

# Chapter 1: Spaces

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## 1 Fields

In Linear Algebra, we will be working with numbers from any type of class/set. Hence, to simplify things and make them more general, we will introduce the idea of fields. A **field** is a set of objects (including numbers) called **scalars** with operations of addition and multiplication that fulfill the following rules (let  $\alpha$  and  $\beta$  be scalars):

(a) **Addition**

- (a) commutativity,  $\alpha + \beta = \beta + \alpha$ .
- (b) associativity,  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .
- (c) additive identity, there is a unique scalar 0 such that for every scalar  $\alpha$ ,  $\alpha + 0 = \alpha$ .
- (d) additive inverse, for each scalar  $\alpha$  there is a unique scalar  $-\alpha$  such that  $\alpha + (-\alpha) = 0$ .

(b) **Multiplication**

- (a) commutativity,  $\alpha\beta = \beta\alpha$ .
- (b) associativity,  $\gamma(\alpha\beta) = (\gamma\alpha)\beta$ .
- (c) multiplicative identity, there is a unique scalar 1 for every scalar  $\alpha$  such that  $1\alpha = \alpha$ .
- (d) multiplicative inverse, for every nonzero scalar  $\beta$ , there is a unique  $\beta^{-1}$  such that  $\beta\beta^{-1} = 1$ .

(c) **Linearity**

- (a) Multiplication is distributive over addition,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ .

For instance, the class of real numbers and the class of complex numbers are fields.

## 1.1 Exercises

**Problem 1.** Almost all the laws of elementary arithmetic are consequences of the axioms defining a field. Prove, in particular, that if  $\mathcal{F}$  is a field, and if  $\alpha, \beta$  and  $\gamma$  belong to  $\mathcal{F}$ , then the following relations hold.

(a)  $0 + \alpha = \alpha$

*Proof.* Due to the commutativity property of addition,  $\alpha = \alpha + 0 = 0 + \alpha$ .  $\square$

(b) If  $\alpha + \beta = \alpha + \gamma$ , then  $\beta = \gamma$ .

*Proof.* Due to the additive inverse, associativity and commutativity,  $\alpha + \beta + (-\alpha) = \alpha + (\beta + (-\alpha)) = (\alpha + (-\alpha)) + \beta = \beta = \gamma$ .  $\square$

(c)  $\alpha + (\beta - \alpha) = \beta$ .

*Proof.* Just like in (b),

$$\begin{aligned}\alpha + (\beta + (-\alpha)) &= \alpha + (-\alpha + \beta) \\ &= (\alpha + (-\alpha)) + \beta = 0 + \beta \\ &= \beta.\end{aligned}$$

$\square$

(d)  $\alpha \cdot 0 = 0 \cdot \alpha = 0$ . (In this case, the dot indicates multiplication).

*Proof.* Note that

$$\begin{aligned}0 \cdot \alpha + (-0 \cdot \alpha) &= 0 = (0 + 0)\alpha + (-0 \cdot \alpha) \\ &= 0 \cdot \alpha + (0 \cdot \alpha + (-0 \cdot \alpha)) = 0 \cdot \alpha \\ &= \alpha \cdot 0\end{aligned}$$

$\square$

(e)  $(-1)\alpha = -\alpha$

*Proof.* Observe that

$$\begin{aligned}\alpha + (-\alpha) &= 0 = 0\alpha \\ &= (1 - 1)\alpha = \alpha + (-1)\alpha.\end{aligned}$$

By (b),  $-\alpha = (-1)\alpha$ .  $\square$

(f)  $(-\alpha)(-\beta) = \alpha\beta$ .

*Proof.* By (e),  $(-\alpha)(-\beta) = ((-1)\alpha)((-1)\beta)$ . Then,

$$\begin{aligned} ((-1)\alpha)((-1)\beta) &= (-1)(\alpha((-1)\beta)) \\ &= (-1)((\alpha(-1))\beta) = (-1)((-1)\alpha)\beta \\ &= ((-1)((-1)\alpha))\beta = (((-1)(-1))\alpha)\beta \\ &= (1\alpha)\beta = \alpha\beta \end{aligned}$$

□

(g)  $\alpha\beta = 0 \implies \alpha = 0$  or  $\beta = 0$ .

*Proof.* Let  $\alpha\beta = 0$ . Note that either  $\alpha = 0$  or  $\alpha \neq 0$ . In the first case, the result is true. In the case of the latter, there is some  $\alpha^{-1}$  and so  $\alpha^{-1}\alpha\beta = 1\beta = \alpha^{-1}0 = 0$ . □

**Problem 2.** (a) Is the set of all positive integers a field? (In familiar systems, such as the integers, we shall almost always use the ordinary operations of addition and multiplication. On the rare occasions when we depart from this convention, we shall give ample warning. As for “positive”, by that word we mean, here and elsewhere in this book, “greater than or equal to zero”. If 0 is to be excluded, we shall say “strictly positive”.)

**Solution** It is not a field. Although the commutativity, associativity and linearity of closed addition and multiplication is maintained, there is an additive identity 0 and the multiplicative identity 1 is present in this set, there are no additive inverses and multiplicative inverses.

(b) What about the set of integers?

**Solution** It is still not a field. It just needs some type of identity multiplicative.

(c) Can the answers to these questions be changed by re-defining addition or multiplication (or both)?

**Solution** We can re-define addition and multiplication so that there are multiplicative identities for every positive integers. Consider some integer  $\alpha$ . Let’s maintain all known properties but make this small change

$$\alpha^2 = \sum^{\alpha} \alpha = 1.$$

Every integer is its own multiplicative inverse.

**Problem 2.** Let  $m$  be an integer,  $m \geq 2$ , and let  $Z_m$  be the set of all positive integers less than  $m$ ,  $Z_m = \{0, 1, \dots, m-1\}$ . If  $\alpha$  and  $\beta$  are in  $Z_m$ , let  $\alpha + \beta$  be the least positive remainder obtained by dividing the (ordinary) sum of  $\alpha$  and  $\beta$  by  $m$ , and, similarly, let  $\alpha\beta$  be the least remainder obtained by dividing the (ordinary) product of  $\alpha$  and  $\beta$  by  $m$ . (Example: if  $m = 12$ , then  $3 + 11 = 2$  and  $3 \cdot 11 = 9$ .)

- (a) Prove that  $Z_m$  is a field if and only if  $m$  is a prime.

*Proof.* Let  $m$  be a prime number. Note that addition and multiplication are both closed, commutative, associative and linear in the set of positive integers and so their least remainder when divided by  $m$  is in  $Z_m$  (recall the equivalence classes in integers modulo  $m$ ). Since  $m \geq 2$ , there is the additive identity 0 and additive multiplicative 1. Also, 0 is its own additive inverse. Now consider some nonzero  $\alpha \in Z_m$ , then  $m > m - \alpha > 0$  and  $m - \alpha \in Z_m$ . Since  $\alpha + (m - \alpha) = m = 0$ , it follows that  $m - \alpha$  is the unique additive inverse of  $\alpha$ .

Now, we show for any nonzero  $x \in Z_m$  that  $x \cdot Z_m = Z_m$  (multiplication of all elements of  $Z_m$  by  $\alpha$ ) to prove the existence of some multiplicative inverse. Consider some nonzero  $x \in Z_m$  and so  $x = m - \beta$  where  $\beta \in \{1, \dots, m - 1\}$ . Note that there are only  $m$  possible remainders when dividing a positive integer by  $m$  and all are contained in  $Z_m$ . Hence, the only way  $x \cdot Z_m \neq Z_m$  is when  $|x \cdot Z_m| < |Z_m|$ , namely, when there are distinct  $y, z \in Z_m$  such that both  $x \cdot y$  and  $x \cdot z$  have the same remainder when divided by  $m$ . We show this is not possible when  $m$  is a prime integer.

Consider two distinct  $y, z \in Z_m$  and so  $y = m - \alpha$  and  $z = m - \gamma$  for  $\alpha, \gamma \in \{1, \dots, m\}$ . Observe that

$$|(m - \beta)(m - \alpha) - (m - \beta)(m - \gamma)| = |(m - \beta)(\gamma - \alpha)|.$$

Since  $y \neq z$ , it follows that  $\gamma \neq \alpha$  and so  $\gamma - \alpha \neq 0$ . Furthermore,  $|\gamma - \alpha| < m$ . Thus,  $(m - \beta)(\gamma - \alpha)$  is the multiplication of two numbers that are not multiples of  $m$  and so  $|(m - \beta)(\gamma - \alpha)|$  is not a multiple of  $m$  since  $m$  is a prime number. Therefore,  $m \nmid |y - z|$  and so  $y$  and  $z$  do not have equal remainder when divided by  $m$ . Thus,  $x \cdot Z_m$  for any  $x \in Z_m$  and so there is some  $y \in Z_m$  such that  $x \cdot y = 1$  (multiplicative inverse). This argument can be used to show that the multiplicative inverse for any nonzero  $\alpha \in Z_m$  is unique.

For the converse, assume that  $Z_m$  is a field. By **Problem 1**, for any elements  $\alpha, \beta \in Z_m$ ,  $\alpha\beta = 0$  if and only if at least one of them is 0. This implies that all possible multiplications between the nonzero positive integers lower than  $m$  are not multiples of  $m$ . Thus,  $m$  is a factorization of itself times 1,  $m$  must be a prime number.  $\square$

- (b) What is  $-1$  in  $Z_5$

**Solution** Let the operations be extended to any integers, not just the ones inside  $Z_5$ . We know that  $-5$  is a multiple of 5 and we must add  $-4$  to  $-1$  to get to  $-5$ . See this as some type of remainder, just as we need to add  $-2$  to 2 to get to 0. Hence,  $-1$  in  $Z_5$  is 4.

- (c) What is  $\frac{1}{3}$  in  $Z_7$ ?

**Solution** It is not defined, since there is no integer in  $\alpha \in Z_7$  such that  $\frac{1}{3} - \alpha$  is divisible by 7.

**Problem 5.** Let  $Q(\sqrt{2})$  be the set of all real numbers of the form  $\alpha + \beta\sqrt{2}$ , where  $\alpha$  and  $\beta$  are rational.

(a) Is  $Q(\sqrt{2})$  a field?

*Proof.* Yes, it is a field. Note that  $Q(\sqrt{2}) \subseteq \mathbb{R}$  and so the properties of commutativity, associativity and linearity of multiplication and addition are present. Also,  $\alpha + \beta\sqrt{2}, -\alpha - \beta\sqrt{2} \in Q(\sqrt{2})$ . Furthermore,  $1 + 0\sqrt{2}, 0 + 0\sqrt{2} \in Q(\sqrt{2})$ . We now show that addition and multiplication are closed. Consider some  $\alpha + \beta\sqrt{2}$  and  $\gamma + \epsilon\sqrt{2}$ , where  $\alpha, \beta, \gamma, \epsilon \in \mathbb{Q}$ . Observe that

$$(\alpha + \beta\sqrt{2}) + (\gamma + \epsilon\sqrt{2}) = (\alpha + \gamma) + (\beta + \epsilon)\sqrt{2} \in Q(\sqrt{2})$$

and

$$\begin{aligned} (\alpha + \beta\sqrt{2})(\gamma + \epsilon\sqrt{2}) &= \alpha\gamma + \alpha\epsilon\sqrt{2} + \gamma\beta\sqrt{2} + 2\beta\epsilon \\ &= (\alpha\gamma + 2\beta\epsilon) + (\alpha\epsilon + \gamma\beta)\sqrt{2} \in Q(\sqrt{2}) \end{aligned}$$

since  $\mathbb{Q}$  is closed under multiplication and addition.

We just have to show that every nonzero element of  $Q(\sqrt{2})$  has a unique inverse in  $Q(\sqrt{2})$ . Consider some  $\alpha + \beta\sqrt{2} \in Q(\sqrt{2})$ . If  $\alpha = 0$  or  $\beta = 0$ , then  $\frac{1}{2\beta}\sqrt{2}$  and  $\frac{1}{\alpha}$  are their inverses, respectively. If  $\alpha, \beta \neq 0$ , then  $\frac{\alpha}{\alpha^2 - 2\beta^2} - \frac{\beta}{\alpha^2 - 2\beta^2}\sqrt{2} \in Q(\sqrt{2})$  is its inverse (Note that  $\alpha^2 - 2\beta^2 = 0$  if and only if  $|\alpha| = \sqrt{2}|\beta| \notin \mathbb{Q}$ ).

Actually, we can use the same argument to show that  $Q(\sqrt{c})$  is a field for any  $c \in \mathbb{Q}$ .  $\square$

(b) What if  $\alpha$  and  $\beta$  are required to be integers?

*Proof.* Then it is not a field since not all members have a multiplicative inverse. For instance, consider  $\alpha + 0\sqrt{2} \in \mathbb{Z}(\sqrt{2})$  for some integer  $|\alpha| > 1$ . Since  $\alpha \in \mathbb{Z}$  its inverse is  $\frac{1}{\alpha}$ , however it is not an integer.  $\square$

**Problem 6.** (a) Does the set of all polynomials with integer coefficients form a field?

*Proof.* No. The unique multiplicative identity is the polynomial  $g(x) = 1$ . We show that there is an infinity of polynomials in our set without inverse multiplicative. Consider some polynomial  $p(x) = 0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n$  where  $n \in \mathbb{N}$ . Multiplying it by any other polynomial  $s(x) = b_0 + b_1x^1 + \cdots + b_kx^k$  for  $k \in \mathbb{N}$ , we get that

$$\begin{aligned} p(x)s(x) &= (0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x^1 + \cdots + b_kx^k) \\ &= 0(b_0 + b_1x^1 + \cdots + b_kx^k) + a_1x^1(b_0 + b_1x^1 + \cdots + b_kx^k) \\ &\quad + \cdots + a_nx^n(b_0 + b_1x^1 + \cdots + b_kx^k). \end{aligned}$$

The polynomial  $p(x)s(x)$  is not a constant and so it can not be the multiplicative identity.  $\square$

(b) What if the coefficients are allowed to be real numbers?

**Solution** The previous argument can still be used to show that it is not a field.

**Problem 7.** Let  $\mathcal{F}$  be the set of all (ordered) pairs  $(\alpha, \beta)$  of real numbers.

(a) If addition and multiplication are defined by

$$(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta)$$

and

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \beta\delta),$$

does  $\mathcal{F}$  become a field?

(b) If addition and multiplication are defined by

$$(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta)$$

and

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma - \beta\delta, \alpha\delta + \beta\gamma),$$

is  $\mathcal{F}$  a field then?

(c) What happens (in both the preceding cases) if we consider ordered pairs of complex numbers instead?