Week 15

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Section 4: Existence Proofs

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During the last sections we have been working with proofs mainly regarding implications with universal quantifiers, namely, $\forall x \in S, R(x)$. Now, its time to check how one can prove an implication with an existence quantifier, which can be known as **Existence theorems**. If one wants to prove the statement $\exists x \in S, R(x)$, then it suffices to come up with some $x \in S$ with the desired property $R(x)$ (An $x \in S$ such that $R(x)$ is true). However, there will be cases where we can not come up with an specific x but be certain and able to show that there exists such x, as David Hilbert said in one of his lectures "That we shall never know; but of his existence we can be absolutely certain.".

Therefore, an **existence proof** may consist of just displaying or constructing an specific x with such property, or showing that such x must exist without the necessity of producing it. All of this with the aid of Results and Theorems.

Problem 40. Show that there exist a rational number a and irrational number b such that a^b is rational.

Proof. An easy example, let $a = 1,0$ and $b = \sqrt{2}$. Then $a^b = 1^{\sqrt{2}}, 0^{\sqrt{2}} = 1,0$. $\square$

Problem 41. Show that there exist a rational number a and an irrational number b such that a^b is irrational.

Proof. Let $a = 2$ and $b = \frac{1}{2}\sqrt{2}$. Then $a^b = 2^{\frac{1}{2}\sqrt{2}} = \left(2^{\frac{1}{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}}$. Remember that it has been proven that $\sqrt{2}^{\sqrt{2}}$ is irrational. $\square$

Problem 42. Show that there exist two distinct irrational numbers a and b such that a^b is rational.

Proof. Let $a = \sqrt{2}$ and $b = 2\sqrt{2}$, both are clearly distinct and irrational. Then $a^b = \sqrt{2}^{2\sqrt{2}}$ can either be rational or irrational.
Case 1. $a^b = \sqrt{2}^{2\sqrt{2}}$ is rational. Then we are set.
Case 2. $a^b = \sqrt{2}^{2\sqrt{2}}$ is irrational. Then we change our irrational numbers so that $a = \sqrt{2}^{2\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$. Thus, $a^b = \left(\sqrt{2}^{2\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}} = (\sqrt{2})^{2\frac{\sqrt{2}}{\sqrt{2}}} = (\sqrt{2})^2 = 2$. $\square$

Problem 43. Show that there exist no nonzero real numbers a and b such that $\sqrt{a^2+b^2} = \sqrt[3]{a^3+b^3}$.

Proof. Assume, to the contrary, that there exist two nonzero real numbers a and b such that $\sqrt{a^2+b^2} = \sqrt[3]{a^3+b^3}$. Then,

$$\begin{aligned}
\sqrt{a^2+b^2} &= \sqrt[3]{a^3+b^3} \\
(\sqrt{a^2+b^2})^6 &= (\sqrt[3]{a^3+b^3})^6 \\
(a^2+b^2)^3 &= (a^3+b^3)^2 \\
a^6+3a^4b^2+3a^2b^4+b^6 &= a^6+2a^3b^3+b^6 \\
3a^4b^2+3a^2b^4 &= 2a^3b^3 \\
3a^4b^2-2a^3b^3+3a^2b^4 &= 0 \\
a^2b^2(3a^2-2ab+3b^2) &= 0 \\
3a^2-2ab+3b^2 &= 0 && \text{Since } a,b \neq 0
\end{aligned}$$

Therefore,

$$3a^2-2ab+3b^2 = a^2-2ab+b^2+2a^2+2b^2 = (a-b)^2+2a^2+2b^2 = 0 \tag{1}$$

However, since $a,b \neq 0$, ti follows that $(a-b)^2+2a^2+2b^2 > 0$, which leads to a contradiciton. $\square$

Problem 44. Prove that there exists a unique real number solution to the equation $x^3 + x^2 - 1 = 0$ between $x = 2/3$ and $x = 1$.

Proof. Let $f(x) = x^3 + x^2 - 1$. Since $f(x)$ is a polynomial, it follows that it is continuos on $\mathbb{R}$. Note that $f(2/3) = -7/27$ and $f(1) = 1$. Therefore, $f(2/3) = -7/27 < 0 < 1 = f(1)$ and so, by the *Intermediate Value Theorem of Calculus*, there exists some $a \in (2/3, 1)$ such that $f(a) = 0$.
Then, assume that there are two real numbers a and b such that $a, b \in (2/3, 1)$ and $f(a) = f(b) = 0$. Thus, $a^3 + a^2 - 1 = b^3 + b^2 - 1 = 0$ implying that $a^3 + a^2 = b^3 + b^2$. Therefore,

$$\begin{aligned}
a^3-b^3+a^2-b^2 = 0& \\
&= (a-b)(a^2+ab+b^2)+(a-b)(a+b) \\
&= (a-b)(a^2+ab+b^2+a+b) = 0
\end{aligned}$$

Since $a, b > 0$, it follows that $a^2 + ab + b^2 + a + b > 0$ and so $a - b = 0$. Therefore, $a = b$. $\square$

Problem 45. Let $R(x)$ be an open sentence over a domain S. Suppose that $\forall x \in S, R(x)$ is a false statement and that the set T of counterexamples is a proper subset of S. Show that there exists a subset W of S such that $\forall x \in W, R(x)$ is true.

Proof. Let $T \subset S$ such that $T = \{x \in S | \sim R(x)\}$, namely, the set of counterexamples of the statement $\forall x \in S, R(x)$. Since $T \subset S$, it follows that there is some $x \in S$ such that $x \notin T$. Let W be some set such that $W = S - T$ and so $W \subseteq S$ and $x \in W$. Because $x \notin T$, it follows that $R(x)$ is true and so $\forall x \in W, R(x)$. $\square$

Problem 46. Prove that there exist four distinct positive integers such that each integer divides the sum of the remaining ones.

Proof. Consider the integers $1, 2, 3$ and 6. $\square$

(b) The previos exercise should suggest another problem to you. State and solve such problem.

Note that $1 + 2 + 3 + 6 = 12$ and $1 \mid (12)$, $2 \mid (12)$, $3 \mid (12)$ and $6 \mid (12)$. Therefore, $1, 2, 3, 6, 12$ are five distinct positive integers such that each integer divides the sum of the remaining ones. In a more general manner, if we have n positive integers such that each divides the sum of the others, we can have $n + 1$ positive integers with the same property dy adding the integer a that represents the sum of all the other n integers.

Problem 48. Prove the equation $cos^2(x) - 4x + \pi = 0$ has a real number solution in the interval $[0, 4]$. (You may assume that $cos^2(x)$ is continuous on $[0, 4]$)

Proof. Let $f(x) = cos^2(x) - 4x + \pi$. The function $f(x)$ is a sum of a polynomial and $cos^2(x)$ both continous on $[0, 4]$ and so $f(x)$ is continuous on $[0, 4]$. Note that, $f(0) = 1 + \pi$ and $f(\pi/2) = -\pi$. Since $f(\pi/2) = -\pi < 0 < 1 + \pi = f(0)$, it follows that, by the *Theorem of Intermediate Value of Calulus*, that there is some $c \in (0, \pi/2)$ such that $f(c) = 0$. Because $(0, \pi/2) \subset [0, 4]$, it follows that $c \in (0, 4)$. $\square$