Section 8.3: Equivalence Relations

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In this section, the concept of **Equivalence Relation** on some set A is introduced. In short words, an **Equivalence Relation** on some set A is one that has is reflexive, symmetric and transitive. One of the best examples is the relation R defined by $x\ R\ y$ if $x = y$. Also, an important subset to understand the behavior of these type of relations is the **equivalence class**. Basically, an **equivalence class** $[a]$ contains all elements $x \in A$ that are related to some specific $a \in A$, namely,

$$[a] = \{x \in A : x\ R\ a\}$$

Note that if $b \in [a]$ (b is related to a), then b and a are "equivalent". Note that $a \in [b]$ and $[b] = [a]$ due to the symmetric and transitive properties of R. Quite interesting!!!

Lemma 8.3.1. Let R be an equivalence relation on an nonempty set A. Then, $a\ R\ b$ for some $a, b \in A$ is a necessary and sufficiente condition for $[a] = [b]$.

Proof. Because R is reflexive, $a \in [a]$ and $b \in [b]$ and so they are nonempty. Consider some $x \in [a]$, then $x\ R\ a$. Note that $a\ R\ b$ and so, by the transtitive property of R, $x\ R\ b$. Hence, $x \in [b]$ which implies that $[a] \subseteq [b]$.

Now consider some $y \in [b]$ and so $y\ R\ b$. Since R is symmetric and $a\ R\ b$, it follows that $b\ R\ a$. Thus, by the transitive property, $y\ R\ a$ and so $y \in [a]$. Therefore, $[b] \subseteq [a]$ and so $[a] = [b]$.

For the converse, assume that $[a] = [b]$. Since R is reflexive, it follows that $a \in [a]$ and so $a \in [b]$. Hence, $a\ R\ b$. $\square$

Note that this implies that the union of all equivalence classes of A is A itself!!!

Corollary 8.3.1. Let R be an equivalence relation on an nonempty set A and consider some $a, b \in A$. Then, $[b] = [a]$ if and only if $b \in [a]$.

Proof. Assume that $[b] = [a]$. By **Lemma 8.3.1**, $b\ R\ a$. Therefore, $b \in [a]$.

For the converse, suppose that $b \in [a]$. Then, $b\ R\ a$. By **Lemma 8.3.1**, $[b] = [a]$. $\square$

Corollary 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \dots, [a_n]$. Then, the equivalence classes are disjoint.

Proof. Suppose, to the contrary, that $[a_i] \cap [a_j] \neq \emptyset$ for some positive integers $i, j \leq n$ such that $i \neq j$. Then, there is at least some $x \in [a_i] \cap [a_j]$ and so $x \in [a_i]$ and $x \in [a_j]$. By **Corollary 8.3.1**, $[a_i] = [x] = [a_j]$. However, this contradicts the assumption that $[a_i] \neq [a_j]$. $\square$

Lemma 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \ldots, [a_n]$. Then,

$$\bigcup_{i=1}^{n} [a_i] = A$$

Proof. Suppose, to the contrary, that

$$\bigcup_{i=1}^{n} [a_i] \neq A.$$

Hence, either

$$\bigcup_{i=1}^{n} [a_i] \not\subseteq A \quad \text{or} \quad \bigcup_{i=1}^{n} [a_i] \not\supseteq A.$$

Suppose the first. Then, there exists some $x \in \bigcup_{i=1}^{n} [a_i]$ such that $x \notin A$. This implies that $x \in [a_k]$ for some positive integer k. However, $x \notin A$ and this contradicts the fact that $[a_k] = \{x \in A : x \; R \; a_k\}$.

Thus, we can assume that $\bigcup_{i=1}^{n} [a_i] \not\supseteq A$. Then, there is some $y \in A$ such that $y \notin \bigcup_{i=1}^{n} [a_i]$. Because $\bigcup_{i=1}^{n} [a_i]$ is the union of all distinct equivalence classes resulting from R, it follows that $y \not R \; a$ for any $a \in A$. Hence $(y, y) \notin R$. However, this contradicts the fact that R is reflexive.

Thus,

$$\bigcup_{i=1}^{n} [a_i] = A.$$

$\square$

Problem 24. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a \; R \; c$, $c \; R \; d$, $d \; R \; g$ and $b \; R \; f$. If there are three distinct equivalence classes resulting from R, then determine these equivalence classes and determine all elements of R.

Solution 24. By repetitive use of **Lemma 8.3.1**, we conclude that $[a] = [c] = [d] = [g]$ and $[b] = [f]$. Also, since e is not related to any element of A, it follows that the remaining equivalence class is $[e]$. Note that the reflexive property of R implies that $g \; R \; g$ and $f \; R \; f$. Therefore, by the transitive property,

$$\begin{aligned}
&[g] = \{a, g, d, c\} = [a] = [c] = [d] \\
&[f] = \{b, f\} = [b] \\
&[e] = \{e\}
\end{aligned}$$

Therefore,

$$\begin{aligned}R = \{&(a,a),(g,a),(d,a),(c,a),(a,c),(g,c),(d,c),\\&(c,c),(a,d),(g,d),(d,d),(c,d),(a,g),(g,g),\\&(d,g),(c,g),(b,b),(f,b),(b,f),(f,f),(e,e)\}.\end{aligned}$$

This is a taste of how useful equivalence classes can be. Wow!!!

Problem 25. Let $A = \{1,2,3,4,5,6\}$. The relation

$$\begin{aligned}R = \{&(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),\\&(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}\end{aligned}$$

is an equivalence relation on A. Determine the distinct equivalence classes.

Solution 25. Since R is an equivalence relation on A, then we can use **Lemma 1.8.3** to determine the equivalence classes. Note that $(1,1),(5,1),(2,2),(6,2),(3,2),(4,4) \in R$. Hence,

$$\begin{aligned}&[1] = \{1,5\} = [5]\\&[2] = \{2,6,3\} = [3] = [6]\\&[4] = \{4\}\end{aligned}$$

Problem 26. Let $A = \{1,2,3,4,5,6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1,4,5\},\{2,6\}$ and $\{3\}$. What is R?

Solution 26. By **Corollary 8.3.1**,

$$\begin{aligned}\{1,4,5\} &= [1] = [4] = [5]\\\{2,6\} &= [2] = [6] \text{ and}\\\{3\} &= [3].\end{aligned}$$

Therefore, the relation

$$\begin{aligned}R = \{&(1,1),(4,1),(5,1),(1,4),(4,4),(5,4),(1,5),\\&(4,5),(5,5),(2,2),(6,2),(2,6),(6,6),(3,3)\}\end{aligned}$$

Corollary 8.3.3. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1],[a_2],[a_3]\ldots,[a_n]$. Then,

$$|R| = \sum_{i=1}^{n} |[a_i]|^2 .$$

Proof. Consider some $[a_k]$ for a $k \leq n$. By **Corollary 8.3.1**, $[x] = [a_k]$ for every $x \in [a_k]$. Then, $|[a_k]|$ elements of A are related to x for every $x \in [a_k]$. Hence, $|[a_k]|^2$ different n-tuples are elements of R. This is the apport of each equivalence class, however we need to be sure that each $x \in A$ is an element of only one equivalence class. Since **Corollary 8.3.2** implies that the different equivalence classes are disjoint and **Lemma 8.3.2** implies that their union is A, it follows that

$$|R| = \sum_{i=1}^{n} |[a_i]|^2$$

$\square$

Problem 27. Let R be a relation defined on $\mathbb{Z}$ by $a\ R\ b$ if $a^3 = b^3$. Show that R is an equivalence relation on $\mathbb{Z}$ and determine the distinct equivalence classes.

Solution 27. We first show that R is an equivalence relation on $\mathbb{Z}$.

Proof. Consider some integer x, then $x^3 = x^3$ and so $x\ R\ x$. Hence, R is reflexive.
Now, consider some $x, y \in \mathbb{Z}$ such that $x\ R\ y$. Therefore, $x^3 = y^3$ and so $y^3 = x^3$, which implies that $y\ R\ x$. Thus, R is symmetric.
Let $x, y, z \in \mathbb{Z}$ such that $x\ R\ y$ and $y\ R\ z$. Then, $x^3 = y^3 = z^3$ and so $x^3 = z^3$, which implies that $x\ R\ z$. Therefore, R is transitive. Hence, R is an equivalence relation on $\mathbb{Z}$. $\square$

Note that $x^3 = y^3 \iff x = y$ for any integers x and y. Therefore, each integer is only related to itself by R namely, $[x] = \{x\}$ whenever $x \in \mathbb{Z}$. Hence, there is an inifinity of different equivalence relations.

Problem 30. Let $H = \{2^m : m \in \mathbb{Z}\}$. A relation R is defined on the set $\mathbb{Q}^+$ of positive rational numbers by $a\ R\ b$ if $a/b \in H$.

(a) Show that R is an equivalence relation.

Proof. Firs, we prove that it is reflexive. Consider some $x \in \mathbb{Q}^+$. Then $x/x = 1 = 2^0 \in H$. Hence, $x\ R\ x$.
Now, we show that it is symmetric. Let $a, b \in \mathbb{Q}^+$ such that $a\ R\ b$, namely, $a/b = 2^m$ for some integer m. Then, $b/a = 1/2^m = 2^{-m}$. Since $-m \in \mathbb{Z}$, it follows that $b\ R\ a$.
Last, we prove that it is transitive. Consider some $a, b, c \in \mathbb{Q}^+$ such that $a\ R\ b$ and $b\ R\ c$, which implies that $a/b = 2^m$ and $b/c = 2^n$ for $m, n \in \mathbb{Z}$. Note that

$$\begin{aligned} \frac{a}{c} &= \frac{a}{b} \cdot \frac{b}{c} \\ &= 2^m \cdot 2^n \\ &= 2^{m+n}. \end{aligned}$$

Since $m + n \in \mathbb{Z}$, it follows that $a\ R\ c$. $\square$

(b) Describe the elements in the equivalence class $[3]$.

Solution b. Consider some positive rational number x such that $x\ R\ 3$. Then, $x/3 = 2^k$ for some integer k. Then, $x = 3 \cdot 2^k$. Thus, $[3] = \{3 \cdot 2^k : k \in \mathbb{Z}\}$.

Problem 31. A relation R on a nonempty set A is defined to be **circular** if whenever $x\ R\ y$ and $y\ R\ z$, then $z\ R\ x$ for all $x, y, z \in A$. Prove that a relation R on A is an equivalence relation if and only if R is circular and reflexive.

Proof. Firts, assume that R is an equivalence relation on some nonempty set A. Hence, R is reflexive, symmetric and transitive. We show that it is circular. Consider some $x, y, z \in A$ such that $x\ R\ y$ and $y\ R\ z$. By the transitive property, $x\ R\ z$. Since R is symmetric, it follows that $z\ R\ x$ and so R is circular.

For the converse, Let R be a **circular** and reflexive relation on some nonempty set A. We show that it is symmetric and transitive. Consider some $x, y \in A$ such that $x\ R\ y$. Since R is reflexive, it follows that $y\ R\ y$, and so $y\ R\ x$ due to the circular property of R. Thus, R is symmetric.
Now, consider some $x, y, z \in A$ such that $x\ R\ y$ and $y\ R\ z$. By the **circular property** of R, $z\ R\ x$. Because R is symmetric, it follows that $x\ R\ z$ and so R is transitive. Hence, R is an equivalence relation on A. □

Problem 32. A relation R is defined on the set $A = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a + b\sqrt{2} \neq 0\}$ by $x\ R\ y$ if $x/y \in \mathbb{Q}$. Show that R is an equivalence relation and determine the distinct equivalence classes.

Solution 32. We first show that R is an equivalence relation.

Proof. Consider some $x \in A$. Because $x \neq 0$, $x/x = 1 \in \mathbb{Q}$ and so R is reflexive.
Let $x, y \in A$ such that $x\ R\ y$. Then, $x/y = c \in \mathbb{Q}$ and so $1/c = y/x \in \mathbb{Q}$. Hence, $y\ R\ x$ and so R is symmetric.
Last, consider some $x, y, z \in A$ such that $x\ R\ y$ and $y\ R\ z$. Then, $x/y = a \in \mathbb{Q}$ and $y/z = b \in \mathbb{Q}$. Note that

$$\frac{x}{y} \cdot \frac{y}{z} = \frac{x}{z} = ab \in \mathbb{Q}.$$

Thus, $x\ R\ z$ and so R is transitive. □

Since R is an equivalence relation, we can proceed to determine the equivalence class for each $y \in A$. Consider some $y \in A$. Then, $y = a + b\sqrt{2} \neq 0$ for some $a, b \in \mathbb{Q}$. For any element $x \in A$, $x\ R\ y$ if $x/y = c$ for some $c \in \mathbb{Q}$. Hence, $x = cy$. Since $y, x \neq 0$, it must be true that $c \neq 0$. Therefore,

$$[y] = \{cy : c \in \mathbb{Q}/\{0\}\}$$

Note that when y is rational, namely, when $a \neq 0$ and $b = 0$, $[y] = \mathbb{Q}/\{0\}$.

Problem 34. Let H be a nonempty subset of $\mathbb{Z}$. Suppose that the relation R defined on $\mathbb{Z}$ by $a\ R\ b$ if $a - b \in H$ is an equivalence relation. Recall that

$$\begin{aligned} R &= \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a\ R\ b\} \\ &= \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \in H\}. \end{aligned}$$

So, whenever it is said *a relation defined by* $a\ R\ b$ *if* $a - b \in H$, it refers to a biconditional, namely $a\ R\ b \iff a - b \in H$.

Verify the following

(a) $0 \in H$

Proof. Since R is an equivalence relation, it follows that it is reflexive. Thus, $x \; R \; x$ for every $x \in \mathbb{Z}$, which implies that $x - x = 0 \in H$. $\square$

(b) If $a \in H$, then $-a \in H$.

Proof. If $a \in H$, then $a - 0 = a \in H$. Thus, $a \; R \; 0$. We know that R is symmetric. Therefore, $0 \; R \; a$ and so $0 - a = -a \in H$. $\square$

(c) If $a, b \in H$, then $a + b \in H$.

Proof. Since $a, b \in H$, it follows that $a \; R \; 0$ and $b \; R \; 0$. By implication (b), $-b \in H$ and so $-b \; R \; 0$. By the symmetry of R, $0 \; R \; -b$. Since (1) $a \; R \; 0$ and $0 \; R \; -b$, (2) R is transitive, it follows that $a \; R \; -b$, which implies that $a - (-b) = a + b \in H$. $\square$

Problem 35. Prove or disprove: There exist equivalence relations R_1 and R_2 on the set $S = \{a, b, c\}$ such that $R_1 \not\subseteq R_2$, $R_1 \not\supseteq R_2$ and $R_1 \cup R_2 = S \times S$.

Solution 35. This is false. Let R_1 and R_2 be equivalence relations on S such that $R_1 \not\subseteq R_2$, $R_1 \not\supseteq R_2$ and $R_1 \cup R_2 = S \times S$. Thus, there exist $(x, y), (m, n) \in S$ such that $(x, y) \in R_1 - R_2$ and $(m, n) \in R_2 - R_1$. Since they are reflexive, $\{(a, a), (b, b), (c, c)\} \subseteq R_1 \cap R_2$ and so we may further assume that $x \neq y$ and $m \neq n$. Also, since S only has three elements, it must be true that one of x, y is equal to at least one of m, n. Without loss of generality, let $y = m$. Due to the symmetric property, $(x, m), (m, x) \in R_1 - R_2$ and $(m, n), (n, m) \in R_2 - R_1$. Since $R_1 \cup R_2 = S \times S$, it must be true that (x, n) belongs to at least one of the two equivalence relations. Let $(x, n) \in R_1$. However, since R_1 is transitive and $(m, x), (x, n) \in R_1$, it follows that $(m, n) \in R_1$ which is not true. A similar case happens if $(x, n) \in R_2$.