The Smallest Possible Refrigerator

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I. Introduction

One of the major idea that helped in progress of the field of thermodynamics was to abstract real machines to a more generalized model independent machines. This helped, as it made ideal machines a sort of model which could be used by the physical machines as a blueprint and thus inherit all the properties of the ideal machines.

In this paper, the main focus was to find the limit to the size of the refrigerator. Thus, one needs to go back to the actual machine as size is very much a parameter that is dependent on the physical construction of the engine.

II. Reasoning

In this section, reasoning for many of the choices taken by the paper will be discussed.

One of the first questions that comes to mind is the definition of size that is used. The paper uses the dimensionality of the Hilbert space of the system. This definition of size is used as it relates itself to thermodynamics as dimensionality can be related to entropy.

Another question that comes to mind is the usefulness of the question being posed in the paper. Some of the possible areas of practical applications where this kind of research can be useful is in biotechnology and nanotechnology where cooling/heating in an extremely constrained space ($O(S) = 10^{-12}m$) might help in some processes. One example given in the literature is the cooling of active sites of a protein that can help in acceleration of catalysis.

This problem can also help in understanding fundamental limits that might arise with decrease in size of the machine. One of the particular questions that is discussed in the paper is the relation between efficiency and size.

III. Details

Most of the quantum heat engines that are discussed and studied till when the paper was written were using an external source of work and/or control. This would not work in the this case as there would be a massive increase in the dimensionality of the Hilbert space and then the discussion of the models using single or multiple qubits and/or qutrits would be trivial and not useful.

Thus a model needs to be used which is self contained. The focus of the paper is on refrigerators, and as no refrigerator can work without a supply of free energy, an equivalent in the form of two heat baths, with two differing temperatures $T_r < T_h$ is taken.

The paper presents three different models who aim to refrigerate a qubit. The models are explained in the sections below.

A. *Two Qubits*

The model in its entirety consists of three qubits, where the first qubit is to be cooled, and the other two constitute the refrigerator.

Two qubits in the same bath

The two qubits, one which is to be cooled and another from the refrigerator initially are taken to be immersed in the same bath. Thus the Hamiltonian for the two qubits would be :

$$H = E_1\Pi^{(1)} + E_2\Pi^{(2)}, \tag{1}$$
$$\Pi^{(i)} = |1\rangle_i \langle 1| \tag{2}$$

In this, the $|1\rangle_i$ represents the excited state for qubit i. It is assumed that $E_2 > E_1$. The density matrix at the Gibbs state (Also known as thermal state), which maximizes entropy for a fixed internal energy, is expressed as given below :

$$\rho_B = \frac{1}{Z} e^{-\beta H}, \tag{3}$$
$$Z = \mathrm{Tr}\, e^{\beta H} \tag{4}$$
$$\beta = \frac{1}{k_B T} \tag{5}$$

Thus, for the given Hamiltonian $H = E_1\Pi^{(1)} + E_2\Pi^{(2)}$ the density matrix would be :

$$\begin{aligned} \tau_i &= \frac{1}{\mathrm{Tr}\, e^{-\beta H}} e^{-\beta H} && (6)\\ &= \frac{1}{\mathrm{Tr}\, e^{-\beta E_i \Pi^{(i)}}} e^{-\beta E_i \Pi^{(i)}} && (7)\\ &= \frac{1}{1 + e^{-\beta E_i}} e^{-\beta E_i \Pi^{(i)}} && (8)\\ &= \frac{1}{1 + e^{-E_i/k_B T_r}} e^{-E_i \Pi^{(i)}/k_B T_r} && (9)\\ &= r_i e^{-E_i \Pi^{(i)}/kT_r} && (10) \end{aligned}$$

Thus, the thermal equilibrium state $\tau_i = r_i e^{-E_i \Pi^{(i)}/kT_r}$. This equation can be further simplified, when $\Pi^{(i)}$ is expanded, and forms the following :

$$\tau_i = r_i e^{-E_i|1\rangle\langle 1|/kT_r} \tag{11}$$
$$= r_i \exp\left\{\begin{bmatrix}-E_i/kT & 0\\ 0 & 0\end{bmatrix}\right\} \tag{12}$$
$$= \begin{bmatrix} r_i e^{-E_i/kT} & 0\\ 0 & r_i\end{bmatrix} \tag{13}$$
$$= \begin{bmatrix} 1-r_i & 0\\ 0 & r_i\end{bmatrix} \tag{14}$$
$$= r_i\,|0\rangle\langle 0| + (1-r_i)\,|1\rangle\langle 1| \tag{15}$$

We can think of r_i and $1-r_i$ as probabilities as they sum up to 1 and they are in the form $\rho = \sum p_i \rho_i$.

Now the task is to cool qubit one, and thus the final temperature of the qubit must be less than the one at equilibrium. let the final temperature be T_1^S, then $T_1^S < T_r$ and thus a larger ground state probability $r_1^S > r_1$. Thus, if an operation increases the probability of ground state of the first qubit, then in essence it has been cooled.

Taking the complete system $|\psi_1\psi_2\rangle$, it can be seen that $P(|\psi_1\psi_2\rangle = |01\rangle) = r_1(1-r_2)$, similarly $P(|\psi_1\psi_2\rangle = |10\rangle) = r_2(1-r_1)$. This is due to the fact that both of the qubits do not interact with each other and thus $P(|\psi_1\psi_2\rangle) = P(|\psi_1\rangle\, P(|\psi_2\rangle)$. The assumption that $E_2 > E_1$ was made earlier and thus $r_2(1-r_1) > r_1(1-r_2)$. Therefore, the system has a higher probability of being in state $|10\rangle$ than $|01\rangle$.

So, if an unitary $\mathcal{U}$ is applied which works as follows :

$$\mathcal{U}\,|01\rangle = |10\rangle \tag{16}$$
$$\mathcal{U}\,|10\rangle = |01\rangle \tag{17}$$

This operation swaps the states $|01\rangle$ and $|10\rangle$. Therefore after the swap, the probability of qubit one being in ground state has increased and the probability of the second qubit being in excited state has increased. Increase in ground state probability means that the energy of qubit one has decreased and thus cooling it. Similarly, the second state has been heated.

As they are in a thermal bath, they would return back to the thermal equilibrium state, so the unitary need to be applied repeatedly. This will now result in them being in a new steady state with temperatures $T_1^S < T_r$ and $T_2^S > T_r$.

Adding the third qubit

This act of performing a unitary operation requires external work to be done, but the machine being built here is required to be self contained. This is where the third qubit comes into play. It acts as an engine, thus providing the work required for the unitary to be performed.

The system now becomes a three particle state. Therefore, the required state change is $|01\psi_3^a\rangle \longleftrightarrow |10\psi_3^b\rangle$. It was required that no external work is to be performed, thus the two states must be degenerate for the external work to be zero. As energies of all the qubits need to be positive, and $E_2 > E_1$, the only state where qubit three can be in the excited state is when the first qubit is excited, and thus, the energy of that qubit is $E_3 = E_2 - E_1$. Therefore, the two degenerate states are :

$$|010\rangle \longleftrightarrow |101\rangle \tag{18}$$

Therefore, the state change can be expressed using a Hamiltonian, which would be as follows :

$$H_{int}\,|010\rangle = g\,|101\rangle \tag{19}$$
$$H_{int}\,|101\rangle = g\,|010\rangle \tag{20}$$
$$\implies H_{int} = g(|010\rangle\langle 101| + |101\rangle\langle 010|) \tag{21}$$

The g used here can be described as interaction strength. As the two states need to remain degenerate under the complete Hamiltonian, and so the eigenvectors and eigenvalues of the total Hamiltonian $H = H_0 + H_{int}$ needs to be extremely close to the free Hamiltonian. Thus $\min_i(E_i) \gg g$. This will also allow the usage of the thermal equilibrium density states τ_i.

Using the Hamiltonian H, it can be seen that flipping of states $|101\rangle$ and $|010\rangle$ can happen at any stage as they are degenerate. But as the qubits one and two are kept in a thermal bath (At T_r), they will remain in the thermal equilibrium state.

The task that was given when defining the unitary $\mathcal{U}$ was that it would be acted repeatedly and thus would result in a new steady state where the probabilities of states $|01\rangle$ and $|10\rangle$ would be flipped. In the current scenario, there is nothing that drives this swap. So for that to occur, the third qubit must be put in another thermal bath with temperature T_h.

To relation between T_h and T_r can be defined by finding out which of the two changes $|010\rangle \to |101\rangle$ or $|101\rangle \to |010\rangle$ is to be preferred. As our task is to increase the probability of qubit one being in ground state, the probability of $|010\rangle$ being the final state must increase. Thus before the flip, the probability of $|101\rangle$ must be higher.

As seen earlier, $\tau_i = r_i\,|0\rangle\langle 0| + (1-r_i)\,|1\rangle\langle 1|$. Thus for the third qubit to drive the change $|101\rangle \to |010\rangle$, the probability $P(|\psi_1\psi_2\psi_3\rangle = |101\rangle > P(|\psi_1\psi_2\psi_3\rangle = |010\rangle)$. Thus $r_1(1-r_2)r_3 > (1-r_1)r_2(1-r_3)$, which leads to $T_h > T_r$.

Mathematical Model

The task is now to model the state change mathematically, keeping in mind that there are thermal baths involved. The conditions of a thermal bath give rise to two possible evolutions, one where it is insulated from the bath and it remains in its original state and the other being where the bath takes it to the τ_i state.

This can be modelled as :

$$\rho_i \mapsto p_i \tau_i \operatorname{Tr}_i \rho_i + (1-p_i)\rho_i \tag{22}$$

$$\implies \frac{\partial \rho_i}{\partial t} = p_i(\tau_i \operatorname{Tr}_i \rho_i - \rho_i) \tag{23}$$

As the three states do not interact, $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$, thus the time evolution of the complete state can be written as :

$$\frac{\partial \rho}{\partial t} = \sum_i p_i(\tau_i \operatorname{Tr}_i \rho - \rho) \tag{24}$$

The time evolution of the states using the Hamiltonian can be written as :

$$\frac{\partial \rho}{\partial t} = -i[H_0 + H_{int}, \rho] + \sum_i p_i(\tau_i \operatorname{Tr}_i \rho - \rho) \tag{25}$$

The appending of H as $H_0 + H_{int}$ usually requires some corrective measures but since the interaction strength g is extremely small, it can be ignored and the master equation can remain consistent.

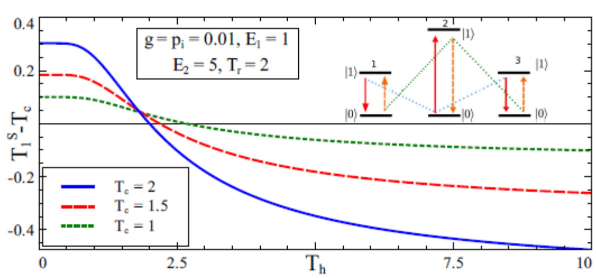
Steady State

The path taken for the ρ to get to the steady state is not the point of the paper but rather the thermodynamic properties of the ρ that is at steady state. Thus the steady state will explored in this section. For the steady state,

$$0 = \partial\rho/\partial t \tag{26}$$

$$= -i[H_0 + H_{int}, \rho^S] + \sum_i p_i(\tau_i \operatorname{Tr}_i \rho^S - \rho^S) \tag{27}$$

The paper, rather than performing analytical analysis, has preferred to perform numerical analysis, as it gives us a better insight into the results. The results are as follows :



Note : T_h is the temperature of the hotter bath, T_c is the temperature of the colder bath and T_r is the temperature at equilibrium

The graph above gives us the temperature difference that the qubit one has cooled to as a function of T_h with the different curves having different T_r. From this, it can be seen that cooling can be achieved when $T_h > T_r$. The dotted lines also tells us that the system manages to cool the qubit even when the temperature of the colder bath is reduced.

Therefore, it can be seen that the system acts as a refrigerator in all conditions.

Cooling other objects

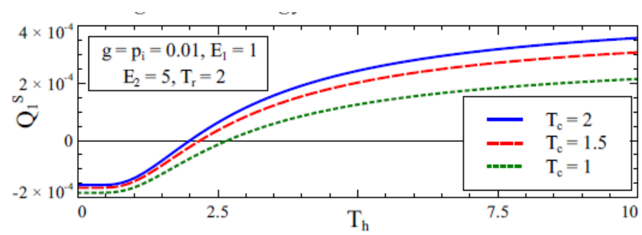
It was proven that the system acts as a refrigerator in all conditions, but in all the cases, the object to be cooled was a qubit. Naturally, the next thing to wonder is that if this system can cool anything other than the qubit.

For this, analysis needs to be done on the heat current between the bath and the first qubit. The heat current is defined as :

$$Q_1^S = \operatorname{Tr}\big(H_1 D_1(\rho^S)\big), \tag{28}$$

$$D_1(\rho^S) = p_1(\tau_1 \oplus \operatorname{Tr}_1 \rho^S - \rho^S) \tag{29}$$

This results in the following graph :



There is a positive flow of current when the colder qubit is cooled below the bath temperature. Thus, as heat can be extracted from the environment and thus the system works as a refrigerator for any arbitrary object.

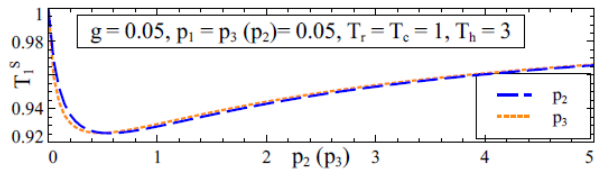
Parameter Influence

The effects of the changes to the different parameters such as p_i will be studied. The paper provided some hypothesis as to what the changes in parameters will result in giving some reasons for it, and providing results and verifying the hypothesis.

The p_i's represent the interaction the qubit i has with the environment. Thus, the demands from different qubits with respect to interaction with the environment are different and depend on the actions performed by the qubit.

Qubits two and three are part of the refrigerator and these are the particles whose interactions will be studied. The second qubit is the spiral and thus more the interaction with the environment, better it is for the performance as the T_1^S will decrease.

Similarly for the third qubit, it is the engine which takes heat from the hot environment to the system and thus, more interaction with the environment will also increase the heat taking ability. So, the third interaction being higher will result in higher performance.



It can be seen that the predictions are followed when the values of p_i are small but goes away from the prediction at higher values. The reason given for this anomaly is the quantum Zeno effect which arises due to the strong coupling between system and environment. Quantum Zeno effect is where the time evolution of the system is arrested when multiple measurements frequently are made.

One more reason that is given for the anomaly is that the interaction Hamiltonian H_{int} will not have enough time to work between successive thermalisations which in turn results in refrigerator not being able to function anymore. Thermalisation is the the process of physical bodies reaching thermal equilibrium through mutual interaction.

The last to check is the first qubit whose dependence is expected as the demand is to cool the qubit beyond the colder bath. Thus the more isolated it is, the better the cooling performance.

Limit to temperature

This section tries to check if there are any fundamental limits to how low such a refrigerator can cool to. For this, the dependence of T_1^S needs to be studied.

The two parameters highlighted in the paper are the imperfect insulation of the cooled qubit to the environment and the ability to cool, assuming perfect insulation. The second part is one in concern as imperfect insulation is not a fundamental limit of nature.

For checking the ability to cool, the parameters that affect it are E_i. Fixing E_1 gives us an idea as to how the difference affects the temperature limit. We also fix T_r. Increasing E_3 and T_h such that E_3/T_h is constant and also $\ll 1$. This results in increase the ground state probability of qubit two and also maintains a large probability for excited state for qubit 3. This results in the forward bias towards flipping the state to $|010\rangle$. Thus, as the cooling keeps increasing rather than slowing down, it can be seen that thus there is no fundamental limit to how low an object can be cooled to.

B. One Qubit, One Qutrit

This model has a qutrit which is three state particle which has two excited states $|1\rangle$ and $|2\rangle$. The energy levels are chosen such that $|020\rangle$ and $|101\rangle$ are degenerate states. The flip from $|101\rangle$ to $|020\rangle$ can cause the first qubit to cool down.

For this, the interaction Hamiltonian is :

$$H_{int}^{(12)} = g(|02\rangle\langle 11| + |11\rangle\langle 02|) \oplus 1^{(3)} \quad (30)$$

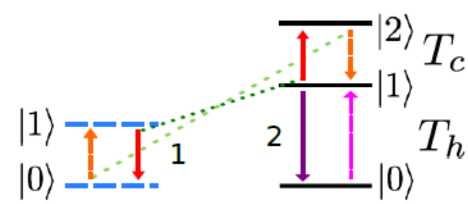
$$H_{int}^{(23)} = h1^{(1)} \oplus (|01\rangle\langle 10| + |10\rangle\langle 01|) \quad (31)$$

The benefit of this system over the other one is that it only has a two body interaction Hamiltonian which is much easier to compute than the three body one. A drawback is that now it requires two Hamiltonian operators to govern the flip which makes it a second order operation.

In essence, while the equations and the parameters might be different, the systems works almost identically to the previous model.

C. One Qutrit

The final model is a single qutrit system which aims to cool a qubit. In this the assumption is that the qutrit has different excitation states at different spatial coordinates thus allowing it to have different baths at different excitation states. Thus the ground state is at T_c whereas the second excited state is at T_h. The system would look like this :



IV. Summary

The paper set out to find fundamental limits on the size of a refrigerator and whether it comes with any caveats or not. They proceeded to present three ways in which a qubit could be cooled using a combination of qubits and qutrits. It was then proved that the qubit which is cooled can extract heat from the surroundings and thus, in essence can cool any arbitrary object. The fundamental limits were then tested and it was found out that there is no limit to how low an object can be cooled, provided that enough energy is provided.

It was also found out that these machines are not particularly efficient and does perform some irreversible work, but it is yet to be known that whether it is due to the size or due to the specific design.

Thus, in this piece, fundamental limits were tested and came away with extremely good results that can be further refined with better models.