Combinatorics HW5

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Counting Derangements A derangement is a permutation of the set where no elements are fixed. We define D_n to be the number of derangements of the cannonical set $[n]$. By the inclusion-exclusion principle, we derive

$$D_n = n!\left(1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

By the alternating series test, we conclude

$$D_n = \left\{\frac{n!}{e}\right\}$$

Posets and Convolutions

Let $(X, \leq)$ be a finite poset. We consider a class of functions that map pairs of the poset X to the reals. Let $f, g : X \times X \to \mathbb{R}$. Define a discrete convolution of the two posets as follows.

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$$

We define three important functions, each corresponding to the identity, the ordering, and the inverse of the ordering. They are called the Kronecker Delta, Zeta, and the Mobius Function.

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

It is trivial to find out that the delta function is the convolutional identity.

Before writing out the Mobius function, we introduce a constructive method to obtain the convolutional inverse of an arbitrary function f. We requre $f(y, y)$ to be nonzero.

Let g be the left inverse of f. We easily observe that for nondistinct paris, g must be the reciprocal of f.

$$g(y, y) = \frac{1}{f(y, y)} \qquad \forall y \in X$$

For distinct pairs, the convolution of f, g must yield zero. If $x > y$, then the convolution is automatically zero. That is, assuming $x < y$,

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y) = 0$$

Break down the sum.

$$f(x, x) \cdot g(x, y) + \sum_{x < z \leq y} f(x, z) \cdot g(z, y) = 0$$

Sove for $g(x,y)$.

$$g(x,y) = -\frac{1}{f(x,x)} \sum_{x<z\leq y} f(x,z) \cdot g(z,y)$$

It is not hard to see that convolutions are associative. This leads us to conclude that the left inverse equals to right inverse.

$$f_l * f * f_r = \delta * f_r = \delta * f_l \quad \text{or} \quad f_r = f_l$$

Finally, we present the Mobius Function. The mobius function is defined as the inverse of the zeta function. plug in $f \mapsto \zeta$.

$$\mu(x,y) = \begin{cases} 1 & \text{if x = y} \\ -\sum_{x<z\leq y} \mu(z,y) & \text{otherwise} \end{cases} \quad \text{then} \quad \mu * \zeta = \delta$$

Proof of Mobius Inversion

Proof. Let ζ be the zeta function of $(X, \leq)$. Using the properties of ζ and μ previously discussed, we calculate as follows for x an arbitrary element in X:

$$\begin{aligned}
\sum_{\{y:y\leq x\}} G(y)\mu(y,x) &= \sum_{\{y:y\leq x\}} \sum_{\{z:z\leq y\}} F(z)\mu(y,x) \\
&= \sum_{\{y:y\leq x\}} \mu(y,x) \sum_{\{z:z\in X\}} \zeta(z,y)F(z) \\
&= \sum_{\{z:z\in X\}} \sum_{\{y:y\leq x\}} \zeta(z,y)\mu(y,x)F(z) \\
&= \sum_{\{z:z\in X\}} \left(\sum_{\{y:z\leq y\leq x\}} \zeta(z,y)\mu(y,x) \right) F(z) \\
&= \sum_{\{z:z\in X\}} \delta(z,x)F(z) \\
&= F(x).
\end{aligned}$$

$\square$

Tips for Mobius Inversion

It is necessary that the cumulative function G is of simple form. If is is not clear what G is, then take the compliment of G's argument with respect to the universal set.

For example, it is horrendous to compute:

$$G(n) = \sum_{i|n} \phi(i)$$

However, consider

$$G(n) = \sum_{i|n} \phi(n/i)$$

Each divisor i uniquely maps to another divisor n/i. If a number ξ is coprime with n/i, $gcd(\xi \cdot i, n) = i$. More precisely, $(\xi, n/i) = 1$ iff $(\xi \cdot i, n) = i$. $\phi(n/i)$ counts the number of such ξ, and this corresponds to the numbers that have a gcd i with n. Each number in $[n]$ must have some gcd that divides n. Thus, $G(n)$ counts all numbers between $1, n$.

Classic Mobius Inversion

Memorize this sum:

$$\sum_{i|n} \mu(n/i)i = \phi(i)$$

Generating Functions and their sums

Proposition Adding two variables in the equations results in multiplication of the generating functions.

$$G\langle e_1, e_2, \dots, e_n \rangle(x) = \prod_{i=1}^{n} G\langle e_i \rangle(x)$$

Proof We prove by induction on n. The base case is trivial. By the inductive hypothesis, we assume

$$G\langle e_1, e_2, \dots, e_n \rangle(x) = \prod_{i=1}^{n} G\langle e_i \rangle(x)$$

Now, we wish to find the generating function

$$G\langle e_1, e_2, \dots, e_{n+1} \rangle(x)$$

In order to find the generating function, we must find the value of the sequence

$$\langle e_1, e_2, \dots, e_{n+1} \rangle_n$$

Which is the number of solutions to the equation

$$e_1 + e_2 + \cdots + e_{n+1} = N$$

We partition all the solution based on the possible values of e_{n+1} Fix the value of $e_{n+1} = l$. The size of the corresponding part will be the number of solutions to

$$e_1 + e_2 + \cdots + e_n = N - l$$

Which is in fact, the value $\langle e_1, e_2, \dots, e_n \rangle_{N-l}$. This value is given my the coefficient of x^{N-l} of the polynomial $G\langle e_1, \dots, e_n \rangle(x)$

Consider the poynomial

$$G\langle e_1, \dots, e_n \rangle(x) G\langle e_{n+1} \rangle = \prod_{i=1}^{n+1} G\langle e_i \rangle(x)$$

where the equality follows by the inductive hypothesis. The coeffieient of x^N of this polynomial will be the sum of the coefficients of x^{N-l} in the polynomial $G\langle e_1, \dots, e_n\rangle(x)$ for all values of l which $G\langle e_{n+1}\rangle$ is nonzero. In symbols, the x^N coefficient is

$$\sum_{l \text{ valid}} \langle e_1, e_2, \dots, e_n\rangle_{N-l} = \langle e_1, e_2, \dots, e_{n+1}\rangle_N$$

We have directly shown that

$$\prod_{i=1}^{n+1} G\langle e_i\rangle(x)$$

Is a generating function of $\langle e_1, e_2, \dots, e_{n+1}\rangle_N$.

$\square$

In light of this powerful machinery, we can find the GFs for variables that are independant.

Preliminary for Q24 To better understand how EGFs can be used, we present the following theorem, which is a slight generalization of Thm 7.3.1 of the textbook.

Theorem Multiplying to EGFs generates the EGF of a sequence that accounts for partitions.

Let $f_i(x)$ be the EGF of the sequence $\{a_n^i\}_{n\in\mathbb{N}}$. The function

$$\prod_{i\leq N} f_i(x)$$

is an EGF of the sequence

$$h_n := \sum_{m_1+\dots m_N=n} \binom{n}{m_1, m_2, \dots, m_N} \prod_{i\leq N} a_i$$

A short proof can be written similarly to that of Thm 7.3.1.

Linear and Homogenous Recurrence Relations

A **Linear recurrence relation** is some relation in the form of:

$$h_{n+k} = a_k(n) \cdot h_{n+k-1} + \dots + a_1(n)h_n + b(n)$$

Where $a_i(n), b(n)$ are some real valued functions dependant on n for all $i \in [k]$. If $a_i(n)$ is constant and $b(n) = 0$, then the function is **homogenous**.

Assume the relation to be homogenous. The **Characteristic Polynomial** of this relation is

$$t^k + a_k t^{k-1} + \cdots a_1 = 0$$

Let $\alpha_1, \dots, \alpha_k$ be the roots. Assuming the roots are distinct, the solutions come in form of

$$h_n = c_1\alpha_1^n + c_2\alpha_2^n + \dots + c_k\alpha_k^n$$

If some of the roots are redundant, multiply by a factor of n. For example, if the characteristic polynomial of some recurrence relation is

$$(t-2)^k$$

then the sequence h_n is in the form of

$$h_n = c_1(2) + c_2(2n) + c_3(2n^2) \cdots (c_k 2n^{k-1})$$

Fibbonacci Numbers

The fibbonacci sequence satisfied the recurrence relation

$$f_{n+2} = f_{n+1} + f_n$$

given the initial conditions $(f_0, f_1) = (0, 1)$. We also define F_n with the same relation but different initial conditions. $(F_0, F_0) = (1, 1)$.

Either using GFs or recurrence relations, we derive a closed form equation for F_n.

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^{n+1} - \psi^{n+1}\right)$$

Where

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2}$$

Notice that the modullus of ψ is less than 1. Hence, for large enough n, the ψ^{n+1} can be ignored. We conclude

$$F_n = \left\{ \frac{1}{\sqrt{5}} \phi^{n+1} \right\}$$

Catalan Numbers

The Catalan numbers satisfy the recurrence

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

along with the initial condition $(C_0, C_1) = (1, 1)$. C_n counts the number of triangulations of a $(n+2)$-gon or Dych words of length $2n$. Using generating functions, it is possible to derive

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

There are five ways to triangulate a pentagon, which means $C_3 = 5$. It is nice to plug this value to confirm the equation.

Difference Operator

Let S be a subset of integers that are closed under succession. Let f_n be a function that maps $S \to \mathbb{R}$.

$$\Delta f(n) := f(n+1) - f(n)$$

We list some functions that behave nicely under Δ

$f(x)$	$\Delta f(x)$	Derivative Analogy
2^x	2^x	e^x
$P(x,n)$	$nP(x,n-1)$	x^n
$\binom{x}{n}$	$\binom{x}{n-1}$	$x^n/n!$
n^x	$(n-1)n^x$	a^x
$x!$	$x \cdot x!$	

We also define the antidifference operator, which is the inverse of the difference operator.

$$h = \Delta g \quad \text{then} \quad g = \Delta^{-1} h$$

An initial condition problem
Let $\{h\}_{n\in\mathbb{N}}$ be a real sequence. We are given an intial condition, that for all integers $k \in [0, N]$,

$$(\Delta^k h)_0 = c_k \quad \text{and} \quad \Delta^{N+1} h = 0$$

where the second equation means that the $N+1$th difference of h is constantly zero. It is possible to write h_n as a closed form formula of n.

$$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_N\binom{n}{N}$$

Pochhammer Symbols We used $P(n,k)$ to denote the kth falling products of n. Alternatively, we can write

$$[n]_k = P(n,k) = n(n-1)\cdots(n-k+1)$$

Bases of the Polynomial space $\mathbb{Z}[x]$ It is not hard to see that the polynomial over the integers can be spanned by the $\mathbb{Z}$ combinations of the base $\{x^k\}$ or $\{[x]_k\}$. In other words, the power set and the Pockhamer set are bases for the $\mathbb{Z}$-module $\mathbb{Z}[x]$

For the Pochhammer set and the power set are both bases, there exists a passive transformation between the two bases. Restrict our space to cover polynomials of degrees strictly less than N. Let $[s(i,j)]_{N\times N}$ denote the matrix of the passive transform from the Pochhammer base to the power base. That is, the colomn vectors of the matrix contain the coefficients of $[n]_i$. In symbols,

$$[n]_i = s(i,0)n^0 + s(i,0)n^1 + \cdots + s(i,N)n^N$$

Note that we have used the row vector convention!

Using the falling product definition of $[n]_k$, we derive the following recurrence relation.

$$s(i+1, j+1) = \begin{cases} 1 & (i = j) \\ 0 & (i < j) \\ s(i,j) - i \cdot s(i,j+1) \end{cases}$$

The initial conditions are $s(0,0) = 1$ and $s(0,n) = s(n,0) = 0$ for all nonzero n.

We call $s(i,j)$ as the **Signed Striling Numbers of the First Kind**.

We also define the matrix $[S(i,j)]_{n\times n}$ where the ith row of the matrix resembles the expansion of n^k in the Pochhammer base. In symbols,

$$n^i = S(i,0)[n]_0 + S(i,1)[n]_1 + \cdots S(i,N)[n]_N$$

Using $n^{(i+1)} = (n-j+j)n^i$, we derive

$$S(i,j) = \begin{cases} 1 & (i = j) \\ 0 & (i < j) \\ S(i-1,j-1) + j \cdot S(i-1,j) \end{cases}$$

The initial conditions are $S(0,0) = 1$ and $S(0,n) = S(n,0) = 0$ for all nonzero n.

We call $S(i,j)$ as the **Striling Numbers of the Second Kind**.

Remember that $S(2,1) = 1$ and $s(2,1) = -1$.

It is nice to memorize

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In[15]:= S4 // MatrixForm
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Out[15]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}$$

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In[14]:= s4 // MatrixForm
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Out[14]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -6 & 11 & -6 & 1 \end{pmatrix}$$

Both Striling numbers depend on the diagonal left entry and the top entry. Either $-i$, the negative row number, or j the column number is multiplied to the top entry and added to the diagonal entry.