1. Let H, I, J be nonzero ideals in dedekind domain D. Given $HI = HJ$, prove $I = J$.

Proof We show $I \subseteq J$. Then, by symmetry, $J \subseteq I$, which shows $I = J$.

We know that any ideal in a dedekind domain has an inverse ideal. The ideal H has some ideal H' such that $H'H = \langle \alpha \rangle$ for some nonzero element $\alpha \in H$. Write:

$$H'HI = H'HJ \quad \text{or} \quad \langle \alpha \rangle I = \langle \alpha \rangle J$$

For any element $i \in I$, we extract $\alpha i = \alpha j$ for some $j \in J$. D is a domain, so by cancellation, $i = j$. We conclude $I \subseteq J$ and thus $I = J$. $\square$

2. Let $R := \mathbb{Z}[\sqrt{-3}]$. Also, define an ideal in R, $I = \langle 2, 1 + \sqrt{-3} \rangle$.

- Prove $I \neq \langle 2 \rangle$
- Prove $I^2 = \langle 2 \rangle I$
- Is R a dedekind domain?

Solution We start with showing that I is not equal to the principal ideal generated by 2. Assume for a contradiction, that indeed $I = \langle 2 \rangle$. Then, it must be $1 + \sqrt{-3} \in \langle 2 \rangle$. There must be some element $r \in R$ such that:

$$2r = 1 + \sqrt{-3} \quad \text{or} \quad r = \frac{1 + \sqrt{-3}}{2}$$

by expanding our search to the field of quotients. However, $r \notin Z[\sqrt{-3}]$, for the field of quotients is indeed a field, and inverses are unique. We reach a contradiction and $I \neq \langle 2 \rangle$

$\square$

We move on to show $I^2 = \langle 2 \rangle I$. By ideal algebra:

$$\langle 2, 1 + \sqrt{-3}^2 \rangle = \langle 4, 2 + 2\sqrt{-3}, (1 + \sqrt{-3})^2 \rangle$$

$$\langle 4, 2 + 2\sqrt{-3}, -2 + 2\sqrt{-3} \rangle = \langle 2 \rangle \langle 2, 1 + \sqrt{-3}, -1 + \sqrt{-3} \rangle$$

Notice that $-1 + \sqrt{-3} = 1 + \sqrt{-3} - 2$. Thus, we conclude:

$$I^2 = \langle 2 \rangle \langle 2, 1 + \sqrt{-3} \rangle = \langle 2 \rangle I$$

as desired. $\square$

Sadly, R is not a dedekind domain. In a dedekind domain, ideals cancel out. Thus $I^2 = \langle 2 \rangle I$ implies $I = \langle 2 \rangle$, which we have proven to be false on the first part. $\lightning$

$\square$

3. Prove that $\langle 3, 1 \pm \sqrt{-5}\rangle$ are prime ideals in the ring $\mathbb{Z}[\sqrt{-5}]$

Proof Denote $I := \langle 3, 1+\sqrt{-5}\rangle$ Consider the following line of Ideal algebra:

$$\langle 3, 1+\sqrt{-5}\rangle^2 = \langle 9, 3+3\sqrt{-5}, -4+2\sqrt{-5}\rangle$$

We can add a ring multiple of one entry and add to another generator and still get the same ideal. Thus:

$$= \langle 9, 3+3\sqrt{-5}+4-2\sqrt{-5}, -4+2\sqrt{-5}\rangle = \langle 9, 7+\sqrt{-5}, -4+2\sqrt{-5}\rangle$$

$$= \langle 9, 7+\sqrt{-5}, -4+2\sqrt{-5}-14-2\sqrt{-5}\rangle = \langle 9, 7+\sqrt{-5}, -18\rangle$$

$$= \langle 9, 7+\sqrt{-5}\rangle = \langle 9, -2+\sqrt{-5}\rangle = \langle -2+\sqrt{-5}\rangle = \langle 2-\sqrt{-5}\rangle$$

In fact, this ideal is a prime ideal. This is because the element $2-\sqrt{-5}$ is prime in the ring $\mathbb{Z}[-5]$. According to the textbook, $\mathbb{Z}[\sqrt{-5}]$ is indeed a UFD, so it suffices to show that $2-\sqrt{-5}$ is irreducible. The element has a norm of 9. Assuming that this element has a nonunit divisor, the norm of the divisor must necessarily be 3.

Assume, for some $(a+b\sqrt{-5})|(2-\sqrt{-5})$:

$$N(a+b\sqrt{-5}) = 3 \quad \text{And} \quad a^2+5b^2 = 3$$

Clearly, there are no integer solutions for a, b. Hence the element is irreducible, and the principal ideal generated by it is also prime. I^2 must be prime, but then, $I|I^2$. This means, by ideal cancellation, $I = R$. (Ideal cancellation is justified for $\mathbb{Z}[\sqrt{-5}]$ is a ring of integers, and all ring of integers are dedekind domains).

We derive a contradiction by demonstrating that I^2 is proper. If $I = R$, $I^2 = R = \langle 1\rangle$. Thus, $1 \in \langle 2-\sqrt{-5}\rangle$, so the multiplicative inverse of $2-\sqrt{-5}$ must be in the ring R. Again, in the ring of quotients,

$$\frac{1}{2-\sqrt{-5}} = \frac{2+\sqrt{-5}}{9}$$

and the latter element is clearly not in the ring $\mathbb{Z}[\sqrt{-5}]$ $\lightning$

For the ideal $I' := \langle 3, 1-\sqrt{-5}\rangle$, it suffices to show that I'^2 is principal of a nonunit element. We can then repeat the argument above. The following lines of algebra concludes the proof:

$$\langle 3, 1-\sqrt{-5}\rangle^2 = \langle 9, 3-3\sqrt{-5}, -4-2\sqrt{-5}\rangle$$

$$= \langle 9, 3-3\sqrt{-5}+4+2\sqrt{-5}, -4-2\sqrt{-5}\rangle = \langle 9, 7-\sqrt{-5}, -4-2\sqrt{-5}\rangle$$

$$= \langle 9, 7-\sqrt{-5}, -4-2\sqrt{-5}-14+2\sqrt{-5}\rangle = \langle 9, 7-\sqrt{-5}, -18\rangle$$

$$= \langle 9, 7-\sqrt{-5}\rangle = \langle 9, -2-\sqrt{-5}\rangle = \langle 2+\sqrt{-5}\rangle$$

$\square$

<u>Book 5.8</u> Let $\mathfrak{p}$ and $\mathfrak{q}$ be