Killing Pairs in a NxN board

Chess Group

Consider a NxN board where N is large enough. We arrange r rooks in the board. We are interested in counting the killing paris of this board. Define the family of arrangement

$$\mathcal{A}_r := \{A_1, ...A_{\binom{N^2}{r}}\}$$

to be the family of all rook arrangements in the NxN board. Use the variable $Kp(A)$ to denote the killing pairs of arrangement A. $\mathbb{E}[Kp(A)]_{A\in A_r}$ is denoted by $Kp[\mathcal{A}_r]$

Problem 1 *Jumping rooks*

Assume $r \ll N^2$. If rooks are allowed to jump pieces, what is the expected value of the killing pairs?

In fact, by a simple combinatorial construction, we can see that the expected value of the killing pair increases linearly as we place additional rooks. For large N, we observe

$$Kp[\mathcal{A}_r] = \frac{r(r-1)}{N}$$

Problem 2 *Non-jumping rooks*

Assume $r \ll N^2$. If rooks are not allowed to jump pieces, what is the expected value of the killing pairs?

We wish to adopt graph theory in our approach. From an arrangement A, we construct a graph as follows.

Constructing graph from a rook arrangement

1. Construct a vertex for every rook.
2. Each pair of rook are connected if they are a killing pair.
3. Add an observer vertex, that is arbitrarily far from the board. Connect each certex to the observer vertex.

Denote this graph by $G(A)$.

Turns out that the crossing lemma needs a condition $E > 4V$... This is a dead end.