Week 1

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Section 1.1: Limits of sequences of sets

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Note that the author uses $\subset$ as a synonym of $\subseteq$. Therefore, this same notation will be used in this document. Also is important to review deeply the definitions of limsup and liminf. Let X be some nonemtpy set and $(A_n : n \in \mathbb{N})$ be some sequence of subsets of X. First, let's consider the definition of liminf, namely,

$$\liminf_{n\to\infty} A_n = \{x \in X : x \in A_n \text{ for all but finitely many } n \in \mathbb{N}\}$$

.

Note that the argument $x \in A_n$ **for all but finitely many** $n \in \mathbb{N}$ is the same as saying $x \in A_n$ **for cofinitely many** $n \in \mathbb{N}$. Basically, this means that there is some infinite set $B \subset \mathbb{N}$ such that its complement is finite and $x \in \bigcap_{n\in B} A_n$. For example, if

$$x \in \bigcap_{n \text{ is an odd positive integer}} A_n,$$

then this does not mean that $x \in \liminf_{n\to\infty} A_n$ (However it does not discard the posibility) since the set of the positive even integers is infinite.

On the other hand, considering the previous example, the fact that

$$x \in \bigcap_{n \text{ is an odd positive integer}} A_n$$

implies that

$$x \in \{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

.

since the set of the odd positive integers is inifinite. Therefore, $x \in \limsup_{n\to\infty} A_n$.

Problem 1. Given two sequences of subsets $(E_n : n \in \mathbb{N})$ and $(F_n : n \in \mathbb{N})$ of a set X.

(a) Show that

$$\begin{aligned}\liminf_{n\to\infty} E_n \cup \liminf_{n\to\infty} F_n &\subset \liminf_{n\to\infty} (E_u \cup F_u) \subset \liminf_{n\to\infty} E_u \cup \limsup F_n \\ &\subset \limsup_{n\to\infty} (E_n \cup F_n) \subset \limsup_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n.\end{aligned}$$

To show this, let $(E_n : n \in \mathbb{N})$ and $(F_n : n \in \mathbb{N})$ be sequences of subsets of some set X. We prove the whole result in individual sections.

1. We show that

$$\liminf_{n\to\infty} E_n \cup \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty} (E_u \cup F_u)$$

Proof. Suppose that

$$x \in \liminf_{n\to\infty} E_n \cup \liminf_{n\to\infty} F_n.$$

Then either $x \in \liminf_{n\to\infty} E_n$ or $x \in \liminf_{n\to\infty} F_n$. In the case of the former, it follows that $x \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k$. Therefore, there is some $n_0 \in \mathbb{N}$ such that

$$x \in \bigcap_{k\geq n_0} E_k \subset \bigcap_{k\geq n_0} (E_k \cup F_k).$$

Hence, $x \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n}(E_k \cup F_k) = \liminf_{n\to\infty} (E_u \cup F_u)$. In the latter case, the same procedure can be applied and thus it is omitted. Therefore,

$$\liminf_{n\to\infty} E_n \cup \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty} (E_n \cup F_n).$$

□

2. We Prove that

$$\liminf_{n\to\infty} (E_u \cup F_u) \subset \liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n$$

Proof. Let $x \in \liminf_{n\to\infty} (E_n \cup F_n)$ and so $x \in (E_n \cup F_n)$ for all but finitely many $n \in \mathbb{N}$. Note that $x \in F_n$ for either finitely or infinitely many $n \in \mathbb{N}$.

Suppose that $x \in F_n$ for infinitely many $n \in \mathbb{N}$, then $x \in \limsup_{n\to\infty} F_n$. On the other hand, if $x \in F_n$ for finitely many $n \in \mathbb{N}$, then $x \in E_n$ for all but finitely many $n \in \mathbb{N}$, since $x \in \liminf_{n\to\infty} (E_n \cup F_n)$. Hence $x \in \liminf_{n\to\infty} E_n$. Therefore, either $x \in \limsup_{n\to\infty} F_n$ or $x \in \liminf_{n\to\infty} E_n$ and so

$$\liminf_{n\to\infty} (E_n \cup F_n) \subset \liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n$$

□

3. Here will be proved that

$$\liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty} (E_n \cup F_n)$$

Proof. Let $x \in \liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} EF_n$. Then, either $x \in F_n$ for infinitely many $n \in \mathbb{N}$ or $x \in E_n$ for all but finitely many $n \in \mathbb{N}$. Suppose the former, then $x \in \limsup_{n\to\infty}(E_n \cup F_n)$ since

$$\limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty}(E_n \cup F_n).$$

This is so because $F_n \subset E_n \cup F_n$ for infinitely many $n \in \mathbb{N}$. Now, assume the latter. Therefore, $x \in \limsup_{n\to\infty}(E_n \cup F_n)$ since

$$\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n \subset \limsup_{n\to\infty} (E_n \cup F_n).$$

Therefore,

$$\liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty} (E_n \cup F_n)$$

□

4. Lastly, we show that

$$\liminf_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty} (E_n \cup F_n)$$

Proof. Let some $x \in \limsup_{n\to\infty} (E_n \cup F_n)$. Then $x \in E_n \cup F_n$ for infinitely many $n \in \mathbb{N}$. Recall that

$$\limsup_{n\to\infty} E_n \subset \limsup_{n\to\infty} (E_n \cup F_n) \text{ and}$$
$$\limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty} (E_n \cup F_n).$$

If $x \in E_n$ for infinitely many $n \in \mathbb{N}$, then $x \in \limsup_{n\to\infty} E_n$. On the other hand, if $x \in E_n$ for finitely many $n \in \mathbb{N}$, then $x \in F_n$ for infinitely many $n \in \mathbb{N}$ since $x \in E_n \cup F_n$ for infinitely many $n \in \mathbb{N}$. Therefore, $x \in \limsup_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n$ and so

$$\limsup_{n\to\infty} (E_n \cup F_n) \subset \limsup_{n\to\infty} E_n \cup \limsup_{n\to\infty} F_n$$

□

(b) Show that

$$\begin{aligned}\liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty} (E_n \cap F_n) \subset \liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n \\ \subset \limsup_{n\to\infty} (E_n \cap F_n) \subset \limsup_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n.\end{aligned}$$

To prove this, let $(E_n : n \in \mathbb{N})$ and $(F_n : n \in \mathbb{N})$ be sequences of subsets of some set X. We proceed by sections.

1. We prove that

$$\liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty} (E_n \cap F_n)$$

Proof. Let some $x \in \liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n$. Then $x \in E_n$ for all but finitely many $n \in \mathbb{N}$ and $x \in F_n$ for all but finitely many $n \in \mathbb{N}$. Therefore, there exists two numbers $n_0, n_1 \in \mathbb{N}$ such that

$$x \in \left(\bigcap_{k\geq n_0} E_k\right) \cap \left(\bigcap_{k\geq n_1} F_k\right).$$

There are three posible exclusive cases: $n_0 > n_1$, $n_0 < n_1$ and $n_0 = n_1$. In the first one, note that

$$\left(\bigcap_{k\geq n_0} E_k\right) \cap \left(\bigcap_{k\geq n_1} F_k\right) = \left(\bigcap_{k\geq n_0} E_k \cap F_k\right) \cap \left(\bigcap_{n_1\leq k<n_0} F_k\right).$$

Hence, $x \in E_k \cap F_k$ for all $k \geq n_0$ and so $x \in \liminf_{n\to\infty}(E_n \cap F_n)$. In the second case we arrive at a same conclusion (just swap the places of n_0 and n_1). In the last case, it clearly follows that $x \in E_k \cap F_k$ for all but finitely many $k \in \mathbb{N}$. Therefore,

$$\liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty}(E_n \cap F_n)$$

□

2. We prove that

$$\liminf_{n\to\infty}(E_n \cap F_n) \subset \liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n$$

Proof. Assume that there is some $x \in \liminf_{n\to\infty}(E_n \cap F_n)$, then $x \in E_n \cap F_n$ for all but finitely many $n \in \mathbb{N}$. Hence, $x \in E_n$ for all but finitely many $n \in \mathbb{N}$ such that $x \in \liminf_{n\to\infty} E_n$ and $x \in \liminf_{n\to\infty} F_n$, and so

$$x \in \liminf_{n\to\infty} E_n \cap \liminf_{n\to\infty} F_n \subset \liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n.$$

Therefore,

$$\liminf_{n\to\infty}(E_n \cap F_n) \subset \liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n$$

□

3. We prove that

$$\liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty}(E_n \cap F_n)$$

Proof. Let $x \in \liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n$. Then, $x \in E_n$ for all but finitely many $n \in \mathbb{N}$ and $x \in F_n$ for infinitely many $n \in \mathbb{N}$. Therefore, $x \in E_n \cap F_n$ for infinitely many n and so

$$\liminf_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n \subset \limsup_{n\to\infty}(E_n \cap F_n).$$

□

4. We show that

$$\limsup_{n\to\infty}(E_n \cap F_n) \subset \limsup_{n\to\infty} E_n \cap \limsup_{n\to\infty} F_n$$

Proof. Assume that there is some $x \in \limsup_{n\to\infty}(E_n \cap F_n)$. Then $x \in E_n \cap F_n$ for infinitely many $n \in \mathbb{N}$. Therefore, $x \in E_n$ for infinitely many $n \in \mathbb{N}$ and $x \in F_n$ for infinitely many $n \in \mathbb{N}$. □

(c) Show that if $\lim_{n\to\infty} E_n$ and $\lim_{n\to\infty} F_n$ exist, then $\lim_{n\to\infty}(E_n \cup F_n)$ and $\lim_{n\to\infty}(E_n \cap F_n)$ exist and moreover

1.

$$\lim_{n\to\infty}(E_n \cup F_n) = \lim_{n\to\infty} E_n \cup \lim_{n\to\infty} F_n.$$

2.

$$\lim_{n\to\infty}(E_n \cap F_n) = \lim_{n\to\infty} E_n \cap \lim_{n\to\infty} F_n$$

Proof. Suppose that both $\lim_{n\to\infty} E_n$ and $\lim_{n\to\infty} F_n$ exist. Then, by definition, it is true that

$$\lim_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n \text{ and}$$
$$\lim_{n\to\infty} F_n = \liminf_{n\to\infty} F_n = \limsup_{n\to\infty} F_n.$$

Note that

$$\lim_{n\to\infty} F_n \cup \lim_{n\to\infty} E_n = \liminf_{n\to\infty} F_n \cup \liminf_{n\to\infty} E_n \subset \liminf_{n\to\infty}(F_n \cup E_n)$$
$$\subset \limsup_{n\to\infty}(F_n \cup E_n) \subset \limsup_{n\to\infty} F_n \cup \limsup_{n\to\infty} E_n = \lim_{n\to\infty} F_n \cup \lim_{n\to\infty} E_n.$$

Therefore, $\liminf_{n\to\infty}(F_n \cup E_n) = \limsup_{n\to\infty}(F_n \cup E_n)$ and so $\lim_{n\to\infty}(F_n \cup E_n)$ exists and it is equal to $\lim_{n\to\infty} F_n \cup \lim_{n\to\infty} E_n$. Also, if you change all union symbols for intersections, you'll get a valid proof for the existence of $\lim_{n\to\infty}(F_n \cap E_n) = \lim_{n\to\infty} F_n \cap \lim_{n\to\infty} E_n$. □

Problem 1.2. **(a)** Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of a set X. Let $(B_n : n \in \mathbb{N})$ be a sequence obtained by dropping finitely many entries in the sequence $(A_n : n \in \mathbb{N})$. Show that $\liminf_{n\to\infty} B_n = \liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} B_n = \limsup_{n\to\infty} A_n$. Show that $\lim_{n\to\infty} B_n$ exists if and only if $\lim_{n\to\infty} A_n$ exists and when they exist they are equal.

Proof. Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of some set X. Also, let $(B_n : n \in \mathbb{N})$ be a sequence of subsets generated by dropping finitely many entires in $(A_n : n \in \mathbb{N})$. Recall that both $\left(\bigcap_{k\geq n} B_n : n \in \mathbb{N}\right)$ and $\left(\bigcap_{k\geq n} A_n : n \in \mathbb{N}\right)$ are increasing sequences and so for any $n_0 \in \mathbb{N}$ there is some integer $n_1 > n_0$ such that $\bigcap_{k\geq n_0} B_k \subset \bigcap_{k\geq n_1} A_k$. Note that the same can be said for $\bigcap_{k\geq n_0} A_k \subset \bigcap_{k\geq n_1} B_k$. Therefore, $\liminf_{n\to\infty} A_n = \liminf_{n\to\infty} B_n$.

Also, note that both $\left(\bigcup_{k\geq n} B_n : n \in \mathbb{N}\right)$ and $\left(\bigcup_{k\geq n} A_n : n \in \mathbb{N}\right)$ are decreasing sequences and so for any $n_0 \in \mathbb{N}$ there is some integer $n_1 > n_0$ such that $\bigcup_{k\geq n_1} A_k \subset \bigcup_{k\geq n_0} B_k$. The same applies for $\bigcup_{k\geq n_1} B_k \subset \bigcup_{k\geq n_0} A_k$. Therefore, $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} B_n$ is a necessary and sufficent condition for $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_n$. Hence, $\limsup_{n\to\infty} A_n =$

$\limsup_{n\to\infty} B_n$.

We now prove that $\lim_{n\to\infty} B_n$ exists if and only if $\lim_{n\to\infty} A_n$. First, assume that $\lim_{n\to\infty} B_n$ exists. Then, $\liminf_{n\to\infty} A_n = \liminf_{n\to\infty} B_n = \limsup_{n\to\infty} B_n = \limsup_{n\to\infty} A_n$ and so $\lim_{n\to\infty} A_n$ exists and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$. The same argument can be used to prove the converse. $\square$

(b) Let $(A_n : n \in \mathbb{N})$ and $(B_n : n \in \mathbb{N})$ be two sequences of subsets of a set X such that $A_n = B_n$ for all but finitely many $n \in \mathbb{N}$. Show that $\liminf_{n\to\infty} B_n = \liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} B_n = \limsup_{n\to\infty} A_n$. Show that $\lim_{n\to\infty} B_n$ exists if and only if $\lim_{n\to\infty} A_n$ exists and when they exist they are equal.

Proof. Let N be the greatest $n \in \mathbb{N}$ such that $B_n \neq A_n$ and so the set of the first N positive integers is finite. Let $(x \in A'_n : n \in \mathbb{N})$ be the sequence created by dropping the first N entries in $(x \in A_n : n \in \mathbb{N})$ and, similarly, let $(x \in B'_n : n \in \mathbb{N})$ be the sequence created by dropping the first N entries in $(x \in B_n : n \in \mathbb{N})$ and so $A'_n = B'_n$ for all $n \in \mathbb{N}$. By the previous proof,

$$\begin{gathered}\liminf_{n\to\infty} A'_n = \liminf_{n\to\infty} A_n,\\ \limsup_{n\to\infty} A'_n = \limsup_{n\to\infty} A_n,\\ \text{and}\\ \liminf_{n\to\infty} B'_n = \liminf_{n\to\infty} B_n,\\ \limsup_{n\to\infty} B'_n = \limsup_{n\to\infty} B_n.\end{gathered}$$

However,

$$\begin{gathered}\liminf_{n\to\infty} A'_n = \liminf_{n\to\infty} B'_n \text{ and}\\ \limsup_{n\to\infty} A'_n = \limsup_{n\to\infty} B'_n\end{gathered}$$

. Therefore, $\liminf_{n\to\infty} A_n = \liminf_{n\to\infty} B_n$ and $\limsup_{n\to\infty} A_n = \limsup_{n\to\infty} B_n$.

By the result shown in the previous proof, the existence of $\lim_{n\to\infty} B_n$ is a necessary and sufficient condition for $\lim_{n\to\infty} A_n$ to exist, and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$. $\square$

Problem 1.3. Let $(E_n : n \in \mathbb{N})$ be a disjoint sequence of subsets of a set X. Show that $\lim_{n\to\infty} E_n$ exists and $\lim_{n\to\infty} E_n = \emptyset$

Proof. Since $(E_n : n \in \mathbb{N})$ is a disjoint sequence of subsets, it follows that there is not some infinite set S such that $x \in \bigcap_{n\in S} E_n$ and so $\limsup_{n\to\infty} E_n = \emptyset$. Also, $\liminf_{n\to\infty} E_n = \emptyset$ since $\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n$. Hence, $\lim_{n\to\infty} E_n = \liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \emptyset$. $\square$

Problem 1.4. Let $a \in \mathbb{R}$ and let $(x_n : n \in \mathbb{N})$ be a sequence of points in $\mathbb{R}$, all distinct from a, such that $\lim_{n\to\infty} x_n = a$. Show that $\lim_{n\to\infty}\{x_n\}$ exists and $\lim_{n\to\infty}\{x_n\} = \emptyset$ and thus $\lim_{n\to\infty}\{x_n\} \neq a$.

Proof. Note that, according to the given description of the sequence of real points $(x_n : n \in \mathbb{N})$, it is evident that $(\{x_n\} : n \in \mathbb{N})$ is a disjoint sequence of subsets of $X = \{x_n : n \in \mathbb{N}\}$. By the previous problem, it follows that $\lim_{n\to\infty} = \liminf_{n\to\infty} = \limsup_{n\to\infty} = \emptyset \neq a$. $\square$

Problem 1.5. For $E \subset \mathbb{R}$ and $t \in \mathbb{R}$, let us write $E + t = \{x + t \in \mathbb{R} : x \in E\}$ and call it the translate of E by t. Let $(t_n : n \in \mathbb{N})$ be a strictly decreasing sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} t_n = 0$ and let $E_n = E + t_n$ for $n \in \mathbb{N}$. Let us investigate the existence of $\lim_{n\to\infty} E_n$.

(a) Let $E = (-\infty, 0)$. Show that $\lim_{n\to\infty} E_n = (-\infty, 0]$.

Proof. Since $(E_n : n \in \mathbb{N}) = ((-\infty, t_n) : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ is a strictly decreasing sequence of real numbers, it follows that $(-\infty, t_n) \supset (-\infty, t_{n+1})$ for all $n \in \mathbb{N}$ and so $(E_n : n \in \mathbb{N})$ is a decreasing sequence of subsets of $\mathbb{R}$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} E_n &= \bigcap_{n\in\mathbb{N}} E_n \\ &= (-\infty, 0]\end{aligned}$$

since $\lim_{n\to\infty} t_n = 0$. $\square$

(b) Let $E = \{a\}$ where $a \in \mathbb{R}$. Show that $\lim_{n\to\infty} E_n = \emptyset$

Proof. Note that $(E_n : n \in \mathbb{N}) = (\{a + t_n\} : n \in \mathbb{N})$. Therefore, $(E_n : n \in \mathbb{N})$ is a disjoint collection of subsets and so $\lim_{n\to\infty} E_n = \emptyset$ (recall **problem 1.3**). $\square$

(c) Let $E = [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. Show that $\lim_{n\to\infty} E_n = (a, b]$

Proof. We know that $(E_n : n \in \mathbb{N}) = ([a + t_n, b + t_n] : n \in \mathbb{N})$. Because $t_n \downarrow 0$, it follows that $a + t_n \downarrow a$ and $b + t_n \downarrow b$ and so $(a, b] \subset E_n$ for all but finitely many $n \in \mathbb{N}$. Therefore,

$$(a, b] \subset \liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n$$

since $\lim_{n\to\infty} t_n = 0$.

Now consider $(-\infty, a]$. Note that $a < a + t_n$ for all $n \in \mathbb{N}$ since $t_n \downarrow 0$. Therefore, $(-\infty, a] \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so

$$(-\infty, a] \cap \limsup_{n\to\infty} E_n = \emptyset.$$

Hence,

$$(-\infty, a] \cap \liminf_{n\to\infty} E_n = \emptyset$$

since $\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n$. Consider (b,∞). We know that $b+t_n \downarrow b$ and so $[a+t_n, b+t_n] = E_n$ for finitely many $n \in \mathbb{N}$ (the distance between b and the upper bound of E_n gets smaller and smaller but never reaches b). Therefore,

$$(b,\infty) \cap \limsup_{n\to\infty} E_n = \emptyset,$$

which implies that

$$(b,\infty) \cap \liminf_{n\to\infty} E_n = \emptyset.$$

Hence, $\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = (a,b]$. $\square$

(d) Let $E = (a,b)$. Show that $\lim_{n\to\infty} E_n = (a,b]$.

Proof. Consider $x \in (a,b]$. Since $(a+t_n, b+t_n) = E_n$ and, $a+t_n \downarrow a$ and $b+t_n \downarrow b$, it follows that $x \in E_n$ for all but finitely many $n \in \mathbb{N}$. Therefore,

$$x \in \liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n.$$

Consider $x \in (-\infty, a]$. It is clear that $x \notin E_n$ for every $n \in \mathbb{N}$ and so $x \notin \limsup_{n\to\infty} E_n$. Now let's take a look at $x \in (b,\infty)$. Hence, $x \in E_n$ for finitely many $n \in \mathbb{N}$ and so $x \notin \limsup_{n\to\infty} E_n$. Therefore,

$$\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = (a,b] = \lim_{n\to\infty} E_n.$$

$\square$

(e) Let $E = \mathbb{Q}$, the set of all rational numbers. Assume that $(t_n : n \in \mathbb{N})$ satisfies the aditional condition that t_n is a rational number for all but finitely many $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} E_n = E$.

Proof. Note that $E_n = (x + t_n : x \in \mathbb{Q}) = \mathbb{Q}$ for all but finitely $n \in \mathbb{N}$ since t_n is a rational number for all but finitely many $n \in \mathbb{N}$. Hence

$$\mathbb{Q} \subset \liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n.$$

Since t_n is an irrational number for finitely many $n \in \mathbb{N}$, it follows that $E_n = (x + t_n : x \in \mathbb{Q}) \subset \mathbb{I}$, the set of all irrational numbers, for finitely many $n \in \mathbb{N}$ and so $\limsup_{n\to\infty} E_n \not\subset \mathbb{I}$. Therefore,

$$\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \mathbb{Q} = E = \lim_{n\to\infty} E_n$$

since $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. $\square$

Problem 1.6. The characteristic function $\mathbf{1}_A$ of a subset A of a set X is a function on X defined by

$$\mathbf{1}_A = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in A^c. \end{cases}$$

Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of X and A be a subset of X.

(a) Show that if $\lim_{n\to\infty} A_n = A$ then $\lim_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_A$ on X.

Proof. Let $\lim_{n\to\infty} A_n = A$. Then, it is true that

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n = A.$$

Note that if $x \in X$, then either $x \in A$ or $x \in A^c$. Consider $x \in A$. Then $\mathbf{1}_A(x) = 1$. Since $\liminf_{n\to\infty} A_n = A$, it follows that $x \in A_n$ for all but finitely many $n \in \mathbb{N}$ and so $\mathbf{1}_{A_n}(x) = 1$ for all but finitely many $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \mathbf{1}_{A_n}(x) = 1$.

Now consider $y \in A^c$. Then $\mathbf{1}_A(y) = 0$. Since $(\limsup_{n\to\infty} A_n)^c = A^c$, it follows that $y \notin \limsup_{n\to\infty} A_n$. Therefore, $y \in A_n$ for finitely many $n \in \mathbb{N}$ and so $\mathbf{1}_{A_n}(y) = 0$ for all but finitely many $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \mathbf{1}_{A_n}(y) = 0$ and so $\lim_{n\to\infty} \mathbf{1}_{A_n}(x) = \mathbf{1}_A(x)$ for any $x \in X$.

PROOF ANALYSIS

This proof seems like an algorithm for any $x \in X$ (like a function). Note that we only needed the fact that $x \in A_n$ for all but finitely many $n \in \mathbb{N}$ to show that $\lim_{n\to\infty} \mathbf{1}_{A_n}(x) = 1$ for any $x \in X$ since all elements of the limit inferior exist in the limit superior. Also we needed the fact that $y \notin A_n$ for infinitely many $n \in \mathbb{N}$ to show that $\lim_{n\to\infty} \mathbf{1}_{A_n}(y) = 0$ since $\lim_{n\to\infty} A_n$ contains elements that belong to A_n for $n \in S$, where $S \subset \mathbb{N}$ is a set that is not finite (this includes cofinite sets). $\square$

(b) Show that if $\lim_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_A$ on X then $\lim_{n\to\infty} A_n = A$.

Proof. Suppose that $\lim_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_A$ on X. Note that if $x \in X$, then either $x \in A$ or $x \in A^c$. Consider $x \in A$. Then $x \in A_n$ for all but finitely many $n \in \mathbb{N}$. Therefore,

$$x \in \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n, \text{ which implies that}$$
$$A \subset \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

Now consider $y \in A^c$. Since $\lim_{n\to\infty} \mathbf{1}_{A_n}(y) = \mathbf{1}_A(y) = 0$, it follows that $y \in A_n$ for only finitely many $n \in \mathbb{N}$. Hence, $y \notin \limsup_{n\to\infty} A_n$ and so

$$A = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n.$$

$\square$

Problem 1.7. Let $\mathfrak{A}$ be a σ-algebra of subsets of a set X and let Y be an arbitrary subset of X. Let $\mathfrak{B} = \{A \cap Y : A \in \mathfrak{A}\}$. Show that $\mathfrak{B}$ is a σ-algebra of subsets of Y.

Proof. To prove that $\mathfrak{B}$ is a σ-algebra of subsets of Y, we will show that it suffices the conditions 1^o and 3^o of **Definition 1.1** and 4^o of **Definition 1.3** with respect to set Y.

1. We show that $Y \in \mathfrak{B}$.

   First, note that $X \cap Y = Y$ and so $\mathfrak{B}$ fulfills condition 1^o.

2. We prove that $B \in \mathfrak{B} \implies B^c \in \mathfrak{B}$.

   Consider some $A \in \mathfrak{A}$. Then, by definition, $A^c \in \mathfrak{A}$. Therefore,

   $$(A \cap Y), (A^c \cap Y) \in \mathfrak{B}.$$

   Note that

   $$\begin{aligned} Y \backslash (A \cap Y) &= (X \cap Y) \backslash (A \cap Y) \\ &= (X \backslash A) \cap Y = A^c \cap Y. \end{aligned}$$

   It follows that condition 3^o is fullfiled.

3. We show that $(B_n : n \in \mathbb{N}) \subset \mathfrak{B} \implies \bigcup_{n \in \mathbb{N}} B_n \in \mathfrak{B}$.

   Let $(A_n : n \in \mathbb{N}) \subset \mathfrak{A}$. Then, by definition, $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A}$. Hence,

   $$\begin{gathered} (A_n \cap Y : n \in \mathbb{N}) = (B_n : n \in \mathbb{N}) \subset \mathfrak{B} \text{ and} \\ \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cap Y = \bigcup_{n \in \mathbb{N}} (A_n \cap Y) = \bigcup_{n \in \mathbb{N}} B_n \in \mathfrak{B} \end{gathered}$$

   Therefore, the condition 4^o is fullfiled and so $\mathfrak{B}$ is a σ-algebra on Y.

$\square$

PROOF ANALYSIS

One could have proven each implication by a simple method of direct proof, namely, for some implication $A \implies B$, you start by assuming A and then conclude B. Some type of unilinear view ofunderstanding the correlation between the statements which could make us think in some type of causality mindframe, namely, the truth of A (pre-existence) is sufficient for B to be true. However, during this process, one uses some set of statements S held to be true in order to conclude B from A.

Does this imply that $(\forall x \in S) \implies (A \implies B)$, which looks like some type of regression problem if we say that other set of assumptions is sufficient for the set S to be true, and so on. If so, then

$$(\forall x \in S) \implies (\sim A \wedge B)$$

However, if $\forall x \in S \implies A$, then it suffices to show that $(\forall x \in S) \implies B$. Something we used for this proof, since, in my opinion, it yielded a more intuitive proof.

Problem 1.8. Let $\mathfrak{A}$ be a collection of subsets of a set X with the following properties:

1. $X \in \mathfrak{A}$,
2. $A, B \in \mathfrak{A} \implies A \backslash B = A \cap B^c \in \mathfrak{A}$.

Show that $\mathfrak{A}$ is an algebra of subsets of a set X.

Proof. We show that $\mathfrak{A}$ has the following properties:

1. $X \in \mathfrak{A}$,
2. $A \in \mathfrak{A} \implies A^c \in \mathfrak{A}$,
3. $A, B \in \mathfrak{A} \implies (A \cup B) \in \mathfrak{A}$.

We prove them in separate sections.

1. Note that $X \in \mathfrak{A}$, by the already given characteristics of the set $\mathfrak{A}$.
2. Consider some $A \in \mathfrak{A}$. Since we know that $X \in \mathfrak{A}$, it follows, by the properties of the set $\mathfrak{A}$, that $X \backslash A = A^c \in \mathfrak{A}$.
3. Suppose that some sets $A, B \in \mathfrak{A}$. Since $X \in \mathfrak{A}$, it follows that $X \backslash A = A^c \in \mathfrak{A}$. Then, by definition, $A^c \backslash B = A^c \cap B^c \in \mathfrak{A}$. Hence, $X \backslash (A^c \cap B^c) \in \mathfrak{A}$. Note that

   $$\begin{aligned} X \backslash (A^c \cap B^c) &= (A^c \cap B^c)^c \\ &= A \cup B. \end{aligned}$$

   Therefore, $A \cup B \in \mathfrak{A}$.

□

Problem 1.9. Let $\mathfrak{A}$ be an algebra of subsets of a set X. Suppose $\mathfrak{A}$ has the property that for every increasing sequence $(A_n : n \in \mathbb{N})$ in $\mathfrak{A}$, we have $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A}$. Show that $\mathfrak{A}$ is a σ-algebra of subsets of the set X.

Proof. Since $\mathfrak{A}$ is an algebra of X, we just need to show that for some sequence $(A_n : n \in \mathbb{N})$ of subsets of X, we have that

$$(A_n : n \in \mathbb{N}) \subset \mathfrak{A} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A}.$$

Let $(A_n : n \in \mathbb{N}) \subset \mathfrak{A}$. We prove that we can construct some increasing sequence from the elements of $(A_n : n \in \mathbb{N})$. Because $\mathfrak{A}$ is an algebra of X, it follows for some $k \in \mathbb{N}$ that

$$\bigcup_{n=1}^{k} A_n \in \mathfrak{A}.$$

Let $\bigcup_{n=1}^{k} A_n = B_k$. Hence,

$$(B_k : k \in \mathbb{N}) \subset \mathfrak{A}.$$

Note that $(B_k : k \in \mathbb{N})$ is an increasing sequence and so, by the characteristics of $\mathfrak{A}$,

$$\bigcup_{k \in \mathbb{N}} B_k \in \mathfrak{A}.$$

However, note that

$$\begin{aligned}
\bigcup_{k \in \mathbb{N}} B_k = \bigcup_{k \in \mathbb{N}} \bigcup_{n=1}^{k} A_n &= A_1 \cup \bigcup_{k \geq 2} \bigcup_{n=1}^{k} A_n \\
&= A_1 \cup \bigcup_{k \geq 2} \left(\bigcup_{n=1}^{k-1} A_n \cup A_k \right) \\
&= A_1 \cup \bigcup_{k \geq 2} \bigcup_{n=1}^{k-1} A_n \cup \bigcup_{k \geq 2} A_k \\
&= \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cup \left(\bigcup_{k \geq 2} \bigcup_{n=1}^{k-1} A_n \right) \\
&= \bigcup_{n \in \mathbb{N}} A_n
\end{aligned}$$

since $\bigcup_{k \geq 2} \bigcup_{n=1}^{k-1} A_n \subset \bigcup_{n \in \mathbb{N}} A_n$. Therefore,

$$\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A},$$

and so $\mathfrak{A}$ is a σ-algebra on X. □

Problem 1.10. Let $(X, \mathfrak{A})$ be a measurable space and let $(E_n : n \in \mathbb{N})$ be an increasing sequence in $\mathfrak{A}$ such that $\bigcup_{n \in \mathbb{N}} E_n = X$.

(a) Let $\mathfrak{A}_n = \mathfrak{A} \cap E_n$, that is, $\mathfrak{A} = \{A \cap E_n : A \in \mathfrak{A}\}$. Show that $\mathfrak{A}_n$ is a σ-algebra of subsets of E_n for each $n \in \mathbb{N}$.

Proof. Since $(X, \mathfrak{A})$ is a measurable space, it follows that $\mathfrak{A}$ is a σ-algebra of subsets of X. Consider some $E_n \in (E_n : n \in \mathbb{N})$. Note that $E_n \subset X$ and $\mathfrak{A}_n = \{A \cap E_n : A \in \mathfrak{A}_\mathfrak{n}\}$. Hence, by the result proven in **Problem 1.7**, $\mathfrak{A}_n$ is a σ-algebra of subsets of E_n. □

(b) Does $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n = \mathfrak{A}$ hold?

Proof. Since $(E_n : n \in \mathbb{N})$ is an increasing sequence, it follows that $\lim_{n \to \infty} E_n = \bigcup_{n \in \mathbb{N}} E_n = X$. Hence, $X \subset E_n$ for all but finitely many $n \in \mathbb{N}$. Note that $E_n \subset X$ for every $n \in \mathbb{N}$ and so $E_n = X$ for all but finitely many $n \in \mathbb{N}$.

Consider some $A \in \mathfrak{A}$. Then,

$$A \cap E_n = A \cap X = A \in \mathfrak{A}_n$$

for all but finitely many $n \in \mathbb{N}$ and so

$$\mathfrak{A} \subset \mathfrak{A}_n = \{A \cap E_n : A \in \mathfrak{A}\}$$

for all but finitely many $n \in \mathbb{N}$ (This implies that $\mathfrak{A} \not\subset \mathfrak{A}_n$ for finitely many $n \in \mathbb{N}$). Note that, $\mathfrak{A}_n \subset \mathfrak{A}$ for every $n \in \mathbb{N}$. Therefore,

$$\mathfrak{A}_n = \mathfrak{A}$$

for all but finitely many $n \in \mathbb{N}$ and so

$$\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n = \mathfrak{A}.$$

Does this imply that $(\mathfrak{A}_n : n \in \mathbb{N})$ is an increasing sequence? □

(c) EXTRA: Show that $(\mathfrak{A}_n : n \in \mathbb{N})$ is an increasing sequence.

Proof. We just need to show for some $k \in \mathbb{N}$ that $\mathfrak{A}_k \subset \mathfrak{A}_{k+1}$. Let $(E_n : n \in \mathbb{N})$ be some increasing sequence of subsets in $\mathfrak{A}$. Also, let $B \in \mathfrak{A}_k$, then there is some $A \in \mathfrak{A}$ such that $A \cap E_k = B$. Since $\mathfrak{A}$ is a σ-algebra of subsets of X and $A, E_k \in \mathfrak{A}$, it follows that

$$A \cap E_k \in \mathfrak{A}$$

and so

$$(A \cap E_k) \cap E_{k+1} = \mathfrak{A}_{k+1}.$$

Note that

$$\begin{aligned}(A \cap E_k) \cap E_{k+1} &= A \cap (E_k \cap E_{k+1}) \\ &= A \cap E_k\end{aligned}$$

since $E_k \subset E_{k+1}$. Thus, $A \cap E_k \in \mathfrak{A}_{k+1}$ and so $\mathfrak{A}_k \subset \mathfrak{A}_{k+1}$. □

Problem 1.11. **(a)** Show that if $(\mathfrak{A}_n : n \in \mathbb{N})$ is an increasing sequence of algebras of subsets of a set X, then $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ is an algebra of subsets of X.

Proof. We will show that $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ fulfills the following conditions:

1 $X \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$.

2 $A \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n \implies A^c \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$.

3 $A, B \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n \implies A \cup B \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$.

We proceed to show each individually:

1 Since each element of $(\mathfrak{A}_n : n \in \mathbb{N})$ is an algebra of X, it follows that each of them contains X. Hence, $X \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$.

2 Consider some $A \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. Then there is some $k \in \mathbb{N}$ such that $A \in \mathfrak{A}_k$ and so $A^c \in \mathfrak{A}_k$, which implies that $A^c \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$.

3 Consider two subsets A and B in $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. Then, $A \in \mathfrak{A}_a$ and $B \in \mathfrak{A}_b$ for $a, b \in \mathbb{N}$. If $a = b$, then $A \cup B \in \mathfrak{A}_a$. On the other hand, WLOG let $a > b$, then $A, B \in \mathfrak{A}_a$ since $\mathfrak{A}_b \subset \mathfrak{A}_a$. Hence, $A \cup B \in \mathfrak{A}_a$. Note that both cases imply $A \cup B \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$.

□

(b) Show that if $(\mathfrak{A}_n : n \in \mathbb{N})$ is a decreasing sequence of algebras of subsets of a set X, then $\bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$ is an algebra of subsets of X.

Proof. We will show that $\bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$ fulfills the following conditions:

1 $X \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$.

2 $A \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n \implies A^c \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$.

3 $A, B \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n \implies A \cup B \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$.

First, note that $X \in \mathfrak{A}_n$ for all $n \in \mathbb{N}$. Hence $X \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$.

Now consider some $A \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$. Then, $A \in \mathfrak{A}_n$ for every $n \in \mathbb{N}$ and so $A^c \in \mathfrak{A}_n$ for all $n \in \mathbb{N}$. Hence, $A^c \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$.

Let some $A, B \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$. Then, $A, B \in \mathfrak{A}_n$ for every $n \in \mathbb{N}$ and so $A \cup B \in \mathfrak{A}_n$ for every $n \in \mathbb{N}$. Thus, $A \cup B \in \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$. Note that we did not use the fact that $(\mathfrak{A}_n : n \in \mathbb{N})$ is a decreasing sequence. This implies for any sequence of algebras that their intersection is an algebra. Remember **Lemma 1.10**, where the intersection of any arbitrary collection of algebras is and algebra. □

(c) **EXTRA:** Show that there exist some sequence of algebras of subsets of a set X with limit inferior and superior.

Proof. Let P be the power set of X and $(\mathfrak{A}_n : n \in \mathbb{N})$ be a sequence of algebras of subsets of X. Since, by definition, P is the largest algebra of X, it follows that $\mathfrak{A}_n \subset P$ for every $n \in \mathbb{N}$. Therefore, $(\mathfrak{A}_n : n \in \mathbb{N})$ is a sequence of subsets of the set P and, by definition, it can have a limit superior and inferior. □

Problem 1.12. Let $(X, \mathfrak{A})$ be a measurable space. Let us call an $\mathfrak{A}$-measurable subset E of X an atom in the measurable space $(X, \mathfrak{A})$ if $E \neq \emptyset$ and $\emptyset$ and E are the only $\mathfrak{A}$-measurable subsets of E. Show that if E_1 and E_2 are two distinct atoms in $(X, \mathfrak{A})$ then they are disjoint.

Proof. □