PHYS 202 Formula Sheet

Daniel Son

Simple Harmonic Oscillators Consider a mass attatched to a spring. Let $x(t)$ be the function of displacement of the mass from the equilibrium position. Suppose that the spring constant is k. By Newton's 2nd Law,

$$m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} = -\frac{k}{m}x$$

Where $\dot{x} := \frac{d}{dt}x$

The following function solves the equation.

$$x(t) = Re(\tilde{A}e^{ii\omega_0 t}) = A\cos(\omega_0 t + \phi)$$

A, ω_0 is referred as the amplitude and the natural frequency of the oscillator. Also,

$$A = \sqrt{\frac{2E}{k}} \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Phase Conventions Velocity leads Displacement by a phase of $\pi/2$. Acceleration leads Velocity by a phase of $\pi/2$

Say $V \sim \cos(\omega t)$ and $I \sim \cos(\omega t + \phi)$ for some positive phase $\phi \leq \pi$. The current leads the voltage and the voltage trails the current.

Simple RLC Consider a circuit where R, L, C is connected in parallel. Let the current be I. By the loop rule,

$$-\frac{q}{C} - RI - L\dot{I} = 0 \quad \text{or} \quad \frac{q}{C} + R\dot{q} + L\ddot{q} = 0$$

Rewrite this in the follwoing form.

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = 0$$

Compare this with the equation for damped driven oscillators.

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

So the following isomorphism holds

$$x \mapsto q \quad \text{then} \quad (m, b, k) \mapsto (L, R, 1/C)$$

In a parallel circuit, the current through each circuit component is identical. By the complexified Ohms Law $\tilde{V} = Z\tilde{I}$. Recall the impedences.

$$Z_R = R \quad \text{and} \quad Z_C = \frac{1}{i\omega C} \quad \text{and} \quad Z_L = i\omega L$$

Thus, the voltage of the resistor leads the voltage on the capacitor by phase $\pi/2$. Likewise, the voltage on the resistor trails the voltage on the Inductor by $\pi/2$.

Circuit Filters and Loglog plot Using inductors and capacitors, it is possible to filter out signals of high or low frequency. Consider the behavior of circuit elemnts in high and low frequency.

$$\lim_{\omega\to 0} Z_L = \lim_{\omega\to 0} i\omega L = 0 \quad \text{and} \quad \lim_{\omega\to\infty} Z_L = \lim_{\omega\to\infty} i\omega L = \infty$$

$$\lim_{\omega\to 0} Z_C = \lim_{\omega\to 0} \frac{1}{i\omega C} = \infty \quad \text{and} \quad \lim_{\omega\to\infty} Z_C = \lim_{\omega\to\infty} \frac{1}{i\omega C} = 0$$

The inductor will block signals of high frequency but allow the passage of signals of low frequency. Hence, inductors act as a **low-pass** filter. On the other hand, capacitors block signals of low frequency and allow the passage of high frequency signals. Hence, capacitors act as a **high-pass** filter.

Corner frequency vs Resonant frequency For an RLC circuit, it is possible to tune the frequency such that the combined impedence reach zero. This frequency is called the resonant frequency. For a circuit involving one R, L, C, the resonant frequency is computed by the following formula.

$$\omega_r = \frac{1}{\sqrt{LC}}$$

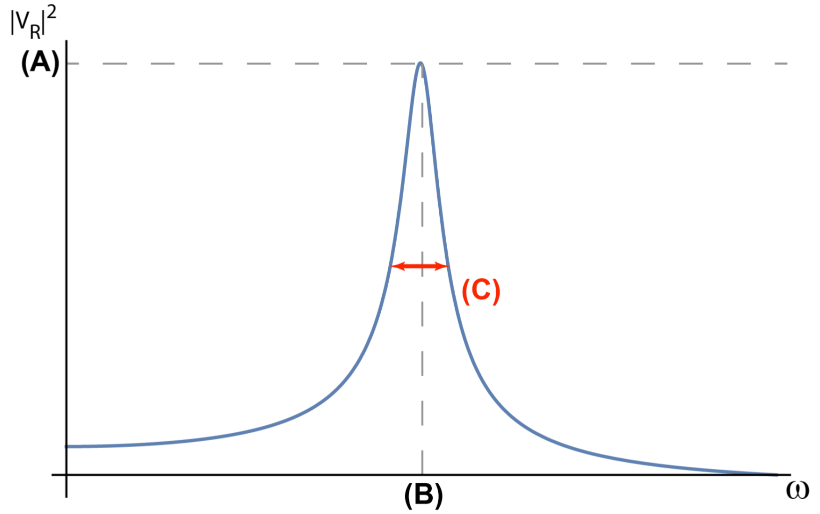
For and RL or an RC circuit, the imaginary part of the impedence cannot reach zero. It is possible to defnine a corner frequency where the circuit behavior changes drastically. It is the freqnecy where $Im(Z_L) = R$ or $Im(Z_C) = R$. In other words,

$$\omega_c = \frac{1}{RC} \quad \text{or} \quad \frac{R}{L}$$

Dimensions of C and L

$$[L] = H = \Omega \cdot s \quad \text{and} \quad [C] = \frac{s}{\Omega}$$

Power Resonance Curve For an RLC circuit, it is possible to plot the relationship between the angular frequency of the imput voltave and the power lost through the resistor. This curve is called the Power Resonance curve.



$P \sim V^2$. The Full-Width-Half-Maximum refers to the width of the curve at the half maximum point. The FWHM of the Power Resonance Curve is exactly $\gamma = R/L$, the damping factor. Be careful of the factor of 2π when the x-axis is set as frequency (Hz).

Also, use the dimension $[\omega] = rad/s$ and $[f] = Hz$. The frequency refers to cycles per second.

Q-factor For damped driven oscillators, the Q-factor is an important number.

$$Q := \frac{\omega_0}{\gamma} = \frac{A_{res}}{A_0}$$

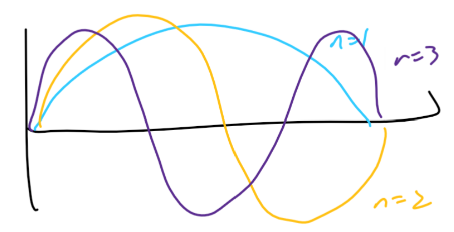
Boundary conditions for N masses The solutions for the n mass oscillators are in the form of:

$$\psi(m,t) = Ae^{i(\omega t + k_m a)} + Be^{i(\omega t - k_m a)}$$

Plugging into the 2nd order DE, we derive an expression for the angular frequency.

$$\omega_m = 2\omega_0 \sin(k_m \frac{a}{2})$$

k_m is the mth wave number. Recall that $k := \frac{2\pi}{\lambda}$. λ_n can be computed by drawing diagrams.



With some algebraic hassle, it is possible to derive the expressions for k_m.

For closed-closed and open-open ends:

$$k_m a = \frac{m\pi}{n+1} \qquad \text{or} \qquad k_m = \frac{m\pi}{a(n+1)} = \frac{m\pi}{L}$$

Where $m \in \mathbb{Z}^+$

For open-open ends:

$$k_m a = \frac{m\pi}{(2n+1)/2} \qquad \text{or} \qquad k_m = \frac{m\pi}{L}$$

Where $m \in \mathbb{Z}^+$

Free end Distance relation Always set the origin to be distance $a/2$ apart from the free end. This is because the two masses at the boundary must be symmetric. Refer to pset 5-1.

Transverse waves From the geometry of the springs, we use approximation. Let theta be the angle between the horizontal axis and the string. $\tan(\theta) \approxeq \theta = \Delta y/a$. Consequently, we arrive at:

$$k \mapsto T/a$$

Dispersion Relation The differential equation of the masses in the center oscillators provide a closed form equation for the angular frequency in terms of the wave number k.

$$\omega(k) = 2\omega_0 \sin(ka/2)$$

where a is the distance between the masses.

Dispersion Relation 2 The dispersion relation defines the wave number k.

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} \qquad \text{then} \qquad \omega^2 = c^2 k^2$$

And c is called the phase velocity of the wave. The wave number depends on the frequency of the wave, which is not necessarily the normal mode frequency

Nodes and Antinodes In a standing wave, nodes are the points that do not move. Max amplitude is achieved at antinodes.

Waves traveling in different medium, same tension Consider a transverse wave moving from one string of linear mass density μ_1 to another string with linear mass density μ_2. The velocity of the waves on each string is entirely determined by the lmd.

$$(v_1, v_2) = \left(\sqrt{\frac{T}{\mu_1}}, \sqrt{\frac{T}{\mu_2}}\right)$$

Let the incident wave to be in the form of

$$\psi_i(x,t) := f_i(t - x/v_1)$$

We make a natural assumption that the combined wavefunction must be smooth with respect to position. Let f_t, f_r be the simplified translated wavefunction and the simplified reflected wavefunciton. We derive

$$\frac{f_r}{f_i} = \frac{v_2 - v_1}{v_1 + v_2} \quad \text{and} \quad \frac{f_t}{f_i} = \frac{2v_2}{v_1 + v_2}$$

Natural conditions on the string displacements We assume that the wavefunction is continuous and differentiable at all positions. Also,

$$\dot{\psi}(x,0) = 0$$

allows us to easily complexify the solution.

Crash-intro to Fourier Transform

$$\boxed{f(x) = \int_{-\infty}^{\infty} C(k) e^{ikx} dk} \quad \text{where} \quad \boxed{C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx} \tag{43}$$

Non-dispersive Waves refers to waves in which the variance of frequency and wavelength of the wave does not affect the velocity of the wave. A good example is a traverse wave on a string.

$$v = \sqrt{\frac{T}{\mu}}$$

where T is the tension of the string and μ refers to mass density.

Mechanical impedence and transmission across different tension We define mechanical impedence of a string as follows.

$$Z = \frac{F}{v} = \sqrt{T\mu}$$

Mechanical impedence is analogous to drag force the string exerts on the ends. That is $Z \mapsto b$. Using impedence, we can write out the reflection and transmission coefficients. Let Z_1, Z_2 denote the mechanical impedence of line 1 and line 2 where the wave is traveling from line 1 to line 2. The reflection and transmission coefficients are written as:

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad \text{and} \quad T = \frac{2Z_1}{Z_1 + Z_2}$$

Reflection happens by a small amount, and transmission is dependant on the original string. Perfect transmission happens when impedences are matched. That is, $Z_1 = Z_2$.

Space-Time relation of traveling waves We know, mathematically, that the solutions to the wave equations are in the form of

$$\psi(x,t) = f(x - vt)$$

Where v is the longditudanal speed of the force. By applying the chain rule, we can deduce

$$\dot{\psi} = -v\psi'$$

This can also be deduced by drawing the following diagram

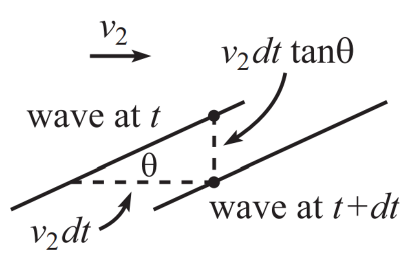


Figure 16

Energy and Power of a traveling wave Considering the string as a conglomerate of segments, we can retrieve a formula for the energy density of the wave. Note that the potential energy of the string is compted by Tdl where dl refers to stretch. The kinetic energy is related to the time derivative, and the potential energy is related to the space derivative. Applying the wave equation, we derive

$$\mathcal{E} = \mu\dot{\psi}^2 = \mu v^2 (\psi')^2$$

So the wave equation somewhat coagulates the space and time dependance of energy. The energy density of a wave is product of the lmd and square of the time derivative of the wave.

It makes sense that the power transmission is the negativ product of linear energy density and transmission speed. This can be derived rigorously using $\dot{\psi} = -v\psi'$.

$$P(x,t) = -v\mathcal{E}$$

Analysis of Standing waves using impedence Fixed ends can be considered as a string connected to another string with infinite impedence. Free ends likewise can be considered as string connected to another spring with zero impedence. Enforcing continuity and smoothness we can derive an equation for standing waves.

Choosing the correct type of wave form The solutions to the wavefunction come in the form of

$$\psi(x,t) = Ae^{i(kx-\omega t)} \quad \text{or} \quad Ae^{i(\omega t - kx)}$$

When imposing the boundary condition which is space-dependant, it is more useful to adapt the latter form.

Crash intro to optics The four Maxwell equations govern the behavior of EM waves. We are especially interested in the case of waves traveling in isotropic media. That means, zero charge density and linear current density. The four Maxwell Equations are presented as below.

$$\begin{aligned} \triangledown \vec{E} &= -\mu_0 \vec{H} & \qquad \triangledown \vec{H} &= -\epsilon_0 \vec{B} \\ \dot{\vec{E}} &= 0 & \dot{\vec{H}} &= 0 \end{aligned}$$

The field $\vec{H}$ is rigorously defined in PM 11.9, but for now, just remember

$$\vec{H} = \frac{1}{\mu_0} \vec{B}$$

μ refers to permeability and ϵ refers to permittivity. The value of permeability is usually fixed to be μ_0 for transparent medium. The value of permittivity depends on the medium. The diaelctric constant is defined as

$$\kappa := \frac{\epsilon}{\epsilon_0}$$

The velocity of an EM wave, aka light, depends on the traveling medium. The index of refraction is defined as

$$n := \frac{c}{v}$$

. Note that the index is defined as the reciprocal of the more intuitive definition.

From E&M, recall that the energy transmission was governed by the Poynting vector,

$$\vec{S} := \vec{E} \times \vec{H}$$

From the Maxwell's equations, we deduce the magnitude of $\vec{H}$.

$$H = \frac{nE}{Z_0}$$

Where $Z_0 := \sqrt{\mu_0/\epsilon_0} = 377\Omega$ is defined as the impedence of free space.

We also deduce that irradiance (or intensity) is proportional to square of the E field.

$$I = \frac{n}{2Z_0}E_0^2$$

<u>Jones Calculus for polarization</u> Consider an EM wave traveling in isotropic media. It is a convention to consider the electric field of the wave for polarization. The electric field has both x and y components which are sinusoids dependant on time and the transmission axis z. That is,

$$\vec{E} = \hat{x}E_x + \hat{y}E_y \qquad \text{where}$$

$$E_x = E_{x0}\cos(kz - \omega t) \quad \text{and} \quad E_y = E_{y0}\cos(kz - \omega t + \phi)$$

Complexifying E_x, E_y, we can write

$$\tilde{E}_x = E_{x0}e^{i(kz-\omega t)} \quad \text{and} \quad \tilde{E}_y = E_{y0}e^{i(kz-\omega t+\phi)}$$

So, the vector $\tilde{\vec{E}}$ can be written as follows

$$\tilde{\vec{E}} = \begin{bmatrix} E_{x0} \\ E_{y0}e^{i\phi} \end{bmatrix} e^{i(kz-\omega t)}$$

The exponential term outside the matrix is insignificant to describe polarizaton. We define the constant matrix as the **Jones vector** of the polarization. Factoring out a global phase factor, a Jones matrix is in the form of

$$L\begin{bmatrix} 1 \\ e^{i\phi} \end{bmatrix} \quad \text{or} \quad R\begin{bmatrix} e^{i\phi} \\ 1 \end{bmatrix}$$

for $\phi \in [0, \pi]$. The first matrix represents a light in which the y-axis has a *slower* phase. Shift the y-sinusoid of the EM wave to deduce that the light is left polarized. Similarly, the second matrix represents a right polarized light.

The effect of an optical element can be represented as a linear transform on the Jones vector. We call this 2x2 matrix a **Jones Matrix**. So far, we have covered two major optical elements, the linear polarizer and the waveplate. Using the bra-ket notation, their Jones matrix can be written as follows.

$$L_{\vec{v}} = |\vec{v}\rangle\langle\vec{v}| \quad \text{and} \quad W(\hat{f}, \hat{s}, \phi) = e^{i\phi}|\hat{s}\rangle\langle\hat{s}| + |\hat{f}\rangle\langle\hat{f}|$$