PHYS 314 Formula Sheet

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Part 1 Quantum Computing

Eigenvalues of the Spin Operators It is a convention to denote the basis state in terms of the z-axis. A particle of spin-s with spin value m is denoted as

$$|\phi\rangle = |s, m\rangle$$

Also, this notation provides insight of the eigenvalues of the spin operators.

$$\hat{S}_z|s, m\rangle = m|s, m\rangle \quad \text{and} \quad \hat{S}^2|s, m\rangle = \sqrt{s(s+1)}|s, m\rangle$$

Behavior of the Spin operators under commutation We are interested in mainly two types of spin operators. One is the spin operator with repect to each axis, and the other is the total angular spin. The following relations hold.

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \quad \text{and} \quad [\hat{S}_z, \hat{S}^2] = 0$$

In other words, the three axes x, y, z behave well under the spin operator, and the total angular spin commutes with the total angular spin.

Singlet States and Superposition

The singlet state is defined as follows.

$$\frac{1}{\sqrt{2}}(|z_+\rangle|z_-\rangle - |z_-\rangle|z_+\rangle)$$

Computing the probability outcomes of the singlet state into all the possible output states, we determine that the particle never collapses to the up-up or down-down state. This means that the spin of one particle decides the spin of the other. This is called *Superposition*.

Raising and Lowering Operators The raising and the lowering operators are conventionally defined on the **z-axis**. The operator bumps up the state by one. Here is the definition along with an example.

$$\hat{S}_+ := \hat{S}_x + i\hat{S}_y \quad \text{and} \quad \hat{S}_- := \hat{S}_x - i\hat{S}_y$$

$$\hat{S}_+|s, j\rangle = \sqrt{s(s+1) - j(j+1)}|s, j+1\rangle$$

Also, spin operators are Hermitian, so $\hat{S_+}^\dagger = \hat{S_-}^\dagger$

Product of spins to Product of Raising/Lowering Ops The sum of the square of all the spin operators of each direction is a natural operator with importance. It is possible to express this operator, which is dependant on all three axes, into a sum of three products that involve only the z-axis operators.

The tensor product of vectors/operators behave nicely. In light of this fact with the definition of the raising/lowering operator, we conclude

$$2\widehat{S_{1x}}\widehat{S_{2x}} + 2\widehat{S_{1y}}\widehat{S_{2y}} + 2\widehat{S_{1z}}\widehat{S_{2z}} = \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\widehat{S_{1z}}\widehat{S_{2z}}$$

Applications It is possible to express a higher order spin, say spin-3/2, as a tensor product of two lower spins, spin-1 and spin-1/2. The idea is to take the lowest and the highest value of the spin states, and to apply the total momentum operator $\hat{J}^2 := (\hat{J}_1 + \hat{J}_2)^2$. The operator $\hat{J}_1\hat{J}_2$ can be expressed entirely in terms of the z-axis operators. It might be tempting to simply apply each of the operators separately, but the base states are eigenvectors of $\hat{J_i}^2$.

$$\hat{J}_1\hat{J}_2 = 2\widehat{J_{1x}}\widehat{J_{2x}} + 2\widehat{J_{1y}}\widehat{J_{2y}} + 2\widehat{J_{1z}}\widehat{J_{2z}}$$

Refer to Townsend Appendix B for details. In conclusion, this equation will show that the tensor product of the states spin-1/2 and spin-1 will yield all the states in spin-3/2 and spin-1/2.

Also, for entangled states, the raising and the lowering operator is defined as follows.

$$\hat{S}_+ := \hat{S}_{1+} \cdot 1_2 + 1_1 \cdot \hat{S}_{2+} \quad \text{and} \quad \hat{S}_- := \hat{S}_{1-} \cdot 1_2 + 1_1 \cdot \hat{S}_{2-}$$

Where S is defined to be the entangled spin of two particles. For an intuitive justification, consider the rotation generator. The sum of the two operators are the terms that survive as $d\theta \to 0$.

Types of Operators and their properties We have the spin operator $\hat{S} := \hat{S}_x + \hat{S}_y + \hat{S}_z$. The nice property is that the magnitude of the combined spin $\hat{S}^2$ commutes with individual axis spin. That is:

$$[\hat{S}^2, \hat{S}_i] = 0$$

We can take the tensor prouct of two entangled particles. That is

$$\hat{S}_1 \cdot \hat{S}_2 := \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$$

$$\hat{\mathbf{S}}^2|s, m\rangle = s(s+1)\hbar^2|s, m\rangle$$

$$\hat{S}_z|s, m\rangle = m\hbar|s, m\rangle$$

Note that any state is a eigenvector of $\hat{S}^2$. Refer to Tonwsend CH5 for more information.

Hyperfine energy structure of the Hydrogen Atom A hydrogen atom can be considered as as a system of two spin-1/2 particles, proton and the electron. The Hamiltonian of the system is defined as $\hat{H} = \frac{2A}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2$. With some computation, we recognize that there are two energy eigenvalues and four energy eigenvectors.

$$\hat{H} = \begin{bmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & 0 \\ 0 & A & -A/2 & 0 \\ 0 & 0 & 0 & A/2 \end{bmatrix}$$

The higher energy eigenvalue is $A/2$, and the corresponding states are the two product states (top/bottom) and the entangled state which has the same signs. These three states are referred as the **triplet states**. In symbols,

$$\lambda = \frac{A}{2} \quad \text{and} \quad |\psi\rangle = |z_+\rangle|z_+\rangle, |z_-\rangle|z_-\rangle, \frac{1}{\sqrt{2}}(|z+\rangle|z-\rangle + |z_-\rangle|z_+\rangle)$$

The lower energy eigenvalue is $-3A/2$, and the corresponding state is the **singlet state**. In symbols,

$$\lambda = -\frac{3A}{2} \quad \text{and} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(-|z+\rangle|z-\rangle + |z_-\rangle|z_+\rangle)$$

Remember that $\frac{2}{\hbar^2}S_1S_2$ has the eigenvalue of $1/2, -3/2$.

The singlet corresponds to spin-0 and the triplet corresponds to spin-1.

Dirac's Spin Exchange The Hamiltonian $\hat{H} := \vec{\sigma}_1 \cdot \vec{\sigma}_2$ can be expressed as

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2P_{\text{spin exchange}} - 1$$

Computing the Bra-ket of product vectors When dealing with the EPR paradox, it is necessary to find the product of to entangled states. We proceed with the principle that spins don't interact with each other in a product state. Suppose

$$|x+, x+\rangle := |x+\rangle_1|x+\rangle_2 \quad \text{and} \quad |y+, y+\rangle := |y+\rangle_1|y+\rangle_2$$

We compute the probability that the first state collapses to the second state.

$$P = |\langle y+, y+ \mid x+, x+\rangle|^2 = |\langle y+ \mid x+\rangle\langle y+ \mid x+\rangle|^2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

Spin probability in terms of angles between Bloch vectors Consider two states $|a+\rangle, |b+\rangle$ which has a bloch vector oriented towards $\vec{a}, \vec{b}$. The probability that one of them will collapse to the other is given as follows.

$$P(a+ \to b+) = \langle b+ \mid a+\rangle^2 = \cos^2(\theta_{ab})$$

and θ_{ab} is the angle between the two vectors. This can be derived by fixing one of the vectors to be the z-axis, then applying the bloch vector formula.

$$|n\rangle = \begin{bmatrix} \sin(\theta/2) \\ \mathrm{Ixp}(\phi)\cos(\theta/2) \end{bmatrix}$$

<u>The Bloch vector and Rotations</u>

The bloch vector is **a unit vector** that describes the direction of the spin. To obtain the physical projection of the spin, the bloch vector must be multiplied by $\hbar/2$. In symbols, that is,

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} r_x$$

r_x refers to the x-component of the Bloch vector. In terms of opertators,

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x$$

And the total projection operator generates the rotation around the axis. That is,

$$\hat{R}_x(\theta) = \exp\left(\frac{\hbar}{2}\sigma_x \cdot \frac{\theta}{i\hbar}\right) = \exp\left(-i\frac{\theta}{2}\sigma_x\right)$$

Computing this matrix exponential, we conclude a following shorthand for the rotational operators.

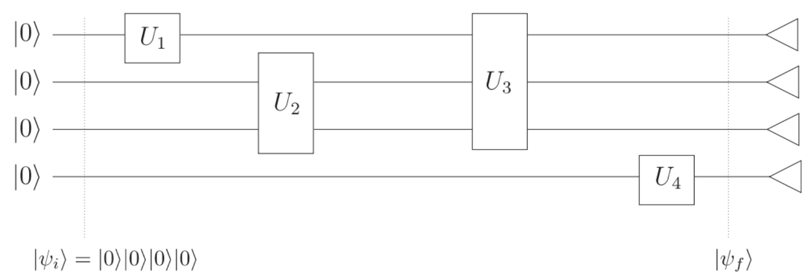
$$\begin{aligned} \hat{R}_x(\theta) &= \cos(\theta/2) - i\sin(\theta/2)\sigma_x \\ &= \cos(\theta/2) + \frac{2}{i\hbar}\sin(\theta/2)\hat{S}_x \end{aligned} \tag{1}$$

This result generalizes to other axes.

Also, it is useful to remember that rotation along the x-axis by a angle of π results in $(a, b) \mapsto (b, a)$. For the y-axis, the same rotation results in $(a, b) \mapsto (-b, a)$.;

Part 2 Quantum Computing

Introducing quantum circits



Here is a sample of a quantum circuit. Unlike traditional circuits where each of the wires take a value of $0, 1$, a quantum circuit can take some entangled state $|\phi\rangle$.

We assume that all quantum gates are **unitary, or reversible**. Without proof, we acknowledge that any unitary operators can be decomposed into rotations. Moreover, any non-unitary operator can be emulated by **ancilla qbits**, which are wires with predetermined values.

Some basic gate definitions

Each quantum gate can be represented by an operator. We define some important quantum gates. Here are a class of gates defined to be the Pauli gates.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X := \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y := \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z := \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that square of any of these operators results in the identity operator. Moreover, for any operator A that satisfies $A^2 = I$,

$$\mathrm{Ixp}(Ax) = \cos(x)I + i\sin(x)A$$

The rotation gates are defined by the rotation opertaors.

$$\hat{R}_z(\theta) := \exp(-\frac{i\theta Z}{2})$$

Decomposing unitary gates Without proof, we present that any unitary operator $\hat{U}$ can be written as follows.

$$\hat{U} = e^{i\alpha}\hat{R}_l(\beta)\hat{R}_m(\gamma)\hat{R}_l(\delta)$$

This decomposition holds for any nonparallel axis l, m and for some real numbers $\alpha, \beta, \gamma, \delta$.

Set $l = z, m = y$ for convinience. Also, we observe

$$X\hat{R}_z(\theta)X = \hat{R}_z(-\theta) \quad \text{and} \quad X\hat{R}_y(\theta)X = \hat{R}_y(-\theta)$$

. A simple proof can be done by direct substitution. Nicely decomposing the rotations, it is possible to decompose $\hat{U}$ as

$$\hat{U} = e^{i\alpha}AXBXC$$

for some A, B, C that satisfies $ABC = I$.

Controlled Gates Imagine a multiqubit gate where there is one control bit and one or more manipulated qbits. The gate performs operation U on the manipulated qbits when the control bit is $|1\rangle$. We call such gates c-U gates. It is depicted in the diagram as follows.

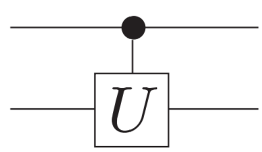


Fig. 4.5 The c-U gate.