Lecture 4: Spectral graph theory and the matrix-tree theorem · 1MA020

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We introduce the basic notions of spectral graph theory, such as the adjacency and incidence matrices of a graph. We build up the theory around these, eventually arriving at the Kirchhoff matrix-tree theorem, which will let us count spanning trees in a graph.

Throughout this lecture, we will assume that $G = (V, E)$ is a graph on n vertices, with vertex set $[n]$, and m edges $\{e_1, \ldots, e_m\}$.

Definition 1. The *adjacency matrix* A of a graph G is the $n \times n$ matrix having entries $A_{ij} = 1$ if i and j are neighbours, and zero otherwise.

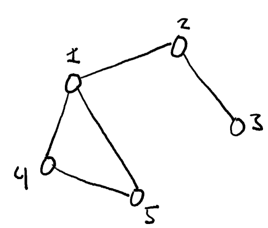

Figure 1: A graph whose adjacency matrix we compute in an example.

Example 2. The graph given in Figure 1 has adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 3. There are a few things we can notice immediately about adjacency matrices. First off, they must be symmetric, since of course i and j being neighbours is the same statement as j and i being neighbours, and the diagonal entries are always zero. Second, the row sums or the column sums give us the degree of each vertex, since a row contains one 1 per edge incident to its vertex.

Finally, it follows from the spectral theorem for symmetric matrices that all eigenvalues of A are real.

There is another way to turn a graph into a matrix that is very useful, but it is easiest to define it first for directed graphs, so we begin by giving a definition of a directed graph.

Definition 4. A *directed graph* $G = (V, E)$ (or *digraph* for short) consists of a set of vertices V and a set of edges E, where each edge is a tuple of two distinct[2] vertices. We call the first vertex in the tuple the *source* and the second vertex the *target* of the edge.

Having said this, we can now define the incidence matrix of a directed graph.

Definition 5. The *incidence matrix* D of a digraph G is the $n \times m$ matrix having entries D_{ij}, where

$$D_{ij} = \begin{cases} 1 & \text{if } i \text{ is the target of } e_j \\ -1 & \text{if } i \text{ is the source of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

For a simple graph G and a matrix D, we say that D is *an* incidence matrix of G if there is a way to direct the edges of G so that the incidence matrix of this directed version is D.

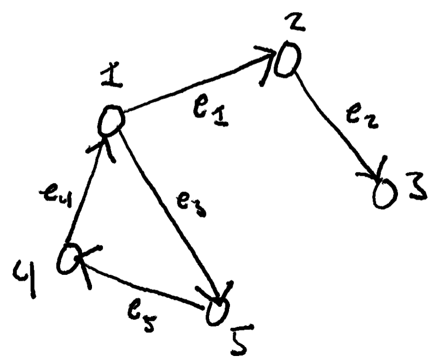

Figure 2: A way of directing the edges of the graph in Figure 1. We have also labelled the edges with their numbers.

Example 6. In Figure 2 we see one way of directing the edges of the graph in Figure 1 to make it a digraph. This digraph has incidence matrix[3]

$$\begin{pmatrix} -1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

and so this matrix is also an incidence matrix for our original simple graph.

Having given these definitions, let us start seeing why these matrices are interesting.

Lemma 7. *Let D be an incidence matrix of a finite graph G. Then*

1. *the sum of any column of D is zero, hence* $\operatorname{rank} D \leq n-1$,
2. *if G is connected,*[4] *then* $\operatorname{rank} D = n-1$,

[2] It is entirely possible to define what we mean by a directed multigraph as well, but for our purposes, we stick to simple directed graphs with no loops or parallel edges.

[3] It is a bit unfortunate that our example graph has exactly one cycle, so it has equally many edges and vertices and the incidence matrix is square – they are of course not in general square.

[4] If you were wondering why we are proving this lemma for incidence matrices of undirected graphs, the reason lies here – there are several notions of connectedness of directed graphs, and most are not equivalent to being connected when you forget the directions of the edges.

3. *and if G has c components, then* $\operatorname{rank} D = n - c$.

Proof. The column sums of D are all zero, since each column contains exactly one 1 and one -1, for the source and target of its associated edge. Therefore, if we take the sum of all the *row*-vectors, each entry will be zero, so we have a non-trivial linear combination of 0, and hence $\operatorname{rank} D \leq n - 1$.

To show the second part, we will show that this is, up to scaling, the only non-trivial linear combination of 0 with the row-vectors. So, let r_i for $i = 1, \ldots, n$ denote the row-vectors of D, and suppose we have a non-trivial linear combination

$$\sum_{i=1}^{n} \alpha_i r_i = 0.$$

Consider a row k for which $\alpha_k \neq 0$ – in this row, there is a non-zero entry in every column corresponding to an edge incident to the vertex k. Each of these columns has one other non-zero entry, and that entry has the opposite sign of the one in our row. So for these to sum to zero, the coefficients in the linear combination must be the same. So what we have seen is that $\alpha_\ell = \alpha_k$ whenever ℓ is adjacent to k.

However, the graph is connected, so this argument in fact extends to showing that all the coefficients must equal α_k, and so this shows the linear combination is indeed just a rescaling of the sum of all the rows.

To see the final part, observe that we can always relabel the vertices and edges in such a way that D is in block-diagonal form, with every block being the incidence matrix of a connected components. So the claim follows from the statement for connected graphs. □

Exercises