Combinatorics HW8
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Q16 Formulate a combinatorial problem for which the generating function is

$$(1 + x + x^2)(1 + x^2 + x^4 + x^6)(1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)$$

Solution Alice decided to spend the weekend in an arcade game plaza "Wonderland." Wonderland has four games. To enter the center, Alice has to pay a quarter. The first game is Dune, and Alice has to pay a quarter to play a round of the game. Nonetheless, due to the popularity of the game, a player is allowed to play a maximum number of two rounds of Dune for each visit. Wonderland also has a Karaoke machine, and Alice has to pay two quarters for each song. A player is allowed to sing maximum three songs every visit. The other two games are pacman and tetris. These classic games cost two quarters for each round. They are less popular than the other two machines, so Alice can play as many games as she want.

The number of ways Alice can spend her time with n quarters in Wonderland generates the generating function in question. We assume that the order in which Alice plays each game does not affect how she spends her time. That is, if Alice plays tetris then pacman, the arrangement is considered equivalent if she plays pacman first then tetris.

□

Q17 Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear. Then find a formula for hn from the generating function.

Solution Upon inspection, we write the generationg function for h_n.

$$G(x) = (1 + x^2 + x^4 + \cdots)(1 + x + x^2)(1 + x^3 + x^6 + \cdots)(1 + x)$$

$$= \frac{1}{1 - x^2} \frac{1 - x^3}{1 - x} \frac{1}{1 - x^3}(1 + x) = \frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty}(n + 1)x^n$$

Q22 Determine the exponential generating function for the sequence of factorials.

Solution

Recall the definition of exponential generating functions. For a sequence $\{a_n\}_{n\in\mathbb{N}}$, the exponential generating function is

$$E(x) := \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

We know $a_n = n!$. Plug in the definition.

$$E(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$\square$

Preliminary for Q24   To better understand how EGFs can be used, we present the following theorem, which is a slight generalization of Thm 7.3.1 of the textbook.

Theorem Multiplying to EGFs generates the EGF of a sequence that accounts for partitions.

Let $f_i(x)$ be the EGF of the sequence $\{a_n^i\}_{n \in \mathbb{N}}$. The function

$$\prod_{i \leq N} f_i(x)$$

is an EGF of the sequence

$$h_n := \sum_{m_1 + \ldots m_N = n} \binom{n}{m_1, m_2, \ldots, m_N} \prod_{i \leq N} a_i$$

A short proof can be written similarly to that of Thm 7.3.1.

Q24

24. Let S denote the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \ldots, \infty \cdot e_k\}$. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where $h_0 = 1$ and, for $n \geq 1$,

   (a) h_n equals the number of n-permutations of S in which each object occurs an odd number of times.

   (b) h_n equals the number of n-permutations of S in which each object occurs at least four times.

   (c) h_n equals the number of n-permutations of S in which e_1 occurs at least once, e_2 occurs at least twice, $\ldots$, e_k occurs at least k times.

   (d) h_n equals the number of n-permutations of S in which e_1 occurs at most once, e_2 occurs at most twice, $\ldots$, e_k occurs at most k times.

Solution

Use the theorem presented in the preliminary. For part a, the EGF of h_n is the product of the EGF for each of the objects. We add the constant 1 to account for $h_0 = 1$.

$$E_a(x) = 1 + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right)^k = 1 + \sinh^k(x)$$

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Similarly for part b,

$$E_b(x) = 1 + \left(\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \right)^k = 1 + \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} \right)^k$$

For part c,

$$E_c(x) = 1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) \cdots \left(\frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} + \cdots \right)$$

For part d, do not add the constant 1.

$$E_c(x) = (1 + x) \left(1 + x + \frac{x^2}{2!} \right) \cdots \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} \right)$$

$\square$

Q25 Let h_n denote the number of ways to color the squares of a I-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, ..., h_n, ...,$ and then find a simple formula for h_n.

Solution The generating function for h_n can be written as

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^2$$

$$= \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^x - e^{-x}}{2} \right) e^{2x} = \frac{1}{4}(e^{4x} - 1)$$

$$= \sum_{k=0}^{\infty} 4^{k-1} x^k - \frac{1}{4} = \sum_{k=1}^{\infty} 4^{k-1} x^k$$

$$= x + 4x^2 + 16x^3 \cdots$$

Thus, we conclude

$$h_n = \begin{cases} 0 & (n = 0) \\ 4^{n-1} & (n \geq 1) \end{cases}$$

Q26 Determine the number of ways to color the squares of a I-by-n chessboard, using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

Solution Same as Q25, we write the generating function.

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^2$$

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$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)e^{2x} = \frac{1}{4}(e^{4x} + 2e^{2x} + 1)$$

$$\frac{1}{4}\sum_{n=0}^{\infty} \left(4^n + 2^{n+1}\right) x^n + 1$$

Thus,

$$h_n = \begin{cases} 1 & (n = 0) \\ 4^{n-1} + 2^{n-1} & (n \geq 1) \end{cases}$$

$\square$