The General Moment of Anticommutator Products involving Block Circulant Matricies

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Definition 1 (Equivalence relation $\approx$ and $\simeq$)**.** *$(i, j) \approx (i', j')$ if and only if*

$$i = j' \quad \text{and} \quad j = i' \tag{1}$$

Also, $(i, j) \simeq (i', j')$ if and only if

$$i - j \equiv j' - i \mod N \tag{2}$$
$$i \equiv j' \quad \text{and} \quad j \equiv i' \mod m \tag{3}$$

The value of N, m are implied from context.

Definition 2 (Product Words)**.** *A product word of length $2k$ is composed of k blocks, where each block is one of $\{AB, BA\}$. We denote the set of all product words of length $2k$ as $PW(2k)$.*

For example, when $k = 3$,

$$W = AB\,BA\,AB\,BA \in PW(8)$$

is an example of a product word of length 6. To refer to the specific index of the word, use the superscript. For example, $W^3 = B$.

Definition 3 (Combining pairings)**.** *Suppose we are given $W \in PW(4k)$ and two pairings $\pi, \delta \in \mathcal{P}[2k]$. We denote the combined pairing of π, δ with respect to the product word W as*

$$\pi *_W \delta$$

where the combined pairing denotes an element in $\mathcal{P}[4k]$ where the composition between A's are specified by π and composition between B's are specified by δ.

For example if

$$\pi = (12)(34) \quad \text{and} \quad \delta = (12)(34)$$

the combined pairing is

$$\pi *_W \delta = (14)(23)(58)(67)$$

We wish to compute $\mu_N^{(2k)}$, the $2k^{th}$ moment of the anticommutator product of ensemble A which is a GOE and ensemble B which is a m-block circulant matrix, where both A, B are of order N. It is straightforward to verify the following.

Proposition 1 (Even moment as product words)**.**

$$\mu_N^{(2k)} = \sum_{W \in PW(2k)} \sum_{1 \le i_1, \dots, i_{4k} \le N} \sum_{\pi \in \mathcal{P}[2k]} \sum_{\delta \in NC(2k)} \mathbb{E}_{(\pi *_W \delta)} \left(\prod_{l=1}^{4k} W^l_{i_l i_{l+1}} \right) \mathbb{1}_{(\pi *_W \delta)} \tag{4}$$

Theorem 1 (GOE times Block Circulant)**.**

$$\mu_N^{(2k)} = \sum_{W\in PW(2k)} \sum_{\pi\in\mathcal{P}[2k]} \sum_{\delta\in NC(2k)} m^{\#((\pi *_W \delta)\circ\gamma_{2k})} \left(\frac{1}{m}\right)^{2k+1} \mathbb{1}_{(\pi *_w \delta)} \tag{5}$$

[1]

Theorem 2 (Block Circulant times Block Circulant)**.**

$$\mu_N^{(2k)} = \sum_{W\in PW(2k)} \sum_{\pi\in\mathcal{P}[2k]} \sum_{\delta\in \mathcal{P}[2k]} m^{\#((\pi *_W \delta)\circ\gamma_{2k})} \left(\frac{1}{m}\right)^{2k+1} \tag{6}$$

To prove these two theorems, we need to establish the following propositions.

Proposition 2 (Rules for pairing)**.** *For a pairing of each word, each of the compositions must match A's to A's and B's to B's. Moreover, the index configurations are confirmed once the two matricies are matched. That is, if $A_{i_s,i_{s+1}}$ is matched with $A_{i_t,i_{t+1}}$, then*

$$(i_s, i_{s+1}) \approx (i_t, i_{t+1}) \tag{7}$$

. Also, if $B_{i_s,i_{s+1}}$ is matched with $B_{i_t,i_{t+1}}$, then

$$(i_s, i_{s+1}) \simeq (i_t, i_{t+1}) \tag{8}$$

Proof. Introduce the signed variable ϵ_j. Adding up the signed difference allows us to find that if any one of the signs are nonzero, the degree of freedom reduces. □

Proposition 3 (GOE pairing rules)**.** *Let A be the GOE ensemble in the anticommutator. The compositions of A must not cross with any other compositions. The compositions between Block Circulant matricies can match, and the crossings do not reduce the degree of freedom.*

Proof. The compositions of A's in each pairing slices the entire word. For example, consider the product word

$$W = ABBABAAB$$

where the pairing is given as

$$\pi = (14)(23)(58)(67)$$

The composition (14) slices the word into

$$W_1 = BB \quad \text{and} \quad W_2' = BAAB$$

[1] γ_n denotes a permutation of the cannonical set $[n]$ where $\gamma_n(x) = x + 1 \mod n$.

where each word is extracted from between (W_1) and outside (W_2) the composition $W^1 = W^4 = A$. Call this composition of A's as the slicing composition. Furthermore, the slicing composition (58) slices W_2' into another word.

$$W_2 \;=\; BB$$

The observation has two implications. The first implication is that any transposition that crosses with the slicing composition reduces a degree of freedom. Hence, crossings with slicing composition, which can be any transposition between A's, result in a vanishing contribution.

The second implication is that any pairing where the slicing compositions do not cross with other compositions always have a positive nonzero contribution. After reducing the entire product word according to all its slicing compositions, we we are left with finite number of sub-words that are comprised solely of B's. For the word W, the remaining words are W_1, W_2.

Composition between B's lose one degree of freedom, regardless of crossings. So these always have a contribution. □

Finally, we present a proof of theorem .

Proof. From proposition 1, we recognize that it suffices to count the number of integer sequences $i_1, \ldots, i_{4k}$ that satisfy the pairing restrictions. Fix a pairing π that pairs all the GOE A's and a pairing δ that pairs the Block Circulnat B's. From We first configure the modular residue of i's mod m. Clearly, by proposition 2, the number of such configurations are [2]

$$m^{\#((\pi *_W \delta)\circ\gamma_{2k})}$$

Move on to choose the value of $\lfloor i/m \rfloor$. We know that as long as the slicing compositions of A do not cross with other compositions, the degree of freedom is not reduced. Otherwise, the contribution is can be ignored at the limit $N \to \infty$. Thus, the ways to configure $\lfloor i/m \rfloor$ is

$$\left(\frac{1}{m}\right)^{2k+1} \mathbb{1}_{(\pi *_w \delta)} \tag{9}$$

where $\mathbb{1}_{(\pi *_w \delta)}$ is defined to be 1 if and only if the pairing $(\pi *_w \delta)$ is non-crossing in the sense of proposition 2 and zero otherwise.

The variance of all the random variables involved in the matricies are fixed to be 1. Thus, from proposition 1, we obtain

$$\mu_N^{(2k)} \;=\; \sum_{W\in\mathrm{PW}(2k)} \sum_{\pi\in\mathcal{P}[2k]} \sum_{\delta\in\mathrm{NC}(2k)} m^{\#((\pi *_W \delta)\circ\gamma_{2k})} \left(\frac{1}{m}\right)^{2k+1} \mathbb{1}_{(\pi *_w \delta)} \tag{10}$$

□

[2] More details to be added from BC paper and FP paper. $\#(\pi)$ denotes the number of orbits of the permutation π

Remark 1. *Though topology, we can rewrite the moment as*

$$\sum_{W,\pi,\delta} m^{-2g} \mathbb{1}_{(\pi *_w \delta)} \tag{11}$$

where g is the minimum genus of the graph correlated to $((\pi *_W \delta) \circ \gamma_{2k})$.

Table 1: Spectral Density of GOE times Block Circulant

Moment	**Value**
2nd moment	2
4th moment	$10 + \frac{2}{m^2}$
6th moment	$66 + \frac{38}{m^2}$
8th moment	$498 + \frac{544}{m^2} + \frac{54}{m^4}$
10th moment	$4066 + \frac{7000}{m^2} + \frac{2086}{m^4}$

Table 2: Normalized Spectral Density of GOE times Block Circulant

Moment	**Value**
Normalized 4th moment	$\frac{5}{2} + \frac{1}{2m^2}$
Normalized 6th moment	$\frac{33}{4} + \frac{19}{4m^2}$
Normalized 8th moment	$\frac{249}{8} + \frac{34}{m^2} + \frac{27}{8m^4}$
Normalized 10th moment	$\frac{2033}{16} + \frac{875}{4m^2} + \frac{1043}{16m^4}$

Theorem 3 (6, 8, 10th moment of GOE times Block Circulant)**.**

Table 3: Normalized Spectral Density of Block Circulant times Block Circulant

Moment	**Value**
Normalized 4th moment	$\frac{10m^4 + 86m^2 + 48}{4m^4 + 8m^2 + 4}$
Normalized 6th moment	$\frac{66m^6 + 1890m^4 + 9084m^2 + 3360}{8m^6 + 24m^4 + 24m^2 + 8}$
Normalized 8th moment	$\frac{498m^8 + 33236m^6 + 529634m^4 + 1759064m^2 + 499968}{16m^8 + 64m^6 + 96m^4 + 64m^2 + 16}$

1 Auxiliary Sequences and Recurrence Relations

Definition 4 ($\sigma_{n,s,k}$)**.** *The auxiliary sequence $\sigma_{n,s,k}$ is defined with the following initial conditions.*

1. $\sigma_{n,s,k} = 0$ *if* $s + k > n$
2. $\sigma_{n,s,0} = (2n-1)!!$*, where C_n is the n-th Catalan number,* $C_n = \frac{1}{n+1}\binom{2n}{n}$
3. $\sigma_{n,s,2k+1} = 0$
4. $\sigma_{n,s,-k} = 0$

The recurrence relation for $\sigma_{n,s,2k}$ is given by:

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} [\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r} + \sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}]$$

Theorem 4 (Moments of GOE times PT)**.** *Let $\mu_N^{(k)}$ be the kth moment of the spectral density of the anticommutator $AB + BA$ where A is a GOE and B is a palindromic topelitz matrix. Then, we obtain the following formula.*

$$\mu_N^{(k)} = \sigma_{N,0,0}$$