Variants of Conway's Soldiers: Monovariant methods and Fibonacci jumping

Conway group

1 Review of notations and preliminaries

We define our fibbonacci sequence with the following initial conditions and recurrence relation.

$$(F_0, F_1) = (0, 1) \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$$

By Binet's Formula, it is possible to write out an explicit equation for the nth Fibonacci number.

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - (1-\varphi)^n\right) \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2}$$

Assuming $n \geq 1$, it is possible to obtain the following inequality.

$$\left| F_n - \frac{\varphi^n}{\sqrt{5}} \right| \leq \frac{1}{\sqrt{5}}(\varphi - 1)^n < .277$$

We are interested in computing the fibbonacci inverse function. Introduce the notation

$$\mathcal{F}^{-1}(m) = n$$

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Proposition 1 Inverse Fibbonacci Inequality

$$\log_\varphi(m - .277) + .167 < \mathcal{F}^{-1}(m) < \log_\varphi(m + .277) + .168$$

Proof. Given an integer m, we wish to compute n that satisfies the following inequality.

$$F_{n-1} < m \leq F_n \tag{1}$$

Using our estimation, we can write

$$\frac{\varphi^{n-1}}{\sqrt{5}} - .277 < m \leq \frac{\varphi^n}{\sqrt{5}} + .277$$

. With some algebra, it is not hard to derive

$$\log_\varphi(m - .277) + .167 < n < \log_\varphi(m + .277) + .168 \tag{2}$$

□

Remark For large enough m, it is reasonable to make the estimation

$$m \approx \log_\varphi(m) + .167$$

2 Conway Soldiers and two related problems

figure to be added... Consider an infinite chessboard. Draw a horizontal line across the board, and fill all the grids under the line with one checkers. [1]. In each turn, we are allowed to take a soldier and jump across a neighboring checker, moving the checker two grids. If a jump occurs, the neighboring checker which is jumped upon is taken away. We continue this game for a finite number of moves.

We are interested in two main problems.

Problem 1 *The M-level Problem*

What is the highest level M which can be reached by the rules of Conway Checkers?

Problem 2 *The Rallying Problem*

Select any grid that is adjacent and below the borderline. What is the maximum number of soldiers we can focus on the target grid?

We reserve the variables M, R for the solution for the two problems. That is, given multiplicity m, M will refer to the optimal solution for the M-level problem, and R will refer to the maximum number of soldiers that can be rallied at the target square.

3 Monovariant Methods to compute upper bounds

More figure, and reference

We start with computing the bound of M using the monovariant method. Assume, after some sequence of moves, it is possible to send a single checker to level M. By the symmetry of the infinite board, we can choose any square that is M grids north from the boundary. At the end of the game, the score of the board must be strictly greater than 1. After a sequence of moves, there must be one piece in the target grid. Also, there must be some other grids in the board, since we are only allowed to conduct finite number of jumps. [2]

[1]In some formulations, the checkers are formulated as soldiers, hence the name Conway Soldiers

[2]The board must have a infinite number of checkers after a finite number of moves.

Theorem *Upper bound for M*

$$\log_{\varphi}(m) + 5 > M$$

Proof. Compute the inital score of the board which we will call T. For simplicity, consider the score of one horizontal strip, and call it s. Remember that there are m checkers in each board initially. Using the geometric series formula, it is possible to derive the following.

$$T = (\varphi - 1)^M \frac{\varphi}{5 - 3\varphi}$$

T must be greater than 1. Thus, we can write

$$m(\varphi - 1)^M \frac{\varphi}{5 - 3\varphi} > 1$$

Which leads to

$$m \frac{\varphi}{5 - 3\phi} > \left(\frac{1}{\varphi - 1}\right)^M = \varphi^M$$

Taking the log base golden ratio gives us the desired result.

$$\log_{\varphi}(m) + 5 > M$$

We note that equality can be achieved when we allow infinite number of steps.

□

Proposition *Fibonacci Climb* Given two grids that each have F_{n+1}, F_n checkers, it is possible to send a single checker n grids away from the beginning location.