Week 1

Juan Patricio Carrizales Torres

Section 7.1: Conjectures in Mathematics

March 24, 2022

Problem 1. Consider the following sequence of equalities:

$$\begin{aligned}
1 &= 0+1 \\
2+3+4 &= 1+8 \\
5+6+7+8+9 &= 8+27 \\
10+11+12+13+14+15+16 &= 27+64
\end{aligned}$$

(a) What is the next equality in this sequence?

Solution (a). The next equality seems to be

$$17+18+19+20+21+22+23+24+25 = 64+125$$

(b) What conjecture is suggested by these equalities?

Solution (b). For any nonnegative integer n, we have that

$$\sum_{i=1}^{2n+1}(n^2+i) = n^3+(n+1)^3$$

(c) Prove the conjecture in (b) by induction?

Proof. We proceed by induction. Since $0^2+(0^2+1) = 1 = 0^3+(0+1)^3$, it follows that the conjecture is true for $n=0$. Assume that

$$\sum_{i=1}^{2k+1}(k^2+i) = k^3+(k+1)^3$$

for some nonnegative integer k. Hence,

$$(k+1)^3 = \sum_{i=1}^{2k+1}(k^2+i) - k^3.$$

We show that

$$\sum_{i=1}^{2(k+1)+1} \left[(k+1)^2 + i\right] = (k+1)^3 + (k+2)^3$$

Note that

$$\begin{aligned}
\sum_{i=1}^{2(k+1)+1} \left[(k+1)^2 + i\right] &= \sum_{i=1}^{2k+3} \left[k^2 + 2k + 1 + i\right] \\
&= \sum_{i=1}^{2k+3} (k^2 + i) + (2k+3)(2k+1) \\
&= \sum_{i=1}^{2k+1} (k^2 + i) + [k^2 + (2k+2)] + [k^2 + (2k+3)] + (2k+3)(2k+1) \\
&= \sum_{i=1}^{2k+1} (k^2 + i) + k^2 + 2k + 2 + k^2 + 2k + 3 + 4k^2 + 8k + 3 \\
&= \sum_{i=1}^{2k+1} (k^2 + i) + 6k^2 + 12k + 8 \\
&= \sum_{i=1}^{2k+1} (k^2 + i) - k^3 + [k^3 + 6k^2 + 12k + 8] \\
&= \sum_{i=1}^{2k+1} (k^2 + i) - k^3 + (k+2)^3 = (k+1)^3 + (k+2)^3,
\end{aligned}$$

according to the inductive hypothesis. By the Principle of Mathematical Induction, it is true that

$$\sum_{i=1}^{2n+1} (n^2 + i) = n^3 + (n+1)^3.$$

for any nonnegative integer n. □

Problem 2. Consider the following statements:

$$\begin{aligned}
(1+2)^2 - 1^2 &= 2^3 \\
(1+2+3)^2 - (1+2)^2 &= 3^3 \\
(1+2+3+4)^2 - (1+2+3)^2 &= 4^3
\end{aligned}$$

(a) Based on the three statements given above, what is the next statement suggested by these?

Solution a. The next statement suggested is

$$(1+2+3+4+5)^2 - (1+2+3+4)^2 = 5^3$$

(b) What conjecture is suggested by these statements?

Solution b. For any integer $n \in \mathbb{N}$,

$$[1+2+\ldots+(n+1)]^2 - (1+2+\ldots+n)^2 = (n+1)^3$$

(c) Verify the conjecture in (b).

Proof. Let $n \in \mathbb{N}$. Note that

$$\begin{aligned}
[1+2+\ldots+(n+1)]^2 - (1+2+\ldots+n)^2 &= \left[(n+1)\left(\frac{n+2}{2}\right)\right]^2 - \left[n\left(\frac{n+1}{2}\right)\right]^2 \\
&= \left(\frac{n^2+3n+2}{2}\right)^2 - \left(\frac{n^2+n}{2}\right)^2 \\
&= \frac{(n^4+9n^2+4+6n^3+4n^2+12n)-(n^4+2n^3+n^2)}{4} \\
&= \frac{4n^3+12n^2+12n+4}{4} = n^3+3n^2+3n+1 \\
&= (n+1)^3.
\end{aligned}$$

(Consider that $1 = 1\frac{1+1}{2}$). $\square$

Problem 3. A sequence $\{a_n\}$ of real numbers is defined recursively by $a_1 = 2$ and for $n \geq 2$,

$$a_n = \frac{2 + 1 \cdot a_1^2 + 2 \cdot a_2^2 + \ldots + (n-1)a_{n-1}^2}{n}.$$

(a) Determine a_2, a_3 and a_4.

Solution a.

$$\begin{aligned}
a_2 &= \frac{2+1\cdot 2^2}{2} = 3 \\
a_3 &= \frac{6+2\cdot 3^2}{3} = 8 \\
a_4 &= \frac{24+3\cdot 8^2}{4} = 54
\end{aligned}$$

(b) Clearly, a_n is a rational number for each $n \in \mathbb{N}$. Based on the information in (a), however, what conjecture does this suggest?

Solution b. For every $n \in \mathbb{N}$, a_n is a positive integer.
This conjecture implies for every $n \in \mathbb{N}$ that

$$\left(2 + 1 \cdot a_1^2 + 2 \cdot a_2^2 + \ldots + (n-1)a_{n-1}^2\right) = n \cdot a$$

for some $a \in \mathbb{N}$.

Problem 5. By an ordered partition of an integer $n \geq 2$ is meant a sequence of positive integers whose sum is n. For example, the ordered partitions of 3 are $3, 1+2, 2+1, 1+1+1$.

(a) Determine the ordered partitions of 4.

Solution a. Let $(a, b, c, \ldots, n)$ represent an n-tuple for the integers of some ordered partition of 4. Then 4 has the following partitions:

$$\begin{aligned}(4)\\(1,3),(3,1)\\(2,2)\\(2,1,1),(1,2,1),(2,1,1)\\(1,1,1,1)\end{aligned}$$

(b) Make a conjecture concerning the number of ordered partitions of an integer $n \geq 2$.

Solution b. Let $M(n)$ be number of ordered partitions of some integer $n \geq 2$. Note that $M(2) = 2, M(3) = 4$ and $M(4) = 8$ are powers of 2 and so one may conjecture the following:
Let some integer $n \geq 2$. Then, the number of ordered partitions of n is 2^{n-1}.

Problem 6. Two recursively defined sequences $\{a_n\}$ and $\{b_n\}$ of positive integers have the same recurrence relation, namely $a_n = 2a_{n-1} + a_{n-2}$ and $b_n = 2b_{n-1} + b_{n-2}$ for $n \geq 3$. The initial values for $\{a_n\}$ are $a_1 = 1$ and $a_2 = 3$, while the initial values for $\{b_n\}$ are $b_1 = 1$ and $b_2 = 2$.

(a) Determine a_3 and a_4.

Solution a.

$$\begin{aligned}a_3 &= 2 \cdot 3 + 1 = 7\\a_4 &= 2 \cdot 7 + 3 = 17.\end{aligned}$$

(b) Determine whether the following is true or false:
Conjecture: $a_n = 2^{n-2} \cdot n + 1$ for every integer $n \geq 2$.

Proof. We proceed by strong induction. Since $a_2 = 3 = 2^0 \cdot 2 + 1$, it follows that the conjecture is true for $n = 2$. Suppose for $2 \leq i \leq k$ that $a_i = 2^{i-2} \cdot i + 1$. We prove that $a_{k+1} \neq 2^{k-1} \cdot (k+1) + 1$. Note that $a_3 = 7 = 2^1 \cdot 3 + 1$ and $a_4 = 17 = 2^2 \cdot 4 + 1$ and so $k \geq 4$. Hence, $k + 1 \geq 5$ and

$$\begin{aligned}a_{k+1} = 2a_k + a_{k-1} &= 2\left(2^{k-2} \cdot k + 1\right) + \left(2^{k-3} \cdot (k-1) + 1\right)\\&= 2^{k-1} \cdot k + 2^{k-3} \cdot (k-1) + 2 + 1\\&= k(2^{k-1} + 2^{k-3}) - 2^{k-3} + 2 + 1\\&\neq 2^{k-1} \cdot k + 2^{k+1} + 1 = 2^{k-1} \cdot (k+1) + 1\end{aligned}$$

since $2^{k-3} > 0$ and so $2^{k-1} + 2^{k-3} \neq 2^{k-1}$. By the Strong Principle of Mathematical Induction, this conjecture is false $((p \implies q, \neg p) \implies \neg q)$. Note that a_{k+1} represents a counterexample to our conjecture. □

(c) Determine b_3 and b_4.

Solution c.

$$b_3 = 2 \cdot 2 + 1 = 5$$
$$b_4 = 2 \cdot 5 + 2 = 12.$$

(d) Determine whether the following is true or false:
Conjecture: $b_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$ for every integer $n \geq 2$.

Proof. We proceed by strong induction. Since

$$\frac{(1+\sqrt{2})^1 - (1-\sqrt{2})^1}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1 = b_1.$$

it follows for $n = 1$ that the conjecture is true. Suppose for $2 \leq i \leq k$ that

$$b_i = \frac{(1+\sqrt{2})^i - (1-\sqrt{2})^i}{2\sqrt{2}}.$$

We show that

$$b_{k+1} = \frac{(1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}}{2\sqrt{2}}.$$

Because

$$\frac{(1+\sqrt{2})^2 - (1-\sqrt{2})^2}{2\sqrt{2}} = \frac{4\sqrt{2}}{2\sqrt{2}} = 2 = b_2,$$

it follows that the conjecture is true for $k \geq 2$ and so $k + 1 \geq 3$. By definition of $\{a_n\}$, we have

$$\begin{aligned}
b_{k+1} = 2b_k + b_{k-1} &= 2\left(\frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}}\right) + \frac{(1+\sqrt{2})^{k-1} - (1-\sqrt{2})^{k-1}}{2\sqrt{2}} \\
&= \frac{2\left(1+\sqrt{2}\right)^k + \left(1+\sqrt{2}\right)^{k-1} - \left[2\left(1-\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^{k-1}\right]}{2\sqrt{2}} \\
&= \frac{2\left(1+\sqrt{2}\right)^{k-1}\left(1+\sqrt{2}\right) + \left(1+\sqrt{2}\right)^{k-1} - \left[2\left(1-\sqrt{2}\right)^{k-1}\left(1-\sqrt{2}\right) + \left(1-\sqrt{2}\right)^{k-1}\right]}{2\sqrt{2}} \\
&= \frac{\left(1+\sqrt{2}\right)^{k-1}\left(2+2\sqrt{2}+1\right) - \left[\left(1-\sqrt{2}\right)^{k-1}\left(2-2\sqrt{2}+1\right)\right]}{2\sqrt{2}} \\
&= \frac{\left(1+\sqrt{2}\right)^{k-1}\left(\sqrt{2}+1\right)^2 - \left(1-\sqrt{2}\right)^{k-1}\left(1-\sqrt{2}\right)^2}{2\sqrt{2}} = \frac{\left(1+\sqrt{2}\right)^{k+1} - \left(1-\sqrt{2}\right)^{k+1}}{2\sqrt{2}}.
\end{aligned}$$

By the Strong Principle of Mathematical Induction,

$$b_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$$

for any $n \in \mathbb{N}$. $\square$

Problem 7. We know that $1 + 2 + 3 = 1 \cdot 2 \cdot 3$; that is, there exist three positive integers whose sum equals their product. Prove or disprove (a) and (b).

(a) There exist four positive integers whose sum equals their product.

Solution a.

$$1 + 1 + 2 + 4 = 1 \cdot 1 \cdot 2 \cdot 4$$

(b) There exist five positive integers whose sum equals their product.

Solution b.

$$1 + 1 + 1 + 2 + 5 = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 5$$

(c) What conjecture does this suggest to you?

Solution c. Let some positive integer $n \geq 2$. Then $\{2, n, 1, 1, 1, \ldots\}$ is the set of n positive integers whose sum is equal to their product.

Proof. Consider some positive integer $n \geq 2$ and let $\{2, n, 1, 1, 1, \ldots\}$ be some set of n positive integers. Then

$$\begin{aligned} 2 + n + 1 + 1 + 1 + \ldots &= 2 + n + (n-2) \\ &= 2n = 2n \cdot 1 \cdot 1 \cdot 1 \cdots . \end{aligned}$$

$\square$

This clearly implies that for $n \geq 2$ there exist n positive integers whose sum equals their product.