1.1 Arc Length

If we work with a "smooth" curve on an interval (derivative is continuous; no abrupt changes in direction), then we can use our Riemann sum model to assist us in finding the length of the curve (arc length).

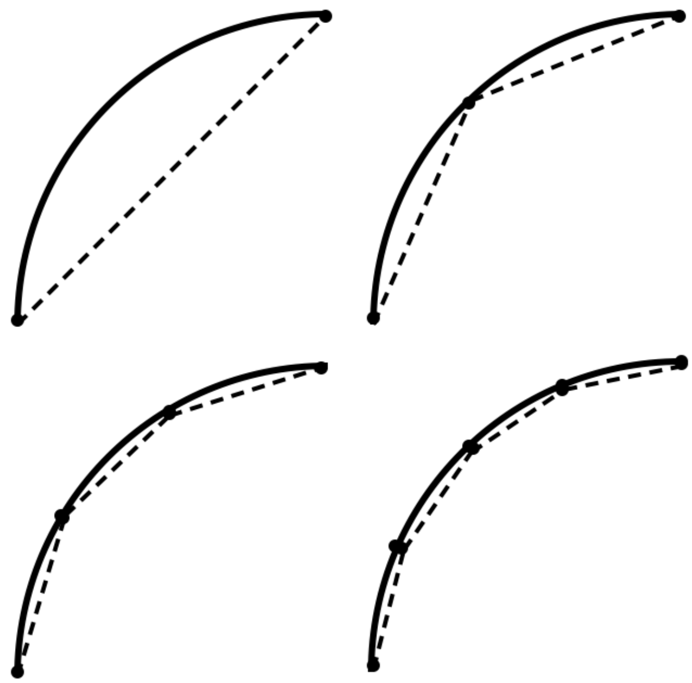

Figure 1

The length of a curve can be approximated by the length of the straight line connecting the endpoints. This approximation can be improved by picking another point on the curve, finding the straight line distances to the endpoints, and adding. We can continue improving the approximation by adding more and more points and straight lines. As the number of line segments increases, the approximation appears to be better - closer and closer to the actual length of the curve.

We will develop three versions of the arc length formula, which can be used when the curve is given to us in different forms.

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Version 1

Let us start with a curve that is the graph of a smooth function $y = f(x)$ from $x = a$ to $x = b$. Dividing $[a, b]$ into n equal subintervals gives us $x_i = a + i(\Delta x)$ where $\Delta x = \frac{b-a}{n}$ and $i = 0, \cdots, n$. Let $P_i = (x_i, f(x_i))$ be $n + 1$ points on the curve.

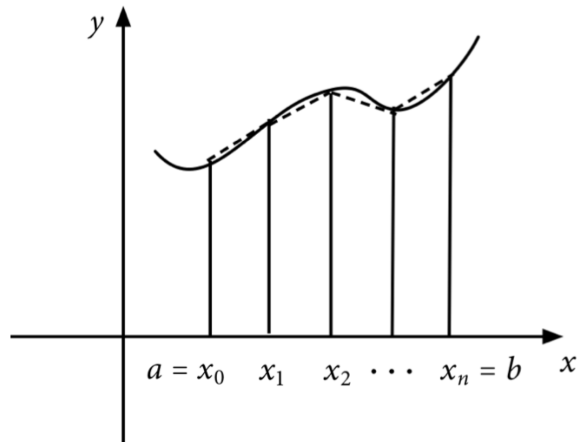

Figure 2

The Pythagorean Theorem gives us lengths of line segments:

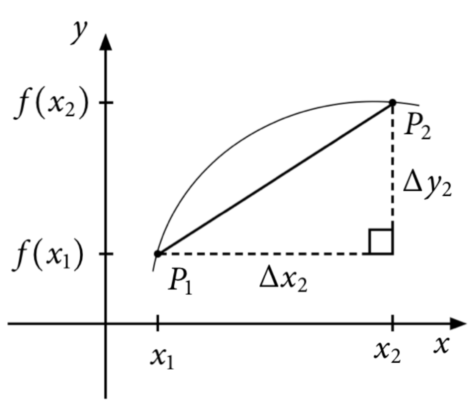

Figure 3

$$d(P_1, P_2) = \sqrt{(\Delta x_2)^2 + (\Delta y_2)^2} = \sqrt{(x_2 - x_1)^2 + (f(x_2) - f(x_1))^2}$$

In general,

$$\begin{aligned} d(P_{i-1}, P_i) &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{\left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)(\Delta x_i)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i \end{aligned}$$

since $\Delta x_i > 0$. Adding these up, we get a Riemann sum approximation of the arc length.

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$$

A good approximation of $\frac{\Delta y_i}{\Delta x_i}$, the slope of a secant line, is the derivative $f'(x_i)$. Thus

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\ \Delta x_i$$

Taking the limit as the number n of points increases without bound gives the exact arc length.

$$\begin{aligned} \text{arc length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\ \Delta x_i \\ &= \int_a^b \sqrt{1 + f'(x)^2}\ dx \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \qquad\qquad a \le b \end{aligned}$$

Version 2

In this case, our curve is the graph of a function $x = f(y)$ from $y = c$ to $y = d$. We are interchanging the roles of x and y. Dividing $[c, d]$ into n equal subintervals gives us $y_i = c + i\Delta y$ where $\Delta y = \frac{d-c}{n}$ and $i = 0, \ldots, n$. Let $P_i = (f(y_i), y_i)$ be $n + 1$ points on the curve.

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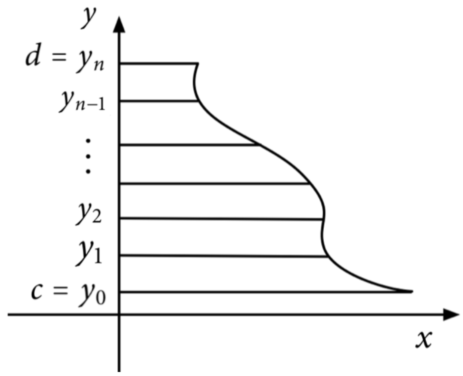

Figure 4

The distance between successive points is

$$\begin{aligned}
d(P_{i-1}, P_i) &= \sqrt{(f(y_i) - f(y_{i-1}))^2 + (y_i - y_{i-1})^2} \\
&= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\
&= \sqrt{\left(\left(\frac{\Delta x_i}{\Delta y_i}\right)^2 + 1\right)(\Delta y_i)^2} \\
&= \sqrt{\left(\frac{\Delta x_i}{\Delta y_i}\right)^2 + 1}\ \Delta y_i
\end{aligned}$$

since $\Delta y_i > 0$. Adding these up, we get a Riemann sum approximation of the arc length.

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta y_i}\right)^2 + 1}\ \Delta y_i$$

Remembering that the roles of x and y are interchanged, a good approximation of $\frac{\Delta x_i}{\Delta y_i}$, the slope of a secant line, is the derivative $f'(y_i)$. Thus

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{(f'(y_i))^2 + 1}\,\Delta y_i$$

Taking the limit as the number n of points increases without bound gives the exact arc length.

$$\begin{aligned}
\text{arc length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(y_i))^2 + 1}\,\Delta y \\
&= \int_c^d \sqrt{(f'(y))^2 + 1}\,dy \\
&= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy
\end{aligned}$$

Note that we are integrating with respect to y from $y = c$ to $y = d$ with $c \le d$.

Version 3

This case is useful when the curve is given parametrically as $x = f(t)$ and $y = g(t)$ with t from a to b. Dividing $[a, b]$ into n equal subintervals gives us $t_i = a + i\Delta t$ where $\Delta t = \frac{b-a}{n}$ and $i = 0, \ldots, n$. Let $P_i = (f(t_i), g(t_i))$ be $n+1$ points on the curve. The distance between successive points is

$$\begin{aligned} d(P_{i-1}, P_i) &= \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2} \\ &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{\left(\frac{(\Delta x_i)^2 + (\Delta y_i)^2}{(\Delta t_i)^2}\right)(\Delta t_i)^2} \\ &= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i \end{aligned}$$

since $\Delta t_i > 0$. Adding these up, we get a Riemann sum approximation of the arc length.

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t$$

When Δt_i is small, $f'(t_i) \approx \frac{\Delta x_i}{\Delta t_i}$ and $g'(t_i) \approx \frac{\Delta y_i}{\Delta t_i}$. Thus

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\ \Delta t_i$$

Taking the limit as the number n of points increases without bound gives the exact arc length.

$$\begin{aligned} \text{arc length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\ \Delta t_i \\ &= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\ dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \end{aligned}$$

Note that we are integrating with respect to t from $t = a$ to $t = b$ with $a \le b$. When we are given parametric equations, we can view this as being given our position as a function of time. This arc length formula gives us our total distance traveled, as seen on an odometer, even if we were to trace the same curve multiple times.

Example 1. *Find the length of the portion of the curve* $f(x) = x^{2/3}$ *from* $(1,1)$ *to* $(8,4)$.

Solution: In this example, we have $y = f(x)$ so we will use Version 1.

$$f'(x) = \frac{2}{3}x^{-1/3} \qquad\qquad (f'(x))^2 = \frac{4}{9}x^{-2/3} = \frac{4}{9x^{2/3}}$$

Since $f(1) = 1^{2/3} = 1$ and $f(8) = 8^{2/3} = 4$, $(1,1)$ and $(8,4)$ are on this curve and we have x from 1 to 8.

$$\begin{aligned}
\text{arc length} &= \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\, dx \\
&= \int_1^8 \sqrt{\frac{4 + 9x^{2/3}}{9x^{2/3}}}\, dx \\
&= \int_1^8 \frac{\sqrt{4 + 9x^{2/3}}}{3x^{1/3}}\, dx \\
&= \frac{1}{3}\int_1^8 x^{-1/3}\sqrt{4 + 9x^{2/3}}\, dx
\end{aligned}$$

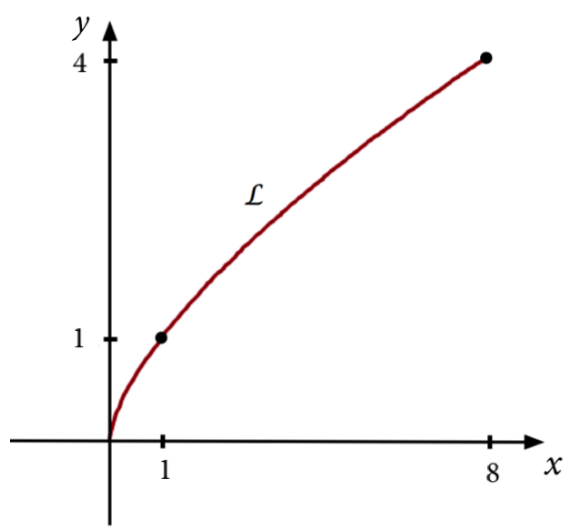

Figure 5

Now let $u = 4 + 9x^{2/3}$ so $du = 9(\frac{2}{3}x^{-1/3})\, dx = 6x^{-1/3}\, dx$. Note that when $x = 1$, $u = 13$, and when $x = 8$, $u = 40$. Then

$$\begin{aligned}
\text{arc length} &= \frac{1}{3}\cdot\frac{1}{6}\int_1^8 \sqrt{4 + 9x^{2/3}}\, 6x^{-1/3}\, dx \\
&= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \\
&= \frac{1}{18}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_{13}^{40} \\
&= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) \approx 7.634
\end{aligned}$$

Alternate Solution: Our function $y = x^{2/3}$ on $[1,8]$ can be rewritten as $x = y^{3/2}$ with y from 1 to 4.

$$\frac{dx}{dy} = \frac{3}{2}y^{1/2} \qquad \left(\frac{dx}{dy}\right)^2 = \frac{9}{4}y$$

Therefore the length of the curve from $y = 1$ to $y = 4$ is

$$\begin{aligned} \text{arc length} &= \int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}y}\, dy \end{aligned}$$

Now let $u = 1 + \frac{9}{4}y$ so $du = \frac{9}{4}\, dy$. Note that when $y = 1$, $u = \frac{13}{4}$, and when $y = 4$, $u = 10$. Then

$$\begin{aligned} \text{arc length} &= \int_1^4 \sqrt{1 + \frac{9}{4}y}\, dy \\ &= \frac{4}{9}\int_{13/4}^{10} u^{1/2}\, du \\ &= \frac{4}{9}\left.\frac{u^{3/2}}{3/2}\right|_{13/4}^{10} \\ &= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) \approx 7.634 \end{aligned}$$

Example 2. *Find the length of the curve* $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ *from* $\left(2, \dfrac{67}{24}\right)$ *to* $\left(1, \dfrac{7}{12}\right)$.

Solution: As in Example 1, we have $y = f(x)$, so we will use Version 1. The arc length from $\left(2, \dfrac{67}{24}\right)$ to $\left(1, \dfrac{7}{12}\right)$ is the same as the arc length from $\left(1, \dfrac{7}{12}\right)$ to $\left(2, \dfrac{67}{24}\right)$. Since Version 1 calls for $a \le b$, we will take x from 1 to 2.

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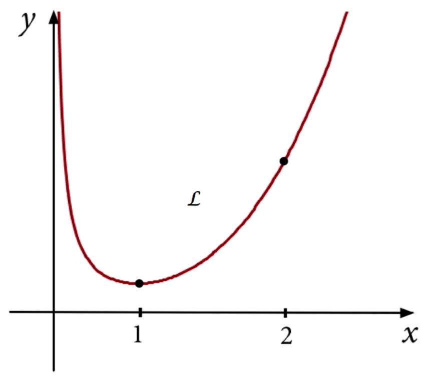

Figure 6

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{3}\left(3x^2\right) + \frac{1}{4}\left(-x^{-2}\right) = x^2 - \frac{1}{4x^2} \\
\left(\frac{dy}{dx}\right)^2 &= \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}
\end{aligned}
$$

$$
\begin{aligned}
\text{arc length} &= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right)}\, dx \\
&= \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx \qquad (1)
\end{aligned}
$$

Detailed steps of the integration

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The result under the radical in Equation (1) happens to be a perfect square. It is $\left(x^2 + \dfrac{1}{4x^2}\right)^2$. So

$$\begin{aligned}
\text{arc length} &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx \\
&= \int_1^2 \left|x^2 + \frac{1}{4x^2}\right|\, dx \\
&= \int_1^2 \left(x^2 + \frac{1}{4x^2}\right)\, dx \\
&= \left(\frac{x^3}{3} + \frac{1}{4}\cdot\frac{x^{-1}}{-1}\right)\Bigg|_1^2 \\
&= \left(\frac{x^3}{3} - \frac{1}{4x}\right)\Bigg|_1^2 \\
&= \frac{59}{24} \approx 2.458
\end{aligned}$$

Example 3. *Find the length of the curve defined parametrically by $x = 3t^2$ and $y = 2t^3$ using t from* 0 *to* 2.

Solution: Since the curve is given parametrically, we will use Version 3.

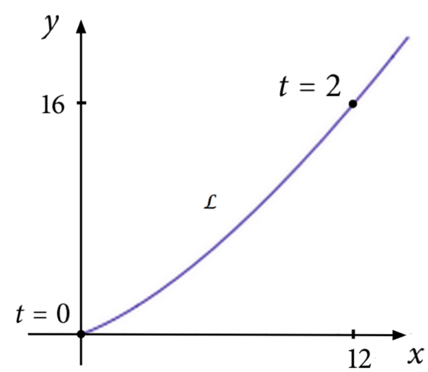

Figure 7

$$\frac{dx}{dt} = 6t \qquad\qquad \frac{dy}{dt} = 6t^2$$

$$\left(\frac{dx}{dt}\right)^2 = 36t^2 \qquad\qquad \left(\frac{dy}{dt}\right)^2 = 36t^4$$

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$$\begin{aligned}
\text{arc length} &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^2 \sqrt{(36t^2) + (36t^4)}\,dt \\
&= \int_0^2 \sqrt{36t^2\left(1+t^2\right)}\,dt \\
&= \int_0^2 \sqrt{1+t^2}\,6t\,dt
\end{aligned}$$

Now let $u = 1 + t^2$ so $du = 2t\,dt$. Then $u = 1$ when $t = 0$ and $u = 5$ when $t = 2$ and

$$\begin{aligned}
\text{arc length} &= 3\int_0^2 \left(1+t^2\right)^{\frac{1}{2}} 2t\,dt \\
&= 3\int_1^5 u^{\frac{1}{2}}\,du \\
&= 3\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^5 \\
&= 2\left(5^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\
&= 2\left(\sqrt{125} - 1\right) \approx 20.361
\end{aligned}$$

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