Section 8.2: Properties of relations

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This chapter mentioned three properties of interested for some relation R on a single set A. Since most of these properties involve implications with universal quantifiers, the easiest way to check wether a relation has certain property is by looking for specific examples for which the implication in question is false.

(a) **Reflexive Property:** if $x \in A$, then $(x, x) \in R$. (x is related to itself)

(b) **Symmetric Property:** $\forall x, y \in A$, if $x \ R \ y$, then $y \ R \ x$ (x is related to y and viceversa). Note that for the relation R to not be symmetric, it must be true that $x \ R \ y$ and $y \not R \ x$. For this to happen, it is necessary that $x \neq y$.

(c) **Transitive Property:** $\forall x, y, z \in A$, if $x \ R \ y$ and $y \ R \ z$, then $x \ R \ z$. Note that for the relation R to not be symmetric, it must be true that $x \ R \ y$, $y \ R \ z$ and $x \not R \ z$. For this to happen, it is necessary that $x \neq y$ and $z \neq y$.

Problem 11. Let $A = \{a, b, c, d\}$ and let

$$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$$

be a relation on A. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 11. The relation is reflexive since $\{(a, a), (b, b), (c, c), (d, d)\} \subset R$. Also, it is transitive since $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in A$ is fulfilled. However, the relation is not symmetric since $(a, b) \in R$ and $(b, a) \notin R$.

Problem 13. Let $S = \{a, b, c\}$. Then $R = \{(a, b)\}$ is a relation on S. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 13. The relation S is transitive since the implication $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in S$ is fulfilled vacuously. However, it is neither reflexive because $(a, a) \notin R$ nor symmetrice since $(a, b) \in R$ but $(b, a) \notin R$.

Problem 14. Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

Solution 14. Let $R = \{(a, b), (b, c)\}$. The relation R is not reflexive since $(a, a) \notin R$, it is not symmetric because $(a, b) \in R$ and $(b, a) \notin R$ and it is not transitive since $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Problem 15. A relation R is defined on $\mathbb{Z}$ by $a \; R \; b$ if $|a - b| \leq 2$. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 15. The relation R is reflexive since $|a - a| = 0 \leq 2$ for any $a \in \mathbb{Z}$ and so $a \; R \; a$. It is symmetric since for any $a, b \in \mathbb{Z}$, if $|a - b| \leq 2$, then $|b - a| = |a - b| \leq 2$. However, it is not transitive since $|3 - 1| = 2$ and $|1 - 0| = 1$ but $|3 - 0| = 3 > 2$.

Problem 16. Let $A = \{a, b, c, d\}$. How many relations defined on A are reflexive, symmetric and transitive and contain the ordered pairs $(a, b), (b, c), (c, d)$?

Solution 16. In order for a relation R on A to be reflexive it must be true that $\{(a, a), (b, b), (c, c), (d, d)\} \subseteq R$. Since $(a, b), (b, c), (c, d) \in R$, it follows that $(b, a), (c, b), (d, c) \in R$ so that R is symmetric. Because, so far

$$\{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c)\} \subseteq R$$

, it follows that $(a, c), (c, a), (b, d) \in R$ for R to be transitive. Since $(b, d) \in R$, it follows that $(d, b) \in R$ so that the symmetric property is mantained. However, $(d, b), (b, a) \in R$ and so $(d, a) \in R$ so that it is transitive. This implies $(a, d) \in R$ since R must be symmetric. Hence,

$$\begin{aligned} R &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c), (a, c), (c, a), (b, d), (d, b), (d, a), (a, d)\} \\ &= A \times A \end{aligned}$$

Since $R \subseteq A \times A$, it follows that there is only one possible relation R on A that fulfills the conditions.

Problem 18. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric but not transitive.

  Solution (a). $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (3, 1), (1, 3)\}$

(b) reflexive and transitive but not symmetric.

  Solution (b). $R = \{(a, a), (b, b), (c, c), (d, d), (b, c)\}$

(c) symmetric and transitive but not reflexive.

  Solution (c). $R = \emptyset$ (the symmetric and transitive logical implications are vacuously true)

(d) reflexive but neither symmetric nor transitive.

Solution (d). $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c)\}$

(e) symmetric but neither reflexive nor transitive.

Solution (e). $R = \{(a,b),(b,a)\}$

(f) transitive but neither reflexive nor symmetric.

Solution (f). $R = \{(a,b)\}$ (The transitive implication follows vacuously)

All of these are counterexamples to the statement that one property implies the other for any relation R on some nonempty set A.

Problem 19. A relation R is defined on $\mathbb{Z}$ by $x \; R \; y$ if $x \cdot y \geq 0$. Prove or disprove the following:

(a) R is reflexive.

Proof. Consider some $x \in \mathbb{Z}$, then $x^2 \geq 0$ and so $x \; R \; x$. The relation R is reflexive. □

(b) R is symmetric.

Proof. Consider some $x, y \in \mathbb{Z}$. Assume that $x \; R \; y$ which implies that $x \cdot y \geq 0$. Since multiplication on real numbers is commutative, it follows that $y \cdot x = x \cdot y \geq 0$ and so $y \; R \; x$. The relation R is symmetric. □

(c) R is transitive.

Solution c. The relation R on $\mathbb{Z}$ is not transitive. Note that $-3 \; R \; 0$ and $0 \; R \; 1$, but $-3 \cdot 1 = -3 < 0$ and so $-3 \not{R} \; 1$.

Problem 20. Determine the maximum number of elements in a relation R on a $3-$*element* set such that R has none of the properties reflexive, symmetric and transitive.

Solution 20. Let R be a relation on a 3-element set B that has none of the properties reflexive, symmetric and transitive. Let's check the maximum number of elements R can contain. Since $R \subseteq B \times B$, it follows that $|R| \leq 9$. However, since R is not reflexive, it follows that $(b,b) \notin R$ for some $b \in B$ and so $|R| \leq 8$.

Because R is not symmetric, it follows that $(b,a) \in R$ and $(a,b) \notin R$ for some different $a, b \in B$ and so $|R| \leq 7$. Also, since R is not transitive, it follows that $(a,b),(b,c) \in R$ and $(a,c) \notin R$ for some $a, b, c \in B$ such that $a \neq b$ and $b \neq c$. Thus, either $c \neq a$ or $c = a$, however note that we already got rid of those two such ordered pairs and so the maximum number of elements in R is 7.

Problem 22. Let S be the set of all polynomials of degree at most 3. An element $s(x)$ of S can then be expressed as $s(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. A relation R is defined on S by $p(x)\ R\ q(x)$ if $p(x)$ and $q(x)$ have a real root in common. (For example, $p(x) = (x-1)^2$ and $q(x) = x^2 - 1$ have the root 1 in common so that $p\ R\ q$.) Determine which of the properties reflexive, symmetric, and transitive are possessed by R.

1. The relation R is reflexive.

   Solution (a). The relation R on S is not reflexive. Consider $p(x) = x^2+1$. Therefore, $p(x) \in S$ but $p(x) \not R\ p(x)$ since $p(x)$ has no real root.

2. The relation R is symmetric.

   Proof. Consider some $p(x), q(x) \in S$. Assume that $p(x)\ R\ q(x)$ and so $p(x)$ and $q(x)$ share some real root c. Therefore, $q(x)$ and $p(x)$ share the real root c which implies that $q(x)\ R\ p(x)$. $\square$

3. The relation R is transitive.

   Solution (c). The relation R is not transitive. Let $p(x) = x^2 - 1$, $q(x) = (x-1)^2$ and $r(x) = (x+1)^2$. Hence, $p(x), q(x), r(x) \in S$. Note that $p(x)$ has real roots -1 and 1, $q(x)$ has only the real root 1 and $r(x)$ only has the real root -1. Then, $r(x)\ R\ p(x)$ and $p(x)\ R\ q(x)$. However, $r(x)$ and $q(x)$ do not have some real root in common and so $r(x) \not R\ q(x)$.

Problem 23. A relation R is defined on $\mathbb{N}$ by $a\ R\ b$ if either $a \mid b$ or $b \mid a$. Determine which of the properties reflexive, symmetric and transitive are possessed by R.