# Recurrence relations with bounded behavior

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Let $x_n$ be a sequence of real numbers. The goal of this paper is to solve the following recurrence.

$$x_n = x_{n-1} + x_{n-2} - x_{n-1}x_{n-2}/M$$

The motivation for this recurrence is a model of a population with one species which asymptotically grows to a population cap $M$. It is natural to assume that $x_0, x_1 \ll M$. [1]

Upon massaging the equation, we can write the following.

$$M - x_n = \frac{(M - x_{n-1})(M - x_{n-2})}{M} \tag{1}$$

We define an auxillary sequence, $y_n$ as follows.

$$y_n := \ln(M - x_n)$$

The recurrence relation of $y$ can be written easily by taking the natural log of (1). Also for convinience, let $m := \ln M$

$$y_n = y_{n-1} + y_{n-2} - m \tag{2}$$

Which is similar to the fibbonacci recurrence. We transcribe this into a matrix relation, presenting the following proposition.

**Proposition 1** The recurrence relation of (2) can be solved by the following matrix recurrence.

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \end{bmatrix} - \begin{bmatrix} m \\ 0 \end{bmatrix} \tag{3}$$

We further simplify the recurrence by introducing following notations.

$$\vec{y}_n := \begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} \quad \text{and} \quad F := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, more succintly,

$$\vec{y}_n = F\vec{y}_{n-1} - m\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

**Proposition 2** The solution for (4) is

$$\vec{y}_n := F^{n-1}\vec{y}_1 - m\begin{bmatrix} F_n - 1 \\ F_{n-1} - 1 \end{bmatrix}$$

for $n \geq 2$.

[1] To derive this recurrence, start from the fibbonacci sequence, and multiply the growth term by an adjustment factor, $\frac{M-x}{M}$

*Proof.* Trivially, the equation holds for the case when $n = 2$. Use induction to proceed. Using the inductive hypothesis, write out a solution for $y_n$ where $n \geq 2$.

$$\vec{y}_n = F^{n-1}\vec{y}_1 - m\begin{bmatrix} F_n - 1 \\ F_{n-1} - 1 \end{bmatrix}$$

We wish to explicitly compute $y_{n+1}$ using the recurrence relation. Write the following.

$$\vec{y}_{n+1} = F\vec{y}_n - m\begin{bmatrix} 1 \\ 0 \end{bmatrix} = F^n\vec{y}_1 - mF\begin{bmatrix} F_n - 1 \\ F_{n-1} - 1 \end{bmatrix} - m\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= F^n\vec{y}_1 - m\begin{bmatrix} F_n - 1 + F_{n-1} - 1 \\ F_n - 1 \end{bmatrix} - m\begin{bmatrix} 1 \\ 0 \end{bmatrix} = F^n\vec{y}_1 - m\begin{bmatrix} F_{n+1} - 1 \\ F_n - 1 \end{bmatrix}$$

□

The matrix power $F^n$ can be expressed in terms of fibbonacci numbers.

$$F^n = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \tag{5}$$

With some algebra, we present the following result.

**Theorem 1** The closed form solution of $\vec{y}_n$ is

$$\vec{y}_n = \begin{bmatrix} y_1F_{n-1} + y_0F_{n-2} - mF_n + m \\ y_1F_{n-2} + y_0F_{n-3} - mF_{n-1} + m \end{bmatrix}$$

and this implies

$$y_n = y_1F_{n-1} + y_0F_{n-2} - mF_n + m \tag{6}$$

[2]

**Corollary** By taking the exponential of (6), it is possible to deduce the following.

$$e^{y_n} = e^{y_1F_{n-1}}e^{y_0F_{n-2}}/e^{mF_n - m}$$

Remember that

$$e^{y_n} = M - x_n \quad \text{and} \quad e^m = M$$

to conclude

$$M - x_n = \frac{(M - x_1)^{F_{n-1}}(M - x_0)^{F_{n-2}}}{M^{F_n - 1}} = M\frac{(M - x_1)^{F_{n-1}}(M - x_2)^{F_{n-2}}}{M^{F_{n-1}}M^{F_{n-2}}}$$

and graciously,

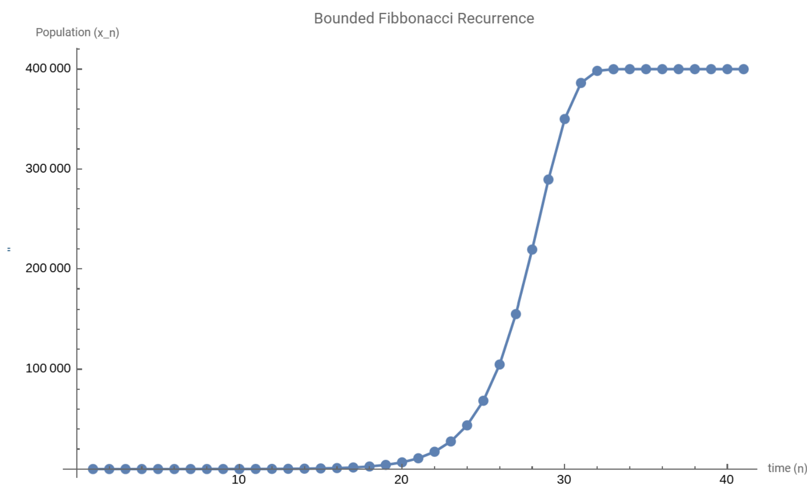
$$M - x_n = M\left(1 - \frac{x_1}{M}\right)^{F_{n-1}}\left(1 - \frac{x_0}{M}\right)^{F_{n-2}}$$

[2] Proof is some tedious lines of algebra. I can email the solution upon request.

. We present a closed form solution for $x_n$.

$$\boxed{x_n = M - M\left(1 - \frac{x_1}{M}\right)^{F_{n-1}}\left(1 - \frac{x_0}{M}\right)^{F_{n-2}}} \tag{7}$$

Here is a mathematica plot that shows that the recurrence is indeed bounded, and that the presented solution of the recurrence matches the computational result.




(* The line plot is the theoretical value computed by the closed - form solution.
The dots show the values computed by computing the recurrence via brute - force
*)

## Numerical Approximations

Recall from the beginning of the paper that it is natural to assume $x_0, x_1 \ll M$. In other words, $x_0/M, x_1/M \approx 0$. We can rewrite our solution in (7) using the Taylor approximation.

$$x_n \approx M - M\left(1 - F_{n-1}\frac{x_1}{M}\right)\left(1 - F_{n-2}\frac{x_0}{M}\right) = F_{n-1}x_1 + F_{n-2}x_0 - \frac{F_{n-1}F_{n-2}x_0x_1}{M}$$

The equation can be even more cleared out under the initial condition $x_0 = x_1$.

$$x_n \approx F_n x_0 - \frac{F_{n-1}F_{n-2}{x_0}^2}{M} = x_0\left(F_n - \frac{F_{n-1}F_{n-2}}{M}x_0\right) \tag{8}$$

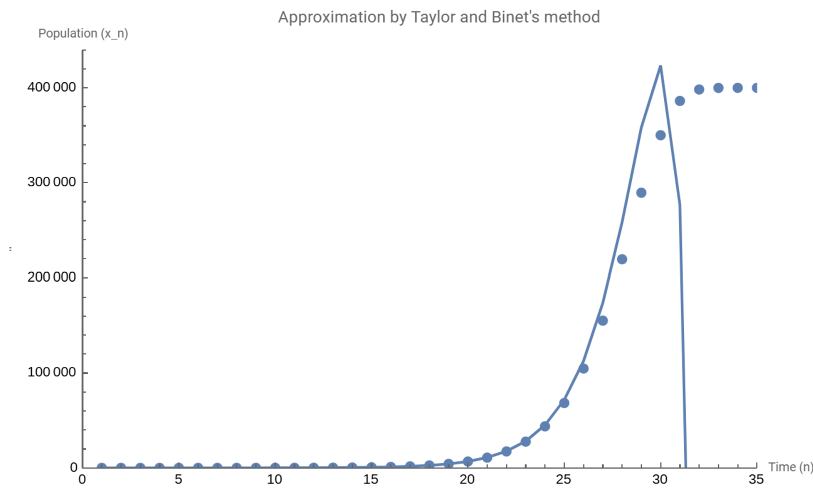
From binet's formula, we can approximate the fibbonacci sequence to a decent accuracy.

$$F_n \approx \frac{\varphi^n}{\sqrt{5}} \quad \text{where} \quad \varphi := \frac{1+\sqrt{5}}{2}$$

Back to our approximation in (8), we write:

$$x_n \approx x_0 \left( \frac{\varphi^n}{\sqrt{5}} - \frac{\varphi^{2n-3}}{5M} x_0 \right) = \boxed{x_0 \frac{\varphi^n}{\sqrt{5}} \left( 1 - \frac{\varphi^{n-3}}{\sqrt{5}M} x_0 \right)}$$

Here is a mathematica plot that compares the approximation with the exact values. The model performs decently until it reaches the maximum populatiion. After equilibrium, the model fails.




(∗The curve shows the approximation and the points show the real value∗)