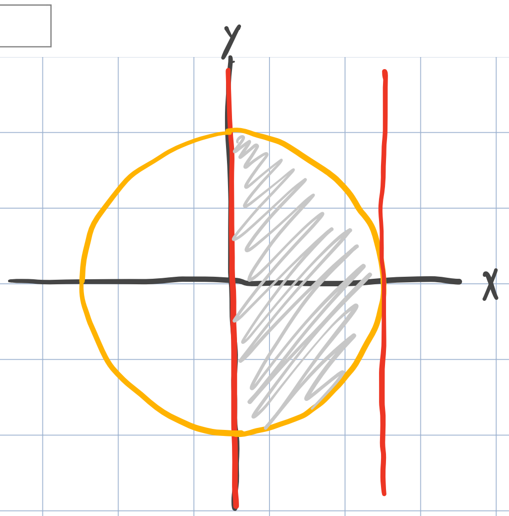
Use a sketch to assist in transforming the following integral to polar coordinates. Then evaluate the resulting integral.

$$\int_0^{\sqrt{6}} \int_{-\sqrt{6-x^2}}^{\sqrt{6-x^2}} e^{x^2+y^2}\, dy\, dx$$



$0 \le x \le \sqrt{6}$

$-\sqrt{6-x^2} \le y \le \sqrt{6-x^2}$

Bounded by

$x = 0$

$x = \sqrt{6}$

$x^2 + y^2 = 6$

Convert to polar

$0 \le r \le \sqrt{6}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Integral

$$\int_0^{\sqrt{6}} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} f(r\cos(\theta), r\sin(\theta))\, r\, d\theta\, dr$$

$$= \int_0^{\sqrt{6}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{r^2\cos^2(\theta)+r^2\sin^2(\theta)}\right) r\, d\theta\, dr$$

$$= \int_0^{\sqrt{6}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r e^{r^2}\, d\theta\, dr$$

$$= \int_0^{\sqrt{6}} r e^{r^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta\, dr$$

$$= \int_0^{\sqrt{6}} r e^{r^2} \left(\frac{\pi}{2} + \frac{\pi}{2}\right) dr$$

$= \pi \int_0^{\sqrt{6}} r e^{r^2} \, dr$

$u = r^2$

Bounds

$\frac{du}{dr} = 2r$

$\sqrt{6} \rightarrow 6$

$\frac{1}{2} du = r \, dr$

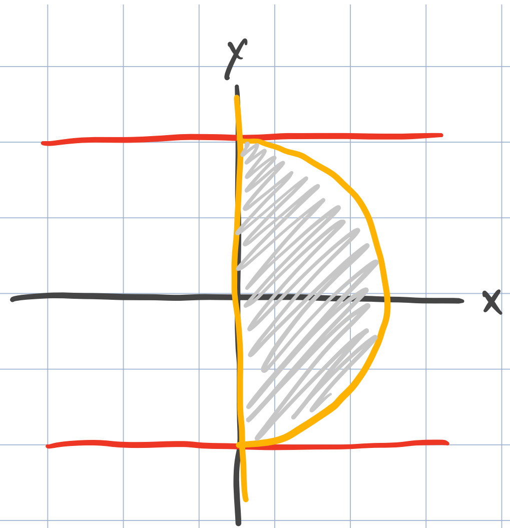
$0 \rightarrow 0$

$= \frac{\pi}{2} \int_0^6 e^u \, du$

$= \frac{\pi}{2} (e^6 - 1)$

Use a sketch to assist in transforming the following integral to polar coordinates. Then evaluate the resulting integral.

$$\int_{-\sqrt{5}}^{\sqrt{5}} \int_0^{\sqrt{5 - y^2}} \cos(x^2 + y^2) \, dx \, dy$$



$-\sqrt{5} \leq y \leq \sqrt{5}$

$0 \leq x \leq \sqrt{5 - y^2}$

Bounds

$x = 0$

$x = \sqrt{5 - y^2}$

To Polar

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Intersection

$x = \sqrt{5 - \sqrt{5}^2}$

$= 0$

$y = \pm\sqrt{5}$ $\qquad$ $0 \le r \le \sqrt{5}$

Integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{5}} \cos(r^2)\, r\, dr\, d\theta$$

$u = r^2$ $\qquad$ Bands

$\frac{1}{2} du = r\,dr$ $\qquad$ $\sqrt{5} \to 5$ $\qquad$ $0 \to 0$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{5} \cos(u)\, du\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(u)\Big|_0^5\, d\theta$$

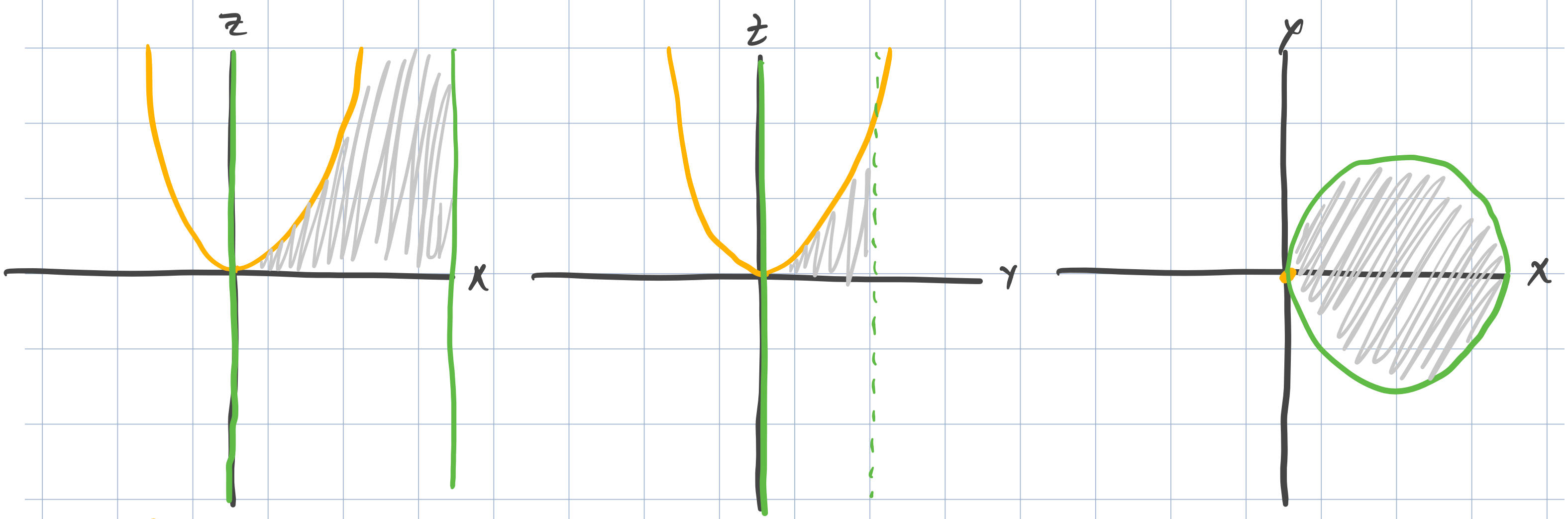
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(5)\, d\theta$$

$$= \frac{1}{2}\sin(5)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta$$

$$= \frac{1}{2}\sin(5)\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}\sin(5)$$

Use polar coordinates to compute the volume of the solid under the surface $z = x^2 + y^2$ and above the region $x^2 + y^2 = 5x$ in the xy-plane.



$z = x^2$

$z = y^2$

$0 = x^2 + y^2$

$x^2 = 5x$

$x^2 - 5x = 0$

$x = 5$

$x = 0$

$y^2 = 0$

$y = 0$

$x^2 + y^2 = 5x$

$y^2 = -x^2 + 5x$

$y = \pm\sqrt{-x^2 + 5x}$

Cylindrical Bands

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

r band

$x^2 + y^2 = 5x$

$r^2 = 5r\cos(\theta)$

$r = 5\cos(\theta)$

$0 \le r \le 5\cos(\theta)$

$0 \le z \le r^2$

Integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{5\cos(\theta)} \int_{0}^{r^2} 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{5\cos(\theta)} r \int_{0}^{r^2} 1 \, dz \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{5\cos(\theta)} r^3 \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_{0}^{5\cos(\theta)} d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (5\cos(\theta))^4 \, d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 5^4 \cos^4(\theta) \, d\theta$$

$$= \frac{5^4}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2(\theta))^2 \, d\theta$$

$$= \frac{5^4}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1}{2}(1+\cos(2\theta))^2 \, d\theta$$

$$= \frac{5^4}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(1+\cos(2\theta))^2 \, d\theta$$

$$= \frac{5^4}{16} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1^2 + 2(\cos(2\theta)) + \cos^2(2\theta) \, d\theta$$

$$= \frac{5^4}{16} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta)) \, d\theta$$

$$= \frac{5^4}{16} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \frac{1}{2} + \frac{1}{2}\cos(4\theta)\, d\theta$$

$$= \frac{5^4}{16} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\, d\theta$$

$$= \frac{5^4}{16}\left(\frac{3}{2}\theta + \frac{2}{2}\sin(2\theta) + \frac{1}{8}\sin(4\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{5^4}{16}\left[\left(\frac{3}{2}\frac{\pi}{2} + \cancel{\sin(\pi)} + \frac{1}{8}\cancel{\sin(2\pi)}\right) - \left(-\frac{3}{2}\frac{\pi}{2} + \cancel{\sin(-\pi)} + \frac{1}{8}\cancel{\sin(-2\pi)}\right)\right]$$

$$= \frac{5^4}{16}\left[\frac{3\pi}{4} + \frac{3\pi}{4}\right]$$

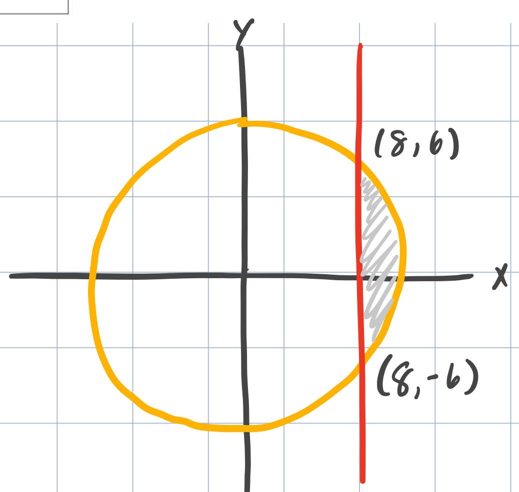
$$= \frac{5^4}{16}\left(\frac{6\pi}{4}\right)$$

$$= \frac{3750\pi}{64}$$

$$= \frac{1875\pi}{32}$$

Use polar coordinates to calculate the area of the region.

$R = \{(x, y) \mid x^2 + y^2 \le 100, x \ge 8\}$



$x^2 + y^2 = 100 \rightarrow r^2 = 10$

$x = 8 \rightarrow r\cos(\theta) = 8$

$r = 8\sec(\theta)$

Intersection of Bounds in Cartesian

$8^2 + y^2 = 100$

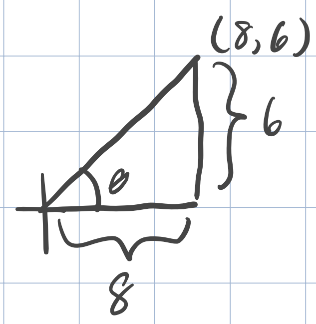
$y^2 = 36$

$y = \pm 6$

Intersection of Bounds in Polar

$\tan(\theta) = \frac{6}{8}$

$\theta = \arctan(\frac{6}{8})$



Bounds in Polar

$-\arctan(\frac{6}{8}) \le \theta \le \arctan(\frac{6}{8})$

$8\sec(\theta) \le r \le 10$

Integral

$$\int_{-\arctan(\frac{6}{8})}^{\arctan(\frac{6}{8})} \int_{8\sec(\theta)}^{10} 1 \cdot r \, dr \, d\theta$$

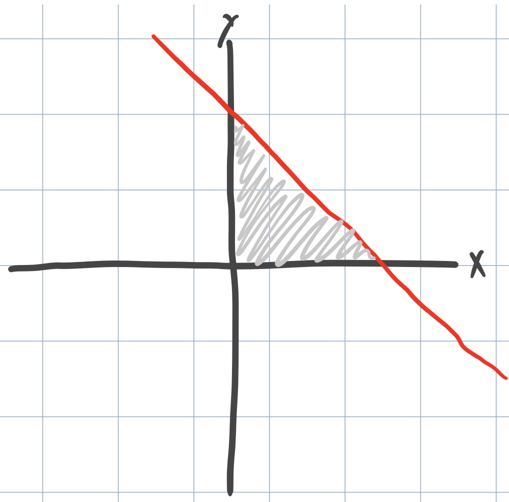
$$= \int_{-\arctan(\frac{6}{8})}^{\arctan(\frac{6}{8})} \left(\frac{1}{2} r^2\right)\Big|_{8\sec(\theta)}^{10} d\theta$$

$$= \frac{1}{2} \int_{-\arctan(\frac{6}{8})}^{\arctan(\frac{6}{8})} 100 - 64\sec^2(\theta) \, d\theta$$

$$= \frac{1}{2}\left(100\theta - 64\tan(\theta)\right)\Big|_{-\arctan\left(\frac{6}{8}\right)}^{\arctan\left(\frac{6}{8}\right)}$$

$$= \frac{1}{2}\Big[\left(100\arctan\left(\tfrac{6}{8}\right) - 64\tan\left(\arctan\left(\tfrac{6}{8}\right)\right)\right) - \Big(-100\arctan\left(\tfrac{6}{8}\right)$$
$$- 64\tan\left(-\arctan\left(\tfrac{6}{8}\right)\right)\Big)\Big)$$

Find the average value of $F(x, y) = \frac{x^2 + 2xy + y^2}{x^2 + y^2}$ over the region in the first quadrant bounded by the coordinate axes and the line $x + y = 4$.



$y = -x + 4$

Bounds

$0 \le x \le 4$

$0 \le y \le -x + 4$

Convert Bounds to Polar

$0 \le \theta \le \frac{\pi}{2}$

r Bound

$r\cos(\theta) + r\sin(\theta) = 4$

$r = \frac{4}{(\cos(\theta) + \sin(\theta))}$

$0 \le r \le \frac{4}{(\cos(\theta) + \sin(\theta))}$

Integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{4}{\cos(\theta)+\sin(\theta)}} F(x,y)\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{4}{\cos(\theta)+\sin(\theta)}} \left(\frac{r^2 + 2(r\cos(\theta))(r\sin(\theta))}{r^2} \right) r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{4}{\cos(\theta)+\sin(\theta)}} \left(\frac{\cancel{r^2} + 2\cancel{r^2}\cos(\theta)\sin(\theta)}{\cancel{r^2}} \right) r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{4}{\cos(\theta)+\sin(\theta)}} r + 2r\cos(\theta)\sin(\theta) \, dr \, d\theta$$

Evaluated with calculator

$$= 4\pi$$

$$\text{Average} = \frac{4\pi}{A}$$

$$= \frac{4\pi}{8}$$

$$= \frac{\pi}{2}$$