Average Matrix of the DFT ensemble

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Definition *Average Matrix*

Let N be a positive integer. We define the average matrix of the DFT ensenble of order N to be a $(N-1)$ square matrix defined as the following.

$$A_{ij} = \begin{cases} 1 & N|ij \\ 0 & N \nmid ij \end{cases}$$

For example, the average matrix of order 6 is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition *Spaced Bases*

Consider the vector space $\mathbb{R}^{(N-1)}$. Define the mth spaced base ν_m as follows.

$$(\nu_m)_i := \begin{cases} 1 & (m|i) \\ 0 & (m \nmid i) \end{cases}$$

For example, in a $(9-1)$ dimensional vector space, the 3rd space base will be

$$\nu_3 = \{0, 0, 1, 0, 0, 1, 0, 0\}$$

Moreover, if $m|N$, we call the vector a divisor vector. Otherwise, the vector is called a nondivisor vector. For convinience ν_1 is identified as a nondivisor vector.

Proposition *The image of divisor vectors under transformation*

Let A be the average matrix of the DFT ensemble of order N, that is a $(N-1)$-by-$(N-1)$ matrix. Let $m|N$. Then, the following formula precisely describes the entries of the image of the mth spaced vector under the transformation induced by A.

$$(A\nu_m)_i = \frac{N}{\operatorname{lcm}(N/\gcd(i,N), m)} - 1$$

Proof. By direct computation, we write the ith entry of $A\nu_m$.

$$A\nu_m = \sum_{j=1}^{N-1} A_{ij}(\nu_m)_j$$

The terms in the sum is nonvanishing if and only if

$$N|ij \quad \text{and} \quad m|j$$

which is equivalent to the condition

$$\frac{N}{\gcd(N,i)}|j \quad \text{and} \quad m|j$$

. We wish to count the number of j's in the range $[1, N-1]$ that satisfy the condition. We conclude

$$(A\nu_m)_i = \frac{N}{\operatorname{lcm}(N/\gcd(i,N), m)} - 1$$

□

Remark *Image of nondivisor vectors*

By a similar argument, when $m \nmid N$,

$$(A\nu_m)_i = \left\lfloor \frac{N-1}{\operatorname{lcm}(N/\gcd(i,N), m)} \right\rfloor$$

Corollary *Divisor vectors map to divisor vectors*

Let $m|N$. Then $A\nu_m$ can be expressed as a linear combination of divisor vectors, i.e

$$A\nu_m = \sum_{d|N} \tilde{a}_{md}\nu_m$$

Proof. Suppose we start from $A\nu_m$ and subtract divisor vectors to make it zero. We demonstrate this with $A\nu_2$ for $N = 8$. We start with the vector

$$\{0, 1, 0, 3, 0, 1, 0\}$$

Subtract ν_2 to obtain

$$\{0, 0, 0, 1, 0, 0, 0\}$$

and subtract ν_4 to obtain the zero vector. We conclude

$$A\nu_2 = \nu_2 + \nu_4$$

Now we prove the theorem in a genereral sense. Suppose we have subtracted k vectors in a sense described above, and call the remainder vector r_k. We induct on k. If $k = 0$, we take p, the smallest prime divisor of N. Look at all the entries j such that $p|j$. We know

$$(A\nu_m)_j = \frac{N}{\operatorname{lcm}(N/\gcd(j,N), m)} - 1 \geq \frac{N}{\operatorname{lcm}(N/\gcd(p,N), m)} - 1$$

and thus subtracting $(A\nu_m)_p\nu_p$ will preserve all the matrix entries positive.

Now, let $k \geq 1$. Let p to be the index of the first positive value occuring in r_i. If $p \nmid N$, then the pth entry of $A\nu_m$ will be zero, so we can assume $p|N$. By subtracting $(r_k)_k \nu_p$, we only affect the entries of r_k that are a multiple of k. It suffices to show that for all $p|j$, $(A\nu_m)_j \geq (A\nu_m)_p$

$$(A\nu_m)_j = \frac{N}{\mathrm{lcm}(N/\gcd(j,N), m)} - 1 \geq \frac{N}{\mathrm{lcm}(N/\gcd(p,N), m)} - 1$$

and the theorem holds. □

Remark *Nondivisor vectors map to divisor vectors*

Replace the formula for divisor vectors to nondivisor vectors, and repeat the same inductive argument used for the divisor vectors.

Theorem *Eigenvalues of A*

The nonzero eigenvalues of A are exactly the eigenvalues of the simplified matrix

$$\tilde{A} := [\tilde{a}_{d_i d_j}]$$

where d_i is the ith proper divisor of A greater than 1. Therefore, if N has $\sigma(N)$ divisors including N, it can have at maximum $\sigma(N) - 2$ eigenvalues, for $\tilde{A}$ is a square matrix of order $\sigma(N) - 2$.

Proof. We first notice that the set of all spacing vectors

$$\{\nu_1, \nu_2, \cdots \nu_{N-1}\}$$

form a basis of $\mathbb{R}^{N-1}$. By the proposition, we notice that the image of this base under the transformation A maps to the space induced by

$$\{\nu_{d_1}, \nu_{d_2}, \cdots \nu_{d_{\sigma(N)-1}}\}$$

. Hence, all nonzero eigenvectors must be in the subspace

$$\mathrm{span}(\nu_{d_1}, \nu_{d_2}, \cdots \nu_{d_{\sigma(N)-1}})$$

and the coefficients $\tilde{a}$ are described in the proposition. □

Remark *The simplified matrix for prime powers*

If $N = 2^k$, the simplified matrix can be simply written as

$$\begin{bmatrix} 1 & 2 & \cdots & 2^{k-2} & 2^{k-1} \\ 1 & 2 & \cdots & 2^{k-2} & 0 \\ & & \vdots & & \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Theorem *Lower bounds of the largest eigenvalue of the DFT matrix*

. Let λ_{max} be the maximum eigenvalue of a $(N-1)$-by-$(N-1)$ average matrix. The lower bound is

$$\lambda_{max} \geq \frac{N}{2\sigma(N)-2}\sqrt{\sum_{d|N}\frac{1}{d^2}}$$

Proof. Since the average matrix is real symmetric, its eigenvalues must be real. We also know that there are only $\sigma(N)$ nonzero eigenvalues of this matrix. Thus,

$$\sum_{i\leq\sigma(N)} \lambda_i^2 \leq \sigma(N)\lambda_{max}^2$$

By the eigenvalue trace lemma,

$$\sum_{i\leq\sigma(N)} \lambda_i^2 = \mathrm{tr}(A^2) = \sum_{1\leq i,j\leq N} a_{ij}^2 = \sum_{d|N}\left(\frac{N}{d}-1\right)\left(\frac{N}{d}-\varphi\left(\frac{N}{d}\right)\right) \geq \sum_{d|N}\left(\frac{N}{d}-\varphi\left(\frac{N}{d}\right)\right)^2$$

Trivially,

$$\sum_{d|N}\left(\frac{N}{d}-\varphi\left(\frac{N}{d}\right)\right)^2 \geq \frac{1}{4}\sum_{d|N}\left(\frac{N}{d}\right)^2$$

Combining the two observations, we conclude

$$\lambda_{max} \geq \frac{N}{2\sigma(N)-2}\sqrt{\sum_{d|N}\frac{1}{d^2}}$$

□

Corollary *When N is a power of 2*

By plugging in $N = 2^k$, we obtain

$$\lambda_{max} = o\left(\frac{2^k}{k}\right)$$