1 Predicate Logic

1.1 First Order Logic

Definitions

- $P(x)$: x is a person
- $F(x)$: x is an ice cream flavor
- $L(x, y)$: x loves y
- $S(x, y)$: x shaves y
- $R(x)$: x is a reindeer
- $N(x)$: x has a red nose
- $W(x)$: x is weird
- $C(x)$: x is a clown
- b: The Barber (constant)
- r: Rudolph (constant)

Solutions

a. **There is an ice cream flavor loved by everyone.**

$$\exists x(F(x) \land \forall y(P(y) \to L(y, x)))$$

b. **It is not true that everyone loves some ice cream flavor.**

$$\neg\forall x(P(x) \to \exists y(F(y) \land L(x, y)))$$

c. **Anyone who does not shave himself must be shaved by the barber.**

$$\forall x(P(x) \land \neg S(x, x) \to S(b, x))$$

d. **Whomever the barber shaves, must not shave himself.**

$$\forall x(P(x) \land S(b, x) \to \neg S(x, x))$$

e. **Rudolph is a reindeer and Rudolph has a red nose.**

$$R(r) \land N(r)$$

f. **Anyone with a red nose is weird or is a clown.**

$$\forall x(N(x) \to (W(x) \lor C(x)))$$

g. **No reindeer is a clown.**

$$\forall x(R(x) \to \neg C(x))$$

1.2 Clausal Forms

a. $\forall x \forall y (R(x,y) \to (R(x,y) \land Q(y)))$

Step 1: Eliminate implication ($A \to B \equiv \neg A \lor B$)

$$\forall x \forall y (\neg R(x,y) \lor (R(x,y) \land Q(y)))$$

Step 2: Distribute $\lor$ over $\land$

$$\forall x \forall y ((\neg R(x,y) \lor R(x,y)) \land (\neg R(x,y) \lor Q(y)))$$

Step 3: Simplify ($\neg A \lor A \equiv \text{True}$)

$$\forall x \forall y (\text{True} \land (\neg R(x,y) \lor Q(y)))$$

$$\forall x \forall y (\neg R(x,y) \lor Q(y))$$

Step 4: Drop universal quantifiers

$$\neg R(x,y) \lor Q(y)$$

Resulting Clause:

$$\{\neg R(x,y), Q(y)\}$$

b. $\forall x \exists y \forall z (P(x,y,z) \to \exists u R(x,u,z))$

Step 1: Eliminate implication

$$\forall x \exists y \forall z (\neg P(x,y,z) \lor \exists u R(x,u,z))$$

Step 2: Prenex Normal Form (move quantifiers left)

$$\forall x \exists y \forall z \exists u (\neg P(x,y,z) \lor R(x,u,z))$$

Step 3: Skolemization

- Replace y with $f(x)$ (depends on $\forall x$)
- Replace u with $g(x,z)$ (depends on $\forall x$ and $\forall z$)

$$\forall x \forall z (\neg P(x,f(x),z) \lor R(x,g(x,z),z))$$

Step 4: Drop universal quantifiers

$$\neg P(x,f(x),z) \lor R(x,g(x,z),z)$$

Resulting Clause:

$$\{\neg P(x,f(x),z), R(x,g(x,z),z)\}$$

c. $\forall x(\neg\exists y P(x,y) \land \neg(Q(x) \land \neg R(x)))$

Step 1: Move negations inward

$$\forall x(\forall y \neg P(x,y) \land (\neg Q(x) \lor \neg\neg R(x)))$$

$$\forall x(\forall y \neg P(x,y) \land (\neg Q(x) \lor R(x)))$$

Step 2: Standardize variables and move quantifiers

$$\forall x \forall y(\neg P(x,y) \land (\neg Q(x) \lor R(x)))$$

Step 3: Drop universal quantifiers and split conjunction

$$\neg P(x,y) \land (\neg Q(x) \lor R(x))$$

Resulting Clauses:

1. $\{\neg P(x,y)\}$
2. $\{\neg Q(x), R(x)\}$

1.3 Knowledge Extraction via Resolution

Conversion to Clause Form (CNF)

Convert the Knowledge Base sentences $S1 - S4$ and the negated goal into a set of clauses.

- Sentence S1: $\forall t \forall x(Observe(t,x) \land Danger(x) \to Suggestion(t, \text{flee}))$
  Elim. Implication: $\neg Observe(t,x) \lor \neg Danger(x) \lor Suggestion(t, \text{flee})$
  Clause C_1: $\{\neg Observe(t,x), \neg Danger(x), Suggestion(t, \text{flee})\}$

- Sentence S2: $\forall t(\neg\exists x(Observe(t,x) \land Danger(x)) \to Suggestion(t, \text{stay}))$
  Elim. Implication: $\exists x(Observe(t,x) \land Danger(x)) \lor Suggestion(t, \text{stay})$
  Skolemization $(x \to h(t))$: $(Observe(t,h(t)) \land Danger(h(t))) \lor Suggestion(t, \text{stay})$
  Clause C_{2a}: $\{Observe(t,h(t)), Suggestion(t, \text{stay})\}$
  Clause C_{2b}: $\{Danger(h(t)), Suggestion(t, \text{stay})\}$

- Sentence S3: $Danger(\text{lion})$
  Clause C_3: $\{Danger(\text{lion})\}$

- Sentence S4: $Observe(\text{now}, \text{lion})$
  Clause C_4: $\{Observe(\text{now}, \text{lion})\}$

- Goal: Prove $\exists x Suggestion(\text{now}, x)$.
  Negated Goal with Answer Literal: $\neg Suggestion(\text{now}, z) \lor Ans(z)$
  Clause C_G: $\{\neg Suggestion(\text{now}, z), Ans(z)\}$

Resolution Trace

- Resolve C_1 and C_4:
  $C_1 : \{\neg Observe(t, x), \neg Danger(x), Suggestion(t, \text{flee})\}$
  $C_4 : \{Observe(\text{now}, \text{lion})\}$
  Substitution: $\theta = \{t/\text{now}, x/\text{lion}\}$
  Resolvent R_1: $\{\neg Danger(\text{lion}), Suggestion(\text{now}, \text{flee})\}$
- Resolve R_1 and C_3:
  $R_1 : \{\neg Danger(\text{lion}), Suggestion(\text{now}, \text{flee})\}$
  $C_3 : \{Danger(\text{lion})\}$
  Substitution: $\theta = \emptyset$
  Resolvent R_2: $\{Suggestion(\text{now}, \text{flee})\}$
- Resolve R_2 and C_G:
  $R_2 : \{Suggestion(\text{now}, \text{flee})\}$
  $C_G : \{\neg Suggestion(\text{now}, z), Ans(z)\}$
  Substitution: $\theta = \{z/\text{flee}\}$
  Resolvent R_3: $\{Ans(\text{flee})\}$

Conclusion

The clause contains only the answer literal. Thus, the Knowledge Base entails the query with the answer:

$$\mathbf{x} = \text{flee}$$

2 Uncertainty

2.1 Probability

a)

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{P(\text{allowed to stay} \mid \text{bribed}) \cdot P(\text{bribed})}{P(\text{allowed to stay})}$$

$$P(\text{allowed to stay} \mid \text{bribed}) = \frac{4}{5}$$

$$P(\text{allowed to stay} \mid \neg\text{bribed}) = \frac{1}{3}$$

$$P(\text{bribed}) = \frac{1}{4}$$

$$P(\text{allowed to stay}) = \sum_{z \in \text{dom}(bribed)} P(\text{allowed to stay} \mid z) \cdot P(z)$$
$$= \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{4}{20} + \frac{3}{12} = \frac{4}{20} + \frac{5}{20} = \frac{9}{20}$$

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{\frac{4}{5} \cdot \frac{1}{4}}{\frac{9}{20}} = \frac{4}{20} \cdot \frac{20}{9} = \frac{4}{9}$$

b)

The participants bribe twice or never

$$\text{Bribing:}\quad P(\text{stays show 1 and 2} \mid \text{bribed}) = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

$$\text{Not bribing:}\quad P(\text{stays show 1 and 2} \mid \neg\text{bribed}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$P(\text{stays show 1 and 2}) = \frac{16}{25} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{4}{25} + \frac{3}{16} = \frac{64}{400} + \frac{75}{400} = \frac{139}{400} \approx 0.34625$$

c)

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{4}{9}$$
$$P(\neg\text{bribed} \mid \text{allowed to stay}) = 1 - \frac{4}{9} = \frac{5}{9}$$

$$P(\text{kicked off in second show}) = \sum_{z \in \text{dom}(bribed)} P(\text{kicked off in second show} \mid z) \cdot P(z)$$
$$= \frac{4}{9} \cdot \frac{1}{5} + \frac{5}{9} \cdot \frac{2}{3} = \frac{4}{45} + \frac{10}{27} \approx 0.4593$$

2.2 Naive Bayes' Classifier

The Naive Bayes' model states that:

$$P(Cause|e) \propto P(Cause).\Pi_j P(e_j|Course)$$

Therefore, by finding the larger value of $P(yes|x)$ and $P(no|x)$ we can find what class x will take, given that x is the case where age $\leq 30, income = high, Married = yesandcredit - rating = fair$.

$$P(yes) = \frac{9}{14}$$
$$P(no) = \frac{5}{14}$$

$$P(yes|x) = P(yes).(P(age \leq 30|yes).P(income = high|yes).P(married = yes|yes).P(cr = fair|yes))$$
$$= \frac{9}{14} * \frac{2}{9} * \frac{2}{9} * \frac{7}{9} * \frac{6}{9} = \frac{12}{729} \approx 0.01646$$

$$P(no|x) = P(no).(P(age \leq 30|no).P(income = high|no).P(married = yes|no).P(cr = fair|no))$$
$$= \frac{5}{14} * \frac{3}{5} * \frac{2}{5} * \frac{1}{5} * \frac{2}{5} = \frac{12}{1750} \approx 0.00686$$

Since $P(yes|x) > P(no|x)$, the class of the given example is yes. $buy\_car = yes$