Week 4

Juan Patricio Carrizales Torres

Section 9: Some Fundamental Properties of Logical Equivalences

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Problem 58. Verify the following laws stated in Theorem 18:

(A) Let P, Q and R be statements. Then

$$P \vee (Q \wedge R) \text{ and } (P \vee Q) \wedge (P \vee R) \text{ are logically equivalent.}$$

Solution a. The logical equivalence $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ is one of the *Distributive Laws* of Theorem 18. Both compound statements $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent since they have the same truth values for all combinations of truth values for component statements P, Q and R. This is shown in the next truth table:

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	F	T	F	F
T	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	T	F	F	F
F	F	F	F	F	F	F	F

(B) Let P and Q be statements. Then

$$\sim (P \vee Q) \text{ and } (\sim P) \wedge (\sim Q) \text{ are logically equivalent.}$$

Solution b. The logical equivalence $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$ is one of *De Morgan's Laws* of Theorem 18. Since the compound statements $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ have the same truth values for all combinations of truth values for the component statements P and Q, these two compound statements are logically equivalent. This can be seen in the truth table below:

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim (P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Problem 59. Write negations of the following open sentences:

(A) Either $x = 0$ or $y = 0$.

Solution a. Consider the following open sentences:

$$P(x) : x = 0 \text{ and } Q(y) : y = 0$$

The open sentence (A) is a disjunction of $P(x)$ and $Q(x)$, $P(x) \vee Q(y)$: either $x = 0$ or $y = 0$. Using *De Morgan's Laws* of Theorem 18, we get the following logical equivalence $\sim (P(x) \vee Q(y)) \equiv (\sim P(x)) \wedge (\sim Q(y))$. Therefore, the negation of the open sentence (A) is:

$$(\sim P(x)) \wedge (\sim Q(y)) : \text{ Both } x \neq 0 \text{ and } y \neq 0$$

(B) The integers a and b are both even.

Solution b. Consider the following open sentences:

$$P(a) : \text{ the integer } a \text{ is even. and } Q(b) : \text{ the integer } b \text{ is even.}$$

The conjunction $P(a) \wedge Q(b)$: The integers a and b are both even. represents the open sentence (B). With the help of *De Morgan's Laws* from Theorem 18, we get the following logical equivalence $\sim (P(a) \wedge Q(b)) \equiv (\sim P(a)) \vee (\sim Q(b))$. Thus, the negation of the open sentence (B) is:

$$(\sim P(a)) \vee (\sim Q(b)) : \text{ Either the integer } a \text{ is odd or the integer } b \text{ is odd.}$$

Problem 60. Consider the implication: If x and y are even, then xy is even.
Before answering the excercises, let's consider the following open sentences:

$$P(x) : x \text{ is even. , } Q(y) : y \text{ is even. and } R(x, y) : xy \text{ is even.}$$

The following open sentence represents the implication of problem 60:

$$(P(x) \wedge Q(y)) \Rightarrow R(x, y) : \text{ If } x \text{ and } y \text{ are even, then } xy \text{ is even.}$$

(A) State the implication using "only if."

Solution a. $(P(x) \wedge Q(y)) \Rightarrow R(x, y)$: Both x and y are even only if xy is even

(B) State the converse of the implication.

Solution b. $R(x, y) \Rightarrow (P(x) \wedge Q(y))$: If xy is even, then x and y are even.

(C) State the implication as a disjunction (see Theorem 17).

Solution c. Using Theorem 17 and *De Morgan's Laws*, the implication of problem 60 can be stated as a disjunction by the following string of logical equivalences:

$$\begin{aligned}(P(x) \wedge Q(y)) \Rightarrow R(x, y) &\equiv \sim (P(x) \wedge Q(y)) \vee R(x, y) \\ &\equiv ((\sim P(x)) \vee (\sim Q(y))) \vee R(x, y)\end{aligned}$$

The implication of problem 60 as a disjunction states the following:

$$((\sim P(x)) \vee (\sim Q(y))) \vee R(x, y) : \text{ Either } x \text{ or } y \text{ is odd, or } xy \text{ is even.}$$

(D) State the negation of the implication as a conjunction (see Theorem 21(a))

Solution d. Using Theorem 21(a), the negation of the implication of problem 60 can be stated as a conjunction:

$$\sim ((P(x) \wedge Q(y)) \Rightarrow R(x,y)) \equiv (P(x) \wedge Q(y)) \wedge (\sim R(x,y))$$

This conjunction declares the following:

$$(P(x) \wedge Q(y)) \wedge (\sim R(x,y)) : \text{Both } x \text{ and } y \text{ are even, and } xy \text{ is odd.}$$

Problem 61. For a real number x, let $P(x) : x^2 = 2.$ and $Q(x) : x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words (see Theorem 21(b)).

Solution . From Theorem 21(b) we can use the following logical equivalence $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$. The biconditional $\sim (P(x) \Leftrightarrow Q(x))$ is logically equivalent to the following:

$$(P(x) \wedge (\sim Q(x))) \vee (Q(x) \wedge (\sim P(x))) : \text{Either } x^2 = 2 \text{ and } x \neq \sqrt{2}, \text{ or } x = \sqrt{2} \text{ and } x^2 \neq 2$$

Problem 62. Let P and Q be statements. Show that $(P \vee Q) \wedge (\sim (P \wedge Q)) \equiv \sim (P \Leftrightarrow Q)$.

Solution . In order to show the logical equivalence $(P \vee Q) \wedge (\sim (P \wedge Q)) \equiv \sim (P \Leftrightarrow Q)$, we will be using the laws of Theorem 18, Theorem 21(b) and the following logical equivalences [1]:

1. *Identity Laws*

   a $P \wedge T \equiv P$

   b $P \vee F \equiv P$

2. *Negation Laws*

   a $P \wedge (\sim P) \equiv F$

   b $P \vee (\sim P) \equiv T$

Due to the commutative properties of the conjunctions and disjunctions, the *Identity Laws* and *Negation Laws* are commutative (e.g., $P \wedge T \equiv T \wedge P \equiv P$). Now we show the logical equivalence:

$$
\begin{aligned}
&\textit{De Morgan's Laws} \\
(P \vee Q) \wedge (\sim (P \wedge Q)) &\equiv (P \vee Q) \wedge ((\sim P) \vee (\sim Q)) \\
&\textit{Distributive Laws} \\
&\equiv ((P \vee Q) \wedge (\sim P)) \vee ((P \vee Q) \wedge (\sim Q)) \\
&\textit{Commutative Laws} \\
&\equiv ((\sim P) \wedge (P \vee Q)) \vee ((\sim Q) \wedge (P \vee Q)) \\
&\textit{Distributive Laws} \\
&\equiv (((\sim P) \wedge P) \vee ((\sim P) \wedge Q)) \vee (((\sim Q) \wedge P) \vee ((\sim Q) \wedge Q)) \\
&\textit{Negation Laws} \\
&\equiv (F \vee ((\sim P) \wedge Q)) \vee (((\sim Q) \wedge P) \vee F) \\
&\textit{Identity Laws} \\
&\equiv ((\sim P) \wedge Q) \vee ((\sim Q) \wedge P) \\
&\textit{Commutative Laws} \\
&\equiv ((\sim Q) \wedge P) \vee ((\sim P) \wedge Q) \\
&\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)) \\
&\textit{Theorem 21(b)} \\
&\equiv \sim (P \Leftrightarrow Q)
\end{aligned}
$$

Problem 63. Let $n \in \mathbb{Z}$. For which implication is its negation the following?
The integer $3n + 4$ is odd and $5n - 6$ is even.

Solution . We consider the logical equivalence $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$ from Theorem 21(a) and we derive the following open sentences from the negation stated in problem 63.

$$P(n) : 3n + 4 \text{ is odd. and } Q(n) : 5n - 6 \text{ is odd.}$$

Therefore, $P(n)$ and $Q(n)$ are the hypothesis and conlusion of the implication in question, respectively.

$$P(n) \Rightarrow Q(n) : \text{If } 3n + 4 \text{ is odd, then } 5n - 6 \text{ is odd.}$$

It's important to note that if we consider the commutative laws of disjunction in the logical equivalence from Theorem 21(a) (e.g., $P \wedge (\sim Q) \equiv (\sim Q) \wedge P \equiv \sim (P \Rightarrow Q)$) we could get 2 possible answers for this problem (including the one we have shown).

Problem 64. For which biconditional is its negation the following?
n^3 and $7n + 2$ are odd or n^3 and $7n + 2$ are even.

Solution . We consider the logical equivalence $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$ from Theorem 21(b) and we derive the following open sentences from the negation stated in problem 64.

$$P(n) : n^3 \text{ is odd. and } Q(n) : 7n + 2 \text{ is even.}$$

Therefore, the biconditional in question is the following:

$$P(n) \Leftrightarrow Q(n) : n^3 \text{ is odd if and only if } 7n + 2 \text{ is even.}$$

Due to the commutative properties of conjunctions and disjunctions, we could get 2 different combinations of open sentences $P(n)$ and $Q(n)$ by changing the order of the conjunctions in the disjunction of the logical equivalence from Theorem 21(b) (e.g., $(P \wedge (\sim Q)) \vee (Q \wedge (\sim P)) \equiv (Q \wedge (\sim P)) \vee (P \wedge (\sim Q)))$. For each of the 2 different ways to order the conjunctions we can get another combination of open sentences $P(n)$ and $Q(n)$ by changing the order of the elements of the first conjunction (e.g., $(Q \wedge (\sim P)) \vee (P \wedge (\sim Q)) \equiv ((\sim P) \wedge Q) \vee (P \wedge (\sim Q)))$. Therefore, $2 + 2 = 4$ combinations of open sentences for the biconditional can be derived. However, due to the commutative nature of the biconditional (e.g., $P \Leftrightarrow Q \equiv Q \Leftrightarrow P$), every two possible combinations of open sentences $P(n)$ and $Q(n)$ will yield the same biconditional. Thus, only 2 possible different answers can be derived for the biconditional.

References

[1] GeeksforGeeks, *Mathematics Propositional Equivalences*, Apr 02, 2019. Retrieved Aug 04, 2021.