PHYS 202 Final Exam

Daniel Son

A1) To Leading Order (Again!) (4)

This didn't go so well for some of you on the midterm, so let's try it again. Find an expression for the following function which is accurate to leading order in x:

$$f(x) = \frac{1 - \ln(1-x)}{e^{-x}}$$

Solution

We know the following Taylor expansions.

$$e^x \approxeq 1 + x + \frac{x^2}{2} \qquad \text{and} \qquad \ln(1-x) \approxeq -\left(x + \frac{x^2}{2}\right)$$

Hence,

$$f(x) = e^x\,(1 - \ln(1-x)) \approxeq \left(1 + x + \frac{x^2}{2}\right)\left(1 + x + \frac{x^2}{2}\right)$$

$$\approxeq 1 + 2x + x^2\left(\frac{1}{2} + 1 + \frac{1}{2}\right) + \cdots = \boxed{1 + 2x + 2x^2}$$

Just up to a leading order, we can also write $1 + 2x$. Plugging in $x = .1$ yields $f(x) = 1.22$ by computing f through a calculator. By our approximation, we also get $f(x) \approxeq 1.22$. $\square$

A2) Spring Equivalencies (6)

The traditional picture of a simple harmonic oscillator from Phys 141 features a mass m coupled to a 'wall' by a single spring. However, most often in the lab or in our classroom demonstrations, our mass was connected to two 'walls,' with a spring on either side of the mass. Take the rest length (unstretched) of each of the two springs to be ℓ_0. Assume they are stretched to a length ℓ when connected with the mass at equilibrium, and each have a spring constant k. You may assume the springs themselves are massless. Working from Newton's second law:

(a) show that our demonstration setup is equivalent to the classic picture.

(b) determine the spring constant k_{eff} that would be required if you wanted to observe the same behaviors using a single spring.

(c) what would happen if you hung the springs vertically, such that gravity tugged downwards and tended to stretch the spring? Would the resonant frequency ω_0 change? Why or why not?

Solution for a, b

We first write out the Newton's 2nd law for oscillators connected to one side of the wall.

$$m\ddot{x} = -k_{eff}x$$

We wish to derive an equation of this form for the two-wall oscillator. Here is a free body diagram for the two-wall oscillators.

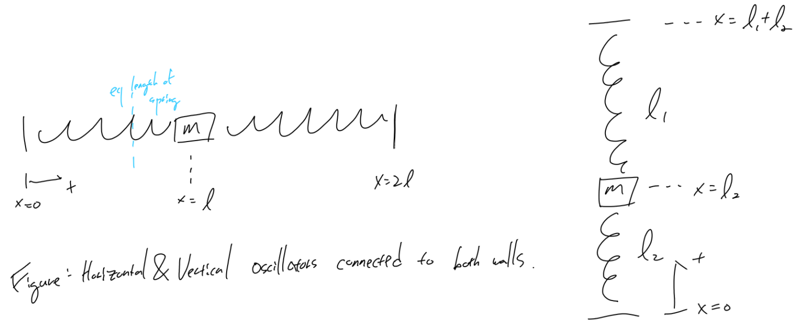


Figure: Horizontal & Vertical oscillators connected to both walls.

Write out Newton's second law. Let x be the raw position away from zero.

$$m\ddot{x} = -k(x - l_0) + k(2l - x)$$

We know that at $x = l$, the force exerted on the oscillator is zero.

$$0 = -k(l - l_0) + k(2l - l)$$

Thus, by subracting zero from the first equation,

$$m\ddot{x} - 0 = -k(x - l_0 - l + l_0) + k(2l - x - 2l + l)$$
$$m\ddot{x} = -k(x - l) - k(x - l) = -2k(x - l)$$

Relabel x to be the displacement. The old $x - l$ becomes the new x.

$$\boxed{m\ddot{x} = -2k}x \quad \text{or} \quad m\ddot{x} - k_{eff}x$$

where $\boxed{k_{eff} = 2k}$. □

<u>**Solution for c**</u> We repeat a similar procedure for the vertical oscilator. Apply Newton's second law. Start with x denoting the raw position of the mass. Suppose, at the equilibrium state, the top spring has length l_1 and the bottom string has length l_2.

$$m\ddot{x} = mg - k(-l_1 - l_2 + l_0 + x) - k(x - l_0)$$

At equilibrium, no force acts on the mass.

$$0 = mg - k(-l_1 - l_2 + l_0 + l_1) - k(l_1 - l_0)$$

Subtracting the second equation from the first, we again obtain,

$$m\ddot{x} = -2k(x - l_1)$$

and relabeling x to be the displacement from the equilibrium position l_1,

$$m\ddot{x} = -2kx$$

Which is exactly the equation that we derived in the previous part. The fundamental frequency of the oscillator is

$$\omega_0 = \sqrt{\frac{2k}{m}}$$

and this value does not change when the oscillator is tilted so that the gravity affects the motion.

A3) Polarization Manipulation (6)

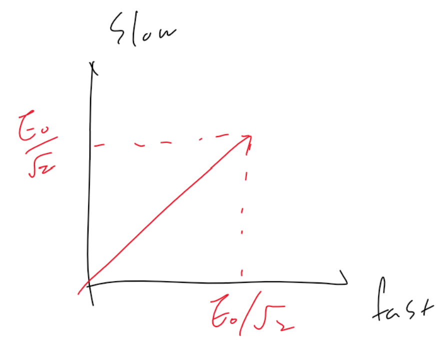
A birefringent material has indices of refraction $n_{\text{fast}} = 1.50$ and $n_{\text{slow}} = 1.51$. Suppose you want to make a device using this material which would convert linearly polarized light into circularly polarized light.

(a) Describe how this device would work.

(b) What would be the minimum thickness of the material required if you plan to use your device with a HeNe laser (λ = 633 nm)? Do not just quote a result; derive any equations that you need from first principles, and explain your reasoning.

(c) Suppose you acquired a piece of the birefringent material with the desired thickness. You also have a HeNe laser and some linear polarizers in rotation stages. How could you use these tools to determine whether or not the thickness you calculated in (b) actually did produce circular polarization?

Solution for a) A wave passing through a birefringent material experiences different speed depending on the axis of measurement. Assume that the light passing through the medium is linearly polarized, entering the birefringent material in a manner where the magnitude of the slow component is equal to the fast component.



The wave component that is passing through the slow axis will experience more phase shift compared to the component passing through the fast axis. Thus, if the phase difference experienced by the slow component is $\pi/2$ more than the phase difference experienced by the fast component, the material acts like a quarter waveplate, circularly polarizing the light.

Solution for b) Let the width of the material be w. We compute the phase difference experienced by the fast and the slow component.

$$\phi_{slow} = k_{slow} w = \frac{2\pi w}{\lambda_{slow}} = \frac{2\pi w n_{slow}}{\lambda}$$

By analogy,

$$\phi_{fast} = \frac{2\pi w n_{fast}}{\lambda}$$

Write out the phase difference. We wish the value to equal $\pi/2$.

$$\phi_{slow} - \phi_{fast} = \frac{\pi}{2} = \frac{2\pi w(n_{slow} - n_{fast})}{\lambda}$$

Fiddling over the constants, we deduce the following.

$$\boxed{w = \frac{\lambda}{4(n_{slow} - n_{fast})} \approxeq 16\mu m}$$

Solutions for c) A circularly polarized light will yield half of the original intensity after passing a linear polarizer. This can be easily demonstrated[1] through a simple Jones Calculus. Write out a Jones vector for some circularly polarized light with intensity 2. [2]

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Now apply a linear polarizer that has a pass axis of $\hat{x}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This light has an intensity 1. Indeed this result generalizes for any axis. So to check circular polarization, place the linear polarizer in any orientation, and check if the intensity is halved.

$\square$

[1] We can indeed show this for an arbitrary linear polarizer. Let M denote a matrix that passively changes the bases to a desired linear polarizer. M must be real and orthogonal. The first entry of $M\begin{bmatrix} 1 \\ i \end{bmatrix}$ describes the magnitude of the electric field. This value equals to $M_{11} + iM_{22}$. The magnitude of this component is $M_{11}^2 + M_{22}^2$, which must equal to 1 because M is known to be orthogonal.

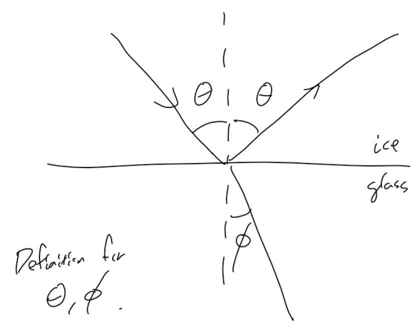
[2] The intensity can be computed by multiplying the complex conjugate of the vector.

A4) Reflections (4)

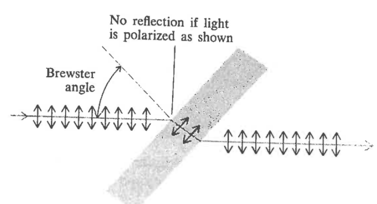
You have a laser, a sheet of glass ($n_{\text{glass}} = 1.5$), and a sheet of ice ($n_{\text{ice}} = 1.31$). By some magic you can manipulate the ice and glass in literally any way you like - if you can dream it, you can do it. Your goal is to transmit as much laser light *into the glass* as possible, i.e. to minimize reflection of the laser off of the glass as it enters.

(a) How might you arrange things to best accomplish your goal? Why?

(b) What fraction of the laser's power might you hope to transmit into the glass? If possible, be quantitatively accurate, but a brief description in words is a good start.



Solution We wish to achieve TM polarizaton.[3] This happens when the magnetic vector of the incident wave is parallel to the boundary plane, that is, perpendicular to the plane of incidence.[4] Here is a nice figure from Fowles p48.



Using our imaginary powers, we orient the dipoles of the ice and the glass to create this configuration.

Send in the light through the Brewster's angle for perfect transmission. We measure the brewster's angle by the following formula.

$$\boxed{\theta_b = \arctan\left(\frac{n_{glass}}{n_{ice}}\right) \approx 49^\circ}$$

Also, we know that at the Brewster's angle, the reflected ray is perpendicular to the refracted ray.[5] Thus, $\phi = 41^\circ$. Fowles also provides a transmission coefficient for TM polarization.

[3]In fact, the magnitude for this specific case is not very different between TE and TM polarization

[4]refer to Fowles p41-42

[5]Or, one can use Snell's law to find the angle of refraction

$$t_s = \frac{2\cos(\theta)\sin\phi}{\sin(\theta+\phi)\cos(\theta-\phi)} \approxeq .87$$

Squaring the transmission coefficient provides the power ratio that is transmitted into the glass.

$$\boxed{P_t/P_0 = t_s^2 \approxeq .76}$$

A5) Key Word (2 pts extra credit)

At one point during the semester I made said during class that there was a word you should make a note of.

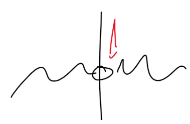
(a) When did we talk about this?

(b) What was the word?

(lec)Bridging to optics

Friday, March 15, 2024 10:00 AM

Special Code: Wingnut



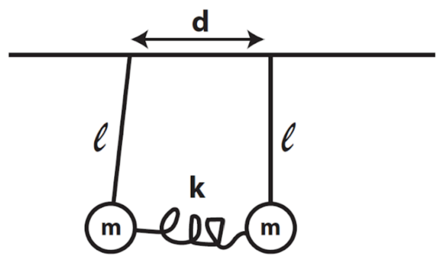
$$F_{string} = -\underbrace{\sqrt{\mu T}}_{Z_{string}}\, v_y$$

Solution The word is Wingnut and it was mentioned in Mar 15, 2024.

B1) Coupled Pendulums (10)

Two pendulum bobs, both of mass m and length ℓ, hang from the ceiling. The bobs are connected together by a spring of constant k and equilibrim length d, which is the same as the separation between the equilibrium positions of the two masses. Assume for the purposes of this problem that any displacements of the masses are small such their motion may be considered to be purely horizontal.

Also: remember that gravity will contribute a restoring force to each pendulum.

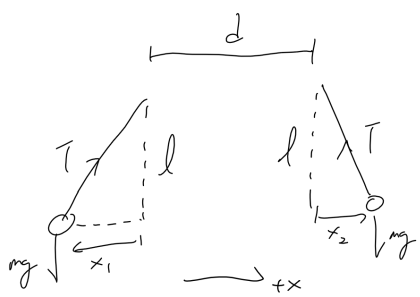


(a) Find the frequencies of all of the normal modes.

(b) Precisely describe the motions of the normal modes.

(c) Taking the positive direction to be to the right, suppose that at $t = 0$ the bobs are released from rest at positions $x_1 = -0.05d, x_2 = +0.1d$. Find an expression for the motion of both masses as a function of time.

Solution for a, b



String + Spring Oscillator FBD

Let x_1, x_2 be the horizontal displacement of the two masses from the equilibrium positon. Assuming the displacements are small, we approximate that all movements of the masses are entirely horizontal. By this approximation, the vertical force component must equal zero. This way, we can measure the sum of the tension and gravity in simple terms.

$$\vec{T} + \vec{F}_g = -mg\frac{x_1}{l}\hat{x}$$

[6]

[6] $\hat{x}$ is a convention used instead of $\hat{i}$. It is a unit vector pointing in the direction of x.

Write out Newton's 2nd law for the two masses in the $\hat{x}$ direction.

$$-k(x_1 - x_2) - \frac{x_1}{l}mg = m\ddot{x}_1$$

$$-k(x_2 - x_1) - \frac{x_2}{l}mg = m\ddot{x}_2$$

Assume the solution to be some complex exponential both for x_1, x_2.

$$x_1 = Ae^{i\omega t} \quad \text{and} \quad x_2 = Be^{i\omega t}$$

Plug in the solutions. After some lines of algebra, we obtain a system of linear equations for A, B.

$$\left(-m\omega^2 + \frac{mg}{l} + k\right) A - kB = 0$$
$$-kA + \left(-m\omega^2 + \frac{mg}{l} + k\right) B = 0$$

In matrix form, the equation nicely reduces to the following.

$$\begin{bmatrix} \left(-m\omega^2 + \frac{mg}{l} + k\right) & -k \\ -k & \left(-m\omega^2 + \frac{mg}{l} + k\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For the amplitudes to be nonzero, the 2x2 matrix must not be invertible. This means that the determinant of the matrix must be zero, which leads to the following identity.

$$\left(-m\omega^2 + \frac{mg}{l} + k\right)^2 - k^2 = 0$$
$$\left(-m\omega^2 + \frac{mg}{l}\right)\left(-m\omega^2 + \frac{mg}{l} + 2k\right) = 0$$

For the identity to hold, positive ω must take one of the two values.

$$\omega = \sqrt{\frac{g}{l}} \quad \text{or} \quad \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

The first frequency, which has a lower value, corresponds to the symmetric mode. The two masses move parallel to each other at the same magnitude. The second frequency, with a higher value, corresponds to the antisymmetric mode. The two masses move opposite to each other at the same magnitude.

□

Solution for c Any general movement of the oscillator can be described by a linear combination of the two normal modes. Let the complexified x_1, x_2 to be written in complex amplitudes A, B. For simplicity, omit the tilde.[7]

[7] In fact, A, B must be real. We know that the velocity of the mass at $t = 0$ must be zero for x_1, x_2. This means $\dot{x}_1 = \dot{x}_2 = 0$. With some algebra, one can deduce $Im(\omega_s A) = Im(\omega_f B) = 0$. All frequencies are real, and thus $Im(A) = Im(B) = 0$.

$$\ddot{x}_1 = Ae^{i\omega_s} + Be^{i\omega_f} \quad \text{and} \quad \ddot{x}_2 = Ae^{i\omega_s} - Be^{i\omega_f}$$

At time $t = 0$, the displacement of the two masses are measured as $-.05d$ and $.1d$. Write down the system in matrix form.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -.05d \\ .1d \end{bmatrix}$$

Multiply the inverse of the 2x2 matrix to compute A, B.

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -.05d \\ .1d \end{bmatrix} = \frac{d}{40} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Finally, plug in the values and take the real part to obtain the displacements x_1, x_2.

$$x_1 = \frac{d}{40}(\cos(\omega_s t) - 3\cos(\omega_f t))$$
$$x_2 = \frac{d}{40}(\cos(\omega_s t) + 3\cos(\omega_f t))$$

where

$$(\omega_s, \omega_f) = \left(\sqrt{\frac{g}{l}}, \sqrt{\frac{g}{l} + \frac{2k}{m}} \right)$$

As a sanity check, we verify that the velocity of both masses at $t = 0$ must vanish, for the time derivative would be some combination of sin waves. It is trivial to see that the displacement at $t = 0$ is also satisfied in our solution. $\square$

B2) Periodically Driven SHO (15)

A simple harmonic oscillator of mass m sits on a table and is connected to a wall by a massless spring of constant k. Suppose that by some mechanism you are able to drive the motion of the mass with an *arbitrary* function $F(t)$. The table is nearly frictionless, so you may ignore damping in your calculations, but you may also assume that any transient behaviors will eventually decay away if one waits long enough.

(a) What is the differential equation for the motion of the mass, including the time-dependent driving force $F(t)$?

(b) Let $F(t)$ be periodic with period T. Assume the mass is driven by this periodic function for a long time so that it settles into a steady-state motion which can be described as $x(t)$. What is $x(t)$?

[Hint: your solution should be clearly stated as $x(t)$ = (something), but it may involve for example integrals or other mathematical expressions involving $F(t)$ that cannot be explicitly calculated unless you were given a specific function for $F(t)$. Use superposition!]

(c) Are there any circumstances for which you can simplify your result from part (b) to yield a result which is approximately correct even if the details of $F(t)$ are not specified beyond its period T?

Solution for a

We ignore damping for our calculations. We deduce the following.

$$m\ddot{x} = -kx + F(t)$$

Solution for b

We first observe that the function F cannot be constantly nonzero. If that is the case, the oscilaltor will not reach a stable state, but will continuously shift to one direction.

Complexify the equation, and suppose the force takes the form of

$$F(t) = c_n e^{i2\pi nt/T}$$

Assume the displacement to take a similar form.

$$x(t) = x_n e^{i2\pi nt/T}$$

The differential equation simplifies to a relation between the constants, independant of time.

$$-m\left(\frac{2\pi n}{T}\right)^2 + k = c_n/x_n$$

$$x_n = \frac{c_n}{k - m\left(\frac{2\pi n}{T}\right)^2}$$

By the fourier series expansion, it is possible to write out any $F(t)$ that is physical (which means continuous, and smooth) as a sum of complex exponentials.

Using Fourier decomposition, decompose $F(t)$ into the following.

$$F(t) = \sum_{n\in\mathbb{Z}} c_n e^{i2\pi nt/T}$$

[8]

[8] To compute $\tilde{c}_n$, refer to Boaz

Then, the displacement function is given as

$$\boxed{x(t) = \sum_{n\in\mathbb{Z}} \frac{c_n}{k - m\left(\frac{2\pi n}{T}\right)^2} e^{i2\pi nt/T}}$$

Solution for c We know one thing about $F(t)$-it is a real valued function. For real valued function, we know that the fourier coefficient c_n satisfies the following relation.

$$c_{-n} = (c_n)^*$$

To see why this is true, consider the equations for c_n. Up to constants,

$$c_n \sim \int_{-T/2}^{T/2} e^{-2\pi int/T} F(t)dt$$

$$c_{-n} \sim \int_{-T/2}^{T/2} e^{2\pi int/T} F(t)dt$$

Since F is real, it is easy to see that the two coefficients are complex conjugates of each other.

Now, write out the equation for $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{c_n e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2} = \frac{c_0}{k} + \sum_{n=1}^{\infty} \frac{c_n e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2} + \sum_{n=1}^{\infty} c_{-n} \frac{e^{-2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2}$$

$$= \frac{c_0}{k} + \sum_{n=1}^{\infty} \frac{c_n e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2} + \sum_{n=1}^{\infty} (c_n)^* \left(\frac{e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2}\right)^* = \frac{c_0}{k} + 2\Re\{\sum_{n=1}^{\infty} \frac{c_n e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2}\}$$

$$= \frac{c_0}{k} + \sum_{n=1}^{\infty} 2\Re\{\frac{c_n e^{2\pi int/T}}{k - m\left(\frac{2\pi n}{T}\right)^2}\} = \boxed{\frac{c_0}{k} + \sum_{n=1}^{\infty} 2|c_n| \frac{\cos(2\pi nt/T + \phi_n)}{k - m\left(\frac{2\pi n}{T}\right)^2}}$$

[9]

Where $\phi_n = \arg(c_n)$. □

[9] Though this result seems more complicated than that of part b, it reduces the number of computations by a factor of 1/2. Suppose we are approximating $x(t)$ up to the first n terms. Each summand of this result corresponds to two summands of the result in b.

B3) Slits with Non-uniform Spacing (15)

Suppose that you have a set of 3 slits, each of width a, positioned with their centers at $x = 0, d$, and $3d$, as shown. Here a and d are constants which each have dimensions of length, with $a \ll d$. You illuminate the slits with monochromatic light from a laser with wavenumber k and look at the far-field illumination that results on a distant wall.

(a) Before doing any calculations, briefly describe in words the 'big picture' here; what are some features you might expect to see in the intensity pattern on the wall? Why?

(b) Find a mathematical expression for I/I_0 on the wall in terms of some parameter, where I_0 is the peak intensity that would be associated with a single slit. You could choose linear position along the wall, or angle, or perhaps some relevant dimensionless quantity. Make sure to clearly define the quantity you have selected, and also what your reference point for the 'zero' of that quantity is (e.g. some point on the wall. Simplify your result if it is practical to do so, but feel free to leave things written in terms of complex exponentials.

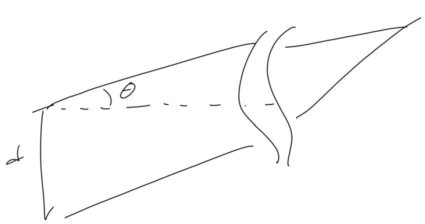
(c) Plot the intensity pattern. To do so,

(*i*) Draw phasor diagrams showing the behavior of the complex electric field amplitude from each of the slits. Make sure to draw diagrams for any 'interesting' values of your parameter of choice which correspond to values where the intensity is near a local minimum or maximum.

(*ii*) Informed by your phasor diagrams, sketch I/I_0. Your sketch should be qualitatively accurate (it should get the number and relative sizes of the maxima or minima correct, and should have axis labels which at least approximately identify key points, etc.), but it need not be quantitatively precise.

Solution for a Considering the three slits as a bundle, there must be a large envelope function that encloses the intensity projected on the wall. It is likely that some sinusoid that decays at the edges is involved. The actual intensity function will be a collection of fringes under the envelope. The fringes are formed by interference.

Solution for b .



Define θ to be the angle between the horizontal and the ray that is being measured. The screen is far away, so we can assume that the rays that reach one point in the screen are parallel.

Also, define I_0 to be the intensity when $\theta = 0$. This will be the point on the center of the wall. That is, if we draw a normal line from the slit, the intersection between the line and the screen will be the point where $\theta = 0$. The interferecne is entirely constructive here. Let k be the wavenumber and define

$$\delta = kd\sin(\theta)$$

For convinience, set the top ray to be the reference ray. We will compute the relative phase and magnitude changes assuming that the top ray to have phase zero at the screen. The two bottom rays will have additional phase difference incurred by extra distance traveled, assuming $\theta > 0$. At the point of the screen corresponding to angle θ, we can write the intensity as follows.

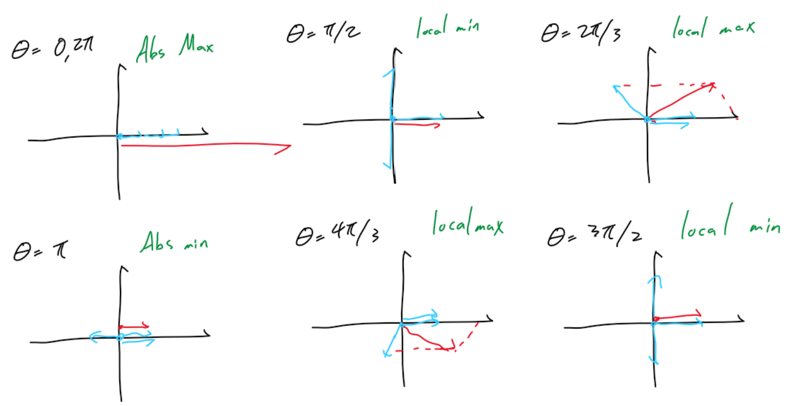
$$\left|\frac{I}{I_0}\right| = \left|\frac{1+e^{i\delta}+e^{3i\delta}}{3}\right|^2 = \left(\frac{1+e^{i\delta}+e^{3i\delta}}{3}\right)\left(\frac{1+e^{-i\delta}+e^{-3i\delta}}{3}\right)$$

With some lines of algebra, we graciously derive the following

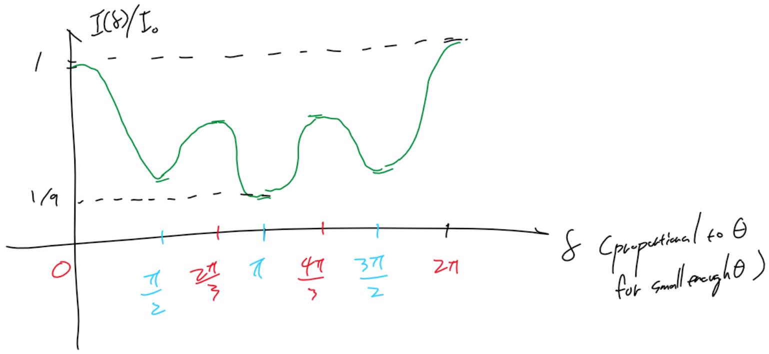
$$\left|\frac{I}{I_0}\right| = \frac{1}{3} + \frac{2}{9}(\cos(\delta) + \cos(2\delta) + \cos(3\delta))$$

Solutiion for c

Here are some phasor diagrams. The blue arrows are component vectors, and the red vector is the sum corresponding to the square root of the intensity.



From these phasor diagrams, we deduce the following intensity plot.

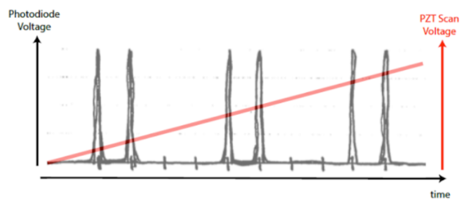


The red points mark the maximum, and the blue points mark the minimum. With some algebra, we can deduce that in the window of $\delta \in [0, 2\pi]$, the intensity ratio takes exactly seven extremas.[10]

[10] Please see B3c-continued in scratchwork.

B4) Laser Mode Measurements (15)

Light from a laser is passed into a confocal Fabry Perot cavity (FP). The spacing between the mirrors of the FP is scanned using a piezoelectric transducer, just as we did in lab. The FP mirrors both have a radius of curvature of *R*. The intensity transmitted by the FP is measured by a photodiode and displayed on an oscilloscope. After the laser has been on for a while, the peaks seen on the oscilloscope stop moving around, and they look something like this:



(a) Sketch the laser's mode structure. Label any interesting features with as much detail as you can given the paramters provided in the problem statement.

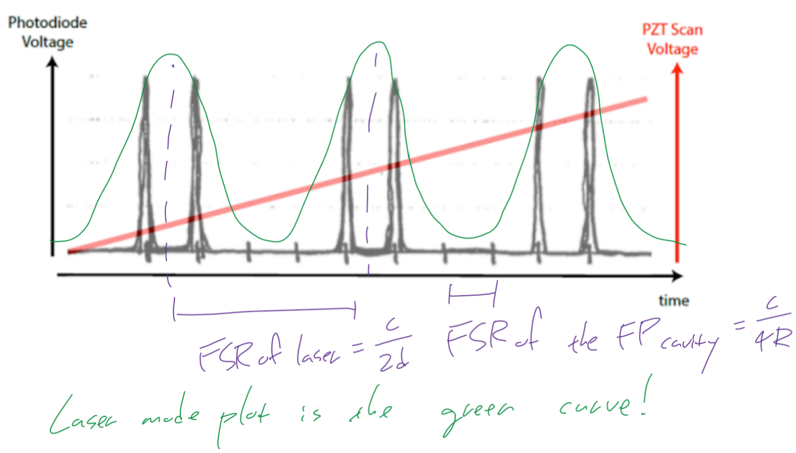
(b) Suppose the laser has a wavelength λ. Over what total distance did the piezoelectric transducer move the mirror of the FP cavity?

(c) In just a few words describe a possible configuration for the optical cavity of the *laser*, and relate the cavity length for your proposed configuration to *R*.

(d) Imagine that you pick off a bit of light from the same laser and send it into a Michelson interferometer. One of the mirrors of the interferometer is on a long translation stage. The arms of the interferometer initially have precisely equal length, and a bullseye interference pattern is observed. Describe, qualitatively *and* quantitatively, what happens to the interference pattern as one of the arms is slowly but continually lengthened using the translation stage.

[Hint: your description should involve *R*, the radius of curvature of the FP cavity you used to conduct your measurements.]

Solution for a



Also, note that

$$FSR_{lazer} = 4FSR_{cavity}$$

Solution for b The light traverses the cavity four times. Also, the spikes occur when the rays interfere constructively. Thus, for one free spectral range movement, the cavity lens either comes close or goes further away by $\lambda/4$.

Over one cycle of the oscilloscope, the PZT voltage is applied for twelve FSR's. $\lambda/4 \cdot 12 = 3\lambda$. We conclude that the PZD moved by 3λ.

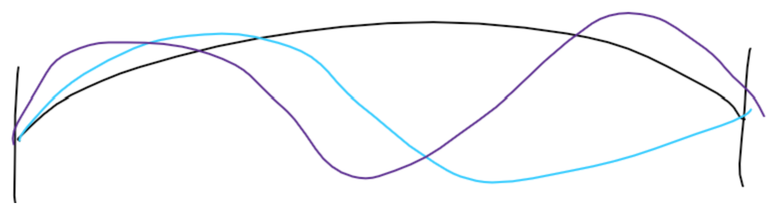
□

Solution for c The laser cavity is made out of a tube surrounded by two high reflectivity mirrors on each end with a gain medium at the center.

Let d be the length of the laser cavity and R the distance between the mirrors for the Faby-Perot cavity. As stated in part a, the FSR of the laser is four times that of the confocal Faby Perot Interferometer[11]. Thus, we derive the following identity.

$$\frac{c}{2d} = 4 \cdot \frac{c}{4R} \quad \text{and} \quad \boxed{R = 2d}$$

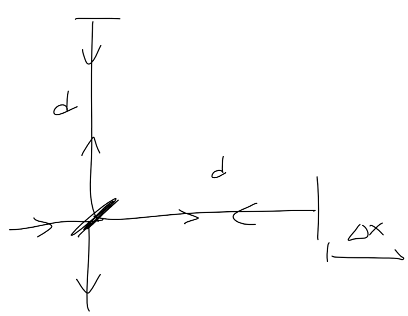
Solution for d We wish to derive an equation for the wavelength of the laser. The laser cavity can be considered as a fixed end string.



From the boundary condition, we write

$$\frac{n\lambda}{2} = d = \frac{R}{2} \quad \text{or} \quad \lambda = \frac{R}{n}$$

Here is a diagram of the Michelson Interferometer.



Simplified diagram of the Michelson Interferometer

[11]We assume confocal, because the question said "just as we did in lab"

The horizontal wave travels by an extra length of Δx, which is displacement of the moving translation stage. The extra phase difference of this ray compared to the vertical ray is

$$\Delta\phi = 2\Delta x k = \frac{4\pi\Delta x}{\lambda} = \frac{4\pi n\Delta x}{R}$$

[12]

Let I_0 be the intensity measured at the bullseye pattern when the length of the two legs are equal, i.e. $\Delta x = 0$. Setting the electric field magnitude of one of the rays as E_0, we deduce that $I_0 \sim (2E_0)^2 = 4E_0^2$. When the translation stage has moved by x, we can compute the intensity ration. For simplicity, write x, ϕ instead of $\Delta x, \Delta\phi$

$$\begin{aligned} I_x/I &= \left|\frac{E_0(1+e^{i\phi})}{2E_0}\right|^2 = \left|\frac{1+e^{i\phi}}{2}\right|^2 = \left(\frac{1+e^{i\phi}}{2}\right)\left(\frac{1+e^{i\phi}}{2}\right)^* \\ &= \left(\frac{1+e^{i\phi}}{2}\right)\left(\frac{1+e^{-i\phi}}{2}\right) = \frac{2+e^{i\phi}+e^{-i\phi}}{4} = \frac{1}{2}+\frac{1}{2}\cos(\phi) \end{aligned}$$

Replace ϕ with the equation that we derived previously.

$$\boxed{\frac{I_x}{I_0} = \frac{1}{2}+\frac{1}{2}\cos\left(\frac{4\pi n x}{R}\right)}$$

Qualitatively, the ratio above describes the sharpness of the fringes in the bullseye pattern. When the Intensity value reaches its peak, for example when $x = 0$, then the distinction between the fringes would be most sharp. However, as x slowly increases, there will be a point where the interference between the two rays would be completely destructive. In that case, the bullseye pattern would be gone, and it would be hard to distinguish between the fringes around this region.

[12] An attentative reader might ask why we are ignoring the phase retardation incurred by the reflection and the transmission of the beam splitter. Note that the vertical ray is once reflected and transmitted, and the horizontal ray is once transmitted and reflected, in that order. This means that the reflections and transmissions do not affect the relative phase difference.

B5) Atomic Clocks and the Shifting Theorem (10)

An atomic clock uses the oscillations of an electromagnetic field as the 'ticks' via which the passage of time is measured, with the oscillation frequency in question precisely tuned to an atomic resonance (which is stable in time). The present definition of the US second is based off of a 'fountain clock' at the National Institute of Standards and Technology in Boulder, Colorado. The details are not important here; what is crucial is that atoms in the clock interact with a microwave cavity *twice* such that they experience an oscillating electric field of the form

$$E(t) = \begin{cases} E_0\cos(\omega_{\mathrm{m}}t) & \text{for} \quad 0 < t < \tau \\ E_0\cos(\omega_{\mathrm{m}}t) & \text{for} \quad T < t < T+\tau \\ 0 & \text{otherwise.} \end{cases}$$

Here ω_{m} is the microwave frequency, τ the time the atoms spend in the microwave cavity on each pass, and T the time the atoms spend 'airborne' between passes through the microwave cavity. The atoms move slowly, so $\omega_{\mathrm{m}} \gg 2\pi/\tau$, i.e. the field oscillates many times while the atoms are in the cavity. Our goal is to determine (and interpret) what the power spectrum of the electric field is from the point of view of the atoms.

(a) Find the power spectrum of the electric field which the atoms see as a consequence of the two interactions with the cavity, and draw a sketch of the spectrum.

[Hint: You may find it helpful as a starting point to determine what the spectrum would be if the atoms saw a *constant* field E_0 when they were inside the cavity, and then to incorporate the effects of the cosine function.]

(b) Based on your results, what timescale (T, τ, or something else) determines

(*i*) the overall width of the spectrum?

(*ii*) the width of the narrowest features of the spectrum?

It is these narrow features which allow one to determine how closely matched ω_{m} is matched to the atomic frequency and the ultimate stabilty of the clock.

Solution for a

This might sound like a boring solution, but simply take the fourier transform of $E(t)$.

$$\hat{E}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikt}E(t)dt = \frac{1}{\sqrt{2\pi}}\left(\int_0^{\tau} e^{-ikt}E(t)dt + \int_T^{T+\tau} e^{-ikt}E(t)dt\right)$$

Using Euler's formula, expand the cos function into complex exponentials. We write the following.

$$= \frac{1}{\sqrt{2\pi}}\left(\int_0^{\tau}\frac{e^{i(\omega_m-k)t}}{2}dt + \int_0^{\tau}\frac{e^{-i(\omega_m+k)t}}{2}dt + \int_T^{T+\tau}\frac{e^{i(\omega_m-k)t}}{2}dt + \int_T^{T+\tau}\frac{e^{-i(\omega_m+k)t}}{2}dt+\right)$$

We wish to unify all the ranges of integration. Focus on the third summand. Apply a simple u-substitution, $u = t - T$.

$$\int_T^{T+\tau}\frac{e^{i(\omega_m-k)t}}{2}dt = \int_{u=0}^{\tau}\frac{e^{i(\omega_m-k)(u+T)}}{2}du = e^{i(\omega_m-k)T}\int_0^{\tau}\frac{e^{i(\omega_m-k)t}}{2}dt$$

We can simplify the last summand in a similar manner. We proceed with

$$\hat{E}(k) = \frac{1}{\sqrt{2\pi}}\left((1+e^{i(\omega_m-k)T})\int_0^{\tau}\frac{e^{i(\omega_m-k)t}}{2}dt + (1+e^{-i(\omega_m+k)T})\int_0^{\tau}\frac{e^{-i(\omega_m+k)t}}{2}dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\Bigg(\cos((\omega_m-k)T/2)e^{i(\omega_m-k)T/2}\int_0^{\tau}e^{i(\omega_m-k)}dt+$$
$$\cos((\omega_m+k)T/2)e^{-i(\omega_m+k)T/2}\int_0^{\tau}e^{-i(\omega_m+k)}dt\Bigg)$$

We now interpret this result. Square the fourier transform. We end up with three terms, which is the square of the first and the second summand, and a cross term which is twice the multiple of the two summands.

Pay attention to the two integrals. The particle experiences enough oscillations within the two short regions. This means that the complex integral

$$\int_0^\tau e^{i(\omega_m - k)} dt \quad \text{and} \quad \int_0^\tau e^{-i(\omega_m + k)} dt$$

is nonzero only if the exponents are close enough to zero. This implies that the cross term can be ignored. This is because ω_m is large enough that the neighborhood of $+\omega_m$ does not intersect with the neighborhood of $-\omega_m$. If the frequency lied outside the neighborhood of one region, one of the two integrals will vanish.

Focus on the two square terms. These two integrals will define an envelope, each located around $k = +\omega_m, -\omega_m$. Inside the two envelopes, the E field magnitude is determined by the two cosine functions. Nonetheless, we want the intensity, so square the two cosine functions. The intensity fringes are described by

$$\cos^2((\omega_m - k)T/2) \quad \text{and} \quad \cos^2((\omega_m + k)T/2)$$

[13] Using trig identity, rewrite the two terms.

$$\frac{\cos((\omega_m - k)T) + 1}{2} \quad \text{and} \quad \frac{\cos((\omega_m + k)T) + 1}{2}$$

Both of the functions have a width of $\frac{2\pi}{T}$ in frequency space. This means that the peak-to-peak distance of the two frequencies that spike are $\frac{2\pi}{T}$ away from each other.

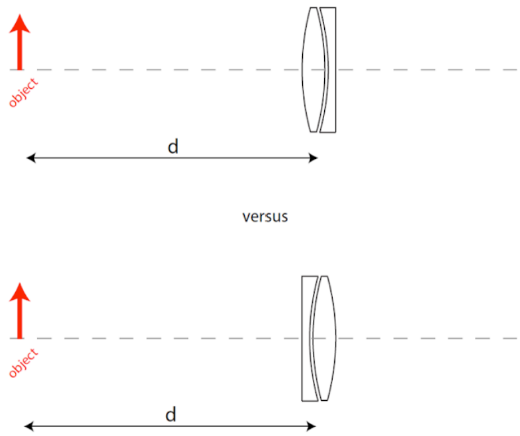
In light of our descoveries, we plot the power specturm.

[13] We can ignore the exponential coefficients, for we are taking the square magnitude. Also, we are not interested in the magnitude of these fringes, but just the frequencies.

Solution for b The asked quantities were all derived in part a. The width of the entire spectrum is about $\boxed{2\omega_m}$, so it is determined by the microwave frequency. The peak-to-peak distance of the fringe is $\boxed{2\pi/T}$, so it is determined by the time the atom is airborne. Also, we note that as the timeslot τ decreases, the peaks of the envelope function becomes more wider. If τ increases, lesser and lesser fringes will fit into the envelope.

B6) Achromatic Doublet (15)

An *achromatic lens* is formed by cementing together the combination of a converging lens with a diverging lens.[1] We can treat the combined pair of lenses - known as singlet lenses - as a 'doublet' with its own effective focal length. Suppose that the converging singlet lens has a focal length f, while the diverging lens has a focal length $-2f$. Our goal here is to determine: does the orientation of the doublet make a significant difference in its focal properties?



Treat the singlet lenses as thin lenses and take the distance between them to be negligible relative to f. If an object is placed a distance $d = 3f$ to the left of the doublet,

(a) Use ray diagrams to show where the image is for both orientations of the doublet. Your diagrams need not be precise quantitatively, but they should be qualitatively accurate. I encourage you to use a straightedge in drawing rays - if you don't have a ruler, use the edge of a piece of paper or the side of a book. Your diagram should:

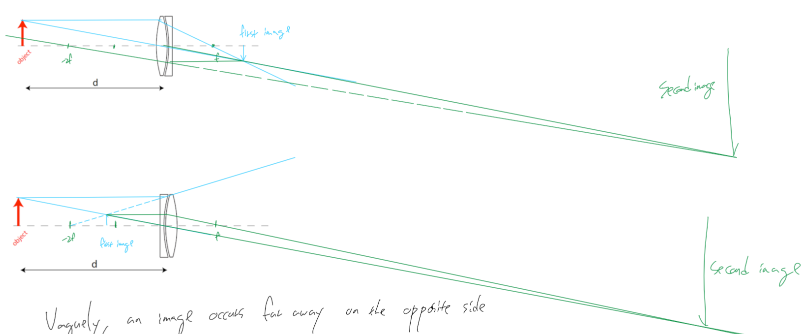
(*i*) indicate where the 'intermediate image,' formed by just the first lens in the doublet, would be located if that lens were all alone

(*ii*) use a suitable set of rays to determine where the actual image is formed by the doublet.

(b) Calculate the image location for the doublet in terms of f, the focal length of the converging singlet lens. Does the orientation make a difference?

(c) If the real object has a height h, what would be the height of the image for each orientation of the doublet? Would it be upright or inverted?

<u>Solution for a</u>



Vaguely, an image occurs far away on the opposite side of the object.

<u>Solution for b</u> The diagram suggests that the image of the two lens occur at the same position. We justify this rigorously by ray matricies. The ray matrix for a lens of focal length f is

$$M_f = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

Thus,

$$M_{-2f} = \begin{bmatrix} 1 & 0 \\ 1/(2f) & 1 \end{bmatrix} \quad \text{and} \quad M_f = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

Label the two different lens configuration as top and bottom, arranged in part a. Compute the corresponding ray matrix for each configuration.

$$M_t = M_{-2f} M_f = \begin{bmatrix} 1 & 0 \\ -1/(2f) & 1 \end{bmatrix}$$

$$M_b = M_f M_{-2f} = \begin{bmatrix} 1 & 0 \\ -1/(2f) & 1 \end{bmatrix}$$

So both configurations act like a converging lens that has a focal length of $2f$. Recall the formula

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

We know the focal length f and the location of the image s. Thus,

$$\frac{1}{3f} + \frac{1}{s'} = \frac{1}{2f} \quad \text{or} \quad 2s' + 6f = 3s' \quad \text{or} \quad s' = 6f$$

So, for both matricies, the image is created at a distance $6f$ from the lens, opposite to the original image. □

<u>Solution for c</u> We know that the magnifying factor m is $-s'/s$. By the formula, our image has a magnifying factor of $-6f/f = -6$. So the image for both orientations would be six times the original height, and the image would be inverted. □