Evaluate $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$
\begin{array}{r l}
 & \quad x^2 + x \\
x-1 \,\big) & x^3 + 0x^2 + x \\
 & -x^3 - x^2 \\
\hline
 & \quad -x^2 + x \\
 & \quad\ \ x^2 - x \\
\hline
 & \qquad\quad 0
\end{array}
$$

$$= \int x^2 + x \, dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Evaluate $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

$$\hookrightarrow \frac{x-9}{(x+5)(x-2)}$$

$$= \frac{A}{(x+5)} + \frac{B}{(x-2)}$$

$$= \frac{A(x-2)}{(x+5)(x-2)} + \frac{B(x+5)}{(x+5)(x-2)}$$

$$\hookrightarrow A(x-2) + B(x+5) = x-9$$

Assume $x = -5$

$$\rightarrow A(-5-2) + B(0) = -5-9$$

$$\rightarrow -7A = -14$$

$\rightarrow A = 2$

Assume $x = 2$

$\rightarrow A(0) + B(2+5) = 2 - 9$

$\rightarrow 7B = -7$

$\rightarrow B = -1$

$$= \frac{2\cancel{(x-2)}}{(x+5)\cancel{(x-2)}} + \frac{-1\cancel{(x+5)}}{\cancel{(x+5)}(x-2)}$$

$$= \frac{2}{x+5} + \frac{-1}{x-2}$$

$$= \int \frac{2}{x+5} + \frac{-1}{x-2}\,dx$$

$$= 2\int (x-5)^{-1}\,dx - \int (x-2)^{-1}\,dx$$

$$= 2\ln(|x-5|) - \ln(|x-2|) + C$$

Evaluate $\int_0^1 \frac{2x+3}{(x+1)^2}\,dx$

$\hookrightarrow \frac{2x+3}{(x+1)(x+1)}$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

$$= \frac{A(x+1)}{(x+1)^2} + \frac{B}{(x+1)^2}$$

$\hookrightarrow A(x+1)+B=2x+3$

Assume $x=-1$

$\rightarrow A(0)+B=2(-1)+3$

$\rightarrow B=-2+3$

$\rightarrow B=1$

Solve for A

$\hookrightarrow A(x+1)+1=2x+3$

$=A(x+1)=2x+2$

$=A=2$

$$=\frac{A(x+1)}{(x+1)^2}+\frac{B}{(x+1)^2}$$

$$=\frac{2\cancel{(x+1)}}{(x+1)^{\cancel{2}}}+\frac{1}{(x+1)^2}$$

$$=\frac{2}{(x+1)}+\frac{1}{(x+1)^2}$$

$$=\int\frac{2}{(x+1)}+\frac{1}{(x+1)^2}\,dx$$

$$=2\int(x+1)^{-1}\,dx+\int(x+1)^{-2}\,dx$$

$= 2\ln(|x+1|) - (x+1)^{-1} + C$

$= 2\ln(|x+1|) - \dfrac{1}{x+1} + C$

Evaluate $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

$\hookrightarrow \dfrac{10}{(x-1)(x^2+9)}$

$= \dfrac{A}{(x-1)} + \dfrac{B}{(x^2+9)}$

$= \dfrac{A(x^2+9)}{(x-1)(x^2+9)} + \dfrac{B(x-1)}{(x-1)(x^2+9)}$

$\hookrightarrow A(x^2+9) + B(x-1) = 10$

Let $x = 1$

$\rightarrow A(1^2+9) + B(0) = 10$

$\rightarrow A(10) = 10$

$\rightarrow A = 1$

Let $x = 0$

$\rightarrow 1(0+9) + B(0-1) = 10$

$\rightarrow 9 - B = 10$

$\rightarrow -B = 1$

$\rightarrow B = -1$

$$= \frac{1\cancel{(x^2+9)}}{(x-1)\cancel{(x^2+9)}} + \frac{-1\cancel{(x-1)}}{\cancel{(x-1)}(x^2+9)}$$

$$= \frac{1}{x-1} - \frac{1}{x^2+9}$$

$$= \int \frac{1}{x-1} - \frac{1}{x^2+9}\, dx$$

$$= \int (x-1)^{-1}\, dx - \int (x^2+9)^{-1}\, dx$$

$$= \ln(|x-1|) - \int (x^2+3^2)^{-1}\, dx$$

$$= \ln(|x-1|) - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Evaluate $\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\, dx$

Use long division because the numerator degree is larger then the denominator degree

$$\begin{array}{r} 1 + 0x \\ x^3-2x^2 \overline{) \, x^3-2x^2+0x-4} \\ \underline{-x^3+2x^2 } \\ 0 + 0x \\ \underline{0 + 0x } \\ 0x - 4 \end{array}$$

$$= \int_3^4 1 - \frac{4}{x^3 - 2x^2}\, dx$$

$$\hookrightarrow \frac{4}{x^2(x-2)}$$

$$= \frac{A}{x^2} + \frac{B}{(x-2)}$$

$$= \frac{A(x-2)}{x^2(x-2)} + \frac{Bx^2}{x^2(x-2)}$$

$$\hookrightarrow A(x-2) + Bx^2 = 4$$

Let $x=2$

$$\rightarrow A(0) + 4B = 4$$

$$\rightarrow B = 1$$

Let $x=0$

$$\rightarrow A(-2) + B(0) = 4$$

$$\rightarrow -2A = 4$$

$$\rightarrow A = -2$$

$$= \frac{-2\cancel{(x-2)}}{x^2\cancel{(x-2)}} + \frac{1\cancel{x^2}}{\cancel{x^2}(x-2)}$$

$$= \frac{-2}{x^2} + \frac{1}{x-2}$$

$$= \int_3^4 1 - \left(\frac{-2}{x^2} + \frac{1}{x-2} \right) dx$$

$$= \int_3^4 1 + \frac{2}{x^2} - \frac{1}{x-2} \, dx$$

$$= \int_3^4 1 \, dx + 2\int_3^4 x^{-2} \, dx - \int_3^4 (x-2)^{-1} \, dx$$

$$= \left(x \Big|_3^4 \right) + 2\left(-\frac{1}{1}x^{-1} \Big|_3^4 \right) - \left(\ln(|x-2|) \Big|_3^4 \right)$$

$$= \left(x \Big|_3^4 \right) - 2\left(\frac{1}{x} \Big|_3^4 \right) - \left(\ln(|x-2|) \Big|_3^4 \right)$$

$$= (4-3) - 2\left(\frac{1}{4} - \frac{1}{3} \right) - \left(\ln(|4-2|) - \ln(|3-2|) \right)$$

$$= 1 - 2\left(-\frac{1}{12} \right) - \left(\ln(|2|) - \ln(|1|) \right)$$

$$= 1 + \frac{1}{6} - \ln(2) + 0$$

$$= \frac{7}{6} - \ln(2)$$

Almost Correct