Section 1.4: Matrix Groups

Juan Patricio Carrizales Torres

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Before describing the matrix group, we must define what a *field* is. A field is a set F with two binary operations $+$ and $\cdot$ such that both $(F, +)$ and $(F/\{0\}, \cdot)$ are abelian groups. Also, the distributive law holds, namely, for any $a, b, c \in F$

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Then, the general linear group $GL_n(F)$ is the set of all $n \times n$ matrices with entries from the field F and nonzero determinant, where the associative matrix multiplication is the binary operation. Two useful results regarding general linear groups are the following:

(a) if F is a finite field, then $|F| = p^m$ for some prime p and integer m.

(b) if $|F| = q < \infty$, then $|GL_n(F)| = (q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{n-1})$.

1 PROBLEMS

Let F be a field and let $n \in \mathbb{Z}^+$.

Problem 1. Prove that $|GL_2(F_2)| = 6$

Proof. This general linear group $GL_2(F_2)$ contains 2×2 matrices

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},$$

where $b_1, b_2, b_3, b_4 \in F_2$ and $b_3 \cdot b_2 - b_4 \cdot b_1 \neq 0$ (nonzero determinant). Then, $b_3 \cdot b_2 \neq b_4 \cdot b_1$ (Recall that $\cdot$ is the binary operation in F_2 such that $(F_2/\{0\}, \cdot)$ is a group). Then, the statement $|GL_2(F_2)| = 6$ is equivalent to saying that there are 6 possible unique equations $b_3 \cdot b_2 \neq b_4 \cdot b_1$ for elements $b_1, b_2, b_3, b_4 \in F_2$. Let's call the instance $b \cdot a$ a *binary multiplication*. Because multiplication is closed, it follows that it is equal to some element inside F_2 and so we must find all ways to accomodate *binary multiplications* in the equation such that one side is 0 and the other is 1. Before doing that, we have to look at the 4 possible *binary* *multiplications*. We know that 0 is the *additive identity* and that the other element 1 is the *multiplicative identity* and its own additive and multiplicative inverse. Then, it follows that

$$\begin{aligned} 0 \cdot 1 &= (1+1) \cdot 1 = 1 \cdot 1 + 1 \cdot 1 \\ &= 0 + 0 = 0 \\ &= 0 \cdot 0 = 0 \cdot (1+1) \\ &= 0 \cdot 1 + 0 \cdot 1 = 0 + 0. \end{aligned}$$

and $1 \cdot 1 = 1$. Then, all binary multiplications, except for $1 \cdot 1$, are equal to 0.
Now, let one side of the equation be 1, which there is only one binary multiplication able to represent that, namely, $1 \cdot 1$. Then, we only have 3 binary multiplications out of the possible 4 that we can place at the other side such that two sides are not equal $(1 \cdot 0, 0 \cdot 1, 0 \cdot 0)$. Hence, per side there are 3 possible non equal equations and so there are 6 possible equations such that the binary multiplications at each side are not equal. ☐

Problem 2. Write out all the elements of $GL_2(F_2)$ and compute the order of each element.

Solution We have the following elements with their respective orders (n):

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, n = 2$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, n = 2$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, n = 1(\text{identity matrix})$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, n = 3$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, n = 3$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n = 2$$

Problem 3. Show that $GL_2(F_2)$ is non-abelian.

Proof. Note that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

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