Week 12

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Section 4: Proofs involving sets

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If one wants to prove $A = B$, it is necessary to show both set inclusions $A \subseteq B$ and $B \subseteq A$, namely, some arbitrary $x \in A$ if and only if $x \in B$. In the case that either $A = \emptyset$ or $B = \emptyset$, their respective set inclusions follow vacuously.
NOTE. Assume all sets discussed belong to some Universal set.

Problem 40. Let A and B be sets. Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Proof. First we show that $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Without loss of generality, let $x \in A$. We consider two cases.
Case 1. $x \in A$ and $x \in B$. Then $x \in A \cap B$ and so $x \in (A - B) \cup (B - A) \cup (A \cap B)$.
Case 2. $x \in A$ and $x \notin B$. Then $x \in A - B$ and so $x \in (A - B) \cup (B - A) \cup (A \cap B)$.
Hence, $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$.

We then show that $(A-B)\cup(B-A)\cup(A\cap B) \subseteq A\cup B$. Let $x \in (A-B)\cup(B-A)\cup(A\cap B)$. Then, $x \in A - B$, $x \in B - A$ or $x \in A \cap B$. In all cases either $x \in A$ or $x \in B$. Therefore, $x \in A \cup B$ and so $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$. $\square$

Problem 41. In result 21, it was proved for sets A and B that $A \cup B = A$ if and only if $B \subseteq A$. Provide another proof of this result by giving a direct proof of the implication "If $A \cup B = A$, then $B \subseteq A$" and a proof by contrapositive of its converse.

Proof. First assume that $A \cup B = A$. Then $A \cup B \subseteq A$ and $A \subseteq A \cup B$. Note that by definition, $B \subseteq A \cup B$. Since $B \subseteq A \cup B$ and $A \cup B \subseteq A$, it follows that $B \subseteq A$.

For the converse, assume that $A \cup B \neq A$. Then either $A \cup B \not\subseteq A$ or $A \not\subseteq A \cup B$. Note that the former leads to a contradiction, thus we only consider that $A \cup B \not\subseteq A$. Then there is some $y \in A \cup B$ such that $y \notin A$. Therefore, $y \in B$. Since $y \notin A$ and $y \in B$, it follows that $B \not\subseteq A$. $\square$

Problem 42. Let A and B be sets. Prove that $A \cap B = A$ if and only if $A \subseteq B$.

Proof. Assume that $A \not\subseteq B$. Then, there is some $x \in A$ such that $x \notin B$. Hence $x \notin A \cap B$. Since $x \notin A \cap B$ and $x \in A$, it follows that $A \not\subseteq A \cap B$ and so $A \neq A \cap B$.

For the converse, assume that $A \cap B \neq A$. Then either $A \not\subseteq A \cap B$ or $A \cap B \not\subseteq A$. Note that the former leads to a contradiction, thus we only consider $A \not\subseteq A \cap B$. Then, there is some $y \in A$ such that $y \notin A \cap B$. Hence, $y \notin B$. Since $y \in A$ and $y \notin B$, it follows that $A \not\subseteq B$. $\square$

Problem 43. (a) Give and example of three sets A, B and C such that $A \cap B = A \cap C$ but $B \neq C$.

Solution a. Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{1, 3\}$. Then, $A \cap B = A \cap C = \{1\}$.

(b) Give an example of three sets A, B and C such that $A \cup B = A \cup C$ but $B \neq C$.

Solution b. Let $A = \{1, 2, 3\}$, $B = \{2\}$ and $C = \{3\}$. Then, $A \cup B = A \cup C = \{1, 2, 3\}$

(c) Let A, B and C be sets. Prove that if $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$.

Proof. Assume $B \neq C$. Then, either $B \not\subseteq C$ or $C \not\subseteq B$. Without loss of generality, let $B \not\subseteq C$. This means that there is some $y \in B$ such that $y \notin C$. We consider the following cases.
Case 1. $y \in A$. Since $y \in A$, $y \in B$ and $y \notin C$, it follows that $y \in A \cap B$ and $y \notin A \cap C$, and so $A \cap B \not\subseteq A \cap C$. Hence, $A \cap B \neq A \cap C$.
Case 2. $y \notin A$. Since $y \notin A$, $y \in B$ and $y \notin C$, it follows that $y \in A \cup B$ and $y \notin A \cup C$, and so $A \cup B \not\subseteq A \cup C$. Therefore, $A \cup B \neq A \cup C$. $\square$

Problem 44. Prove that if A and B are sets such that $A \cup B \neq \emptyset$, then $A \neq \emptyset$ or $B \neq \emptyset$.

Proof. Let $A = \emptyset$ and $B = \emptyset$. Then, $A \cup B = \emptyset \cup \emptyset = \emptyset$. $\square$

Problem 45. Let $A = \{n \in \mathbb{Z} : n \equiv 1 \pmod 2\}$ and $B = \{n \in \mathbb{Z} : n \equiv 3 \pmod 4\}$. Prove that $B \subseteq A$.

Proof. Let some $x \in B$. Then, $x \in \mathbb{Z}$ and $x \equiv 3 \pmod 4$, and so $x = 4m+3$ for some integer m. Note that $x = 4m + 3 = 4m + 2 + 1 = 2(2m + 1) + 1$. Since $2m + 1 \in \mathbb{Z}$, it follows that $2 \mid (x - 1)$ and so $x \equiv 1 \pmod 2$. So $x \in A$. Therefore, $B \subseteq A$. $\square$

Problem 46. Let A and B be sets. Prove that $A \cup B = A \cap B$ if and only if $A = B$.

Proof. First assume $A \neq B$. We show that $A \cup B \neq A \cap B$. Then, either $A \not\subseteq B$ or $B \not\subseteq A$, say the former. Thus, there is some $a \in A$ such that $a \notin B$. Therefore, $a \in A \cup B$ and $a \notin A \cap B$. Hence $A \cup B \neq A \cap B$.
For the converse, suppose $A = B$. Therefore, $A \cup B = A \cap B = A = B$. $\square$

Problem 47. Let $A = \{n \in \mathbb{Z} : n \equiv 2 \pmod 3\}$ and $B = \{n \in \mathbb{Z} : n \equiv 1 \pmod 2\}$.
(a) Describe the elements of the set $A - B$.

Solution a. If $x \in A$, then $x \in \mathbb{Z}$ and $x \equiv 2 \pmod 3$, and so $x = 3m + 2$ where $m \in \mathbb{Z}$. If $x \in B$, then $x \in \mathbb{Z}$ and $x \equiv 1 \pmod 2$, and so $x = 2q + 1$ where $q \in \mathbb{Z}$ (the odd integers). Note that $x = 3m + 2$ is odd when m is odd, and it is even when m is even. The set $A - B$ contains all integers x such that $x \equiv 2 \pmod 3$ and x is even. Thus, $A - B = \{n \in \mathbb{Z} : n = 3m + 2, \text{ where } m \text{ is an even integer}\}$. Since m is an even integer, it follows that $m = 2b$ for some integer b. Then, $3m + 2 = 3(2b) + 2 = 6b + 2$. Therefore, $A - B = \{n \in \mathbb{Z} : n = 6b + 2, \text{ where } b \in \mathbb{Z}\}$

(b) Prove that if $n \in A \cap B$, then $n^2 \equiv 1 \pmod{12}$.

Proof. Assume $n \in A \cap B$. Then, $n \in \mathbb{Z}$, $n \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 2$. This means that $n = 3q + 2$ and n is odd. Thus, q is odd and so $q = 2m + 1$ for some $m \in \mathbb{Z}$. Then, $n = 3(2m + 1) + 2 = 6m + 3 + 2 = 6m + 5$ and so

$$\begin{aligned} n^2 - 1 &= (6m + 5)^2 - 1 \\ &= 36m^2 + 60m + 25 - 1 \\ &= 12(3m^2 + 5m + 2) \end{aligned}$$

Since $3m^2 + 5m + 2 \in \mathbb{Z}$, it follows that $12 \mid (n^2 - 1)$ and so $n^2 \equiv 1 \pmod{12}$. $\square$

Problem 48. Let $A = \{n \in \mathbb{Z} : 2 \mid n\}$ and $B = \{n \in \mathbb{Z} : 4 \mid n\}$. Let $n \in \mathbb{Z}$. Prove that $n \in A - B$ if and only if $n = 2k$ for some odd integer k.

Proof. First, let $n \in A - B$. Then, $n \in \mathbb{Z}$, $2 \mid n$ and $4 \nmid n$. Then, n is even and so $n = 2k$ where $k \in \mathbb{Z}$. Since $4 \nmid n$, it follows that k must not be even. Thus, k is odd.

For the converse, assume $n = 2k$ for some odd integer k. Then, $k = 2m + 1$ where $m \in \mathbb{Z}$ and so $n = 2(2m + 1) = 4m + 2$. Since $n = 2(2m + 1)$, it follows that $2 \mid n$ and so $n \in A$. Also, since $n = 4m + 2$, it follows that $4 \nmid n$ and so $n \notin B$. Hence, $n \in A - B$.

$\square$

Problem 49. Prove for every two sets A and B that $A = (A - B) \cup (A \cap B)$.

Proof. First we prove that $A \subseteq (A - B) \cup (A \cap B)$. Let $x \in A$. We consider two cases.
Case 1. $x \in B$. Since $x \in A$ and $x \in B$, it follows that $x \in A \cap B$. Thus, $x \in (A - B) \cup (A \cap B)$.
Case 2. $x \notin B$. Since $x \in A$ and $x \notin B$, it follows that $x \in A - B$. Hence, $x \in (A - B) \cup (A \cap B)$.
Therefore, $A \subseteq (A - B) \cup (A \cap B)$

We then prove that $(A - B) \cup (A \cap B) \subseteq A$. Let $y \in (A - B) \cup (A \cap B)$. Then, either $y \in A - B$ or $y \in A \cap B$. Both cases imply that $y \in A$. Therefore, $(A - B) \cup (A \cap B) \subseteq A$. Hence, $A = (A - B) \cup (A \cap B)$. $\square$

Problem 50. Prove for every two sets A and B that $A - B$, $B - A$ and $A \cap B$ are pairwise disjoint.

Proof. Let $x \in A - B$. Then, $x \in A$ and $x \notin B$. Therefore, $x \notin B - A$ and $x \notin A \cap B$. Hence, $(A - B) \cap (B - A) = \emptyset$ and $(A - B) \cap (A \cap B) = \emptyset$.

Let $y \in B - A$. Then, $y \in B$ and $y \notin A$. Therefore, $y \notin A \cap B$ and so $(B - A) \cap (A \cap B) = \emptyset$. Thus, $A - B$, $B - A$ and $A \cap B$ are pairwise disjoint. $\square$