with $|r'_j - j| + 1 \le \rho(j)$ for $1 \le j < n$. Hence $s_j = |r_j - j| + 1 \le \rho(j) + 1$. Now $s_n = |n - n| + 1 = 1$ and so if we put $\pi(j) = \rho(j) + 1$ for $j < n$ and $\pi(n) = 1$ then π is the required permutation.

Note that in neither (a) nor (b) do we need strict monotonicity of h.

Editorial comment. The proposer used the assumption of strict inequality to guarantee that the sum would have a maximum attained at a single point. Most solvers succumbed to the temptation to simplify the argument by characterizing the point $(x_1, \ldots, x_n)$ at which the maximum is attained.

Solved also by M. V. Bjelica (Yugoslavia), M. Bowron, R. B. Israel (Canada), M. Mócsy (Hungary), K. Schilling, and the proposer.

An Integral Infinite Sum

10231 [1992, 570]. *Proposed by Adrian Riskin, Northern Arizona University, Flagstaff, AZ.*

For positive integers m and n, let

$$f(m,n) = \sum_{k=1}^{\infty} k^n \left(\tfrac{m}{m+1}\right)^k.$$

(a) Prove that $f(m,n)$ is an integer.
(b) Show that the last digit of the decimal expansion of $f(1,n)$ can only be 0, 2, or 6.

Solution to part (a) by David Beckwith, Sag Harbor, NY. Let

$$g_n(x) = f(x,n) = \sum_{k=1}^{\infty} k^n \left(\tfrac{x}{x+1}\right)^k.$$

The series converges uniformly on a positive interval; termwise differentiation yields

$$g'_n(x) = \sum_{k=1}^{\infty} k^n k \left(\tfrac{x}{x+1}\right)^{k-1} \tfrac{1}{(x+1)^2} = \tfrac{g_{n+1}(x)}{x(x+1)}.$$

Hence $g_{n+1}(x) = x(x+1)g'_n(x)$. By explicit computation, $g_0(x) = x$. By the recurrence, every $g_n(x)$ is a polynomial in x with integer coefficients. Hence every $f(m,n) = g_n(m)$ is an integer.

Solution to part (b) by Richard Holzsager, American University, Washington, DC. Starting with $f(m,1) = g_1(m) = m^2 + m$ and applying the recurrence, we obtain the additional polynomials

$$\begin{aligned}
f(m,2) &= 2m^3 + 3m^2 + m,\\
f(m,3) &= 6m^4 + 12m^3 + 7m^2 + m,\\
f(m,4) &= 24m^5 + 60m^4 + 50m^3 + 15m^2 + m,\\
f(m,5) &= 120m^6 + 360m^5 + 390m^4 + 180m^3 + 31m^2 + m.
\end{aligned}$$

Reducing the coefficients modulo 10, we have $f(m,5) \equiv f(m,1) \bmod 10$. Hence the succeeding polynomials repeat mod 10 with a period of 4, for any fixed m. For $m = 1$, the cycle is 2, 6, 6, 0.

Editorial comment. A popular method of solution was to show that

$$f(m,n) = (m+1)\sum_{k=1}^{n} k!S(n,k)m^k,$$

where $S(n,k)$ denotes the Stirling numbers of the second kind. This establishes (a), and $f(1,n) \equiv 2\sum_{k=1}^{4} k!S(n,k) \bmod 10$ leads to (b).

Gerry Meyerson noted that $f(m,n)$ is an integer multiple of $m(m+1)$ for $n > 0$. This also follows from the selected proof. He also located $f(1,n)/2$ as sequence #1191 in N. J. A. Sloane, *Handbook of Integer Sequences*, Academic Press, 1973, where the sequence is traced back to Cayley. The number $f(1,n)/2$ is equal to the number of distributions of n distinct objects into ordered cells such that no occupied cell is above an unoccupied cell. A proof of part (b) using this interpretation can be found in O. A. Gross, "Preferential arrangements", this MONTHLY 69 (1962), 4-8. This latter reference was mentioned by István Nemes.

William Y. C. Chen gave further references dealing with the question of periodicity of the $f(1,n)$ modulo primes.

Solved by 54 solvers and the proposer.

Mutually Convergent Series

10291 [1993, 290]. *Proposed by Howard Morris, Chatsworth, CA.*

Let k be a positive integer and let $\langle x_n \rangle$ be a nondecreasing sequence of real numbers for which $\sum(1/x_n)$ converges. Show that $\sum(\ln x_n)^k/x_n$ converges if and only if $\sum(\ln n)^k/x_n$ converges.

Solution by Frank Schmidt, Arlington, VA. Since $x_n \le x_{n+1}$ and $\sum(1/x_n) < \infty$, we have $\lim_{n\to\infty}(n/x_n) = 0$ (see *Editorial Comment* below). In particular, there is a real number K with $n \le Kx_n$ for all n, so that $\sum(\log x_n)^k/x_n < \infty$ implies $\sum(\log n)^k/x_n < \infty$. To prove the converse, split the indices into two subsets: (I) $x_n \le n^{k+2}$; (II) $x_n > n^{k+2}$. On subset I, $\log x_n \le (k+2)(\log n)$, hence $\sum_{\mathrm{I}}(\log n)^k/x_n < \infty$ implies $\sum_{\mathrm{I}}(\log x_n)^k/x_n < \infty$. As for subset (II), for sufficiently large n (depending on k), we have

$$\frac{(\log x_n)^k}{x_n} < \frac{(x_n)^{\frac{k}{k+1}}}{x_n} = \frac{1}{(x_n)^{\frac{1}{k+1}}} < \frac{1}{n^{\frac{k+2}{k+1}}}.$$

Since $(k+2)/(k+1) > 1$, $\sum_{\mathrm{II}}(\log x_n)^k/x_n < \infty$.

Editorial comment. The upper bound on x_n may be obtained in many ways: most readers gave a simple *ad hoc* proof; three readers appeared to treat it as *obvious*; and three readers refered to a well-known result (theorems of Abel, Kroneker or Pringsheim) without citing the statement that the solver had in mind. An editor supplied another approach: paraphrasing theorem 3.27 (attributed to Cauchy) of Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976, we find that given conditions imply $\sum 2^k/x_{2^k}$ converges, from which the result follows.

H.-J. Seiffert observed that the upper bound on x_n is also part of the general theory of *convergence exponent* λ of $\langle x_n \rangle$. For x_n as in this problem, λ is characterized by $\sum x_n^{-\sigma}$ converging for $\sigma > \lambda$ and diverging for $\sigma < \lambda$, so that $\lambda \le 1$ in this case. In G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. II, Springer-Verlag, 1972–76, pp. 25–26, entry 113, one finds that $\lambda = \limsup_{n\to\infty} \frac{\log n}{\log x_n}$. Although this leads to slightly weaker inequalities than have been mentioned above, it is strong enough for the present needs.

Solved also by V. Božin (student, Yugoslavia), D. A. Darling, E. Hertz, R. Holzsager, G. L. Isaacs, I. Kastanas, A. D. Melas (Greece), A. Pedersen (Denmark), H.-J. Seiffert (Germany), R. Stong, A. A. Tarabay (Lebanon), R. B. Tucker, H. V. Vu (student, Hungary), A. N. 't Woord (the Netherlands), and the proposer.