Lecture 2: Eulerianity, simple graphs, and subgraphs · 1MA020

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We formalize the ideas we started with in the first exercise session, giving a proof of Euler's result on Eulerian circuits. We then make some more definitions about simple graphs and subgraphs, and we state some elementary results about these notions.

The very first definition we give in this course will actually be of a *multigraph*, not of a graph.

Definition 1. A *multigraph* G is a tuple (V, E), consisting of a set V of *vertices*, and a multiset[2] E of *edges*. Each edge is a multiset containing two vertices from V, called its *endpoints*. We say two vertices are *adjacent* if there is an edge between them, and a vertex v is *incident* to an edge e if $v \in e$.

[2] A multiset is just like a set, except an element may occur more than once.

If the same edge occurs more than once in E, we say that these edges are *parallel*. If the two endpoints of an edge are equal, we call it a *loop*.

Unless explicitly stated, we always assume that both V and E are finite sets. Otherwise, we say the graph is *infinite*.[3]

[3] It may sometimes be the case that our proofs work without modification also for infinite graphs – thinking about whether they do may be a useful thing to do when reading the proofs, to understand them better.

Next, continuing to formalize the things we learned thinking about the bridges of Königsberg, let us define what a walk is.

Definition 2. Let $G = (V, E)$ be a multigraph. A *walk* of length k is a sequence of $k + 1$ vertices $v_0 v_1 v_2 \ldots v_k$ and a sequence of k edges[4] $e_1 e_2 \ldots e_k$ such that $e_i = \{v_{i-1}, v_i\}$ for all i. A *trail* is a walk that uses no edge twice, and a *path* is a walk that uses no vertex twice. A *circuit* is a trail where the first and last vertices coincide, and a *cycle* is a circuit where these are the only vertices that coincide.

[4] So the length of a walk is the number of edges, not the number of vertices.

We have one example of a walk in Figure 1 – it does not repeat any of the edges, so it is a trail, but it repeats the central vertex we have labelled with both v_2 and v_6, so it is not a path.

Having introduced walks, we can give a definition of another very natural property, namely connectedness.[5]

[5] You've probably already seen the notion of connectedness of a subset of $\mathbb{R}^2$ in a calculus course, and if you've looked at other geometry it appears there as well. This is the same notion, just discretized.

Definition 3. We say that a graph G is *connected* if there is, for any two vertices $u, v \in G$, a walk from u to v. We say that two vertices in a graph are connected to each other if there is a walk between them.

Notice that there is a trivial "lazy" walk connecting every vertex to itself, so the relation of connectedness is an equivalence relation. The equivalence classes of this equivalence relation are called the *connected components* of the graph.[6]

[6] We could equivalently have defined the connected components as the maximal connected subgraphs of the graph – when we get to a formal definition of subgraph, think about why this is true.

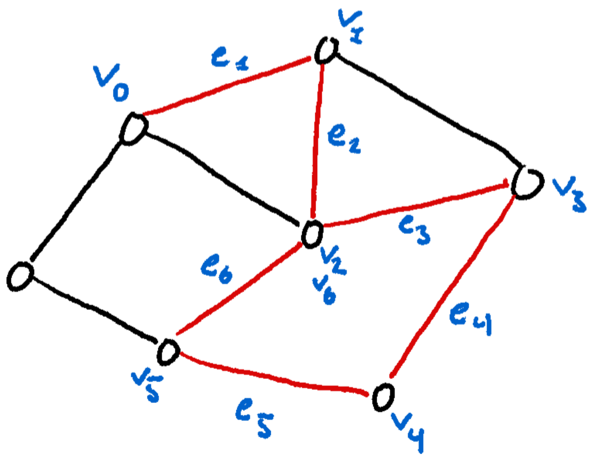

Figure 1: A walk in a graph, which is a trail but not a path.

Let us now define the thing we were studying when we thought about the bridges of Königsberg.

Definition 4. An *Eulerian trail* is a trail that uses every edge in the graph exactly once – if additionally it has the same starting and ending vertex, we call it an *Eulerian circuit*. If there is an Eulerian circuit in a graph, we call the graph *Eulerian*.

The problem we were studying was thus to find a simple condition for when a graph is Eulerian. The condition we found[7] involved the number of edges incident to a vertex, so let us also give this notion a name.

[7] Hopefully.

Definition 5. The *degree* of a vertex v, denoted d_v, is the number of edges a vertex is incident to, with loops counted twice.

We now have all the language we need to formally state and prove the theorem that started graph theory all those nearly three hundred years ago.

Theorem 6 (Euler (1736))**.** *A finite connected graph is Eulerian if and only if all its vertices have even degree.*