Section 1.3: Axiom of Completeness

Juan Patricio Carrizales Torres

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I find it interesting that the author wants to follow and historical approach in the book about Real Analysis. The completness of the Real Numbers is stated as an axiom and the set $\mathbb{R}$ is defined as an ordered field. Naturally, these properties can be proven from more fundamental principles but this may be misleading and terse for a first exposure to Real Analysis. Also, once seen the most important theorems of the 1800s, one can fully appreciate the construction of $\mathbb{R}$ from $\mathbb{Q}$.

Problem 1.3.1. (a) Write a formal definition in the style of Definition 1.3.2 for the $infimum$ or *greatest lower bound* of a set.

Solution a. A real number s is the *greatest lower bound* of a set $A \subseteq \mathbb{R}$ if the following criteria are met:

1) s is a lower bound of A;
2) if b is a lower bound of A, then $b \leq s$.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution b. Assume s is some lower bound of $A \subseteq \mathbb{R}$. Then, $\inf(A) = s$ if and only if for any $\varepsilon > 0$, it is true that $a < s + \varepsilon$ for some $a \in A$.

Proof. Assume that $\inf(A) = s$. Then, s is the *greatest lower bound* of A and so $s + \varepsilon > s$, where $\varepsilon > 0$, is not a lower bound of A. Hence, $s + \varepsilon > a$ for some $a \in A$.

For the converse, let s be a lower bound for A and suposse for any $\varepsilon > 0$ that $a < s + \varepsilon$ for some $a \in A$. Since $s + \varepsilon > s$, it follows that any real number greater than s will not be an upper bound. Hence, any lower bound b of A will be lower or equal to s, and so $s = \inf A$. □

Problem 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

Solution (a). This is not possible, since $\inf B \geq \sup B$ implies that $\inf B$ is an upper bound of A and so $\inf A \geq a$ for all $a \in A$. This contradicts the fact that $\inf A \leq a$ for all $a \in A$.

(b) A finite set that contains its infimum but not its supremum.

Solution (b). This is not possible, since real numbers are ordered, a finite set will contain a greatest element x. Then $x \geq a$ for all $a \in A$ and at the same time $x \in A$, which implies that $x = \sup A$.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

Solution (c). This is possible. Let $A = \{x \in \mathbb{Q} : 2 < x \leq 4\}$. Then $A \subseteq \mathbb{Q}$. Note that $\inf A = 2 \notin A$ and $\sup A = 4 \in A$.

Problem 1.3.3. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

Proof. Since B is the set of all lower bounds for A, it follows that $a \geq b$ for every $a \in A$ and $b \in B$, and so B is bounded above. By the Axiom of Completeness, there exists some $\sup B = s$. Note that all elements of A are upper bounds of B and so $s \leq a$ for every $a \in A$ since $s = \sup B$. Hence, s is a lower bound for A. Also, $s \geq b$ for all $b \in B$. Therefore, $s = \sup B = \inf A$. $\square$

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution b. Note that for any set A that is bounded below, one can obtain the *supremum* of the set of all lower bounds of A, which we previously showed that is the $infimum$ of A. Hence, one can derive the theorem of *greatest lower bound* from the axiom of completeness.

Problem 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

Solution (a). First we show the following result:

Lemma 1. Let A_1 and A_2 be nonempty sets of real numbers such that both are bounded above. Then $\sup(A_1 \cup A_2) = \sup\{\sup A_1, \sup A_2\}$.

Proof. Since A_1 and A_2 are bounded above, it follows that $\sup A_1$ and $\sup A_2$ exist. Since real numbers are ordered, it follows that $\max\{\sup A_1, \sup A_2\} = s$ exists. By definition, s is greater than or equal to all elements of A_1 and A_2 and so it is considered and upper bound for $A_1 \cup A_2$, this also implies that $A_1 \cup A_2$ is bounded above. Note that if b is an upper bound for $A_1 \cup A_2$, then b is an upper bound for A_1 and A_2 and so $b \geq s$. Therefore, $\max\{\sup A_1, \sup A_2\} = \sup\{\sup A_1, \sup A_2\} = \sup(A_1 \cup A_2)$. $\square$

We can expand this to a finite quantity of set, namely:

Theorem 1. Let $A_1, A_2, A_3, \ldots, A_n$ be collection of $n \geq 2$ nonempty sets, each of which is bounded above. Then, $\sup\left(\bigcup_{i=1}^{n} A_k\right) = \sup\{\sup A_1, \sup A_2, \sup A_3, \ldots, \sup A_n\}$.

Proof. We proceed by induction. Since, by **Lemma 1**, $\sup(A_1 \cup A_2) = \sup\{\sup A_1, \sup A_2\}$ is true, it follows that the result is true for $n = 2$. Assume for a collection of $k \geq 2$ nonempty sets $B_1, B_2, B_3, \ldots, B_k$, each of them being bounded above, that

$$\sup\left(\bigcup_{i=1}^{k} B_i\right) = \sup\{\sup B_1, \sup B_2, \sup B_3, \ldots, \sup B_k\}.$$

We show for a collection of $k+1$ nonempty sets $C_1, C_2, C_3, \ldots, C_{k+1}$, each of them being bounded above, that

$$\sup\left(\bigcup_{i=1}^{k+1} C_i\right) = \sup\{\sup C_1, \sup C_2, \sup C_3, \ldots, \sup C_{k+1}\}.$$

Note that

$$\begin{aligned}
\sup\left(\bigcup_{i=1}^{k+1} C_i\right) &= \sup\left(\left(\bigcup_{i=1}^{k} C_i\right) \cup C_{k+1}\right) \\
&= \sup\left\{\sup\left(\bigcup_{i=1}^{k} C_i\right), \sup C_{k+1}\right\} \text{ (}\textbf{Lemma 1}\text{)} \\
&= \sup\left\{\sup\{\sup C_1, \sup C_2, \sup C_3, \ldots, \sup C_k\}, \sup C_{k+1}\right\} \text{ (}\textbf{Inductive Hypothesis}\text{).} \\
&= \sup\left\{\sup C_1, \sup C_2, \sup C_3, \ldots, \sup C_{k+1}\right\}
\end{aligned}$$

Since we are taking the *supremum* of a finite set of real numbers, namely, the *maximum* for this specific case. □

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution (b). In order for our result to extend for a general infinite case, the *supremum* of the *supremums* of the sets must exist. However, the set of *supremums* of the sets can be an inifinite set of real numbers not bounded above. This could be a counterexample.

Problem 1.3.5. As in **Example 1.3.7**, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

Proof. Consider some real number $c \geq 0$. If $c = 0$, then $cA = \{0\}$ and so $\sup(cA) = 0 = c \sup A$. Hence, we may assume that $c > 0$.
Since A is bounded above, it follows that $\sup A$ exists. Now, consider the set $cA = \{ca : a \in A\}$. Note that $a \leq \sup A$ for every $a \in A$ and so $ca \leq c \sup A$ for every $a \in A$. Therefore, cA is bounded above by $c \sup A$.

Consider some upper bound b for cA, then $ca \leq b$ for all $a \in A$. Then, $a \leq b/c$ (recall that $c > 0$). Therefore, b/c is an upper bound for A and so $\sup A \leq b/c$. This implies that $c \sup A \leq b$. However, recall that $c \sup A$ is an upper bound for cA. Thus, $\sup(cA) = c \sup A$. □

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution b. For this, we need to further assume that A is boundedd below. To illustrate this take for example the set $A = (-\infty, 0]$, which is bounded above and let $c = -1$. However, $cA = [0, \infty)$ is not bounded above and so $\sup(cA)$ does not exist.

If $c < 0$, then $\sup(cA) = c \inf A$.

Proof. Let $c < 0$. Since A is bounded belowe, it follows that $\inf A$ exists and so $a \geq \inf A$ for every $a \in A$. Note that $ca \leq c \inf A$ since $c < 0$, which implies that $c \inf A$ is an upper bound for cA.
Now, consider some upper bound b for cA. Then, $b \geq ca$ for every $a \in A$. Thus, $b/c \leq a$ since $1/c < 0$, and so b/c is a lower bound for A. Thus, $b/c \leq \inf A$ and so $b \geq c \inf A$. Thus, $c \inf A = \sup(cA)$. □

Problem 1.3.6. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

Proof. Consider some $a \in A$ and $b \in B$. We know that $a \leq s$ and $b \leq t$. Hence,

$$a + b \leq s + b \leq s + t$$

and so $a + b \leq s + t$. Since $a + b \in A + B$, it follows that $s + t$ is an upper bound for $A + B$. □

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

Proof. Since u is an upper bound for $A + B$, it follows that $a + b \leq u$ for any $a \in A$ and $b \in B$. Then, $b \leq u - a$ which implies that $u - a$, where a is any element of A, is an upper bound for B. Thus, $t \leq u - a$ for all $a \in A$. □

(c) Finally, show $\sup(A+B) = s+t$.

Proof. According to (a), the number $s+t$ is an upper bound for $A+B$. Now we just need to prove that if u is an upper bound for $A+B$, then $s+t \leq u$.
According to (b), $t \leq u-a$ for any $a \in A$ and upper bound u for $A+B$. Then, $a \leq u-t$ which implies that $u-t$ is an upper bound for A. Hence, $s \leq u-t$ and so $s+t \leq u$ for any upper bound u of $A+B$. Thus, $\sup(A+B) = s+t = \sup A + \sup B$. $\square$

(d) Construct another proof of this same fact using Lemma 1.3.8.

Proof. By **Lemma 1.3.8**, for some real number $\varepsilon > 0$, it is true that $a > s - \varepsilon/2$ for some $a \in A$ and $b > t - \varepsilon/2$ for some $b \in B$ since $\varepsilon/2 > 0$. Note that $a+b \in A+B$ and

$$\begin{aligned} a+b &> (s-\varepsilon/2)+b > (s-\varepsilon/2)+(t-\varepsilon/2) \\ &= (t+s)-\varepsilon. \end{aligned}$$

By **Lemma 1.3.8**, $s+t = \sup(A+B)$. $\square$

It's kind of interesting how sup() is multiplicative for nonnegative real numbers ($\sup(cA) = c \sup A$) and it is additive ($\sup(A+B) = \sup A + \sup B$). This implies that

$$\sup(cA + dB) = c \sup A + d \sup B$$

for nonnegative real numbers c and d, and sets $A, B \subseteq \mathbb{R}$ that are bounded above.

Problem 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Proof. Let some $c \in A$ be an upper bound for A. Consider some upper bound b for A, then, by definition, $c \leq b$ since $c \in A$. Hence, $\sup A = c$. $\square$

Before working with **problem 1.3.8**, we first prove the following lemma.

Lemma 1.3.8. Let $A = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. Then, $\sup A = b$ and $\inf A = a$.

Proof. Since $b > x$ and $a < x$ for every $x \in A$, it follows that b and a are an upper bound and lower bound for A, respectively. Consider some real number $\varepsilon > 0$, then, by the denisty of real numbers in $\mathbb{R}$, there are $x, y \in \mathbb{Q}$ such that

$$\begin{aligned} a < x < a + \varepsilon, \\ b - \varepsilon < y < b \end{aligned}$$

and $a < x, y < b$. Therefore, $\inf A = a$ and $\sup A = b$. $\square$

Problem 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $A = \{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$

Solution (a). Since $m, n \in \mathbb{N}$, it follows that $m/n \in \mathbb{Q}$. Note that $m < n$ implies that $m/n < 1$, and $0 < m, n$ implies that $0 < m/n$. Hence, $A = \{x \in \mathbb{Q} : 0 < x < 1\}$ and so 0 and 1 are a lower and upper bound for A, respectively. Consider any $\varepsilon > 0$, by the density of rational numbers, there are some $x, y \in \mathbb{Q}$ such that

$$\begin{aligned} 0 < x < 0 + \varepsilon, \\ 1 - \varepsilon < y < 1. \end{aligned}$$

and $0 < x, y < 1$. Hence, $\inf A = 0$ and $\sup A = 1$

(b) $B = \{(-1)^{m/n} : m, n \in \mathbb{N}\}$

Solution (b). Note that $(-1)^{m/n} = \left[(-1)^{1/n}\right]^m$. Recall that $(-1)^{1/n} \in \mathbb{R}$ if n is odd. Thus, $(-1)^{1/n} = -1$ for every odd integer n and so $(-1)^m \in \{-1, 1\}$ since $m \in \mathbb{N}$. Therefore, $B = \{-1, 1\}$ is a finite set and so $\inf B = -1$ and $\sup B = 1$.

(c) $C = \{n/(3n + 1) : n \in \mathbb{N}\}$

Solution (c). For any $n \in \mathbb{N}$ we have that

$$\begin{aligned} \frac{n}{3n+1} &= \frac{n}{n\left(3 + \frac{1}{n}\right)} \\ &= \frac{1}{3 + \frac{1}{n}}. \end{aligned}$$

Hence, for two positive integers m and n, if $n < m$, then $3 + \frac{1}{m} < 3 + \frac{1}{n}$ and so

$$\frac{1}{3 + \frac{1}{n}} < \frac{1}{3 + \frac{1}{m}}.$$

Hence, all elements of C can be seen as some type of strictly increasing sequence, where $\frac{1}{3+1} = \frac{1}{4}$ is the lowest element of A. Hence, $\inf A = \frac{1}{4}$.

Let $n \in \mathbb{N}$, then $n/(3n+1) < n/3n = 1/3$ and so $1/3$ is an upper limit for C. We show that it is the *supremum* of C. Consider some real number $\varepsilon > 0$. Note that

$$\varepsilon = \frac{1}{\frac{1}{\varepsilon}}$$

By the Archimedean Property, there exists some $x \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < x$ and so $0 < \frac{1}{\varepsilon} < 3(3x + 1)$. Thus,

$$\frac{1}{3(3x+1)} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Hence,

$$\frac{1}{3} - \frac{1}{3(3x+1)} > \frac{1}{3} - \varepsilon.$$

Note that

$$\begin{aligned}\frac{1}{3} - \frac{1}{3(3x+1)} &= \frac{3x+1-1}{3(3x+1)} \\ &= \frac{3x}{3(3x+1)} = \frac{x}{3x+1} \\ &> \frac{1}{3} - \varepsilon.\end{aligned}$$

Since $\frac{x}{3x+1} \in C$, it follows that $\sup C = \frac{1}{3}$.

(d) $D = \{m/(m+n) : m, n \in \mathbb{N}\}$

Solution d. Note that

$$\frac{m}{m+n} = \frac{m}{m(1+\frac{n}{m})} = \frac{1}{1+\frac{n}{m}}.$$

Since $n, m \in \mathbb{N}$, it follows that $n/m \in \mathbb{Q}$ and $n/m > 0$. Hence, $D = \left\{\frac{1}{1+x} : x \in \mathbb{Q} \text{ and } x > 0\right\}$. Note that for any rational number $x > 0$, it is true that $1 + x > 1$ and so $\frac{1}{1+x} < 1$. Thus, 1 is an upper bound for D and we proceed to show that it is the suprema of D. Consider any $\varepsilon > 0$ and so

$$\varepsilon = \frac{1}{\frac{1}{\varepsilon}}$$

. Due to the density of rational numbers, we know there exists some $\frac{1}{x} > \frac{1}{\varepsilon} > 0$ for some $x \in \mathbb{Q}$ $(\varepsilon < x)$. Therefore, $1 + \frac{1}{x} > \frac{1}{\varepsilon}$ and so

$$\frac{1}{\frac{1}{\varepsilon}} > \frac{1}{1+\frac{1}{x}}.$$

Note that

$$\begin{aligned}1 - \frac{1}{1+\frac{1}{x}} &= \frac{1+\frac{1}{x}-1}{1+\frac{1}{x}} \\ &= \frac{\frac{1}{x}}{1+\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{1}{x}(x+1)} \\ &= \frac{1}{1+x} > 1 - \frac{1}{\frac{1}{\varepsilon}} \\ &= 1 - \varepsilon.\end{aligned}$$

Thus, $\frac{1}{1+x} \in D$ and so $\sup D = 1$. Since $1/(1+x) > 0$ for any positive rational number x, it follows that 0 is a lower bound for D. We show that it is the infima of

D. Consider any $\varepsilon > 0$. By the density of rational numbers, there exists some positive rational number x such that $\varepsilon > \frac{1}{1+x} > 0$. Hence,

$$\varepsilon + 0 > \frac{1}{1+x} + 0$$

and so $\inf D = 0$ since $\frac{1}{1+x} \in D$.

The last two solutions are a little bit too laborious. Is there an easy way to solve them?. Namely, another perspective for suprema and infima of sets of real numbers rather than the one given by $\varepsilon > 0$.

Problem 1.3.9. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Since $\sup A < \sup B$, it follows that $\sup B - \sup A > 0$. By **Lemma 1.3.8**, there exists some $b \in B$ such that $b > \sup B - (\sup B - \sup A) = \sup A$. Thus, b is an upper bound for A. $\square$

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution b. Let $A = [0, 1]$ and $B = [0, 1)$. Then, $\sup A \leq \sup B$ since $\sup A = \sup B = 1$. However, since $\sup B \notin B$ and $\sup B \geq b$ for all $b \in B$, it follows that $\sup B = \sup A > b$ for every $b \in B$ which implies that no element of b is an upper bound for A (recall that $\sup A$ is the lowest upper bound).

Problem 1.3.10. (Cut Property). The *Cut Property* of the real numbers is the following: If A and B are nonempty, disjoint sets such that $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

Proof. Since $a < b$ for all $a \in A$ and $b \in B$, it follows that every $b \in B$ is an upper bound of A and so A is bounded above. By the **Axiom of Completeness**, there is some real number c that is the least upper bound of A. Hence, $c \leq x$ for every $x \in B$ and c is an upper bound of A, namely, $c \geq x$ for every $x \in A$. $\square$

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

Proof. Let E be a nonempty set of real numbers that is bounded above. Let $B = \{x \in \mathbb{R} : \forall e \in E, x \geq e\}$ and $A = B^c = \{x \in \mathbb{R} : \exists e \in E, x < e\}$. Then A and B are disjoint sets such that $A \cup B = \mathbb{R}$ and $a < b$ whenever $a \in A$ and $b \in B$. By the **Cut Property**, there exists some $c \in \mathbb{R}$ such that $c \geq x$ for every $x \in A$ and $x \geq c$ for every $x \in B$.

Note that either $c \in A$ or $c \in B$. If $c \in A$, then there is some $y \in E$ such that $c < y$. However, by the density of rational numbers, there is some rational number q such that $c < q < y$. Hence, $q \in A$, but this contradicts the fact that c is greater or equal to every element in A. Therefore, $c \in B$ and so $c \geq e$ whenever $e \in E$ which implies that c is an upper bound. Also, note that every element of B is an upper bound of E and $c \leq b$ whenever $b \in B$. Therefore, c is the lowest upper bound of E. $\square$

(c) The punchline of parts (a) and (b) is that the Cute Property could be used in place of the Zxiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid estatement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.