Week 2

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Section 5: More on Implication

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Problem 30. Consider the open sentences $P(n) : 5n{+}3$ is prime. and $Q(n) : 7n{+}1$ is prime., both over the domain $\mathbb{N}$.

(A) State $P(n) \implies Q(n)$ in words.

Solution a. $P(n) \implies Q(n)$: If $5n+3$ is prime, then $7n+1$ is prime.

(B) State $P(2) \implies Q(2)$ in words. Is this statement true or false?

Solution b. $P(2) \implies Q(2)$: If 13 is prime, then 15 is prime. This statement is false because the hypothesis $P(2)$ is true and the conclusion $Q(2)$ is false.

(C) State $P(6) \implies Q(6)$ in words. Is this statement true or false?

Solution c. $P(6) \implies Q(6)$: If 33 is prime, then 43 is prime. This statement is true, because $P(6)$ is false and $Q(6)$ is true.

Problem 31. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine the truth value of $P(x) \implies Q(x)$ for each $x \in S$.

(A) $P(x) : |x| = 4;\ Q(x) : x = 4;\ S = \{-4, -3, 1, 4, 5\}$.

Solution a. $P(-4) \implies Q(-4)$: if $4 = 4$, then $-4 = 4$. $P(-4)$ is true and $Q(-4)$ is false. This implication is false.
$P(-3) \implies Q(-3)$: if $3 = 4$, then $-3 = 4$. Both $P(-3)$ and $Q(-3)$ are false. This implication is true.
$P(1) \implies Q(1)$: if $1 = 4$, then $1 = 4$. Both $P(1)$ and $Q(1)$ are false. This implication is true.
$P(4) \implies Q(4)$: if $4 = 4$, then $4 = 4$. Both $P(4)$ and $Q(4)$ are true. This statement is true.
$P(5) \implies Q(5)$: if $5 = 4$, then $5 = 4$. Both $P(5)$ and $Q(5)$ are false. This statement is true.

(B) $P(x) : x^2 = 16;\ Q(x) : |x| = 4;\ S = \{-6, -4, 0, 3, 4, 8\}$.

Solution b. $P(-6) \implies Q(-6)$: if $36 = 16$, then $6 = 4$. Both the hypothesis and conclusion are false. This statement is true.
$P(-4) \implies Q(-4)$: if $16 = 16$, then $4 = 4$. Both the hypothesis and conclusion are true. This statement is true.
$P(0) \implies Q(0)$: if $0 = 16$, then $0 = 4$. Both the hypothesis and conclusion are false. This statement is true.
$P(3) \implies Q(3)$: if $9 = 16$ then $3 = 4$. Both the hypothesis and conclusion are false. This implication is true.
$P(4) \implies Q(4)$: if $16 = 16$, then $4 = 4$. Both the hypothesis and conclusion are true. This implication is true.
$P(8) \implies Q(8)$: if $64 = 16$, then $8 = 4$. Both the hypothesis and conclusion are false. This implication is true.
$P(x) \implies Q(x)$ is true for all $x \in S$.

(C) $P(x) : x > 3;\ Q(x) : 4x - 1 > 12;\ S = \{0, 2, 3, 4, 6\}$.

Solution c. $P(0) \implies Q(0)$: if $0 > 3$, then $-1 > 12$. Both the hypothesis and conclusion are false. This statement is true.
$P(2) \implies Q(2)$: if $2 > 3$, then $7 > 12$. Both the hypothesis and conclusion are false. This statement is true.
$P(3) \implies Q(3)$: if $3 > 3$, then $11 > 12$. Both the hypothesis and conclusion are false. This statement is true.
$P(4) \implies Q(4)$: if $4 > 3$, then $15 > 12$. Both the hypothesis and conclusion are true. This statement is true.
$P(6) \implies Q(6)$: if $6 > 3$, then $23 > 12$. Both the hypothesis and conclusion are true. This statement is true.
$P(x) \implies Q(x)$ is true for all $x \in S$.

Problem 32. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \implies Q(x)$ is a true statement.

(A) $P(x) : x - 3 = 4;\ Q(x) : x \geq 8;\ S = \mathbb{R}$.

Solution a. We must find a subset M of S for whose elements the implication $P(x) \implies Q(x)$ is true, which is the same as saying that for all $x \in M$ the open sentence $(\sim P(x)) \vee Q(x)$ is true. $(\sim P(x)) \vee Q(x)$: $x - 3 \neq 4$ or $x \geq 8$, and by simplifying $\sim P(x)$ we get $(\sim P(x)) \vee Q(x)$: $x \neq 7$ or $x \geq 8$. Thus, the subset M is as follows: $M = \{x \in \mathbb{R} : x \geq 8 \text{ or } x \neq 7\}$. Since the elements x must satisfy a disjunction and all real numbers except the number 7 make either the statement $Q(x)$ or $\sim P(x)$ true, the subset M contains all $x \in \mathbb{R}$ except the 7.

(B) $P(x) : x^2 \geq 1;\ Q(x) : x \geq 1;\ S = \mathbb{R}$.

Solution b. For all elements in the subset M of S the open sentence $(\sim P(x)) \vee Q(x)$ must be true so that they also make the implication $P(x) \implies Q(x)$ true. $(\sim P(x)) \vee Q(x) : x^2 <$

1 or $x \geq 1$, and after solving for x in $P(x)$ the disjunction becomes $(\sim P(x)) \vee Q(x) : -1 < x < 1$ or $x \geq 1$. The subset M would be the following: $M = \{x \in \mathbb{R} : -1 < x < 1 \text{ or } x \geq 1\}$, which means that $M = (-1, \infty)$.

(C) $P(x) : x^2 \geq 1;\ Q(x) : x \geq 1;\ S = \mathbb{N}$.

Solution c. The two open sentences $P(x)$ and $Q(x)$ are the same as the ones from the section (B), but now $S = \mathbb{N}$. The disjunction for every $x \in M$ to fullfil is $(\sim P(x)) \vee Q(x) : -1 < x < 1$ or $x \geq 1$. However, since M is a subset of S and there are no negative integers in $\mathbb{N}$, for every $x \in M$ the disjunction $0 \leq x < 1$ or $x \geq 1$ must be true. Thus, $M = \{x \in \mathbb{N} : x \geq 0\}$, which is the same as $M = \{0, 1, 2, 3, \dots\}$.

(D) $P(x) : x \in [-1, 2];\ Q(x) : x^2 \leq 2;\ S = [-1, 1]$.

Solution d. Every element in the subset M of S must make the disjunction $(\sim P(x)) \vee Q(x) : x \notin [-1, 2]$ or $x^2 \leq 2$ true. It's important to remark that $S \subset [-1, 2]$, thus for every $x \in S$ the open sentence $\sim P(x)$ will be false. Aditionally, every element of S makes the open sentence $Q(x)$ true, this means that $M = S$.

Problem 33. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \implies Q(x, y)$ for the given values of x and y.

(A) $P(x, y) : x^2 - y^2 = 0.$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

Solution a. $P(1, -1) \implies Q(1, -1)$: if $0 = 0$, then $1 = -1$. This implication is false, because the hypothesis is true and the conclusion is false.
$P(3, 4) \implies Q(3, 4)$: if $-7 = 0$, then $3 = 4$. The implication is true, because both the premise and conclusion are false.
$P(5, 5) \implies Q(5, 5)$: if $0 = 0$, then $5 = 5$. This implication is true, because both the premise and conclusion are true.

(B) $P(x, y) : |x| = |y|.$ and $Q(x, y) : x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

Solution b. $P(1, 2) \implies Q(1, 2)$: if $1 = 2$, then $1 = 2$. This implication is true, because both the premise and conclusion are false.
$P(2, -2) \implies Q(2, -2)$: if $2 = 2$, then $2 = -2$. This implication is false, because the premise is true and the conclusion is false.
$P(6, 6) \implies Q(6, 6)$: if $6 = 6$, then $6 = 6$. This implication is true, since both the premise and conclusion are true.

(C) $P(x, y) : x^2 + y^2 = 1.$ and $Q(x, y) : x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$

Solution c. $P(1,-1) \implies Q(1,-1)$: if $2 = 1$, then $0 = 1$. The implication is true, since both the premise and conclusion are false.
$P(-3,4) \implies Q(-3,4)$: if $25 = 1$, then $1 = 1$. This implication is true, because the premise is false and the conclusion is true.
$P(0,-1) \implies Q(0,-1)$: if $1 = 1$, then $-1 = 1$. The implication is false, because the premise is true and the conclusion is false true.
$P(1,0) \implies Q(1,0)$: if $1 = 1$, then $1 = 1$. Both the premise and conclusion are true, which means that the implication is true.

Problem 34. Each of the following describes an implication. Write the implication in the form "if, then."

(A) Any point on the straight line with equation $2y + x - 3 = 0$ whose x-coordinate is an integer also has an integer for its y-coordinate.

Solution a. If the x-coordinate of a point on the straight line with equation $2y+x-3 = 0$ is an integer, then its y-coordinate is an integer.

(B) The square of every odd integer is odd.

Solution b. If x is odd, then x^2 is odd.

(C) Let $n \in \mathbb{Z}$. Whenever $3n + 7$ is even, n is odd.

Solution c. If $3n + 7$ is even, then n is odd.

(D) The derivative of the function $f(x) = \cos x$ is $f'(x) = -\sin x$.

Solution d. If $f(x) = \cos x$, then $f'(x) = -\sin x$.

(E) Let C be a circle of circumference 4π. Then the area of C is also 4π.

Solution e. If C is a circle and C has a circumference of 4π, then the area of C is 4π.

(F) The integer n^3 is even only if n is even.

Solution f. If n^3 is even, then n is even.