Rallying Soldiers in a Conway Game

Conway group

So previously, we have studied on how to send a single soldier to the highest level possible, given a board multiplicity m. In this paper, we focus on a similar but different problem: given a board multiplicty m, what is the maximum amount of soldiers we can rally on a single grid?

Problem 1 *The M-level problem*

Assume we have a two dimensional checkerboard that extends infinitely both in the horizontal and the vertical direction. A borderline is drawn, and we are provided with $m$soldiers on each grid of the checkerboard under the reference line. What is the highest level M which can be reached by the rules of Conway Checkers?

Problem 2 *The Rallying Problem*

Consider the same board from Problem 1. Select any grid that is adjacent and below the borderline. What is the maximum number of soldiers we can focus on the target grid?

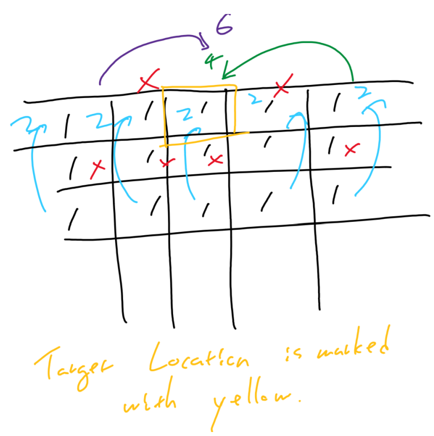


Figure 1. The rallying problem for m = 1

It is easier to study this problem on a one dimensional strip. From the 2D board, focus on one vertical strip and ignore the rest of the board. Assuming $x = \psi - 1$ [1], we compute that the monovariant of the single strip is

$$\frac{m}{1-x} = \frac{m}{2-\psi} = (1+\psi)m$$

Where the last equality follows from the formula $\psi^2 + \psi - 1 = 0$.

Assume, after a series of manipulations, we end up with r soldiers in the target square. The score of the board will be r. The monovariant is invariant

[1] ψ is the golden ratio $\frac{\sqrt{5}-1}{2}$

under a Conway move. Hence, we obtain the following inequality.

$$r < (1 + \psi)m$$

[2]

We have shown the following.

Proposition 1 The theoretical upper bound of r in a 1D strip is

$$\lceil(1 + \psi)\rceil - 1$$

.

In a 1D strip, it is possible to come up with an optimal strategy that asymptotically approaches this limit. For utility, define the following.

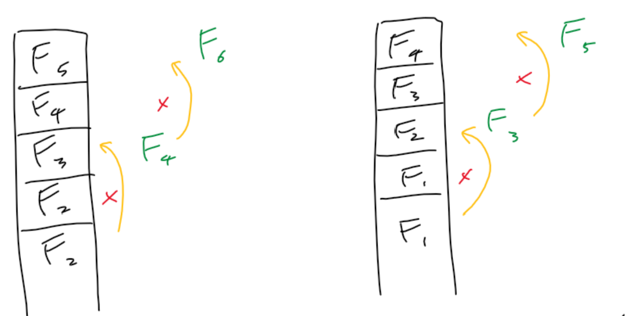
Definiton *Fibonacci inversion function*
Let

$$\mathcal{F}^{-1}(N) = n$$

where n is the index of the largest fibonacci number less than or equal to N. That is, $F_n \leq N < F_{n+1}$.

Proposition 2 *The heavy-headed stick* Suppose $n = \mathcal{F}(m)$. In a 1D strip of the Conway game with multiplicity m, it is possible to rally F_{n+1} Soldiers to one target grid.

Proof. Here is a visual demonstration.

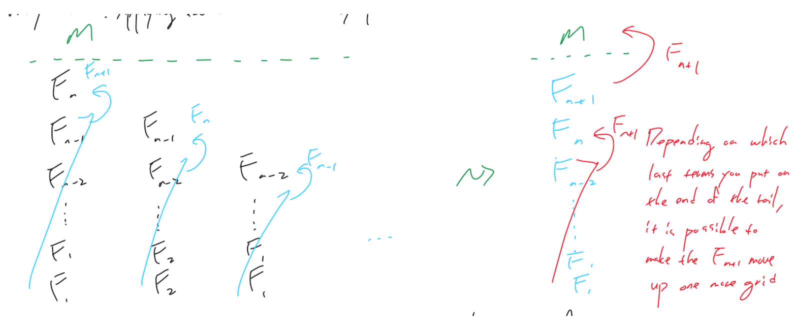


The heavy head arrangement. Given a strip of multiplicity F_n, it is possible to rally F_{n+1} soldiers to the target/top square.

Proposition 3 *Multiple Heavy Heads*
Suppose $n - 1 = \mathcal{F}^{-1}(m/2)$. Consider a 1D strip of the Conway game with multiplicity m. Suppose that the strip has a fixed head(or the top grid) and extends infintely below. It is possible to rally $m + F_{n+1}$ Soldiers to the head/top grid.

[2]equality holds assuming we allow infinite number of moves

Proof. Here is a visual demonstration.



The construction of n follows from the fact that the superposed board needs the following multiplicity m.

$$m = \max_{0 \leq j < N} \{(j+1)F_{N-j}\}$$

$\square$

Corollary We can approximate the number of added soldiers, F_{n+1}.

$$F_{n+1} \approx \varphi^{n+1} \approx \varphi^{\log_\varphi(m/2)+2} = \frac{m\varphi^2}{2}$$

$\square$