Week 13

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Section 5: Fundamental properties of set operations

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Problem 52. Prove that $A \cap B = B \cap A$ for every two sets A and B (Theorem 22(1b)).

Proof. First we prove $A \cap B \subseteq B \cap A$. Let $x \in A \cap B$. Then, $x \in A$ and $x \in B$. By the commutative law of conjunction of two statements, we conclude that $x \in B$ and $x \in A$. Hence, $x \in B \cap A$ and so $A \cap B \subseteq B \cap A$.
We prove $B \cap A \subseteq A \cap B$ using the same argument and therefore the proof is omitted. $\square$

Problem 53. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for every three sets A, B and C (Theorem 22(3b)).

Proof. First we prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then, $x \in A$ and $x \in (B \cup C)$. Hence, $x \in A$ and either $x \in B$ or $x \in C$. Applying the distributive law, we conclude that either $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. Thus, $x \in (A \cap B) \cup (A \cap C)$.

We then prove $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. Then, either $x \in A \cap B$ or $x \in A \cap C$. Say the former, then $x \in A$ and $x \in B$, and so $x \in B \cup C$. Hence, $x \in A \cap (B \cup C)$. Therefore, $x \in A \cap (B \cup C)$ and so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. $\square$

Problem 54. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ for every two sets A and B (Theorem 22(4b)).

Proof. First we prove $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cap B}$. Then, $x \notin A \cap B$. Thus, $x \notin A$ or $x \notin B$. Assume that $x \notin A$. Hence, $x \in \overline{A}$, and so $x \in \overline{A} \cup \overline{B}$.

We now prove $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Let $x \in \overline{A} \cup \overline{B}$. Then either $x \in \overline{A}$ or $x \in \overline{B}$. Say the latter. Hence, $x \in \overline{B}$; so $x \notin B$. Therefore, $x \notin A \cap B$ and so $x \in \overline{A \cap B}$. $\square$

Problem 55. Let A, B and C be sets. Prove that $(A - B) \cap (A - C) = A - (B \cup C)$.

Proof. First we prove that $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. Let $x \in (A - B) \cap (A - C)$. Then $x \in A - B$ and $x \in A - C$. The former implies that $x \in A$ and $x \notin B$ and the latter implies that $x \in A$ and $x \notin C$. Since $x \notin C$ and $x \notin B$, it follows that $x \notin B \cup C$. Because $x \in A$ and $x \notin B \cup C$, $x \in A - (B \cup C)$.

We then prove that $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. Let $x \in A - (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Since $x \notin B \cup C$, it follows that $x \notin B$ and $x \notin C$. Since $x \notin B$, $x \in (A - B)$. And, since $x \notin C$, $x \in (A - C)$.Therefore, $x \in (A - B) \cap (A - C)$ $\square$

Problem 56. Let A, B and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Proof. We first prove that $(A - B) \cup (A - C) \subseteq A - (B \cap C)$. Let $x \in (A - B) \cup (A - C)$. Then, either $x \in A - B$ or $x \in A - C$, say the former. Therefore, $x \in A$ and $x \notin B$; so $x \notin B \cap C$. Since $x \in A$ and $x \notin B \cap C$, it follows that $x \in A - (B \cap C)$.

We then show that $A - (B \cap C) \subseteq (A - B) \cup (A - C)$. Let $x \in A - (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Since $x \notin B \cap C$, it follows that $x \notin B$ or $x \notin C$, say the former. Because $x \in A$ and $x \notin B$, $x \in A - B$ and so $x \in (A - B) \cup (A - C)$. $\square$

Problem 57. Let A, B and C be sets. Use Theorem 22 to prove that $\overline{\overline{A} \cup (\overline{B} \cap C)} = (A \cap B) \cup (A - C)$.

Proof. Let's use Theorem 22 to prove that $\overline{\overline{A} \cup (\overline{B} \cap C)} = (A \cap B) \cup (A - C)$.

$$
\begin{aligned}
\overline{\overline{A} \cup (\overline{B} \cap C)} &= A \cap \overline{(\overline{B} \cap C)} && \text{By De Morgan's Laws (Theorem 22 (4))}\\
&= A \cap (B \cup \overline{C}) && \text{By De Morgan's Laws (Theorem 22(4))}\\
&= (A \cap B) \cup (A \cap \overline{C}) && \text{By Distributive law (Theorem 22(3))}\\
&= (A \cap B) \cup (A - C)
\end{aligned}
$$

As desired. $\square$

Problem 58. Let A, B and C be sets. Prove that $A \cap \overline{(B \cap \overline{C})} = \overline{(\overline{A} \cup B) \cap (\overline{A} \cup \overline{C})}$

Proof. We show with Theorem 4.22 that $\overline{(\overline{A} \cup B) \cap (\overline{A} \cup \overline{C})} = A \cap (\overline{B} \cup C)$.

$$
\begin{aligned}
\overline{(\overline{A} \cup B) \cap (\overline{A} \cup \overline{C})} &= \overline{(\overline{A} \cup B)} \cup \overline{(\overline{A} \cup \overline{C})}\\
&= (\overline{\overline{A}} \cap \overline{B}) \cup (\overline{\overline{A}} \cap \overline{\overline{C}}) = (A \cap \overline{B}) \cup (A \cap C)\\
&= A \cap (\overline{B} \cup C) = A \cap \overline{(B \cap \overline{C})}
\end{aligned}
$$

As desired. $\square$