Decentralized Systems as Network Games

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1 Overview

A wide spectrum of natural, social and economic systems can be studied by means of networks (a.k.a. graphs), which are structures consisting of vertices connected by links. If the information only flows in one direction, a link is called an arc and a network is *directed* or a *digraph*. On the contrary, if either of two vertices in a link can establish an interaction with the other, then a link is called an edge and the network a *non-directed* network (Figure 1).

The network structure (arc/edge formation) primarily depends on whether the network is the design of a central actor or the result of autonomous decisions among many of them. Many distribution or service networks fall into the category of centralized networks. For example, given a list of cities (vertices), an airline decides on the routing of planes, and thus, an arc is formed if a plane can fly between an origin and destination. Decentralized networks arise in situations where a number of individuals (vertices) may have an incentive to create connections (edges) with others to fulfill their own utility function. These networks include labor markets, political alliances, and in general any peer-to-peer network.

Every network, whether centralized or not, operates under some previously agreed principles (e.g., safety, fairness, efficiency, etc.), under which agents join and interact in a network. Measuring whether these principles are met and maintained in a decentralized network requires a closer look to their constantly evolving nature. In this discussion, we will make use of *game theory* and *optimal control* to evaluate static and dynamic approaches to decentralized networks' evaluation criteria. Our goal is to develop intuition and conceptual understanding, without sacrificing the rigor of the scientific method.

The remainder of this discussion is as follows: Section 2 defines games on networks, Section 3 introduces the concept of Nash Equilibrium, Section 4 shows an example of an incomplete information game and its Nash Equilibrium and Section 6 addresses dynamic games and notions of stability.

2 Defining Network Games

Following the seminal work by Jackson and Wolinsky (1996), let $N = [1, ..., n]$ be a finite set of agents who may be involved in a network. The network relations among these agents are

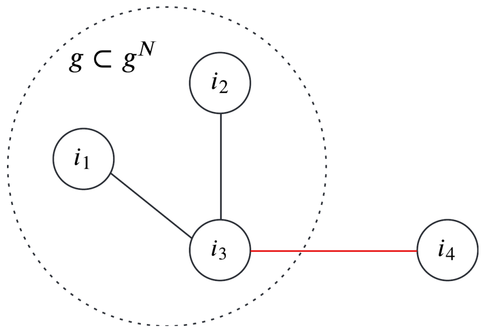


Figure 1: A non-directed graph with $n = 4$ agents. A pair of vertices (circles) is connected by a black edge if both parties agree to establish a connection. Red edges are not yet established connections.

formally represented by graphs whose vertices are identified with the agents and whose edges capture the pairwise relations. Let g^N be the graph with all possible pairwise connections, and $g \subseteq g^N$ a sub-graph representing some of those connections, such as in Figure 1. An agent's payoff is the incentive driving his/her willingness to form or sever a link and it is captured by the trade-off between the benefits and costs associated to either decision.

To analyze games on networks, we first must specify how the network is allowed to change and what information is available to agents to make a connection. In other words, we need to define the "rules of the game". These rules are typically known as the network formation process.

A model of network formation needs to specify whether changes in the network's connections take place one link/several links at a time, whether we consider a static network (fixed number of agents) or a dynamic network where agents arrive and leave over-time, whether agents have complete/incomplete information about the payoffs or other relevant restrictions, together with a stability or network equilibrium concept compatible with this process.

Since our goal is to evaluate whether the network structures that are likely to arise in a decentralized system, allow agents to participate under the principles the system was created for, we need to define whether interactions within the network game reach a point of "convergence" or "equilibrium". A widely known solution concept for non-cooperative games is Nash Equilibrium, which we review in the following section.

3 Nash Equilibrium

Nash Equilibrium assumes that the action set of each agent is common knowledge and that each agent acts optimally in response to other agents' actions. Nash Equilibrium can be defined as follows:

Definition 1 (Nash Equilibrium)**.** *A situation where no agent could increase their payoff by unilaterally changing the strategy (action) they've chosen, while all other agents' strategies are fixed, is a Nash Equilibrium.*

Suppose we were interested in finding a Nash Equilibrium in Figure 1, thus let's assume a static model, allowing multiple link formations or removals at the same time and perfect information is available to all agents, including payoffs. In the worst-case scenario, an agent must know all possible graph structures to compute his/her payoff, that is, 2^n graphs on N to decide on his/her best response given other agents' actions, which is an exponential number of graph structures as n grows.

Real-world networks can have thousands of agents and in such large networks it may be unrealistic to think that each agent considers all the other agents he/she could potentially interact with. Thus, even if these agents were perfectly rational, their information set may be restricted to connections within some proximity. Moreover, Nash Equilibrium assumes there is no collaboration (collusion) among agents, but in some scenarios (e.g., voting) this assumption may not hold. Another difficulty is that Nash Equilibrium may not exist. Suppose agent i_4, in Figure 1, would like to form a link with agent i_3 but the former is not reciprocated by the latter, since such an alliance would be strictly detrimental to agent i_3's payoff.

In the next section we review a specific case where depending on uncertain parameters in the network formation of the game, our confidence on a Nash Equilibrium solution depends on our confidence on the true uncertain parameters.

4 Bayesian Games

A Bayesian, or incomplete information, game is a generalization of a complete-information game. That is, in a Bayesian game, in addition to what is common knowledge, agents may have private information. This private information is captured by the notion of a *type*, which describes an agent's knowledge or state about the game.

Thus, in a Bayesian game, in addition to agents, actions, and payoffs, an agent i can have a type $t_i \in T_i$, where T_i is the type set of that agent. There is also a common prior or belief, which is a probability distribution over type profiles. Payoff functions then depend not only on the action set of agents, but on their types as well.

Suppose now, that we only have two agents from Figure 1. One of them (agent 1) is a user trying to get a miner (agent 2) to generate a cryptographic number that is within some proximity to a number set by the network's difficulty algorithm, so that agent 1's transaction can be validated and recorded in a block. Depending on this algorithm setup, validating a transaction, and thus, generating the cryptographic number may be classified as "low cost" or "high cost" for the miner. Let's assume that with probability p the validation of agent's 1 transaction is difficult, and with probability $1 - p$ the validation has low cost. agent 1 pays a fee to the miner for his/her services. Let's assume the algorithm setup, determining the difficulty level of a task cannot be anticipated by the miner. The miner only knows its prior, and must decide on either strategy: accept (A) or reject (R) the validation of agent 1's transaction.

Depending on the difficulty of the task, a miner can obtain a reward or incur in costs for validating agent 1's transaction. On the other hand, agent 1 can decide to either submit a petition to agent 2 (P^{E} if task has low cost or P^{D} if task is difficult) or to find another miner, in which case, agent 1 does not seek help in agent 2 (N^{E} if task is easy or N^{D} if task has high cost). However, if agent 1 does submit a validation request to the miner, and this one rejects

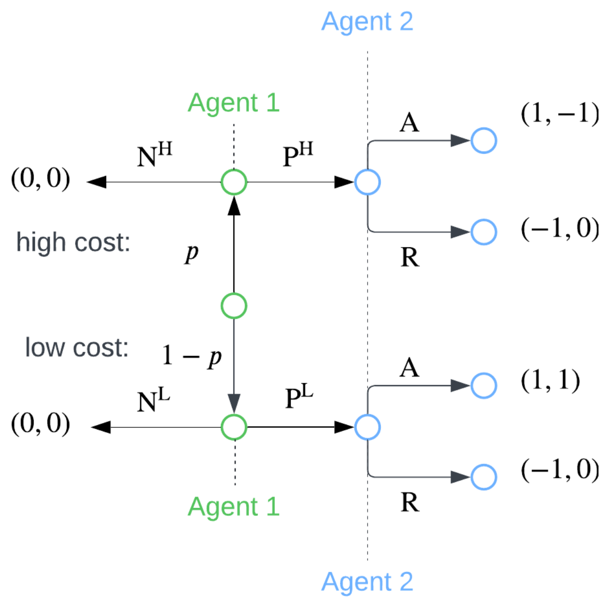


Figure 2: Extensive form of a Bayesian Game with $n = 2$ agents. The first element of the tuple is the payoff of agent 1 and the second element, the payoff of agent 2.

it, then agent 1 incurs in costs too. Just as in the games we have reviewed so far, each agent acts unilaterally to optimize his/her own payoff. The extensive form of the game is shown in Figure 2.

Table 1 shows the expected payoff for each possible scenario. For example, ($\mathrm{P^L N^H}$,A) implies that agent 1 submits a petition if the task is low cost, and does not when the task is high cost and in both cases agent 2 accepts to process the petition. When agent 1 does not seek help in agent 2 and the latter accepts or rejects, it implies no action on agent 2's side.

Table 1: Expected payoffs for the Bayesian Game with $n = 2$ agents.

		Agent 2	
		A	R
Agent 1	$\mathrm{N^H N^L}$	$(0, 0)$	$(0, 0)$
	$\mathrm{N^H P^L}$	$(1 - p, 1 - p)$	$(p - 1, 0)$
	$\mathrm{P^H N^L}$	$(p, -p)$	$(-p, 0)$
	$\mathrm{P^H P^L}$	$(1, -2p + 1)$	$(-1, 0)$

What is the (pure strategy) Nash Equilibrium of this game? Depending on the probability p, such equilibrium may not exist. If $p \leq \frac{1}{2}$, then ($\mathrm{P^H P^L}$,A) is a Nash Equilibrium. Otherwise, the miner would rather reject agent 1's petition although this is the scenario that would benefit agent 1 the most. A slight deviation in the prior could inadvertently prevent agents from making transactions in the system. Not only a Nash Equilibrium (Bayesian or not) may not exist, not only it is limited to non-cooperative games, but a Bayesian-Nash Equilibrium can be even more intractable than Nash Equilibrium in a complete information game, since under no special assumptions on the game (Mossel et al., 2016), the computational burden of each agent updating their believes based on the information of every other individual in the network, is prohibitive for adopting Bayesian learning, even in relatively simple networks.

We then, face the questions: what other notions of equilibria could be considered? and how could we characterize the conditions for a network game to converge to such equilibria? Thinking whether a network game can reach a stationary state is another way of thinking about equilibrium. For instance, instead of framing the equilibrium as a best response, we could think of it as whether the system achieves a point of stability by repeatedly playing the game. In the example just provided, we could think of such a point as the game reaching a *bound* on the accepted transactions as time goes to infinity. That is, the evolution of the game may still converge to a stable point regardless of whether that point will correspond to every agent's best response to the other agents' responses.

Cooperatives games deal with behavior dynamics and so they are a good example to understand other notions of stability. In the next section, we review some of the works in this domain, their network formation games and the conditions under which they define stability.

5 Cooperative Games

Unlike the non-cooperative game we just examined, in a cooperative game, agents find incentive in forging links because the payoff of interacting with other agents is not conditioned on other agents' connections and is considered positive regardless of the agents who share a link. Interaction can also be the result of a tacit or an explicit social protocol that all agents share. Examples of these networks include consensus networks, co-authorship networks, even disease transmission networks and in general, any network where forging a link is a natural behavior or its payoff is considered positive.

Consider the cooperative game proposed by Jadbabaie et al. (2012). The authors develop a dynamic model of opinion formation in a static social network. Social learning occurs in two steps. First, agents receive private observations conditioned on the true state of the environment and perform a *local* Bayesian update (as opposed to global). Then, the agents average the intermediate beliefs of their neighbors using linear combination rules. Subsequent works include variations, such as geometric combination rules (Nedić et al., 2017), learning under randomized collaboration (İnan et al., 2022), partial information sharing (Bordignon et al., 2023), asynchronous learning (Cemri et al., 2023) and community structured networks where each community admits their own true hypothesis (Shumovskaia et al., 2024). Thus, in these cases, the notion of equilibrium is linked to the evolution of the network to a stationary state, where all agents learn some underlying truth.

It is worth noting that the previous works address a unique network game under which long-term stability is achieved: social learning. Since we aim to explore methodologies that allow us to study stability for general network games, not just social learning and not just cooperative/non-cooperative games, we shift our attention to dynamical systems. A big part of our motivation also relies on the fact that for many examples of games with a Nash equilibrium, the dynamics fail to converge (Sandholm, 2010, Chapter 9). In particular, we would like to determine where the dynamic leads when set in motion from various initial conditions.

6 Dynamical Systems for Network Games

Dynamical systems is a field of mathematics that aims to explain the behavior of complex dynamical systems by employing differential equations (continuous time) or difference equations (discrete time). The goal is not on finding exact solutions to the frequently very complex equations defining the dynamical system, but rather to answer questions about whether the system will settle down to a steady state in the long term, determining what those steady states are if they exist, and understanding the dependency between the long-term behavior of the system and the initial conditions.

We start by introducing some of the most established notions of stability in dynamical systems, but we note such notions are not static since as the literature evolves new notions may be known. We then present a case study to put into perspective stability analysis of dynamical systems within the context of network games.

6.1 Notions of stability in dynamical systems

In what follows $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^n$. Consider the following time-invariant, discrete-time dynamical system.

$$x(t+1) = f(x(t)),\ x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ denotes the system state vector and initial condition $x(0)$. Let x_e be an equilibrium point, which then characterizes the stability of the system as stable in the sense of Lyapunov, asymptotically stable, globally asymptotically stable and exponentially stable. We define such notions of stability as follows (Astolfi, 2021):

Definition 2 (Lyapunov Stability)**.** *The equilibrium is stable (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists $\|x(0) - x_e\| < \delta$ which implies $\|x(t) - x_e\| < \epsilon$ for all $t \geq 0$.*

The definition of Lyapunov stability can be interpreted as follows. An equilibrium point x_e is stable if regardless of the value taken by a tolerable deviation ϵ, there exists a neighborhood of the equilibrium x_e such that all initial conditions in this neighborhood result in trajectories which are within the tolerable deviation ϵ.

Definition 3 (Asymptotic Stability)**.** *The equilibrium is asymptotically stable if it is stable and if $\exists \delta_a > 0$ such that $\|x(0) - x_e\| < \delta_a$ implying $\lim_{t\to\infty}\|x(t) - x_e\| = 0$.*

An equilibrium point is asymptotically stable if it is stable and, whenever the initial perturbation is within a certain neighborhood of x_e, the trajectory converges, asymptotically, to the equilibrium point x_e.

Definition 4 (Global Asymptotic Stability)**.** *The equilibrium is globally asymptotically stable if it is stable and if, for all $x(0)$, $\lim_{t\to\infty}\|x(t) - x_e\| = 0$.*

Definition 5 (Exponential Stability)**.** *For discrete-time systems, the equilibrium is exponentially stable if there exists $0 < \lambda < 1$, such that for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\|x(0) - x_e\| < \delta_\epsilon$ implies that $\|x(t) - x_e\| < \epsilon\lambda^t\ \forall t \geq 0$.*

We now present an illustrative case study where we characterize the stability of the system.

6.2 Case Study: A Consensus Network Game

Consensus problems have a long history in computer science and in the field of distributed computing (Lynch, 1996). In networks of agents, such as multi-agent networked systems, "consensus" means to reach an agreement regarding a quantity of interest that depends on the state of all agents (Olfati-Saber et al., 2007). Consensus is then reached through an algorithm or protocol that specifies the information exchange between an agent and all of its neighbors on the network.

Consider a network of agents with an *initial* opinion $x_i(0) \in \mathbb{R}$ on a certain matter, i.e., an opinion is represented by an scalar. These agents are interested in reaching a consensus via local communication with their neighbors on a graph $G = (V, A)$. Agent i communicates with agent j if j is a neighbor of i. The set of all arcs defines the arc set A. A consensus is reached if all agents' opinions asymptotically converge to the same agreement space characterized by the following equation:

$$x_1 = x_2 = \ldots = x_n$$

Following Olfati-Saber et al. (2007), an agent linearly updates her opinion at time $t+1$ based on her opinion and that of her neighbors at time t according to the following dynamics:

$$x_i(t+1) = x_i(t) + \epsilon \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)),\ x_i(0) = z_i \tag{1}$$

The set of neighbors of an agent i is $N_i = \{j \in V : a_{ij} = 1\}$. Agent i communicates with agent j if j is a neighbor of i or $a_{ij} = 1$ (e.g., in Figure 3, j is a neighbor of i because there is an arc emanating from i and coming into j, thus $a_{ij} = 1$). A step size is defined by $0 < \epsilon < \frac{1}{\Delta}$ where $\Delta = \max_i(\sum_{i \neq j} a_{ij})$ is the vertex with maximum out-degree in the network.

The discrete-time collective dynamics of the network under this algorithm can be written in terms of the *Perron* matrix ($\mathcal{P}$) of a graph G

$$x(t+1) = \mathcal{P}x(t) \tag{2}$$

with $\mathcal{P} = I - \epsilon L$. Here, I is the identity matrix, and L is known as the graph Laplacian of G. The graph Laplacian is defined as

$$L = D - \mathcal{A}$$

where $D = diag(d_1, ..., d_{|V|})$ is the degree matrix of G with elements $d_i = \sum_{i \neq j} a_{ij}$and zero off-diagonal elements. Matrix $\mathcal{A}$ is the adjacency matrix of G.

Theorem 1 (Olfati-Saber et al. (2007))**.** *Consider a strongly connected digraph G of agents applying the distributed consensus algorithm defined in* (1) *and* $0 < \epsilon < \frac{1}{\Delta}$. *Then,*

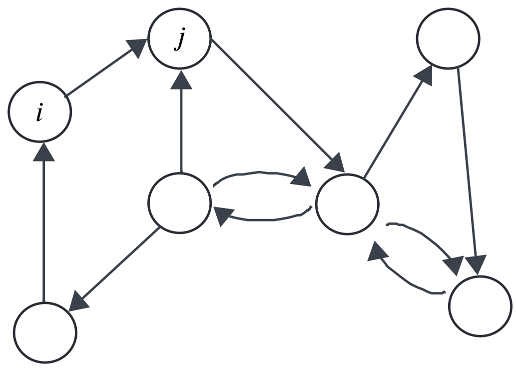


Figure 3: A Consensus Network with $|V| = 7$ agents.

1. *A consensus is asymptotically reached for all initial states*
2. *The group decision value is* $\alpha = \sum_{i \in V} w_i x_i(0)$ *with* $\sum_{i \in V} w_i = 1$

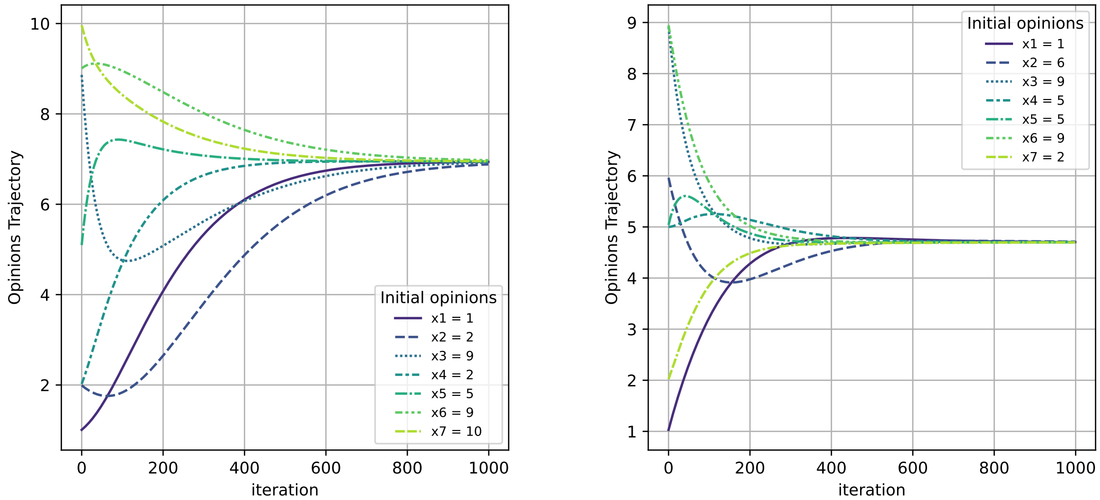
Proof. If G is a strongly connected digraph and a correct step size $0 < \epsilon < 1/\Delta$ is selected, the Perron matrix $\mathcal{P}$ becomes a primitive non-negative matrix, for which $\lim_{t \to \infty} \mathcal{P}^t = vw^T$, with left and right eigenvectors w and v, respectively, satisfying $\mathcal{P}v = v$, $w^T \mathcal{P} = w^T$, and $v^T w = 1$. Thus, $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \mathcal{P}^t x(0) = v(w^T x(0))$ with $v = \mathbf{1}$. Hence the group decision value is $\sum_{i \in V} w_i x_i(0)$ with $\sum_{i \in V} w_i = 1$ due to $v^T w = 1$. $\square$

A special case occurs when G is balanced, i.e., $\sum_j a_{ij} = \sum_j a_{ji} \forall i \in V$, case in which an average consensus is asymptotically reached and $\alpha = \frac{\sum_{i \in V} x_i(0)}{|V|}$. We refer the reader to (Olfati-Saber et al., 2007) for the corresponding proof.

Figure 4 shows the evolution of the group decision for the consensus network G in Figure 3 under two sets of initial conditions. The initial agents' opinions were generated following a uniform distribution between 0 and 10 and then rounded down to the nearest integer. As observed, asymptotic stability is reached in both cases. The initial conditions of the system not only determine the ultimate group decision but they also seem to affect the speed of the consensus process itself. Since the group decision depends on the initial conditions, the consensus game is *not globally* asymptotically stable.

7 Conclusions

In this discussion we reviewed some of the most adopted notions of stability in game theory and dynamical systems within the context of games on networks. The selection of the notion of stability is linked to the network formation game and the ultimate goal of the system. For many real systems it is the case that a network game is played over a time horizon and agents adapt their behavior as a result of repeated interactions with other agents. When a game is intrinsically dynamic, it seems reasonable to think of notions of stability that can provide insight on the state of the system in the long term.



(a) Experiment 1 (b) Experiment 2

Figure 4: Asymptotic stability for the consensus game in Figure 3 with $\epsilon = \frac{1}{11} - \frac{1}{12}$

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