Section 9.3: One-To-One and Onto Functions

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We have seen that a function from A to B is a relation that fulfills the following condition:

$$a = b \implies f(a) = f(b).$$

Furthermore, functions can posses to important properties. A function $f : A \to B$ is said to be **One-to-One** if every image is unique to its respective $x \in A$, namely,

$$\begin{aligned} f(a) = f(b) &\implies a = b \\ \equiv a \neq b &\implies f(a) \neq f(b). \end{aligned}$$

Obviously, for this to be true, B must contain at least the same number of elements as A, namely, $|A| \leq |B|$. On the other hand, the function f is said to be **Onto** if every element in B is the image of some element of A, namely,

$$b \in B \implies \exists a \in A, f(a) = b.$$

Hence, $f(A) = B$. Clearly, $|B| \leq |A|$, otherwise, there would be not enough elements of A to cover all elements of B. Then, if a function is both one-to-one and onto, then $|A| = |B|$.

Problem 20. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 2n + 1$. Determine whether f is injective, surjective.

Solution First we show that it is injective. Consider two $f(a) = f(b)$ for some $a, b \in \mathbb{Z}$. Then, $2a + 1 = 2b + 1$. Substracting 1 to both sides, we get $2a = 2b$. Dividing by 2, we obtain $a = b$. However, it is not surjective. Consider any even integer r and so there is no integer n such that $f(n) = 2n + 1 = r$.

Problem 21. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n - 3$. Determine whether f is injective, surjective.

Solution The function is both injective and surjective. Consider some some $f(a) = f(b)$ for $a, b \in \mathbb{Z}$. Then, $a - 3 = b - 3$ and so $a = b$. Now, let $y \in \mathbb{Z}$. Note that $x = b + 3$ is an integer. Then $f(x) = (y + 3) - 3 = b$.

Problem 23. Prove or disprove: For every nonempty set A, there exists an injective function $f : A \to \mathcal{P}(A)$.

Proof. Let $g : A \to \mathcal{P}(A)$ be deifned by $f(n) = \{n\}$. We show that it is injective. Consider some element $f(a) = f(b)$ for $a, b \in A$, then $\{a\} = \{b\}$, which implies that $a = b$. Due to the individuality of each element of A, $f(n)$ is injective. $\square$

Note that it is impossible to define a function $f : A \to \mathcal{P}(A)$ that is surjective since $|A| < 2^{|A|} = |\mathcal{P}(A)|$.

Problem 24. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 9$ is one-to-one, onto.

Solution We show that it is not one-to-one. Consider some $f(x) = f(y)$ for $x, y \in \mathbb{R}$. Then, $x^2 + 4x + 9 = y^2 + 4y + 9$ and so $x^2 + 4x - (y^2 + 4y) = 0$. Note that

$$\begin{aligned} x^2 + 4x - (y^2 + 4y) &= (x^2 - y^2) + 4(x - y) \\ &= (y + x)(x - y) + 4(x - y) = (x - y)(y + x + 4) = 0. \end{aligned}$$

Hence, either $x - y = 0$ or $y + x + 4 = 0$. In the latter, $y = -(x + 4)$. For instace, if $x = 3$ and $y = 1$, then $f(x) = f(y)$.
Also, it is not surjective. Note that

$$\begin{aligned} x^2 + 4x + 9 &= \left(x^2 + 4x + 4\right) - 4 + 9 \\ &= (x + 2)^2 + 5 \geq 5. \end{aligned}$$

Thus, there is no $x \in \mathbb{R}$ such that $f(x) < 4$.

Problem 25. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one? Explain your answer.

Solution Yes, there is such function. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(n) = \begin{cases} n, \text{ if } n \leq -\frac{\pi}{2} \\ \tan(n), \text{ if } -\frac{\pi}{2} < n < \frac{\pi}{2} \\ n, \text{ if } n \geq \frac{\pi}{2}. \end{cases}$$

Clearly, $\mathrm{dom}(g) = \mathbb{R}$. Note that, the function $\varphi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined by $\varphi(n) = \tan(n)$ is by itself injective and surjective. However, by adding identity relations for the lower and upper bounds, namely, $\left(-\infty, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \infty\right)$, we make sure that g is not injective, in other words, there are $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $a \neq b$.

Problem 26. Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is

(a) one-to-one and onto

Solution Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n$.

(b) one-to-one but not onto

Solution Let f be defined by $f(n) = 2n$. We show that it is one-to-one. Consider some $f(a) = f(b)$ for positive integers a, b. Then, $2a = 2b$ and so $a = b$. However, note that $\{2n + 1 : n \in \mathbb{N}\} \not\subseteq \text{range}(f)$. The function f is not surjective.

(c) onto but not one-to-one

Solution Let f be defined by $f(1) = 1$ and $f(n) = n - 1$ if $n \geq 2$. Clearly, $f(1) = f(2) = 1$ and so it is not injective. We prove that it is surjective. Consider any $b \in \mathbb{N}$, then $b + 1 \in \mathbb{N}$ and $f(b + 1) = (b + 1) - 1 = b$.

(d) neither one-to-one nor onto

Solution Let f be defined by $f(n) = 1$. Note that $f(a) = f(b)$ for any $a, b \in \mathbb{N}$ and $\text{range}(f) = \{1\}$. Hence, f is neither onto nor one-to-one.

Problem 28. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 4, 7, 9\}$. A relation f is defined from A to B by $a\ f\ b$ if 5 divides $ab + 1$. Is f a one-to-one function?

Solution We now that $f = \{(2, 7), (4, 1), (6, 4), (6, 9)\}$. Since 6 is related to two number, namely 4 and 9, it follows that f is not a function.

Problem 29. Let f be a function with $\text{dom}(f) = A$ and let C and D be subsets of A. Prove that if f is one-to-one, then $f(C \cap D) = f(C) \cap f(D)$.

Proof. Since f is a function, it follows that $x \in A \implies f(x) \in f(A)$. However, it is also one-to-one, which means that if $f(a) = f(b) \implies a = b$ for any $a, b \in A$. Therefore, $x \in A \iff f(x) \in f(A)$. Note that

$$\begin{aligned} f(C \cap D) &= \{f(x) : x \in C \cap D\} \\ &= \{f(x) : x \in C \text{ and } x \in D\} \\ &= \{f(x) : f(x) \in f(C) \text{ and } f(x) \in f(D)\} \\ &= f(C) \cap f(D) \end{aligned}$$

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