Solve $\frac{dy}{dx} = xe^{-y}$

$$\frac{dy}{dx} = x\frac{1}{e^y}$$

$$e^y\,dy = x\,dx$$

$$\int e^y\,dy = \int x\,dx$$

$$e^y = \frac{1}{2}x^2 + C$$

$$\ln(e^y) = \ln\left(\frac{1}{2}x^2 + C\right)$$

$$y = \ln\left(\frac{1}{2}x^2 + C\right)$$

Find the family of implicit solutions for $\frac{dy}{dx} = \frac{x}{y}$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 = x^2 + C$$

Solve $2xy + 10x + (x^2-4)\frac{dy}{dx} = 0$

$$(x^2-4)\frac{dy}{dx} = -2xy - 10x$$

$$(x^2-4)\frac{dy}{dx} = x(-2y-10)$$

$$\frac{1}{-2y-10}\,dy = \frac{x}{x^2-4}\,dx$$

$$-\frac{1}{2}\int \frac{1}{y+5}\,dy = \int \frac{x}{x^2-4}\,dx$$

$$-\frac{1}{2}\ln(|y+5|) = \underline{\int \frac{x}{x^2-4}\,dx}$$

$$\rightarrow \int \frac{x}{x^2-4}\,dx$$

$$u = x^2-4$$

$$\frac{du}{dx} = 2x$$

$$\frac{1}{2}du = x\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln(|u|) + C$$

$$= \frac{1}{2}\ln(|x^2-4|) + C$$

$$-\cancel{\frac{1}{2}}\ln(|y+5|) = \cancel{\frac{1}{2}}\ln(|x^2-4|) + C$$

$$-\ln(|y+5|) = \ln(|x^2-4|) + C$$

Find the unique solution to $\frac{dy}{dx} = \frac{y^2}{x^3}$ where $y(1) = 1$

$$\frac{1}{y^2}\,dy = \frac{1}{x^3}\,dx$$

$$\int y^{-2}\,dy = \int x^{-3}\,dx$$

$$-y^{-1} + C = -\frac{1}{2}x^{-2} + C$$

$$-\frac{1}{y} = -\frac{1}{2x^2} + C$$

$$\frac{1}{y} = \frac{1}{2x^2} + C$$

$$y = \frac{1}{\frac{1}{2x^2} + C}$$

$$y(1) = 1$$

$$1 = \frac{1}{\frac{1}{2(1)^2} + C}$$

$$\frac{1}{2} + C = 1$$

$$C = \frac{1}{2}$$

$$y = \frac{1}{\frac{1}{2x^2} + \frac{1}{2}}$$

$$Y = \frac{1}{\frac{1}{2x^2} + \frac{x^2}{2x^2}}$$

$$Y = \frac{2x^2}{1 + x^2}$$

Find the unique solution to $\frac{dy}{dx} = \frac{2x + \sec^2(x)}{2y}$ with $y(0) = -6$

$$2y\, dy = 2x + \sec^2(x)\, dx$$

$$2\int y\, dy = \int 2x + \sec^2(x)\, dx$$

$$2\tfrac{1}{2}y^2 + C = 2\tfrac{1}{2}x^2 + \tan(x) + C$$

$$y^2 = x^2 + \tan(x) + C$$

$$y = \pm\sqrt{x^2 + \tan(x) + C}$$

$$y(0) = -6$$

$$-6 = \pm\sqrt{0^2 + \tan(0) + C}$$

$$-6 = \pm\sqrt{C}$$

$$36 = C$$

$$y = -\sqrt{x^2 + \tan(x) + 36}$$

Find the unique solution to $\frac{dy}{dx} = \frac{\ln(x)}{xy}$ with $y(1) = 7$

$$y\,dy = \frac{1}{x}\ln(x)\,dx$$

$$\int y\,dy = \int \frac{1}{x}\ln(x)\,dx$$

$$u = \ln(x)$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\,dx$$

$$\int y\,dy = \int u\,du$$

$$\cancel{\frac{1}{2}}y^2 + C = \cancel{\frac{1}{2}}u^2 + C$$

$$y^2 = \ln(x)^2 + C$$

$$y = \sqrt{\ln(x)^2 + C}$$

$$y(1) = 7$$

$$7 = \sqrt{\ln(1)^2 + C}$$

$$C = 49$$

$$y = \sqrt{\ln(x)^2 + 49}$$

Find the orthogonal trajectory for $x^2+5y^2=k$

$2x+10y\frac{dy}{dx}=0$

$10y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=-\frac{2x}{10y}$

$\frac{dy}{dx}=-\frac{x}{5y}$

Flip and negate $\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{5y}{x}$

$\frac{1}{5y}\,dy=\frac{1}{x}\,dx$

$\frac{1}{5}\int\frac{1}{y}\,dy=\int\frac{1}{x}\,dx$

$\frac{1}{5}\ln(y)=\ln(x)+C$

$\ln(y^{\frac{1}{5}})=\ln(x)+C$

$e^{\ln(y^{1/5})}=e^{\ln(x)+C}$

$y^{1/5}=e^{C}x$

$y=Cx^5$

Find the orthogonal trajectory for $xy=k$

$x\frac{dy}{dx}+y=0$

$x\frac{dy}{dx}=-y$

$\frac{dy}{dx}=-\frac{y}{x}$

Flip and negate $\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{x}{y}$

$y\,dy=x\,dx$

$\int y\,dy=\int x\,dx$

$\frac{1}{2}y^2=\frac{1}{2}x^2+C$

$y^2=x^2+C$

Find the orthogonal trajectory for $y=ke^{-x}$

$\ln(y)=\ln(ke^{-x})$

$\ln(y)=\ln(k)-x\ln(e)$

$\ln(y)=\ln(k)-x$

$\frac{1}{y} \frac{dy}{dx} = -1$

$\frac{dy}{dx} = -y$

Flip and negate $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{y}$

$y\,dy = dx$

$\int y\,dy = \int dx$

$\frac{1}{2} y^2 = x + C$

$y^2 = 2x + C$