \tag{1.2}$$

1

$\square$

Part B. Two objects in space that interact only through Newtonian gravity must orbit one another either in an elliptic orbit or a circular orbit.

$\square$

Part C. The following is the formula for inertia tensor elements.

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \vec{r} \cdot \vec{r} - r_i r_j\right) \tag{1.3}$$

$\square$

Part D. Here are the Hamilton's equations.

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{1.4}$$

$\square$

2 Von Braun's Space Station

Figure 2: Von Braun's Space Station.

Part A. Assume the space station to be a ring of radius R, and ignore the width and height. We compute the inertia tensor of the space station. First, observe that the off diagonal elements are zero.

$$\int xz \, dm = \int yz \, dm = 0 \tag{2.1}$$

These integrals vanish because $z = 0$ around the entirety of the space station.

$$\int xy \, dm \tag{2.2}$$

[1]The corresponding period is about $1.1s$ per cycle.

This integral vanishes, since every infinesimal chunck of the space station as a corresponding chunk which is symmetric to the x-axis.

Secondly, compute the following integrals.

$$\int z^2 \ dm \ = \ 0 \quad \text{and} \quad \int y^2 \ dm = \int x^2 \ dm \tag{2.3}$$

The first integral vanishes by the same argument above, and the latter two integrals agree by symmetry. Evaluate the x integral. Denote the linear mass density of the station as $\lambda = M/(2\pi R)$

$$\int x^2 dm \ = \ \int_0^{2\pi} R^2 \cos^2(\theta) \ \lambda R d\theta \ = \ \lambda R^3 \pi \ = \ \frac{MR^2}{2} \tag{2.4}$$

This allows us to compute the diagonal elements of the inertia tensor.

$$I_{xx} = \int y^2 \ dm + \int z^2 \ dm \ = \ \frac{MR^2}{2} \tag{2.5}$$

$$I_{yy} = \int x^2 \ dm + \int z^2 \ dm \ = \ \frac{MR^2}{2} \tag{2.6}$$

$$I_{zz} = \int x^2 \ dm + \int y^2 \ dm \ = \ \frac{MR^2}{2} + \frac{MR^2}{2} \ = \ MR^2 \tag{2.7}$$

Therefore the inertia matrix is

$$I \ = \ \begin{bmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{MR^2}{2} & 0 \\ 0 & 0 & MR^2 \end{bmatrix}. \tag{2.8}$$

□

Part B. The magnitude angular velocity vector ω must satisfy the following.

$$\frac{v^2}{R} = R\omega^2 \ = \ g \tag{2.9}$$

$$\omega = \sqrt{\frac{g}{R}} \tag{2.10}$$

Upon inspection $\vec{\omega} = \omega \hat{z}$. Also, the angular momentum can be computed as the image of the angular velocity vector under the inertia tensor.

$$\vec{L} \ = \ I\vec{\omega} \ = \ \omega I_{zz} \hat{z} \ = \ \sqrt{\frac{g}{R}} MR^2 \hat{z} \ = \ \sqrt{gM^2R^3} \hat{z} \tag{2.11}$$

Here is the answer.

$$\boxed{(\vec{\omega}, \vec{L}) \ = \ \left(\sqrt{\frac{g}{R}} \hat{z}, \sqrt{gM^2R^3} \hat{z} \right)} \tag{2.12}$$

□

Part C. From Euler's equation of torque, we can write out the time evolution of the angular momentum operator as follows.

$$\frac{d}{dt}\vec{L} + \vec{\omega} \times \vec{L} \;=\; \vec{\tau} \;=\; \vec{r} \times \vec{F} \tag{2.13}$$

Assuming that the time Δt and the force F is small enough, we can assume that $\vec{\omega} \times \vec{L} \approx 0$ since both vectors are approximately parallel to $\hat{z}$.[2] Therefore,

$$\frac{d}{dt}\vec{L} \;=\; R\hat{x} \times (-zF) \;=\; RF\hat{y} \tag{2.14}$$

and the angular momentum changes at a constant rate. After time Δt, the final angular momentum is

$$\vec{L}_{\Delta t} \;=\; \vec{L} + \Delta t RF\hat{y} \;=\; \sqrt{gM^2R^3}\hat{z} + \Delta t RF\hat{y} \tag{2.15}$$

Though the perturbation of the angular momentum is small, $\vec{L}$ points in a slightly different direction along the yz-axis after the spacecraft leaves.

It remains to justify that the cross product $\vec{\omega} \times \vec{L}$ is negligible. Take some time $t_1 \in (0, \Delta t)$. The angular momentum can be extracted by changing Δt to t_1, and the angular velocity by taking the inverse image of the inertia tensor.

$$\vec{L}_{t_1} \;=\; t_1 RF\hat{y} + \sqrt{gM^2R^3}\hat{z} \tag{2.16}$$

$$\vec{\omega}_{t_1} \;=\; \frac{1}{MR^2}\left(2t_1 RF\hat{y} + \sqrt{gM^2R^3}\hat{z}\right) \tag{2.17}$$

$$\vec{\omega}_{t_1} \times \vec{L}_{t_1} \;=\; t_1 F\sqrt{gR} \tag{2.18}$$

Under the assumption $\Delta t F \ll 1$, our assumption is justified. □

3 Jiggly Pendulum

Part A. The crux of constructing the Lagrangian is to find the kinetic energy of the swinging mass. Let d be the displacement of the cart. By vector addition, we obtain the displacement of the swinging mass which we name $\vec{r}$.

$$\vec{r} \;=\; d\hat{x} + l\sin(\phi)\hat{x} - l\cos(\phi)\hat{y} \tag{3.1}$$

Take the derivative to retrieve the velocity, and retrieve the squred magnitude.

$$\frac{d}{dt}\vec{r} = \; \dot{d}\hat{x} + l\dot{\phi}\cos(\phi)\hat{x} + l\dot{\phi}\sin(\phi)\hat{y} \tag{3.2}$$

$$\left\|\frac{d}{dt}\vec{r}\right\|^2 \;=\; (\dot{d})^2 + (l\dot{\phi})^2 + 2l\dot{d}\dot{\phi}\cos(\phi) \tag{3.3}$$

[2]Another way to justify this assumption is to ignore the relativistic correction. If the spaceship leaves quickly enough, the motion of the spaceship at the time of departure is approximately the same in both frames.

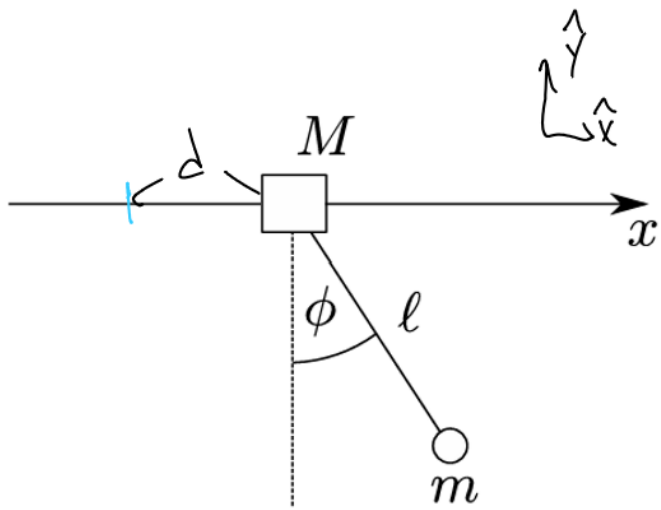

Figure 3: Free body diagram of a Jiggly Pendulum.

From here, simple algebra yields the kinetic energy of the system, which is the sum of the kinetic energy of the swinging mass and the cart. The potential energy comes from gravitational potential of the mass. Let M, m be the mass of the cart and the swinging mass respectively.

$$\begin{aligned}
T = \; T_{\text{cart}} + T_{\text{mass}} \; &= \; \frac{1}{2}(M(\dot{d})^2 + m(\dot{r})^2) && (3.4)\\
&= \; \frac{M+m}{2}(\dot{d})^2 + \frac{m}{2}(l\dot{\phi})^2 + m(\dot{d})(\dot{\phi})l\cos(\phi) && (3.5)\\
U = \; -mgl\cos(\phi) & && (3.6)\\
\mathcal{L} = \; T - U \; &= \; \boxed{\frac{M+m}{2}(\dot{d})^2 + \frac{m}{2}(l\dot{\phi})^2 + m(\dot{d})(\dot{\phi})l\cos(\phi) + mgl\cos(\phi)} && (3.7)
\end{aligned}$$

□

Part B. The Euler-Lagrange equation provides two identities that describes the motion.

$$\frac{\partial \mathcal{L}}{\partial d} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{d}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{3.8}$$

Bash out the algebra to obtain the two equations.

$$\begin{aligned}
0 \; &= \; (M+m)\ddot{d} + m\ddot{\phi}l\cos(\phi) - m(\dot{\phi})^2 l\sin(\phi) && (3.9)\\
-2(\dot{d})(\dot{\phi})l\sin(\phi) - gl\sin(\phi) \; &= \; l^2\ddot{\phi} + (\ddot{d})l\cos(\phi) - (\dot{d})(\dot{\phi})l\sin(\phi) && (3.10)
\end{aligned}$$

In fact, the Lagrangian does not have an explicit dependence on the displacement d. Therefore, the general momentum in the d direction is invariant.

$$\frac{d}{dt}\Big((M+m)\dot{d} + m(\dot{\phi})l\cos(\phi)\Big) \;=\; 0 \tag{3.11}$$

□

Part C. Lets simplify the equations from part B. In the small angle limit, all higher order terms other than the constant and linear terms can be ignored. Moreover, $\sin(\phi) = \phi$ and $\cos(\phi) = 1$. We obtain the follwing.

$$0 \;=\; (M+m)\ddot{d} + ml\ddot{\phi} \tag{3.12}$$

$$-gl\phi \;=\; l^2\ddot{\phi} + l\ddot{d} \tag{3.13}$$

Convert $\ddot{d}$ to $\ddot{\phi}$ and plug in to the second equation.

$$\frac{M}{M+m}\ddot{\phi} \;=\; -\frac{g}{l}\phi \tag{3.14}$$

$$\ddot{\phi} \;=\; -\frac{g(M+m)}{lM}\phi \tag{3.15}$$

Thus, the frequency is

$$\omega \;=\; \sqrt{\frac{g(M+m)}{lM}} \tag{3.16}$$

Clearly, if $M \gg m$, then $\frac{M+m}{M} = 1$ and we obtain the frequency

$$\omega_{M\gg m} \;=\; \sqrt{\frac{g}{l}} \tag{3.17}$$

which is the frequency of a regular fixed pendulum. □

Part D. The general solution for $\phi(t), d(t)$ is described as follows.

$$\phi(t) = \;\phi_0\cos(\omega t + a) \tag{3.18}$$

$$d(t) = \;-\frac{ml}{M+m}\phi_0\cos(\omega t + a) \tag{3.19}$$

We wish to draw a state space diagram for $\phi(t)$. Assume the initial condition $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$. Then, the angular displacement and velocity is

$$\phi(t) = \;\phi_0\cos(\omega t) \tag{3.20}$$

$$\dot{\phi}(t) = \;-\omega\phi_0\sin(\omega t) \tag{3.21}$$

Below is a sketch of the solution in time. Note that the angular displacement of the mass has an opposite sign as the linear displacement of the cart.

□

Part E. The hidden normal mode is the symmetric mode where both the cart and the mass move/swing in the same direction. This is a natural assumption, since we have deduced an antisymmetric mode with a π phase shift between d and ϕ. □

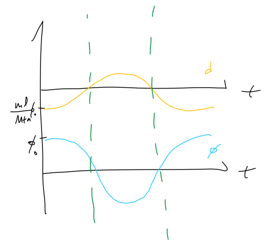

Figure 4: Time progression of $d(t), \phi(t)$.

4 Rotating Spring Revisited

Part A. We compute the Hamiltonian from the Lagrangian. From the sum of genaral velocities and general momentum, subtract the Lagrangian.

$$\begin{aligned}\mathcal{H} = \sum_i p_i q_i - \mathcal{L} &= \dot{r}(m\dot{r}) + \dot{\phi}(mr^2\dot{\phi}) - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}k(r-a)^2 \quad (4.1)\\ &= \boxed{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}k(r-a)^2} \quad (4.2)\end{aligned}$$

The generalized momentum is computed by taking the partial derivative of the lagrangian with respect to generalized velocity. Upon inspecting Hamilton's equations, we notice that the Hamiltonian itself and the generalized angular momentum is preserved. The Lagrangian does not depend on time. Therefore, invoke Beltrami's identity to see that the Hamiltonian is constant. [3] Also, the Hamiltonian does not have an explicit ϕ dependence. Therefore,

$$\frac{\partial \mathcal{H}}{\partial \phi} = -\dot{p}_\phi = 0 \quad (4.3)$$

and p_ϕ is invariant. □

Part B. The invariant angular momentum is

$$p_\phi = mr^2\dot{\phi}. \quad (4.4)$$

Let r_0, ω be the equilibrium radius and angular velocity. Then,

$$p_\phi = mr_0^2\omega \quad (4.5)$$

which yields

$$(r_0, \omega) = \left(\sqrt{\frac{p_\phi}{m\omega}}, \frac{p_\phi}{mr_0^2}\right) \quad (4.6)$$

□

[3] Or, one can take the time derivative of the Hamiltonian and invoke the chain rule. Invoking Euler-Lagrange equation on the partials will yield the same result.

Part C. As long as we get the results, it is better to do less work. Apply Lagrange's equation w.r.t r.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad \text{or} \quad mr(\dot{\phi})^2 - k(r-a) = \frac{d}{dt}(m\dot{r}) \tag{4.7}$$

The RHS of the equation vanishes, assuming circular motion. Solve for $\dot{\phi}$.

$$\dot{\phi} = \sqrt{\frac{k}{m}\frac{r-a}{r}} \tag{4.8}$$

Since a is a positive length, the fraction $(r-a)/r$ is strictly less that 1 and asymptotes to 1 as $r \to \infty$. Thus, the maximum value of $\dot{\phi}$ is attained when $a/r \to 0$. Finally,

$$\dot{\phi}_{\max} = \sqrt{\frac{k}{m}} \tag{4.9}$$

as desired. □

5 Wobble of the Earth

Part A. Assuming the inner core has three times the density of the outward shell, we wish to find the density of the shell, namely ρ. The mass can be computed by adding an oblate spheroid with dimensions (a, b, c), density ρ with a sphere of radius $c/2$, density 2ρ. Equate the sum of the two masses to the mass of the earth.

$$\rho V_{ob} + 2\rho V_{sp} = M_E \tag{5.1}$$

$$\frac{4\pi}{3}\rho\left(abc + \frac{2c^3}{8}\right) = M_E \tag{5.2}$$

$$\rho = \frac{3}{4\pi}M_E \Big/ \left(a^2c + c^3/4\right) = \boxed{4416kg/m^3} \tag{5.3}$$

```
ME = 5.972*^24; (* kg *)
a = 6378*^3; (* m *)
c = 6357*^3; (* m *)
ρ = ME / (a^2 c * 4 π / 3 + c^3 * π / 3) (* kg / m^3 *)
4416.43
```

□

Part B. The Inertia tensors add up just like masses. Therefore, take the inertia tensor of the oblate spheroid, and add it with the tensor of the sphere. Remember that the core has twice the density as the shell.

Remember the formula for the inertia of an oblate spheroid.

$$I = \begin{bmatrix} b^2 + a^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix} \frac{M}{5} \tag{5.4}$$

The off diagonal entries of the inertia tensor of earth vanish. Below are the diagonal entries of the inertia tensor of the entire earth.

$$(I_{xx}, I_{yy}, I_{zz}) = (8.24 \cdot 10^{37}, 8.24 \cdot 10^{37}, 8.26 \cdot 10^{37}) \tag{5.5}$$
$$I_{zz}/I_{xx} = 1.00311 \tag{5.6}$$

so the theoretical inertias match with the experimental inertias up to the first signifigant figure.

```
(* Compute the inertia tensor *)
Ishell = {a^2 + c^2, a^2 + c^2, 2 a^2} 4 / 15 π a^2 c * ρ;
Icore = {c^2, c^2, c^2} / 2 * 4 / 15 / 8 * π * c^3 * (2 ρ);
Itot = Ishell + Icore;
Itot
Itot[[3]] / Itot[[1]]
{8.23868×10^37, 8.23868×10^37, 8.26426×10^37}

1.00311
```

□

Part C. Invoke (t10.93) to compute the frequency of the free precession.

$$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \tag{5.7}$$

The angular velocity of the earth points towards the z-axis, and the magnitude can be computed by

$$\omega = \frac{2\pi \ rad}{24 \cdot 3600 sec}. \tag{5.8}$$

Plug and chug into mathematica.

$$\boxed{T = 307.7 \text{days}} \tag{5.9}$$

□

[4]

[4] Thank you for a wonderful sememster, Professor Jensen and Professor Strauch. I feel more confident about solving problems with where the details of the system are unknown. See you again next sememster, and Merry Christmas!

```
(*Computation directly from corrected B*)
(*Compute ω, the angular velocity of earth rotation*)
ω = 2 π / (24 * 3600) (*rad/s*);
(*The precession frequency is given in T10.93*)
Ωb = (1.00325 - 1) / 1 * ω;
(*Find the period in seconds*)
T = 2 π / Ωb;
(*Convert to days*)
Tdays = T / 3600 / 24
```

307.692