RMT results

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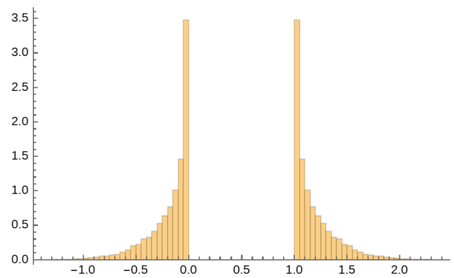
Proposition *Eigenvalues of R*

By a simple computation,

$$\lambda = \frac{1 + \sqrt{4a^2 + 1}}{2}$$

Observation *Spectral density of R*

Experimentally, the spectral density of the eigenvalues resemble the following.



Theorem *Eqn for spectral density*

The spectral density of R is given as the following.

$$f(x) = \begin{cases} 0 & x \in (0,1) \\ \frac{2x-1}{\sqrt{2\pi}\sqrt{x^2-x}} e^{\frac{-\sqrt{x^2-x}}{2}} & x < 0 \\ -\frac{2x-1}{\sqrt{2\pi}\sqrt{x^2-x}} e^{\frac{-\sqrt{x^2-x}}{2}} & x > 1 \end{cases}$$

Proof. The strategy is to compute the cumulative distribution function for the eigenvalues. Let a denote the random variable and λ the desired eigenvalue. x denotes the bound for the eigenvalue. Note that, given $\lambda < 0$,

$$\lambda < x \leftrightarrow a \in (\sqrt{x^2 - x}, \infty) \cup (-\infty, -\sqrt{x^2 - x})$$

and also, given $\lambda > 1$,

$$\lambda < x \leftrightarrow a \in (-\sqrt{x^2 - x}, \sqrt{x^2 - x})$$

Let $g(x)$ denote the PDF of the Gaussian distribution with mean zero and variance 1. It is possible to show, that

for $x < 0$

$$f(x) = -\frac{2x - 1}{\sqrt{x^2 - x}} g(\sqrt{x^2 - x})$$

for $x > 1$

$$f(x) = \frac{2x-1}{\sqrt{x^2-x}} g(\sqrt{x^2-x})$$

Trivially, it is impossible for the eigenvalue to value between zero and one. This is a consequence of elementary algebra. Finally, substituting

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

yields the desired result. $\square$

Proposition *Kronecker Product of an ensenble times a constant 2 by 2 matrix*

Let $\mathcal{A}$ be a matrix ensemble with spectral density $f(x)$. The spectral density of the ensemble

$$\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \otimes \mathcal{A}$$

is

$$\frac{f(x/a) + f(x/b)}{2}$$

.

Proof. Decompose the function $f(x)$ into a δ bins.

$$f(x) = \int_{t=-\infty}^{\infty} f(t)\delta(x-t)dt$$

Move the bins appropriately to get the desired result. $\square$

Conjecture *Kronecker product of two ensembles*

Let $\mathcal{A}, \mathcal{B}$ be two marix ensembles with spectral densities $f(x)$ and $g(x)$. The spectral density of the Kronecker product of the two ensembles are given by

$$h(x) = \int_{t\in\mathbb{R}} f(t)g(tx)dt$$

.

Definition *Red sea ensenble*

Let a $N(0,1)$. Consider the following matrix ensemble.

$$R := \begin{bmatrix} 0 & a \\ a & 1 \end{bmatrix}$$

Question *lemma for computing trace of the Topelitz distb*

Consider finite sequence of integers (i_n) which have k integers. For all $n \in [k]$, $i_n \in [N]$. [1] Also, for convinience, set $i_0 = i_k$ This sequence satisfies

$$|i_{n-1} - i_n| = x_i$$

for any $n \in [k]$. x_n is an auxillary sequence of integers that is also drawn from $[N]$, i.e. $x_n \in [N] \forall n \in [k]$. Also, x_i satisfies the following condition. Given any integer $v \in [N]$, there are either two or zero values that satisfiy

$$x_i = v$$

. Can we compute the number of solutions of this system as a function of N, k?

Remark *It suffices to find the asymptotic behavior of the number of solutions as* $N \to \infty$

Definition *Spaced Bases*

Consider the vector space $\mathbb{R}^{(N-1)}$. Define the mth spaced base ν_m as follows.

$$(\nu_m)_i := \begin{cases} 1 & (m|i) \\ 0 & (m \nmid i) \end{cases}$$

For example, in a $(9-1)$ dimensional vector space, the 3rd space base will be

$$\nu_3 = \{0, 0, 1, 0, 0, 1, 0, 0\}$$

Moreover, if $m|N$, we call the vector a divisor vector. Otherwise, the vector is called a nondivisor vector. For convinience ν_1 is identified as a nondivisor vector.

Proposition *The image of divisor vectors under transformation*

Let A be the average matrix of the DFT ensemble of order N, that is a $(N-1)$-by-$(N-1)$ matrix. Let $m|N$. Then, the following formula precisely describes the entries of the image of the mth spaced vector under the transformation induced by A.

$$(A\nu_m)_i = \frac{N}{\operatorname{lcm}(N/\gcd(i, N), m)} - 1$$

Proof. By direct computation, we write the ith entry of $A\nu_m$.

$$A\nu_m = \sum_{j=1}^{N-1} A_{ij}(\nu_m)_j$$

[1]Let the cannonical set $[N]$ denote the set that includes all natural numbers less than or equal to N.

The terms in the sum is nonvanishing if and only if

$$N|ij \quad \text{and} \quad m|j$$

which is equivalent to the condition

$$\frac{N}{\gcd(N,i)}|j \quad \text{and} \quad m|j$$

. We wish to count the number of j's in the range $[1, N-1]$ that satisfy the condition. We conclude

$$(A\nu_m)_i = \frac{N}{\operatorname{lcm}(N/\gcd(i,N), m)} - 1$$

□

Remark *Image of nondivisor vectors*

By a similar argument, when $m \nmid N$,

$$(A\nu_m)_i = \left\lfloor \frac{N-1}{\operatorname{lcm}(N/\gcd(i,N), m)} \right\rfloor$$

Corollary *Divisor vectors map to divisor vectors*

Let $m|N$. Then $A\nu_m$ can be expressed as a linear combination of divisor vectors, i.e

$$A\nu_m = \sum_{d|N} \tilde{a}_{md}\nu_m$$

Proof. Suppose we start from $A\nu_m$ and subtract divisor vectors to make it zero. We demonstrate this with $A\nu_2$ for $N = 8$. We start with the vector

$$\{0, 1, 0, 3, 0, 1, 0\}$$

Subtract ν_2 to obtain

$$\{0, 0, 0, 1, 0, 0, 0\}$$

and subtract ν_4 to obtain the zero vector. We conclude

$$A\nu_2 = \nu_2 + \nu_4$$

Now we prove the theorem in a genereral sense. Suppose we have subtracted k vectors in a sense described above, and call the remainder vector r_k. We induct on k. If $k = 0$, we take p, the smallest prime divisor of N. Look at all the entries j such that $p|j$. We know

$$(A\nu_m)_j = \frac{N}{\operatorname{lcm}(N/\gcd(j,N), m)} - 1 \geq \frac{N}{\operatorname{lcm}(N/\gcd(p,N), m)} - 1$$

and thus subtracting $(A\nu_m)_p\nu_p$ will preserve all the matrix entries positive.

Now, let $k \geq 1$. Let p to be the index of the first positive value occuring in r_i. If $p \nmid N$, then the pth entry of $A\nu_m$ will be zero, so we can assume $p|N$. By subtracting $(r_k)_k \nu_p$, we only affect the entries of r_k that are a multiple of k. It suffices to show that for all $p|j$, $(A\nu_m)_j \geq (A\nu_m)_p$

$$(A\nu_m)_j = \frac{N}{\mathrm{lcm}(N/\gcd(j,N),m)} - 1 \geq \frac{N}{\mathrm{lcm}(N/\gcd(p,N),m)} - 1$$

and the theorem holds. □

Remark *Nondivisor vectors map to divisor vectors*

Replace the formula for divisor vectors to nondivisor vectors, and repeat the same inductive argument used for the divisor vectors.

Theorem *Eigenvalues of A*

The nonzero eigenvalues of A are exactly the eigenvalues of the simplified matrix

$$\tilde{A} := [\tilde{a}_{d_i d_j}]$$

where d_i is the ith proper divisor of A greater than 1. Therefore, if N has $\sigma(N)$ divisors including N, it can have at maximum $\sigma(N) - 2$ eigenvalues, for $\tilde{A}$ is a square matrix of order $\sigma(N) - 2$.

Proof. We first notice that the set of all spacing vectors

$$\{\nu_1, \nu_2, \cdots \nu_{N-1}\}$$

form a basis of $\mathbb{R}^{N-1}$. By the proposition, we notice that the image of this base under the transformation A maps to the space induced by

$$\{\nu_{d_1}, \nu_{d_2}, \cdots \nu_{d_{\sigma(N)-1}}\}$$

. Hence, all nonzero eigenvectors must be in the subspace

$$\mathrm{span}(\nu_{d_1}, \nu_{d_2}, \cdots \nu_{d_{\sigma(N)-1}})$$

and the coefficients $\tilde{a}$ are described in the proposition. □

Remark *The simplified matrix for prime powers*

If $N = 2^k$, the simplified matrix can be simply written as

$$\begin{bmatrix} 1 & 2 & \cdots & 2^{k-2} & 2^{k-1} \\ 1 & 2 & \cdots & 2^{k-2} & 0 \\ & & \vdots & & \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Theorem *Lower bounds of the largest eigenvalue of the DFT matrix*

. Let λ_{max} be the maximum eigenvalue of a $(N-1)$-by-$(N-1)$ average matrix. The lower bound is

$$\lambda_{max} \geq \sqrt{\frac{N}{\sigma(N)} \sum_{d|N} \frac{1}{d} - 1}$$

Proof. Since the average matrix is real symmetric, its eigenvalues must be real. We also know that there are only $\sigma(N)$ nonzero eigenvalues of this matrix. Thus,

$$\sum_{i \leq \sigma(N)} \lambda_i^2 \leq \sigma(N) \lambda_{max}^2$$

By the eigenvalue trace lemma,

$$\sum_{i \leq \sigma(N)} \lambda_i^2 = \mathrm{tr}(A^2) = \sum_{1 \leq i,j \leq N} a_{ij}^2 = \sum_{d|N} \left(\frac{N}{d} - 1 \right) = \sum_{d|N} \left(\frac{N}{d} \right) - \sigma(N)$$

Combining the two observations, we conclude

$$\lambda_{max}^2 \geq \frac{N}{\sigma(N)} \left(\sum_{d|N} \frac{1}{d} \right) - 1$$

□

Corollary *When N is a power of 2*

By plugging in $N = 2^k$, we obtain

$$\lambda_{max} = o\left(\frac{2^k}{\sqrt{k}} \right)$$