Chapter 1: Spaces

Juan Patricio Carrizales Torres

Sep 29, 2022

1 Fields

In Linear Algebra, we will be working with numbers from any type of class/set. Hence, to simplify things and make them more general, we will introduce the idea of fields. A **field** is a set of objects (including numbers) called **scalars** with operations of addition and multiplication that fulfill the following rules (let α and β be scalars):

(a) **Addition**

- (a) commutatitivity, $\alpha + \beta = \beta + \alpha$.
- (b) associativity, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- (c) additive identity, there is a unique scalar 0 such that for every scalar α, $\alpha + 0 = \alpha$.
- (d) additive inverse, for each scalar α there is a unique scalar $-\alpha$ such that $\alpha + (-\alpha) = 0$.

(b) **Multiplication**

- (a) commutativity, $\alpha\beta = \beta\alpha$.
- (b) associativity, $\gamma(\alpha\beta) = (\gamma\alpha)\beta$.
- (c) multiplicative identity, there is a unique scalar 1 for every scalar α such that $1\alpha = \alpha$.
- (d) multiplicative inverse, for every nonzero scalar β, there is a unique β^{-1} such that $\beta\beta^{-1} = 1$.

(c) **Linearity**

- (a) Multiplication is distributive over addition, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

For instance, the class of real numbers and the class of complex numbers are fields.

1.1 Excercises

Problem 1. Almost all the laws of elementary arithmetic are consequences of the axioms defining a field. Prove, in particular, that if $\mathcal{F}$ is a field, and if α, β and γ belong to $\mathcal{F}$, then the following relations hold.

(a) $0 + \alpha = \alpha$

Proof. Due to the commutativity property of addition, $\alpha = \alpha + 0 = 0 + \alpha$. □

(b) If $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

Proof. Due to the additive inverse, associativity and commutativity, $\alpha + \beta + (-\alpha) = \alpha + (\beta + (-\alpha)) = (\alpha + (-\alpha)) + \beta = \beta = \gamma$. □

(c) $\alpha + (\beta - \alpha) = \beta$.

Proof. Just like in (b),

$$\begin{aligned} \alpha + (\beta + (-\alpha)) &= \alpha + (-\alpha + \beta) \\ &= (\alpha + (-\alpha)) + \beta = 0 + \beta \\ &= \beta. \end{aligned}$$

□

(d) $\alpha \cdot 0 = 0 \cdot \alpha = 0$. (In this case, the dot indicates multiplication).

Proof. Note that

$$\begin{aligned} 0 \cdot \alpha + (-0 \cdot \alpha) &= 0 = (0 + 0)\alpha + (-0 \cdot \alpha) \\ &= 0 \cdot \alpha + (0 \cdot \alpha + (-0 \cdot \alpha)) = 0 \cdot \alpha \\ &= \alpha \cdot 0 \end{aligned}$$

□

(e) $(-1)\alpha = -\alpha$

Proof. Observe that

$$\begin{aligned} \alpha + (-\alpha) &= 0 = 0\alpha \\ &= (1 - 1)\alpha = \alpha + (-1)\alpha. \end{aligned}$$

By (b), $-\alpha = (-1)\alpha$. □

(f) $(-\alpha)(-\beta) = \alpha\beta$.

Proof. By (e), $(-\alpha)(-\beta) = ((-1)\alpha)((-1)\beta)$. Then,

$$\begin{aligned}
((-1)\alpha)((-1)\beta) &= (-1)(\alpha((-1)\beta)) \\
&= (-1)((\alpha(-1))\beta) = (-1)(((-1)\alpha)\beta) \\
&= ((-1)((-1)\alpha))\beta = (((-1)(-1))\alpha)\beta \\
&= (1\alpha)\beta = \alpha\beta
\end{aligned}$$

$\square$

(g) $\alpha\beta = 0 \implies \alpha = 0$ or $\beta = 0$.

Proof. Let $\alpha\beta = 0$. Note that either $\alpha = 0$ or $\alpha \neq 0$. In the first case, the result is true. In the case of the latter, there is some α^{-1} and so $\alpha^{-1}\alpha\beta = 1\beta = \alpha^{-1}0 = 0$. $\square$