Lecture 8: Hamilton cycles, matchings, independent sets · 1MA020

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We begin by continuing to pursue consequences of the Ford-Fulkerson theorem, proving König's theorem.

König's theorem

In our last lecture, we proved the Ford-Fulkerson theorem relating minimum cuts and maximal flows, and used it to prove the Hall marriage theorem on matchings in bipartite graphs. We begin this lecture by proving another result we can derive from Ford-Fulkerson, namely König's theorem about vertex covers of bipartite graphs.

Definition 1. Let G be a finite simple graph. A *vertex cover* of G is a subset $S \subseteq V$ such that every edge has an endpoint in S. The *covering number* of G, denoted $\beta(G)$, is the minimum size of any vertex cover of G.

Example 2. A star graph has covering number 1, while a complete graph on n vertices has covering number $n - 1$. A cycle graph on $2n$ vertices has covering number n.

Theorem 3 (König's theorem)**.** *Let G be a bipartite graph. Then the maximum cardinality of a matching on G equals $\beta(G)$, the minimum cardinality of a vertex cover of G.*

Proof. Let M be a maximal matching in G, and like in the proof of the marriage theorem, construct a flow network G' from G. As we saw in the proof of that theorem, this matching M corresponds to a maximal flow in G' of value $|M|$. By Ford-Fulkerson, this means there is a minimum cut S, T on G' of capacity $|M|$.

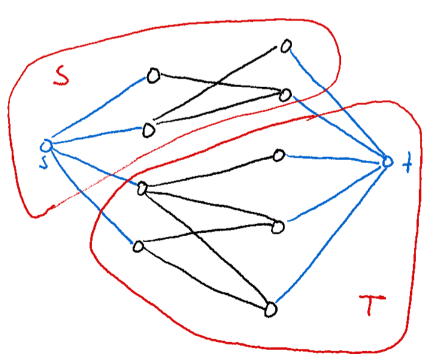

Figure 1: A non-trivial minimal s-t-cut in a flow network created from a bipartite graph.

Now, given this cut, let us construct a vertex cover. In particular, we let

$$C = (A \cap T) \cup (B \cap S).$$

That this C is a vertex cover of G is precisely the statement that there is no edge $a \to b$ with $a \in A \cap S$ and $b \in B \cap T$. This, however, is something we already saw is true for all minimum cuts in the previous proof – because if there were such an edge, it would contribute ∞ to the capacity of the cut. So C is indeed a vertex cover, and it is easy to convince oneself, looking at Figure 1, that $|C| = c(S,T) = |f| = |M|$. So what we have seen is that the size of a maximal matching upper bounds the minimum vertex cover size, that is, $\beta(G) \leq |M|$.

The other direction of the inequality in fact holds for all graphs, not just bipartite graphs – every edge of a matching has to be covered by a vertex in the cover, and no vertex can cover more than one edge of the matching at a time. So any vertex cover has to have at least as many vertices in it as a maximal matching has edges. □

Hamilton cycles

We studied, at the very beginning of the course, the notion of Eulerian circuits. The twin notion of Hamilton cycles appeared in our exercise session, but less us quickly repeat what we said about those.

Definition 4. Let $G = (V, E)$ be a finite simple graph. A *Hamilton cycle* in G is a cycle that visits every vertex of G.[2] If G admits a Hamilton cycle, we call it *Hamiltonian*.

[2] Recall that a *cycle* by definition is a walk that starts and ends at the same vertex, and does not reuse any vertices other than the one it started with.

Unlike for Eulerianity, there is no simple condition to determine whether a graph is Hamiltonian. In fact, computing whether a graph is Hamiltonian is an NP-Complete problem.

Remark 5. Given a Hamiltonian cycle C in G, we can easily construct a perfect matching of G by just picking every second edge in the cycle. We can of course not in general turn a perfect matching into a Hamilton cycle, so Hamiltonicity is a stronger condition than having a perfect matching.

That Hamiltonicity is an NP-Complete problem means we probably will never have any necessary and sufficient conditions for it. However, we can still prove results of the form "if condition so-and-so holds, G is Hamiltonian". Let us prove the most famous such result.[3]

[3] It is named after a different Dirac than the one of quantum mechanics fame.

Theorem 6 (Dirac's theorem)**.** *Let $G = (V, E)$ be a simple graph on $n \geq 3$ vertices, such that every vertex has degree at least $\frac{n}{2}$. Then G is Hamiltonian.*

Proof. Assume G is a graph satisfying the conditions of the theorem. Let us begin by observing that this G cannot be disconnected – if it were, the vertices in the smallest component would have fewer than half of the other vertices to connect to, so they would have too low degree.

Now, consider a path P of maximum length in G, say

$$P = v_0\, e_1\, v_1\, \ldots\, v_{k-1}\, e_k\, v_k.$$

Since this path is maximal, it contains all neighbours of v_0 and of v_k, since otherwise it could be extended. By our degree condition, there are (counting with multiplicity) at least n such neighbours – and so by essentially the pigeonhole principle, there must be an edge $e_i = \{v_{i-1}, v_i\}$ on our path such that v_i is a neighbour of v_0 and v_{i-1} is a neighbour of v_k, as in Figure 2.

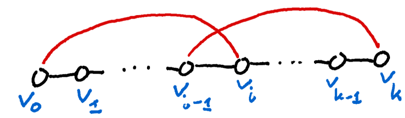

Figure 2: The path P along with the configuration of two edges that must exist by the pigeonhole principle.

Now we notice that this in fact lets us turn our path into a cycle – we start at v_0, walk until v_{i-1}, then use the edge to v_k, walk backwards to v_i, and finally use the edge to v_0. This gives us a cycle C.

It remains to see that this C is in fact a Hamilton cycle. So, suppose for contradiction that it is not, so that there is some vertex v not on the cycle. Since G is connected, we can in fact assume that this vertex is adjacent to some vertex of C. This, however, means we can create a longer path than P, by starting at v, walking onto C, and then following the cycle, as indicated in Figure 3. However, we assumed P was a maximal path, so we have a contradiction. So C must be a Hamilton cycle. $\square$

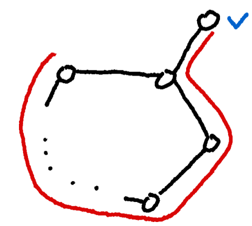

Figure 3: A longer path than P created from the cycle C, assuming it was not a Hamilton cycle.

Exercises