MATH 403 Pset 1

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Problem *Section 2.1 Q2*

a) Show that the intervals in the definition of outer measure may be assumed to be closed.

b) Show that for any δ, the intervals in the definition of outer measure may be assumed to be open and of length less than δ.

c) Show that if A is contained in an interval K, then we can assume that all intervals I_i in the cover are contained in K.

part a). Recall the definition of the outer measure.

$$\lambda^*(A) \ := \ \inf\left\{ \sum_{s\in S} |s| : A \subseteq \bigcup_{s\in S} s, \ \text{S is a collection of intervals} \right\} \tag{1}$$

We define a custom outer measure where all intervals are closed.

$$\alpha(A) \ := \ \inf\left\{ \sum_{s\in S} |s| : A \subseteq \bigcup_{s\in S} s, \ \text{S is a collection of closed intervals} \right\} \tag{2}$$

Clealy, $\alpha(A)$ takes the infimum of a more restricted set compared to the regular outer measure. Hence, $\lambda^*(A) \leq \alpha(A)$. It suffices to show $\lambda^*(A) \geq \alpha(A)$ to show that the two measures are indeed equivalent.

Any fully open or half open interval has a corresponding closed interval that has the same length but includes the original interval. Consider the following.

$$(a,b) \subseteq (a,b] \subseteq [a,b] \quad \text{and} \quad (a,b) \subseteq [a,b) \subseteq [a,b] \tag{3}$$

Consider a collection of a interval S, either closed or open, that is included in the infimum for the outer measure $\lambda^*(A)$. By substituting all the intervals in S with the fully-closed interval, we obtain a collection $\widetilde{S}$ where all intervals are closed, the sum of the length is preserved, and includes A. In symbols, that is

$$A \ \subseteq \bigcup_{s\in\widetilde{S}} s \quad \text{and} \quad \sum_{s\in S} |s| \ = \ \sum_{s\in\widetilde{S}} |s|. \tag{4}$$

The existance of the collection $\widetilde{S}$ shows that $\lambda^*(A) \geq \alpha(A)$ which concludes the proof □

part b). We start with a simple observation that we can split any interval $s = (a,b)$ into a collection of intervals $\dot{s}$ that have a length strictly less than δ and $\bigcup \dot{s} = s$. Let k be the greatest integer that satisfies

$$a + k\frac{\delta}{2} \ < \ b \tag{5}$$

where k's uniqueness is guaranteed by the well-ordering theorem. Let our collection $\dot{s}$ be written as follows.

$$\dot{s} \ := \ \left\{(a, a+\frac{\delta}{2}), [a+\frac{\delta}{2}, a+\frac{\delta}{2}), \ldots, [a+k\frac{\delta}{2}, b)\right\} \tag{6}$$

Apparently $\bigcup \dot{s} = s$ and the number of collection in $\dot{s}$ is finite. Also, change the open/closedness of the first and the last interval depending on the open/closedness of the inital interval s.

Now, define the custom outer measure $\alpha(A)$ as in part a, where this time the length of each interval is restricted to be strictly less than δ. Again, it suffices to show that $\alpha(A) \leq \lambda^*(A)$. For each collection S included in the infimum computation of $\lambda^*(A)$, we define a new collection

$$\dot{S} \ := \ \{\dot{s} : s \in S\} \tag{7}$$

$\dot{S}$ sees witness to $\alpha(A) \leq \lambda^*(A)$ □

part c). We repeat the procedure in the previous parts. Let the custom measure $\alpha(A)$ to restrict the collections such that the union of the collections are contained in the interval K. For collection S associated with $\lambda^*(A)$, we define

$$\dot{S} \ := \ \{s \cap K : s \in S\} \tag{8}$$

Since $A \subseteq K$ by assumption, $A \subseteq \bigcup \dot{S} = K \cap \bigcup S$ so $\alpha(A) \geq \lambda^*(A)$ which concludes the proof. □

<u>Problem</u> *Section 2.1 Q7*

Show that the union of countably many null sets is a null set.

Proof. For simplicity, we call S the collection of pairwise disjoint intervals that cover the set A [1] as **coverings** and denote its length $|S|$ as the sum of the length of its component intervals.

A null set is a set with an outer measure zero. This means, that for any $\epsilon > 0$, there exists some covering of the null set with a length shorter than ϵ.

Now, we move on to the proof of the proposition. Let $\{A_n\}_{n\in\mathbb{Z}_{pos}}$ be a collection of null sets. For each A_n, it is possible to obtain a covering of the set, namely S_n such that $|S_n| < \frac{\epsilon}{2^n}$. From the collection of coverings $\{S_n\}_{n\in\mathbb{Z}_{pos}}$, we take the union $\bigcup_{n\in\mathbb{Z}_{pos}} \bigcup_{s\in S_n} s$ and disassemble the resulting subset of $\mathbb{R}$ into disjoint intervals. Call this new covering S_ϵ. For example, if

$$S_1 \ = \ \{(1, 1.5), (4, 6)\}, S_2 \ = \ \{(.5, 2)\} \tag{9}$$

then the total union of the coverings are

$$\bigcup_{n\in\mathbb{Z}_{pos}} \bigcup_{s\in S_n} s \ = \ (1, 2) \cup (4, 6) \tag{10}$$

[1] i.e. $A \subseteq \bigcup S$

and we obtain

$$S_\epsilon \;=\; \{(1,2),(4,6)\} \tag{11}$$

Clearly, the length of the covering S_ϵ is less than ϵ.

$$|S_\epsilon| \;\le\; |S_1| + |S_2| + |S_3| + \cdots \;\le\; \epsilon\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\cdots\right) \;=\; \epsilon \tag{12}$$

The union of countable number of countable sets are countable, so there must be a contable number of intervals in the collection S_ϵ $\square$