THE LIMITING SPECTRAL MEASURE OF VARIOUS MATRIX ENSEMBLES UNDER THE ANTICOMMUTATOR OPERATOR

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ABSTRACT. We introduce the anticommutator operator $\{\cdot,\cdot\}$, where $\{A_N, B_N\} = A_N B_N + B_N A_N$, to various real symmetric random matrix ensembles, including the Gaussian orthogonal ensemble (GOE), the real symmetric palindromic Toeplitz ensemble (PTE), the k-checkerboard ensemble, and the real symmetric block k-circulant ensemble (k-BCE). By using classic combinatorial techniques related to the non-crossing and free-matching properties of the cyclic product, respectively of the GOE and PTE, we obtain recursive formulae for the moments of the limiting spectral distribution of $\{$GOE, GOE$\}$, $\{$PTE, PTE$\}$, $\{$GOE, PTE$\}$ and the bulk moments of $\{$GOE, k-checkerboard$\}$ and $\{k$-checkerboard, j-checkerboard$\}$. For the anticommutator of the m-BCE matrices with other ensembles the combinatorics is more complicated so we develop a genus expansion formulae for these cases. For $\{$GOE, k-checkerboard$\}$ and $\{k$-checkerboard, j-checkerboard$\}$, we observe vastly different blip behaviors: while $\{$GOE, k-checkerboard$\}$ has two blips each containing k eigenvalues near $\pm\frac{N^{3/2}}{k}$, $\{k$-checkerboard, j-checkerboard$\}$ has one largest eigenvalue near $\frac{2N^2}{jk}$, two intermediary blips each containing $k-1$ eigenvalues near $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}$ and two intermediary blips each containing $j-1$ eigenvalues near $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}$. We prove that both blips of $\{$GOE, k-checkerboard$\}$ converge to the $k \times k$ GOE up to a constant factor and the largest blip of $\{k$-checkerboard, j-checkerboard$\}$ converges to some distribution dependent on k and j. After developing an appropriate weight function, we highlight the combinatorial difficulties of finding the moments of each intermediary blip due to inability to separate out the contribution from other blips. Using the moments we can use traditional methods to show almost-sure convergence for the bulk in all of these cases as well as the largest blip in all cases.

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1. Introduction

1.1. **Background.** Random matrix theory was first introduced by Wishart [Wis] in the 1920s. Since then, the eigenvalues of random matrices have been widely studied, with important applications to various fields including physics, number theory, and computer science. One cornerstone result in random matrix theory is the semi-circle law, which was discovered by Wigner while he was investigating nuclear resonance levels [Wig1, Wig2]. The semi-circle law states that the normalized eigenvalue distribution of certain matrix ensembles converges to a semi-circle. The limiting spectral distribution of different types of random matrix ensembles is now extensively researched with several surveys relating to the topic [BasBo1, BasBo2, BLMST, BHS1, BHS2, FM, GKMN, HM, McK, Me].

Although many matrix ensembles follow the semi-circle law, their true distributions and strengths of convergence often depend on their symmetric properties. Imposing certain symmetries on the entries of matrices in an ensemble gives rise to different eigenvalue distributions. Examples of such ensembles with imposed symmetries include Toeplitz matrices [HM], k-checkerboard matrices [BCDHMSTPY], adjacency matrices of d-regular graphs [GKMN], and block circulant matrices [KKMSX]. The moments and formula of these distributions are not guaranteed to have a nice closed form, many methods have been developed to compute the former and approximate the closed form of the latter. Typically, these methods leverage the symmetries of these matrix ensembles and use combinatorics in their calculations.

A natural question that arises pertains to how different matrix ensembles can be combined under various operations. This paper expands upon previous research [BBDLMSWX, DFJKRSSW] that combines different matrix ensembles under a "disco" or a "swirl" operation. We combine matrix ensembles using the anticommutator operator $\{\cdot,\cdot\}$, defined as $\{A, B\} := AB + BA$. The anticommutator of two random matrix ensembles is a common object in random matrix theory and has previously been studied in [NR]. Employing tools from free probability, their method involves calculations of free cumulants via lattice of non-crossing partitions and a combinatorial Fourier transform that converts R-transform into S-transform, which yields the distribution of the anticommutator of two matrix ensembles given their respective distributions. This method, however, applies only for certain distributions such as semicircle, free Poisson, arsine, and Bernoulli, etc., due to the intractability of free cumulants calculations, especially combined with analytic transforms.

Unlike most paper in the literature that heavily relies on free probability, our paper employs combinatorial and topological tools such as recurrence and genus expansion to directly compute the moments of the anticommutator of various ensembles with additional symmetries: GOE, palindromic Toeplitz, block circulant, and k-checkerboard, and use these moments to prove convergence.

1.2. **Preliminaries.** We study the anticommutators of the matrix ensembles defined below. The distributions of all these matrices have been studied using the moment method, but not all of them have closed form expressions for their moments. For all of these ensembles, we use the empirical spectral measure $\nu_{A,N}$ for an $N \times N$ matrix A and we normalize eigenvalues by $\sqrt{N}$. Formally, the spectral measure is defined as

$$\nu_{A,N} \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i}{\sqrt{N}}\right), \tag{1.1}$$

where $\{\lambda_i\}_{i=1}^{N}$ are the eigenvalues of A.

Definition 1.1 (Limiting Spectral Distribution)**.** *For the spectral distribution $\nu_{A,N}$ the limiting spectral distribution is $\lim_{N\to\infty}\nu_{A,N}$ where the convergence can be shown as weak converge, almost-sure convergence, or convergence in probability depending on the specific ensemble considered. The measure $\nu_{A,N}$ in the case of the Gaussian Orthogonal Ensemble was* (1.1)*, but in this paper we consider some different measures for different anticommutators.*

To find the limiting spectral distribution, we use the eigenvalue trace lemma to calculate the m^{th} moment of the distribution, taking the limit as $N \to \infty$.

Definition 1.2 (Gaussian Orthogonal Ensemble (GOE))**.** *The GOE is a random matrix ensemble whose matrices have entries defined by $a_{ij} = a_{ji} \sim \mathcal{N}(0,1)$ for $i \neq j$ and $a_{ii} \sim \mathcal{N}(0,\sqrt{2})$.*

As discovered by Wigner in [Wig1], the limiting distribution of the normalized eigenvalues of the GOE is the semi-circle. The proof of the semicircle law uses the moment method, with odd moments being 0 and even moments being the Catalan numbers, which matches with the moments of the semi-circle density.

Definition 1.3 (Palindromic Toeplitz)**.** [MMS] *An $N \times N$ real symmetric palindromic Toeplitz matrix (where N is assumed to be even for simplicity) is a matrix A_N whose entries are paramatrized by $b_0, b_1, \dots, b_{N/2-1}$, where the b_i's are i.i.d. random variables with mean 0 and variance 1:*

$$a_{ij} = \begin{cases} b_{|i-j|}, & \text{if } 0 \leq |i-j| \leq \frac{N}{2}-1 \\ b_{N-1-|i-j|}, & \text{if } \frac{N}{2} \leq |i-j| \leq N-1. \end{cases} \tag{1.2}$$

This matrix is therefore of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & \cdots & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & \cdots & b_4 & b_3 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \cdots & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & \cdots & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \end{pmatrix}. \tag{1.3}$$

Random Toeplitz matrices were first studied in [HM] where the authors noticed that there were some "Diophantine obsructions" which made the caulcation of moments more difficult in many cases. For this reason the palindromic Toeplitz ensemble was first introduced in [MMS], adding extra combinatorial structure to the original Toeplitz ensemble in order to get nicer combinatorial structure in the calculation. The distribution of the normalized eigenvalues of a palindromic Toeplitz matrix was established in [MMS] to be Gaussian. This was done by proving that in the expansion of the moments using eigenvalue trace lemma all of the matchings are free, which gives that the $2m^{\text{th}}$ moment is $(2m-1)!!$ and all the odd moments are 0. These are exactly the moments of the Gaussian.

Definition 1.4 (**Block Circulant**)**.** [KKMSX] *Let $k|N$. An $N \times N$ real symmetric k-block circulant matrix is of the form*

$$\begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{\frac{N}{k}-1} \\ B_{-1} & B_0 & B_1 & \cdots & B_{\frac{N}{k}-2} \\ B_{-2} & B_{-1} & B_0 & \cdots & B_{\frac{N}{k}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{1-\frac{N}{k}} & B_{2-\frac{N}{k}} & B_{3-\frac{N}{k}} & \cdots & B_0 \end{pmatrix}, \tag{1.4}$$

where each B_i is an $k \times k$ real matrix, each $B_{-i} = B_i^T$, and specifically B_0 is symmetric.

The block circulant matrix ensemble is studied in [KKMSX], where they use the genus expansion to express the moments of the spectral distribution in terms of the number of pairings of the edges of a polygon giving rise to a genus g surface. The moments are then used to find the exact limiting spectral distribution.

Definition 1.5 (**(k, w)-checkerboard**)**.** [BCDHMSTPY] *An $N \times N$ matrix from a (k, w)-checkerboard ensemble over $\mathbb{R}$ for $k \in \mathbb{Z}_{>0}$ and $w \in \mathbb{R}$ is given by $M = (m_{ij})$ such that*

$$m_{ij} = \begin{cases} a_{ij}, & \text{if } i \not\equiv j \mod k \\ w, & \text{if } i \equiv j \mod k, \end{cases} \tag{1.5}$$

where $a_{ij} = a_{ji}$ and all of the distinct a_{ij} terms are sampled from a distribution with mean 0 and variance 1. We refer to the $(k, 1)$-checkerboard ensemble as the k-checkerboard ensemble. Unless specified otherwise, we always assume the weight of the checkerboard to be 1.

Remark 1.6. *We say that $f(m) = O(g(m))$ if there exist a positive real number C and a real number M such that $f(m) \leq Cg(m)$ for all $m \geq M$. If $f(m) = O(g(m))$ and $g(m) = O(f(m))$, then we say that $f(m) = \Theta(g(m))$.*

The k-checkerboard matrix ensemble is studied in [BCDHMSTPY], where it is observed that the spectral distribution is split into a bulk of order $O(\sqrt{N})$ containing $N - k$ eigenvalues (with the largest eigenvalue of the bulk $\Theta(\sqrt{N})$) and a blip of order $\Theta(N)$ containing k eigenvalues. It is shown that the bulk distribution is semi-circular while the blip distribution is that of a $k \times k$ hollow GOE.

Definition 1.7 (**Anticommutator**)**.** *The anticommutator*[1] *ensemble of two matrix ensembles is defined as $\{A_N, B_N\} = A_N B_N + B_N A_N$, where A_N is an $N \times N$ matrix sampled from an ensemble A and B_N an $N \times N$ matrix sampled from an ensemble B.*

1.3. **Results.** In this paper we study the spectral distribution of the anticommutators of the different ensembles above. The most natural example is the anticommutator of the GOE with another GOE, or {GOE, GOE}. This was previously studied using free probability in [NR] without providing a moment formula. By exploiting the non-crossing properties of the cyclic product of GOE, we are able to obtain the even moments of {GOE, GOE} (the odd moments are 0, similar to other ensembles), as shown below.

Lemma 1.8. *The $2m^{th}$ moment M_{2m} of* {GOE, GOE} *is given by $M_{2m} = 2f(m)$, where $f(0) = f(1) = 1$, $g(1) = 1$, and*

$$f(m) \;=\; 2\sum_{j=1}^{m-1} g(j)f(m-j) + g(m), \tag{1.6}$$

$$g(m) \;=\; 2f(m-1) + \sum_{\substack{0 \leq x_1, x_2 \leq m-2 \\ x_1 + x_2 \leq m-2}} (1 + \mathbb{1}_{x_1>0})(1 + \mathbb{1}_{x_2>0}) f(x_1) f(x_2) g(m-1-x_1-x_2). \tag{1.7}$$

[1]The commutator is alternatively defined as $AB - BA$, but we do not study the commutator as it is not necessarily real symmetric even if A and B are individually real symmetric.

Lemma 1.8 can be naturally extended to finding the moments of the bulk distribution of {GOE, k-checkerboard} and {k-checkerboard, j-checkerboard}. In general, the method for the moments of the anticommutator of GOE and a different ensemble immediately gives us the bulk moments of the anticommutator of k-checkerboard and that same ensemble, since we can always extract the constant matrix of finite rank and apply the following result from [Tao1]. This reduces to the case of a $(k, 0)$-checkerboard, with a factor of $\left(1 - \frac{1}{k}\right)$ to some power.

Lemma 1.9. [Tao1] *Let $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ be a sequence of random Hermitian matrix ensembles such that $\{\nu_{\mathcal{A}_N,N}\}_{N\in\mathbb{N}}$ converges weakly almost surely to a limit ν. Let $\{\tilde{\mathcal{A}}_N\}_{N\in\mathbb{N}}$ be another sequence of random matrix ensembles such that $\frac{1}{N}$rank$(\tilde{\mathcal{A}}_N)$ converges almost surely to zero. Then $\{\nu_{\mathcal{A}_N+\tilde{\mathcal{A}}_N,N}\}_{N\in\mathbb{N}}$ converges weakly to ν.*

Similarly, by using the non-crossng properties and the free-matching properties of the cyclic product of respectively of GOE and palindrmoic Toeplitz, we obtain the even moments of {GOE, PTE}:

Theorem 1.10. *The $2m^{th}$ moment of* {GOE, PTE} *is given by $\sigma_{2m,0,m}$, where $\sigma_{n,s,k}$ is given by the conditions:*

(1) $\sigma_{n,s,k} = 0$ *if* $k < 0$,
(2) $\sigma_{n,s,k} = 0$ *if* $s + k > n$,
(3) $\sigma_{n,s,2k+1} = 0$,
(4) $\sigma_{n,s,0} = (2n-1)!!$,

and the recurrence relation

$$\sigma_{n,s,2k} \;=\; \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}\right]. \tag{1.8}$$

For the block circulant and GOE we use the genus expansion to compute the moments of their anticommutator. For the anticommutator of the palindromic Toeplitz and block circulant ensembles the genus expansion cannot handle the complexity so we are unable to calculate the moments in this case.

For anticommutator ensembles involving block circulant ensembles, due to different weights associated with different types of matchings in the cyclic product, we are unable to provide recurrence relations for the moments. Instead, we give the genus expansion formulae for these moments. We also give the genus expansion formulae for the other ensembles:

	k-Block Circulant
GOE	$\sum_{C\in\mathcal{C}_{2,4m}} \sum_{\pi_C\in NCF_{2,C}(4m)} k^{\#(\gamma_{4m}\pi)-(2m+1)}$
k-Block Circulant	$\sum_{C\in\mathcal{C}_{2,4m}} \sum_{\pi_C\in\mathcal{P}_{2,C}(4m)} k^{\#(\gamma_{4m}\pi)-(2m+1)}$

TABLE 1. Moments of {GOE, k-BCE} and {k-BCE, k-BCE}.

All the anticommutator ensembles mentioned above have eigenvalues on the order $O(N)$. However, as we take the anticommutator of an ensemble with the k-checkerboard ensemble, we begin to see different regimes of eigenvalues. In the case of {GOE, k-checkerboard}, there are $N - 2k$ eigenvalues of order $O(N)$, k blip eigenvalues at $\frac{N^{3/2}}{k} + O(N)$ and k eigenvalues at $-\frac{N^{3/2}}{k} + O(N)$. In the case of {$k$-checkerboard, j-checkerboard} where $\gcd(k, j) = 1$ and $jk \mid N$, there are $N - 2k - 2j + 3$ eigenvalues of order $O(N)$, two intermediary blips each containing $k - 1$ eigenvalues at $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2} + O(N)$, two intermediary blips each containing $j - 1$ eigenvalues at $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2} + O(N)$, and one blip eigenvalue at $\frac{2}{jk}N^2 + O(N)$. These can be proven using Weyl's inequality.

Lemma 1.11 (Weyl's inequality)**.** [HJ] *Let H, P be two $N \times N$ Hermitian matrices and let the eigenvalues of H, P, and $H + P$ be arranged in increasing order (i.e., $\lambda_N(H) \geq \lambda_{N-1}(H) \geq \cdots \geq \lambda_1(H)$). Then for any pair of integers j, k such that $1 \leq j, k \leq n$ and $j + k \geq n + 1$ we have*

$$\lambda_{j+k-n}(H + P) \leq \lambda_j(H) + \lambda_k(P), \tag{1.9}$$

and for any pair of integers j, k such that $1 \leq j, k \leq n$ and $j + k \leq n + 1$

$$\lambda_j(H) + \lambda_k(P) \leq \lambda_{j+k-1}(H + P). \tag{1.10}$$

After choosing an appropriate weight function that concentrates on one blip at a time, we define the empirical blip measure for {GOE, k-checkerboard} and {$k-$-checkerboard, j-checkerboard} as follows.

Definition 1.12. *Let $w_s = \frac{(-1)^{s+1}}{k}$ for $s \in \{1, 2\}$. Then the **empirical blip spectral measure** associated to the anticommutator of an $N \times N$ GOE and k-checkerboard $\{A_N, B_N\}$ around $w_s N^{3/2}$ is*

$$\mu_{\{A_N,B_N\},s}(x) \ = \ \frac{1}{k} \sum_{\lambda \text{ eigenvalues}} f_s^{2n}\left(\frac{\lambda}{w_s N^{3/2}}\right) \delta\left(\frac{x - \left(\lambda - w_s N^{3/2}\right)}{N}\right), \tag{1.11}$$

where $n(N)$ is a function satisfying $\lim_{N\to\infty} n(N) = \infty$ and $n(N) = \log\log(N)$ and

$$f_s^{2n}(x) \ := \ \left(\frac{x(2-x)(x+1)(3-x)}{4}\right)^{2n}. \tag{1.12}$$

Definition 1.13. *The **empirical largest blip spectral measure** associated to the anticommutator of an $N \times N$ k-checkerboard and j-checkerboard $\{A_N, B_N\}$, where $\gcd(k, j) = 1$ and $jk \mid N$, is*

$$\mu_{\{A_N,B_N\}}(x) = \sum_{\lambda \text{ eigenvalues}} g_0^{2n}\left(\frac{jk\lambda}{2N^2}\right) \delta\left(x - \left(\frac{\lambda - \frac{2}{jk}N^2}{N}\right)\right). \tag{1.13}$$

*Let $w_s = \frac{(-1)^{s+1}}{k}\sqrt{1 - \frac{1}{j}}$ and $h_s = k$ for $s \in \{1, 2\}$ and $w_s = \frac{(-1)^{s+1}}{j}\sqrt{1 - \frac{1}{k}}$ and $h_s = j$ for $s \in \{3, 4\}$ and g_s as defined in the Appendix E. Then the **empirical intermediary blip spectral measure** associated to $\{A_N, B_N\}$ around $w_s N^{3/2}$ is*

$$\mu_{\{A_N,B_N\},s}(x) \ = \ \frac{1}{h_s} \sum_{\lambda \text{ eigenvalues}} g_s^{2n}\left(\frac{\lambda}{w_s N^{3/2}}\right) \delta\left(\frac{x - \left(\lambda - w_s N^{3/2}\right)}{N}\right). \tag{1.14}$$

We again require that $n(N)$ is a function satisfying $\lim_{N\to\infty} n(N) = \infty$ and $n(N) = \log\log(N)$.

After applying the eigenvalue trace lemma to the empirical blip measures and identifying the configuration that has the highest contribution to the moment, we obtain an explicit formula for the moments of the blips of {GOE, k-checkerboard} and the largest blip of {k-checkerboard, j-checkerboard}.

Theorem 1.14. *The expected m^{th} moment associated to the empirical blip spectral measure is*

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\},1}\right] \ = \ 2\left(\frac{1}{k}\right)^{m+1} \mathbb{E}_k[\text{Tr}\, C^m]. \tag{1.15}$$

Theorem 1.15. *The m^{th} moment of the largest blip spectral measure is*

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] \ = \sum_{\substack{m_{1a}+m_{1b}+m_{2a}+m_{2b}=m;\\ m_{1a},m_{1b} \text{ even}}} C(m, m_{1a}, m_{2a}, m_{1b}, m_{2b}) \left(k\sqrt{1 - \frac{1}{k}}\right)^{m_{1a}+2m_{2a}} \left(j\sqrt{1 - \frac{1}{j}}\right)^{m_{1b}+2m_{2b}},$$

where $C(m, m_{1a}, m_{2a}, m_{1b}, m_{2b}) := m!\left(\frac{2}{jk}\right)^m \frac{2^{\frac{m_{1a}+m_{1b}}{2} - 2(m_{2a}+m_{2b})} m_{1a}!!m_{1b}!!}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!}$.

2. MOMENTS OF THE ANTICOMMUTATOR OF RANDOM MATRICES

In this section, we provide formulae for the moments of the following anticommutator ensembles: $\{$GOE, GOE$\}$, $\{$PTE, PTE$\}$, $\{$GOE, PTE$\}$, $\{$GOE, k-BCE$\}$, and $\{k$-BCE, k-BCE$\}$. Due to the similarities of the structures of GOE and checkerboards, we leave the bulk moment calculation of $\{$GOE, k-checkerboard$\}$ and $\{k$-checkerboard, j-checkerboard$\}$ in Appendix A. To better understand the combinatorics of moment calculation, we first go through some notations and results (see [MS] for more details).

Definition 2.1 ((ℓ, m)-configurations)**.** *Let $\Sigma = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ be a finite alphabet and $\phi_\ell := \alpha_1\alpha_2 \dots \alpha_\ell$. Define the action of the symmetric group S_ℓ on ϕ_ℓ as $\sigma \circ \phi_\ell := \alpha_{\sigma(1)}\alpha_{\sigma(2)} \cdots \alpha_{\sigma(\ell)}$ for all $\sigma \in S_\ell$. Then, a (ℓ, m)-**configuration** is a string of length m comprised of the concatenation of group actions on some ϕ_ℓ, i.e., $\sigma_1 \circ \phi_\ell\sigma_2 \circ \phi_\ell \cdots \sigma_m \circ \phi_\ell$. We denote the set of all (ℓ, m)-configurations as $\mathcal{C}_{\ell,\ell m}$.*

For our purpose, we simply set $\Sigma = \{a, b\}$ and restrict our attention to $(2, m)$-configurations.

Definition 2.2 (Partitions w.r.t. a configuration)**.** *For positive integers n and m, let $[n] = \{1, 2, \dots, n\}$ and $C = c_1c_2 \cdots c_m$ be a $(2, m)$-configuration, where $c_i \in \{a_i, b_i\}$ for all $1 \le i \le m$ under the restriction in Definition 2.1 that $(c_{2s-1}, c_{2s}) \in \{(a_{2s-1}, b_{2s}), (b_{2s-1}, a_{2s})\}$. Then, a partition with respect to C, $\pi_C = (V_1, \dots, V_t)$, is a tuple of subsets of $[2n]$ such that the following holds:*

1. *$V_i \neq \emptyset$ for all $1 \le i \le t$,*
2. *$V_1 \cup \cdots \cup V_t = [2n]$,*
3. *$V_i \cap V_j = \emptyset$ for $i \neq j$,*
4. *For all $1 \le i \le t$ and $i_1, i_2 \in V_i$, $\{c_{i_1}, c_{i_2}\} \in \{\{a_{i_1}, a_{i_2}\}, \{b_{i_1}, b_{i_2}\}\}$.*

*Let $\mathcal{P}_C(2n)$ denote the set of all partitions with respect to C of $[2n]$. We call $V_1, V_2, \dots, V_t$ **blocks** of π_C. A partition is called a **pairing** (or matchings) if each block is of size 2. We denote all the pairings with respect to C of $[2n]$ as $\mathcal{P}_{2,C}(2n)$.*

Note that this definition can be easily extended to any arbitrary subset $S \subseteq [n]$ and configuration C_S (indexed by S), where S is not necessarily equal to $[k]$ for any $k \in \mathbb{N}$.

Definition 2.3 (Non-crossing partitions w.r.t. a configuration)**.** *A partition with respect to C, $\pi = (V_1, \dots, V_t)$, of $[2n]$ is **crossing** if there exists blocks V and W with $i, k \in V$ and $j, l \in W$ such that $i < j < k < l$. We denote the set of non-crossing partitions with respect to C of $[2n]$ by $NC_C(2n)$ and the set of non-crossing pairings with repsect to C of $[2n]$ by $NC_{2,C}(2n)$.*

Proposition 2.4 (Wick's formula)**.** *For $\pi \in \mathcal{P}_2(2m)$, let $\mathbb{E}_\pi(X_1, \dots, X_{2m}) = \prod_{(r,s)\in\pi} \mathbb{E}(X_rX_s)$. Let $(X_1, \dots, X_n)$ be a real Gaussian random vector. Then*

$$\mathbb{E}[X_{i_1}, \dots, X_{i_{2m}}] = \sum_{\pi\in\mathcal{P}_2(2m)} \mathbb{E}_\pi[X_{i_1}, \dots, X_{i_{2m}}] \tag{2.1}$$

for any $i_1, \dots, i_{2m} \in [n]$.

In moment calculation, it is often important to characterize the pairings that contribute in the limit, as it not only simplifies the calculation, but also allows us to view the moments as known combinatorial objects (i.e. the m^{th} Catalan number is the $2m^{\text{th}}$ moment of the limiting distribution of the GOE). To facilitate this characterization, we extend the method of genus expansion originally used in the moment calculation of the GUE (see [MS] section 1.7 for more details) to our ensembles.

2.1. Moments of $\{$GOE, GOE$\}$.

In general, due to the structure of the anticommutator, a cyclic product $c_{i_1i_2}c_{i_2i_3} \cdots c_{i_{2m}i_1}$ in the m^{th} moment of the anticommutator of any two arbitrary random matrices has the restriction that $(c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell}i_{2\ell+1}}) \in \{(a_{i_{2\ell-1}i_{2\ell}}, b_{i_{2\ell}i_{2\ell+1}}), (b_{i_{2\ell-1}i_{2\ell}}, a_{i_{2\ell}i_{2\ell+1}})\}$ for all $1 \le \ell \le m$. In other words, a cyclic product $c_{i_1i_2}c_{i_2i_3} \cdots c_{i_{2m}i_1}$ is of a valid configuration iff it is a 2-configuration of the alphabet $\Sigma = \{a, b\}$. Now, let $A_N = (a_{ij})$ and $B_N = (b_{ij})$ be independent $N \times N$ GOE's with $\mathbb{E}[a_{ij}^2] = \mathbb{E}[b_{ij}^2] = 1$

and consider the m^{th} moment of $A_NB_N + B_NA_N$. With a slight abuse of notation, we identify a $(2, 2m)$-configuration $C = c_1c_2\cdots c_{2m}$ with the cyclic product $c_{i_1i_2}c_{i_2i_3}\cdots c_{i_{2m}i_1}$. Then,

$$M_m(N) \;=\; \frac{1}{N^{m+1}}\mathbb{E}[\mathrm{Tr}(A_NB_N + B_NA_N)^m] \;=\; \frac{1}{N^{m+1}}\sum_{C\in\mathcal{C}_{2,2m}}\ \sum_{1\le i_1,\cdots,i_m\le N}\mathbb{E}[c_{i_1i_2}c_{i_2i_3}\dots c_{i_{2m}i_1}], \tag{2.2}$$

We apply genus expansion to each $\mathbb{E}[c_{i_1i_2}c_{i_2i_3}\dots c_{i_{2m}i_1}]$. The argument that follows is essentially the same argument as the genus expansion of the $2m^{\text{th}}$ moment of the GUE, since treating a's and b's both as c's while ensuring that they are matched within themselves preserves the "non-crossing" property of pairings that contribute in the limit.

Now, as $N\to\infty$, we have $M_m(N) = 0$ when m is odd, since by standard argument the contribution from each type of configuration is $O(N^m)$, but the number of types of configurations depends only on m. When m is even, by Wick's formula

$$\mathbb{E}[c_{i_1i_2}c_{i_2i_3}\cdots c_{i_{2k}i_1}] \;=\; \sum_{\pi_C\in\mathcal{P}_{2,C}(2m)}\mathbb{E}_{\pi_C}[c_{i_1i_2}, c_{i_2i_3}, \dots, c_{i_{2m}i_1}]. \tag{2.3}$$

Since $\mathbb{E}[c_{i_ri_{r+1}}c_{i_si_{s+1}}] = 1$ when $i_r = i_{s+1}$ and $i_{r+1} = i_s$ and is 0 otherwise, then $\mathbb{E}[c_{i_1i_2}c_{i_2i_3}\cdots c_{i_{2m}i_1}]$ is the number of pairings π_C with respect to C of $[2m]$ such that $i_r = i_{s+1}$, $i_{r+1} = i_s$, and a's and b's are matched within themselves (i.e., an a is not matched with a b). Now, we think of a tuple of indices $(i_1, \dots, i_{2m})$ as a function $i : [2m] \to [N]$ and write the pairing $\pi_C = \{(r_1, s_1), (r_2, s_2), \cdots, (r_k, s_m)\}$, as the product of transpositions $(r_1, s_1)(r_2, s_2)\dots(r_k, s_m)$. We also take γ_{2m} to be the cycle $(1, 2, 3, \dots, 2m)$. If $\pi_C C$ is a pairing of $[2m]$ and (r, s) is a pair of π_C, then we express our conditions $i_r = i_{s+1}$ and $i_s = i_{r+1}$ as $i(r) = i(\gamma_{2m}(\pi_C(r)))$ and $i(s) = i(\gamma_{2k}(\pi_C(s)))$ respectively. Hence, $\mathbb{E}_{\pi_C}[c_{i_1i_2}c_{i_2i_3}\cdots, c_{i_{2m}i_1}] = 1$ if i is constant on the orbits of $\gamma_{2m}\pi_C$ (e.g. $i(r) = i(s + 1)$) and 0 otherwise. Let $\#(\sigma)$ denote the number of cycles of a permutation σ, then

$$M_m(N) \;=\; \frac{1}{N^{m+1}}\mathbb{E}[\mathrm{Tr}(A_NB_N + B_NA_N)^m] \;=\; \frac{1}{N^{m+1}}\sum_{C\in\mathcal{C}_{2,2m}}\ \sum_{\pi_C\in\mathcal{P}_{2,C}(2m)} N^{\#(\gamma_{2m}\pi_C)}. \tag{2.4}$$

Proposition 2.5 ([MS]). *If π is a pairing of $[2m]$ then $\#(\gamma_{2m}\pi) \le m - 1$ unless π is non-crossing in which case $\#(\gamma_{2m}\pi) = m + 1$.*

Corollary 2.6. *As $N\to\infty$,*

$$M_m \;=\; \lim_{N\to\infty} M_m(N) \;=\; \sum_{C\in\mathcal{C}_{2,2m}}\ \sum_{\pi_C\in NC_{2,C}(2m)} 1. \tag{2.5}$$

Given genus expansion formula 2.5, we are then able to obtain a recurrence relation for the even moment of {GOE, GOE}.

Lemma 2.7. *The $2m^{th}$ moment M_{2m} of* {GOE, GOE} *is given by $M_{2m} = 2f(m)$, where $f(0) = f(1) = 1$, $g(1) = 1$, and*

$$f(m) \;=\; 2\sum_{j=1}^{m-1} g(j)f(m-j) + g(m), \tag{2.6}$$

$$g(m) \;=\; 2f(m-1) + \sum_{\substack{0\le x_1,x_2\le m-2\\ x_1+x_2\le m-2}} (1 + \mathbb{1}_{x_1>0})(1 + \mathbb{1}_{x_2>0})f(x_1)f(x_2)g(m-1-x_1-x_2). \tag{2.7}$$

Proof. Let $f(m)$ be the number of non-crossing pairings with respect to all $(2, 4m)$-configurations starting with an a, and $g(m)$ be the number of non-crossing pairings with respect to all $(2, 4m)$-configurations starting and ending with an a such that these two a's are matched together (i.e., a configuration $a_{i_1i_2}b_{i_2i_3}\cdots b_{i_{4m-1}i_{4m}}a_{i_{4m}i_1}$ with $i_{4m}=i_2$).

We first find the recurrence relation for $f(m)$. We know that $a_{i_1i_2}$ is matched with some $a_{i_{4j}i_{4j+1}}$ with $j\leq m$ (in case when $j=m$, we identify $4m+1$ as 1) since there should be an even number of both a and b terms between $a_{i_1i_2}$ and $a_{i_{4j}i_{4j+1}}$ to ensure non-crossing pairings. When $j=m$, the number of non-crossing pairings is just $g(m)$ by definition. When $j<m$, the number of non-crossing pairings within $a_{i_1,i_2}b_{i_2,i_3}\cdots b_{i_{4j-1}i_{4j}}a_{i_{4j}i_{4j+1}}$ is $g(j)$. We multiply this by the number of non-crossing pairings within the rest of the cyclic product which have no restrictions and is therefore simply $2f(m-j)$, with the 2 accounting for starting with either an a or b. Thus, summing over all possible j's, we have

$$f(m)\ =\ 2\sum_{j=1}^{m-1}g(j)f(m-j)+g(m). \tag{2.8}$$

Similarly, we know that either $b_{i_2i_3}$ is matched with $b_{i_{4m-1}i_{4m}}$, or $b_{i_2i_3}$ is matched with $b_{i_{4x_1+3}i_{4x_1+4}}$ and $b_{i_{4m-1}i_{4m}}$ is matched with $b_{i_{4m-4x_2-2}i_{4m-4x_2-1}}$, with $4x_1+4<4m-4x_2-2$, or $x_1+x_2\leq m-2$. In the first case, since there are no restrictions on the $4k-4$ terms between $b_{i_2i_3}$ and $b_{i_{4m-1}i_{4m}}$, the number of non-crossing pairings is $2f(m-1)$. In the second case, the number of non-crossing pairings of terms between $b_{i_2i_3}$ and $b_{i_{4x_1+3}i_{4x_1+4}}$ is $(1+\mathbb{1}_{x_1>0})f(x_1)$, the number of non-crossing pairings of terms between $b_{i_{4m-4x_2-2}i_{4m-4x_2-1}}$ and $b_{i_{4m-1}i_{4m}}$ is $(1+\mathbb{1}_{x_2>0})f(x_2)$, with the indicator functions accounting for the intermediary terms starting with either an a or a b, and lastly the number of non-crossing pairings of terms between $b_{i_{4x_1+3}i_{4x_1+4}}$ and $b_{i_{4m-4x_2-2}i_{4m-4x_2-1}}$ is $g(m-1-x_1-x_2)$. Thus,

$$g(m)\ =\ 2f(m-1)+\sum_{\substack{0\leq x_1,x_2\leq m-2\\ x_1+x_2\leq m-2}}(1+\mathbb{1}_{x_1>0})(1+\mathbb{1}_{x_2>0})f(x_1)f(x_2)g(m-1-x_1-x_2). \tag{2.9}$$

We have now defined our recurrence for $f(m)$, which counts the number of non-crossing pairings with respect to $(2,4m)$-configurations starting with an a. Since general $(2,4m)$-configurations can start with either an a or a b, we multiply $f(m)$ by 2 to get all possible non-crossing pairings of $(2,4m)$-configurations, and we arrive at the even moments being $M_{2m}=2f(m)$. □

A natural extension of the anticommutator $\{A_N,B_N\}$ is the **ℓ- anticommutator** of ℓ matrix ensembles $A_N^{(1)},A_N^{(2)},\dots,A_N^{(\ell)}$, defined as

$$\{A_N^{(1)},A_N^{(2)},\dots,A_N^{(\ell)}\}\ :=\ \sum_{\sigma\in S_\ell}A_N^{(\sigma(1))}A_N^{(\sigma(2))}\cdots A_N^{(\sigma(\ell))}. \tag{2.10}$$

By employing the same method as in the proof of Lemma 2.7, we are able to obtain a recurrence relation for the moments of the ℓ-anticommutator. Now, however, instead of two interdependent recurrence relations, we have ℓ interdependent recurrence relations. We state the results below and leave the details of the proof in Appendix A.

2.2. **Moments of {PTE, PTE}.** The palindromic Toeplitz ensemble is introduced by Massey-Miller-Sinsheimer in [MMS] to remove the Diophantine obstruction encountered in the moment calculation of Toeplitz ensemble in [HM]. Essentially, the additional symmetry in the structure of palindromic Toeplitz allows almost all free matching of all the terms in the cyclic product to have consistent choice of indexing and contribute in the limit. Interestingly, the $2m^{\text{th}}$ moment of palindromic Toeplitz is $(2m-1)!!$, which is exactly the $2m^{\text{th}}$ moment of standard Gaussian. The moment calculation of palindromic Toeplitz can be naturally extended to that of {Palindromic Toeplitz, Palindromic Toeplitz}. By standard argument, the odd moments of {palindromic Toeplitz, palindromic Toeplitz} vanish in the limit. For even moments, we can view each cyclic product in the $2m^{\text{th}}$ moment of {palindromic Toeplitz, palindromic Toeplitz} as a cyclic product in the $4m^{\text{th}}$ moment of palindromic Toeplitz. Thus, the matching in each cyclic product is again free, giving us the following genus expansion formula:

$$M_m \ = \ \lim_{N\to\infty} M_m(N) \ = \ \sum_{C\in\mathcal{C}_{2,2m}} \sum_{\pi_C\in P_{2,C}(2m)} 1. \tag{2.11}$$

Theorem 2.8. *The $2m^{th}$ moment M_{2m} of* $\{\text{PTE, PTE}\}$ *is* $2^m((2m-1)!!)^2$.

Proof. Since there are the number of $(2,m)$ configurations is 2^m and each configuration has $(2m-1)!!$ ways of matching up the a's and $(2m-1)!!$ ways of matching up the b's, then by (2.11) we have

$$M_{2m} \ = \ 2^m((2m-1)!!)^2. \tag{2.12}$$

□

2.3. **Moments of $\{$GOE, PTE$\}$.** So far, we've only been looking at **homogeneous** anticommutator ensembles $\{A_N, B_N\}$, i.e., A_N and B_N are the same ensembles. Genus expansions of $\{$GOE, GOE$\}$ and $\{$PTE, PTE$\}$ suggest that in general, genus expansion of a homogeneous anticommutator ensemble $\{A_N, B_N\}$ is an easy generalization of the genus expansion of A_N (or B_N). A natural question to ask is: what does genus expansion of an **inhomogeneous** anticommutator ensembles $\{A_N, B_N\}$ (i.e., A_N and B_N are different ensembles) look like? In this section, we turn our attention to an inhomogeneous anticommutator ensemble, namely $\{$GOE, PTE$\}$. Interestingly, we shall see that the matching properties of GOE and palindromic Toeplitz are well preserved under the anticommutator operator, that is, the contributions to the moments of $\{$GOE, PTE$\}$ in the limit come solely from non-crossing matchings of the GOE terms and free matchings of the palindromic Toeplitz terms that don't cross the matchings of the GOE terms.

Similar to the previous examples, we have that the m^{th} moment of $\{A_N, B_N\}$ is given by

$$M_m(N) \ = \ \frac{1}{N^{m+1}} \sum_{C\in\mathcal{C}_{2,2m}} \sum_{1\le i_1,\cdots,i_m\le N} \mathbb{E}[c_{i_1i_2}c_{i_2i_3}\cdots c_{i_{2m}i_1}]. \tag{2.13}$$

Definition 2.9. *For positive integers n and m, let $C = c_1c_2\cdots c_{2m}$ be a $(2,m)$-configuration, where $c_i \in \{a_i, b_i\}$ for all $1 \le i \le 2m$ under the restriction that $(c_{2s-1}, c_{2s}) \in \{(a_{2s-1}, b_{2s}), (b_{2s-1}, a_{2s})\}$ and $S \subseteq [2n]$ be the set of all the indices of the a's. Let π_S be a partition of S. Then a* ***layer of** $[2n]$ **with respect to*** *S is a maximal subset $B_S^{(i)} \subseteq [2n] \setminus S$ such that for any $j,k \in B_S^{(i)}$, there doesn't exist $(p,q) \in \pi_S$ such that $j<p<k<q$ or $p<j<q<k$. It's clear from definition that distinct layers must be disjoint. Then we denote the union of all the layers with respect to S by $B_S := \cup_{i=1}^t B_S^{(i)}$, where t is the total number of layers.*

Lemma 2.10. *Consider a cyclic product in* (2.13)*. Let S be the set of all the indices of the a's, π_S be a matching of the a's and $\pi_{[2n]\setminus S}$ be a matching of the b's. If π_S is non-crossing, then the matching $\pi_S \circ \pi_{[2n]\setminus S}$ contribute to* (2.13) *in the limit if and only if there exists i such that $j,k \in B_S^{(i)}$ for each $(j,k) \in \pi_{[2n]\setminus S}$, i.e. every layer is matched within itself. For the $2m^{th}$ moment, the number of ways to assign indices for all the t layers is $N^{m+t} + O(N^{m+t-1})$.*

Proof. First, observe that each layer $B_S^{(i)}$ can be thought of as a cyclic product. For example, consider the following layer $B_S^{(i)}$ consisting of 2ℓ b's. For clarity, we include some of the a's to highlight how the matching the a's give rise to the cyclic product:

$$\begin{aligned}&a_{i_{j_1-1}i_{j_1}} b_{i_{j_1}i_{j_1+1}} a_{i_{j_1+1}i_{j_1+2}} \cdots a_{i_{j_2-1}i_{j_2}} b_{i_{j_2}i_{j_2+1}} a_{i_{j_2+1}i_{j_2+2}} \cdots \\ &a_{i_{j_3-1}i_{j_3}} b_{i_{j_3}i_{j_3+1}} a_{i_{j_3+1}i_{j_3+2}} \cdots b_{i_{j_{2\ell}}i_{j_{2\ell}+1}} a_{i_{j_{2\ell}+1}i_{j_{2\ell}+2}} \cdots .\end{aligned} \tag{2.14}$$

Since the b's form a layer, then for every neighboring two b's, the inner adjacent two a's must be paired together. For example, $a_{i_{j_1+1}i_{j_1+2}}$ and $a_{i_{j_2-1}i_{j_2}}$, which are adjacent $b_{i_{j_1}i_{j_1+1}}$ and $b_{i_{j_2}i_{j_2+1}}$, must be paired together to ensure that all the b's form a layer. Hence, the indices must satisfy the relations $i_{j_1} = i_{j_{2\ell+1}}$,

$i_{j_1+1} = i_{j_2}$, $i_{j_2+1} = i_{j_3}$, …, $i_{j_{2\ell-1}+1} = i_{j_{2\ell}}$, which allows us to think of $B_S^{(i)}$ as $b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{2\ell} i_1}$. Let $\#(B_S^{(i)})$ be the number of b's in the layer $B^{(i)}$. For each cyclic product, if the matching of all the b's is within each layer, then the number of ways to choose indices for all the b's is $\prod_{i=1}^{t}(N^{\#(B_S^{(i)})/2+1} + O(N^{\#(B_S^{(i)})/2})) = N^{m+t} + O(N^{m+t-1})$ by [MMS].

We move on to the case where the matchings of the b's are across different layers. Now, for two arbitrary layers $B_S^{(i_1)}$ and $B_S^{(i_2)}$, suppose that all the b's are paired within these two layers except for at least two b's that are paired across these two layers. Due to the special structure of palindromic Toeplitz, if $b_{i_j i_{j+1}}$ and $b_{i_k i_{k+1}}$ are paired together, then the indices must satisfy the equation $i_{j+1} - i_j + i_{k+1} - i_k = C_j$ for some $C_j \in \{0, \pm(N-1)\}$. Hence, similar to [MMS], we can think of the matching of all the indices as a system of $M := (m(B_S^{(i_1)}) + m(B_S^{(i_2)})/2$ equations, where each index appears exactly twice. After labeling these equations, we pick any equation as eq(M), and choose an index that has occured only once. Then, we select the equation in which this index first appeared and label this equation as eq$(M-1)$. This index is one of our dependent indices and guarantees consistency choice of indices for the other indices in eq$(M-1)$. We can continue this process, and at stage s, if at least one index of eq$(M-s)$ has occured only once among eq$(M-s)$, eq$(M-s+1), \ldots,$ eq(M), then we can choose such an index as one of our dependent indices and continue this process. The only way to terminate this process at stage $s < M-1$ is for all the indices among eq$(M-s)$, eq$(M-s+1), \ldots,$ eq(M) to occur twice, which implies that each layer is paired within itself, a contradiction. Hence, if at least two b's are paired across these two layers, then the number of dependent indices is $M/2-1$ and the number of ways to choose indices for all the b's is $N^{M/2+1}$. This is a lower order term compared to the case where each layer is paired within itself, which gives that the number of ways to choose indices for all the b's is $N^{M/2+2}$.

Finally, we consider the case where the matchings of the a's cross each other. If a matching of two a's cross another matching of two a's, then we automatically have three layers $B_S^{(i_1)}, B_S^{(i_2)}$, and $B_S^{(i_3)}$. Due to the mismatch, we can no longer view different layers as independent cyclic products, but all three layers as a single cyclic product. The total number of ways to assign the indices for the three layers is $N^{(m(B_S^{(i_1)})+m(B_S^{(i_2)}+m(B^{(i_3)}))/2)+1}$, which is a lower order term compared to the case where each of the three layers is matched within itself. Thus, the number of ways to assign indices for all the layers is $O(N^{m+t-1})$, which is again a lower order term. □

Lemma 2.11. *With the same notation as in Lemma 2.10, regardless of whether π_S is non-crossing or not, the number of ways to assign the remaining indices for the $2m^{th}$ moment is $N^{m+1-t} + O(N^{m-t})$.*

Proof. For a fixed m, when a cyclic product has only one layer, the only possible configurations for the cyclic product are $abba \cdots abba$ or $baab \cdots baab$; moreover, all the a's must be matched in adjacent pairs. Since there is one free index for each adjacent pair of a, then the number of ways to assign the remaining indices is $N^m = N^{(m+1)-1}$. This proves the base case.

When a cyclic product C has two layers $B_S^{(1)}$ and $B_S^{(2)}$, suppose that layer $B_S^{(1)}$ is contained in the cyclic product C_1 with $2k_1$ total b's and layer $B_S^{(2)}$ is contained in the cyclic product C_2 with $2k_2$ total b's, where $C_1 \cap C_2 = \emptyset$. We can think of C as inserting C_2 into C_1. From the base case, C_1 and C_2 are either $abba \cdots abba$ or $baab \cdots baab$. Then, without loss of generalilty, suppose that C_1 has the configuration $abba \cdots abba$. If C_2 is inserted between two a's in C_1, then it must have the configuration $abba \cdots abba$, otherwise C has only one layer instead of two layers. Let C_2 be $a_{i'_1 i'_2} b_{i'_2 i'_3} \cdots b_{i'_{4k-1} i'_{4k}} a_{i'_{4k} i'_1}$ and surrounded by $a_{i_\ell i_{\ell+1}}$ and $a_{i_{\ell+1} i_{\ell+2}}$ in C_1. Since $a_{i'_1 i'_2}$ and $a_{i'_{4k} i'_1}$ as well as $a_{i_\ell i_{\ell+1}}$ and $a_{i_{\ell+1} i_{\ell+2}}$ are no longer adjacent, then we lose one additional degrees of freedom and the number of ways to assign the remaining indices is $k_1 + k_2 - 1$. If C_2 is inserted between two b's in C_1, then it can either be $abba \cdots abba$ or $baab \cdots baab$. Similarly, we can see that the number of ways to assign the remaining indices is $k_1 + k_2 - 1$. Similar constructions follow when we have an arbitrary number of layers in the cyclic product, and whenever we get another layer we lose one extra degree of freedom, giving us $N^{(m+1)-t} + O(N^{m-t})$ ways of assigning the remaining indices for t layers.

$\square$

By (2.10) and (2.11), for the $2m^{\text{th}}$ moment, the number of ways to assign all the indices is $N^{2m+1} + O(N^{2m})$ when π_S is non-crossing and every layer is matched within itself, and $O(N^{2m})$ otherwise. In other words, in the limit, the only contributions come from non-crossing matchings of the a's and matchings of the b's within the same layer. This leads us to Theorem 2.12, as follows.

Theorem 2.12. *The $2m^{th}$ moment of* {GOE, PTE} *is given by $\sigma_{2m,0,m}$, where $\sigma_{n,s,k}$ is given by the conditions*

(1) $\sigma_{n,s,k} = 0$ *if* $k < 0$,
(2) $\sigma_{n,s,k} = 0$ *if* $s + k > n$,
(3) $\sigma_{n,s,2k+1} = 0$,
(4) $\sigma_{n,s,0} = (2n-1)!!$,

and the recurrence relation

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}\right]. \tag{2.15}$$

Proof. Let $\sigma_{n,s,k}$ be the total number of matchings of any cyclic products of a's and b's of length $2n$ that starts with at least s adjacent pairs of bb and has k adjacent pairs of ab and ba in total. It's clear that the $2m^{\text{th}}$ moment of {GOE, PTE} is given by $\sigma_{2m,0,m}$ and conditions (1), (2), (3) trivially follows from the definition. Moreover, since $\sigma_{n,s,0}$ is the number of matchings of cyclic products of b's of length $2n$, where the matchings of b's are free. Then, $\sigma_{n,s,0} = (2n-1)!!$.

Now, we move on to prove the recurrence relation for $\sigma_{n,s,2k}$. Suppose that the p^{th} adjacent pair is the first occurrence of ab or ba pair and that the a is paired with another a in the q^{th} block. Since no matchings can cross the matching of two a's, then if the p^{th} block is ab, the q^{th} block must be ba, and vice versa. In both cases, the matching of the two a's split the cyclic product into two smaller cyclic product, as illustrated in the following example:

Example 2.13. *If $(n,s,k) = (5,1,2)$, then an example of a cyclic product with $p = 3$ and $q = 5$ is $b_{i_1i_2}b_{i_2i_3}b_{i_3i_4}b_{i_4i_5}b_{i_5i_6}a_{i_6i_7}b_{i_7i_8}b_{i_8i_9}a_{i_9i_{10}}b_{i_{10}i_1}$. The matching of the a's partitions the cyclic product into two smaller cyclic products $b_{i_7i_8}b_{i_8i_9}$ (inner cyclic product) and $b_{i_{10}i_1}b_{i_1i_2}b_{i_2i_3}b_{i_3i_4}b_{i_4i_5}b_{i_5i_6}$ (outer cyclic product), where a term from either smaller cyclic product is paired with another term in the same smaller cyclic product.*

If the p^{th} and the q^{th} adjacent pairs are both ba, then the matching of the a's partitions the cyclic product into two smaller cyclic products C_1' (inner cyclic product) and C_2' (outer cyclic product), where C_1' is of length $2(q-p-1)$ and has no restrictions on the number of starting adjacent pairs of bb and C_2' is of length $2(n-(q-p))$ and starts with at least p adjacent pairs of bb. Then the total number of matchings is $\sigma_{q-p-1,0,r}$ for C_1' and $\sigma_{n-q+p,p,2k-2-r}$ for some r. Similarly, if p^{th} and the q^{th} adjacent pairs are both ab, then the total number of matchings is $\sigma_{q-p,1,r}$ for C_1' and $\sigma_{n-q+p-1,p-1,2k-2-r}$ for C_2'.

Summing over all possible p's, q's and r's, we have

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}\right]. \tag{2.16}$$

$\square$

2.4. **Moments of $\{$GOE, k-BCE$\}$ and $\{k$-BCE, k-BCE$\}$.** The real symmetric k-block circulant ensemble is introduced by Koloğlu-Kopp-Miller in [KKMSX] and possesses an even more complicated symmetry structure than the palindromic Toeplitz ensemble: not only do entries on different diagonals satisfy relations analogous to the palindromic Toeplitz ensemble, but the entries on the same diagonal also appear periodically due to the k-block structure. Because of its complicated structure, the $2m^{\text{th}}$ moments of the spectral distribution are not given explicitly, but expressed in terms of the number of pairings of the edges of a $2m$-gon which give rise to a genus g surface. Similar to the palindromic case, we can extend the moment calculation of k-block circulant ensemble to that of $\{k\text{-BCE}, k\text{-BCE}\}$ and $\{\text{GOE}, k\text{-BCE}\}$.

Suppose that $b_{i_s i_{s+1}}$ and $b_{i_t i_{t+1}}$ are entries from an $N \times N$ real symmetric k-block circulant matrix, then $b_{i_s i_{s+1}}$ and $b_{i_t i_{t+1}}$ are matched iff either of the following relations hold:

(1) $i_{s+1} - i_s = i_{t+1} - i_t + C_s$ and $i_s \equiv i_t \pmod k$, or
(2) $i_{s+1} - i_s = -(i_{t+1} - i_t) + C_s$ and $i_s \equiv i_{t+1} \pmod k$,

where $C_s \in \{0, \pm N\}$. The difference in sign in the two relations above allows us to think of the matching of $(s, s+1)$ and $(t, t+1)$ as having the same or different orientations. For both $\{k\text{-BCE}, k\text{-BCE}\}$ and $\{$GOE, k-BCE$\}$, we can apply the same argument from [HM], [MMS], and [KKMSX] to show that the total contribution of all the pairings with at least one matching of the same orientation is $O(1/N)$. Hence, it suffices to consider those pairings with matchings of the same orientation, i.e. $i_{s+1} - i_s = -(i_{t+1} - i_t) + C_s$ and $i_s \equiv i_t \pmod k$. By assumption $k = o(N)$, then the modular restrictions do not reduce the total degrees of freedom. Hence, analogous to the palindromic Toeplitz case, we can think of the pairing of terms of in the $2m^{\text{th}}$ moment of an $N \times N$ real symmetric k-block circulant matrix as a system of m linear equations each of the form $i_{s+1} - i_s = -(i_{t+1} - i_t) + C_s$. This gives us $m + 1$ free indices with $m - 1$ dependent indices and constants $C_s \in \{0, \pm N\}$ uniquely determined, except for a lower order term of choices of free indices.

Using the idea of layers developed in subsection 2.3, we see that if π is a pairing of $\{\text{GOE}, k\text{-BCE}\}$, then π contributes to the moment of $\{\text{GOE}, k\text{-BCE}\}$ iff the GOE terms are matched non-crossing and the real symmetric k-block circulant terms are matched within each layer. Now, consider a pairing π in the $2m^{\text{th}}$ moment of $\{\text{GOE}, k\text{-BCE}\}$. We identify the matched indices in the same congruence class modulo k by the equivalence relation $\sim$. For example, if $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are matched, i.e. $i_s = i_{t+1}$ and $i_{s+1} = i_t$, then $i_s \sim i_{t+1}$ and $i_{s+1} \sim i_t$. If $b_{i_s i_{s+1}}$ and $b_{i_t i_{t+1}}$ are matched, i.e. $i_{s+1} - i_s = -(i_{t+1} - i_t) + C_s$ and $i_s \equiv i_{t+1} \pmod k$, then we also have $i_{s+1} \equiv i_t \pmod k$. Hence, we still have $i_s \sim i_{t+1}$ and $i_{s+1} \sim i_t$. We see that the number of equivalence classes of indices of the pairing π is $\#(\gamma_{4m}\pi)$. For each equivalence class, there are k ways to choose congruence classes. So the number of ways to choose congruence class for all the indices is $k^{\#(\gamma_{4m}\pi)}$. Since there are N/k choices for indices for each congruence class, then $2m^{\text{th}}$ moment of $\{\text{GOE}, k\text{-BCE}\}$ is given by

$$M_{2m} \;=\; \sum_{C \in \mathcal{C}_{2,4m}} \sum_{\pi_C \in NCF_{2,C}(4m)} k^{\#(\gamma_{4m}\pi)-(2m+1)},$$

where $NCF_{2,C}(4m)$ denotes the set of all the pairings with respect to C of $[4m]$ where the GOE terms are matched non-crossingly and the palindromic Toeplitz terms are matched freely without crossing the matchings of the GOE terms. Similarly, the $2m^{\text{th}}$ moment of $\{k\text{-BCE}, k\text{-BCE}\}$ is given by

$$M_{2m} \;=\; \sum_{C \in \mathcal{C}_{2,4m}} \sum_{\pi_C \in \mathcal{P}_{2,C}(4m)} k^{\#(\gamma_{4m}\pi)-(2m+1)}.$$

3. The Blip Spectral Measure of anticommutators

In this section, we consider the blip spectral measures of two ensembles: $\{$GOE, k-checkerboard$\}$ and $\{k$-checkerboard, j-checkerboard$\}$. We always assume that $\gcd(k, j) = 1$ and $jk \mid N$, which as we will see later on is crucial to the structure of the anticommutator. Even though the bulk eigenvalues of these two ensembles are both of order $O(N)$ (with largest and smallest eigenvalues $\Theta(N)$), we observe drastically different splitting behaviors: while {GOE, k-checkerboard} has blip eigenvalues only of order $\Theta(N^{3/2})$, {k-checkerboard, j-checkerboard} has blip eigenvalues of order $\Theta(N^2)$ and $\Theta(N^{3/2})$. Specifically, {GOE, k-checkerboard} has $2k$ eigenvalues of order $\Theta(N^{3/2})$ (called the **blip** eigenvalues), of which k are $\frac{N^{3/2}}{k}+O(N)$ and k are $-\frac{N^{3/2}}{k}+O(N)$; by contrast {k-checkerboard, j-checkerboard} has 1 eigenvalue of order $\Theta(N^2)$ (called the **largest blip** eigenvalue) at $\frac{2}{jk}N^2+O(N)$ and $2k+2j-4$ eigenvalues of order $\Theta(N^{3/2})$ (called the **intermediary blip** eigenvalues), of which two intermediary blips each containing $k-1$ eigenvalues are $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$ and two intermediary blips each containing $j-1$ eigenvalues are $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$. For proofs of different regimes, see Appendix B.

3.1. **Structural Preliminaries.** We first define the empirical blip spectral measure using appropriate weight functions and reduce the blip moment calculation to combinatorics. Next, using the language developed in [BCDHMSTPY], we identify the types of cyclic products that contribute to the expected m^{th} moments of the blip spectral measures of both ensembles. Then, we explicitly obtain the expected m^{th} moments of the blip spectral measures of {GOE, k-checkerboard} and of {k-checkerboard, j-checkerboard} around the far away blip. Finally, we highlight the combinatorial challenge that we encounter in the calculation of the expected m^{th} moments of the blip spectral measures of {k-checkerboard, j-checkerboard} around the intermediary blips.

Definition 3.1. *Let $w_s=\frac{(-1)^{s+1}}{k}$ for $s\in\{1,2\}$. Then the **empirical blip spectral measure** associated to the anticommutator of an $N\times N$ GOE and k-checkerboard $\{A_N,B_N\}$ around $w_sN^{3/2}$ is*

$$\mu_{\{A_N,B_N\},s}(x) \;=\; \frac{1}{k}\sum_{\lambda\text{ eigenvalues}} f_s^{2n}\left(\frac{\lambda}{w_sN^{3/2}}\right)\delta\left(\frac{x-\left(\lambda-w_sN^{3/2}\right)}{N}\right), \tag{3.1}$$

where $n(N)$ is a function satisfying $n(N)=\log\log(N)$ (note that when we use n in this section we are referring to $n(N)$) and

$$f_s^{2n}(x) \;:=\; \left(\frac{x(2-x)(x+1)(3-x)}{4}\right)^{2n}. \tag{3.2}$$

Definition 3.2. *The **empirical largest blip spectral measure** associated to the anticommutator of an $N\times N$ k-checkerboard and j-checkerboard $\{A_N,B_N\}$, where $\gcd(k,j)=1$ and $jk\mid N$, is*

$$\mu_{\{A_N,B_N\}}(x)=\sum_{\lambda\text{ eigenvalues}} g_0^{2n}\left(\frac{jk\lambda}{2N^2}\right)\delta\left(x-\left(\frac{\lambda-\frac{2}{jk}N^2}{N}\right)\right), \tag{3.3}$$

*where $g_0^{2n}(x):=x^{2n}(2-x)^{2n}$. Let $w_s=\frac{(-1)^{s+1}}{k}\sqrt{1-\frac{1}{j}}$ and $h_s=k$ for $s\in\{1,2\}$ and $w_s=\frac{(-1)^{s+1}}{j}\sqrt{1-\frac{1}{k}}$ and $h_s=j$ for $s\in\{3,4\}$ and g_s as defined in Appendix E. Then the **empirical intermediary blip spectral measure** associated to $\{A_N,B_N\}$ around $w_sN^{3/2}$ is*

$$\mu_{\{A_N,B_N\},s}(x) \;=\; \frac{1}{h_s}\sum_{\lambda\text{ eigenvalues}} g_s^{2n}\left(\frac{\lambda}{w_sN^{3/2}}\right)\delta\left(\frac{x-\left(\lambda-w_sN^{3/2}\right)}{N}\right). \tag{3.4}$$

We again require that $n(N)$ is a function satisfying $\lim_{N\to\infty}n(N)=\infty$ and $n(N)=\log\log(N)$.

Remark 3.3. *Note that we never use this blip spectral measure since the combinatorics needed to do the calculations become too complex. We would need some strong algebraic and combinatorial tools to simplify the large sums we have so we were unable to complete the analysis of the intermediate blip despite creating a weight function with the desired properties. The difficulties of the calculations needed to reduce these sums is discussed in Appendix E.*

We first consider empirical blip spectral measure associated to $\{A_N, B_N\}$ around $\frac{N^{3/2}}{k}$. As we shall see later in this section and by symmetry and from 3.5 with 3.20, the empirical blip spectral measure associated to $\{A_N, B_N\}$ around $-\frac{N^{3/2}}{k}$ is the same as that around $\frac{N^{3/2}}{k}$. Since the weight polynomial $f_1^{2n}(x)$ can be written as $\sum_{\alpha=2n}^{8n} c_\alpha x^\alpha$, where $c_\alpha \in \mathbb{R}$, then by eigenvalue trace lemma, the expected m^{th} moment of the empirical blip spectral measure associated to $\{A_N, B_N\}$ around $\frac{N^{3/2}}{k}$ is

$$\begin{aligned}
\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\},1}\right] &= \mathbb{E}\left[\frac{1}{k}\sum_{\lambda \text{ eigenvalues}}\sum_{\alpha=2n}^{8n} c_\alpha \left(\frac{k\lambda}{N^{3/2}}\right)^\alpha \left(\frac{\lambda - w_1 N^{3/2}}{N}\right)^m\right] \\
&= \mathbb{E}\left[\frac{1}{k}\sum_{\alpha=2n}^{8n} c_\alpha \left(\frac{k}{N^{3/2}}\right)^\alpha \frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{N^{3/2}}{k}\right)^{m-i} \operatorname{Tr}(\{A_N, B_N\}^{\alpha+i})\right] \\
&= \frac{1}{k}\sum_{\alpha=2n}^{8n} c_\alpha \left(\frac{k}{N^{3/2}}\right)^\alpha \frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{N^{3/2}}{k}\right)^{m-i} \mathbb{E}[\operatorname{Tr}(\{A_N, B_N\}^{\alpha+i})]. \quad (3.5)
\end{aligned}$$

Let $g_0^{2n}(x) = \sum_{\beta=2n}^{4ln} d_\beta x^\beta$, then similarly by eigenvalue trace lemma, the expected m^{th} moment of the empirical largest blip spectral measure associated to $\{A_N, B_N\}$ is

$$\begin{aligned}
\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] &= \mathbb{E}\left[\sum_{\lambda \text{ eigenvalues}}\sum_{\beta=2n}^{4nl} d_\beta \left(\frac{jk\lambda}{2N^2}\right)^\beta \left(\frac{\lambda - \frac{2}{jk}N^2}{N}\right)^m\right] \\
&= \mathbb{E}\left[\sum_{\beta=2n}^{4nl} d_\beta \left(\frac{jk}{2N^2}\right)^\beta \frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{2}{jk}N^2\right)^{m-i} \operatorname{Tr}(\{C_N, D_N\}^{\beta+i})\right] \\
&= \sum_{\beta=2n}^{4nl} d_\beta \left(\frac{jk}{2N^2}\right)^\beta \frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{2}{jk}N^2\right)^{m-i} \mathbb{E}[\operatorname{Tr}\{C_N, D_N\}^{\beta+i}]. \quad (3.6)
\end{aligned}$$

We know that for an $N \times N$ anticommutator ensemble $\{X_N, Y_N\}$, the $(\alpha+i)^{\text{th}}$ expected moment is

$$\mathbb{E}[\operatorname{Tr}(\{X_N, Y_N\}^{\alpha+i})] = \sum_{1 \le i_1, \cdots, i_2 m \le N} \mathbb{E}[c_{i_1 i_2} \cdots c_{i_{2m} i_1}]. \quad (3.7)$$

Hence, the calculation of the blip moment has now been transformed into a combinatorics problem of counting different types of products of entries. For the rest of this section, we use a to denote a non-weight term of A_N, w a weight term of A_N, b a non-weight term of B_N, v a weight term of B_N, and c any term of A_N or B_N.

Definition 3.4. *A **block** is a set of adjacent a's and b's surrounded by w's and v's in a cyclic product, where the last term of a cyclic product is considered to be adjacent to the first. We refer to a block of length ℓ as an ℓ block or sometimes a block of size ℓ.*

Definition 3.5. *A **weight block** is a set of adjacent w's and v's surrounded by a's and b's in a cyclic product. We similarly refer to a weight block of length ℓ as an ℓ weight block or sometimes a weight block of size ℓ.*

Definition 3.6. *An **adjacent pair** is a pair of adjacent entries of the form $c_{i_{2\ell-1}i_{2\ell}}$, where the first term starts with an odd index.*

Definition 3.7. *A **weight pair** is a pair of adjacent weight terms $c_{i_{2\ell-1}i_{2\ell}} c_{i_{2\ell}i_{2\ell+1}}$. Due to the structure of anticommutator, $\{c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell}i_{2\ell+1}}\} \in \{\{w_{i_{2\ell-1}i_{2\ell}}, v_{i_{2\ell}i_{2\ell+1}}\}, \{v_{i_{2\ell-1}i_{2\ell}}, w_{i_{2\ell}i_{2\ell+1}}\}\}$.*

Definition 3.8. *An **mixed pair** is a pair of adjacent weight and non-weight terms $c_{i_{2\ell-1}i_{2\ell}}c_{i_{2\ell}i_{2\ell+1}}$. Due to the structure of anticommutator,*

$$\{c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell}i_{2\ell+1}}\} \in \{\{a_{i_{2\ell-1}i_{2\ell}}, v_{i_{2\ell}i_{2\ell+1}}\}, \{v_{i_{2\ell-1}i_{2\ell}}, a_{i_{2\ell}i_{2\ell+1}}\}, \{b_{i_{2\ell-1}i_{2\ell}}, w_{i_{2\ell}i_{2\ell+1}}\}, \{w_{i_{2\ell-1}i_{2\ell}}, b_{i_{2\ell}i_{2\ell+1}}\}\}. \tag{3.8}$$

Definition 3.9. *A **configuration** is a set of all cyclic products for which it is specified (a) how many blocks there are and what each of them compose of (e.g., a block of abba); and (b) in what order these blocks appear (up to cyclic permutation); However, it is not specified how many w's and v's there are between each block.*

Definition 3.10. *A **congruence configuration** is a configuration together with a choice of the congruence class modulo k every index.*

Definition 3.11. *Given a configuration, a **matching** is an equivalence relation $\sim$ on the a's and b's in the cyclic product which constrains the way of indexing: for any $c_{i_\ell i_{\ell+1}}$ and $c_{i_t i_{t+1}}$, if $\{c_{i_\ell i_{\ell+1}}, c_{i_t i_{t+1}}\} \in \{\{a_{i_\ell i_{\ell+1}}, a_{i_t i_{t+1}}\}, \{b_{i_\ell i_{\ell+1}}, b_{i_t i_{t+1}}\}\}$, then $\{i_\ell, i_{\ell+1}\} = \{i_t, i_{t+1}\}$ if and only if $c_{i_\ell i_{\ell+1}} \sim c_{i_t i_{t+1}}$.*

Definition 3.12. *Given a configuration, matching, and length of the cyclic product, then an **indexing** is a choice of*

(1) *the (positive) number of w's and v's between each pair of adjacent blocks (in the cyclic sense), and*
(2) *the integer indices of each a, b, w, v in the cyclic product.*

Definition 3.13. *A **configuration equivalence** $\sim_C$ is an equivalence relation on the set of all configurations such that for any configurations C_1, C_2, $C_1 \sim_C C_2$ if and only they have the same blocks (but they may have different number and arrangement of w's and v's between their blocks). Every equivalence class under $\sim_C$ is called an S**-class**, specified by the blocks in all the configurations in the equivalence class.*

The following lemma characterizes the S-class with the highest degree of freedom and boils down the blip moment calculation for both {GOE, k-checkerboard} and {k-checkerboard, j-checkerboard} to consideration of some nice configurations.

Lemma 3.14. *Fix $m \geq 1$, consider all the S-classes with $|S| = m$. Then a S-class with a matching $\sim$ yields the highest degrees of freedom iff it satisfies the following conditions.*

(1) *It consists only of the following blocks: (i) 1-block of a; (ii) 1-block of b; (iii) 2-block of aa, (iv) 2-block of bb.*
(2) *Each 1-block is paired up to another 1-block and the two terms in each 2-block are paired up with each other.*

Proof. Similar to Lemma 3.14 of [BCDHMSTPY], we see that when a 1-block of a (resp. b) is paired up with another 1-block of a (resp. b) or when the letters in a 2-block of a's (resp. b's) are paired up with each other, there is one degree of freedom lost per block. Now, fix a configuration $\mathcal{C}$ with α from the a's and β from the b's and a matching $\sim$. Suppose that $\sim$ partitions all the a's into equivalence classes $\mathcal{E}_1, \cdots, \mathcal{E}_{s_a}$ and $\mathcal{E}'_1, \cdots, \mathcal{E}'_{s_b}$. Then, without any matching restrictions, the degrees of freedom of $\mathcal{C}$ is

$$\tilde{\mathcal{F}}_{\mathcal{C}} \;=\; \sum_{\text{blocks } \mathcal{B}} (\text{len}(\mathcal{B}) + 1) \;=\; \alpha + \beta + m. \tag{3.9}$$

To find the actual degree of freedom $\mathcal{F}_{\mathcal{C}}$ of $\mathcal{C}$, we can choose two indices from each equivalence class. However, adjacent a's and b's from different equivalence classes (which we call cross-overs) place restrictions on the indices and cause additional loss of degrees of freedom. Let the loss of degrees of freedom due to cross-overs be γ, then $\mathcal{F}_{\mathcal{C}} = 2s_a + 2s_b - \gamma$. Thus, the degree of freedom lost per block is

$$\overline{\mathcal{L}}_{\mathcal{C}} \;=\; \frac{\tilde{\mathcal{F}}_{\mathcal{C}} - \mathcal{F}_{\mathcal{C}}}{m} \;=\; 1 + \frac{\alpha + \beta + \gamma - 2s_a - 2s_b}{m}. \tag{3.10}$$

Since $|\mathcal{E}_i|, |\mathcal{E}'_j| \geq 2$ for $1 \leq i \leq s_a$ and $1 \leq j \leq s_b$, then $s_a \leq \frac{\alpha}{2}$ and $s_b \leq \frac{\beta}{2}$, and so $\overline{\mathcal{L}} \geq 1$. We've shown that if $\mathcal{C}$ satisfies the conditions (1) and (2), then $\overline{\mathcal{L}}_{\mathcal{C}} = 1$. Hence, it suffices to show that if $\mathcal{C}$ with a matching $\sim$ loses one degree of freedom per block (or equivalently, satisfies $\frac{\alpha+\beta+\gamma}{s_a+s_b} = 2$), then it must satisfy the conditions (1) and (2). Since $|\mathcal{E}_i|, |\mathcal{E}'_j| \geq 2$, we get $\alpha \geq 2s_a$ and $\beta \geq 2a_b$. Hence, if some $|\mathcal{E}_i| > 2$ or $|\mathcal{E}'_j| > 2$, then $\frac{\alpha+\beta+\gamma}{s_a+s_b} > 2$. Moreover, if $\gamma > 0$, then $\frac{\alpha+\beta+\gamma}{s_a+s_b} > 2$. Therefore, if $\mathcal{C}$ with a matching $\sim$ loses one degree of freedom per block, then all the blocks are paired up and there can be no cross-overs from different equivalence classes. Thus, the only possible S-classes and matchings are those satisfying conditions (1) and (2). □

Due to the structure of anticommutator, there are eight possibilities for each adjacent pair: ab, ba, av, va, bw, wb, vw, wv. Hence, after specifying the 1-block, we know the other term in the mixed pair, i.e. if the 1-block is of the form $c_{i_{2\ell-1}i_{2\ell}}$ and $c_{i_{2\ell-1}i_{2\ell}} = a_{i_{2\ell-1}i_{2\ell}}$ (resp. $c_{i_{2\ell-1}i_{2\ell}} = b_{i_{2\ell-1}i_{2\ell}}$), then $c_{i_{2\ell}i_{2\ell+1}} = v_{i_{2\ell}i_{2\ell+1}}$ (resp. $c_{i_{2\ell}i_{2\ell+1}} = w_{i_{2\ell}i_{2\ell+1}}$); similar conclusion holds for 1-block of the form $c_{i_{2\ell}i_{2\ell+1}} = a_{i_{2\ell}i_{2\ell+1}}$ (resp. $c_{i_{2\ell}i_{2\ell+1}} = b_{i_{2\ell}i_{2\ell+1}}$). Moreover, after specifying the 2-block, we know both of its adjacent terms (or the two mixed pairs the 2-block belongs to), i.e. every 2-block is of the form $c_{i_{2\ell}i_{2\ell+1}}c_{i_{2\ell+1}i_{2\ell+2}}$, and if $\{c_{i_{2\ell}i_{2\ell+1}}, c_{i_{2\ell+1}i_{2\ell+2}}\} = \{a_{i_{2\ell}i_{2\ell+1}}, a_{i_{2\ell+1}i_{2\ell+2}}\}$, then $\{c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell+2}i_{2\ell+3}}\} = \{v_{i_{2\ell-1}i_{2\ell}}, v_{i_{2\ell+2}i_{2\ell+3}}\}$; if $\{c_{i_{2\ell}i_{2\ell+1}}, c_{i_{2\ell+1}i_{2\ell+2}}\} = \{b_{i_{2\ell}i_{2\ell+1}}, b_{i_{2\ell+1}i_{2\ell+2}}\}$, then $\{c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell+2}i_{2\ell+3}}\} = \{w_{i_{2\ell-1}i_{2\ell}}, w_{i_{2\ell+2}i_{2\ell+3}}\}$. We also know that a weight pair $\{c_{i_{2\ell-1}i_{2\ell}}, c_{i_{2\ell}i_{2\ell+1}}\} \in \{\{w_{i_{2\ell-1}i_{2\ell}}, v_{i_{2\ell}i_{2\ell+1}}\}, \{v_{i_{2\ell-1}i_{2\ell}}, w_{i_{2\ell}i_{2\ell+1}}\}\}$. Thus, we can view specifying a cyclic product of length 2η as only specifying η terms, where specifying one term of the pair $c_{i_{2\ell-1}i_{2\ell}}c_{i_{2\ell}i_{2\ell+1}}$ and whether the pair is weight/non-weight or not uniquely determines the other term of the pair.

3.2. $\{$**GOE, k-checkerboard**$\}$.

Lemma 3.15. *For $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, the total contribution to $\mathbb{E}[\text{Tr}\{A_N, B_N\}^\eta]$ of an S-class with m_1 1-blocks of a's and m_2 2-blocks of a is*

$$\frac{p(\eta) N^{\frac{3}{2}\eta - \frac{1}{2}m_1}}{k^\eta} \mathbb{E}_k[\text{Tr}\, C^{m_1}] + O\left(N^{\frac{3}{2}\eta - \frac{1}{2}m_1 - 1}\right), \tag{3.11}$$

where $p(\eta) = \frac{2\eta^{m_1}}{m_1!} + O(\eta^{m_1-1})$ and C is a $k \times k$ Gaussian Wigner matrix.

Proof. We begin by noting that by 3.14, an S-class containing any b would have fewer degrees of freedom and hence would contribute at most $O(N^{\frac{3}{2}\eta - \frac{m_1}{2} - 1})$. Thus, it suffices to consider the case when $m_2 = \frac{\eta - m_1}{2}$ and there are no b's. The rest of the proof is divided into two parts: we first count the number of ways to arrange a prescribed number of blocks into a cyclic product of length 2η; we then count the number of ways to pair together 1-blocks and assign indices that are consistent throughout the cyclic product.

Given $m_1 = o(\eta)$, we claim that the number of ways $q(\eta)$ of arranging m_1 1-blocks and $\frac{\eta - m_1}{2}$ 2-blocks of a's into a cyclic product of length 2η is $\frac{2\eta^{m_1}}{m_1!} + O(\eta^{m_1-1})$.

Indeed, there are two ways to choose the $\frac{\eta - m_1}{2}$ 2-blocks since we can either start with aw or wa, and there are $2^{m_1}\binom{\frac{\eta - m_1}{2}}{m_1} = \frac{\eta^{m_1}}{m_1!} + O(\eta^{m_1-1})$ ways to choose the m_1 1-blocks between adjacent 2-blocks assuming that $m_1 = o(\eta)$. Moreover, as we shall see later in the proof, we require that mixed pairs containing the 1-blocks are not placed adjacent to each other, which is possible since the number of ways of having at least one 2-block formed from adjacent mixed pairs is $2\binom{\frac{\eta - m_1}{2}}{}\binom{\frac{\eta - m_1}{2} - 2}{m_1 - 2} = O(\eta^{m_1-1})$ and thus a lower order term. Hence, overall we get $\frac{2\eta^{m_1}}{m_1!} + O(\eta^{m_1-1})$ ways to arrange the prescribed blocks.

Now, we observe that the second and the first index respectively of two adjacent 1-blocks are congruent mod k, as illustrated in the example below.

Example 3.16. *Consider the configuration*

$$\cdots v_{i_1i_2}a_{i_2i_3}v_{i_3i_4}v_{i_4i_5}a_{i_5i_6}a_{i_6i_7}v_{i_7i_8}a_{i_8i_9}\cdots.$$

Since a's within a 2-block are matched together, we have $i_5 = i_7$ with i_6 being free, and that

$$i_3 \equiv i_4 \equiv i_5 \equiv i_7 \equiv i_8 \text{ (mod } k).$$

Thus, all the indices of terms between a pair of 1-blocks, except for those within 2-blocks, share a congruence class. The number of ways to specify the congruence classes of 1-blocks and to pair the 1-blocks up is

$$\sum_{1 \leq i_1, i_2, \ldots, i_{m_1} \leq k} \mathbb{E}[a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{m_1} i_1}], \tag{3.12}$$

where each $c_{ij} \sim \mathcal{N}(0,1)$. The above expression is simply the $m_1{}^{\text{th}}$ expected moment of $k \times k$ GOE.

Since the $2m_1$ 1-blocks are paired together, then there are only m_1 free indices. As the congruence class of these indices are fixed, the number of choices of these indices is $(\frac{N}{k})^{m_1}$. Similarly, the number of choices of indices for all the 2-blocks is $(\frac{N^2}{k})^{\frac{\eta - m_1}{2}}$, since the indices of each 2-block $a_{i_\ell i_{\ell+1}} a_{i_{\ell+1} i_{\ell+2}}$ must satisfy $i_\ell = i_{\ell+2}$, and there are $\frac{N}{k}$ choices for $i_\ell = i_{\ell+2}$ whose congruence class is fixed and N choices for i_2 that is free. The remaining indices are those of the weight blocks, which must satisfy congruence mod k and hence are each restricted to $\frac{N}{k}$ choices. By the structure imposed by the anticommutator, the total number of indices of all weight blocks is $\eta - \left(\left(\frac{\eta - m_1}{2}\right) + m_1\right) = \frac{\eta - m_1}{2}$. Thus, the total number of ways to assign indices is $\left(\frac{N}{k}\right)^{m_1} \left(\frac{N^2}{k}\right)^{\frac{\eta - m_1}{2}} \left(\frac{N}{k}\right)^{\frac{\eta - m_1}{2}} = \frac{N^{\frac{3}{2}\eta - \frac{1}{2}m_1}}{k^\eta}$. After combining all these pieces, we arrive at the desired result for the contribution of a fixed S-class. $\square$

In the expected m^{th} moment calculation, the following two combinatorial equalities from [BCDHMSTPY] are extremely useful for cancelling the contribution of S-classes with fewer than m blocks.

Lemma 3.17. *For any $0 \leq p < m$,*

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} i^p \;=\; 0, \tag{3.13}$$

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i^m \;=\; m!. \tag{3.14}$$

Observe that if $m_1 > m$, then by Lemma 3.15 the contribution of an S class with m_1 1-block is

$$\begin{aligned}
&\frac{1}{k} \sum_{\alpha=2n}^{8n} c_\alpha \left(\frac{k}{N^{3/2}}\right)^\alpha \frac{1}{N^m} \sum_{i=0}^{m} \binom{m}{i} \left(-\frac{N^{3/2}}{k}\right)^{m-i} p(\alpha+i) \frac{N^{\frac{3}{2}(\alpha+i) - \frac{1}{2}m_1}}{k^{\alpha+i}} \\
&= \frac{C_{k,m}}{N^{\frac{1}{2}(m_1 - m)}} \sum_{\alpha=2n}^{8n} c_\alpha \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} p(\alpha+i) \\
&= \frac{C_{k,m}}{N^{\frac{1}{2}(m_1 - m)}} \sum_{\alpha=2n}^{8n} c_\alpha \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \left(\frac{2(\alpha+i)^{m_1}}{m_1} + O\left((\alpha+i)^{m_1 - 1}\right)\right) \\
&= \frac{C_{k,m,m_1}}{N^{\frac{1}{2}(m_1 - m)}} \sum_{\alpha=2n}^{8n} c_\alpha \alpha^{m_1}.
\end{aligned} \tag{3.15}$$

Since $f_1^{2n}(x) = \left(\frac{x(2-x)(x+1)(3-x)}{4}\right)^{2n}$, then $|c_\alpha| \ll C_0^{2n}$ for some $C_0 > 0$. Moreover, $\alpha \ll \log\log(N)$, then for some $\epsilon > 0$

$$\sum_{\alpha=2n}^{8n} c_\alpha \alpha^{m_1} \ll n^{m_1+1} C_0^{2n} \ll (\log\log(N))^{m_1+1} \log(N) \ll N^{1/2(m_1 - m) - \epsilon} \tag{3.16}$$

Hence, as $N \to 0$, the contribution of an S-class with $m_1 > m$ total a blocks and m_2 total aa blocks is negligible. Moreover, if $m_1 < m$, then the contribution of an S-class with m_1 total a blocks is

$$\begin{aligned}
&\frac{1}{k}\sum_{\alpha=2n}^{8n} c_\alpha \left(\frac{k}{N^{3/2}}\right)^{\alpha}\left(\frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{N^{3/2}}{k}\right)^{m-i} p(\alpha+i)\left(\frac{N^{\frac{3}{2}(\alpha+i)-\frac{1}{2}m_1}}{k^{\alpha+i}}\right)\right)\\
&= \frac{C_{k,m}}{N^{\frac{1}{2}(m_1-m)}}\sum_{\alpha=2n}^{8n} c_\alpha \sum_{i=0}^{m}\binom{m}{i}(-1)^i p(\alpha+i)\\
&= \frac{C_{k,m}}{N^{\frac{1}{2}(m_1-m)}}\sum_{\alpha=2n}^{8n} c_\alpha \sum_{q=0}^{m_1} c_q \alpha^{m_1-q}\sum_{i=0}^{m}(-1)^i\binom{m}{i} i^q \;=\; 0. \qquad (3.17)
\end{aligned}$$

Thus, we must have $m_1 = m$.

Theorem 3.18. *The expected m^{th} moment associated to the empirical blip spectral measure is*

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\},1}\right] \;=\; 2\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}\ C^m]. \qquad (3.18)$$

Proof. By the discussion above, we know that $m_1 = m$. Then

$$\begin{aligned}
&\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\},1}\right] \qquad (3.19)\\
&= \frac{1}{k}\sum_{\alpha=2n}^{8n} c_\alpha\left(\frac{k}{N^{3/2}}\right)^{\alpha}\frac{1}{N^{m+\frac{1}{2}m}}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\left(\frac{N^{3/2}}{k}\right)^{m+\alpha}\frac{2(\alpha+i)^m}{m!}\mathbb{E}_k[\mathrm{Tr}C^m]\\
&= \frac{2}{m!}\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}C^m]\sum_{\alpha=2n}^{8n} c_\alpha\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}(\alpha+i)^m\\
&= \frac{2}{m!}\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}C^m]\sum_{\alpha=2n}^{8n} c_\alpha\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\sum_{p=0}^{m}\binom{m}{p}\alpha^p i^{m-p}\\
&= \frac{2}{m!}\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}C]\sum_{\alpha=2n}^{8n}\sum_{p=0}^{m}\binom{m}{p}c_\alpha\alpha^p\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}i^{m-p}. \qquad (3.20)
\end{aligned}$$

Since the inner sum is 0 if $p > 0$ and $m!$ if $p = 0$ by Lemma 3.17 and $f_1^{(2n)}(1) = \sum_{\alpha=2n}^{8n} c_\alpha = 1$, then

$$\begin{aligned}
\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\},1}\right] &= \frac{2}{m!}\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}C^m]\sum_{\alpha=2n}^{8n} c_\alpha m!\\
&= 2\left(\frac{1}{k}\right)^{m+1}\mathbb{E}_k[\mathrm{Tr}C^m]. \qquad (3.21)
\end{aligned}$$

□

3.3. Moments of $\{k$-checkerboaord, j-checkerboard$\}$.

Proposition 3.19. *For $m_{1a}, m_{2a}, m_{1b}, m_{2b} \in \mathbb{Z}_{\geq 0}$, define $m_1 := m_{1a} + m_{1b}$ and $m_2 := m_{2a} + m_{2b}$. If $m_1 + m_2 = o(\eta)$, then the total contribution to $\mathbb{E}[\mathrm{Tr}\{A_N, B_N\}^\eta]$ of an S-class with m_{1a} 1-blocks of a, m_{1b} 1-blocks of b, m_{2a} 2-blocks of a, and m_{2b} 2-blocks of b is*

$$\frac{2^{\eta-2m_2}\eta^{m_1+m_2}}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!}2^{\frac{m_1}{2}}(m_{1a})!!(m_{1b})!!\left(\tfrac{1}{k}\right)^{\eta-m_{1a}-2m_{2a}}\left(\tfrac{1}{j}\right)^{\eta-m_{1b}-2m_{2b}}\left(1-\tfrac{1}{k}\right)^{\frac{m_{1a}}{2}+m_{2a}}\left(1-\tfrac{1}{j}\right)^{\frac{m_{1b}}{2}+m_{2b}}N^{2\eta-(m_1+m_2)}.$$

Proof. The proof is divided into two parts. First, we count the number of ways to arrange the prescribed blocks into the cyclic product of length 2η and assign the weight pairs; second, we count the number of ways to pair up all the 1-blocks and assign indices that ensures consistent indexing throughout the cyclic product.

Given $m_1 + m_2 = o(\eta)$, we claim that the number of ways $q(\eta)$ of arranging the prescribed blocks into the cyclic product of length 2η and assign the adjacent weight pairs is $\frac{2^{\eta-2m_2}\eta^{m_1+m_2}}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!} + O(2^\eta \eta^{m_1+m_2-1})$. Naively, $q(\eta)$ is simply

$$\binom{\eta - m_2}{m_1 + m_2}\binom{m_1 + m_2}{m_2} 2^{\eta-2m_2}\binom{m_1}{m_{1a}}\binom{m_2}{m_{2a}} \; = \; \frac{2^{\eta-2m_2}\eta^{m_1+m_2}}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!} + O(2^\eta \eta^{m_1+m_2-1}). \quad (3.22)$$

That is, we first choose all the $m_1 + m_2$ blocks, viewing each of the m_2 2-block as a 1-block (where $\eta - m_2$ comes from), which can be done in $\binom{\eta-m_2}{m_1+m_2}$ ways. Then, we choose the m_2 2-blocks from all the $m_1 + m_2$ blocks, m_{1a} 1-blocks of a from all the m_1 1-blocks, m_{2a} 2-blocks of a from all the m_2 2-blocks, and finally specifying the mixed pairs the 1-blocks belong to and the weight pairs, all of which can be done in $\binom{m_1+m_2}{m_2}2^{\eta-2m_2}\binom{m_1}{m_{1a}}\binom{m_2}{m_{2a}}$ ways.

However, this naive counting method fails to account for the restriction that different mixed pairs of a and v and of b and w cannot be placed adjacent to each other to form a 2-block (e.g. if two mixed pairs va and av are adjacent to each other, then we have a 2-block of a). The number of ways of having at least one 2-block formed from different mixed pairs of a and v and of b and w is $2^{\eta-2m_2-1}(\eta - m_2)\binom{\eta-m_2-2}{m_1+m_2-2}\binom{m_1+m_2}{m_2}\binom{m_1}{m_{1a}}\binom{m_2}{m_{2a}} = O(2^\eta \eta^{m_1+m_2-1})$ if $m_1 + m_2 = o(\eta)$. Thus,

$$q(\eta) \; = \; \frac{2^{\eta-2m_2}\eta^{m_1+m_2}}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!} + O(2^\eta \eta^{m_1+m_2-1}). \quad (3.23)$$

Similarly, we can also guarantee that none of the 1-blocks are adjacent to each other, since the number of ways of having at least two adjacent 1-blocks is $2^{\eta-2m_2}(\eta - m_2)\binom{\eta-m_2-2}{m_1+m_2-2}\binom{m_1+m_2}{m_2}\binom{m_1}{m_{1a}}\binom{m_2}{m_{2a}} = O(2^\eta \eta^{m_1+m_2-1})$ if $m_1 + m_2 = o(\eta)$.

Now, we count the number of ways to assign the indices for the S-class. In contrast to [BCDHMSTPY] where there are restrictions on the indices of the 1-blocks, we demonstrate in the example below that we can remove such restrictions.

Example 3.20. *Consider a weight block surrounded by two non-weight terms $c_{i_{\ell-1}i_\ell}c_{i_\ell i_{\ell+1}} \cdots c_{i_t i_{t+1}}c_{i_{t+1}i_{t+2}}$, where $t - \ell$ is sufficiently large. We assume without loss of generality that $c_{i_{\ell-1}i_\ell} = a_{i_{\ell-1}i_\ell}$ and $c_{i_{t+1}i_{t+2}} = a_{i_{t+1}i_{t+2}}$. After specifying i_ℓ, if $c_{i_\ell i_{\ell+1}} = w_{i_\ell i_{\ell+1}}$, then $i_\ell \equiv i_{\ell+1} \pmod k$ and there are $\frac{N}{k}$ choices of indices for $i_{\ell+1}$; if $c_{i_\ell i_{\ell+1}} = v_{i_\ell i_{\ell+1}}$, then $i_\ell \equiv i_{\ell+1} \pmod j$ and there are $\frac{N}{j}$ choices of indices for $i_{\ell+1}$. After specifying i_{t+1}, there are similar number of choice of indices for i_t. Since $t - \ell$ is sufficiently large, then with high probability there exists $\ell + 1 \le s \le t - 2$ such that $\{c_{i_s i_{s+1}}, c_{i_{s+1}i_{s+2}}\} = \{w_{i_s i_{s+1}}, v_{i_{s+1}i_{s+2}}\}$. We can specify the indices $i_{\ell+2}, \dots, i_s$ and $i_{s+2}, \dots, i_{t-1}$ the same way as before. Then we have $i_{s+1} \equiv i_s \pmod k$ and $i_{s+1} \equiv i_{s+2} \pmod j$. Since $\gcd(k, j) = 1$ and $jk \mid N$, then by Chinese remainder theorem, there are $\frac{N}{kj}$ choices of indices i_{s+1}. If the number of w's and v's in the weight block is r and $t - \ell + 1 - r$, respectively, then there are $\left(\frac{1}{k}\right)^r \left(\frac{1}{j}\right)^{t-\ell+1-r} N^{t-\ell}$ ways of specifying the $i_{\ell+1}, \dots, i_t$. Thus, regardless of the indices we specify for the two non-weight terms surrounding a weight block, we can guarantee with high probability consistency of indexing throughout the weight block.*

We know that the total number of w's and v's in a cyclic product of length 2η is $\eta - m_{1a} - 2m_{2a}$ and $\eta - m_{1b} - 2m_{2b}$, respectively. Then the number of choices of congruence classes of indices for all the w's and v's has the corresponding factors $\left(\frac{1}{k}\right)^{\eta-m_{1a}-2m_{2a}}$ and $\left(\frac{1}{j}\right)^{\eta-m_{1b}-2m_{2b}}$. Now, by Lemma 3.14, each 1-block is paired up with another 1-block and the two terms of each 2-block are paired up with each other. Moreover, the indices of any non-weight terms $a_{i_\ell i_{\ell+1}}$ and $b_{i_t i_{t+1}}$ must satisfy the modular restrictions

$i_\ell \not\equiv i_{\ell+1}$ (mod k) and $i_t \not\equiv i_{t+1}$ (mod j). Then similarly, the number of choices of congruence classes of indices for all the a's and b's has the corresponding factors $\left(1-\frac{1}{k}\right)^{\frac{m_{1a}}{2}+m_{2a}}$ and $\left(1-\frac{1}{j}\right)^{\frac{m_{1b}}{2}+m_{2b}}$. Since the loss of degrees of freedom per block in a contributing configuration is 1, then the contribution of actually specifying all the indices is $N^{2\eta-(m_1+m_2)}$. Thus, the number of ways of assigning the indices that guarantees consistent indexing is

$$\left(\frac{1}{k}\right)^{\eta-m_{1a}-2m_{2a}}\left(\frac{1}{j}\right)^{\eta-m_{1b}-2m_{2b}}\left(1-\frac{1}{k}\right)^{\frac{m_{1a}}{2}+m_{2a}}\left(1-\frac{1}{j}\right)^{\frac{m_{1b}}{2}+m_{2b}}N^{2\eta-(m_1+m_2)}. \tag{3.24}$$

Finally, since there are m_{1a} 1-block of a and m_{1b} 1-block of b, and there are no restrictions on their indices, then the number of ways of matching up all the 1-blocks is $2^{\frac{m_1}{2}}(m_{1a})!!(m_{1b})!!$. Note that the $2^{\frac{m_1}{2}}$ factor is due to the fact that for any two paired 1-blocks $c_{i_\ell i_{\ell+1}}, c_{i_t,i_{t+1}}$, we either have $i_\ell = i_{t+1}$ and $i_{\ell+1} = i_t$ or $i_\ell = i_t$ and $i_{\ell+1} = i_{t+1}$. This completes the proof. □

Theorem 3.21. *The m^{th} moment of the largest blip spectral measure is*

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] = \sum_{\substack{m_{1a}+m_{1b}+m_{2a}+m_{2b}=m;\\ m_{1a},m_{1b}\text{ even}}} C(m,m_{1a},m_{2a},m_{1b},m_{2b})\left(k\sqrt{1-\frac{1}{k}}\right)^{m_{1a}+2m_{2a}}\left(j\sqrt{1-\frac{1}{j}}\right)^{m_{1b}+2m_{2b}},$$

where $C(m,m_{1a},m_{2a},m_{1b},m_{2b}) := m!\left(\frac{2}{jk}\right)^m \frac{2^{\frac{m_{1a}+m_{1b}}{2}-2(m_{2a}+m_{2b})}m_{1a}!!m_{1b}!!}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!}$.

Proof. By Lemma 3.14, it suffices to consider the contributions from S-classes with 1-blocks and 2-blocks. Applying Proposition 3.19 to Equation (3.6), we have

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] = \sum_{\beta=2n}^{4nl} d_\beta\left(\frac{jk}{2N^2}\right)^\beta \frac{1}{N^m}\sum_{i=0}^m\binom{m}{i}\left(-\frac{2}{jk}N^2\right)^{m-i}\left(\frac{2}{kj}\right)^{\beta+i}N^{2(\beta+i)-(m_1+m_2)}$$
$$\sum_{\substack{m_{1a},m_{1b},m_{2a},m_{2b}\\ m_{1a},m_{1b}\text{ even}}}\frac{2^{-2m_2+m_1/2}(m_{1a})!!(m_{1b})!!}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!}\left(k\sqrt{1-\frac{1}{k}}\right)^{m_{1a}+2m_{2a}}\left(j\sqrt{1-\frac{1}{j}}\right)^{m_{1b}+2m_{2b}}(\beta+i)^{m_1+m_2}. \tag{3.25}$$

Similar to the blip moment calculation of {GOE, k-checkerbaord}, we require $m_1+m_2 \le m$, since otherwise the contribution from $m_1+m_2 > m$ vanishes in the limit. Moreover, by Lemma 3.17, the sum $\sum_{i=0}^m\binom{m}{i}(-1)^{m-i}(\beta+i)^{m_1+m_2}$ vanishes except for the m^{th} power of i. Hence, the contribution to the moment in the limit only comes from $m_1+m_2=m$. After combining terms and canceling out the dependency on i, and noting that $g_0^{2n}(1)=\sum_{\beta=2n}^{4ln}d_\beta x^\beta$, we have

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] = \sum_{\substack{m_{1a}+m_{1b}+m_{2a}+m_{2b}=m;\\ m_{1a},m_{1b}\text{ even}}} C(m,m_{1a},m_{2a},m_{1b},m_{2b})\left(k\sqrt{1-\frac{1}{k}}\right)^{m_{1a}+2m_{2a}}\left(j\sqrt{1-\frac{1}{j}}\right)^{m_{1b}+2m_{2b}}. \tag{3.26}$$

□

APPENDIX A. MOMENTS OF ANTICOMMUTATORS

A.1. Moments of ℓ anticommutators.

Proposition A.1. *For sequences $f_0, f_1, \dots, f_\ell$ defined such that $f_0(0) = 1$, and $f_0(1) = f_1(1) = f_2(1) = \cdots = f_\ell(1) = 1$, and with recurrence relations for $m > 1$ given by*

$$f_0(m) = f_1(m) + \ell! \sum_{j=1}^{m-1} f_1(j) f_0(m-j),$$

$$f_k(m) = f_{k+1}(m) + \sum_{\substack{1 \le x_1, x_2 \le m \\ x_1 + x_2 \le m}} (\ell - k)!(k-1)! f_{k+1}(x_1) f_{k+1}(x_2) f_{\ell-k-1}(m - x_1 - x_2 + 1) \tag{A.1}$$

for any $0 < k < \ell - 1$, and by

$$f_{\ell-1}(m) = f_\ell(m) + \sum_{\substack{0 \le x_1, x_2 < m-1 \\ x_1 + x_2 < m-1}} (\ell-1)!(1 + (\ell! - 1) \cdot \mathbb{1}_{x_1 > 0})(1 + (\ell! - 1) \cdot \mathbb{1}_{x_2 > 0}) f_0(x_1) f_0(x_2) f_1(m - x_1 - x_2 - 1),$$

$$f_\ell(m) = \ell! \cdot f_0(m-1), \tag{A.2}$$

the $2m^{th}$ moment of the ℓ-anticommutator is

$$M_{2m} = \ell! \cdot f_0(m) \tag{A.3}$$

Proof. The proof follows similarly as with the 2-anticommutator. Let $f_0(m)$ be the number of non-crossing matchings with respect to all $(\ell, 2\ell m)$-configurations starting with $a^{(1)}_{i_1 i_2} a^{(2)}_{i_2 i_3} \cdots a^{(\ell)}_{i_\ell i_{\ell+1}}$ and let $f_k(m)$ be the number of such matchings where the first k terms are paired with the last k terms in a nested fashion (i.e. for $k = 3$, we would have configurations of the form $a^{(1)}_{i_1 i_2} a^{(2)}_{i_2 i_3} a^{(3)}_{i_3 i_4} \cdots a^{(3)}_{i_{2\ell m-2} i_{2\ell m-1}} a^{(2)}_{i_{2\ell m-1} i_{2\ell m}} a^{(1)}_{i_{2\ell m} i_1}$ and matchings such that $i_2 = i_{2\ell m}$, $i_3 = i_{2\ell m-1}$, and $i_4 = i_{2\ell m-2}$).

We first find the recurrence relation for $f_0(m)$. To ensure non-crossing matchings, $a^{(1)}_{i_1 i_2}$ must be paired with some $a^{(1)}_{i_{2\ell j} i_{2\ell j+1}}$ with $j \le m$ (in the case when $j = m$, we identify $2\ell m + 1$ as 1). When $j = m$, the number of non-crossing matchings is simply $f_1(m)$ by definition. When $j < m$, the number of non-crossing matchings within $a^{(1)}_{i_1 i_2} \cdots a^{(1)}_{i_{2\ell j} i_{2\ell j+1}}$ is $f_1(j)$, while the number of non-crossing matchings within the rest of the cyclic product for which we have no restrictions is $\ell! f_0(m-j)$, with the $\ell!$ accounting for different possible arrangements of the first ℓ terms. Multiplying these together and summing over all possible j's gives

$$f_0(m) = f_1(m) + \ell! \sum_{j=1}^{m-1} f_1(j) f_0(m-j). \tag{A.4}$$

Now turning to $f_k(m)$, we look separately at when $0 < k < \ell - 1$, when $k = \ell - 1$, and when $k = \ell$.

When $0 < k < \ell - 1$, we have either that $a^{(k+1)}_{i_{k+1} i_{k+2}}$ is paired with $a^{(k+1)}_{i_{2\ell m-k} i_{2\ell m-k+1}}$, or that $a^{(k+1)}_{i_{k+1} i_{k+2}}$ is paired with $a^{(k+1)}_{i_{2\ell x_1-k} i_{2\ell x_1-k+1}}$ and $a^{(n)}_{i_{2\ell m-k} i_{2\ell m-k+1}}$ is paired with $a^{(n)}_{i_{2\ell(m-x_2)+k+1} i_{2\ell(m-x_2)+k+2}}$, where $n \in \{k+1, \dots, \ell\}$, with both $x_1, x_2 \ge 1$ and $2\ell x_1 - k + 1 < 2\ell(m - x_2) + k + 1$, or $x_1 + x_2 \le m$. The first case is precisely the definition of $f_{k+1}(m)$. In the second case, the number of non-crossing matchings of terms between $a^{(k+1)}_{i_{k+1} i_{k+2}}$ and $a^{(k+1)}_{i_{2\ell x_1-k} i_{2\ell x_1-k+1}}$ is $f_{k+1}(x_1)$, the number of non-crossing matchings of terms between $a^{(n)}_{i_{2\ell(m-x_2)+k+1} i_{2\ell(m-x_2)+k+2}}$ and $a^{(n)}_{i_{2\ell m-k} i_{2\ell m-k+1}}$ is $(\ell - k)! f_{k+1}(x_2)$, with the $(\ell - k)!$ accounting for different possible arrangements of the last ℓ terms, and the number of non-crossing matchings of terms between $a^{(k+1)}_{i_{2\ell x_1-k} i_{2\ell x_1-k+1}}$ and $a^{(n)}_{i_{2\ell(m-x_2)+k+1} i_{2\ell(m-x_2)+k+2}}$ is $(k-1)! f_{\ell-k+1}(m - x_1 - x_2 + 1)$.

The last statement follows from viewing the $2\ell(m-x_1-x_2)+2k+2$ terms between $a^{(k+1)}_{i_{2\ell x_1-k}i_{2\ell x_1-k+1}}$ and $a^{(n)}_{i_{2\ell(m-x_2)+k+1}i_{2\ell(m-x_2)+k+2}}$ as $2\ell(m-x_1-x_2+1)$ terms where the $(l-k-1)$ terms on both end are matched to each other and hence fixed, with the $(k-1)!$ accounting for different permutations of the remaining $k+1$ of the first ℓ terms. We sum over all possible x_1 and x_2's to get the desired result:

$$f_k(m) = f_{k+1}(m) + \sum_{\substack{1\le x_1,x_2\le m\\ x_1+x_2\le m}} (\ell-k)!(k-1)!f_{k+1}(x_1)f_{k+1}(x_2)f_{\ell-k-1}(m-x_1-x_2+1). \tag{A.5}$$

When $k=\ell-1$, we either have that $a^{(\ell)}_{i_\ell i_{\ell+1}}$ is paired with $a^{(\ell)}_{i_{2\ell m-\ell+1}i_{2\ell m-\ell+2}}$, which is simply $f_\ell(m)$ by definition, or that $a^{(\ell)}_{i_\ell i_{\ell+1}}$ is paired with $a^{\ell}_{i_{2\ell x_1+\ell+1}i_{2\ell x_1+\ell+2}}$ and $a^{(n)}_{i_{2\ell m-\ell+1}i_{2\ell m-\ell+2}}$ is paired with $a^{(n)}_{i_{2\ell(m-x_2)-\ell}i_{2\ell(m-x_2)-\ell+1}}$, with both $x_1,x_2\ge 0$. The number of non-crossing matchings of terms between $a^{(\ell)}_{i_\ell i_{\ell+1}}$ and $a^{\ell}_{i_{2\ell x_1+\ell+1}i_{2\ell x_1+\ell+2}}$ is $(1+(\ell!-1)\mathbb{1}_{x_1>0})f_0(x_1)$, the number of non-crossing matchings of terms between $a^{(n)}_{i_{2\ell(m-x_2)-\ell}i_{2\ell(m-x_2)-\ell+1}}$ and $a^{(n)}_{i_{2\ell m-\ell+1}i_{2\ell m-\ell+2}}$ is $(1+(\ell!-1)\mathbb{1}_{x_2>0})f_0(x_2)$, with a factor of $\ell!$ when either $x_1,x_2>0$ due to different possible arrangements of the first ℓ terms starting at $a^{(n)}_{i_{\ell+1}i_{\ell+2}}$, where $n\in\{1,\dots,\ell-1\}$, and the number of non-crossing matchings of terms between $a^{\ell}_{i_{2\ell x_1+\ell+1}i_{2\ell x_1+\ell+2}}$ and $a^{(n)}_{i_{2\ell(m-x_2)-\ell}i_{2\ell(m-x_2)-\ell+1}}$ is $(\ell-1)!f_1(m-x_1-x_2-1)$. For the last statement, we view the $2\ell(m-x_1-x_2)-2\ell-2$ terms between $a^{\ell}_{i_{2\ell x_1+\ell+1}i_{2\ell x_1+\ell+2}}$ and $a^{(n)}_{i_{2\ell(m-x_2)-\ell}i_{2\ell(m-x_2)-\ell+1}}$ as $2\ell(m-x_1-x_2-1)$ terms with the first and last term matched with each other, with $(l-1)!$ accounting for different arrangements of the remaining $l-1$ terms. We once again sum over all possible x_1 and x_2's to reach the desired result:

$$\begin{aligned} f_{\ell-1}(m) &= f_\ell(m) + \\ &\sum_{\substack{0\le x_1,x_2<m-1\\ x_1+x_2<m-1}} (\ell-1)!(1+(\ell!-1)\cdot\mathbb{1}_{x_1>0})(1+(\ell!-1)\cdot\mathbb{1}_{x_2>0})f_0(x_1)f_0(x_2)f_1(m-x_1-x_2-1). \end{aligned} \tag{A.6}$$

Lastly, when $k=\ell$, with no matching conditions on the terms between the first and last ℓ terms, for each possible permutation of the next ℓ terms, we have $f_0(m-1)$ non-crossing matchings, amounting to $\ell!\cdot f_0(m-1)$ total non-crossing matchings.

We have now fully defined our recurrences for $f_0(k)$, which represents the number of non-crossing matchings with respect to $(\ell,2\ell m)$-configurations where the first ℓ terms are fixed to be $a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(\ell)}_{i_\ell i_{\ell+1}}$. Applying any permutation to these ℓ terms preserves the non-crossing property of these matchings. Hence, we multiply $f_0(m)$ by $\ell!$ to obtain all possible non-crossing partitions with respect to $(\ell,2\ell m)$-configurations, and we arrive at the even moments being $M_{2k}=\ell!\cdot f_0(k)$.

□

A.2. **Bulk Moments of $\{$GOE, k-checkerboard$\}$.**

Proposition A.2. *The $2m^{th}$ bulk moment of* $\{$GOE, k-checkerboard$\}$ *is $M_{2m}=2(1-\frac{1}{k})^m f(m)$, where $f(0)=f(1)=1$, $g(1)=1$, and*

$$f(m) \;=\; 2\sum_{j=1}^{m-1} g(j)f(m-j)+g(m), \tag{A.7}$$

and

$$g(m) \ = \ 2f(m-1) + \sum_{\substack{0 \le x_1, x_2 \le m-2 \\ x_1+x_2 \le m-2}} (1+\mathbb{1}_{x_1>0})(1+\mathbb{1}_{x_2>0}) f(x_1) f(x_2) g(m-1-x_1-x_2) \qquad \text{(A.8)}$$

Proof. By a result in [Tao1], the limiting distribution of the bulk of {GOE, $(k,1)$-checkerboard} is given by the limiting distribution of {GOE, $(k,0)$-checkerboard}. Because in a contributing cyclic product, every term $c_{i_\ell i_{\ell+1}}$ from the $(k,0)$-checkerboard must be non-weight with the modular restriction $i_\ell \not\equiv i_{\ell+1}$ (mod k), then the $2m^{\text{th}}$ bulk moment of {GOE, k-checkerboard} is essentially the $2m^{\text{th}}$ moment {GOE, GOE}, except that we have to account for all the modular restrictions. Since the $2m$ non-weight terms are paired together, the probability that all the terms from the $(k,0)$-checkerboard are non-weights is $\left(1-\frac{1}{k}\right)^m$. This completes the proof.

□

A.3. **Bulk Moments of $\{k$-checkerboard, j-checkerboard$\}$.**

Corollary A.3. *The $2m^{th}$ bulk moment of* {k-checkerboard, j-checkerboard} *is* $M_{2m} = 2\left(1-\frac{1}{k}\right)^m \left(1-\frac{1}{j}\right)^m f(m)$, *where*

$$f(m) \ = \ 2\sum_{j=1}^{m-1} g(j) f(m-j) + g(m) \qquad \text{(A.9)}$$

and

$$g(m) \ = \ 2f(m-1) + \sum_{\substack{0 \le x_1, x_2 \le m-2 \\ x_1+x_2 \le m-2}} (1+\mathbb{1}_{x_1>0})(1+\mathbb{1}_{x_2>0}) f(x_1) f(x_2) g(m-1-x_1-x_2). \qquad \text{(A.10)}$$

Proof. The proof is essentially the same as the proof of Proposition A.2. □

Appendix B. Proof of Multiple Regimes

In this section, we prove the existence of multiple regimes of eigenvalues for {GOE, k-checkerboard} and {k-checkerbord, j-checkerboard}. Our method involves decomposing each checkerboard matrix into the sum of its mean matrix and perturbation matrix and applying Weyl's inequality to bound the eigenvalue of the matrix ensemble in terms of the eigenvalue of its components. For the sake of simplicity, throughout this section we assume that the weight $w = 1$ and that $k|N$ for {GOE, k-checkerboard} and $jk|N$ for k{-checkerboard, j-checkerboard}.

Definition B.1 (Mean Matrix)**.** *The mean matrix $\overline{A}_N$ of the k-checkerboard matrix $A_N = (a_{ij})$ is given by*

$$\bar{a}_{ij} = \begin{cases} 0, & \text{if } i \not\equiv j \mod k \\ 1, & \text{if } i \equiv j \mod k. \end{cases} \qquad \text{(B.1)}$$

We note that the rank of $\overline{A}_N$ is k.

Definition B.2 (Perturbation Matrix)**.** *The perturbation matrix $\tilde{A}_N$ of the k-checkerboard matrix $A_N = \tilde{A}_N = (\tilde{a}_{ij})$ is given by*

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & \text{if } i \not\equiv j \mod k \\ 0, & \text{if } i \equiv j \mod k. \end{cases} \qquad \text{(B.2)}$$

Thus, we can write the k-checkerboard matrix simply as $A_N = \overline{A}_N + \tilde{A}_N$. As shown in [BCDHMSTPY], all the eigenvalues of $\tilde{A}_N$ are $O(N^{1/2})$. Moreover, $\overline{A}_N$ has k eigenvalues at $\frac{N}{k}$ and $N-k$ eigenvalues at 0.

Lemma B.3. *Let $\tilde{A}_N$ be the perturbation matrix as defined above and B_N an $N \times N$ GOE matrix, then with probability $1 - o(1)$, all the eigenvalues of $\{\tilde{A}_N, B_N\}$ are $O(N)$.*

Proof. We know that the maximum eigenvalue of a matrix is equal to the operator norm of the matrix. Since $||\tilde{A}_N|| = O(N^{1/2})$ and $||B_N|| = O(N^{1/2})$, then by submultiplicativity of the matrix norm, $||\tilde{A}_N B_N|| \leq ||\tilde{A}_N|| ||B_N|| = O(N)$. Similarly, $||B_N \tilde{A}_N|| = O(N)$. By Weyl's inequality, $\lambda_N(\{\tilde{A}_N, B_N\}) \leq \lambda_N(\tilde{A}_N B_N) + \lambda_N(B_N \tilde{A}_N) = O(N)$. The argument for the smallest negative eigenvalues follows from considering $-\tilde{A}_N$ and $-B_N$. □

Lemma B.4. *Let $\overline{A}_N$ be the mean matrix as defined above and B_N the $N \times N$ GOE matrix, then the largest eigenvalue of $\{\overline{A}_N, B_N\}$ is bounded above by $\frac{4N^{3/2}}{k}$, the smallest eigenvalue is bounded below by $-\frac{4N^{3/2}}{k}$, and there are at least $N - 2k$ eigenvalues at 0.*

Proof. First, observe that $\text{rank}(\overline{A}_N B_N) \leq \min(\text{rank}(\overline{A}_N), \text{rank}(B_N)) = k$. Similarly, $\text{rank}(B_N \overline{A}_N) \leq k$. By the subadditivity of rank, $\text{rank}(\{\overline{A}_N, B_N\}) \leq 2k$. Thus, at least $N - 2k$ eigenvalues are 0. For the highest eigenvalues, we see that $||\{\overline{A}_N, B_N\}|| \leq 2||\overline{A}_N|| ||B_N|| = 2 \cdot \frac{N}{k} \cdot 2N^{1/2} = \frac{4N^{3/2}}{k}$. Similarly, the smallest eigenvalue is bounded below by $-\frac{4N^{3/2}}{k}$. □

Empirically, we observe that $\overline{A}_N$ has k blip eigenvalues at $\frac{N^{3/2}}{k} + O(N)$ and k blip eigenvalues at $-\frac{N^{3/2}}{k} + O(N)$. By assuming this, we are able to prove the existence of multiple regimes for $\{A_N, B_N\}$, as follows:

Lemma B.5. *Let A_N be a k-checkerboard matrix and B_N an $N \times N$ GOE matrix, then $\{A_N, B_N\}$ has a blip containing k eigenvalues at $\frac{N^{3/2}}{k} + O(N)$, a blip containing k eigenvalues at $-\frac{N^{3/2}}{k} + O(N)$, and $N - 2k$ eigenvalues of $O(N)$.*

Proof. First note that we can write $\{A_N, B_N\} = \{\overline{A}_N, B_N\} + \{\tilde{A}_N, B_N\}$. By Weyl's inequality, we see that

$$\lambda_{N-k+1}(\{A_N, B_N\}) \ \geq \ \lambda_{N-k+1}(\{\overline{A}_N, B_N\}) + \lambda_1(\{\tilde{A}_N, B_N\}) \ = \ \frac{1}{k} N^{3/2} + O(N) \tag{B.3}$$

and

$$\lambda_N(\{A_N, B_N\}) \ \leq \ \lambda_N(\{\overline{A}_N, B_N\}) + \lambda_N(\{\tilde{A}_N, B_N\}) \ = \ \frac{1}{k} N^{3/2} + O(N). \tag{B.4}$$

So this proves the existence of k blip eigenvalues at $\frac{N^{3/2}}{k}$. Similarly, we can use Weyl's inequality to show the existence of blip eigenvalues at $-\frac{N^{3/2}}{k}$. For the bulk, we see that

$$\lambda_{N-k}(\{A_N, B_N\}) \ \leq \ \lambda_{N-k}(\{\overline{A}_N, B_N\}) + \lambda_N(\{\tilde{A}_N, B_N\}) \ = \ O(N), \tag{B.5}$$

and

$$\lambda_{k+1}(\{A_N, B_N\}) \ \geq \ \lambda_{k+1}(\{\overline{A}_N, B_N\}) + \lambda_1(\{\tilde{A}_N, B_N\}) \ = \ O(N). \tag{B.6}$$

This completes the proof for the existence of three different regimes. □

Now we consider $\{A_N, B_N\}$, where A_N is a k-checkerboard matrix and B_N is a j-checkerboard matrix. We assume $\gcd(k, j) = 1$, $N \mid kj$. Then we can write $\{A_N, B_N\} = \{\tilde{A}_N, \tilde{B}_N\} + \{\overline{A}_N, \tilde{B}_N\} + \{\tilde{A}_N, \overline{B}_N\} + \{\overline{A}_N, \overline{B}_N\}$. Similarly, we see that all eigenvalues of $\{\tilde{A}_N, \tilde{B}_N\}$ are of $O(N)$. Empirically, $\{\overline{A}_N, \tilde{B}_N\}$ has k eigenvalues at $\frac{1}{k}\sqrt{1 - \frac{1}{j}} N^{3/2}$, k eigenvalues at $-\frac{1}{k}\sqrt{1 - \frac{1}{j}} N^{3/2}$, and the remaining $N - 2k$ eigenvalues at 0. Heuristically, this can be seen from the fact that $\overline{A}_N$ has k eigenvalues at $\frac{1}{k} N$ and the eigenvalues of $\tilde{B}_N$ are bounded above and below by $\pm 2\sqrt{1 - \frac{1}{j}} N^{1/2}$. Similarly, empirically $\{\tilde{A}_N, \overline{B}_N\}$ has j eigenvalues at $\frac{1}{j}\sqrt{1 - \frac{1}{k}} N^{3/2}$, j eigenvalues at $-\frac{1}{j}\sqrt{1 - \frac{1}{k}} N^{3/2}$, and the remaining $N - 2j$ eigenvalues are at 0. For $\{\overline{A}_N, \tilde{B}_N\} + \{\tilde{A}_N, \overline{B}_N\}$, we observe that the largest eigenvalue is of $O(N^{3/2})$ but larger than $\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}$ and $\frac{1}{j}\sqrt{1-\frac{1}{j}}N^{3/2}$, the smallest eigenvalue is of $O(N^{3/2})$ but smaller than $-\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}$ and $-\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}$. Furthermore, There are $k-1$ eigenvalues at each of $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}$, and $j-1$ eigenvalues at each of $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}$, and the remaining $N-2k-2j+3$ eigenvalues of $O(N)$.

Lemma B.6. *Let $\overline{A}_N$ and $\overline{B}_N$ be average matrices as defined above, then $\{\overline{A}_N, \overline{B}_N\}$ has 1 eigenvalue exactly at $\frac{2N^2}{jk}$ and $N-1$ eigenvalues at 0.*

Proof. Since j and k are relatively prime, then from matrix multiplication, we see that $\overline{A}_N\overline{B}_N$ and $\overline{B}_N\overline{A}_N$ are both the constant matrix where every entry is $\frac{N}{kj}$. Hence, $\{\overline{A}_N, \overline{B}_N\}$ is the constant matrix where every entry is $\frac{2N}{kj}$. Such matrix has 1 eigenvalue exactly at $\frac{2N^2}{kj}$ and $N-1$ eigenvalues at 0. ☐

Lemma B.7. *Let A_N be an $N\times N$ k-checkerboard matrix, and B_N an $N\times N$ j-checkerboard matrix such that $\gcd(j,k)=1$ and $jk|N$. Assume without loss of generality that $2\le k<j$. Then the eigenvalues of $\{A_N, B_N\}$ are distributed as follows:*

(1) 1 *eigenvalue at* $\frac{2}{jk}N^2 + O(N^{3/2})$,
(2) $k-1$ *eigenvalues at each of* $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$,
(3) 1 *eigenvalue between* $\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$ *and* $\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$ *and* 1 *eigenvalue between* $-\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$ *and* $-\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$,
(4) $j-2$ *eigenvalues at* $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$,
(5) *1 eigenvalue between $O(N)$ (positive) and* $\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$ *and 1 eigenvalue between $O(N)$ (negative) and* $-\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$,
(6) *The remaining $N-2k-2j+1$ eigenvalues of $O(N)$.*

Proof. By assumption, we have $2\le k<j$, so $\frac{1}{k}\sqrt{1-\frac{1}{j}} > \frac{1}{j}\sqrt{1-\frac{1}{k}}$. Since $\{A_N, B_N\} = \{\tilde{A}_N, \tilde{B}_N\} + \{\overline{A}_N, \tilde{B}_N\} + \{\tilde{A}_N, \overline{B}_N\} + \{\overline{A}_N, \overline{B}_N\}$, then

$$\begin{aligned}\lambda_N(\{A_N, B_N\}) \ &\geq\ \lambda_N(\{\overline{A}, \overline{B}\}) + \lambda_1(\{\overline{A}, \tilde{B}\} + \{\tilde{A}, \overline{B}\} + \{\tilde{A}, \tilde{B}\}) \\ &\geq\ \lambda_N(\{\overline{A}, \overline{B}\}) + \lambda_1(\{\overline{A}, \tilde{B}\}) + \lambda_1(\{\tilde{A}, \overline{B}\}) + \lambda_1(\{\tilde{A}, \tilde{B}\}) \ =\ \frac{2N^2}{jk} + O(N^{3/2}).\end{aligned} \tag{B.7}$$

This establishes the existence of the largest blip. Then, we establish the existence of the intermediary blip containing $k-1$ eigenvalues at $\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$:

$$\begin{aligned}\lambda_{N-1}(\{A_N, B_N\}) \quad &\leq\ \lambda_{N-1}(\{\overline{A}_N, \overline{B}_N\}) + \lambda_N(\{\overline{A}, \tilde{B}\} + \{\tilde{A}, \overline{B}\} + \{\tilde{A}, \tilde{B}\}) \\ &\leq\ \lambda_N(\{\overline{A}_N, \tilde{B}_N\} + \{\tilde{A}_N, \overline{B}_N\}) + \lambda_N(\{\tilde{A}_N, \tilde{B}_N\}) \ =\ \frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2} + O(N), \\ \lambda_{N-k+1}(\{A_N, B_N\}) \ &\geq\ \lambda_1(\{\overline{A}_N, \overline{B}_N) + \lambda_{N-k+1}(\{\overline{A}, \tilde{B}\} + \{\tilde{A}, \overline{B}\} + \{\tilde{A}, \tilde{B}\}) \\ &\geq\ \lambda_{N-k+1}(\{\overline{A}_N, \tilde{B}_N\} + \{\tilde{A}_N, \overline{B}_N\}) + \lambda_1(\{\tilde{A}_N, \tilde{B}_N) = \frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2} + O(N).\end{aligned} \tag{B.8}$$

Next, we show that there is one eigenvalue between $\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$ and $\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N)$ as well as $j-2$ eigenvalues at $\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N)$:

$$\begin{aligned}
\lambda_{N-k}(\{A_N,B_N\}) &\leq \lambda_{N-k+1}(\{\overline{A}_N,\tilde{B}_N\}+\{\tilde{A}_N,\overline{B}_N\})+\lambda_{N-1}(\{\overline{A}_N,\overline{B}_N\}+\{\tilde{A}_N,\tilde{B}_N\}) \\
&= \frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2}+O(N), \\
\lambda_{N-k-1}(\{A_N,B_N\}) &\leq \lambda_{N-k}(\{\overline{A}_N,\tilde{B}_N\}+\{\tilde{A}_N,\overline{B}_N\})+\lambda_{N-1}(\{\overline{A}_N,\overline{B}_N\}+\{\tilde{A}_N,\tilde{B}_N\}) \\
&= \frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N), \\
\lambda_{N-k-j+2}(\{A_N,B_N\}) &\geq \lambda_{1}(\{\overline{A}_N,\overline{B}_N\}+\{\tilde{A}_N,\tilde{B}_N\})+\lambda_{N-j-k+2}(\{\overline{A}_N,\tilde{B}_N\}+\{\tilde{A}_N,\overline{B}_N\}) \\
&= \frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N). \qquad \text{(B.9)}
\end{aligned}$$

By symmetry argument, we can use Weyl's inequality to establish the existence of their negative counterpart. Finally, for the bulk, we have

$$\begin{aligned}
\lambda_{N-j-k+1}(\{A_N,B_N\}) &\leq \lambda_{N-j-k+2}(\{\overline{A}_N,\tilde{B}_N\}+\{\tilde{A}_N,\overline{B}_N\})+\lambda_{N-1}(\{\overline{A}_N,\overline{B}_N\}+\{\tilde{A}_N,\tilde{B}_N\}) \\
&\leq \frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2}+O(N), \\
\lambda_{N-j-k}(\{A_N,B_N\}) &\leq \lambda_{N-j-k+1}(\{\overline{A}_N,\tilde{B}_N\}+\{\tilde{A}_N,\overline{B}_N\})+\lambda_{N-1}(\{\overline{A}_N,\overline{B}_N\}+\{\tilde{A}_N,\tilde{B}_N\}) \\
&= O(N). \qquad \text{(B.10)}
\end{aligned}$$

By symmetry argument, we can use Weyl's inequality to bound the bulk from the below. This completes the proof. □

Note that in the proof of Lemma B.7, Weyl's inequality fails to provide an accurate bound on the four eigenvalues between different regimes. Empirically, we observe that among those eigenvalues, the positive ones belong to the regimes corresponding to their lower bound, and the negative ones belong to the regimes corresponding to their upper bound.

Appendix C. Almost Sure Convergence

The traditional way to show weak convergence of empirical spectral measures to a limiting spectral measure (in probability or almost-surely) is to show that the variance (resp. fourth moment) of the m^{th} moment, averaged over the $N\times N$ ensemble, is $O(\frac{1}{N})$ (resp. $O\left(\frac{1}{N^2}\right)$). In the case of the blip spectral measure, neither of these methods works properly. However, for a k-checkerboard matrix there are only k eigenvalues in the blip, so each blip spectral measure is just a collection of k isolated delta spikes distributed according to the limiting spectral computed in Theorem 3.18. As such, for fixed k the variance and fourth moment over the ensemble of the general m^{th} moment do not go to 0 which means we cannot use the general methods. We therefore define a modified spectral measure which averages over the eigenvalues of many matrices in order to extend standard techniques, in particular we link the moments to the moments of the $k\times k$ Gaussian Wigner matrix using similar methods as [BCDHMSTPY].

In order to facilitate the proof of the main convergence result (Theorem C.5) we first introduce some new notation. In all that follows we fix k and suppress k-dependence in our notation. Let Ω_N be the probability space of $N\times N$ k-checkerboard matrices with the natural probability measure. Then we define the product probability space

$$\Omega \ := \ \prod_{N\in\mathbb{N}} \Omega_N. \tag{C.1}$$

By Kolmogorov's extension theorem (see [Tao2]), this is equipped with a probability measure which agrees with the probability measures on Ω_N when projected to the N^{th} coordinate. Given $\{A_N\}_{N\in\mathbb{N}} \in \Omega$, we denote by A_N the $N \times N$ matrix given by projection to the N^{th} coordinate. In what follows, we suppress the subscript $N \in \mathbb{N}$ on elements of Ω, writing them as $\{A_N\}$.

Remark C.1. [KKMSX] *employs a similar construction using product space, while* [HM] *views elements of* Ω *as infinite matrices and the projection map* $\Omega \to \Omega_N$ *as simply choosing the upper left* $N \times N$ *minor.*

Previously we treated the m^{th} moment of an empirical spectral measure $\mu_{A,N}^{(m)}$ as a random variable on Ω_N, but we may equivalently treat it as a random variable on Ω. To highlight this, we define the random variable $X_{m,N}$ on Ω

$$X_{m,N}(\{A_N\}) \ := \ \mu_{A_N,N}^{(m)}. \tag{C.2}$$

These have centered r^{th} moment

$$X_{m,N}^{(r)} \ := \ \mathbb{E}[(X_{m,N} - \mathbb{E}[X_{m,N}])^r]. \tag{C.3}$$

Per our motivating discussion at the beginning of this section, because we wish to average over a growing number of matrices of the same size, it is advantageous to work over $\Omega^{\mathbb{N}}$; this again is equipped with a natural probability measure by Kolmogorov's extension theorem. Its elements are sequences of sequences of matrices, and we denote them by $\overline{A} = \{A^{(i)}\}_{i\in\mathbb{N}}$ where $A^{(i)} \in \Omega$. We now give a more abstract definition of the averaged blip spectral measure.

Definition C.2. *Fix a function* $g : \mathbb{N} \to \mathbb{N}$*. The* ***averaged empirical blip spectral measure*** *associated to* $\overline{A} \in \Omega^{\mathbb{N}}$ *is*

$$\mu_{N,g,\overline{A}} \ := \ \frac{1}{g(N)} \sum_{i=1}^{g(N)} \mu_{A_N^{(i)},N}. \tag{C.4}$$

In other words, we project onto the N^{th} coordinate in each copy of Ω and then average over the first $g(N)$ of these $N \times N$ matrices.

Remark C.3. *If one wishes to avoid defining an empirical spectral measure which takes eigenvalues of multiple matrices, one may use the (rather contrived) construction of a* $\mathbb{N} \times \mathbb{N}$ *block matrix with independent* $N \times N$ *checkerboard matrix blocks.*

Analogously to $X_{m,N}$, we denote by $Y_{m,N,g}$ the random variable on $\Omega^{\mathbb{N}}$ defined by the moments of the averaged empirical blip spectral measure

$$Y_{m,N,g}(\overline{A}) \ := \ \mu_{N,g,\overline{A}}^{(m)}. \tag{C.5}$$

The centered r^{th} moment (over $\Omega^{\mathbb{N}}$) of this random variable is denoted by $Y_{m,N,g}^{(r)}$.

We now prove almost-sure weak convergence of the averaged blip spectral measures under a growth assumption on g. Recall the following definition.

Definition C.4. *A sequence of random measures* $\{\mu_N\}_{N\in\mathbb{N}}$ *on a probability space* Ω *converges* ***weakly almost-surely*** *to a fixed measure* μ *if, with probability* 1 *over* $\Omega^{\mathbb{N}}$*, we have*

$$\lim_{N\to\infty} \int f d\mu_N = \int f d\mu \tag{C.6}$$

for all $f \in \mathcal{C}_b(\mathbb{R})$ *(continuous and bounded functions).*

Theorem C.5. *Let $g : \mathbb{N} \to \mathbb{N}$ be such that there exists an $\delta > 0$ for which $g(N) = \omega(N^{\delta})$. Then, as $N \to \infty$, the averaged empirical spectral measures $\mu_{N,g,\overline{A}}$ of the k-checkerboard ensemble converge weakly almost-surely to the measure with moments $M_{k,m} = \frac{1}{k}\mathbb{E}Tr[B^m]$, the limiting expected moments computed in Theorem 3.18.*

Proof. For simplicity of notation, we suppress k and denote $M_{k,m}$ by M_m. By the triangle inequality, we have

$$|Y_{m,N,g} - M_m| \ \leq \ |Y_{m,N,g} - \mathbb{E}[Y_{m,N,g}]| + |\mathbb{E}[Y_{m,N,g}] - M_m|. \tag{C.7}$$

From Theorem 3.18, we know that $\mathbb{E}[X_{m,N}] \to M_m$, and it follows that $\mathbb{E}[Y_{m,N,g}] \to M_m$. Hence to show that $Y_{m,N,g} \to M_m$ almost surely, it suffices to show that $|Y_{m,N,g} - \mathbb{E}[Y_{m,N,g}]| \to 0$ almost surely as $N \to \infty$. We show that the limit as $N \to \infty$ of all moments over Ω_N of any arbitrary moment of the empirical spectral measure exists, and that we may always choose a sufficiently high moment[2] such that the standard method of Chebyshev's inequality and the Borel-Cantelli lemma gives that $|Y_{m,N,g} - \mathbb{E}[Y_{m,N,g}]| \to 0$. Finally, the moment convergence theorem gives almost-sure weak convergence to the limiting averaged blip spectral measure.

Lemma C.6. *Let $X_{m,N}$ be as defined in* (C.2). *Then for any $t \in \mathbb{N}$, the r^{th} centered moment of $X_{m,N}$ satisfies*

$$X_{m,N}^{(r)} \ = \ \mathbb{E}[(X_{m,N} - \mathbb{E}[X_{m,N}])^r] \ = \ O_{m,r}(1) \tag{C.8}$$

as N goes to infinity.

Proof. Firstly, we have

$$\begin{aligned}
\mathbb{E}[(X_{m,N} - \mathbb{E}[X_{m,N}])^r] \ &= \ \mathbb{E}\left[\sum_{\ell=0}^{r}\binom{r}{\ell}(X_{m,N})^{\ell}\,(\mathbb{E}[X_{m,N}])^{r-\ell}\right] \\
&= \ \sum_{\ell=0}^{r}\binom{r}{\ell}\mathbb{E}\left[(X_{m,N})^{\ell}\right](\mathbb{E}[X_{m,N}])^{r-\ell}.
\end{aligned} \tag{C.9}$$

From the moments given in Section 3.18, we have $\mathbb{E}[X_{m,N}] = O_m(1)$ hence $(\mathbb{E}[X_{m,N}])^{r-\ell} = O_{m,r,\ell}(1)$ for all ℓ. As such, it suffices to show that $\mathbb{E}[(X_{m,N})^{\ell}] = O_{m,\ell}(1)$. By (3.18), we have that

$$\begin{aligned}
&\mathbb{E}[X_{m,N,1}]^l \\
&= \ \mathbb{E}\left[\frac{1}{k_1}\sum_{\alpha=2n}^{8n} c_\alpha\left(\frac{k}{N^{3/2}}\right)^{\alpha}\left(\frac{1}{N^m}\sum_{i=0}^{m}\binom{m}{i}\left(-\frac{N^{3/2}}{k}\right)^{m-i}\mathrm{Tr}[\{A_N,B_N\}^{\alpha+i}]\right)^l\right] \\
&= \ \mathbb{E}\left[\sum_{\substack{2n\le\alpha_1\le 8n\\0\le i_1\le m}}\ \sum_{\substack{2n\le\alpha_2\le 8n\\0\le i_2\le m}}\cdots\sum_{\substack{2n\le\alpha_l\le 8n\\0\le i_l\le m}}\frac{1}{N^{m\ell}}\prod_{\nu=1}^{l}c_{\alpha_v}\binom{m}{i_v}(-1)^{m-i_v}\left(\frac{N^{3/2}}{k}\right)^{l-i_v}\mathrm{Tr}[\{A_N,B_N\}^{\alpha+i_v}]\right] \\
&= \ \sum_{\substack{2n\le\alpha_1\le 8n\\0\le i_1\le m}}\ \sum_{\substack{2n\le\alpha_2\le 8n\\0\le i_2\le m}}\cdots\sum_{\substack{2n\le\alpha_l\le 8n\\0\le i_l\le m}}\frac{1}{N^{m\ell}}\prod_{\nu=1}^{l}c_{\alpha_v}\binom{m}{i_v}(-1)^{m-i_v}\left(\frac{N^{3/2}}{k}\right)^{l-i_v}\mathbb{E}\left[\prod_{v=1}^{l}\mathrm{Tr}[\{A_N,B_N\}^{\alpha+i_v}]\right].
\end{aligned} \tag{C.10}$$

[2]Note the difference between this and the standard technique of, for instance, [HM], which uses only the fourth moment.

Now, recall that

$$\mathbb{E}\left[\prod_{v=1}^{\ell}\mathrm{Tr}A^{2n+i_v}\right] = \sum_{\alpha_1^1,\dots,\alpha_{2n+i_1}^1\le N}\cdots\sum_{\alpha_1^\ell,\dots,\alpha_{2n+i_\ell}^\ell\le N}\mathbb{E}\left[\prod_{j=1}^{\ell}a_{\alpha_1^j,\alpha_2^j}\dots a_{\alpha_{2n+i_j}^j,\alpha_1^j}\right]. \tag{C.11}$$

We have now reached a combinatorial problem similar to the one we encounter in Section 2. For each j, since the length of the cyclic product $a_{\alpha_1^j,\alpha_2^j}\dots a_{\alpha_{2n+i_j}^j,\alpha_1^j}$ is fixed at $2n+i_j$, we can choose the number of blocks (determining the class), the location of the blocks (determining the configuration), the matchings and indexings. By Lemma 3.14 and 3.15, we have that the main contribution from configurations of length $(2n+i_j)$ in B_j-class is $\frac{(2n+i_j)^{B_j}}{B_j!}$. By the same arguments made in §2, the number of ways we can choose the number of blocks having one a and two a's as well as the number of ways to choose matchings across the ℓ cyclic products are independent of N, j's and i_j's, so for simplicity, we are denoting them as C. Finally, the contribution from choosing the indices of all the blocks and w's is $O_k(N^{2n\ell+i_1+\cdots+i_\ell-B_1-\cdots-B_\ell})$. As such, if $B_1,\dots,B_\ell\ge m$, the total contribution is $O_{m,k}(1)$. If there exists $B_{j'}<m$, then the overall contribution is

$$CN^{\ell m-B_1-\cdots-B_\ell}\prod_{u=1}^{\ell}\left[\sum_{j_u=0}^{2n}\binom{2n}{j_u}\sum_{i_u=0}^{m+j_u}\binom{m+j_u}{i_u}(-1)^{m-i_u}\frac{(2n+i_u)^{B_u}}{B_u!}\right] = 0, \tag{C.12}$$

since the sum over $j_u=j'$ is equal to 0 by by Lemma 3.17. As such, the total contribution of $\mathbb{E}[X_{m,N}^\ell]$ is simply $O_{m,\ell}(1)$ (suppressing k), as desired. □

We apply the following theorem (Theorem 1.2 of [Fer]) with $X=X_{m,N}-\mathbb{E}[X_{m,N}]$, $s=g(N)$ and $\mu_i=X_{m,N}^{(i)}$.

Theorem C.7. *Let $r\in\mathbb{N}$ and let $X_1,\dots,X_s$ be i.i.d. copies of some mean-zero random variable X with absolute moments $\mathbb{E}[|X|^\ell]<\infty$ for all $\ell\in\mathbb{N}$. Then*

$$\mathbb{E}\left[\left(\sum_{i=1}^{s}X_i\right)^r\right] = \sum_{1\le m\le\frac{r}{2}}B_{m,r}(\mu_2,\mu_3,\dots,\mu_r)\binom{s}{m} \tag{C.13}$$

where the μ_i are the moments of X and $B_{m,r}$ is a function independent of s, the details of which are given in [Fer].

We must first show boundedness of the absolute moments of $X_{m,N}$. By Cauchy-Schwarz,

$$\left(\int|x^{2\ell+1}|d\mu_{X_{m,N}}\right)^2 \le \int|x|^2d\mu_{X_{m,N}}\cdot\int|x|^{4\ell}d\mu_{X_{m,N}}, \tag{C.14}$$

where $\mu_{X_{m,N}}$ is the probability measure on Ω given by the density of $X_{m,N}$. Since, for fixed N, the even moments of $X_{m,N}$ are finite by (C.10), the previous bound shows that all odd absolute moments are finite as well. Hence Theorem C.7 applies, yielding

$$\mathbb{E}\left[\left(\sum_{i=1}^{g(N)}X_{m,N,i}-\mathbb{E}[X_{m,N,i}]\right)^r\right] = \sum_{1\le m\le\frac{r}{2}}B_{m,r}(X_{m,N}^{(2)},X_{m,N}^{(3)},\dots,X_{m,N}^{(r)})\binom{g(N)}{m}. \tag{C.15}$$

where the $X_{m,N,i}$ are i-indexed i.i.d. copies of $X_{m,N}$. By Lemma C.6, for sufficiently high N, $X_{m,N}^{(t)}$ are uniformly bounded above by some constant K for $1\le t\le m$, so there exists C such that

$B_{m,r}(X^{(2)}_{m,N}, X^{(3)}_{m,N}, \dots, X^{(r)}_{m,N}) < C$ for all sufficiently large N and for all $1 \le m \le r/2$. Hence

$$\mathbb{E}\left[\left(\sum_{i=1}^{g(N)} X_{m,N,i} - \mathbb{E}[X_{m,N,i}]\right)^r\right] \le \sum_{1 \le m \le \frac{r}{2}} C\binom{g(N)}{m}. \tag{C.16}$$

As such, we have

$$Y^{(r)}_{m,N,g} = \frac{1}{g(N)^r}\mathbb{E}\left[\left(\sum_{i=1}^{g(N)} X_{m,N,i} - \mathbb{E}[X_{m,N,i}]\right)^r\right] \le \sum_{1 \le m \le \frac{r}{2}} \frac{C}{g(N)^r}\binom{g(N)}{m} = O\left(\frac{1}{g(N)^{r/2}}\right). \tag{C.17}$$

Since $g(N) = \omega(N^\delta)$, we may choose r sufficiently large so that

$$Y^{(r)}_{m,N,g} \ = \ O\left(\frac{1}{N^2}\right). \tag{C.18}$$

Then by Chebyshev's inequality,

$$\mathbb{P}(|Y_{m,N,g} - \mathbb{E}[Y_{m,N,g}]| > \epsilon) \ \le \ \frac{\mathbb{E}[(Y_{m,N,g} - \mathbb{E}[Y_{m,N,g}])^r]}{\epsilon^r} \ = \ \frac{Y^{(r)}_{m,N,g}}{\epsilon^r} \ = \ O\left(\frac{1}{N^2}\right). \tag{C.19}$$

We now apply the following (see for example [Can]).

Lemma C.8 (Borel-Cantelli)**.** *Let B_i be a sequence of events with $\sum_i \mathbb{P}(B_i) < \infty$. Then*

$$\mathbb{P}\left(\bigcap_{j=1}^{\infty}\bigcup_{\ell=j}^{\infty} B_\ell\right) \ = \ 0. \tag{C.20}$$

Define the events

$$B_N^{(m,d,g)} \ := \ \left\{A \in \Omega^{\mathbb{N}} : |Y_{m,N,g}(A) - \mathbb{E}[Y_{m,N,g}]| \ \ge \ \frac{1}{d}\right\}. \tag{C.21}$$

Then $\mathbb{P}(B_N^{(m,d,g)}) \ \le \ \frac{C_m d^r}{N^2}$, so for fixed m, d, the conditions of the Borel-Cantelli lemma are satisfied. Hence

$$\mathbb{P}\left(\bigcap_{j=1}^{\infty}\bigcup_{\ell=j}^{\infty} B_\ell^{(m,d,g)}\right) \ = \ 0. \tag{C.22}$$

Taking a union of these measure-zero sets over $d \in \mathbb{N}$ we have

$$\mathbb{P}\left(Y_{m,N,g} \neq \mathbb{E}[Y_{m,N,g}] \text{ for infinitely many } N\right) \ = \ 0, \tag{C.23}$$

and taking the union over $m \in \mathbb{Z}_{\ge 0}$,

$$\mathbb{P}\left(\exists m \text{ such that } Y_{m,N,g} \neq \mathbb{E}[Y_{m,N,g}] \text{ for infinitely many } N\right) \ = \ 0. \tag{C.24}$$

Therefore with probability 1 over $\Omega^{\mathbb{N}}$, $|Y_{m,N,g} - \mathbb{E}[Ym,N,g]| \to 0$ for each m. This, together with (C.7) and the discussion following it, yields that the moments $\mu^{(m)}_{N,g} = Y_{m,N,g} \to M_m$ almost surely. We now use the following to show almost-sure weak convergence of measures (see for example [Ta]).

Theorem C.9 (Moment Convergence Theorem)**.** *Let μ be a measure on $\mathbb{R}$ with finite moments $\mu^{(m)}$ for all $m \in \mathbb{Z}_{\ge 0}$, and $\mu_1, \mu_2, \dots$ a sequence of measures with finite moments $\mu_n^{(m)}$ such that $\lim_{n\to\infty}\mu_n^{(m)} = \mu^{(m)}$ for all $m \in \mathbb{Z}_{\ge 0}$. If in addition the moments $\mu^{(m)}$ uniquely characterize a measure, then the sequence μ_n converges weakly to μ.*

To show Carleman's condition is satisfied for the limiting moments M_m, we show that M_m are bounded above by the moments of the Gaussian. The odd moments vanish, and by Theorem 3.18 the even moments are given by

$$M_{2m} \;=\; \frac{2}{k}\mathbb{E}[\text{Tr } A^{2m}] \;=\; \sum_{1\le i_1,\dots,i_{2m}\le k} 2\mathbb{E}[a_{i_1i_2}a_{i_2i_3}\dots a_{i_{2m}i_1}], \tag{C.25}$$

and as $\mathbb{E}[a_{i_1i_2}a_{i_2i_3}\dots a_{i_ni_1}]$ is maximized when all $a_{i_\ell i_{\ell+1}}$ are equal,

$$M_{2m} \;\le\; \sum_{1\le i_1,\dots,i_{2m}\le k} 2(2m-1)!! \;=\; 2k^{2m}(2m-1)!!. \tag{C.26}$$

These are bounded by the moments of $\mathcal{N}(0,2k)$ so Carleman's condition is satisfied, thus we let $\overline{\mu}$ be the unique measure determined by the moments M_m. Choose $\overline{A}\in\Omega^{\mathbb{N}}$. Then the preceding argument showed that, with probability 1 over $\overline{A}$ chosen from $\Omega^{\mathbb{N}}$, all moments $\mu^{(m)}_{N,g,\overline{A}}$ of the measures $\mu_{N,g,\overline{A}}$ converge to M_m. Then by Theorem C.9 the measures $\mu_{N,g,\overline{A}}$ converge weakly to $\overline{\mu}$ with probability 1, completing the proof. □

Appendix D. Almost-Sure Convergence of the Bulk

Theorem D.1. *For A_N and B_N both $N\times N$ $(k,0)$-checkerboard matrices we get that for any fixed ℓ*

$$\lim_{N\to\infty} \mathit{Var}[\nu^{(\ell)}_{\{A_N,B_N\}}] \;=\; O\left(\frac{1}{N^2}\right). \tag{D.1}$$

Proof. We know that by the eigenvalue trace lemma, we have

$$\begin{aligned}\text{Var}\left[\nu^{(\ell)}_{\{A_N,B_N\}}\right] \;&=\; \left|\mathbb{E}[(\nu_{\{A_N,B_N\}^{(\ell)})^2}] - \left[\mathbb{E}[\nu_{\{A_N,B_N\}}]\right]^2\right| \\ &= \frac{1}{N^{2\ell+2}}\left|\mathbb{E}[\text{Tr}(\{A_N,B_N\}^\ell)^2] - (\mathbb{E}[\text{Tr}\{A_N,B_N\}^\ell])^2\right| \;=\; \frac{1}{N^{2\ell+2}}\sum_{I,I'}|\mathbb{E}[\zeta_I\zeta_{I'}] - \mathbb{E}[\zeta_I]\mathbb{E}[\zeta_{I'}]|\end{aligned} \tag{D.2}$$

where the ζ_I and $\zeta_{I'}$ are stand ins for a product $c_{i_1,i_2}c_{i_2,i_3}\cdots c_{i_{2\ell},i_1}$ where every c is a or b and is some expansion of $(AB+BA)^\ell$. So then we know that from the proof in [BCDHMSTPY] that for any choices of A's and B's this is $O(1/N^2)$ and since we consider ℓ as fixed we know that there are 2^ℓ different configurations which are constant and for each configuration from the paper we know that they are $O_m(1/N^2)$ which means that if we add up all of these different cases and configurations we get that it is still $O(1/N^2)$ which proves the theorem. This theorem proves convergence when combined with Chebyshev's inequality and the Borel-Cantelli lemma. □

By Chebyshev's inequality we get the first inequality and by Theorem D.1 we get that the sum of variances is finite, giving,

$$\sum_{N=1}^{\infty}\Pr\left(\left|\nu^{(\ell)}_{\{A_N,B_N\}} - \mathbb{E}[\nu^{(\ell)}_{\{A_N,B_N\}}] > \epsilon\right|\right) \;\le\; \frac{1}{\epsilon^2}\sum_{N=1}^{\infty}\text{Var}(\nu_{\{A_N,B_N\}}) \;<\; \infty. \tag{D.3}$$

So then by the Borel-Cantelli lemma and Theorem D.1 we get that the moments converge almost surely.

Theorem D.2. *For A_N and B_N both $N\times N$ palindromic Toeplitz matrices we get that for any fixed ℓ*

$$\lim_{N\to\infty}\mathbb{E}[|M_m(\{A_N,B_N\}) - \mathbb{E}[M_m(\{A_N,B_N\})]|^4] \;=\; O_m\left(\frac{1}{N^2}\right) \tag{D.4}$$

Proof. Expanding this yields

$$\begin{aligned}&\mathbb{E}[M_m(\{A_N,B_N\})^4] \;-\; 4\mathbb{E}[M_m(\{A_N,B_N\})^3]\mathbb{E}[M_m(\{A_N,B_N\})] \\ &\quad+\; 6\mathbb{E}[M_m(\{A_N,B_N\})^2]\mathbb{E}[M_m(\{A_N,B_N\})]^2 \;-\; 3\mathbb{E}[M_m(\{A_N,B_N\})]\mathbb{E}[M_m(\{A_N,B_N\})]^3\end{aligned} \tag{D.5}$$

As the odd moments are all 0 their expected value is always 0l so we only need to consider even moments. So then we can write the terms as

$$\mathbb{E}[M_{2m}(\{A_N, B_N\})^4] \;=\; \frac{1}{N^{8m+4}} \sum_i \sum_j \sum_k \sum_\ell \mathbb{E}[c_{is}c_{js}c_{ks}c_{\ell s}] \tag{D.6}$$

where

$$\mathbb{E}[c_{is}c_{js}c_{\ell s}c_{ks}] = \mathbb{E}[c_{|i_1-i_2|}\cdots c_{|i_{4m}-i_1|}c_{|j_1-j_2|}\cdots c_{|j_{4m}-j_1|}c_{|k_1-k_2|}\cdots c_{|k_{4m}-k_1|}c_{|\ell_1-\ell_2|}\cdots c_{|\ell_{4m}-\ell_1|}], \tag{D.7}$$

where the c's are all a's or b's and a's can only match with a's and b's can only match with b's, however, it suffices to allow matches to be free since the upper bound of $O(1/N^2)$ holds in any case. Then we know that from [HM], given a fixed expansion of a's and b's of the terms in the binomial expansion are $O(N^{8m+2})$. So then since we know that there are 2^{8m} expansions of the anticommutator and m is a fixed constant we get that the total contribution of all of the terms in the anticommutator expansion are $2^{8m} \cdot O(N^{8m+2}) = O(N^{8m+2})$ since 2^{8m} is a fixed constant. $\square$

Remark D.3. *Note that the proof of Theorem D.2 also applies when A_N is a palindromic Toeplitz and B_N is a GOE. This is due to the fact that all matchings of a palindromic Toeplitz matrix and GOE anticommutator are valid in the palindromic Toeplitz case, so whenever a degree of freedom is lost in the palindromic Toeplitz and palindromic-Toeplitz anticommutator it is also lost in the palindromic Toeplitz - GOE case since the GOE case is strictly more restricted and having the same indices implies their differences are equal. So the same argument proves convergence for the case where A_N is palindromic Toeplitz and B_N is a GOE.*

Appendix E. Polynomial Weight Functions for Intermediary Blips

In theory, the exact choice of the polynomial weight function shouldn't affect the moment of the intermediary blips, as long as it satisfies all the required conditions. For the sake of completeness, we include here the expression for the weight functions for the intermediary blips of $\{k$-checkerboard, j-checkerboard$\}$. Let $w_s = \frac{(-1)^{s+1}}{k}\sqrt{1-\frac{1}{j}}$ for $s \in \{1,2\}$ and $w_s = \frac{(-1)^{s+1}}{j}\sqrt{1-\frac{1}{k}}$ for $s \in \{3,4\}$. Then, the weight function for the intermediary blip at $w_s N^{3/2}$ is

$$g_s^{2n}(x) \;=\; \frac{x^{2n}\prod_{i=1;i\neq s}^{4}\left(x-\frac{w_i}{w_s}\right)^{2n}\left(x-\frac{w_5\sqrt{N}}{w_s}\right)^{10n}(x-A)^{2n}}{B^{2n}\left(1-\frac{w_5\sqrt{N}}{w_s}\right)^{10n}(1-A)^{2n}}, \tag{E.1}$$

where $A = 1+\left(1+\sum_{i=1;i\neq s}^{4}\frac{1}{1-w_i/w_s}\right)^{-1}$ and $B = \prod_{i=1;i\neq s}^{4}\left(1-\frac{w_i}{w_s}\right)$. It's clear that g_s^{2n} has zeros of order $2n$ at 0, $\frac{w_i}{w_s}$ for $i \neq s$, and zero of order $10n$ at $\frac{w_5\sqrt{N}}{w_s}$. We shall prove that g_s^{2n} vanishes at $O\left(\frac{1}{\sqrt{N}}\right)$, $\frac{w_i}{w_s}+O\left(\frac{1}{\sqrt{N}}\right)$ for $i \neq s$, and $\frac{w_5\sqrt{N}}{w_s}+O\left(\frac{1}{\sqrt{N}}\right)$, is equal to 1 at $1+O\left(\frac{1}{\sqrt{N}}\right)$, and has a critical point at 1. The key to the proof is the evaluation of the expression $\lim_{N\to\infty}(1+O(1/\sqrt{N}))^{2n}$. Since $\lim_{y\to\infty}(1+1/y)^y = e$ and $n \ll \log\log(N)$, then $1 \leq \lim_{N\to\infty}(1+O(1/\sqrt{N}))^{2n} \leq \lim_{N\to\infty}\left((1+O(1/\sqrt{N}))^{O(\sqrt{N})}\right)^{2n/O(\sqrt{N})} = \lim_{N\to\infty} e^{2n/O(\sqrt{N})} = 1$. Hence, $\lim_{N\to\infty}(1+O(1/\sqrt{N}))^{2n} = 1$.

Now, we first evaluate the function at $x = 1 + O\left(\frac{1}{N}\right)$ as $N \to \infty$,

$$\begin{aligned}
&\lim_{N\to\infty} g_s^{2n}\left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \\
&= \lim_{N\to\infty} \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)^{2n} \cdot \frac{\left(1 + O\left(\frac{1}{\sqrt{N}}\right) - \frac{w_5\sqrt{N}}{w_s}\right)^{10n}}{\left(1 - \frac{w_5\sqrt{N}}{w_s}\right)^{10n}} \cdot \frac{\left(1 + O\left(\frac{1}{\sqrt{N}}\right) - A\right)^{2n}}{(1-A)^{2n}} \\
&\cdot \frac{\prod_{i=1;i\neq s} 4\left(1 + O\left(\frac{1}{\sqrt{N}}\right) - \frac{w_i}{w_s}\right)^{2n}}{B^{2n}} \\
&= \lim_{N\to\infty} \frac{\prod_{i=1;i\neq s}^{4}\left(1 + O\left(\frac{1}{\sqrt{N}}\right) - \frac{w_i}{w_s}\right)^{2n}}{B^{2n}} \;=\; 1.
\end{aligned} \tag{E.2}$$

Note that we repeatedly apply the evaluation $\lim_{N\to\infty}(1 + O(1/\sqrt{N}))^{2n} = 1$ above to simplify the expression. As an example, $\left(\frac{1+O(1/\sqrt{N})-A}{1-A}\right)^{2n} = \left(1 + \frac{O(1/\sqrt{N})}{1-A}\right)^{2n} = \left(1 + O(1/\sqrt{N})\right)^{2n} = 1$.

Then, we consider the evaluation of the function at $O\left(\frac{1}{\sqrt{N}}\right)$, and the evaluation of the function at other vanishing points similarly follows,

$$\begin{aligned}
&\lim_{N\to\infty} g_s^{2n}\left(O\left(\frac{1}{\sqrt{N}}\right)\right) \\
&= \lim_{N\to\infty} \frac{\prod_{i=1;i\neq s}^{4}\left(O\left(\frac{1}{\sqrt{N}}\right) - \frac{w_i}{w_s}\right)^{2n}}{B^{2n}} \cdot \frac{\left(O\left(\frac{1}{\sqrt{N}}\right) - \frac{w_5\sqrt{N}}{w_s}\right)^{10n}}{\left(1 - \frac{w_5\sqrt{N}}{w_s}\right)^{10n}} \cdot \frac{\left(O\left(\frac{1}{\sqrt{N}}\right) - A\right)^{2n}}{(1-A)^{2n}} \cdot \left(O\left(\frac{1}{\sqrt{N}}\right)\right)^{2n} \\
&= \lim_{N\to\infty} C_N \cdot \left(O\left(\frac{1}{\sqrt{N}}\right)\right)^{2n} \;=\; 0.
\end{aligned} \tag{E.3}$$

Finally, we prove that g_s^{2n} has a critical point at 1. Using logarithmic derivative, we obtain

$$\frac{(g_s^{2n})'(x)}{g_s^{2n}(x)} \;=\; 2n \cdot \left(\frac{1}{x} + \sum_{i=1;i\neq s}^{4} \frac{1}{x - \frac{w_i}{w_s}} + \frac{5}{x - \frac{w_5\sqrt{N}}{w_s}} + \frac{1}{x - A}\right) \tag{E.4}$$

Since $g_s^{2n}(1) = 1$ and $A = 1 + \left(1 + \sum_{i=1;i\neq s}^{4} \frac{1}{1 - \frac{w_i}{w_s}}\right)$, then

$$(g_s^{2n})'(1) \;=\; \lim_{N\to\infty} 2n \cdot \left(1 + \sum_{i=1;i\neq s} \frac{1}{1 - \frac{w_i}{w_s}} + \frac{5}{1 - \frac{w_5\sqrt{N}}{w_s}} + \frac{1}{1 - A}\right) \;=\; \lim_{N\to\infty} \frac{10n}{1 - \frac{w_5\sqrt{N}}{w_s}} \;=\; 0. \tag{E.5}$$

Thus, g_s^{2n} is the desired weight function for the intermediary blip at $w_s N^{3/2}$.

Appendix F. Lower even moments of $\{\text{GOE}, k\text{-BCE}\}$ and $\{k\text{-BCE}, k\text{-BCE}\}$

We provide here explicit expression of lower even moments of $\{\text{GOE}, k\text{-BCE}\}$ and $\{k\text{-BCE}, k\text{-BCE}\}$ based on genus expansion formulae, where distributions are rescaled and normalized to have mean zero and variance one. This means that the second method of both distributions are 1. Theoretically, with enough computing power, we should be able to obtain closed form expressions for any even moments for the two distribution. However, in reality, the computation is extremely complicated and time-consuming. Hence, we only provide the moments of $\{\text{GOE}, k\text{-BCE}\}$ up to the 10^{th} moment and $\{k\text{-BCE}, k\text{-BCE}\}$ up to the 8^{th} moment.

TABLE 2. Lower Even Moments of $\{\text{GOE}, k\text{-BCE}\}$

Moment	Value
4^{th}	$\frac{5}{2} + \frac{1}{2k^2}$
6^{th}	$\frac{33}{4} + \frac{19}{4k^2}$
8^{th}	$\frac{249}{8} + \frac{34}{k^2} + \frac{27}{8k^4}$
10^{th}	$\frac{2033}{16} + \frac{875}{4k^2} + \frac{1043}{16k^4}$

TABLE 3. Lower Even Moments of $\{k\text{-BCE}, k\text{-BCE}\}$

Moment	Value
4^{th}	$\frac{10k^4 + 86k^2 + 48}{4k^4 + 8k^2 + 4}$
6^{th}	$\frac{66k^6 + 1890k^4 + 9084k^2 + 3360}{8k^6 + 24k^4 + 24k^2 + 8}$
8^{th}	$\frac{498k^8 + 33236k^6 + 529634k^4 + 1759064k^2 + 499968}{16k^8 + 64k^6 + 96k^4 + 64k^2 + 16}$

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