# Section 9.3: One-To-One and Onto Functions

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We have seen that a function from $A$ to $B$ is a relation that fulfills the following condition:

$$a = b \implies f(a) = f(b).$$

Furthermore, functions can posses to important properties. A function $f : A \to B$ is said to be **One-to-One** if every image is unique to its respective $x \in A$, namely,

$$\begin{gathered} f(a) = f(b) \implies a = b \\ \equiv a \neq b \implies f(a) \neq f(b). \end{gathered}$$

Obviously, for this to be true, $B$ must contain at least the same number of elements as $A$, namely, $|A| \leq |B|$. On the other hand, the function $f$ is said to be **Onto** if every element in $B$ is the image of some element of $A$, namely,

$$b \in B \implies \exists a \in A, f(a) = b.$$

Hence, $f(A) = B$. Clearly, $|B| \leq |A|$, otherwise, there would be not enough elements of $A$ to cover all elements of $B$. Then, if a function is both one-to-one and onto, then $|A| = |B|$.

**Problem 20.** A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 2n + 1$. Determine whether $f$ is injective, surjective.

**Solution** First we show that it is injective. Consider two $f(a) = f(b)$ for some $a, b \in \mathbb{Z}$. Then, $2a + 1 = 2b + 1$. Substracting 1 to both sides, we get $2a = 2b$. Dividing by 2, we obtain $a = b$. However, it is not surjective. Consider any even integer $r$ and so there is no integer $n$ such that $f(n) = 2n + 1 = r$.

**Problem 21.** A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n - 3$. Determine whether $f$ is injective, surjective.

**Solution** The function is both injective and surjective. Consider some some $f(a) = f(b)$ for $a, b \in \mathbb{Z}$. Then, $a - 3 = b - 3$ and so $a = b$. Now, let $y \in \mathbb{Z}$. Note that $x = b + 3$ is an integer. Then $f(x) = (y + 3) - 3 = b$.

**Problem 23.** Prove or disprove: For every nonempty set $A$, there exists an injective function $f : A \to \mathcal{P}(A)$.

*Proof.* Let $g : A \to \mathcal{P}(A)$ be deifned by $f(n) = \{n\}$. We show that it is injective. Consider some element $f(a) = f(b)$ for $a, b \in A$, then $\{a\} = \{b\}$, which implies that $a = b$. Due to the individuality of each element of $A$, $f(n)$ is injective. $\square$

Note that it is impossible to define a function $f : A \to \mathcal{P}(A)$ that is surjective since $|A| < 2^{|A|} = |\mathcal{P}(A)|$.

**Problem 24.** Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 9$ is one-to-one, onto.

**Solution** We show that it is not one-to-one. Consider some $f(x) = f(y)$ for $x, y \in \mathbb{R}$. Then, $x^2 + 4x + 9 = y^2 + 4y + 9$ and so $x^2 + 4x - (y^2 + 4y) = 0$. Note that

$$\begin{aligned} x^2 + 4x - (y^2 + 4y) &= (x^2 - y^2) + 4(x - y) \\ &= (y + x)(x - y) + 4(x - y) = (x - y)(y + x + 4) = 0. \end{aligned}$$

Hence, either $x - y = 0$ or $y + x + 4 = 0$. In the latter, $y = -(x + 4)$. For instace, if $x = 3$ and $y = 1$, then $f(x) = f(y)$.
Also, it is not surjective. Note that

$$\begin{aligned} x^2 + 4x + 9 &= \left(x^2 + 4x + 4\right) - 4 + 9 \\ &= (x + 2)^2 + 5 \geq 5. \end{aligned}$$

Thus, there is no $x \in \mathbb{R}$ such that $f(x) < 4$.