PHYS 201 Problemset 10

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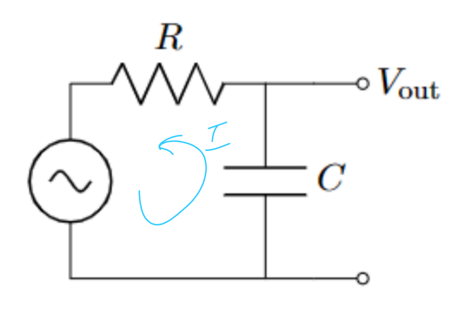


Figrure for Q1

Q1-a Let $V_{in} = V_0 cos(\omega t)$. Use complex impedence to find the amplitude of V_{out}

Solution We know that the loop rule applies to impedences. Write:

$$V_0 = \tilde{I} z_R + \tilde{I} z_C$$

where z_R, z_C denotes the impedence of the capacitor and the resistor. By the impedence formula:

$$V_0 = \tilde{I}(z_R + z_C) \quad \text{and} \quad \tilde{I} = \frac{V_0}{R - \frac{i}{\omega C}}$$

To compute the impedence of V_out:

$$\tilde{V}_{out} = R\tilde{I} = \frac{RV_0}{R - \frac{i}{\omega C}}$$

Finally, take the modullus of $\tilde{V}_{out}$ to compute the amplitude.

$$\boxed{V_{out} = \left|\frac{RV_0}{R - \frac{i}{\omega C}}\right| = \frac{RV_0}{\sqrt{(R - \frac{i}{\omega C})(R + \frac{i}{\omega C})}} = \frac{V_0}{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}}}$$

Q1-b Compute the phase shift of V_{out}

Solution The phase shift can be easily computed by dividing the impedence by amplitude.

$$e^{i\theta} = \tilde{V}_{out}/V_{out} = R\tilde{I} = \frac{RV_0}{R - \frac{i}{\omega C}} \Bigg/ \frac{V_0}{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}} = \frac{\sqrt{1 + \frac{1}{\omega^2 C^2 R^2}}}{1 - \frac{i}{\omega CR}}$$

Thus:

$$e^{i\theta} = \sqrt{\frac{1+\frac{i}{\omega CR}}{1-\frac{i}{\omega CR}}} \quad \text{or} \quad e^{2i\theta} = \frac{1+\frac{i}{\omega CR}}{1-\frac{i}{\omega CR}}$$

With a little bit of geometry, it is possible to deduce:

$$2\theta = 2arctan(\frac{1}{\omega CR}) \quad \text{and} \quad \boxed{\theta = arctan(\frac{1}{\omega CR})}$$

□

<u>Q1-c</u> Repeat the analysis for a circuit where the capacitor is replaced by an inductor.

<u>Solution</u> Rewrite the loop rule as:

$$V_0 = \tilde{I}z_L + \tilde{I}z_R \quad \text{and} \quad V_0 = \tilde{I}(z_L + z_R) \quad \text{and} \quad \tilde{I} = \frac{V_0}{z_L + z_R}$$

Applying the impedence formula:

$$\tilde{I} = \frac{V_0}{i\omega L + R} \quad \text{and} \quad \tilde{V}_{out} = \frac{V_0 R}{i\omega L + R} = \frac{V_0}{i\omega L/R + 1}$$

Compute the modullus and the argument for the amplitude and phase.

$$\boxed{V_{out} = |V_{out}| = \frac{V_0}{1 + \omega^2 L^2/R^2}}$$

With rationalization, $\tilde{V}_{out}$ reduces to:

$$\tilde{V}_{out} = \frac{V_0(1 - i\omega L/R)}{(1 + i\omega L/R)(1 - i\omega L/R)} = \frac{V_0(1 - i\omega L/R)}{1 + \omega^2 L^2/R^2}$$

For the denominator is a real value, it suffices to compute the argument of the denominator to compute the argument of the impedence. We conclude:

$$\boxed{\theta = -arctan(\omega L/R)}$$

<u>Q1-d</u> A capacitor acts as a short circuit at high-frequency and an open circuit at low-frequency. Make an analogous statement for inductors.

<u>Statement</u> An inductor acts as an open circuit at high-frequency and a short circuit at low-frequency. The justification comes from observing V_{out} and its dependency on the angular frequency, ω. □

P&M 8.27 *RLC Parallel Circuit* A resistor, inductor, capacitor each of resistance 1k ohms, 500p farads, 2m henries are connected in parallel. The frequency is given as 10k cycles per second. Compute the combined impedence of the circuit. Also, compute the frequency where the magnitude of impedence is maximal.

Solution For the junction rule holds for impedence, we compute the combined impedence by taking the sum of the reciprocals of the impedence of each circuit components, and again dividing it from 1. In symbols:

$$z_{eff} = \frac{1}{1/z_c + 1/z_l + 1/z_r}$$

where z_c, z_l, z_r denotes the impedence of the capacitor, inductor, and resistor respectively. Applying the impedence formula:

$$z_{eff} = \frac{1}{\frac{1}{R} + \frac{1}{i\omega L} - \frac{\omega C}{i}} = \frac{i\omega LR}{i\omega L + R - \omega^2 CLR}$$

The frequency is given as $1kHz$ and $10mHz$, so the angular frequency is:

$$\omega_1 = 2\pi \cdot 10^4 Hz \quad \text{and} \quad \omega_2 = 2\pi \cdot 10^7 Hz$$

By the problem conditions, we have:

$$R = 1000\Omega \quad \text{and} \quad C = 5 \cdot 10^{-7} F \quad \text{and} \quad L = 2 \cdot 10^{-3} H$$

Also, the following unit conversions are useful:

$$[C] = F = \Omega^{-1} \cdot s \quad \text{and} \quad [L] = H = \Omega \cdot s$$

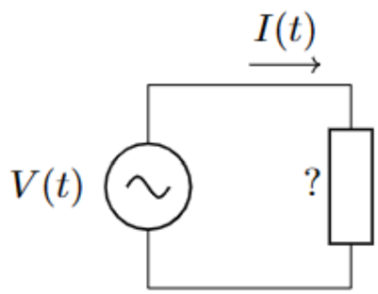
As a sidenote, to derive the two unit conversions, apply dimensional analysis on the following formulas:

$$Q = CV \quad \text{and} \quad \mathcal{E} = L\frac{dI}{dt}$$

With some algebra, with help of python, we conclude:

$$\boxed{z_{eff} \approxeq 15.7 + 124.2i \text{ for 10kHz} \quad \text{and} \quad z_{eff} \approxeq 1.01 - 31.8i \text{ for 10MHz}}$$

Q3 A mystery device is connected to an AC circuit with sinusodial input voltage. The frequency is 100kHz. The amplitude of the voltage and current is given as $V_0 = 5V, I_0 = 2mA$. The voltage leads the current by a phase $\pi/6$.



Circuit image for Q3

i) Compute the impedence of the mystery device

Solution
Write:

$$\tilde{V} = 5V \quad \text{and} \quad \tilde{I} = 2mAe^{-i\pi/6}$$

Thus:

$$z = \tilde{V}/\tilde{I} = 2.5 \cdot 10^3 e^{i\pi/6} \Omega$$

Or, in cartesian form:

$$\boxed{z = (2170 + 1250i)\Omega}$$

ii) What is a potential candidate for this mystery device?

Solution An equivalent of the device can be constructed by connecting a resistor with a resistor. Denote the resistance and the inductor as R, L. The combined impedence of connecting the two components in series is:

$$z_{cmb} = R + i\omega L$$

We deduce:

$$R = 2170\Omega \quad \text{and} \quad \omega L = 2\pi f L = 1250\Omega$$

Ergo:

$$\boxed{R = 2170\Omega \quad \text{and} \quad L = 2mH}$$

iii) Assume that the mystery device is indeed composed of the elements that were guessed in the previous problem. How will the amplitude of current change as the frequency is increased? Moreover, how will the change of frequency affect the phase difference between the voltage and current?

Solution As frequency increases, ω increases. This results in an increase of both the argument and the magnitude of the impedence. Recall:

$$z = \tilde{V}/\tilde{I} \quad \text{or} \quad \tilde{I} = \tilde{V}/z$$

Hence, larger magnitude of impedence leads to decrease of the amplitude of the current. Also, the larger the argument of the impedence, the larger the phase difference between voltage and current. □

Prelude to Q4 When talking about power in AC circuits, it is convinient to use root mean squared values for voltage and current. Note that RMS is NOT EQUAL to the simple mean.

Two following equations come in handy:

$$\bar{P} = V_{rms}^2/R \quad \text{and} \quad \bar{P} = V_{rms} I_{rms} cos(\phi)$$

As part of being physicists, we leave it as an exercise to the mathematicians to justify this equation.

To compute the rms value of a sinusodial function, remember:

$$A_{rms} = A_0/\sqrt{2}$$

where A_0 denotes the amplitude of the quantity.

Q4 An incandecent bulb consumes $60W$ of power if connected to a standard $120V, 60Hz$ AC circuit. We wish to connect this bulb to a circuit with $240V, 60Hz$ voltage and frequency. By connecting an inductor in series with the bulb, it is possible to make the bulb consume the same amount of power in average. What must be the inductance of the inductor?

Solution The bulb has an internal resistance. This can be easily computed by the power formula. Write:

$$P = V_{rms}^2/R \quad \text{and} \quad R = V_{rms}^2/P = (120V)^2/60W = 240\Omega$$

If the bulb is connected in series with an inductor, the magnitude of the combined impedence increases. Let L denote the inductance of the inductor. Write:

$$z_{eff} = i\omega L + R$$

$$\tilde{I} = V/z_{eff} \quad \text{and} \quad \tilde{V_b} = \tilde{I}R = \frac{VR}{z_{eff}}$$

Thus

$$V_b = \frac{VR}{|i\omega L + R|} = \frac{VR}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V}{\sqrt{1 + \omega^2 (L/R)^2}}$$

where V_b denotes the voltage through the bulb.

If $V_b = 120V$, then the bulb consumes the same power as connected to the standard circuit. $V = 240V$ so we wish to fix the denominator as 2. Then:

$$\sqrt{1+\omega^2(L/R)^2} = 2 \quad \text{or} \quad 1+\omega^2(L/R)^2 = 4$$

So:

$$L^2 = 3R^2/\omega^2 \quad \text{and} \quad L = \sqrt{3}R/\omega$$

Ergo:

$$\boxed{L \approxeq 1.1H}$$

P&M 9.18

Find the appropriate magnetic field for the given electric field below:

$$\vec{E} = E_0(\hat{x} + \hat{y})sin[2\pi/\lambda(z + ct)] \quad \text{and} \quad E_0 = 20V/m$$

Solution We tilt the x and y axis around the z axis. Consider the following change of axis from $(\hat{x}, \hat{y}, \hat{z}) \mapsto (\hat{i}, \hat{j}, \hat{z})$:

$$\hat{i} := \frac{\hat{x} + \hat{y}}{sqrt2} \quad \text{and} \quad \hat{j} := \frac{\hat{y} - \hat{x}}{sqrt2} \quad \text{and} \quad \hat{z} := \hat{z}$$

With some algebra, it is not hard to verify that $\hat{i} \times \hat{j} = \hat{z}$.

The electric field reduces to:

$$\vec{E} = \sqrt{2}E_0\hat{i}sin[2\pi/\lambda(z + ct)]$$

The magnetic field must be in the form of:

$$\vec{B} = B_0\hat{j}sin[2\pi/\lambda(z + ct)]$$

To compute the relation between E_0 and B_0, we invoke the differential version of Faraday's law, written formally as:

$$\Delta \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Plugging in the appropriate values:

$$-\hat{j}\sqrt{2}E_0\frac{2\pi}{\lambda}cos[(2\pi/\lambda)(z + ct)] = -\hat{j}\frac{2\pi}{\lambda}cB_0cos[(2\pi/\lambda)(z + ct)]$$

And by cancellation:

$$cB_0 = \sqrt{2}E_0 \quad \text{or} \quad B_0 = \frac{\sqrt{2}}{c}E_0$$

Substituting back to the original axis system, we obtain:

$$\boxed{\vec{B} = \frac{E_0}{c}(\hat{x} + \hat{y})sin[2\pi/\lambda(z + ct)]}$$

Additionaly:

$$\boxed{\frac{E_0}{c} \approxeq 6.67 \cdot 10^{-8}T}$$

$\square$

P&M 9.23

An electromagnetic field is given as follows:

$$\vec{E} = E_0 \hat{z} \cos(kx)\cos(ky)\cos(\omega t)$$

$$\vec{B} = B_0(\hat{x}\cos(kx)\sin(ky) - \hat{y}\sin(kx)\cos(ky))\sin(\omega t)$$

The relationship between E_0, B_0 is given as $E_0 = \sqrt{2}cB_0$ and $\omega = \sqrt{2}ck$. Show that this electormagnetic field satisfies the Maxwell's equations. Also, describe how this field looks like inside a box where $x, y \in [-\pi/2k, \pi/2k], z \in \mathbb{R}$.

Solution

Upon inspection, we observe that the divergence of both the electric and magnetic field is zero everywhere. The electric field only has a $\hat{z}$ component, but the component has no z dependance. As for the magnetic field, the partial derivatives related to the x, y component cancel out each other.

The divergence is a little complicated. But through many lines of algebra (which I have included as a snapshot at the end of the problem),

we can make the following relationship between E_0, B_0 from the Maxwell equations. As long as the following equalities hold, so does the Maxwell's equations.

$$kE_0 = \omega B_0 \quad \text{and} \quad B_0 = \frac{\mu_0 \epsilon_0 \omega E_0}{2k} = \frac{\omega E_0}{2kc^2}$$

Recall $E_0 = \sqrt{2}cB_0$ and $\omega = \sqrt{2}ck$. By plugging in, we notice that the two conditions above reduce into an identity.

Begin with the analysis on how the electric field looks like. Cutting the box by some plane perpendicular to the z axis, we notice that the field magnitudes increase as we move to the center of the box. The direction is parallel or opposite to the z axis. Also, the maximum magnitude, which is achieved at the intersection of the cut and the $z - axis$, oscillates in time.

The magnetic field has no z dependence. Looking at the same cut from the previous analysis, we note that the vector has some form of swirl around the center axis. At $x = 0$ and $y = 0$ the vector is zero. Also, all vectors oscilate in time. $\square$

Appendix Messy lines of algebra

$$\vec{E} = E_0 \hat{z} \cos(kx)\cos(ky)\cos(\omega t)$$

$$\vec{B} = B_0\left(\hat{x}\cos(kx)\sin(ky) - \hat{y}\sin(kx)\cos(ky)\right)\sin\omega t$$

$$\frac{\partial \vec{E}}{\partial t} = E_0 \hat{z}\cos(kx)\cos(ky)\left[-\sin(\omega t)\cdot\omega\right]$$

$$= -\omega E_0 \hat{z}\cos(kx)\cos(ky)\sin(\omega t)$$

$$\frac{\partial \vec{B}}{\partial t} = B_0\left(\hat{x}\cos(kx)\sin(ky) - \hat{y}\sin(kx)\cos(ky)\right)\omega\cos\omega t$$

$$\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \vec{E}/\hat{z} \end{vmatrix}$$

$\vec{E}/\hat{z}$ has both x,y dependance...

$$= \hat{x}\frac{\partial}{\partial y}\frac{\vec{E}}{\hat{z}} - \hat{y}\frac{\partial}{\partial x}\frac{\vec{E}}{\hat{z}}$$

Note, $\frac{\partial}{\partial y}\vec{E}/\hat{z} = E_0 \cancel{\hat{z}}\,(-k)\cos(kx)\sin(ky)\cos(\omega t)$

$$\frac{\partial}{\partial x}\vec{E}/\hat{z} = -E_0 \cancel{\hat{z}}\, k \sin(kx)\cos(ky)\cos(\omega t)$$

Thus, $\nabla\times\vec{E} = E_0\cos(\omega t)\left[\hat{x}\left[-k\cos(kx)\sin(ky)\right] + \hat{y}\left[k\sin(kx)\cos(ky)\right]\right]$

$$= -kE_0\cos(\omega t)\left[\hat{x}\left(\cos(kx)\sin(ky)\right) - \hat{y}\left(\sin(kx)\cos(ky)\right)\right]$$

So ~~the~~ Faraday implies $kE_0 = \omega B_0$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_0 \cos(kx)\sin(ky)\sin(\omega t) & -B_0 \sin(kx)\cos(ky)\sin(\omega t) & 0 \end{vmatrix}$$

$$= \left[-B_0 \frac{\partial}{\partial x}\left[\sin(kx)\cos(ky)\sin(\omega t)\right] - B_0 \frac{\partial}{\partial y}\left[\cos(kx)\sin(ky)\sin(\omega t)\right] \right] \cdot \hat{z}$$

$$= -B_0 k \left[\cos(kx)\cos(ky)\sin(\omega t) \cdot 2 \right] \hat{z}$$

The modified Ampere's Law in diff form dictates

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E} \iff$$

$$-2B_0 k \hat{z} \left[\cos(kx)\cos(ky)\sin(\omega t)\right] =$$
$$\mu_0 \varepsilon_0 \left[-\omega E_0 \hat{z} \cos(kx)\cos(ky)\sin(\omega t) \right]$$

and by cancellation:

$$2B_0 k \cancel{\hat{z}} = \mu_0 \varepsilon_0 \omega E_0 \cancel{\hat{z}} \quad \text{and}$$

$$B_0 = \frac{\mu_0 \varepsilon_0 \omega E_0}{2k}$$

P&M 9.26 An electromagnetic field in free space is described by the following two equations:

$$\vec{E} = \hat{y}E_0 \sin(kx + \omega t)$$
$$\vec{B} = -\hat{z}(E_0/c) \sin(kx + \omega t)$$

i) Show that the field satisfies the Maxwell's equations

Solution It is easy to observe that the divergence of both field equals to zero. The $\hat{y}$ component of the electric field has no y dependance and the $\hat{z}$ component of the magnetic field has no z dependance.

It suffices to verify the relationship regarding the divergence.

With some computation, we deduce:

$$\Delta \times \vec{E} = \hat{z}E_0 k \cos(kx + \omega t)$$
$$\Delta \times \vec{B} = \hat{y}E_0(k/c) \cos(kx + \omega t)$$

$$\frac{\partial}{\partial t}\vec{E} = \omega E_0 \cos(kx + \omega t)$$

$$\frac{\partial}{\partial t}\vec{B} = -\omega(E_0/c) \cos(kx + \omega t)$$

Recall the Faraday's Law and Ampere's Law in differential form:

$$\Delta \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad \text{and} \quad \Delta \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$$

With cancellation of the cosine terms, the two equations convert to:

$$E_0 k = \frac{E_0 \omega}{c} \quad \text{and} \quad \frac{E_0 k}{c} = \frac{\omega E_0}{c^2}$$

With more cancellation, both equations imply:

$$\boxed{c = \frac{\omega}{k}}$$

ii) Suppose $\omega = 10^{10} s^{-1}$ and $E_0 = 1kV/m$. Compute the energy density of the field per cubic meter and the rate of energy transfer per square meter.

Solution The energy density of an electromagnetic field is given as:

$$\frac{E^2 \epsilon_0}{2} + \frac{B^2}{2\mu_0}$$

Where E and B denotes the magnitude of the electric and magnetic field.

The field is given as:

$$\vec{E} = \hat{y}E_0 \sin(kx + \omega t)$$

$$\vec{B} = -\hat{z}(E_0/c)\sin(kx+\omega t)$$

The energy density at a point can thus be computed by:

$$\frac{\epsilon_0 E_0^2 \sin^2(kx+\omega t)}{2} + \frac{(E_0/c)^2 \sin^2(kx+\omega t)}{2\mu_0}$$

For sufficiently large enough range of time and space, the square of the sine term averges out to 1/2. The following relationship between μ_0, ϵ_0, c comes handy:

$$\mu_0 \epsilon_0 = 1/c^2$$

The average energy density is:

$$\frac{E_0^2 \epsilon_0}{4} + \frac{1}{c^2 \mu_0} \frac{E_0^2}{4} = \boxed{\frac{E_0^2 \epsilon_0}{2} \approxeq 4.4 \cdot 10^{-6} J/m^3}$$

To compute the power flow, it is useful to use the method of Poynting vectors. The Poynting vector is a vector field defined as:

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Thus:

$$\begin{aligned}\vec{S} &= \hat{y} E_0 \sin(kx+\omega t) \times -\hat{z}(E_0/c)\sin(kx+\omega t)/\mu_0 \\ &= -\hat{x} E_0^2/(c^2\mu_0)\sin^2(kx+\omega t)\end{aligned}$$

Take a large surface perpendicular to $\hat{x}$. The power transfer along a surface can be computing by computing the surface integral of the Poyinting vector. Given that the area is sufficiently large enough, the average charge density will equal to the average of the magnitude of the Poynting vectors.

We write the average power density as:

$$\boxed{\left|\frac{\partial}{\partial t} U\right| = \frac{\epsilon_0 E_0^2}{2c} \approxeq 1300 J/(m^2 s)}$$

P&M 9.27 A sinusodial wave reflects at the surface of a medium which absorbs the half of the energy of the incident wave. Compute VSWR(Voltage Standing Wave Ratio) of the standing EM wave created by the reflection.

Solution From the Poynting vector equation, we observe that the magnitude of energy is proportional to the product of the magnitude of the electric and magnetic field. The magnetic field is a constant multiple of the electric field for traveling waves in free space. Hence, we conclude that the energy of a traveling wave is proportional to the square amplitude of the electric field.

If the medium absorbs half the energy of the incident wave, the reflection will have an amplitude reduced by a factor of $1/\sqrt{2}$. VSWR, by definition, is the maximum amplitude of the standing wave divided by the minimum amplitude. The standing wave reaches its maximum amplitude when the incident and the reflection interacts constructively. Minimum amplitude is achieved when the two waves cancel each other out. Thus, we write:

$$\boxed{VSWR = \frac{(1 + 1/\sqrt{2})E}{(1 - 1/\sqrt{2})E} = \frac{(\sqrt{2}+1)^2}{2-1} \approxeq 5.83}$$