

1 Predicate Logic

1.1 First Order Logic

Definitions

- $P(x)$: x is a person
- $F(x)$: x is an ice cream flavor
- $L(x, y)$: x loves y
- $S(x, y)$: x shaves y
- $R(x)$: x is a reindeer
- $N(x)$: x has a red nose
- $W(x)$: x is weird
- $C(x)$: x is a clown
- b : The Barber (constant)
- r : Rudolph (constant)

Solutions

a. There is an ice cream flavor loved by everyone.

$$\exists x(F(x) \wedge \forall y(P(y) \rightarrow L(y, x)))$$

b. It is not true that everyone loves some ice cream flavor.

$$\neg \forall x(P(x) \rightarrow \exists y(F(y) \wedge L(x, y)))$$

c. Anyone who does not shave himself must be shaved by the barber.

$$\forall x(P(x) \wedge \neg S(x, x) \rightarrow S(b, x))$$

d. Whomever the barber shaves, must not shave himself.

$$\forall x(P(x) \wedge S(b, x) \rightarrow \neg S(x, x))$$

e. Rudolph is a reindeer and Rudolph has a red nose.

$$R(r) \wedge N(r)$$

f. Anyone with a red nose is weird or is a clown.

$$\forall x(N(x) \rightarrow (W(x) \vee C(x)))$$

g. No reindeer is a clown.

$$\forall x(R(x) \rightarrow \neg C(x))$$

1.2 Clausal Forms

a. $\forall x \forall y (R(x, y) \rightarrow (R(x, y) \wedge Q(y)))$

Step 1: Eliminate implication ($A \rightarrow B \equiv \neg A \vee B$)

$$\forall x \forall y (\neg R(x, y) \vee (R(x, y) \wedge Q(y)))$$

Step 2: Distribute \vee over \wedge

$$\forall x \forall y ((\neg R(x, y) \vee R(x, y)) \wedge (\neg R(x, y) \vee Q(y)))$$

Step 3: Simplify ($\neg A \vee A \equiv \text{True}$)

$$\forall x \forall y (\text{True} \wedge (\neg R(x, y) \vee Q(y)))$$

$$\forall x \forall y (\neg R(x, y) \vee Q(y))$$

Step 4: Drop universal quantifiers

$$\neg R(x, y) \vee Q(y)$$

Resulting Clause:

$$\{\neg R(x, y), Q(y)\}$$

b. $\forall x \exists y \forall z (P(x, y, z) \rightarrow \exists u R(x, u, z))$

Step 1: Eliminate implication

$$\forall x \exists y \forall z (\neg P(x, y, z) \vee \exists u R(x, u, z))$$

Step 2: Prenex Normal Form (move quantifiers left)

$$\forall x \exists y \forall z \exists u (\neg P(x, y, z) \vee R(x, u, z))$$

Step 3: Skolemization

- Replace y with $f(x)$ (depends on $\forall x$)
- Replace u with $g(x, z)$ (depends on $\forall x$ and $\forall z$)

$$\forall x \forall z (\neg P(x, f(x), z) \vee R(x, g(x, z), z))$$

Step 4: Drop universal quantifiers

$$\neg P(x, f(x), z) \vee R(x, g(x, z), z)$$

Resulting Clause:

$$\{\neg P(x, f(x), z), R(x, g(x, z), z)\}$$

c. $\forall x(\neg\exists y P(x, y) \wedge \neg(Q(x) \wedge \neg R(x)))$

Step 1: Move negations inward

$$\forall x(\forall y \neg P(x, y) \wedge (\neg Q(x) \vee \neg \neg R(x)))$$

$$\forall x(\forall y \neg P(x, y) \wedge (\neg Q(x) \vee R(x)))$$

Step 2: Standardize variables and move quantifiers

$$\forall x \forall y (\neg P(x, y) \wedge (\neg Q(x) \vee R(x)))$$

Step 3: Drop universal quantifiers and split conjunction

$$\neg P(x, y) \wedge (\neg Q(x) \vee R(x))$$

Resulting Clauses:

1. $\{\neg P(x, y)\}$
2. $\{\neg Q(x), R(x)\}$

1.3 Knowledge Extraction via Resolution

Conversion to Clause Form (CNF)

Convert the Knowledge Base sentences $S1 - S4$ and the negated goal into a set of clauses.

- Sentence S1: $\forall t \forall x(Observe(t, x) \wedge Danger(x) \rightarrow Suggestion(t, flee))$
 Elim. Implication: $\neg Observe(t, x) \vee \neg Danger(x) \vee Suggestion(t, flee)$
 Clause $C_1: \{\neg Observe(t, x), \neg Danger(x), Suggestion(t, flee)\}$
- Sentence S2: $\forall t(\neg\exists x(Observe(t, x) \wedge Danger(x)) \rightarrow Suggestion(t, stay))$
 Elim. Implication: $\exists x(Observe(t, x) \wedge Danger(x)) \vee Suggestion(t, stay)$
 Skolemization ($x \rightarrow h(t)$): $(Observe(t, h(t)) \wedge Danger(h(t))) \vee Suggestion(t, stay)$
 Clause $C_{2a}: \{Observe(t, h(t)), Suggestion(t, stay)\}$
 Clause $C_{2b}: \{Danger(h(t)), Suggestion(t, stay)\}$
- Sentence S3: $Danger(lion)$
 Clause $C_3: \{Danger(lion)\}$
- Sentence S4: $Observe(now, lion)$
 Clause $C_4: \{Observe(now, lion)\}$
- Goal: Prove $\exists x Suggestion(now, x)$.
 Negated Goal with Answer Literal: $\neg Suggestion(now, z) \vee Ans(z)$
 Clause $C_G: \{\neg Suggestion(now, z), Ans(z)\}$

Resolution Trace

- Resolve C_1 and C_4 :

$$C_1 : \{\neg\text{Observe}(t, x), \neg\text{Danger}(x), \text{Suggestion}(t, \text{flee})\}$$

$$C_4 : \{\text{Observe}(\text{now}, \text{lion})\}$$

Substitution: $\theta = \{t/\text{now}, x/\text{lion}\}$

$$\text{Resolvent } R_1 : \{\neg\text{Danger}(\text{lion}), \text{Suggestion}(\text{now}, \text{flee})\}$$

- Resolve R_1 and C_3 :

$$R_1 : \{\neg\text{Danger}(\text{lion}), \text{Suggestion}(\text{now}, \text{flee})\}$$

$$C_3 : \{\text{Danger}(\text{lion})\}$$

Substitution: $\theta = \emptyset$

$$\text{Resolvent } R_2 : \{\text{Suggestion}(\text{now}, \text{flee})\}$$

- Resolve R_2 and C_G :

$$R_2 : \{\text{Suggestion}(\text{now}, \text{flee})\}$$

$$C_G : \{\neg\text{Suggestion}(\text{now}, z), \text{Ans}(z)\}$$

Substitution: $\theta = \{z/\text{flee}\}$

$$\text{Resolvent } R_3 : \{\text{Ans}(\text{flee})\}$$

Conclusion

The clause contains only the answer literal. Thus, the Knowledge Base entails the query with the answer:

$$\mathbf{x} = \text{flee}$$

2 Uncertainty

2.1 Probability

a)

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{P(\text{allowed to stay} \mid \text{bribed}) \cdot P(\text{bribed})}{P(\text{allowed to stay})}$$

$$P(\text{allowed to stay} \mid \text{bribed}) = \frac{4}{5}$$

$$P(\text{allowed to stay} \mid \neg\text{bribed}) = \frac{1}{3}$$

$$P(\text{bribed}) = \frac{1}{4}$$

$$\begin{aligned} P(\text{allowed to stay}) &= \sum_{z \in \text{dom}(\text{bribed})} P(\text{allowed to stay} \mid z) \cdot P(z) \\ &= \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{4}{20} + \frac{3}{12} = \frac{4}{20} + \frac{5}{20} = \frac{9}{20} \end{aligned}$$

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{\frac{4}{5} \cdot \frac{1}{4}}{\frac{9}{20}} = \frac{4}{20} \cdot \frac{20}{9} = \frac{4}{9}$$

b)

The participants bribe twice or never

$$\text{Bribing: } P(\text{stays show 1 and 2} \mid \text{bribed}) = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

$$\text{Not bribing: } P(\text{stays show 1 and 2} \mid \neg\text{bribed}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$P(\text{stays show 1 and 2}) = \frac{16}{25} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{4}{25} + \frac{3}{16} = \frac{64}{400} + \frac{75}{400} = \frac{139}{400} \approx 0.34625$$

c)

$$P(\text{bribed} \mid \text{allowed to stay}) = \frac{4}{9}$$

$$P(\neg\text{bribed} \mid \text{allowed to stay}) = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\begin{aligned} P(\text{kicked off in second show}) &= \sum_{z \in \text{dom}(\text{bribed})} P(\text{kicked off in second show} \mid z) \cdot P(z) \\ &= \frac{4}{9} \cdot \frac{1}{5} + \frac{5}{9} \cdot \frac{2}{3} = \frac{4}{45} + \frac{10}{27} \approx 0.4593 \end{aligned}$$

2.2 Naive Bayes' Classifier