PHYS 202 HW5

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Question 3 Closed-open ended oscillators Consider N masses attatched by a string of constant k, each separated by a distance a apart. One side of the string is attatched to a wall, and the other end is free. What are the normal modes of this system?

a) What are the boundary conditions?

Add two imaginary masses at the wall and at the end of mass N. Label their displacements to be x_0, x_{N+1}. The boundary conditions are as follows.

$$\boxed{x_0(t) = 0 \quad \text{and} \quad x_{N+1}(t) = x_N}$$

b) Find the normal mode frequencies.

We know that the solutions are in the form of

$$x_n(t) = \tilde{B}e^{i(\omega t+kan)} + \tilde{C}e^{i(\omega t-kan)}$$

Enforce $x_0(t) = 0$, which converts to

$$x_0(t) = e^{i\omega t}(\tilde{B} + \tilde{C}) = 0 \quad \text{and} \quad \tilde{C} = -\tilde{B}$$

. $x_N(t) - x_{N-1}(t) = 0$ for any t. This converts to

$$x_N(t) - x_{N-1}(t) = \tilde{B}e^{i(\omega t+kaN)}(e^{ika} - 1) + \tilde{C}e^{i(\omega t-kaN)}(e^{i-ka} + 1)$$

$$= \tilde{B}\left(2ie^{i(\omega t+kaN+ka/2)}\sin(ka/2) - 2ie^{i(\omega t-kaN-ka/2)}\sin(-ka/2)\right)$$

$$= 2i\tilde{B}\sin(ka/2)\left(e^{i(\omega t+kaN+ka/2)} + e^{i(\omega t-kaN-ka/2)}\right)$$

$$= 4i\tilde{B}\sin(ka/2)\cos(kaN + ka/2)e^{i\omega t} = 0$$

If the summand inside the cosine function is in the form of $\frac{2m-1}{2}\pi$, the entire function vanishes, and the boundary condition is satisfied. Note that $m = 1, 2, 3, ...$ Thus,

$$\frac{2m-1}{2}\pi = \frac{ka(2N+1)}{2}$$

Depending on the value of m, the value of k varies. Relabel k as k_m.

$$k_m = \frac{2m-1}{(2N+1)a}\pi$$

We have previously derived the frequency dependant on k_m.

$$\boxed{\omega = 2\omega_0\sin(k_m a/2) = 2\omega_0\sin\left[\frac{m-1/2}{N+1/2}\frac{\pi}{2}\right]}$$

Question 4 Driven system of N masses

A system of N masses is connected with strings. One end of the system is driven with a displacement of $A\cos(\omega_d t)$. Each of the masses are distance a apart.

a) Imagine two imaginary masses at the left of mass zero and right of mass n. What boundary conditions can be enforced on these two masses?

Mass zero can be the driver, and mass $N+1$ can be a phantom mass that trails the movement of the N th mass. Quantituatively,

$$x_0(t) = A\cos(\omega_d t) \quad \text{and} \quad x_{N+1}(t) = x_N(t)$$

b) Find the displacements of each of the masses.

Complexifying the displacements, we know that the solutions must be in the form of

$$\tilde{x}_n(t) = \tilde{B}e^{i(\omega t + kz_n)} + \tilde{C}e^{i(\omega t - kz_n)}$$

. We define z_n to be zero on the right $a/2$ position of the Nth mass. Algebraically, z_n can be expressed as

$$z_n = \frac{2n - 2N - 1}{2}$$

Enforce the boundary condition on the $N+1$th mass.

$$\tilde{x}_{N+1}(t) - \tilde{x}_N(t) = 0 \quad \text{or} \quad \tilde{B}e^{i(\omega t + kz_{N+1})} + \tilde{C}e^{i(\omega t - kz_{N+1})} - \tilde{B}e^{i(\omega t + kz_N)} - \tilde{C}e^{i(\omega t - kz_N)} = 0$$

Divide by $e^{i\omega t}$ both sides and collect $\tilde{B}, \tilde{C}$

$$\tilde{B}(e^{ikz_{N+1}} - e^{ikz_N}) + \tilde{C}(e^{i(-kz_{N+1})} - e^{i(-kz_N)}) = 0$$

By our placement of the origin, $z_N = -ka/2, z_{N+1} = ka/2$. Apply Euler's formula to obtain

$$\tilde{B}\sin(ka/2) + \tilde{C}\sin(-ka/2) = 0 \quad \text{or} \quad (\tilde{B} - \tilde{C})\sin(ka/2) = 0$$

If $\sin(ka/2) = 0$, then all the masses will be stationary. Thus, we conclude $\tilde{B} = \tilde{C}$. Returning to our expression of the displcaement, we collect $\tilde{B}e^{i\omega t}$ to write

$$\boxed{\tilde{x}_n(t) = \tilde{B}e^{i\omega t}(e^{ikz_n} + e^{i(-kz_n)}) = \tilde{D}e^{i\omega t}\cos(kz_n)}$$

c) Show that the amplitude diverges if the oscillator is driven at a normal mode frequency.

Now, we know that $\tilde{x_0} = Ae^{i\omega t}$. Plugging $n = 0$ to our general solution,

$$\tilde{x}_0(t) = \tilde{D}e^{i\omega t}\cos(kz_0) = \tilde{D}e^{i\omega t}\cos(k\frac{-2N-1}{2}a) = Ae^{i\omega t}$$

The complex amplitude $\tilde{D}$ can be expressed in terms of A.

$$\tilde{D} = A/\cos(ka\frac{2N+1}{2})$$

The wavenumber k is dependant on the drive frequency by the dispersion relation.

$$\omega_d = 2\omega_0 \sin(ka/2) \quad \text{or} \quad k = \frac{2}{a}\sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right)$$

Thus,

$$\tilde{D} = A/\cos\left((2N+1)\sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right)\right)$$

From Q3, we know the ratio $\omega_d/2\omega_0$ for the normal modes.

$$\frac{\omega_d}{2\omega_0} = \sin\left[\frac{2m-1}{2N+1}\frac{\pi}{2}\right] \quad \text{and} \quad \sin^{-1}\left(\frac{\omega_d}{2\omega_0}\right) = \frac{2m-1}{2N+1}\frac{\pi}{2}$$

At the normal mode frequency, our amplitude simplifies to

$$\tilde{D} = A/\cos\left((2N+1)\frac{2m-1}{2N+1}\frac{\pi}{2}\right) = A/\cos\left((2m-1)\frac{\pi}{2}\right) = A/0 = \pm\infty$$

And the amplitude diverges. □

d) Write out a closed form expression for $x_1(t)$ where there is only one mass.
We define an angle for convinience.

$$\theta := ka/2$$

Write the dispersion relation and our amplitude $\tilde{D}$ in terms of θ.

$$\frac{\omega_d}{2\omega_0} = \sin(\theta) \quad \text{and} \quad \tilde{D} = A/\cos((2N+1)\theta) = A/\cos(3\theta)$$

The displacement x_1 is

$$x_1(t) = \tilde{D}e^{i\omega t}\cos(kz_1) = \tilde{D}e^{i\omega t}\cos(\theta(2n-2N-1)) = \tilde{D}e^{i\omega t}\cos(\theta)$$

Plugging in our expression for $\tilde{D}$,

$$x_1(t) = A\frac{\cos(\theta)}{\cos(3\theta)}e^{i\omega t}$$

We pull out an interesting trig identity

$$\frac{\cos(\theta)}{\cos(3\theta)} = \frac{1}{1-4sin^2(\theta)}$$

The derivation can be completed by using Euler's formula. We know $\sin(\theta) = \omega_d/2\omega_0$. Thus

$$\frac{1}{1-4\sin^2(\theta)} = \frac{1}{1-\omega_d^2/\omega_0^2} = \frac{\omega_0^2}{\omega_0^2-\omega_d^2}$$

Finally,

$$\boxed{x_1(t) = \frac{\omega_0^2}{\omega_0^2 - \omega_d^2} A e^{i\omega t}}$$

Q5 Rod of atoms

a) Consider an oscillator comprised of n masses where both ends are free. Compute the lowest normal mode frequency, and plot it using mathematica.

We reuse our results from Q4. Use the same coordinate system to define the equilibrium positon z_n. Enforcing the right boundary condition, we have

$$\tilde{x}_n(t) = \tilde{D} e^{i\omega t} \cos(k z_n)$$

The left boundary condition would be that the imaginary mass 0 shadows the motion of the real mass 1. In symbols, $x_0(t) = x_1(t)$. With some messy lines of algebra, this translates to

$$\cos(kaN + ka/2) = \cos(kan - ka/2)$$

Or substituting $ka/2 \mapsto \theta$, we can write

$$\cos(kaN + \theta) = \cos(kaN - \theta)$$

We reason that $\cos(kaN - x)$ must be an even function. Hence, $kaN = m\pi$ for any integer m. Thus

$$k = \frac{m\pi}{aN}$$

And now, plug this into the dispersion relation.

$$\frac{\omega_m}{\omega_0} = 2 \sin\left(\frac{ka}{2}\right) = 2 \sin\left(\frac{m\pi}{2N}\right)$$

We are interested in the lowest frequency, $m = 1$.

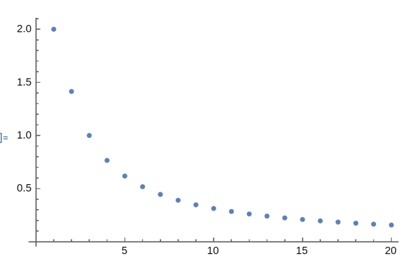
$$\frac{\omega_1}{\omega_0} = 2 \sin\left(\frac{\pi}{2N}\right)$$

Plotting this into mathematica, we get the following result.

In[9]:=

```
ω1ratio[N_] := 2 Sin[π / (2 N)];
ω1Table = Table[{i, ω1ratio[i]}, {i, 1, 20}];
ListPlot[ω1Table]
```

Out[11]=



b) Approximate the frequency as $N \to \infty$
The sin function can be approximated as the identity around the origin.

$$\boxed{\frac{\omega_1}{\omega_0} = 2\sin\left(\frac{\pi}{2N}\right) \approxeq 2\frac{\pi}{2N} = \frac{\pi}{N}}$$

c) Plot the relative displacements of the masses for $N = 3, 6, 20$.

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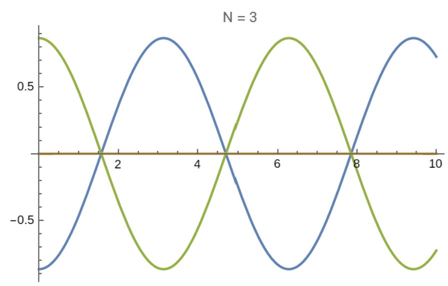
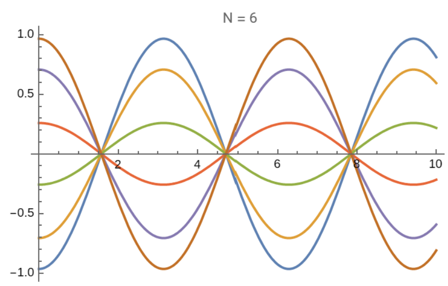
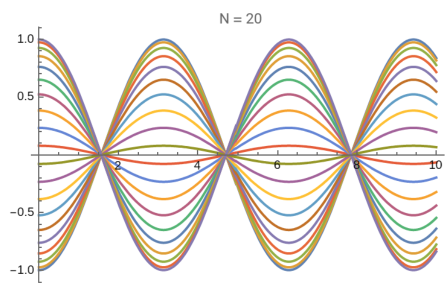
In[90]:= (*Here is a function for the relative frequency. N is the total number of masses. n is the specific mass number
 A is a constant for the amplitude*)
Displacement[n_, N_] := A Cos[ω t] Cos[(2 n - 2 N - 1) π / (2 N)]

(*Specify N, the number of masses, and obtain a list of functions*)
(*Assume ω = 1, A = 1 for convinience*)
A = 1;
ω = 1;
caseN3 = Table[Displacement[i, 3], {i , 3}];
caseN6 = Table[Displacement[i, 6], {i , 6}];
caseN20 = Table[Displacement[i, 20], {i , 20}];
Plot[caseN3, {t, 0, 10}]
Plot[caseN6, {t, 0, 10}]
Plot[caseN20, {t, 0, 10}]
```







d) Plot the three lowest normal mode frequencies for $N = 3, 6, 20$. Do they agree with Morin 2-84?

But this is our good ol' simple-harmonic-oscillator equation, so the solution is

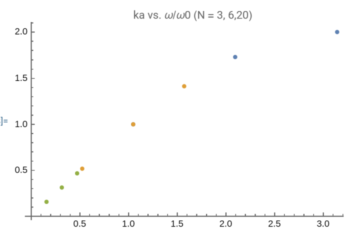
$$a(x) = Ae^{\pm ikx} \qquad \text{where} \quad \boxed{k \equiv \omega\sqrt{\frac{\rho}{E}}} \tag{84}$$

So morin claims that the normal mode frequencies have a linear relationship with the wavenumber k. We can confirm that this relationship holds for large N.

```
In[237]:= kvsω = Table[{km[m, N] * a, ω[km[m, N]] / ω0}, {N, {3, 6, 20}}, {m, 3}]
ListPlot[
 kvsω, PlotLabel → "ka vs. ω/ω0 (N = 3, 6,20)"
 , PlotLegends → "Expressions"
]
```

Out[237]= $\left\{\left\{\left\{\frac{\pi}{3}, 1\right\}, \left\{\frac{2\pi}{3}, \sqrt{3}\right\}, \{\pi, 2\}\right\}, \left\{\left\{\frac{\pi}{6}, \frac{-1+\sqrt{3}}{\sqrt{2}}\right\}, \left\{\frac{\pi}{3}, 1\right\}, \left\{\frac{\pi}{2}, \sqrt{2}\right\}\right\}, \left\{\left\{\frac{\pi}{20}, 2\,\mathrm{Sin}\left[\frac{\pi}{40}\right]\right\}, \left\{\frac{\pi}{10}, 2\,\mathrm{Sin}\left[\frac{\pi}{20}\right]\right\}, \left\{\frac{3\pi}{20}, 2\,\mathrm{Sin}\left[\frac{3\pi}{40}\right]\right\}\right\}\right\}$

Out[238]=

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(* There exists an intersection at ka = π/3 *)
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The green dots indicate that for $N = 20$, the relationship between the wavenumber and the frequency is linear.