# Section 8.3: Equivalence Relations

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This chapter reviews some properties that we realized and proved in the problems of **Section 8.3**. However, there's something worth noting. Let $R$ be some relation on some nonempty set $A$. I previously showed that the union of the equivalence classes by $R$ is $A$ and they all are pairwise disjoint. Nevertheless, I didn't ponder on it much to realize what this meant, namely, that the set of these distinct equivalence classes is a partition of $A$!!!!
This was proven by the authors by just showing that each $x \in A$ belongs to exaclty one equivalence class by $R$.

**Problem 36.** Give an example of an equivalence relation $R$ on the set $A = \{v, w, x, y, z\}$ such that there are exactly three distinct equivalence classes. What are the equivalence classes for your example?

**Solution 36.** Consider the parition $P = \{\{v\}, \{w\}, \{x, y, z\}\}$ of $A$. By **Theorem 4**, the relation $R$ definded by $a\ R\ b$ if $a, b \in X$ for some $X \in P$ is an equivalence relation. Hence, the distinct equivalance classe are

$$\begin{aligned} a_1 &= \{x, y, z\} \\ a_2 &= \{w\} \\ a_3 &= \{v\} \end{aligned}$$

**Problem 37.** A relation $R$ is defined on $\mathbb{N}$ by $a\ R\ b$ if $a^2 + b^2$ is even. Prove that $R$ is an equivalence relation. Determine the distinct equivalence classes.

*Proof.* We first prove that $R$ is an equivalence relation. Consider some positive integer $c$. Then, $c^2 + c^2 = 2c^2$. Since $c^2$ is an integer, it follows that $2c^2$ is even and so $c\ R\ c$. Hence, $R$ is reflexive. Let $a, b \in \mathbb{N}$. By the commutative property of sums on real numbers, it follows that if $a^2 + b^2$ is even, then $b^2 + a^2$ is equal to the same even number. Therefore, $a\ R\ b$ implies $b\ R\ a$ and so $R$ is symmetric. Consider $x, y, z \in \mathbb{Z}$ such that $x\ R\ y$ and $y\ R\ z$. Hence, $x^2 + y^2 = 2m$ and $y^2 + z^2 = 2n$ for $m, n \in \mathbb{Z}$. Thus, $x^2 = 2m - y^2$ and $z^2 = 2n - y^2$. Therefore,

$$\begin{aligned} x^2 + z^2 &= (2m - y^2) + (2n - y^2) \\ &= 2m + 2n - 2y^2 = 2(m + n - y^2). \end{aligned}$$

Because $m + n - y^2 \in \mathbb{Z}$, it follows that $x^2 + z^2$ is even and so $x \; R \; z$, which implies that $R$ is transitive.

Once $R$ is shown to be an equivalence relation, we now determine the dsitinct equivalence classes. Let $x$ be an even positive integer. Then $x^2$ is even. Consider some $y \in \mathbb{N}$. Note that $y^2 + x^2$ is even if and only if $y^2$ is even. We also know that $y^2$ is even if and only if $y$ is even. Therefore,

$$[x] = \{n \in \mathbb{N} : n \text{ is even}\}.$$

Consider positive integers $y$ and $z$. If $y$ is and odd positive integer, then $z^2 + y^2$ is odd if and only if $z^2$ is odd. Hence, $z$ must be odd.

$$[y] = \{n \in \mathbb{N} : n \text{ is odd}\}.$$

Since the set of even and odd positive integers is a partition of $\mathbb{N}$, it follows that there only two distinct equivalence classes. $\square$

**Problem 38.** Let $R$ be a relation defined on the set $\mathbb{N}$ by $a \; R \; b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: $R$ is and equivalence relation.

**Solution 38.** The relation $R$ on $\mathbb{N}$ is not an equivalence relation. Consider the positive integers 2, 3 and 5. Since $2 \mid (2 \cdot 3)$ and $2 \mid (2 \cdot 5)$, it follows that $3 \; R \; 2$ and $2 \; R \; 5$. However, $3 \nmid (2 \cdot 5)$ and $5 \nmid (2 \cdot 3)$. Hence, $3 \not{R} \; 5$ and so $R$ is not transitive. This implies that $R$ is not an equivalence relation.

**Problem 39.** Let $S$ be a nonempty subset of $\mathbb{Z}$ and let $R$ be a relation defined on $S$ by $x \; R \; y$ if $3 \mid (x + 2y)$.

(a) Prove that $R$ is an equivalence relation.

*Proof.* Let $S$ be some nonempty subset of $\mathbb{Z}$ and $R$ some relation on $S$ defined by $x \; R \; y$ if $3 \mid (x + 2y)$. For some integer $x \in S$, $x + 2x = 3x$ and so $3 \mid 3x$. Hence, $x \; R \; x$ is reflexive.
Let $x, y \in S$ such that $x \; R \; y$. Hence, $x + 2y = 3c$ for some integer $c$. Then, $x = 3c - 2y$ and so

$$\begin{aligned} y + 2x &= y + 2(3c - 2y) \\ &= y + 6c - 4y \\ &= 3(2c - y). \end{aligned}$$

Since $2c - y \in \mathbb{Z}$, it follows that $3 \mid (y + 2x)$ and so $y \; R \; x$ ($R$ is symmetric).
Consider some $x, y, z \in S$ sucht that $x \; R \; y$ and $y \; R \; z$. Therefore, $x + 2y = 3a$ and $y + 2z = 3b$ for $a, b \in \mathbb{Z}$. Then, $x = 3a - 2y$ and $2z = 3b - y$. Note that

$$\begin{aligned} x + 2z &= 3a - 2y + 3b - y \\ &= 3(a - y + b). \end{aligned}$$

Since $a - y + b \in \mathbb{Z}$, it follows that $3 \mid (x + 2z)$ and so $x \; R \; z$ ($R$ is transitive). $\square$

(b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

**Solution (b).** The distinct equivalence classes are:

$$\begin{aligned} A_1 &= \{-6, 0\} = [-6] = [0] \\ A_2 &= \{5, -7\} = [-7] = [5] \\ A_3 &= \{-2, 1, 4, 7\} = [-2] = [1] = [4] = [7] \end{aligned}$$

**Problem 40.** A relation $R$ is defined on $\mathbb{Z}$ by $x\ R\ y$ if $3x - 7y$ is even. Prove that $R$ is an equivalence relation. Determine the distinct equivalence classes.

**Solution 40.** First, we show that $R$ is an equivalence relation.

*Proof.* We show that $R$ is reflexive. Consider some integer $x$. Then, $3x - 7x = -4(x) = 2(-2x)$, where $-2x \in \mathbb{Z}$ and so it is even. Hence, $x\ R\ x$.
We prove that $R$ is symmetric. Consider two integers $x$ and $y$ such that $x\ R\ y$. Hence, $3x - 7y = 2c$ for some integer $c$. Then, $3y - 7x = 2c + 10y - 10x = 2(c + 5y - 5x)$. Since $c + 5y - 5x \in \mathbb{Z}$, it follows that $3y - 7x$ is even and so $y\ R\ x$.
Now, consider three integers $x, y, z$ such that $x\ R\ y$ and $y\ R\ z$. Thus, $3x - 7y = 2a$ and $3y - 7z = 2b$ for some $a, b \in \mathbb{Z}$. Note that $(3x - 7y) + (3y - 7z) = 2a + 2b$ and so $3x - 7z = 2a + 2b + 4y = 2(a + b + y)$. Since $a + b + y \in \mathbb{Z}$, it follows that $3x - 7y$ is even and so $x\ R\ z$. $\square$

Now that it has been proven that $R$ is an equivalence relation. We proceed to determine its equivalence classes. We first determine the equivalence class for some even integer, say 0. Then

$$\begin{aligned} [0] &= \{x \in \mathbb{Z} : x\ R\ 0\} \\ &= \{x \in \mathbb{Z} : 3x - 70 \text{ is even}\} \\ &= \{x \in \mathbb{Z} : 3x \text{ is even}\} \\ &= \{x \in \mathbb{Z} : x \text{ is even}\}. \end{aligned}$$

Now, consider some odd integer, say 1. Then

$$\begin{aligned} [1] &= \{x \in \mathbb{Z} : x\ R\ 1\} \\ &= \{x \in \mathbb{Z} : 3x - 7 \text{ is even}\} \\ &= \{x \in \mathbb{Z} : 3x \text{ is odd}\} \\ &= \{x \in \mathbb{Z} : x \text{ is odd}\}. \end{aligned}$$

Therefore, there are two distinct equivalence classes, namley, the set of even integers and the set of odd ones.

**Problem 41.** contenidos...