Week 3

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Section 7: Tautologies and Contradictions

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The following logical equivalences and properties of logical equivalence will be used for the proofs without truth tables [1] [2]:

1. *Logical equivalence for implication*
   $P \Rightarrow Q \equiv (\sim P) \vee Q$

2. *Double negation Law*
   $\sim (\sim P) \equiv P$

3. *Commutative Laws*

   a $P \wedge Q \equiv Q \wedge P$

   b $P \vee Q \equiv Q \vee P$

4. *Distributative Laws*

   a $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

   b $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

5. *Associative Laws*

   a $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

   b $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

6. *De Morgan's Laws*

   a $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

   b $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

7. *Identity Laws*

   a $P \wedge T \equiv P$

   b $P \vee F \equiv P$

8. *Domination Laws*

a $P \wedge F \equiv F$

b $P \vee T \equiv T$

9. *Negation Laws*

a $P \wedge (\sim P) \equiv F$

b $P \vee (\sim P) \equiv T$

Due to the commutative property of the conjunctions and disjunctions, the *Identitiy Laws*, *Dominant Laws* and *Negation Laws* are commutative (e.g., $P \wedge F \equiv F \wedge P \equiv F$)

Problem 46. For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology.

Solution . The implication $P \Rightarrow (P \vee Q)$ can only be false when the premise is true and the conclusion is false. In this case, if P is true, then $P \vee Q$ is true. Thus, $P \Rightarrow (P \vee Q)$ is a tautology, because this compound statement is true for all combinations of truth values of its component statements. This can be seen in the following truth table:

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Proving that $P \Rightarrow (P \vee Q)$ is a tautology without a truth table:

Proof.

Logical equivalence for implication.

$$P \Rightarrow (P \vee Q) \equiv (\sim P) \vee (P \vee Q)$$

Associative Laws

$$\equiv ((\sim P) \vee P) \vee Q$$

Negation Laws

$$\equiv T \vee Q$$

Dominant Laws

$$\equiv T$$

□

Problem 47. For statements P and Q, show that $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiciton.

Solution . The compound statement $(P \wedge (\sim Q)) \wedge (P \wedge Q)$, which is a conjunction, can only be true when both conjunctions $P \wedge (\sim Q)$ and $P \wedge Q$ are true. In the case where P is true, both conjunctions $P \wedge (\sim Q)$ and $P \wedge Q$ have opposite truth values because they contain $(\sim Q)$ and Q respectively. Thus, the compound statement $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiction because it is false for all the combinations of truth values of P and Q. This can be seen in the following table

P	Q	$\sim Q$	$P \wedge Q$	$P \wedge (\sim Q)$	$(P \wedge (\sim Q)) \wedge (P \wedge Q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	F	F	F
F	F	T	F	F	F

Proving that $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradicition without a truth table:

Proof.

$$
\begin{aligned}
&\textit{Associative Laws} \\
(P \wedge (\sim Q)) \wedge (P \wedge Q) &\equiv P \wedge ((\sim Q) \wedge (P \wedge Q)) \\
&\textit{Commutative Laws} \\
&\equiv P \wedge ((\sim Q) \wedge (Q \wedge P)) \\
&\textit{Associative Laws} \\
&\equiv P \wedge (((\sim Q) \wedge Q) \wedge P) \\
&\textit{Negation Laws} \\
&\equiv P \wedge (F \wedge P) \\
&\textit{Dominant Laws} \\
&\equiv P \wedge F \\
&\equiv F
\end{aligned}
$$

□

Problem 48. For statements P and Q, show that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology. Then state $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ in words. (This is an important logical argument form, called **modus ponens**.)

Solution . The compound statement $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology because it is true for all combinations of truth values of the component statements P and Q. This is shown in the following truth table:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The compound statement in words:
$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$: If P and P implies Q, then Q.

Proving that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology without a truth table:

Proof.

$$\begin{aligned}
&\textit{Logical equivalence for implication} \\
(P \wedge (P \Rightarrow Q)) \Rightarrow Q &\equiv (P \wedge ((\sim P) \vee Q)) \Rightarrow Q \\
&\equiv (\sim (P \wedge ((\sim P) \vee Q))) \vee Q \\
\textit{De Morgan's Laws}& \\
&\equiv ((\sim P) \vee (\sim ((\sim P) \vee Q))) \vee Q \\
&\equiv ((\sim P) \vee ((\sim (\sim P)) \wedge (\sim Q))) \vee Q \\
\textit{Double Negation}& \\
&\equiv ((\sim P) \vee (P \wedge (\sim Q))) \vee Q \\
\textit{Distributative Laws}& \\
&\equiv (((\sim P) \vee P) \wedge ((\sim P) \vee (\sim Q))) \vee Q \\
\textit{Negation Laws}& \\
&\equiv (T \wedge ((\sim P) \vee (\sim Q))) \vee Q \\
\textit{Identity Laws}& \\
&\equiv ((\sim P) \vee (\sim Q)) \vee Q \\
\textit{Associative Laws}& \\
&\equiv (\sim P) \vee ((\sim Q) \vee Q) \\
\textit{Negation Laws}& \\
&\equiv (\sim P) \vee T \\
\textit{Domination Laws}& \\
&\equiv T
\end{aligned}$$

□

Problem 49. For statements P, Q and R, show that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism.**)

Solution . The compound statement $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology because it is true for all combinations of truth values of its component statements P, Q and R. This can be seen in the following truth table:

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The compound statement in words:
$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$: If P implies Q and Q implies R, then P implies R.

Proving that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology without a truth table:

Proof.

$$
\begin{aligned}
&\textit{Logical equivalence for implication} \\
((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R) &\equiv (((\sim P) \vee Q) \wedge ((\sim Q) \vee R)) \Rightarrow ((\sim P) \vee R) \\
&\equiv (\sim (((\sim P) \vee Q) \wedge ((\sim Q) \vee R))) \vee ((\sim P) \vee R) \\
\textit{De Morgan's Laws} & \\
&\equiv ((\sim ((\sim P) \vee Q)) \vee (\sim ((\sim Q) \vee R))) \vee ((\sim P) \vee R) \\
&\equiv (((\sim (\sim P)) \wedge (\sim Q)) \vee ((\sim (\sim Q)) \wedge (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Double negation Law} & \\
&\equiv ((P \wedge (\sim Q)) \vee (Q \wedge (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Distributative Laws} & \\
&\equiv (((P \wedge (\sim Q)) \vee Q) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Commutative Laws} & \\
&\equiv ((Q \vee (P \wedge (\sim Q))) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Distributative Laws} & \\
&\equiv (((Q \vee P) \wedge (Q \vee (\sim Q))) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Negation Laws} & \\
&\equiv (((Q \vee P) \wedge T) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Identity Laws} & \\
&\equiv ((Q \vee P) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \vee ((\sim P) \vee R) \\
\textit{Commutative Laws} & \\
&\equiv ((\sim P) \vee R) \vee ((Q \vee P) \wedge ((P \wedge (\sim Q)) \vee (\sim R))) \\
\textit{Distributative Laws} & \\
&\equiv (((\sim P) \vee R) \vee (Q \vee P)) \wedge (((\sim P) \vee R) \vee ((P \wedge (\sim Q)) \vee (\sim R))) \\
\textit{Commutative Laws} & \\
&\equiv ((R \vee (\sim P)) \vee (P \vee Q)) \wedge (((\sim P) \vee R) \vee ((\sim R) \vee (P \wedge (\sim Q)))) \\
\textit{Associative Laws} & \\
&\equiv (((R \vee (\sim P)) \vee P) \vee Q) \wedge ((((\sim P) \vee R) \vee (\sim R)) \vee (P \wedge (\sim Q))) \\
&\equiv ((R \vee ((\sim P) \vee P)) \vee Q) \wedge (((\sim P) \vee (R \vee (\sim R))) \vee (P \wedge (\sim Q))) \\
\textit{Negation Laws} & \\
&\equiv ((R \vee T) \vee Q) \wedge (((\sim P) \vee T) \vee (P \wedge (\sim Q))) \\
\textit{Domination Laws} & \\
&\equiv (T \vee Q) \wedge (T \vee (P \wedge (\sim Q))) \\
&\equiv T \wedge T \\
&\equiv T
\end{aligned}
$$

$\square$

Problem 50. Let R and S be compound statements involving the same component statements. If R is a tautology and S is a contradiciton, then what can be said of the following?

(A) $R \vee S$

Solution a. For all combinations of component statements, the compound statement R is true, while S is false. Thus, this disjunction is a tautology since one of its compound statements is true for all combinations of component statements.

(B) $R \wedge S$

Solution b. Since the compound statement S is a contradiction, both compound statements of this conjunction are not true for all the combinations of component statements. This conjunction is a contradiction.

(C) $R \Rightarrow S$

Solution c. The compound statement R is a tautology and S is a contradiction. For all combinations of the component statements, the premise and conclusion are therefore true and false, respectively. This implication is a contradiction.

(D) $S \Rightarrow R$

Solution d. For all combinations of the component statements, the premise and conclusion are false and true, respectively. This is because S is a contradiction and R is a tautology. This implication is a tautology.

References

[1] G. Chartrand, A. Polimeni, P. Zhang, *Mathematical Proofs: A Transition to Advanced Mathematics*, Pearson, 2014.

[2] GeeksforGeeks, *Mathematics Propositional Equivalences*, Apr 02, 2019. Retrieved Aug 04, 2021.