Recurrence relations with bounded behavior

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Let $x, y \in [0,1]$ be a real value that describes the population proportion of the prey and predator respectively. The Lokta-Volterra relation that describes the predator-prey model is a 2nd order differential equation as the following.

$$\frac{d}{dx}x' := \alpha x + \beta xy$$
$$\frac{d}{dy}y' := \gamma y - \delta xy$$

It would be nice to solve the difference equation related to the differential equation. Given some small discrete timestep, we can use Euler's method to write a system of difference equations that approximates the predator-prey model.

$$x' := x + t\alpha x + t\beta xy$$
$$y' := y + t\gamma y - t\delta xy$$

We do not know how to solve this equation directly. Instead, we wish to solve the a related difference equation of the second order. We hope that our solution for this related system potrays similar behavior as the Lokta-Volterra system.

$$\begin{aligned} (ax' + by' + c) &= M(ax + by + c)(\bar{a}x + \bar{b}y + \bar{c}) \\ (\bar{a}x' + \bar{b}y' + \bar{c}) &= \bar{M}(\bar{a}x + \bar{b}y + \bar{c}) \end{aligned} \tag{1}$$

At equilibrium, $(x', y') = (x, y)$. Denote the value at equilibrium as $(x, y) = (x_{eq}, y_{eq})$. Plugging this condition into (1), we deduce

$$\begin{aligned} (ax_{eq} + by_{eq} + c) &= 0 \\ (\bar{a}x_{eq} + \bar{b}y_{eq} + \bar{c}) &= 0 \end{aligned}$$

which determines the value of $c, \bar{c}$ as

$$\begin{aligned} c &= -(ax_{eq} + by_{eq}) \\ \bar{c} &= -(\bar{a}x_{eq} + \bar{b}y_{eq}) \end{aligned}$$

In light these relations, it is natural to look at a normalized quantity of the variables. Subtract a constant factor from both x, y and define the following quantities.

$$\Delta x := (x - x_{eq}) \quad \text{and} \quad \Delta y := (y - y_{eq})$$

The system in (1) is converted into a purely second order system.

$$\begin{aligned} (a\Delta x' + b\Delta y') &= M(a\Delta x + b\Delta y)(\bar{a}\Delta x + \bar{b}\Delta y) \\ (\bar{a}\Delta x' + \bar{b}\Delta y') &= \bar{M}(\bar{a}\Delta x + \bar{b}\Delta y) \end{aligned} \tag{2}$$

The values of x, y over time create a real value sequence $\{x_n\}, \{y_n\}$ where

$$x_n := x_0^{(n)} \quad \text{and} \quad y_n := y_0^{(n)}$$

[1] where x_1, y_1 are the initial values of the population ratio. Define an auxillary sequence as the following.

$$U_n := a\Delta x + b\Delta y = a(x_n - x_{eq}) + b(y_n - y_{eq})$$
$$V_n := \bar{a}\Delta x + \bar{b}\Delta y = a(x_n - x_{eq}) + b(y_n - y_{eq})$$

The system (2) simplifies even more in terms of U_n and V_n.

$$U_{n+1} = MU_nV_n \quad \text{and} \quad V_{n+1} = \bar{M}V_n$$

Taking the natural logarithm both sides reduces the system to

$$\begin{aligned} \ln(U_{n+1}) &= \ln(M) + \ln(U_n) + \ln(V_n) \\ \ln(V_{n+1}) &= \ln(\bar{M}) + \ln(V_n) \end{aligned}$$

Adopt more shorthands.

$$\ln(U_n) := u_n \quad \text{and} \quad \ln(V_n) := v_n$$

$$\ln(M) := m \quad \text{and} \quad \ln(\bar{M}) := \bar{m}$$

Graciously, we arrive at the following equation.

$$\begin{aligned} u_{n+1} &= m + u_n + v_n \\ v_{n+1} &= \bar{m} + v_n \end{aligned}$$

In matrix form, this equation can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = F \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} m \\ \bar{m} \end{bmatrix} \tag{3}$$

where $F := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the fibbonacci matrix.

[1]The power of (n) means that we operate the time difference operation n times. e.g. $x^(2) = x''$.

Proposition The solution for (3) is

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} F_{n+1}(u_0 + \bar{m} + m) + F_n(v_0 + m) - \bar{m} - m \\ F_n\,(u_0 + \bar{m} + m) + F_{n-1}\,(v_0 + m) - m \end{bmatrix}$$

where F_n is the nth fibbonacci number with the initial conditions $F_0 = F_1 = 1$.

Proof. It is easy to verify the equation for the nth power of F for $n > 0$.

$$F^n := \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

Adopt the shorthand $\vec{v}_n := \begin{bmatrix} u_n \\ v_n \end{bmatrix}$ and $\vec{m} := \begin{bmatrix} m \\ \bar{m} \end{bmatrix}$.
We have the recurrence

$$\vec{u}_{n+1} = F\vec{u}_n + \vec{m}$$

which indicates the formula

$$\vec{u}^{(n)} = F^n\vec{u} + \sum_{k=0}^{n-1} F^k\vec{m}$$

. Using the geometric series formula, we deduce the follows. Note that $(F - I)^{-1} = F$ where I is the identity matrix.

$$\sum_{k=0}^{n-1} F^k = F(F^n - I)$$

Thus,

$$\vec{u}^{(n)} = F^n\vec{u} + F^nF\vec{m} - F\vec{m}$$

which indicates the result

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} F_{n+1}(u_0 + \bar{m} + m) + F_n(v_0 + m) - \bar{m} - m \\ F_n\,(u_0 + \bar{m} + m) + F_{n-1}\,(v_0 + m) - m \end{bmatrix}$$

□

Corollary 1 Taking the exponential of u_n, v_n, it is possible to obtain the following.

$$\begin{bmatrix} U_n \\ V_n \end{bmatrix} = \begin{bmatrix} (U_0M\bar{M})^{F_{n+1}}(V_0M)^{F_n}/(M\bar{M}) \\ (U_0M\bar{M})^{F_n}\,(V_0M)^{F_{n-1}}\,/M \end{bmatrix}$$

Remark Eventually, by taking linear combinations, we can recover a closed form formula for x, y .