# Control Systems
# Module I: Overview

# Control Systems Basics

## Explains

- Overview
- Stability
- Performance
- Robustness

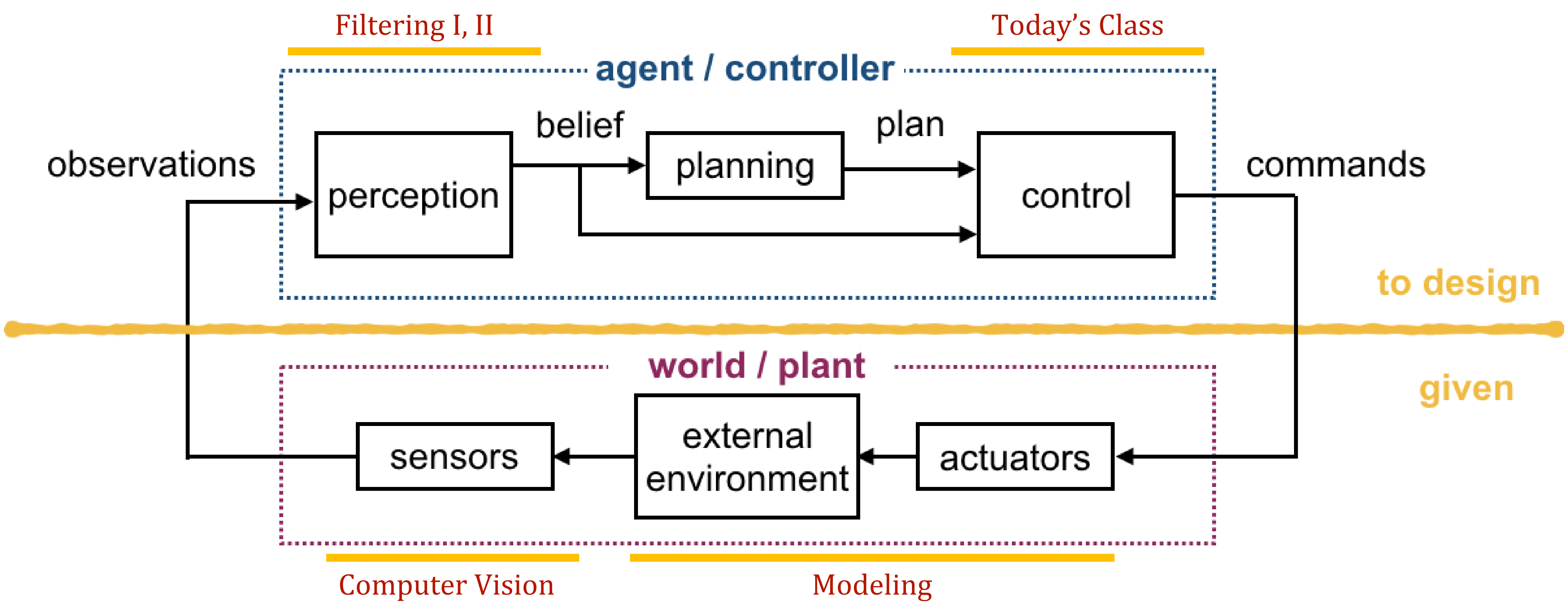
## Prerequisites

- Linear Algebra
- Fundamentals of modeling and control

## Credits

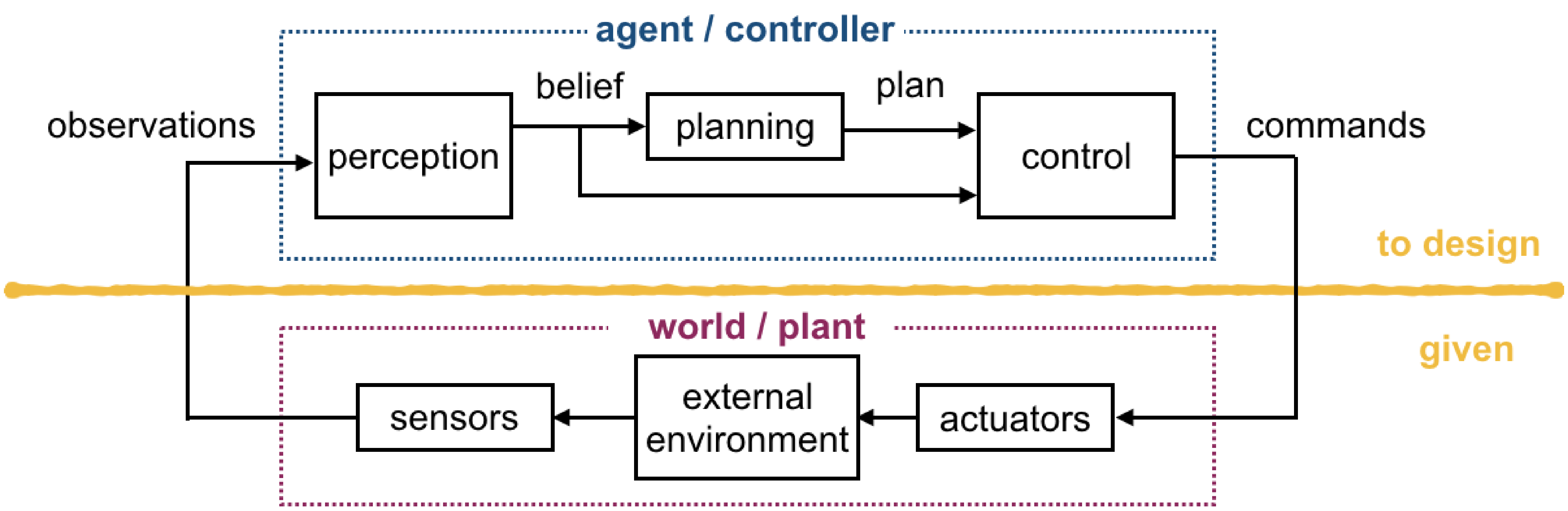
- Jacopo Tani – ETHZ – 1st of November 2017

# Big picture




Today we are going to talk about control systems

# Terminology

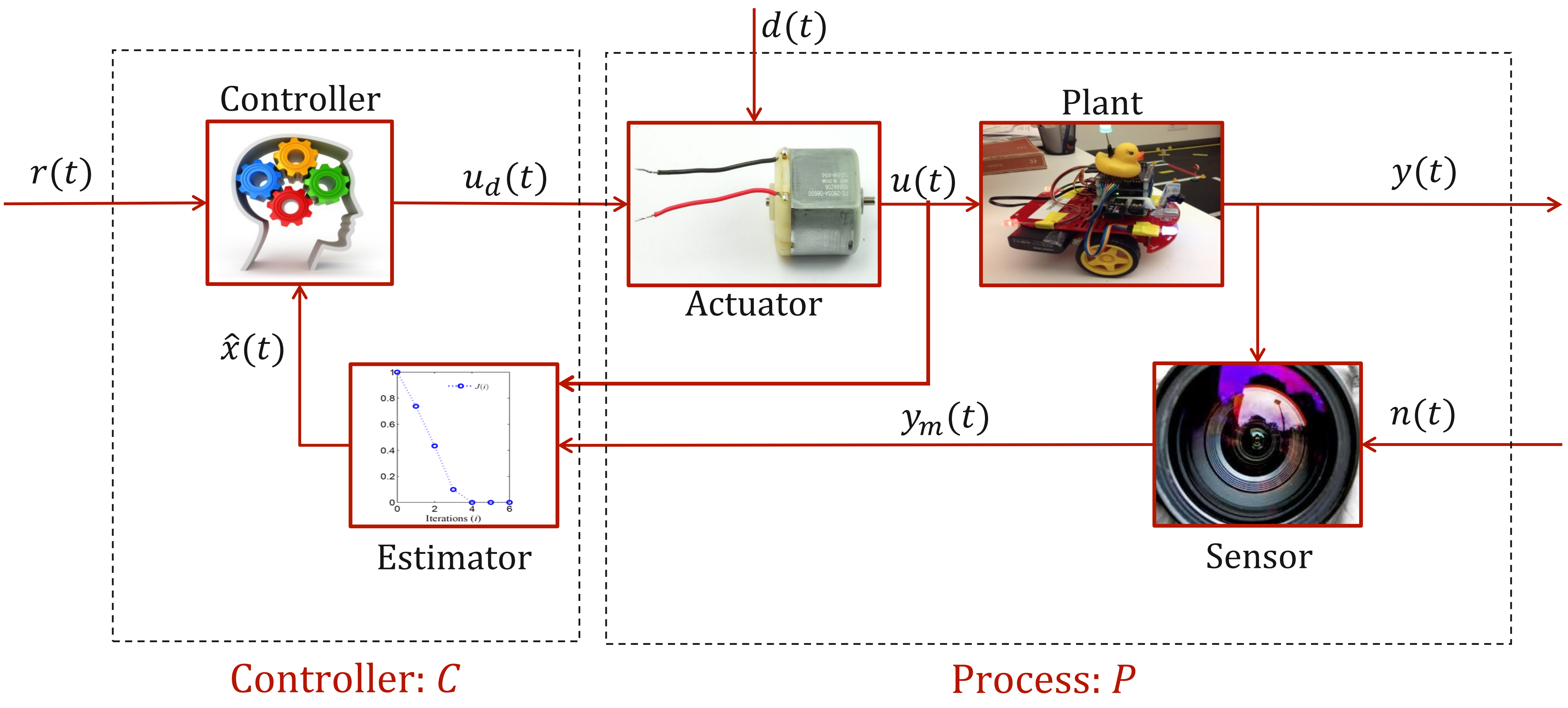



- Belief ~ Estimate: $\hat{x}(\mathrm{t})$
- Plan ~ Reference trajectory: $r(\mathrm{t})$ or $y_{ref}(t)$
- Commands ~ (Ideal) Inputs: $u_d(t)$
- External Environment ~ System/Plant: $\dot{x}(t) = f(x,u,t) + w(t)$ Process Noise

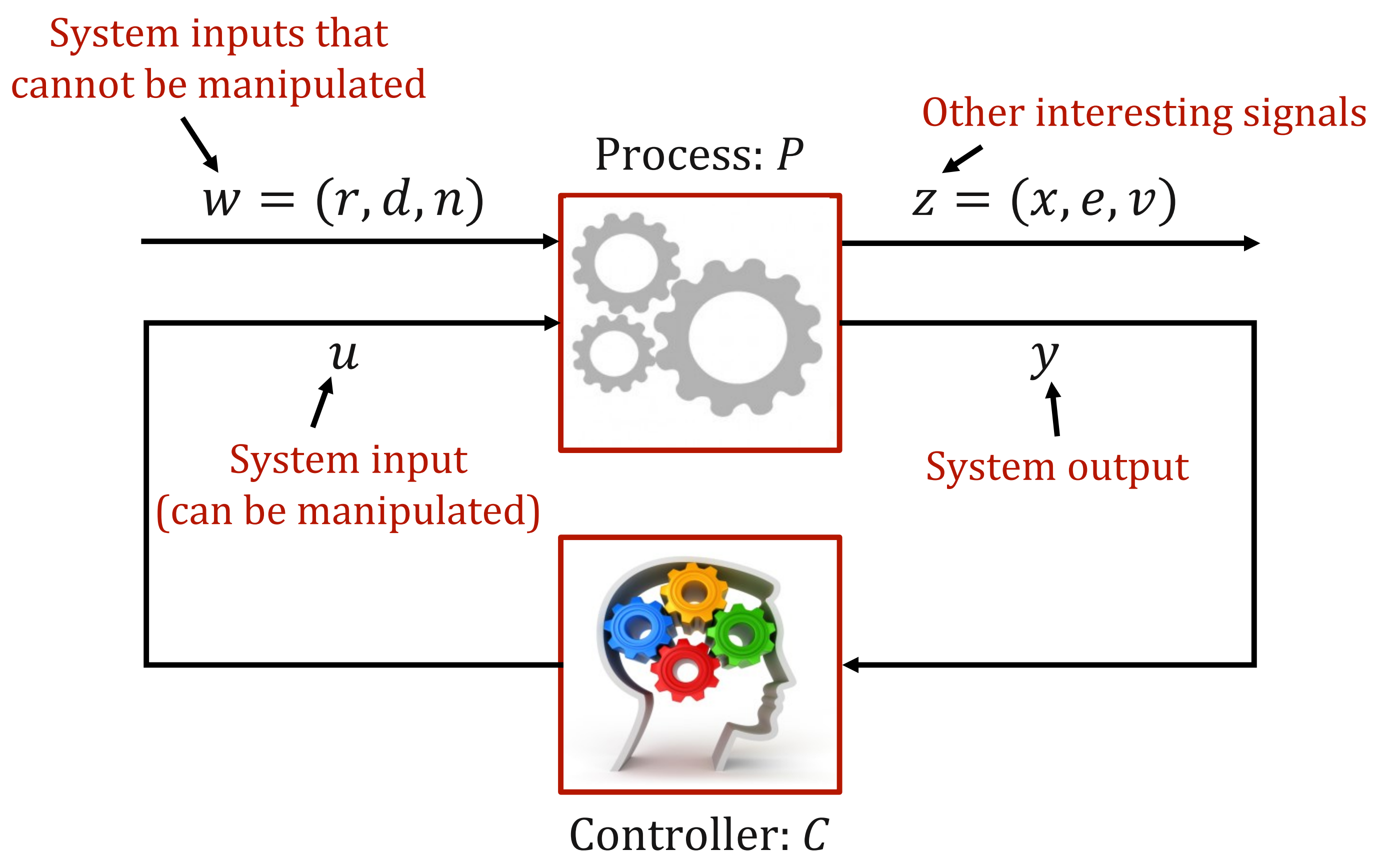
$$y(t) = h(x,u,t)$$

- Observations ~ Measured Output: $y_m(t) = g\big(y(t)\big) + n(t)$ Measurement Noise

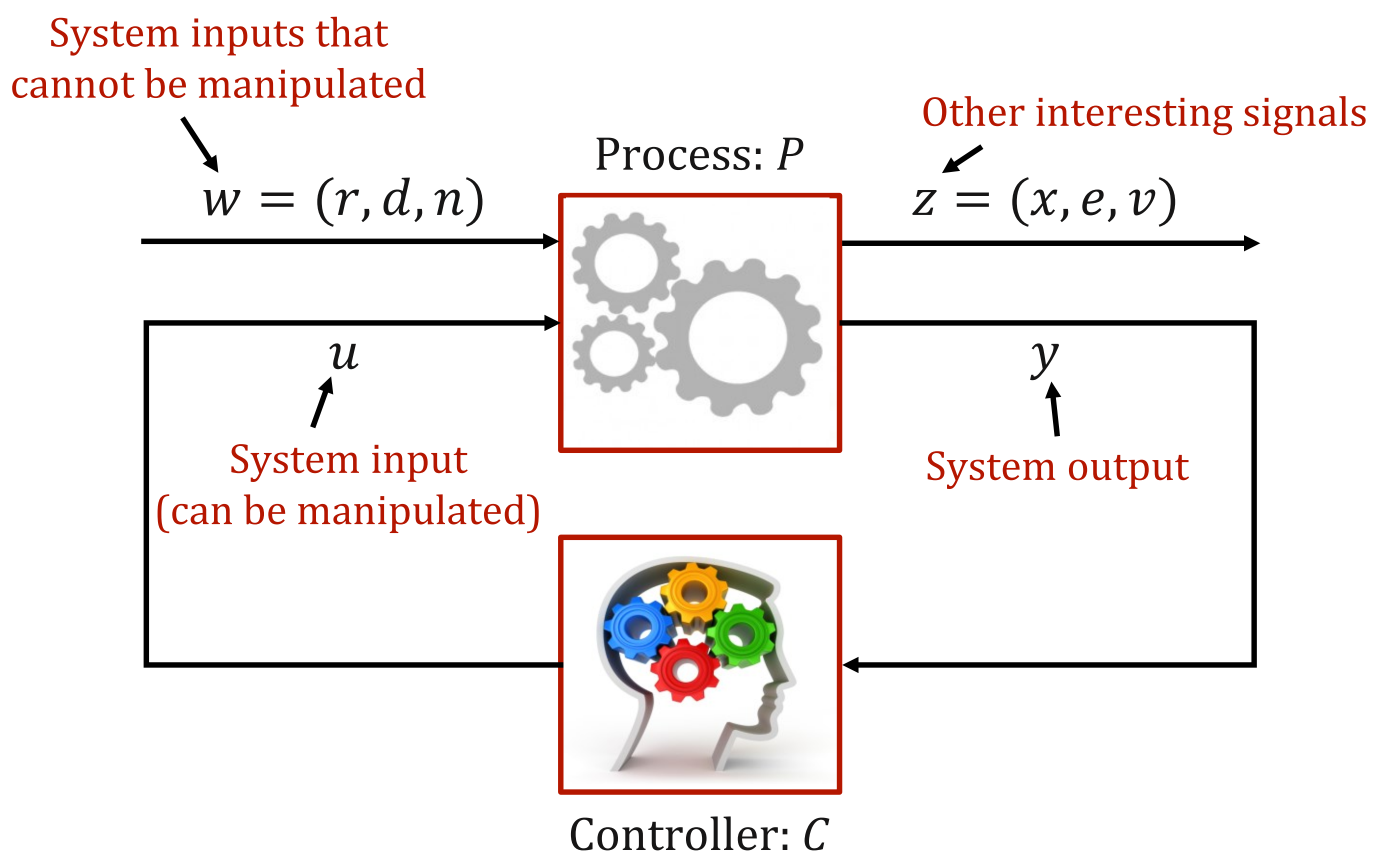
# Big picture in Control Systems Language



# Big picture adult's version

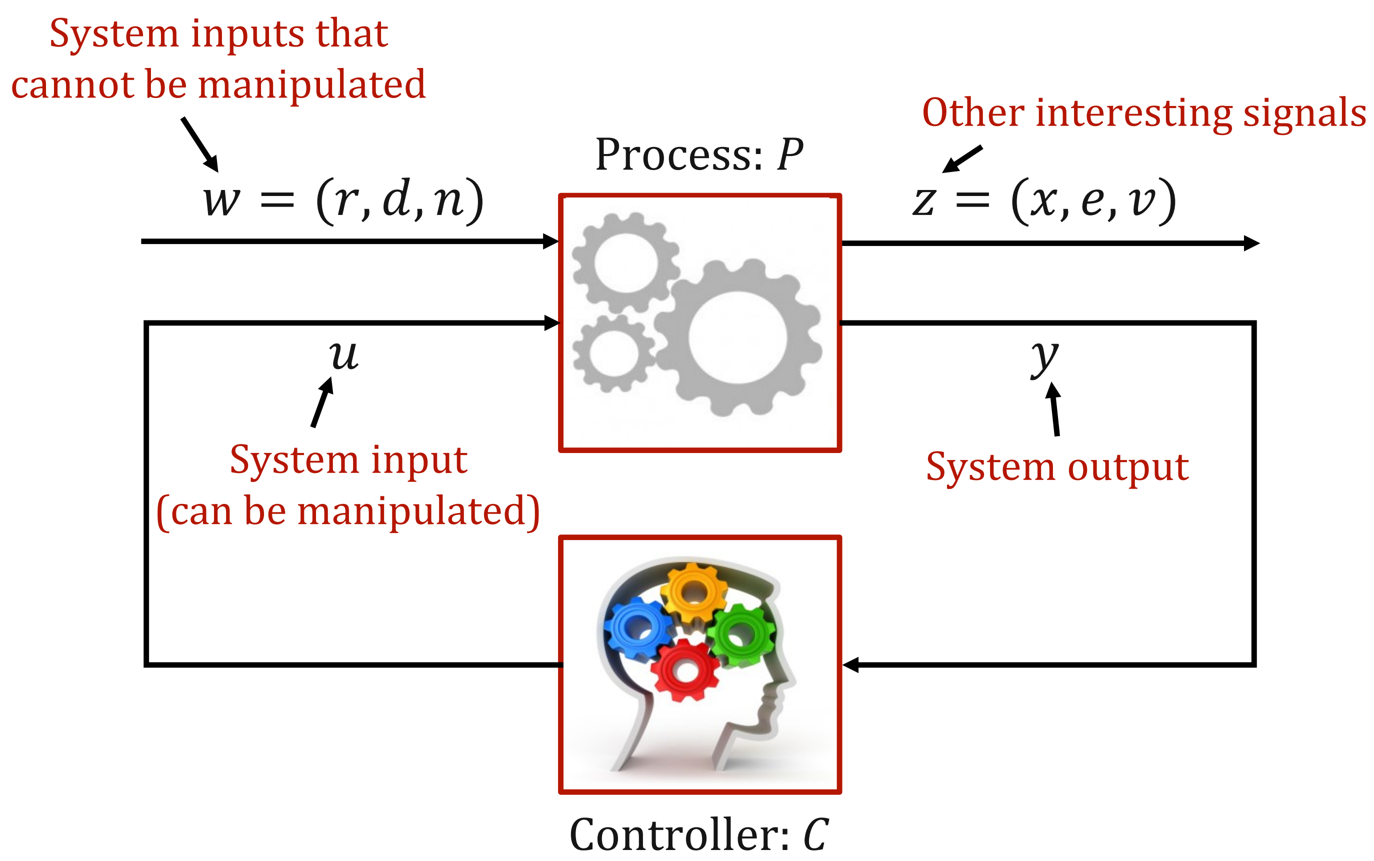


# Objectives of a controller




- Stability
  - Things don't blow up
- Performance
  - Track reference signals "well"
  - Load disturbance attenuation
  - Measurement noise attenuation
- Robustness
  - Model uncertainty

# Objectives of a controller

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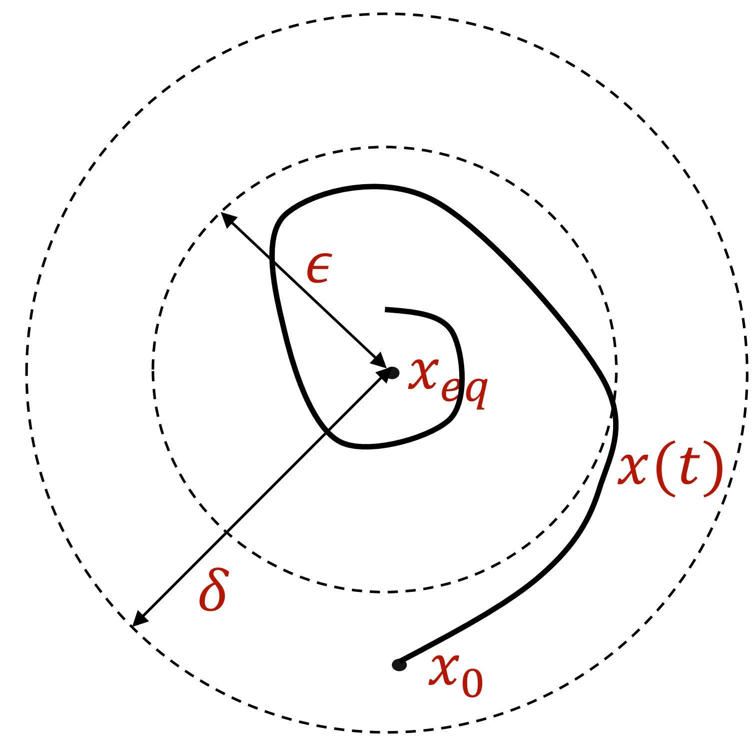
# Stability

TLDR: "Things don't blow up"

Too long, don't read




- Lyapunov stability

$$\forall \epsilon > 0, \forall \delta > 0: \; || x_0 - x_{eq} || < \delta \Rightarrow || x(t) - x_{eq} || < \epsilon, \forall t \geq 0$$

where:

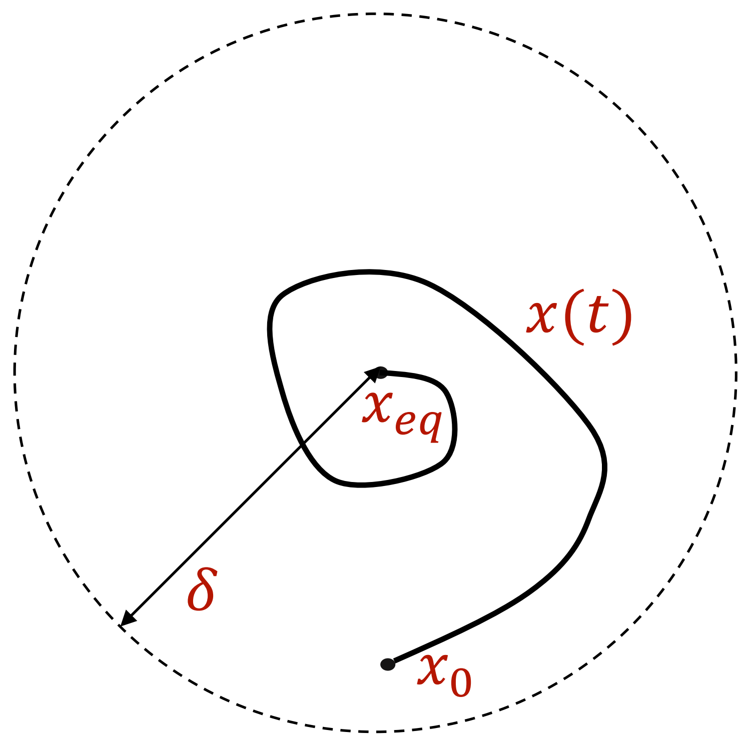
$$\dot{x}(t) = f(x), \qquad x(0) = x_0, \qquad x \in X \subseteq \mathbb{R}^n$$
$$f: X \to \mathbb{R}^n, \qquad f\left(x_{eq}\right) = 0$$

What starts close (to an equilibrium point), stays close forever

# Asymptotic Stability

TLDR: "Things converge to an equilibrium point"

Too long, don't read




- Lyapunov stability

$$\forall \delta > 0:\ ||\, x_0 - x_{eq}|| < \delta \Rightarrow \lim_{t \to \infty} ||x(t) - x_{eq}|| = 0$$

where:

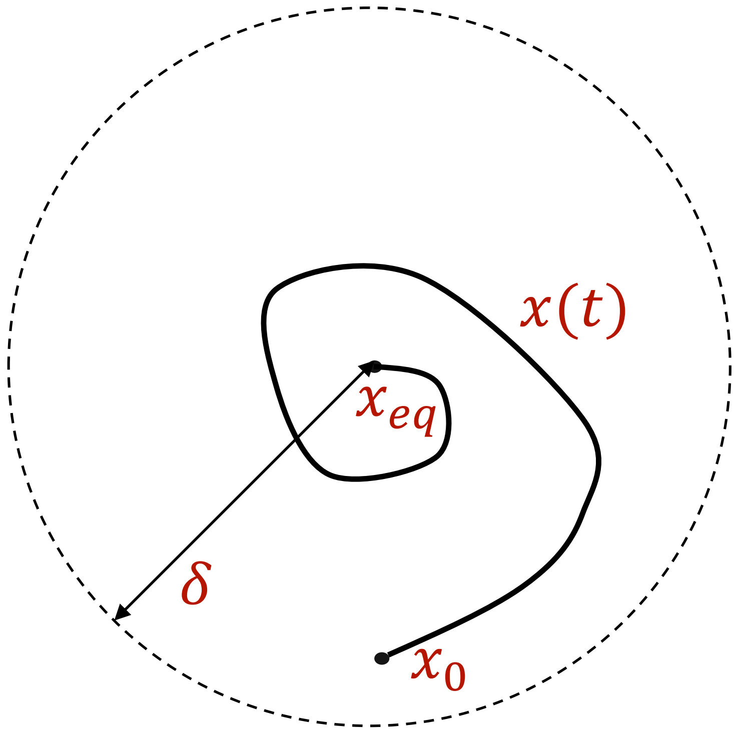
$$\dot{x}(t) = f(x), \qquad x(0) = x_0, \qquad x \in X \subseteq \mathbb{R}^n$$
$$f: X \to \mathbb{R}^n, \qquad f\left(x_{eq}\right) = 0$$

What starts close (to an equilibrium point) eventually reaches the equilibrium point

# Exponential Stability

TLDR: "Things converge to an equilibrium point, fast"

Too long, don't read




- Lyapunov stability

$$\forall\, \alpha, \beta, \delta > 0\text{: } ||\, x_0 - x_{eq}|| < \delta \Rightarrow ||x(t) - x_{eq}|| \leq \alpha||x_0 - x_{eq}||e^{-\beta t}$$

where:

$$\dot{x}(t) = f(x), \qquad x(0) = x_0, \qquad x \in X \subseteq \mathbb{R}^n$$
$$f\colon X \to \mathbb{R}^n, \qquad f\left(x_{eq}\right) = 0$$

What starts close (to an equilibrium point) eventually reaches the equilibrium point at guaranteed minimum rate

# When is a system stable?

A system is (asymptotically) stable when the Jacobian, evaluated at an equilibrium point, is Hurwitz

All eigenvalues have negative real part

- A Jacobian is: $J = \partial f / \partial x|_{x=x_{eq}}, J_{ij} = \partial f_i / \partial x_j|_{x=x_{eq}}$

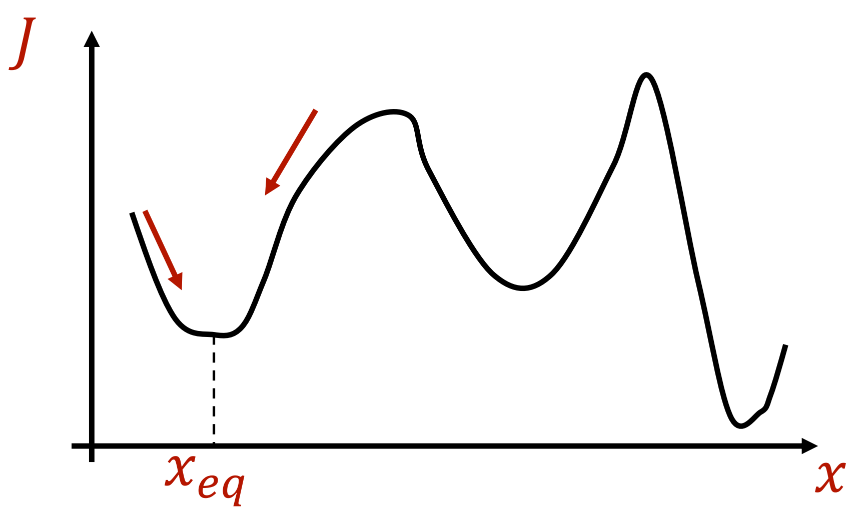
For linear systems: $f = Ax \Rightarrow J = A$

- Another way to check: Lyapunov Stability Criterion

Define a Lyapunov function such that:

$$V(x - x_{eq}) = 0 \iff x = x_{eq}$$
$$V(x - x_{eq}) > 0 \iff x \neq x_{eq}$$

$\dot{V}(\mathrm{x} - \mathrm{x}_{\mathrm{eq}}) \leq 0, \forall x \neq x_{eq} \rightarrow$ stability
$\dot{V}(\mathrm{x} - \mathrm{x}_{\mathrm{eq}}) < 0, \forall x \neq x_{eq} \rightarrow$ asymptotic stability



# Can all systems be stabilized?

No, only those that are controllable (reachable)

- Controllability (reachability) answers the question:

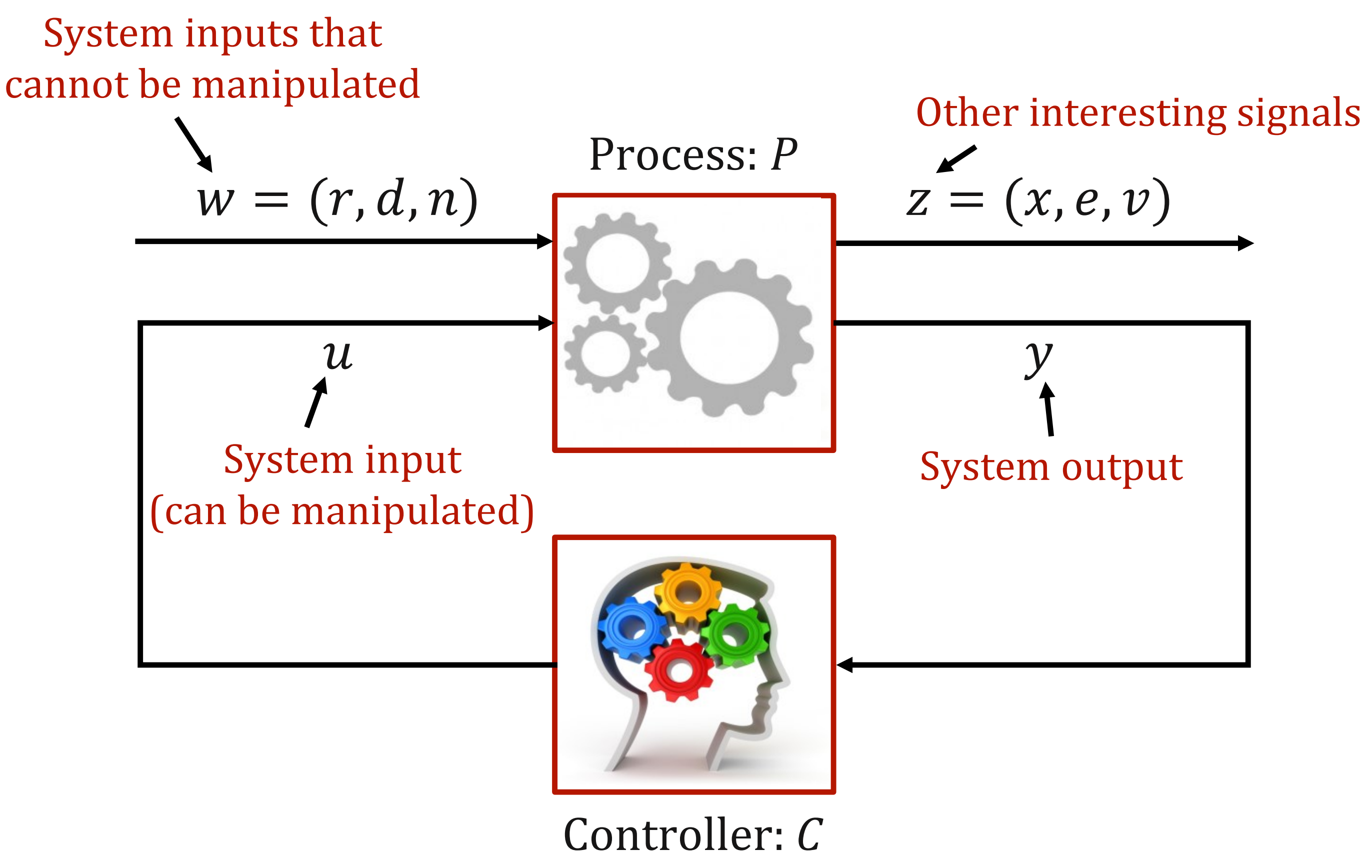
  Is it always possible to change the system's states by changing the input sequence?

  (For a state variable, does an input sequence exist such that the state can be brought from $x_0$ to $x_1$ in a finite time?)

- For linear systems: verify $\mathrm{rank}(A) = \mathrm{rank}([B, AB, \ldots, A^{n-1}B]) = n$

- For nonlinear systems: Use Lie derivates to build controllability matrix and check rank

# Objectives of a controller




- Stability
  - Things don't blow up
- Robustness
  - Model uncertainty
- Performance
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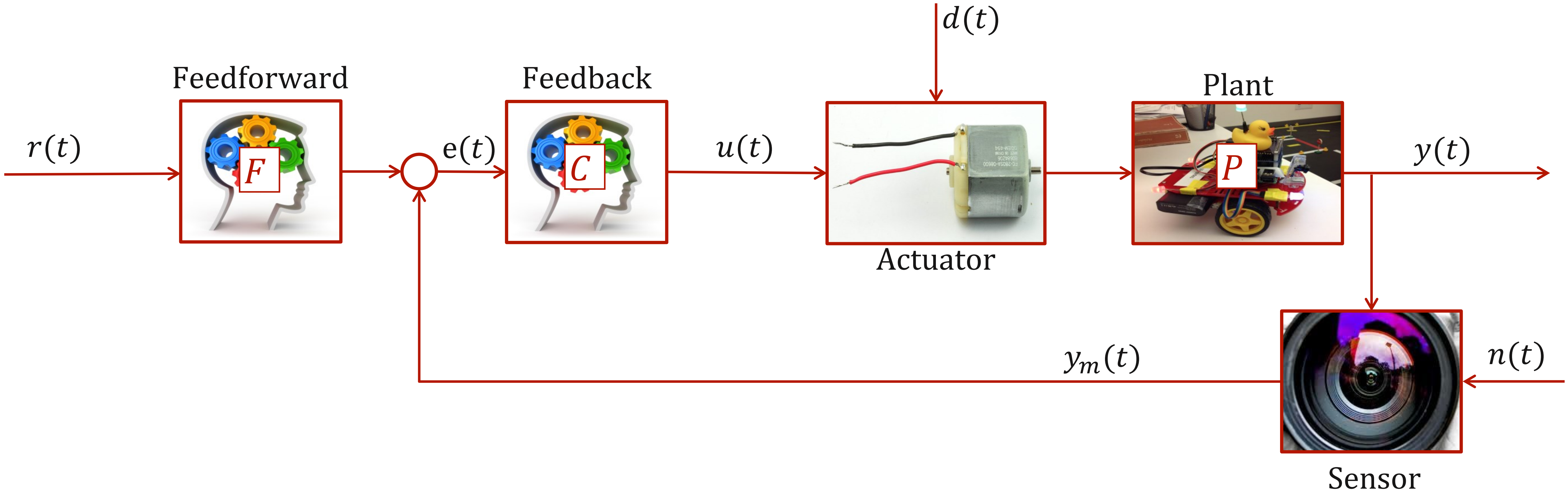
# Performance - Linear Systems




$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

| | $D$ | $N$ | $R$ |
|---|---|---|---|
| $X$ | $H_{xd}$ | $H_{xn}$ | $H_{xr}$ |
| $Y$ | $H_{yd}$ | $H_{yn}$ | $H_{yr}$ |
| $U$ | $H_{ud}$ | $H_{un}$ | $H_{ur}$ |

$$X = X(s) = \mathcal{L}[x(t)](s)$$

# Performance

The Gang of Six (Four, when $F = 1$)

- Sensitivity: $H_{yn} = S = \frac{1}{1+PC}$
- Complimentary Sensitivity: $H_{xn} = H_{ud} = T = \frac{PC}{1+PC} = PCS$
- Load disturbance Sensitivity: $H_{xd} = H_{yd} = \frac{P}{1+PC} = PS$
- Noise Sensitivity: $H_{un} = \frac{C}{1+PC}$

$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

| | $D$ | $N$ | $R$ |
|---|---|---|---|
| $X$ | $H_{xd}$ | $H_{xn}$ | $H_{xr}$ |
| $Y$ | $H_{yd}$ | $H_{yn}$ | $H_{yr}$ |
| $U$ | $H_{ud}$ | $H_{un}$ | $H_{ur}$ |

$X = X(s) = \mathcal{L}[x(t)](s)$

# Performance – Typical Scenario

- Performance
- Design spec.
  - Track reference ($r$) signals "well" → high magnitude of $H_{yr}$ at low frequencies
  - Load disturbance ($d$) attenuation → low $PS$ magnitude at low frequencies
  - Measurement noise ($n$) attenuation → low magnitude of $H_{un}$ and $S$ at high frequency

- Design constraint: **waterbed effect**

$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

- Sensitivity: $H_{yn} = S = \frac{1}{1+PC}$
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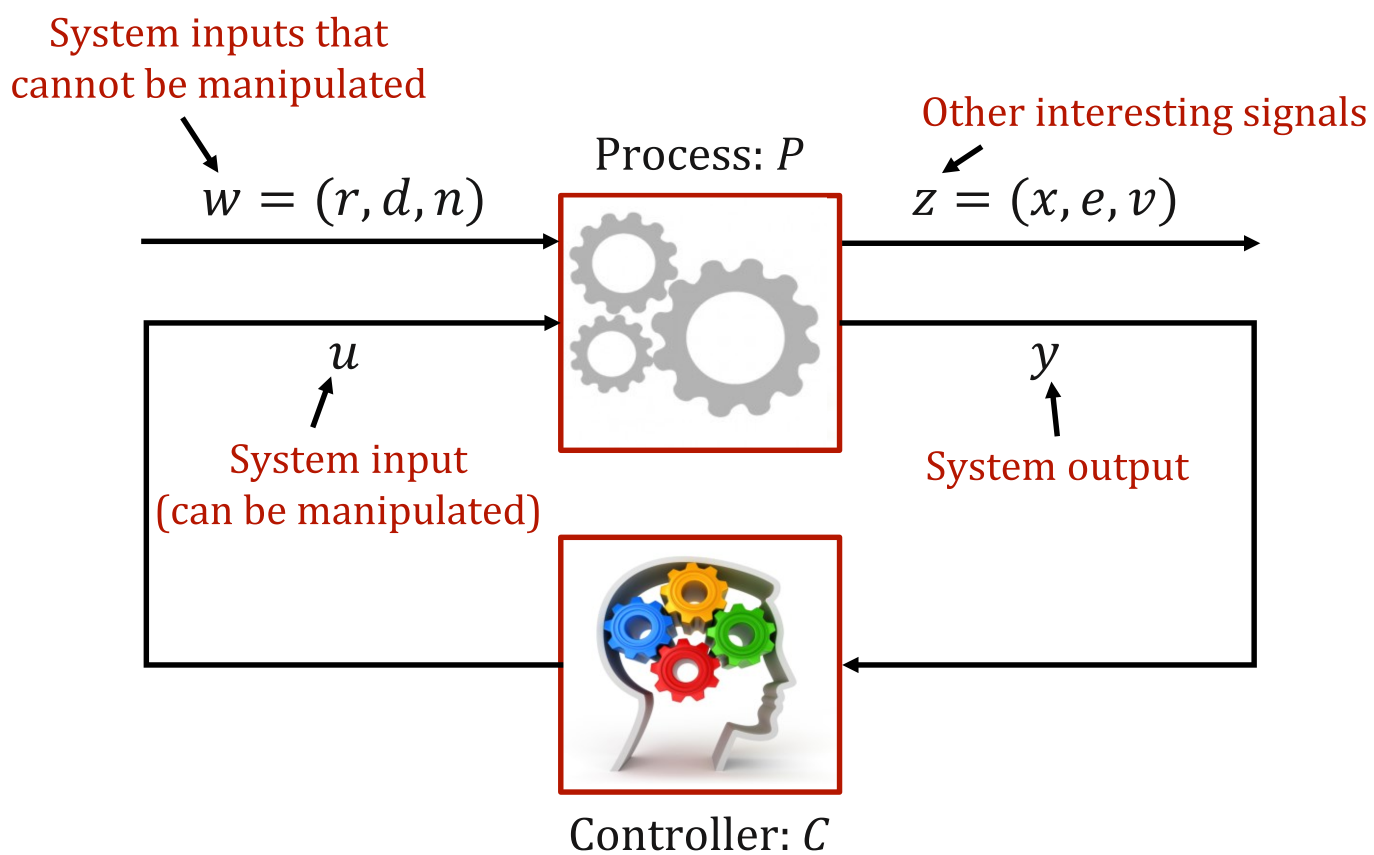
# Objectives of a controller

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# Robustness

- How to fight model uncertainty? (Step by step)

1. We have a controller that works well for motion tracking applications. I.e., $H_{yr} = T$ is good.

2. What is "model uncertainty"? A (bounded) variation in $P$.

3. How does $H_{yr}$ change with $P$, i.e., what is $\frac{dT}{dP}$?

The reason for its name..

Easy: $\frac{dT}{dP} = S\frac{T}{P}$

Even better: $\frac{d\log T}{d\log P} = S$

For a controller to be robust, the magnitude of $S$ should be small (at all frequencies)

# Robustness – Waterbed effect

Hendrik Wade Bode

- Unfortunately it is not possible to shape $S$'s magnitude arbitrarily

Bode's Equation: $$\int_0^\infty \log|S(i\omega)|\, d\omega = constant$$

- Moreover: $S + T = 1$, so the effect has repercussions on the complementary sensitivity too

The interested reader may find a formal derivation in [1](https://www.cds.caltech.edu/~murray/courses/cds101/fa02/caltech/astrom-ch5.pdf).

# Other considerations

## On feedback

- Feedback injects measurement noise in the system;
- Real actuators can suffer from (rate) saturation;
- A high open loop gain can provide good tracking, but amplifies noise. Noises are typically high frequency, therefore fast variations in the command inputs will follow. The actuator rate saturation may clip off the command resulting in unpredicted performance.

## On the process dynamics

- We looked at continuous time, while implementations are in discrete time. Approximations degrade performance.

- Especially in robotics, latencies are a big deal. A system cannot respond faster than it's delay. Non minimum phase zeros (right hand plane) are equivalent to delays. Low zeros, high delays.

- Handling instabilities (poles in the right hand plane) requires fast controllers.

# Conclusions

- Controllers are design to provide:

  Stability
  Performance
  Robustness

- These requirements often conflict and tradeoffs are required

- There are some fundamental limitations in feedback that stem from:

  Measurement noise
  Actuator saturation
  Process Dynamics