Recursion on Finite Field Matricies

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From Lidi's result, we know that a recursive sequence over any finite field is periodic. We wish to stretch this result to matricies.

Proposition Let F be a finite field. Consider a fixed depth recursive relation over matricies of dimension N by N. Denote the sequence as

$$X_n := \begin{bmatrix} x_n^{11} & \cdots & x_n^{1N} \\ \vdots & \ddots & \vdots \\ x_n^{N1} & \cdots & x_n^{NN} \end{bmatrix}$$

and the depth of the recursion d. The recursive relation can be written as

$$X_n = f(X_{n-1}, X_{n-2}, \ldots, X_{n-d})$$

where f is some polynomial involving d constants.

Any such finite recursion must be periodic.

Proof. We apply the pigeonhole principle. The nth matrix is entirely determined by the previous d matricies, $X_{n-1}, \ldots, X_{n-d}$. In other words, if the previous d matricies agree, then the two matricies must agree. In symbols, if

$$(X_{m-1}, \ldots, X_{m-d}) = (X_{n-1}, \ldots, X_{n-d})$$

then,

$$X_m = X_n$$

. Call the tuple previous d matricies that determine X_n to be the **seed** of X_n. Since F is a finite field, there are only finite number of different seeds. By the principle of multiplication, the number of possible seeds are

$$|F|^{N^2 d}$$

because there are $|F|^{N^2}$ different N by N matricies over F and d of such matricies determine the seed.

Though large, the number of different possible seeds are finite. This implies that in an infinite sequence, the series must be periodic. $\square$

Remark Note that this result applies only to recurrences with fixed depth. For example, the catalan recurrence does not fall under this umbrella, and its periodicity is undetermined by this result.