Section 8.3: Equivalence Relations

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In this section, the concept of **Equivalence Relation** on some set A is introduced. In short words, an **Equivalence Relation** on some set A is one that has is reflexive, symmetric and transitive. One of the best examples is the relation R defined by $x \ R \ y$ if $x = y$. Also, an important subset to understand the behavior of these type of relations is the **equivalence class**. Basically, an **equivalence class** $[a]$ contains all elements $x \in A$ that are related to some specific $a \in A$, namely,

$$[a] = \{x \in A : x \ R \ a\}$$

Note that if $b \in [a]$ (b is related to a), then b and a are "equivalent". Note that $a \in [b]$ and $[b] = [a]$ due to the symmetric and transitive properties of R. Quite interesting!!!

Lemma 8.3.1. Let R be an equivalence relation on an nonempty set A. Then, $a \ R \ b$ for some $a, b \in A$ is a necessary and sufficiente condition for $[a] = [b]$.

Proof. Because R is reflexive, $a \in [a]$ and $b \in [b]$ and so they are nonempty. Consider some $x \in [a]$, then $x \ R \ a$. Note that $a \ R \ b$ and so, by the transtitive property of R, $x \ R \ b$. Hence, $x \in [b]$ which implies that $[a] \subseteq [b]$.

Now consider some $y \in [b]$ and so $y \ R \ b$. Since R is symmetric and $a \ R \ b$, it follows that $b \ R \ a$. Thus, by the transitive property, $y \ R \ a$ and so $y \in [a]$. Therefore, $[b] \subseteq [a]$ and so $[a] = [b]$.

For the converse, assume that $[a] = [b]$. Since R is reflexive, it follows that $a \in [a]$ and so $a \in [b]$. Hence, $a \ R \ b$. $\square$

Note that this implies that the union of all equivalence classes of A is A itself!!!

Corollary 8.3.1. Let R be an equivalence relation on an nonempty set A and consider some $a, b \in A$. Then, $[b] = [a]$ if and only if $b \in [a]$.

Proof. Assume that $[b] = [a]$. By **Lemma 8.3.1**, $b \ R \ a$. Therefore, $b \in [a]$.
For the converse, suppose that $b \in [a]$. Then, $b \ R \ a$. By **Lemma 8.3.1**, $[b] = [a]$. $\square$

Corollary 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \ldots, [a_n]$. Then, the equivalence classes are disjoint.

Proof. Suppose, to the contrary, that $[a_i] \cap [a_j] \neq \emptyset$ for some positive integers $i, j \leq n$ such that $i \neq j$. Then, there is at least some $x \in [a_i] \cap [a_j]$ and so $x \in [a_i]$ and $x \in [a_j]$. By **Corollary 8.3.1**, $[a_i] = [x] = [a_j]$. However, this contradicts the assumption that $[a_i] \neq [a_j]$. $\qquad\square$

Lemma 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \ldots, [a_n]$. Then,

$$\bigcup_{i=1}^{n} [a_i] = A$$

Proof. Suppose, to the contrary, that

$$\bigcup_{i=1}^{n} [a_i] \neq A.$$

Hence, either

$$\bigcup_{i=1}^{n} [a_i] \not\subseteq A \quad \text{or} \quad \bigcup_{i=1}^{n} [a_i] \not\supseteq A.$$

Suppose the first. Then, there exists some $x \in \bigcup_{i=1}^{n} [a_i]$ such that $x \notin A$. This implies that $x \in [a_k]$ for some positive integer k. However, $x \notin A$ and this contradicts the fact that $[a_k] = \{x \in A : x \, R \, a_k\}$.

Thus, we can assume that $\bigcup_{i=1}^{n} [a_i] \not\supseteq A$. Then, there is some $y \in A$ such that $y \notin \bigcup_{i=1}^{n} [a_i]$. Because $\bigcup_{i=1}^{n} [a_i]$ is the union of all distinct equivalence classes resulting from R, it follows that $y \not\mathrel{R} a$ for any $a \in A$. Hence $(y, y) \notin R$. However, this contradicts the fact that R is reflexive.

Thus,

$$\bigcup_{i=1}^{n} [a_i] = A.$$

$\qquad\square$

Problem 24. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a \, R \, c$, $c \, R \, d$, $d \, R \, g$ and $b \, R \, f$. If there are three distinct equivalence classes resulting from R, then determine these equivalence classes and determine all elements of R.

Solution 24. By repetitive use of **Lemma 8.3.1**, we conclude that $[a] = [c] = [d] = [g]$ and $[b] = [f]$. Also, since e is not related to any element of A, it follows that the remaining equivalence class is $[e]$. Note that the reflexive property of R implies that $g \, R \, g$ and $f \, R \, f$. Therefore, by the transitive property,

$$[g] = \{a, g, d, c\} = [a] = [c] = [d]$$
$$[f] = \{b, f\} = [b]$$
$$[e] = \{e\}$$

Therefore,

$$R = \{(a, a), (g, a), (d, a), (c, a), (a, c), (g, c), (d, c),$$
$$(c, c), (a, d), (g, d), (d, d), (c, d), (a, g), (g, g),$$
$$(d, g), (c, g), (b, b), (f, b), (b, f), (f, f), (e, e)\}.$$

This is a taste of how useful equivalence classes can be. Wow!!!

Problem 25. Let $A = \{1, 2, 3, 4, 5, 6\}$. The relation

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3),$$
$$(3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

is an equivalence relation on A. Determine the distinct equivalence classes.

Solution 25. Since R is an equivalence relation on A, then we can use **Lemma 1.8.3** to determine the equivalence classes. Note that $(1, 1), (5, 1), (2, 2), (6, 2), (3, 2), (4, 4) \in R$. Hence,

$$[1] = \{1, 5\} = [5]$$
$$[2] = \{2, 6, 3\} = [3] = [6]$$
$$[4] = \{4\}$$

Problem 26. Let $A = \{1, 2, 3, 4, 5, 6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1, 4, 5\}, \{2, 6\}$ and $\{3\}$. What is R?

Solution 26. By **Corollary 8.3.1**,

$$\{1, 4, 5\} = [1] = [4] = [5]$$
$$\{2, 6\} = [2] = [6] \quad \text{and}$$
$$\{3\} = [3].$$

Therefore, the relation

$$R = \{(1, 1), (4, 1), (5, 1), (1, 4), (4, 4), (5, 4), (1, 5),$$
$$(4, 5), (5, 5), (2, 2), (6, 2), (2, 6), (6, 6), (3, 3)\}$$

Corollary 8.3.3. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \ldots, [a_n]$. Then,

$$|R| = \sum_{i=1}^{n} |[a_i]|^2.$$

Proof. Consider some $[a_k]$ for a $k \leq n$. By **Corollary 8.3.1**, $[x] = [a_k]$ for every $x \in [a_k]$. Then, $|[a_k]|$ elements of A are related to x for every $x \in [a_k]$. Hence, $|[a_k]|^2$ different n-tuples are elements of R. This is the apport of each equivalence class, however we need to be sure

that each $x \in A$ is an element of only one equivalence class. Since **Corollary 8.3.2** implies that the different equivalence classes are disjoint and **Lemma 8.3.2** implies that their union is A, it follows that

$$|R| = \sum_{i=1}^{n} |[a_i]|^2$$

$\square$

Problem 27. Let R be a relation defined on $\mathbb{Z}$ by $a \ R \ b$ if $a^3 = b^3$. Show that R is an equivalence relation on $\mathbb{Z}$ and determine the distinct equivalence classes.

Solution 27. We first show that R is an equivalence relation on $\mathbb{Z}$.

Proof. Consider some integer x, then $x^3 = x^3$ and so $x \ R \ x$. Hence, R is reflexive.
Now, consider some $x, y \in \mathbb{Z}$ such that $x \ R \ y$. Therefore, $x^3 = y^3$ and so $y^3 = x^3$, which implies that $y \ R \ x$. Thus, R is symmetric.
Let $x, y, z \in \mathbb{Z}$ such that $x \ R \ y$ and $y \ R \ z$. Then, $x^3 = y^3 = z^3$ and so $x^3 = z^3$, which implies that $x \ R \ z$. Therefore, R is transitive. Hence, R is an equivalence relation on $\mathbb{Z}$. $\square$

Note that $x^3 = y^3 \iff x = y$ for any integers x and y. Therefore, each integer is only related to itself by R namely, $[x] = \{x\}$ whenever $x \in \mathbb{Z}$. Hence, there is an inifinity of different equivalence relations.