Strong Induction for AM-GM

Benevolent Tomato, ChatGPT

Theorem 1 (AM-GM Inequality)**.** *Let $x_1, x_2, \ldots, x_N$ be non-negative real numbers. Then the following inequality holds:*

$$\frac{x_1 + x_2 + \cdots + x_N}{N} \geq \sqrt[N]{x_1 x_2 \cdots x_N}$$

with equality if and only if $x_1 = x_2 = \cdots = x_N$.

Proof. We will use strong induction on N to prove the AM-GM inequality.

Base Case: For $N = 2$, the inequality is the classical AM-GM inequality for two numbers:

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}$$

This holds by the trivial identity derived from squaring both sides:

$$(x_1 - x_2)^2 \geq 0$$

Thus, the base case holds.

Inductive Step: Suppose the AM-GM inequality holds for $n < N$. We will prove it for N.

For N even, let $x_1, \ldots, x_N$ be non-negative real numbers. Then,

$$\frac{x_1 + \cdots + x_{N/2}}{N/2} \geq \sqrt[N/2]{x_1 \cdots x_{N/2}} \quad \text{and} \quad \frac{x_{N/2+1} + \cdots + x_N}{N/2} \geq \sqrt[N/2]{x_{N/2+1} \cdots x_N}$$

Taking the arithmetic mean of these inequalities, we obtain:

$$\frac{x_1 + \cdots + x_N}{N} \geq \sqrt[N]{x_1 \cdots x_N}$$

Now, if N is odd, we can show that for any natural number N, adding an additional variable x_{N+1}, we have:

$$\frac{x_1 + \cdots + x_N + x_{N+1}}{N+1} \geq \sqrt[N+1]{x_1 \cdots x_N x_{N+1}}$$

(denote this as $(*)$).

If we set $x_{N+1} = k(x_1 + \cdots + x_N)$, then:

$$\sum_{i \leq N} x_i + x_{N+1} = (k+1) \sum_{i \leq N} x_i$$

and $(*)$ simplifies to:

$$\frac{k+1}{N+1} \left(\sum_{i \leq N} x_i \right) \geq \sqrt[N+1]{x_1 \cdots x_N \cdot k \sum_{i \leq N} x_i}$$

Taking the logarithm of both sides, we obtain:

$$\frac{1}{k}\left(\frac{k+1}{N+1}\right)^{N+1}\left(\sum_{i\leq N} x_i\right) \geq \sqrt[N]{x_1\cdots x_N}$$

Thus, by the intermediate value theorem, there exists a $k>0$ such that:

$$\frac{(k+1)^{N+1}}{k(N+1)^{N+1}} = 1$$

Therefore, the inequality holds.

If $k \to 0$, the left-hand side blows up and hence guarantees the inequality. For $k = N$, the left-hand side is $\frac{N}{N+1} < 1$, so by the intermediate value theorem, there exists a k satisfying the condition.

Thus, the AM-GM inequality is proved by induction. $\square$