PHYS 202 HW6

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Q1 Matching initial conditions for the wave equation

We know that the 1D wave equation

$$y'' = v_p^2 \ddot{y}$$

is satisfied by the solution

$$y(x,t) = f(x - v_p t) + g(x + v_p t)$$

for arbitrary functions f, g. Note that f is the forward moving wave and g is the backward moving wave.

A wave is known to have an initial displacement

$$y(x,0) = \frac{.001}{1+x^2}$$

Compute the solution for the wave moving in $10m/s$ given that:

a) the wave is moving forwards

b) the wave is moving backwrads

c) the wave is traveling in opposite directions

Solution The case a, b can be dispatched with ease. We know that $g = 0$ and $f = 0$ for case a, b, respectively. Thus, for a,

$$f(x - v_p t)\bigg|_{t=0} = \frac{.001}{1+x^2} \qquad \text{or} \qquad f(x) = \frac{.001}{1+x^2}$$

. Likewise, for b,

$$g(x + v_p t)\bigg|_{t=0} = \frac{.001}{1+x^2} \qquad \text{or} \qquad g(x) = \frac{.001}{1+x^2}$$

. Now we write the solution for the wave equation.

$$y_a(x,t) = f(x - v_p t) = \frac{.001}{1 + (x - 10m/s \cdot t)^2}$$

$$y_b(x,t) = g(x + v_p t) = \frac{.001}{1 + (x + 10m/s \cdot t)^2}$$

Case 3 can be resolved as considering the wave as a superposition of the two waves y_a, y_b with half amplitudes. In symbols,

$$y_c = \frac{1}{2}(y_a + y_b) = .0005\left(\frac{1}{1 + (x - 10m/s \cdot t)^2} + \frac{1}{1 + (x + 10m/s \cdot t)^2}\right)$$

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ya[x_, t_] := .001/(1+ (x - 10 t)^2);
yb[x_, t_] := .001/(1+ (x + 10 t)^2);
yc[x_, t_] := ( ya[x, t] + yb[x, t])/2

Animate[
 Plot[yc[x, t], {x, -10, 10}, PlotRange → {0, .001}],
 {t, 0, 1},
 AnimationRunning → False
]
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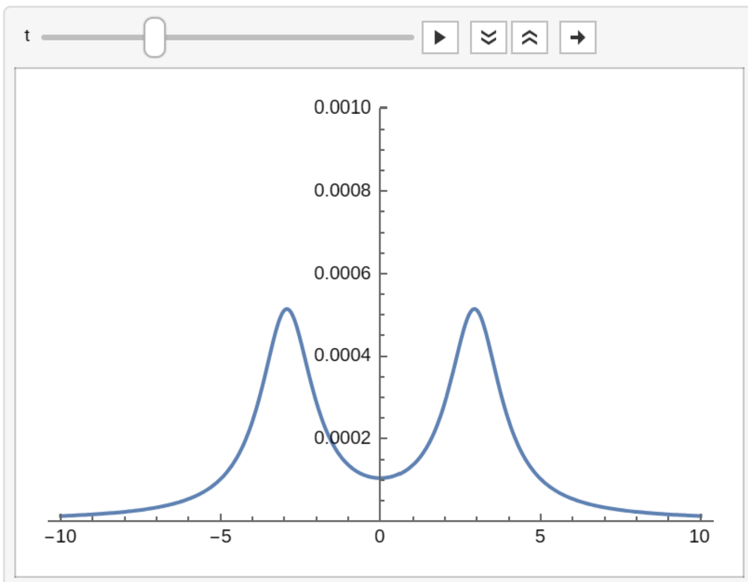


Q2 Schrodinger's Equation

Recall the general Schrodinger's Equation.

$$i\hbar|\dot{\psi}\rangle = \hat{H}|\psi\rangle$$

The Hamiltonian of the 1D particle can be written as the sum of the kinetic and potential energy.

$$\hat{H} = \hat{P} + \hat{V} = \frac{\hat{p}^2}{2m} + \hat{V}$$

The operator $\hat{p}$ denotes the momentum of the particle.

$$\hat{p} = -i\hbar\nabla$$

For a free particle, there is no potential energy. Rewrite the Hamiltonian.

$$\hat{H}_{free} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\Delta$$

where Δ denotes the Laplacian.

Now, consider a free particle in 1 dimension. The equation simplifies to:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

a) Can $\psi(x,t)$ be in the form of $e^{i(kx-\omega t)}$? If so, what is the dispersion relation?

For the complex exponential form of ψ, the time derivative acts as multiplying $-i\omega$ and the space derivative ik. Hence, the 1D Schrodinger equation can be rewritten as

$$i\hbar(-i\omega) = -\frac{\hbar^2}{2m}(ik)^2 \quad \text{or} \quad \boxed{\frac{\omega}{k^2} = \frac{\hbar}{2m}}$$

We also take note of the phase velocity

$$v_p = \frac{\omega}{k} = \frac{\hbar k}{2m}$$

b) Show that for an arbitrary function f, the Schrodinger's equation cannot be satisfied by $\psi(x,t) = f(x-vt)$ for any velocity v.

Plug the candidate solution into the Schrodinger's equation. We get

$$i\hbar(-v)f'(x-vt) = -\frac{\hbar^2}{2m}f''(x-vt)$$

This implies that the ratio f''/f' is a constant. This is not true in general. Hence, $f(x-vt)$ is not a solution.

c) Let $\tilde{\psi}(x,t)$ be a solution to TDSE. Is the realified version, $\Re\tilde{\psi} := \psi$ always a solution for TDSE?

Unfortunately, no. Let $\tilde{\psi} := Ae^{ikx-\omega t}$ be a solution to the TDSE. We take the real part and plug it into the TDSE.

$$\psi = \Re(Ae^{ikx-\omega t}) = A\cos(kx-\omega t)$$

Try ψ as a candidate solution for TDSE.

$$i\hbar\frac{\partial(A\cos(kx-\omega t))}{\partial t} = -\frac{\hbar^2}{2m}A\cos(kx-\omega t)''$$

The LHS of the equation is imaginary while the RHS is real. Thus, both sides must identically equal zero, implying $A = 0$. This is not true for a general $\tilde{\psi}$. $\lightning$

d) In quantum mechanics, energy and momentum can be interpreted as a multiple of frequency and wavenumber. We know that the energy of a photon is determined by $E = hf$. Thus

$$E = \hbar\omega$$

By the DeBroglie equation, we have $p = h/\lambda$. $k = 2\pi/\lambda$ so

$$p = \hbar$$

Recall

$$\frac{\omega}{k^2} = \frac{\hbar}{2m}$$

. Replace ω, k with energy and momentum.

$$\frac{E}{p^2} = \frac{1}{2m} \quad \text{or} \quad \boxed{E = \frac{p^2}{2m}}$$

e) The Klein-Gordon Equation reads

$$-\ddot{\phi} + c^2\phi'' = \left(\frac{mc^2}{\hbar}\right)^2 \phi$$

Let $\phi = Ae^{i(kx-\omega t)}$. Again, the time derivative operator and the space derivative operator corresponds to $-i\omega$, ik respectively. Thus, the equation simplifies to

$$-(i\omega)^2\phi + c^2(ik)^2\phi = \left(\frac{mc^2}{\hbar}\right)^2 \phi$$

. Divide by ϕ and simplify to obtain the dispersion relation.

$$\omega^2 - (kc)^2 = \left(\frac{mc^2}{\hbar}\right)^2$$

Now utilize $E = \hbar\omega$ and $p = \hbar k$. Multiply both sides by $\hbar^2$

$$E^2 = (pc)^2 + (mc^2)^2$$

We have obtained the equation for relativistic energy.

f) Consider the 1D diffusion equation,

$$\dot{\phi} = \gamma^2\phi''$$

Does $\phi(x,t) = Ae^{i(kx-\omega t)}$ satisfy this equation if k, ω are both real? What if ω is not necessarily real? How does the temperature distribution change over time?

Again, using differential operators, we rewrite the equation as

$$(-i\omega)\phi = \gamma^2(ik)^2 \qquad \text{or} \qquad \omega = -i\gamma^2 k^2$$

If $k \in \mathbb{R}$, necessarily ω must be purely imaginary. So assume this to be the case. Rewrite the temperature distribution $\tilde{\phi}$.

$$\tilde{\phi} = Ae^{ikx}e^{-i(-i\gamma^2k^2t)} = Ae^{-\gamma^2k^2t}e^{ikx}$$

Take the real part to retrieve ϕ

$$\phi = Ae^{-\gamma^2k^2t}\cos(kx)$$

At any time, the temperature is distributed sinusoidially along the x-axis. As time goes on, the amplitude of the distribution decreases exponentially. Eventually, the temperature is distributed equally along the whole plane.

Q3 Multiple interferences

An infinite string has a wave number of k_1. A short segment of length L has a wave number of k_2. We distinguish the string into three regions, region I, II, III. A wave travels through this string from region I. Some reflection/transmission occurs at two boundaries. Let A_1, A_2, A_3 be the three waves traveling in the forward direction for each region. Likewise, let B_1, B_2 be the backward traveling wave.

a) Compute $|A_{III}|/|A_I|$ and $|B_I|/|A_I|$ given $k_2L = 2\pi$.

b) Compute the two qualities when $k_2L = \pi$

Solution The wave must be continuous and smooth at the boundaries, i.e. $x = 0, L$. Thus, we write

$$\begin{aligned} A_1(0,t) + B_1(0,t) &= A_2(0,t) + B_2(0,t) \\ (A_1(0,t) + B_1(0,t))' &= (A_2(0,t) + B_2(0,t))' \\ A_2(L,t) + B_2(L,t) &= A_3(L,t) \\ (A_2(L,t) + B_2(L,t))' &= (A_3(L,t))' \end{aligned}$$

We also know that each complexified A, B must be in the follwing form.

$$\tilde{A}_1(x,t) = a_1e^{i(\omega t - k_1x)} \qquad \text{and} \qquad \tilde{A}_2(x,t) = a_2e^{i(\omega t - k_2x)} \qquad \text{and} \qquad \tilde{A}_3(x,t) = a_3e^{i(\omega t - k_1x)}$$

$$\tilde{B}_1(x,t) = b_1e^{i(\omega t + k_1x)} \qquad \text{and} \qquad \tilde{B}_2(x,t) = b_2e^{i(\omega t + k_2x)}$$

Note that it is convinient to use $\omega t - kx$ in the exponent, for we wish to extract the $e^{i\omega t}$ for any time t!

Realifying the forms and plugging in yields the following four formulas.

$$\begin{aligned} a_1e^{i\omega t} + b_1e^{i\omega t} &= a_2e^{i\omega t} + b_2e^{i\omega t} \\ -k_1a_1e^{i\omega t} + k_1b_1e^{i\omega t} &= -k_2a_2e^{i\omega t} + k_2b_2e^{i\omega t} \\ a_2e^{i(\omega t - k_2L)} + b_2e^{i(\omega t + k_2L)} &= a_3e^{i(\omega t - k_1L)} \\ -k_2a_2e^{i(\omega t - k_2L)} + k_2b_2e^{i(\omega t + k_2L)} &= -k_1a_3e^{i(\omega t - k_1x)} \end{aligned} \tag{1}$$

Let $k_2L = 2\pi$. Equations in (1) reduce to

$$\begin{aligned} a_1e^{i\omega t} + b_1e^{i\omega t} &= a_2e^{i\omega t} + b_2e^{i\omega t} \\ -k_1a_1e^{i\omega t} + k_1b_1e^{i\omega t} &= -k_2a_2e^{i\omega t} + k_2b_2e^{i\omega t} \\ a_2e^{i(\omega t)} + b_2e^{i(\omega t)} &= a_3e^{i(\omega t - k_1L)} \\ -k_2a_2e^{i\omega t} + k_2b_2e^{i\omega t} &= -k_1a_3e^{i(\omega t - k_1x)} \end{aligned} \tag{2}$$

Let $t = 0$. Comparing the equations in pairs as of the 1st 3rd, and 2nd 4th, we deduce

$$a_1 + b_1 = a_3e^{ik_1L}$$

$$-a_1 + b_1 = -a_3e^{ik_1L}$$

Thus, $b_1 = 0$ and $|a_3|/|a_a| = |e^{ik_1L}| = 1$

Let $k_2L = \pi$. Equations in (1) reduce to

$$\begin{aligned} a_1e^{i\omega t} + b_1e^{i\omega t} &= a_2e^{i\omega t} + b_2e^{i\omega t} \\ -k_1a_1e^{i\omega t} + k_1b_1e^{i\omega t} &= -k_2a_2e^{i\omega t} + k_2b_2e^{i\omega t} \\ a_2e^{i(\omega t)} + b_2e^{i(\omega t)} &= -a_3e^{i(\omega t - k_1L)} \\ -k_2a_2e^{i\omega t} + k_2b_2e^{i\omega t} &= k_1a_3e^{i(\omega t - k_1x)} \end{aligned} \tag{3}$$

Again, compare the equation in pairs as we did for the previous case.

$$a_1 + b_1 = -a_3\cos(k_1L)$$

$$-a_1 + b_1 = a_3\cos(k_1L)$$

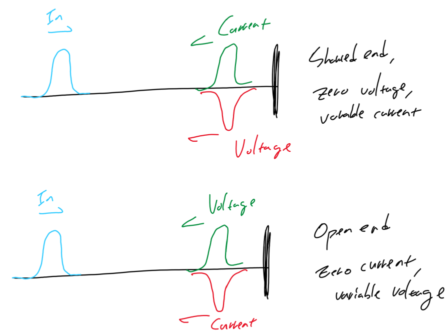
Thus, $b_1 = 0$ and $|a_3/a_1| = 1$

To sum up, the wave is amplified by a factor of $\sec(k_1L) > 1$ after passing the middle segment. If $k_2L = 2\pi$, the phase of the wave is preserved. If $k_2L = \pi$, the phase of the wave is shifted by π. In both cases, there is no reflection in the original string. Regardless of k_2L,

$$\boxed{|B_I|/|A_I| = 0 \quad \text{and} \quad |A_{III}|/|A_I| = 1}$$

Q4 BNC line

a) Describe the reflected pulse on a BNC cable where one of the ends are open or shorted.



When the end is shorted, the voltage of the end must be zero, but the current is allowed to change. The current is preserved and the voltage is reversed. When the end is open, the voltage difference can change but the current between the two wires are fixed to be zero. Hence the voltage is preserved and the current is reversed.

b) Use the notion of Impedence to explain your results above

The voltage wave of the shorted end or the current wave of the open end can be considered as a string connected to another string with infinite impedence.

$$Z_2 = \infty$$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -1$$

So the wave is reversed.

The current wave of the shorted end or the voltage wave of the open end can be considered as a string connected to another string with zero impedence.

$$Z_2 = 0$$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = 1$$

So the amplitude is preserved.

If we fix $Z_open = \infty$ and $Z_short = 0$ for both current and voltage, we see from the results above that the reflection formula works only for the current wave but not the voltage wave.

Remark Image waves To remember fixed and free ends, consider image waves traveling on the other side of the string.

c) Use the differential equations that relate the voltage and the current wave to find an impedence value that works for the voltage current.

From our results above, for the current wave,

$$R_I = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

We also know that the following differential equation holds by charge conservation.

$$-\frac{\partial I}{\partial x} = C_0 \frac{\partial V}{\partial t}$$

The incident and reflect current wave is known to us.

$$\tilde{I}_i = I_0 e^{i(kx-\omega t)} \quad \text{and} \quad \tilde{I}_r = R_I I_0 e^{i(kx+\omega t)}$$

Likewise, we can express the incident and reflect voltage wave.

$$\tilde{V}_i = V_0 e^{i(kx-\omega t)} \quad \text{and} \quad \tilde{V}_r = R_V V_0 e^{i(kx+\omega t)}$$

Relate I_i, V_i and I_r, V_r using the differential equation.

$$-kI_0 = -\omega C_0 V_0 \quad \text{and} \quad -kR_I I_0 = \omega C_0 R_V V_0$$

Divide RHS by LHS and multiply by -1.

$$\boxed{R_V = -R_I = -\frac{Z_1 - Z_2}{Z_1 + Z_2}}$$

Q5 Dispersion relation and LC transmission line

Consider a LC transmission line comprised of capacitors connected in parallel. Each of the capacitors are connected with an inductor. Let the inductance and the capacitance be L and C. Remember the mechanical analog of the circuit.

$$\begin{aligned} q &\mapsto x \\ L &\mapsto m \\ C &\mapsto 1/k \\ \text{k string constant} \end{aligned}$$

We can compute the angular frequency of the circuit using the dispersion relation. Beware that k is the wavenumber, not the spring constant.

$$\omega = 2\omega_0 \sin(ka/2)$$

The natural frequency of the circuit is

$$\omega_0 = \sqrt{\frac{k}{m}} \mapsto \sqrt{\frac{1}{LC}}$$

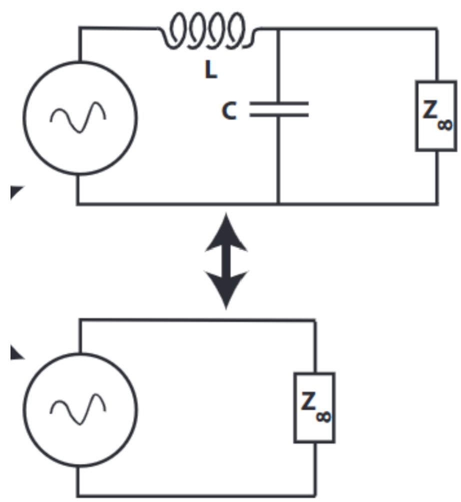
At continum, $a \to 0$. We approximate the sin as an identity function.

$$\omega \approxeq 2\omega_0 \frac{ka}{2} = \sqrt{\frac{1}{LC}} ak = \sqrt{\frac{a^2}{LC}} k = \sqrt{\frac{1}{L_0 C_0}} k$$

The phase velocity is

$$\boxed{v_p = \omega/k = \sqrt{\frac{1}{L_0 C_0}}}$$

Q6 Alternate derivattion of the impedence of BNC cable



Let the impedence of the upper and the lower circuits be Z_1 and Z_2 respectively. We know that $Z_1 = Z_2$. Also,

$$Z_1 = i\omega L + \frac{1}{i\omega C + \frac{1}{Z_\infty}} \quad \text{and} \quad Z_2 = Z_\infty$$

The equality of the two impedences imply

$$i\omega L + \frac{1}{i\omega C + \frac{1}{Z_\infty}} = Z_\infty$$

And with some algebra, we deduce

$$Z_\infty^2 - i\omega L Z_\infty - \frac{L}{C} = 0$$

Apply the quadratic formula to compute Z_∞.

$$\boxed{Z_\infty = \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \left(\frac{\omega L}{2}\right)^2}}$$

Now, we wish to take the continum limit of this impedence to find the impedence of the BNC cable. Let L_0, C_0 be the impedence and capacitence per unit length of the cable. Consider a 1m segment of the cable. The total inductance and the capacitance is

$$L_{tot} = (1m)L_0 \quad \text{and} \quad C_{tot} = (1m)C_0$$

The continum case can be thought as disecting this 1m cable into n segments with inductance and capacitance L_{tot}/n and C_{tot}/n. Plug this into our result from (a).

$$Z_\infty = \frac{i\omega L_{tot}}{2n} + \sqrt{\frac{L_{tot}/n}{C_{tot}/n} - \left(\frac{\omega L_{tot}}{2n}\right)^2}$$

Send $n \to \infty$. Observe that $L_{tot}/C_{tot} = L_0/C_0$.

$$\boxed{Z_\infty = \sqrt{\frac{L_0}{C_0}}}$$

Q7 Malus' Law

a) A light passes through three polarizers. The pass axis of the polarizers are arranged 0, 45, 90 degrees with respect to the electric field of the light. Compute the intensity of the light after it passes the polarizers.

The first polarizer does not affect the light. Apply Malus's Law twice.

$$I'/I = (\cos(\pi/4))^2 = \boxed{1/4}$$

b) Consider $N+1$ polarizers that rotate π/N each.

Let I_n be the intensity of the light after passing the nth polarizer. By Malus' Law,

$$I_n = (\cos(\pi/2N)^2)I_{n-1}$$

The first polarizer does not affect intensity. After passing N polarizers, the intensity is multiplied by the cos square term N times.

$$\boxed{I_{N+1}/I_0 = \cos(\pi/2N)^{2N}}$$

c)

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(* Problem 6 *)
R[N_] := Cos[π / (2 (N - 1))]^(2 (N - 1));
R[3]
N[R[4]]
N[R[31]]
N[R[9999]]
```

```
1
─
4

0.421875

0.92101

0.999753
```

d) Approximate the intensity at the limit $N \to \infty$

$$
\begin{aligned}
I_{N+1}/I_0 &= \cos(\pi/2N)^{2N} = (1 - \sin(\pi/2N)^2)^{2N} \\
&\approxeq (1 - (\pi/2N)^2)^{2N} \approxeq 1 + (\pi/2N)^2 \cdot 2N \approxeq 1
\end{aligned}
$$

So the intensity does not change.