Section 9.1: The Definition of Function

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A very famous type of relation is the function. For some sets A, B, a function f is a relation from A to B, expressed as $f : A \to B$, such that for every $a \in A$, $(a, b) \in f$ for only one $b \in B$. Hence, $|A| = |f|$. Also, $\text{dom}(f) = A$ and $\text{codom}(f) = B$. For a function $f : A \to B$, Consider some $(a, b) \in f$. Because every ordered pair in f is adscribed to only one $a \in A$, it follows that $(a, b), (a, c) \in f$ implies $b = c$. Thus, $b = f(a)$ is considered as the **image** of a. In fact this is known as **mapping**. For instance, f is said to map a into b. Hence, the **range** of this relation f can be expressed as

$$\begin{aligned} \text{range}(f) &= \{b \in B : (a, b) \in f,\ a \in A\} \\ &= \{f(a) : a \in A\}. \end{aligned}$$

Now, suppose that we have some subset C of A. Then,

$$f(C) = \{f(x) : x \in C\}$$

is known as the **image** of C. Obviously, if $C = A$, then $f(C) = \text{range}(f)$. Furthermore, for some subeset D of B, its **inverse image** is denoted as

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

Due to the definition of a function, $f^{-1}(B) = A$.

Problem 3. Let A be a nonempty set. If R is a relation from A to A that is both an equivalence relation and a function, then what familiar function is R?

Solution 3. It is some type of identity linear function, namely, $R = \{(x, x) : x \in A\}$. Recall that for it to be a function, each $x \in A$ must be paired with only one $c \in A$. However, each x must be paired with itself, since the reflexive property requires each element of A be related to itself. Note that the property of symmetry follows directly and the transitivity follows vacuously. (Remember that this linear relation is the smallest equivalence relation for some nonempty set).

Problem 4. For the given subset A_i of $\mathbb{R}$ and the relation $R_i (1 \leq i \leq 3)$ from A_i to $\mathbb{R}$, determine whether R_i is a function from A_i to $\mathbb{R}$.

(a) $A_1 = \mathbb{R}, R_1 = \{(x, y) : x \in A_1, y = 4x - 3\}$

Solution a. It is true that $\mathrm{dom}(R_1) = A_1$. Now, consider two $y_1, y_2 \in \mathbb{R}$ for some $x_1 = x_2 \in A_1$. Then, $4x_1 - 3 = y_1 = y_2 = 4x_2 - 3$. Hence, R_1 is a function.

(b) $A_2 = [0, \infty), R_2 = \{(x, y) : x \in A_2, (y+2)^2 = x\}$

Solution b. Consider two $y_1, y_2 \in \mathbb{R}$ such that $(y_1 + 2)^2 = (y_2 + 2)^2$ (they have the same preimage, namely, the same $x \geq 0$). Then, $|y_1 + 2| = |y_2 + 2|$. Note that $y_2 = -(y_1 + 4)$ fulfills the previous equality. For instance, $(2 + 2)^2 = (2 - 6)^2 = 16$. Thus, R_2 is not a function.

(c) $A_3 = \mathbb{R}, R_3 = \{(x, y) : x \in A_3, (x + y)^2 = 4\}$

Solution c. Consider two $y_1, y_2 \in \mathbb{R}$ such that $x_1 = x_2 = c$. Then, $(c + y_1)^2 = (c + y_2)^2 = 4$ and so $|c + y_1| = |c + y_2|$. Note that for $y_2 = -(y_1 + 2c)$ the previous equation is fulfuilled. For instance, $(2+0)^2 = (2-4)^2 = 4$. Hence, R_3 is not a function.

Problem 5. Let A and B be nonempty sets and let R be a nonempty relation from A to B. Show that there exists a subset A' of A and a subset f of $\mathbb{R}$ such that f is a function from A' to B.

Proof. Let A and B be two nonempty sets and let R be a nonempty relation from A to B. Since R is nonempty, let $A' = \{a : (a, b) \in R \text{ for some } b \in B\}$. Now, for each $a' \in A'$ select only one b such that $(a, b) \in R$, and let $B' \subset B$ be the set containing these elements. Then, let $f = \{(a, b) : a \in A', b \in B'\}$. Thus, $\mathrm{dom}(f) = A'$ and every a is only related to only one b through f. Therefore, $f : A' \to B$ is a function. $\square$