Section 9.4: Bijective Functions

Juan Patricio Carrizales Torres

Jul 28, 2022

As it was mentioned in the previous section, for finite sets A and B, $|A| \geq |B|$ is a necessary and sufficient condition for an onto function $f : A \to B$ to exist. The same can be said for $|A| \leq |B|$ and some one-to-one function $g : A \to B$. Since we are talking about positive integers, it must be true that $|A| = |B|$ is a necessary and sufficient condition for an onto and one-to-one function $\varphi : A \to B$ to exist, knwon as a bijective function.
In fact, for finite sets B and C such that $|B| = |C| = n$, there are $n!$ distinct bijective functions from B to C. Namely, every bijective function is a permutation of the elements of $|C|$ for n spaces. Furthermore, for any function f from B to C, f is onto if and only if f is one-to-one. All this makes sense for finite sets, we must make sure to pair all elements of C with the constriction of assigning one unique element to every element of B. However, this intuition does not work for analyzing the cases with infinite ones.
Let A, B be sets. So far, we defined the function $f : A \to B$ as a relation from A to B such that

(a) $x \in A \implies \exists b \in B, (a, b) \in f$

(b) $(a, b), (a, c) \in f \implies b = c$

If a relation satisfies (b), then it is called **well-defined**.
Lastly, the identity function i_S on ANY nonempty set S defined by $i_S(n) = n$ for all $n \in S$ is bijective.

Problem 31. Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ be a function defined by $f([a]) = [2a + 3]$.

(a) Show that f is well-defined.

Proof. Consider two $[a] = [b]$ such that $[a], [b] \in \mathbb{Z}_5$. Then, $a \equiv b \pmod 5$ which implies that $a - b = 5k$ for some $k \in \mathbb{Z}$. Then, $f([a]) = [2a + 3]$ and $f([b]) = [2b + 3]$. Note that

$$(2a + 3) - (2b + 3) = 2(a - b) = 5(2k).$$

Therefore, $(2a + 3) \equiv (2b + 3) \pmod 5$ and so $f([a]) = f([b])$ $\square$

(b) Determine wheter f is bijective.

Proof. We know that $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$. Note that $f([0]) = [3], f([1]) = [5] = [0], f([2]) = [7] = [2], f([3]) = [4]$ and $f([4]) = [11] = [1]$. Hence, all elements of $\mathbb{Z}_5$ are paired with a unique element of $\mathbb{Z}_5$. The function is bijective. $\square$

Problem 33. Let $A = [0, 1]$ denote the closed interval of real numbers between 0 and 1. Give an example of two different bijective functions f_1 and f_2 from A to A, neither of which is the identity function.

(a) $f : A \to A$ defined by $f(n)$

Problem 34. Give a proof of Theorem 7 using mathematical induction.

Solution If A and B are sets with $|A| = |B| = n$, then there are $n!$ bijective functions from A to B.

Proof. We proceed by induction. Let A and B be sets with $|A| = |B| = 1$, then there is only $1 = 1!$ bijective function from A to B, namely, the pairing of the only element of A with the only element of B. In fact, this is the only function from A to B since $\left|B^A\right| = 1$.
Suppose for sets A_1 and B_1 with $|A_1| = |B_1| = k$ that there are $k!$ bijective functions from A to B. We prove for sets A_2 and B_2 with $|A_2| = |B_2| = k + 1$ that there are $(k + 1)!$ bijective functions.
By our inductive hypothesis, we can only create $k!$ distinct bijective functions by fixing an element (a_{k+1}, b_{k+1}) in all of them since the remaining elements correspond to a bijective function from $\{a_1, a_2, \ldots, a_k\}$ to $\{b_1, b_2, \ldots, b_k\}$. Note that we can do this with $(a_{k+1}, b_k), (a_{k+1}, b_{k-1}), \ldots, (a_{k+1}, b_2), (a_{k+1}, b_1)$. Therefore, for each of the possible $k+1$ images of a_{k+1}, there are only $k!$ distinct bijective functions. By the Principle of Mathematical Induction, there are $(k + 1)k! = (k + 1)!$ bijective functions from A_2 to B_2. $\square$