4.4.1 Exponential representation of complex numbers

The right-hand side of Equation 4.17 can be interpreted as the Cartesian representation of a complex number z with real and imaginary parts

$$Re(z) = \cos(\theta), \quad Im(z) = \sin(\theta).$$

The magnitude and phase of z are (see Equation 4.7, page 192)

$$\begin{aligned} |z| &= \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1. \\ \theta(z) &= \arctan\left(\frac{\sin(\theta)}{\cos(\theta)}\right) = \arctan\left(\tan(\theta)\right) = \theta. \end{aligned}$$

(Remember to always check the proper quadrant in calculations of the phase!) In general, let $z = x + jy$ be an arbitrary complex number. Let

$$\begin{aligned} |z| &\triangleq \sqrt{x^2 + y^2} \\ \sin(\theta) &\triangleq \frac{y}{\sqrt{x^2 + y^2}} \\ \cos(\theta) &\triangleq \frac{x}{\sqrt{x^2 + y^2}} \end{aligned}$$

Using the above equations, we can rewrite

$$
\begin{aligned}
z &= x + jy = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}(x+jy), \\
&= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + j\frac{y}{\sqrt{x^2+y^2}}\right), \\
&= |z|[\cos(\theta) + j\sin(\theta)], \qquad (4.23) \\
&= |z|e^{j\theta}.
\end{aligned}
$$

where in the last equation we used Theorem 4.1.

Definition: The equation

$$z = |z| \cdot e^{j\theta} \qquad (4.24)$$

is called the *exponential representation* of the complex number z. $|z|$ is the magnitude and θ is the phase of z.

△