

Section 8.3: Equivalence Relations

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This chapter reviews some properties that we realized and proved in the problems of **Section 8.3**. However, there's something worth noting. Let R be some relation on some nonempty set A . I previously showed that the union of the equivalence classes by R is A and they all are pairwise disjoint. Nevertheless, I didn't ponder on it much to realize what this meant, namely, that the set of these distinct equivalence classes is a partition of A !!!! This was proven by the authors by just showing that each $x \in A$ belongs to exactly one equivalence class by R .

Problem 36. Give an example of an equivalence relation R on the set $A = \{v, w, x, y, z\}$ such that there are exactly three distinct equivalence classes. What are the equivalence classes for your example?

Solution 36. Consider the partition $P = \{\{v\}, \{w\}, \{x, y, z\}\}$ of A . By **Theorem 4**, the relation R defined by $a R b$ if $a, b \in X$ for some $X \in P$ is an equivalence relation. Hence, the distinct equivalence classes are

$$\begin{aligned}a_1 &= \{x, y, z\} \\a_2 &= \{w\} \\a_3 &= \{v\}\end{aligned}$$

Problem 37. A relation R is defined on \mathbb{N} by $a R b$ if $a^2 + b^2$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Proof. We first prove that R is an equivalence relation. Consider some positive integer c . Then, $c^2 + c^2 = 2c^2$. Since c^2 is an integer, it follows that $2c^2$ is even and so $c R c$. Hence, R is reflexive. Let $a, b \in \mathbb{N}$. By the commutative property of sums on real numbers, it follows that if $a^2 + b^2$ is even, then $b^2 + a^2$ is equal to the same even number. Therefore, $a R b$ implies $b R a$ and so R is symmetric. Consider $x, y, z \in \mathbb{Z}$ such that $x R y$ and $y R z$. Hence, $x^2 + y^2 = 2m$ and $y^2 + z^2 = 2n$ for $m, n \in \mathbb{Z}$. Thus, $x^2 = 2m - y^2$ and $z^2 = 2n - y^2$. Therefore,

$$\begin{aligned}x^2 + z^2 &= (2m - y^2) + (2n - y^2) \\&= 2m + 2n - 2y^2 = 2(m + n - y^2).\end{aligned}$$

Because $m + n - y^2 \in \mathbb{Z}$, it follows that $x^2 + z^2$ is even and so $x R z$, which implies that R is transitive.

Once R is shown to be an equivalence relation, we now determine the distinct equivalence classes. Let x be an even positive integer. Then x^2 is even. Consider some $y \in \mathbb{N}$. Note that $y^2 + x^2$ is even if and only if y^2 is even. We also know that y^2 is even if and only if y is even. Therefore,

$$[x] = \{n \in \mathbb{N} : n \text{ is even}\}.$$

Consider positive integers y and z . If y is an odd positive integer, then $z^2 + y^2$ is odd if and only if z^2 is odd. Hence, z must be odd.

$$[y] = \{n \in \mathbb{N} : n \text{ is odd}\}.$$

Since the set of even and odd positive integers is a partition of \mathbb{N} , it follows that there are only two distinct equivalence classes. \square

Problem 38. Let R be a relation defined on the set \mathbb{N} by $a R b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is an equivalence relation.

Solution 38. The relation R on \mathbb{N} is not an equivalence relation. Consider the positive integers 2, 3 and 5. Since $2 \mid (2 \cdot 3)$ and $2 \mid (2 \cdot 5)$, it follows that $3 R 2$ and $2 R 5$. However, $3 \nmid (2 \cdot 5)$ and $5 \nmid (2 \cdot 3)$. Hence, $3 \not R 5$ and so R is not transitive. This implies that R is not an equivalence relation.

Problem 39. Let S be a nonempty subset of \mathbb{Z} and let R be a relation defined on S by $x R y$ if $3 \mid (x + 2y)$.

(a) Prove that R is an equivalence relation.

Proof. Let S be some nonempty subset of \mathbb{Z} and R some relation on S defined by $x R y$ if $3 \mid (x + 2y)$. For some integer $x \in S$, $x + 2x = 3x$ and so $3 \mid 3x$. Hence, $x R x$ is reflexive.

Let $x, y \in S$ such that $x R y$. Hence, $x + 2y = 3c$ for some integer c . Then, $x = 3c - 2y$ and so

$$\begin{aligned} y + 2x &= y + 2(3c - 2y) \\ &= y + 6c - 4y \\ &= 3(2c - y). \end{aligned}$$

Since $2c - y \in \mathbb{Z}$, it follows that $3 \mid (y + 2x)$ and so $y R x$ (R is symmetric).

Consider some $x, y, z \in S$ such that $x R y$ and $y R z$. Therefore, $x + 2y = 3a$ and $y + 2z = 3b$ for $a, b \in \mathbb{Z}$. Then, $x = 3a - 2y$ and $2z = 3b - y$. Note that

$$\begin{aligned} x + 2z &= 3a - 2y + 3b - y \\ &= 3(a - y + b). \end{aligned}$$

Since $a - y + b \in \mathbb{Z}$, it follows that $3 \mid (x + 2z)$ and so $x R z$ (R is transitive). \square

- (b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

Solution (b). The distinct equivalence classes are:

$$\begin{aligned} A_1 &= \{-6, 0\} = [-6] = [0] \\ A_2 &= \{5, -7\} = [-7] = [5] \\ A_3 &= \{-2, 1, 4, 7\} = [-2] = [1] = [4] = [7] \end{aligned}$$

Problem 40. A relation R is defined on \mathbb{Z} by $x R y$ if $3x - 7y$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Solution 40. First, we show that R is an equivalence relation.

Proof. We show that R is reflexive. Consider some integer x . Then, $3x - 7x = -4(x) = 2(-2x)$, where $-2x \in \mathbb{Z}$ and so it is even. Hence, $x R x$.

We prove that R is symmetric. Consider two integers x and y such that $x R y$. Hence, $3x - 7y = 2c$ for some integer c . Then, $3y - 7x = 2c + 10y - 10x = 2(c + 5y - 5x)$. Since $c + 5y - 5x \in \mathbb{Z}$, it follows that $3y - 7x$ is even and so $y R x$.

Now, consider three integers x, y, z such that $x R y$ and $y R z$. Thus, $3x - 7y = 2a$ and $3y - 7z = 2b$ for some $a, b \in \mathbb{Z}$. Note that $(3x - 7y) + (3y - 7z) = 2a + 2b$ and so $3x - 7z = 2a + 2b + 4y = 2(a + b + y)$. Since $a + b + y \in \mathbb{Z}$, it follows that $3x - 7z$ is even and so $x R z$. \square

Now that it has been proven that R is an equivalence relation. We proceed to determine its equivalence classes. We first determine the equivalence class for some even integer, say 0. Then

$$\begin{aligned} [0] &= \{x \in \mathbb{Z} : x R 0\} \\ &= \{x \in \mathbb{Z} : 3x - 7 \cdot 0 \text{ is even}\} \\ &= \{x \in \mathbb{Z} : 3x \text{ is even}\} \\ &= \{x \in \mathbb{Z} : x \text{ is even}\}. \end{aligned}$$

Now, consider some odd integer, say 1. Then

$$\begin{aligned} [1] &= \{x \in \mathbb{Z} : x R 1\} \\ &= \{x \in \mathbb{Z} : 3x - 7 \text{ is even}\} \\ &= \{x \in \mathbb{Z} : 3x \text{ is odd}\} \\ &= \{x \in \mathbb{Z} : x \text{ is odd}\}. \end{aligned}$$

Therefore, there are two distinct equivalence classes, namely, the set of even integers and the set of odd ones.

Problem 41. (a) Prove that the intersection of two equivalence relations on a nonempty set is an equivalence relation.

Proof. Let R_1 and R_2 be two equivalence relations on some nonempty set A . Let their intersection be the set K . Since both R_1 and R_2 are reflexive, it follows that if $x \in A$, then $(x, x) \in R_1, R_2$, and so $(x, x) \in K$. Hence, K is reflexive. Consider some $a, b \in A$ such that $a K b$ (Recall that $a K b$ is the same as saying $(a, b) \in K$). Then, $a R_1 b$ and $a R_2 b$. Since both are symmetric, $b [R_1, R_2] a$ (b is related to a by both R_1 and R_2) and so $b K a$, which implies that K is symmetric.

Now consider some $a, b, c \in A$ such that $a K b$ and $b K c$. Therefore, $a [R_1, R_2] b$ and $b [R_1, R_2] c$. Since both relations are transitive, it follows that $a [R_1, R_2] c$. Therefore, $a K c$, which implies that K is transitive. Thus, K , namely, the intersection of two equivalence relations on a nonempty set, is an equivalence relation. \square

Lemma 8.4.1. Let a, b be integers. $a \equiv b \pmod{2}$ and $a \equiv b \pmod{3}$ is a necessary and sufficient condition for $a \equiv b \pmod{6}$

Proof. Assume that $a \equiv b \pmod{6}$, then $a = 6(c) + b = 2(3c) + b = 3(2c) + b$ for some integer c . Since $2c, 3c \in \mathbb{Z}$, it follows that $a \equiv b \pmod{3}$ and $a \equiv b \pmod{2}$.

Suppose that $a \equiv b \pmod{3}$ and $a \equiv b \pmod{2}$. Hence, $a = 3x + b = 2y + b$ and so $3x = 2y$ for some $x, y \in \mathbb{Z}$. Hence, $3x$ is even and so $2 \mid x$. Hence $3x = 3 \cdot 2(c)$ for some $c \in \mathbb{Z}$. Therefore, $a = 6(c) + b$, which implies that $a \equiv b \pmod{6}$. \square

- (b) Consider the equivalence relations R_2 and R_3 defined on \mathbb{Z} by $a R_2 b$ if $a \equiv b \pmod{2}$ and $a R_3 b$ if $a \equiv b \pmod{3}$. By (a), $R_1 = R_2 \cap R_3$ is an equivalence relation on \mathbb{Z} . Determine the distinct equivalence classes in R_1 .

Solution b. Note that R_1 is defined by $a R_1 b$ if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{3}$. Since both 2 and 3 are prime, by the previous Lemma, $a \equiv b \pmod{2}$ and $a \equiv b \pmod{3} \iff a \equiv b \pmod{6}$. Thus,

$$\begin{aligned} [a] &= \{x \in \mathbb{Z} : x R_1 a\} \\ &= \{x \in \mathbb{Z} : x \equiv a \pmod{6}\} \\ &= \{x \in \mathbb{Z} : x = 6m + a, m \in \mathbb{Z}\}. \end{aligned}$$

Recall that any integer can be expressed as $6c + b$ for exactly one (c, b) , where $c \in \mathbb{Z}$ and $b \in \{0, 1, 2, 3, 4, 5\}$ by the **Division Algorithm**. Hence,

$$\begin{aligned} [0] &= \{6x + 0 : x \in \mathbb{Z}\} \\ [1] &= \{6x + 1 : x \in \mathbb{Z}\} \\ [2] &= \{6x + 2 : x \in \mathbb{Z}\} \\ [3] &= \{6x + 3 : x \in \mathbb{Z}\} \\ [4] &= \{6x + 4 : x \in \mathbb{Z}\} \\ [5] &= \{6x + 5 : x \in \mathbb{Z}\} \end{aligned}$$