Week 8

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Section 1: Trivial and Vacuous Proofs

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Problem 1. Let $x \in \mathbb{R}$. Prove that if $0 < x < 1$, then $x^2 - 2x + 2 \neq 0$.

Proof. Note that

$$x^2 - 2x + 1 = (x-1)^2 \geq 0$$

Thus, $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1 > 0$. Hence, $x^2 - 2x + 2 \neq 0$ is true for all $x \in \mathbb{R}$ and the implication is true trivially. □

Problem 2. Let $n \in \mathbb{N}$. Prove that if $|n-1| + |n+1| \leq 1$, then $|n^2 - 1| \leq 4$.

Proof. Note that for $n \in \mathbb{N}$, $|n+1| \geq 2 > 1$. Thus, $|n-1| + |n+1| \leq 1$ is false for all $n \in \mathbb{N}$ and the implication follows vacuously. □

Problem 3. Let $r \in \mathbb{Q}^+$. Prove that if $\frac{r^2+1}{r} \leq 1$, then $\frac{r^2+2}{r} \leq 2$.

Proof. Note that $\frac{r^2+1}{r} = r + \frac{1}{r}$. If $r \geq 1$, then $r + \frac{1}{r} > 1$. On the other hand, if $0 < r < 1$, then $\frac{1}{r} > 1$ and $r + \frac{1}{r} > 1$. Hence, $\frac{r^2+1}{r} \leq 1$ is false for all $r \in \mathbb{Q}^+$ and the implication follows vacuously. □

Problem 4. Let $x \in \mathbb{R}$. Prove that if $x^3 - 5x - 1 \geq 0$, then $(x-1)(x-3) \geq -2$.

Solution . Note that $(x-1)(x-3) = x^2 - 4x + 3$. Also,

$$x^2 - 4x + 4 = (x-2)^2 \geq 0$$

Thus, $x^2 - 4x + 3 = (x-2)^2 - 1 \geq -1$. Therefore, $(x-1)(x-3) \geq -2$ is true for all $x \in \mathbb{R}$ and this implication is true trivially.

Problem 5. Let $n \in \mathbb{N}$. Prove that if $n + \frac{1}{n} < 2$, then $n^2 + \frac{1}{n^2} < 4$.

Solution . For $n = 1$, $n + \frac{1}{n} = 2$. On the other cases, $n \geq 2$. Thus, $n + \frac{1}{n} < 2$ is false for all $n \in \mathbb{N}$ and this implication follows vacuously.

Problem 6. Prove that if a, b and c are odd integers such that $a + b + c = 0$, then $abc < 0$. (You are permitted to use well-known properties of integers here.)

Solution . Let $a = 2m + 1$, $b = 2k + 1$ and $c = 2n + 1$ for some $m, k, n \in \mathbb{Z}$. Note that $a+b+c = 2(m+k+n)+3$. The sum of three odd integers is an odd integer. Since 0 is an even integer, it follows that $a+b+c = 0$ is false for all odd integers a, b, c. This implication is true vacuously.

Problem 7. Prove that if x, y and z are three real numbers such that $x^2 + y^2 + z^2 < xy + xz + yz$, then $x + y + z > 0$.

Solution . Since $(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0$, it follows that $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz \geq 0$. Therefore, $x^2 + y^2 + z^2 \geq xy + xz + yz$ for all $x, y, z \in \mathbb{R}$. This implication is true vacuously.