Midterm II-part I

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1. Prove or disprove: there is an entire analytic function with real part $x - xy$. If there is such an analytic function, find all such functions. Also, find the series expansion of the function of z around the origin.

Solution Let function $f(z) = f(x + iy)$ be such a function that satisfies the condition. Analytic functions are necessarily holomorphic and vice versa. Hence, it is possible to apply the Cauchy-Riemann Equations in this context. Define:

$$u := Re(f(x + iy)) \quad \text{and} \quad v := Im(f(x + iy))$$

It is given that $u = x - xy$. We compute:

$$u_x = 1 - y \quad \text{and} \quad u_y = -x$$

By the Cauchy-Riemann Equations, we deduce:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$v_x = -u_y = x \quad \text{and} \quad v_y = u_x = 1 - y$$

The function $v(x, y)$ must be expressed as the following:

$$v(x, y) = x^2/2 + C(y) = y - y^2/2 + D(x)$$

Where C, D are functions that map real values to real values that depend solely on y and x respectively. The two expressions of $v(x, y)$ must equate each other. Write:

$$C(y) - y + y^2/2 = D(x) - x^2/2$$

Recognize that the LHS is independent of x and the RHS independent of y. Thus, we conclude that both expressions equal to a constant, say C.

$$D(x) = x^2/2 + C \quad \text{and} \quad v(x, y) = x^2/2 + y - y^2/2 + C$$

Compute the complex derivative of f by differentiating it over the real axis. The holomorphicity of f guarantees that the derivative is unique. Write:

$$\frac{d}{dz} f(z) = \frac{\partial}{\partial x} u(x, y) + \frac{\partial}{\partial x} v(x, y) i$$

$$= (1 - y) + xi = 1 - iz$$

Taking the antiderivative, we conclude, for some complex constant C',

$$f(z) = z - iz^2/2 + C'$$

The real part of f does not contain a constant. Hence, we narrow down $C' = Ci$ where C is a real value.

We have shown that a function f that satisfies $Re(f) = x - xy$ must be in the form of:

$$f(x) = Ci + z - iz^2/2 \qquad (C \in \mathbb{R})$$

Indeed all such functions must be holomorphic, for f is a complex polynomial of order two. Moreover, by some algebra, we notice that such functions always have a real part $x - xy$. We conclude that the functions of the form above are all the analytic entire functions that have a real part of $x - xy$. The function is already written as its series expansion about the origin.

□

2. Compute four integrals.
i) Compute:

$$I := \int_{-\infty}^{\infty} \frac{dx}{cosh(x)} = \int_{-\infty}^{\infty} \frac{2dx}{e^x + e^{-x}}$$

Soultion The integrand is an even function. Hence we write:

$$I = 4\int_{0}^{\infty} \frac{dx}{e^x + e^{-x}} \qquad \text{and} \qquad I/4 = \int_{-\infty}^{\infty} \frac{e^x dx}{e^{2x} + 1}$$

Apply the u-subsitution, $u = e^x$:

$$I/4 = \int_{-\infty}^{\infty} \frac{du}{u^2 + 1} = arctan(u)\Big|_{-\infty}^{\infty} = \pi$$

Hence:

$$I = 4\pi$$

$\square$

ii) Let ζ be any real number and $a > 0$. Evaluate:

$$I := \int_{-\infty}^{\infty} \frac{e^{-2\pi\zeta x}}{x^2 + a^2} dx$$

Solution Define a holomorphic function $f(z)$ as follows:

$$f(z) = \frac{e^{-2\pi\zeta z}}{z^2 - a^2}$$

The numerator and the denominator are known to be holomorphic. Thus the function is holomorphic everywhere other than the poles which are located at $z = \pm a$. Draw a semicircular contour centered at the origin that occupies quadrant I and IV. Call this contour γ, and denote the radius as R.

Take the contour integral of $f(z)$ over γ. Let the straight segment of the contour be called S, and the circular region C.

We claim that the integral over the circular region vanishes. That is, a as $R \to \infty$, $\oint_C f = 0$

Notice:

$$\left| \oint_C f \right| = \left| \int_{z \in C} \frac{e^{-2\pi\zeta z}}{z^2 + a^2} dz \right| \leq \int_{z \in C} \frac{max|e^{-2\pi\zeta z}|}{R^2 - a^2} dz$$

Note that the modulus of an exponent is the exponent of the modulus of the argument. That is:

$$|e^{-z}| = e^{Re(-2\pi\zeta z)}$$

And for $z \in C$, the quality is bounded under 1. Thus:

$$\left| \oint_C f \right| \leq \frac{2\pi R}{R^2 - a^2}$$

And the upper bound converges to zero as R approaches infinity. This shows that the circular region converges to zero. ✓

By the residue theorem:

$$\oint_C f + \oint_S f = 2\pi i Res_f(a)$$

The first summand of the LHS vanishes. The second summand can be computed with some algebra. We write:

$$\oint_S f = \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix} \cdot (-i)dx}{(xi)^2 - a^2} = i \int_{x=-\infty}^{\infty} \frac{e^{-2\pi\zeta ix} dx}{x^2 + a^2} dx = iI$$

The residue can be computed with ease:

$$Res_f(a) = \lim_{z \to a} \frac{e^{-2\pi\zeta z}(z - a)}{z^2 - a^2} = \lim_{z \to a} \frac{e^{-2\pi\zeta z}}{z + a} = \frac{e^{-2\pi\zeta a}}{2a}$$

Combining the results, we write:

$$iI = 2\pi i \frac{e^{-2\pi\zeta a}}{2a} \qquad \text{or} \qquad \boxed{I = \frac{\pi e^{-2\pi\zeta a}}{a}}$$

iii) Compute:

$$\frac{I}{2\pi i} = \frac{1}{2\pi i} \oint_{|z|=2} \frac{zdz}{z^2 - 1}$$

Solution The function

$$f(z) = \frac{z}{z^2 - 1}$$

is holomorphic outisde the two poles $z = \pm 1$. By the residue theorem, the integral I equals to the sum of the residues multiplied by $2\pi i$. Our answer is the following sum:

$$Res_f(1) + Res_f(-1)$$

Write:

$$Res_f(1) = \lim_{z \to 1} \frac{z(z-1)}{z^2 - 1} = z/(z+1)\Big|_{z=1} = 1/2$$

$$Res_f(-1) = \lim_{z \to -1} \frac{z(z+1)}{z^2 - 1} = z/(z-1)\Big|_{z=-1} = 1/2$$

Thus:

$$\boxed{\frac{I}{2\pi i} = 1}$$

iv) Compute:

$$I := \int_0^\infty \frac{x^{-1/2}}{x+1} dx$$

Solution We use two identities about the beta function. Recall the definiton:

$$B(n,m) := \int_0^1 x^n (1-x)^m dx$$

And the two identities:

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \quad \text{and} \quad B(n,m) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$$

By the seond condition, the integral simplies to:

$$I = B(1/2, 1/2)$$

And by the first identity:

$$B(1/2,1/2) = \Gamma(1/2)^2/\Gamma(1) = \pi$$

We conclude:

$$I = \pi$$

$\square$

3. Consider the following infinite products:

$$I_1(a) := \prod_{n=1}^{\infty}(1+a_n) \quad \text{and} \quad I_2(b) := \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}(1+b_{mn})$$

a) State the definition of convergence of $I_1(a)$. Give an example of a product that converges to a finite, nonzero number, and an example that diverges.

Definition We define the partial product S_N as follows:

$$S_N := \sum_{n=1}^{N}(1+a_n)$$

If the partial product converges as $N \to \infty$, then the infinite product $I_1(a)$ is defined to converge.

Consider the case where $a_n = 0$ identically. Trivially, $S_N = 1$ regardless of N. The infinite series converges to 1.

Now, let $a_n = 1/n$. By induction, it is possible to show $S_N = N + 1$. For the base case, $S_1 = 1 + a_1 = 2$. For the inductive case:

$$S_{N+1} = \prod_{n=1}^{N+1}(1+\frac{1}{n}) = \frac{N+2}{N+1}S_N = N+2$$

which proves the claim. Ergo, S_{N+1} diverges to infinity.

b) State the definition for the convergence of the infinite product $I_2(b)$.

Definition It would be nice if the nested products all converge. That is: $I_1(b_k)$ converges for any k. The sequence b_k denotes the sequence:

$$b_{k1}, b_{k2}, b_{k3}, \ldots, b_{kn}, \ldots$$