Section 9.4: Bijective Functions

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As it was mentioned in the previous section, for finite sets A and B, $|A| \geq |B|$ is a necessary and sufficient condition for an onto function $f : A \to B$ to exist. The same can be said for $|A| \leq |B|$ and some one-to-one function $g : A \to B$. Since we are talking about positive integers, it must be true that $|A| = |B|$ is a necessary and sufficient condition for an onto and one-to-one function $\varphi : A \to B$ to exist, knwon as a bijective function.
In fact, for finite sets B and C such that $|B| = |C| = n$, there are $n!$ distinct bijective functions from B to C. Namely, every bijective function is a permutation of the elements of $|C|$ for n spaces. Furthermore, for any function f from B to C, f is onto if and only if f is one-to-one. All this makes sense for finite sets, we must make sure to pair all elements of C with the constriction of assigning one unique element to every element of B. However, this intuition does not work for analyzing the cases with infinite ones.
Let A, B be sets. So far, we defined the function $f : A \to B$ as a relation from A to B such that

(a) $x \in A \implies \exists b \in B, (a, b) \in f$

(b) $(a, b), (a, c) \in f \implies b = c$

If a relation satisfies (b), then it is called **well-defined**.
Lastly, the identity function i_S on ANY nonempty set S defined by $i_S(n) = n$ for all $n \in S$ is bijective.

Problem 31. Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ be a function defined by $f([a]) = [2a + 3]$.

(a) Show that f is well-defined.

Proof. Consider two $[a] = [b]$ such that $[a], [b] \in \mathbb{Z}_5$. Then, $a \equiv b \pmod 5$ which implies that $a - b = 5k$ for some $k \in \mathbb{Z}$. Then, $f([a]) = [2a + 3]$ and $f([b]) = [2b + 3]$. Note that

$$(2a + 3) - (2b + 3) = 2(a - b) = 5(2k).$$

Therefore, $(2a + 3) \equiv (2b + 3) \pmod 5$ and so $f([a]) = f([b])$ $\square$

(b) Determine wheter f is bijective.

Proof. We know that $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$. Note that $f([0]) = [3], f([1]) = [5] = [0], f([2]) = [7] = [2], f([3]) = [4]$ and $f([4]) = [11] = [1]$. Hence, all elements of $\mathbb{Z}_5$ are paired with a unique element of $\mathbb{Z}_5$. The function is bijective. □

Problem 33. Let $A = [0, 1]$ denote the closed interval of real numbers between 0 and 1. Give an example of two different bijective functions f_1 and f_2 from A to A, neither of which is the identity function.

(a) $f : A \to A$ defined by $f(n)$

Problem 34. Give a proof of Theorem 7 using mathematical induction.

Solution If A and B are sets with $|A| = |B| = n$, then there are $n!$ bijective functions from A to B.

Proof. We proceed by induction. Let A and B be sets with $|A| = |B| = 1$, then there is only $1 = 1!$ bijective function from A to B, namely, the pairing of the only element of A with the only element of B. In fact, this is the only function from A to B since $\left|B^A\right| = 1$.
Suppose for sets A_1 and B_1 with $|A_1| = |B_1| = k$ that there are $k!$ bijective functions from A to B. We prove for sets A_2 and B_2 with $|A_2| = |B_2| = k + 1$ that there are $(k + 1)!$ bijective functions.
By our inductive hypothesis, we can only create $k!$ distinct bijective functions by fixing an element (a_{k+1}, b_{k+1}) in all of them since the remaining elements correspond to a bijective function from $\{a_1, a_2, \ldots, a_k\}$ to $\{b_1, b_2, \ldots, b_k\}$. Note that we can do this with $(a_{k+1}, b_k), (a_{k+1}, b_{k-1}), \ldots, (a_{k+1}, b_2), (a_{k+1}, b_1)$. Therefore, for each of the possible $k + 1$ images of a_{k+1}, there are only $k!$ distinct bijective functions. By the Principle of Mathematical Induction, there are $(k + 1)k! = (k + 1)!$ bijective functions from A_2 to B_2. □

Problem 35. For two finite nonempty sets A and B, let R be a relation from A to B such that range$(R) = B$. Define the domination number $\gamma(R)$ of R as the smallest cardinality of a subset $S \subseteq A$ such that for every element y of B, there is an element $x \in S$ such that x is related to y.

(a) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{a, b, c, d, e, f, g\}$ and let

$$R = \{(1, c), (1, e), (2, c), (2, f), (2, g), (3, b), (3, f), (4, a), \\ (4, c), (4, g), (5, a), (5, b), (5, c), (6, d), (6, e), (7, a), (7, g)\}.$$

Determine $\gamma(R)$.

Solution Observe that B has 7 elements and each element of A is related to either 2 or 3 distinct elements of B. Therefore, wihtout looking at R, we can assure that $S \subseteq A$ has $|S| \geq 3$. For instance, $S = \{3, 4, 6\}$. Therefore, $\gamma(R) = 3$.

(b) If R is an equivalence relation defined on a finite nonempty set A (and so $B = A$), then what is $\gamma(R)$?

Solution Since R is an equivalence relation on the finite set A, it follows that there are n distinct equivalence classes. We know that the union of the distinct equivalence classes is A, they are pairwise disjoint and any element belonging to one of them is related to itself and all elements inside of the equivalence class. Hence, $\gamma(R) = n$.

(c) If f is a bijective function from A to B, then what is $\gamma(f)$?

Solution Clearly, $\gamma(f) = |A|$ due to the onto and one-to-one properties of bijective functions.

Problem 36. Let $A = \{a, b, c, d, e, f\}$ and $B = \{u, v, w, x, y, z\}$. With each element $r \in A$, there is associated a list or subset $L(r) \subseteq B$. The goal is to define a "list function" $\varphi : A \to B$ with the property that $\varphi(r) \in L(r)$ for each $r \in A$.

(a) For $L(a) = \{w, x, y\}$, $L(b) = \{u, z\}$, $L(c) = \{u, v\}$, $L(d) = \{u, w\}$, $L(e) = \{u, x, y\}$, $L(f) = \{v, y\}$, does there exist a bijective list function $\varphi : A \to B$ for these lists?

Solution Let $\varphi = \{(a, x), (b, z), (c, v), (d, w), (e, u), (f, y)\}$. Then φ is a bijective list function from A to B.

(b) For $L(a) = \{u, v, x, y\}$, $L(b) = \{v, w, y\}$, $L(c) = \{v, y\}$, $L(d) = \{u, w, x, z\}$, $L(e) = \{v, w\}$, $L(f) = \{w, y\}$, does there exist a bijective list function $\varphi : A \to B$ for these lists?

Solution Note that the only list that contains z is $L(d)$. Hence, $(d, z) \in \varphi$. However, u and x are contained only in $L(d), L(a)$ and a can only have one image. Hence, there is no onto (bijective) list function $\varphi : A \to B$. Also, note that $\varphi(b), \varphi(c), \varphi(e), \varphi(f) \in \{v, w, y\}$.