A spring with a mass of 2 kg has damping constant 14 kg/s. A force of 3.6 N is required to keep the spring stretched 0.3 m beyond its natural length. The spring is stretched 0.8 m beyond natural length and released. If an external force of $F_e(t) = 10e^{-t}$ N is applied to the spring, find the position at time t.

$$mx'' + cx' + kx = F_e$$

$$F = kx$$

$$2x'' + 14x' + 12x = 10e^{-t}$$

$$k = \frac{F}{x}$$

$$= 12$$

Auxiliary Equation

$$2r^2 + 14r + 12 = 0$$

$$r^2 + 7r + 6 = 0$$

$$(r+1)(r+6) = 0$$

$$r = -1 \qquad r = -6$$

Complementary Equation

$$x_c(t) = C_1 e^{-x} + C_2 e^{-6x}$$

Assume $x_p(t)$ is exponential

$$x_p(t) = Ae^{-t}$$

$x_p'(t) = -Ae^{-t}$

$x_p''(t) = Ae^{t}$

Sub and solve

$2(A\cancel{e^{-t}}) + 14(-A\cancel{e^{-t}}) + 12(A\cancel{e^{-t}}) = 10\cancel{e^{-t}}$

$0 = 10$

Multiply by t

$x_p(t) = Ate^{-t}$

$x_p'(t) = A(t \cdot -e^{-t} + e^{-t})$

$= A(-te^{-t} + e^{-t})$

$= -Ate^{-t} + Ae^{-t}$

$x_p''(t) = -A(t \cdot -e^{-t} + e^{-t}) - Ae^{-t}$

$= -A(-te^{-t} + e^{-t}) - Ae^{-t}$

$= Ate^{-t} - Ae^{-t} - Ae^{-t}$

$= Ate^{-t} - 2Ae^{-t}$

Sub and solve

$$2(Ate^{-t}-2Ae^{-t})+14(-Ate^{-t}+Ae^{-t})+12(Ate^{-t})=10e^{-t}$$

$$2At\cancel{e^{-t}}-4A\cancel{e^{-t}}-14At\cancel{e^{-t}}+14\cancel{e^{-t}}+12At\cancel{e^{-t}}=10\cancel{e^{-t}}$$

$$\cancel{2At}-4A-\cancel{14At}+14A+\cancel{12tA}+10$$

$$10A=10$$

$$A=1$$

$$x_p(t)=te^{-t}$$

General Solution

$$x(t)=C_1e^{-t}+C_2e^{-6t}+te^{-t}$$

$$x'(t)=-C_1e^{-t}-6C_2e^{-6t}+t\cdot-e^{-t}+e^{-t}$$

$$=-C_1e^{-t}-6C_2e^{-6t}-te^{-t}+e^{-t}$$

Initial Condition $x(0)=0.8m$ and $x'(0)=0$

$$0.8=C_1e^0+C_2e^0+0$$
$$0=-C_1e^0-6C_2e^0-0+1$$

$0.8 = C_1 + C_2$
$0 = -C_1 - 6C_2 + 1$

$0.8 = -5C_2 + 1$

$-0.2 = -5C_2$

$C_2 = 0.04$

$C_1 = 0.76$

Unique Solution

$x(t) = 0.76e^{-t} + 0.04e^{-6t} + te^{-t}$

A spring with a mass of 1kg has a spring constant 43 kg/s². If the spring begins at equilibrium position and is given a velocity of 3 m/s, find the damping constant that would produce critical damping.

$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + kx = 0$

Critical Damping: $C^2 - 4mk = 0$

$C^2 = 4mk$

$C = \sqrt{4mk}$

$= 2\sqrt{43}$

A spring with a 4 kg mass will stretch 0.5 m past rest with a force of 12 N. The spring starts at equillibrum and is pushed to the right with an initial value of 3 m/s. Find position at time t.

$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + kx = 0$

$k = \frac{F}{x}$

$4x'' + 0 + 24x = 0$

$= 24$

Auxiliary Equation

$4r^2 + 24 = 0$

$r^2 + 6 = 0$

$r^2 = -6$

$r = 0 \pm \sqrt{6}\,i \qquad \alpha = 0 \qquad \beta = \sqrt{6}$

General Solution

$x(t) = e^0\left(C_1 \sin(\sqrt{6}t) + C_2 \cos(\sqrt{6}t)\right)$

$= C_1 \sin(\sqrt{6}t) + C_2 \cos(\sqrt{6}t)$

$x'(t) = \sqrt{6}C_1 \cos(\sqrt{6}t) - \sqrt{6}C_2 \sin(\sqrt{6}t)$

Initial Values $x(0)=0m$ and $x'(0)=3\,m/s$

$$0=C_1\sin(0)+C_2\cos(0)$$
$$3=\sqrt{6}C_1\cos(0)-\sqrt{6}C_2\sin(0)$$

$$0=C_2$$
$$3=\sqrt{6}C_1$$

$$C_2=0$$
$$C_1=\frac{3}{\sqrt{6}}$$

Unique Solution

$$x(t)=\frac{3}{\sqrt{6}}\sin(\sqrt{6}\,t)$$

A series circuit consists of a resistor with $R=20\,\Omega$, an inductor with $L=1H$, a capacitor with $C=0.01F$, and a 9v battery. Its initial charge and current are initially 0. Find charge and current as a function of time.

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=19$$

$$1Q''+20Q'+100Q=19$$

Auxiliary Equation

$$r^2+20r+100=0$$

$$r = \frac{-20 \pm \sqrt{400 - 400}}{2}$$

$$= -10$$

Complementary Solution

$$Q_c(t) = C_1 e^{10t} + C_2 t e^{10t}$$

Assume the particular solution is a constant

$$Q_p(t) = A$$

$$Q_p'(t) = 0$$

$$Q''_p(t) = 0$$

Sub and solve

$$0 + 0 + A\,100 = 19$$

$$A = \frac{19}{100}$$

$$Q_p(t) = \frac{19}{100}$$

General Solution

$$Q(x) = C_1 e^{-10t} + C_2 t e^{-10t} + \frac{19}{100}$$

$$I(x) = -10C_1e^{-10t} + C_2(-10te^{-10t} + e^{-10t})$$

$$= -10C_1e^{-10t} - 10C_2te^{-10t} + C_2e^{-10t}$$

Initial Condition $Q(0)=0$ and $I(0)=0$

$$0 = C_1e^0 + \frac{19}{100}$$
$$0 = -10C_1e^0 + C_2e^0$$

$$0 = C_1 + \frac{19}{100}$$
$$0 = -10C_1 + C_2$$

$$C_1 = -\frac{19}{100}$$

$$0 = -10\left(-\frac{19}{100}\right) + C_2$$

$$0 = \frac{190}{100} + C_2$$

$$C_2 = -\frac{19}{10}$$

$$= -10C_1e^{-10t} - 10C_2te^{-10t} + C_2e^{-10t}$$

Unique Solution

$$Q(t) = -\frac{19}{100}e^{-10t} - \frac{19}{10}te^{-10t} + \frac{19}{100}$$

$$I(t) = -10\left(-\frac{19}{100}\right)e^{-10t} - 10\left(-\frac{19}{10}\right)te^{-10t} + \left(-\frac{19}{10}\right)e^{-10t}$$

$$= \frac{19}{10}e^{-10t} + 19te^{-10t} - \frac{19}{10}e^{-10t}$$

A series circuit consists of a resistor with $R = 20\,\Omega$, an inductor with $L = 1\,H$, a capacitor with $C = 0.01\,F$, and a generator producing a voltage of $E(t) = 160\sin(10t)$. Initial charge and current are 0. Find charge and current at time t.

Let the general solution of the problem above be the complementary solution

$$Q_c(x) = C_1 e^{-10t} + C_2 t e^{-10t}$$

Assume the particular solution is trigonomic.

$$Q_p(x) = A\sin(10t) + B\cos(10t)$$

$$Q_p'(x) = 10A\cos(10t) - 10B\sin(10t)$$

$$Q''p(x) = -100A\sin(10t) - 100B\cos(10t)$$

Sub and solve

$$(-100A\sin(10t) - 100B\cos(10t)) + 20(10A\cos(10t) - 10B\sin(10t) + 100(A\sin(10t) + B\cos(10t)) = 160\sin(10t)$$

$$\cancel{-100A\sin(10t)} \cancel{-100B\cos(10t)} + 200A\cos(10t) - 200B\sin(10t) + \cancel{100A\sin(10t)} + \cancel{100B\cos(10t)} = 160\sin(10t)$$

$\rightarrow -200B\sin(10t) = 160\sin(10t)$

$200A\cos(10t) = 0$

$-200B = 160$

$A = 0$

$B = -\frac{160}{200}$

$= -\frac{4}{5}$

$Q_p(x) = -\frac{4}{5}\cos(10t)$

General Solution

$Q(x) = C_1 e^{-10t} + C_2 t e^{-10t} - \frac{4}{5}\cos(10t)$

$Q'(x) = -10C_1 e^{-10t} + C_2(-10te^{-10t} + e^{-10t}) + 8\sin(10t)$

$= -10C_1 e^{-10t} - 10C_2 t e^{-10t} + C_2 e^{-10t} + 8\sin(10t)$

Initial Values $Q(0) = 0$ and $Q'(0) = 0$

$0 = C_1 e^0 + 0 - \frac{4}{5}\cos(0)$

$0 = -10C_1 e^0 - 0 + C_2 e^0 - 8\sin(0)$

$0 = C_1 - \frac{4}{5}$

$0 = -10C_1 + C_2$

$C_1 = \frac{4}{5}$

$0 = -10\left(\frac{4}{5}\right) + C_2$

$0 = -\frac{40}{5} + C_2$

$C_2 = \frac{40}{5}$

$= 8$

Unique Solution

$Q(x) = \frac{4}{5}e^{-10t} + 8te^{-10t} - \frac{4}{5}\cos(10t)$

$I(x) = -10\left(\frac{4}{5}\right)e^{-10t} - 10(8)te^{-10t} + (8)e^{-10t} + \frac{4}{5}\sin(10t)$

$= -8e^{-10t} - 80te^{-10t} + 8e^{-10t} + 8\sin(10t)$

A series circuit consists of a resistor with $R = 20\,\Omega$, a capacitor with $C = 0.01$ F, and a decaying battery with $E(t) = 500e^{-5t}$. If the initial charge is 0, find charge as a function of time.

$LQ'' + RQ' + \frac{1}{C}Q = E(t)$

$20Q' + 100Q = 500e^{-5t}$

$Q' + 5Q = 25e^{-5t}$

Auxiliary Equation

$$r + S = 0$$

$$r = -S$$

Complementary Solution

$$Q_c(t) = C_1 e^{-st}$$

Assume the particular solution is exponential

$$Q_p(t) = Ae^{-st}$$

$$Q_p'(t) = -SAe^{-st}$$

Sub and solve

$$(\cancel{-S})(\cancel{A}\cancel{e^{-st}}) + \cancel{S}(\cancel{A}\cancel{e^{-st}}) = 500\cancel{e^{-st}}$$

$$0 = 500$$

Multiply by x

$$Q_p(t) = Ate^{-st}$$

$$Q_p'(t) = A(t \cdot -Se^{-st} + e^{-st})$$

$$= -SAte^{-St} + Ae^{-St}$$

Sub and solve

$$(\cancel{-SAte^{-St}} + Ae^{-St}) + \cancel{S(Ate^{-St})} = 2Se^{-St}$$

$$A\cancel{e^{-St}} = 2S\cancel{e^{-St}}$$

$$A = 2S$$

$$Q_p(t) = 2S\,te^{-St}$$

General Solution

$$Q(t) = C_1 e^{-St} + 2S\,te^{-St}$$

Initial Value $Q(0) = 0$

$$0 = C_1 e^0 + 0$$

$$C_1 = 0$$

Unique Solution

$$Q(t) = 2S\,te^{-St}$$