PHYS 202 Formula Sheet

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Simple Harmonic Oscillators Consider a mass attatched to a spring. Let $x(t)$ be the function of displacement of the mass from the equilibrium position. Suppose that the spring constant is k. By Newton's 2nd Law,

$$m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} = -\frac{k}{m}x$$

Where $\dot{x} := \frac{d}{dt}x$

The following function solves the equation.

$$x(t) = Re(\tilde{A}e^{i\omega_0 t}) = A\cos(\omega_0 t + \phi)$$

A, ω_0 is referred as the amplitude and the natural frequency of the oscillator. Also,

$$A = \sqrt{\frac{2E}{k}} \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Phase Conventions Velocity leads Displacement by a phase of $\pi/2$. Acceleration leads Velocity by a phase of $\pi/2$

Say $V \sim \cos(\omega t)$ and $I \sim \cos(\omega t + \phi)$ for some positive phase $\phi \leq \pi$. The current leads the voltage and the voltage trails the current.

Simple RLC Consider a circuit where R, L, C is connected in parallel. Let the current be I. By the loop rule,

$$-\frac{q}{C} - RI - L\dot{I} = 0 \quad \text{or} \quad \frac{q}{C} + R\dot{q} + L\ddot{q} = 0$$

Rewrite this in the follwoing form.

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = 0$$

Compare this with the equation for damped driven oscillators.

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

So the following isomorphism holds

$$x \mapsto q \quad \text{then} \quad (m, b, k) \mapsto (L, R, 1/C)$$

In a parallel circuit, the current through each circuit component is identical. By the complexified Ohms Law $\tilde{V} = Z\tilde{I}$. Recall the impedences.

$$Z_R = R \quad \text{and} \quad Z_C = \frac{1}{i\omega C} \quad \text{and} \quad Z_L = i\omega L$$

Thus, the voltage of the resistor leads the voltage on the capacitor by phase $\pi/2$. Likewise, the voltage on the resistor trails the voltage on the Inductor by $\pi/2$.

Circuit Filters and Loglog plot Using inductors and capacitors, it is possible to filter out signals of high or low frequency. Consider the behavior of circuit elemnts in high and low frequency.

$$\lim_{\omega\to 0} Z_L = \lim_{\omega\to 0} i\omega L = 0 \quad \text{and} \quad \lim_{\omega\to\infty} Z_L = \lim_{\omega\to\infty} i\omega L = \infty$$

$$\lim_{\omega\to 0} Z_C = \lim_{\omega\to 0} \frac{1}{i\omega C} = \infty \quad \text{and} \quad \lim_{\omega\to\infty} Z_C = \lim_{\omega\to\infty} \frac{1}{i\omega C} = 0$$

The inductor will block signals of high frequency but allow the passage of signals of low frequency. Hence, inductors act as a **low-pass** filter. On the other hand, capacitors block signals of low frequency and allow the passage of high frequency signals. Hence, capacitors act as a **high-pass** filter.

Corner frequency vs Resonant frequency For an RLC circuit, it is possible to tune the frequency such that the combined impedence reach zero. This frequency is called the resonant frequency. For a circuit involving one R, L, C, the resonant frequency is computed by the following formula.

$$\omega_r = \frac{1}{\sqrt{LC}}$$

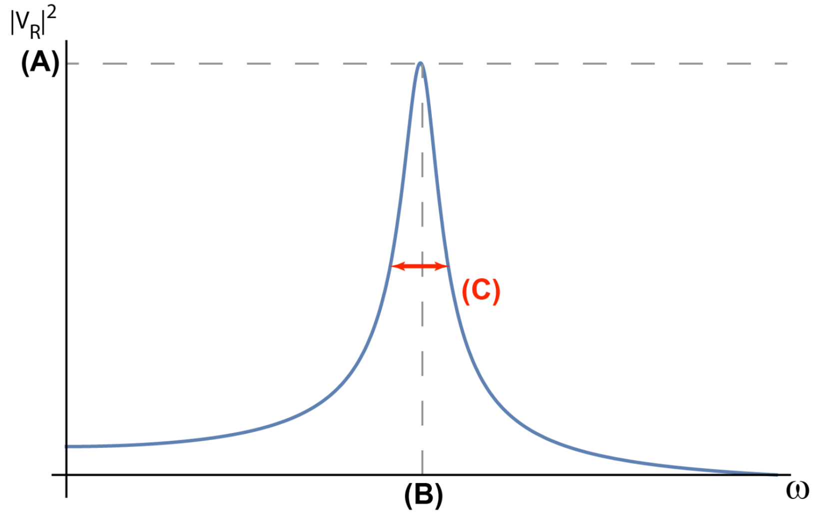
For and RL or an RC circuit, the imaginary part of the impedence cannot reach zero. It is possible to defnine a corner frequency where the circuit behavior changes drastically. It is the freqnecy where $Im(Z_L) = R$ or $Im(Z_C) = R$. In other words,

$$\omega_c = \frac{1}{RC} \quad \text{or} \quad \frac{R}{L}$$

Dimensions of C and L

$$[L] = H = \Omega \cdot s \quad \text{and} \quad [C] = \frac{s}{\Omega}$$

Power Resonance Curve For an RLC circuit, it is possible to plot the relationship between the angular frequency of the imput voltave and the power lost through the resistor. This curve is called the Power Resonance curve.



$P \sim V^2$. The Full-Width-Half-Maximum refers to the width of the curve at the half maximum point. The FWHM of the Power Resonance Curve is exactly $\gamma = R/L$, the damping factor. Be careful of the factor of 2π when the x-axis is set as frequency (Hz).

Also, use the dimension $[\omega] = rad/s$ and $[f] = Hz$. The frequency refers to cycles per second.

Q-factor For damped driven oscillators, the Q-factor is an important number.

$$Q := \frac{\omega_0}{\gamma} = \frac{A_{res}}{A_0}$$

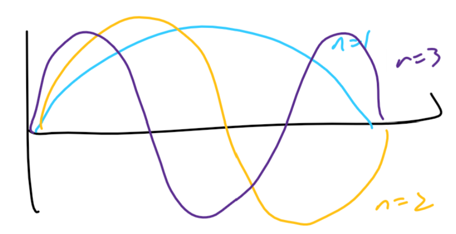
Boundary conditions for N masses The solutions for the n mass oscillators are in the form of:

$$\psi(m,t) = Ae^{(\omega t + k_m a)} + Be^{(\omega t - k_m a)}$$

Plugging into the 2nd order DE, we derive an expression for the angular frequency.

$$\omega_m = 2\omega_0 \sin(k_m \frac{a}{2})$$

k_m is the mth wave number. Recall that $k := \frac{2\pi}{\lambda}$. λ_n can be computed by drawing diagrams.



With some algebraic hassle, it is possible to derive the expressions for k_m.

For closed-closed and open-open ends:

$$k_m a = \frac{m\pi}{n+1} \quad \text{or} \quad k_m = \frac{m\pi}{a(n+1)} = \frac{m\pi}{L}$$

Where $m \in \mathbb{Z}^+$

For open-open ends:

$$k_m a = \frac{m\pi}{(2n+1)/2} \quad \text{or} \quad k_m = \frac{m\pi}{L}$$

Where $m \in \mathbb{Z}^+$

Free end Distance relation Always set the origin to be distance $a/2$ apart from the free end. This is because the two masses at the boundary must be symmetric. Refer to pset 5-1.

Transverse waves From the geometry of the springs, we use approximation. Let theta be the angle between the horizontal axis and the string. $\tan(\theta) \approxeq \theta = \Delta y/a$. Consequently, we arrive at:

$$k \mapsto T/a$$

Dispersion Relation The differential equation of the masses in the center oscillators provide a closed form equation for the angular frequency in terms of the wave number k.

$$\omega(k) = 2\omega_0 \sin(ka/2)$$

where a is the distance between the masses.

Dispersion Relation 2 The dispersion relation defines the wave number k.

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} \quad \text{then} \quad \omega^2 = c^2k^2$$

And c is called the phase velocity of the wave. The wave number depends on the frequency of the wave, which is not necessarily the normal mode frequency

Nodes and Antinodes In a standing wave, nodes are the points that do not move. Max amplitude is achieved at antinodes.

Waves traveling in different medium Consider a transverse wave moving from one string of linear mass density μ_1 to another string with linear mass density μ_2. The velocity of the waves on each string is entirely determined by the lmd.

$$(v_1, v_2) = \left(\frac{T}{\mu_1}, \frac{T}{\mu_2}\right)$$

Let the incident wave to be in the form of

$$\psi_i(x,t) := f_i(t - x/v_1)$$

We make a natural assumption that the combined wavefunction must be smooth with respect to position. Let f_t, f_r be the simplified translated wavefunction and the simplified reflected wavefunciton. We derive

$$\frac{f_r}{f_i} = \frac{v_2 - v_1}{v_1 + v_2} \quad \text{and} \quad \frac{f_t}{f_i} = \frac{2v_2}{v_1 + v_2}$$

Natural conditions on the string displacements We assume that the wavefunction is continuous and differentiable at all positions. Also,

$$\dot{\psi}(x,0) = 0$$

allows us to easily complexify the solution.