Section 1.6: Cantor's Theorem

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This chapter deals with some excercises that aid in proving a transcendental Theorem of Cantor:

Problem 1.6.1. Show that $(0,1)$ is uncountable if and only if $\mathbb{R}$ is uncountable.

Proof. The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \tan(\pi x - \pi/2)$ is bijective. Thus, $(0,1) \sim \mathbb{R}$. $\square$

Problem 1.6.2. (a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.

Solution (a). Because the real number with decimal expansion $.3\ldots \neq .2\ldots$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.

Solution (b). In general, for any $n \in \mathbb{N}$, $a_{n,n} \neq b_n$. This difference is what makes the decimal expansion of $f(n)$ to be different from x.

(c) Point out the contradiciton that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution (c). Since $x \neq f(n)$ for any $n \in \mathbb{N}$, it follows that function f is not onto. This is a contradiction of the assumption.

Problem 1.6.3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1 ($\mathbb{R} \sim (0,1)$).

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rationl numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

Solution (a). Recall that rational numbers are expressed by periodic decimal expansions. Carrying out the diagonalization does not assure the creation of a periodic decimal expansion. This sounds very unlikely, in fact the periodic decimal expansion repeats a finite sequence, I may not have a proof, but it must be that the number created is irrational (lacks periodic decimal expansion).

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $4.999\ldots$. Doesn't this cause some problems?

Solution (b). Since 2 and 3 are used to generate the real number b, we should see the case where this can be considered a problem. Let a be some real number paired with the positive integer n such that it terminates in the position n with 3, namely, $a = .\ \ldots 3$ and so $a = .\ \ldots 299999\ldots$. Then, the integer in the n'th place of the decimal expansion of a corresponds to the one in the decimal expansion of b. However, note that the integer in the $n+1$ place of the decimal expansion of b is either 2 or 3 which clearly does not correspond with 0 or 9. Thus, $a \neq b$ remains true.

Problem 1.6.4. Let S be the set consisting of all sequences of $0's$ and $1's$. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \cdots : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, 1, \ldots)$. Give a rigorous argument showing that S is uncountable.

Solution 1.6.4. We can apply the a similar argument to the one given by Cantor; namley, suppose, to the contrary, that there is a bijection $f : N \to S$, where $f(n) = (a_{n1}, a_{n2}, a_{n3}, a_{n4}, \ldots)$. Thus, one can construct the following matrix of values.

$$\begin{aligned}
f(1) &\to (a_{\mathbf{11}}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, \ldots) \\
f(2) &\to (a_{21}, a_{\mathbf{22}}, a_{23}, a_{24}, a_{25}, a_{26}, \ldots) \\
f(3) &\to (a_{31}, a_{32}, a_{\mathbf{33}}, a_{34}, a_{35}, a_{36}, \ldots) \\
f(4) &\to (a_{41}, a_{42}, a_{43}, a_{\mathbf{44}}, a_{45}, a_{46}, \ldots) \\
f(5) &\to (a_{51}, a_{52}, a_{53}, a_{54}, a_{\mathbf{55}}, a_{56}, \ldots) \\
f(6) &\to (a_{61}, a_{62}, a_{63}, a_{64}, a_{65}, a_{\mathbf{66}}, \ldots) \\
&\vdots
\end{aligned}$$

Now, define a sequence $b = (b_1, b_2, b_3, b_4, b_5, \ldots)$ by

$$b_n = \begin{cases} 1, & \text{if } a_{nn} = 0 \\ 0, & \text{if } a_{nn} \neq 0. \end{cases}$$

Thus, $b \neq f(n)$ for all $n \in \mathbb{N}$ and so f is not onto. This is a contradiction. Therefore, S is uncountable.
Using this same argument, one can prove that the countable cartesian product of $\mathbb{N}$ is uncountable. Which also implies that the countable product of countable sets is uncountable.

Problem 1.6.5. If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution 1.6.5. Consider some finite set A with k elements. Hence, we can order them finitely with some bijective function $f : \mathbb{N} \to A$. Now, consider the set S of all possible k-tuples of 0's and 1's. For instance, the k-tuple $(1, 1, 0, 1, \ldots, 0) \in S$. Since each place can be either 1 or 0, and their values are independent of each other, it follows that S has 2^k elements. Let some function $\varphi : S \to P(A)$ be defined by

$$\varphi(s) = \{f(n) : \text{ for all n'th positions in } s \text{ that are equal to } 1\}\,.$$

Hence, every subset of A is represented by a unique sequence in S, and so φ is bijective. Therefore, $|S| = |A| = 2^k$.

Problem 1.6.6. (a) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.

Solution (a). Define the funciton g by

$$g(c) = \{c\}\,.$$

(b) Explain why, in part (a), it is impossible to construct mappings that are *onto*.

Solution (b). Consider some finite set C. Then, any function $f : C \to P(C)$ is a relation where each $c \in C$ is paired with only one $p \in P(C)$. In fact, this is true for any function. Therefore, $|f(C)| = |C| < |P(C)| = 2^{|C|}$. Hence, there are elements in $P(C)$ that are not the image of any element of c, and so it is impossible to build an *onto* function from any finite set to its power set.

Problem 1.6.8. (a) First, show that the case $a' \in B$ leads to a contradiction.

Solution (a). We know that $f(a') = B$. If $a' \in B$, then $a' \in f(a')$ which contradicts our definition of B, namely, $B = \{x \in A : x \notin f(x)\}$ (It only contains every a that is not an element of its image subsets).

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution (b). If $a' \notin B$, then $a' \notin f(a')$. Thus, this also contradicts our definition of B (It must contain every a that is not contained in its image subsets). Therefore, there is no $a \in A$, such that $f(a) = B$ and so f is not onto.