Predator-Prey Model using Leslie matricies and optimal predation strategies

PP Group

1 Abstract

In this paper, we introduce a new predator-prey model based the Lotka-Voltera model. Replacing the constant coefficients to Leslie matricies motivates the study of dominant eigenvalues which can be conducted using techniques in Complex Analysis. Using the theory of dominating eigenvalues, we provide a bound for maximum predation rate for population survival in a long term. We also discuss the competetive model, and prove the last species standing theorem, which describes the unlikelyhood of stable equilibrium between two competetive species.

2 The Leslie model on a single specie population

Leslie matricies characterize the change of population with different age groups, given the survival rate and the fertility rate of the species.

We focus on a specific class of Leslie matricies with a fixed fertility rate f and a survival rate 1.

Definition 1 (Simple Leslie Matricies). *Suppose $N \in \mathbb{Z}_{pos}$. A simple Leslie matrix that characterizes the population evolution is defined as follows.*

$$(L_f)_{ij} = \begin{cases} f & (i = 0) \\ 1 & (i \neq 0 \wedge j = i + 1) \\ 0 & Otherwise \end{cases}$$

Or writing the matrix out,

$$L_f := \begin{bmatrix} f & f & \cdots & f \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \vdots & \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

.

The maximum eigenvalue of this Matrix describes the asymptotic behavior of the population. The first apporach is to compute the characteristic equation and find the roots to derive properties about the eigenvalues.

Theorem 1 (Lotka-Euler Equation). *The characteristic equation of a simple leslie matrix L_f of order N is*

$$ch_N(x) := x^N - f(x^{N-1} + \cdots + x + 1)$$

which, using the geometric series formula, can be simplified as

$$x^N - f\frac{x^N - 1}{x - 1}$$

Proof. Induct on N.

Using methods from Complex Analysis, it is possible to derive the following two theorems.

Theorem 2 (Complex Roots of the Characteristic Equation). *The complex roots of the characteristic equation ch_N is always lie inside the unit circle.*

Theorem 3 (Real Root of the Characteristic Equation). *The real root of the characteristic equation has a magnitude greater than* 1*, given that* $1 - fN \leq 0$*.*

With a little more analysis, we provide a lower bound and the upper bound of the maximum eigenvalues of L_f.

Theorem 4 (Bounds for the maximum eigenvalue). *Given that* $1 - fN \leq 0$*, the maximum eigenvalue of L_f of order N is given by*

$$1 + \frac{f}{N+1} \leq \lambda_{max} \leq 1 + f$$

3 The predator-prey model

Definition 2 (Leslie Predator-Prey). *Let α_n, β_n be the population vectors of the predator and prey at timestep n. The Leslie Predator-Prey model is defined by the following system of matrix differences.*

$$\begin{aligned} \alpha_{n-1} &= \max(L_\alpha \alpha_n + km\beta_n, \vec{0}) \\ \beta_{n-1} &= \max(L_\beta \beta_n - k\alpha_n, \vec{0}) \end{aligned} \quad (1)$$

k, m are predation ratio and nurturing ratios, both between $0, 1$*.*

We assume that the x-value of L_α is less than $1/2$ and that the x-value of L_β is greater than $1/2$. In other words, the predator population decays in absense of the prey and the prey populatin explodes in absence of the predator.

Moreover, the population is fixed to be nonnegative.

Question 1 (Optimal Predation Strategy). *For what ranges of the real value k guarantees exponential growth of the predator? Moreover, what value of k is necessary to guarantee maximum growth?*

Theorem 5 (Coulpled 1st order to 2nd order). *The predator population satisfies the following second order difference equation.*

$$\alpha_n = (L_\alpha + L_\beta)\alpha_{n-1} - L_\beta L_\alpha \alpha_{n-2} - mk^2\alpha_{n-2}$$

$$\beta_n = (L_\beta + L_\alpha)\beta_{n-1} - L_\alpha L_\beta \beta_{n-2} - mk^2\beta_{n-2}$$

4 Special case 1: when L is a scalar

The following three propositions properly models the population where the dimension of the Leslie matrix is 1. That is, the population growth is characterized by a exponential of a scalar without interaction. To emphasize the scalarness, write $l_a < 1$ and $l_b > 1$ instead of L_a, L_b.

Theorem 6 (Eigenvalues of the companion matrix)**.** *Using Lemma 1, it is possible to obtain a companion matrix that describes the population.*

$$\begin{bmatrix} l_a + l_b & -l_a l_b - k^2 m \ 1 & 0 \end{bmatrix}$$

The eigenvalue of this matrix is purely real if and only if

$$k \leq \frac{l_a - l_b}{2\sqrt{m}}$$

Otherwise, the eigenvalues of these maticies are complex conjugates of each other.

Theorem 7 (Exponential growth of population for small predation)**.** *The following condition guarantees that the predator and prey population to not vanish as $n \to \infty$.*

$$k < \sqrt{\frac{(1 - l_b)(l_a - 1)}{m}}$$

Proof. Compute the eigenvalues of the companion matrix directly, and set it to be less than one.

Theorem 8 (Complex eigenvalue implies extinction)**.** *If*

$$k \geq \frac{l_a - l_b}{2\sqrt{m}}$$

then the population is guaranteed to be extinct.

Proof. Take the eigendecomposition and notice that the rotation eventually takes the population to some zero value. □

5 Special case 2: when $L_a = \rho L_b$

To solve the second order matrix recurrence related to the predator-prey model, we solve a characteristic equation where the coefficients are matricies.

We wish to find a matrix Λ such that

$$\Lambda^2 - (\rho + 1) L_\beta \Lambda + \rho L_\beta^2 + mk^2 I = 0$$

Since the only matricies involved on this equation are I and L_β which commute, we can use the quadratic equation to solve this equation.

Theorem 9. *The population of the predator in (1) can be characterized as*

$$\vec{\alpha}_n = \vec{v}_1 \Lambda_1^n + \vec{v}_2 \Lambda_2^n$$

for some vectors $\vec{v}_1$ and $\vec{v}_2$. Moreover, the growth of $\vec{\alpha}_n$ is dominated by the maximum eigenvalue of Λ_1. Call the maximum eigenvalue of L_f as λ_{max}. We write

$$\lambda_{max} = \frac{(\rho+1) + \sqrt{(\rho+1)^2\lambda_{max}^2 - 4mk^2}}{2}$$

Withs some more analysis, we provide a bound for k that guarantees the survival of both the predator and k.

Theorem 10. *Given that $\rho \geq 1 - \frac{2f}{1+N+f}$, the following condition guarantees population growth.*

$$k \leq \frac{1-\rho}{2\sqrt{m}}\left(1 + \frac{f}{1+N}\right)^2$$

6 The Competetive Model

We can slightly modify one of the sign of the model and study the following system.

Definition 3 (Leslie Competetive Model)**.** *Let α_n, β_n be the population vectors of the predator and prey at timestep n. The competetive model is defined by the following system of matrix differences.*

$$\begin{aligned} \alpha_{n-1} &= \max(L_\alpha \alpha_n - km\beta_n, \vec{0}) \\ \beta_{n-1} &= \max(L_\beta \beta_n - k\alpha_n, \vec{0}) \end{aligned} \tag{2}$$

k, m are interaction ratio and competetive advantage, both between $0, 1$.

A similar analysis used for the predator-prey model can be applied to yield the following result.

Theorem 11 (Last Species Standing)**.** *In a Leslie competetive model, one of the two species are likely to vanish as $n \to \infty$ Suppose $\vec{\alpha}_0 = \alpha_0(1, \cdot, 1)$ and $\vec{\beta}_0 = \beta_0(1, \cdot, 1)$ the fate of the species is determined by the sign of the term*

$$D := \alpha_0 - \sqrt{m}\beta_0$$

. To qualify, if $D > 0$, then the population β vanishes. If $D < 0$, then the population α vanishes. If $D = 0$, either both species vanish or grow exponentially together.

7 Difficulties for the general case solution

It turns out that solving the recurrence for the general case where L_a, L_b is extremely challenging. Suppose we wish to solve the PP model where the Leslie matricies are degree k-by-k, where $k > 1$. If we adopt the scalar solution, we have to compute the eigenvalues of a $2k$-by-$2k$ matrix, and show that the eigenvector corresponding to the dominating eigenvalue is positive. Another attempted solution was to consider the following characteristic equation of the 2nd order recurrence

$$\Lambda^2 - (L_\alpha + L_\beta)\Lambda + (L_\alpha L_\beta - k^2 mI) = 0$$

In general, L_α and L_β do not commute. This imposes hardships when applying the quadratic formula to solve this equation.