1.3 Applications of Integration in Engineering and Physics

Work, **pressure**, and **center of mass** are important concepts in physics and engineering applications. In this section we will apply the integral calculus to compute these quantities. Continuing with our standard approach, these physical quantities will be broken down into small pieces and approximated. We then find the sum of the approximations and take the limit as the number of pieces increases without bound. This process will generate a definite integral in each application.

1.3.1 Work

A critical component of work, in the physics and engineering sense, is **force**. Force can be thought of as an influence (a push or pull) that tends to change the motion of an object. Simply speaking, it takes ten pounds of force to lift a ten-pound object. In general, the force F acting on an object moving along a linear path with position function $s(t)$ is given by Newton's Second Law of Motion to be the product of its mass m and its acceleration $\frac{d^2s}{dt^2}$:

$$F = m\left(\frac{d^2s}{dt^2}\right)$$

We need to be prepared to operate in two systems of units. In the SI (Système International) metric system the following units are used:

- kg kilograms for mass,
- m meters for length,
- s seconds for time,
- N Newtons for force, and $1\text{ N} = 1\ (\text{kg m})/\text{s}^2$.

1 N is the amount of force needed to accelerate 1 kg of mass at the rate of 1 meter per second per second. In the US system, the fundamental unit of force is the pound. If a constant force F is applied to an object, moving it a fixed distance d in the direction of the force, then the work W done on the object is Work = Force × distance

$$W = F \times d$$

In the SI system, work is measured in Newton-meters (1 N-m = 1 J, where J is in Joules). In the US system, work is measured in foot-pounds (ft-lb).

Example 7. *If a 10 kg dumbbell is lifted 1.3 m and placed on a rack, how much work is done?*

Solution: The force exerted is equal to and opposite from gravity, which has acceleration $g = 9.8\ \text{m}/\text{s}^2$. So the force exerted is

$$F = m\,g = (10\text{ kg})\left(9.8\ \frac{\text{m}}{\text{s}^2}\right) = 98\text{ N}$$

and the work done would be

$$W = (98\text{ N})(1.3\text{ m}) = 127.4\text{ J}.$$

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Example 8. *If a 22 lb dumbbell is lifted 4.26 ft and placed on a rack, how much work is done?*

Solution: In this problem the weight, rather than the mass, of the dumbbell has been specified. The dumbbell's weight, 22 lb, is a force rather than a mass, and the acceleration of gravity has already been included in the weight of the dumbbell. Recall that in the US system, force F is in pounds, distance d is in feet, and so work is in foot-pounds (1 ft-lb = 1.36 J). The work done is

$$W = F \times d = (22 \text{ lbs})(4.26 \text{ ft}) = 93.72 \text{ ft-lbs}.$$

Actually, these two examples are approximately the same since 10 kg weighs approximately 22 lbs and 1.3 m $\approx$ 4.26 ft. Converting the work done from foot-pounds to Joules gives

$$(93.72 \text{ ft-lbs})\left(\frac{1.36 \text{ J}}{\text{ft-lb}}\right) \approx 127.46 \text{ J}$$

These two examples were straight forward since the force was constant. Our integral calculus becomes necessary when the force is variable.

Let's assume that the object moves along a straight line from $x = a$ to $x = b$. Let's also assume a variable but continuous force $f(x)$ acts on the object as it moves from a to b. Subdividing the interval $[a, b]$ into n equal subintervals gives us $x_i = a + i\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $i = 0, \cdots, n$.

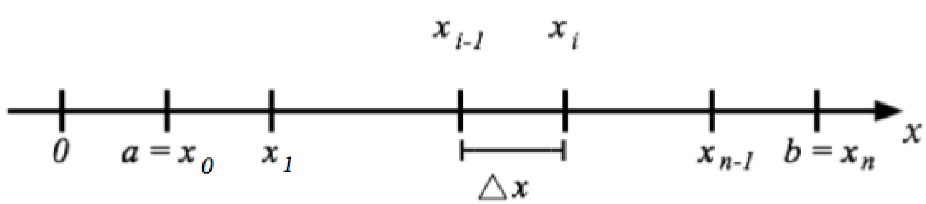

Figure 12

The work done in moving the object from x_{i-1} to x_i is approximately the force at x_i times the length of the subinterval:

$$W_i \approx f(x_i)\Delta x$$

Since f is continuous, this approximation improves as the number n increases and Δx gets smaller. Adding these approximations gives an approximation of the total work done:

$$W \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

To get the exact amount of work done we let the number of subintervals increase without bound.

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

We should note that if the force is a constant F, when

$$W = \int_a^b F\,dx = F\int_a^b dx = F \times (b-a) = \text{force} \times \text{distance}$$

Let's examine the work needed to stretch (or compress) a spring as our first example. Before doing so, we need a law from physics to assist us in determining the variable force.

Hooke's Law

The force required to maintain a spring stretched (or compressed) x units beyond its natural length is proportional to x, so $f(x) = kx$ where the positive constant k is called the **spring constant**. The restoring force supplied by the spring is $f(x) = -kx$ (which we will revisit in chapter 3 on differential equations).

Example 9. *A spring has a natural length of* 20 *cm. A force of* 35 *N is required to keep it stretched to a length of* 30 *cm. How much work is required to stretch it from* 25 *cm to* 35 *cm?*

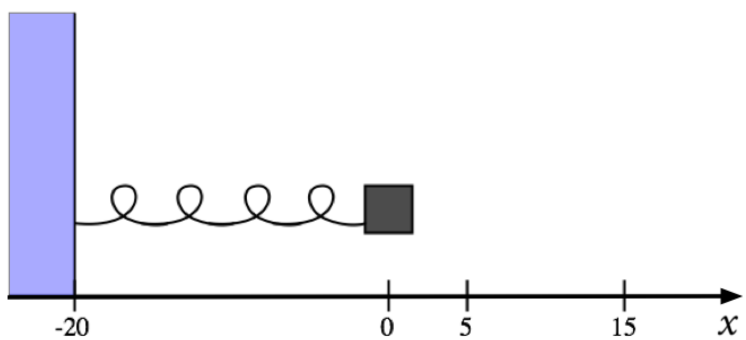

Figure 13

Solution: The first step is to find the spring constant. A force of 35 N is required to keep it stretched to a length of 30 cm. This is an elongation of 10 cm beyond its natural length of 20 cm. Since 10 cm = 0.1 m, Hooke's Law gives $35 = 0.1k$. Hence $k = \frac{35}{0.1} = 350$ N/m. Thus $f(x) = 350x$. The work done in stretching the spring from 25 cm (.05 m beyond its natural length) to 35 cm (.15 m beyond its natural length) is

$$\begin{aligned} W = \int_{.05}^{.15} 350x\,dx &= \left(350\frac{x^2}{2}\right)\Bigg|_{.05}^{.15} \\ &= \frac{350}{2}\left((.15)^2 - (.05)^2\right) \\ &= 175(.0225 - .0025) \\ &= 3.5 \text{ N-m} = 3.5 \text{ J} \end{aligned}$$

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Example 10. *A tank has the shape of a hemisphere (circular part at the top) with a radius of* 5 *meters. If the tank is full of water, find the work required to pump all of the water to the top of the tank. (The mass density of water is* 1000 kg/m^3*).*

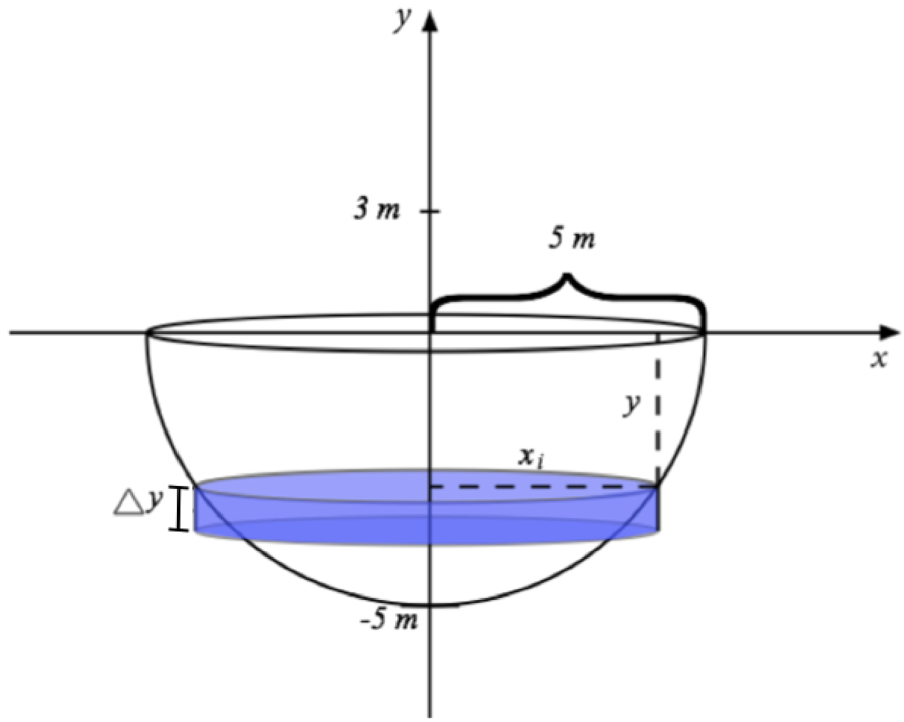

Figure 14

Solution: There are several ways we can put a coordinate system on our graph. We choose to put the origin at the center of the circular top and y as the vertical axis since it simplifies the equation of the sphere. Since different slices of the water are at different depths and the disks are of different volumes, it will take varying amounts of work to lift the slices to the top. All the water in a thin horizontal disk will have approximately the same depth. Therefore we will slice up the y-axis from $y = -5$ to $y = 0$ into n equal subintervals giving us $y_i = -5 + i\Delta y$ with $\Delta y = (0 - (-5))/n$ and $i = 0, \cdots, n$. These subintervals are used as one side of a rectangle with the other side going out to the hemisphere. We can choose to keep these rectangles in the xy-plane. The equation of the sphere is $x^2 + y^2 + z^2 = 25$. In the xy plane, $z = 0$ so we have $x^2 + y^2 = 25$ for the equation of the circle our rectangles touch. Since x_i is the length of our horizontal rectangle,

$$x_i = \sqrt{25 - y_i^2}$$

It is also the horizontal dimension of the i^{th} rectangle. Each of these rectangles is revolved around the y axis producing a disk of volume

$$V_i = \pi(x_i)^2 \Delta y = \pi(25 - y_i^2)\Delta y$$

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The mass density of water is 1000 kg/m^3 and the mass of a volume V_i is the mass density times the volume. The mass of the water in the i^{th} disk is

$$m_i = 1000 \times \pi(25 - y_i^2)\Delta y$$

The force of gravity on this mass of water is

$$F_i = m_i \times g = 9800\pi(25 - y_i^2)\Delta y$$

This disk of water is lifted from approximately y_i to 0. Thus the distance traveled is

$$d_i = 0 - y_i = -y_i$$

The work done on this disk to get it to the top is

$$W_i = F_i \times d_i = 9800\pi(25 - y_i^2)\Delta y\,(-y_i)$$

Adding these we get an approximation to the work done:

$$W \approx \sum_{i=1}^{n} 9800\pi(25 - y_i^2)(-y_i)\,\Delta y$$

The exact work done is then obtained by letting n increase without bound

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi(25 - y_i^2)(-y_i)\,\Delta y = 9800\pi \int_{-5}^{0} (y^3 - 25y)\,dy$$

Let's check our units, anticipating an appropriate answer for work, remembering that dy is measured in meters. The units of each factor are in the color blue.

$$\int \left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\pi\left((25 - y^2)\,\mathrm{m}^2\right)(-y\,\mathrm{m})dy\,\mathrm{m}$$

We get $\frac{(\mathrm{m})(\mathrm{kg})}{\mathrm{s}^2} \cdot \frac{(\mathrm{m}^2)(\mathrm{m})(\mathrm{m})}{\mathrm{m}^3} = \mathrm{N}\cdot\mathrm{m}$, or N-m = J.
Now we integrate and find the final solution.

$$\begin{aligned}
W &= 9800\pi \int_{-5}^{0} (y^3 - 25y)\,dy \\
&= 9800\pi\left(\frac{y^4}{4} - \frac{25y^2}{2}\right)\Bigg|_{-5}^{0} \\
&= 9800\pi\left((0) - \left(\frac{(-5)^4}{4} - \frac{25(-5)^2}{2}\right)\right) \\
&= 9800\pi\left(-\left(\frac{625}{4} - \frac{625}{2}\right)\right) \\
&= 9800\pi\left(\frac{625}{4}\right) \\
&= 1,531,250\pi \text{ J} \approx 4.81 \times 10^6 \text{ J}
\end{aligned}$$

Example 11. *A tank has the shape of a cylinder with a height of 3 meters and radius 5 meters that sits on top of a hemisphere with a radius of 5 meters. If the tank is full of water to the top of the hemisphere (bottom of the cylinder), find the work required to pump all of the water to the top of the cylinder. (The mass density of water is 1000 kg/m^3).*

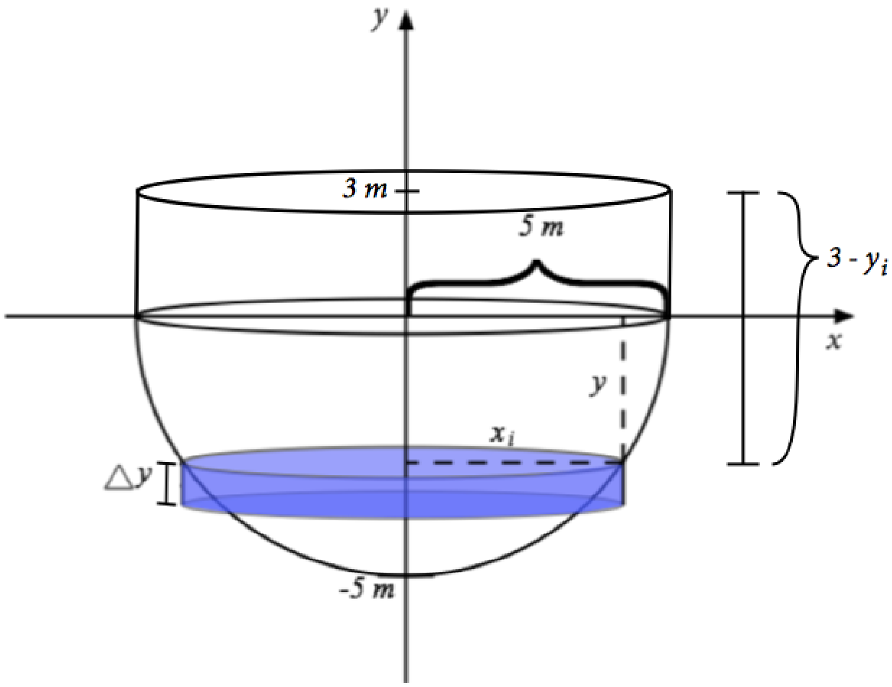

Figure 15

Solution: This example has many of the same properties as the last example. The major difference is that each disk of water needs to be lifted a distance of

$$d_i = 3 - y_i$$

Each disk of water has the same mass and hence the force of gravity on each is the same as in the last example.

$$F_i = m_i \times g = 9800\pi(25 - y_i^2)\Delta y$$

The work done to lift the disk of water to the top of the cylinder is

$$W_i = F_i \times d_i = 9800\pi(25 - y_i^2)\Delta y\,(3 - y_i)$$

Adding these we get an approximation to the work done:

$$W \approx \sum_{i=1}^{n} 9800\pi(25 - y_i^2)(3 - y_i)\,\Delta y$$

The exact work done is then obtained by letting n increase without bound

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi(25 - y_i^2)(3 - y_i)\,\Delta y = 9800\pi \int_{-5}^{0} (25 - y^2)(3 - y)\,dy$$

Now we integrate

$$\begin{aligned}
W &= 9800\pi \int_{-5}^{0} (25 - y^2)(3 - y)\,dy \\
&= 9800\pi \int_{-5}^{0} (75 - 25y - 3y^2 + y^3)\,dy \\
&= 9800\pi \left(75y - \frac{25y^2}{2} - \frac{3y^3}{3} + \frac{y^4}{4}\right)\Bigg|_{-5}^{0} \\
&= 9800\pi \left((0) - \left(75(-5) - \frac{25(-5)^2}{2} - \frac{3(-5)^3}{3} + \frac{(-5)^4}{4}\right)\right) \\
&= 9800\pi \left(\frac{1625}{4}\right) \\
&= 3,981,250\pi \text{ J} \approx 1.25 \times 10^7 \text{ J}
\end{aligned}$$

Example 12. *A cable* 60 *feet long and weighing* 90 *pounds hangs vertically from the top of a building. If a* 400*-lb generator is attached to the end of the cable, how much work is required to pull it to the top?*

Solution: First we draw a sketch and choose a coordinate system. We choose y for the vertical direction and place the origin 60 feet below the top of the building.

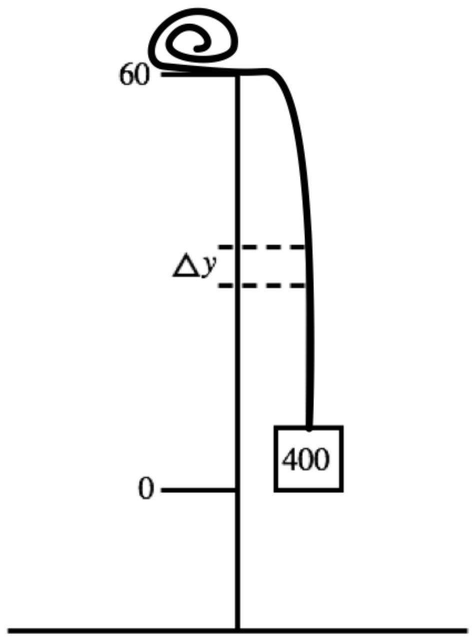

Figure 16

We divide the interval from $y = 0$ to $y = 60$ into n equal subintervals $[y_{i-1}, y_i]$ of length Δy. When the lifting begins, the initial force required is 490 pounds (all of the cable and the generator). As the cable is pulled up, there is less cable remaining to be lifted up; therefore, less force is required, lessened by 1.5 pounds for each linear foot (90 pounds/60 feet = 1.5 lb/ft). As a result, the variable force required at y_i is $F(y_i) = 490 - 1.5y_i$. The work done in lifting the weight from y_{i-1} to y_i is approximately

$$W_i \approx F(y_i)\Delta y = (490 - 1.5y_i)\,\Delta y$$

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} (490 - 1.5y_i)\,\Delta y \\ &= \int_0^{60} (490 - 1.5y)\,dy \\ &= 26700 \text{ ft-lb} \end{aligned}$$