1. Find a functional equation for the following function:

$$G(s) := \int_0^\infty \mathrm{Exp}[-x^2]x^{s-1}dx$$

Solution We first observe the following derivative:

$$\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$$

Hence:

$$\int xe^{-x^2}dx = -\frac{e^{-x^2}}{2} + C$$

With this indefinite integral in mind, we integrate G by parts. Apply the following substitutions:

$$u = x^{s-2} \quad \text{and} \quad du = (s-2)x^{s-3}$$

$$dv = x\mathrm{Exp}[-x^2]dx \quad \text{and} \quad v = -\frac{\mathrm{Exp}[-x^2]}{2}$$

Integrate G:

$$G(s) = uv\Big|_{x=0}^{\infty} - \int_{x=0}^{\infty} vdu$$

$$= \left[-\frac{x^{s-2}\mathrm{Exp}[-x^2]}{2}\right]_0^\infty - \int_0^\infty \left(-\frac{(s-2)\mathrm{Exp}[-x^2]x^{s-3}}{2}\right)dx$$

$$= \left[-\frac{x^{s-2}\mathrm{Exp}[-x^2]}{2}\right]_0^\infty + \frac{(s-2)}{2}\int_0^\infty \left(\mathrm{Exp}[-x^2]x^{s-3}\right)dx$$

For the sake of convergence of the first summand, we assume $s \geq 2$. As for $s = 2$, we evaluate:

$$G(2) = \int_0^\infty xe^{-x^2}dx = -\frac{e^{-x^2}}{2}\Big|_0^\infty = 1$$

As for $s > 2$, we compute the equality obtained by the by parts integration. Notice that the first summand converges to zero. Hence:

$$G(s) = \frac{s-2}{2}G(s-2)$$

□

2. Find an analytic continuation of the function:

$$H(x) = 1 + z^2 + z^4 + z^6 + \cdots + z^{2n} + \cdots$$

For which value is the continuation undefined? What is the value of the continuation at $z = 2$?

<u>Solution</u> The natural response after observing the function is to consider the geometric series. We define the continuation to be $\bar{H}(z)$. Write:

$$\bar{H}(z) = \frac{1}{1 - z^2}$$

Clearly, for values where $|z^2| < 1$, H and $\bar{H}$ agrees. It is possible to obtain a sequence that accumulate on some point in the unit circle that is not the center. For example, consider:

$$z_n := \frac{1}{2} e^{\pi i/n}$$

The series of points converge to $1/2$, and $H, \bar{H}$ agrees for any point z_n.

Upon inspection, we notice that $\bar{H}(z)$ is defined everywhere other than $z = \pm 1$. Other then at these two poles, $\bar{H}$ is holomorphic, for $z^2 - 1$ is holomorphic and the reciprocal of a holomorphic function must be holomorphic as long as the function is defined. □

3. For which value of s is the following integral defined?

$$L(s) := \int_0^\infty \frac{x^s}{x^2+1} dx$$

Claim $L(s)$ is defined for $s \in (0, 1)$

Proof It is trivial to notice that the integrand blows up at $x = 0$ when s is negative. We only consider positive s. Divide the integral into two intervals.

$$\int_0^\infty \frac{x^s}{x^2+1} dx = \int_0^1 + \int_1^\infty$$

In the first interval, the integrand takes some finite value, regardless of the value of s, as long as it is positive. Thus, we focus on the latter summand. Notice:

$$0 < \int_1^\infty \frac{x^s}{x^2+1} dx \leq \int_1^\infty \frac{x^s}{x^2} dx = \int_1^\infty x^{s-2} dx$$

For $s < 1$, $s - 2 < -1$ hence:

$$\int_1^\infty x^{s-2} dx = \left. \frac{x^{s-1}}{s-1} \right|_{x=1}^\infty = \frac{1}{1-s}$$

Thus, the integral is bounded between zero and some positive value. For the integrand is always positive in the interval $(1, \infty)$, the integral monotonically increases as the upper bound is sent to infinity. Bounded monotone sequences must converge, and hence we conclude the the integral converges for $s < 1$.

It remains to show that the integral diverges for $s \geq 1$. Consider the following inequality:

$$0 < \int_1^\infty \frac{x^s}{2x^2} dx \leq \int_1^\infty \frac{x^s}{x^2+1} dx$$

Assume $s \geq 1$. We evaluate the left integral:

$$\int_1^\infty \frac{x^s}{2x^2} dx = \int_1^\infty \frac{x^{s-2}}{2} dx$$

If $s = 1$, the following integral equals to:

$$\left. ln(x)/2 \right|_1^\infty$$

which diverges. If $s > 1$, then:

$$\left. \frac{x^{s-1}}{2(s-1)} \right|_1^\infty$$

which also diverges. We have verified that the integral diverges to infinity for $s \geq 1$. $\square$