Combinatorics HW9

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Exercise 1

1. Fill in these difference tables.

0		0		0		1		4		10
	0		0		1		3		6	
		0		1		2		3		
			1		1		1			
				0		0				

0		1		2		4		8		15
	1		1		2		4		7	
		0		1		2		3		
			1		1		1			
				0		0				

Exercise 3 Find a value of n that satisfies

$$\Delta n^x = n^x$$

Solution Upon inspection, $n = 2$ works.

$$\Delta 2^x = 2^{x+1} - 2^x = 2 \cdot 2^x - 2^x = 2^x$$

$\square$

Exercise 5a Prove the linearity of the difference operator. That is, for all real sequences f, g and real numbers c_1, c_2,

$$\Delta(c_1 f + c_2 g) = c_1 \Delta f + c_2 \Delta g$$

Solution

We prove the preservation of addition and scalar multiplication separately. Let $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ be real sequences. We wish to show, for any natural number n,

$$\Delta(f_n + g_n) = \Delta f_n + \Delta g_n$$

. Directly evaluate the LHS.

$$\Delta(f_n + g_n) = (f_{n+1} + g_{n+1} - f_n - g_n) = (f_{n+1} - f_n + g_{n+1} - g_n) = \Delta f_n + \Delta g_n$$

Now, move on to show the preservation of scalar multiplication. Let $c \in \mathbb{R}$ and n be any natural number. We wish to show

$$\Delta(cf_n) = c\Delta f_n$$

Again, by directly evaluating the LHS,

$$\Delta(cf_n) = (cf_{n+1} - cf_n) = c(f_{n+1} - f_n) = c\Delta f_n$$

Finally, consider the following line of algebra.

$$\Delta(c_1 f + c_2 g) = \Delta(c_1 f) + \Delta(c_2 g) = c_1 \Delta f + c_2 \Delta g$$

$\square$

Exercise 5b Prove

$$\Delta(fg) = \Delta f \Delta g - f \Delta g - g \Delta f$$

Solution

$\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ be real sequences. We wish to show, for any $n \in \mathbb{N}$,

$$\Delta(fg)_n = \Delta f_n \Delta g_n + f_n \Delta g_n + g_n \Delta f_n$$

We prove this by directly evaluating the RHS.

$$RHS = (f_{n+1} - f_n)(g_{n+1} - g_n) + f_n(g_{n+1} - g_n) + g_n(f_{n+1} - f_n)$$

$$= f_{n+1}g_{n+1} - f_n g_n = \Delta(fg)_n$$

$\square$

Preliminary for Q24 To better understand how EGFs can be used, we present the following theorem, which is a slight generalization of Thm 7.3.1 of the textbook.

Theorem Multiplying to EGFs generates the EGF of a sequence that accounts for partitions.

Let $f_i(x)$ be the EGF of the sequence $\{a_n^i\}_{n\in\mathbb{N}}$. The function

$$\prod_{i\le N} f_i(x)$$

is an EGF of the sequence

$$h_n := \sum_{m_1+\ldots m_N=n} \binom{n}{m_1, m_2, \ldots, m_N} \prod_{i\le N} a_i$$

A short proof can be written similarly to that of Thm 7.3.1.