Problem (4b) gives 5 points, the others give 10 each. In problem 1, it suffices to state the answer as the difference of two values of a suitable function, that is, something of the form: arc length $= F(y)|_{y=a}^{y=b}$.

IMPORTANT: remember to include the units of measure in the answers to problems (2)–(4).

(1) Find the arc length of the curve defined by $2xy^3 - y^8 = \frac{1}{15}$ from $y = 1$ to $y = 2$.

(2) A mass is attached to a spring with spring constant $k = 20$ N/m. How much force is required to stretch the spring to 20 cm past its resting position? How much work is made to stretch the spring by an *additional* 80 cm?

(3) An inverted cone-shaped tank is 3 ft tall and is filled with water (weight density $\rho = 62.4$ lb/ft^3) till 1 ft from the top; the radius of the circle at the very top is 2 ft long. How much work is required to pump all of the water 2 ft above the top of the tank?

(4) On one of the walls of a tank full of a liquid with weight density $\rho = 55$ lb/ft^3 there is a gate in the shape of an isosceles triangle (vertical and with the base at the bottom). The height and the base of the triangle measure 5 ft and 4 ft respectively.

  (a) Measure the total force exerted by the liquid on the gate if the water level is 1 ft above the triangle.

  (b) (for bonus points) Measure the total force exerted by the liquid on the gate if the water level is L ft above the triangle, where L is a generic positive number.

(5) Find the moments about the X and Y axes and the center of mass of the homogeneous lamina with density $\rho = 1$ and delimited by $x = 1$, $x = 2$, $y = 0$, and $y = \frac{1}{x}$.

(1) Find the arc length of the curve defined by $2xy^3 - y^8 = \frac{1}{15}$ from $y = 1$ to $y = 2$.

$$2xy^3 = \frac{1}{15} + y^8$$

$$x = \frac{\frac{1}{15} + y^8}{2y^3}$$

$$x = \frac{1}{15} \cdot \frac{1}{2y^3} + \frac{y^8}{2y^3}$$

$$x = \frac{1}{30y^3} + \frac{y^5}{2}$$ from $y = 1$ to $y = 2$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = \frac{1}{30} y^{-3} + \frac{1}{2} y^5$$

$$\frac{dx}{dy} = \frac{1}{30}(-3y^{-4}) + \frac{1}{2}(5y^4)$$

$$= -\frac{1}{10} y^{-4} + \frac{5}{2} y^4$$

$$= -\frac{1}{10y^4} + \frac{5y^4}{2}$$

$$= \frac{-2 + 50y^8}{20y^4}$$

$$= \frac{-1 + 25y^8}{10y^4}$$

$$= \frac{25y^8 - 1}{10y^4}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{25y^8 - 1}{10y^4}\right)^2$$

$$= \left(\frac{25y^8 + 1}{10y^4}\right)^2$$

$$\text{Arc Length} = \int_1^2 \sqrt{\left(\frac{25y^8+1}{10y^4}\right)^2} \, dy$$

$$= \int_1^2 \frac{25y^8+1}{10y^4} \, dy$$

$$= \int_1^2 \frac{\overset{5}{\cancel{25}}\,\overset{4}{\cancel{y^8}}}{\underset{2}{\cancel{10}}\,\cancel{y^4}} + \frac{1}{10y^4} \, dy$$

$$= \int_1^2 \frac{5}{2}y^4 + \frac{1}{10}y^{-4} \, dy$$

$$= \frac{1}{2}\int_1^2 5y^4 + \frac{1}{5}y^{-4} \, dy$$

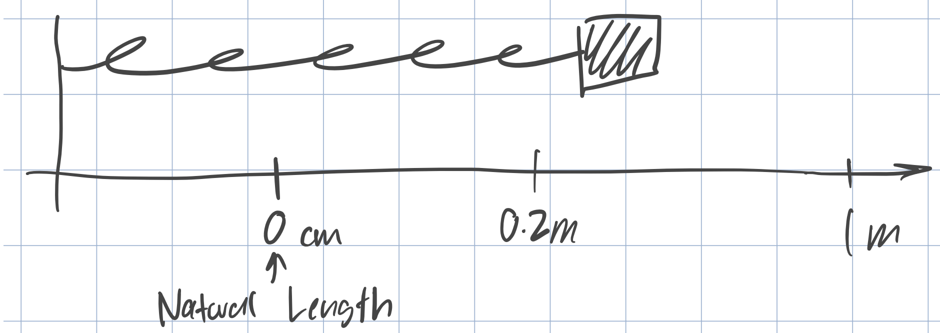
$$= \frac{1}{2}\left(5 \cdot \frac{1}{5}y^5 + \frac{1}{5} \cdot \frac{1}{-3}y^{-3} \Big|_1^2\right)$$

$$= \frac{1}{2}\left(y^5 - \frac{1}{15}y^{-3} \Big|_1^2\right)$$

$$= \frac{1}{2}\left[\left((2)^5 - \frac{1}{15}(2)^{-3}\right) - \left((1)^5 - \frac{1}{15}(1)^{-3}\right)\right]$$

$$= \frac{3727}{240}$$

(2) A mass is attached to a spring with spring constant $k = 20$ N/m. How much force is required to stretch the spring to 20 cm past its resting position? How much work is made to stretch the spring by an *additional* 80 cm?



$$W_i = F_i \cdot d_i$$

$$F_i = kx$$
$$d = \Delta x$$

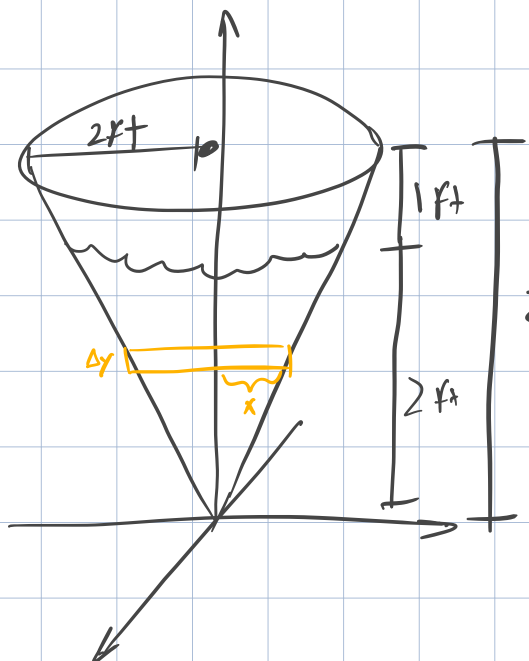
$$W_i = kx\,\Delta x$$

$$W = k\int_a^b x\,dx$$

$$F = kx$$
$$= 20 \cdot 0.2$$
$$= 4\text{ N}$$

$$W = 20\int_{0.2}^{1} x\,dx$$
$$= 20\left(\frac{1}{2}x^2\Big|_{0.2}^{1}\right)$$
$$= 10\left(x^2\Big|_{0.2}^{1}\right)$$
$$= 10(1^2 - 0.2^2)$$
$$= 9.6\text{ J}$$

(3) An inverted cone-shaped tank is 3 ft tall and is filled with water (weight density $\rho = 62.4\ \text{lb}/\text{ft}^3$) till 1 ft from the top; the radius of the circle at the very top is 2 ft long. How much work is required to pump all of the water 2 ft above the top of the tank?



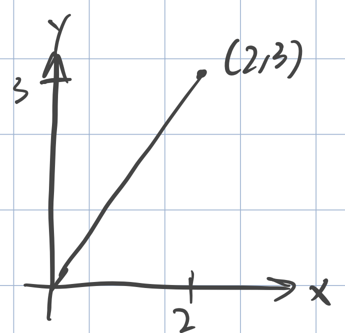
$W_i = F_i \cdot d_i$

$d_i = 3 + 2 - y_i$

$= 5 - y_i$

$F_i = \rho V$

$= \rho\left(\pi \frac{4}{9} y^2\right)\Delta y$



$m = \frac{3-0}{2-0} = \frac{3}{2}$

$y = \frac{3}{2}x$

$x = \frac{2}{3}y$

$W_i = \rho\left(\pi \frac{4}{9} y^2\right)(5-y)\,\Delta y$

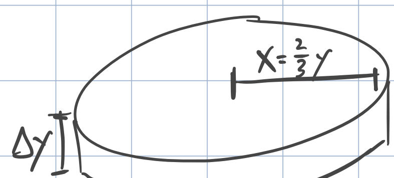
$W = \rho \pi \frac{4}{9} \int_0^2 y^2(5-y)\,dy$

$= \frac{4\rho\pi}{9} \int_0^2 5y^2 - y^3\,dy$

$= \frac{4\rho\pi}{9}\left(\frac{5}{3}y^3 - \frac{1}{4}y^4 \Big|_0^2\right)$

$= \frac{4\rho\pi}{9}\left(\frac{5}{3}(2^3) - \frac{1}{4}(2^4)\right)$

$= 813.184\ J$



$V_i = \pi r^2 h$

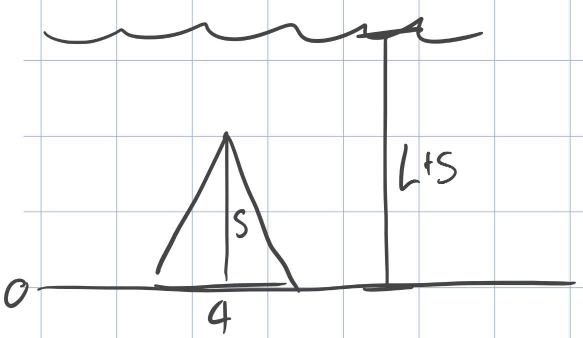
$= \pi\left(\frac{2}{3}y\right)^2 \Delta y$

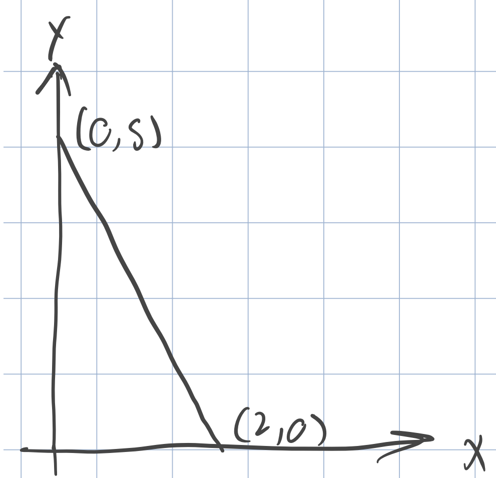
$= \pi \frac{4}{9} y^2 \Delta y$

(4) On one of the walls of a tank full of a liquid with weight density $\rho = 55$ lb/ft^3 there is a gate in the shape of an isosceles triangle (vertical and with the base at the bottom). The height and the base of the triangle measure 5 ft and 4 ft respectively.

(a) Measure the total force exerted by the liquid on the gate if the water level is 1 ft above the triangle.

(b) (for bonus points) Measure the total force exerted by the liquid on the gate if the water level is L ft above the triangle, where L is a generic positive number.





$m = \frac{0-5}{2-0} = -\frac{5}{2}, \; b = 5$

$y = -\frac{5}{2}x + 5$

$-\frac{5}{2}x = y - 5$

$x = -\frac{2(y-5)}{5}$

$F_i = A_i d_i \rho$

$d_i = L + 5 - y_i$

$A = 2x_i \Delta y$

$= 4(1 - \frac{1}{5}y)\Delta y$

$F_i = \rho(4(1-\frac{1}{5}y))(L+5-y_i)\Delta y$

$= 4\rho(1-\frac{1}{5}y)(L+5-y_i)\Delta y$

$F = 4\rho \cdot \int_0^5 (1-\frac{1}{5}y)(L+5-y_i)\Delta y$

$F = 4(55)\cdot \int_0^5 (1-\frac{1}{5}y)(6+y)\,dy$

$= 220 \cdot \int_0^5 \frac{1}{5}y^2 - \frac{11}{5}y + 6 \; dy$

$X = -2\left(\frac{1}{5}Y - 1\right)$

$X = 2\left(1 - \frac{1}{5}Y\right)$

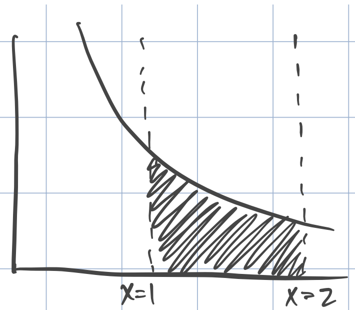
$= 220 \cdot \left(\frac{1}{5}\left(\frac{1}{3}y^3\right) - \frac{11}{5}\left(\frac{1}{2}y^2\right) + 6y \Big|_0^5\right)$

$= 220\left(\frac{1}{15}y^3 - \frac{11}{10}y^2 + 6y \Big|_0^5\right)$

$= 220\left(\frac{1}{15}(5)^3 - \frac{11}{10}(5)^2 + 6(5)\right)$

$= 2383.333$

(5) Find the moments about the X and Y axes and the center of mass of the homogeneous lamina with density $\rho = 1$ and delimited by $x = 1$, $x = 2$, $y = 0$, and $y = \frac{1}{x}$.



$M_y = \int_a^b x f(x)\, dx$

$= \int_1^2 x\left(\frac{1}{x}\right) dx$

$= \int_1^2 1\, dx$

$= 2 - 1 = 1$

$M_x = \int_a^b \frac{1}{2} f(x) f(x)\, dx$

$= \frac{1}{2}\int_1^2 \left(\frac{1}{x}\right)^2 dx$

$m = \int_a^b f(x)\, dx$

$= \int_1^2 x^{-1}\, dx$

$= \ln(x)\Big|_1^2$

$= \ln(2) - \ln(1)$

$= \ln(2)$

$\bar{X} = \frac{M_Y}{m}$

$= \frac{1}{\ln(2)}$

$\bar{Y} = \frac{M_X}{m}$

$$= \frac{1}{2} \int_1^2 x^{-2}\, dx$$

$$= \frac{1}{2} \left(-\frac{1}{1} x^{-1} \Big|_1^2 \right)$$

$$= -\frac{1}{2} \left(\frac{1}{x} \Big|_1^2 \right)$$

$$= -\frac{1}{2} \left(\frac{1}{2} - 1 \right)$$

$$= -\frac{1}{2} \left(-\frac{1}{2} \right.$$

$$= \frac{1}{4}$$

$$= \frac{\frac{1}{4}}{\ln(2)}$$

$$= \frac{1}{4} \cdot \frac{1}{\ln(2)}$$

$$= \frac{1}{4\ln(2)}$$

$$COM = \left(\frac{1}{\ln(2)}, \frac{1}{4\ln(2)} \right)$$

All problems are worth 10 points. For problems 2 and 5, the acceleration of gravity is 9.8 m/s^2.

1. Compute the arc length of the curve defined parametrically by $x = e^t \cos t$ and $y = e^t \sin t$ from $t = 0$ to $t = \pi$.

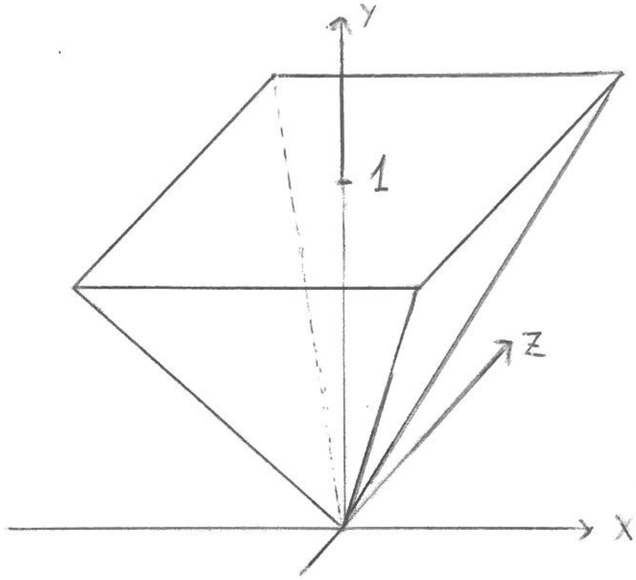
2. A 0.5 kg box is suspended 10 cm above the floor by a 0.09 kg wire hanging from the top of a 1 m tall table. Compute the work needed to lift the box up to the top of the table (ignore the size of the box).

3. A spring has natural length of 0.5 ft and its spring constant is $k = 2,000$ lb/ft. How much force does it take to hold the spring so that its length is doubled? How much work is required to double the length of the spring?

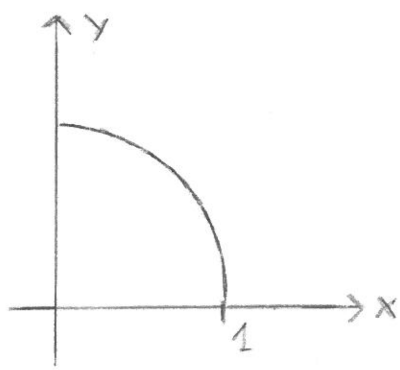
4. A lamina has the shape of the upper-right quarter of a circle with radius 1. Assuming the center of the circle is at $(0, 0)$, compute the center of mass of the lamina.
   HINT: use elementary geometry to compute the area of the lamina.

5. A tank has the shape of an upside-down pyramid with square base. The height of the pyramid is 1 m and its base measures 4 m^2. If the whole tank is full of water, how much work is needed to pump all of the water to the top of the tank? The density of water is 1000 kg/m^3. Set up the formula (the integral) for regular score, solve the problem for a 5-point bonus.

5.



4.



1. Compute the arc length of the curve defined parametrically by $x = e^t \cos t$ and $y = e^t \sin t$ from $t = 0$ to $t = \pi$.

$$\text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = e^t \cdot -\sin(t) + \cos(t) \cdot e^t$$

$$= -e^t \sin(t) + e^t \cos(t)$$

$$= e^t(-\sin(t) + \cos(t))$$

$$\frac{dy}{dt} = e^t \cdot \cos(t) + \sin(t) \cdot e^t$$

$$= e^t(\cos(t) + \sin(t))$$

$$= e^t(\cos(t) + \sin(t))$$

$$\left(\frac{dx}{dt}\right)^2 = (e^t)^2(-\sin(t) + \cos(t))^2$$

$$= e^{2t}(\sin^2(t) - 2\sin(t)\cos(t) + \cos^2(t))$$

$$\left(\frac{dy}{dt}\right)^2 = (e^t)^2(\sin(t) + \cos(t))^2$$

$$= e^{2t}(\sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t))$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\sin^2(t) \cancel{- 2\sin(t)\cos(t)} + \cos^2(t) + \sin^2(t) + \cancel{2\sin(t)\cos(t)} + \cos^2(t))$$

$$= e^{2t}(2\sin^2(t) + 2\cos^2(t))$$

$$= 2e^{2t}(\sin^2(t) + \cos^2(t))$$

$$= 2e^{2t}$$

$$\text{Arc Length} = \int_0^{\pi} \sqrt{2e^{2t}}\, dt$$

$$= \int_0^{\pi} e^t \sqrt{2}\, dt$$

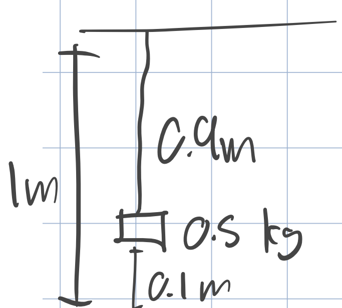
$$= \sqrt{2} \int_0^{\pi} e^t\, dt$$

$$= \sqrt{2}\left(e^t \Big|_0^{\pi}\right)$$

$$= \sqrt{2}\left(e^{\pi} - e^0\right)$$

$$= \sqrt{2}\left(e^{\pi} - 1\right)$$

2. A 0.5 kg box is suspended 10 cm above the floor by a 0.09 kg wire hanging from the top of a 1 m tall table. Compute the work needed to lift the box up to the top of the table (ignore the size of the box).



$$W_i = F_i \cdot d_i$$

$$d_i = \Delta y$$

$$F_i = 9.8\, m_i$$

$$m_i = 0.5 + \rho(1 - y_i)$$

$$\rho = \frac{0.09\,kg}{0.9\,m} = 0.1\,\frac{kg}{m}$$

$$m_i = 0.5 + 0.1(1 - y_i)$$

$$= 0.5 + 0.1 - 0.1y_i$$

$$= 0.6 - 0.1y_i$$

$$F_i = 9.8(0.6 - 0.1y_i)$$

$$W_i = 9.8(0.6 - 0.1y_i)\Delta y$$

$$W = 9.8\int_{0.1}^{1} 0.6 - 0.1y\ dy$$

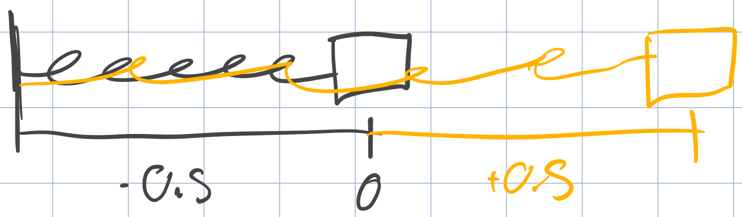
$$= 0.98\int_{0.1}^{1} 6 - y\ dy$$

$$= 0.98\left(6y - \frac{1}{2}y^2 \Big|_{0.1}^{1}\right)$$

$$= 0.98\left[\left(6(1) - \frac{1}{2}(1)^2\right) - \left(6(0.1) - \frac{1}{2}(0.1)^2\right)\right]$$

$$= 4.8069\ J$$

3. A spring has natural length of 0.5 ft and its spring constant is $k = 2,000$ lb/ft. How much force does it take to hold the spring so that its length is doubled? How much work is required to double the length of the spring?



$F = kx$

$= 2000 \cdot 0.5$

$= 1000 \text{ lb}$

$W_i = F_i \cdot d_i$

$d_i = \Delta x$

$F_i = k x_i$

$W_i = k x_i \Delta x$

$W = 2000 \int_0^{0.5} x \, dx$

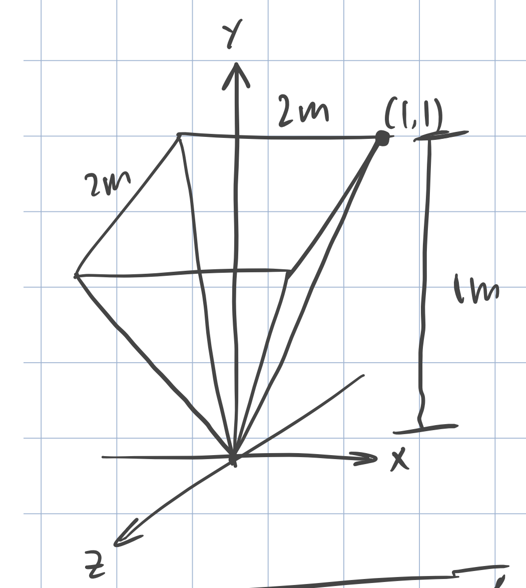
$= 2000 \left(\frac{1}{2} x^2 \Big|_0^{0.5} \right)$

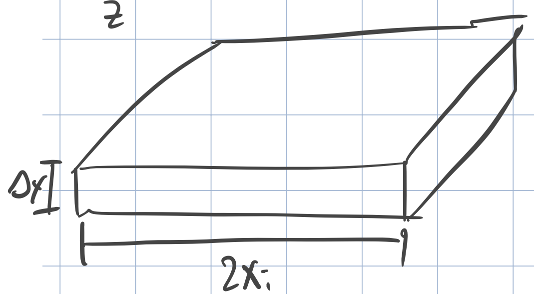
$= 1000 \left(x^2 \Big|_0^{0.5} \right)$

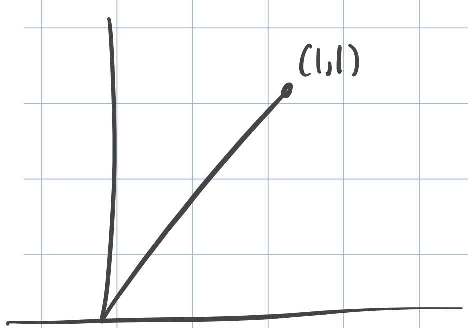
$= 1000 [0.5^2 - 0^2]$

$= 250 \text{ ft-lb}$

5. A tank has the shape of an upside-down pyramid with square base. The height of the pyramid is 1 m and its base measures 4 m^2. If the whole tank is full of water, how much work is needed to pump all of the water to the top of the tank? The density of water is 1000 kg/m^3. Set up the formula (the integral) for regular score, solve the problem for a 5-point bonus.







$Y = X$

$X = Y$

$W_i = F_i \cdot d_i$

$d_i = 1 - y_i$

$F_i = \rho g V_i$

$V_i = (2x_i)^2 \Delta y$

$= (2y)^2 \Delta y$

$= 4y^2 \Delta y$

$F_i = 9.8 \cdot 1000 (4y^2 \Delta y)$

$= 4000 y^2 \Delta y$

$W_i = (39200 y^2 \Delta y)(1 - y_i)$

$W = \int_0^1 (39200 y^2)(1 - y)\, dy$

$= 39200 \int_0^1 y^2 (1 - y)\, dy$

$$= 39200 \int_0^1 y^2 - y^3 \, dy$$

$$= 39200 \left(\frac{1}{3} y^3 - \frac{1}{4} y^4 \Big|_0^1 \right)$$

$$= 39200 \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$= 3{,}266.667 \text{ J}$$