Notes as of Aug 22

Benevolent Tomato

1 Free Probability

We have proved two theorems regarding the expected trace of GOE's

Theorem 7 *Even moments of the GOE are Catalan Numbers*

$$\mathbb{E}[\operatorname{tr}(X^n)] \;=\; c_n \;=\; \begin{cases} 0 & 2 \nmid k \\ C_{k/2} & 2|k \end{cases} \tag{1}$$

Theorem 10 *Product of the centered moments converge to zero*
Suppose $X_1, X_2, \ldots, X_m$ are independent N-by-N GOEs. As $N \to \infty$, the following value converges to zero.

$$\mathbb{E}\left[\operatorname{tr}\left(X_1^{r_1} - c_{r_1}\right)\cdots\left(X_m^{r_m} - c_{r_m}\right)\right] \;=\; 0 \tag{2}$$

We can generalize the concept of expected trace. Let φ denote the expected trace in an abstract sense. We call the function to be a **state function** under two additional conditions.

$$\begin{aligned} \varphi(1) &\;=\; 1 \\ \varphi(a * a) &\;\geq a \qquad \forall a \in \mathcal{A} \end{aligned} \tag{3}$$

The second condition is necessary to make $\mathcal{A}$ a $*-$algebra.

Motivated by the space of all random matricies, we define a probability space $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is some unital algebra. The state function links the algebra element to some complex number.

A $*-$ probability space $(\mathcal{A}, \varphi)$ is **faithful** if

$$\varphi(x * x) = 0 \text{ iff } x = 0 \tag{4}$$

ity space $(\mathcal{A}, \varphi)$ is **non-degenerate** if

$$\begin{aligned} \varphi(yx) = 0 \;\forall y \in \mathcal{A} \;&\to\; x = 0 \\ \varphi(xy) = 0 \;\forall y \in \mathcal{A} \;&\to\; x = 0 \end{aligned} \tag{5}$$

Also, by the means of induction, it is possible to deduce the following.

Proposition 13 *Recovering the global state from local states*
Suppose $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are free subalgebras of a $*-$probability space $\mathcal{A}$. The local state functions $\varphi|_{\mathcal{A}_i}$ for $1 \leq i \leq m$ determines the global state function φ

2 Putnam Problems

1991B6 *Hyperbolic sine inequality*
Let a, b be positive numbers. Find the largest c, in terms of a, b that satisfies the inequality

$$a^x b^{1-x} \leq a\frac{\sinh(ux)}{\sinh(u)} + b\frac{\sinh(u(1-x))}{\sinh(u)} \tag{6}$$

Apply the substitution $v := e^u$ and $r := a/b$. Then, guess an appropriate value of c where the inequality turns to an equality. Afterwards, show that any value of c greater than the guessed value has a x that bears a witness to the failiture of the inequality.

1991B3 *Using the Postage Stamp Theorem*
Given $a, b \in \mathbb{Z}_{pos}$, we know that for any equation

$$ax + by = s \tag{7}$$

where $\gcd(a, b) = 1$, there exists some $N > 0$ where there always exists a solution for the equation for any $s \geq N$. We can prove this by considering the residues of the set

$$\{0, a, 2a, \ldots, (b-1)a\} \tag{8}$$

which must be a compete set of residues mod b.

1991B2 *Cauchy's Lemma*
If the function $f : \mathbb{R} \to \mathbb{R}$ is an additive continuous automorphism of the reals, that is for all $a, b \in \mathbb{R}$

$$f(a+b) = f(a) + f(b) \tag{9}$$

$f(x)$ must be in the form of

$$f(x) = cx \tag{10}$$

Proof. Applying the additive automorphism, it is easy to verify that this must be true for all the rationals. Thus, the function $f(x) - cx$ must vanish at all the rationals, and by continuity, at all the reals. □