Solutions for CH2

Daniel Son

Ex 2.1

For convergence, we present the **Hadamard's formula** that provides the radius of convergence for any generating function. Let $g_a(x)$ be the generating function of the sequence a_n. We define

$$\frac{1}{R} := \limsup_{n\to\infty} |a_n|^{1/n}$$

. For $x < R$, the series converges, and for $x > R$ the series diverges. The behavior of the function at $x = R$ depends on the sequence. The convergence/divergence can be proved by comparing the partial sums to geometric series. [1]

Using the formula, we write out the radius of convergence for both a_n, b_n.

$$\frac{1}{R_a} = \limsup_{n\to\infty} |a^n|^{1/n} \quad \text{and} \quad \frac{1}{R_b} = \limsup_{n\to\infty} |n!|^{1/n}$$

To compute R_b, recall the Stirling's formula which provides an asymptotic estimation [2] of $n!$.

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

We finally present the values of R_a, R_b.

$$R_a = 1/|a| \quad \text{and} \quad R_b = \lim_{n\to\infty} e/n = 0$$

$R_b = 0$ means that $g_b(x)$ diverges for all values of nonzero x. □

Ex2.2

Make use of Hadamard's formula. We wish to compute the following.

$$\frac{1}{R} = \limsup_{n\to\infty} \binom{2n}{n}^{1/n} = \limsup_{n\to\infty} \left(\frac{(2n)!}{(n!)^2}\right)^{1/n} \tag{1}$$

Using Stirling's formula, it is possible to write the multiplicative identity in two different ways.

$$\lim_{n\to\infty} \left(\frac{n!}{(n/e)^n\sqrt{2\pi n}}\right)^2 = \lim_{n\to\infty} \frac{(2n)!}{(2n/e)^{2n}\sqrt{4\pi n}} = 1$$

[1] To learn more, visit `https://en.wikipedia.org/wiki/Cauchy-Hadamard_theorem`

[2] Asymptotic estimation means that the ratio of the estimate to the value converges to zero at the $n \to \infty$ limit.

Multiply and divide by 1 in (1). We obtain the following.

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{(2n/e)^{2n}\sqrt{4\pi n}}{(n/e)^{2n} 2\pi n} \right)^{1/n} = 4$$

Thus, the radius of convergence is

$$\boxed{R = \frac{1}{4}}$$

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<u>Ex2.5</u>

Let $g_k(x)$ be the generating function of $a_n = n^k$ Take the derivative of $g_k(x)$ to derive the recurrence

$$x(G_k(x))' = G_{k+1}(x)$$

Notice that g_0 is the geometric series. Thus

$$g_0(x) = \frac{1}{1-x}$$

Some computation shows that:

$$\begin{aligned} g_1(x) &= \frac{x}{(1-x)^2} \\ g_2(x) &= \frac{(x+x^2)}{(1-x)^3} \\ g_3(x) &= \frac{(x+4x^2+x^3)}{(1-x)^4} \end{aligned}$$

□

<u>Ex2.10</u>

We will show the first identity and leave the proof of the second and third as an exercise to the reader. So the decimal expansion is created so that each digit is a fibbonacci number. Call the desired fraction d. We write the following.

$$d = .\overline{F_1F_2F_3\cdots} = F_1(.1) + F_2(.1)^2 + F_3(.1)^3 + \cdots$$

Write d in sigma notation.

$$d = \sum_{n=0}^{\infty} F_n(.1)^n$$

In fact, d is the generating function of the fibbonacci sequence evaluated at $x = .1$.

$$d = g_f(.1) = \left.\frac{x}{1-x-x^2}\right|_{x=.1} = \frac{10}{89}$$

□