PHYS 202 HW3

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Q1 Identify the faulty labeling from the power resonance curve.

Solution

The peak of the curve occurs at $f = 22.5Hz$ so $\omega_r es \approxeq 172kHz$. The FWHM of the curve is about $\gamma = 2\pi \cdot 500Hz \approxeq 3kHz$. We claim that the wrong label is the inductance, and hence the correct label is $(R, L, C) = (30mH, 1nF, 100\Omega)$.

Assuming the purported label is correct, we compute the theoretical resonant frequency and γ.

$$\gamma_{th} = R/L = 3.3kHz$$
$$\omega_{th} = 1/\sqrt{RC} = 180kHz$$

Which is close enough to the experimental value. $\square$

Q2 A resistor, inductor, and conductor is connected in parallel. Assuming that the a varying input current is entering the circuit in the rate of $I(t) = I_0 \cos(\omega t)$,

a) What is the resonant frequency?

b) What is the FWHM of the power resonance curve?

c) Define $Q := \omega_0 / FWHM$. What is the relationship between the Q-factors of the parallel and series circuit?

Solution

We start off with writing the complexified voltage across each component.

$$\tilde{I}_{in} = \sum I = \tilde{V}_{in}\left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)$$

Thus

$$\tilde{V}_{in} = \tilde{I}_{in} / \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)$$

a)

Resonance occurs when the complex impedences cancel out. That is

$$\omega C = 1/(\omega L) \quad \text{or} \quad \omega^2 = \frac{1}{LC} \quad \text{or} \quad \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Recovering the real value for V_{in}

$$V_{in}(t) = \frac{I_0 \cos(\omega t + \phi)}{\sqrt{1/R^2 + (\omega C - 1/(\omega L))^2}}$$

we can write an expression for $P(t)$.

$$P(t) = V_{in}(t)^2/R = \frac{I_0^2 \cos^2(\omega t + \phi)}{R(1/R^2 + (\omega C - 1/(\omega L))^2)}$$

The time average of the squared cosine function is 1/2. Thus

$$\langle P(t)\rangle = \frac{I_0^2}{2R(1/R^2 + (\omega C - 1/(\omega L))^2)}$$

Ignoring the constants, we recognize that the FWHM occurs when

$$1/R^2 = (\omega C - 1/(\omega L))^2$$

Multiply both sides by $L^2\omega^2$

$$\omega^2/\gamma^2 = (\omega^2 LC - 1)^2 \quad \text{and} \quad \pm\,\omega/\gamma = \omega^2 LC - 1$$

Call the solutions to the two equations as ω_1 and ω_2.

$$\omega_1/\gamma = \omega_1^2 LC - 1 \quad \text{and} \quad -\,\omega_2/\gamma = \omega_2^2 LC - 1$$

Subtracting the bottom equation from the top

$$(\omega_1 + \omega_2)/\gamma = (\omega_1^2 - \omega_2^2)LC \quad \text{or} \quad (\omega_1 - \omega_2) = \frac{1}{\gamma LC} = \boxed{\omega_0^2/\gamma}$$

c) We compute the Q-factor for the parallel circuit.

$$Q_{par} = \omega_0/\omega_0^2/\gamma = \gamma/\omega_0$$

We also know the Q-factor for series circuits.

$$Q_{ser} = \omega_0/\gamma$$

Thus

$$\boxed{Q_{ser} = 1/Q_{par}}$$

Q3 Reduced Mass Two masses are connected to each other with a string with constant k.

a) Write down the equation of Newton's II law for the two masses. Write the matrix form of the equation.

Solution We write the two equations.

$$m_1\ddot{x}_1 = -k(x_1 - x_2) \quad \text{and} \quad m_2\ddot{x}_2 = -k(x_2 - x_1)$$

We wish to obtain a homogeneous system.

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \quad \text{and} \quad m_2\ddot{x}_2 + k(x_2 - x_1) = 0$$

We guess the solutions to be in the form of $\tilde{x}_1 = Ae^{i\omega t}, \tilde{x}_2 = Be^{i\omega t}$. Plugging in and cancelling the exponential term, we write

$$-\omega^2 m_1 A + kA - kB = 0 \quad \text{and} \quad -\,\omega^2 m_2 B + kB - kA = 0$$

In matrix form

$$\begin{bmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) Determine the normal modes of the system

Solution

The normal modes occur when the determinant of the coefficient matrix equals zero. That is

$$\begin{vmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{vmatrix} = (k - \omega^2 m_1)(k - \omega^2 m_2) - k^2 = 0$$

$$\omega^4 m_1 m_2 - k(m_1 + m_2)\omega^2 = 0 \qquad \text{or} \qquad \omega^2(\omega^2 m_1 m_2 - k m_1 - k m_2) = 0$$

The solutions to this equation are

$$\boxed{\omega = 0 \qquad \text{or} \qquad \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{k/\mu}}$$

c) Determine the eigenvector which corresponds to each normal mode frequency. For the vibrational mode, how does the ratio of the two masses' ranges of motion relate to their respective masses?

Solution Write the coefficient matrix for the two possible cases of ω. If the frequency is zero and nonzero, the coefficient matricies are respectively

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k - m_1 k/\mu & -k \\ -k & k - m_2 k/\mu \end{bmatrix} = -\begin{bmatrix} m_1 k/m_2 & k \\ k & m_2 k/m_1 \end{bmatrix}$$

Upon inspection, the eigenvectors that correspond to each of the matricies are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} m_2/m_1 \\ -1 \end{bmatrix}$$

The former mode is the stationary mode and the latter mode is the vibrational mode. For the vibrational mode, if the mass of the two bodies are comparable to each other, the amplitude of their oscillations will also be comparable. Nonetheless, if one of the bodies are much heavier, the oscillation amplitude of the lighter mass will be larger than the lighter mass. Also, the phase of the two oscillations will differ by π.

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