411T Midterm

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1 Falling Stick

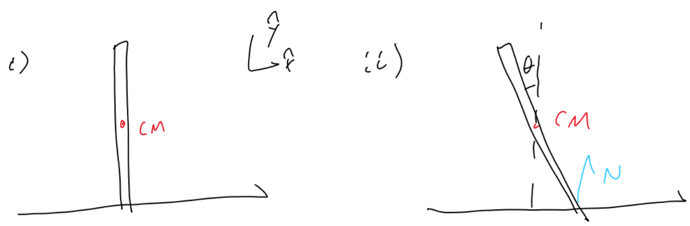

Figure 1: Setup for Q1, the length of the rod is $2l$, and θ is the angular displacement measured CCW from the vertical.

Part a. In order to compute the potential U, we must consider the conservative forces acting on the rod. The gravitational force exerts a fixed force of of $-mg\hat{y}$.

Set the potential at the surface to be zero. The potential of the system is a linear equation with respect to the height y.

$$U(x, y) = mgy \tag{1.1}$$

[1] We note that the cases where $y < 0$ is unphysical, since the object is bound to exist over the surface. □

Remark 1. *We note that the forces acting on the rod is entirely parallel to the y-axis. Hence, the motion of the rod is characterized by one parameter, either height or the angular displacement from the vertical. The relationship between the two parameters are*

$$y = l\cos(\theta) \tag{1.2}$$

Part b. Compute the Lagrangian in terms of angular displacement, θ.

$$U = mgy = mgl\cos(\theta) \tag{1.3}$$

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}ml^2\sin(\theta)^2\dot{\theta}^2 + \frac{1}{6}ml^2\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2\left(\sin(\theta)^2 + \frac{1}{3}\right) \tag{1.4}$$

$$\boxed{\mathcal{L} = T - U = \frac{1}{2}ml^2\dot{\theta}^2\left(\sin(\theta)^2 + \frac{1}{3}\right) - mgl\cos(\theta)} \tag{1.5}$$

[1] $y \neq \hat{y}$. The former is the height, the latter is a unit vector pointing upwards.

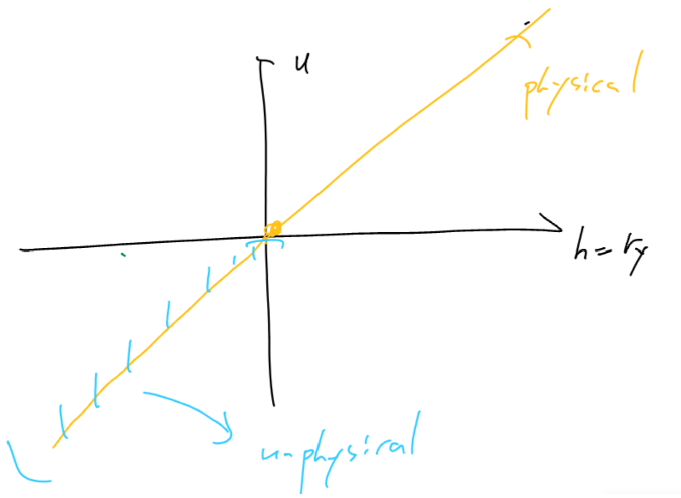

Figure 2: A sketch of the potential. $r_y = y$

Apply the Lagrange equation wrt the angular displacement.

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \tag{1.6}$$

Faithfully applying the derivatives, we obtain the following.

$$ml^2\dot{\theta}^2 \sin(\theta)\cos(\theta) + mgl\sin(\theta) = \frac{d}{dt} ml^2\dot{\theta}^2(\sin(\theta)^2 + 1/3) \tag{1.7}$$

$$\dot{\theta}^2 \sin(\theta)\cos(\theta) + \frac{g}{l}\sin(\theta) = \ddot{\theta}(\sin(\theta)^2 + 1/3) + \dot{\theta}^2(2\sin(\theta)\cos(\theta)) \tag{1.8}$$

$$\boxed{\ddot{\theta} = \frac{3g\sin(\theta) - 3l\dot{\theta}^2\sin(\theta)\cos(\theta)}{3l\sin(\theta)^2 + l}} \tag{1.9}$$

□

Part c. Lets use Newtonian mechanics to solve for $\ddot{\theta}$. We obtain an equation for force and torque.

$$\sum F = -mg + N = m\ddot{y} \tag{1.10}$$

$$\tau = I\alpha = \frac{ml^2}{3}\ddot{\theta} = Nl\sin(\theta) \tag{1.11}$$

Solving for N and canceling out redundant terms, we obtain

$$l\ddot{\theta} = 3g\sin(\theta) - 3l\ddot{\theta}\sin(\theta)^2 - 3l\dot{\theta}^2\sin(\theta)\cos(\theta) \tag{1.12}$$

$$\boxed{\ddot{\theta} = \frac{3g\sin(\theta) - 3l\dot{\theta}^2\sin(\theta)\cos(\theta)}{3l\sin(\theta)^2 + l}} \tag{1.13}$$

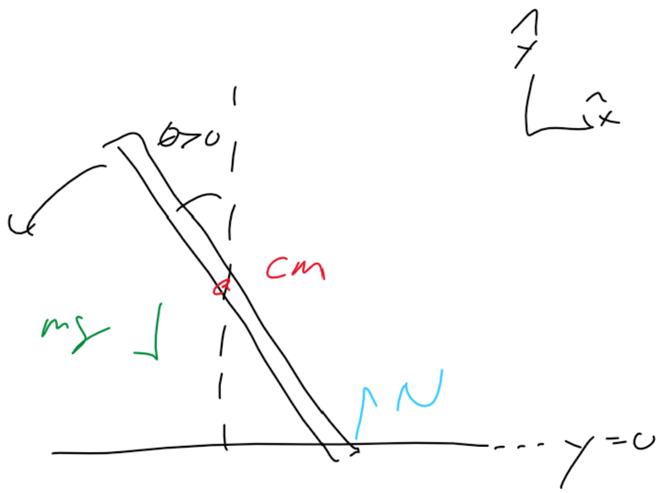

Figure 3: Diagram with force labels

which is great, since the result agrees with Part c. □

[2]

Part d. Use conservation of energy to compute the value of $\dot{\theta}$.

$$E = \frac{1}{2}ml^2\dot{\theta}^2\left(\sin(\theta)^2 + \frac{1}{3}\right) + mgl\cos(\theta) = mgl \tag{1.14}$$

Plug in $\theta = \pi/2$ to get angular velocity.

$$\dot{\theta}_1 = \sqrt{\frac{3g}{2l}} \tag{1.15}$$

Angular acceleration follows from (1.12).

$$\ddot{\theta}_1 = \frac{3g}{4l} \tag{1.16}$$

Compute linear velocity and linear acceleration by taking consecutive derivatives of y.

$$\boxed{\dot{y}_1 = -l\dot{\theta}_1\sin(\pi/2) = -\sqrt{\frac{3}{2}gl}} \tag{1.17}$$

$$\boxed{\ddot{y}_1 = -l\ddot{\theta}_1\sin(\pi/2) - l\dot{\theta}_1^2\cos(\pi/2) = -\frac{3}{4}g} \tag{1.18}$$

[2]Thanks for answering my questions, pf Jensen, pf Strauch. It helped a lot.

The negative signs indicate that the velocity and acceleration points towards the negative y-direction. □

2 Rotating Spring

Figure 4: Setup for Q2

Part a. At equilibrium, the centrepetal force matches the force exerted by the string.

$$\frac{mv^2}{r} = kx \tag{2.1}$$

Rewrite the velocity and displacement in terms of r, ϕ.

$$x = r - l_0 \tag{2.2}$$

$$v = \dot{\phi}(l_0 + x) \tag{2.3}$$

Solve for x. Also, $\dot{\phi} = \omega$ constantly.

$$\frac{m\dot{\phi}^2(l_0 + x)^2}{l_0 + x} = kx \tag{2.4}$$

$$m\dot{\phi}^2(l_0 + x) = kx \tag{2.5}$$

$$x = \frac{m\omega^2 l_0}{k - m\omega^2} \tag{2.6}$$

$$\boxed{r_0 = l_0 + x = \frac{kl_0}{k - m\omega^2}} \tag{2.7}$$

□

Part b. To compute the Lagrangian, we compute the kinetic and potential energy. Recall that

$$v = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}. \tag{2.8}$$

Write out T, U.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \tag{2.9}$$

$$U = \frac{1}{2}k(r - l_0)^2 \tag{2.10}$$

Thus

$$\mathcal{L}(t, r, \dot{r}, \phi, \dot{\phi}) = T - U = \frac{1}{2}\left(m(\dot{r}^2 + r^2\dot{\phi}^2) - k(r - l_0)^2\right). \tag{2.11}$$

Faithfully apply the Lagrange equation for both the radial and angular displacement.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad &\text{or} \quad mr\dot{\phi}^2 - k(r - l_0) = \frac{d}{dt}m\dot{r} \\ &\text{or} \quad \boxed{\ddot{r} = \left(\dot{\phi}^2 - \frac{k}{m}\right)r + \frac{k}{m}l_0} \end{aligned} \tag{2.12}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad &\text{or} \quad 0 = \frac{d}{dt}(mr^2\dot{\phi}) \\ &\boxed{2r\dot{r}\dot{\phi} + r^2\ddot{\phi} = 0} \end{aligned} \tag{2.13}$$

□

Part c. Set $r(t) = r_0 + \epsilon(t)$. Also suppose $\dot{\phi} \approx \omega$ constantly. The Lagrange equation with respect to radial displacemnt simplifies as follows.

$$\begin{aligned} \ddot{\epsilon} &= \left(\omega^2 - \frac{k}{m}\right)(r_0 + \epsilon) + \frac{k}{m}l_0 = \left(\frac{m\omega^2 - k}{m}\right)\left(\frac{kl_0}{k - m\omega^2} + \epsilon\right) + \frac{k}{m}l_0 \\ &= -\frac{k}{m}l_0 + \frac{k}{m}l_0 + \left(\omega^2 - \frac{k}{m}\right)\epsilon = \left(\omega^2 - \frac{k}{m}\right)\epsilon \end{aligned} \tag{2.14}$$

$$\boxed{\ddot{\epsilon} = \left(\omega^2 - \frac{k}{m}\right)\epsilon} \tag{2.15}$$

Assuming $\omega^2 - k/m < 0$, ϵ is a solution for a simple harmonic oscillator. A particular solution for $\epsilon(0) = A, \dot{\epsilon}(0) = 0$ is

$$\epsilon(t) = A\cos\left(\sqrt{\frac{k}{m} - \omega^2}\right) \tag{2.16}$$

Thus

$$\boxed{\Omega \;=\; \sqrt{\frac{k}{m} - \omega^2}}. \tag{2.17}$$

□

Part d. Without the assumption $\omega^2 - k/m < 0$, ϵ displays exponential decay. Under the initial condition $\epsilon(0) = A$, a particular solution is

$$\epsilon(t) \;=\; \frac{Ae^{-qt} + Be^{+qt}}{2} \tag{2.18}$$

where

$$q \;=\; \sqrt{\omega^2 - \frac{k}{m}} \tag{2.19}$$

Physically, it is not plausible that ϵ displays exponential growth. Thus, we set $B = 0$ We write the solution in one clean formula.

$$\boxed{\epsilon(t) \;=\; A\exp\left(-t\sqrt{\omega^2 - \frac{k}{m}}\right)} \tag{2.20}$$

The formula tells us that the displacement of the spring reaches the equilibrium position in an asymptotic manner as $t \to \infty$. □

Part e. We investigate the change of ϕ over time. From the Lagrange equation w.r.t. angular displacement, solve for $\ddot{\phi}$.

$$\ddot{\phi} \;=\; -2\frac{\dot{r}}{r}\dot{\phi} \tag{2.21}$$

Set $\dot{\phi}_0 \;=\; \omega$. Then, apply the method of successive approximations.

$$\ddot{\phi}_1 \;=\; -2\frac{\dot{r}}{r}\dot{\phi}_0 \;=\; -2\omega\frac{\dot{r}}{r} \;=\; \frac{2\omega q}{r_0}Ae^{-tq} \tag{2.22}$$

Suppose $\phi(\infty) = \omega$. Then,

$$\dot{\phi}_1 \;=\; \omega + Ce^{-qt} \tag{2.23}$$

[3] satisfies the differential equation and the limiting condition. Regardless of the initial amplitude of the oscillation, the angular velocity will display exponential decay either from above or below and reach a constant terminal velocity of ω.

□

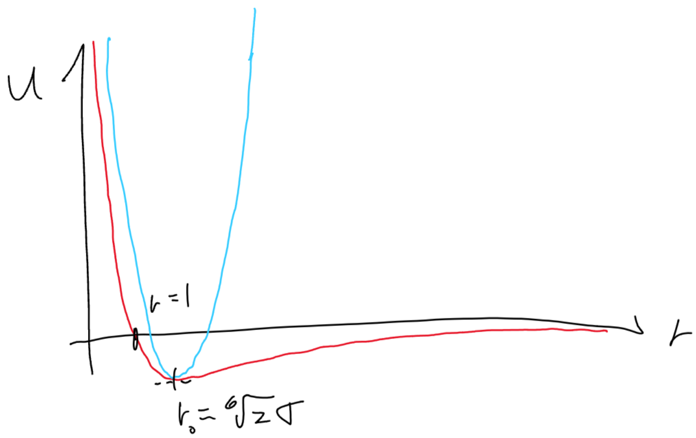

Figure 5: Plot of $U(r)$ with "stiffer spring" potential. The potential at the valley is $-\epsilon$. Also, the potential reaches zero at $r = \sigma$, not 1.

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(* Q3, potentials *)
U[r_] := 4 ϵ ((σ / r)^12 - (σ / r)^6);
U''[r]

4 ϵ (-42 σ^6/r^8 + 156 σ^12/r^14)

Solve[U'[r] == 0, r]

{{r → -2^(1/6) σ}, {r → 2^(1/6) σ}, {r → -(-1)^(1/3) 2^(1/6) σ}, {r → (-1)^(1/3) 2^(1/6) σ}, {r → -(-1)^(2/3) 2^(1/6) σ}, {r → (-1)^(2/3) 2^(1/6) σ}}

(* We set r to be the peak value *)
sol = {r → 2^(1/6) σ};
U''[r] /. sol

36 2^(2/3) ϵ / σ^2

(* Equilibrium energy *)
U[r] /. sol

-ϵ
```

Figure 6: Supplemental Mathematica code

3 Modeling Matter

Part a. The stationary point is attained when $\frac{dU}{dr} = 0$. With the help of mathematica, we notice that the equilibrium position is

$$r_0 = 2^{1/6}\sigma \tag{3.1}$$

and

$$U(r_0) = -\epsilon \tag{3.2}$$

. Upon inspection, we recognize tht the potential reaches zero at $r = \sigma$.

$$U(\sigma) = 0 \tag{3.3}$$

[3]The exact expression of constant C is not of our interest.

$\square$

Part b. The effective spring constant is the value of the second derivative of the potential at the valley. Mathematica computation shows the following.

$$U''(r_0) \;=\; \frac{36 \cdot 2^{2/3}\epsilon}{\sigma^2} \tag{3.4}$$

Thus, the spring constant increases for higher ϵ and lesser σ. $\square$

Part c. We notice that for $E \geq 0$, the motion of the particle is unbounded. For $E < 0$, the horizontal energy limit must intersect with the potential, since $\lim_{r\to\infty} U(r) = 0$, and for this case, the particle is bounded. We expect small oscilaltions around the valley, i.e. $E \approx -\epsilon$. [4] $\square$

Part d. $\square$

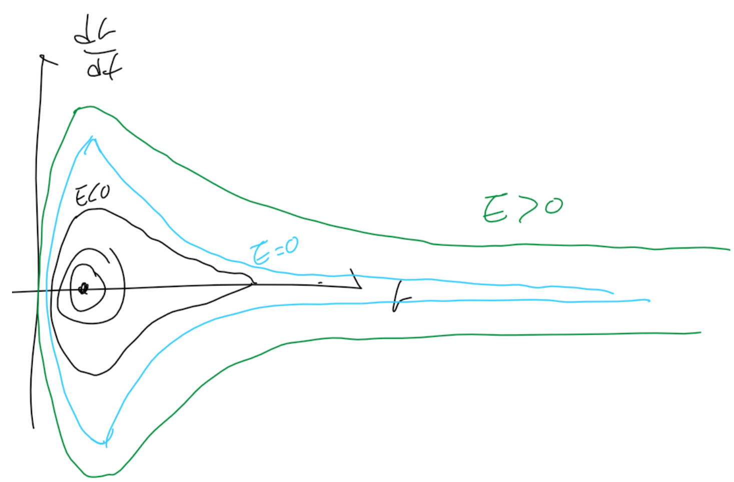

Figure 7: Phase space diagram. Particle is bounded for trajectories corresponding to $E < 0$. For particles with $E \geq 0$, the trajectories display unbounded behavior. The center dot is a stationary trajectory with $E = -\epsilon$.

4 Bending Light

Part a. Fermat's principle states that the ray of light follows the trajectory which takes the minimal amount of time to travel between two points. We set

[4] E must be slightly greater than ϵ, since negative kinetic energy is not allowed.

up an integral to describe time elapsed between two points.

$$t = \int_1^2 \frac{n(y)ds}{c} = \frac{1}{c}\int_0^{x_t} \left(1+\frac{y(x)}{l}\right)\sqrt{1+y'(x)^2}dx \tag{4.1}$$

We wish to apply calculus of variations. We set up our function f as follows.

$$f(x,y,y') = \left(1+\frac{y}{l}\right)\sqrt{1+y'(x)^2} \tag{4.2}$$

The function f has no explicit x dependency. Hence, we apply Beltrami's identity. Suppose C is a constant.

$$f - y'\frac{\partial f}{\partial y'} = C \tag{4.3}$$

$$f - y'\left(1+\frac{y}{l}\right)\frac{2y'}{2\sqrt{1+y'^2}} = C \tag{4.4}$$

$$\frac{1+y/l}{\sqrt{1+y'^2}} = C \tag{4.5}$$

We solve for y' and apply seperation of variables to obtain an equation of x with respect to y.

$$y'(x) = \sqrt{\left(\frac{1}{C}+\frac{y}{Cl}\right)^2 - 1} \tag{4.6}$$

$$x = \int_{y_0}^{y_t} \frac{dy}{\sqrt{\left(\frac{1}{C}+\frac{y}{Cl}\right)^2 - 1}} \tag{4.7}$$

Apply u-substitution where u is the quantity within the parentheses.

$$x = \int_{y_0/Cl+1/C}^{y_t/Cl+1/C} \frac{Cl du}{\sqrt{u^2-1}} \tag{4.8}$$

$$x = Cl\tanh^{-1}\left(\frac{u}{\sqrt{u^2-1}}\right)\Bigg|_{y_0/Cl+1/C}^{y_t/Cl+1/C} \tag{4.9}$$

Solving for y_t, we deduce that the solution must be in the following form.

$$\boxed{y_t = y(x) = Cl\cosh\left(\frac{x}{Cl}+B\right) - l} \tag{4.10}$$

Imposing $y(0) = y_0$ and $y'(0) = y_0'$, we compute B, C

$$C = \frac{y_0/l+1}{\sqrt{1+y_0'^2}} \tag{4.11}$$

$$B = -\cosh^{-1}\left(\sqrt{1+y_0'^2}\right) \tag{4.12}$$

□

Part b. Upon inspection, the reflection point occurs when the argument of the cosh function vanishes. This occurs in the following value of x.

$$\boxed{x_0 \;=\; -lBC \;=\; \frac{y_0 + l}{\sqrt{1 + y_0'^2}} \cosh^{-1}\left(\sqrt{1 + y_0'^2}\right)} \tag{4.13}$$

□

Part c. Taylor expand the following two functions for $x \ll 1$.

$$\cosh(x) \;\approx\; 1 + \frac{x^2}{2} \tag{4.14}$$

$$\sqrt{1 + x^2} \;\approx\; 1 + \frac{x^2}{2} \tag{4.15}$$

So, for $x \ll 1$,

$$\cosh^{-1}\left(\sqrt{1 + x^2}\right) \;\approx\; x \tag{4.16}$$

which leads us to approximate (4.13) as

$$\boxed{x_0 \;=\; (y_0 + l)(y_0')} \tag{4.17}$$

□