Week 8

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Section 5: Partitions of Sets

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Problem 46. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$

Solution a. The set S_1 is a partition of A.

(b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$

Solution b. The set S_2 is not a partition of A since $\bigcup_{X \in S_2} X \neq A$. The letter g belongs to no subset in S_2.

(c) $S_3 = \{A\}$

Solution c. The set S_3 is a partition of A.

(d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$

Solution d. The set S_4 is not a partition of A because $\emptyset \in S_4$.

(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$

Solution e. The set S_5 is not a partition of A since b belongs to two distinct subsets in S_5, namely $\{b, g\}$ and $\{b, f\}$.

Problem 47. Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?

(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$

Solution a. The set S_1 is not a partition of A since the number 4 belongs to no subset in S_1 ($\bigcup_{X \in S_1} X \neq A$).

(b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}\}$

Solution b. The set S_2 is a partition of A.

(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$

Solution c. The set S_3 is not a partition of A because each number 2, 3 and 4 belongs to two distinct sets. The set S_3 is not pairwise disjoint.

(d) $S_4 = A$

Solution d. The set S_4 is not a partition of A since it is not a collection subsets of A.

Problem 48. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$.

Solution . The partition S of A must contain only 3 subsets. One such example is $S = \{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$.

Problem 49. Give an example of a set A with $|A| = 4$ and two disjoint partitions of S_1 and S_2 of A with $|S_1| = |S_2| = 3$.

Solution . Let $A = \{1, 2, 3, 4\}$, $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$. Therefore, $|A| = 4$, the sets S_1 and S_2 are disjoint partitions of A, and $|S_1| = |S_2| = 3$.

Problem 50. Give an example of a partition of $\mathbb{N}$ into three subsets.

Solution . Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbb{N} : x < 3\}$, $A_2 = \{3\}$ and $A_3 = \{x \in \mathbb{N} : x > 3\}$.
Another solution to this problem:
Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbb{N} : x = 3k, \text{ for some } k \in \mathbb{Z}\}$, $A_2 = \{x \in \mathbb{N} : x = 3k + 1, \text{ for some } k \in \mathbb{Z}\}$ and $A_3 = \{x \in \mathbb{N} : x = 3k + 2, \text{ for some } k \in \mathbb{Z}\}$.

Problem 51. Give an example of a partition of $\mathbb{Q}$ into three subsets.

Solution . Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbb{Q} : x < 2\}$, $A_2 = \{2\}$ and $A_3 = \{x \in \mathbb{Q} : x > 2\}$. The set S is a partition of $\mathbb{Q}$ and $|S| = 3$.

Problem 52. Give an example of three sets A, S_1 and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1 and $|S_2| < |S_1| < |A|$.

Solution . Let $A = \{1, 2, 3\}$, $S_1 = \{\{1, 2\}, \{3\}\}$ and $S_2 = \{S_1\}$. Thus, S_1 is a partition of A, the set S_2 is a partition of S_1 and $|S_2| = 1 < |S_1| = 2 < |A| = 3$.

Problem 53.

Problem 54.

Problem 55. A set S is partitioned into two subsets S_1 and S_2. This produces a partition $\mathcal{P}_1$ of S where $\mathcal{P}_1 = \{S_1, S_2\}$ and so $|\mathcal{P}_1| = 2$. One of the sets in $\mathcal{P}_1$ is then partitioned into two subsets, producing a partition $\mathcal{P}_2$ with $|\mathcal{P}_2| = 3$. A total of $|\mathcal{P}_1|$ sets in $\mathcal{P}_2$ are partitioned into two subsets each, producing a partition $\mathcal{P}_3$ of S. Next, a total of $|\mathcal{P}_2|$ sets in $\mathcal{P}_3$ are partitioned into two subsets each, producing a partition $\mathcal{P}_4$ of S. This is continued until a partition $\mathcal{P}_6$ of S is produced. What is $|\mathcal{P}_6|$?

Solution . A pattern can be seen in the description of the problem. For all natural numbers $n > 2$, a new partition $\mathcal{P}_n$ is produced by partitioning $|\mathcal{P}_{n-2}|$ sets in $\mathcal{P}_{n-1}$ into two subsets each. This means that $\mathcal{P}_n$ has $|\mathcal{P}_{n-2}|$ more elements than $\mathcal{P}_{n-1}$. Therefore, $|\mathcal{P}_n| = |\mathcal{P}_{n-1}| + |\mathcal{P}_{n-2}|$ for every natural number $n > 2$.
The cardinality

$$\begin{aligned}
|\mathcal{P}_6| &= |\mathcal{P}_4| + |\mathcal{P}_5| \\
&= 2|\mathcal{P}_4| + |\mathcal{P}_3| \\
&= 2(|\mathcal{P}_3| + |\mathcal{P}_2|) + |\mathcal{P}_3| \\
&= 3|\mathcal{P}_3| + 2|\mathcal{P}_2| \\
&= 5|\mathcal{P}_2| + 3|\mathcal{P}_1| \\
&= 21
\end{aligned}$$

Problem 56. We mentioned that there are three ways that a collection $\mathcal{S}$ of subsets of a nonempty set A is defined to be a partition of A.
Definition 1 The collection $\mathcal{S}$ consists of pairwise disjoint nonempty subsets of A and every element of A belongs to a subset in $\mathcal{S}$.
Definition 2 The collection $\mathcal{S}$ consists of nonempty subsets of A and every element of A belongs to exactly one subset in $\mathcal{S}$.
Definition 3 The collection $\mathcal{S}$ consists of subsets of A satisfying the three properties (1) every subset in $\mathcal{S}$ is nonempty, (2) every two subsets of A are equal or disjoint and (3) the union of all subsets in $\mathcal{S}$ is A.

(a) Show that any collection $\mathcal{S}$ of subsets of A satisfying Definition 1 satisfies Definition 2.

Solution a. Let the collection $\mathcal{S}$ of subsets of A satisfy **Definition 1**. Then the sets in $\mathcal{S}$ are nonempty. Every element of A belongs to a subset in $\mathcal{S}$. However, if some element of A belonged to more than one subset in $\mathcal{S}$, then the sets in $\mathcal{S}$ would not be pairwise disjoint. Therefore, the set $\mathcal{S}$ satisfies **Definition 2**.

(b) Show that any collection $\mathcal{S}$ of subsets of A satisfying **Definition 2** satisfies **Definition 3**.

Solution b. Let the collection $\mathcal{S}$ of subsets of A satisfy **Definition 2**. Then the set $\mathcal{S}$ consists of nonempty subsets of A. If two different subsets $A_1, A_2 \in \mathcal{S}$ were not disjoint, then there would be some $a \in A$ such that $A_1 \cap A_2 = a$. There would be some $a \in A$ that belongs to more than one subset. This does not satisfy **Definition 2**. Also, if $\bigcup_{X \in \mathcal{S}} X \neq A$, then some $x \in A$ does not belong to any subset in $\mathcal{S}$. This does not satisfy **Definition 2**. Therefore, the set $\mathcal{S}$ satisfies conditionss (1), (2) and (3) of **Definition 3**.

(c) Show that any collection $\mathcal{S}$ of subsets of A satisfying **Definition 3** satisfies **Definition 1**.

Solution c. Let collection $\mathcal{S}$ of subsets of A satisfy **Definition 3**. By condition (1) and (2), the subsets in $\mathcal{S}$ are nonempty and disjoint. Also, since condition (3) states that $\bigcup_{X \in \mathcal{S}} X = A$, it follows that every $x \in A$ belongs to a subset in $\mathcal{S}$. Thus, $\mathcal{S}$ satisfies **Definition 1**.