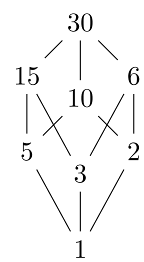
Combinatorics HW4

Daniel Son

Section 4.6: 36, 37, 43, 50
Section 5.7: 3, 4, 7, 9, 15, 23, 24

Additional Problem 1. Draw the Hasse diagram for the divides rela- tion on the positive divisors of 30. Then explain in your own words the relationship between that diagram, and the diagram in Figure 4.7 from the textbook.

Solution



The Hasse diagram of the divisors of 30 are isomorphic to the Hasse diagram of the subsets of $\{1, 2, 3\}$. We can describe the isomorphism as follows. Let define function f to as $f(1) = 2, f(2) = 3, f(3) = 5$. The isomorphism ϕ that maps subsets to numbers is

$$\phi(S) = \prod_{k \in S} f(k)$$

where $S \subseteq \{1, 2, 3\}$. For example, $\phi(\{1, 3\}) = f(1) \cdot f(3) = 2 \times 5 = 10$. By the fundamental theorem of arithmetic, it is easy to see that each divisor correspond to a unique subset via the inverse of ϕ. $\square$

Sec4.6Q36 Let X be a set of n elements. How many different relations on X are there? How many of these relations are reflexive? Symmetric? Antisymmetric? Reflexive and symmetric? Reflexive and anti-symmetric?

Solution We distinguish any ordered pairs of two integers into distinct and nondistinct pairs. The former refers to the pairs which have distinct entries. That is (a, b) where $a \neq b$. The latter refers to (a, a).

Relations can be considered as a set of ordered pairs. If the relation is symmetric, the choice of a nondistinct pair of the relation forces the relation to include the corresponding pair. That is, if $(a, b) \in R$ for where $a \neq b$, then $(b, a) \in R$. The choice of nondistinct pairs does not affect the symmetry of the relation. For a relation to be reflexive, it must include all the nondistinct pairs. For a relation to be antisymmetric it must not include any of the nondistinct pairs.

From the observations made above, we construct relations that satisfy the given conditions. If the relation is defined from the cannonical set $[n]$ to $[n]$, there exists n nondistinct pairs and $\binom{n}{2}$ couples of distinct pairs.

To construct all the symmetric relations, we either choose or leave each nondistinct pair or a distinct couple of pairs. There are a total of $n + \binom{n}{2}$ such objects. Thus, the number of symmetric relations are $\boxed{2^{n(n+1)/2}}$.

As for the reflexive relations, we choose all the nondistinct pairs and choose or leave the distinct pairs. There are $n(n+1)$ distinct pairs so we conclude that there are $\boxed{2^{n(n-1)}}$ reflexive relations.

For antisymmetric relations, we can either choose or leave all the nondistinct pairs. For the distinct pair couples, we are allowed to choose one of the two pairs, or include both of them from the relation. Thus, there are three possible choices for each distinct couple. We count $\boxed{3^{\binom{n}{2}}2^n}$.

For reflexive and symmetric relations, we choose or leave the distinct couples and choose all the nondistinct pairs. There are $\boxed{2^{n(n-1)/2}}$ such relations.

For reflexive and antisymmetric relations, we choose all the nondistinct pairs and choose between the three options for each distinct couple. We count $\boxed{3^{\binom{n}{2}}}$.

Sec4.6Q37 Let R', R'' be partial orders on a set X. Define the intersection R such that $xRy \leftrightarrow (xR'y) \wedge (xR''y)$. Prove that R is a partial order.

Proof. We demonstrate symmetry, reflexivity, and transitivity of the relation R. For R', R'' are both partial orders, they must be reflexive. Hence, $xR'x$ and $xR''x$ for any $x \in X$. By the definition of the intersect R, xRx. R is reflexive.

To demonstrate symmetry, assume xRy. We deduce $xR'y$ and $xR''y$ from definition. R' and R'' are symmetric, so $yR'x$ and $yR''x$. Thus, yRx and R is symmetric.

By the same logic, we demonstrate transitivity. Assume xRy and yRz. $xR'y$ and $yR'z$ from the definition of R and we infer $xR'z$ from the transitivity of R'. Likewise, $xR''z$. Thus, xRz which concludes the proof. $\square$

Sec4.6Q43 Let $X = a, b, c, d, e, f$ and let the relation R on X be defined by $aRb, bRc, cRd, aRe, eRf, fRd$. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.