4.8 Taylor and Maclaurin Series

In this section we will take a more in-depth look at power series. Let's begin with the function f where $f(x)$ is represented by a power series about $x = a$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned} \tag{3}$$

The domain of f is the interval of convergence of the series. This function is always defined at $x = a$. When $x = a$ is substituted in, all the terms are 0 except the first, so

$$f(a) = c_0$$

Let's see if there are such specific formulas for all of the coefficients c_n. We assume the power series has a positive radius of convergence and use Theorem 22 to differentiate both sides of Equation (3).

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n \cdot c_n (x-a)^{n-1} \\ &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \\ f'(a) &= c_1 \end{aligned}$$

The pattern is not evident yet, but becomes clearer as we continue the process.

$$\begin{aligned} f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n (x-a)^{n-2} \\ &= 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots \\ f''(a) &= 2\,c_2 \end{aligned}$$

$$\begin{aligned} f'''(x) &= \sum_{n=2}^{\infty} n(n-1)(n-2)c_n (x-a)^{n-3} \\ &= 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots \\ f'''(a) &= 2\cdot 3\,c_3 \end{aligned}$$

Examining these together in more detail we find

$$\begin{aligned} f(a) &= c_0 = 1\cdot c_0 = 0!\,c_0 \\ f'(a) &= c_1 = 1\cdot c_1 = 1!\,c_1 \\ f''(a) &= 2\cdot c_2 = 2\cdot 1\cdot c_2 = 2!\,c_2 \\ f'''(a) &= 2\cdot 3\cdot c_3 = 3\cdot 2\cdot 1\cdot c_3 = 3!\,c_3 \\ f^{(4)}(a) &= 2\cdot 3\cdot 4\cdot c_4 = 4\cdot 3\cdot 2\cdot 1\cdot c_4 = 4!\,c_4 \end{aligned}$$

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Thus the following pattern emerges for the coefficients.

$$c_0 = \frac{f(a)}{0!} \qquad c_1 = \frac{f'(a)}{1!} \qquad c_2 = \frac{f''(a)}{2!} \qquad c_3 = \frac{f'''(a)}{3!} \qquad c_4 = \frac{f^{(4)}(a)}{4!}$$

In general,

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

By convention $0! = 1$ and $f^{(0)}(x) = f(x)$. This serves as proof for the following theorem.

Theorem 1. Derivative Formulas for Power Series Coefficients
If f is a function represented by a power series centered at $x = a$ with radius of convergence $R > 0$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

then the coefficients c_n are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Hence

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \end{aligned}$$

A special case of this theorem is when $a = 0$.

Corollary 3.
If f is a function represented by a power series centered at $x = 0$ with radius of convergence $R > 0$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{for all } x \text{ in } (-R, R)$$

then the coefficients c_n are given by

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Hence

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \end{aligned}$$

Definition 4. Taylor and Maclaurin Series for a Function
If f has derivatives of all orders at $x = a$, then we say the **Taylor** *series of f about $x = a$ is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a = 0$ this is called the **Maclaurin** *series for f.*

A function f and its Taylor series about $x = a$ do not have to be equal anywhere except at $x = a$. Later in this section we will discuss a method of determining the values of x for which they are equal.

Example 47. *Find the Maclaurin series for the function $f(x) = e^x$, and find its interval of convergence.*

Solution: If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x$ and so on. So

$$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = e^0 = 1 \quad \text{for all } n$$

Thus the Maclaurin series for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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We find the radius and the interval of convergence by examining the absolute value of the ratio of consecutive terms

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x\cancel{x^n}\,\cancel{n!}}{\cancel{x^n}(n+1)\cancel{n!}}\right| = |x|\cdot\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$$

Thus the Maclaurin series for $f(x) = e^x$ converges for all x. Hence the radius of convergence is $R = \infty$.

In the previous example we found the Maclaurin series for $f(x) = e^x$. However, we have not shown that the series converges to $f(x) = e^x$. We will prove this in two different ways. We give the first proof here, and the second proof in the next section. Using the theorem from the last section we can differentiate a power series term-by-term. We use it to determine if the derivative of the function g given by the Maclaurin series for e^x is equal to g itself.

$$\begin{aligned}
g(x) &= \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{d}{dx}g(x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{n\cdot x^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{n\cdot x^{n-1}}{n(n-1)!} \\
&= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!} = g(x)
\end{aligned}$$

Hence $g'(x) = g(x)$. Thus $g(x)$ solves the differential equation $y' = y$ and the initial condition $y(0) = 1$. From our work on differential equations, we know that e^x is the only solution of this initial value problem. Thus $g(x) = e^x$ and indeed e^x is equal to its Maclaurin series.

Remark: The examples in the rest of this section involve functions that are equal to their Taylor series on their respective intervals of convergence. The method to show this will be established in the next section.

Example 48. *Find the Taylor series for $f(x) = e^x$ centered at $a = 4$.*

Solution: Since $f^{(n)}(x) = e^x$ for every n we have that the Taylor series for e^x about $a = 4$ is

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(4)}{n!}(x-4)^n = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \cdots \\
&= \sum_{n=0}^{\infty}\frac{e^4}{n!}(x-4)^n \qquad \text{for all } x
\end{aligned}$$

The radius of convergence is again $R = \infty$.

We now have two different representations of e^x as a power series: the Maclaurin series and the Taylor series centered at $a = 4$. In fact, we can generate infinitely many power series expansions by taking different values for the center. In general, if we are interested in the values of a function near some particular number a, it is useful to consider the Taylor series centered at that number a.

Example 49. *Find the Taylor series representation for $f(x) = e^{x^2}$ centered at $x = 0$.*

Solution: Since we know the Taylor series representation

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we can use it to find the representation for e^{x^2} by substituting x^2 for x.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

Example 50. *Find the Maclaurin series for $f(x) = \sin x$ and find its radius of convergence.*

Solution: Since $\sin x$ has derivatives of all orders, it has a Maclaurin series.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1
\end{aligned}$$

The cycle now repeats itself.

$$\begin{aligned}
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\sin x &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= 0 + \frac{x}{1!} - \frac{0\,x^2}{2!} - \frac{1\,x^3}{3!} + \frac{0\,x^4}{4!} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}$$

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This is an alternating series. The terms have odd powers of x and odd factorials. Conveniently, $\sin x$ is an odd function.
We will use the ratio test to find the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{(-1)^n x^{2n+1}}\cdot\frac{(2n+1)!}{(2(n+1)+1)!}\right| \\
&= x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= 0
\end{aligned}$$

This power series converges for every x. The radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$. In the next section we will show that $\sin x$ equals its Maclaurin series.

Example 51. *Find the Maclaurin series for $f(x) = \cos x$. (HINT: Differentiate the series generated for $f(x) = \sin x$ in Example 4.)*

Solution:

$$\begin{aligned}
\cos x &= \frac{d}{dx}\sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x\in\mathbb{R}
\end{aligned}$$

This is an alternating series. The terms have even powers of x and even factorials. This corresponds to $\cos x$ being an even function.

Example 52. *Find a Maclaurin series for the hyperbolic cosine function*

$$f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

Solution:

$$\begin{aligned}
f(x) &= \frac{1}{2}(e^x + e^{-x}) & f(0) &= \frac{1}{2}(1+1) = 1 \\
f'(x) &= \frac{1}{2}(e^x - e^{-x}) & f'(0) &= \frac{1}{2}(1-1) = 0 \\
f''(x) &= \frac{1}{2}(e^x + e^{-x}) & f''(0) &= \frac{1}{2}(1+1) = 1 \\
f'''(x) &= \frac{1}{2}(e^x - e^{-x}) & f'''(0) &= \frac{1}{2}(1-1) = 0 \\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
\cosh x = \frac{1}{2}(e^x + e^{-x}) &= f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= 1 + 0 \cdot x + \frac{x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots
\end{aligned}$$

These terms are all absolute values of the terms in the Maclaurin series for $\cos x$.

Example 53. *Find the Maclaurin series for $f(x) = x^2 \sin 2x$.*

Solution: Since we know the Maclaurin series for $\sin x$, we can substitute $2x$ for x and then multiply by x^2 to obtain the result.

$$\begin{aligned}
\sin x &= & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots & &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
\sin 2x &= & (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots & &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \\
\sin 2x &= & 2^1x^1 - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \cdots & &= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}\, x^{2n+1}}{(2n+1)!} \\
x^2 \sin 2x &= & x^2(2^1x^1) - \frac{x^2(2^3x^3)}{3!} + \frac{x^2(2^5x^5)}{5!} - \frac{x^2(2^7x^7)}{7!} + \cdots & &= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}\, x^{2n+1}}{(2n+1)!} \\
x^2 \sin 2x &= & 2^1x^3 - \frac{2^3x^5}{3!} + \frac{2^5x^7}{5!} - \frac{2^7x^9}{7!} + \cdots & &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+3}}{(2n+1)!}
\end{aligned}$$

Example 54. *Find the Taylor series for $f(x) = \cos x$ centered at $a = \pi/6$.*

Solution:

$$\begin{aligned} f(x) &= \cos x & f(\pi/6) &= \sqrt{3}/2 \\ f'(x) &= -\sin x & f'(\pi/6) &= -1/2 \\ f''(x) &= -\cos x & f''(\pi/6) &= -\sqrt{3}/2 \\ f'''(x) &= \sin x & f'''(\pi/6) &= 1/2 \end{aligned}$$

The repetition begins here.

$$\begin{aligned} f^{(4)}(x) &= \cos x & f^{(4)}(\pi/6) &= \sqrt{3}/2 \\ f^{(5)}(x) &= -\sin x & f^{(5)}(\pi/6) &= -1/2 \\ &\vdots \end{aligned}$$

$$\begin{aligned} \cos x &= f(\pi/6) + \frac{f'(\pi/6)}{1!}(x-\pi/6)^1 + \frac{f''(\pi/6)}{2!}(x-\pi/6)^2 + \frac{f'''(\pi/6)}{3!}(x-\pi/6)^3 + \cdots \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}(x-\pi/6) - \frac{\sqrt{3}/2}{2!}(x-\pi/6)^2 + \frac{1/2}{3!}(x-\pi/6)^3 + \cdots \end{aligned}$$

or

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2(2n)!}(x-\pi/6)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2(2n+1)!}(x-\pi/6)^{2n+1}$$

Binomial Series

The binomial expansion for $f(x) = (1+x)^k$ is

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

where k is a positive integer greater than or equal to n, and the **binomial coefficient** is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

While this formula for the binomial coefficient is usually used when k and n are nonnegative integers, we use this formula as our definition of $\binom{k}{n}$ for any real number k and nonnegative integer n.

The function $f(x) = (1+x)^k$ where k is a **real number** is important in many applications. To find its Maclaurin series, which we will call the **binomial series**, we need its derivatives.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots\\
f^{(n)}(x) &= k(k-1)\dots(k-(n-1))(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\dots(k-(n-1))\\
f^{(n)}(x) &= k(k-1)\dots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\dots(k-n+1)
\end{aligned}$$

$$(1+x)^k = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{k(k-1)\dots(k-n+1)}{n!}x^n$$

Hence the binomial series is

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{1\cdot 2}x^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

Does the binomial series converge for all values of x? Let's find out using the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{k(k-1)\dots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\right|\cdot\left|\frac{n!}{k(k-1)\dots(k-n+1)x^n}\right|\\
&= \lim_{n\to\infty}\frac{|k-n|}{n+1}|x| = |x|\lim_{n\to\infty}\frac{|n-k|}{n+1} = |x|\lim_{n\to\infty}\left|\frac{n}{n+1} - \frac{k}{n+1}\right| = |x|
\end{aligned}$$

Thus, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. The radius of convergence is 1. Thus although $f(x) = (1+x)^k$ is defined for all $x > -1$, it doesn't equal its Maclaurin series when $x > 1$. In the next section we will develop a method for showing

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

for $x \in (-1,1)$.

Example 55. *Use the binomial series expansion to find a Maclaurin series for* $f(x) = \sqrt[3]{8-x}$

Solution: Let's first rewrite in the form of $(1+x)^k$.

$$\begin{aligned}
\sqrt[3]{8-x} &= (8+(-x))^{1/3} \\
&= \left(8\left(\frac{8}{8}+\frac{-x}{8}\right)\right)^{1/3} = 2\left(1+\left(\frac{-x}{8}\right)\right)^{1/3}
\end{aligned}$$

Let $k = 1/3$ and replace x with $\dfrac{-x}{8}$ in the binomial series.

$$\begin{aligned}
2\left(1+\left(\frac{-x}{8}\right)\right)^{1/3} &= 2\left(1+\frac{\left(\frac{1}{3}\right)}{1!}\left(\frac{-x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}\left(\frac{-x}{8}\right)^2+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}\left(\frac{-x}{8}\right)^3+\right. \\
&\qquad \left. +\cdots+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\cdots\left(\frac{1}{3}-n+1\right)}{n!}\left(\frac{-x}{8}\right)^n+\cdots\right) \\
&= 2\left(1-\frac{x}{24}-\frac{\overset{1}{\cancel{2}}}{9\cdot 2}\,\frac{x^2}{\underset{32}{\cancel{64}}}+\frac{\overset{5}{\cancel{10}}\,x^3}{27\cdot 6\cdot \underset{256}{\cancel{512}}}+\cdots\right) \\
&= 2\left(1-\frac{x}{24}-\frac{x^2}{576}+\frac{5x^3}{41,472}+\cdots\right)
\end{aligned}$$

This converges when $\left|\dfrac{-x}{8}\right| < 1$, i.e., when $|x| < 8$. The radius of convergence is $R = 8$.

Manipulating Power Series

The next few examples illustrate how power series can be used to integrate functions that we were previously unable to integrate and to calculate complicated limits. We will also investigate how to multiply and divide power series.

Example 56. *Using a Maclaurin series, estimate* $\int_0^1 \sin x^2\, dx$ *accurate to four decimal places.*

Solution: Substituting x^2 for x in the Maclaurin series for $\sin x$ yields

$$\sin x^2 = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots \quad = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \quad \text{for all } x$$

The function $\sin x^2$ does not have an anti-derivative that can be expressed in terms of our elementary functions. But since it has a power series representation, we can integrate using power series.

$$\begin{aligned}
\int_0^1 \sin x^2\,dx &= \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) dx \\
&= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\Big]_0^1 \\
&= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots \\
&= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75,600} + \cdots
\end{aligned}$$

Since the terms are alternating and their magnitudes are decreasing to 0, the Alternating Series Estimation Theorem says that the magnitude of the error in using n terms in the series to approximate the integral is less than the magnitude of a_{n+1}.

$$\frac{1}{15\cdot 7!} = \frac{1}{75,600} < .000014$$

$$\frac{1}{3} - \frac{1}{42} + \frac{1}{1320} = 0.310\overline{281385}$$

Hence the actual value is between 0.31026 and 0.31030. All these numbers round to 0.3103 to 4 decimal places.

Theorem 2. Algebra of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *with a common interval of convergence I. Then:*

1. *The power series for f and g can be added or subtracted to obtain a new power series for their sum or difference with interval of convergence at least as large as I:*

$$(f \pm g)(x) = f(x) \pm g(x) = \sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

2. *The power series for f and g can be multiplied to obtain a new power series for their product with interval of convergence at least as large as I:*

$$\begin{aligned}
(f\cdot g)(x) = f(x)\cdot g(x) &= \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots
\end{aligned}$$

3. *The power series for f and g can be divided provided $b_0 \neq 0$ to obtain a new power series for their quotient. The interval of convergence is generally difficult to determine.*

Example 57. *Evaluate* $\displaystyle\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$ *using the Maclaurin series for* $\cos x$.

Solution:

$$\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0}\frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1 + \frac{x^2}{2}}{x^4}$$

Since we can add and subtract convergent power series, we obtain

$$= \lim_{x\to 0}\frac{\left(\frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)}{x^4}$$

Dividing convergent power series yields

$$= \lim_{x\to 0}\left(\frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \cdots\right)$$

$$= \frac{1}{24}$$

Example 58. *Find the first three non-zero terms in the Maclaurin series for the product* $\left(\frac{1}{1-x}\right)(\tan^{-1} x)$.

Solution: Recall we derived the Maclaurin series for the quotient $\frac{1}{1-x}$ and for $\tan^{-1} x$ in Section 4.7: Equation (4.7.1) and Example 6, respectively.

$$\left(\frac{1}{1-x}\right)(\tan^{-1} x) = \left(1 + x + x^2 + x^3 + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$$

The multiplication is done in a manner similar to multiplying polynomials.
We multiply all the terms in one power series by each of the terms in the other.
Then add, grouping like powers of x.

$$\begin{array}{l} 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\ x \quad - \frac{x^3}{3} \quad + \frac{x^5}{5} - \cdots \\ \hline x + x^2 + x^3 + x^4 + x^5 + \cdots \\ \quad - \frac{x^3}{3} - \frac{x^4}{3} - \frac{x^5}{3} - \cdots \\ \qquad\qquad \frac{x^5}{5} + \frac{x^6}{5} + \frac{x^7}{5} \cdots \\ \hline x + x^2 + \frac{2x^3}{3} + \frac{2x^4}{3} + \cdots \end{array}$$

$$\left(\frac{1}{1-x}\right)(\tan^{-1} x) = x + x^2 + \frac{2x^3}{3} + \cdots$$

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Example 59. *Find the first four nonzero terms in the Maclaurin series for the quotient* $\dfrac{e^x}{\cos x}$.

Solution: We use the Maclaurin series for e^x and $\cos x$ and use "polynomial-like" long division.

$$\frac{e^x}{\cos x} = \frac{1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots}$$

$$\begin{array}{r|cccccccc}
 & 1 & +x & +x^2 & +\frac{2x^3}{3} & & & & +\cdots \\ \hline
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots & 1 & +x & +\frac{x^2}{2!} & +\frac{x^3}{3!} & +\frac{x^4}{4!} & +\frac{x^5}{5!} & +\frac{x^6}{6!} & +\cdots \\
 & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & -\frac{x^6}{6!} & +\cdots \\ \hline
 & & x & +x^2 & +\frac{x^3}{3!} & & +\frac{x^5}{5!} & +\frac{x^6}{360} & +\cdots \\
 & & x & & -\frac{x^3}{2!} & & +\frac{x^5}{4!} & & +\cdots \\ \hline
 & & & x^2 & +\frac{2x^3}{3} & & -\frac{x^5}{30} & +\frac{x^6}{360} & +\cdots \\
 & & & x^2 & & -\frac{x^4}{2!} & & +\frac{x^6}{4!} & +\cdots \\ \hline
 & & & & \frac{2x^3}{3} & +\frac{x^4}{2} & -\frac{x^5}{30} & -\frac{7x^6}{180} & +\cdots \\
 & & & & \frac{2x^3}{3} & & -\frac{x^5}{3} & & +\cdots \\ \hline
 & & & & & \frac{x^4}{2} + \cdots & & &
\end{array}$$

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \cdots$$

We have done this by hand, but using Maple to do the division is preferable.

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4.8.1 Exercises

For problems 1-7, find the Maclaurin series for $f(x)$, assuming that $f(x)$ has a power series expansion.

1. $f(x) = \sin(4x)$

2. $f(x) = e^{3x}$

3. $f(x) = \cos(2x)$

4. $f(x) = (1-x)^{-3}$

5. $f(x) = x^2 e^x$

6. $f(x) = \cos(x^2)$

7. $f(x) = \int_0^x e^{t^2}\,dt$

For problems 8-13, find the Taylor series for $f(x)$ centered at the given value of a, assuming that $f(x)$ has a power series expansion about a.

8. $f(x) = \sin(x) \quad a = \frac{\pi}{4}$

9. $f(x) = \cos(x) \quad a = \frac{\pi}{3}$

10. $f(x) = \frac{1}{x} \quad a = 2$

11. $f(x) = e^x \quad a = 4$

12. $f(x) = \frac{1}{\sqrt{x}} \quad a = 9$

13. $f(x) = x - x^3 \quad a = -1$

For problems 14-17, use the binomial series to expand the function as a power series.

14. $f(x) = \sqrt{1+x}$

15. $f(x) = \frac{1}{(1+x)^5}$

16. $f(x) = \frac{1}{(3-x)^2}$

17. $f(x) = (1-x)^{\frac{3}{4}}$

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For problems 18 and 19, find the Maclaurin series for $f(x)$.

18. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

19. $f(x) = \arcsin x$

For problems 20-25, use an infinite series to approximate to four decimal places.

20. $\sqrt{e}$

21. $\sin(1^\circ)$

22. $\displaystyle\int_0^1 e^{-x^2}\,dx$

23. $\displaystyle\int_0^{.4} \sqrt{1+x^3}\,dx$

24. $\displaystyle\int_0^1 \frac{\sin(x)}{x}\,dx$

25. $\cos\left(\dfrac{\pi}{24}\right)$

26. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{7^n n!}$.

27. Find the first three non-zero terms of the Maclaurin series (using multiplication of power series) for $f(x) = e^{-5x^2}\sin 3x$.

4.8.2 Answers to Selected Exercises

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(4x)^{2n+1}}{(2n+1)!}$

3. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}$

5. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+2}}{n!}$

7. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}$

9.
$$\cos\frac{\pi}{3}-\frac{\sin\frac{\pi}{3}\,(x-\frac{\pi}{3})}{1!}-\frac{\cos\frac{\pi}{3}\,(x-\frac{\pi}{3})^2}{2!}+\frac{\sin\frac{\pi}{3}\,(x-\frac{\pi}{3})^3}{3!}+\frac{\cos\frac{\pi}{3}\,(x-\frac{\pi}{3})^4}{4!}+\cdots=$$
$$\frac{1}{2}-\frac{\sqrt{3}(x-\frac{\pi}{3})}{2}-\frac{(x-\frac{\pi}{3})^2}{4}+\frac{\sqrt{3}(x-\frac{\pi}{3})^3}{12}+\frac{(x-\frac{\pi}{3})^4}{48}+\cdots$$

11. $\displaystyle\sum_{n=0}^{\infty}\frac{e^4(x-4)^n}{n!}$

13. $-2(x+1)+3(x+1)^2-(x+1)^3$

15. $\displaystyle\sum_{n=0}^{\infty}\binom{-5}{n}x^n=1-5x+\frac{(-5)(-6)x^2}{2!}+\frac{(-5)(-6)(-7)x^3}{3!}+\cdots$

17. $\displaystyle\sum_{n=0}^{\infty}\binom{3/4}{n}(-x)^n=1-\frac{3x}{4}+\frac{(3/4)(-1/4)x^2}{2!}-\frac{(3/4)(-1/4)(-5/4)x^3}{3!}+\cdots$

19. $\displaystyle\sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{x^{2n+1}}{2n+1}$

21. 0.0175

23. 0.4032

25. 0.9914

27. $3x-\dfrac{39}{2}x^3+\dfrac{2481}{40}x^5$

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