Speed dates with optimization problems

10 problems in 10 minutes

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Introduction

Understand $\leftrightarrow$ **Search** $\leftrightarrow$ Solve

- Efficient algorithms exist for many problems.
- Implementations of these algorithms are also available.
- The remaining difficulty is often having enough prior knowledge to recognize a problem.

Figure: Johann Bernoulli, circa 1740

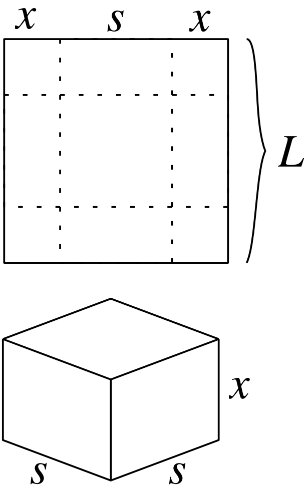
(1) The paper box problem

Introduction

Given a square piece of paper, cut the paper at a location x to maximize the volume of the resulting box.

This problem can be formulated as

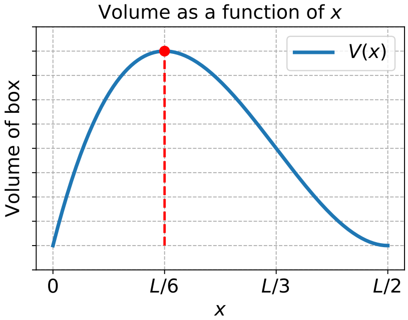
$$\begin{aligned} \text{minimize} \quad & V(x,s) = s^2 x \\ \text{subject to} \quad & 2x + s = L \\ & s, x \geq 0. \end{aligned}$$



(1) The paper box problem

Problem instance

$$\begin{aligned} \text{minimize} \quad & V(x) = (L-2x)^2 x \\ \text{subject to} \quad & x \geq 0 \\ & x \leq L/2 \end{aligned}$$



Problem generalization

Given a smooth function $f: \mathbb{R} \to \mathbb{R}$.

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \geq a \\ & x \leq b \end{aligned}$$

Solved using differentiation.

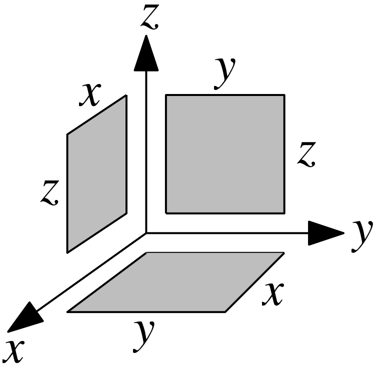
(2) The industrial box problem

Introduction

Given a square meter price for the material of a box with no lid, construct the cheapest box having unit volume.

This problem may be formulated as

$$\begin{aligned} \text{minimize} \quad & P(x,y,x) = xy + 2yz + 2xz \\ \text{subject to} \quad & V(x,y,z) = xyz = 1 \\ & x,y,z \geq 0. \end{aligned}$$

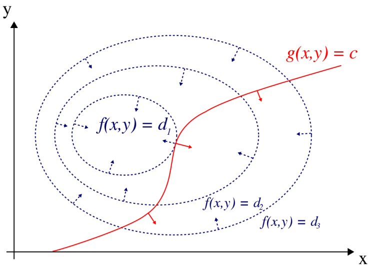


(2) The industrial box problem

Problem instance

Construct a *Lagrange function* $L(x,y,z,\lambda)$, solve the equations

$$\begin{aligned} L_x &= y + 2z + \lambda yz &&= 0 \\ L_y &= x + 2z + \lambda xz &&= 0 \\ L_z &= 2(x+y) + \lambda xy &&= 0 \\ L_\lambda &= xyz - 1 &&= 0 \end{aligned}$$



Problem generalization

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) = 0 \\ & i = 1, 2, \ldots \end{aligned}$$

Introduction

A company wants equal exposure to 4 sub-populations. Given 2 advertisement channels and their associated reach units of views/dollar, create an optimal budget.

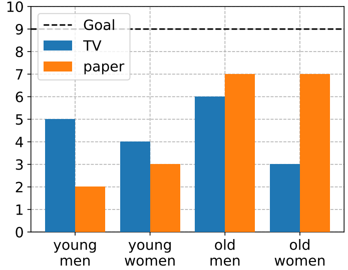
$$\begin{pmatrix}5\\4\\6\\3\end{pmatrix}x_1+\begin{pmatrix}2\\3\\7\\7\end{pmatrix}x_2=\begin{pmatrix}9\\9\\9\\9\end{pmatrix}+\begin{pmatrix}e_1\\e_2\\e_3\\e_4\end{pmatrix}$$

This problem can be formulated as

$$\text{minimize} \qquad \sum_{i=1}^{4} e_i^2 = \mathbf{e}^T\mathbf{e} = \|\mathbf{Ax}-\mathbf{b}\|_2^2 .$$

Views per unit of money

$$A = \begin{array}{c} \\ \text{y} \ ♀ \\ \text{y} \ ♂ \\ \text{o} \ ♀ \\ \text{o} \ ♂ \end{array} \begin{array}{c} \begin{array}{cc} \text{tv} & \text{paper} \end{array} \\ \begin{pmatrix} 5 & 2 \\ 4 & 3 \\ 6 & 7 \\ 3 & 7 \end{pmatrix} \end{array}$$



(3) The advertisement problem

Problem instance

$$\text{minimize} \quad \mathbf{e}^T\mathbf{e} = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

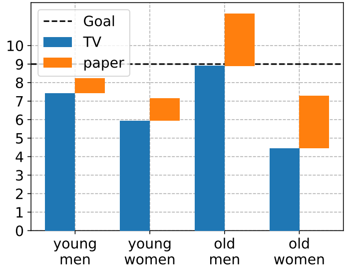
Problem generalization

Minimizing $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$ is a *least squares problem*, solved analytically by the equation

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}.$$

Solution

$$\begin{pmatrix}5\\4\\6\\3\end{pmatrix} \mathbf{1.5} + \begin{pmatrix}2\\3\\7\\7\end{pmatrix} \mathbf{0.4} = \begin{pmatrix}9\\9\\9\\9\end{pmatrix} + \begin{pmatrix}-0.7\\-1.8\\2.8\\-1.7\end{pmatrix}$$



Introduction

Same as before, but constrained by a budget of 1 unit of money.

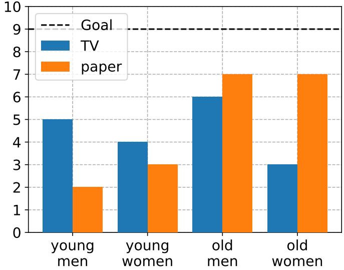
$$\begin{pmatrix}5\\4\\6\\3\end{pmatrix}x_1+\begin{pmatrix}2\\3\\7\\7\end{pmatrix}x_2=\begin{pmatrix}9\\9\\9\\9\end{pmatrix}+\begin{pmatrix}e_1\\e_2\\e_3\\e_4\end{pmatrix}$$

This problem can be formulated as

$$\begin{aligned}\text{minimize}\quad & \sum_{i=1}^{4} e_i^2=\mathbf{e}^T\mathbf{e}=\|\mathbf{Ax}-\mathbf{b}\|_2^2\\ \text{subject to}\quad & x_1+x_2=1\end{aligned}$$

Views per unit of money

$$A=\begin{array}{c}\\ \text{y}\ ♀\\ \text{y}\ ♂\\ \text{o}\ ♀\\ \text{o}\ ♂\end{array}\begin{array}{c}\begin{array}{cc}\text{tv} & \text{paper}\end{array}\\ \begin{pmatrix}5 & 2\\4 & 3\\6 & 7\\3 & 7\end{pmatrix}\end{array}$$



(4) The *constrained* advertisement problem

Problem instance

$$\begin{array}{rl} \text{minimize} & \mathbf{e}^T\mathbf{e} = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 \\ \text{subject to} & \mathbf{x}^T\mathbf{1} = 1 \end{array}$$

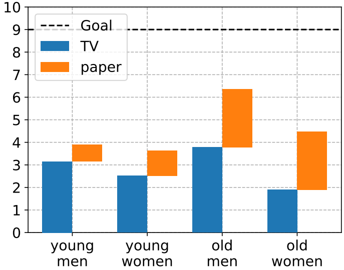
Problem generalization

Constrained least squares problem, solved by Lagrange multipliers and linear algebra.

$$\begin{pmatrix} 2\mathbf{A}^T\mathbf{A} & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} 2\mathbf{A}^T\mathbf{b} \\ \mathbf{d} \end{pmatrix}$$

Solution

$$\begin{pmatrix} 5 \\ 4 \\ 6 \\ 3 \end{pmatrix} \mathbf{0.6} + \begin{pmatrix} 2 \\ 3 \\ 7 \\ 7 \end{pmatrix} \mathbf{0.4} = \begin{pmatrix} 9 \\ 9 \\ 9 \\ 9 \end{pmatrix} + \begin{pmatrix} -5.1 \\ -5.4 \\ -2.6 \\ -4.5 \end{pmatrix}$$



(5) The worker-assignment problem

Introduction

Assign 4 workers to 4 tasks, given a matrix C specifying to which degree workers enjoy each task.

This amounts to specifying X with entries in $X_{ij} \in \{0,1\}$, i.e.

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

The above yields a satisfaction of

$$6+4+8+1 = 19.$$

Problem data

$$C = \begin{array}{c} \\ \text{ole} \\ \text{åse} \\ \text{dag} \\ \text{lise} \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 5 & 6 & 1 & 6 \\ 4 & 5 & 0 & 1 \\ 1 & 2 & 6 & 8 \\ 7 & 2 & 1 & 1 \end{pmatrix} \end{array}$$

Solution space growth

n	digits in $n!$
1	1
10	7
25	26
50	65

(5) The worker-assignment problem

Problem instance

$$\begin{aligned} \text{minimize} \quad & -\sum_i \sum_j C_{ij} X_{ij} \\ \text{subject to} \quad & \sum_i X_{ij} = 1 \text{ for every } j \\ & \sum_j X_{ij} = 1 \text{ for every } i \end{aligned}$$

Solution

$$C_{ij}\widehat{X}_{ij} = \begin{array}{c} \\ \text{ole} \\ \text{åse} \\ \text{dag} \\ \text{lise} \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 5 & 6 & 1 & \mathbf{6} \\ 4 & \mathbf{5} & 0 & 1 \\ 1 & 2 & \mathbf{6} & 8 \\ \mathbf{7} & 2 & 1 & 1 \end{pmatrix} \end{array}$$

$$-\sum_i \sum_j C_{ij}\widehat{X}_{ij} = 6+4+6+7 = 23$$

Problem generalization

This is the *assignment problem*, solved by the Hungarian algorithm. Solved in $\mathcal{O}(n^3)$ time, not $\mathcal{O}(n!)$.

(6) The diet problem

Introduction

Minimize the total cost of the diet, subject to the dietary constraints.

$$
\begin{aligned}
\text{minimize} \quad & p_1 x_1 + p_2 x_2 + p_3 x_3 \\
\text{subject to} \quad & 16x_1 + 5x_2 + 12x_3 \geq 100 \\
& 150x_1 + 100x_2 + 40x_3 \geq 2000 \\
& 150x_1 + 100x_2 + 40x_3 \leq 2500
\end{aligned}
$$

"Minimize cost, but get 100 grams of protein, and between 2000 and 2500 calories."

Problem data

Food	Price	Protein	Calories
x_1 eggs	p_1	16	150
x_2 bread	p_2	5	100
x_3 milk	p_3	12	40

The numbers above are fictitious.

(6) The diet problem

Problem instance

$$
\begin{array}{rl}
\text{minimize} & \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T \\
\text{subject to} & \begin{pmatrix} 16 & 5 & 12 \\ 150 & 100 & 40 \\ -150 & -100 & -40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 100 \\ 2000 \\ -2500 \end{pmatrix} \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

Problem generalization

This is the *linear programming* problem. Efficient algorithms exist.

$$
\begin{array}{rl}
\text{minimize} & \mathbf{p}^T \mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{array}
$$

(7) The hotel problem

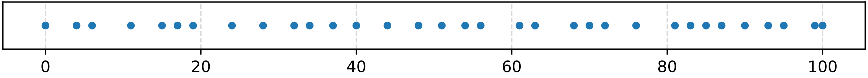
Introduction

We wish to travel 100 units of distance. There are many hotels along the way. Pick hotels to travel ~10 units per day.

Problem instance

Set $M = 10$. Find a sequence $h_1, h_2, \ldots, h_n$ to

$$\text{minimize} \quad \sum_j (M - (h_j - h_{j-1}))^2 .$$



Examples

Traveling from $x = 0$ to $x = 6$ incurs a penalty of $(10 - (6 - 0))^2 = 4^2$.
Traveling from $x = 0$ to $x = 11$ incurs a penalty of $(10 - (11 - 0))^2 = 1^2$.
There are 31 hotels above, and $2^{31} = 2\,147\,483\,648$ possibilities.

(7) The hotel problem

Problem

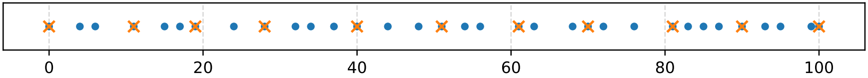
Let $P(j)$ be the minimal penalty at stop j. Realize that

$$P(j) = \min_{0 \leq i < j} \left(P(i) + (M - (h_j - h_i))^2 \right).$$

Solved in $\mathcal{O}(n^2)$ time, not $\mathcal{O}(2^n)$.

Problem generalization

The solution technique is called *dynamic programming*, and depends on an *optimal substructure* property.



To apply dynamic programming we must (1) identify recursive relationship, (2) define initial conditions and (3) solve problems in correct order.

(8) The magnet problem

Introduction

We are given 6 magnets. Choose $x_i \in \{-1, 1\}$ to minimize the total energy

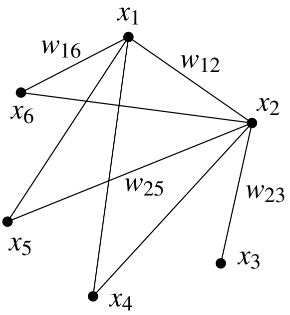
$$E(\mathbf{x}) = w_{12}x_1x_2 + w_{13}x_1x_3 + \cdots + w_{56}x_5x_6.$$

The problem can be formulated as

$$\begin{array}{rl} \text{minimize} & E(\mathbf{x}) = \mathbf{x}^T\mathbf{W}\mathbf{x} \\ \text{subject to} & x_i \in \{-1, 1\}. \end{array}$$

There are 2^{6-1} states, and $E(\mathbf{x})$ is not differentiable. A difficult problem.

Problem



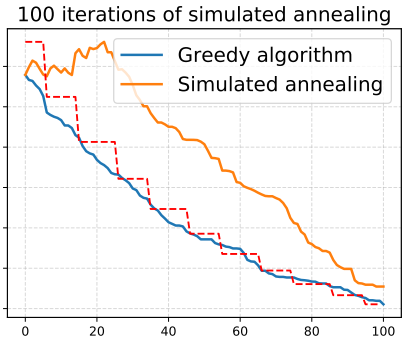
(8) The magnet problem

Problem instance

$$\begin{aligned} \text{minimize} \quad & E(\mathbf{x}) = \mathbf{x}^T \mathbf{W} \mathbf{x} \\ \text{subject to} \quad & x_i \in \{-1, 1\}. \end{aligned}$$

Problem generalization

When we have a (1) non-differentiable function with (2) a vast search space and (3) a notion of neighborhoods, use *simulated annealing* to balance exploitation and exploration.



(9) The egg boiling problem

Introduction

Let b be the boiling time of an egg, c be the cooling time, and s be the amount of salt used. Let $f(b,c,s) : \mathbb{R}^3 \to \mathbb{R}$ be the quality of a boiled egg.

This problem can be formulated as

$$\begin{array}{rl} \text{minimize} & -f(b,c,s) \\ \text{subject to} & b,c,s \geq 0 \end{array}$$

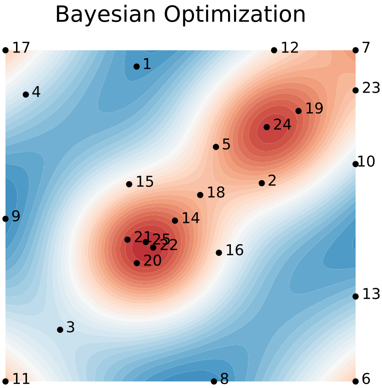
Evaluating $f(b,c,s)$ is expensive.

Problem generalization

Given a smooth function $f:\mathbb{R}^n \to \mathbb{R}$ which is expensive to evaluate.

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{a} \leq \mathbf{x} \leq \mathbf{b} \end{aligned}$$

Clever sampling via *bayesian optimization*, which builds a probability distribution over functions. Exploration vs. exploitation.



(10) The brachistochrone problem

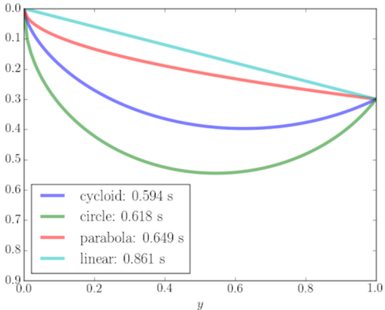
Introduction

A *functional* is a function from a function to a real number.

$$\text{functional} : \text{function} \to \text{real number}$$

Problem

Find the path (i.e. function) minimizing the travel time of a bead.



(10) The brachistochrone problem

Problem generalization

The problem amounts to minimizing a functional. In the space of all functions, find a function minimizing a functional. This is the domain of *calculus of variations*.

Johann Bernoulli solved the problem in 1696. The solution is a cycloid.

References (1/2)

- Strang, Gilbert. *Introduction to Applied Mathematics*. Wellesley, Mass: Wellesley-Cambridge Press, 1986.
  - Chapter 3 – “The brachistochrone problem”
  - Chapter 7 – “The worker-assignement problem”
  - Chapter 8 – “The diet problem”
- Boyd, Stephen, and Lieven Vandenberghe. *Introduction to Applied Linear Algebra*. Cambridge University Press, 2018.
  - Chapter 12 – “The advertisement problem”
  - Chapter 16 – “The constrained advertisement problem”
- Dasgupta, Sanjoy, Christos H. Papadimitriou, and Umesh Virkumar. Vazirani. *Algorithms*. Boston, Mass: McGraw Hill, 2008.
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- Duda, Richard O., Peter E. Hart, and David G. Stork. *Pattern Classification*. 2 edition. New York: Wiley-Interscience, 2000.
  - Chapter 7 – “The magnet problem”
- Jasper Snoek, Hugo Larochelle, Ryan P. Adams. *Practical Bayesian Optimization of Machine Learning Algorithms*. arXiv.org, 2012.
  - “The egg boiling problem”

Thank you for your attention.

Slides, LaTeX source and Python code solving the problems and generating plots:

`github.com/tommyod/10_optimization_problems`