Lecture 2: Eulerianity, simple graphs, and subgraphs · 1MA020

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We formalize the ideas we started with in the first exercise session, giving a proof of Euler's result on Eulerian circuits. We then make some more definitions about simple graphs and subgraphs, and we state some elementary results about these notions.

The very first definition we give in this course will actually be of a *multigraph*, not of a graph.

Definition 1. A *multigraph* G is a tuple (V, E), consisting of a set V of *vertices*, and a multiset[2] E of *edges*. Each edge is a multiset containing two vertices from V, called its *endpoints*. We say two vertices are *adjacent* if there is an edge between them, and a vertex v is *incident* to an edge e if $v \in e$.

[2] A multiset is just like a set, except an element may occur more than once.

If the same edge occurs more than once in E, we say that these edges are *parallel*. If the two endpoints of an edge are equal, we call it a *loop*.

Unless explicitly stated, we always assume that both V and E are finite sets. Otherwise, we say the graph is *infinite*.[3]

[3] It may sometimes be the case that our proofs work without modification also for infinite graphs – thinking about whether they do may be a useful thing to do when reading the proofs, to understand them better.

Next, continuing to formalize the things we learned thinking about the bridges of Königsberg, let us define what a walk is.

Definition 2. Let $G = (V, E)$ be a multigraph. A *walk* of length k is a sequence of $k + 1$ vertices $v_0v_1v_2 \dots v_k$ and a sequence of k edges[4] $e_1e_2 \dots e_k$ such that $e_i = \{v_{i-1}, v_i\}$ for all i. A *trail* is a walk that uses no edge twice, and a *path* is a walk that uses no vertex twice.

[4] So the length of a walk is the number of edges, not the number of vertices. Can I change this definition to fix this?

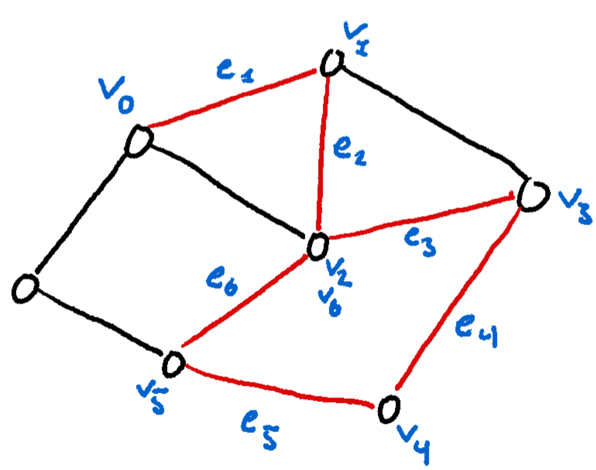

Figure 1: A walk in a graph, which is a trail but not a path.

We have one example of a walk in Figure 1 – it does not repeat any of the edges, so it is a trail, but it repeats the central vertex we have labelled with both v_2 and v_6, so it is not a path.