1. Find a functional equation for the following function:

$$G(s) := \int_0^\infty \text{Exp}[-x^2] x^{s-1} dx$$

Solution We first observe the following derivative:

$$\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$$

Hence:

$$\int xe^{-x^2} dx = -\frac{e^{-x^2}}{2} + C$$

With this indefinite integral in mind, we integrate G by parts. Apply the following substitutions:

$$u = x^{s-2} \quad \text{and} \quad du = (s-2)x^{s-3}$$

$$dv = x\text{Exp}[-x^2]dx \quad \text{and} \quad v = -\frac{\text{Exp}[-x^2]}{2}$$

Integrate G:

$$G(s) = uv\Big|_{x=0}^{\infty} - \int_{x=0}^{\infty} vdu$$

$$= \left[-\frac{x^{s-2}\text{Exp}[-x^2]}{2}\right]_0^\infty - \int_0^\infty \left(-\frac{(s-2)\text{Exp}[-x^2]x^{s-3}}{2}\right) dx$$

$$= \left[-\frac{x^{s-2}\text{Exp}[-x^2]}{2}\right]_0^\infty + \frac{(s-2)}{2} \int_0^\infty \left(\text{Exp}[-x^2]x^{s-3}\right) dx$$

For the sake of convergence of the first summand, we assume $s \geq 2$. As for $s = 2$, we evaluate:

$$G(2) = \int_0^\infty xe^{-x^2} dx = -\frac{e^{-x^2}}{2}\Big|_0^\infty = 1$$

As for $s > 2$, we compute the equality obtained by the by parts integration. Notice that the first summand converges to zero. Hence:

$$G(s) = \frac{s-2}{2} G(s-2)$$

□

2. Find an analytic continuation of the function:

$$H(x) = 1 + z^2 + z^4 + z^6 + \cdots + z^{2n} + \cdots$$

For which value is the continuation undefined? What is the value of the continuation at $z = 2$?

<u>**Solution**</u> The natural response after observing the function is to consider the geometric series. We define the continuation to be $\bar{H}(z)$. Write:

$$\bar{H}(z) = \frac{1}{1 - z^2}$$

Clearly, for values where $|z^2| < 1$, H and $\bar{H}$ agrees. It is possible to obtain a sequence that accumulate on some point in the unit circle that is not the center. For example, consider:

$$z_n := \frac{1}{2} e^{\pi i/n}$$

The series of points converge to $1/2$, and $H, \bar{H}$ agrees for any point z_n.

Upon inspection, we notice that $\bar{H}(z)$ is defined everywhere other than $z = \pm 1$. Other then at these two poles, $\bar{H}$ is holomorphic, for $z^2 - 1$ is holomorphic and the reciprocal of a holomorphic function must be holomorphic as long as the function is defined. $\square$

3. For which value of s is the following integral defined?

$$L(s) := \int_0^\infty \frac{x^s}{x^2+1}dx$$

Claim $L(s)$ is defined for $s \in (0,1)$

Proof It is trivial to notice that the integrand blows up at $x = 0$ when s is negative. We only consider positive s. Divide the integral into two intervals.

$$\int_0^\infty \frac{x^s}{x^2+1}dx = \int_0^1 + \int_1^\infty$$

In the first interval, the integrand takes some finite value, regardless of the value of s, as long as it is positive. Thus, we focus on the latter summand. Notice:

$$0 < \int_1^\infty \frac{x^s}{x^2+1}dx \le \int_1^\infty \frac{x^s}{x^2}dx = \int_1^\infty x^{s-2}dx$$

For $s < 1$, $s - 2 < -1$ hence:

$$\int_1^\infty x^{s-2}dx = \left.\frac{x^{s-1}}{s-1}\right|_{x=1}^\infty = \frac{1}{1-s}$$

Thus, the integral is bounded between zero and some positive value. For the integrand is always positive in the interval $(1, \infty)$, the integral monotonically increases as the upper bound is sent to infinity. Bounded monotone sequences must converge, and hence we conclude the the integral converges for $s < 1$.

It remains to show that the integral diverges for $s \ge 1$. Consider the following inequality:

$$0 < \int_1^\infty \frac{x^s}{2x^2}dx \le \int_1^\infty \frac{x^s}{x^2+1}dx$$

Assume $s \ge 1$. We evaluate the left integral:

$$\int_1^\infty \frac{x^s}{2x^2}dx = \int_1^\infty \frac{x^{s-2}}{2}dx$$

If $s = 1$, the following integral equals to:

$$\left. ln(x)/2 \right|_1^\infty$$

which diverges. If $s > 1$, then:

$$\left.\frac{x^{s-1}}{2(s-1)}\right|_1^\infty$$

which also diverges. We have verified that the integral diverges to infinity for $s \ge 1$. □

4. i) Define:

$$\zeta_{alt}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

Prove that for $Re(s) > 1$, this series converges

Solution It suffices to show the absolute convergence of the sequence. We show the convergence of the following sequence.

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \quad \text{or} \quad \sum_{n=1}^{\infty} \left| n^{-s} \right| = \sum_{n=1}^{\infty} \left| e^{-sln(n)} \right| = \sum_{n=1}^{\infty} e^{-Re(s)ln(n)} = \sum_{n=1}^{\infty} n^{-Re(s)}$$

Which, in fact equals to:

$$\sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}}$$

This series is known to converge by the p-series test, for it is given that $Re(s) > 1$. $\square$

ii) Prove:

$$\zeta_{alt}(s) = \zeta(s) - (2/2^s)\zeta(s)$$

Solution

Denote the nth partial sum of $\zeta(s), \zeta_{alt}(s)$ as $P_n(s), P'_n(s)$. That is:

$$P_n(s) := \sum_{k=1}^{n} \frac{1}{k^s} \quad \text{and} \quad P'_n(s) := \sum_{k=1}^{n} \frac{(-1)^k}{k^s}$$

$\zeta(s)$ converges absolutely for $Re(s) > 1$. Thus, as $n \to \infty$, $P_n(s) \to \zeta(s)$ and $P'_n(s) \to \zeta_{alt}(s)$.

For finite n, notice the following equality:

$$P_{2n}(s) - \frac{2}{2^s} P_n(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{(2n)^s}$$

$$- \frac{2}{2^s} - \frac{2}{4^s} - \cdots - \frac{2}{(2n)^s}$$

$$= 1 - \frac{1}{2^s} + \frac{1}{3^s} \cdots - \frac{1}{(2n)^s} = P'_{2n}(s)$$

Set $n \to \infty$. The equality converts to:

$$\zeta(s) - \frac{2}{2^s}\zeta(s) = \zeta_{alt}(s)$$

$\square$

Evaluate the integral:

$$I := \int_0^\infty \frac{x^4}{x^8+1}$$

Solution We take a semicircular toy contour centered at the origin with an angle of $\pi/4$, radius of $R \to \infty$. Call the contour γ.

First show that the circular region of the contour converges to zero as $R \to \infty$. Call the circular region γ_c. Notice:

$$\begin{aligned}\left|\int_{\gamma_c} \frac{z^4}{z^8+1} dz\right| &\leq \int_{\gamma_c} \frac{|z^4|}{R^8|(z/R)^8 + 1/R^8|} R\frac{dz}{R} \\ &= \frac{1}{R^3}\int_{\gamma_c} \frac{|dz/R|}{|(z/R)^8 + 1/R^8|}\end{aligned}$$

Now substitute:

$$z/R = e^{i\theta} \quad \text{and} \quad dz/dR = ie^{i\theta}d\theta$$

The circular contour is bounded by:

$$\frac{1}{R^3}\int_{\theta=0}^{\pi} \frac{d\theta}{|e^{8i\theta} + 1/R^8|}$$

And for sufficiently large R, we can bound this integral by:

$$\frac{\pi}{R^3(1 - 1/R^8)}$$

As $R \to \infty$, the bound converges to zero. By the squeeze limit theorem, we conclude that the circular region of the integral converges to zero.

By the residue theorem:

$$\int_{-R}^{R} \frac{x^4}{1+x^8}dx + \int_{\gamma_c} \frac{z^4}{1+z^8}dz = 2\pi i \sum_{k=0}^{n} Res(z_k)$$

where z_k denotes the poles.

Notice that the integrand is even in the reals. As $R \to \infty$, write:

$$\int_0^\infty \frac{x^4}{1+x^8}dx = \pi i \sum_{k=0}^{n} Res(z_k)$$

It remains to compute the residues in the selected toy contour. The poles are the 8th roots of unity that lies on the first and the first and second quadrant. They all have a modulus of 1, and an argument of $\frac{1+2k}{8}\pi$ for $k = 0, 1, 2, 3$.

The order of each pole is one. denote the pole as z_k. Compute the residue as follows:

$$Res(z_k) = \lim_{z\to z_k} \frac{z^4(z - z_k)}{1+z^8} =^H \lim_{z \to z_k} \frac{5z^4 - 4z^3 z_k}{8z^7} = \frac{z_k^4}{8z_k^7} = \frac{1}{8z_k^3}$$

Add up the residues:

$$\sum_{k=0}^{3} Res(z_k) = \frac{1}{8}[z_0^{-3} + z_1^{-3} + z_2^{-3} + z_3^{-3}]$$

$$= \frac{1}{8}[e^{-3\pi i/8} + e^{-9\pi i/8} + e^{-15\pi i/8} + e^{-21\pi i/8}]$$

And through some geometry:

$$= \frac{1}{4}(sin(\pi/8) - cos(\pi/8))i$$

We must compute $(sin(\pi/8) - cos(\pi/8))$. Square it:

$$(sin(\pi/8) - cos(\pi/8))^2 = 1 - 2sin(\pi/8)cos(\pi/8)$$
$$= 1 - sin(\pi/4) = 1 - 1/\sqrt{2}$$

The value of cos is greater than sin. Thus:

$$(sin(\pi/8) - cos(\pi/8)) = -\sqrt{1 - 1/\sqrt{2}}$$

Finally, we write:

$$I = \pi i \sum_{k=0}^{3} Res(z_k) = \frac{\pi i^2}{4}(-\sqrt{1 - 1/\sqrt{2}}) = \frac{\pi}{4}\sqrt{1 - 1/\sqrt{2}}$$

□

5. Prove:

$$|\Gamma(1/2+it)| = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}$$

Proposition The complex conjugate of the gamma function is the value of the gamma function evaluated at the conjugate of the argument. That is:

$$\overline{\Gamma(z)} = \Gamma(\bar{z})$$

Proof We expand the gamma function back to its definiton.

$$\overline{\Gamma(z)} = \overline{\int_{x=0}^{\infty} x^{z-1}e^{x}dx} = \int_{x=0}^{\infty} \overline{x^{z-1}}e^{x}dx$$

Rewrite the base of x as a base of e.

$$= \int_{0}^{\infty} \overline{e^{ln(x)(z-1)}}e^{x}dx$$

Observe that $ln(x)$ is always a real value. The conjugate of an exponent is the exponent of the conjugate of its argument. Hence:

$$= \int_{0}^{\infty} e^{ln(x)(\bar{z}-1)}e^{x}dx = \int_{0}^{\infty} x^{\bar{z}-1}e^{x}dx = \Gamma(\bar{z})$$

□

Solution We are given the reflection formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin(\pi s)}$$

Plug in $s = 1/2+it$. Write:

$$\Gamma(1/2+it)\Gamma(1/2-it) = \frac{\pi}{sin(\pi(1/2+it))}$$

With a litte manipulation, we rewrite the LHS.

$$\begin{aligned} LHS &= \Gamma(1/2+it)\Gamma(\overline{1/2+it}) = \Gamma(1/2+it)\overline{\Gamma(1/2+it)} \\ &= |\Gamma(1/2+it)|^2 \end{aligned}$$

Rewrite the RHS using the basic trig identity and Euler's formula:

$$RHS = \frac{\pi}{cos(\pi it)} = \pi\frac{2}{e^{i^2\pi t}+e^{-i^2\pi t}} = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

$LHS = RHS$ converts to:

$$|\Gamma(1/2 + it)|^2 = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}$$

Finally:

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}$$

$\square$