Modeling a Leslie population with constant migration

PP Group

1 Abstract

The goal of this paper is to model a population growth that has constant migration. We use the Leslie matrix model, dividing the population into three age groups, and deduce conditions for the stability assuming constant migration.

2 Setup

We consider a population with three age groups. The model is discrete, and we measure the population after each discrete time staes. Denote the population vector at time n as

$$\vec{p}_n := (p_1, p_2, p_3)$$

which implies that the total population at time n to be

$$P_n := p_1 + p_2 + p_3$$

Given the initial population p_0, we model the evolution of the population vector by the following recurrence relations.

$$\vec{p}_{n+1} = L\vec{p}_n + \vec{m} \quad \text{or} \quad \vec{p}_{n+1} = L\vec{p}_n - \vec{e}$$

The first equation describes a model with constant immigration into the system, and the second equation describes a model with constant emigration. Assume $\vec{m}, \vec{e}$ to be vectors in $\mathbb{R}^3_{pos}$.

For simplicity, we consider a N-byN leslie matrix with perfect survival and fertility rate f. For example, if $N = 3$,

$$L := \begin{bmatrix} f & f & f \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{1}$$

3 Results

<u>Theorem</u> *Limiting behavior of the system*

If the operator norm of L is less than 1, the population vector converges to

$$\vec{p}_{eq} = (L - I)^{-1}\vec{m} \quad \text{or} \quad \vec{p}_{eq} = (L - I)^{-1}\vec{e}$$

where I is the identity matrix of order 3. The convergence depends on the value $1 - fs - fs^2$. That is, if $fs^2 + fs - 1 < 0$, then a constant positive influx of population is necessary to maintain an equilibrium. Otherwise, if $fs^2 + fs - 1 > 0$, then a constant positive outflux is required.

Proof. Without loss of generality, choose the recurrence

$$\vec{p}_{n+1} = L\vec{p}_n + \vec{m}$$

and by simply applying the recurrence n times, we obtain

$$\vec{p}_n = L^n\vec{p}_0 + \left(\sum_{i=0}^{n-1} L^i\right)\vec{m}$$

which, by the geometric series formula, again simplifies to

$$\vec{p}_n = L^n\vec{p}_0 + (L-I)^{-1}(L^n - I)\vec{m}$$

. Assuming that the operator norm of L is strictly less than 1, as $n \to \infty$, L^n converges to the zero matrix. Hence,

$$\vec{p}_{eq} = -(L-I)^{-1}\vec{m}$$

. In order for this equilibrium population to be positive, the matrix $-(L-I)^{-1}$ must yield a positive result when multiplied with the migration vector m. Recall the adjoint inverse formula.

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)^T$$

Thus, we must obtain

$$-|(L-I)^{-1}| > 0 \quad \text{or} \quad -\frac{1}{fs^2 + fs - 1} > 0$$

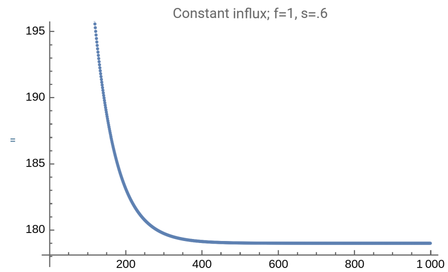
This inequality is achieved when

$$fs^2 + fs - 1 < 0$$

When the model satisfies the recurrence relation with emigration out of the system, substitude $\vec{m}$ with $-\vec{e}$ and repeat the argument

$\square$

Example *Numerical Example*



Here is a plot of the case with constant immigration into the system. Notice that the population ocverges after $n = 400$.

Model *Leslie Predator-Prey*
Let α_n, β_n be the population vectors of the predator and prey at timestep n. The Leslie Predator-Prey model is defined by the following system of matrix differences.

$$\begin{aligned}\alpha_{n-1} &= \max(L_\alpha \alpha_n + km\beta_n, \vec{0}) \\ \beta_{n-1} &= \max(L_\beta \beta_n - k\alpha_n, \vec{0})\end{aligned}$$

k, m are predation ratio and nurturing ratios, both between $0, 1$.

We assume that the x-value of L_α is less than $1/2$ and that the x-value of L_β is greater than $1/2$. In other words, the predator population decays in absense of the prey and the prey populatin explodes in absence of the predator.

Moreover, the population is fixed to be nonnegative.

Problem *Optimal Predation Strategy*
For what ranges of the real value k guarantees exponential growth of the predator? Moreover, what value of k is necessary to guarantee maximum growth?

Lemma 1 *Coulpled 1st order to 2nd order*
The predator population satisfies the following second order difference equation.

$$\alpha_n = (L_\alpha + L_\beta)\alpha_{n-1} - L_\beta L_\alpha \alpha_{n-2} - mk^2\alpha_{n-2}$$

$$\beta_n = (L_\beta + L_\alpha)\beta_{n-1} - L_\alpha L_\beta \beta_{n-2} - mk^2\beta_{n-2}$$

For simplicity, consider a leslie matrix of order 1. That is, a scalar.

The following three propositions properly models the population where the dimension of the Leslie matrix is 1. That is, the population growth is characterized by a exponential of a scalar without interaction. To emphasize the scalarness, write $l_a < 1$ and $l_b > 1$ instead of L_a, L_b.

Prop *Eigenvalues of the companion matrix*
Using Lemma 1, it is possible to obtain a companion matrix that describes the population.

$$\begin{bmatrix} l_a + l_b & -l_a l_b - k^2 m \\ 1 & 0 \end{bmatrix}$$

The eigenvalue of this matrix is purely real if and only if

$$k \leq \frac{l_a - l_b}{2\sqrt{m}}$$

Otherwise, the eigenvalues of these maticies are complex conjugates of each other.

Prop *Exponential growth of population for small predation*
The following condition guarantees that the predator and prey population to not vanish as $n \to \infty$.

$$k < \sqrt{\frac{(1 - l_b)(l_a - 1)}{m}}$$

Proof. Compute the eigenvalues of the companion matrix directly, and set it to be less than one.

Prop *Complex eigenvalue implies extinction*
If

$$k \geq \frac{l_a - l_b}{2\sqrt{m}}$$

then the population is guaranteed to be extinct.

Proof. Take the eigendecomposition and notice that the rotation eventually takes the population to some zero value. $\square$

It turns out that solving the recurrence for the general case where L_a, L_b is extremely challenging. Suppose we wish to solve the PP model where the Leslie matricies are degree k-by-k, where $k > 1$. If we adopt the scalar solution, we have to compute the eigenvalues of a $2k$-by-$2k$ matrix, and show that the eigenvector corresponding to the dominating eigenvalue is positive. Another attempted solution was to consider the following characteristic equation of the 2nd order recurrence

$$\Lambda^2 - (L_\alpha + L_\beta)\Lambda + (L_\alpha L_\beta - k^2 mI) = 0$$

In general, L_α and L_β do not commute. This imposes hardships when applying the quadratic formula to solve this equation. Also, by the natural condition of the predator prey model, it is impossible to set $L_\alpha = L_\beta$, for the former matrix describes a vanishing population while the latter describes a growing population.

Two go-arounds for this problem are as follows:

1. Consider a competitive model where $L_\alpha = L_\beta$
2. Suppose $L_\alpha = \rho L_\beta$ for some scalar ρ

For both cases, we can obtain a closed form formula for the population.