Evaluate the following iterated integral.

$$\int_0^{\pi/3} \int_{\pi/3}^{\pi/2} \int_0^{\cos(\varphi)} \rho^6 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$\int_0^{\frac{\pi}{3}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\phi) \int_0^{\cos(\phi)} \rho^6\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\phi) \left(\frac{1}{7}\rho^7\right)\Big|_0^{\cos(\phi)} d\phi\, d\theta$$

$$= \frac{1}{7} \int_0^{\frac{\pi}{3}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\phi) \left(\cos^7(\phi)\right) d\phi\, d\theta$$

$$= \frac{1}{7} \int_0^{\frac{\pi}{3}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\phi) \cos^7(\phi)\, d\phi\, d\theta$$

$$= \frac{1}{7} \int_0^{\frac{\pi}{3}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\phi) \cos^7(\phi)\, d\phi\, d\theta$$

$u = \cos(\phi)$ Bounds

$\frac{du}{d\phi} = -\sin(\phi)$ $\frac{\pi}{2} \to 0$

$-du = \sin(\phi)\, d\phi$ $\frac{\pi}{3} \to \frac{1}{2}$

$$= -\frac{1}{7} \int_0^{\frac{\pi}{3}} \int_{\frac{1}{2}}^{0} u^7\, du\, d\theta$$

$$= -\frac{1}{7} \int_0^{\frac{\pi}{3}} \left(\frac{1}{8}u^8\right)\Big|_{\frac{1}{2}}^{0} d\theta$$

$$= -\frac{1}{7} \int_0^{\frac{\pi}{3}} \left(0 - \frac{1}{8}\left(\frac{1}{2}\right)^8\right) d\theta$$

$$= -\frac{1}{7} \int_0^{\frac{\pi}{3}} -\frac{1}{8 \cdot 2^8}\, d\theta$$

$$= \frac{1}{56 \cdot 2^8} \int_0^{\frac{\pi}{3}} 1 \, d\theta$$

$$= \frac{\pi}{168 \cdot 2^8}$$

Find the spherical coordinate expression for the function $F(x, y, z)$.

$$F(x, y, z) = x^2 y^4 \sqrt{x^2 + y^2 + z^2}$$

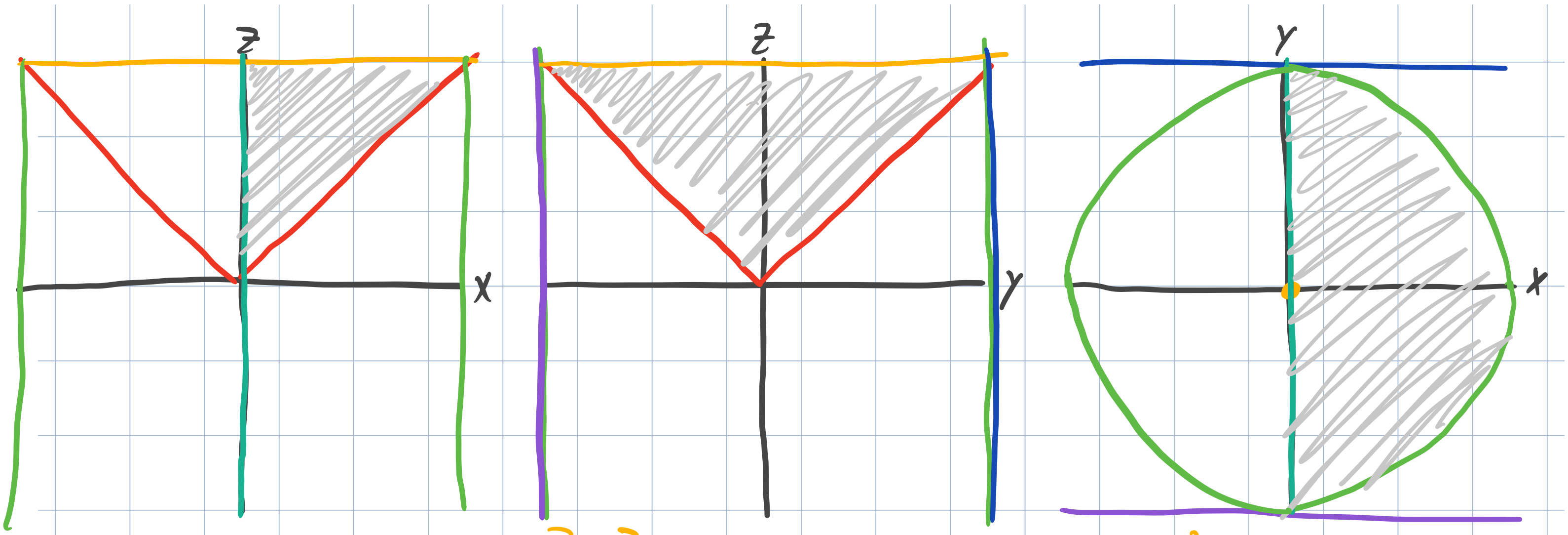
$f(\rho, \theta, \varphi) =$

$F(x, y, z)$

$= (\rho \sin(\phi) \cos(\theta))^2 (\rho \sin(\phi) \sin(\theta))^4 \rho$

Evaluate the following integral by first changing to spherical coordinates.

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{3} z \, dz \, dx \, dy$$




| | | |
|---|---|---|
| $z = 3$ | $z = 3$ | $0 = x^2 + y^2$ |
| $z = \sqrt{x^2}$ | $z = \sqrt{y^2}$ | $x = 0$ |
| $x = 0$ | $0 = 9 - y^2$ | $x = \sqrt{9 - y^2}$ |
| $x = \sqrt{9}$ | $y = \pm 3$ | $x^2 + y^2 = 9$ |

$= \pm 3$ $\quad y=3 \quad y=3$

$y=-3 \quad y=-3$

Bounded By

$x^2+y^2=9$

$z^2=x^2+y^2$

$z=3$

$z=0$

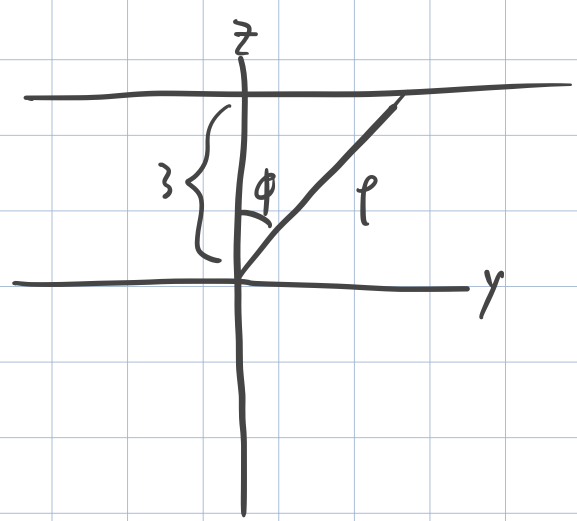
$x=0$

Bounds in spherical

$0 \le \theta \le \pi$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \rho \le 3\sec(\phi)$




$\cos(\phi) = \frac{3}{\rho} \qquad \rho = 3\sec(\phi)$

Integral

$$\int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_0^{3\sec(\phi)} (\rho\cos(\phi))(\rho^2\sin(\phi))\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{4}} \cos(\phi)\sin(\phi)\int_0^{3\sec(\phi)} \rho^3\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{4}} \cos(\phi)\sin(\phi)\left(\tfrac{1}{4}\rho^4\right)\Big|_0^{3\sec(\phi)}\,d\phi\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\int_0^{\frac{\pi}{4}} \cos(\phi)\sin(\phi)(3\sec(\phi))^4\,d\phi\,d\theta$$

$$= \frac{3^4}{4}\int_0^{\pi}\int_0^{\frac{\pi}{4}} \cancel{\cos(\phi)}\sin(\phi)\frac{1}{\cos^{\cancel{4}\,3}(\phi)}\,d\phi\,d\theta$$

$$= \frac{3^4}{4}\int_0^{\pi}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3(\phi)}\sin(\phi)\,d\phi\,d\theta$$

$U = \cos(\phi)$ Bounds

$\frac{du}{d\phi} = -\sin(\phi)$ $\frac{\pi}{4} \to \frac{\sqrt{2}}{2}$

$-du = \sin(\phi)\,d\phi$ $0 \to 1$

$$= -\frac{3^4}{4}\int_0^{\pi}\int_1^{\frac{\sqrt{2}}{2}} U^{-3}\,d\phi\,d\theta$$

$$= -\frac{3^4}{4}\int_0^{\pi}\left(-\tfrac{1}{2}U^{-2}\right)\Big|_1^{\frac{\sqrt{2}}{2}}\,d\theta$$

$$= \frac{3^4}{8}\int_0^{\pi}\left(\frac{1}{U^2}\right)\Big|_1^{\frac{\sqrt{2}}{2}}\,d\theta$$

$$= \frac{3^4}{8}\int_0^{\pi}\left(\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} - 1\right)$$

$$= \frac{3^4}{8}\int_0^{\pi} \frac{4}{2} - 1\,d\theta$$

$$= \frac{3^4}{8} \int_0^{\pi} 1 \, d\theta$$

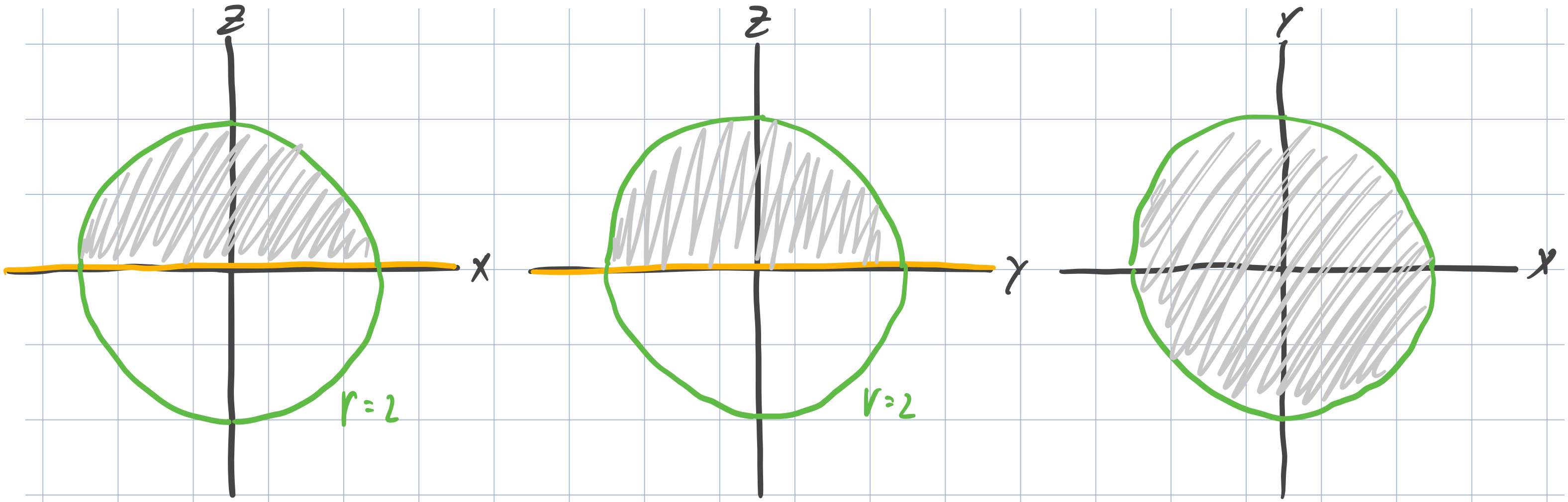
$$= \frac{3^4}{8} \cdot \pi$$

$$= \frac{81\pi}{8}$$

A solid body with mass density $\sigma(x, y, z) = \sqrt{x^2 + y^2}$ kg/m$^3$ occupies the region in space below the sphere $x^2 + y^2 + z^2 = 4$ and above the $xy$-plane. Find the total mass $M$ and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the solid.

$M =$ [ ] kg

$(\bar{x}, \bar{y}, \bar{z}) =$ ( [ ] )




$z=0$ $\qquad$ $z=0$ $\qquad$ $x^2+y^2=4$

$z^2+x^2=4$ $\qquad$ $z^2+y^2=4$

Bounds

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

$0 \le \rho \le 2$

# Part One

$\sigma(x,y,z) = \rho \sin(\phi)$

$M = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (\rho\sin(\phi))(\rho^2\sin(\phi))\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 \rho^3\sin^2(\phi)\,d\rho\,d\phi\,d\theta$

Evaluated with calculator

$= 2\pi^2$

# Part Two

Simple center of mass things, skipped to save time

$(\bar{x},\bar{y},\bar{z}) = (0,0,\frac{32}{15\pi})$

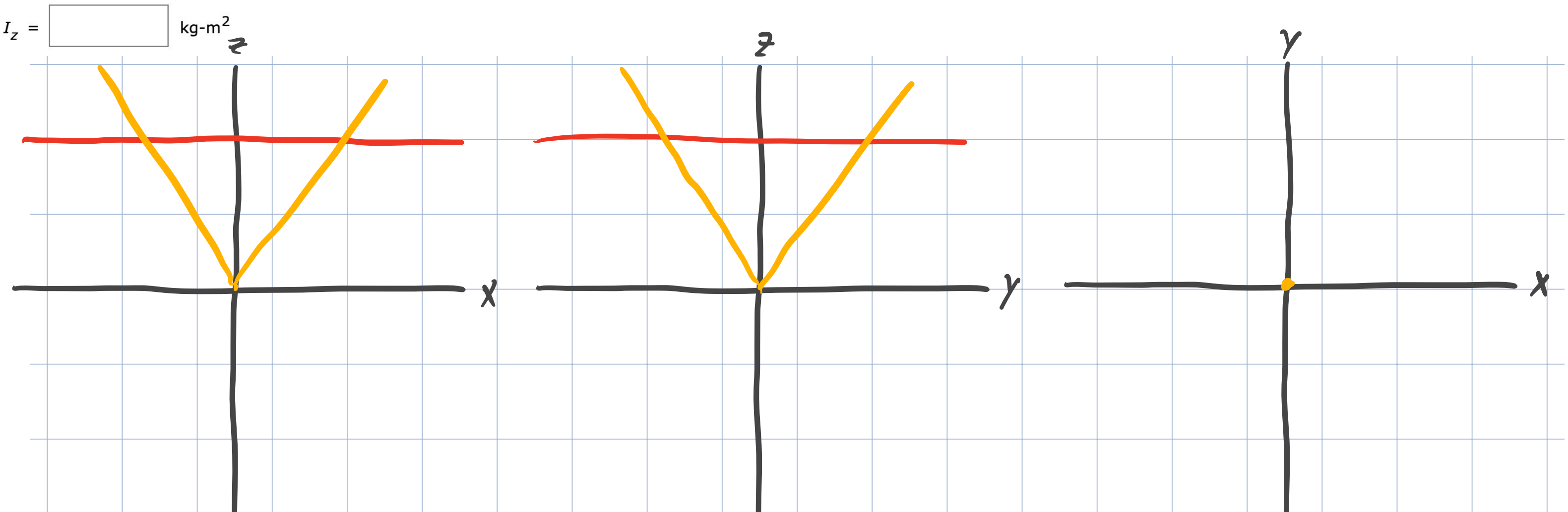
Use spherical coordinates to find the total mass $M$ and the moments of inertia $I_x$, $I_y$, and $I_z$ of the solid bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and the plane $z = 4$ if the mass density of the solid is $\sigma(x, y, z) = z$ kg/m$^3$.

$M$ = [ ] kg

$I_x$ = [ ] kg-m$^2$

$I_y$ = [ ] kg-m$^2$

$I_z$ = [ ] kg-m$^2$



$z=4$

$z=4$

$0=3x^2+3y^2$

$z=\sqrt{3x^2}$

$z=\sqrt{3y^2}$

$0=x^2+y^2$

$z=\sqrt{3}\sqrt{x^2}$

$z=\sqrt{3}\sqrt{y^2}$

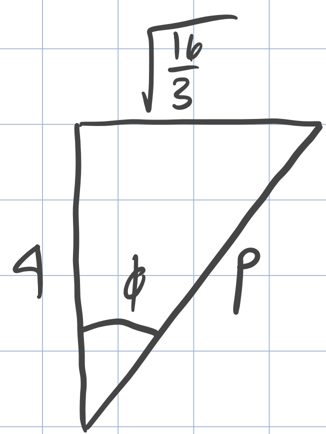
Intersection of $z=4$ and $z=\sqrt{3x^2}$

$4=\sqrt{3x^2}$

$16=3x^2$

$x=\pm\sqrt{\frac{16}{3}}$

Bands




$0\le\theta\le 2\pi$

$0\le\phi\le \arctan\left(\frac{\sqrt{\frac{16}{3}}}{4}\right)$

$0\le\rho\le 4\sec(\phi)$

$\tan(\phi)=\frac{\sqrt{\frac{16}{3}}}{4}$

$\phi=\arctan\left(\frac{\sqrt{\frac{16}{3}}}{4}\right)$

$\cos(\phi)=\frac{4}{\rho}$

$\rho=4\sec(\phi)$

Part One

$$M=\int_0^{2\pi}\int_0^{\arctan\left(\frac{\sqrt{\frac{16}{3}}}{4}\right)}\int_0^{4\sec(\phi)}(\rho\cos(\phi))(\rho^2\sin(\phi))\,d\rho\,d\phi\,d\theta$$

Evaluated with calculator

$$= \frac{64\pi}{3}$$

Find the volume of the region in the first octant cut from the solid sphere $\rho \le 5$ by the half-planes $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$.

cubic units

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$$

$$0 \le \phi \le \frac{\pi}{2}$$

$$0 \le \rho \le 5$$

$$V = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \int_0^5 1(\rho^2 \sin(\phi))\, d\rho\, d\phi\, d\theta$$

Evaluated with calculator

$$= \frac{125\pi}{36}$$

Find the average value of $F(x, y, z) = z$ over the region bounded below by the $xy$-plane, on the sides by the sphere $x^2 + y^2 + z^2 = 16$, and bounded above by the cone $\varphi = \frac{\pi}{3}$.

Bounds

$$0 \le \theta \le \frac{\pi}{2}$$

$$\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$$

$$0 \le \rho \le 4$$

$$\text{Avg} = \frac{\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^4 (\rho\cos(\phi))(\rho^2\sin(\phi))\, d\rho\, d\phi\, d\theta}{\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^4 1\,(\rho^2\sin(\phi))\, d\rho\, d\phi\, d\theta}$$

Evaluated with Calculator

$$= \frac{3}{4}$$

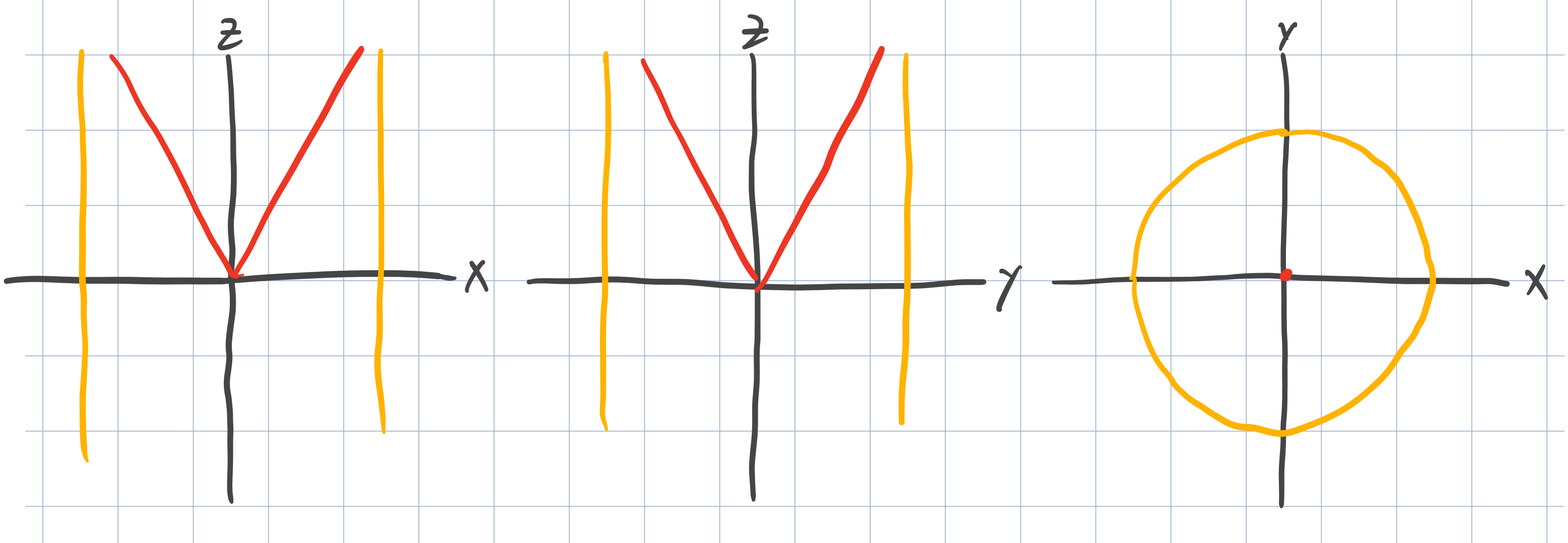
Use set notation and spherical coordinates as in the equation

$$F = \{(\rho, \theta, \varphi) \mid \theta_1 < \theta < \theta_2, \varphi_1 < \varphi < \varphi_2, g(\theta, \varphi) < \rho < h(\theta, \varphi)\}$$

to describe each region.

The region in the first octant that is below the cone $z = \sqrt{3x^2 + 3y^2}$ and inside the cylinder $x^2 + y^2 = 49$.

$F = \left\{(\rho, \theta, \varphi) \mid 0 < \theta < \underline{\quad\quad}, \underline{\quad\quad} < \varphi < \frac{\pi}{2}, 0 < \rho < \underline{\quad\quad}\right\}$




$z = \sqrt{3}\sqrt{x^2}$ $\quad$ $z = \sqrt{3}\sqrt{y^2}$ $\quad$ $C = x^2 + y^2$

$x = \pm 7$ $\qquad$ $y = \pm 7$ $\qquad$ $x^2 + y^2 = 49$

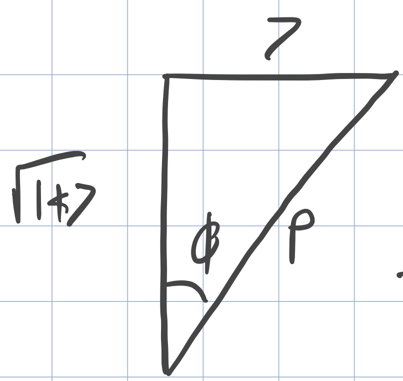
Intersection of $x = 7$ and $z = \sqrt{3x^2}$

$z = \sqrt{3 \cdot 49}$

$z = \sqrt{147}$

$x = 7$

Bounds



$0 \le \theta \le \frac{\pi}{2}$

$0 \le \phi \le \arctan\left(\frac{7}{\sqrt{147}}\right)$

$0 \le \rho \le 7\csc(\phi)$

$\tan(\phi) = \left(\frac{7}{\sqrt{147}}\right)$

$\phi = \arctan\left(\frac{7}{\sqrt{147}}\right)$

$\sin(\phi) = \frac{7}{\rho}$

$\rho = 7\csc(\phi)$