Exponent Growth and Decay

$\frac{dy}{dt} = ky$ where k is the relative growth constant

Solve for y

$\frac{1}{y}\, dy = k\, dt$

$\int \frac{1}{y}\, dy = \int k\, dt$

$\ln(|y|) = kt + C$

$e^{\ln(|y|)} = e^{kt+C}$

$y = e^{kt} e^{C}$

$y = Ce^{kt}$

Initial Condition $y(0) = y_0$

$y(t) = Ae^{kt}$ where A is some constant

$y_0 = Ae^{k \cdot 0}$

$y_0 = Ae^{0}$

$y_0 = A$

$Y(t) = Y_0 e^{kt}$ $\frac{dy}{dt} = yt$

<u>Example</u>: A bacteria culture initially contains 100 cells and grows at a rate porportional to its size. After an hour the population has increased to 420 cells.

$(0, 100)$

$y = Y_0 e^{kt}$

$Y = 100e^{kt}$

$(1, 420)$

$420 = 100e^{k(1)}$

$4.2 = e^{k}$

$\ln(4.2) = \ln(e^{k})$

$k = \ln(4.2)$

Final Equation

$y = 100e^{\ln(4.2)t}$

$Y = 100e^{\ln(4.2^{t})}$

$Y = 100(4.2)^t$ $\qquad \frac{dy}{dt} = \ln(4.2)y$

Find the number of bacteria after 3 hours:

$Y = 100(4.2)^3$

$= 7408.8 \ \frac{\text{bacteria}}{\text{hour}}$

Find the rate of growth after 3 hours

$\frac{dy}{dt} = \ln(4.2)y$

$= 10632.25$

When will the population reach 10,000?

$Y = 100(4.2)^t$

$10{,}000 = 100(4.2)^t$

$100 = (4.2)^t$

$(4.2)^{3.209} = (4.2)^t$

$t = 3.209$ hours

Example: Radium decays exponentially and has a half life of 1600 years. Find a formula for the amount of radium remaining from 50 mg after t years. When will there be 20 mg left?

$Y = Y_0 e^{kt}$

$(0, 50)$

$Y = 50e^{kt}$

$(1600, 25)$

$25 = 50e^{k(1600)}$

$\frac{1}{2} = e^{k(1600)}$

$\ln(\frac{1}{2}) = \ln(e^{k(1600)})$

$\ln(\frac{1}{2}) = k(1600)$

$k = \frac{\ln(\frac{1}{2})}{1600}$

Final Equation

$Y = 50e^{\frac{\ln(\frac{1}{2})}{1600}t}$

Solve for 20 mg left

$$20 = 50e^{\frac{\ln(\frac{1}{2})}{1600}t}$$

$$\frac{2}{5} = e^{\frac{\ln(\frac{1}{2})}{1600}t}$$

$$\ln(\tfrac{2}{5}) = \ln(e^{\frac{\ln(\frac{1}{2})}{1600}t})$$

$$\ln(\tfrac{2}{5}) = \frac{\ln(\frac{1}{2})}{1600}t$$

$$t = \frac{\ln(\frac{2}{5})1600}{\ln(\frac{1}{2})}$$

$t = 2115.08$ hours