Section 1.4: Consequences of Completeness

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One important "corollary" of the Axiom of Completeness in $\mathbb{R}$ is the **Archimedean Property**, which states that there is no real number that bounds above the set $\mathbb{N}$. Interetingly, this implies the density of $\mathbb{Q}$ in $\mathbb{R}$, which is a powerful property that can be used to determine the *supremums* and *infimums* of some bounded sets (as we have seen in the **Problem 1.3.8** of the previous section). Note that $\mathbb{Q}$ is dense in itself.

Problem 1.4.1. Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

Proof. Consider two rational numbers a and b. Hence, $a = m/n$ and $b = x/y$ for some $m, x \in \mathbb{Z}$ and $y, n \in \mathbb{N}$. Note that

$$\frac{m}{n} + \frac{x}{y} = \frac{my + xn}{ny} \quad \text{and}$$
$$\frac{m}{n} \cdot \frac{x}{y} = \frac{mx}{ny}.$$

Since $my, xn, mx \in \mathbb{Z}$ and $ny \in \mathbb{N}$, it follows that $a + b$ and ab are rationals. □

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Consider some nonzero $a \in \mathbb{Q}$ and $t \in \mathbb{I}$. Recall that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is closed under addition and multiplication. Hence, assume, to the contrary, that $a + t$ and at are rationals. Then, $a + t = m \in \mathbb{Q}$ and $at = n \in \mathbb{Q}$. Therefore,

$$t = m + (-a) \quad \text{and}$$
$$t = n \cdot \frac{1}{a}.$$

Since, $(-a), 1/a \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition and multiplication, it follows that in both cases, t ends up being a rational number, which contradicts our assumption that it is an irrational one. □

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrationl numbers s and t, what can we say about $s + t$ and st?

Solution c. Let's examine some specific examples of multiplication and addition of irrational numbers:
In the case of multiplication, consider $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$ and $(1+\sqrt{2}) \cdot (1+\sqrt{2}) = 2\sqrt{2} + 3 \in \mathbb{I}$.
In the case of addition, consider $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$ and $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2} \in \mathbb{I}$.
Hence, the sum and addition of irrational numbers can result in either a rational or an irrational number. Thus, $\mathbb{I}$ is neither closed under multiplication nor under addition.

Problem 1.4.2. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Proof. First, we will show that s is an upper bound for A. Assume, to the contrary, that s is not an upper bound for A, namely, there is some $a \in A$ such that $s < a$. By the **Archimedean Property**, there is some $k \in \mathbb{N}$ such that $0 < 1/k < a - s$ and so $s < s + \frac{1}{k} < a$ which contradicts our assumption that $s + \frac{1}{k}$ $(k \in \mathbb{N})$ is an upper bound for A. Thus, s is an upper bound for A.

Now, consider some $\varepsilon > 0$. By the **Archimedean Property**, there is some $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Hence,

$$s - \varepsilon < s - \frac{1}{k}.$$

Because $s - \frac{1}{k}$ is not an upper bound for A, there is some $a \in A$ such that $s - \frac{1}{k} < a$ and so $s - \varepsilon < a$. Hence, $s = \sup A$. □

Problem 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Proof. Note that $(0, 1/n) \supseteq (0, 1/(n+1))$ for any $n \in \mathbb{N}$ and so $(0, 1/n) : n \in \mathbb{N}$ is a nested sequence of sets. Assume, to the contrary, that there exists some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$ and so $0 < x$. However, by the **Archimedean Property**, there exist some $k \in \mathbb{N}$ such that $0 < 1/k < x$ and so $x \notin \bigcap_{n \geq k}(0, 1/k)$. This contradicts our assumption. □

Problem 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

Proof. Note that

$$\begin{aligned} T &= \mathbb{Q} \cap [a, b] \\ &= \{x : x \in \mathbb{Q}, a \leq x \leq b\} \\ &= \{x \in \mathbb{Q} : a \leq x \leq b\}. \end{aligned}$$

Hence, b is an upper bound for T. We show that it is the least one. Consider some $\varepsilon > 0$. Then, $b - \varepsilon < b$. By **The Density of Rational Numbers in** $\mathbb{R}$, there exists some rational number c such that $b - \varepsilon < c < b$. If $a \leq c$, then $c \in T$. On the other hand, if $c < a$, recall that there is some rational number d such that $a < d < b$ and so $b - \varepsilon < d \in T$. Hence, $\sup T = b$. □

Problem 1.4.5. Using **Problem 1.4.1**, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. Recall that

Corollary 1.4.4. Given any two real number $a < b$, there exists an irrational number t satisfying $a < t < b$.

Proof. Consider two real numbers a and b such that $a < b$. Then, $a - \sqrt{2} < b - \sqrt{2}$ and, by the density of rational numbers in $\mathbb{R}$, there exists some rational q such that $a - \sqrt{2} < q < b - \sqrt{2}$. Thus, $a < q + \sqrt{2} < b$. Note that $q + \sqrt{2}$ is the sum of a rational and irrational number and so $q + \sqrt{2} \in \mathbb{I}$. $\square$

Problem 1.4.6. Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

Proof. This set is not dense in $\mathbb{R}$. Let $a = 0$ and $b = 1/100$. Consider some rational number $x = p/q$ for $p \in \mathbb{Z}$ and for the natural number $q \leq 10$. For $p \leq 0$, $a \leq 0$ and so we may further assume that $p > 0$. Note that $1/100 < 1/10$ and so $1/100 \leq p/100 < p/10$ for any positive integer p. Thus,

$$\frac{1}{100} < \frac{p}{10} < \frac{p}{9} < \frac{p}{8} < \cdots < \frac{p}{1}.$$

Therefore, there is no such x such that $0 < a < 1/100$. $\square$

(b) The set of all rational numbers p/q with q a power of 2.

Proof. This set is dense in $\mathbb{R}$. Consider some rational numbers $a < b$. By the **Archimedean Property**, there is some $n \in \mathbb{N}$ such that $1/n < b - a$ and so $1/2^n < 1/n < b - a$. Now we can proceed with the same argument from the proof of **Theorem 1.3.4**. Consider some integer m such that

$$m - 1 \leq 2^n \cdot a < m.$$

Since $1/2^n < b - a$, it follows that $a < b - 1/2^n = \frac{1}{2^n}(2^n \cdot b - 1)$. Therefore,

$$2^n \cdot a + 1 < 2^n \cdot b$$

and so $m \leq 2^n \cdot a + 1 < 2^n \cdot b$. Thus, $2^n \cdot a < m < 2^n \cdot b$, which implies that

$$a < \frac{m}{2^n} < b.$$

$\square$

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Proof. This set is not dense in $\mathbb{R}$. Let $a = 0$ and $b = 1/11$. If the integer $p < 0$, then $p/q < 0$. Also, there is no element in this set for $p = 0$ since $q \leq 10 \cdot 0$ contradicts the fact that $q \in \mathbb{N}$. Hence, we may assume that $p > 0$ and so $|p| = p$ and $10p \geq q$. Note that

$$\frac{1}{10p} < \frac{1}{10p-1} < \frac{1}{10p-2} < \frac{1}{10p-3} < \cdots < \frac{1}{1},$$

which implies that

$$\frac{1}{10} < \frac{p}{10p-1} < \frac{p}{10p-2} < \frac{p}{10p-3} < \cdots < p.$$

Since $1/11 < 1/10$ it follows that $1/11 < p/q$ for any positive integer p. Therefore, there is no element in this set that lies between 0 and $1/11$. $\square$

Problem 1.4.7. Finsih the proof of **Theorem 1.4.5**

Problem 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

Solution (a). Let $A = \{x \in \mathbb{Q} : x < 1\}$ and $B = \{x \in \mathbb{I} : x < 1\}$. By the desnities of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$, we can show that $\sup A = \sup B = 1$ (the number 1 is an upper bound for both sets and for any $x < 1$ we can find some rational and irrational number between x and 1). Also, 1 is neither in A nor in B, and $A \cap B = \emptyset$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containig only a finite number of elements.

Solution (b). Let $J_n = (-\frac{1}{n}, \frac{1}{n})$ for some $n \in \mathbb{N}$. Note that

$$\frac{1}{n+1} < \frac{1}{n} \quad \text{and}$$
$$-\frac{1}{n} < -\frac{1}{n+1}.$$

Hence $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$. Since for any $k \in \mathbb{N}$ it is true that $-\frac{1}{k} < 0 < \frac{1}{k}$, it follows that $0 \in \bigcap_{n=1}^{\infty} J_n$. Also, $-\frac{1}{k} < -\frac{1}{k+1} < 0 < \frac{1}{k+1} < \frac{1}{k+1}$ and so we can find another . Thus

$$\bigcap_{n=1}^{\infty} J_n = \{0\}$$

is finite.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbouned closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

Solution (c). Let $L_n = [n, \infty)$ for any $n \in \mathbb{N}$. Hence, $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ and so $\{L_n : n \in \mathbb{N}\}$ is a sequence of nested unbounded closed intervals.
We now show that $\bigcap_{n=1}^{\infty} L_n = \emptyset$. Assume to the contrary, that $\bigcap_{n=1}^{\infty} L_n \neq \emptyset$. Then, there is some real number $x \in \bigcap_{n=1}^{\infty} L_n$. By **The Archimedean Property**, there exists some $k \in \mathbb{N}$ such that $x < k$ and so $x \notin [k, \infty) = L_k$. This contradicts our assumption that x is contained in every L_n. Therefore, $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

(d) A sequence of closed (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution (c). Since each I_n is closed and $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, it follows that each $K_N = \bigcap_{n=1}^{N} I_n \neq \emptyset$ is a closed set and so $S = \{K_n : n \in \mathbb{N}\}$ is a sequence of closed sets. Since we are talking about consecutive intersections, it follows that either S is decreasing ($K_N \supseteq K_{N+1}$), increasing ($K_N \subseteq K_{N+1}$) or constant ($K_N = K_{N+1}$), where $N \in \mathbb{N}$. If S is decreasing (nested sequence), then

$$\begin{aligned} \bigcap_{n=1}^{\infty} K_n &= \bigcap_{N=1}^{\infty} \left(\bigcap_{n=1}^{N} I_n \right) \\ &= \bigcap_{n=1}^{\infty} I_n \neq \emptyset \end{aligned}$$

by the **Nested Interval Property** of closed sets of real numbers. If S is increasing, then it is easy to understand that

$$\bigcap_{n=1}^{\infty} I_n = I_1$$

which, by assumption, is nonempty. And if it is constant, then every $K_n = X$ for some closed set of real numbers X and so

$$\bigcap_{n=1}^{\infty} I_n = X.$$