Convergence of Anticommutator Spectral Densities

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Definition 1 (Dependency of Pairings)**.** *Let $\mathcal{P}_2[n \cdot 2k]$ to denote the set of all pairings of the cannonical set $[n \cdot 2k]$. Consider $\pi \in \mathcal{P}_2[n \cdot 2k]$. Partition the set $[n \cdot 2k]$ into n blocks, namely*

$$\begin{aligned} B_1 &= \{1, 2, \ldots, 2k\} \\ B_2 &= \{2k+1, 2k+2, \ldots, 4k\} \\ &\vdots \\ B_n &= \{(n-1)2k+1, (n-1)2k+2, \ldots, n \cdot 2k\} \end{aligned}$$

A block B_i is called to be dependent, if the image of B_i under the paring π is not a subset of B_i. The dependency of the pairing is the number of dependent blocks of a pairing.

For example, the pairing $\pi \in \mathcal{P}[8]$ defined as

$$\pi = (12)(34)(58)(67)$$

has two dependent blocks, namely B_3, B_4. Hence, its dependency is 2.

From the works of Hammond and Miller , we employ the fourth moment method. With slight modification of the normalization coefficient, we establish the following.

Theorem 1 (Fourth Moment Method for Convergence)**.** *If, for any positive integer m*

$$\mathbb{E}\left[\frac{1}{N^{4m+4}}\left|\mathrm{Tr}(AB+BA)^m - \mathbb{E}[\mathrm{Tr}(AB+BA)^m]\right|^4\right] = O\left(\frac{1}{N^2}\right) \tag{1}$$

then spectral density of the anticommutator product converges almost surely.

We bound the lefthand side of (1) appropriately. In order to do this, we first expand out the forth power by the binomial expansion. For convinience, introduce the following shorthand.

$$M_j = \mathbb{E}\left[(\mathrm{Tr}(AB+BA)^m)^j\right]$$

Proposition 1. *The lefthand side of (1) can be rewritten as*

$$\frac{1}{N^{4m+4}}\left(M_4 - 4M_3M_1 + 6M_2M_1^2 - 3M_1^4\right) \tag{2}$$

Proposition 2. *Define D_j to be the contribution from the summands of the trace expansion that involves pairings of maximum dependency. In symbols,*

$$D_j \;=\; \sum_{W\in PW(j\cdot 2m)} \sum_{a_1,a_2,\ldots,a_j} \sum_{\substack{\pi\in\mathcal{P}_2[j\cdot 2k] \\ \pi \text{ dependency j}}} \mathbb{E}_\pi[W_{a_1 s}W_{a_2 s}\cdots W_{a_j s}] \tag{3}$$

[1] *where $a_1, a_2, \ldots, a_j$ are finite sequences of $2m$ integers between 1 and N. M_j can be rewritten as a sum involving pairings of different dependencies. Namely,*

$$\begin{aligned} M_2 &= D_2 + M_1^2 \\ M_3 &= D_3 + 3M_1D_2 + M_1^3 \\ M_4 &= D_4 + 3D_2^2 + 6D_2M_1^2 + 4D_3M_1 + M_1^4 \end{aligned} \tag{4}$$

Proof. We show that the proposition must hold for $j = 2$, for the simplicity of notation. For higher values of j, the proof is similar up to a slight modifictation, and we provide a verbal reasoning without the symbolic manipulation. We start with the principal definition of M_2.

$$M_2 \;=\; \mathbb{E}\left[(\mathrm{Tr}(AB+BA)^m)^2\right] \;=\; \sum_{W\in PW(2m)} \sum_{V\in PW(2m)} \mathbb{E}\left[\mathrm{Tr}(W)\mathrm{Tr}(V)\right] \tag{5}$$

Introducing two finite sequences a, b of $2m$ integers between 1 and N, we can further expand this equation. For convinience, set $a_{2m+1} = a_{2m}$ and $b_{2m+1} = b_{2m}$.

$$M_2 \;=\; \sum_{W\in PW(2m)} \sum_{V\in PW(2m)} \sum_{a,b} \mathbb{E}\left[W_{a_1a_2}W_{a_2a_3}\cdots W_{a_{2m}a_1}V_{b_1b_2}V_{b_2b_3}\cdots V_{b_{2m}b_1}\right] \tag{6}$$

For W, V iterates through the set of all product words of length $2m$, We can consider the product WV to iterate through all the product words of length $2\cdot 2m$. Furthermore, all the random variables are Gaussian. Hence, we can use Wick's formula. We progress to the following equation.

$$M_2 \;=\; \sum_{W\in \mathrm{PW}(4m)} \sum_{a,b} \sum_{\pi\in\mathcal{P}[4m]} \mathbb{E}_\pi\left[\prod_{i=1}^{2m} W^{(i)}_{a_ia_{i+1}} \prod_{i=1}^{2m} W^{(i+2m)}_{b_ib_{i+1}}\right] \tag{7}$$

Finally, split the sum of involving the pairings with respect to the dependency. For $j = 2$, $\mathcal{P}[4m]$ can either have a dependency of two or zero.

[1]The subscript s is an abuse of notation. See page 14 of Hammond and Miller.

$$
\begin{aligned}
M_2 &= \sum_{W \in \mathrm{PW}(4m)} \sum_{a,b} \sum_{\substack{\pi \in \mathcal{P}[4m] \\ \pi \text{ dependency } 0}} \mathbb{E}_\pi \left[\prod_{i=1}^{2m} W^{(i)}_{a_i a_{i+1}} \prod_{i=1}^{2m} W^{(i+2m)}_{b_i b_{i+1}} \right] \\
&+ \sum_{W \in \mathrm{PW}(4m)} \sum_{a,b} \sum_{\substack{\pi \in \mathcal{P}[4m] \\ \pi \text{ dependency } 2}} \mathbb{E}_\pi \left[\prod_{i=1}^{2m} W^{(i)}_{a_i a_{i+1}} \prod_{i=1}^{2m} W^{(i+2m)}_{b_i b_{i+1}} \right] \qquad (8)
\end{aligned}
$$

$$
= M_1^2 + D_2 \qquad (9)
$$

The last equality follows from the nature of the paired expectations. [2] If the pairing π has zero dependency, it The pairied expectation of π can be considered as products of the paired expectation of two independent blocks. The second summand is the principal definition of D_2

From the case of $j = 2$, we observe that the decomposition of M_j is determined by the property of the pairings $\pi \in \mathcal{P}[j \cdot 2m]$. For $j = 3$, each pairings can be categorized as dependency 3 (completely dependent) dependency 2 (one independent block with two blocks depending on each other) or dependency zero (all blocks independent). Upon inspection of the ways the blocks can be paired to each other, we conclude that there are only one configuration for complete dependence or independence. If the dependency is 2, choosing an independent block decides the other two dependent blocks, so there are 3 ways such pairings can occur.

A similar argument can be carried out to the $j = 4$ case.

$\square$

Lemma 1. *If*

$$
\frac{1}{N^{4m+4}}(3D_2^2 + D_4) = O\left(\frac{1}{N^2}\right) \qquad (10)
$$

then the spectral density of the anticommutator product converges almost surely.

Proof. Plug in equation (4) to equation (2). Simple algebra proves the result.

$\square$

Theorem 2. *Suppose A, B are random matricies drawn from Topelitz ensembles. Then, equation (1) indeed holds and the spectral density converges almost surely.*

Proof. It suffices to show

$$
D_2 = O(N^{2m+1}) \quad \text{and} \quad D_4 = O(N^{4m+1}) \qquad (11)
$$

Start with proving the first inequality. To compute D_2, we must consider two finite sequences a, b. Recall that the paired expectation $\mathbb{E}_\pi$ is a product of the expected values in the form of

$$
\mathbb{E}[W^{(s)}_{i_s i_{s+1}} W^{(\pi(s))}_{i_{\pi(s)} i_{\pi(s)+1}}] \qquad (12)
$$

[2] Refer to Mingo and Speicher CH1

where the sequence i is either a or b. The modular restriction of the Topelitz ensenble dictates that this term is 1 if and only if

$$i_s - i_{s+1} \equiv i_{\pi(s)} - i_{\pi(s)+1} \mod N \tag{13}$$

and vanishes to zero otherwise. The anticommutator structure imposes an additional restriction that the letter $W^{(s)}$ and $W^{(\pi(s))}$ must both be A or both be B's in order for the expected value to not vanish.

We wish to overcount the number of pairings that produces a nonvanishing expectation. In order to do this, we choose all the modular differences of a, b. Using the difference notation, we note that there are $2m$ copies of Δa and $2m$ copies of Δb to be chosen. By the nature of the pairing, we observe that choosing one of the differences decides another paired difference. We also have the following restriction.

$$\begin{aligned} \sum_{i=1}^{2m} \Delta a_i &= a_{2m+1} - a_1 &= 0 \\ \sum_{i=1}^{2m} \Delta b_i &= b_{2m+1} - a_1 &= 0 \end{aligned} \tag{14}$$

The two equations takes away one degree of freedom from the original $2m$ degrees of freedom from to choose the paired differences. The fact that pairing π is dependant accounts for the fact that condition (14) cannot be naturally met without losing a degree of freedom. Also, we only lose one degree of freedom because the pairedness of the differences imply

$$\sum_{i=1}^{2m} \Delta a_i + \sum_{i=1}^{2m} \Delta b_i = 0 \tag{15}$$

Finally, choosing a_1, b_1 determines both sequences a, b, and this adds two degrees of freedom. Thus, there are $2m - 1 + 2 = 2m + 1$ degrees of freedom to choose D_2 and we conclude

$$D_2 = O(N^{2m+1}) \tag{16}$$

Similarly, for D_4, we lose three degrees of freedom by the dependence for the four sequences, and recieve four degrees of freedom by the choice of a_1, b_2, c_1, d_1, where the four sequence a, b, c, d show up in the sum index of D_4. The total degree of freedom is $4m - 3 + 4 = 4m + 1$ which leads to the bound

$$D_4 = O(N^{4m+1}) \tag{17}$$

$\square$

Corollary 1. *Suppose A, B are random matricies drawn either drawn from GOE, Palindromic Toeplitz, or Block Circulant ensembles. Then, equation (1) indeed holds and the spectral density converges almost surely.*

Proof. The derivation of lemma 1 does not involve the structure of individual matrix ensembles. All the three ensenbles listed in the corollary have either equal or more strict modular restrictions for the paired expectations to be non-vanishing, and therefore the value of D_2, D_4 are strictly larger for the case of the anticommutator of different products other than Topelitz anticommuted with Topelitz. Therefore the bounds in theorem 2 carries out directly. $\square$