Section 9.1: The Definition of Function

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July 7, 2022

A very famous type of relation is the function. For some sets A, B, a function f is a relation from A to B, expressed as $f : A \to B$, such that for every $a \in A$, $(a, b) \in f$ for only one $b \in B$. Hence, $|A| = |f|$. Also, $\text{dom}(f) = A$ and $\text{codom}(f) = B$. For a function $f : A \to B$, Consider some $(a, b) \in f$. Because every ordered pair in f is adscribed to only one $a \in A$, it follows that $(a, b), (a, c) \in f$ implies $b = c$. Thus, $b = f(a)$ is considered as the **image** of a. In fact this is known as **mapping**. For instance, f is said to map a into b. Hence, the **range** of this relation f can be expressed as

$$\begin{aligned} \text{range}(f) &= \{b \in B : (a, b) \in f,\ a \in A\} \\ &= \{f(a) : a \in A\}. \end{aligned}$$

Now, suppose that we have some subset C of A. Then,

$$f(C) = \{f(x) : x \in C\}$$

is known as the **image** of C. Obviously, if $C = A$, then $f(C) = \text{range}(f)$. Furthermore, for some subeset D of B, its **inverse image** is denoted as

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

Due to the definition of a function, $f^{-1}(B) = A$.

Problem 3. Let A be a nonempty set. If R is a relation from A to A that is both an equivalence relation and a function, then what familiar function is R?

Solution 3. It is some type of identity linear function, namely, $R = \{(x, x) : x \in A\}$. Recall that for it to be a function, each $x \in A$ must be paired with only one $c \in A$. However, each x must be paired with itself, since the reflexive property requires each element of A be related to itself. Note that the property of symmetry follows directly and the transitivity follows vacuously. (Remember that this linear relation is the smallest equivalence relation for some nonempty set).