1. Define:

$$f_n(x) := \frac{n}{1+nx^2}$$

Prove that f_n is uniformly continuous.
Is the family of function $\mathcal{F} := \{f_n(x)|n \in \mathbb{Z}_{pos}\}$ equicontinuous?

Proposition $\frac{x}{1+nx^2}$ is bounded.

Proof First, consider the function in the range $x \in \mathbb{R} \setminus [-1, 1]$. Within the range, we have $|x| > 1$ and hence $x^2 > 1$. Write:

$$|\frac{x}{1+nx^2}| < |\frac{x}{nx^2}| = |\frac{1}{nx}| < 1$$

Now, consider the range $x \in [-1, 1]$. Write:

$$|\frac{x}{1+nx^2}| < |\frac{x}{1}| < 1$$

This shows that our function is bounded for $x \in \mathbb{R}$ $\square$

Claim $f_n(x)$ is uniformly continuous. Given any $\epsilon > 0$, we wish to obtain a δ_{max} where for any δ such that $|\delta| < \delta_{max}$ satisfies:

$$|f_n(x) - f_n(x+\delta)| < \epsilon$$

Or equivalently

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta)^2}| < \epsilon$$

Notice:

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta)^2}| < |\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta_{max})^2}|$$

It suffices to construct δ_{max} that satisfies:

$$|\frac{n}{1+nx^2} - \frac{n}{1+n(x+\delta_{max})^2}| < \epsilon$$

Through some algebra:

$$|\frac{n\left[1+n(x+\delta_{max})^2 - (1+nx^2)\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

$$|\frac{n\left[2nx\delta_{max} + n\delta_{max}^2\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

$$|\frac{n^2\delta_{max}\left[2x+\delta_{max}\right]}{(1+nx^2)(1+n(x+\delta_{max})^2)}| < \epsilon$$

Set $\delta_{max} < 1$. Also, notice that the terms in the denominators are both greater than 1. We construct a δ_{max} that satisfies a stronger condition:

$$|\frac{n^2\delta_{max}(2x+1)}{1+nx^2}| < \epsilon \quad \text{or} \quad |\delta_{max}||n^2||\frac{2x}{1+nx^2} + \frac{1}{1+nx^2}| < \epsilon$$

It is easy to see that the function $\frac{1}{1+nx^2}$ is bounded. The denominator is always greater than 1, so the function is bounded by 1. We have shown that the second summand is bounded for any real x. Again, we construct a stronger δ_{max} that satisfies:

$$|\delta_{max}|B < \epsilon$$

Where B is the maximum bound of the other terms. If $B < 0$, the statement is a tauology. Otherwise, set $\delta = \epsilon/(2B)$. This concludes the proof. $\square$

Claim The family $\mathcal{F}$ is not equicontinuous

Proof We claim that equicontinuity is violated at $x = 0$. Notice that $f_n(0) = n$. Assume for a contradiction, that $\mathcal{F}$ is equicontinuous at $x = 0$. For $\epsilon = 1$, it must be possible to obtain a δ_{max} where for all δ such that $|\delta| < \delta_{max}$, δ satisfies:

$$|f_n(0) - f_n(\delta)| < 1 \quad \text{or} \quad |n - f_n(\delta)| < 1$$

So

$$|f_n(\delta)| > n - 1$$

$\delta \neq 0$ by assumption, so as $n \to \infty$, $|f_n(\delta)| \to 1$. This is a contradiction. $\square$

2. Define:

$$f_n(x) := \begin{cases} 1 - nx & (x \in (0, 1/n)) \\ 0 & (\text{Otherwise}) \end{cases}$$

Show that the limit of this function exists and find the limit.

Solution Consider the function:

$$f(x) := \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases}$$

We claim that f_n converges to f pointwise. We must show that for any $x_0 \in \mathbb{R}$, the following equality holds:

$$\lim_{n \to \infty} f_n(x_0) = f(x_0)$$

If $x_0 \leq 0$, the problem becomes trivial. If the inequality is strict, $f_n(x_0) = 0$ regardless of the value of n. Also computing the value of $f_n(0)$, we notice that the value is identically 1, regardless of the value of n.

It remains to demonstrate the equality for $x_0 > 0$. Recall that $\lim_{n \to \infty} 1/n = 0$. Hence, it is possible to obtain a sufficiently large integer N such that for any $n > N$, we have $1/n < x_0$. By the construction of $f_n(x)$, $f_n(x_0) = 0$ for any $n > N$. This concludes the proof. $\square$

3. Define:

$$f_n(x) := \begin{cases} 1 - nx & (x \in (0, 1/n)) \\ 0 & (\text{Otherwise}) \end{cases}$$

and,

$$\mathcal{F} := \{f_n(x) | n \in \mathbb{Z}_{pos}\}$$

Is the family $\mathcal{F}$ normal?

Claim No, $\mathcal{F}$ is not normal.

Proof Assume for a contradiction, that indeed the family is normal. Then, the entire family $\mathcal{F}$ must have some subsequence of functions that converge uniformly. Let the sequence of functions $\{f_{m_1}, f_{m_2}, f_{m_3}, \dots\}$ be such a sequence of functions.

For the value $\epsilon = 1/4$, we extract some integer N such that for any $n > N$, the function achieves:

$$|f_{m_n}(x) - f(x)| < 1/4$$

For any real value x. $f(x)$ is some imaginary function that the subsequence uniformly converges to. Extract another arbitrary integer $k > N$ that satisfies the same condition. Adding the two inequalities, we obtain:

$$|f_{m_n}(x) - f(x)| + |f_{m_k}(x) - f(x)| < 1/2$$

Which implies, by the triangle inequality:

$$|f_{m_n}(x) - f_{m_k}(x)| < 1/2$$

And this is for any values of $n, k > N$. We explicitly construct a value x_0 that violates this inequality.

Take any integer n greater than N. Obtain k such that $m_k > 2m_n$. This is possible because m is a strictly increasing sequence of integers. Set $x_0 = 1/m_k$. Write:

$$|f_{m_n}(x_0) - f_{m_k}(x_0)| = |f_{m_n}(1/m_k) - f_{m_k}(1/m_k)|$$

Notice that the latter summand vanishes. Also the fraction $1/m_k$ is between zero and $1/m_n$. We proceed to write:

$$|f_{m_n}(x_0) - f_{m_k}(x_0)| = |1 - m_n/m_k| > 1/2$$

by construction. But then again, this whole absolute value must be less than 1/2, which is a contradiction. □

4. Solve the integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx$$

Solution Define the integand as a complex function. That is:

$$f(z) := \frac{z^2}{z^4 + z^2 + 1}$$

We look at the semicircular contour with radius R that is centered at the origin. The contour occupies the first and the second quadrant. Call the contour γ.

The circular part of the contour vanishes as $R \to \infty$. Write:

$$\left| \oint_{\gamma_c} f \right| = \left| \int_{\theta=0}^{\pi} \frac{R^2 e^{2i\theta}}{R^4 e^{4i\theta} + R^2 e^{2i\theta} + 1} Rie^{i\theta} d\theta \right| < 2\pi/R$$

and as $R \to \infty$, clearly the integral converges to zero.

The problem reduces down to identifying the poles of the function f. We must identify all the zeros of the denominator. Notice:

$$(x^2 - 1)(x^4 + x^2 + 1) = (x^6 - 1)$$

Ergo, the zeros of the denominators are the four complex roots of $z^6 - 1$. The two real roots $z = \pm 1$ can be excluded by computing $f(1) = 3, f(-1) = 3$. In the contour γ, the two poles are:

$$z_0 = e^{i\pi/3} \quad \text{and} \quad z_1 = e^{2i\pi/3}$$

Both poles are of order 1. To compute the residue, apply L'Hopital's rule. For any of the poles $p \in z_0, z_1$,

$$Res_f(p) = \lim_{z \to p} \frac{(z-p)z^2}{z^4 + z^2 + 1}$$

By taking derivatives in both the numerator and the denominator:

$$\lim_{z \to p} \frac{(z-p)z^2}{z^4 + z^2 + 1} = \lim_{z \to p} \frac{3z^2 - 2zp}{4z^3 + 2z} = \frac{3p - 2p}{4p^2 + 2} = \frac{1}{4p + 2p^{-1}}$$

Plugging in the appropriate values of p, we write:

$$Res_f(e^{i\pi/3}) = \frac{1}{4e^{i\pi/3} + 2e^{-i\pi/3}} = \frac{1}{2 + 2\sqrt{3}i + 1 - \sqrt{3}i} = \frac{1}{3 + \sqrt{3}i} = \frac{3 - \sqrt{3}i}{12}$$

$$Res_f(e^{2i\pi/3}) = \frac{1}{4e^{2i\pi/3} + 2e^{-2i\pi/3}} = \frac{1}{-2 + 2\sqrt{3}i - 1 - \sqrt{3}i} = \frac{1}{-3 + \sqrt{3}i} = \frac{-3 - \sqrt{3}i}{12}$$

By the residue theorem, we evaluate the contour integral:

$$\oint_{\gamma} f = 2\pi i [Res_f(z_0) + Res_f(z_1)] = 2\pi i \frac{-2\sqrt{3}i}{12} = \frac{\pi}{\sqrt{3}}$$

Finally we conclude

$$\boxed{\int_{-\infty}^{\infty} \frac{x^2}{x^4 + x^2 + 1} = \frac{\pi}{\sqrt{3}}}$$

5. Solve the integral:

$$I := \int_0^{2\pi} \frac{1}{a + bsin(\theta)} d\theta$$

Solution First, consider when the integral is valid. The integrand must be finite, that is, the numerator must be nonzero. The magnitude of the function $bsin(\theta)$ must not be greater than a. Otherwise, the numerator will hit zero at some point, and the integral will be invalid. From now on, assume $|b| < |a|$.

Also, if $b = 0$, the integral becomes trivial. $I = 2\pi/a$ given that a is nonzero.

Recall Euler's formula:

$$sin(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

Now manipuate the integral:

$$I = \int_0^{2\pi} \frac{2i}{2ia + 2ibsin(\theta)} d\theta = \int_0^{2\pi} \frac{2i}{2ia + b(e^{i\theta} + e^{-i\theta})} d\theta$$

$$I = \frac{2}{b} \int_0^{2\pi} \frac{ie^{i\theta}}{e^{2i\theta} + 2iae^{i\theta}/b + 1} d\theta$$

Let $\zeta := e^{i\theta}$. ζ forms a unit circle in the range $\theta \in [0, 2\pi]$. Rewrite the integral as:

$$I = \frac{2}{b} \oint_{\zeta \in C} \frac{d\zeta}{\zeta^2 + 2ik\zeta + 1}$$

Where $k := a/b$ and C is the unit circle. By the quadratic formula, the integrand has poles at:

$$\zeta = -ik \pm \sqrt{-k^2 + 1}$$

$k^2 > 1$ since $|a/b| > 1$. ζ is purely imaginary. Note that one of the poles necessarily fall into the circle and that the other does not. The pole p that falls into the unit circle is:

$$p = -ik + \sqrt{-k^2 + 1}$$

Denote the integrand of the contour function f.

$$f(z) = \frac{1}{z^2 + 2ikz + 1}$$

Compute the residue at p. The pole is a simple pole.

$$Res_f(p) = \lim_{z \to p} \frac{z - p}{z^2 + 2ikz + 1} = \lim_{z \to p} \frac{1}{2z + 2ik}$$

$$= \frac{1}{2} \frac{1}{-ik + \sqrt{-k^2 + 1} + ik} = \frac{1}{2} \frac{1}{\sqrt{1 - k^2}}$$

By the residue theorem:

$$I = \frac{2}{b} 2\pi i [Res_f(p)] = \frac{2\pi}{b\sqrt{k^2-1}} = \pm \frac{2\pi}{\sqrt{a^2-b^2}}$$

The $\pm$ sign depends on the sign of b. As inserting b into the square root, we must seperate its sign.

We conclude, for $|b| > |a|$

$$\boxed{I = \begin{cases} \frac{2\pi}{\sqrt{a^2-b^2}} & (b > 0) \\ -\frac{2\pi}{\sqrt{a^2-b^2}} & (b < 0) \end{cases}}$$