

1 Adversarial Search and Games

1.1 Minimax Search

1.2 Games of Chance

1.3 Correctness of $\alpha - \beta$ -pruning

2 Propositional Logic

2.1 Who is lying?

We define the variables:

- J : John tells the truth.
- P : Peter tells the truth.
- E : Emma tells the truth.

(a)

The statements:

John says: 'Peter always lies.'

This means J is true if and only if P is false. Formula: $(J) \leftrightarrow \neg(P)$

Peter says: 'Either John is a liar or Emma is a liar, but not both.'

There are two possibilities for this statement to be true:

- Possibility 1: John lies ($\neg J$) AND Emma tells the truth (E).
- Possibility 2: John tells the truth (J) AND Emma lies ($\neg E$).
- We connect these two possibilities with an OR.

Formula: $P \leftrightarrow ((\neg J \wedge E) \vee (J \wedge \neg E))$

Emma says: 'If John is a liar, then Peter is also a liar.'

This is a classical implication.

Formula: $E \leftrightarrow (\neg J \rightarrow \neg P)$

(b)

The set of formulae is:

$$1. \quad J \leftrightarrow \neg P \tag{1}$$

$$2. \quad P \leftrightarrow ((\neg J \vee \neg E) \wedge \neg(\neg J \wedge \neg E)) \tag{2}$$

$$3. \quad E \leftrightarrow (\neg J \rightarrow \neg P) \tag{3}$$

Goal: Find a truth assignment that satisfies all three.

Step 1: Analysis of John's formula

Formula: $J \leftrightarrow \neg P$

Using the definition $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$, we have:

$$(J \leftrightarrow \neg P) \equiv (J \rightarrow \neg P) \wedge (\neg P \rightarrow J)$$

Now rewrite the implications:

$$\begin{aligned} J \rightarrow \neg P &\equiv \neg J \vee \neg P \\ \neg P \rightarrow J &\equiv P \vee J \end{aligned}$$

So John's statement becomes:

$$(\neg J \vee \neg P) \wedge (P \vee J)$$

From this, we see the required relation:

$$J = \neg P$$

Step 2: Analysis of Emma's formula

Formula: $E \leftrightarrow (\neg J \rightarrow \neg P)$

Rewrite the implication using $A \rightarrow B \equiv \neg A \vee B$:

$$\neg J \rightarrow \neg P \equiv J \vee \neg P$$

Now substitute John's relation $J = \neg P$:

$$J \vee \neg P \equiv (\neg P) \vee \neg P \equiv \neg P$$

$$E \leftrightarrow \neg P$$

We now know:

$$J = \neg P \quad \text{and} \quad E = \neg P$$

Therefore:

$$J = E$$

Interim result: John and Emma must have the same truth value.

Step 3: Analysis of Peter's formula

Peter states his 'one lies but not both' condition:

$$P \leftrightarrow ((\neg J \vee \neg E) \wedge \neg(\neg J \wedge \neg E))$$

3.1 Substitute $J = E$

Since we know $J = E$, then $\neg J = \neg E$. Thus the formula becomes:

$$P \leftrightarrow ((\neg J \vee \neg J) \wedge \neg(\neg J \wedge \neg J))$$

Simplify step by step:

(a) $\neg J \vee \neg J = \neg J$

(b) $\neg J \wedge \neg J = \neg J$

(c) $\neg(\neg J) = J$

Therefore Peter's statement reduces to:

$$P \leftrightarrow (\neg J \wedge J)$$

3.2 Simplify the contradiction

The term $(\neg J \wedge J)$ is a contradiction, meaning it is always false:

$$\neg J \wedge J \equiv \text{false}$$

So:

$$P \leftrightarrow \text{false}$$

Thus:

$$P = \text{false}$$

Step 4: Determine J and E

We previously established:

$$J = \neg P \quad \text{and} \quad E = \neg P$$

Since $P = \text{false}$, we conclude:

$$\begin{aligned} J &= \neg \text{false} = \text{true} \\ E &= \neg \text{false} = \text{true} \end{aligned}$$

Final answer

- John tells the truth
- Peter lies
- Emma tells the truth

2.2 Knowledge Bases

We are given the knowledge base:

$$K = \{ A \vee (B \vee \neg C), A \leftrightarrow B, (C \wedge A) \Rightarrow D \}.$$

(a)

1. The formula $A \vee (B \vee \neg C)$ simplifies to:

$$A \vee B \vee \neg C.$$

2. The equivalence $A \Leftrightarrow B$ is rewritten as:

$$(A \rightarrow B) \wedge (B \rightarrow A),$$

which becomes:

$$(\neg A \vee B) \wedge (\neg B \vee A).$$

3. The implication $(C \wedge A) \Rightarrow D$ becomes:

$$\neg(C \wedge A) \vee D = \neg C \vee \neg A \vee D.$$

Thus, the CNF of the entire knowledge base is:

$$(A \vee B \vee \neg C) \wedge (\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg C \vee \neg A \vee D).$$

(b)

The clauses in the CNF are:

$$\{A \vee B \vee \neg C, \neg A \vee B, \neg B \vee A, \neg C \vee \neg A \vee D\}.$$

As sets of literals:

$$\{\{A, B, \neg C\}, \{\neg A, B\}, \{\neg B, A\}, \{\neg C, \neg A, D\}\}.$$

(c)

A definite clause contains exactly one positive literal.

Checking each clause:

- $A \vee B \vee \neg C$: two positive literals \rightarrow not definite.
- $\neg A \vee B$: one positive literal $B \rightarrow$ definite.
- $\neg B \vee A$: one positive literal $A \rightarrow$ definite.
- $\neg C \vee \neg A \vee D$: one positive literal $D \rightarrow$ definite.

Thus, the definite clauses are:

$$\neg A \vee B, \quad \neg B \vee A, \quad \neg C \vee \neg A \vee D.$$

2.3 Models

Consider the four propositions A, B, C , and D . The total number of possible truth assignments (interpretations) for 4 binary variables is $2^4 = 16$. We need to find how many of these 16 assignments make the following formulae true.

(a) $B \vee \neg C$

This formula depends only on B and C . The variables A and D are 'don't cares' (irrelevant to the truth value, but must be counted).

- **Analyze** $B \vee \neg C$: A disjunction is FALSE only if both parts are false.
 - B is False (0)
 - $\neg C$ is False $\Rightarrow C$ is True (1)

Combinations for (B, C) :

- $(0, 0) \rightarrow 0 \vee 1 = 1$ (True)
- $(0, 1) \rightarrow 0 \vee 0 = 0$ (False)
- $(1, 0) \rightarrow 1 \vee 1 = 1$ (True)
- $(1, 1) \rightarrow 1 \vee 0 = 1$ (True)

So, there are **3** valid assignments for the pair (B, C) .

- **Account for A and D** : For each valid combination of B and C , the variables A and D can be either True or False ($2 \times 2 = 4$ variations).
- **Calculation:**

$$\text{Models} = 3 \times 2^2 = 3 \times 4 = 12$$

Answer: 12 models.

(b) $A \wedge \neg(\neg B \vee C) \wedge D$

This is a conjunction. For the formula to be True, **every** part must be True.

- **Analyze the components:**
 - Part 1: A must be **True** (1).
 - Part 2: D must be **True** (1).
 - Part 3: $\neg(\neg B \vee C)$ must be **True**.
- **Simplify Part 3 (De Morgan's Law):**

$$\neg(\neg B \vee C) \equiv \neg(\neg B) \wedge \neg C \equiv B \wedge \neg C$$

For this to be True:

- B must be **True** (1).
- C must be **False** (0).

- **Calculation:** We have fixed values for all four variables:

$$A = 1, \quad B = 1, \quad C = 0, \quad D = 1$$

There is only 1 specific assignment that satisfies this.

Answer: 1 model.

(c) $((B \rightarrow D) \vee (A \rightarrow D)) \wedge C$

This is a conjunction of a complex term and C .

- **Constraint on C :** Since it is a conjunction ($\dots \wedge C$), C must be **True** (1). This reduces our search space. We effectively assume $C = 1$ and look for satisfying assignments of A, B, D ($2^3 = 8$ possibilities).

- **Analyze the complex term:** Formula: $(B \rightarrow D) \vee (A \rightarrow D)$

Rewrite implication ($X \rightarrow Y \equiv \neg X \vee Y$):

$$(\neg B \vee D) \vee (\neg A \vee D)$$

Simplify (Associativity and Idempotence of \vee):

$$\neg A \vee \neg B \vee D \vee D \equiv \neg A \vee \neg B \vee D$$

This formula is a disjunction. It is **True** in all cases *except* when all literals are False.

- **Find the failing case:** The expression $\neg A \vee \neg B \vee D$ is False if:

- $\neg A$ is False $\Rightarrow A = 1$
- $\neg B$ is False $\Rightarrow B = 1$
- D is False $\Rightarrow D = 0$

There is exactly **1** failing combination for (A, B, D) out of 8 possibilities.

- **Calculation:**

- Total combinations for (A, B, D) : 8
- Failing combinations: 1
- Valid combinations: $8 - 1 = 7$

Since C is fixed to True, we do not multiply further.

Answer: 7 models.