Week 14

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Section 3: A Review of Three Proof Techniques

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Problem 34. Prove that if n is an odd integer, then $7n - 5$ is even by

(a) Direct Proof

Solution a. Let n be an odd integer. Then $n = 2c + 1$ for some $c \in \mathbb{Z}$. Therefore, $7n - 5 = 7(2c + 1) - 5 = 14c + 2 = 2(7c + 1)$. Since $7c + 1 \in \mathbb{Z}$, it follows that $7n - 5$ is even. Note that $7n$ is odd (odd integer times an odd integer) and -5 is also odd; so their sum must be an even integer.

(b) Proof by Contrapositive

Solution b. Let $7n - 5$ be an odd integer. Then $7n - 5 = 2c + 1$, where $c \in \mathbb{Z}$. Note that

$$n = (7n - 5) + (-6n + 5) = 2c + 1 - 6n + 5 = 2(c - 3n + 3)$$

Since $c - 3n + 3 \in \mathbb{Z}$, it follows that n is even.

ALTERNATE SOLUTION

Let $7n - 5$ be an odd integer. Then $7n - 5 = 2c + 1$, where $c \in \mathbb{Z}$. Note that $7n = 2c + 1 + 5 = 2(c + 3)$. Since $c + 3 \in \mathbb{Z}$, it follows that $7n$ is even. By theorem, either 7 or n are even. Then $2 \mid n$ since $2 \nmid 7$.

(c) Proof by Contradiction

Solution c. Assume, to the contrary, that there is an odd integer n such that $7n - 5$ is odd. Then $n = 2c + 1$ for some integer c. Therefore, $7n - 5 = 7(2c + 1) - 5 = 14c + 7 - 5 = 14c + 2 = 2(7c + 1)$. Since $7c + 1$ is an integer, $7n - 5$ is even. This contradicts our initial assumption.

Problem 35. Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$ by

(a) Direct Proof

Proof. Let x be a positive real number such that $x - \frac{2}{x} > 1$. Since $x > 0$, we can multiply both sides of the inequality $x - \frac{2}{x} > 1$ by x. Therefore, $x^2 - 2 > x$ and so $x^2 - x - 2 > 0$. Factorizing we find that $(x+1)(x-2) > 0$. Since $x + 1 > 0$, it follows, by dividing both sides by $x + 1$, that $x - 2 > 0$. Therefore, $x > 2$. □

(b) Proof by Contrapositive

Proof. Let x be a positive real number such that $x \leq 2$. Then $x - 2 \leq 0$. Since $x > 0$, it follows that $x + 1$ is a positive number. Multiplying both sides of the inequality $x - 2 \leq 0$ by the positive number $x + 1$ yields $(x+1)(x-2) \leq 0$. Note that

$$\begin{aligned} (x+1)(x-2) &\leq 0 \\ x^2 - x - 2 &\leq 0 \\ x^2 - 2 &\leq x \end{aligned}$$

Since x is positive, it follows, by dividing both sides by x, that $x - \frac{2}{x} \leq 1$. □

(c) Proof by Contradiction

Proof. Assume, to the contrary, that there is a positive real number x such that $x - \frac{2}{x} > 1$ and $x \leq 2$. Since $x \leq 2$, it follows that $x - 2 \leq 0$ and by multiplying both sides by the positive number $x+1$ we get that $(x+1)(x-2) = x^2 - x - 2 \leq 0$. Dividing by the positive real number x yields $x - \frac{2}{x} - 1 \leq 0$ and so $x - \frac{2}{x} \leq 1$. This clearly leads to a contradiction. □

Problem 36. Let $a, b \in \mathbb{R}$. Prove that if $ab \neq 0$, then $a \neq 0$ by using as many of the three proof techniques as posible.

(a) Proof by Contrapositive

Proof. Assume that $a = 0$. Then $ab = 0b = 0$. Therefore, $ab = 0$. □

(b) Proof by Contradiction

Proof. Assume, to the contrary, that there exist some real numbers a and b such that $ab \neq 0$ and $a = 0$. Then, $ab = 0b = 0$, which contradicts our initial assumption. □

Problem 37. Let $x, y \in \mathbb{R}^+$. Prove that if $x \leq y$, then $x^2 \leq y^2$ by

(a) Direct Proof

Proof. Let $x, y \in \mathbb{R}^+$. Assume that $x \leq y$. Multiplying both sides by the positive real numbers x and y, respectively, we get that $x^2 \leq xy$ and $xy \leq y^2$. Therefore, $x^2 \leq xy \leq y^2$ and so $x^2 \leq y^2$.

□

(b) Proof by Contrapositive

Proof. Let $x, y \in \mathbb{R}^+$. Assume that $x^2 > y^2$. Then $x^2 - y^2 > 0$ and so $(x+y)(x-y) > 0$. Since $x + y > 0$, we can divide both sides by $x + y$. Therefore, $x - y > 0$ and so $x > y$. □

(c) Proof by Contradiction

Proof. Assume, to the contrary, that there exist two positive real numbers x and y such that $x \leq y$ and $x^2 > y^2$. Multiplying both sides of $x \leq y$ by the positive x we get that $x^2 \leq xy$. Then, multiplying $x \leq y$ by the positive real number y we get that $xy \leq y^2$. Thus, $x^2 \leq xy \leq y^2$ and so $x^2 \leq y^2$, which leads to a contradiction. □

Problem 38. Prove the following statement using more than one method of proof.
Let $a, b \in \mathbb{Z}$. If a is odd and $a + b$ is even, then b is odd and ab is odd.

(a) Direct Proof

Proof. Let $a, b \in \mathbb{Z}$ such that a is odd and $a + b$ is even. Since $a + b$ is even, it follows, by *Theorem 3.16*, that a and b are of the same parity and so b is odd. Therefore, $a = 2n + 1$ and $b = 2m + 1$ where $n, m \in \mathbb{Z}$. Then, $ab = (2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$. Since $2nm + n + m \in \mathbb{Z}$, it follows that ab is odd. □

(b) Proof by Contradiction

Proof. Assume, to the contrary, that there exist two integers a and b such that a is odd and $a + b$ is even, and either b or ab are even. Since $a + b$ is even, it follows, by *Theorem 3.16*, that a and b are of the same parity and so b is odd. Therefore, $a = 2n + 1$ and $b = 2m + 1$ where $n, m \in \mathbb{Z}$. Then, $ab = (2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$. Since $2nm + n + m \in \mathbb{Z}$, it follows that ab is odd. Because b and ab are odd, this leads to a contradiction □

Problem 39. Prove the following statement using more than one method of proof.
For every three integers a, b and c, exactly two of the integers ab, ac and bc cannot be odd.

(a) Direct Proof

Proof. Let a, b and c be integers. We have to show that exactly two of the integers ab, ac and bc cannot be odd. If the all of a, b, c are odd, then the three integers ab, ac and cb are odd. If one of them is even, say b, then ab and cb are even. Therefore, exactly two of the integers ab, ac and bc can not be odd. □

(a) Proof by Contradiction

Proof. Assume, to the contrary, that there are 3 integers a, b and c such that exactly two of the integers ab, ac and bc are odd. Note that for every possible pair of the integers ab, ac and bc, an integer of a, b and c will be multiplied by the other two, respectively (i.e., ab and ac). Since two of them are odd, this implies that the three integers a, b and c must be odd by the previous reason. However the three integers ab, ac and bc end up being odd, leading to a contradiction. □