Week 5

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Section 10: Quantified Statements

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Problem 65. Let S denote the set of odd integers and let

$$P(x) : x^2 + 1 \text{ is even. and } Q(x) : x^2 \text{ is even.}$$

be open sentences over the domain S. State $\forall x \in S, P(x)$ and $\exists x \in S, Q(x)$ in words.

Solution . Quantified statements stated in words:
$\forall x \in S, P(x)$: For every odd integer x, $x^2 + 1$ is even.
$\exists x \in S, Q(x)$: There exists an odd integer x such that x^2 is even.

Problem 66. Define an open sentence $R(x)$ over some domain S and then state $\forall x \in S, R(x)$ and $\exists x \in S, R(x)$ in words.

Solution . Let S be the set of positive integers and consider the following open sentence over the domain S:

$$R(x) : x - 1 \text{ is not a positive integer.}$$

Quantified statements stated in words:
$\forall x \in S, R(x)$: For every positive integer x, $x - 1$ is not a positive integer.
$\exists x \in S, R(x)$: There exists a positive integer x such that $x - 1$ is not a positive integer.

Problem 67. State the negations of the following quantified statements, where all sets are subsets of some universal set U:
(a) For every set A, $A \cap \overline{A} = \emptyset$

Solution a. There exists a set A such that $A \cap \overline{A} \neq \emptyset$

(b) There exists a set A such that $\overline{A} \subseteq A$.

Solution b. For every set A, we have $\overline{A} \nsubseteq A$.

Problem 68. State the negations of the following quantified statements:
(a) For every rational number r, the number $1/r$ is rational.

Solution a. There exists a rational number r such that $1/r$ is irrational.

(b) There exists a rational number r such that $r^2 = 2$.

Solution b. For every rational number r, the number $r^2 \neq 2$.

Problem 69. Let $P(n) : (5n-6)/3$ is an integer. be an open sentence over the domain $\mathbb{Z}$. Determine, with explanations, whether the following statements are true:
(a) $\forall n \in \mathbb{Z}, P(n)$

Solution a. For this quantified statement to be true, all possible integers $5n-6$ from all n must be divisible by 3. However, this is not possible. As an example, consider $(5(1)-6) = -1$ which is not divisible by 3. Therefore, this quantified statement is false.

(b) $\exists n \in \mathbb{Z}, P(n)$

Solution b. A way to find one integer n, for which $5n-6$ is divisible by 3, is by solving for n in the following equation $5n-6 = 9$ (the number 9 was chosen because $9+6 = 15$ is a multiple of 3 and 5). In this case, we get that $n = 3$ and $P(3)$ is true. Thus, this quantified statement is true.
Consider the equation $5n+6 = b$. The number $5n$ must be a multiple of 3 so that the sum $5n+6$ is a multiple of 3. This means that the integer n must be a multiple of 3 in order for b to be a multiple of 3. Thus, $P(n)$ is true for all integers n that are multiples of 3.

Problem 70. Determine the truth value of each of the following statements.

(a) $\exists x \in \mathbb{R},\ x^2 - x = 0.$

Solution a. This statement basically says that there is at least one root of the equation $x^2 - x = 0$. This is true, since $x = 1$ is a root of the aformentioned equation $((1)^2 - (1) = 0)$.

(b) $\forall n \in \mathbb{N},\ n+1 \geq 2.$

Solution b. The inequality $n+1 \geq 2$ is true for the number 1 $(2 \geq 2)$. Since the number 1 is the element with the lowest value in the set $\mathbb{N}$, if $n \in \mathbb{N}$ and $n \neq 1$, then $n > 1$. By adding one to both sides we obtain $n+1 > 2$ for all $n \neq 1$. The inequality $n+1 \geq 2$ is true for 1 and all other $n \in \mathbb{N}$ different to 1. This quantified statement is true.

(c) $\forall x \in \mathbb{R},\ \sqrt{x^2} = x.$

Solution c. The equation $\sqrt{x^2} = x$ holds for all $x \in \mathbb{R}^+$ and 0. However, it does not hold for every $x \in \mathbb{R}^-$ (e.g., $\sqrt{(-1)^2} = 1 \neq -1$). This quantified statement is false.

(d) $\exists x \in \mathbb{Q},\ 3x^2 - 27 = 0.$

Solution d. We can try to solve for x in the equation $3x^2 - 27 = 0$.

$$\begin{aligned} 3x^2 - 27 &= 0 \\ 3x^2 &= 27 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \\ |x| &= 3 \end{aligned}$$

Both 3 and -3 are rational numbers ($\{3, -3\} \subset \mathbb{Q}$) and are roots of the equation $3x^2-27 = 0$. Thus, there are two rational numbers for which the aforementioned equation holds and this quantified statement is true.

(e) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y + 3 = 8$.

Solution e. The equation of the open sentence in this quantified statement can be expresed as the linear equation $y = -x + 5$. Thus, there is an infinity of ordered pairs $(x, y) \in \mathbb{R} \times \mathbb{R}$ that satisfy the linear equation (e.g., $(3, 2)$, and more generally $(x, -x + 5)$). The quantified statement is true.

(f) $\forall x, y \in \mathbb{R},\ x + y + 3 = 8$.

Solution f. Since the ordered pairs that satisfy the linear equation $x+y+3 = 8$ are limited to the general format $(x, -x + 5)$, there will be numbers $x, y \in R$ that does not satisfy it (e.g., $(3, 10)$ does not satisfy the linear equation). Although there is an infinity of ordered pairs $(x, y) \in \mathbb{R} \times \mathbb{R}$ which satisfy the linear equation, not every ordered pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ will satisfy it. Therefore, this quantified statement is false.

(g) $\exists x, y \in \mathbb{R},\ x^2 + y^2 = 9$.

Solution g. There exists an ordered pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ that satisfies the equation of the circle $x^2 + y^2 = 9$, this ordered pair is $(0, 3)$. Therefore, this quantified statement is true.

(h) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R},\ x^2 + y^2 = 9$.

Solution h. A counterexample to this quantified statement is the ordered pair $(10, 0)$ which belongs to the set $\mathbb{R} \times \mathbb{R}$, but does not satisfy the equation of the circle $x^2 + y^2 = 9$. Thus, this quantified statement is false.

Problem 71. The statement

For every integer m, either $m \leq 1$ or $m^2 \geq 4$.

can be expresed using a quantifier as:

$$\forall m \in \mathbb{Z}, m \leq 1 \text{ or } m^2 \geq 4.$$

Do this for the following two statements.

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

Solution a. $\exists a, b \in \mathbb{Z}, ab < 0$ and $a + b > 0$.

(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

Solution b. $\forall x, y \in \mathbb{R}, x \neq y$ implies that $x^2 + y^2 > 0$.

(c) Express in words the negation of the statements in (a) and (b).

Solution c. (a) For every integer a and b either $ab \geq 0$ or $a + b \leq 0$.
(b) There exist real numbers x and y such that $x \neq y$ and $x^2 + y^2 \leq 0$.

(d) Using quantifiers, express in symbols the negations of the statements in both (a) and (b).

Solution d. (a) $\forall a, b \in \mathbb{Z}, ab \geq 0$ or $a + b \leq 0$.
(b) $\exists x, y \in \mathbb{R}, x \neq y$ and $x^2 + y^2 \leq 0$.

Problem 72. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is S. Which of the following implies that $(\sim P(x)) \Rightarrow Q(x)$ is false for some $x \in S$?
We must check whether the next statements imply the falseness of $(\sim P(x)) \Rightarrow Q(x)$ for some $x \in S$, which is the same as saying that it implies the truthness of $\sim ((\sim P(x)) \Rightarrow Q(x))$ for some $x \in S$:

$$\begin{aligned}
&\text{Theorem 17} \\
\sim ((\sim P(x)) \Rightarrow Q(x)) &= \sim (\sim (\sim P(x)) \vee Q(x)) \\
&\text{Double Negation} \\
&= \sim (P(x) \vee Q(x)) \\
&\text{De Morgan's Laws} \\
&= (\sim P(x)) \wedge (\sim Q(x))
\end{aligned}$$

The quantified statement $(\sim P(x)) \Rightarrow Q(x)$ is false for some $x \in S$. can be stated symbolically as:

$$\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$$

(a) $P(x) \wedge Q(x)$ is false for all $x \in S$.

Solution a. By using the *De Morgan's Laws* on the open sentence $P(x) \wedge Q(x)$ (the quantified statement declares that it is false for all $x \in S$) we derive the following implication to be checked:

$$(\forall x \in S, (\sim P(x)) \vee (\sim Q(x))) \Rightarrow (\exists x \in S, (\sim P(x)) \wedge (\sim Q(x)))$$

The quantified statement $\forall x \in S, (\sim P(x)) \vee (\sim Q(x))$ can be implied by multiple quantified statements. Some of them are the following 3:

1. $\forall x \in S, (\sim P(x)) \wedge (Q(x))$.
2. $\forall x \in S, (P(x)) \wedge (\sim Q(x))$.
3. $\forall x \in S, (\sim P(x)) \wedge (\sim Q(x))$.

Not all of the aforementioned quantified statements implies the quantified statement $\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$ (syllogism). Therefore, the quantified statement $\forall x \in S, (\sim P(x)) \vee (\sim Q(x))$ being true does not mean that $\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$ will be true. The quantified statement $\forall x \in S, (\sim P(x)) \vee (\sim Q(x))$ does not imply $\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$.

(b) $P(x)$ is true for all $x \in S$.

Solution b. The following implication

$$(\forall x \in S, P(x)) \Rightarrow (\exists x \in S, (\sim P(x)) \wedge (\sim Q(x)))$$

is not true, because $P(x)$ being true for all $x \in S$ means that $(\sim P(x)) \wedge (\sim Q(x))$ will be false for all $x \in S$. This is so since $\sim P(x)$ will be false for all $x \in S$.

(c) $Q(x)$ is true for all $x \in S$.

Solution c. The implication

$$(\forall x \in S, Q(x)) \Rightarrow (\exists x \in S, (\sim P(x)) \wedge (\sim Q(x)))$$

is false since $Q(x)$ being true for all $x \in S$ means that the conjunction $(\sim P(x)) \wedge (\sim Q(x))$ will be false for all $x \in S$. This is so, because $\sim Q(x)$ will be false for all $x \in S$.

(d) $P(x) \vee Q(x)$ is false for some $x \in S$.

Solution d. By applying *De Morgan's Laws* on the open sentence $P(x) \vee Q(x)$ we derive this implication:

$$(\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))) \Rightarrow (\exists x \in S, (\sim P(x)) \wedge (\sim Q(x)))$$

The premise and conclusion contain exactly the same quantified statement. Thus, this implication is true.

(e) $P(x) \wedge (\sim Q(x))$ is false for all $x \in S$.

Solution e. After obtaining the negation of $P(x) \wedge (\sim Q(x))$ with the aid of *De Morgan's Laws* we formulate the following implication:

$$(\forall x \in S, (\sim P(x)) \vee (Q(x))) \Rightarrow (\exists x \in S, (\sim P(x)) \wedge (\sim Q(x)))$$

The premise $\forall x \in S, (\sim P(x)) \vee (Q(x))$ can be implied by multiple quantified statements. We show 3 of them:

1. $\forall x \in S, (\sim P(x)) \wedge (\sim Q(x))$
2. $\forall x \in S, (P(x)) \wedge (Q(x))$
3. $\forall x \in S, (\sim P(x)) \wedge (Q(x))$

Not all of them imply the quantified statement $\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$ (syllogism). Therefore, the truthness of the quantified statement $\forall x \in S, (\sim P(x)) \vee (Q(x))$ does not imply the quantified statement $\exists x \in S, (\sim P(x)) \wedge (\sim Q(x))$.

Problem 73. Let $P(x)$ and $Q(x)$ be open sentences where the domain of the variable x is T. Which of the following implies that $P(x) \Rightarrow Q(x)$ is true for all $x \in T$?
The statement $P(x) \Rightarrow Q(x)$ is true for all $x \in T$. can be expressed in symbols with the aid of Theorem 17:

$$\forall x \in T, (\sim P(x)) \vee Q(x)$$

(a) $P(x) \wedge Q(x)$ is false for all $x \in T$.

Solution a. After applying *De Morgan's Laws* to obtain the negation of $P(x) \wedge Q(x)$ (the quantified statement declares it is false for all $x \in T$) we formulate the implication

$$(\forall x \in T, (\sim P(x)) \vee (\sim Q(x))) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

The premise $\forall x \in T, (\sim P(x)) \vee (\sim Q(x))$ can be implied by multiple quantified statements. We show 3 of them:

1. $\forall x \in T, (\sim P(x)) \wedge Q(x)$
2. $\forall x \in T, P(x) \wedge (\sim Q(x))$
3. $\forall x \in T, (\sim P(x)) \wedge (\sim Q(x))$

Not all of the aforementioned statements implies $\forall x \in T, (\sim P(x)) \vee Q(x)$. Therefore, $\forall x \in T, (\sim P(x)) \vee (\sim Q(x))$ does not imply $\forall x \in T, (\sim P(x)) \vee Q(x)$.

(b) $Q(x)$ is true for all $x \in T$.

Solution b. The implication

$$(\forall x \in T, Q(x)) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

is true since the statement $Q(x)$ being true for all $x \in T$ means that the disjunction $(\sim P(x)) \vee Q(x)$ will be true for all $x \in T$.

(c) $P(x)$ is false for all $x \in T$.

Solution c. The implication

$$(\forall x \in T, \sim P(x)) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

is true, because the disjunction $(\sim P(x)) \vee Q(x)$ is true for all $x \in T$. This is so since $\sim P$ is true for all $x \in T$.

(d) $P(x) \wedge (\sim Q(x))$ is true for some $x \in T$.

Solution d. The following implication

$$(\exists x \in T, P(x) \wedge (\sim Q(x))) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

is false. The premise states that for some $x \in T$ the negation of $(\sim P(x)) \vee Q(x)$ will be true, which means that $(\sim P(x)) \vee Q(x)$ will not be true for all $x \in T$.

(e) $P(x)$ is true for all $x \in T$.

Solution e. The implication

$$(\forall x \in T, P(x)) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

is false. Some quantified statements imply $\forall x \in T, P(x)$. We show 2 of them:

1. $\forall x \in T, P(x) \wedge Q(x)$
2. $\forall x \in T, P(x) \wedge (\sim Q(x))$

Not all of them imply $\forall x \in T, (\sim P(x)) \vee Q(x)$ (syllogism).

(f) $(\sim P(x)) \wedge (\sim Q(x))$ is false for all $x \in T$.

Solution f. After applying *De Morgan's Laws* to obtain an open sentence logically equivalent to the negation of $(\sim P(x)) \wedge (\sim Q(x))$, we formulate the following implication

$$(\forall x \in T, P(x) \vee Q(x)) \Rightarrow (\forall x \in T, (\sim P(x)) \vee Q(x))$$

The premise $\forall x \in T, P(x) \vee Q(x)$ can be implied by multiple quantified statements. Some of them are the following 3:

1. $\forall x \in T, (\sim P(x)) \wedge Q(x)$
2. $\forall x \in T, P(x) \wedge (\sim Q(x))$
3. $\forall x \in T, P(x) \wedge Q(x)$

Not all of the aforementioned quantified statements imply $\forall x \in T, (\sim P(x)) \vee Q(x)$ (syllogism). Therefore, the quantified statement $\forall x \in T, P(x) \vee Q(x)$ does not imply $\forall x \in T, (\sim P(x)) \vee Q(x)$.

Problem 74. Consider the open sentence

$$P(x, y, z) : (x - 1)^2 + (y - 2)^2 + (z - 2)^2 > 0.$$

where the domain of each of the variables x, y and z is $\mathbb{R}$.

(a) Express the quantified statement $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, P(x, y, z)$ in words.

Solution a. For all real numbers x, y and z, $(x - 1)^2 + (y - 2)^2 + (z - 2)^2 > 0$.

(b) Is the quantified statement in (a) true or false? Explain.

Solution b. It is false. One counterexample to this quantified statement is $(x, y, z) = (1, 2, 2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The inequality $(x - 1)^2 + (y - 2)^2 + (z - 2)^2 > 0$ does not hold for all ordered triples $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(c) Express the negation of the quantified statement in (a) in symbols.

Solution c. The negation of the quantified statement in (a) is

$$\begin{aligned} \sim (\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, P(x, y, z)) &\equiv \exists x \in \mathbb{R}, \sim (\forall y \in \mathbb{R}, \forall z \in \mathbb{R}, P(x, y, z)) \\ &\equiv \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \sim (\forall z \in \mathbb{R}, P(x, y, z)) \\ &\equiv \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}, \sim P(x, y, z). \end{aligned}$$

(d) Express the negation of the quantified statement in (a) in words.

Solution d. There exist real numbers x, y and z such that $(x-1)^2+(y-2)^2+(z-2)^2 \leq 0$.

(e) Is the negation of the quantified statement in (a) true or false? Explain.

Solution e. The negation of the quantified statement in (a) is true since we've already found a counterexample for the quantified statemetn in (a). The inequality $(x - 1)^2 + (y - 2)^2 + (z - 2)^2 \leq 0$ is true for the ordered triple $(x, y, z) = (1, 2, 2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Problem 75. Consider the quantified statement

For every $s \in S$ and $t \in S, st - 2$ is prime.

where the domain of the variables s and t is $S = \{3, 5, 11\}$.

(a) Express this quantified statement in symbols.

Solution a. Let $P(s, t) : st - 2$ is prime.
The quantified statement in (a) expressed in symbols is
$\forall s, t \in S, P(s, t)$.

(b) Is the quantified statement in (a) true or false? Explain.

Solution b. Due to the commutative properties of multiplication and the fact that s and t can have the same value since they have the same domain S there will be [1]

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!} = \frac{(3+2-1)!}{2!(3-1)!} = 6$$

different results from the multiplication st for all possible combinations of values for s and t. If we susbtrac 2 from all of these 6 results we then obtain the following statements:

1. $(3)(3) - 2 = 7$ is prime.
2. $(3)(5) - 2 = 13$ is prime.
3. $(3)(11) - 2 = 31$ is prime.
4. $(5)(5) - 2 = 23$ is prime.
5. $(5)(11) - 2 = 53$ is prime.
6. $(11)(11) - 2 = 119$ is prime.

The statement $P(11, 11) : 119$ is prime. is false since 119 is a composite number. The ordered pair $(s, t) = (11, 11)$ represent a counterexample to the quantified statement in (a). Thus, the quantified statement $\forall s, t \in S, P(s, t)$. is false.

(c) Express the negation of the quantified statement in (a) in symbols.

Solution c. The negation of the quantified statement in (a) is

$$\begin{aligned} \sim (\forall x \in S, \forall t \in S, P(s,t)). &\equiv \exists s \in S, \sim (\forall t \in S, P(s,t)). \\ &\equiv \exists s \in S, \exists t \in S, \sim P(s,t). \\ &\equiv \exists s, t \in S, \sim P(s,t). \end{aligned}$$

(d) Express the negation of the quantified statement in (a) in words.

Solution d. There exist $s \in S$ and $t \in S$ such that $st - 2$ is not prime.

(e) Is the negation of the quantified statement in (a) true or false? Explain.

Solution e. It is true since the original quantified statement in (a) is false. Not all numbers $st - 2$ for all combinations of values of s and t will be prime.

Problem 76. Let A be the set of circles in the plane with center $(0, 0)$ and let B be the set of circles in the plane with center $(1, 1)$. Furthermore, let

$$P(C_1, C_2) : C_1 \text{ and } C_2 \text{ have exactly two points in common.}$$

be an open sentence where the domain of C_1 is A and the domain of C_2 is B.

(a) Express the following quantified statement in words:

$$\forall C_1 \in A, \exists C_2 \in B, P(C_1, C_2) \tag{1}$$

Solution a. For every circle $C_1 \in A$, there exists a circle $C_2 \in B$ such that C_1 and C_2 have exactly two points in common.

(b) Express the negation of the quantified statement in **(1)** in symbols.

Solution b. The negation of **(1)** is

$$\begin{aligned} \sim (\forall C_1 \in A, \exists C_2 \in B, P(C_1, C_2)) &\equiv \exists C_1 \in A, \sim (\exists C_2 \in B, P(C_1, C_2)). \\ &\equiv \exists C_1 \in A, \forall C_2 \in B, \sim P(C_1, C_2). \end{aligned}$$

(c) Express the negation of the quantified statement in **(1)** in words.

Solution c. There exists a circle $C_1 \in A$ such that for every circle $C_2 \in B$, C_1 and C_2 don't have exactly two points in common.

Problem 77. For a triangle T, let $r(T)$ denote the ratio of the length of the longest side of T to the length of the smallest side of T. Let A denote the set of all triangles and let

$$P(T_1, T_2) : r(T_2) \geq r(T_1).$$

be an open sentence where the domain of both T_1 and T_2 is A.

(a) Express the following quantified statement in words

$$\exists T_1 \in A, \forall T_2 \in A, P(T_1, T_2). \tag{2}$$

Solution a. There exists a triangle $T_1 \in A$ such that for every triangle $T_2 \in A$, $r(T_2) \geq r(T_1)$.

(b) Express the negation of the quantified statement in **(2)** in symbols.

Solution b. The negation of **(2)** is

$$\begin{aligned} \sim (\exists T_1 \in A, \forall T_2 \in A, P(T_1, T_2)) &\equiv \forall T_1 \in A, \sim (\forall T_2 \in A, P(T_1, T_2)) \\ &\equiv \forall T_1 \in A, \exists T_2 \in A, \sim P(T_1, T_2). \end{aligned}$$

(c) Express the negation of the quantified statement in **(2)** in words.

Solution c. For every triangle $T_1 \in A$, there exists a triangle $T_2 \in A$ such that $r(T_2) < r(T_1)$

Problem 78. Consider the open sentence $P(a, b) : a/b < 1$. where the domain of a is $A = \{2, 3, 5\}$ and the domain of b is $B = \{2, 4, 6\}$.

(a) State the quantified statement $\forall a \in A, \exists b \in B, P(a, b)$. in words.

Solution a. For every $a \in A$, there exists $b \in B$ such that $a/b < 1$.

(b) Show the quantified statement in (a) is true.

Solution b. For the inequality $a/b < 1$ to hold, $a < b$ must be true. The integer b with the greatest value in B is greater than the integer a with the greatest value in A. Thus, every integer $a \in A$ can be divided by the integer b with the greatest value in B yielding an integer lower that 1 in all cases. The quantified statement in (a) is true.

Problem 79. Consider the open sentence $Q(a, b) : a - b < 0.$ where the domain of a is $A = \{3, 5, 8\}$ and the domain of b is $B = \{3, 6, 10\}$.

(a) State the quantified statement $\exists b \in B, \forall a \in A, Q(a, b)$ in words.

Solution a. There exists $b \in B$ such that for every $a \in A$, $a - b < 0$.

(b) Show the quantified statement in (a) is true.

Solution b. For the inequality $a - b < 0.$ to hold, $a < b$ must be true. The integer b with the greatest value in B is greater than the integer a with the greatest value in A. Therefore, substracting the integer b with the greatest value in B from every integer $a \in A$ yields an integer lower than 0 in all cases. The quantified statement in (a) is true.

References

[1] J. Roirdan, *An Introduction to Combinatorial Analysis*, John Wiley & Sons, INC., 1967.