Section 9.6: Inverse Functions

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A part from its properties, functions come with an interesting concept, namley, the **inverse function**. Let $f : A \to B$ be some function. Then, the inverse f^{-1} is a relation defined by

$$f^{-1} = \{(b, a) : (a, b) \in f\}.$$

In fact, f^{-1} is a function from B to A if and only if f is bijective. Furthermore, f being bijective implies that f^{-1} is bijective. This points out that all **inverse functions** are bijective. Also, for some function f from A to B, if $f \circ f^{-1} = B$ and $f^{-1} \circ f = A$, then f is bijective. In fact, for functions $f : A \to B$ and $g : B \to A$ such that $f \circ g = i_A$ and $g \circ f = i_B$, both g and f are bijective and $g = f^{-1}$.

Moreover, for any function $f : A \to B$, let g be some function such that $f \circ g = i_B$. Then, g is known as the **right inverse** of f. In fact, if $h \circ f = i_A$ for some function h, then h is the left inverse of f. The following can be proven:

(a) f is surjective $\iff$ function g exists.

(b) f is injective $\iff$ funciton h exists.

Problem 51. Show that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x}{x-3}$ is bijective and determine $f^{-1}(x)$ for $x \in \mathbb{R} - \{5\}$.

Proof. We first show that $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$ is bijective. Consider some $a, b \in \mathbb{R} - \{3\}$ such that $f(a) = f(b)$. Then, $\frac{5a}{a-3} = \frac{5b}{b-3}$. Multiplying by $(a-3)(b-3)$ we have $(5a)(b-3) = (5b)(a-3)$. Hence, $5ab - 15a = 5ba - 15b$. Substracting $5ba$ and then dividing by -15 results in $a = b$. The function f is one-to-one. Now, consider any $y \in \mathbb{R} - \{5\}$. Then, $r = \frac{-3y}{5-y}$ is defined and $r \neq 3$ (otherwise $15 = 0$). Hence, $r \in \mathbb{R} - \{3\}$ and so

$$\begin{aligned} f(r) = \frac{5r}{r-3} &= \frac{5\left(\frac{-3y}{5-y}\right)}{\left(\frac{-3y}{5-y}\right) - 3} \\ &= \frac{\frac{-15y}{5-y}}{\frac{-3y-15+3y}{5-y}} \\ &= \frac{\frac{-15y}{5-y}}{\frac{-15}{5-y}} = y. \end{aligned}$$

Thus, f is onto and so bijective.
Since f is bijective, it follows that f^{-1} is a bijective function. We determine $f^{-1}(x)$ for any $x \in \mathbb{R} - \{5\}$. Consider some $x \in \mathbb{R} - \{5\}$. Because f is onto, it follows that there is some $a \in \mathbb{R} - \{3\}$ such that $f(a) = x$ and so $f^{-1}(x) = a$. Hence,

$$f\left(f^{-1}(x)\right) = \frac{5f^{-1}(x)}{f^{-1}(x) - 3} = x.$$

Hence, $5f^{-1}(x) = xf^{-1} - 3x$ and so $f^{-1}(x)(5 - x) = -3x$. This implies that $f^{-1}(x) = \frac{3x}{x-5}$. $\square$

Problem 53. Let A and B be sets with $|A| = |B| = 3$. How many functions from A to B have inverse functions?

Solution Recall that a function has an inverse function if and only if it is bijective. Since there are 3! bijective functions from A to B, it follows that only $3! = 6$ functions from A to B have inverse functions.

Problem 56. Let A, B and C be nonempty sets and let f, g and h be functions such that $f : A \to B, g : B \to C$ and $h : B \to C$. For each of the following, prove or disprove:

(a) If $g \circ f = h \circ f$, then $g = h$.

  Solution This is false. Let $A = \{1\}, B = \{1, 2, 3\}$ and $C = \{b, c\}$. Also, let $f = \{(1, 1)\}, g = \{(1, b), (2, c), (3, c)\}$ and $h = \{(1, b), (2, b), (3, b)\}$. Then, $g \circ f = \{(1, b)\} = h \circ f$ and $h \neq g$.

(b) If f is one-to-one and $g \circ f = h \circ f$, then $g = h$.

  Solution This is also false. Note that in the previous counterexample, $f = \{(1, 1)\}$ is one-to-one.

Problem 57. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x < 1 \\ \sqrt{x-1} & \text{if } x \geq 1. \end{cases}$$

(a) Show that f is a bijection.

  Proof. Note that for all real numbers $a < 1$ and $b \geq 1$ we have $a - 1 < 0$ and $0 \leq b - 1$. Hence, $f(a) = 1/(a-1) < 0$ and $f(b) = \sqrt{b-1} \geq 0$. Also, $\sqrt{b-1} \geq 0 \implies b \geq 1$ and $1/(a-1) < 0 \implies a < 1$. Thus, $f(a) < 0 \iff a < 1$ and $f(b) \geq 0 \iff b \geq 1$. We first show that f is injective. Let $f(a) = f(b)$. We consider two cases.

  Case1. $f(a) = f(b) < 0$ and so $a, b < 1$.Then, $1/(a-1) = 1/(b-1)$. Multiplying both sides by $(a-1)(b-1)$ we have $b - 1 = a - 1$ and so $a = b$.
  Case2. $f(a) = f(b) \geq 0$ and so $a, b \geq 1$. Then, $\sqrt{a-1} = \sqrt{b-1}$. Squaring both sides

we have $a-1=b-1$ and so $a=b$.

Now, we show that f is surjective. Consider some real number b.
Case1. $b<0$. Let $r=1/b+1$. Hence,

$$\begin{aligned} f(r) &= \frac{1}{r-1} \\ &= \frac{1}{\frac{1}{b}+1-1} = \frac{1}{\frac{1}{b}} \\ &= b. \end{aligned}$$

Note that the fact that there is some real number r such that $f(r)<0$ implies that $r<1$.

Case2. Consider some real number $b \geq 0$. Let $r=b^2+1$. Thus,

$$\begin{aligned} h(r) &= \sqrt{(b^2+1)-1} \\ &= \sqrt{b^2} = b. \end{aligned}$$

Therefore, f is bijective. $\square$

(b) Determine the inverse f^{-1} of f

Solution Since $f:\mathbb{R}\to\mathbb{R}$ is bijective, we know that $f\circ f^{-1}(x)=x$ for all $x\in\mathbb{R}$. Let $x\in\mathbb{R}$. We considr the following cases.

Case1 $x<0$. Then, there is some real number $a<1$ such that $f(a)=x$. Thus, $f^{-1}(x)=a$ and so $f^{-1}(x)<1$. Then,

$$f\circ f^{-1}(x)=\frac{1}{f^{-1}(x)-1}=x.$$

Multiplying both sides by $f^{-1}(x)-1$ and adding x. we have $1+x=xf^{-1}(x)$. Dividing by x we get $f^{-1}(x)=\frac{1+x}{x}$ for all real numbers $x<0$.

Case2 $x\geq 0$. Then, there is some real number $a\geq 1$ such that $f(a)=x$. Thus, $f^{-1}(x)=a$ and so $f^{-1}(x)\geq 1$. Then,

$$f\circ f^{-1}(x)=\sqrt{f^{-1}(x)-1}=x.$$

Squaring both sides and adding 1, we have $f^{-1}(x)=x^2+1$ for all real numbers greater than or equal to zero.

Hence, the function $f^{-1}:\mathbb{R}\to\mathbb{R}$ is defined by

$$f^{-1}(x)=\begin{cases} \frac{1+x}{x} & \text{if } x<0 \\ x^2+1 & \text{if } x\geq 0 \end{cases}$$

for all $x\in\mathbb{R}$.

Problem 58. Suppose, for a function $f : A \to B$, that there is a function $g : B \to A$ such that $f \circ g = i_B$. Prove that if g is surjective, then $g \circ f = i_A$.

Proof. Consider some $x \in A$. Since g is surjective, it follows that there is some $b \in B$ such that $g(b) = x$. Because $f(x) = f(g(b)) = (f \circ g)(b)$ and $f \circ g = i_B$, we have $f(x) = b$. Then, $(g \circ f)(x) = g(f(x)) = g(b) = x$. Therefore, $g \circ f = i_A$. $\square$

Problem 59. Let $f : A \to B$, $g : B \to C$ and $h : B \to C$ be functions where f is a bijection. Prove that if $g \circ f = h \circ f$, then $g = h$.

Proof. Consider some $b \in B$. Since $f : A \to B$ is bijective, it follows that f is surjective and so there is some $a \in A$ such that $f(a) = b$. Then, $(g \circ f)(a) = g(f(a)) = g(b) = (h \circ f)(a) = h(f(a)) = h(b)$. Hence, $g(b) = h(b)$ for each $b \in B$ and so $g = h$. $\square$