Section 8.3: Equivalence Relations

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We know that a function is a special kind of relation and for nonempty sets A and B, that the set of all posible relations is $\mathcal{P}(A \times B)$. One may ask how many of those subsets are functions. In other words, we are looking for the set of all functions from A to B denoted by $B^A = \{f : f : A \to B\}$. Its symbolical representation aludes to its cardinality, namly, $|B^A| = |B|^{|A|}$. This is so since for each function $f : A \to B$, every $a \in A$ must be paired with only one $b \in B$, and so each $a \in A$ can be paired with $|B|$ possible choices in an independly manner. It is like obtaining all possible combinations of $|B|$ repetible elements in $|A|$ ordered places. Namely,

1	2	3	4	$\ldots$	$\lvert A\rvert$
b_1	b_1	b_1	b_1	$\ldots$	b_1
b_2	b_1	b_1	b_1	$\ldots$	b_1
b_3	b_1	b_1	b_1	$\ldots$	b_1
$\vdots$					
$b_{\lvert B\rvert}$	b_1	b_1	b_1	$\ldots$	b_1
b_1	b_2	b_1	b_1	$\ldots$	b_1
b_2	b_2	b_1	b_1	$\ldots$	b_1
b_3	b_2	b_1	b_1	$\ldots$	b_1
$\vdots$					
$b_{\lvert B\rvert}$	b_2	b_1	b_1	$\ldots$	b_1
b_1	b_3	b_1	b_1	$\ldots$	b_1
$\vdots$					
$b_{\lvert B\rvert}$	$b_{\lvert B\rvert}$	$b_{\lvert B\rvert}$	$b_{\lvert B\rvert}$	$\ldots$	$b_{\lvert B\rvert}$

Problem 13. Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Determine B^A.

Solution

$$\begin{aligned} B^A &= \{f : f : A \to B\} \\ &= \{f_{xxx}, f_{yxx}, f_{xyx}, f_{yyx}, f_{xxy}, f_{yxy}, f_{xyy}, f_{yyy}\}, \end{aligned}$$

where $f_{abc} = \{(1, a), (2, b), (3, c)\}$.

Problem 16. (a) Give an example of two sets A and B such that $|B^A| = 8$

Solution It suffices to have a set B with 2 elements and a set A with 3. For instance, $B = \{a, b\}$ and $A = \{1, 2, 3\}$.However this is not necessary. An alternate example is $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{0\}$.

(b) Give an example of an element in B^A for the sets A and B given in (a).

Solution One example is $\{(1, a), (2, a), (3, a)\}$.

Problem 17. (a) For nonempty sets A, B and C, what is a possible interpretation of th notation C^{B^A}?

Solution One possible interpretation is that C^{B^A} is the set of all functions from B^A to C, namely, $C^{B^A} = \left\{f : f : B^A \to C\right\}$, where $B^A = \{g : g : A \to B\}$. Thus, $\{(g_1, c_1), (g_2, c_1), (g_3, c_1), \ldots, (g_k, c_1)\} \in C^{B^A}$.

(b) According to the definition given in (a), determine C^{B^A} for $A = \{0, 1\}$, $B = \{a, b\}$ and $C = \{x, y\}$.

Solution We know that

$$B^A = \{g_{aa},\ g_{ba},\ g_{bb},\ g_{ab}\},$$

where $g_{aa} = \{(0, a), (1, a)\}$. Let g_{aa} the element 1, g_{ba} be 2, g_{bb} be 3 and g_{ab} be 4. Therefore,

$$C^{B^A} = \{\varphi_{ijkl} : i, j, k, l \in \{x, y\}\}$$

where the order of i, j, k, l represents the order of the elements in B^A. For instance, $\varphi_{xxxx} = \{(g_{aa}, x), (g_{ba}, x), (g_{bb}, x), (g_{ab}, x)\}$ and $\varphi_{xyyx} = \{(g_{aa}, x), (g_{ba}, y), (g_{bb}, y), (g_{bb}, x)\}$. (Although, the notation is "long" it gives the idea of combination of independent and repetible elements).