

PHYS 314 HW9

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Problem 2

The Quantum Fourier Transform is defined as follows.

$$QFT_m|x\rangle := \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i xy/m} |y\rangle$$

a) Verify the inverse QFT of which the formula is given as

$$(QFT_m)^{-1}|x\rangle = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{-2\pi i xy/m} |y\rangle$$

Solution Naively, we wish to show, for any quantum state vector $|x\rangle$,

$$(QFT_m)^{-1}QFT_m|x\rangle = |x\rangle$$

Using the linearity of quantum operators, we deduce that the equation is equivalent to the following expression.

$$\frac{1}{m} \sum_{y,z < m} e^{2\pi i y(x-z)/m} |z\rangle = |x\rangle$$

We will separate the double sum in terms of the basis vectors $|z\rangle$ and show that its coefficient equals to zero unless $z = x$. Rewrite the LHS as follows.

$$\frac{1}{m} \sum_{y,z < m} e^{2\pi i y(x-z)/m} |z\rangle = \sum_{z < m} \xi_z |z\rangle$$

where

$$\xi_z := \frac{1}{m} \sum_{y < m} e^{2\pi i y(x-z)/m}$$

We claim that $\xi_z = 1$ if and only if $z = x$ and the term is zero otherwise.

$$\xi_x = \frac{1}{m} \sum_{y < m} e^{2\pi i y(0)/m} = \frac{m}{m} = 1$$

If $z \neq x$, then the complex exponentials would be a sum of a unit vectors that are rotated by a phase. Geometrically, they must sum to zero. Suppose $k = x - z \neq 0$. In symbols,

$$\xi_z = \frac{1}{m} \sum_{y < m} e^{2\pi i y(k/m)} = 0$$

In conclusion,

$$(QFT_m)^{-1}QFT_m|x\rangle = \sum_{z_m} \xi_z|z\rangle = |x\rangle$$

as desired. □

b) Concisely describe $(QFT_m)^2$. How about $(QFT_m)^4$.

Solution We proceed with the same steps we used for the previous problem. The coefficient of $|z\rangle$ of the transform simplifies to the following expression.

$$\xi_z = \frac{1}{m} \sum_{y < m} e^{2\pi i y(x+z)/m}$$

If $2\pi \frac{x+z}{m}$ is a nonzero phase, the sum equals zero by geometry in the complex plane. Otherwise, if the phase is zero, then the coefficient sums up to 1. The phase is zero if and only if $z = m - x$. Thus,

$$(QFT_m)^2|x\rangle = |m - x\rangle$$

So the double Quantum Fourier Transform inverts the state $|x\rangle$ with respect to the mean.

Consequently,

$$(QFT_m)^4|x\rangle = (QFT_m)^2|m - x\rangle = |x\rangle$$

b-2) Find the Eigenvalues of QFT_m

Solution Let λ be an eigenvalue of (QFT_m) .

$$(QFT_m)^4 = \lambda^4 I = I$$

This implies

$$\lambda^4 = 1$$

So

$$\lambda \in \{1, i, -1, -i\}$$

We observe that if $m = 2$, then the eigenvalues are ± 1 . Otherwise, we observe that all the eigenvalues show up.

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(*Define N, and compute W, the corresponding phase*)
For[NN := 2^1, NN < 2^6, NN = 2 * NN,
W := e^(2 π i / NN);
Wpower[n_] := W^n;

(* Create Vandermonde Matrix, which is the matrix of the QFT *)
rg = Range[0, NN - 1];
Seed = Map[Wpower, rg];
Vmat = (1/Sqrt[NN]) VandermondeMatrix[Seed];
MatrixForm[Vmat]

(* Compute the Eigensystem *)
Print[FullSimplify[Eigenvalues[Vmat]] // MatrixForm]
1

$$\begin{pmatrix} -1 \\ 1 \\ i \\ 1 \\ 1 \\ 1 \\ 1 \\ i \\ i \\ -i \\ -1 \\ -1 \end{pmatrix}$$


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Problem 3

Before continuing to determining $\omega = 0.x_1x_2\cdots$ for more complicated states, make note of the following very useful identity.

$$\begin{aligned}
\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i \omega y} |y\rangle &= \left(\frac{|0\rangle + e^{2\pi i (2^{n-1}\omega)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (2^{n-2}\omega)} |1\rangle}{\sqrt{2}} \right) \otimes \cdots \\
&\quad \cdots \otimes \left(\frac{|0\rangle + e^{2\pi i (\omega)} |1\rangle}{\sqrt{2}} \right) \\
&= \left(\frac{|0\rangle + e^{2\pi i (0.x_n x_{n+1} \cdots)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_{n-1} x_n x_{n+1} \cdots)} |1\rangle}{\sqrt{2}} \right) \otimes \cdots \\
&\quad \cdots \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_1 x_2 \cdots)} |1\rangle}{\sqrt{2}} \right). \tag{7.1.18}
\end{aligned}$$

Prove this identity

Solution Before showing that this equation is true, first consider what it means. The equation tells us that an QFT of a n-qbit system can be considered as a superposition of rotations of each individual quantum states. The degree of rotation is determined by the rank of the bit among the entire collection of bits along with the frequency ω in question.

Label the rank of the n-qubits as $1, 2, \dots, n$. When taking the tensor product of all the qubits, the rank of the qubit determines the magnitude in which it will affect the digit space. For example, consider a 3-qbit system and suppose there are two states $|0\rangle|0\rangle|1\rangle$ and $|1\rangle|0\rangle|1\rangle$. In digit space, these two states correspond to $|1\rangle, |5\rangle$ respectively. The difference between the two states is the first qubit which has a rank of 1. The rank 1 qubit affects the state by magnitude 4.

In light of the relation between rank and magnitude difference, foil the right tensor product of the given identity.

$$\begin{aligned} & \left(\frac{|0\rangle + e^{2\pi i(2^{n-1})\omega}|1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i(2^{n-2})\omega}|1\rangle}{\sqrt{2}} \right) \\ & \otimes \dots \otimes \left(\frac{|0\rangle + e^{2\pi i(2^0)\omega}|1\rangle}{\sqrt{2}} \right) = \frac{1}{\sqrt{2^n}} \sum_{y < 2^n} \xi_y |y\rangle \end{aligned} \quad (1)$$

Focus on one basis state $|y\rangle$. y , in digits, is some nonnegative integer less than 2^n . Write $|y\rangle$ as

$$y = 2^{n-b_1} + 2^{n-b_2} + \dots + 2^{n-b_m}$$

assuming that y has m states that are 1 in binary notation. We also assume that b_k is a strictly increasing interger sequence where all entries are bounded between 1 and n . Inspecting the tensor product, we notice that the coefficient of $|y\rangle$ is

$$\begin{aligned} \xi_y &= e^{2\pi i 2^{n-b_1}\omega} e^{2\pi i 2^{n-b_2}\omega} \dots e^{2\pi i 2^{n-b_m}\omega} \\ &= e^{2\pi i 2^n (2^{-b_1} + 2^{-b_2} + \dots + 2^{-b_m})\omega} = e^{2\pi i y \omega} \end{aligned}$$

And plugging in the coefficient to the sum, we conclude

$$\begin{aligned} & \left(\frac{|0\rangle + e^{2\pi i(2^{n-1})\omega}|1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i(2^{n-2})\omega}|1\rangle}{\sqrt{2}} \right) \\ & \otimes \dots \otimes \left(\frac{|0\rangle + e^{2\pi i(2^0)\omega}|1\rangle}{\sqrt{2}} \right) = \frac{1}{\sqrt{2^n}} \sum_{y < 2^n} e^{2\pi i y \omega} |y\rangle \end{aligned} \quad (2)$$

□

b We know that the inverse QTF reverts a quantum state correctly to its corresponding state if the frequency value is a multiple of $1/2^n$. The inverse QTF is also called the phase estimation algorithm. Let ω be some frequency that is not a integer multiple of $1/2^n$. Compute $p(x)$, the probability that the quantum state will collapse to $|x\rangle$ after the phase estimation algorithm.

Solution Consider state $|\psi\rangle$, which corresponds to sending through the state ω through the regular QFT.

$$|\psi\rangle := \frac{1}{\sqrt{2^n}} \sum_{y < 2^n} e^{2\pi i y \omega} |y\rangle$$

Where $\omega = .x_1 x_2 \dots x_n \dots$

Recall that the inverse QTF for pure states is defined as follows.

$$(QFT_{2^n})^{-1} |x\rangle := \frac{1}{\sqrt{2^n}} \sum_{y < 2^n} e^{-2\pi i y x / 2^n} |y\rangle$$

Apply the inverse QFT to state ψ and exploit linearity.

$$\begin{aligned} (QFT_{2^n})^{-1}|\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_{y < 2^n} e^{2\pi i \omega y / 2^n} (QFT_{2^n})^{-1}|y\rangle \\ &= \frac{1}{2^n} \sum_{y, z < 2^n} e^{2\pi i y(\omega - z/2^n)} |z\rangle := \sum_{z < 2^n} \xi_z |z\rangle \end{aligned}$$

We wish to compute the coefficient ξ_z . Write

$$\xi_z = \frac{1}{2^n} \sum_{y < 2^n} e^{2\pi i y(\omega - z/2^n)}$$

This sum can be computed by the geometric series formula with ease.

$$\xi_z = \frac{1}{2^n} \frac{e^{2\pi i(\omega - z/2^n)} - 1}{e^{2\pi i(\omega - z/2^n)} - 1}$$

Ignore the phase and apply the Euler's formula.

$$|\xi_z| = \frac{1}{2^n} \frac{\sin(\pi(\omega - z/2^n)2^n)}{\sin(\pi(\omega - z/2^n))} = \frac{1}{2^n} \frac{\sin(\pi(2^n\omega - z))}{\sin(\pi(\omega - z/2^n))}$$

Use Born's probability rule to compute $p(x)$.

$$p(x) = |\xi_x|^2 = \frac{1}{2^{2n}} \frac{\sin^2(\pi(2^n\omega - x))}{\sin^2(\pi(\omega - x/2^n))}$$

□

c) Here is a plot of $p(x)$ for $n = 3, 8, 10$ where $\omega = .7$.



