Week 7

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Section 4: Indexed Collctions of Sets

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Problem 37. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$ and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

Solution . The set of all elements that belong to at least one of the sets in S.
$\bigcup_{X \in S} X = A \cup B \cup C = \{0, 1, \ldots, 5\}$.
The set of all elements that belong to every set in S.
$\bigcap_{X \in S} X = A \cap B \cap C = \{2\}$.

Problem 38. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1, r+1]$ and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$.

Solution a. $A_1 = \{1\}$, $A_2 = \{4\}$ and $A_4 = \{16\}$.
$\bigcup_{\alpha \in S} A_\alpha = A_1 \cup A_2 \cup A_4 = \{1, 4, 16\}$.
$\bigcap_{\alpha \in S} A_\alpha = A_1 \cap A_2 \cap A_4 = \emptyset$.

(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$.

Solution b. $B_1 = [0, 2]$, $B_2 = [1, 3]$ and $B_4 = [3, 5]$.
$\bigcup_{\alpha \in S} B_\alpha = B_1 \cup B_2 \cup B_4 = [0, 5]$.
$\bigcap_{\alpha \in S} B_\alpha = B_1 \cap B_2 \cap B_4 = \emptyset$ (since $B_1 \cap B_4 = \emptyset$).

(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

Solution c. $C_1 = (1, \infty)$, $C_2 = (2, \infty)$ and $C_4 = (4, \infty)$.
$\bigcup_{\alpha \in S} C_\alpha = C_1 \cup C_2 \cup C_4 = (1, \infty)$.
$\bigcap_{\alpha \in S} C_\alpha = C_1 \cap C_2 \cap C_4 = (4, \infty)$.

Problem 39. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

Solution . Since $|A| = 26$ and $|A_\alpha| = 3$ for every $\alpha \in S$, at least 9 subsets A_α are needed (the greatest multiple of 3 nearest to 26 is 27) for their union to be the set A. Let, $S = \{a, d, g, j, m, p, s, v, y\}$. A majority of 8 subsets contains three different letters and one contains 2 different and one repeated letters, namely $A_y = \{y, z, a\}$.

Problem 40. For $i \in \mathbb{Z}$, let $A_i = \{i-1, i+1\}$. Determine the following:

(a) $\bigcup\limits_{i=1}^{5} A_{2i}$

Solution a. Since each set $A_{2i} = \{2i-1, 2i+1\}$, it follows that $\bigcup_{i=1}^{5} A_{2i}$ contains the odd numbers that precede and follow each of the first 5 positive multiples of 2.

$$\bigcup_{i=1}^{5} A_{2i} = \{1, 3, 5, 7, 9, 11\}$$

(b) $\bigcup\limits_{i=1}^{5} (A_i \cap A_{i+1})$

Solution b. Because $A_i = \{i-1, i+1\}$ and $A_{i+1} = \{i, i+2\}$, it follows that $A_i \cap A_{i+1} = \emptyset$ for every $i \in \mathbb{Z}$. Thus,

$$\bigcup_{i=1}^{5} (A_i \cap A_{i+1}) = \emptyset$$

(c) $\bigcup\limits_{i=1}^{5} (A_{2i-1} \cap A_{2i+1})$

Solution c. Since $A_{2i-1} = \{2(i-1), 2i\}$ and $A_{2i+1} = \{2i, 2(i+1)\}$, it follows that $A_{2i-1} \cap A_{2i+1} = \{2i\}$. Let $B = \{1, 2, \ldots, 5\}$. Therefore,

$$\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1}) = \{2i : i \in B\} = \{2, 4, 6, 8, 10\}$$

Problem 41. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbb{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

(a) $\bigcap\limits_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup\limits_{n=1}^{\infty} A_n = [0, 1]$

Solution a. Let the indexed collection of sets be $\{A_n\}_{n \in \mathbb{N}}$, where $A_n = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\} = \left[0, \frac{1}{n}\right]$.
The intersection $\bigcap_{n=1}^{\infty} A_n = \{0\}$ since $A_n = \left[0, \frac{1}{n}\right]$ and

$$\lim_{n \to \infty} A_n = \{0\}$$

The union $\bigcup_{n=1}^{\infty} A_n = [0, 1]$ mainly because $A_1 = [0, 1]$ (For all positive integers $n > 1$, the positive number $1/n < 1$).

(b) $\bigcap\limits_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup\limits_{n=1}^{\infty} A_n = \mathbb{Z}$

Solution b. Let the indexed collection of sets be $\{A_n\}_{n\in\mathbb{N}}$, where $A_n = \{x \in \mathbb{Z} : |x| \leq n\}$. The intersection $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ since $A_n = \{x \in \mathbb{Z} : -n \leq x \leq n\}$ and $A_1 = \{-1, 0, 1\}$.
The union $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$ mainly because

$$\lim_{n\to\infty} A_n = \{\ldots, -1, 0, 1, \ldots\} = \mathbb{Z}$$

Problem 42. For each of the following collections of sets, define a set A_n for each $n \in \mathbb{N}$ such that the indexed collection $\{A_n\}_{n\in\mathbb{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collections of sets.

(a) $\{[1, 2+1), [1, 2+1/2), [1, 2+1/3), \ldots\}$

Solution a. Let the indexed collection be $\{A_n\}_{n\in\mathbb{N}}$, where $A_n = \{x \in \mathbb{R} : 1 \leq x < 2 + 1/n\} = [1, 2+1/n)$.
The union $\bigcup_{n\in\mathbb{N}} A_n = [1, 2+1) = [1, 3)$ since $A_1 = [1, 2+1) = [1, 3)$. The value of the positive number $1/n$ decreases as the positive integer n increases ($n \in \mathbb{N}$).
The intersection $\bigcap_{n\in\mathbb{N}} A_n = [1, 2)$ because $A_n = [1, 2+1/n)$ and

$$\lim_{n\to\infty} A_n = [1, 2+0) = [1, 2)$$

(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$

Solution b. Let the indexed collection be $\{A_n\}_{n\in\mathbb{N}}$, where

$$A_n = \left\{x \in \mathbb{R} : \frac{-2n+1}{n} < x < 2n\right\} = \left(\frac{-2n+1}{n}, 2n\right)$$

Certainly, for $n \in \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \left(\lim_{n\to\infty} \frac{-2n+1}{n}, \lim_{n\to\infty} 2n\right) \\
&= \left(-2 + \lim_{n\to\infty} \frac{1}{n}, \infty\right) \\
&= (-2, \infty)
\end{aligned}$$

It is understood that for $a, b \in \mathbb{N}$, if $a > b$, then $A_b \subseteq A_a$.
Therefore, the union $\bigcup_{n\in\mathbb{N}} A_n = (-2, \infty)$.
Also, the intersection $\bigcap_{n\in\mathbb{N}} A_n = A_1 = (-1, 2)$.

Problem 43. For $r \in \mathbb{R}^+$, let $A_r = \{x \in \mathbb{R} : |x| < r\}$. Determine $\bigcup_{r\in\mathbb{R}^+} A_r$ and $\bigcap_{r\in\mathbb{R}^+} A_r$.

Solution . Certainly, for every $r \in \mathbb{R}^+$,

$$A_r = \{x \in \mathbb{R} : |x| < r\} = \{x \in \mathbb{R} : -r < x < r\} = (-r, r)$$

It is understood that for $a, b \in \mathbb{R}^+$, if $a > b$, then $A_b \subseteq A_a$; In fact, for $r \in \mathbb{R}^+$,

$$\lim_{r\to\infty} A_r = (-\infty, \infty)$$

Therefore, the union $\bigcup_{r\in\mathbb{R}^+} A_r = (-\infty, \infty) = \mathbb{R}$.
Also, the intersection $\bigcap_{r\in\mathbb{R}^+} A_r = \{0\}$ since $0 \in A_r$ for every $r \in \mathbb{R}^+$.

Problem 44. Each of the following sets is a subset of $A = \{1, 2, \ldots, 10\}$:
$A_1 = \{1, 5, 7, 9, 10\}$, $A_2 = \{1, 2, 3, 8, 9\}$, $A_3 = \{2, 4, 6, 8, 9\}$,
$A_4 = \{2, 4, 8\}$, $A_5 = \{3, 6, 7\}$, $A_6 = \{3, 8, 10\}$, $A_7 = \{4, 5, 7, 9\}$,
$A_8 = \{4, 5, 10\}$, $A_9 = \{4, 6, 8\}$, $A_{10} = \{5, 6, 10\}$,
$A_{11} = \{5, 8, 9\}$, $A_{12} = \{6, 7, 10\}$, $A_{13} = \{6, 8, 9\}$.
Find a set $I \subseteq \{1, 2, \ldots, 13\}$ such that for every two distinct elements $j, k \in I$, $A_j \cap A_k = \emptyset$ and $|\bigcup_{i\in I} A_i|$ is maximum.

Problem 45. For $n \in \mathbb{N}$, let $A_n = \left(-\frac{1}{n}, 2 - \frac{1}{n}\right)$. Determine $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.

Solution . For every $a, b \in \mathbb{N}$, if $a > b$, then $1/b > 1/a$. Also, for $n \in \mathbb{N}$,

$$A_1 = (-1, 1) \quad \text{and} \quad \lim_{n\to\infty} A_n = (0, 2)$$

Therefore, as n increases, the left and right endpoints of the interval A_n approach from the left, respectively, 0 and 2.
Certainly, the union $\bigcup_{n\in\mathbb{N}} A_n = (-1, 2)$.
Also, the intersection $\bigcap_{n\in\mathbb{N}} A_n = [0, 1)$ since $[0, 1) \subseteq A_n$ for every $n \in \mathbb{N}$.