PHYS 314 HW7

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Q1 Classical Circuits

a) Express the OR gate in terms of AND and XOR.
Adding up the entries of the truth table yields the desired result. Thus,

$$a \vee b = (a \wedge b) \oplus (a \oplus b)$$

b) Express the XOR gate in terms NOT, AND, OR gates

$$a \oplus b = [\neg(a \wedge b)] \wedge (a \vee b)$$

c) Express AND, OR, XOR, solely in terms of NAND and FANOUT.
We first establish the NOT gate.

$$\neg a = \mathrm{NAND}(1, a)$$

The AND gate is a negation of the NAND gate.

$$a \wedge b = \mathrm{NAND}(1, \mathrm{NAND}(a, b))$$

The OR gate can be easily deduced by DeMorgan's Law.

$$a \vee b = \mathrm{NAND}(\mathrm{NAND}(1, a), \mathrm{NAND}(1, b))$$

In part b, we have shown how to write an XOR gate with NOT, AND, OR gates. Thus, write

$$a \oplus b = \mathrm{NAND}(a, b) \wedge \mathrm{NAND}(\mathrm{NAND}(1, a), \mathrm{NAND}(1, b))$$

$$= \mathrm{NAND}(1, \mathrm{NAND}(\mathrm{NAND}(a, b), \mathrm{NAND}(\mathrm{NAND}(1, a), \mathrm{NAND}(1, b))))$$

Q2 no-cloning theorem a) Consider a qualtum controlled-NOT gate. This gate seems to copy the states for

$$|\psi\rangle = |0\rangle, |1\rangle$$

. Does this gate violate the no-cloning theorem?

Solution No, the no-cloning theorem introduced in Townsend tells us that there does not exist a unitary operator the copies a general quantum state. The c-NOT gate successfully clones the $|0\rangle, |1\rangle$ state, but it fails for an entangled state, for example

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

. An attempt to copy $|\psi\rangle$ through the c-NOT gate results in a state

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The correct copy must result in a state

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

And clearly the two states do not match which leads to a contradiction. $\lightning$

b) By using the method of Quantum Teleportation, Alice can send a quantum state exactly by using entanglement and sending two classical bits. Now, assume Bob recieved a cubit from Alice and Bob made a measurement. How much information about $\{\theta, \phi\}$ can Bob retrieve from this experiemnt?

Solution Suppose Bob recieves a state

$$|\psi\rangle = \begin{bmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix}$$

We can retrieve the probability that $|\psi\rangle$ will collapse to either $|0\rangle$ or $|1\rangle$.

$$P(0) = \cos^2(\theta/2) \quad \text{and} \quad P(1) = \sin^2(\theta/2)$$

Depending on Bob's measurement, we can claim that the probability that $|\psi\rangle$ will collapse to the measured state is more likely. If Bob measures 1, then it is likely that

$$\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]$$

. This method does not allow us to make any claims about the phase ϕ.

c) What if Bob is allowed many duplicates of the same qubit?

Solution It would be possible to narrow down the exact value of θ. Still, it would be impossible to recover the value of ϕ

Q4 The controlled Z-gate has the following matrix form.

$$Z_c := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a) Show that, up to a global phase, this gate can be generated by the time evolution operator of the following Hamiltonian.

$$\hat{H} := \omega_1(Z \otimes I + I \otimes Z) + \omega_2 Z \otimes Z$$

for some appropriate value $\omega_1 t, \omega_2 t$.

Solution We wish to accomplish the identity

$$\exp\left(\frac{\hat{H}t}{i\hbar}\right) = e^{i\phi} Z_c$$

for some phase factor ϕ. Expanding both sides into matricies, we rewrite the identity as follows.

$$\begin{pmatrix} \exp\left(-\frac{it}{\hbar}(2\omega_1+\omega_2)\right) & & & \\ & \exp\left(\frac{it}{\hbar}\omega_2\right) & & \\ & & \exp\left(\frac{it}{\hbar}\omega_2\right) & \\ & & & \exp\left(-\frac{it}{\hbar}(-2\omega_1+\omega_2)\right) \end{pmatrix} = \begin{pmatrix} e^{i\phi} & & & \\ & e^{i\phi} & & \\ & & e^{i\phi} & \\ & & & -e^{i\phi} \end{pmatrix}$$

Comparing the diagonal entries, we obtain three distinct equations. Upon inspection, we guess a solution.

$$\boxed{\omega_1 t = \frac{\pi}{4} \quad \text{and} \quad \omega_2 t = -\frac{\pi}{4}}$$

where $\phi = -\frac{\pi}{4}$.

b The Hadamard Gate is defined as

$$H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note that the Hadamard gate can be considered as an active change of the basis vectors.

$$\hat{z} \mapsto \hat{x} \quad \text{and} \quad \hat{x} \mapsto \hat{z}$$

The transformation goes both directions. Thus, the Hadamard gate can be considered a "flip" of $\hat{x}$ and $\hat{z}$. Show that a X_c gate is equivalent to HZ_cH

Solution It suffices to show $HZH = X$. Plug the equation into mathematica.

In[7]:=

```
H := (1/√2) {{1, 1}, {1, -1}}
Z := {{1, 0}, {0, -1}}
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In[19]:=

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H.Z.H // MatrixForm
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Out[19]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[24]:=

```
X := H.Z.H;
H.X.H // MatrixForm
```

Out[25]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

□

c) Recall the Hamiltonian for the hyperfine problem.

$$\hat{H} := \omega_1(X \otimes X + Y \otimes Y + Z \otimes Z)$$

evolution of this Hamiltonian for time $t = \pi\hbar/(4\omega_1)$ corresponds to the swap gate. Prove and justify this claim.

Solution The swap gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The time evolution operator of the Hamiltonian is

$$\hat{U}(t) = \exp\left(\frac{\hat{H}t}{i\hbar}\right)$$

Simplify the argument of the matrix exponential. Also, remember the condition of t.

$$\frac{\hat{H}t}{i\hbar} = \frac{\omega_1 t(X \otimes X + Y \otimes Y + Z \otimes Z)}{i\hbar} = -\frac{i\pi}{4}(X \otimes X + Y \otimes Y + Z \otimes Z)$$

Thus,

$$\hat{U}(t) = \exp\left(-\frac{i\pi}{4}(X \otimes X + Y \otimes Y + Z \otimes Z)\right)$$

which can be computed by plugging in to Mathematica.

In[58]:=

```
(* Q3 - c *)
X := {{0, 1}, {1, 0}};
Y := {{0, -ⅈ}, {ⅈ, 0}};
Z := {{1, 0}, {0, -1}};
H := KroneckerProduct[X, X] + KroneckerProduct[Y, Y] + KroneckerProduct[Z, Z];
H // MatrixForm
MatrixExp[-ⅈ π H / 4] // MatrixForm
```

Out[62]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

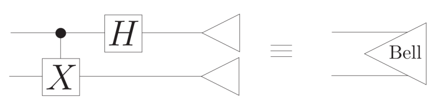
Out[63]//MatrixForm=

$$\begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1-i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{i\pi}{4}} \end{pmatrix}$$

Up to a global phase of $e^{-i\pi/4}$, the time evolution operator is a swap gate.

BSM and Teleportation

a Figure 4.11 of KLM shows a circuit that conducts a Bell State Measurement. Prove this claim.



The physical system of this two wire quantum circuit can be descirbed as a tensor product of two Hilbert spaces. Let $|0\rangle_u, |1\rangle_u$ be the basis for the upper channel and $|0\rangle_d, |1\rangle_d$ be the basis for the lower channel. The combined state has four bases, namely

$$|00\rangle := |0\rangle_u|0\rangle_d \qquad |01\rangle := |0\rangle_u|1\rangle_d$$
$$|10\rangle := |1\rangle_u|0\rangle_d \qquad |11\rangle := |1\rangle_u|1\rangle_d$$

The circuit first applies the Hadamard gate to the first channel, then applies a controlled-X gate to the second channel which is controlled by the first channel. The operator of this circuit is

$$B := (H \otimes I)X_c$$

Considering the operator B as an passive linear transformation, we find the basis of the quantum state after the signal passes the gate. We must obtain the inverse of B.

$$B^\dagger = X_c^\dagger (H \otimes I)^\dagger = X_c (H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

The four column vectors represent the basis of this passive transform.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle \qquad \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle$$
$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle \qquad \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle$$

These four bases are exactly the Bell Bases. □

c) Suppose Alice witheld the result of the classical measurement after the measurement was made. Describe Bob's state.

Solution All four Bell basis states are equally likely. Hence, the solution is the classical average of all the four Bell states.

$$|\phi\rangle = \frac{1}{4}(|\Psi^+\rangle + |\Psi^-\rangle + |\Phi^+\rangle + |\Phi^+\rangle)$$
$$\boxed{= \frac{1}{2\sqrt{2}}(|00\rangle + |01\rangle)}$$