Combinatorics HW2

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Question 1 For each of the subsets of property a, b, count the number of four digit numbers whose digits are either 1, 2, 3, 4, 5.

a) The digits are distinct

b) The number is even

Proof. Start off with counting the digits that need not satisfy both a or b. By principle of multiplication, we count $5^4 = 625$.

Now we count the digits that satisfy only condition a. Again, by principle of multiplication, $5 \cdot 4 \cdot 3 \cdot 2 = 120$.

For the digits that only satisfy condition b, we start with choosing the last digit. If the last digit is even, the whole number is even. There are two even numbers within the set $\{1, 2, 3, 4, 5\}$. So, by principle of multiplication, we count $2 \cdot 5 \cdot 5 \cdot 5 = 250$.

Finally, for the digits that satisfy both of the conditions, we again count from the last digit. By principle of multiplication, the answer is $2 \cdot 4 \cdot 3 \cdot 2 = 48$. □

Question 2 How many orderings are there for a deck of cards if all the cards in the same suite are together?

Solution First, ignore the order of the cards but only consider the ordering of the suites. There are four possible suites, so there are $4! = 24$ ways of ordering. For each suite, there are 13 cards. The 13 cards can be arranged each in 13! ways. Thus, by principle of multiplication, there are a total of $\boxed{4!(13!)^4}$ ways of ordering. □

Question 4 How many distinct positive divisors does the each of the following numbers have: $3^4 \cdot 5^2 \cdot 7^6 \cdot 11, 620, 10^{10}$

Proposition Let n be a positive integer. By the Fundamental Theorem of Arithmetic, it is possible to write n as:

$$n = \prod_{i=0}^{k} {p_i}^{a_i}$$

where p_i's are prime and a_i's are positive integers. The number of positive divisiors of n is

$$\prod_{i=0}^{k} (a_i + 1)$$

Proof. The divisors of n must involve only the prime numbers $\{p_1, ..., p_n\}$. The power of the prime p_i can range from $a_i + 1$. Then, we proceed with the principle of multiplication to obtain the answer. $\square$

Solution In light of the proposition, the question boils down to finding the prime factorization of the three numbers, which is given as follows.

$$3^4 \cdot 5^2 \cdot 7^6 \cdot 11, \quad 2^2 \cdot 5 \cdot 31, \quad 2^{10} \cdot 5^{10}$$

The number of their divisors are $5 \cdot 3 \cdot 7 \cdot 2 = 210$, $3 \cdot 2 \cdot 2 = 12$, $11 \cdot 11 = 121$. $\square$

Question 7 In how many ways can four men and eight women be seated around a round table if there are two women between consecutive men around the table?

Solution Take any three consecutive seats of the table. We observe that there must be a men sitting in one of the three tables. Otherwise, there will be a pair of two consecutive men where there is only one women in between the men. Moreover, there can be exactly one men sitting on one of these three seats.

Designate one of the three seats to be a seat for men. Such a choice will fix the seats where the men must sit. Afterwards, we count the number of ways to permute the four men and eight women, which equals $4! \cdot 8!$.

By principle of multiplication, we deduce that the total possible seatings are $\boxed{3 \cdot 4! \cdot 8!}$.

$\square$

Question 8 How many ways can six women and six men could be seated if men and women are to sit in alternate seats?

Solution Apply the reasoning that we used for the previous problem. From two consecutive seats, choose one seat for the men, which decides all the seats where the men should sit. Multiply by the permutations for men and women. We conclude that the number of total possible sittings are $\boxed{2 \cdot (6!)^2}$. $\square$