

Section 8.3: Equivalence Relations

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In this section, the concept of **Equivalence Relation** on some set A is introduced. In short words, an **Equivalence Relation** on some set A is one that has is reflexive, symmetric and transitive. One of the best examples is the relation R defined by $x R y$ if $x = y$. Also, an important subset to understand the behavior of these type of relations is the **equivalence class**. Basically, an **equivalence class** $[a]$ contains all elements $x \in A$ that are related to some specific $a \in A$, namely,

$$[a] = \{x \in A : x R a\}$$

Note that if $b \in [a]$ (b is related to a), then b and a are "equivalent". Note that $a \in [b]$ and $[b] = [a]$ due to the symmetric and transitive properties of R . Quite interesting!!!

Lemma 8.3.1. Let R be an equivalence relation on an nonempty set A . Then, $a R b$ for some $a, b \in A$ is a necessary and sufficient condition for $[a] = [b]$.

Proof. Because R is reflexive, $a \in [a]$ and $b \in [b]$ and so they are nonempty. Consider some $x \in [a]$, then $x R a$. Note that $a R b$ and so, by the transitive property of R , $x R b$. Hence, $x \in [b]$ which implies that $[a] \subseteq [b]$.

Now consider some $y \in [b]$ and so $y R b$. Since R is symmetric and $a R b$, it follows that $b R a$. Thus, by the transitive property, $y R a$ and so $y \in [a]$. Therefore, $[b] \subseteq [a]$ and so $[a] = [b]$.

For the converse, assume that $[a] = [b]$. Since R is reflexive, it follows that $a \in [a]$ and so $a \in [b]$. Hence, $a R b$. \square

Note that this implies that the union of all equivalence classes of A is A itself!!!

Corollary 8.3.1. Let R be an equivalence relation on an nonempty set A and consider some $a, b \in A$. Then, $[b] = [a]$ if and only if $b \in [a]$.

Proof. Assume that $[b] = [a]$. By **Lemma 8.3.1**, $b R a$. Therefore, $b \in [a]$.

For the converse, suppose that $b \in [a]$. Then, $b R a$. By **Lemma 8.3.1**, $[b] = [a]$. \square

Corollary 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3] \dots, [a_n]$. Then, the equivalence classes are disjoint.

Proof. Suppose, to the contrary, that $[a_i] \cap [a_j] \neq \emptyset$ for some positive integers $i, j \leq n$ such that $i \neq j$. Then, there is at least some $x \in [a_i] \cap [a_j]$ and so $x \in [a_i]$ and $x \in [a_j]$. By **Corollary 8.3.1**, $[a_i] = [x] = [a_j]$. However, this contradicts the assumption that $[a_i] \neq [a_j]$. \square

Lemma 8.3.2. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3], \dots, [a_n]$. Then,

$$\bigcup_{i=1}^n [a_i] = A$$

Proof. Suppose, to the contrary, that

$$\bigcup_{i=1}^n [a_i] \neq A.$$

Hence, either

$$\bigcup_{i=1}^n [a_i] \subsetneq A \quad \text{or} \quad \bigcup_{i=1}^n [a_i] \supsetneq A.$$

Suppose the first. Then, there exists some $x \in \bigcup_{i=1}^n [a_i]$ such that $x \notin A$. This implies that $x \in [a_k]$ for some positive integer k . However, $x \notin A$ and this contradicts the fact that $[a_k] = \{x \in A : x R a_k\}$.

Thus, we can assume that $\bigcup_{i=1}^n [a_i] \supsetneq A$. Then, there is some $y \in A$ such that $y \notin \bigcup_{i=1}^n [a_i]$. Because $\bigcup_{i=1}^n [a_i]$ is the union of all distinct equivalence classes resulting from R , it follows that $y \not R a$ for any $a \in A$. Hence $(y, y) \notin R$. However, this contradicts the fact that R is reflexive.

Thus,

$$\bigcup_{i=1}^n [a_i] = A.$$

\square

Problem 24. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a R c$, $c R d$, $d R g$ and $b R f$. If there are three distinct equivalence classes resulting from R , then determine these equivalence classes and determine all elements of R .

Solution 24. By repetitive use of **Lemma 8.3.1**, we conclude that $[a] = [c] = [d] = [g]$ and $[b] = [f]$. Also, since e is not related to any element of A , it follows that the remaining equivalence class is $[e]$. Note that the reflexive property of R implies that $g R g$ and $f R f$. Therefore, by the transitive property,

$$\begin{aligned} [g] &= \{a, g, d, c\} = [a] = [c] = [d] \\ [f] &= \{b, f\} = [b] \\ [e] &= \{e\} \end{aligned}$$

Therefore,

$$R = \{(a, a), (g, a), (d, a), (c, a), (a, c), (g, c), (d, c), \\ (c, c), (a, d), (g, d), (d, d), (c, d), (a, g), (g, g), \\ (d, g), (c, g), (b, b), (f, b), (b, f), (f, f), (e, e)\}.$$

This is a taste of how useful equivalence classes can be. Wow!!!

Problem 25. Let $A = \{1, 2, 3, 4, 5, 6\}$. The relation

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), \\ (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

is an equivalence relation on A . Determine the distinct equivalence classes.

Solution 25. Since R is an equivalence relation on A , then we can use **Lemma 1.8.3** to determine the equivalence classes. Note that $(1, 1), (5, 1), (2, 2), (6, 2), (3, 2), (4, 4) \in R$. Hence,

$$\begin{aligned} [1] &= \{1, 5\} = [5] \\ [2] &= \{2, 6, 3\} = [3] = [6] \\ [4] &= \{4\} \end{aligned}$$

Problem 26. Let $A = \{1, 2, 3, 4, 5, 6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1, 4, 5\}$, $\{2, 6\}$ and $\{3\}$. What is R ?

Solution 26. By **Corollary 8.3.1**,

$$\begin{aligned} \{1, 4, 5\} &= [1] = [4] = [5] \\ \{2, 6\} &= [2] = [6] \text{ and} \\ \{3\} &= [3]. \end{aligned}$$

Therefore, the relation

$$R = \{(1, 1), (4, 1), (5, 1), (1, 4), (4, 4), (5, 4), (1, 5), \\ (4, 5), (5, 5), (2, 2), (6, 2), (2, 6), (6, 6), (3, 3)\}$$

Corollary 8.3.3. Let R be an equivalence relation on A with a total of $n \in \mathbb{N}$ different equivalence classes $[a_1], [a_2], [a_3], \dots, [a_n]$. Then,

$$|R| = \sum_{i=1}^n |[a_i]|^2.$$

Proof. Consider some $[a_k]$ for a $k \leq n$. By **Corollary 8.3.1**, $[x] = [a_k]$ for every $x \in [a_k]$. Then, $|[a_k]|$ elements of A are related to x for every $x \in [a_k]$. Hence, $|[a_k]|^2$ different n-tuples are elements of R . This is the apport of each equivalence class, however we need to be sure

that each $x \in A$ is an element of only one equivalence class. Since **Corollary 8.3.2** implies that the different equivalence classes are disjoint and **Lemma 8.3.2** implies that their union is A , it follows that

$$|R| = \sum_{i=1}^n |[a_i]|^2$$

□

Problem 27. Let R be a relation defined on \mathbb{Z} by $a R b$ if $a^3 = b^3$. Show that R is an equivalence relation on \mathbb{Z} and determine the distinct equivalence classes.

Solution 27. We first show that R is an equivalence relation on \mathbb{Z} .

Proof. Consider some integer x , then $x^3 = x^3$ and so $x R x$. Hence, R is reflexive. Now, consider some $x, y \in \mathbb{Z}$ such that $x R y$. Therefore, $x^3 = y^3$ and so $y^3 = x^3$, which implies that $y R x$. Thus, R is symmetric.

Let $x, y, z \in \mathbb{Z}$ such that $x R y$ and $y R z$. Then, $x^3 = y^3 = z^3$ and so $x^3 = z^3$, which implies that $x R z$. Therefore, R is transitive. Hence, R is an equivalence relation on \mathbb{Z} . □

Note that $x^3 = y^3 \iff x = y$ for any integers x and y . Therefore, each integer is only related to itself by R namely, $[x] = \{x\}$ whenever $x \in \mathbb{Z}$. Hence, there is an infinity of different equivalence relations.

Problem 30. Let $H = \{2^m : m \in \mathbb{Z}\}$. A relation R is defined on the set \mathbb{Q}^+ of positive rational numbers by $a R b$ if $a/b \in H$.

(a) Show that R is an equivalence relation.

Proof. First, we prove that it is reflexive. Consider some $x \in \mathbb{Q}^+$. Then $x/x = 1 = 2^0 \in H$. Hence, $x R x$.

Now, we show that it is symmetric. Let $a, b \in \mathbb{Q}^+$ such that $a R b$, namely, $a/b = 2^m$ for some integer m . Then, $b/a = 1/2^m = 2^{-m}$. Since $-m \in \mathbb{Z}$, it follows that $b R a$.

Last, we prove that it is transitive. Consider some $a, b, c \in \mathbb{Q}^+$ such that $a R b$ and $b R c$, which implies that $a/b = 2^m$ and $b/c = 2^n$ for $m, n \in \mathbb{Z}$. Note that

$$\begin{aligned} \frac{a}{c} &= \frac{a}{b} \cdot \frac{b}{c} \\ &= 2^m \cdot 2^n \\ &= 2^{m+n}. \end{aligned}$$

Since $m + n \in \mathbb{Z}$, it follows that $a R c$. □

(b) Describe the elements in the equivalence class $[3]$.

Solution b. Consider some positive rational number x such that $x R 3$. Then, $x/3 = 2^k$ for some integer k . Then, $x = 3 \cdot 2^k$. Thus, $[3] = \{3 \cdot 2^k : k \in \mathbb{Z}\}$.

Problem 31. A relation R on a nonempty set A is defined to be **circular** if whenever $x R y$ and $y R z$, then $z R x$ for all $x, y, z \in A$. Prove that a relation R on A is an equivalence relation if and only if R is circular and reflexive.

Proof. First, assume that R is an equivalence relation on some nonempty set A . Hence, R is reflexive, symmetric and transitive. We show that it is circular. Consider some $x, y, z \in A$ such that $x R y$ and $y R z$. By the transitive property, $x R z$. Since R is symmetric, it follows that $z R x$ and so R is circular.

For the converse, Let R be a **circular** and reflexive relation on some nonempty set A . We show that it is symmetric and transitive. Consider some $x, y \in A$ such that $x R y$. Since R is reflexive, it follows that $y R y$, and so $y R x$ due to the circular property of R . Thus, R is symmetric.

Now, consider some $x, y, z \in A$ such that $x R y$ and $y R z$. By the **circular property** of R , $z R x$. Because R is symmetric, it follows that $x R z$ and so R is transitive. Hence, R is an equivalence relation on A . \square

Problem 32. A relation R is defined on the set $A = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a + b\sqrt{2} \neq 0\}$ by $x R y$ if $x/y \in \mathbb{Q}$. Show that R is an equivalence relation and determine the distinct equivalence classes.

Solution 32. We first show that R is an equivalence relation.

Proof. Consider some $x \in A$. Because $x \neq 0$, $x/x = 1 \in \mathbb{Q}$ and so R is reflexive.

Let $x, y \in A$ such that $x R y$. Then, $x/y = c \in \mathbb{Q}$ and so $1/c = y/x \in \mathbb{Q}$. Hence, $y R x$ and so R is symmetric.

Last, consider some $x, y, z \in A$ such that $x R y$ and $y R z$. Then, $x/y = a \in \mathbb{Q}$ and $y/z = b \in \mathbb{Q}$. Note that

$$\frac{x}{y} \cdot \frac{y}{z} = \frac{x}{z} = ab \in \mathbb{Q}.$$

Thus, $x R z$ and so R is transitive. \square

Since R is an equivalence relation, we can proceed to determine the distinct equivalence classes. Consider some $x \in A$. Then, $x = a + b\sqrt{2} \neq 0$ for some $a, b \in \mathbb{Q}$. There are two possible cases:

Case 1. x is rational, namely, $b = 0$. Then, for y/x to be rational ($y R x$), it must be true that $y = d + 0\sqrt{2}$ for some $d \in \mathbb{Q}$. Thus, $[x] = \mathbb{Q}$

Case 2. x is irrational, namely, $b \neq 0$. Then, for y/x to be rational ($y R x$), it must be true that $y = cx$ for some $c \in \mathbb{Q}$. Hence, $[x] = \{ax : a \in \mathbb{Q}\}$.

By **Corollary 8.3.1**, the only three equivalence classes are: ???????

(a) \mathbb{Q}

(b) $\{c + c\sqrt{2} : \text{for some nonzero rational number } c\}$

(c) $\{c(a + b\sqrt{2}) : \text{for some nonzero } a, b, c \in \mathbb{Q} \text{ such that } a \neq b\}$

Problem 34. Let H be a nonempty subset of \mathbb{Z} . Suppose that the relation R defined on \mathbb{Z} by $a R b$ if $a - b \in H$ is an equivalence relation. Verify the following

- (a) $0 \in H$
- (b) If $a \in H$, then $-a \in H$.
- (c) If $a, b \in H$, then $a + b \in H$.

Problem 35. Prove or disprove: There exist equivalence relations R_1 and R_2 on the set $S = \{a, b, c\}$ such that $R_1 \not\subseteq R_2$, $R_1 \not\supseteq R_2$ and $R_1 \cup R_2 = S \times S$.