PHYS 202 HW3

Daniel Son

Q1 Identify the faulty labeling from the power resonance curve.

Solution

The peak of the curve occurs at $f = 22.5Hz$ so $\omega_{res} \approxeq 172kHz$. The FWHM of the curve is about $\gamma = 2\pi \cdot 500Hz \approxeq 3kHz$. We claim that the wrong label is the inductance, and hence the correct label is $(L, C, R) = (30mH, 1nF, 100\Omega)$.

Assuming the purported label is correct, we compute the theoretical resonant frequency and γ.

$$\gamma_{th} = R/L = 3.3kHz$$
$$\omega_{th} = 1/\sqrt{LC} = 180kHz$$

Which is close enough to the experimental value. $\square$

Q2 A resistor, inductor, and conductor is connected in parallel. Assuming that the a varying input current is entering the circuit in the rate of $I(t) = I_0 \cos(\omega t)$,

a) What is the resonant frequency?

b) What is the FWHM of the power resonance curve?

c) Define $Q := \omega_0/FWHM$. What is the relationship between the Q-factors of the parallel and series circuit?

Solution

We start off with writing the complexified voltage across each component.

$$\tilde{I}_{in} = \sum I = \tilde{V}_{in}\left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)$$

Thus

$$\tilde{V}_{in} = \tilde{I}_{in}/\left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)$$

a)

Resonance occurs when the complex impedences cancel out. That is

$$\omega C = 1/(\omega L) \quad \text{or} \quad \omega^2 = \frac{1}{LC} \quad \text{or} \quad \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Recovering the real value for V_{in}

$$V_{in}(t) = \frac{I_0 \cos(\omega t + \phi)}{\sqrt{1/R^2 + (\omega C - 1/(\omega L))^2}}$$

we can write an expression for $P(t)$.

$$P(t) = V_{in}(t)^2/R = \frac{I_0^2 \cos^2(\omega t + \phi)}{R(1/R^2 + (\omega C - 1/(\omega L))^2)}$$

The time average of the squared cosine function is 1/2. Thus

$$\langle P(t)\rangle = \frac{I_0^2}{2R(1/R^2 + (\omega C - 1/(\omega L))^2)}$$

Ignoring the constants, we recognize that the FWHM occurs when

$$1/R^2 = (\omega C - 1/(\omega L))^2$$

Multiply both sides by $L^2\omega^2$

$$\omega^2/\gamma^2 = (\omega^2 LC - 1)^2 \quad \text{and} \quad \pm\,\omega/\gamma = \omega^2 LC - 1$$

Call the solutions to the two equations as ω_1 and ω_2.

$$\omega_1/\gamma = \omega_1^2 LC - 1 \quad \text{and} \quad -\,\omega_2/\gamma = \omega_2^2 LC - 1$$

Subtracting the bottom equation from the top

$$(\omega_1 + \omega_2)/\gamma = (\omega_1^2 - \omega_2^2)LC \quad \text{or} \quad (\omega_1 - \omega_2) = \frac{1}{\gamma LC} = \boxed{\omega_0^2/\gamma}$$

c) We compute the Q-factor for the parallel circuit.

$$Q_{par} = \omega_0/\omega_0^2/\gamma = \gamma/\omega_0$$

We also know the Q-factor for series circuits.

$$Q_{ser} = \omega_0/\gamma$$

Thus

$$\boxed{Q_{ser} = 1/Q_{par}}$$

Q3 Reduced Mass Two masses are connected to each other with a string with constant k.

a) Write down the equation of Newton's II law for the two masses. Write the matrix form of the equation.

Solution We write the two equations.

$$m_1\ddot{x}_1 = -k(x_1 - x_2) \quad \text{and} \quad m_2\ddot{x}_2 = -k(x_2 - x_1)$$

We wish to obtain a homogeneous system.

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \quad \text{and} \quad m_2\ddot{x}_2 + k(x_2 - x_1) = 0$$

We guess the solutions to be in the form of $\tilde{x}_1 = Ae^{i\omega t}, \tilde{x}_2 = Be^{i\omega t}$. Plugging in and cancelling the exponential term, we write

$$-\omega^2 m_1 A + kA - kB = 0 \quad \text{and} \quad -\,\omega^2 m_2 B + kB - kA = 0$$

In matrix form

$$\begin{bmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Determine the normal modes of the system

Solution

The normal modes occur when the determinant of the coefficient matrix equals zero. That is

$$\begin{vmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{vmatrix} = (k - \omega^2 m_1)(k - \omega^2 m_2) - k^2 = 0$$

$$\omega^4 m_1 m_2 - k(m_1 + m_2)\omega^2 = 0 \quad \text{or} \quad \omega^2(\omega^2 m_1 m_2 - k m_1 - k m_2) = 0$$

The solutions to this equation are

$$\boxed{\omega = 0 \quad \text{or} \quad \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{k/\mu}}$$

c) Determine the eigenvector which corresponds to each normal mode frequency. For the vibrational mode, how does the ratio of the two masses' ranges of motion relate to their respective masses?

Solution Write the coefficient matrix for the two possible cases of ω. If the frequency is zero and nonzero, the coefficient matricies are respectively

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k - m_1 k/\mu & -k \\ -k & k - m_2 k/\mu \end{bmatrix} = -\begin{bmatrix} m_1 k/m_2 & k \\ k & m_2 k/m_1 \end{bmatrix}$$

Upon inspection, the eigenvectors that correspond to each of the matricies are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} m_2/m_1 \\ -1 \end{bmatrix}$$

The former mode is the stationary mode and the latter mode is the vibrational mode. For the vibrational mode, if the mass of the two bodies are comparable to each other, the amplitude of their oscilations will also be comparable. Nonetheless, if one of the bodies are much heavier, the oscillation amplitude of the lighter mass will be larger than the lighter mass. Also, the phase of the two oscillations will differ by π.

$\square$

Q4 a) Derive the differential equations that governs the evolution of the coupled LC oscillator.

Solution Call the current that goes to the leftmost and the rightmost part of the circuit I_1, I_2, and the current through the center I_C. We mark all currents to point from bottom to top. By the juction rule, we deduce

$$I_1 + I_2 + I_C = 0$$

Integrating over time, and assuming there is no input voltage

$$Q_1 + Q_2 + Q_C = 0$$

Also, the voltage difference across all the circuit elements must be the same. Hence

$$L_1\ddot{Q}_1 + Q_1/C_1 = L_2\ddot{Q}_2 + Q_2/C_2 = Q_C/C_C$$

Substituting $Q_C = -(Q_1 + Q_2)$, we obtain

$$L_1\ddot{Q}_1 + Q_1/C_1 = -(Q_1 + Q_2)/C_C \quad \text{and} \quad L_2\ddot{Q}_2 + Q_2/C_2 = -(Q_1 + Q_2)/C_C$$

The inductance of each component relates to the mass and the reciprocal of the capacity relates to the spring constant. Formally:

$$\boxed{(L_1, L_2) \leftrightarrow (m_1, m_2), (1/C_1, 1/C_2, 1/C_C) \leftrightarrow (k_1, k_2, \kappa)}$$

Complexify Q_1, Q_2 and try the solutions $\tilde{Q}_1 = Ae^{i\omega t}, \tilde{Q}_2 = Be^{i\omega t}$. Dividing by the exponential terms, we obtain

$$-L_1\omega^2 A + A/C_1 + (A + B)/C_C = 0$$
$$-L_2\omega^2 B + B/C_2 + (A + B)/C_C = 0$$

Divide by inductance and write in matrix form

$$\begin{bmatrix} -\omega^2 + 1/(L_1C_1) + 1/(L_1C_C) & 1/(L_1C_C) \\ 1/(L_2C_C) & -\omega^2 + 1/(L_2C_2) + 1/(L_2C_C) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Assuming $L_1 = L_2 = L, C_1 = C_2 = C_C$, compute the two resonant frequencies.

<u>Solution 1</u> Using matricies

Plug in the conditions to the matrix formula

$$\begin{bmatrix} -\omega^2 + 2/(LC) & 1/(LC) \\ 1/LC & -\omega^2 + 2/(LC) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The normal modes occur when the determinant of the coefficient matrix equals zero. That is

$$(-\omega^2 + 2/(LC))^2 = (1/(LC))^2$$

$$-\omega^2 = (-2 \pm 1)/LC \quad \text{or} \quad \boxed{\omega = \sqrt{(2 \pm 1)/(LC)} = \sqrt{1/(LC)}, \sqrt{3/(LC)}}$$

We now compute the modes. Plug in the resonant frequency to the coefficient matrix and find the eigenvectors.

If $\omega = \sqrt{1/(LC)}$, the coefficient matrix is

$$1/(LC)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and the corresponding eigenvector is $(A, B) = (1, -1)$.
If $\omega = \sqrt{3/(LC)}$, the coefficient matrix is

$$1/(LC)\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the corresponding eigenvector is $(A, B) = (1, 1)$.
We conclude that the two modes are

$$\boxed{(\omega, Q_1, Q_2) = (1/\sqrt{LC}, A\cos(\omega t), -A\cos(\omega t))}$$

$$\boxed{(\omega, Q_1, Q_2) = (\sqrt{3/LC}, A\cos(\omega t), A\cos(\omega t))}$$

The I_2 in this solution is defined to be opposite to that of defined in the problemset!

Solution 2 3x3 matricies

Let Q_1, Q_2, Q_C be three unknowns. Instead of applying the restraint that their sum must be zero, we claim that the specific solution $Q_1 = Ae^{i\omega t}, Q_2 = Be^{i\omega t}, Q_3 = Ce^{i\omega t}$ works. In matrix form, the equation simplifies to

$$\begin{bmatrix} i\omega & i\omega & i\omega \\ -\omega^2 L_1 + 1/C_1 & 0 & -1/C_C \\ 0 & -\omega^2 L_2 + 1/C_2 & -1/C_C \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

And assuming all the inductance and the capacitance is identical

$$\begin{bmatrix} i\omega & i\omega & i\omega \\ -\omega^2 L + 1/C & 0 & -1/C \\ 0 & -\omega^2 L + 1/C & -1/C \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix turns out to be

$$i\omega(\omega^2 L - 1/C)(\omega^2 L - 3/C)$$

This value equals zero when $\boxed{\omega = 1/\sqrt{LC}, \sqrt{3/(LC)}}$

The normal modes can be derived equally as from the previous solution.