Section 2.4: The Monotone Convergence Theorem and Infinite Series

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In this chapter we are introduced to the Monotone Convergence Theorem, which is very useful in cheecking the convergence of sequences of partial sums. Let (a_n) be a sequence. This theorem states that if (a_n) is monotone (either increasing or decreasing), namely $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ respectively, and it is bounded, then it converges to some limit. Its usfulness comes in two "flavors". First, the fact that partial sums of positive real numbers are elements of an increasing sequence. Second, it suffices to show that a sequence is increasing and bounded to conclude that converges without the necessity to come up with a particular limit. We are interested in the convergence of partial sums, since an infinite series

$$\sum_{n \in \mathbb{N}} a_n$$

is said to converge (equal) some number N if the sequence of its partial sums $(s_n) = (a_1 + a_2 + \cdots + a_n)$ converges to N. One way to show that an increasing sequence of partial sums is bounded is by proving that every element is lower or equal to other element from a bounded sequence. On the other hand, a sequence of partial sums (s_n) is not bounded if for every element k of some unbounded sequence (p_n) there is an element in (s_n) that is greater or equal to p_k. For instance, one can extract another sequence (m_n) from (s_n) such that $m_k \geq p_k$ for all $k \in \mathbb{N}$.

Let's state this in a clear and clean way. Let (a_n) and (b_n) be sequences. If for every $n \in \mathbb{N}$ there is some positive integer k such that $a_n \leq b_k$, then $(a_n) \leq (b_n)$. Now, let (s_n) and (p_n) be bounded and unbounded sequences, respectively. Then, the increasing sequence (a_n) is bounded if $(a_n) \leq (s_n)$. On the other hand, (a_n) is unbounded if $(p_n) \leq (a_n)$.

For example, the **Cauchy Condensation Test** uses the infinite series

$$\sum_{n \in \mathbb{N}} 2^n b_{2^n}.$$

to check the converge or divergence of the infinite series of some decreasing sequence (b_n) of nonengative real numbers since $(s_{2^n b_{2^n}}) \leq (s_{b_n})$ and viceversa.

Problem 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

Proof. We proceed by induction. We show that $3 \geq x_n > x_{n+1} > 0$ for all $n \in \mathbb{N}$. Note that $x_1 = 3$ and $x_2 = 1/(4-3) = 1$. Hence, $3 \geq x_1 > x_2 > 0$. Now, assume for some $k \in \mathbb{N}$ that $3 \geq x_k > x_{k+1} > 0$. We prove that $3 \geq x_{k+1} > x_{k+2} > 0$. Note that

$$3 \geq x_k > x_{k+1} > 0 \implies$$
$$1 \leq 4 - x_k < 4 - x_{k+1} < 4 \implies$$
$$1 \geq \frac{1}{4 - x_k} > \frac{1}{4 - x_{k+1}} > \frac{1}{4}.$$

Therefore, $3 \geq x_{k+1} > x_{k+2} > 0$. By the Principle of Mathmatical Induction, $3 \geq x_k > x_{k+1} > 0$ for all $k \in \mathbb{N}$. Thus, x_n is decreasing and bounded. It converges to some a, and according to the given argument, it seems that $a = \frac{1}{4}$. $\qquad \square$

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Solution Recall that when dealing with convergence of sequences, we are mostly intersted in the "tail", namely, how infinitely but finite many of them behave. Note that $x_{n+1} = (x_n : n \geq 2)$ is the same sequence as x_n minus the first term. We keep infinitely many of them (tail). Hence, for any ϵ such that for any $n \geq N$ we have $|x_n - a| < \epsilon$, there is still some $K \geq N$ such that $|x_{n+1} - a| < \epsilon$ for all $n \geq K$. Thus, $\lim x_n = \lim x_{n+1} = a$.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly xompute $\lim x_n$.

Solution The sequence (x_{n+1}) is recursively defined by

$$x_{n+1} = \frac{1}{4 - x_n}$$

for all $n \in \mathbb{N}$. Taking the limit in both sides we get that

$$\lim x_{n+1} = \lim \frac{1}{4 - x_n} \implies$$
$$a = \frac{1}{4 - a},$$

by the Algebraic Limit Theorem. Hence, $(x_{n+1}) \to a$ and so $(x_{n+1}) = \left(\frac{1}{4 - x_n}\right) \to \frac{1}{4-a} = a$. Then, $a^2 - 4a + 1 = 0$. Using the quadratic formula, we get $a = 2 \pm \sqrt{3}$. Since $3 \geq x_n + 1$ for all $n \in \mathbb{N}$, it follows that $3 \geq a$ and so $a = 2 - \sqrt{3}$. Note that $a \approx 0.267949$ which is very near to our initial guess of $\frac{1}{4}$.

Problem 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+}) have the same limit, taking the limit across the the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

Solution Computing the first 5 terms of (y_n) we realize that $(y_n) = (2, 1, 2, 1, 2, \ldots)$ is a sequence that alternates between 2 values and so it does not converges. The argument is good if the assumption is true, however the stament $y = \lim y_n$ is false.

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution In order to be sure that the strategy in (a) can be applied to this example we must show that the recursive sequence converges. One way is using the Monotone Convergence Theorem.

Proof. We procced by induction. We show that $3 > y_{n+1} > y_n \geq 1$ for all $n \in \mathbb{N}$. Observe that $y_1 = 1$ and $y_2 = 3 - 1 = 2$. Hence, $3 > y_2 > y_1 \geq 1$. Suppose for some $k \in \mathbb{N}$ that $3 > y_{k+1} > y_k \geq 1$. We prove that $3 > y_{k+2} > y_{k+1} > 1$. Note that $1/3 < 1/y_{k+1} \leq 1/y_k < 1$ and so $3 - 1/3 > 3 - 1/y_{k+1} > 3 - 1/y_k \geq 3 - 1$. Therefore,

$$3 > \frac{8}{3} > y_{k+2} > y_{k+1} \geq 2 > 1.$$

Thus, by the Principle of Mathematical Induction, $3 > y_{n+1} > y_n \geq 1$ for all $n \in \mathbb{N}$. Then, (y_n) is an increasing and bounded sequence, and so, by the Monotone Convergence sequence, $\lim y_n = y$ for some $y \in \mathbb{R}$. $\qquad \square$

Then, it is valid to apply the argument of (a) in this case. Hence,

$$\lim y_{n+1} = \lim \left(3 - \frac{1}{y_n} \right) \implies$$
$$y = 3 - \frac{1}{y}$$

since $y_n > 0$ for all $n \in \mathbb{N}$. Thus, $y^2 - 3y + 1 = 0$ and so, using the cuadratic equation, $y = \frac{3 \pm \sqrt{5}}{2}$. Since $y_n \geq 1$ for all $n \in \mathbb{N}$, then $y \geq 1$ and so $y = \frac{3 + \sqrt{5}}{2} \approx 2.6180$.

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