PHYS 202 HW5

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Question 1 a) A RLC circuit is constructed by connecting four inductors and five capacitors. Using the analogy of the open-ended spring oscillator, find the four mode frequencies of this system.

The dispersion relation gives the frequency with respect to the wave number. For the spring oscillators,

$$\omega = 2\omega_0 \sin(k_n a/2) \quad \text{and} \quad k_n = \frac{2\pi}{\lambda_n}$$

For free ends, the normal modes occur at

$$\lambda_n = \frac{2L}{n}$$

Plugging this in to the dispersion relation,

$$\omega = 2\omega_0 \sin\left(\frac{n\pi a}{2L}\right)$$

L/a is the number of springs in the system, which is known to be five.

$$\omega = 2\omega_0 \sin\left(\frac{n\pi}{10}\right)$$

```
ω0 = 10^6;
Table[N[2 ω0 Sin[n π / 10]], {n, 1, 4}]
```

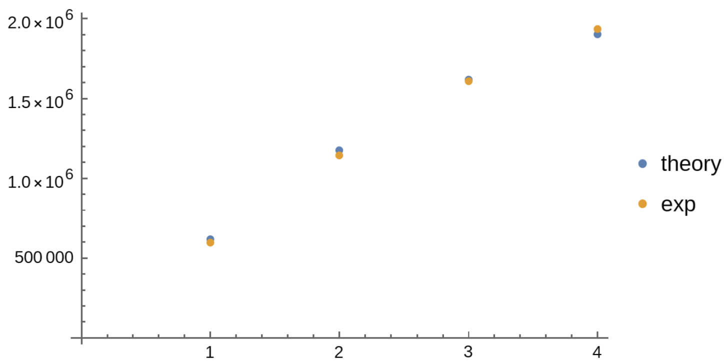
$\{618\,034., 1.17557\times10^6, 1.61803\times10^6, 1.90211\times10^6\}$

And there are the frequencies in Hz.

b)

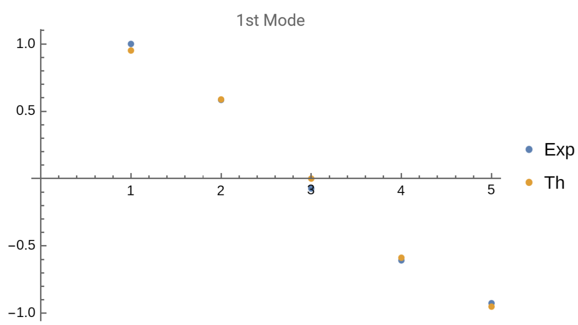
```
ω0 = 10^6;
ωtheory := Table[N[2 ω0 Sin[n π / 10]], {n, 1, 4}];
ff := {95, 182, 256, 308};
ωexp := N[ff * 2 π * 1000];

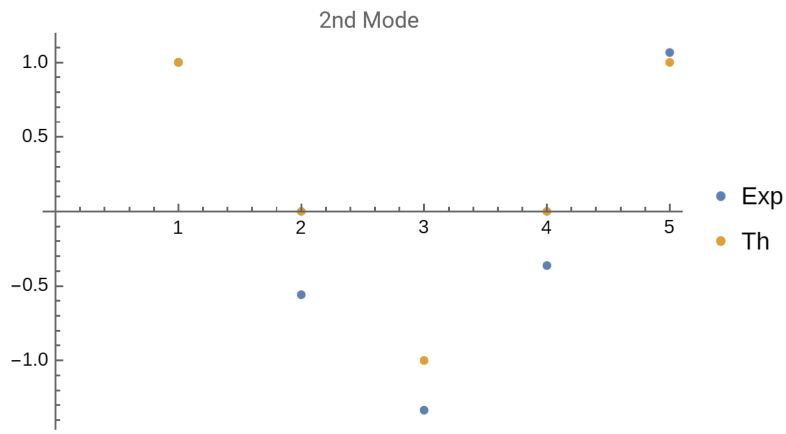
ListPlot[{ωtheory, ωexp}, PlotLegends → {"theory", "exp"}]
```

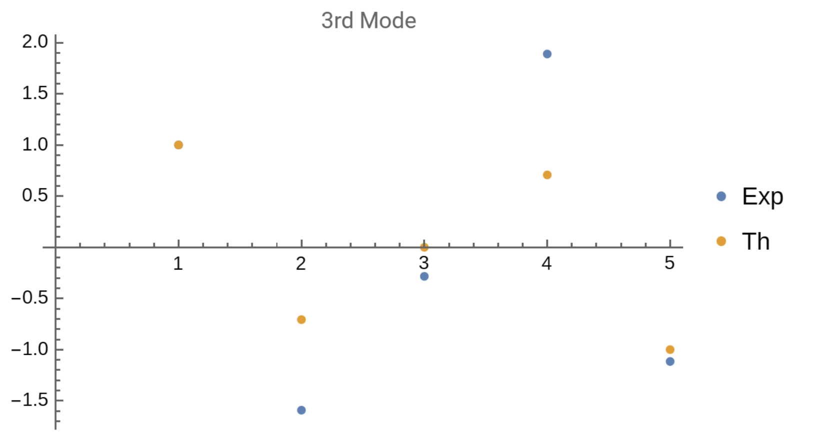


c) Compare the experimental measurement of the string displacement at the certain time with the theoretical results.

```
(* Experimental data for the lowest frequency *)
x1Exp = {+1, +5.84 / 10, -.76 / 10.3, -6.44 / 10.6, -9.44 / 10.2};
x1Th = Table[Cos[π / 5 (n + .5)], {n, 0, 4}];
ListPlot[{x1Exp, x1Th}, PlotLegends → {"Exp", "Th"}, PlotLabel → "1st Mode"]
```







Question 2 Fixed end Waves

Consider string connected with N beads, each distance a apart. The bead 1, N are distance $a/2$ apart from the walls.

a) Consider the wave as a segment of an infinite wave. What are the boundary conditions?

Assume there is bead zero on the left of bead one, and bead N+1 next to bead N. The string must be fixed on the boundaries. Thus,

$$y_0(t) = -y_1(t) \quad \text{and} \quad y_N(t) = -y_N(t)$$

b) Derive an expression for $y_n(t)$.

We know that the complexified solutions come in the form of:

$$\tilde{y}_n(t) = \tilde{A}e^{i(\omega t + kz_n)} + \tilde{B}e^{i(\omega t - kz_n)}$$

The time derivative of this function is

$$\dot{\tilde{y}}_n(t) = i\omega(\tilde{A}e^{i(\omega t + kz_n)} + \tilde{B}e^{i(\omega t - kz_n)})$$

Realify this solution, and set it equal to zero at time zero.

$$\dot{y}_n(0) = Re(i\omega\tilde{y}_n(t)) = 0 \qquad \text{or} \qquad Im(\tilde{y}_n(0)) = 0$$

Impose $y_0(t) + y_1(t) = 0$.

$$\tilde{A}e^{i(\omega t + kz_0)} + \tilde{B}e^{i(\omega t - kz_0)} + \tilde{A}e^{i(\omega t + kz_1)} + \tilde{B}e^{i(\omega t - kz_1)} = 0$$

$$\tilde{A}e^{i(\omega t - ka/2)} + \tilde{B}e^{i(\omega t + ka/2)} + \tilde{A}e^{i(\omega t + ka/2)} + \tilde{B}e^{i(\omega t - ka/2)} = 0$$

$$2e^{i\omega t}(\tilde{A}\cos(ka/2) + \tilde{B}\cos(ka/2)) = 0$$

Technically, the real part of the LHS is zero, but we can adjust t to align the phase so that the maximum amplitude is achieved. We can rule out $\sin(ka/2) = 0$. If this is the case, the displacement will be identically zero for every bead.

$$\tilde{B} = -\tilde{A}$$

The general solution is rewritten as:

$$\tilde{y}_n(t) = \tilde{C}e^{i\omega t}\sin(kz_n)$$

Enforce the right boundary condition.

$$\tilde{A}e^{i\omega t}(\sin(kz_{N+1}) + \sin(kz_N)) = 0$$

$$\sin(kz_N) + \sin(kz_{N+1}) = 0$$

$$\sin(ka(N - 1/2)) + \sin(ka(N + 1/2)) = 0$$

Use substitution $\theta := ka/2$.

$$\sin(kaN - \theta) = -\sin(kaN + \theta)$$

The fuction $\sin(kaN - \theta)$ is odd with respect to θ. Hence $kaN = m\pi$ and $k = m\pi/(aN)$. $(m = 1, 2, 3, ...)$. The dispersion relation provides a closed form expression for the norml mode frequency.

$$\omega_m = \omega_0 \sin(ka/(2N)) = \omega_0 \sin(m\pi/(2N))$$

We note that for $m > N$, the frequencies end up being redundant due to the summetry of the sin function. In conclusion,

$$\boxed{y_n(t) = G\cos(\omega_m t) sin\left(\frac{(2n-1)m\pi}{N}\right)}$$

$$\boxed{\omega_m = \omega_0 \sin(m\pi/(2N))}$$

And $m = 1, 2, ...N$.

Question 3 Closed-open ended oscillators Consider N masses attatched by a string of constant k, each separated by a distance a apart. One side of the string is attatched to a wall, and the other end is free. What are the normal modes of this system?
a) What are the boundary conditions?

Add two imaginary masses at the wall and at the end of mass N. Label their displacements to be x_0, x_{N+1}. The boundary conditions are as follows.

$$\boxed{x_0(t) = 0 \quad \text{and} \quad x_{N+1}(t) = x_N}$$

b) Find the normal mode frequencies.

We know that the solutions are in the form of

$$x_n(t) = \tilde{B}e^{i(\omega t+kan)} + \tilde{C}e^{i(\omega t-kan)}$$

Enforce $x_0(t) = 0$, which converts to

$$x_0(t) = e^{i\omega t}(\tilde{B} + \tilde{C}) = 0 \quad \text{and} \quad \tilde{C} = -\tilde{B}$$

. $x_N(t) - x_{N-1}(t) = 0$ for any t. This converts to

$$x_N(t) - x_{N-1}(t) = \tilde{B}e^{i(\omega t+kaN)}(e^{ika} - 1) + \tilde{C}e^{i(\omega t-kaN)}(e^{i-ka} + 1)$$

$$= \tilde{B}\left(2ie^{i(\omega t+kaN+ka/2)}\sin(ka/2) - 2ie^{i(\omega t-kaN-ka/2)}\sin(-ka/2)\right)$$

$$= 2i\tilde{B}\sin(ka/2)\left(e^{i(\omega t+kaN+ka/2)} + e^{i(\omega t-kaN-ka/2)}\right)$$

$$= 4i\tilde{B}\sin(ka/2)\cos(kaN + ka/2)e^{i\omega t} = 0$$

If the summand inside the cosine function is in the form of $\frac{2m-1}{2}\pi$, the entire function vanishes, and the boundary condition is satisfied. Note that $m = 1, 2, 3, ...$ Thus,

$$\frac{2m-1}{2}\pi = \frac{ka(2N+1)}{2}$$

Depending on the value of m, the value of k varies. Relabel k as k_m.

$$k_m = \frac{2m-1}{(2N+1)a}\pi$$

We have previously derived the frequency dependant on k_m.

$$\boxed{\omega = 2\omega_0 \sin(k_m a/2) = 2\omega_0 \sin\left[\frac{m-1/2}{N+1/2}\frac{\pi}{2}\right]}$$

Question 4 Driven system of N masses

A system of N masses is connected with strings. One end of the system is driven with a displacement of $A\cos(\omega_d t)$. Each of the masses are distance a apart.

a) Imagine two imaginary masses at the left of mass zero and right of mass n. What boundary conditions can be enforced on these two masses?

Mass zero can be the driver, and mass $N+1$ can be a phantom mass that trails the movement of the N th mass. Quantituatively,

$$x_0(t) = A\cos(\omega_d t) \quad \text{and} \quad x_{N+1}(t) = x_N(t)$$

b) Find the displacements of each of the masses.

Complexifying the displacements, we know that the solutions must be in the form of

$$\tilde{x}_n(t) = \tilde{B}e^{i(\omega t + kz_n)} + \tilde{C}e^{i(\omega t - kz_n)}$$

. We define z_n to be zero on the right $a/2$ position of the Nth mass. Algebraically, z_n can be expressed as

$$z_n = \frac{2n - 2N - 1}{2}$$

Enforce the boundary condition on the $N+1$th mass.

$$\tilde{x}_{N+1}(t) - \tilde{x}_N(t) = 0 \quad \text{or} \quad \tilde{B}e^{i(\omega t + kz_{N+1})} + \tilde{C}e^{i(\omega t - kz_{N+1})} - \tilde{B}e^{i(\omega t + kz_N)} - \tilde{C}e^{i(\omega t - kz_N)} = 0$$

Divide by $e^{i\omega t}$ both sides and collect $\tilde{B}, \tilde{C}$

$$\tilde{B}(e^{ikz_{N+1}} - e^{ikz_N}) + \tilde{C}(e^{i(-kz_{N+1})} - e^{i(-kz_N)}) = 0$$

By our placement of the origin, $z_N = -ka/2, z_{N+1} = ka/2$. Apply Euler's formula to obtain

$$\tilde{B}\sin(ka/2) + \tilde{C}\sin(-ka/2) = 0 \quad \text{or} \quad (\tilde{B} - \tilde{C})\sin(ka/2) = 0$$

If $\sin(ka/2) = 0$, then all the masses will be stationary. Thus, we conclude $\tilde{B} = \tilde{C}$. Returning to our expression of the displcaement, we collect $\tilde{B}e^{i\omega t}$ to write

$$\boxed{\tilde{x}_n(t) = \tilde{B}e^{i\omega t}(e^{ikz_n} + e^{i(-kz_n)}) = \tilde{D}e^{i\omega t}\cos(kz_n)}$$

c) Show that the amplitude diverges if the oscillator is driven at a normal mode frequency.

Now, we know that $\tilde{x_0} = Ae^{i\omega t}$. Plugging $n = 0$ to our general solution,

$$\tilde{x}_0(t) = \tilde{D}e^{i\omega t}\cos(kz_0) = \tilde{D}e^{i\omega t}\cos(k\frac{-2N-1}{2}a) = Ae^{i\omega t}$$

The complex amplitude $\tilde{D}$ can be expressed in terms of A.

$$\tilde{D} = A/\cos(ka\frac{2N+1}{2})$$

The wavenumber k is dependant on the drive frequency by the dispersion relation.

$$\omega_d = 2\omega_0 \sin(ka/2) \qquad \text{or} \qquad k = \frac{2}{a}\sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right)$$

Thus,

$$\tilde{D} = A/\cos\left((2N+1)\sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right)\right)$$

From Q3, we know the ratio $\omega_d/2\omega_0$ for the normal modes.

$$\frac{\omega_d}{2\omega_0} = \sin\left[\frac{2m-1}{2N+1}\frac{\pi}{2}\right] \qquad \text{and} \qquad \sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right) = \frac{2m-1}{2N+1}\frac{\pi}{2}$$

At the normal mode frequency, our amplitude simplifies to

$$\tilde{D} = A/\cos\left((2N+1)\frac{2m-1}{2N+1}\frac{\pi}{2}\right) = A/\cos\left((2m-1)\frac{\pi}{2}\right) = A/0 = \pm\infty$$

And the amplitude diverges. □

d) Write out a closed form expression for $x_1(t)$ where there is only one mass.

We define an angle for convinience.

$$\theta := ka/2$$

Write the dispersion relation and our amplitude $\tilde{D}$ in terms of θ.

$$\frac{\omega_d}{2\omega_0} = \sin(\theta) \qquad \text{and} \qquad \tilde{D} = A/\cos((2N+1)\theta) = A/\cos(3\theta)$$

The displacement x_1 is

$$x_1(t) = \tilde{D}e^{i\omega t}\cos(kz_1) = \tilde{D}e^{i\omega t}\cos(\theta(2n-2N-1)) = \tilde{D}e^{i\omega t}\cos(\theta)$$

Plugging in our expression for $\tilde{D}$,

$$x_1(t) = A\frac{\cos(\theta)}{\cos(3\theta)}e^{i\omega t}$$

We pull out an interesting trig identity

$$\frac{\cos(\theta)}{\cos(3\theta)} = \frac{1}{1-4sin^2(\theta)}$$

The derivation can be completed by using Euler's formula. We know $\sin(\theta) = \omega_d/2\omega_0$. Thus

$$\frac{1}{1-4\sin^2(\theta)} = \frac{1}{1-\omega_d^2/\omega_0^2} = \frac{\omega_0^2}{\omega_0^2-\omega_d^2}$$

Finally,

$$\boxed{x_1(t) = \frac{\omega_0^2}{\omega_0^2 - \omega_d^2} A e^{i\omega t}}$$

Q5 Rod of atoms

a) Consider an oscillator comprised of n masses where both ends are free. Compute the lowest normal mode frequency, and plot it using mathematica.

We reuse our results from Q4. Use the same coordinate system to define the equilibrium positon z_n. Enforcing the right boundary condition, we have

$$\tilde{x}_n(t) = \tilde{D} e^{i\omega t} \cos(k z_n)$$

The left boundary condition would be that the imaginary mass 0 shadows the motion of the real mass 1. In symbols, $x_0(t) = x_1(t)$. With some messy lines of algebra, this translates to

$$\cos(kaN + ka/2) = \cos(kan - ka/2)$$

Or substituting $ka/2 \mapsto \theta$, we can write

$$\cos(kaN + \theta) = \cos(kaN - \theta)$$

We reason that $\cos(kaN - x)$ must be an even function. Hence, $kaN = m\pi$ for any integer m. Thus

$$k = \frac{m\pi}{aN}$$

And now, plug this into the dispersion relation.

$$\frac{\omega_m}{\omega_0} = 2\sin\left(\frac{ka}{2}\right) = 2\sin\left(\frac{m\pi}{2N}\right)$$

We are interested in the lowest frequency, $m = 1$.

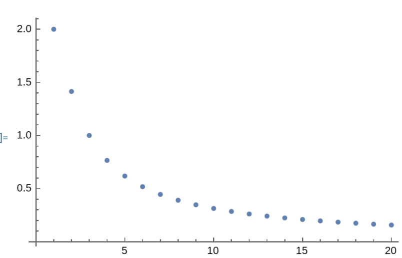
$$\frac{\omega_1}{\omega_0} = 2\sin\left(\frac{\pi}{2N}\right)$$

Plotting this into mathematica, we get the following result.

In[9]:=

```
ω1ratio[N_] := 2 Sin[π / (2 N)];
ω1Table = Table[{i, ω1ratio[i]}, {i, 1, 20}];
ListPlot[ω1Table]
```

Out[11]=



b) Approximate the frequency as $N \to \infty$

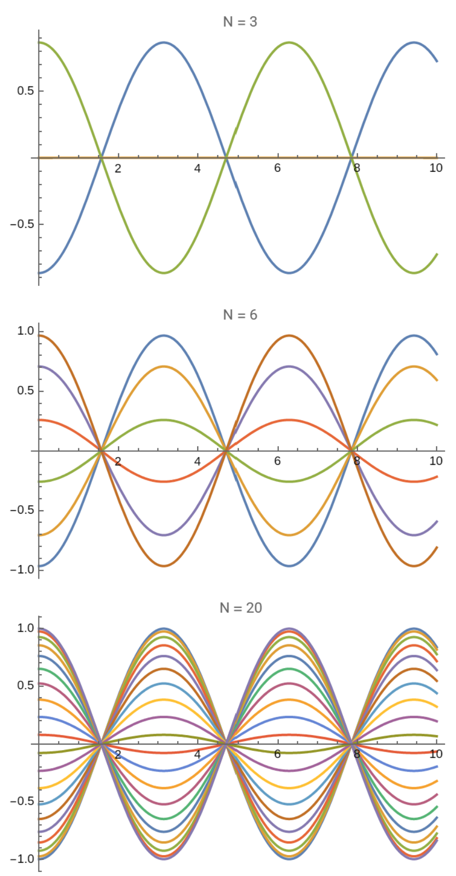
The sin function can be approximated as the identity around the origin.

$$\boxed{\frac{\omega_1}{\omega_0} = 2\sin\left(\frac{\pi}{2N}\right) \approx 2\frac{\pi}{2N} = \frac{\pi}{N}}$$

c) Plot the relative displacements of the masses for $N = 3, 6, 20$.

```
In[90]:= (*Here is a function for the relative frequency. N is the total number of masses. n is the specific mass number
 A is a constant for the amplitude*)
Displacement[n_, N_] := A Cos[ω t] Cos[(2 n - 2 N - 1) π / (2 N)]

(*Specify N, the number of masses, and obtain a list of functions*)
(*Assume ω = 1, A = 1 for convinience*)
A = 1;
ω = 1;
caseN3 = Table[Displacement[i, 3], {i , 3}];
caseN6 = Table[Displacement[i, 6], {i , 6}];
caseN20 = Table[Displacement[i, 20], {i , 20}];
Plot[caseN3, {t, 0, 10}]
Plot[caseN6, {t, 0, 10}]
Plot[caseN20, {t, 0, 10}]
```



d) Plot the three lowest normal mode frequencies for $N = 3, 6, 20$. Do they agree with Morin 2-84?

But this is our good ol' simple-harmonic-oscillator equation, so the solution is

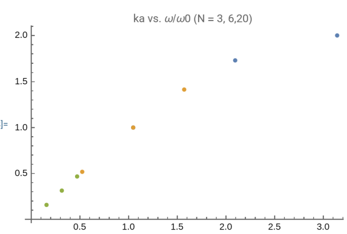
$$a(x) = Ae^{\pm ikx} \qquad \text{where} \qquad \boxed{k \equiv \omega\sqrt{\frac{\rho}{E}}} \tag{84}$$

So morin claims that the normal mode frequencies have a linear relationship with the wavenumber k. We can confirm that this relationship holds for large N.

```
In[237]:= kvsω = Table[{km[m, N] * a, ω[km[m, N]] / ω0}, {N, {3, 6, 20}}, {m, 3}]
          ListPlot[
           kvsω, PlotLabel → "ka vs. ω/ω0 (N = 3, 6,20)"
           , PlotLegends → "Expressions"
          ]
```

Out[237]= $\left\{\left\{\left\{\frac{\pi}{3}, 1\right\}, \left\{\frac{2\pi}{3}, \sqrt{3}\right\}, \{\pi, 2\}\right\}, \left\{\left\{\frac{\pi}{6}, \frac{-1+\sqrt{3}}{\sqrt{2}}\right\}, \left\{\frac{\pi}{3}, 1\right\}, \left\{\frac{\pi}{2}, \sqrt{2}\right\}\right\}, \left\{\left\{\frac{\pi}{20}, 2\,\mathrm{Sin}\left[\frac{\pi}{40}\right]\right\}, \left\{\frac{\pi}{10}, 2\,\mathrm{Sin}\left[\frac{\pi}{20}\right]\right\}, \left\{\frac{3\pi}{20}, 2\,\mathrm{Sin}\left[\frac{3\pi}{40}\right]\right\}\right\}\right\}$

Out[238]=

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(* There exists an intersection at ka = π / 3 *)
```

The green dots indicate that for $N = 20$, the relationship between the wavenumber and the frequency is linear.

Question 6

(a) Using ideas of normal mode analysis, explain this surprising result, and also find the value of τ in terms of the other parameters above.

(b) Explain why the inverse of the original shape, that is, the one shown in the lower part of the figure, is never observed.

a) Expression for τ

We know that the wave travels a distance of $2L$ every τ time. Also, the velocity of the wave is determined by $\sqrt{T/\mu}$. Thus

$$v = \frac{2L}{\tau} = \sqrt{\frac{T}{\mu}} \qquad \text{and} \qquad \boxed{\tau = 2L\sqrt{\frac{\mu}{T}}}$$

b) Is there a time when the original signal is inverted?

No, the signal is never inverted. Since the ends are fixed, the displacement function can be written as follows.

$$y(x,t) = Re\left(\sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right) e^{i\omega_m t}\right) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)\cos(\omega_m t)$$

The snapshot of the function at time $t = 0$ can be computed by plugging in $t = 0$.

$$y(x,0) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)\cos(0) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)$$

The inversion will look like the following.

$$-y(x,0) = -\sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)$$

We wish, for some time τ'

$$y(x,\tau) = -y(x,0) \quad \text{or} \quad \sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)\cos(\omega_m t) = -\sum_{m=1}^{\infty} C_m \sin\left(\frac{mx\pi}{L}\right)$$

Thus, for all $m \in \mathbb{Z}^+$,

$$\cos(\omega_m t) = -1$$

Invoke the dispersion relation, and assume $N \to \infty$.

$$\omega_m = 2\omega_0 \sin\left(\frac{m\pi}{2N}\right) \approxeq \omega_0 \frac{m\pi}{N}$$

Say $\cos(\omega_1 t) = -1$. The angular frequency is almost linear in relation with m. Thus, $\omega_2 = 2\omega_1$.

$$\cos(\omega_2 t) = \cos(2\omega_1 t) = 2\cos(\omega_1 t)^2 - 1 = 1$$

But $\cos(\omega_2 t) = -1$. We reach a contradiction.

$\square$

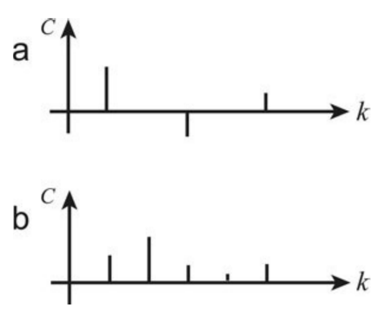

Figure 7.P.3 Two possible k-space graphs.

Smith 7.17 A string with fixed ends is once plucked at the quarter end position and at the half position. Two k-space graphs are shown. Find which pluck corresponds to which graph.

Solution The position where the string is plucked must be an antinode. Let this position be d. We observe

$$\frac{m\lambda}{2} + \frac{\lambda}{4} = d \quad \text{or} \quad (2m+1)\lambda = 4d \quad \text{or} \quad \lambda = \frac{4d}{2m+1}$$

for $m \in \mathbb{Z}^+$.

Now write the wavenumber.

$$k = \frac{2\pi}{\lambda} = \frac{2\pi(2m+1)}{4d} = \frac{\pi(2m+1)}{2d}$$

Let L be the length of the string. $d = L/2, L/4$.

$$k_{L/2} = \frac{\pi(2m+1)}{L} \quad \text{and} \quad k_{L/4} = \frac{2\pi(2m+1)}{L}$$

We notice that the spacings between the k values are more sparce for $d = L/4$. Thus, plot a corresponds to $d = L/4$ andf plot b corresponds to $d = L/2$. □