Week ??

Juan Patricio Carrizales Torres

Section 4: The Strong Principle of Mathematical Induction

January 27, 2022

Another form of mathematical induction is commonly known as **The Strong Principle of Mathematical Induction**. This technique is quite similar to the Principle of Mathematical Induction (in fact, if you use either to prove a theorem, it is possible to prove the same result with the remaining technique). The general Theorem goes as

Theorem 20. The Strong Principle of Mathematical Induction For a fixed integer m, let $S = \{a \in \mathbb{Z} : a \geq m\}$. For each $n \in S$, let $P(n)$ be a statement. If

1. $P(m)$ is true and
2. the quantified statement

   For any $k \in S$, if $P(i)$ for every integer i with $m \leq i \leq k$, then $P(k+1)$.

   is true,

then $\forall n \in S, P(n)$

Note that in the inductive hypothesis went from $\forall k \in S, P(k) \implies P(k+1)$ to $\forall k \in S, P(m) \wedge P(m+1) \wedge \ldots \wedge P(k) \implies P(k+1)$. Basically, with the **Strong Principle of Mathematical Induction** you have a more "inclusive" hypothesis, namely, you assume that $P(m) \wedge P(m+1) \wedge \ldots \wedge P(k)$ for any $k \in S$.

Problem 41. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$ and $a_2 = 2a_{n-1}$ for $n \geq 2$. Conjecture a formula for a_n and verify that your conjecture is correct.

Result . Let the sequence $\{a_n\}$ be defined recursively by $a_1 = 1$ and $a_2 = 2a_{n-1}$ for $n \geq 2$. Then $a_n = 2^{n-1}$, where $n \in \mathbb{N}$.

Proof. We proceed by induction. Since $a_1 = 2^0 = 1$, the result is true for $n = 1$. Assume that $a_k = 2^{k-1}$ for some $k \in \mathbb{N}$. We show that $a_{k+1} = 2^k$. Note that,

$$\begin{aligned} a_{k+1} &= 2a_k \\ &= 2(2^{k-1}) \\ &= 2^{k-1+1} = 2^k. \end{aligned}$$

By the Principle of Mathematical Induction, this result is true. □

Problem 42. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for a_n and verify that your conjecture is correct.

Result . Let $\{a_n\}$ be a sequence defined recursively by $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Then, $a_n = 2^{n-1}$ for any positive integer n.

Proof. We proceed by strong induction. Since $a_1 = 2^{1-1} = 1$, it follows that the result is true for $n = 1$. Suppose that $a_i = 2^{i-1}$ for $1 \leq i \leq k$, where $k \in \mathbb{N}$. We show that $a_{k+1} = 2^k$. Note that $a_2 = a_{1+1} = 2^1 = 2$ and so $a_{k+1} = 2^k$ is true for $k = 1$ and $k \geq 2$. Since $k + 1 \geq 3$, it follows that

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} \\ &= 2^{k-1} + 2^{k-1} = 2^k. \end{aligned}$$

By the Strong Principle of Mathematical Induction, this result is true. □

Problem 43. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 4, a_3 = 9$ and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$$

for $n \geq 4$. Conjecture a formula for a_n and prove that your conjecture is correct.

Result . Let $\{a_n\}$ be a sequence defined by $a_1 = 1, a_2 = 4, a_3 = 9$ and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$$

for $n \geq 4$. Then, $a_n = n^2$ for all $n \in \mathbb{N}$.

Proof. We proceed by the Strong Principle of Mathematical Induction. Because $a_1 = 1 = 1^2$, it follows that $a_n = n^2$ when $n = 1$. Assume that $a_i = i^2$ for $1 \leq i \leq k$ for some $k \in \mathbb{N}$. We prove that $a_{k+1} = (k+1)^2$. Since $a_{1+1} = 4 = (1+1)^2$ and $a_{2+1} = 9 = (2+1)^2$, it follows that $a_{k+1} = (k+1)^2$ is true for $k = 1, 2$ and so $k \geq 3$. Because $k + 1 \geq 4$, it follows that

$$\begin{aligned} a_{k+1} &= a_k - a_{k-1} + a_{k-2} + 2(2(k+1) - 3) \\ &= k^2 - (k-1)^2 + (k-2)^2 + 4k - 2 \\ &= k^2 - k^2 + 2k - 1 + k^2 - 4k + 4 + 4k - 2 \\ &= k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

By the Principle of Mathematical Induction, this result is true. □

Problem 44. Consider the sequence $F_1, F_2, F_3, \ldots,$ where

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5 \text{ and } F_6 = 8.$$

The terms of this sequence are called **Fibonacci numbers**.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.

We can express the sequence of Fibonacci numbers by means of the recursively defined sequence $\{a_n\}$, where $a_1 = 1, a_2 = 1$ and

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$.

(b) Prove that $2 \mid F_n$ if and only if $3 \mid n$.

Proof. Let $P(n) : 3 \mid n$, where $n \in \mathbb{N}$, if and only if F_n is even. We proceed by the Strong Principle of Mathematical Induction. Since $3 \nmid 1$ and $F_1 = 1$, it follows that $P(1)$ is true. Assume that $P(i)$ is true for $1 \leq i \leq k$, where $k \in \mathbb{N}$. We show that $P(k+1)$ is true. Because $3 \nmid (1+1)$ and $F_{1+1} = 1$, it follows that $P(k+1)$ is true for $k = 1$ and so $k \geq 2$. Hence, $k+1 \geq 3$.

Let $3 \mid (k+1)$. Then, $k+1 = 3c$ for some integer c and so $k = 3c - 1 = 3(c-1) + 2$ and $k - 1 = 3c - 2 = 3(c-1) + 1$. Therefore, k and $k+1$ are not divisible by 3, and so, by the inductive hypothesis, $F_{k+1} = F_k + F_{k-1}$ is the sum of two odd integers, which leads to an even integer.

We now show the converse. Suppose that $3 \nmid k+1$. Then, either $k+1 = 3c+1$ or $k+1 = 3c+2$ for some integer c. It is easy to see that in both cases, either $3 \mid k$ or $3 \mid (k-1)$, but not both. Hence, by our inductive assumption, $F_{k+1} = F_k + F_{k-1}$ is the sum of two integers of opposite parity which implies that F_{k+1} is odd.

By the Strong Principle of Mathematical Induction, the implication $P(n)$ is true for all $n \in \mathbb{N}$.

$\square$

Problem 45. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$.

Proof. We proceed by using the Strong Principle of Mathematical Induction. Since $12 = 3(4) + 7(0)$, it follows that the result is true for $n = 12$. Assume that $i = 3a + 7b$, where a and b are some arbitrary nonnegative integers, for $12 \leq i \leq k$. We show that the result is true for $k+1$. Because

$$13 = 3(2) + 7(1) \text{ and}$$
$$14 = 3(0) + 7(2),$$

it follows that the result is true for $k = 13, 14$ and so $k \geq 14$. Hence, $k+1 \geq 15$. Note that

$$k + 1 = (k-2) + 3$$

and, by the inductive hypothesis, we have

$$\begin{aligned} k+1 &= (3x+7y)+3 \\ &= 3(x+1)+7(y). \end{aligned}$$

where x and y are some arbitrary nonnegative integers. Since $x+1$ and y are nonnegative integers, it follows that, by the Strong Principle of Mathematical Induction, that the for each integer $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$. □

Problem 46. Use the Strong Principle of Mathematical Induction to prove the following. Let $S = \{i \in \mathbb{Z} : i \geq 2\}$ and let P be a subset of S with the properties that $2, 3 \in P$ and if $n \in S$, then either $n \in P$ or $n = ab$, where $a, b \in S$. Then every element of S either belongs to P or can be expressed as a product of elements of P. [Note: You might recognize the set P of primes. This is an important theorem in mathematics.]

Proof. We proceed by strong induction. Assume that $S = \{i \in \mathbb{Z} : i \geq 2\}$ and let P be a subset of S with the desired properties. Since $2 \in P$, it follows that the result is true for $n = 2$. Assume that either $i \in P$ or i is equal to the product of elements of P for $2 \leq i \leq k$, where $k \geq 2$. We show that either $k+1 \in P$ or $k+1$ is a product of elements of P. .

Since $k+1 \in S$, it follows that either $k+1 \in P$ or $k+1 = ab$, where $a, b \in S$. In the former, the result is satisfied. In the case of the latter, $k \notin P$ and $k+1 = ab$ where $a, b \in S$. However, we know that $2 \leq a \leq k$ and $2 \leq b \leq k$, which implies, by our inductive hypothesis, that each integer of a and b is either in P or is a product of elements of P. In all possible cases, the integer $k+1$ ends up being a product of elements of P. By the Strong Principle of Mathematical Induction, this result is true. □

Problem 47. Prove that there exists an odd integer m such that every odd integer n with $n \geq m$ can be expressed either as $3a + 11b$ or as $5c + 7d$ for nonnegative integers a, b, c and d.

Proof. Let $m = 17$. Since $17 = 3(2) + 11(1)$, it follows that the result is true for $n = 17$. Assume for $17 \leq i \leq k$ that if i is odd, then i can be expressed either as $3a + 11b$ or as $5c + 7d$ for nonnegative integers a, b, c and d. We show that if $k+1$ is odd, then it can be expressed either as $3e + 11f$ or as $5g + 7h$ for nonnegative integers e, f, g and h. Note that

$$\begin{aligned} 19 &= 5(1) + 7(2), \\ 21 &= 3(7) + 11(0), \\ 23 &= 3(4) + 11(1), \\ 25 &= 3(1) + 11(2), \\ 27 &= 5(4) + 7(1), \\ 29 &= 3(6) + 11(1). \end{aligned}$$

Hence $k \geq 29$ and so $k+1 \geq 30$. Suppose that $k+1$ is odd. Then,

$$k+1 = (k-11) + 12.$$

Since $17 \leq k-11 \leq k$ and $k-11$ is odd, it follows that either $k-11 = 3e+11f$ or $k-11 = 5g+7h$ for some nonnegative integers e, f, g and h. Note that $12 = 3(4) = 5+7$. Hence, if $k-11 = 3e+11f$, then $k+1 = 3e+11f+3(4) = 3(e+4)+11f$. On the other hand, if $k-11 = 5g+7h$, then $k+1 = 5g+7h+(5+7) = 5(g+1)+7(h+1)$.

By the Strong Priciple of Mathematical Induction, every odd integer n with $n \geq 17$ can be expressed either as $3a+11b$ or as $5c+7d$ for nonnegative integers a, b, c and d. $\square$