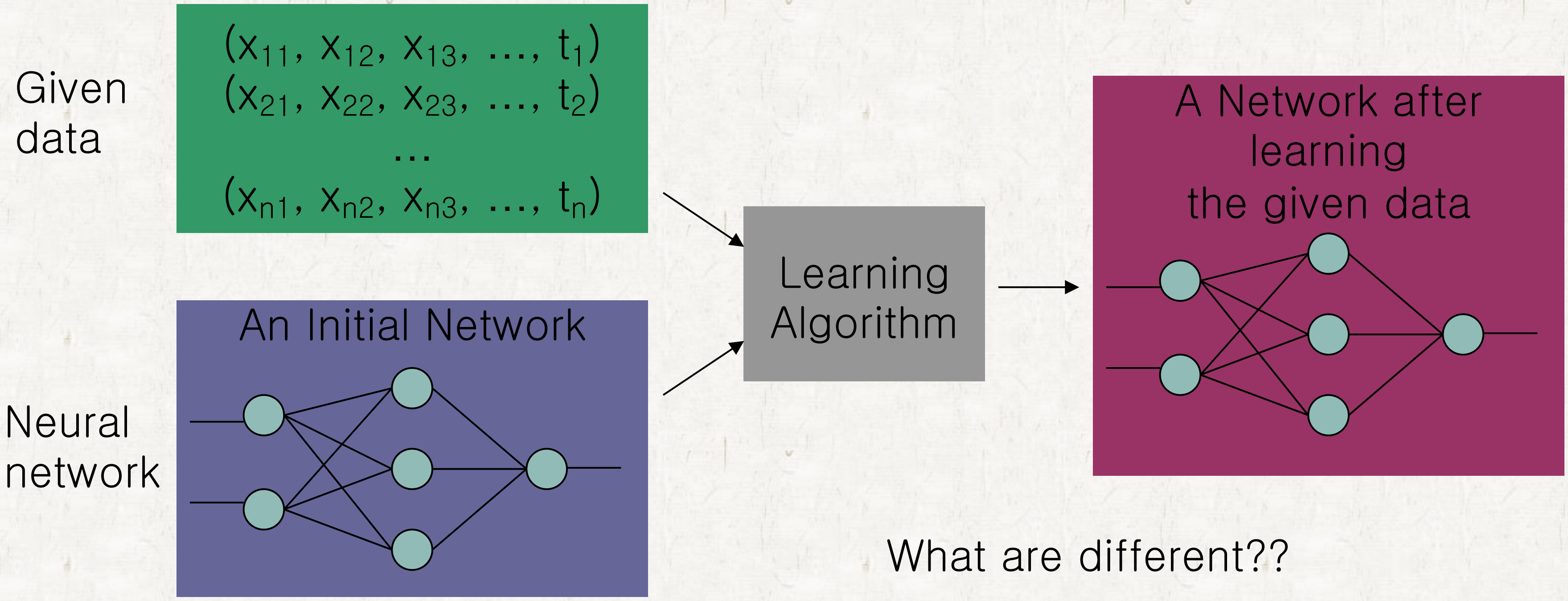
Error Back Propagation

Learning Algorithm (1)

- Preparation for Learning
  - Given input-output data of the target function to learn
  - Given structure of network (# of nodes in hidden layer)
  - Randomly initialized weights



What are different??

Learning Algorithm (2)

- Basic Idea of Learning
  - Find weights $\boldsymbol{w} = (w_1, w_2, \dots, w_n)$ so that

$$NN(\boldsymbol{w}, \boldsymbol{x}) = \boldsymbol{t} \quad \text{for all } (\boldsymbol{x}, t)$$

  - Find weights $\boldsymbol{w} = (w_1, w_2, \dots, w_n)$ which minimize

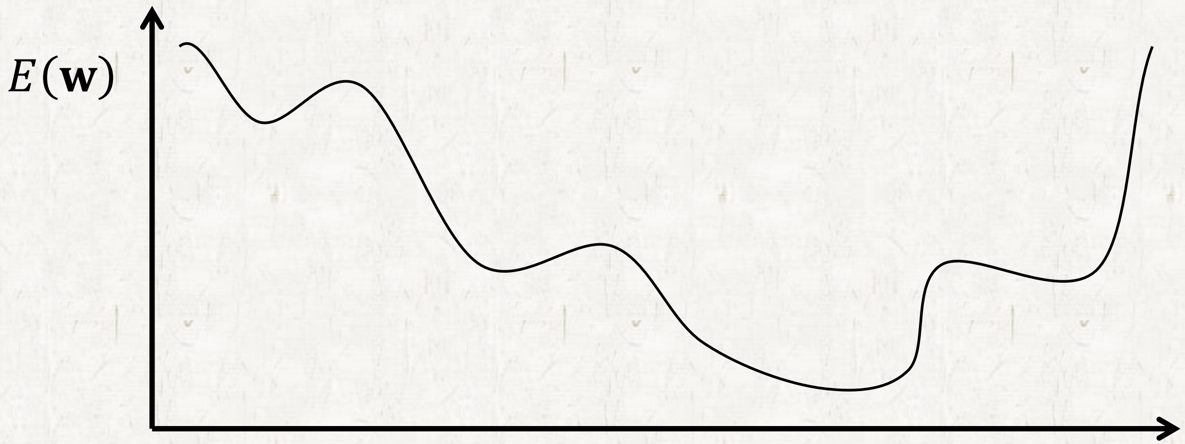
$$E(\boldsymbol{w}) = \sum_{(\boldsymbol{x},t) \in Data} \left(t - NN(\boldsymbol{w}, \boldsymbol{x})\right)^2 \qquad \boldsymbol{w} = (w_1, w_2, \dots, w_n)$$

Gradient Descent Method (1)

- How?

Find weights $\boldsymbol{w} = (w_1, w_2, \dots, w_n)$ which minimize

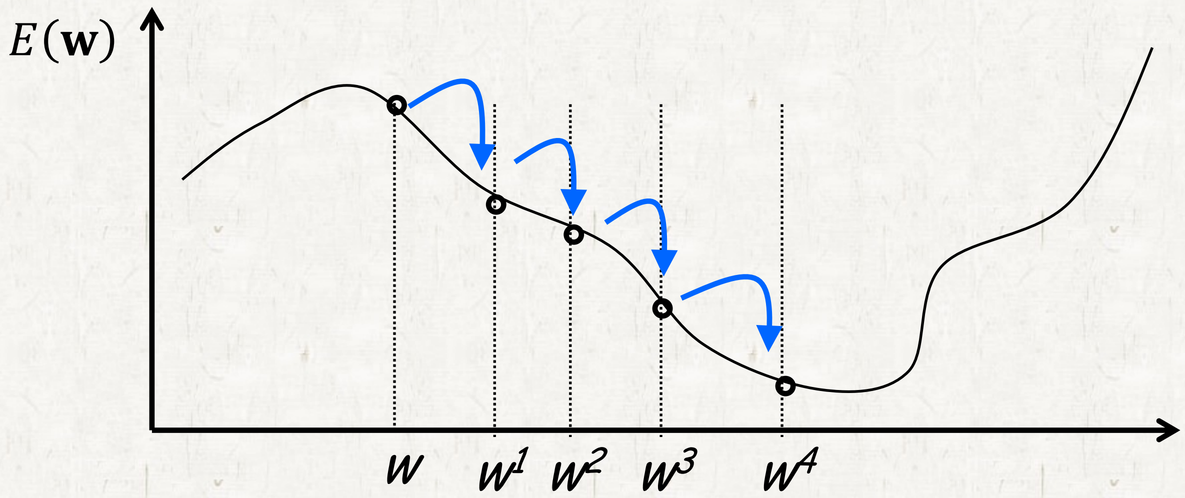
$$E(\mathbf{w}) = \sum_{(\mathbf{x},t)\in Data} \big(y - NN(\mathbf{x};\mathbf{w})\big)^2$$



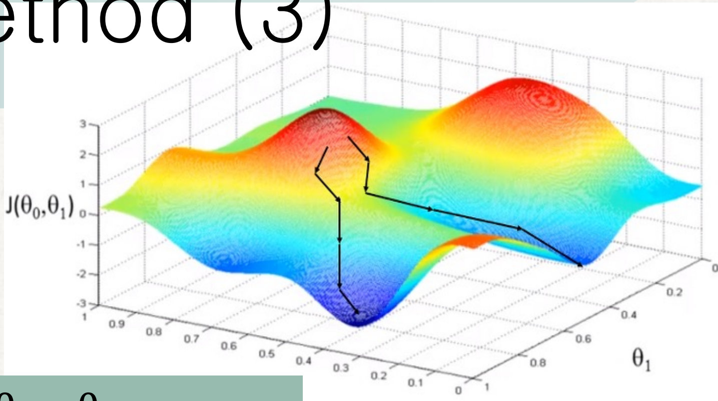
Gradient Descent Method (2)

4. Repeat until the gradient is zero

$$w^{t+1} = w^t - \eta \left. \frac{\partial E}{\partial w} \right|_{w=w^t}$$



Gradient Descent Method (3)



- Multi-variable case

Randomly choose an initial solution, w_0^0 w_1^0

Repeat

$$w_0^{t+1} = w_0^t - \eta \left.\frac{\partial E}{\partial w_0}\right|_{w_0=w_0^t, w_1=w_1^t}$$

$$w_1^{t+1} = w_1^t - \eta \left.\frac{\partial E}{\partial w_1}\right|_{w_0=w_0^t, w_1=w_1^t}$$

Until stopping condition is satisfied

Composite Function Differentiation (1)

- Evaluate $\left.\dfrac{dy}{dw}\right|_{w=0}$ when $y=\dfrac{1}{x}$ $\quad x=v^2 \quad v=e^w$

$$w \longrightarrow v \longrightarrow x \longrightarrow y \qquad \frac{dy}{dw}=\frac{dy}{dx}\frac{dx}{dv}\frac{dv}{dw}$$

$$\frac{dy}{dw}=-\frac{1}{x^2}\cdot 2v\cdot e^w$$

$$w=0$$
$$v=1$$
$$x=1$$

$$\left.\frac{dy}{dw}\right|_{w=0}=-\frac{1}{1^2}\cdot 2\cdot 1\cdot e=-e$$

Composite Function Differentiation (2)

- Evaluate $\left.\dfrac{dy}{dv}\right|_{v=0}$ when $y = \dfrac{1}{(x+t)}$ $\quad x = v^2 \quad t = e^v$

$$\begin{tikzcd} & x \arrow[dr] & \\ v \arrow[ur] \arrow[dr] & & y \\ & t \arrow[ur] & \end{tikzcd}$$

$$\frac{dy}{dv} = \frac{\partial y}{\partial x}\frac{dx}{dv} + \frac{\partial y}{\partial t}\frac{dt}{dv}$$

$$\frac{dy}{dv} = -\frac{1}{(x+t)^2}\cdot 2v - \frac{1}{(x+t)^2}\cdot e^v$$

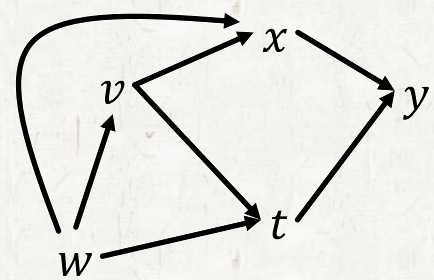
$$v = 0$$
$$t = 1$$
$$x = 0$$

$$\left.\frac{dy}{dv}\right|_{v=0} = -\frac{1}{1^2}\cdot 2\cdot 0 - \frac{1}{1^2}\cdot e^0 = -1$$

Composite Function Differentiation (3)

- Evaluate $\left.\dfrac{dy}{dw}\right|_{w=0}$ when $y = \dfrac{1}{x+t}$ $\quad x = v^2 + w \quad t = \sin(w+v)\pi \quad v = e^w$

$$w = 0$$
$$v = 1$$
$$t = 0$$
$$x = 1$$



$$\frac{dy}{dw} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dw} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v}\frac{dv}{dw} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial w}$$

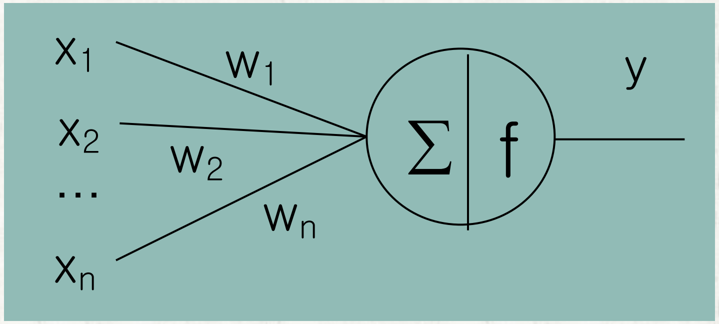
$$\frac{\partial y}{\partial w} = -\frac{1}{(x+t)^2}\cdot 1 - \frac{1}{(x+t)^2}\cdot 2v\cdot e^w - \frac{1}{(x+t)^2}\cdot \pi\cos(w+v)\pi\cdot e^w - \frac{1}{(x+t)^2}\cdot \pi\cos(w+v)\pi$$

$$\left.\frac{\partial y}{\partial w}\right|_{w=0} = -\frac{1}{(1+0)^2} - \frac{1}{(1+0)^2}\cdot 2\cdot 1\cdot e^0 - \frac{1}{(1+0)^2}\cdot \pi\cos(0+1)\pi\cdot e^0 - \frac{1}{(1+0)^2}\cdot \pi\cos(0+1)\pi$$

$$\left.\frac{\partial y}{\partial w}\right|_{w=0} = -1 - 2 - \pi\cdot(-1) - \pi\cdot(-1) = 2\pi - 3$$

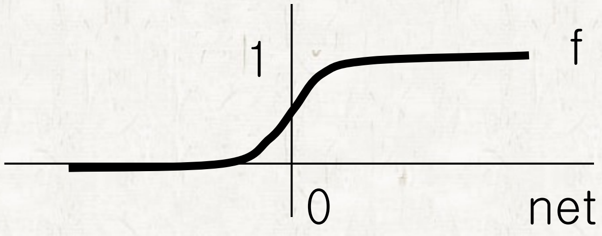
Activation Function (1)

- Error Back Propagation
  - ANN learning algorithm based on gradient descent method
  - Using derivatives to change the weights so that the error is minimized
  - Hard limit is not differentiable. -> Sigmoid



$net = x_1w_1 + x_2w_{j2} + \ldots + x_nw_n$

Sigmoid

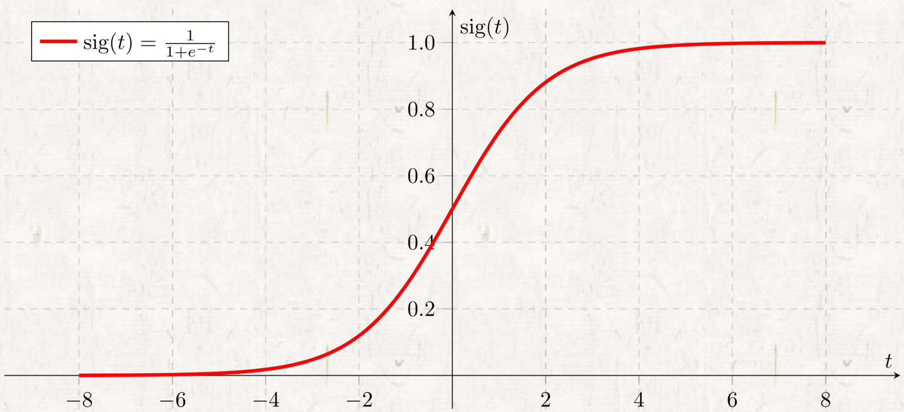


$y = sigmoid(net)$

Activation Function (2)

- Sigmoid function

$$y = \frac{1}{1+e^{-x}}$$



$$y' = \left(\frac{1}{1+e^{-x}}\right)^2 e^{-x}$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

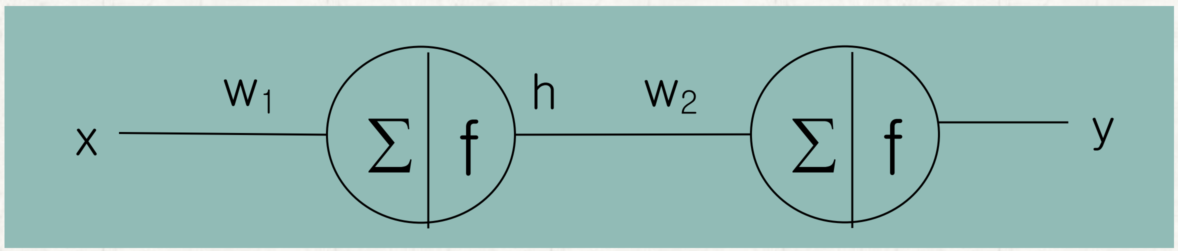
$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$$

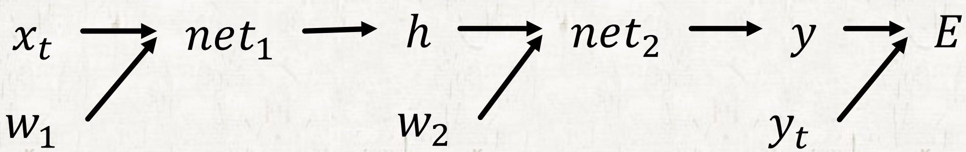
$$= \left(\frac{1}{1+e^{-x}}\right)\left(1-\frac{1}{1+e^{-x}}\right)$$

$$= y(1-y)$$

Simple Examples (1)

- Training of a Simple Neural Network
  - Let's assume that there is one training data (x_t, y_t)





$$net_1 = x_t \cdot w_1$$

$$h = sigmoid(net_1)$$

$$net_2 = h \cdot w_2$$

$$y = sigmoid(net_2)$$

$$E = \frac{1}{2}(y_t - y)^2$$

$$\frac{\partial E}{\partial w_2} = \frac{\partial E}{\partial y}\frac{\partial y}{\partial net_2}\frac{\partial net_2}{\partial w_2}$$

$$\frac{\partial E}{\partial w_1} = \frac{\partial E}{\partial y}\frac{\partial y}{\partial net_2}\frac{\partial net_2}{\partial h}\frac{\partial h}{\partial net_1}\frac{\partial net_1}{\partial w_1}$$

Simple Examples (2)

- Training of a Simple Neural Network

$$net_1 = x_t \cdot w_1$$

$$h = sigmoid(net_1)$$

$$net_2 = h \cdot w_2$$

$$y = sigmoid(net_2)$$

$$E = \frac{1}{2}(y_t - y)^2$$

$$\frac{\partial net_1}{\partial w_1} = x_t$$

$$\frac{\partial h}{\partial net_1} = h(1-h)$$

$$\frac{\partial net_2}{\partial w_2} = h \quad \frac{\partial net_2}{\partial h} = w_2$$

$$\frac{\partial y}{\partial net_2} = y(1-y)$$

$$\frac{\partial E}{\partial y} = -(y_t - y)$$

$$\frac{\partial E}{\partial w_2} = \frac{\partial E}{\partial y}\frac{\partial y}{\partial net_2}\frac{\partial net_2}{\partial w_2}$$

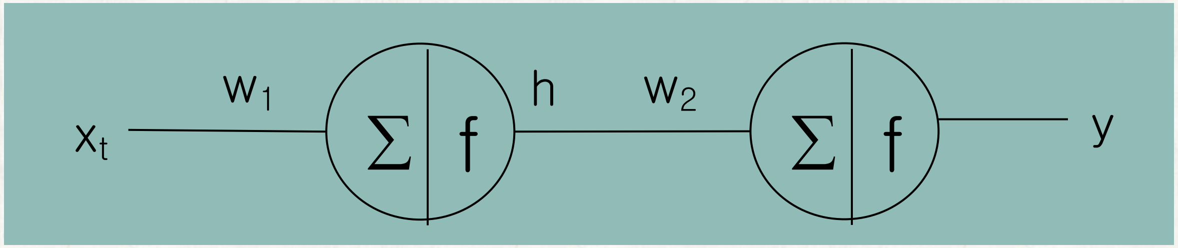
$$= -(y_t - y)y(1-y)h$$

$$\frac{\partial E}{\partial w_1} = \frac{\partial E}{\partial y}\frac{\partial y}{\partial net_2}\frac{\partial net_2}{\partial h}\frac{\partial h}{\partial net_1}\frac{\partial net_1}{\partial w_1}$$

$$= -(y_t - y)y(1-y)w_2 h(1-h)x_t$$

Simple Examples (3)

- Training of a Simple Neural Network



$$(x_t, y_t) = (1,1), w_1 = 1, w_2 = 1, \eta = 0.1$$

$$net_1 = x_t \cdot w_1 = 1$$

$$h = sigmoid(net_1) = 0.731$$

$$net_2 = h \cdot w_2 = 0.731$$

$$y = sigmoid(net_2) = 0.675$$

$$E = \frac{1}{2}(y_t - y)^2 = \frac{0.343^2}{2}$$

$$\frac{\partial E}{\partial w_2} = -(y_t - y)y(1-y)h$$
$$= -0.343 \cdot 0.675 \cdot 0.325 \cdot 0.731$$

$$\frac{\partial E}{\partial w_1} = -(y_t - y)y(1-y)w_2 h(1-h)x_t$$
$$= -0.343 \cdot 0.675 \cdot 0.325 \cdot 1 \cdot 0.731 \cdot 0.269 \cdot 1$$

Simple Examples (4)

- Training of a Simple Neural Network

$$\frac{\partial E}{\partial w_2} = -(y_t - y)y(1-y)h$$
$$= -0.343 \cdot 0.675 \cdot 0.325 \cdot 0.731 = -0.055$$

$$\frac{\partial E}{\partial w_1} = -(y_t - y)y(1-y)w_2 h(1-h)x_t$$
$$= -0.343 \cdot 0.675 \cdot 0.325 \cdot 1 \cdot 0.731 \cdot 0.269 \cdot 1 = -0.015$$

$$(x_t, y_t) = (1,1)$$
$$w_1^0 = 1$$
$$w_2^0 = 1$$
$$\eta = 1$$

$$w_1^1 = w_1^0 - \eta \frac{\partial E}{\partial w_1}$$

$$1.0015 = 1 + 0.1 \cdot 0.015$$

$$w_2^1 = w_2^0 - \eta \frac{\partial E}{\partial w_2}$$

$$1.0055 = 1 + 0.1 \cdot 0.055$$

Randomly choose an initial solution, $w_1^0\ w_2^0$

Repeat

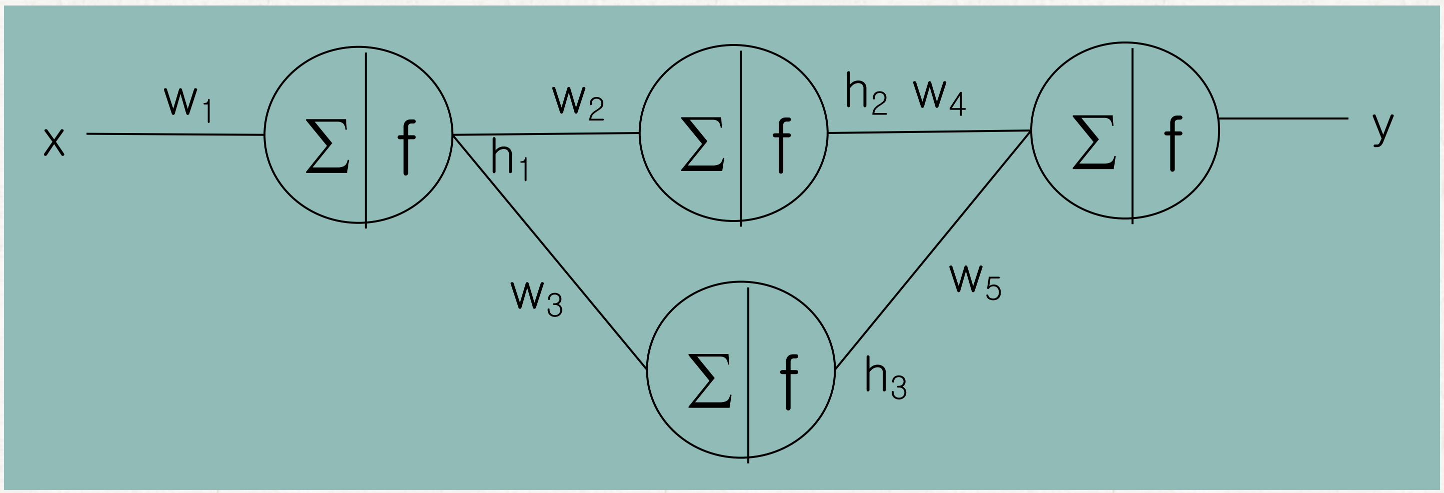
$$w_1^{t+1} = w_1^t - \eta \left.\frac{\partial E}{\partial w_0}\right|_{w_1=w_1^t, w_2=w_2^t}$$

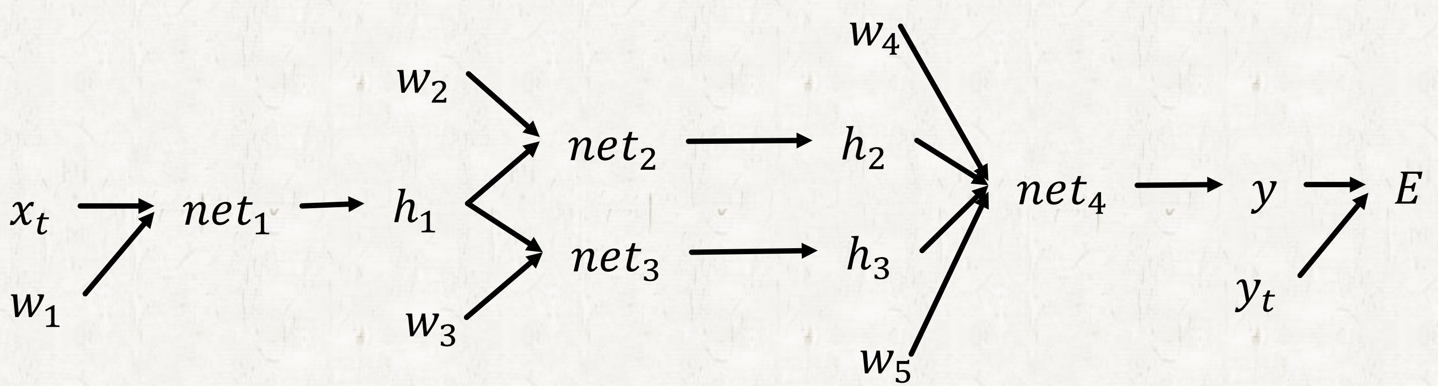
$$w_2^{t+1} = w_2^t - \eta \left.\frac{\partial E}{\partial w_2}\right|_{w_1=w_1^t, w_2=w_2^t}$$

Until stopping condition is satisfied

Simple Examples (4)

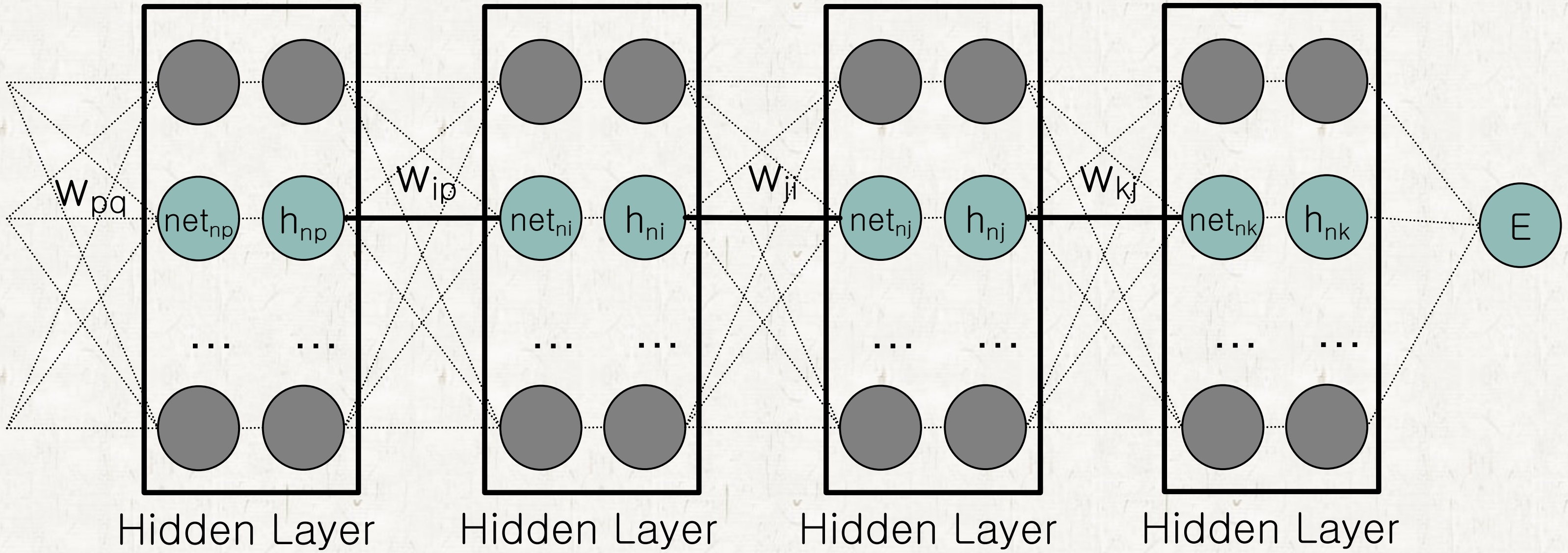
- A little complex one





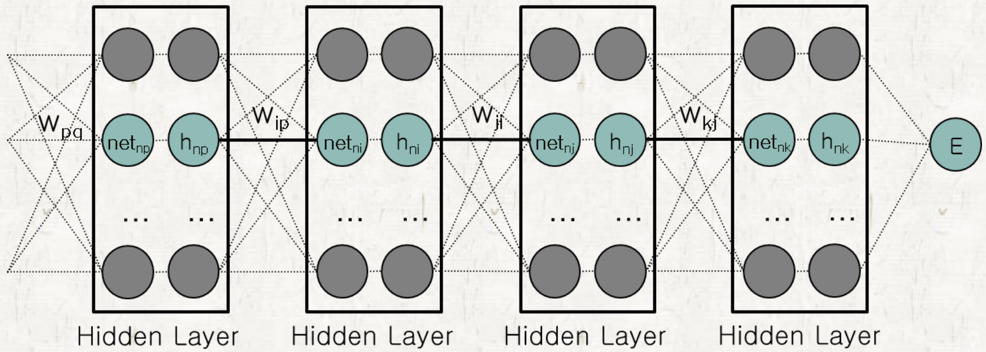
Error Back Propagation (1)

- Weights between deep layers
  - For D_n=(x_{n1}, x_{n2}, …, x_{nd}, t_{n1} , t_{n2}, …, t_{nm})



Error Back Propagation (2)

- Weights between deep layers



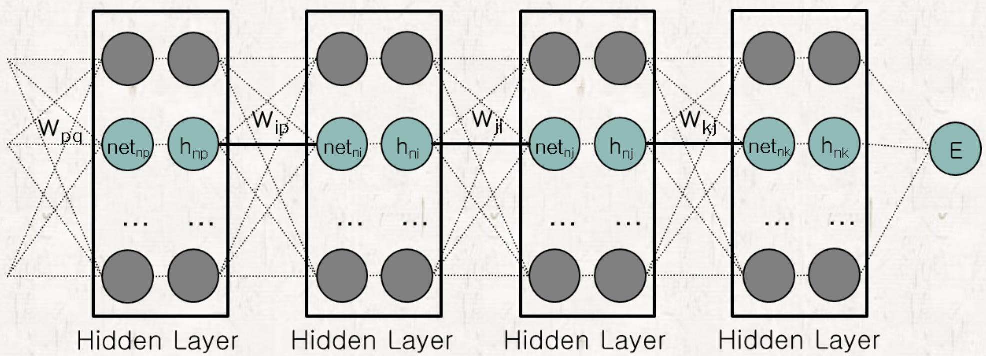
$$\frac{\partial E}{\partial w_{kj}} = \frac{\partial E}{\partial net_{nk}} \frac{\partial net_{nk}}{\partial w_{kj}} = \delta_{nk} h_{nj} \qquad \delta_{nk} = \frac{\partial E}{\partial net_{nk}}$$

$$\frac{\partial E}{\partial w_{ji}} = \frac{\partial E}{\partial net_{nj}} \frac{\partial net_{nj}}{\partial w_{ji}} = \delta_{nj} h_{ni} \qquad \delta_{nj} = \frac{\partial E}{\partial net_{nj}}$$

$$\frac{\partial E}{\partial w_{ip}} = \frac{\partial E}{\partial net_{ni}} \frac{\partial net_{ni}}{\partial w_{ip}} = \delta_{ni} h_{np} \qquad \delta_{ni} = \frac{\partial E}{\partial net_{ni}}$$

Error Back Propagation (3)

- Weights between deep layers



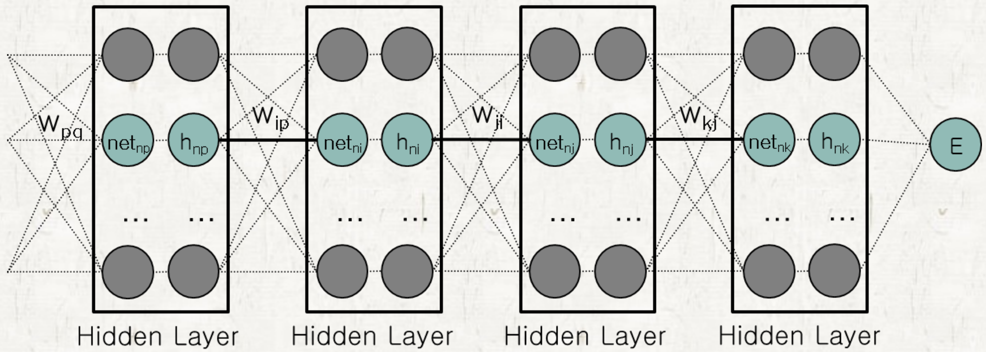
$$\delta_{nk} = \frac{\partial E}{\partial net_{nk}} = \frac{\partial E}{\partial h_{nk}} \frac{\partial h_{nk}}{\partial net_{nk}}$$

$$\delta_{nj} = \frac{\partial E}{\partial net_{nj}} = \frac{\partial E}{\partial h_{nj}} \frac{\partial h_{nj}}{\partial net_{nj}} = \left(\sum_{k=1}^{K} \frac{\partial E}{\partial net_{nk}} \frac{\partial net_{nk}}{\partial h_{nj}} \right) \frac{\partial h_{nj}}{\partial net_{nj}} = \left(\sum_{k=1}^{K} \delta_{nk} w_{kj} \right) \frac{\partial h_{nj}}{\partial net_{nj}}$$

$$\delta_{ni} = \frac{\partial E}{\partial net_{ni}} = \frac{\partial E}{\partial h_{ni}} \frac{\partial h_{ni}}{\partial net_{ni}} = \left(\sum_{j=1}^{J} \frac{\partial E}{\partial net_{nj}} \frac{\partial net_{nj}}{\partial h_{ni}} \right) \frac{\partial h_{ni}}{\partial net_{ni}} = \left(\sum_{j=1}^{J} \delta_{nj} w_{ji} \right) \frac{\partial h_{ni}}{\partial net_{ni}}$$

Error Back Propagation (4)

- Weights between deep layers



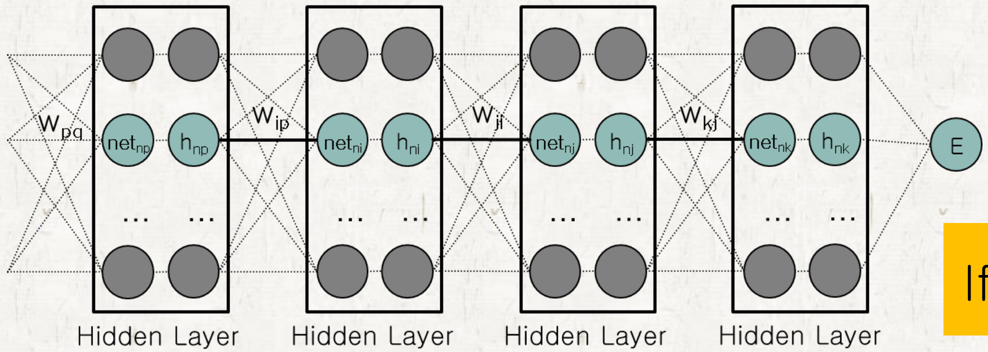
$$\frac{\partial E}{\partial w_{kj}} = \frac{\partial E}{\partial net_{nk}} \frac{\partial net_{nk}}{\partial w_{kj}} = \delta_{nk} h_{nj} \qquad \delta_{nk} = \frac{\partial E}{\partial h_{nk}} \frac{\partial h_{nk}}{\partial net_{nk}}$$

$$\frac{\partial E}{\partial w_{ji}} = \frac{\partial E}{\partial net_{nj}} \frac{\partial net_{nj}}{\partial w_{ji}} = \delta_{nj} h_{ni} \qquad \delta_{nj} = \left(\sum_{k=1}^{K} \delta_{nk} w_{kj} \right) \frac{\partial h_{nj}}{\partial net_{nj}}$$

$$\frac{\partial E}{\partial w_{ip}} = \frac{\partial E}{\partial net_{ni}} \frac{\partial net_{ni}}{\partial w_{ip}} = \delta_{ni} h_{np} \qquad \delta_{ni} = \left(\sum_{j=1}^{J} \delta_{nj} w_{ji} \right) \frac{\partial h_{ni}}{\partial net_{ni}}$$

Error Back Propagation (5)

- Weights between deep layers



If $h = Sigmoid(net)$

$$\frac{\partial E}{\partial w_{kj}} = \frac{\partial E}{\partial net_{nk}} \frac{\partial net_{nk}}{\partial w_{kj}} = \delta_{nk} h_{nj} \qquad \delta_{nk} = \frac{\partial E}{\partial h_{nk}} \frac{\partial h_{nk}}{\partial net_{nk}} = -(t_n - h_{nk}) h_{nk} (1 - h_{nk})$$

$$\frac{\partial E}{\partial w_{ji}} = \frac{\partial E}{\partial net_{nj}} \frac{\partial net_{nj}}{\partial w_{ji}} = \delta_{nj} h_{ni} \qquad \delta_{nj} = \left(\sum_{k=1}^{K} \delta_{nk} w_{kj} \right) \frac{\partial h_{nj}}{\partial net_{nj}} = \left(\sum_{k=1}^{K} \delta_{nk} w_{kj} \right) h_{nj} (1 - h_{nj})$$

$$\frac{\partial E}{\partial w_{ip}} = \frac{\partial E}{\partial net_{ni}} \frac{\partial net_{ni}}{\partial w_{ip}} = \delta_{ni} h_{np} \qquad \delta_{ni} = \left(\sum_{j=1}^{J} \delta_{nj} w_{ji} \right) \frac{\partial h_{ni}}{\partial net_{ni}} = \left(\sum_{j=1}^{J} \delta_{nj} w_{ji} \right) h_{ni} (1 - h_{ni})$$