Lower moments of Anticommutators with Block Circulant Ensembles

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Before we move on to computing the lower moments, of the anticommutator products, we simplify the computation using the cyclicity of trace. Let A and B be square matrices. We observe the following.

Proposition 1 (Trace Expansion for the second and the forth Moment)**.** *The trace of the power of the anticommutator can be simplified as follows.*

$$Tr[(AB+BA)^2] \;=\; 2(Tr(ABAB)+Tr(AABB))$$

$$\begin{aligned} Tr[(AB+BA)^4] \; &= 2\,Tr(ABABABAB)+4\,Tr(ABABABBA)+2tr(ABBAABBA) \\ &+4\,Tr(ABABBABA)+4\,Tr(ABBABABA) \end{aligned}$$

Proof. We demostrate for the second power and leave the proof for the forth power as an exercise.

First, we expand $(AB+BA)^2$:

$$(AB+BA)^2 \;=\; ABAB+ABBA+BAAB+BABA$$

Using the cyclic property of the trace, $\mathrm{Tr}(XY)=\mathrm{Tr}(YX)$, we have:

$$\mathrm{Tr}(ABBA) \;=\; \mathrm{Tr}(AABB) \quad \text{and} \quad \mathrm{Tr}(BAAB)=\mathrm{Tr}(AABB)$$

And thus

$$\mathrm{Tr}[(AB+BA)^2] \;=\; 2(\mathrm{Tr}(ABAB)+\mathrm{Tr}(AABB))$$

.

□

We proceed with computing the second moment of the anticommutator product of an anticommutator matrix and a GOE. Let A be a GOE and B be a m-circulant matrix. Also, set the order of both matricies to be N. Let μ_N denote the spectral density of the anticommutator product $AB+BA$ and $\mu_N^{(k)}$ the kth moment. Using the eigenvalue trace lemma, we obtain the following.

$$\mu_N^{(k)} \;=\; \frac{1}{N^{k+1}}\mathbb{E}(Tr[(AB+BA)^k]) \tag{1}$$

Theorem 1 (2nd and 4th moment of GOE times Block Circulant)**.**

$$\mu_N^{(2)} \;=\; 2 \quad \textit{and} \quad \mu_N^{(4)} \;=\; 10+\frac{2}{m^2}$$

Proof. Start with the second moment. We use the eigenvalue trace lemma along with the trace expansion for $k = 2$. Also note that the expected value is linear.

$$\mu_N^{(2)} = \frac{1}{N^3}\mathbb{E}(Tr(ABAB)) + \frac{1}{N^3}\mathbb{E}(Tr(AABB)) \quad (2)$$

We compute each of the summands independantly. Focus on the first summand, and use Wick's formula to rewrite the summand in tractable form. [1]

$$\frac{1}{N^3}\mathbb{E}(Tr(ABAB)) = \frac{1}{N^3}\sum_{1\le i_1,i_2,i_3,i_4\le N}\sum_{\pi\in\mathcal{P}[4]}\mathbb{E}_\pi(A_{i_1i_2}B_{i_2i_3}A_{i_3i_4}B_{i_4i_1}) \quad (3)$$

It is trivial that the pairings that match A's with B's vanish, for the two matricies A, B are assumed to be indepent. Thus, the permutation π must be

$$\pi = (13)(24)$$

and the double sum simplifies to

$$\frac{1}{N^3}\mathbb{E}(Tr(ABAB)) = \frac{1}{N^3}\sum_{1\le i_1,i_2,i_3,i_4\le N}\mathbb{E}(A_{i_1i_2}A_{i_3i_4})\mathbb{E}(B_{i_2i_3}B_{i_4i_1}) \quad (4)$$

Since A is a GOE and B is a block circulant matrix, the indicies i must satisfy the following condition.

$$\begin{aligned} i_1 &= i_4 \quad \text{and} \quad i_2 = i_3 && (5)\\ i_2 - i_3 &\equiv i_4 - i_1 \mod N && (6)\\ i_2 &\equiv i_1 \mod m && (7)\end{aligned}$$

Notice that the choice of i_1, i_2 determines both i_3, i_4. Hence, there are a maximum N^2 sequences of i's where the expected value is nonvanishing. So as $N \to \infty$,

$$\frac{1}{N^3}\mathbb{E}(Tr(ABAB)) = 0 \quad (8)$$

Repeat the procedure for $ABAB$.

$$\frac{1}{N^3}\mathbb{E}(Tr(AABB)) = \frac{1}{N^3}\sum_{1\le i_1,i_2,i_3,i_4\le N}\mathbb{E}(A_{i_1i_2}A_{i_2i_3})\mathbb{E}(B_{i_3i_4}B_{i_4i_1}) \quad (9)$$

For the expected value to be nonvanishing, the sequence i must satisfy

$$\begin{aligned} i_1 &= i_3 \quad \text{and} \quad i_2 \ \text{free} && (10)\\ i_3 - i_4 &\equiv i_1 - i_4 \mod N && (11)\\ i_3 &\equiv i_1 \mod m && (12)\end{aligned}$$

[1] We adopt the notion from the free probability book

The conditions simplify to $i_1 = i_3$ and other variables are free. Thus, there are N^3 sequences of i where the expected value is nonvanishing. In the limit $N \to \infty$,

$$\frac{1}{N^3}\mathbb{E}(Tr(AABB)) = 1 \tag{13}$$

Finally, from (2),

$$\mu_N^{(2)} = 2(0+1) = 2$$

As for the forth moment, we notice that there are five summands in the trace expansion. However, by a degree of freedom argument, the pairings which have a crossings of A's vanish. Hence, we deduce

$$\mu_N^{(4)} = \frac{2}{N^5}\mathbb{E}(Tr(ABBAABBA)) + \frac{4}{N^5}\mathbb{E}(Tr(ABABBABA)) \tag{14}$$

Focus on the first summand. Use Wick's formula and rewrite as the following.

$$\frac{2}{N^5}\sum_{1\le i_1,\dots,i_8\le N}\sum_{\pi\in\mathcal{P}[8]}\mathbb{E}_\pi\left(A_{i_1i_2}B_{i_2i_3}B_{i_3i_4}A_{i_4i_5}A_{i_5i_6}B_{i_6i_7}B_{i_7i_8}A_{i_8i_1}\right) \tag{15}$$

With some brute-force condition checking, it possible to verify that any pairings that have a crossing with A's do not contribute to the sum. So, the following two pairings have zero contribution as $N \to \infty$

$$(15)(23)(48)(67) \tag{16}$$

$$(14)(27)(36)(58) \tag{17}$$

The first permutation has a crossing $(15)(48)$ where both transposition pair two A's. For the first permutation, the crossing is $(14)(27)$ and the first transposition pairs two A's while the second pairs two B's.

Note that the crossings between pairings of B's do contribute to the sum. To demonstrate the fact, we compute the contribution of the pairing

$$\pi = (18)(26)(37)(45)$$

which is

$$\frac{2}{N^5}\sum_{1\le i_1,\dots,i_8\le N}\mathbb{E}\left(A_{i_1i_2}A_{i_8i_1}\right)\mathbb{E}\left(B_{i_2i_3}B_{i_6i_7}\right)\mathbb{E}\left(B_{i_3i_4}B_{i_7i_8}\right)\mathbb{E}\left(A_{i_4i_5}A_{i_5i_6}\right) \tag{18}$$

We wish to count the number of finite sequences i of length 8 that satisfies the conditions below.

$$i_2 = i_8 \tag{19}$$

$$i_2 - i_3 \equiv i_7 - i_6 \mod N \tag{20}$$

$$i_3 - i_4 \equiv i_8 - i_7 \mod N \tag{21}$$

$$i_4 = i_6 \tag{22}$$

$$i_2 \equiv i_7, i_3 \equiv i_6 \mod m \tag{23}$$

$$i_3 \equiv i_8, i_4 \equiv i_7 \mod m \tag{24}$$

Determine the residue of i's by mod m first. Notice that i_1, i_5 are free to be any value mod m, and all other values must be congruent to each other mod m. As for the value $\lfloor i/m \rfloor$, we determine that t here are five degrees of freedom where the $i's$ split into the following equivalence classes.

$$\{i_2, i_8\}, \{i_4, i_6\}, \{i_3\}, \{i_5\}, \{i_1\}$$

The index i_7 is determined by the conditions. Thus, there are 3 degrees of freedom to choose $i \mod m$ and 5 degrees of freedom for $\lfloor i/m \rfloor$. The total contribution in the limit is

$$\frac{1}{N^5} m^3 \left(\frac{N}{m}\right)^5 = \frac{1}{m^2}$$

If there are no crossings in the pairings, the contribution equals exactly one. Thus, by (14), we write

$$\mu_N^{(4)} \;=\; 2\left(3 + \frac{1}{m^2}\right) + 4(1) \;=\; 10 + \frac{2}{m^2}$$

□

Theorem 2 (2nd moment of Block Circulant times Block Circulant)**.**

$$\mu_N^{(2)} \;=\; 2 + \frac{2}{m^2}$$

The proof is similar to the case of GOE times Block Circulant.