Week 15

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Section 4: Existence Proofs

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Let $R(x)$ be an open sentence over the set S. In order to disprove the existence statement $\exists x \in S, R(x)$, one must prove it's negations, namely, $\sim (\exists x \in S, R(x)) \equiv \forall x \in S, \sim R(x)$. Basically, one must show that the open sentence $R(x)$ is false for all $x \in S$.

Problem 49. Disprove the statement: There exist odd integers a and b such that $4 \mid (3a^2 + 7b^2)$.

Solution . We show that for every odd integers a and b, $4 \nmid (3a^2 + 7b^2)$. Let a and b be odd integers. Then $a = 2m + 1$ and $b = 2n + 1$ for some integers m, n. Therefore,

$$\begin{aligned} 3a^2 + 7b^2 &= 3(2m+1)^2 + 7(2n+1)^2 \\ &= 3(4m^2 + 4m + 1) + 7(4n^2 + 4n + 1) = 4(3m^2 + 3m) + 4(7n^2 + 7n) + 10 \\ &= 4(3m^2 + 3m + 7n^2 + 7n + 2) + 2 \end{aligned}$$

Since $3m^2 + 3m + 7n^2 + 7n + 2 \in \mathbb{Z}$, it follows that $4 \nmid (3a^2 + 7b^2)$.

Problem 50. Disprove the statement: There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

Solution . We show that for every $x \in \mathbb{R}$, $x^6 + x^4 + 1 \neq 2x^2$. Note that, $x^6 + x^4 - 2x^2 + 1 = x^6 + (x^2 - 1)^2 = 0$. Since $x^6 \geq 0$ and $(x^2 - 1)^2 \geq 0$, it follows that $x^6 + (x^2 - 1)^2 = 0$ iff $x^6 = (x^2 - 1)^2 = 0$. Note that, $x^6 = 0$ iff $x = 0$. However, $(x^2 - 1)^2 = (-1)^2 = 1 \neq 0$ for $x = 0$. Thus, there is no real solution for $x^6 + x^4 + 1 = 2x^2$.

Problem 51. Disprove the statement: There is an integer n such that $n^4 + n^3 + n^2 + n$ is odd.

Solution . We show that for any integer n, the integer $n^4 + n^3 + n^2 + n$ is even. Let $n \in \mathbb{Z}$. We consider two cases.
Case 1. n is even. Then $n = 2a$ for some integer a. Therefore, $n^4 + n^3 + n^2 + n = n(n^3 + n^2 + n + 1) = 2[a(n^2 + n^2 + n + 1)]$. Since $a(n^2 + n^2 + n + 1)$ is an integer, it follows that $n^4 + n^3 + n^2 + n$ is even.
Case 2. n is odd. According to *Theorem 3.17*, ab is odd if and only if a and b are odd. Therefore, $n^4 = n^2 \cdot n^2$, $n^3 = n^2 \cdot n$ and n^2 are odd. Let $n^4 = 2a + 1$, $n^3 = 2b + 1$, $n^2 = 2c + 1$ and $n = 2d + 1$ for some integers a, b, c, d. Thus, $n^4 + n^3 + n^2 + n = 2a + 2b + 2c + 2d + 4 = 2(a + b + c + d + 2)$. Since $a + b + c + d + 2 \in \mathbb{Z}$, it follows that $n^4 + n^3 + n^2 + n$ is even.