Combinatorics HW7

Daniel Son

Q1 sums of fibbonacci numbers

Find a closed form solution for all the partial sums defined below.

a)

$$s_n := \sum_{i=1}^{n} f_{2i-1}$$

b)

$$s_n := \sum_{i=0}^{n} f_{2i}$$

c)

$$s_n := \sum_{i=0}^{n} (-1)^i f_i$$

d)

$$s_n := \sum_{i=0}^{n} f_i^2$$

Solution

1.

$$s_1 = 1 \quad \text{and} \quad s_n = f_{2n}$$

2.

$$s_0 = 0 \quad \text{and} \quad s_n = f_{2n+1} - 1$$

3.

$$s_n = (-1)^n f_{n-1} - 1 \quad \text{and} \quad f_{-1} := 1$$

4.

$$s_n = f_{n+1}^2 - f_n^2 - (-1)^n$$

Proof a The base case holds trivially. $s_1 = f_1 = 1$. For $n > 1$,

$$s_n = f_{2n-1} + s_{n-1} = f_{2n-1} + f_{2n-2}$$

by the inductive hypothesis. Use the recursive relation for f_n.

$$\boxed{s_n = f_{2n}}$$

Proof b The base case holds trivially. $s_0 = f_1 - 1 = 0$. For $n > 0$,

$$s_n = f_{2n} + s_{n-1} = f_{2n} + f_{2n-1} - 1$$

by the inductive hypothesis. Use the recursive relation for f_n.

$$\boxed{s_n = f_{2n+1} - 1}$$

Proof c Apply a similar argument for a, b. $s_0 = (-1)^0 f_{-1} - 1 = 1 - 1 = 0$ for the base case. For the inductive case,

$$s_{n+1} = (-1)^{n+1} f_{n+1} + s_n = (-1)^{n+1} f_{n+1} + (-1)^n f_{n-1} - 1$$
$$= (-1)^{n+1}(f_{n+1} - f_{n-1}) - 1 = (-1)^{n+1} f_n - 1$$

which concludes the proof. □

Proof d We present an arbitrary polynomial division that will come in handy later.

$$x^2 - x - 1 \parallel x^4 - 3x^2 + 1$$

and this is over the field of real numbers. This implies that the roots of x^2-x-1 are also roots for $x^4 - 3x^2 + 1$.

Now we prove the identity

$$3f_{n+1}^2 - f_{n+2}^2 - f_n^2 - 2(-1)^n = 0 \tag{1}$$

Recall that the nth fibbonacci number can be represented as follows.

$$f_n = \frac{p^n - q^n}{\sqrt{5}}$$

p, q are $(1+\sqrt{5})/2, (1-\sqrt{5})/2$ respectively. They are roots of the polynomial $x^2 - x - 1$. To prove the desired identiy in (1), we evaluate the left hand side directly and show that it is identically zero . We obtain

$$\begin{aligned}\frac{3}{5}\left[p^{2n+2} + q^{2n+2} - 2(pq)^{n+1}\right] - \frac{1}{5}\left[p^{2n+4} + q^{2n+4} - 2(pq)^{n+2}\right]\\ -\frac{1}{5}\left[p^{2n} + q^{2n} - 2(pq)^n\right] - 2(-1)^n\end{aligned} \tag{2}$$

$$\begin{aligned}&= -\frac{1}{5}\left[(p^4 - 3p^2 + 1)p^{2n} + (q^4 - 3q^2 + 1)q^{2n}\right]\\ &\quad -\frac{6}{5}(pq)^{n+1} + \frac{2}{5}(pq)^{n+2} + \frac{2}{5}(pq)^n - 2(-1)^n\end{aligned} \tag{3}$$

We know that p, q are roots to the quintic polynomial. Also, we know $pq = 1$.

$$= 0 - \frac{6}{5}(-1)^{n+1} + \frac{2}{5}(-1)^{n+2} + \frac{2}{5}(-1)^n - 2(-1)^n$$
$$= \frac{6+2+2}{5}(-1)^n - 2(-1)^n = 0$$

Identity (1) can be slightly manipulated as

$$f_{n+2}^2 - f_{n+1}^2 - (-1)^{n+1} = 2f_{n+1}^2 - f_n^2 - (-1)^n \tag{4}$$

Now, we purport

$$\boxed{\sum_{i=0}^{n} f_i^2 = f_{n+1}^2 - f_n^2 - (-1)^n}$$

We prove this formula by induction on the integer n. For the base case $n = 1$,

$$0 = f_1^2 - f_0^2 - (-1)^0 \quad \text{or} \quad 0 = 1 - 0 - 1 \quad \text{or} \quad 0 = 0$$

Which is a tautology.

Assume the formula holds for n. We prove the formula for $n + 1$. That is, assuming (4), we prove

$$\sum_{i=0}^{n+1} f_i^2 = f_{n+2}^2 - f_{n+1}^2 - (-1)^{n+1}$$

We evaluate the left sum to show equality.

$$\begin{aligned}\sum_{i=0}^{n+1} f_i^2 = f_{n+1}^2 + \sum_{i=0}^{n} f_i^2 &= f_{n+1}^2 + f_{n+1}^2 - f_n^2 - (-1)^n \\ &= 2f_{n+1}^2 - f_n^2 - (-1)^n\end{aligned}$$

Apply equation (4).

$$\sum_{i=0}^{n+1} f_i^2 = f_{n+2}^2 - f_{n+1}^2 - (-1)^{n+1}$$

$\square$

Q8 A 1xn chessboard is given. Each block can be colored either red or blue. No two consecutive squares can be colored red. Compute h_n, the number of distinct colorings of the chessboard.

Solution We define two auxillary sequances r_n, b_n which count the number of colorings that satisfy the coloring rule and has either a red or a blue coloring on the nth block. We notice $r_1 = b_1 = 1$. Also, by the nature of the coloring

$$r_{n+1} = b_n \quad \text{and} \quad b_{n+1} = r_n + b_n$$

To color the last square red, the previous square must be colored blue. To complete the board with blue, the previous coloring does not matter. Also, we can write

$$b_{n+1} = b_n + b_{n-1}$$

Which is the linear recurrence for the fibbonacci sequence. Also, $b_1 = 1, b_2 = 2$ so we conclude $b_n = F_n$. It follows that $r_n = F_{n-1}$.

$$\boxed{h_n = b_n + r_n = F_n + F_{n-1} = F_{n+1}}$$

Q10 Resolve Fibbonacci's rabbit problem assuming that the cage has 2 pairs of rabbits in the beginning.

Solution A simple solution is to consider the new cage as two cages that each include one rabbit pair. It follows that the number of pairs in the nth month is $2F_n$.

In mathematical language, the linear recurrence relation is linear. Let G_n be the number of rabbits in the nth month in the new cage. We know $G_n = 2F_n$ satisfies the recurrence relation $G_{n+2} = G_{n+1} + G_n$ for any $n \geq 0$. Also, $G_0 = 2 \cdot F_0 = 2$ satisfies the initial condition. Hence, $G_n = 2F_n$.

So the answer is $\boxed{2F_{12} = 466}$

Q12 Define a sequence $h_n := n^3$. Verify the recurrence relation

$$h_n = h_{n-1} + 3n^2 - 3n + 1$$

Proof Prove the relation directly by subsituting h_n with its definition. We evaluate the RHS.

$$h_{n-1} + 3n^2 - 3n + 1 = (n-1)^3 + 3n^2 - 3n + 1$$

Apply the binomial theorem to expand the cubic.

$$= (n^3 - 3n^2 + 3n - 1) + 3n^2 - 3n + 1 = n^3 = h_n$$

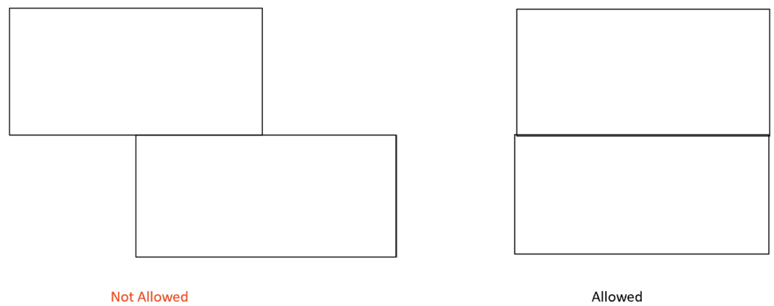
□

Additional Problem 1 Let sequence g_n count the number of coverings of a 2xn chessboard with dominos. Prove that g_n satisfies the fibbonacci recurrence relation by a bijective proof.

Solution We know that the number of ways to disect a word of length n into bead of length 1 and 2 satisfied the fibbonacci recurrence relation. That is, if f_n counts the number of ways to break down a word of length n into beads smaller than 2,

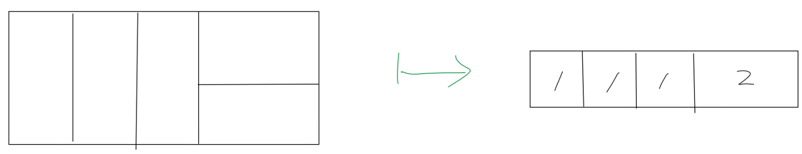
$$f_{n+2} = f_{n+1} + f_n$$

Consider a domino tiling of a 2xn board. Either the domino must be arraged horizontal or vertical. If a domino is arranged horizontally, a parallel domino must be placed on top of it or under it



The slanted orientation would have to repeat itself to produce a perfect cover. As the pattern reaches the boundary of the board, it is evident that a perfect cover is not attainable.

Map the two horizontal dominos as a long bead and a single vertical domino as a short bead. Below is an example.



From any covering, it is possible to obtain a unique arrangement of beads. Also, from any bead arrangement, it is possible to create a covering. Hence, this mapping is bijective. The number of bead arrangements satisfy the fibbonacci recurrence relation, and so does the number of domino coverings.

$$g_{n+2} = g_{n+1} + g_n$$

$\square$

Additional Problem 2 Find the generating functions for the given sequences.

Solution The general strategy is to use derivatives and series sums in a clever way to obtain a nice closed form for the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

The algebra behind deriving the sums are mostly trivial. We demonstrate the process for one sample sequence

$$a_n = n^2$$

From the geometric sequence formula,

$$1 + x + x^2 + \cdots + x^n \cdots = \frac{1}{1-x}$$

Or, in sigma notation,

$$\sum_{x=0}^{\infty} x^n = \frac{1}{1-x}$$

Take the derivative once to obtain

$$\sum_{x=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \tag{5}$$

Multiply both sides by x.

$$\sum_{x=0}^{\infty} nx^{n} = \frac{x}{(1-x)^2} \tag{6}$$

From (5), take the derivative again.

$$\sum_{x=0}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}$$

Multiply both sides by x^2

$$\sum_{x=0}^{\infty} n(n-1)x^{n} = \frac{2x^2}{(1-x)^3} \tag{7}$$

Add (6), (7) and write

$$\sum_{n=0}^{\infty} (n(n-1)+n)x^{n} = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

Which simplifies to

$$\sum_{n=0}^{\infty} n^2 x^{n} = \frac{x^2+x}{(1-x)^3}$$

Here is a table of sequences and their corresponding generating functions.

a_n	GF
2	$\frac{2}{1-x}$
2^n	$\frac{1}{1-2x}$
$n-1$	$\frac{2x-1}{1-x^2}$
$(n\%3=0)$	$\frac{1}{1-x^3}$
$\binom{n}{2}$	$\frac{x^2}{(1-x)^3}$
n^2	$\frac{x^2+x}{(1-x)^3}$

Mathematica confirms some of these results

In[2]:=

```
Series[
 1/(1 - 2x),
 {x, 0, 10}
]
```

Out[2]= $1+2x+4x^2+8x^3+16x^4+32x^5+64x^6+128x^7+256x^8+512x^9+1024x^{10}+O[x]^{11}$

In[4]:=

```
Series[
 (2x - 1)/(1-x)^2,
 {x, 0, 10}
]
```

Out[4]= $-1+x^2+2x^3+3x^4+4x^5+5x^6+6x^7+7x^8+8x^9+9x^{10}+O[x]^{11}$

In[5]:=

```
Series[
 1/(1-x^3),
 {x, 0, 10}
]
```

Out[5]= $1+x^3+x^6+x^9+O[x]^{11}$

In[6]:=

```
Series[
 x^2/(1-x)^3,
 {x, 0, 10}
]
```

Out[6]= $x^2+3x^3+6x^4+10x^5+15x^6+21x^7+28x^8+36x^9+45x^{10}+O[x]^{11}$

In[7]:=

```
Series[
 (x^2 + x)/(1-x)^3,
 {x, 0, 10}
]
```

Out[7]= $x+4x^2+9x^3+16x^4+25x^5+36x^6+49x^7+64x^8+81x^9+100x^{10}+O[x]^{11}$

Additonal Problem 3 Find the generating function for the number of ways to make change equal to $n$cents out of nickels, dimes, and silver dollars.

Solution Provided an infinite number of nickels, the generating function to make n cents is as follows.

$$1+x^5+x^{10}+\cdots=\frac{1}{1-x^5}$$

and similarly, we write the generating functions for only using the dimes and dollars.

$$\frac{1}{1-x^{10}} \quad \text{and} \quad \frac{1}{1-x^{100}}$$

To find the generating function of using all three kinds of coins, we multiply the generating functions.

$$\boxed{G(x) = \frac{1}{(1-x^5)(1-x^{10})(1-x^{100})}}$$

Additional Problem 4 Find a GF for the number of solutions to the equation

$$e_1 + e_2 + e_3 + e_4 = n$$

given the restrictions on the variable e_i.

Solution Adding a little rigor to our notion of generating functions would help. We prove the following claim. For the rest of this problem, let e_i be some variable that takes integer values.

Notation We intorduce a notation to express the number of solutions to a particular equation. Given variables e_i for some integers i, the number of solutions of the equation

$$\sum e_i = n$$

is denoted by

$$\langle e_1, e_2, \ldots, e_i \rangle_n$$

Depending on the value of n, this value can be interpreted as a sequence. Also, we provide a notation for the generating function of this sequence.

$$G\langle e_1, e_2, \ldots, e_i \rangle(x)$$

The restrictions on e_i determine the generating function $G\langle e_i \rangle(x)$. For example, if e_i is restricted to be a positive integer,

$$G\langle e_i \rangle(x) = x + x^2 + \cdots = \frac{x}{1-x}$$

This fact trivially follows from the definition. $e_i = n$ has a solution if and only if e_i is allowed to be the value n.

Proposition Adding two variables in the equations results in multiplication of the generating functions.

$$G\langle e_1, e_2, \ldots, e_n \rangle(x) = \prod_{i=1}^{n} G\langle e_i \rangle(x)$$

Proof We prove by induction on n. The base case is trivial. By the inductive hypothesis, we assume

$$G\langle e_1, e_2, \ldots, e_n \rangle(x) = \prod_{i=1}^{n} G\langle e_i \rangle(x)$$

Now, we wish to find the generating function

$$G\langle e_1, e_2, \ldots, e_{n+1}\rangle(x)$$

In order to find the generating function, we must find the value of the sequence

$$\langle e_1, e_2, \ldots, e_{n+1}\rangle_n$$

Which is the number of solutions to the equation

$$e_1 + e_2 + \cdots + e_{n+1} = N$$

We partition all the solution based on the possible values of e_{n+1} Fix the value of $e_{n+1} = l$. The size of the corresponding part will be the number of solutions to

$$e_1 + e_2 + \cdots + e_n = N - l$$

Which is in fact, the value $\langle e_1, e_2, \ldots, e_n\rangle_{N-l}$. This value is given my the coefficient of x^{N-l} of the polynomial $G\langle e_1, \ldots, e_n\rangle(x)$

Consider the poynomial

$$G\langle e_1, \ldots, e_n\rangle(x)G\langle e_{n+1}\rangle = \prod_{i=1}^{n+1} G\langle e_i\rangle(x)$$

where the equality follows by the inductive hypothesis. The coeffieient of x^N of this polynomial will be the sum of the coefficients of x^{N-l} in the polynomial $G\langle e_1, \ldots, e_n\rangle(x)$ for all values of l which $G\langle e_{n+1}\rangle$ is nonzero. In symbols, the x^N coefficient is

$$\sum_{l \text{ valid}} \langle e_1, e_2, \ldots, e_n\rangle_{N-l} = \langle e_1, e_2, \ldots, e_{n+1}\rangle_N$$

We have directly shown that

$$\prod_{i=1}^{n+1} G\langle e_i\rangle(x)$$

Is a generating function of $\langle e_1, e_2, \ldots, e_{n+1}\rangle_N$.

□

In light of this powerful machinery, we can find the GFs for variables that are independant.

<u>Question</u> Find $G\langle e_1, e_2, e_3, e_4\rangle(x)$ for

a) $e_i \in \mathbb{Z}^+$ for all $i \leq 4$

b) $e_i \in \mathbb{Z}^+$ for all $i \leq 4$ and e_i is a multiple of i

c) $e_i \in \mathbb{N}$ for all $i \leq 4$ and $e_2 = 2e_1$, $e_3 + e_4 = 4$

<u>Answer</u>

a)

$$G\langle e_1, e_2, e_3, e_4\rangle(x) = \left(\frac{x}{1-x}\right)^4$$

b)

$$\begin{aligned} G\langle e_1, e_2, e_3, e_4\rangle(x) &= \frac{x}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\frac{x^4}{1-x^4} \\ &= \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \end{aligned}$$

c) This is trickier since the variables are interrelated. Write the equation

$$e_1 + e_2 + e_3 + e_4 = N$$

Applying the substitution, this condition simplifies to

$$3e_1 = N - 4 \tag{8}$$

e_2 is entirely determined by e_1. Also, there are five solutions to the eqation $e_3 + e_4 = 4$ regardless of the value N.

We directly compute the generating function by counting the values of Nm which yields a solution for (8). $e_1 \in \mathbb{N}$, so allowed values of N are $N = 4, 7, 10, 13, \ldots$. Finally, we write the generating function.

$$G\langle e_1, e_2, e_3, e_4\rangle(x) = 5(x^4 + x^7 + x^{10} + \cdots) = \frac{5x^4}{1-x^3}$$

<u>Additional Problem 5</u> A certain flower shop has 2 black roses, 2 white roses, and an unlimited number of red roses. Let b_n be the number of ways to form a bouquet of n roses, where $n \in N$.

a) Find a formula for b_n

b) Find the GF of b_n

<u>Solution</u> In fact, it is easy to write out a GF for b_n using our Proposition proved in the previous problem.

$$\begin{aligned} \boxed{G(x) = \frac{(1+x+x^2)^2}{1-x}} &= \frac{9}{1-x} - (8 + 6x + 3x^2 + x^3) \\ &= 1 + 3x + 6x^2 + 8x^3 + 9x^4 + \cdots \end{aligned}$$

Note that the coefficient for x^i for all $i \geq 4$ is 9. Thus, we write the formula for b_n.

$$\boxed{(b_0, b_1, b_2, b_3) = (1, 3, 6, 8) \quad \text{and} \quad b_i = 9 \ \forall i \geq 4}$$