Section 2.4: The Monotone Convergence Theorem and Infinite Series

Juan Patricio Carrizales Torres

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In this chapter we are introduced to the Monotone Convergence Theorem, which is very useful in cheecking the convergence of sequences of partial sums. Let (a_n) be a sequence. This theorem states that if (a_n) is monotone (either increasing or decreasing), namely $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ respectively, and it is bounded, then it converges to some limit. Its usfulness comes in two "flavors". First, the fact that partial sums of positive real numbers are elements of an increasing sequence. Second, it suffices to show that a sequence is increasing and bounded to conclude that converges without the necessity to come up with a particular limit. We are interested in the convergence of partial sums, since an infinite series

$$\sum_{n \in \mathbb{N}} a_n$$

is said to converge (equal) some number N if the sequence of its partial sums $(s_n) = (a_1 + a_2 + \cdots + a_n)$ converges to N. One way to show that an increasing sequence of partial sums is bounded is by proving that every element is lower or equal to other element from a bounded sequence. On the other hand, a sequence of partial sums (s_n) is not bounded if for every element k of some unbounded sequence (p_n) there is an element in (s_n) that is greater or equal to p_k. For instance, one can extract another sequence (m_n) from (s_n) such that $m_k \geq p_k$ for all $k \in \mathbb{N}$.
Let's state this in a clear and clean way. Let (a_n) and (b_n) be sequences. If for every $n \in \mathbb{N}$ there is some positive integer k such that $a_n \leq b_k$, then $(a_n) \leq (b_n)$. Now, let (s_n) and (p_n) be bounded and unbounded sequences, respectively. Then, the increasing sequence (a_n) is bounded if $(a_n) \leq (s_n)$. On the other hand, (a_n) is unbounded if $(p_n) \leq (a_n)$.

For example, the **Cauchy Condensation Test** uses the infinite series

$$\sum_{n \in \mathbb{N}} 2^n b_{2^n}.$$

to check the converge or divergence of the infinite series of some decreasing sequence (b_n) of nonengative real numbers since $(s_{2^n b_{2^n}}) \leq (s_{b_n})$ and viceversa.

Problem 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

Proof. We proceed by induction. We show that $3 \geq x_n > x_{n+1} > 0$ for all $n \in \mathbb{N}$. Note that $x_1 = 3$ and $x_2 = 1/(4-3) = 1$. Hence, $3 \geq x_1 > x_2 > 0$. Now, assume for some $k \in \mathbb{N}$ that $3 \geq x_k > x_{k+1} > 0$. We prove that $3 \geq x_{k+1} > x_{k+2} > 0$. Note that

$$\begin{aligned}
&3 \geq x_k > x_{k+1} > 0 \implies \\
&1 \leq 4 - x_k < 4 - x_{k+1} < 4 \implies \\
&1 \geq \frac{1}{4-x_k} > \frac{1}{4-x_{k+1}} > \frac{1}{4}.
\end{aligned}$$

Therefore, $3 \geq x_{k+1} > x_{k+2} > 0$. By the Principle of Mathmatical Induction, $3 \geq x_k > x_{k+1} > 0$ for all $k \in \mathbb{N}$. Thus, x_n is decreasing and bounded. It converges to some a, and according to the given argument, it seems that $a = \frac{1}{4}$. $\square$

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

Solution Recall that when dealing with convergence of sequences, we are mostly intersted in the "tail", namely, how infinitely but finite many of them behave. Note that $x_{n+1} = (x_n : n \geq 2)$ is the same sequence as x_n minus the first term. We keep infinitely many of them (tail). Hence, for any ϵ such that for any $n \geq N$ we have $|x_n - a| < \epsilon$, there is still some $K \geq N$ such that $|x_{n+1} - a| < \epsilon$ for all $n \geq K$. Thus, $\lim x_n = \lim x_{n+1} = a$.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly xompute $\lim x_n$.

Solution The sequence (x_{n+1}) is recursively defined by

$$x_{n+1} = \frac{1}{4-x_n}$$

for all $n \in \mathbb{N}$. Taking the limit in both sides we get that

$$\begin{aligned}
\lim x_{n+1} = \lim \frac{1}{4-x_n} &\implies \\
a = \frac{1}{4-a},&
\end{aligned}$$

by the Algebraic Limit Theorem. Hence, $(x_{n+1}) \to a$ and so $(x_{n+1}) = \left(\frac{1}{4-x_n}\right) \to \frac{1}{4-a} = a$. Then, $a^2 - 4a + 1 = 0$. Using the quadratic formula, we get $a = 2 \pm \sqrt{3}$. Since $3 \geq x_n + 1$ for all $n \in \mathbb{N}$, it follows that $3 \geq a$ and so $a = 2 - \sqrt{3}$. Note that $a \approx 0.267949$ which is very near to our initial guess of $\frac{1}{4}$.

Problem 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+}) have the same limit, taking the limit across the the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

Solution Computing the first 5 terms of (y_n) we realize that $(y_n) = (2, 1, 2, 1, 2, \dots)$ is a sequence that alternates between 2 values and so it does not converges. The argument is good if the assumption is true, however the stament $y = \lim y_n$ is false.

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution In order to be sure that the strategy in (a) can be applied to this example we must show that the recursive sequence converges. One way is using the Monotone Convergence Theorem.

Proof. We procced by induction. We show that $3 > y_{n+1} > y_n \geq 1$ for all $n \in \mathbb{N}$. Observe that $y_1 = 1$ and $y_2 = 3 - 1 = 2$. Hence, $3 > y_2 > y_1 \geq 1$. Suppose for some $k \in \mathbb{N}$ that $3 > y_{k+1} > y_k \geq 1$. We prove that $3 > y_{k+2} > y_{k+1} > 1$. Note that $1/3 < 1/y_{k+1} \leq 1/y_k < 1$ and so $3 - 1/3 > 3 - 1/y_{k+1} > 3 - 1/y_k \geq 3 - 1$. Therefore,

$$3 > \frac{8}{3} > y_{k+2} > y_{k+1} \geq 2 > 1.$$

Thus, by the Principle of Mathematical Induction, $3 > y_{n+1} > y_n \geq 1$ for all $n \in \mathbb{N}$. Then, (y_n) is an increasing and bounded sequence, and so, by the Monotone Convergence sequence, $\lim y_n = y$ for some $y \in \mathbb{R}$. $\square$

Then, it is valid to apply the argument of (a) in this case. Hence,

$$\begin{aligned} \lim y_{n+1} &= \lim \left(3 - \frac{1}{y_n}\right) \implies \\ y &= 3 - \frac{1}{y} \end{aligned}$$

since $y_n > 0$ for all $n \in \mathbb{N}$. Thus, $y^2 - 3y + 1 = 0$ and so, using the cuadratic equation, $y = \frac{3 \pm \sqrt{5}}{2}$. Since $y_n \geq 1$ for all $n \in \mathbb{N}$, then $y \geq 1$ and so $y = \frac{3+\sqrt{5}}{2} \approx 2.6180$. This technique is possible thanks to the **Algebraic Theorem of Limits**.

Problem 2.4.3. (a) Show that

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \dots$$

converges and find the limit.

Proof. Note that this is can be expressed as a recursive sequence defined by $y_1 = \sqrt{2}$ and $y_{n+1} = \sqrt{2 + y_n}$. We first show that it is bounded and an increasing sequence. Hence, we proceed by induction and prove that $1 < y_n < y_{n+1} < 2$ for all $n \in \mathbb{N}$. Note that $1 < 2 < 2 + \sqrt{2} < 4$ and, by taking the root, $1 < \sqrt{2} < \sqrt{2 + \sqrt{2}} < 2$. Hence, $1 < y_1 < y_2 < 2$.
Now, assume that $1 < y_k < y_{k+1} < 2$ for some $k \in \mathbb{N}$. We show that $1 < y_{k+1} < y_{k+2} < 2$.
Observe that $1 < 2 + y_k < 2 + y_{k+1} < 4$ and, by taking the square root,

$$1 < y_{k+1} < \sqrt{2 + y_{k+1}} = y_{k+2} < 2.$$

By the Principle of Mathematical Induction, $1 < y_n < y_{n+1} < 2$ for all $n \in \mathbb{N}$ and so (y_n) is a bounded and increasing sequence. Thus, it converges to some limit, by the Bounded Monotone Convergence Theorem.

Now, we proceed to find its limit. Note that $\lim(y_{n+1}) = \lim(y_n) = y$ and so

$$\lim y_{n+1} = \sqrt{2 + y} \implies$$
$$y^2 - y - 2 = 0$$

Thus, $(y - 2)^2 = 0$ and so $y = 2$. $\square$

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Proof. Note that sequence can be expressed as a recursive sequence defined by $y_1 = \sqrt{2}$ and $y_{n+1} = \sqrt{2y_n}$. We show that it is an increasing bounded sequence. Therefore, we proceed by induction and show that $1 < y_n < y_{n+1} < 2$ for all $n \in \mathbb{N}$. First, note that $1 < 2 < 2\sqrt{2} < 4$ and, by taking the square root, $1 < \sqrt{2} < \sqrt{2\sqrt{2}} < 2$. Hence, $1 < y_1 < y_2 < 2$.
Now, suppose that $1 < y_k < y_{k+1} < 2$ for some $k \in \mathbb{N}$. We show that $1 < y_{k+1} < y_{k+2} < 2$.
Observe that $1 < 2y_k < 2y_{k+1} < 4$ and, by taking the square root, $1 < y_{k+1} < y_{k+2} < 2$. By the Principle of Mathematical Induction, $1 < y_n < y_{n+1} < 2$ for all $n \in \mathbb{N}$. Hence, (y_n) is an increasing bounded sequence and, by the Monotone Bounded Sequence, (y_n) converges to some y.
We have that $\lim y_{n+1} = \lim \sqrt{2y_n}$ and so $y^2 - 2y = y(y - 2) = 0$. Thus, either $y = 0$ or $y = 2$. However, since $1 < y_n$ for all $n \in \mathbb{N}$, it follows that $1 < y$, and so $y = 2$. $\square$

Lemma 1. Let (a_n) be an increasing sequence ($a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$) that converges to a. Then, $a \geq a_n$ for all $n \in \mathbb{N}$.

Proof. Assume, to the contrary, that there is some $N \in \mathbb{N}$ such that $a_N > a$. Thus, $\epsilon = a_N - a > 0$ and so $|a_n - a| \geq \epsilon$ for all $n \geq N$ (recall (a_n) is increasing). This contradicts the fact that a is the limit of (a_n). □

Problem 2.4.4. (a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC

Proof. First, let (a_n) be a sequence defined by $a_n = n$. Note that $a_n = n < n+1 = a_{n+1}$ for all $n \in \mathbb{N}$ and so it is an increasing sequence. Just like the proof from **Section 1.4**, we proceed by contradiction. Suppose, to the contrary, that there is some real number y such that $a_n < y$ for all $n \in \mathbb{N}$. Thus, $0 < a_n < y$ for all $n \in \mathbb{N}$ and it is a bounded increasing sequence. By the Bounded Monotone Convergence Sequence, there is some $\lim a_n = a$. Since (a_n) is an increasing sequence, it follows that $a_n \leq a$ for all $n \in \mathbb{N}$.

Now, consider some $m \in \mathbb{N}$ and so there is some $N \in \mathbb{N}$ such that $|a_n - a| = |n - a| < m$ for all $n \geq \mathbb{N}$. Thus, $a - n < m$ and so $a < m + n$. Because $\mathbb{N}$ is closed under addition, $m + n \in \mathbb{N}$ and so $a < a_{m+n}$ for all $n \geq N$, which contradicts the fact that a is the limit of an increasing sequence. □

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (**Theorem 1.4.1**) that doesn't make use of AoC.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and every b_n serves as an upper bound for A. Also, note that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ since each $I_{n+1} = [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] = I_n$ for all $n \in \mathbb{N}$ (nested). Hence, (a_n) is an increasing and bounded sequence, and so, by the Bounded Monotone Convergence Theorem, it converges to some x. Thus, $a_n \leq x$ for all $n \in \mathbb{N}$ since x is the limit of an increasing sequence.

As we have previously seen, $\lim a_n = x$ must the lowest upper bound of A (l.u.b for all elements in the sequence), otherwise, it leads to a contradiction. Therefore, $x \leq b_n$ for all $n \in \mathbb{N}$. Now, consider some I_n and so $a_n \leq x \leq b_n$. Hence, $x \in I_n$ and $x \in \bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. □

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building theory of the real numbers.

Problem 2.4.5 (Calculating Square Roots). Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

Proof. We proceed by induction to prove $2 \le x_n^2$ for all $n \in \mathbb{N}$. Note that $2 \le 4 = x_1^2$. Now, suppose that $2 \le x_k^2$ for some $k \in \mathbb{N}$. We show that $2 \le x_{k+1}^2$. Note that

$$\begin{aligned} x_{k+1}^2 &= \frac{1}{4}\left(x_k + \frac{2}{x_k}\right)^2 = \frac{1}{4}\left(x_k^2 + 4 + \frac{4}{x_k^2}\right) = \frac{1}{4}\left(x_k^2 - 4 + \frac{4}{x_k^2}\right) + 2 \\ &= \frac{1}{4}\left(x_k - \frac{2}{x_k}\right)^2 + 2 \ge 2. \end{aligned}$$

By the Principle of Mathematical Induction, $x_n \ge 2$ for every $n \in \mathbb{N}$.
Furthermore, $x_1 = 2 > 0$ and if $x_k > 0$, then $x_{k+1} = \frac{1}{2}\left(x_k + \frac{2}{x_k}\right) > 0$. This implies that $x_n > 0$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned} x_n - x_{n+1} &= x_n - \left(\frac{1}{2}x_n + \frac{1}{x_n}\right) \\ &= \frac{1}{2}x_n - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n} \\ &\ge \frac{2-2}{2x_n} = 0 \end{aligned}$$

Hence, $2 \ge x_n \ge x_{n+1} > 0$ and so x_n is a bounded decreasing sequence that converges to some limit a. Since $a > 0$, we have

$$\begin{aligned} \lim x_{n+1} &= \frac{1}{2}\left(\lim x_n + \lim \frac{2}{x_n}\right) \implies \\ a &= \frac{1}{2}a + \frac{1}{a}. \end{aligned}$$

Substracting $a/2$ and multiplying by $2a$, we get $a^2 = 2$ and so $a = \sqrt{2}$. $\square$

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution Let (x_n) be a sequence defined by $x_1 = c$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

Using the same argument and procedure in (a), it can be shown that $(x_n) \to \sqrt{c}$.

Problem 2.4.6 (Arithmetic-Geometric Mean). (a) Explain why $\sqrt{xy} \le (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

Proof. Let $\sqrt{x} = a$ and $\sqrt{y} = b$. Then, $x = a^2$ and $y = b^2$. Because, $0 \le (a-b)^2$, it follows that $2ab \le a^2 + b^2$ and so $ab = \sqrt{xy} \le (a^2 + b^2)/2 = (x+y)/2$. $\square$

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \qquad \text{and} \qquad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Proof. Since $0 \leq x_1 \leq y_1$ and, by the arithmetic-geometric mean, $0 \leq \sqrt{x_n y_n} \leq (x_n + y_n)/2$ for any $n \in \mathbb{N}$, it follows that $0 \leq x_n \leq y_n$ for every $n \in \mathbb{N}$. Hence, $0 \leq x_n + y_n \leq 2y_n$ and so $0 \leq (x_n + y_n)/2 \leq y_n$ for any $n \in \mathbb{N}$. This implies that (y_n) is a decreasing bounded sequence.

Furthermore, $0 \leq \sqrt{x_n} \leq \sqrt{y_n}$ and so $0 \leq x_n \leq \sqrt{y_n x_n}$. Then, x_n is an increasing sequence. Note that $0 \leq x_n \leq y_n$ and so each x_n can be "enclosed" in the limits of the bounded decreasing sequence (y_n). By the Bounded Monotone Theorem, both (x_n) and (y_n) converge to some limit x and y, respectively. Therefore,

$$\begin{aligned} \lim y_{n+1} &= \lim \left(\frac{x_n + y_n}{2} \right) \\ &= \frac{\lim x_n + \lim y_n}{2} \implies \\ y &= \frac{x+y}{2}. \end{aligned}$$

This implies that $y = x$. $\square$

Problem 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Proof. Consider some $y_n = \sup\{a_k : k \geq n\}$. Note that $y_{n+1} = \sup\{a_k : k \geq n+1\}$ is the supremum of the set $\{a_k : k \geq n+1\}/\{a_n\}$. Thus, $y_{n+1} \leq y_n$ which implies that (y_n) is a decreasing sequence. Furthermore, since $|a_n| \leq M$ for some $M \in \mathbb{R}$ (bounded), it follows that each $y_n \leq M$ (supremum) and $-M \leq a_n \leq y_n$.
By the Bounded Monotone Sequence, (y_n) converges. $\square$

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this excercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

Solution The *limit inferior* of (a_n) can be defined by

$$\liminf a_n = \lim y_n$$

where (y_n) is a sequence defined by $y_n = \inf\{a_k : k \geq n\}$. Note that y_n is an increasing sequence. Furthermore, if (a_n) is bounded by M, then each $y_n \geq -M$ (infimum) and $M \geq a_n \geq y_n$.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

Proof. Let $x_n = \inf\{a_k : k \geq n\}$ and $y_n = \sup\{a_k : k \geq n\}$. Note that $x_n \leq y_n$ for all $n \in \mathbb{N}$ and so $\lim x_n = \liminf a_n \leq \lim y_n = \limsup a_n$. □

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. Suppose that $\liminf a_n = \limsup a_n$ and let $x_n = \inf a_n$ and $y_n = \sup a_n$. By definition, $x_n \leq a_n \leq y_n$ for all $n \in \mathbb{N}$. By the **Squeeze Theorem** proved in the previous section, the sequence (a_n) converges and $\lim a_n = \liminf a_n = \limsup a_n$.

For the converse, assume that $\lim a_n = a$ exists. Consider some $\epsilon > 0$, then there is some $N \in \mathbb{N}$ such that $a_n \in (a - \epsilon, a + \epsilon)$ for all $n \geq N$. Consider some $k \geq N$, then y_k and x_k are the gratest lower and lowest upper bounds of $\{a_n : n \geq k\}$, respectively. This implies that $a - \epsilon < y_k \leq a_k$ and $a_k \leq x_k < a + \epsilon$. Hence, $y_k, x_k \in (a - \epsilon, a + \epsilon)$ for all $k \geq N$ and so $\liminf a_n = \limsup a_n = \lim a_n = a$. □

Problem 2.4.10 (Infinite Products). A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots_m .$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \ldots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

Solution We consider the case where $a_n = 1/n$. Then, let's define the sequence (x_n) of partial products by

$$\begin{aligned} x_n &= \prod_{m=1}^{n} (1 + 1/m) = \prod_{m=1}^{n} \left(\frac{m+1}{m}\right) \\ &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n} = n + 1 \end{aligned}$$

Clearly, this sequence diverges. Now, let's consider the case when $a_n = \frac{1}{n^2} = \left(\frac{1}{n}\right)^2$. The first 6 terms are $(2, 2.5, 2.7777, 2.951, 3.069, 3.154, \dots)$. It seems that this sequence slowly diverges, however, calculating the first 6000 terms with a program, it seems that the sequence approaches 3.675443.

Another way to prove the divergence when $a_n = 1/n$.

Let the sequence (x_n) be defined by $a_1 = 2$ and

$$a_{n+1} = a_n \left(1 + \frac{1}{n+1}\right) = a_n \left(\frac{n+2}{n+1}\right).$$

Clearly, this is an increasing sequence and $1 < x_n$ for all $n \in \mathbb{N}$. However, it is not bounded above by any real number. Consider any real number y such that $x_n < y$ for some $n \in \mathbb{N}$. By the Archimedean principle, there is some positive integer b such that $y < x_n b$. Now, consider the positive integer $k = (n+1)b$ and observe that

$$\begin{aligned} x_k &= x_n \left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)\left(\frac{n+4}{n+3}\right)\cdots\left(\frac{k}{k-1}\right)\left(\frac{k+1}{k}\right) \\ &= x_n \left(\frac{k+1}{n+1}\right) = x_n \left(b + \frac{1}{n+1}\right) > x_n b > y. \end{aligned}$$

Hence, (x_n) diverges and so $\prod_{n=1}^{\infty}(1+a_n)$ diverges.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Proof. First, assume that the sequence of partial products $(\prod_{i=1}^{n}(1+a_i))$ converges. Note that $\left(\sum_{i\leq n} a_i\right) = (y_n)$ is an increasing sequence since $0 \leq a_n$. Furthermore, observe that

$$\begin{aligned} \prod_{i=1}^{n}(1+a_i) &= (1+a_1)(1+a_2)(1+a_3)\dots(1+a_n) \\ &= \sum_{i=1}^{n} a_i + b + 1 > \sum_{i=1}^{n} a_i \end{aligned}$$

for som $0 \leq b$. Hence, (y_n) is bounded above and so it converges.

For the converse, suppose that $\sum_{n\in\mathbb{N}} a_n$ converges. Hence, the sequence of partial sums $\left(\sum_{i\leq n} a_i\right)$ converges and so it is bounded by some M. We know that $1 + x \leq 3^x$ is true for every real number $x \geq 0$. Hence, for any positive integer n,

$$\begin{aligned} \prod_{i=1}^{n}(1+a_i) &\leq \prod_{i=1}^{n} 3^{a_i} \\ &= 3^{\sum_{i=1}^{n} a_i} \leq 3^M \end{aligned}$$

since $a_n \geq 0$ for every $n \in \mathbb{N}$. Furthermore, $(\prod_{i=1}^{n}(1+a_i))$ is an increasing sequence and so, by the Monotone Bounded Theorem, it converges. $\square$