PHYS 314 Formula Sheet

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Singlet States and Superposition

The singlet state is defined as follows.

$$\frac{1}{\sqrt{2}}(|z_+\rangle|z_-\rangle - |z_-\rangle|z_+\rangle)$$

Computing the probability outcomes of the singlet state into all the possible output states, we determine that the particle never collapses to the up-up or down-down state. This means that the spin of one particle decides the spin of the other. This is called *Superposition*.

Raising and Lowering Operators The raising and the lowering operators are conventionally defined on the **z-axis**. The operator bumps up the state by one. Here is the definition along with an example.

$$\hat{S}_+ := \hat{S}_x + i\hat{S}_y \quad \text{and} \quad \hat{S}_- := \hat{S}_x - i\hat{S}_y$$

$$\hat{S}_+|s,j\rangle = \sqrt{s(s+1) - j(j+1)}|s,j+1\rangle$$

Also, spin operators are Hermitian, so $\hat{S_+}^\dagger = \hat{S_-}^\dagger$

Product of spins to Product of Raising/Lowering Ops The sum of the square of all the spin operators of each direction is a natural operator with importance. It is possible to express this operator, which is dependant on all three axes, into a sum of three products that involve only the z-axis operators. The tensor product of vectors/operators behave nicely. In light of this fact with the definition of the raising/lowering operator, we conclude

$$2\hat{S_{1x}}\hat{S_{2x}} + 2\hat{S_{1y}}\hat{S_{2y}} + 2\hat{S_{1z}}\hat{S_{2z}} = \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S_{1z}}\hat{S_{2z}}$$

Applications It is possible to express a higher order spin, say spin-3/2, as a tensor product of two lower spins, spin-1 and spin-1/2. The idea is to take the lowest and the highest value of the spin states, and to apply the total momentum operator $\hat{J}^2 := (\hat{J}_1 + \hat{J}_2)^2$. The operator $\hat{J}_1\hat{J}_2$ can be expressed entirely in terms of the z-axis operators. It might be tempting to simply apply each of the operators separately, but the base states are eigenvectors of $\hat{J_i}^2$.

$$\hat{J}_1\hat{J}_2 = 2\hat{J_{1x}}\hat{J_{2x}} + 2\hat{J_{1y}}\hat{J_{2y}} + 2\hat{J_{1z}}\hat{J_{2z}}$$

Refer to Townsend Appendix B for details. In conclusion, this equation will show that the tensor product of the states spin-1/2 and spin-1 will yield all the states in spin-3/2 and spin-1/2.

Also, for entangled states, the raising and the lowering operator is defined as follows.

$$\hat{S}_+ := \hat{S}_{1+} \cdot 1_2 + 1_1 \cdot \hat{S}_{2+} \quad \text{and} \quad \hat{S}_- := \hat{S}_{1-} \cdot 1_2 + 1_1 \cdot \hat{S}_{2-}$$

Where S is defined to be the entangled spin of two particles. For an intuitive justification, consider the rotation generator. The sum of the two operators are the terms that survive as $d\theta \to 0$.

Types of Operators and their properties We have the spin operator $\hat{S} := \hat{S}_x + \hat{S}_y + \hat{S}_z$. The nice property is that the magnitude of the combined spin $\hat{S}^2$ commutes with individual axis spin. That is:

$$[\hat{S}^2, \hat{S}_i] = 0$$

We can take the tensor prouct of two entangled particles. That is

$$\hat{S}_1 \cdot \hat{S}_2 := \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$$

$$\hat{\mathbf{S}}^2|s, m\rangle = s(s+1)\hbar^2|s, m\rangle$$

$$\hat{\mathbf{S}}_z|s, m\rangle = m\hbar|s, m\rangle$$

Note that any state is a eigenvector of $\hat{S}^2$. Refer to Tonwsend CH5 for more information.

Hyperfine energy structure of the Hydrogen Atom A hydrogen atom can be considered as as a system of two spin-1/2 particles, proton and the electron. The Hamiltonian of the system is defined as $\hat{H} = \frac{2A}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2$. With some computation, we recognize that there are two energy eigenvalues and four energy eigenvectors.

$$\hat{H} = \begin{bmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & 0 \\ 0 & A & -A/2 & 0 \\ 0 & 0 & 0 & A/2 \end{bmatrix}$$

The higher energy eigenvalue is $A/2$, and the corresponding states are the two product states (top/bottom) and the entangled state which has the same signs. In symbols,

$$\lambda = \frac{A}{2} \quad \text{and} \quad |\psi\rangle = |z_+\rangle|z_+\rangle, |z_-\rangle|z_-\rangle, \frac{1}{\sqrt{2}}(|z+\rangle|z-\rangle + |z_-\rangle|z_+\rangle)$$

The lower energy eigenvalue is $-3A/2$, and the corresponding state is the singlet state. In symbols,

$$\lambda = -\frac{3A}{2} \quad \text{and} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(-|z+\rangle|z-\rangle + |z_-\rangle|z_+\rangle)$$

Remember that $\frac{2}{\hbar^2}S_1S_2$ has the eigenvalue of $1/2, -3/2$.

Dirac's Spin Exchange The Hamiltonian $\hat{H} := \vec{\sigma}_1 \cdot \vec{\sigma}_2$ can be expressed as

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2P_{\text{spin exchange}} - 1$$