Section 2.2: The Limit of a Sequence

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The section started with some examples as arguments for the idea that our intuition regarding the properties and manipulations (i.e. reordering the terms, splitting it into finite sums) of finite sums are ambiguous in the field of infinite sums. In fact saying "This infinite sum *equals* . . . " for any inifnite sum is kind of problematic. That's why, one must first check the concepts of sequences and convergence, which are very related to the infinite sums.

A *sequence* in real analysis is defined as a function from $\mathbb{N}$ to $\mathbb{R}$, which orders some real numbers. Recall that a function is a relation and a sequence is a subset of $\mathbb{N} \times \mathbb{R}$. The contents of a sequence define an ordered list of ordered real numbers. For instance, the list of ordered real numbers $(0, 1, 2, 3, \dots)$ is defined by the sequence with rule $f(n) = n - 1$.

The ordered list of real numbers (a_n) defined by a sequence is said to converge to a if for any real number $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$. This states that as we go higher in the sequence at some point, the distance between the next elements and the limit a gets eventually smaller and smaller approaching 0. The elements eventually get arbitrarily close to the limit a. Arbitrarily in the sense that we are choosing any positive real number ε and eventually since there is some positive integer N after which the sequence gets closer to the limit with a distance lower than ε.
Also, one can define convergence in a topological manner, namely, an ordered list of real number (a_n) defind by some sequence is said to converge to a if for every ε-neighbourhood $V_\varepsilon(a)$, there is some $N \in \mathbb{N}$ such that $a_n \in V_\varepsilon(a)$ for every $n \geq \mathbb{N}$. A neighbourhood $V_\varepsilon(a)$ is defined as

$$\begin{aligned} V_\varepsilon(a) &= \{x \in \mathbb{R} : |x - a| < \varepsilon\} \\ &= (a - \varepsilon, a + \varepsilon). \end{aligned}$$

for some $a \in \mathbb{R}$ and positive real number ε. This definition says that every ε-neighbourhood $V_\varepsilon(a)$ contains all but finitely many elements of the sequence (a_n).

Problem 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* some $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq \mathbb{N}$ implies $|x_n - x| < \varepsilon$. Give an example of a vercongent sequence. Is there an example of a vercongent sequnce that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution First, note that "*for all* $N \in \mathbb{N}$ it is true that $n \geq \mathbb{N}$ implies $|x_n - x| < \varepsilon$" is equivalent to "*for all* $n \in \mathbb{N}$, $|x_n - x| < \varepsilon$". Now, a very simple example of a vercongent sequence is $(1, 0, 1, 0, 1, 0, 1, \dots)$, namely, the number 1 (0) is assigned to the odd (even) places. Now consider any real number a and let $m = \max(\{|0 - a|, |1 - a|\})$. Thus, $0 \leq |x_n - a| \leq m < m + 1$ for all $n \in \mathbb{N}$ and so $m + 1$ is the ε that we were looking for. In fact, this sequence is divergent and verconges to any real number. The interesting part of this definition is that it suggests that if some sequence (x_n) verconges to some real value a, then it can be "enclosed" inside some ε-neighborhood $V_\varepsilon(a)$. Actually, one could further conjecture that if one can show that $m = \sup(\{|x_n - a| : n \in \mathbb{N}\}$ exists for some real number a and sequence (a_n), then the sequence verconges to any real value by mere adjustement of ε.

Problem 2.2.2. Verify, using the dfinition of convergence of a sequence, that the following sequence converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

Solution Consider some real number $\varepsilon > 0$. Note that

$$\begin{aligned} \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{(10n+5) - (10n+8)}{5(5n+4)}\right| \\ &= \left|\frac{-3}{25n+20}\right| = \frac{3}{25n+20}, \end{aligned}$$

for every positive integer n. Consider some $N \in \mathbb{N}$ such that $\frac{3}{\varepsilon} < N$ and so $\frac{1}{N} < \frac{\varepsilon}{3}$. Thus, $\frac{3}{25N+20} < \frac{3}{N} < \varepsilon$ and so $\frac{3}{25n+20} \leq \frac{3}{25N+20} < \varepsilon$ for any $n \geq N$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

Solution Consider some real number $\varepsilon > 0$. Note that $\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3}$ for any postive integer n. Then, $\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$. Then, choose som $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$, which further implies that $\frac{1}{N} < \frac{\varepsilon}{2}$ and so

$$\frac{2n^2}{n^3+3} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon$$

for every $n \geq N$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution Consider some positive real number ε. Note that $\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}}$. Consider some $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon^3}$. Now, let $n \geq N$. Hence, $n > \frac{1}{\varepsilon^3}$ and so $\sqrt[3]{n} > \frac{1}{\varepsilon}$. Therefore, $\frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} < \varepsilon$. Thus, $|x_n - 0| < \varepsilon$ for all $n \geq N$.

Problem 2.2.4. Give an example of each or state that the request is impossible. For any that are imposible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

Solution This is impossible. Since the lowest infinity is card($\mathbb{N}$) and a sequence is defined as a function from $\mathbb{N}$ to $\mathbb{R}$, then $(a_n = 1 : n \in \mathbb{N})$ is the only possible sequence with an infinity of ones. Note that for any positive real number ε, $|a_n - 1| = 0 < \varepsilon$ for every $n \in \mathbb{N}$. Therefore, the sequence converges to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

Solution The former argument also shows that such sequence is impossible to create.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution Let $S = (a_n)$ be a sequence defined by $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$. Then,

$$S = (1, 3, 6, 10, \dots).$$

Now, define the sequence $M = (b_n)$ by $b_n = 0$ for $n \in S$ and $b_k = 1$ for $k \in \mathbb{N} - S$. Then,

$$M = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots).$$

We first show that one can find k consecutive ones in the sequence. Suppose that $k = 1$, then $b_1 = 0, b_2 = 1, b_3 = 0$ and so there is 1 consecutive one. We may further assume that $k \geq 2$. Hence, $a_k = a_{k-1} + k$ and $a_{k+1} = a_k + k + 1$. Note that $a_{k+1} - a_k = k + 1$ and so in between a_{k+1} and a_k there are k consecutive positive integers such that they are not elements of S. Thus, these k spaces are ones. Also, M is divergent, since one can only find two discrete values inside it.