Week 1

Juan Patricio Carrizales Torres

Section 7.2: Revisiting Quantified Statements

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Problem 10. Express the following quantified statement in symbols:

For every odd integer n, the integer $3n + 1$ is even.

and prove it true.

Solution . Let T be the set of odd integers and $P(n)$: the integer $3n + 1$ is even.

$\forall n \in T, P(n)$.

Proof. Since n is odd, it follows that $3n$ is odd. Then, $3n + 1$ is the sum of two odd integers, which is even. □

Problem 11. Express the following quantified statement in symbols:

There exists a positive even integer n such that $3n + 2^{n-2}$ is odd.

and prove it true.

Solution . Let S^+ be the set of positive even integers and $P(n) : 3n + 2^{n-2}$ is odd. Then

$\exists n \in S^+, P(n)$.

Proof. Consider $n = 2$. Then $3(2) + 2^{2-2} = 6 + 1 = 7$ is odd. □

Problem 12. Express the following quantified statement in symbols:

For every positive integer n, the integer n^{n-1} is even.

and prove it false.

Solution . Let $P(n)$: is even. Then
$\forall n \in \mathbb{N}, P(n)$.

This statement is false. Consider $n = 1$. Then $1^{1-1} = 1$ is odd. Also, let $n \geq 3$ be some odd number. Then n^{n-1} is the multiplication of odd numbers, which is odd. (a and c are odd $\iff$ ab is odd).

Problem 13. Express the following quantified statement in symbols:

There exists an integer n such that $3n^2 - 5n + 1$ is an even integer.

and prove it false.

Solution . Lemma ODD. Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a finite set of n integers, where the integer $n \geq 2$. Then $\prod_{i=1}^{n} a_i$ is odd if and only if every integer a_i is odd.

Proof. We prove this by induction. First, suppose that all integers considered are odd. Since ab is odd $\iff$ a and b are odd, it follows that the result is true for $n = 2$. Assume for some set $\{b_1, b_2, b_3, \ldots, b_k\}$ of $k \geq 2$ odd integers that $\prod_{i=1}^{k} b_i$ is odd. We show for some set $\{c_1, c_2, c_3, \ldots, c_{k+1}\}$ of $k+1$ odd integers that $\prod_{i=1}^{k+1} c_i$ is odd. Note that

$$\prod_{i=1}^{k+1} c_i = \left(\prod_{i=1}^{k} c_i\right) \cdot c_{k+1}$$

is odd since it is a multiplication of two odd integers according to our inductive hypothesis. By the Principle of Mathematical Induction, if every a_i is odd, then $\prod_{i=1}^{n} a_i$ is odd.

For the converse, suppose that at least some a_i is even. Then, the multiplication of two integers, where one of them is even, is even since ab is odd $\iff$ a and b are odd. By the Principle of Mathematical Induction, if some a_i is even, then $\prod_{i=1}^{n} a_i$ is even. $\square$

We proceed with the problem. Let $P(n) : 3n^2 - 5n + 1$ is an even integer. Then,
$\forall n \in \mathbb{Z}, P(n)$.

We prove this statement false. Let n be odd. Note that $3n^2$ and $5n$ are a multiplication of 3 and 2 odd integers, respectively. By **Lemma ODD**, $3n^2 + 5n$ is a sum of two odd integers, which is an even number. Therefore, $(3n^2 + 5n) + 1$ is the sum of an even and odd number, which is odd.

Suppose n is even. Then, by **Lemma ODD**, $3n^2$ and $5n$ are even and so $(3n^2 + 5n) + 1$ is the sum of an even number and an odd number, which is odd.

Problem 14. Express the following quantified statement in symbols:

For every integer $n \geq 2$, there exists an integer m such that $n < m < 2n$

and prove it true.

Solution . Let $A = \{x \in \mathbb{Z} : x \geq 2\}$ and $P(n, m) : n < m < 2n$. Then,
$\forall n \in A, \exists m \in \mathbb{Z}, P(n, m)$

Proof. Consider some integer $n \geq 2$. Then $n < n + 1 = m < n + 2 \leq 2n$. $\square$

Problem 24. Express the following quantified statement in symbols:

For every three odd integers a, b and c, their product abc is odd.

and prove it true.

Solution . Let $P(a,b,c) : abc \in T$. Then
$\forall a, b, c \in T, P(a,b,c)$
Remember that $\forall a \in B, \forall b \in B \equiv \forall a, b \in B$.

Proof. Let a, b and c be odd integers. Then, $a = 2m+1, b = 2n+1$ and $c = 2l+1$. Therefore,

$$\begin{aligned} abc &= (2m+1)(2n+1)(2l+1) \\ &= (4mn+2m+2n+1)(2l+1) \\ &= 8lmn+4ml+4nl+2l+4mn+2m+2n+1 \\ &= 2(4lmn+2ml+2nl+l+2mn+m+n)+1. \end{aligned}$$

Thus, abc is odd. □

Problem 25. Consider the following statement.

R : There exists a real number L such that for every positive real number e, there exists a positive real number d such that if x is a real number with $|x| < d$, then $|3x - L| < e$.

Use $P(x,d) : |x| < d$ and $Q(x,L,e) : |3x - L| < e$ to express the statement R in symbols. Prove R true.

Solution . $\exists L \in \mathbb{R}, \forall e \in \mathbb{R}^+, \exists d \in \mathbb{R}^+, \forall x \in \mathbb{R}, P(x,d) \implies Q(x,L,e)$

Proof. Let $L = 0$ and $d = \frac{e}{3}$. Now, consider some real number x such that $|x| < \frac{e}{3}$ since $e > 0$. Therefore, $3|x| = |3x| < 3\frac{e}{3} = e$ and so $|3x - 0| = |3x - L| < e$. □

Problem 26. Prove the following statement. For every positive real number a and positive rational number b, there exist a real number c and irrational number d such that $ac+bd = 1$.

Proof. Let $a \in \mathbb{R}^+$ and $b \in \mathbb{Q}^+$. Then $d = \frac{1-r}{b}$ and $c = \frac{r}{a}$, where r is any irrational number. This is posible since $a, b > 0$. Note that d is irrational since b is rational and $1-r$ is irrational. Therefore,

$$ac + bd = a\left(\frac{r}{a}\right) + b\left(\frac{1-r}{b}\right) = 1$$

is the sum of two irrational numbers that equals to 1.
. □

Problem 27. Prove the following statement. For every integer a, there exist integers b and c such that $|a - b| > cd$ for every integer d.

Proof. Let $a \in \mathbb{Z}$. Consider some integer $b \neq a$ and let $c = 0$. Therefore, $|a - b| > 0 = cd$ for every integer d. □