PHYS 202 HW4

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Q1 Two masses m_1, m_2 are connected by a spring with a spring constant k.
a) Write out a general solution for the position of the two masses.
b) Find a specific solution for $x_1(0) = x_2(0) = 0$ and $v_1(0) = v$
c) Sketch $x_1(t)$ and $x_2(t)$.

Solution

Each of the masses must have a linear term which describes the constant velocity of the center of mass. Also, each mass must have a oscillating term. By Newton's 2nd law, we write

$$\ddot{x_1} = -k(x_1 - x_2) \quad \text{and} \quad \ddot{x_2} = -k(x_2 - x_1)$$

Also, in light of our previous observation, we guess the solution to be in the form of

$$x_1 = Re[Ae^{i\omega t} + Ct] x_2 = Re[Be^{i\omega t} + Ct]$$

It is possible to include a constant term, but it is possible to redefine x_1, x_2 to be zero at the defined place of axis, so we discard the extra variables for simplicity.

Complexify the equation and plug them into the coupled differential eqation. We obtain the following matrix equation.

$$k\begin{bmatrix} -1/m_1 & 1/m_1 \\ 1/m_2 & -1/m_2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = -\omega^2 \begin{bmatrix} A \\ B \end{bmatrix} \quad \text{or} \quad k\begin{bmatrix} 1/m_1 & -1/m_1 \\ -1/m_2 & 1/m_2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$$

The eigensystem of the 2x2 matrix narrows down the candidates of the angular frequency and the amplitude A, B. According to mathematica, the eigensystem is as follows.

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In[6]:= Coeff2 = {{k / m, -k / m}, {-k / M, k / M}};
        Eigensystem[Coeff2] // MatrixForm
Out[7]//MatrixForm=
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$$\begin{pmatrix} 0 & \frac{k\,(m+M)}{m\,M} \\ \{1, 1\} & \left\{-\frac{M}{m}, 1\right\} \end{pmatrix}$$

Thus, we obtain the two possible solutions.

$$(\omega, A, B) = (0, G, G) \quad \text{or} \quad (\sqrt{\frac{k(m+M)}{mM}}, -MG/m, G)$$

Where G is an arbitrary constant. The two restrictions does not provide constraints on C. Also, we defined the initial position to be zero at $t = 0$. Thus, we compute the imaginary part of the complexified equation instead of the real part. We write the general solution as follows.

$$\boxed{(x_1, x_2) = (Ct, Ct) \qquad \text{or} \qquad (-\frac{MG}{m}\sin(\sqrt{\frac{k(m+M)}{mM}}t) + Ct, G\sin(\sqrt{\frac{k(m+M)}{mM}}t) + Ct)}$$

We simplify the equation by using the definition of reduced mass.

$$\mu := \frac{mM}{m+M}$$

$$\boxed{(x_1, x_2) = (Ct, Ct) \qquad \text{or} \qquad (-\frac{MG}{m}\sin(\sqrt{k\mu}t) + Ct, G\sin(\sqrt{\frac{k(m+M)}{mM}}t) + Ct)}$$

To compute the particular solution, compute the time derivative of $x1$ and apply the condition $\dot{x_1}(0) = v$.

$$\dot{x_1}(0) = -\frac{MG}{m}\sqrt{\frac{k(m+M)}{mM}} = v$$

With some algebra,

$$G = -v\sqrt{\frac{m^3}{kM(m+M)}}$$

Thus

$$G = -vm$$