Lecture 3: Common graph families, trees, and Cayley's theorem · 1MA020

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We start by introducing a few named families of graphs. Then we introduce the class of *trees*, and prove some results about them. The main result is Cayley's theorem, which counts the number of labelled trees on n vertices.

Common graph families

We warm up today by giving names to some common families of simple graphs that we will see reappearing throughout the course. They are illustrated in Figure 1.

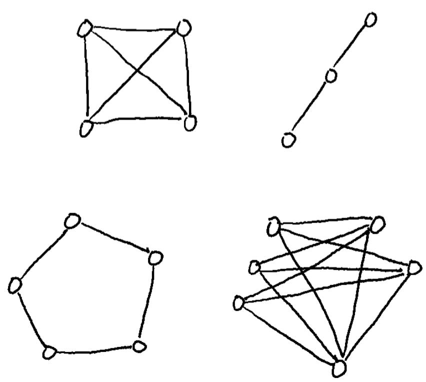

Figure 1: Four graphs: K_4, P_2, C_5, and $K_{3,2,1}$.

1. The complete graphs on n vertices, denoted K_n. These contain all the $\binom{n}{2}$ potential edges. These are also called *cliques* when we see them as subgraphs of a bigger graph.

2. The paths of length ℓ, denoted P_ℓ. If we take the set $\{0, 1, \ldots, \ell\}$ to be our vertex set, the edges are precisely of the form $\{i-1, i\}$ for $i \in [\ell]$.[2]

3. The cycle graphs on n vertices, denoted C_n. We can think of these as a path of length $n-1$ with an extra edge joining the first and last vertex.

4. The complete bipartite graphs on $a + b$ vertices, denoted $K_{a,b}$. These have as vertex set the disjoint union of two sets L and R,[3] with $|L| = a$ and $|R| = b$, and there is an edge between v and w whenever $v \in L$ and $w \in R$. When we see these graphs as subgraphs of a bigger graph, we sometimes also call them *bicliques*.

[2] This is another notation you might not have seen before: For an integer n, we write $[n]$ for the set $\{1, 2, \ldots, n\}$

[3] Think of these as the "left" and "right" vertices.

5. Generalizing the complete bipartite graphs, the complete multipartite graph on r parts with sizes $a_1, a_2, \ldots a_r$, denoted $K_{a_1,a_2,\ldots,a_r}$, has as its vertex set the disjoint union of r sets $V_1, V_2, \ldots, V_r$, where $|V_i| = a_i$, and there is an edge between two vertices whenever they are not in the same part. We can notice that when $r = 2$ this is a complete bipartite graph, and when all the parts are of size 1 this is a complete graph.

For most of these, it is obvious how many edges they will have. Let us state a lemma that shows how many the complete multipartite graphs have.

Lemma 1. *The complete multipartite graph $K_{a_1,a_2,\ldots,a_r}$ has $\frac{1}{2}\left(n^2 - a_1^2 - \ldots - a_r^2\right)$ edges.*

Proof. We use the handshake lemma from the previous lecture. Since a vertex in V_i has one edge to every vertex not in V_i, it has degree $n - a_i$, and there are a_i such vertices. Thus we can compute that

$$\begin{aligned}
2|E| &= \sum_{v \in V} d_v = \sum_{i=1}^{r} \sum_{v \in V_i} d_v \\
&= \sum_{i=1}^{r} \sum_{v \in V_i} (n - a_i) = \sum_{i=1}^{r} a_i (n - a_i) \\
&= n \left(\sum_{i=1}^{r} a_i \right) - \sum_{i=1}^{r} a_i^2 = n^2 - \sum_{i=1}^{r} a_i^2
\end{aligned}$$

proving the claim. □

Corollary 2. *The complete bipartite graph $K_{a,b}$ has $\frac{1}{2}\left(n^2 - a^2 - b^2\right) = ab$ edges.*

Trees

Exercises