Section 7.3: Testing Statements (QUIZ)

Juan Patricio Carrizales Torres

April 025, 2022

Prove or disprove each of the following statements.

Problem 1. If n is a positive integer and s is an irrational number, then n/s is an irrational number.

Proof. Let $n \in \mathbb{N}$ and $s \in \mathbb{R}/\mathbb{Q}$. Since $n = \frac{n}{1}$, it follows that $n \in \mathbb{Q}$. Now, assume, to the contrary, that $n/s = q$ for some rational number $q \neq 0$ since $n > 0$. Therefore, $n = s \cdot q$ is irrational, which leads to a contradiction. (Note that this implies that $s \in \mathbb{R}/\mathbb{Q} \iff s^{-1} \in \mathbb{R}/\mathbb{Q}$). □

Problem 2. For every integer b, there exists a positive integer a such that $|a - |b|| \leq 1$.

Proof. Let $b \in \mathbb{Z}$ and $a = |b| + 1$. Thus, $a \geq 1 > 0$ and $|a - |b|| = |1| = 1$. □

Problem 3. If x and y are integers of the same parity, then xy and $(x+y)^2$ are of the same parity.

Solution 3. This statement is false. Let x and y be arbitrary odd integers. Therefore, xy is odd (multiplication of odd integers; refer to **Lemma ODD** in pdf of Section7.2) and $x+y$ is even (sum of two odd integers), which implies that $(x+y)^2$ is even. Hence, xy and $(x+y)^2$ are of opposite parity.

Problem 4. Let $a, b \in \mathbb{Z}$. If $6 \nmid ab$, then either (1) $2 \nmid a$ and $3 \nmid b$ or (2) $3 \nmid a$ and $2 \nmid b$.

Solution 4. This statement is false. Let $a = 3$ and $b = 9$. Then, $ab = 27$ and $6 \nmid ab$. However, $3 \mid a$ and $3 \mid b$. Another example, let $a = 2$ and $b = 4$. Then, $ab = 8$ and $6 \nmid ab$. However, $2 \mid a$ and $2 \mid b$.

This is so since $6 = 3 \cdot 2$. Therefore, for $6 \nmid ab$ to be true, it suffices that a and b are not divisible by either 2 or 3.

Problem 5. For every positive integer n, $2^{2^n} \geq 4^{n!}$.

Solution 5. This statement is false. Let $n = 4$. Then $2^{2^4} = 2^{16} = 4^8$ and $4^{4!} = 4^{24}$. Thus, $4^8 < 4^{24}$ and so $n = 4$ represents a counterexample.

Problem 6. If A, B and C are sets, then $(A - B) \cup (A - C) = A - (B \cup C)$.

Solution 6. This statement is false. Let $A = \{1, 2\}, B = \emptyset$ and $C = \{1\}$. Then, $(A - B) \cup (A - C) = \{1, 2\} \cup \{2\} = A$ and $A - (B \cup C) = \{1, 2\} - \{2\} = \{1\} \neq A$. Therefore, these specific sets A, B and C represent a counterexample.

In general, let $C \subseteq A$ and $B = \emptyset$. Hence,

$$(A - B) \cup (A - C) = A \cup (A - C) = A$$

and

$$A - (B \cup C) = A - C = A \cap \overline{C} \subset A$$

since $C \subseteq A$.

Problem 7. Let $n \in \mathbb{N}$. If $(n+1)(n+4)$ is odd, then $(n+1)(n+4) + 3^n$ is odd.

Proof. Let $n \in \mathbb{N}$. Hence, n is either odd or even. If n is even, then $(n+4)$ is even (same parity) and so $(n+1)(n+4)$ is even. On the other hand, if n is odd, then $(n+1)$ is even (same parity) and so $(n+1)(n+4)$ is even. Therefore, there is no positive integer such that $(n+1)(n+4)$ is odd and so the statement follows vacuously.

Curiously, if someone tried to prove the implication directly it would lead to a false conclusion. 3^n is odd (multiplication of n odd numbers **Theorem ODD**) and so $(n+1)(n+4) + 3^n$ is even (sum of numbers with same parity). One must understand that proof techniques deal with the deduction process but not guarantee that the premises are true. $\square$

Problem 8. (a) There exist distinct rational numbers a and b such that $(a-1)(b-1) = 1$.

Proof. Consider some **nonzero** rational number r such that $|r| \neq 1$. Let $a = 1 + \frac{1}{r}$ and $b = 1 + r$. Then $a \neq b$ and

$$\begin{aligned}(a-1)(b-1) &= \left(\left(1 + \frac{1}{r}\right) - 1\right)((1+r) - 1) \\ &= \frac{1}{r} \cdot r = 1\end{aligned}$$

$\square$

(b) There exist distinct rational numbers a and b such that $\frac{1}{a} + \frac{1}{b} = 1$.

Proof. Let $a = \frac{3}{2}$ and $b = 3$. Then, $a \neq b$ and

$$a^{-1} + b^{-1} = \frac{2}{3} + \frac{1}{3} = 1.$$

Note that, $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab} = 1$ implies $b + a = ab$. Therefore, $0 = ab - b - a$ and so $1 = 1 + ab - b - a$. Thus,

$$\begin{aligned}1 &= b(a-1) + 1 - a = b(a-1) - (a-1) \\ &= (b-1)(a-1),\end{aligned}$$

which is the statement (a). Hence, (a) $\iff$ (b) (They are logically equivalent). $\square$

Problem 9. Let $a, b, c \in \mathbb{Z}$. If every two of a, b and c are of the same parity, then $a+b+c$ is even.

Solution 9. This statement is false. Let a, b and c be odd. Then, every two of a, b, c are of the same parity. Note that $a+b$ is even (sum of two odd numbers). However, $(a+b)+c$ is odd, since it is the sum of an even number with and odd one.

Problem 10. If n is a nonnegative integer, then 5 divides $2 \cdot 4^n + 3 \cdot 9^n$.

Proof. Note that

$$\begin{aligned} 2 \cdot 4^n + 3 \cdot 9^n &= 2 \cdot 2^{2n} + 3 \cdot 3^{2n} \\ &= 2^{2n+1} + 3^{2n+1}. \end{aligned}$$

Let $n \geq 0$. We proceed by induction. Since $2^1 + 3^1 = 2 + 3 = 5$, it follows that the result is true for $n = 0$. Assume that $5 \mid \left(2^{2k+1} + 3^{2k+1}\right)$ for some $k \geq 0$. We show that $5 \mid \left(2^{2k+3} + 3^{2k+3}\right)$. Note that $2^{2k+1} + 3^{2k+1} = 5c$ for some integer c and so $2^{2k+1} = 5c - 3^{2k+1}$. Therefore,

$$\begin{aligned} 2^{2k+3} + 3^{2k+3} &= 2^2 \cdot 2^{2k+1} + 3^2 \cdot 3^{2k+1} \\ &= 2^2 \left(5c - 3^{2k+1}\right) + 3^2 \cdot 3^{2k+1} \\ &= 2^2 \cdot 5c - 2^2 \cdot 3^{2k+1} + 3^2 \cdot 3^{2k+1} \\ &= 2^2 \cdot 5c + (3^2 - 2^2) \cdot 3^{2k+1} \\ &= 2^2 \cdot 5c + 5 \cdot 3^{2k+1} = 5\left(2^2 c + 3^{2k+1}\right). \end{aligned}$$

Since $2^2 c + 3^{2k+1}$ is an integer, it follows that $5 \mid \left(2^{2k+3} + 3^{2k+3}\right)$. By the Principle of Mathematical Induction, if $n \geq 0$, then

$$5 \mid \left(2^{2n+1} + 3^{2n+1}\right).$$

An interesting observation is that both 2 and 3 are raised to the same odd power. Note that

$$\begin{aligned} 2^1 &= 2 & 3^1 &= 3 \\ 2^3 &= 8 & 3^3 &= 27 \\ 2^5 &= 32 & 3^5 &= 243 \\ 2^7 &= 128 & 3^7 &= 2187 \\ 2^9 &= 512 & 3^9 &= 19,683 \\ 2^{11} &= 2048 & 3^{11} &= 177,147 \end{aligned}$$

Some type of pattern seems to hold for the last digits for both integers, namely, 8 and 2 alternate in 2 raised to an odd power, and 7 and 3 alternate in 3 raised to an odd power $n \geq 1$. If the power is the same for both, then either the last digits are 8 for 2^n and 7 for 3^n, or 2 for 2^n and 3 for 3^n. Note that their sum give a number that ends in 5, which is the last digit of $2^n + 3^n$. $\square$