PHYS 201 Problemset 9

Daniel Son

1. Find the electromotive force induced on a square wire when the wire is passing through the center.

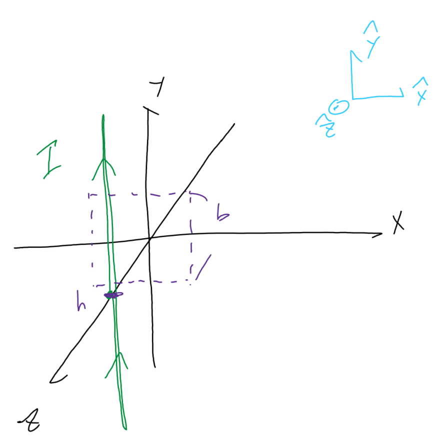


Figure 1: setup

To begin with, we determine the magnetic force along the xy-plane. Notice that by symmetry, the magnetic field is independent of the y axis. Thus, fix $y = z = 0$. We want to derive an equation for $\vec{B}(x)$.

Recall:

$$B = \frac{\mu_0 I}{2\pi r}$$

And that the direction of B is determined by the right hand rule. We write:

$$\vec{B}(x) = \frac{\mu_0 I}{2\pi\sqrt{x^2 + h^2}} - \overline{\langle h, 0, x\rangle}$$

Where the bar denotes the unit vector. As to understand how the directional vetor is derived refer to the following figure:

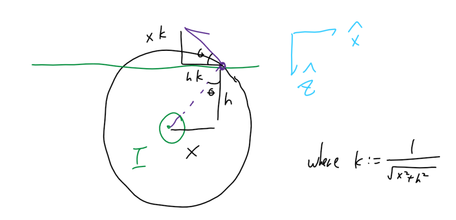


Figure 2: horizontal cut

Moreover, note that:

$$\vec{B} \cdot \vec{z} = \frac{\mu_0 I}{2\pi(x^2 + h^2)}(-x)$$

Now, compute the electromotive force by Faraday's Law. Write:

$$\mathcal{E} = -\frac{d\Phi}{dt}\bigg|_{x=0}$$

We can compute:

$$\frac{d}{dx}\oint_\gamma \vec{B} d\vec{a}\bigg|_{x=0} = b(\vec{B} \cdot \vec{z}\bigg|_{x=b/2} - \vec{B} \cdot \vec{z}\bigg|_{x=-b/2})$$

As we have established earlier, we write:

$$= b\frac{\mu_0 I}{2\pi(b^2/4 + h^2)}\left[-\frac{b}{2} - \frac{b}{2}\right] = -b^2\frac{\mu_0 I}{2\pi(b^2/4 + h^2)}$$

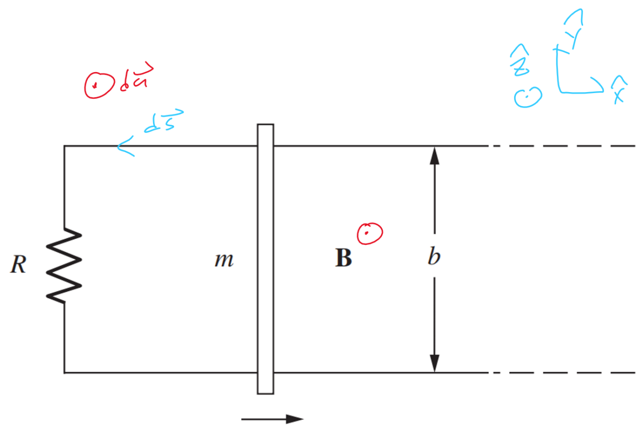
By the chain rule:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}\bigg|_{x=0} = -\frac{dx}{dt}\frac{d\Phi}{dx}\bigg|_{x=0} = \frac{2vb^2\mu_0 I}{\pi(b^2 + 4h^2)}}$$

$\square$

2. A rod connected to a circuit is passing through a uniform magnetic field. Answer the following problems:

1. When does the rod stop?
2. How much has the rod traveled?
3. What can be said about the energy of the system?



Solution It seems reasonable to first come up with a time dependent equation of the vertical velocity of the rod. Let $v(t)$ be a scalar function that describes the x direction velocity. Denote the initial velocity as v_0.

We want to compute the current induced on the rod. The induced current will be perpendicular to the direction of the magnetic field. This in turn will create a Lorentz force that is opposite of the direction of movement.

Define the direction of the surface vector $d\vec{a}$ to point out of the surface. Also, let $d\vec{s}$ to be counterclockwise.

Now that all the directions are established move on to compute the integrals. To invoke Faraday's law, compute the magnetic flux. Write:

$$\Phi = \oint_{loop} \vec{B} d\vec{a} = lbB$$

Where l denotes the horizontal distance from the resistor to the rod.

By Faraday:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -vbB$$

This derivative is justified by observing that the time derivative of l is v.

Apply Ohm's law to compute the current.

$$\mathcal{E} = IR$$
$$I = -\frac{vbB}{R}$$

The current flows clockwise. Applying the right hand rule, we deduce that the magnetic field is towards the $-x$ direction.

To compute the magnitude of the force exerted on the rod, consider the following equation:

$$qv_{drift} = Ib$$

where q is the total moving charge inside the rod.

Assuming that there is no electric field, we write:

$$F = qv_{drift}B$$

We consider only the magnitudes, and v_{drift}, B are perpendicular. Thus the equation is justified. Ignore self inductance. Also, $F = ma$. We write:

$$ma = -IbB = -\frac{vb^2B^2}{R}$$

$$a = -\frac{vb^2B^2}{mR}$$

$$\frac{dv}{dt} \cdot \frac{1}{v} = -\frac{b^2B^2}{mR}$$

Noticd that the left side is the logarithmic derivative. Write:

$$ln(v) = \frac{-b^2B^2}{mR}t + C$$

The initial condition $v(0) = v_0$ must be met. We conclude:

$$v = v_0 \text{Exp}\left[-\frac{b^2B^2}{mR}t\right]$$

To answer the first question, the rod theoretically never stops. As for the distance traveled we take the integral:

$$\int_{t=0}^{\infty} v_0 \text{Exp}\left[-\frac{b^2B^2}{mR}t\right] = -\frac{v_0 mR}{b^2B^2}\text{Exp}\left[-\frac{b^2B^2}{mR}t\right]\Bigg|_{t=0}^{\infty} = \boxed{\frac{v_0 mR}{b^2B^2}}$$

The total kinetic energy lost is:

$$\Delta K = \frac{1}{2}mv_0^2$$

The magnetic field has done no work. Nonetheless, energy is dissipated through the resistor. Recall $P = VI$ and we integrate the power over time to measure the energy lost through the resistor. Write:

$$\Delta E = \int_{t=0}^{\infty} Pdt = \int_{t=0}^{\infty} I^2 Rdt$$

Recall the relationship between the current and the rod speed:

$$I = -\frac{vbB}{R} = -\frac{v_0bB}{R}\text{Exp}\left[-\frac{b^2B^2}{mR}t\right]$$

Thus:

$$\Delta E = \int_{t=0}^{\infty}\left(-\frac{v_0bB}{R}\text{Exp}\left[-\frac{b^2B^2}{mR}t\right]\right)^2 Rdt = \int_{t=0}^{\infty}\frac{v_0^2b^2B^2}{R}\text{Exp}\left[-\frac{2b^2B^2}{mR}t\right]dt$$

$$= -\frac{mv_0^2}{2}\text{Exp}\left[-\frac{-2b^2B^2}{mR}t\right]\Bigg|_0^{\infty} = \frac{1}{2}mv_0^2$$

And hence, $\Delta E = \Delta K$ as desired.

$\square$

3. A ring-shaped wire of radius r is placed in a solenoid of infinite length. The current through the solenoid is given as a function of time as $I(t) = I_0cos(\omega t)$. The solenoid has a radius of b and has n turns per unit. Answer the following questions.

- What is the current through the wire?
- For what values does the magnetic force reach maximum?
- How does the magnetic force affect the wire?

Solution Recall the formula for magnetic force inside the solenoid. The field is uniform and the magnitude is given by:

$$\vec{B} = \mu_0 nI(t)$$

Also, by the right hand rule, the magnetic field points upwards.

Define the positive current direction to be counterclockwise. Also denote the induced current as I_r and the solenoid current as I.

$$\mathcal{E} = \frac{d\Phi}{dt} = -\mu_0nr^2\pi\frac{dI}{dt} = \mu_0nr^2\pi\omega I_0sin(\omega t)$$

By Ohm's law:

$$I_rR = \mu_0nr^2\pi\omega I_0sin(\omega t)$$

So,

$$\boxed{I_r = \frac{\mu_0nr^2\pi\omega I_0sin(\omega t)}{R}}$$

Move on to compute the magnitude of magnetic force. Assume $\vec{E} = 0$. By the Lorentz force law:

$$\vec{F} = q\vec{v} \times \vec{B}$$

We observe that $q\vec{v}$ is parallel to I_r and B is perpendicular to I. In terms of magnitude:

$$\vec{F} = CI_r\vec{B}$$

Moreover, the magnetic field is a constant multiple of I. We write:

$$\vec{F} = CI_rI$$

Where C is a constant independent of time. Combining previous results, we again simplify:

$$\vec{F} = Csin(\omega t)cos(\omega t) = Csin(\omega 2t)$$

Thus, $\vec{F}$ achieves maximum magnitude when $2t = (n + 1/2)\pi$ or

$$\boxed{t = (2n+1)\pi/4 \quad \text{where} \quad n = (0, 1, 2, \dots)}$$

Finally, consider the direction of the magnetic force. $q\vec{b} \parallel I$ and applying the right hand rule on I and B shows that the magnetic force directs radially outwards. Hence, depending when the force is measured, the magnetic force might strech or shrink the wire. However, the vertical force will always be zero. $\square$

4. A rectangular wire is moving away from an infinite wire with current 100A. Compute the electromotive force on the wire when the loop is 15 cm away from the wire. How large must the resistance of the wire be in order that self inductance can be ignored? The dimensions of the rectangular wire is of width 10cm and length 15cm. The loop is moving away in a speed of 5m/s.

Solution First, ignore self inductance. By Faraday's law, it suffices to compute the change of magnetic flux through the wire. We compute the change of flux depending on the displacement from the wire (say x). Then, apply the chain rule to compute EMF.

Define the x-direction to be the direction where the loop is moving. Also, let the suface vector to point upwards. Write:

$$\frac{d\Phi}{dx} = w(-B_{r=x} + B_{r=x+l})$$

where $w = 8cm$, $l = 10cm$. Moving on, recall the formula for the magnitude of the magnetic field induced by a straight wire:

$$B = \frac{\mu_0 I}{2\pi r}$$

Thus,

$$\frac{d\Phi}{dx} = w\left(-\frac{\mu_0 I}{2\pi x} + \frac{\mu_0 I}{2\pi(x+l)}\right) = w\frac{\mu_0 I}{2\pi}\left(\frac{1}{x+l} - \frac{1}{x}\right) = w\frac{\mu_0 I}{2\pi}\left(\frac{l}{x(x+l)}\right)$$

By Faraday:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dx}\frac{dx}{dt} = -wv\frac{\mu_0 I}{2\pi}\left(\frac{l}{x(x+l)}\right) - \approxeq 2.1 \cdot 10^{-5}V$$

The flux is decreasing in magnitude, but the signs are negative. Hence, absolute flux increases. The EMF has an opposite sign. $d\vec{s}$ is counterclockwise, so the current is clockwise.

Now, move on to compute the resistance which makes self inductance ignorable. Let Φ denote the magnetic flux induced by the wire, and Φ_l the self-induced magnetic flux. We can rigorously compute Φ_l by Biot-Savat's law, but we use a dummy estimate.

Assume magnetic field inside the loop to be uniformly:

$$B_l = \frac{\mu_0 I_l}{2\pi d}$$

where I_l is the loop current, and d is a characteristic radius, say $d = 4cm$. Computing the change of the flux:

$$\frac{d\Phi_l}{dt} = -\frac{dB_l}{dt}(lw) =\approxeq \frac{\mu_0 I_l wl}{2\pi d}\frac{v}{h}$$

Where h is the mean distance from the wire to the loop. $h \approxeq 20cm$. We wish that the self-induced EMF is smaller compared to the wire-induced EMF. Write:

$$|\frac{d\Phi}{dt}| \gg |\frac{d\Phi_l}{dt}|$$

$$wv\frac{\mu_0 I}{2\pi}\left(\frac{l}{x(x+l)}\right) \gg \frac{\mu_0 I_l wl}{2\pi d}\frac{v}{h}$$

By cancellation, we write:

$$\frac{I}{x(x+l)} \gg \frac{I_l}{dh} = \frac{\mathcal{E}/R}{dh}$$

We want:

$$R \gg \frac{\mathcal{E}x(x+l)}{Idh}$$

Through a rough computation, we realize that the left hand side is in the order of $10^{-7}\Omega$. It is reasonable to assume that the loop has much larger resistance than this value. $\square$.

5.

Estimate the inductance of a solenoid with a length of 2m and a diameter of 10cm. The solenoid is single layered, and has 1200 turns. What is the approximate error?

Solution

Recall the definition of inductance:

$$L := \Phi/I$$

Where Φ is the magnetic field, and I is the current through the solenoid. The linear dependance between the flux and the current is implied by the Biot-Savat law. This motivates the definition of inductance.

Ignore edge effects. The magnetic field is uniformly:

$$B = \mu_0 nI$$

where n is the density of the turns. Multiply the magnetic field magnitude by the total area created by the solenoid loop.

$$\Phi = \mu_0 nI(2\pi r^2)N = \frac{\mu_0 N^2 I(2\pi r^2)}{l}$$

where r is the radius.

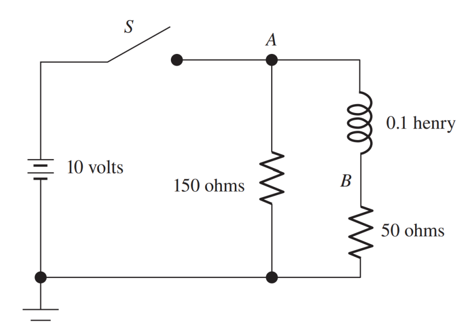
Applying the definition of inductance, again write:

$$\boxed{L = \frac{\mu_0 N^2(2\pi r^2)}{l} \approxeq 7.11 \cdot 10^{-3} T/A \text{ or } H}$$

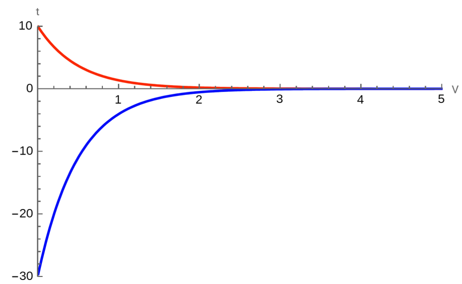
The wider the radius, the shorter the length of the solenoid, the error will be greater. The magnetic flux is overestimated in our idealized example. Simply assume the error to be in the order or radius divided by the length. This is:

$$\epsilon = 10cm/2m = 5\%$$

6. A circuit is presented below. The switch is remained closed for a few seconds. At $t = 0$, the switch is opened. Plot the voltage at point A, B as a function of time.



Solution Here is the plot:



And here comes the reasoning. A few seconds is enough to bring the current to an equilibrium state. Thus, the solenoid can be ignored. It becomes trivial to observe, that at time $t = 0$, $V_A = V_B = 0$. By Ohm's law, compute the current through the solenoid at equilibrium.

$$V_0 = I_0 \cdot R_{eff} \quad \text{and} \quad I_0 = 15V/50\Omega = .2A$$

Now, we consider the current after the switch has been open. Combining Kirhihoff's loop rule along with Faraday's law, write:

$$\mathcal{E} - I(200\Omega) = 0$$

Furthermore:

$$-\frac{d\Phi}{dt} = -L\frac{dI}{dt} = 200\Omega I$$

And thus:

$$\frac{1}{I}\frac{dI}{dt} = -\frac{200\Omega}{L}$$

The left side is the logarithmic derivative. We have previously deduced an initial condition, $I(0) = .2A$. Ergo:

$$I(t) = .2e^{-200\Omega t/L}\text{Amps}$$

It is easy to deduce V_A, V_B via Ohm's law:

$$V_A = 50I(t)\text{Volts}$$
$$V_B = -150I(t)\text{Volts}$$

And the plot illustrates this result.

7. An RL circuit is presented. The inductance of the solenoid is $.5mH$ and the resistance is $.1\Omega$. The power supply provides a potential difference of $12V$. Assuming the power supply has no internal resistance, compute the time t_1 when the current reaches 90% of its full capacity. Also, compute the energy stored in the solenoid at $t = t_1$.

Utility

$$(1 - e^{-x}) = .9$$

has a solution at $x = ln(10)$

Proof Follow the algebra:

$$e^{-x} = .1$$

$$e^{x} = 10$$

$$x = ln(10)$$

□

Solution We know that the current is given as a function of time as:

$$I(t) = \frac{V_0}{R}(1 - e^{-t/\tau})$$

where $\tau = L/R$. After a sufficient amount of time, the current reaches $I_{eq} = V_0/R$. We wish to compute the time t_1 where the current is 90% of this value. Write:

$$I(t_1) = \frac{V_0}{R}(1 - e^{-t_1/\tau}) = \frac{V_0}{R}(.9)$$

By the utility equation that we had solved above, we deduce:

$$t_1/\tau = ln(10) \quad \text{and} \quad t_1 = ln(10)\tau = ln(10)L/R$$

After some computation:

$$t_1 \approxeq .115s$$

To compute the energy, recall the formula:

$$E(t) = \frac{1}{2}LI(t)^2$$

Compute:

$$I(t_1) = .9V_0/R = 1080A$$

Ergo:

$$\boxed{E(t_1) = \frac{1}{2}(.5mH)(1080A)^2 \approxeq 292J}$$