Approximating the cosine function

Daniel Son

Problem Find the average error of the following approximation in the range $x \in [0, 2\pi]$:

Given an integer N, theta $n < N$ is defined as:

$$\theta_n := 2\pi \frac{n}{N}$$

We approximate the function $cos(x)$ by taking the largest θ_n that is less than x.

Lemma

$$\sum_{n=0}^{N-1} cos(2\theta_n) = 0 \qquad \sum_{n=0}^{N-1} sin(2\theta_n) = 0$$

Proof The idea is to apply the Euler's formula and consider the sum as a geometric series. Write:

$$\sum_{n=0}^{N-1} cos(2\theta_n) = \sum_{n=0}^{N-1} \frac{e^{2i\theta_n} + e^{-2i\theta_n}}{2} = \sum_{n=0}^{N-1} \frac{e^{n(2i\theta_1)} + e^{n(-2i\theta_1)}}{2}$$

The sum can be computed as two seperate geometric series. We obtain:

$$\sum_{n=0}^{N-1} cos(2\theta_n) = \frac{1}{2}\left[\sum_{n=0}^{N} e^{n(2i\theta_1)} + \sum_{n=0}^{N} e^{n(-2i\theta_1)}\right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{N2i\theta_1}}{1 - e^{2i\theta_1}} + \frac{1 - e^{-N2i\theta_1}}{1 - e^{-2i\theta_1}}\right]$$

Note that $N2i\theta_1 = 4i\pi$. Thus, $e^{N2i\theta_1} = 1$ Furthermore, the numerators of the fractions reduce to zero. This concludes the proof of the first equation. Repeat a similar process for the second equation.

$$\sum_{n=0}^{N-1} sin(2\theta_n) = \sum_{n=0}^{N-1} \frac{e^{2i\theta_n} - e^{-2i\theta_n}}{2i} = \sum_{n=0}^{N-1} \frac{e^{n(2i\theta_1)} - e^{n(-2i\theta_1)}}{2i}$$

The sum can be computed as two seperate geometric series. We obtain:

$$\sum_{n=0}^{N-1} sin(2\theta_n) = \frac{1}{2i}\left[\sum_{n=0}^{N} e^{n(2i\theta_1)} - \sum_{n=0}^{N} e^{n(-2i\theta_1)}\right]$$

$$= \frac{1}{2i}\left[\frac{1 - e^{N2i\theta_1}}{1 - e^{2i\theta_1}} - \frac{1 - e^{-N2i\theta_1}}{1 - e^{-2i\theta_1}}\right]$$

And again, both fractions are zero. $\square$

Solution The idea to compute the approximation is to cut the function into N regions, each of length θ_1.

Define the following integral:

$$I_n := \int_{\theta_n}^{\theta_{n+1}} [cos(x) - cos(\theta_n)]^2 dx$$

Through some algebra, we reduce the integrand as:

$$cos^2(x) - 2cos(\theta_n)cos(x) + cos^2(\theta_n)$$

$$= \frac{1 + cos(2x)}{2} - 2cos(\theta_n)cos(x) + \frac{1 + cos(2\theta_n)}{2}$$

Integrate over the range $x \in [\theta_n, \theta]$. This results in three integrals.

$$I_n = \int_{\theta_n}^{\theta_{n+1}} \frac{1 + cos(2x)}{2} dx - 2cos(\theta_n) \int_{\theta_n}^{\theta_{n+1}} cos(x)dx + \int_{\theta_n}^{\theta_{n+1}} \frac{1 + cos(2\theta_n)}{2} dx$$

Label them as:

$$I_n := A_n - 2B_n + C_n$$

Adding up all the I_n's, we get:

$$I = \sum_{n=0}^{N-1} (A_n - 2B_n + C_n) = \sum_{n=0}^{N-1} A_n - 2\sum_{n=0}^{N-1} B_n + \sum_{n=0}^{N-1} C_n$$

Evaluate each integrals. Start with A:

$$A_n = \frac{1}{2}[x + sin(2x)/2]_{\theta_n}^{\theta_{n+1}} = \frac{1}{2}[\theta_1 + sin(2\theta_{n+1})/2 - sin(2\theta_n)/2]$$

$$\sum_{n=0}^{N-1} A_n = \frac{N\theta_1}{2} + sin(2\theta_N)/2 - sin(0)/2 = \pi$$

For C_n, notice that the integrand is constant:

$$C_n = \theta_1 \left[\frac{1 + cos(2\theta_n)}{2}\right]$$

$$\sum_{n=0}^{N-1} C_n = \frac{\theta_1}{2} \sum_{n=0}^{N-1} [1 + cos(2\theta_n)]$$

And notice that the latter term vanishes by the lemma. Thus:

$$\sum_{n=0}^{N-1} C_n = N\theta_1/2 = \pi$$

B is tricky. Before we proceed, compute the following sum:

$$cos(\theta_n)sin(\theta_{n+1}) = \frac{e^{\theta_1 ni} + e^{-\theta_1 ni}}{2} \frac{e^{\theta_1(n+1)i} - e^{-\theta_1(n+1)i}}{2i}$$
$$= \frac{e^{\theta_1(2n+1)i} + e^{\theta_1 i} - e^{-\theta_1 i} - e^{-\theta_1(2n+1)i}}{4i}$$
$$= \frac{e^{\theta_1(2n+1)i} - e^{-\theta_1(2n+1)i}}{4i} + \frac{e^{\theta_1 i} - e^{-\theta_1 i}}{2 \cdot 2i}$$
$$= \frac{e^{\theta_1(2n+1)i} - e^{-\theta_1(2n+1)i}}{4i} + \frac{sin(\theta_1)}{2}$$

Applying sums:

$$\sum_{n=0}^{N-1} cos(\theta_n)sin(\theta_{n+1}) = \sum_{n=0}^{N-1} \left(\frac{e^{\theta_1(2n+1)i} - e^{-\theta_1(2n+1)i}}{4i} \right) + \frac{Nsin(\theta_1)}{2}$$

$$= \frac{1}{4i} \left[e^{\theta_1 i} \sum_{n=0}^{N-1} e^{2\theta_1 ni} - e^{-\theta_1 i} \sum_{n=0}^{N-1} e^{-2\theta_1 ni} \right] + \frac{Nsin(\theta_1)}{2}$$

$$= \frac{1}{4i} \left[e^{\theta_1 i} \frac{1 - e^{2\theta_1 Ni}}{1 - e^{2\theta_1 i}} - e^{-\theta_1 i} \frac{1 - e^{-2\theta_1 Ni}}{1 - e^{-2\theta_1 i}} \right] + \frac{Nsin(\theta_1)}{2}$$

$$= \frac{Nsin(\theta_1)}{2}$$

Now, compute B_n and its sum:

$$B_n = cos(\theta_n) \int_{\theta_n}^{\theta_{n+1}} cos(x)dx = cos(\theta_n) \left[sin(x) \right]_{\theta_n}^{\theta_{n+1}}$$

$$= cos(\theta_n)sin(\theta_{n+1}) - cos(\theta_n)sin(\theta_n)$$
$$= cos(\theta_n)sin(\theta_{n+1}) - sin(2\theta_n)/2$$

$$\sum_{n=0}^{N-1} B_n = \sum_{n=0}^{N-1} cos(\theta_n)sin(\theta_{n+1}) - \sum_{n=0}^{N-1} sin(2\theta_n)/2$$

By the lemma and our established result above:

$$\sum_{n=0}^{N-1} B_n = \frac{Nsin(\theta_1)}{2}$$

Finally, compute I:

$$I = \sum_{n=0}^{N-1} A_n - 2\sum_{n=0}^{N-1} B_n + \sum_{n=0}^{N-1} C_n$$

$$= \pi - 2\frac{Nsin(\theta_1)}{2} + \pi = 2\pi - Nsin(\theta_1)$$

To compute the average error over the region $x \in [0, 2\pi]$, divide I by 2π. The average error is thus:

$$1 - \frac{Nsin(\theta_1)}{2\pi} = 1 - \frac{sin(\theta_1)}{\theta_1}$$

As a sanity check, we observe that as $N \to \infty$, $\theta_1 \to 0$ and the error also goes to zero, for $sin(\theta_1) \to \theta_1$ □