Algebraically Linked GOEs

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1 Statement of the Result

Definition 1 (Algebraic Link)**.** *Let X, Z be independant GOEs of dimension N-by-N. We define an algebraically linked GOE with link variable $l \in [0, 1]$ as follows.*

$$Y = lX + (1 - l)Z$$

Definition 2 (Spectral Density of the Anticommutator)**.** *Let X, Y be two GOEs that are not necessarily independant. Let Λ denote the set of all eigenvalues of the matrix $XY + YX$. The spectral density of the Anticommutator is defined as the following.*

$$\mu(x) \ := \ \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{\lambda\in\Lambda}\delta\left(x - \frac{\lambda}{N}\right)\right]$$

It is well known that we can compute the (n)th moment by using the trace.

Theorem 1 (Normalized Spectral Density of the Anticommutator)**.** *Denote the kth moment of this probability distribution as $\mu^{(k)}$. Then,*

$$\mu^{(n)} \ = \ Tr\left(\left[\frac{1}{\sqrt{N}}(XY + YX)\right]^n\right)$$

.

Our result provides a formula for μ^n that involves a constant defined via a reasonable recursion.

Theorem 2 (Moments of the anticommutator of two algebraically linked GOEs)**.**

$$\mu^{(n)} \ = \ \sum_{k=0}^{n} \nu_{n,k}(2l)^k(1 - l)^{n-k}$$

Here are the moments up to $n = 7$.

$\mu^{(1)} = 2l$

$\mu^{(2)} = 2 - 4l + 10l^2$

$\mu^{(3)} = 18l - 36l^2 + 58l^3$

$\mu^{(4)} = 10 - 40l + 204l^2 - 328l^3 + 378l^4$

$\mu^{(5)} = 170l - 680l^2 + 2140l^3 - 2920l^4 + 2634l^5$

$\mu^{(6)} = 66 - 396l + 3054l^2 - 9576l^3 + 22014l^4 - 25932l^5 + 19218l^6$

$\mu^{(7)} = 1666l - 9996l^2 + 46830l^3 - 120680l^4 + 222558l^5 - 230412l^6 + 144946l^7$

2 Combinatorial Preliminaries

Definition 3 (Special Words)**.** *A special word of length $2k$ is composed of k blocks, where each block is one of $\{XX, ZX, XZ\}$. The characteristic of a special word w, denoted by $\chi(w)$, is the number of blocks XX used in the word.*

For example, when $k = 3$,

$$XX\,ZX\,XZ$$

is an example of a special word of length 6 with characteristic 1.

Definition 4 (Set of Special Words)**.** *$H_{n,k}$ is defined as the set of all special words of length $2n$ with characteristic k.*

For example, if $n = 2$ and $k = 1$, then

$$H_{2,1} = \{XX\ ZX,\ XX\ XZ,\ ZX\ XX,\ XZ\ XX\}.$$

Definition 5 (Valid Pairings)**.** *A valid pairing is a partition of the indices of the word into pairs such that each pair contains the same type of letter.*

For example, for the word $XXZZ$, a valid pairing is $\{\{1, 2\}, \{3, 4\}\}$.

Definition 6 (Non-Crossing Pairings)**.** *A non-crossing pairing is a valid pairing where for any two pairs $\{i, k\}$ and $\{j, l\}$, it is not the case that $i < j < k < l$.*

For example, for the word $XXZZ$, the pairing $\{\{1, 2\}, \{3, 4\}\}$ is non-crossing, while the pairing $\{\{1, 3\}, \{2, 4\}\}$ is crossing because $1 < 2 < 3 < 4$.

Definition 7 (Pairing number of a property n, k)**.** *Call $\nu_{n,k}$ to be the pairing number of the property n, k [1] $\nu_{n,k}$ is defined as the number of valid, non-crossing pairings for all special words in $H_{n,k}$. To compute $\nu_{n,k}$:*

(i) *Consider all special words in $H_{n,k}$.*

(ii) *For each word, count the number of valid, non-crossing pairings of the indices.*

(iii) *Sum these counts for all words in $H_{n,k}$.*

Mathematically, $\nu_{n,k}$ is given by:

$$\nu_{n,k} = \sum_{w \in H_{n,k}} \phi(w)$$

Where $\phi(w)$ counts valid, non-crossing pairings of w.

[1] Technically, would be accurate to say the the Pairing number of the words with property n, k, but the word is clearly implied by the context

For example, if $n = 2$ and $k = 0$:

$$H_{2,0} = \{ZX\ ZX,\ ZX\ XZ,\ XZ\ ZX,\ XZ\ XZ\}$$

For $ZX\ ZX$, no valid non-crossing pairings.
For $ZX\ XZ$, valid non-crossing pairing: $\{\{1,4\},\{2,3\}\}$.
For $XZ\ ZX$, valid non-crossing pairing: $\{\{1,4\},\{2,3\}\}$.
For $XZ\ XZ$, no valid non-crossing pairings. $\{\{1,3\},\{2,4\}\}$.

Therefore, $\nu_{2,0} = 1 + 1 = 2$.

3 Counting Valid Pairings by σ-recurrences

In this section, we provide a method to compute Pairing numbers with a property n, k. We introduce an additional quantity to the property of the word, s, that denotes the number of XX blocks at the beginning of the word.

Definition 8. *Define the pairing number of the property n, s, k as the following.*

$$\sigma_{n,s,k} = \sum_{w \in H_{n,s,k}} \phi(w)$$

. $H_{n,s,k}$ denotes the set of all words composed of n blocks that have at least s XX blocks in the beginning of the word, and k blocks of XY, YX.

Theorem 3 ($\sigma_{n,s,k}$)**.** *The auxiliary sequence $\sigma_{n,s,k}$ is defined with the following initial conditions.*

1. $\sigma_{n,s,k} = 0$ *if* $s + k > n$
2. $\sigma_{n,s,0} = C_n$*, where C_n is the n-th Catalan number,* $C_n = \frac{1}{n+1}\binom{2n}{n}$
3. $\sigma_{n,s,2k+1} = 0$
4. $\sigma_{n,s,-k} = 0$

The recurrence relation for $\sigma_{n,s,2k}$ is given by:

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} [\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}] \tag{1}$$

Proof. Initial conditions 1, 3, 4 trivially follows from the nature of valid pairings. $s + k \leq n$ in any block. Also, if there are $2k+1$ blocks of the type XY, YX, the number of Y's in the word is odd, and hence there exists no valid pairing. Clearly, the number of XY, YX blocks cannot be negative.

Consider initial condition 2. If $k = 0$, then the word is entirely composed of XX blocks, so the number of non-crossing pairngs can be easily counted by the Catalan numbers. This concludes the proof for the four initial conditions.

We move on to prove the recurrence relation. Let p be the first occurence of any block that has a Y and q the block in which the Y in the pth block matches to. For example, if $(n, s, k) = (5, 1, 2)$, here is an example word with the pairing with $p = 3, q = 5$.

$$W \;=\; XX\,XX\,XY\,XX\,YX$$

Note that the pairing between the Y blocks divide into two types. Type 1 pairing is $XY\,YX$ and Type 2 paring is $YX\,XY$ both in order. Pairing the two Y's split the word into two sub-words, the word outside the YY block and thw word between the YY block. So for the previous example,

$$W_1 \;=\; XX\,XX\,XX \quad \text{and} \quad W_2 \;=\; XX$$

where W_1 is outside the Y pairing and W_2 is between the Y pairings.

For pairing Type 1, the value of s increases by 1 after the splitting for the outer word. For pairing Type 2, the value of s increases from zero to 1 after the split. The inner word and the outer word can be considered independent. The important obsevation to deduce the latter fact is to observe that the pairing number is equivalent for the following two blocks.

$$X\,\text{[Some Blocks]}\,X$$

$$XX\,\text{[Some Blocks]}$$

With these fact in mind, we count the contribution of Type 1 and Type 2 matchings for fixed p, q. For Type 1, the contribution is

$$\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r}$$

For Type 2, the contribution is

$$\sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}$$

This proves the recursive relation

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}\right]$$

. □

Remark 1 (Relation to $\nu_{n,k}$)**.** *$\nu_{n,k}$ is related to $\sigma_{n,s,k}$ by the following:*

$$\nu_{n,k} = \sigma_{n,0,n-k}$$

This means that to compute $\nu_{n,k}$, we compute $\sigma_{n,0,n-k}$ using the above initial conditions and recurrence relation.

Remark 2 (Case where X's are allowed to cross)**.** *When computing the moments for GOE anticommutated with Palindromic Topelitz, we can modify the initial condition as*

$$\sigma_{n,s,0} \;=\; (2n-1)!!$$

to obtain the Pairing Number appropriate for this case.