Week 9

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Section 4: Proof by Cases

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Let S be the domain of some open sentence $P(x)$. It can be useful to observe that an arbitrary $x \in S$ may have one of two or more properties (i.e. belong to a particular subset of S). If one proves that $P(x)$ is true for each property that x can have, then $P(x)$ is shown to be true. Then the proof of $P(x)$ can be divided into cases, and, if necessary, each case into subcases. This method is called Proof by Cases.
When the proof of two cases is similar, one can use **without loss of generality** (WLOG) to indicate that only the proof of one situation is needed.

Problem 26. Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

Proof. We proceed by cases, according to whether n is even or odd.
Case 1. Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$(2k)^2 - 3(2k) + 9 = 4k^2 - 6k + 8 + 1 = 2(2k^2 - 3k + 4) + 1$$

Since $2k^2 - 3k + 4 \in \mathbb{Z}$, $n^2 - 3n + 9$ is odd.
Case 2. Assume n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Thus,

$$(2k+1)^2 - 3(2k+1) + 9 = 4k^2 + 4k + 1 - 6k - 3 + 9 = 4k^2 - 2k + 1 + 6 = 2(2k^2 - k + 3) + 1$$

Because $2k^2 - k + 3 \in \mathbb{Z}$, $n^2 - 3n + 9$ is odd. $\square$

Problem 27. Prove that if $n \in \mathbb{Z}$, then $n^3 - n$ is even.

Proof. For this proof we consider two cases.
Case 1. Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Hence,

$$(2k)^3 - 2k = 8k^3 - 2k = 2(4k^3 - k)$$

Since $4k^3 - k$ is an integer, $n^3 - n$ is even.
Case 2. Let n be odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}(2k+1)^3 - 2k - 1 &= 8k^3 + 12k^2 + 6k + 1 - 2k - 1 \\ &= 8k^3 + 12k^2 + 4k \\ &= 2(4k^3 + 6k^2 + 2k)\end{aligned}$$

Because $4k^3 + 6k^2 + 2k$ is an integer, it follows that $n^3 - n$ is even. $\square$

Problem 28. Let $x, y \in \mathbb{Z}$. Prove that if xy is odd, then x and y are odd.

Proof. Assume that x or y is even. Without loss of generality, let x be even. Then $x = 2a$ for some $a \in \mathbb{Z}$. Therefore, $2a(y) = 2(ay)$.
Since $ay \in \mathbb{Z}$, it follows that xy is even. $\square$

Problem 29. Let $a, b \in \mathbb{Z}$. Prove that if ab is odd, then $a^2 + b^2$ is even.

Proof. Assume ab is odd. By result 28, a and b are odd. Therefore $a = 2m+1$ and $b = 2n+1$ for some $m, n \in \mathbb{Z}$. Hence,

$$\begin{aligned}(2m+1)^2 + (2n+1)^2 &= 4m^2 + 4m + 1 + 4n^2 + 4n + 1 \\ &= 4m^2 + 4m + 4n^2 + 4n + 2 \\ &= 2(2m^2 + 2m + 2n^2 + 2n + 1)\end{aligned}$$

Since $2m^2 + 2m + 2n^2 + 2n + 1 \in \mathbb{Z}$, it follows that $a^2 + b^2$ is even. $\square$

Problem 30. Let $x, y \in \mathbb{Z}$. Prove that $x - y$ is even if and only if x and y are of the same parity.

Lemma If $n \in \mathbb{Z}$, then n and $-n$ are of the same parity.

Proof. We consider two cases.
Case 1. Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore, $-n = -(2k) = 2(-k)$. Since $-k$ is an integer, it follows that n is even.
Case 2. Let n be odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Thus, $-n = -(2k+1) = -2k-1 = -2k-2+1 = 2(-k-1)+1$. Since $-k-1$ is an integer, $-n$ is odd. $\square$

Proof. We now proceed to prove the result.
Let x and y be of opposite parity. By lemma, $x + (-y)$ is a sum of two integers of opposite parity. Therefore, by theorem 16, $x - y$ is odd.
For the converse, assume x and y are of the same parity. By lemma, $x + (-y)$ is a sum of two integers of the same parity. Therefore, by theorem 16, $x - y$ is even. $\square$

Problem 31. Let $a, b \in \mathbb{Z}$. Prove that if $a + b$ and ab are of the same parity, then a and b are even.

Proof. Assume that a or b are odd. We consider two cases.
Case 1. Without loss of generality, let b be odd and a be even. Then $b = 2m+1$ and $a = 2k$ for some $m, k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}a + b &= 2k + 2m + 1 = 2(k+m) + 1 \\ ab &= (2k)(2m+1) = 4km + 2k = 2(2km + k)\end{aligned}$$

Since both $2km + k$ and $k + m$ are integers, $a + b$ is odd and ab is even. They are of opposite parity.

Case 2. Let a and b be odd. Then $a = 2k+1$ and $b = 2m+1$ for some $k, m \in \mathbb{Z}$. Thus,

$$\begin{aligned} a+b &= 2k+1+2m+1 = 2(k+m+1) \\ ab &= (2k+1)(2m+1) = 4km+2k+2m+1 = 2(2km+k+m)+1 \end{aligned}$$

Because both $k+m+1$ and $2km+k+m$ are integers, $a+b$ is even and ab is odd. They are of opposite parity. □

Problem 32. (a) Let x and y be integers. Prove that $(x+y)^2$ is even if and only if x and y are of the same parity.

Proof. For this proof we will use the following two theorems:
Theorem 12 Let $x \in \mathbb{Z}$. Then x^2 is even if and only if x is even.
Theorem 16 Let $x, y \in \mathbb{Z}$. Then x and y are of the same parity if and only if $x+y$ is even.

Assume $(x+y)^2$ is even. By theorem 12, $x+y$ is even. Thus, by theorem 16, x and y are of the same parity.
For the converse, let x and y be of the same parity. By theorem 16, $x+y$ is even and so, by theorem 12, $(x+y)^2$ is even. □

(b) Restate the result in (a) in terms of odd integers.

Solution . Let $x, y \in \mathbb{Z}$. Then $(x+y)^2$ is odd if and only if x and y are of opposite parity.

Problem 33. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$ be subsets of $S = \{1,2,3,4\}$. Let $n \in S$. Prove that $2n^2 - 5n$ is either (a) positive and even or (b) negative and odd if and only if $n \notin A \cap B$.

Proof. Let $n \in S$ such that $n \in A \cap B$. Then $n \in \{2,3\}$. If $n = 2$, then $2n^2 - 5n = -2$ is even and negative; while, if $n = 3$, then $2n^2 - 5n = 3$ is odd and positive.
For the converse, let $n \in S$ such that $n \notin A \cap B$. Then $n \in \{1,4\}$. If $n = 1$, then $2n^2 - 5n = -3$ is negative and odd. If $n = 4$, then $2n^2 - 5n = 12$ is positive and even. □

Problem 34. Let $A = \{3,4\}$ be a subset of $S = \{1,2,\ldots,6\}$. Let $n \in S$. Prove that if $\frac{n^2(n+1)^2}{4}$ is even, then $n \in A$.

Proof. Let $n \in S$ such that $n \notin A$. Then $n = \{1,2,5,6\}$. If $n = 1$, then $n^2(n+1)^2/4 = 1$ is odd. If $n = 2$, then $n^2(n+1)^2/4 = 9$ is odd. If $n = 5$, then $n^2(n+1)^2/4 = 225$ is odd. If $n = 6$, then $n^2(n+1)^2/4 = 441$ is odd. □

Problem 35. Prove for every nonnegative integer n that $2^n + 6^n$ is an even integer.

Proof. Let n be a nonnegative integer. We consider two cases for this proof.
Case 1. Assume $n \in \mathbb{Z}$ such that $n = 0$. Therefore, $2^0 + 6^0 = 2$ is an even integer.
Case 2. Let $n > 0$. Then $n - 1 \geq 0$. Note that

$$\begin{aligned} 2^n + 6^n &= 2(2^{n-1}) + 6(6^{n-1}) \\ &= 2(2^{n-1} + 3(6^{n-1})) \end{aligned}$$

Since $n - 1 \geq 0$, it follows that $2^{n-1} + 3(6^{n-1}) \in \mathbb{Z}$. Thus, $2^n + 6^n$ is an even integer. □