Week 13

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Section 6: Proofs involving cartesian products of sets

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Problem 60. For $A = \{x, y\}$, determine $A \times \mathcal{P}(A)$.

Solution . The power set of A is $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Hence, $A \times \mathcal{P}(A) = \{(x, \emptyset), (x, \{x\}), (x, \{y\}), (x, \{x, y\}), (y, \emptyset), (y, \{x\}), (y, \{y\}), (y, \{x, y\})\}$.

Problem 61. For $A = \{1\}$ and $B = \{2\}$, determine $\mathcal{P}(A \times B)$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

Solution . Since $A \times B = \{(1, 2)\}$, it follows that $\mathcal{P}(A \times B) = \{\emptyset, \{(1, 2)\}\}$.
On the other hand, $\mathcal{P}(A) \times \mathcal{P}(B) = \{\emptyset, \{1\}\} \times \{\emptyset, \{2\}\} = \{(\emptyset, \emptyset), (\emptyset, \{2\}), (\{1\}, \emptyset), (\{1\}, \{2\})\}$.

Problem 62. Let A and B be sets. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

Proof. First assume that either $A = \emptyset$ or $B = \emptyset$, say the former. Since $A = \emptyset$, it follows that there is no (a, b) such that $a \in A$ and $b \in B$. Hence, $A \times B = \emptyset$.
For the converse, assume that $A \neq \emptyset$ and $B \neq \emptyset$. Then, there is some $a \in A$ and $b \in B$. Therefore, $(a, b) \in A \times B$ and so $A \times B \neq \emptyset$. $\square$

Problem 63. For sets A and B, find a necessary and sufficient condition for $A \times B = B \times A$.

Solution . Result Let A and B be sets. $A \times B = B \times A$ if and only if $A = B$ or one of them is the empty set.

Proof. First suppose $A = B$ or one of them is empty. If $A = B$ then $A \times B = A \times A = A \times A = B \times A$. If either $A = \emptyset$ or $B = \emptyset$, then $A \times B = B \times A = \emptyset$.
For the converse, assume $A \neq B$ and that they are nonempty sets. Then, either $A \not\subseteq B$ or $B \not\subseteq A$, say the latter. Since $B \not\subseteq A$, there is some $b \in B$ such that $b \notin A$. Also, since A is a nonempty set, there must be some $x \in A$. Thus, $(b, x) \in B \times A$ and $(b, x) \notin A \times B$. Hence, $A \times B \neq B \times A$. $\square$

Problem 64. For sets A and B, find a necessary and sufficient condition for $(A \times B) \cap (B \times A) = \emptyset$. Verify that this condition is necessary and sufficient.

Solution . Result Let A and B be sets. Then $(A \times B) \cap (B \times A) = \emptyset$ if and only if $A \cap B = \emptyset$.

Proof. First assume that $A \cap B \neq \emptyset$. Then, there is some $y \in A \cap B$, and so $y \in A$ and $y \in B$. Therefore, $(y, y) \in (A \times B)$ and $(y, y) \in (B \times A)$ and so $(y, y) \in (A \times B) \cap (B \times A)$. For the converse assume that $(A \times B) \cap (B \times A) \neq \emptyset$. Therefore, there is some $(x, y) \in (A \times B) \cap (B \times A)$. Then $(x, y) \in (A \times B)$ and $(x, y) \in (B \times A)$. Therefore, $x \in A, B$ and $y \in A, B$ and so $x, y \in A \cap B$. □

Problem 65. Let A, B and C be nonempty sets. Prove that $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

Proof. First, assume that $A \not\subseteq B$. Then, there is some $x \in A$ such that $x \notin B$. Since B is nonempty, there is some $y \in B$. Therefore, $(x, y) \in A \times C$ but $(x, y) \notin B \times C$. Therefore $A \times C \not\subseteq B \times C$.
For the converse, suppose that $A \times C \not\subseteq B \times C$. Then, there is some $(x, y) \in A \times C$ such that $(x, y) \notin B \times C$. Therefore, $x \in A$ and $x \notin B$ and so $A \not\subseteq B$. □

Problem 66. Result 23 states that if A, B, C and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

(a) Show that the converse of Result 23 is false.

Result Let A, B, C and D be sets. If $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

Proof. Let $A = \emptyset$ and $B \not\subseteq D$. Then $A \times B = \emptyset \subseteq C \times D$. However we know that $B \not\subseteq D$ and so the implication is false. This is a counterexample. □

(b) Under what added hypothesis is the converse true? Prove your assertion.

Result Let A, B, C and D be sets such that A and B are nonempty. If $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

Proof. Since A and B are nonempty sets, let $x \in A$ and $y \in B$. Then, $(x, y) \in A \times B$. Since $A \times B \subseteq C \times D$, it follows that $(x, y) \in C \times D$. Therefore, $x \in C$ and $y \in D$. □

Problem 67. Let A, B and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. First we prove that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. Since $y \in B \cap C$, it follows that $y \in B$ and $y \in C$. Because $x \in A$ and $y \in B$, $(x, y) \in A \times B$. Also, since $x \in A$ and $y \in C$, $(x, y) \in A \times C$. Therefore, $(x, y) \in (A \times B) \cap (A \times C)$.
Then we shall prove that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Let $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Since $(x, y) \in A \times B$, it follows that $x \in A$ and $y \in B$. Also, since $(x, y) \in A \times C$, $x \in A$ and $y \in C$. Therefore, $y \in B \cap C$ and so $(x, y) \in A \times (B \cap C)$. □

Problem 68. Let A, B, C and D be sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. First we prove that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. Let $(x, y) \in (A \times B) \cap (C \times D)$. Then $(x, y) \in A \times B$ and $(x, y) \in C \times D$; so $x \in A$, $x \in C$, $y \in B$ and $y \in D$. Therefore, $x \in A \cap C$ and $y \in B \cap D$, and so $(x, y) \in (A \cap C) \times (B \cap D)$.
We then prove that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$. Let $(x, y) \in (A \cap C) \times (B \cap D)$. Then $x \in A \cap C$ and $y \in B \cap D$; so $x \in A$, $x \in C$, $y \in B$ and $y \in D$. Since $x \in A$ and $y \in B$, it follows that $(x, y) \in A \times B$. Also, since $x \in C$ and $y \in D$, it follows that $(x, y) \in C \times D$. Thus, $(x, y) \in (A \times B) \cap (C \times D)$. $\square$

Problem 69. Let A, B, C and D be sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof. Suppose there is some $(x, y) \in (A \times B) \cup (C \times D)$. Then, either $(x, y) \in A \times B$ or $(x, y) \in C \times D$, say the former. Then, $x \in A$ and $y \in B$, and so $x \in A \cup C$ and $y \in B \cup D$. Thus, $(x, y) \in (A \cup C) \times (B \cup D)$. $\square$

Problem 70. Let A and B be sets. Show, in general, that $\overline{A \times B} \neq \overline{A} \times \overline{B}$

Proof. We show that $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$ is true or false depending on the case. Let $(x, y) \in \overline{A \times B}$. Then $(x, y) \notin A \times B$ and so either $x \notin A$ or $y \notin B$. Without loss of generality, let $x \notin A$; so $x \in \overline{A}$. We consider two cases. If $y \in B$, then $y \notin \overline{B}$ and so $(x, y) \notin \overline{A} \times \overline{B}$. On the other hand, if $y \notin B$, then $y \in \overline{B}$ and so $(x, y) \in \overline{A} \times \overline{B}$.

We show that $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$ is true. Let $(x, y) \in \overline{A} \times \overline{B}$. Then $x \notin A$ and $y \notin B$. Therefore, $(x, y) \notin A \times B$ and so $(x, y) \in \overline{A \times B}$.
Therefore, this implication is not always true and some hypothesis must be added (**SPECULATION:** $A = B \neq \emptyset$???). $\square$