Combinatorics HW5

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Section 2.7: 42, 50, 54
Section 5.7: 27, 29, 36

Sec2.7Q42 Determine the number of ways to distribute 10 orange drinks, 1 lemon drink, and 1 lime drink to four thirsty students so that each student gets at least one drink, and the lemon and lime drinks go to different students.

Solution Apply the principle of multiplication. Give out the lemon and the lime juice to two distinct students. Then, give out two orange juices to the two student who has not yet received any juice. Then, distribute the eight orange juices by random.

$$P(4,2)\left(\binom{4}{8}\right) = 4 \cdot 3\binom{11}{3} = \boxed{1980}$$

Sec2.7Q50 In how many ways can five identical rooks be placed on the squares of an 8-by-8 board so that four of them form the corners of a rectangle with sides parallel to the sides of the board?

Solution Choosing two rows and two columns determine the position of the four rooks that form a rectangle. Choose the last rook from any of the $64-4=60$ positions. We count the answer as follows.

$$\binom{8}{2}^2 \cdot 60 = \boxed{47040}$$

Sec2.7Q54 Determine the number of towers of the form $\emptyset \subseteq A \subseteq B \subseteq [n]$.

Solution Notice that each tower correspond to a unique partition of the canonical set $[n]$, that is $A, B \setminus A, [n] \setminus B$. Calling these parts X, Y, Z respectively, we notice that $(A, B) = (X, X \cup Y)$. We have demonstrated an injective mapping from the tower to the partition, and from partition to the tower. Hence, the mapping is bijective.

The number of ways to partition $[n]$ to three parts is 3^n. Thus, there are $\boxed{3^n}$ towers. We list out all the towers for $n = 2$.

$$(A,B) = (\emptyset,\emptyset), (\emptyset,\{1\}), (\emptyset,\{2\}), (\emptyset,\{1,2\}), (\{1\},\{1,2\}), (\{1\},\{1\}), (\{2\},\{2\}), (\{2\},\{1,2\}), (\{1,2\},\{1,2\})$$

Sec5.7Q27 Give a combinatorial proof of the identity

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}$$

Solution Consider the following scenario. A toy shop has n different toys. John plans to buy some set of the toys for his children Amy and Bob. After John has bought $k \leq n$ toys from the shop, he takes it home where Amy and Bob choose to play with one of the toys each. We assume Amy and Bob to behave nicely, meaning that even if they choose the same toy, they will not fight each other.

If John brings k toys from home, there are k^2 scenarios where Amy and Bob chooses a single toy. John brings k toys from a total of n toys from the shop. Hence, the total configuration of the toys at the end of the day is

$$\sum_{k=1}^{n} k^2 \binom{n}{k}$$

Suppose Charles, the owner of the toy shop, computes the total possible arrangement of the toys. The first possible scenario is that Amy and Bob both choose the same toy. The chosen toy must be taken by John, and all the remaining $n-1$ toys can be taken home or stay in the shop. There are

$$n \cdot 2^{n-1}$$

such cases.

The alternate scenario is when the two children choose different toys. By a similar argument there are

$$P(n,2)2^{n-2}$$

such cases.

Adding up the size of the two parts, we obtain

$$n \cdot 2^{n-1} + P(n,2)2^{n-2} = (2n + n(n-1))2^{n-2} = n(n+1)2^{n-2}$$

For we have double counted all the possible toy arrangements, the combinatorial sum must equal to the term that we have obtained.

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}$$

$\square$

Sec5.7Q29 Find and prove a formula for

$$\sum_{\substack{r,s,t \geq 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$$

Solution Use a similar argument used for Vandermonde Convolutions. From a set of $m_1 + m_2 + m_3$ elements, imagine choosing a total of n elements. Partitioning the orignal set into three sets of size m_1, m_2, m_3, we observe that the total ways to choose n element from the original set is to count the number of ways to choose r, s, t elements from the respective parts, such that the scripts range under the condition $r + s + t = n$.

On the other hand, it is trivial that there are $\binom{m_1+m_2+m_3}{n}$ ways to choose a subset of size n from the original set. For we have counted the same quantity twice, we establish the following identity.

$$\sum_{\substack{r,s,t\geq 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t} = \binom{m_1+m_2+m_3}{n}$$

□

Sec5.7Q36 Prove the Vandermonde Convolutions using the Binomail Theorem.

Solution We start with the identity

$$(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2}$$

Foil both sides using the Binomial Theorem.

$$\left(\sum_{k=0}^{m_1}\binom{m_1}{k}x^k\right)\left(\sum_{j=0}^{m_2}\binom{m_2}{j}x^j\right) = \sum_{l=0}^{m_1+m_2}\binom{m_1+m_2}{l}x^l$$

Focus on the coefficient of x^l for some fixed l. For the term x^k, the corresponding multiple must be x^{l-k}. The coefficients of x^l must be equal for both sides. Hence,

$$\sum_{\substack{k+j=l \\ 0\leq k\leq m_1 \\ 0\leq j\leq m_2}} \binom{m_1}{k}\binom{m_2}{j} = \binom{m_1+m_2}{l}$$

Note that the binomial term vanishes if the bottom term is less than equal to zero. That is $\binom{a}{b} = 0$ for any $b \leq 0$. Rewrite the LHS.

$$\sum_{k=0}^{l}\binom{m_1}{k}\binom{m_2}{l-k} = \binom{m_1+m_2}{l}$$

□