Section 9.6: Inverse Functions

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A part from its properties, functions come with an interesting concept, namley, the **inverse function**. Let $f : A \to B$ be some function. Then, the inverse f^{-1} is a relation defined by

$$f^{-1} = \{(b, a) : (a, b) \in f\}.$$

In fact, f^{-1} is a function from B to A if and only if f is bijective. Furthermore, f being bijective implies that f^{-1} is bijective. This points out that all **inverse functions** are bijective. Also, for some function f from A to B, if $f \circ f^{-1} = B$ and $f^{-1} \circ f = A$, then f is bijective. In fact, for functions $f : A \to B$ and $g : B \to A$ such that $f \circ g = i_A$ and $g \circ f = i_B$, both g and f are bijective and $g = f^{-1}$.

Moreover, for any function $f : A \to B$, let g be some function such that $f \circ g = i_B$. Then, g is known as the **right inverse** of f. In fact, if $h \circ f = i_A$ for some function h, then h is the left inverse of f. The following can be proven:

(a) f is surjective $\iff$ function g exists.

(b) f is injective $\iff$ funciton h exists.

Problem 51. Show that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x}{x-3}$ is bijective and determine $f^{-1}(x)$ for $x \in \mathbb{R} - \{5\}$.

Proof. We first show that $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$ is bijective. Consider some $a, b \in \mathbb{R} - \{3\}$ such that $f(a) = f(b)$. Then, $\frac{5a}{a-3} = \frac{5b}{b-3}$. Multiplying by $(a-3)(b-3)$ we have $(5a)(b-3) = (5b)(a-3)$. Hence, $5ab - 15a = 5ba - 15b$. Substracting $5ba$ and then dividing by -15 results in $a = b$. The function f is one-to-one. Now, consider any $y \in \mathbb{R} - \{5\}$. Then, $r = \frac{-3y}{5-y}$ is defined and $r \neq 3$ (otherwise $15 = 0$). Hence, $r \in \mathbb{R} - \{3\}$ and so

$$\begin{aligned} f(r) = \frac{5r}{r-3} &= \frac{5\left(\frac{-3y}{5-y}\right)}{\left(\frac{-3y}{5-y}\right) - 3} \\ &= \frac{\frac{-15y}{5-y}}{\frac{-3y-15+3y}{5-y}} \\ &= \frac{\frac{-15y}{5-y}}{\frac{-15}{5-y}} = y. \end{aligned}$$

Thus, f is onto and so bijective.
Since f is bijective, it follows that f^{-1} is a bijective function. We determine $f^{-1}(x)$ for any $x \in \mathbb{R} - \{5\}$. Consider some $x \in \mathbb{R} - \{5\}$. Because f is onto, it follows that there is some $a \in \mathbb{R} - \{3\}$ such that $f(a) = x$ and so $f^{-1}(x) = a$. Hence,

$$f\left(f^{-1}(x)\right) = \frac{5f^{-1}(x)}{f^{-1}(x) - 3} = x.$$

Hence, $5f^{-1}(x) = xf^{-1} - 3x$ and so $f^{-1}(x)(5-x) = -3x$. This implies that $f^{-1}(x) = \frac{3x}{x-5}$. $\square$