Week 11

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Section 3: Proofs involving real numbers

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Problem 25. Let $x, y \in \mathbb{R}$. Prove that if $x^2 - 4x = y^2 - 4y$ and $x \neq y$, then $x + y = 4$.

Proof. Assume $x^2 - 4x = y^2 - 4y$ and $x \neq y$. Note that,

$$\begin{aligned} x^2 - y^2 - 4x + 4y &= 0 \\ (x+y)(x-y) - 4(x-y) &= 0 \\ (x-y)[(x+y) - 4] &= 0 \end{aligned}$$

Since $x - y = 0$ if and only if $x = y$, it follows that $x + y - 4 = 0$ and so $x + y = 4$. $\square$

Problem 26. Let a, b and m be integers. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.

Proof. Assume $a < 3m + 1$ and $b < 2m + 1$. Since $a, b \in \mathbb{Z}$, $a \leq 3m$ and $b \leq 2m$. Hence

$$2a + 3b \leq 12m < 12m + 1$$

Thus, $2a + 3b < 12m + 1$. $\square$

Problem 27. Let $x \in \mathbb{R}$. Prove that if $3x^4 + 1 \leq x^7 + x^3$, then $x > 0$.

Proof. Assume $x \leq 0$. We consider the following two cases.
Case 1. $x = 0$. Then, $3x^4 + 1 = 1 > 0 = x^7 + x^3$.
Case 2. $x < 0$. Then $x^7 < 0$, $x^3 < 0$ and $x^4 > 0$. Therefore,

$$\begin{aligned} x^7 - 3x^4 + x^3 - 1 &< 0 \\ x^7 + x^3 &< 3x^4 + 1 \end{aligned}$$

Therefore, $3x^4 + 1 > x^7 + x^3$. $\square$

Problem 28. Prove that if r is a real number such that $0 < r < 1$, then $\frac{1}{r(1-r)} \geq 4$.

Proof. Assume $0 < r < 1$. Note that $(2r-1)^2 \geq 0$. Thus,

$$\begin{aligned} (2r-1)^2 &\geq 0 \\ 4r^2 - 4r + 1 &\geq 0 \\ 1 &\geq -4r^2 + 4r = 4[r(1-r)] \end{aligned}$$

Since $0 < r < 1$, it follows that $r(1-r) > 0$. Thus, $\frac{1}{r(1-r)} \geq 4$. $\square$

Problem 29. Prove that if r is a real number such that $|r-1|<1$, then $\frac{4}{r(4-r)} \geq 1$.

Proof. Let $r \in \mathbb{R}$ such that $|r-1|<1$. Then $-1<r-1<1$ and so $0<r<2$. Note that, for any $r \in \mathbb{R}$, $(r-2)^2 \geq 0$. Thus,

$$\begin{aligned} r^2-4r+4 &\geq 0 \\ 4 &\geq -r^2+4r = r(4-r) \end{aligned}$$

Since $0<r<2$, it follows that $r(4-r)>0$ and so we can divide both sides by $r(4-r)$. Hence, $\frac{4}{r(4-r)} \geq 1$, as desired. □

Problem 30. Let $x, y \in \mathbb{R}$. Prove that $|xy| = |x| \cdot |y|$.

Proof. Let $x, y \in \mathbb{R}$. First, observe that when $x=y=0$ the equation $|xy| = |x| \cdot |y|$ holds. Then, we consider the following cases when x and y are nonzero.
Case 1. $x>0$ and $y>0$. Then, $|xy| = xy$ and $|x| \cdot |y| = xy$. Thus, $|xy| = |x| \cdot |y|$.
Case 2. $x<0$ and $y<0$. Then, $|xy| = xy$ and $|x| \cdot |y| = (-x)(-y) = xy$. Therefore, $|xy| = |x| \cdot |y|$.
Case 3. Exactly one of x and y is greater than zero and the other is lower than zero. Without loss of generality, let $x>0$ and $y<0$. Then, $|xy| = -xy$ and $|x| \cdot |y| = (x)(-y) = -xy$. Thus, $|xy| = |x| \cdot |y|$. □

Problem 31. Prove for every two real numbers x and y that $|x+y| \geq |x| - |y|$.

Proof. □

Problem 32. (a) Recall that $\sqrt{r} > 0$ for every positive real number r. Prove that if a and b are positive real numbers, then $0 < \sqrt{ab} \leq \frac{a+b}{2}$. (The number $\sqrt{ab}$ is called the **geometric mean** of a and b, while $(a+b)/2$ is called the **arithmetic mean** or **average**.)

Proof. Let $a, b \in \mathbb{R}$ such that $a>0$ and $b>0$. We know that $(a-b)^2 \geq 0$. Then,

$$\begin{aligned} a^2-2ab+b^2 &\geq 0 \\ a^2-2ab+4ab+b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab \end{aligned}$$

Since $ab>0$, we can square both sides. Then, $|a+b| \geq 2\sqrt{ab}$. Note that $a+b>0$ and so $a+b = |a+b|$. Therefore, $0 < \sqrt{ab} \leq \frac{a+b}{2}$. □

(b) Under what conditions does $\sqrt{ab} = (a+b)/2$ for positive real numbers a and b? Justify your answer.

Solution b. For positive real numbers a and b such that $a=b$.

Problem 34. Prove for every three real numbers x, y and z that $|x-z| \leq |x-y| + |y-z|$.

Proof. Let $x, y, z \in \mathbb{Z}$. Then, by the Triangle Inequality,

$$\begin{aligned} |(x-y)+(y-z)| &\leq |x-y|+|y-z| \\ |x-z| &\leq |x-y|+|y-z| \end{aligned}$$

As desired. $\square$

Problem 35. Prove that if x is a real number such that $x(x+1) > 2$, then $x < -2$ or $x > 1$.

Proof. Let $x \in \mathbb{R}$ such that $x(x+1) > 2$. Note that

$$\begin{aligned} x^2 + x &> 2 \\ x^2 + x + \frac{1}{4} &> \frac{9}{4} \\ \left(x + \frac{1}{2}\right)^2 &> \frac{9}{4} \\ \left|x + \frac{1}{2}\right| &> \frac{3}{2} \end{aligned}$$

Hence, $x + \frac{1}{2} < -\frac{3}{2}$ or $x + \frac{1}{2} > \frac{3}{2}$. Then, $x < -2$ or $x > 1$, as desired. $\square$

Problem 36. Prove for every positive real number x that $1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3}$.

Proof. Let $x \in \mathbb{R}$ such that $x > 0$. Let's consider $(x^3-1)(x-1)$. If $0 < x < 1$, then $x^3 - 1 < 0$ and $x - 1 < 0$. If $x = 1$, then $x^3 - 1 = x - 1 = 0$. Also, if $x > 1$, then $x^3 - 1 > 0$ and $x - 1 > 0$. Therefore, $(x^3-1)(x-1) \geq 0$. Then,

$$\begin{aligned} (x^3-1)(x-1) &\geq 0 \\ x^4 - x^3 - x + 1 &\geq 0 \\ x^4 + 1 &\geq x^3 + x \end{aligned}$$

Since $x^4 > 0$, $\frac{x^4+1}{x^4} = 1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3} = \frac{x^3+x}{x^4}$, as desired. $\square$

Problem 37. Prove for $x, y, z \in \mathbb{R}$ that $x^2 + y^2 + z^2 \geq xy + xz + yz$.

Proof. Let $x, y, z \in \mathbb{R}$. We know that $(x-y)^2 + (x-z)^2 + (z-y)^2 \geq 0$. Then

$$\begin{aligned} x^2 - 2xy + y^2 + x^2 - 2xz + z^2 + z^2 - 2zy + y^2 &\geq 0 \\ 2x^2 + 2y^2 + 2z^2 &\geq 2xy + 2xz + 2zy \\ x^2 + y^2 + z^2 &\geq xy + xz + zy \end{aligned}$$

As desired. $\square$

Problem 38. Let $a, b, x, y \in \mathbb{R}$ and $r \in \mathbb{R}^+$. Prove that if $|x-a| < r/2$ and $|y-b| < r/2$, then $|(x+y)-(a+b)| < r$.

Proof. Assume $|x - a| < r/2$ and $|y - b| < r/2$. Then $|x - a| + |y - b| < r$ and, by the Triangle Inequality, $|(x-a)+(y-b)| \leq |x-a|+|y-b|$. Therefore, $|(x+y)-(a+b)| < r$ $\square$

Problem 39. Prove that if $a, b, c, d \in \mathbb{R}$, then $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

Proof. Let $a, b, c, d \in \mathbb{R}$. Note that,

$$\begin{aligned} (ab + cd)^2 &\leq (ab + cd)^2 + (cb - ad)^2 \\ (ab + cd)^2 + (cb - ad)^2 &= a^2b^2 + 2abcd + c^2d^2 + c^2b^2 - 2abcd + a^2d^2 \\ &= (a^2 + c^2)(b^2 + d^2) \end{aligned}$$

Therefore, $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$. $\square$