Section 1.5: Cardinality

Juan Patricio Carrizales Torres

May 31, 2022

This chapter introduces to the concept of cardinality and the classification of infinite sets exclusively as either countable or uncountable ones.
The term **Cardinality** accounts, hand-wavy speaking, for the "quantity" of elements of some set. This informal definition is intuitive for finite ones, however lacks clarity for infinite sets. Nevertheless, it gives some proper foundation to understand the human questioning that catalized it, namely, how can we compare the "size" of two infinite sets (Check Galileo's Paradox).

The equivalence relation $\sim$ is defined by $A \sim B$ for some sets A and B, if they have the same cardinality, namely, the same "size". By definition, two sets A and B have the same cardinality if there exists some $f : A \to B$ that is *one-to-one* and *onto*. In fact, this function $f : A \to B$ represents the **1-1 correspondence** between A and B, and so $A \sim B$. Basically, every element of B is assigned to one unique element of A and every element of A is paired with one unique element of B.

This is quite interesting since one can demonstarte that the the set of positive even integers has the same cardinality as $\mathbb{N}$!!! The parts are not necesarily "smaller" than the whole.

Just like in physical sciences, one can use some "standard" or "norm" to compare sizes between two sets, namely, $A \sim C \wedge C \sim B \implies A \sim B$ (transitive property). A very useful "norm" is the set $\mathbb{N}$. In fact, for any infinite set A, if $A \sim \mathbb{N}$, then A is **countable**. Three interesting theorems state the following:

(a) $A \subseteq C \wedge C \sim \mathbb{N} \implies A \sim \mathbb{N}$ or A is finite.

(b) For some sequence of n countable sets $A_1, A_2, A_3, \ldots, A_n$, the union $\bigcup_{i=1}^{n} A_i$ is countable.

(c) For some sequence of countable sets $\{B_n : n \in \mathbb{N}\}$, the union $\bigcup_{n\in\mathbb{N}} B_n$ is countable.

Using the **Nested Interval Property**, one can show that $\mathbb{R}$ is not countable. That's why, one concludes that the uncountability of $\mathbb{R}$ is another consequence of the **Axiom of Completeness**. Since $\mathbb{Q}$ is countable and $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is uncountable, it follows that $\mathbb{I}$ is uncountable. Otherwise, it will lead to a contradiction (The union of two countable sets is countable). This is quite interesting!!! The set of irrational numbers is of greater "size" than the set of rational ones.

Problem 1.5.1. Finish the following proof for **Theorem 1.5.7**.
Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.
Let $n_1 = min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbb{N}$ onto A.

Solution 1.5.1. By the **Well-ordering Principle**, we can define $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Now, let $n_2 = \min(\{n \in \mathbb{N} : f(n) \in A\}/\{n_1\})$. In fact, let

$$n_k = \min(\{n \in \mathbb{N} : f(n) \in A\}/\{n_1, n_2, n_3, \ldots, n_{k-1}\})$$

for any integer $k \geq 2$. We first prove that $\{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : f(n) \in A\}$.

Proof. Assume, to the contrary, that $\{n_k : k \in \mathbb{N}\} \neq \{n \in \mathbb{N} : f(n) \in A\}$. Then, either

$$\{n_k : k \in \mathbb{N}\} \not\subseteq \{n \in \mathbb{N} : f(n) \in A\} \qquad \text{or} \qquad \{n_k : k \in \mathbb{N}\} \not\supseteq \{n \in \mathbb{N} : f(n) \in A\}$$

Consider the earlier. Then, there exists some integer x such that $n_x \notin \{n \in \mathbb{N} : f(n) \in A\}$. However, this contradicts the fact that $n_x = \min(\{n \in \mathbb{N} : f(n) \in A\}/\{n_1, n_2, n_3, \ldots, n_{x-1}\})$. Hence, we may assume that $\{n_k : k \in \mathbb{N}\} \not\supseteq \{n \in \mathbb{N} : f(n) \in A\}$. Then, there is some $x \in \{n \in \mathbb{N} : f(n) \in A\}$ such that $x \notin \{n_k : k \in \mathbb{N}\}$. This implies that there is no $k \in \mathbb{N}$ such that $x = \min(\{n \in \mathbb{N} : f(n) \in A\}/\{n_1, n_2, n_3, \ldots, n_{k-1}\})$. Hence, $x > a$ for all $a \in \{n \in \mathbb{N} : f(n) \in A\}$. This means that $x \notin \{n \in \mathbb{N} : f(n) \in A\}$, which is a contradiction. □

Let the function $g : \mathbb{N} \to A$ be defined by

$$g(x) = f(n_x)$$

for every $x \in \mathbb{N}$. Then, $g : \mathbb{N} \to A$ is a 1-1 function.

Proof. Since $\{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : f(n) \in A\}$ and $f : \mathbb{N} \to B$ is biijective, it follows for every $a \in A$ that $a = f(n_l) = g(l)$ for some positive integer l (onto), and for every $k \in \mathbb{N}$, $f(n_k) = g(k)$ is equal to a unique element of A (one-to-one). □

Problem 1.5.2. Review the proof of **Theorem 1.5.6** part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:
Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \in I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

Solution 1.5.2. It is true that $\bigcap_{n=1}^{\infty} I_n = \emptyset$, however, it is also true that **NIP** is a consequence of **AoC**. In this closed intervals, we are just considering elements of $\mathbb{Q}$ and we know it is not complete (It lacks the cut property).

Problem 1.5.3. Use the following outline to supply proofs for the statements in **Theorem 1.5.8**.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example **1.5.3** (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \backslash A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?). Then explain how the more general statement in (i) follows.

Solution (a). We first show for any two countable sets A_1 and A_2 that their union is countable.

Proof. Since A_1 and A_2 are countable, it follows that there are biijective functions $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to A_2$. Let $B_2 = A_2 \backslash A_1$ and so $A_1 \cap B_2 = \emptyset$. Also, consider the following sets $S_1 = \{n \in \mathbb{N} : f(n) \in A_1\}$ and $S_2 = \{n \in \mathbb{N} : g(n) \in B_2\}$ since $B_2 \subseteq A_2$. If $B_2 = \emptyset$, then $A_1 \cup A_2 = A_1$ and so $f : \mathbb{N} \to A_1 \cup A_2$ is a biijective function and $A_1 \cup A_2$ is countable. Hence, we may assume that $B_2 \neq \emptyset$. By the **Well Ordering Principle**, $s_2 = \min(S_2)$ exists. Note that $S_1 = \mathbb{N}$ and $1 \leq s_2$.

Therefore, there are two possible cases. If B_2 is finite with k elements, then define the function $h : \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) \begin{cases} g(s_2 + (n-1)), & \text{if } n \leq k \\ f(n-k), & \text{if } n > k. \end{cases}$$

Therefore, $A_1 \cup B_2 = A_1 \cup A_2$ is countable.

If B_2 is infinite, then define some function $h : \mathbb{N} \to A_1 \cup B_2$ by

$$h(n) = \begin{cases} f(n/2), & \text{if } n \text{ is even.} \\ g(s_2 + (n-1)/2), & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $A_1 \cup B_2 = A_1 \cup A_2$ is countable. □

Maybe this proof is more complicated than it has to be, however, it's just to understand the "joints and bolts" of what's happening. We now show for $n \geq 2$ countable sets $A_1, A_2, A_3, \ldots, A_n$ that

$$\bigcup_{i=1}^{n} A_i$$

is countable.

Proof. We proceed by induction. Since the union of any two countable sets is countable, then the statement is true for $n = 2$. Suppose for $k \geq 2$ countable sets $A_1, A_2, A_3, \ldots, A_k$ that $\bigcup_{i=1}^{k} A_i$ is countable. We show for $k+1$ countable sets that $B_1, B_2, B_3, \ldots, B_{k+1}$ that $\bigcup_{i=1}^{k+1} B_i$ is countable. Note that

$$\bigcup_{i=1}^{k+1} B_i = \left(\bigcup_{i=1}^{k} B_i \right) \cup B_{k+1}.$$

Since $\bigcup_{i=1}^{k} B_i$ is a countable set according to our inductive hypothesis, it follows that we have a union of two countable sets. Therefore, $\bigcup_{i=1}^{k+1} B_i$ is countable. $\square$

(b) Explain why induction *cannot* be used to prove part (ii) of **Theorem 1.5.8** from part (i).

Solution (c). Because with induction we just can prove, in this case, that it is true for a finite quantity of countable sets (it is true for any positive integer $n \geq 2$). This always leads to finite cases.

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of **Theorem 1.5.8** (ii).

Proof. Consider some sequence of countables sets $\{A_n; n \in \mathbb{N}\}$. Since the sequence is countable and every set in it is countable, we can arrange the elements of every $S \in \{A_n; n \in \mathbb{N}\}$ into the two-dimensional array

$$\begin{array}{cccccc} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & \cdots \\ S_{21} & S_{22} & S_{23} & S_{24} & \cdots & \\ S_{31} & S_{32} & S_{33} & \cdots & & \\ S_{41} & S_{42} & \cdots & & & \\ S_{51} & \cdots & & & & \\ \vdots & & & & & \end{array},$$

where S_{ij} represents the jth element of the ith set. However, we know that we can arrange all elements of $\mathbb{N}$ in a same fashion, namely,

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}.$$

Hence, the union of the sets in $\{A_n; n \in \mathbb{N}\}$ will be at most infinitely countable. In the case that they share elements in common, we just don't count the copies, that's why we say **AT MOST** infinitely countable. $\square$

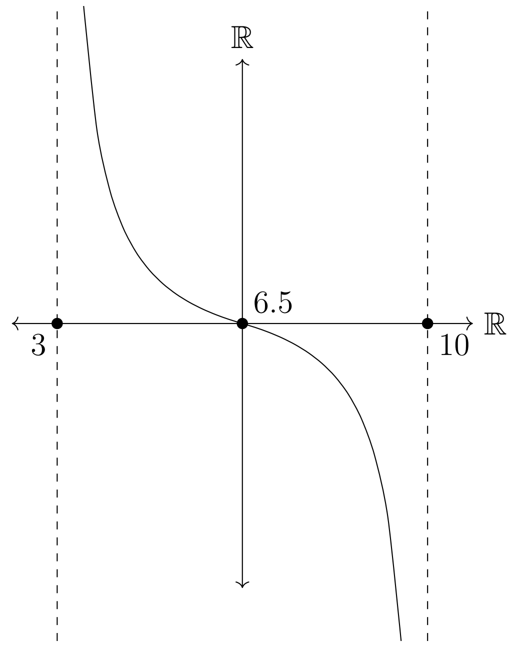

Figure 1: $(3, 10) \sim \mathbb{R}$ USING $f(x) = (x - 6.5)/(|x - 6.5| - 3.5)$

Problem 1.5.4. (a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Proof. Consider some open interval $(a, b) \subset \mathbb{R}$. Then, we can define a function $f : (a, b) \to \mathbb{R}$ that is *one-to-one* and *onto* by

$$f(x) = \frac{(x - \varphi)}{|x - \varphi| - \Delta},$$

where $\varphi = \frac{a+b}{2}$ and $\Delta = \frac{b-a}{2}$ represent the middle point and half the difference of (a, b), respectively. For example, the function $f(x) = (x - 6.5)/(|x - 6.5| - 3.5)$ takes the interval $(3, 10)$ onto $\mathbb{R}$ in a $1 - 1$ fashion (Figure (a)).

Hence, $(a, b) \sim \mathbb{R}$ for any $a, b \in \mathbb{R}$. This can be proven using a little calculus, but I don't provide it since I lack theoretical knowledge in this field. □

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof. Consider some open interval (a, ∞) for some $a \in \mathbb{R}$. Then, we can define a function $g : (a, \infty) \to \mathbb{R}$ that is *one-to-one* and *onto* by

$$g(x) = \log(x - a).$$

For example, the function $f(x) = \log(x - 5)$ takes the interval $(5, \infty)$ onto $\mathbb{R}$ in a $1 - 1$ fashion (Figure (b)).

□

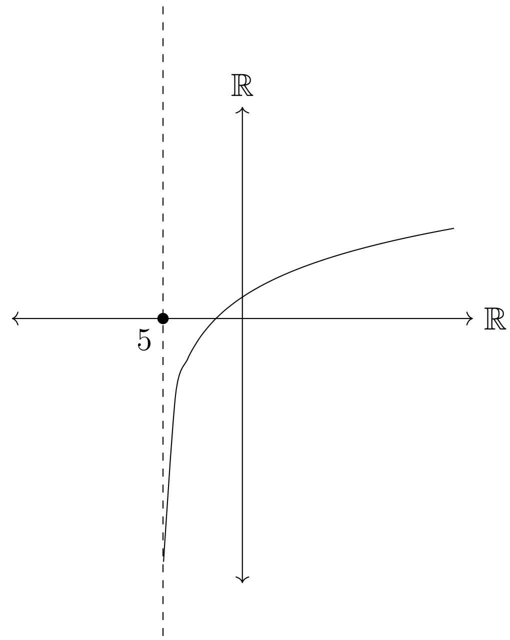

Figure 2: $(5, \infty) \sim \mathbb{R}$ USING $f(x) = \log(x - 5)$

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1)$ $(0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Proof. We know that $\mathbb{R}$ is the union of an uncountable set $\mathbb{I}$ and a countable set $\mathbb{Q}$. Let $r = \{r_n : n \in \mathbb{N}\}$ be the sequence of all rational numbers in $(0, 1)$ and let $r_0 = 0$. Then, define some function φ by

$$\varphi(x) = \begin{cases} x, \ x \in \mathbb{I}. \\ r_n, \ x = r_{n-1} \text{ for all } n \in \mathbb{N}. \end{cases}$$

Hence, $\varphi : [0, 1) \to (0, 1)$ is a 1-1 onto function and so $[0, 1)$ $(0, 1)$. $\square$

Problem 1.5.5. (a) Why is $A \sim A$ for every set A?

Solution (a). Just consider the function $g(x) = x$ for every $x \in A$ (some type of identity function) which clearly is bijective.

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Solution (b). If there is a bijective function $\varphi : A \to B$, then there is an inverse function $\varphi^{-1} : B \to A$ that is also bijective.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation.*

Solution (c). If there are bijective functions $\varphi : A \to B$ and $\gamma : B \to C$, there is a composite function $\varphi \circ \gamma : A \to C$ that is bijective.

Problem 1.5.6. (a) Give an example of a countable collection of disjoint open intervals.

Solution (a). Consider the collection $S = \{(n, n+1) : n \in \mathbb{N}\}$. Then, S is disjoint and countable.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution (b). No such collection exists. Although, I don't have a proof, I can give some hand-wavy argument of why. This has to do with the density property of rational numbers in $\mathbb{R}$. For every open interval, there will be a rational point in it and since all are disjoint, one can pair each interval with a distinct rational number. Since $\mathbb{Q}$ is countable, it follows that any collection of open intervals is countable.

Problem 1.5.7. Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

1. Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S.

   Solution (a). Let the function $\varphi : (0, 1) \to S$ be defined by

   $$\varphi(x) = (x, 0).$$

   This is clearly 1-1 but not onto.

2. Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expanison such as .235 represents the same real number as $.234999\ldots$).

   Solution (b). Consider some $(x, y) \in S$. Since $x, y \in (0, 1)$, it follows that both have unique decimal expansions that can be expressed as $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$. Then, let the function $\varphi : S \to (0, 1)$ be defined by

   $$\varphi((x, y)) = 0.x_1y_1x_2y_2x_3y_3\ldots,$$

   which clearly is 1-1 since each decimal expansion represents one unique real number. This function seems to be onto. Consider some real number $z = 0.z_1z_2z_3z_4z_5z_6\ldots$ contained in $(0, 1)$, then one can obtain two real numbers $x = 0.z_1z_3z_5\ldots$ and $y = 0.z_2z_4z_6\ldots$ such that $\varphi((x, y)) = z$. However, one may come up with some counterexample, namely $z = 0.101010101010\ldots$, where the even decimal places are 0, which implies that $(x, y) \notin S$. So φ is not onto.

Problem 1.5.9. A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

Solution (a). The number $\sqrt{2}$ is a root of x^2-2, $\sqrt[3]{2}$ is a root of x^3-2 and $\sqrt{3}+\sqrt{2}$ is a root of $-x^5+10x^3-x$.

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

Solution (b). First let's show that for any positive integer $n \geq 2$, $\mathbb{N} \sim \mathbb{N}^n$.

Proof. We proceed by induction. We know that $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$. Thus, our statement is true for $n = 2$. Assume that $\mathbb{N} \sim \mathbb{N}^k$ for some $k \geq 2$. We show that $\mathbb{N} \sim \mathbb{N}^{k+1}$. By our inductive hypothesis, there is some bijective function $f : \mathbb{N}^k \to \mathbb{N}$. Consider the set, $A = \{(f(a), b) : a \in \mathbb{N}^k, b \in \mathbb{N}\}$, which is equal to $\mathbb{N} \times \mathbb{N}$ since $\mathbb{N} \sim \mathbb{N}^k$. Hence, $A \sim \mathbb{N}$. Now, if we pair each $((a_1, a_2, \dots, a_k), b)$ with $(a_1, a_2, \dots, a_k, b)$ it follows that $\mathbb{N}^{k+1} \sim A$ and so $\mathbb{N}^{k+1} \sim \mathbb{N}$. $\square$

Note that the set P of all polynomials of degree n with integer coefficients can be shown to be correspondent with the set $\mathbb{Z}^{n+1}$ if we pair each polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $(a_n, a_{n-1}, \dots, a_1, a_0)$. Furthermore, each element $(a_n, a_{n-1}, \dots, a_1, a_0)$ can be paired with $(\varphi(a_n), \varphi(a_{n-1}), \dots, \varphi(a_1), \varphi(a_0))$ for the bijective funciton $\varphi : \mathbb{Z} \to \mathbb{N}$. Hence, $P \sim \mathbb{Z}^{n+1} \sim \mathbb{N}^{n+1} \sim \mathbb{N}$ and so $P \sim \mathbb{N}$. Therefore, the set A_n contains the roots of countably many polynomials of degree n with integer coefficients (A_n is the union of countably many sets with finite elements), and so A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution (c). Consider the collection $\{P_n : n \in \mathbb{N}\}$, where each P_n contains all algebraic numbers as roots for polynomials of degree n with integer coefficients. Then, $A = \bigcup_{n \in \mathbb{N}} P_n$ is the set of all algebraic numbers. Note that A is the union of countably many countable sets, which implies that A is countable. Since each real number is either algebraic or transcendental, it follows that $\mathbb{R} = A \cup T$, where T is the set of transcendentals. Therefore, T must be uncountable.

Problem 1.5.10. (a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

Solution (a). Choose any $a \in (0,1)$ such that $a < \sup(C)$. Since $(a, \sup(C))$ has inifinitely many uncountable elements, it follows that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is countable, and set $a = \sup A$. Is $C \cap [a,1]$ an uncountable set?

Solution (b). It is not since $\sup A = \sup C$. Therefore, either $C \cap [a,1] = \{\sup C\}$ or $C \cap [a,1] = \emptyset$ depending on wether $\sup C \in C$ or not.

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution (c). Yes, since a set being uncountable already implies that it is infinite.

Problem 1.5.11 (Schröder-Bernstein Theorem).. Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The stratgey is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

1. Explain how achieving this would lead to a proof that $X \sim Y$.

   Solution (a). By achieving this, one would show that $f : A \to B$ and $g : B' \to A'$ are bijective functions. Then, one can define the function $\varphi : X \to Y$ by

   $$\varphi(x) = \begin{cases} f(x), \ x \in A \\ g^{-1}(x), \ x \in A' \end{cases}$$

   which is 1-1 and onto. This suggests that if $A \sim B$ and $C \sim D$, then $(A \cup C) \sim (B \cup D)$.

2. Set $A_1 = X \backslash g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

   Solution (b). First we prove the following lemma:

   Lemma 1.5.11.b. Let $f : A \to B$ be a 1-1 function and A_1, A_2 be two subsets of A. Then, $f(A_1) \cap f(A_2) \neq \emptyset$ if and only if $A_1 \cap A_2 \neq \emptyset$.

Proof. Let $A_1 \cap A_2 \neq \emptyset$. Then, there is some $x \in A_1 \cap A_2$ and so $f(x) \in A_1 \cap A_2$. For the converse, assume that $f(A_1) \cap f(A_2) \neq \emptyset$. Hence, there are $f(a) \in f(A_1)$ and $f(b) \in f(A_2)$ such that $f(a) = f(b)$ for $a \in A_1$ and $b \in A_2$. Due to the injective nature of f, it follows that $a = b \in A_1 \cap A_2$. □

We now proceed to show the required result:

Proof. Suppose, to the contrary, that there are distinct positive integers k, p such that $A_k \cap A_p \neq \emptyset$. Note that $A_1 \cap A_n = \emptyset$ for any integer $n \geq 2$ since $A_1 = X \backslash g(Y)$ and $A_n \subseteq g(Y)$. Then, we may further assume that $k, p \geq 2$. Hence, $g\left(f\left(A_{k-1}\right)\right) \cap g\left(f\left(A_{p-1}\right)\right) \neq \emptyset$. Note that $g \circ f$ is injective and so, by **Lemma 1.5.11.b**, $A_{k-1} \cap A_{p-1} \neq \emptyset$. We can use the same argument recursively in a finite fashion, until we get to the statement that $A_1 \cap A_n \neq \emptyset$ for some $n \geq 2$, which is a contradiction. Therefore, the collection $\{A_n : n \in \mathbb{N}\}$ is pairwise disjoint. Also, by **Lemma 1.5.11.b**, the collection $\{f(A_n) : n \in \mathbb{N}\}$ is pairwise disjoint. □

3. Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

   Proof. Consider some $b \in B$. Since $\bigcup_{n=1}^{\infty} f(A_n)$ is a union of pairwise disjoint sets, it follows that $b \in f(A_k)$ for one unique $k \in \mathbb{N}$. Thus, there is some $a \in A_n$ such that $f(a) = b$. Hence, $f : A \to B$ is onto. □

4. Let $A' = X \backslash A$ and $B' = Y \backslash B$. SHow g maps B' onto A'.

   Proof. Note that $X/g(Y) \subseteq A$ and so A' is a partition of $g(Y)$. Assume, to the contrary, that there is some $a \in A'$ such that for all $b \in B'$, $g(b) \neq a$. Hence, there is some $c \in B$, where $c \in f(A_n)$ for some positive integer n. However, this implies that $g(c) \in A_{n+1}$, which contradicts the fact that $A' = X \backslash A$. Thus, $g : B' \to A'$ is onto. □