Section 8.2: Properties of relations

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This chapter mentioned three properties of interested for some relation R on a single set A. Since most of these properties involve implications with universal quantifiers, the easiest way to check wether a relation has certain property is by looking for specific examples for which the implication in question is false.

(a) **Reflexive Property:** if $x \in A$, then $(x, x) \in R$. (x is related to itself)

(b) **Symmetric Property:** $\forall x, y \in A$, if $x \; R \; y$, then $y \; R \; x$ (x is related to y and viceversa). Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$ and $y \not R \; x$. For this to happen, it is necessary that $x \neq y$.

(c) **Transitive Property:** $\forall x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$. Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$, $y \; R \; z$ and $x \not R \; z$. For this to happen, it is necessary that $x \neq y$ and $z \neq y$.

Problem 11. Let $A = \{a, b, c, d\}$ and let

$$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$$

be a relation on A. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 11. The relation is reflexive since $\{(a, a), (b, b), (c, c), (d, d)\} \subset R$. Also, it is transitive since $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in A$ is fulfilled. However, the relation is not symmetric since $(a, b) \in R$ and $(b, a) \notin R$.

Problem 13. Let $S = \{a, b, c\}$. Then $R = \{(a, b)\}$ is a relation on S. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 13. The relation S is transitive since the implication $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in S$ is fulfilled vacuously. However, it is neither reflexive because $(a, a) \notin R$ nor symmetrice since $(a, b) \in R$ but $(b, a) \notin R$.

Problem 14. Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

Solution 14. Let $R = \{(a, b), (b, c)\}$. The relation R is not reflexive since $(a, a) \notin R$, it is not symmetric because $(a, b) \in R$ and $(b, a) \notin R$ and it is not transitive since $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Problem 15. A relation R is defined on $\mathbb{Z}$ by $a \; R \; b$ if $|a - b| \leq 2$. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 15. The relation R is reflexive since $|a - a| = 0 \leq 2$ for any $a \in \mathbb{Z}$ and so $a \; R \; a$. It is symmetric since for any $a, b \in \mathbb{Z}$, if $|a - b| \leq 2$, then $|b - a| = |a - b| \leq 2$. However, it is not transitive since $|3 - 1| = 2$ and $|1 - 0| = 1$ but $|3 - 0| = 3 > 2$.

Problem 16. Let $A = \{a, b, c, d\}$. How many relations defined on A are reflexive, symmetric and transitive and contain the ordered pairs $(a, b), (b, c), (c, d)$?

Solution 16. In order for a relation R on A to be reflexive it must be true that $\{(a, a), (b, b), (c, c), (d, d)\} \subseteq R$. Since $(a, b), (b, c), (c, d) \in R$, it follows that $(b, a), (c, b), (d, c) \in R$ so that R is symmetric. Because, so far

$$\{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c)\} \subseteq R$$

, it follows that $(a, c), (c, a), (b, d) \in R$ for R to be transitive. Since $(b, d) \in R$, it follows that $(d, b) \in R$ so that the symmetric property is mantained. However, $(d, b), (b, a) \in R$ and so $(d, a) \in R$ so that it is transitive. This implies $(a, d) \in R$ since R must be symmetric. Hence,

$$\begin{aligned} R &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c), (a, c), (c, a), (b, d), (d, b), (d, a), (a, d)\} \\ &= A \times A \end{aligned}$$

Since $R \subseteq A \times A$, it follows that there is only one possible relation R on A that fulfills the conditions.

Problem 18. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric but not transitive.

Solution (a). $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (3, 1), (1, 3)\}$

(b) reflexive and transitive but not symmetric.

Solution (b). $R = \{(a, a), (b, b), (c, c), (d, d), (b, c)\}$

(c) symmetric and transitive but not reflexive.

Solution (c). $R = \emptyset$ (the symmetric and transitive logical implications are vacuously true)

(d) reflexive but neither symmetric nor transitive.

Solution (d). $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c)\}$

(e) symmetric but neither reflexive nor transitive.

Solution (e). $R = \{(a,b),(b,a)\}$

(f) transitive but neither reflexive nor symmetric.

Solution (f). $R = \{(a,b)\}$ (The transitive implication follows vacuously)

All of these are counterexamples to the statement that one property implies the other for any relation R on some nonempty set A.

Problem 19. A relation R is defined on $\mathbb{Z}$ by $x\ R\ y$ if $x \cdot y \geq 0$. Prove or disprove the following:

(a) R is reflexive.

Proof. Consider some $x \in \mathbb{Z}$, then $x^2 \geq 0$ and so $x\ R\ x$. The relation R is reflexive. □

(b) R is symmetric.

Proof. Consider some $x, y \in \mathbb{Z}$. Assume that $x\ R\ y$ which implies that $x \cdot y \geq 0$. Since multiplication on real numbers is commutative, it follows that $y \cdot x = x \cdot y \geq 0$ and so $y\ R\ x$. The relation R is symmetric. □

(c) R is transitive.

Solution c. The relation R on $\mathbb{Z}$ is not transitive. Note that $-3\ R\ 0$ and $0\ R\ 1$, but $-3 \cdot 1 = -3 < 0$ and so $-3 \not R\ 1$.

Problem 20. Determine the maximum number of elements in a relation R on a $3-$*element* set such that R has none of the properties reflexive, symmetric and transitive.

Solution 20. Let R be a relation on a 3-element set B that has none of the properties reflexive, symmetric and transitive. Let's check the maximum number of elements R can contain. Since $R \subseteq B \times B$, it follows that $|R| \leq 9$. However, since R is not reflexive, it follows that $(b,b) \notin R$ for some $b \in B$ and so $|R| \leq 8$.

Because R is not symmetric, it follows that $(b,a) \in R$ and $(a,b) \notin R$ for some different $a, b \in B$ and so $|R| \leq 7$. Also, since R is not transitive, it follows that $(a,b),(b,c) \in R$ and $(a,c) \notin R$ for some $a, b, c \in B$ such that $a \neq b$ and $b \neq c$. Thus, either $c \neq a$ or $c = a$, however note that we already got rid of those two such ordered pairs and so the maximum number of elements in R is 7.

Problem 22. Let S be the set of all polynomials of degree at most 3. An element $s(x)$ of S can then be expressed as $s(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. A relation R is defined on S by $p(x)\ R\ q(x)$ if $p(x)$ and $q(x)$ have a real root in common. (For example, $p(x) = (x-1)^2$ and $q(x) = x^2 - 1$ have the root 1 in common so that $p\ R\ q$.) Determine which of the properties reflexive, symmetric, and transitive are possessed by R.

1. The relation R is reflexive.

   Solution (a). The relation R on S is not reflexive. Consider $p(x) = x^2+1$. Therefore, $p(x) \in S$ but $p(x) \not R\ p(x)$ since $p(x)$ has no real root.

2. The relation R is symmetric.

   Proof. Consider some $p(x), q(x) \in S$. Assume that $p(x)\ R\ q(x)$ and so $p(x)$ and $q(x)$ share some real root c. Therefore, $q(x)$ and $p(x)$ share the real root c which implies that $q(x)\ R\ p(x)$. □

3. The relation R is transitive.

   Solution (c). The relation R is not transitive. Let $p(x) = x^2 - 1$, $q(x) = (x-1)^2$ and $r(x) = (x+1)^2$. Hence, $p(x), q(x), r(x) \in S$. Note that $p(x)$ has real roots -1 and 1, $q(x)$ has only the real root 1 and $r(x)$ only has the real root -1. Then, $r(x)\ R\ p(x)$ and $p(x)\ R\ q(x)$. However, $r(x)$ and $q(x)$ do not have some real root in common and so $r(x) \not R\ q(x)$.

Problem 23. A relation R is defined on $\mathbb{N}$ by $a\ R\ b$ if either $a \mid b$ or $b \mid a$. Determine which of the properties reflexive, symmetric and transitive are possessed by R.

Solution 23. The reflexive property follows instantly, every positive integer is divisible by itself. The symmetric property follows immeaditly too since, by the condition of the relation, if $a\ R\ b$ it is assured that $b\ R\ a$.
However, this relations is not transitive (this has to do with the disjunction). Consider the positive integers $4, 3$ and 1. Then, $4\ R\ 1$ and $1\ R\ 3$ (recall that $(a, b) \in \mathbb{R} \iff$ either $a \mid b$ or $b \mid a$). However, $3 \nmid 4$ and $4 \nmid 3$ and so $4 \not R\ 3$.