PHYS 201 Problemset 10

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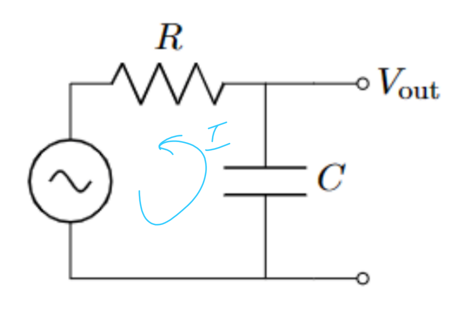


Figrure for Q1

Q1-a Let $V_{in} = V_0 cos(\omega t)$. Use complex impedence to find the amplitude of V_{out}

Solution We know that the loop rule applies to impedences. Write:

$$V_0 = \tilde{I} z_R + \tilde{I} z_C$$

where z_R, z_C denotes the impedence of the capacitor and the resistor. By the impedence formula:

$$V_0 = \tilde{I}(z_R + z_C) \quad \text{and} \quad \tilde{I} = \frac{V_0}{R - \frac{i}{\omega C}}$$

To compute the impedence of V_out:

$$\tilde{V}_{out} = R\tilde{I} = \frac{RV_0}{R - \frac{i}{\omega C}}$$

Finally, take the modullus of $\tilde{V}_{out}$ to compute the amplitude.

$$\boxed{V_{out} = \left| \frac{RV_0}{R - \frac{i}{\omega C}} \right| = \frac{RV_0}{\sqrt{(R - \frac{i}{\omega C})(R + \frac{i}{\omega C})}} = \frac{V_0}{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}}}$$

Q1-b Compute the phase shift of V_{out}

Solution The phase shift can be easily computed by dividing the impedence by amplitude.

$$e^{i\theta} = \tilde{V}_{out}/V_{out} = R\tilde{I} = \frac{RV_0}{R - \frac{i}{\omega C}} \Bigg/ \frac{V_0}{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}} = \frac{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}}{1 - \frac{i}{\omega CR}}$$

Thus:

$$e^{i\theta} = \sqrt{\frac{1+\frac{i}{\omega CR}}{1-\frac{i}{\omega CR}}} \quad \text{or} \quad e^{2i\theta} = \frac{1+\frac{i}{\omega CR}}{1-\frac{i}{\omega CR}}$$

With a little bit of geometry, it is possible to deduce:

$$2\theta = 2arctan(\frac{1}{\omega CR}) \quad \text{and} \quad \boxed{\theta = arctan(\frac{1}{\omega CR})}$$

$\square$

Q1-c Repeat the analysis for a circuit where the capacitor is replaced by an inductor.

Solution Rewrite the loop rule as:

$$V_0 = \tilde{I}z_L + \tilde{I}z_R \quad \text{and} \quad V_0 = \tilde{I}(z_L + z_R) \quad \text{and} \quad \tilde{I} = \frac{V_0}{z_L + z_R}$$

Applying the impedence formula:

$$\tilde{I} = \frac{V_0}{i\omega L + R} \quad \text{and} \quad \tilde{V}_{out} = \frac{V_0 R}{i\omega L + R} = \frac{V_0}{i\omega L/R + 1}$$

Compute the modullus and the argument for the amplitude and phase.

$$\boxed{V_{out} = |V_{out}| = \frac{V_0}{1 + \omega^2 L^2/R^2}}$$

With rationalization, $\tilde{V}_{out}$ reduces to:

$$\tilde{V}_{out} = \frac{V_0(1 - i\omega L/R)}{(1 + i\omega L/R)(1 - i\omega L/R)} = \frac{V_0(1 - i\omega L/R)}{1 + \omega^2 L^2/R^2}$$

For the denominator is a real value, it suffices to compute the argument of the denominator to compute the argument of the impedence. We conclude:

$$\boxed{\theta = -arctan(\omega L/R)}$$

Q1-d A capacitor acts as a short circuit at high-frequency and an open circuit at low-frequency. Make an analogous statement for inductors.

Statement An inductor acts as an open circuit at high-frequency and a short circuit at low-frequency. The justification comes from observing V_{out} and its dependency on the angular frequency, ω. $\square$

P&M 8.27 *RLC Parallel Circuit* A resistor, inductor, capacitor each of resistance 1k ohms, 500p farads, 2m henries are connected in parallel. The frequency is given as 10k cycles per second. Compute the combined impedence of the circuit. Also, compute the frequency where the magnitude of impedence is maximal.

Solution For the junction rule holds for impedence, we compute the combined impedence by taking the sum of the reciprocals of the impedence of each circuit components, and again dividing it from 1. In symbols:

$$z_{eff} = \frac{1}{1/z_c + 1/z_l + 1/z_r}$$

where z_c, z_l, z_r denotes the impedence of the capacitor, inductor, and resistor respectively. Applying the impedence formula:

$$z_{eff} = \frac{1}{\frac{1}{R} + \frac{1}{i\omega L} - \frac{\omega C}{i}} = \frac{i\omega LR}{i\omega L + R - \omega^2 CLR}$$

The frequency is given as $1kHz$ and $10mHz$, so the angular frequency is:

$$\omega_1 = 2\pi \cdot 10^4 Hz \quad \text{and} \quad \omega_2 = 2\pi \cdot 10^7 Hz$$

By the problem conditions, we have:

$$R = 1000\Omega \quad \text{and} \quad C = 5 \cdot 10^{-7} F \quad \text{and} \quad L = 2 \cdot 10^{-3} H$$

Also, the following unit conversions are useful:

$$[C] = F = \Omega^{-1} \cdot s \quad \text{and} \quad [L] = H = \Omega \cdot s$$

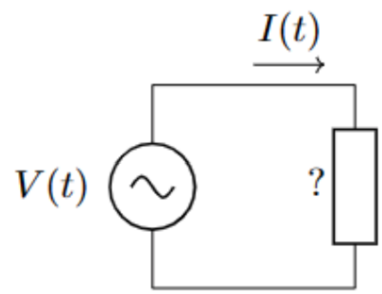
As a sidenote, to derive the two unit conversions, apply dimensional analysis on the following formulas:

$$Q = CV \quad \text{and} \quad \mathcal{E} = L\frac{dI}{dt}$$

With some algebra, with help of python, we conclude:

$$\boxed{z_{eff} \approxeq 15.7 + 124.2i \text{ for 10kHz} \quad \text{and} \quad z_{eff} \approxeq 1.01 - 31.8i \text{ for 10MHz}}$$

Q3 A mystery device is connected to an AC circuit with sinusodial input voltage. The frequency is 100kHz. The amplitude of the voltage and current is given as $V_0 = 5V, I_0 = 2mA$. The voltage leads the current by a phase $\pi/6$.



Circuit image for Q3

i) Compute the impedence of the mystery device

Solution
Write:

$$\tilde{V} = 5V \quad \text{and} \quad \tilde{I} = 2mAe^{-i\pi/6}$$

Thus:

$$z = \tilde{V}/\tilde{I} = 2.5 \cdot 10^3 e^{i\pi/6}\Omega$$

Or, in cartesian form:

$$\boxed{z = (2170 + 1250i)\Omega}$$

ii) What is a potential candidate for this mystery device?

Solution An equivalent of the device can be constructed by connecting a resistor with a resistor. Denote the resistance and the inductor as R, L. The combined impedence of connecting the two components in series is:

$$z_{cmb} = R + i\omega L$$

We deduce:

$$R = 2170\Omega \quad \text{and} \quad \omega L = 2\pi f L = 1250\Omega$$

Ergo:

$$\boxed{R = 2170\Omega \quad \text{and} \quad L = 2mH}$$

iii) Assume that the mystery device is indeed composed of the elements that were guessed in the previous problem. How will the amplitude of current change as the frequency is increased? Moreover, how will the change of frequency affect the phase difference between the voltage and current?

Solution As frequency increases, ω increases. This results in an increase of both the argument and the magnitude of the impedence. Recall:

$$z = \tilde{V}/\tilde{I} \quad \text{or} \quad \tilde{I} = \tilde{V}/z$$

Hence, larger magnitude of impedence leads to decrease of the amplitude of the current. Also, the larger the argument of the impedence, the larger the phase difference between voltage and current. □

Prelude to Q4 When talking about power in AC circuits, it is convinient to use root mean squared values for voltage and current. Note that RMS is NOT EQUAL to the simple mean.

Two following equations come in handy:

$$\bar{P} = V_{rms}^2/R \quad \text{and} \quad \bar{P} = V_{rms}I_{rms}cos(\phi)$$

As part of being physicists, we leave it as an exercise to the mathematicians to justify this equation.

To compute the rms value of a sinusodial function, remember:

$$A_{rms} = A_0/\sqrt{2}$$

where A_0 denotes the amplitude of the quantity.

Q4 An incandecent bulb consumes $60W$ of power if connected to a standard $120V, 60Hz$ AC circuit. We wish to connect this bulb to a circuit with $240V, 60Hz$ voltage and frequency. By connecting an inductor in series with the bulb, it is possible to make the bulb consume the same amount of power in average. What must be the inductance of the inductor?

Solution The bulb has an internal resistance. This can be easily computed by the power formula. Write:

$$P = V_{rms}^2/R \quad \text{and} \quad R = V_{rms}^2/P = (120V)^2/60W = 240\Omega$$

If the bulb is connected in series with an inductor, the magnitude of the combined impedence increases. Let L denote the inductance of the inductor. Write:

$$z_{eff} = i\omega L + R$$

$$\tilde{I} = V/z_{eff} \quad \text{and} \quad \tilde{V}_b = \tilde{I}R = \frac{VR}{z_{eff}}$$

Thus

$$V_b = \frac{VR}{|i\omega L + R|} = \frac{VR}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V}{\sqrt{1 + \omega^2 (L/R)^2}}$$

where V_b denotes the voltage through the bulb.

If $V_b = 120V$, then the bulb consumes the same power as connected to the standard circuit. $V = 240V$ so we wish to fix the denominator as 2. Then:

$$\sqrt{1+\omega^2(L/R)^2} = 2 \quad \text{or} \quad 1+\omega^2(L/R)^2 = 4$$

So:

$$L^2 = 3R^2/\omega^2 \quad \text{and} \quad L = \sqrt{3}R/\omega$$

Ergo:

$$\boxed{L \approxeq 1.1H}$$