Section 1.7: Group Actions

Juan Patricio Carrizales Torres

Jul 10, 2023

Intuitively speaking, a **Group Action** by some group G on a set A, is some type of "action" that permutes the elements of A in such a manner that group operations are maintained. More precisely, a **group action** by some group G on a set A is a map $\varphi : G \times A \to A$ such that for every $(g, a) \in G \times A$,

(a) $\varphi((g, a)) = ga \in A$.

(b) $g_1 \circ (g_2 a) = (g_1 \circ g_2)a$.

(c) $1a = a$.

Furthermore, if we let $\sigma_g : A \to A$ be a representation of the group action og the element g on A, defined by $\sigma_g(a) = ga$, then

(a) $\sigma_g \in S_A$, namely, a symmetric permutation of A.

(b) The map $G \to S_A$ defined by $g \to sigma_g$ is homomorphic.

This map $G \to S_A$ is called *permutation representation* associated to the given action. Where there is a bijective correspondence between the group action $G \to A$ and its permutation representation $G \times A \to A$. This is very usefull, since it implies that both are the same thing but expressed differently.

Other important concepts are the "faithful" characteristic and the kernel. A permutation representation is **faithfull** if it is injective. A kernel of a group action is the set

$$A = \{b \in G : gb = b, b \in A\} \supset \sigma_1 = i_A.$$

Note that the left cancellation law in groups ($a \circ b = a \circ c \implies b = c$) implies that the group action of a group G over itself defined by *left multplication* is **faithful**.