Section 8.3: Equivalence Relations

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This chapter reviews some properties that we realized and proved in the problems of **Section 8.3**. However, there's something worth noting. Let R be some relation on some nonempty set A. I previously showed that the union of the equivalence classes by R is A and they all are pairwise disjoint. Nevertheless, I didn't ponder on it much to realize what this meant, namely, that the set of these distinct equivalence classes is a partition of A!!!! This was proven by the authors by just showing that each $x \in A$ belongs to exaclty one equivalence class by R.

Problem 36. Give an example of an equivalence relation R on the set $A = \{v, w, x, y, z\}$ such that there are exactly three distinct equivalence classes. What are the equivalence classes for your example?

Solution 36. Consider the parition $P = \{\{v\}, \{w\}, \{x, y, z\}\}$ of A. By **Theorem 4**, the relation R definded by $a\ R\ b$ if $a, b \in X$ for some $X \in P$ is an equivalence relation. Hence, the distinct equivalance classe are

$$\begin{aligned} a_1 &= \{x, y, z\} \\ a_2 &= \{w\} \\ a_3 &= \{v\} \end{aligned}$$

Problem 37. A relation R is defined on $\mathbb{N}$ by $a\ R\ b$ if $a^2 + b^2$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Proof. We first prove that R is an equivalence relation. Consider some positive integer c. Then, $c^2 + c^2 = 2c^2$. Since c^2 is an integer, it follows that $2c^2$ is even and so $c\ R\ c$. Hence, R is reflexive. Let $a, b \in \mathbb{N}$. By the commutative property of sums on real numbers, it follows that if $a^2 + b^2$ is even, then $b^2 + a^2$ is equal to the same even number. Therefore, $a\ R\ b$ implies $b\ R\ a$ and so R is symmetric. Consider $x, y, z \in \mathbb{Z}$ such that $x\ R\ y$ and $y\ R\ z$. Hence, $x^2 + y^2 = 2m$ and $y^2 + z^2 = 2n$ for $m, n \in \mathbb{Z}$. Thus, $x^2 = 2m - y^2$ and $z^2 = 2n - y^2$. Therefore,

$$\begin{aligned} x^2 + z^2 &= (2m - y^2) + (2n - y^2) \\ &= 2m + 2n - 2y^2 = 2(m + n - y^2). \end{aligned}$$

Because $m+n-y^2 \in \mathbb{Z}$, it follows that x^2+z^2 is even and so $x \; R \; z$, which implies that R is transitive.

Once R is shown to be an equivalence relation, we now determine the dsitinct equivalence classes. Let x be an even positive integer. Then x^2 is even. Consider some $y \in \mathbb{N}$. Note that y^2+x^2 is even if and only if y^2 is even. We also know that y^2 is even if and only if y is even. Therefore,

$$[x] = \{n \in \mathbb{N} : n \text{ is even}\}.$$

Consider positive integers y and z. If y is and odd positive integer, then z^2+y^2 is odd if and only if z^2 is odd. Hence, z must be odd.

$$[y] = \{n \in \mathbb{N} : n \text{ is odd}\}.$$

Since the set of even and odd positive integers is a partition of $\mathbb{N}$, it follows that there only two distinct equivalence classes. $\square$

Problem 38. Let R be a relation defined on the set $\mathbb{N}$ by $a \; R \; b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is and equivalence relation.

Solution 38. The relation R on $\mathbb{N}$ is not an equivalence relation. Consider the positive integers 2, 3 and 5. Since $2 \mid (2 \cdot 3)$ and $2 \mid (2 \cdot 5)$, it follows that $3 \; R \; 2$ and $2 \; R \; 5$. However, $3 \nmid (2 \cdot 5)$ and $5 \nmid (2 \cdot 3)$. Hence, $3 \not\!R \; 5$ and so R is not transitive. This implies that R is not an equivalence relation.