# Part I Corrections

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ii) Let $\zeta$ be any real number and $a > 0$. Evaluate:

$$I(\zeta) := \int_{-\infty}^{\infty} \frac{e^{-2\pi\zeta x}}{x^2 + a^2} dx$$

**Solution**

If $\zeta < 0$, then apply the substitution $x \to -x$.

The integral converts to:

$$I(\zeta) = \int_{\infty}^{-\infty} \frac{e^{-2\pi(-\zeta)x}}{(-x)^2 + a^2}(-dx) = \int_{-\infty}^{\infty} \frac{e^{-2\pi(-\zeta)x}}{x^2 + a^2} dx = I(-\zeta)$$

So the kernel is an even function with respect to $\zeta$.

Define a holomorphic function $f(z)$ as follows:

$$f(z) = \frac{e^{-2\pi\zeta z}}{z^2 - a^2}$$

The numerator and the denominator are known to be holomorphic. Thus the function is holomorphic everywhere other than the poles which are located at $z = \pm a$. Draw a semicircular contour centered at the origin that occupies quadrant I and IV. Call this contour $\gamma$, and denote the radius as $R$.

Take the contour integral of $f(z)$ over $\gamma$. Let the straight segment of the contour be called $S$, and the circular region $C$.

We claim that the integral over the circular region vanishes. That is, a as $R \to \infty$, $\oint_C f = 0$

Notice:

$$\left| \oint_C f \right| = \left| \int_{z \in C} \frac{e^{-2\pi\zeta z}}{z^2 + a^2} dz \right| \leq \int_{z \in C} \frac{max|e^{-2\pi\zeta z}|}{R^2 - a^2} dz$$

Note that the modulus of an exponent is the exponent of the modulus of the argument. That is:

$$|e^{-z}| = e^{Re(-2\pi\zeta z)}$$

And for $z \in C$, the quality is bounded under 1. Thus:

$$\left| \oint_C f \right| \leq \frac{2\pi R}{R^2 - a^2}$$

And the upper bound converges to zero as $R$ approaches infinity. This shows that the circular region converges to zero. ✓

By the residue theorem:

$$\oint_C f + \oint_S f = 2\pi i Res_f(a)$$

The first summand of the LHS vanishes. The second summand can be computed with some algebra. We write:

$$\oint_S f = \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix} \cdot (-i)dx}{(xi)^2 - a^2} = i \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix}dx}{x^2 + a^2} dx = iI$$

The residue can be computed with ease:

$$Res_f(a) = \lim_{z \to a} \frac{e^{-2\pi\zeta z}(z-a)}{z^2 - a^2} = \lim_{z \to a} \frac{e^{-2\pi\zeta z}}{z+a} = \frac{e^{-2\pi\zeta a}}{2a}$$

Combining the results, we write, for $\zeta > 0$:

$$iI(\zeta) = 2\pi i \frac{e^{-2\pi\zeta a}}{2a} \qquad \text{or} \qquad I(\zeta) = \frac{\pi e^{-2\pi\zeta a}}{a}$$

Finally, for any real value $\zeta$, we conclude:

$$\boxed{I(\zeta) = \frac{\pi e^{-2\pi|\zeta|a}}{a}}$$

**Q3** Consider the following infinite products:

$$I_1(a) := \prod_{n=1}^{\infty}(1+a_n) \quad \text{and} \quad I_2(b) := \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}(1+b_{mn})$$

b) State the definition for the convergence of the infinite product $I_2(b)$.

**Definiton** Define a double partial product $S_{ab}$ as follows:

$$S_{ab} = \prod_{m=1}^{a}\prod_{n=1}^{b}(1+b_{mn})$$

We claim that if the limit of the partial product converges regardless of the relative growth of a, b, then the infinite product converges. That is:

$$\lim_{a,b\to\infty} S_{ab} = L$$

for some $L \in \mathbb{R}$. $\square$