# Week 9

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Section 3: Proof by Contrapositive

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Let $P(x)$ and $Q(x)$ be two open sentences over some domain $S$. Suppose one wants to prove that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$, but it is not directly clear how or whether a direct proof can be applied. Then one can prove directly that the contrapositive of $P(x) \Rightarrow Q(x)$, namely $\sim Q(x) \Rightarrow \sim P(x)$, is true for each $x \in S$ since an implication and its contrapositive are logically equivalent. One assumes that $\sim Q(x)$ is true for an arbitrary $x \in S$ and proves that $\sim P(x)$ is true for this same $x$. This method is known as a Proof by Contrapositive.
Note that if one proves the truth of $\forall x \in S, P(x) \Rightarrow Q(x)$, then the truth of $\forall x \in S, \sim Q(x) \Rightarrow \sim P(x)$ is also proven, and vice versa. Therefore it is understood that for all $x \in S$ for which $P(x)$ is true, $Q(x)$ is true. Also, for all $x \in S$ for which $Q(x)$ is false, $P(x)$ is false.

**Problem 16.** Let $x \in \mathbb{Z}$. Prove that if $7x + 5$ is odd, then $x$ is even.

*Proof.* Assume that $x$ is odd. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. Hence,

$$7(2k + 1) + 5 = 14k + 12 = 2(7k + 6)$$

Since $7k + 6$ is an integer, it follows that $7x + 5$ is even. □

**Problem 17.** Let $n \in \mathbb{Z}$. Prove that if $15n$ is even, then $9n$ is even.

*Proof.* Let $15n$ be even. Then $15n = 2k$ for some $k \in \mathbb{Z}$. Note that

$$9n = 15n - 6n = 2k - 6n = 2(k - 3n)$$

Because $k - 3n$ is an integer, it follows that $9n$ is even. □

**Problem 18.** Let $x \in \mathbb{Z}$. Prove that $5x - 11$ is even if and only if $x$ is odd.

*Proof.* Assume $x$ is even. Then $x = 2m$ for some $m \in \mathbb{Z}$. Therefore,

$$5(2m) - 11 = 10m - 12 + 1 = 2(5m - 6) + 1$$

Since $5m - 6$ is an integer, $5x - 11$ is odd.
For the converse, let $x$ be odd. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. Hence,

$$5(2k + 1) - 11 = 10k + 5 - 11 = 10k - 6 = 2(5k - 3)$$

Since $5k - 3$ is an integer, it follows that $5x - 11$ is even. □

**Problem 19.** Let $x \in \mathbb{Z}$. Use a lemma to prove that if $7x+4$ is even, then $3x-11$ is odd.

**Lemma** Let $x \in \mathbb{Z}$. If $7x+4$ is even, then $x$ is even.

*Proof.* Assume $x$ is odd. Then $x = 2n+1$ for some $n \in \mathbb{Z}$. Therefore,

$$7(2n+1)+4 = 14n+7+4 = 14n+6+4+1 = 2(7n+5)+1$$

Since $7n+5$ is an integer, $7x+4$ is odd. $\square$

We are now ready to prove the result in problem 19.

*Proof.* Let $7x+4$ be even. Then by lemma, $x = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$3(2k)-11 = 2(3k)-12+1 = 2(3k-6)+1$$

Since $3k-6$ is an integer, it follows that $3x-11$ is odd. $\square$

**Problem 20.** Let $x \in \mathbb{Z}$. Prove that $3x+1$ is even if and only if $5x-2$ is odd.

*Proof.* Assume $3x+1$ is even. Then $3x+1 = 2k$ for some $k \in \mathbb{Z}$. Note that

$$5x-2 = (3x+1)+2x-4+1 = 2k+2x-4+1 = 2(k+x-2)+1$$

Since $k+x-2$ is an integer, $5x-2$ is odd.
For the converse, assume $5x-2$ is odd. Then $5x-2 = 2k+1$ for some $k \in \mathbb{Z}$. Note that

$$3x+1 = (5x-2)-2x+3 = 2k+1-2x+3 = 2(k-x+2)$$

Since $k-x+2$ is an integer, it follows that $3x+1$ is even. $\square$

**Problem 21.** Let $n \in \mathbb{Z}$. Prove that $(n+1)^2-1$ is even if and only if $n$ is even.

*Proof.* Assume $n$ is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Therefore

$$(n+1)^2-1 = (2k+2)^2-1 = 4k^2+8k+4-1 = 4k^2+8k+3 = 2(2k^2+4k+1)+1$$

Because $2k^2+4k+1$ is an integer, $(n+1)^2-1$ is odd.
For the converse, assume $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$(2k+1)^2-1 = 4k^2+4k+1-1 = 2(2k^2+2k)$$

Since $2k^2+2k$ is an integer, $(n+1)^2-1$ is even. $\square$

**Problem 22.** Let $S = \{2,3,4\}$ and let $n \in S$. Use a proof by contrapositive to prove that if $n^2(n-1)^2/4$ is even, then $n^2(n+1)^2/4$ is even.

*Proof.* Let $n \in S$ such that $n^2(n+1)^2/4$ is odd. Then $n = 2$ and so $n^2(n-1)^2/4 = 1$ is odd. $\square$

**Problem 23.** Let $A = \{0, 1, 2\}$ and $B = \{4, 5, 6\}$ be subsets of $S = \{0, 1, \ldots, 6\}$. Let $n \in S$. Prove that if $\frac{n(n-1)(n-2)}{6}$ is even, then $n \in A \cup B$.

*Proof.* Assume $n \notin A \cup B$, where $n \in S$. Then $n = 3$ and so $\frac{n(n-1)(n-2)}{6} = 1$ is odd. $\square$

**Problem 24.** Let $n \in \mathbb{Z}$. Prove that $2n^2 + n$ is odd if and only if $\cos \frac{n\pi}{2}$ is even.

*Proof.* Assume $\cos \frac{n\pi}{2}$ is odd. Then $\cos \frac{n\pi}{2} = \pm 1$ and $n$ is even. Therefore, $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$2(2k)^2 + 2k = 8k^2 + 2k = 2(4k^2 + k)$$

Since $4k^2 + k$ is an integer, $2n^2 + n$ is even.
For the converse, let $\cos \frac{n\pi}{2}$ be even. Then $\cos \frac{n\pi}{2} = 0$ and $n$ is odd. Therefore, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Thus,

$$2(2k+1)^2 + 2k + 1 = 2(4k^2 + 4k + 1) + 2k + 1 = 2(4k^2 + 5k + 1) + 1$$

Because $4k^2 + 5k + 1$ is an integer, it follows that $2n^2 + n$ is odd. $\square$

**Problem 25.** Let $\{A, B\}$ be a partition of the set of $S = \{1, 2, \ldots, 7\}$, where $A = \{1, 4, 5\}$ and $B = \{2, 3, 6, 7\}$. Let $n \in S$. Prove that if $\frac{n^2+3n-4}{2}$ is even, then $n \in A$.

*Proof.* Let $n \in S$ such that $n \notin A$. Then $n \in B = \{2, 3, 6, 7\}$. Note that $\frac{2^2+3(2)-4}{2} = 3$, $\frac{3^2+3(3)-4}{2} = 7$, $\frac{6^2+3(6)-4}{2} = 25$ and $\frac{7^2+3(7)-4}{2} = 33$ are all odd integers. $\square$