301 Midterm

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1 Section A

A1) Math Fact

Part a. Split and switch order. It is easier done than said.

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{-a}^{0} f(x)dx \tag{1.1}$$

$$= \int_{0}^{a} f(x)dx + \int_{a}^{0} f(-x)(-dx) \tag{1.2}$$

$$= \int_{0}^{a} \left(f(x) + f(-x)\right) dx \ = \ 0 \tag{1.3}$$

The integrand of the last integral vanishes since $f(x)$ is odd. $\square$

Part b. Set $u = x - b$, then invoke the result from part a.

$$\int_{b-a}^{b+a} f(x-b)dx \ = \ \int_{-a}^{a} f(u)du \ = \ 0 \tag{1.4}$$

$\square$

A2) Photon Absorbtion

We first claim that we can ignore relativity for this problem. Upon inspection, the rest energy of the Rb atom is about 8 orders or magnitude greater than the kinetic energy added by the photon. Equate kinetic energy with the energy of a single photon.

$$\frac{1}{2}mv^2 \ = \ \frac{hc}{\lambda} \tag{1.5}$$

$$v \ = \ \sqrt{\frac{2ch}{m\lambda}} \tag{1.6}$$

Also, the following list of constants come in handy.

$$m = \ 87 \text{ amu} \ = \ 87 \cdot 1.66 \cdot 10^{-27}\text{kg}$$
$$c = \ 3 \cdot 10^{8} m/s$$
$$h = \ 6.626 \cdot 10^{-34} J/s$$
$$\lambda = \ 780 \ nm \ = \ 7.8 \cdot 10^{-7} m$$

Thus,

$$\boxed{v \ = \ 1879 m/s} \tag{1.7}$$

A3) Solar Radiation

As a preliminary, recall the Plank distribution.

$$\rho(v, T) \;=\; \frac{8\pi v^2}{c^3}\frac{hv}{e^{hb/k_B T} - 1} \tag{1.8}$$

Part a.

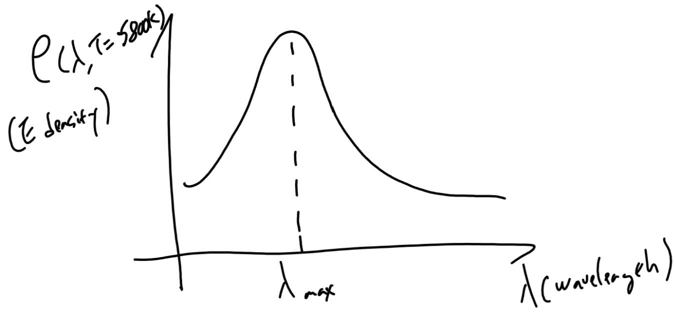

Figure 1: Sketch of the spectrum for part a

Part b. We wish to find the peak wavelength. From PS1, we have computed the peak frequency.

$$\nu_{\text{pk}} \;\approx\; \frac{2.82 k_B T}{h} \tag{1.9}$$

We derive the following expression for peak wavelength.

$$\boxed{\lambda_{\text{pk}} \;=\; \frac{hc}{2.82 k_B T} \;=\; 880\ nm} \tag{1.10}$$

□

Part c. If the wavelength ranges from 500 nm to 1100 nm, then the frequency ranges from $\nu_1 = c/1100\ nm$ and $\nu_2 = c/500\ nm$. We set up an integral over the Plank distribution to find the fraction of energy that will be absorbed.

$$\boxed{\int_{c/1100\ nm}^{c/500\ nm} \rho(\nu, 5800K) d\nu} \tag{1.11}$$

□

2 Section B

B1) Single Photon Interference

Part a. A HeNe laser emits a light of wavelength 633 nm with a power efficiency of $P = 3\ mw$. The ray of light interacts with the intereferometer, and reaches the APD where it is detected. The loss is incurred by:

1. Iris, loss of 50%
2. ND filter, loss of 10^{-d}
3. APD detection efficiency, loss of 50%

We first count how much photons the laser emits. A single photon has an energy of

$$E = pc = \frac{hc}{\lambda} \tag{2.1}$$

so the numerical frequency without losses are

$$P/E = \frac{P\lambda}{hc}. \tag{2.2}$$

Finally, we set up an equation, taking account of losses.

$$\frac{10^{-d}}{4}\frac{P\lambda}{hc} = 500 \text{ Hz} \tag{2.3}$$

Solve for d, the necessary index of the ND filter.

$$\boxed{d = -\log\left(\frac{hc}{P\lambda} 2000 \text{ Hz}\right) \approx 12.68} \tag{2.4}$$

□

Part b-i. The linear gate accepts input from the function generator and the APD. Whenever a photon is detected, a pulse is sent to the linear gate, and the linear gate sends the voltage of the function generator at the moment when of detection. The function generator is also connected to the speaker, and the voltage of the function generator corresponds to the displacement of the moving leg.

The APD tallys the voltage information recieved from the linear gate and outputs an histogram that displays the information of photon counts for each bin of voltages[1]. □

Part b-ii. The plots of the background signal will not change significantly. However, the interference signal would have an additional cycle, i.e. the existing plot would be squished to include about 2/3 of an additional sin function. This is because the increase of the function generator voltage increases the scan distance.

□

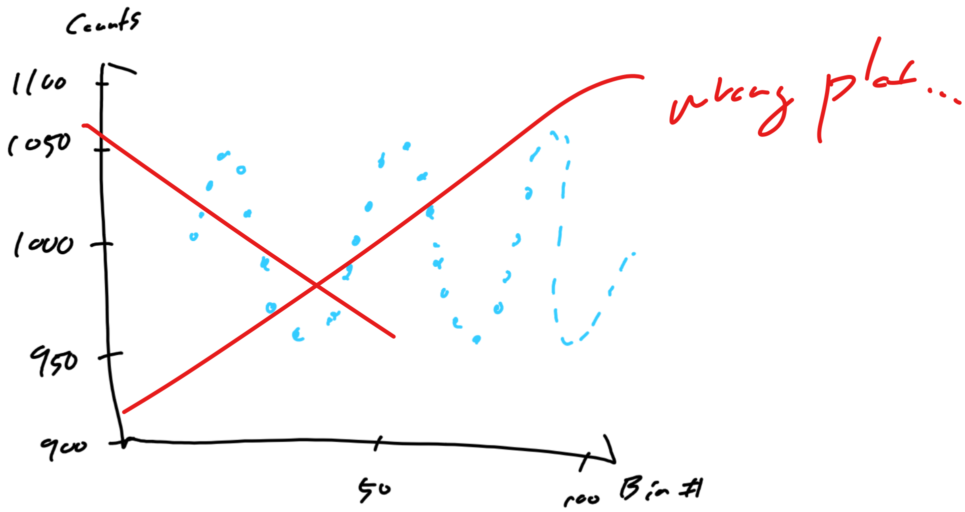

Figure 2: Expected measurement for increased scan (20%)

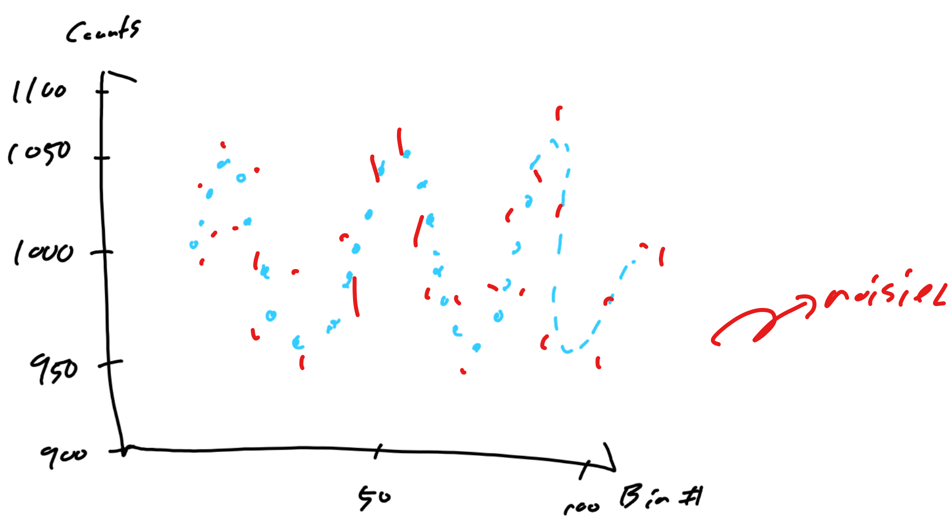

Figure 3: Expected measurement for additional attenuation

Part b-iii. The peak to peak count of the interference signal before attenuation is about 10000 counts. An additional attenuation of ND2 will decrease this count by a factor of 100. Hence, the new peak to peak difference of the count would be around 100. Moreover, the background signal ranges around 1000 counts in average. ☐

~~*Proof.*~~ Part C Consider Feynman's description of the single slit experiment. Single photons without interaction can be considered as the distribution of bullets as a gunner randomly shoots toward the screen. Without interference, the distribution would be uniform, without an identifiable peak or valley, but only random noise. ☐

3 Section C

C1) Free Particles and Fourier Transforms

Part a. ☐

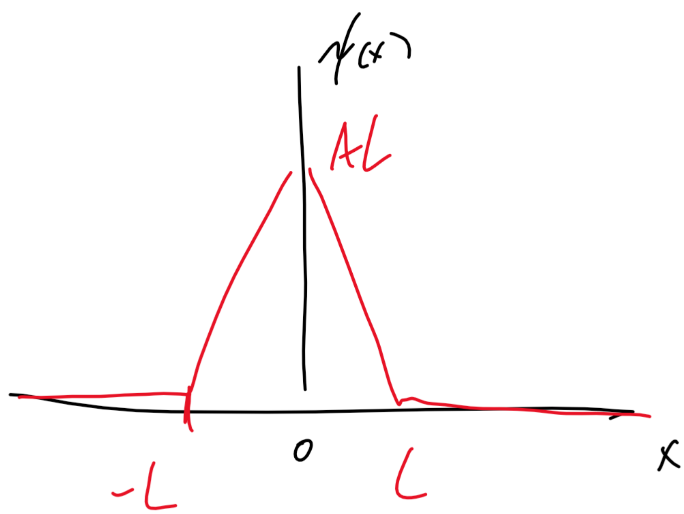

Figure 4: Sketch of the wavefunction at $t = 0$

Part b. We compute the normalization constant by imposing the condition that

[1]each bin is a small interval of voltage

the integral of the square of the wavefunction must equal 1.

$$\int_{\mathbb{R}} \psi(x)^2 dx = 1 \tag{3.1}$$

$$2\int_0^L (L-x)^2 A^2 dx = 1 \tag{3.2}$$

$$2A^2 \left[-\frac{(L-x)^3}{3}\right]_0^L = 1 \tag{3.3}$$

$$\frac{2A^2L^3}{3} = 1 \tag{3.4}$$

$$\boxed{A = \sqrt{\frac{3}{2L^3}}} \tag{3.5}$$

□

Part c. Clearly, $\psi(x)^2 x$ is odd. Thus, integrating this function over the entirety of $\mathbb{R}$ should yield zero.

$$\langle x \rangle = 0 \tag{3.6}$$

The momentum can be computed via by-parts integration. Recall the form of the momentum operator in position space.

$$p \equiv -i\hbar\nabla \quad \text{and} \quad p^2 \equiv -\hbar^2\nabla^2 \tag{3.7}$$

Our particle is 1D, so $\nabla = \frac{\partial}{\partial x}$. Consider the following lines of algebra.

$$\langle p^2 \rangle = \int_{\mathbb{R}} \psi(x)(-\hbar^2\nabla^2)\psi(x)dx \tag{3.8}$$

$$= -\hbar^2 \int_{\mathbb{R}} \psi(x)\psi''(x)dx \tag{3.9}$$

Set $u = \psi(x)$, $dv = \psi''(x)dx$.

$$\langle p^2 \rangle = -\hbar^2 \left(uv - \int vdu\right) = -\hbar^2\psi(x)\psi'(x)\Big|_{-\infty}^{\infty} + \hbar^2 \int_{\mathbb{R}} \psi'(x)^2 dx$$

$$= \hbar^2(A^2 2L) = \boxed{\frac{3\hbar^2}{L^2}} \tag{3.10}$$

The last equality of (3.10) follows from the fact that $\psi'(x)$ is nonvanishing only at the region $(-L, L)$.[2] □

[2]To the timid mathematician who worries that $\psi'(x)$ is undefined at $x = 0, \pm L$, we assure that we can smoothen the function by infinesimal amount and take limits.

Part d. Take the Fourier transform of the wavefunction to obtain an expression for $\phi(k)$.

$$\phi(k) \;=\; \mathcal{F}\{\psi(x)\}(k) \tag{3.11}$$

Unfold the definition and evaluate the integral.

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{-ikx} dx & (3.12)\\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^L A(L-x) e^{-ikx} dx + \int_{-L}^0 A(L-x) e^{-ikx} dx \right] & (3.13)\\
&= \frac{1}{\sqrt{2\pi}} \int_0^L 2A(L-x)\cos(kx) dx & (3.14)
\end{aligned}$$

Set up a by-parts integration table.

sign	u	dv
$+$	$L-x$	$\cos(kx)$
$-$	-1	$\frac{\sin(kx)}{k}$
$+$	0	$-\frac{\cos(kx)}{k^2}$

Finally, we have a closed form expression for $\psi(x)$.

$$\begin{aligned}
\phi(k) &= \frac{2A}{\sqrt{2\pi}} \left[\frac{L-x}{k} \sin(kx) - \frac{\cos(kx)}{k^2} \right]_0^L \;=\; \frac{2A}{\sqrt{2\pi}} \left[\frac{1}{k^2} - \frac{\cos(kL)}{k^2} \right] & (3.15)\\
&= \boxed{\sqrt{\frac{3L}{\pi}} \left(\frac{1-\cos(kL)}{(kL)^2} \right)} & (3.16)
\end{aligned}$$

□

Part e. The wavefunction starts as a $\wedge$ shape then disperses to the entirety of $\mathbb{R}$ in form of multiple sinuoidial bumps with different amplitudes. This is because the wavefunction at time $t = 0$ is composed of multiple energy eigenstates. Eigenstates with higer energy disperse away from the center faster, and the others move slower. □

C2) Superpositions in the infinite well

Remember that in an ISW, the nth energy eigenstate is the following.

$$\psi_n(x) \;=\; \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \tag{3.17}$$

Part a. Invoke orthonormality of the energy eigenstates.

$$\int_{\mathbb{R}} \psi_A(x)^2 dx = 1 \tag{3.18}$$

$$N^2 \left[\int 3\psi_1^2(x)dx + \int \psi_3^2(x)dx - \sqrt{3} \int \psi_1(x)\psi_3(x)dx \right] = 1 \tag{3.19}$$

$$4N^2 = 1 \tag{3.20}$$

$$\boxed{N = \frac{1}{2}} \tag{3.21}$$

□

Part b. There are no hopes that either of ψ_A or ψ_B are eigenstates of $\hat{p} \equiv i\hbar\nabla$. The spacial derivative converts the sin terms to cosine, and a constant multiple of the sin function cannot generate a cosine function.

ψ_B is an eigenstate of $\hat{p}^2$, while ψ_A is not. ψ_B is an energy eigenstate, and within the well, the potential is zero so $E = p^2/2m$ and the energy operator is a constant multiple of $\hat{p}^2$. ψ_A is not an energy eigenstate, so it cannot be an eigenstate of $\hat{p}^2$. □

Part c. For a mixed state, the time dependent wavefunction can be retrieved by decomposing the time independent solutions into energy eigenstates and invoking time evolution for each energies. We already have an eigendecomposition.

The time evolution function is

$$T_n(t) = \exp\left(-\frac{iE_n t}{\hbar}\right) \tag{3.22}$$

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2} \tag{3.23}$$

for the nth energy eigenstate.

We write out the time dependent wavefunctions.

$$\Psi_A(x,t) = \sqrt{\frac{1}{2a}} \left[-\sqrt{3}\sin\left(\frac{\pi x}{a}\right)\exp\left(-\frac{i\hbar\pi^2 t}{2ma^2}\right) + \sin\left(\frac{3\pi x}{a}\right)\exp\left(-\frac{9i\hbar\pi^2 t}{2ma^2}\right)\right] \tag{3.24}$$

$$\Psi_B(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\exp\left(-\frac{2i\hbar\pi^2 t}{ma^2}\right) \tag{3.25}$$

□

Part d. The particle collapses into one of the energy eigenstates, either ψ_1 or ψ_3. The possible energy measurements are

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2} \quad \text{and} \quad E_2 = \frac{2\hbar^2\pi^2}{ma^2} \tag{3.26}$$

We compute the probability of this probability by the Born probability rule.

$$P(A \to 1) = \langle \psi_A | \psi_1 \rangle^2 = \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4} \tag{3.27}$$

$$P(A \to 3) = \langle \psi_A | \psi_3 \rangle^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4} \tag{3.28}$$

The state of the particle after measurement is the first or the third energy eigenstate, which is either

$$\psi_1(x)T_1(t) \quad \text{or} \quad \psi_3(x)T_3(t). \tag{3.29}$$

□

Part e. Classical probability comes to the rescue.

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \tag{3.30}$$

We know that $E_3 = 3E_1$ and that the probability of the wavefunction collapsing into the third energy eigenstate is $1/4$.

$$\langle E \rangle = \frac{3}{4}E_1 + \frac{1}{4}E_3 = \frac{3+9}{4}E_1 = 3E_1 \tag{3.31}$$

$$\langle E^2 \rangle = \frac{3}{4}E_1^2 + \frac{1}{4}E_3^2 = \frac{3+81}{4}E_1^2 = 21E_1^2 \tag{3.32}$$

$$\boxed{\Delta E = \sqrt{21-9}E_1 = 2\sqrt{3}E_1 = \frac{\sqrt{3}\hbar^2\pi^2}{ma^2}} \tag{3.33}$$

□

Part f. The probability densities of Ψ_A are time dependent, while the probability density of Ψ_B is time independent. This is because ψ_B is in a pure state and the wavefunction can be brought to the same shape under some phase shift. However, for Ψ_B, the two energy eigenstate evolve in a different speed, and the interaction between the two deforms the wavefunction such that a constant phase shift cannot bring the wavefunction to the initial form.

The frequency of the oscillation of the amplitude $|\Psi_A|^2$ is determined by the frequency of the lowest eigenstate.[3] Let τ denote the period, and f the frequency.

$$\frac{E_1\tau}{\hbar} = 2\pi \tag{3.34}$$

$$\boxed{f = \frac{1}{\tau} = \frac{E_1}{2\pi\hbar} = \frac{\hbar\pi}{4ma^2}} \tag{3.35}$$

□

[3]This does not hold in full generality, but only because 1 is an integer multiple of $1/3^2$ in this case.

Part g. Both the position and momentum are time independent. Relabel the axis such that the new zero is the center of the well. Then, both the first and the third energy eigenstates have an even symmetry. The same goes for their derivates. The superposition of the first and the third eigenstate, regarless of the phase difference imposed by time evolution, must also be even. Hence the expected value of both of these observables are zero in the relabeled space. □

C3) Uncertainty and Bound States

Part a. Write out the time independent Schrodinger equation to provide a condition for the energy eigenstates.

$$E\psi(x) \;=\; -\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) \tag{3.36}$$

Solve for $V(x)$ and impose even symmetry.

$$V(x) = \; E + \frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} \tag{3.37}$$

$$V(x) = \; V(-x) \quad \text{ or } \quad E + \frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} \;=\; \frac{\hbar^2}{2m}\frac{\psi''(-x)}{\psi(-x)} \tag{3.38}$$

We obtain a nice condition for $\psi(x)$.

$$\psi''(x)\psi(-x) \;=\; \psi''(-x)\psi(x) \tag{3.39}$$

Since $\psi(x)$ is a bounded particle, it is possible to expand the function into sum of sines and cosines. For simplicity, suppose the particle is bounded in the region $[-\pi, \pi]$.

$$\psi(x) = \; \sum_n a_n \sin(nx) + \sum_n b_n \cos(nx) + C \tag{3.40}$$

$$\psi''(x) = \; -\sum_n a_n n^2 \sin(nx) - \sum_n b_n n^2 \cos(nx) \tag{3.41}$$

Suppose the constant offset C is nonzero, and compare the coefficient of $\sin(nx)$ from (3.39).

$$-C a_n n^2 \;=\; C a_n n^2 \tag{3.42}$$

Hence the a_n terms vanish and $\psi(x)$ is a sum of cosines, thus even. Now, suppose there exists some $a_n \neq 0$. Then, we compare the coefficient of $\sin(nx)\cos(mx)$ for any m.

$$-a_n b_m n^2 m^2 \;=\; a_n b_m n^2 m^2 \tag{3.43}$$

So b_m vanishes and $\psi(x)$ is a sum of sines, hence odd.

This means that the probability amplitude $\psi(x)^2$ is always even. The expected value of x over any even probability distribution defined over $\mathbb{R}$ must be zero. Thus,

$$\langle x \rangle \;=\; 0 \tag{3.44}$$

Extra Credit: Suppose $\Psi(x,t)$ is a general state that is not necessarily pure. Then, the time average expectation value can be computed as follows.

$$\langle x \rangle = \int_T \int_{\mathbb{R}} \Psi(x,t)^* x \Psi(x,t) dx\ dt \tag{3.45}$$

$$\int_T \int_{\mathbb{R}} \left(\sum_n \psi_n(x) e^{iE_n/\hbar t} \right) x \left(\sum_m \psi_m(x) e^{-iE_m/\hbar t} \right) dx\ dt \tag{3.46}$$

The cross terms involving $\psi_n(x)\psi_m(x)$ dies out when taking long time averages, since the oscillations do not cancel out, i.e. $e^{-i(E_n - E_m)/\hbar t}$ is not constant.

$$\langle x \rangle \;=\; \sum_n \int_{\mathbb{R}} x \psi_n(x)^2 dx \;=\; 0 \tag{3.47}$$

Each of the component integrals are zero, for they are the expected position of each energy eigenstate. $\square$

Part b.

$$\Delta x \;=\; \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \;=\; \sqrt{\langle x^2 \rangle} \;=\; x_{\text{RMS}} \tag{3.48}$$

The crux of the argument is that $\langle x \rangle = 0$. $\square$

Part c. Considering the Fourier expansion of a function, we notice that a derivative of an even function must be odd and the derivative of an odd function must be even. With this in mind, compute $\langle p \rangle$ for an eigenstate.

$$\langle p \rangle \;=\; \int_{\mathbb{R}} \psi(x)(-i\hbar)\frac{d}{dx}\psi(x) dx \;=\; -i\hbar \int_{\mathbb{R}} \psi(x)\psi'(x) dx \;=\; 0 \tag{3.49}$$

The last equality follows because integrating an odd function over $\mathbb{R}$ yields zero. For nonpure states, we can repeat the argument in part a to obtain the same result.

$$\Delta p \;=\; \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \;=\; \sqrt{\langle p^2 \rangle} \;=\; x_{\text{typical}} \tag{3.50}$$

Note that for nonpure states, this result holds only for time average values of uncertainty.

$$\Delta p \;\sim\; p_{\text{typical}} \tag{3.51}$$

$\square$

Part d-i. The quantity $\Delta x \Delta p = x_{\mathrm{RMS}} p_{\mathrm{typical}}$ resembles the angular momentum of the particle. The lowest energy eigenstate has the lowest energy, and it is reasonable to say that a particle in this state must have the lowest angular momentum. □

Part d-ii. Bash out the expected value computation.

$$
\begin{aligned}
E = &\ \langle p^2 \rangle / 2m + \langle V(x) \rangle && (3.52)\\
= &\ p_{\mathrm{typical}}^2 / 2m + \frac{m\omega^2}{2} \langle x^2 \rangle && (3.53)\\
= &\ p_{\mathrm{typical}}^2 / 2m + \frac{m\omega^2}{2} x_{\mathrm{RMS}}^2 && (3.54)\\
= &\ \frac{\hbar}{2m x_{\mathrm{RMS}}^2} + \frac{m\omega^2 x_{\mathrm{RMS}}^2}{2} && (3.55)
\end{aligned}
$$

We wish to find the minimum value of this energy. Consider the following function $f : \mathbb{R}_{\mathrm{pos}} \to \mathbb{R}$.

$$f(u) \ = \ u + \frac{1}{u} \tag{3.56}$$

This function achieves minimum of 2 at $u = 1$.[4] For some constants α, β, we can rewrite the energy as follows.

$$E(x_{\mathrm{RMS}}^2) \ = \ \alpha \left(\frac{1}{\beta x_{\mathrm{RMS}}^2} + \beta x_{\mathrm{RMS}}^2 \right) \ = \ \alpha f(\beta x_{\mathrm{RMS}}^2) \tag{3.57}$$

By comparison with (3.55), we write

$$
\begin{aligned}
\alpha\beta \ = \ \frac{m\omega^2}{2} \quad \text{and} \quad \beta/\alpha \ = \ \frac{2m}{\hbar^2} && (3.58)\\
\alpha \ = \ \sqrt{\frac{m\omega^2\hbar^2}{4m}} \ = \ \frac{\hbar\omega}{2} && (3.59)
\end{aligned}
$$

The minimum energy is 2α.

$$\boxed{E \ = \ \hbar\omega} \tag{3.60}$$

□

C4) Half Infinite Well

Part a, b. Both of the three eigenstates vanish at region I where the potential is infinite. On region II, it displays oscillatory behavior, where the first mode has one node, second mode has two, and the third has three. On region III, the potentials die out exponentially. □

[4] This is a calculus proof. $f'(1) = 0$ and indeed $u = 1$ is a global minimum upon inspection.

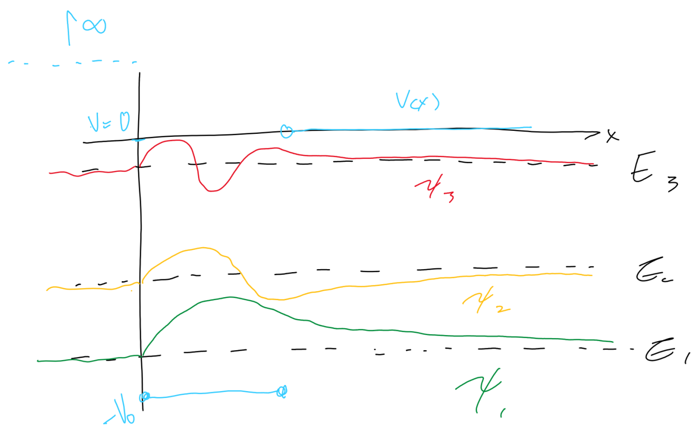

Figure 5: The Half infinite potential and the first three energy eigenstates.

Part c.

$$E\psi = -\frac{\hbar^2}{2m}\psi'' + \infty\psi \qquad \text{Region I} \tag{3.61}$$
$$E\psi = -\frac{\hbar^2}{2m}\psi'' - V_0\psi \qquad \text{Region II} \tag{3.62}$$
$$E\psi = -\frac{\hbar^2}{2m}\psi'' \qquad \text{Region III} \tag{3.63}$$

□

The wavefunction vanishes at Region I.[5]

Since $E < 0$, we notice that the equation for Region II is exactly a SHO, and has a complex exponential solution. Region III is similar, but the solution is a regular exponential. Exponential growth at $x \to \infty$ is not plausible, so the solution must be a negative exponential. Thus, the time independent solution must be in the following form.

$$\psi(x) = \begin{cases} 0 & (x < 0) \\ A\sin(kx) & (0 \le x \le L) \\ Be^{-qx} & (x \ge L) \end{cases} \tag{3.64}$$

Part d. The wavefunction is continuous and differentiable at point $x = L$.

$$\boxed{\psi(L_-) = \psi(L_+) \quad \text{and} \quad \psi'(L_-) = \psi'(L_+)} \tag{3.65}$$

□

[5] It feels queasy to write ∞ as a number, but it shows the fact that ψ cannot be nonvanishing at the region.

We do some more work before we provide a solution for the last part. The boundary condition can be solved as follows.

$$A\sin(kL) = Be^{-qL} \tag{3.66}$$

$$A\cos(kL) = -\frac{q}{k}Be^{-qL} \tag{3.67}$$

Also, the energy can be written in terms of k, q by (3.61).

$$E = -\frac{\hbar^2 q^2}{2m} \tag{3.68}$$

$$E = \frac{\hbar^2 k^2}{2m} - V_0 \tag{3.69}$$

Introduce the following subsititution.

$$R = L\sqrt{k^2+q^2} \tag{3.70}$$

$$y = kL \tag{3.71}$$

Note that R depends on V_0.

$$R = \frac{L}{\hbar}\sqrt{2mV_0} \tag{3.72}$$

Part e-i. We suspect that if $L \to 0$ or $V_0 \to \infty$, then the wavefunction must approach an infinite square well.[6] □

Part e-ii. We wish to show that the energies of the wavefunction approach those of an infinite square well as $V_0 \to \infty$. From (3.69), we notice that the energies already resemble the ISW case but with a constant offset of V_0. For the previous solution of the ISW, we assumed that the bottom of the well has a potential of zero. For the limiting case of the half infinite well, the bottom potential is V_0, so we can ignore this constant offset. It suffices to show that $k = n\pi/L$ for eigenstates.

From the boundary conditions stated in (3.66), we can deduce

$$A\sin(kL) + \frac{Ak}{q}\cos(kL) = 0 \tag{3.73}$$

$$\frac{y}{\sqrt{R^2-y^2}} = -\tan(kL) \tag{3.74}$$

$$-y\cot(y) = \sqrt{R^2-y^2} \tag{3.75}$$

As $V_0 \to \infty$, $R \to \infty$, and the RHS of (3.75) becomes an infinitely large semicircle, which can be considered as a horizontal line at infinity. $\cot(y)$ approaches infinity at $y = n\pi$. Therefore,

$$\boxed{k = \frac{n\pi}{L}} \tag{3.76}$$

[6] The first condition $L \to 0$ is disputable, since if the well width goes to zero, then the well disappears.

and this yields

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2} - V_0 \tag{3.77}$$

which is exactly the energy levels of the nth eigenstate of a ISW where the bottom potential is V_0 and the well width is L.

In the limiting case of the Half Infinite Well with width L,

$$\boxed{L = l} \tag{3.78}$$

where l is the width of an ISW. That is, the width of the two wells match. $\square$