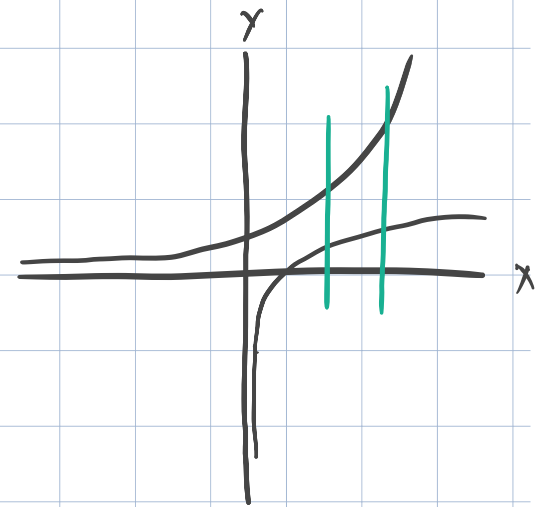
Integrate the region bounded by $f(x)=e^x$, $g(x)=\ln(x)$, $x=2$, and $x=3$



$2 \le x \le 3$

$\ln(x) \le y \le e^x$

$$\int_2^3 \int_{\ln(x)}^{e^x} 1 \, dy \, dx$$

$$= \int_2^3 (y)\Big|_{\ln(x)}^{e^x} dx$$

$$= \int_2^3 (e^x) - (\ln(x)) \, dx$$

$$= (e^x - x\ln(x) + x)\Big|_2^3$$

$$= (e^3 - 3\ln(3) + 3) - (e^2 - 2\ln(2) + 2)$$

$$= e^3 - 3\ln(3) + 3 - e^2 + 2\ln(2) - 2$$

$$= e^3 - e^2 - 3\ln(3) + 2\ln(2) + 1$$

Integrate the region bounded by $f(x) = 7\sin(x)$ and $g(x) = \frac{14}{\pi} x$

Find the points of intsection

$$7\sin(x) = \frac{14}{\pi} x$$

$$\pi \sin(x) = 2x$$

$$x = \pm \frac{\pi}{2}$$

Integral when $x > 0$

$$0 \leq x \leq \frac{\pi}{2}$$

$$g(x) \leq y \leq f(x)$$

$$A_1 = \int_0^{\frac{\pi}{2}} \int_{\frac{14}{\pi}x}^{7\sin(x)} 1 \, dy \, dx$$

$$= \int_0^{\frac{\pi}{2}} \left(7\sin(x) - \frac{14}{\pi} x\right) dx$$

$$= \left(-7\cos(x) - \frac{14}{\pi}\left(\frac{1}{2}x^2\right)\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(-7\cos(x) - \frac{7}{\pi}x^2\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(-7\cos\left(\frac{\pi}{2}\right) - \frac{7}{\pi}\left(\frac{\pi}{2}\right)^2\right) - \left(-7\cos(0) - \frac{7}{\pi}(0)^2\right)$$

$$= -\frac{2\pi}{4} + 7$$

The area is symetrical so double it

$$A = 2\left(-\frac{2\pi}{4} + 7\right)$$

Recall from Calculus I that the integral of an **odd** continuous function between symmetric limits about the origin is zero. Use this fact to evaluate $\iint_D (5 + 7x^3 + 9y^7)\, dA$ where D is the square $D = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}$.

$$\int_{-1}^{1}\int_{-1}^{1} (5+7x^3+9y^7)\,dy\,dx$$

$$= \int_{-1}^{1} \left(5y + 7x^3y + \frac{9}{8}y^8\right)\Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left(5 + 7x^3 + \frac{9}{8}\right) - \left(-5 - 7x^3 + \frac{9}{8}\right) dx$$

$$= \int_{-1}^{1} 5 + 7x^3 + \frac{9}{8} + 5 + 7x^3 - \frac{9}{8}\, dx$$

$$= \int_{-1}^{1} 10 + 14x^3\, dx$$

$$= \left(10x + \frac{14}{4}x^4\right)\Big|_{-1}^{1}$$

$$= \left(10 + \frac{14}{4}\right) - \left(-10 + \frac{14}{4}\right)$$

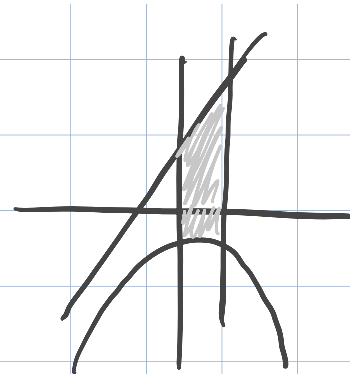
$$= 10 + 10 + \frac{14}{4} - \frac{14}{4}$$

$$= 20$$

Find the average value of the function $h(x, y) = x^3y^2$ on the region bounded by $f(x) = 2x + 1$, $g(x) = -x^2 - 1$, $x = 0$, and $x = 1$.

$$0 \le x \le 1$$

$$g(x) \le y \le f(x)$$



$$\text{Avg} = \frac{\int_0^1 \int_{g(x)}^{f(x)} h(x,y)\,dy\,dx}{\int_0^1 \int_{g(x)}^{f(x)} 1\,dy\,dx}$$

$$\rightarrow \int_0^1 \int_{g(x)}^{f(x)} h(x,y)\,dy\,dx$$

$$= \int_0^1 \int_{-x^2-1}^{2x+1} x^3 y^2\,dy\,dx$$

$$= \int_0^1 \left(\tfrac{1}{3} x^3 y^3\right)\Big|_{-x^2-1}^{2x+1}\,dx$$

$$= \int_0^1 \tfrac{1}{3} x^3 \left(y^3\right)\Big|_{-x^2-1}^{2x+1}\,dx$$

$$= \int_0^1 \tfrac{1}{3} x^3 \left[(2x+1)^3 - (-x^2-1)^3\right]\,dx$$

$$= \int_0^1 \tfrac{1}{3} x^3 \left[(2x+1)^3 + (x^2+1)^3\right]\,dx$$

$$= \tfrac{1}{3} \int_0^1 x^3 \left[(4x^2+4x+1)(2x+1) + (x^4+2x^2+1)(x^2+1)\right]\,dx$$

$$\begin{array}{c|cc|} & 2x & 1 \\ \hline 4x^2 & 8x^3 & 4x^2 \\ 4x & 8x^2 & 4x \\ 1 & 2x & 1 \\ \hline \end{array} \qquad \begin{array}{c|cc|} & x^2 & 1 \\ \hline x^4 & x^6 & x^4 \\ 2x^2 & 2x^4 & 2x^2 \\ 1 & x^2 & 1 \\ \hline \end{array}$$

$$8x^3 + 12x^2 + 6x + 1 \qquad x^6 + 3x^4 + 3x^2 + 1$$

Add both

$$x^6 + 8x^3 + 3x^4 + 15x^2 + 6x + 2$$

$$= \frac{1}{3}\int_0^1 x^3(x^6 + 8x^3 + 3x^4 + 15x^2 + 6x + 2)\, dx$$

$$= \frac{1}{3}\int_0^1 x^9 + 8x^6 + 3x^7 + 15x^5 + 6x^4 + 2x^3 \, dx$$

$$= \frac{1}{3}\left(\frac{1}{10}x^{10} + \frac{8}{7}x^7 + \frac{3}{8}x^8 + \frac{15}{6}x^6 + \frac{6}{5}x^5 + \frac{2}{4}x^4\right)\Big|_0^1$$

$$= \frac{1}{3}\left(\frac{1}{10} + \frac{8}{7} + \frac{3}{8} + \frac{15}{6} + \frac{6}{5} + \frac{1}{2}\right)$$

$$= \frac{543}{280}$$

$$\rightarrow \int_0^1 \int_{g(x)}^{f(x)} 1 \, dy\, dx$$

$$= \int_0^1 \int_{-x^2-1}^{2x+1} 1 \, dy \, dx$$

$$= \int_0^1 (2x+1) - (-x^2-1)\, dx$$

$$= \int_0^1 2x + 1 + x^2 + 1 \, dx$$

$$= \int_0^1 x^2 + 2x + 2 \, dx$$

$$= \left(\frac{1}{3}x^3 + \frac{2}{2}x^2 + 2x\right)\Big|_0^1$$

$$= \frac{1}{3} + 1 + 2$$

$$= \frac{10}{3}$$

$$\text{Avg} = \frac{\frac{543}{280}}{\frac{10}{3}}$$

$$= \frac{1629}{2800}$$

Find the value of *a* such that the average value of $f(x, y) = 1 + x + y$ on the region $D = \{(x, y) \mid 0 \le x \le a, 0 \le y \le 4\}$ is equal to 6.

$0 \le x \le a$

$0 \le y \le 4$

$$\text{avg} = \frac{\int_0^a \int_0^4 1+x+y \, dy\,dx}{\int_0^a \int_0^4 1 \, dy\,dx}$$

$$\rightarrow \int_0^a \int_0^4 1+x+y \, dy\, dx$$

$$= \int_0^a \left(y + xy + \tfrac{1}{2}y^2\right)\Big|_0^4 \, dx$$

$$= \int_0^a \left(4 + 4x + \tfrac{1}{2}(4)^2\right) dx$$

$$= \int_0^a (4 + 4x + 8)\, dx$$

$$= \left(4x + \tfrac{4}{2}x^2 + 8x\right)\Big|_0^a$$

$$= 12a + \tfrac{1}{2}a^2$$

$$\to \int_0^a \int_0^4 1 \, dy \, dx$$

The bounds are constant and make a rectangle, so simply multiply side lengths.

$$= 4a$$

$$avg = \frac{12a + \frac{1}{2}a^2}{4a}$$

$$avg = \frac{12 + \frac{1}{2}a}{4}$$

Plug in avg

$$6 = \frac{12 + \frac{1}{2}a}{4}$$

$$24 = 12 + \frac{1}{2}a$$

$$12 = \frac{1}{2}a$$

$$a = 6$$

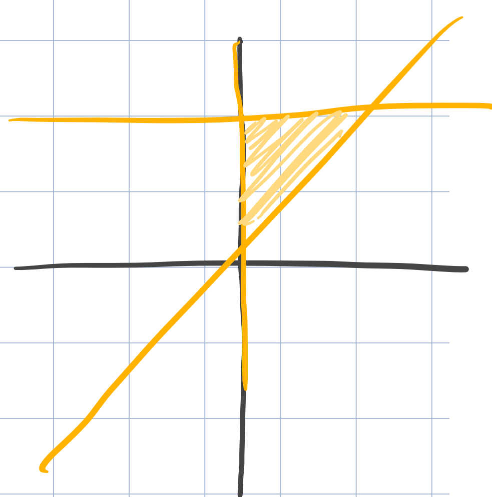
Find the volume of the solid region F.

The region F is the region below the graph of $f(x, y) = xy\sqrt{x^2 + y^2}$ and above the region in the xy-plane bounded by the lines $y = x$, $y = 5$, and $x = 0$.

$$0 \le y \le 5$$

$$0 \le x \le y$$

$$V = \int_0^5 \int_0^y xy\sqrt{x^2+y^2} \, dx \, dy$$



$U = x^2 + y^2$

$\frac{dU}{dx} = 2x$

$\frac{1}{2} dU = x\,dx$

Bounds

$y \to y^2 + y^2$

$= 2y^2$

$0 \to y^2$

$$V = \int_0^5 \int_{y^2}^{2y^2} \frac{1}{2} y \sqrt{U} \, dx \, dy$$

$$= \int_0^5 \frac{1}{2} y \int_{y^2}^{2y^2} U^{\frac{1}{2}} \, dx \, dy$$

$$= \frac{1}{2} \int_0^5 y \left(\frac{2}{3} U^{\frac{3}{2}} \right) \Big|_{y^2}^{2y^2} dy$$

$$= \frac{1}{2} \cdot \frac{2}{3} \int_0^5 y \left[(2y^2)^{\frac{3}{2}} - (y^2)^{\frac{3}{2}} \right] dy$$

$$= \frac{1}{3} \int_0^5 y \left(\sqrt{8} y^3 - y^3 \right) dy$$

$$= \frac{1}{3} \int_0^5 \sqrt{8} y^4 - y^4 \, dy$$

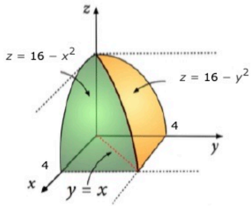
$$= \frac{1}{3} \left(\frac{\sqrt{8}}{5} y^5 - \frac{1}{5} y^5 \right) \Big|_0^5$$

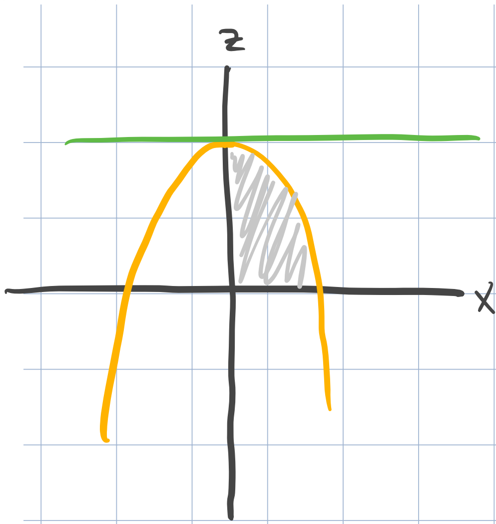
$$= \frac{1}{15} \left(\sqrt{8}\, y^5 - y^5 \right)$$

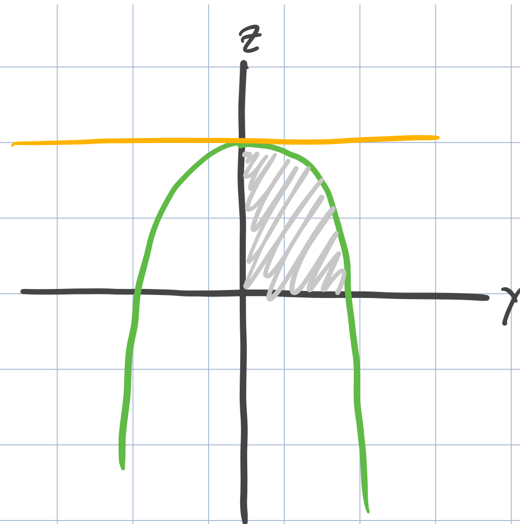
$$= \frac{1}{15} \left(\sqrt{8}\, 5^5 - 5^5 \right)$$

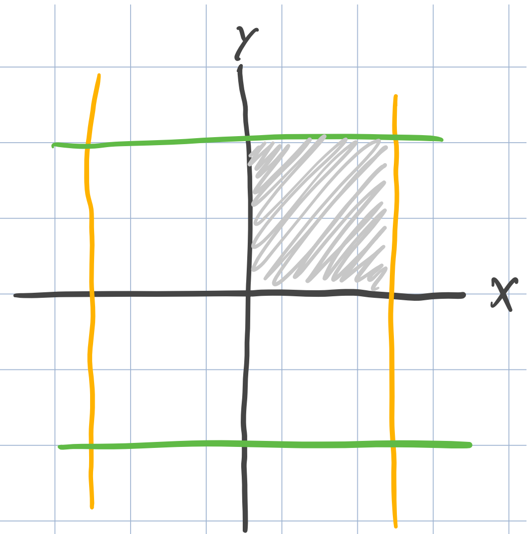
Find the volume of the solid region F.

The region F is the region in the first octant that is bounded by the two parabolic cylinders $z = 16 - y^2$ and $z = 16 - x^2$.









$z = 16 - x^2$

$z = 16$

$z = 16 - y^2$

$z = 16$

$0 = 16 - y^2$

$y = \pm 4$

$0 \le z \le 16$

$0 \le x \le \sqrt{16 - z}$

$0 \le y \le \sqrt{16 - z}$

$0 = 16 - x^2$

$x = \pm 4$

zx plane

$x^2 = 16 - z$

$x = \sqrt{16 - z}$

zy plane

$y^2 = 16 - z$

$y = \sqrt{16 - z}$

$$V = \int_0^{16}\int_0^{\sqrt{16-z}}\int_0^{\sqrt{16-z}} 1 \, dy \, dx \, dz$$

$$= \int_0^{16}\int_0^{\sqrt{16-z}} \sqrt{16-z} \, dx \, dz$$

$$= \int_0^{16} \sqrt{16-z} \int_0^{\sqrt{16-z}} 1 \, dx \, dz$$

$$= \int_0^{16} \sqrt{16-z}^{\,2} \, dz$$

$$= \int_0^{16} 16 - z \, dz$$

$$= \left(16z - \frac{1}{2}z^2\right)\Big|_0^{16}$$

$$= 16^2 - \frac{16^2}{2}$$

A plane lamina with mass density $\sigma(x, y) = 3 + x^2y^2$ occupies the region in the xy-plane bounded by $x = 0$, $y = e^x$, and $y = e$. Find the total mass M of the lamina, the moments M_x and M_y, and the center of mass $(\bar{x}, \bar{y})$ of the lamina.

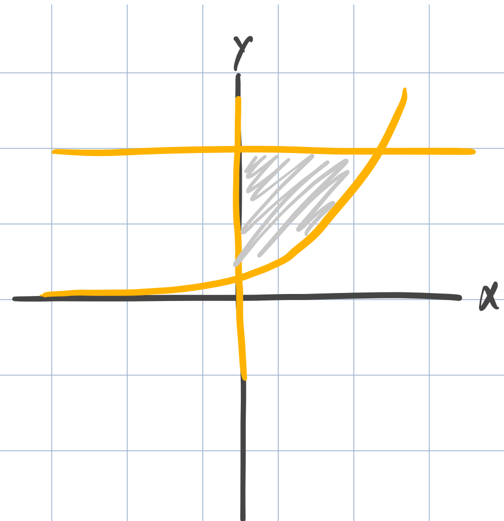
M =

M_x =

M_y =

$(\bar{x}, \bar{y})$ = ( )

Part One



Find intersection of $y = e^x$ and $y = e$

$$e = e^x$$

$$x = 1$$

$y = e$

Bounds

$0 \le x \le 1$

CR

$0 \le y \le e$

$0 \le x \le \ln(y)$

$y = e^x$

$\ln(y) = x$

$e^x \le y \le e$

$$M = \int_0^1 \int_{e^x}^{e} 3 + x^2 y^2 \, dy \, dx$$

$$= \int_0^1 \left(3y + \tfrac{1}{3} x^2 y^3\right)\Big|_{e^x}^{e} \, dx$$

$$= \int_0^1 \left(3e + \tfrac{1}{3} x^2 e^3\right) - \left(3e^x + \tfrac{1}{3} x^2 (e^x)^3\right) dx$$

$$= \int_0^1 \left(3e + \tfrac{e^3}{3} x^2\right) - \left(3e^x + \tfrac{1}{3} e^{3x} x^2\right) dx$$

$$= \int_0^1 3e + \tfrac{e^3}{3} x^2 - 3e^x - \tfrac{1}{3} e^{3x} x^2 \, dx$$

$$= 3e \int_0^1 1 \, dx + \frac{e^3}{3} \int_0^1 x^2 \, dx - 3\int_0^1 e^x \, dx - \frac{1}{3} \int_0^1 e^{3x} x^2 \, dx$$

$\rightarrow \int_0^1 e^{3x} x^2 \, dx$

$u = x^2$ $\qquad dv = e^{3x} dx$

$\frac{du}{dx} = 2x$ $\qquad \frac{dv}{dx} = e^{3x}$

$du = 2x \, dx$ $\qquad v = \frac{1}{3} e^{3x}$

$$= (x^2)\left(\tfrac{1}{3}e^{3x}\right) - \int_0^e \tfrac{1}{3}e^{3x}\,2x\,dx$$

$$= \tfrac{1}{3}e^{3x}x^2 - \tfrac{2}{3}\int_0^e e^{3x}x\,dx$$

$$\rightarrow \int e^{3x}x\,dx$$

$$u = x \qquad dv = e^{3x}dx$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^{3x}$$

$$du = dx \qquad v = \tfrac{1}{3}e^{3x}$$

$$= \tfrac{1}{3}e^{3x}x - \int \tfrac{1}{3}e^{3x}dx$$

$$= \tfrac{1}{3}e^{3x}x - \tfrac{1}{3}\cdot\tfrac{1}{3}e^{3x}$$

$$= \tfrac{1}{3}e^{3x}x - \tfrac{1}{9}e^{3x}$$

$$= \tfrac{1}{3}e^{3x}x^2 - \tfrac{2}{3}\left(\tfrac{1}{3}e^{3x}x - \tfrac{1}{9}e^{3x}\right)$$

$$\rightarrow \int_0^1 e^{3x}x^2\,dx$$

$$= \left(\tfrac{1}{3}e^{3x}x^2 - \tfrac{2}{3}\left(\tfrac{1}{3}e^{3x}x - \tfrac{1}{9}e^{3x}\right)\right)\Big|_0^1$$

$$= \left(\tfrac{1}{3}e^3 - \tfrac{2}{3}\left(\tfrac{1}{3}e^3 - \tfrac{1}{9}e^3\right)\right) - \left(\cancel{\tfrac{1}{3}(0)} - \tfrac{2}{3}\left(\cancel{\tfrac{1}{3}(0)} - \tfrac{1}{9}e^0\right)\right)$$

$$= \tfrac{1}{3}e^3 - \tfrac{2}{3}\left(\tfrac{1}{3}e^3 - \tfrac{1}{9}e^3\right) - \tfrac{2}{3}\cdot\tfrac{1}{9}$$

$$= \frac{1}{3}e^3 - \frac{2}{3}\left(\frac{1}{3}e^3 - \frac{1}{9}e^3\right) - \frac{2}{27}$$

$$= 3e + \frac{e^3}{3}\left(\frac{1}{3}x^3\right)\Big|_0^1 - 3\left(e^x\right)\Big|_0^1 - \frac{1}{3}\left(\frac{1}{3}e^3 - \frac{2}{3}\left(\frac{1}{3}e^3 - \frac{1}{9}e^3\right) - \frac{2}{27}\right)$$

$$= 3e + \frac{e^3}{3}\cdot\frac{1}{3} - 3(e-1) - \frac{1}{3}\left(\frac{1}{3}e^3 - \frac{2}{3}\left(\frac{1}{3}e^3 - \frac{1}{9}e^3\right) - \frac{2}{27}\right)$$

$$= \cancel{3e} + \cancel{\frac{1}{9}e^3} - \cancel{3e} + 3 - \cancel{\frac{1}{9}e^3} + \frac{2}{9}\left(\frac{2}{9}e^3\right) + \frac{2}{81}$$

$$= 3 + \frac{4}{81}e^3 + \frac{2}{81}$$

$$= 3 + \frac{4e^3+2}{81}$$

Part Two

$$M_x = \int_0^1 \int_{e^x}^{e} y\left(3 + x^2y^2\right)\,dy\,dx$$

Requires too much time

Part Three

$$M_y = \int_0^1 \int_{e^x}^{e} x\left(3 + x^2y^2\right)\,dy\,dx$$

Requires too much time

Part Four

$$\bar{x} = \frac{M_x}{M}$$

$$\bar{x} = \frac{M_y}{M}$$

Requires too much time