Week 3

Juan Patricio Carrizales Torres

Section 6: The Biconditional

August 18, 2021

Problem 35. Let P : 18 is odd. and Q : 15 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?

Solution . $P \Leftrightarrow Q$: The integer 18 is odd if and only if 15 is even.
This biconditional is true since both statements are false.

Problem 36. Let $P(x)$: x is odd. and $Q(x)$: x^2 is odd. be open sentences over the domain $\mathbb{Z}$. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "If and only if" and (2) using "necessary and sufficient".

Solution . $P(x) \Leftrightarrow Q(x)$: The integer x is odd if and only if x^2 is odd.
$P(x) \Leftrightarrow Q(x)$: The condition x is odd is necessary and sufficient for x^2 is odd.

Problem 37. For the open sentences $P(x)$: $|x-3| < 1$. and $Q(x)$: $x \in (2,4)$. over the domain $\mathbb{R}$, state the biconditional $P(x) \Leftrightarrow Q(x)$ in two different ways.

Solution . $P(x) \Leftrightarrow Q(x)$: The real number $|x-3| < 1$ is equivalent to $x \in (2,4)$.
$P(x) \Leftrightarrow Q(x)$: The condition $|x-3| < 1$ is necessary and sufficient for $x \in (2,4)$.

Problem 38. Consider the open sentences:

$$P(x) : x = -2. \text{ and } Q(x) : x^2 = 4.$$

over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statements are true.

(A) $\sim P(x)$: The integer $x \neq -2$.
$\sim P(0)$ and $\sim P(2)$ are true statements, while $\sim P(-2)$ is false.

(B) $P(x) \vee Q(x)$: Either $x = -2$ or $x^2 = 4$.
$P(-2) \vee Q(-2)$ and $P(2) \vee Q(-2)$ are true, while $P(0) \vee Q(0)$ is false.

(C) $P(x) \wedge Q(x)$: The integer $x = -2$ and $x^2 = 4$.
Just $P(-2) \wedge Q(-2)$ is true, while $P(0) \wedge Q(0)$ and $P(2) \wedge Q(2)$ are false.

(D) $P(x) \Rightarrow Q(x)$: If $x = -2$, then $x^2 = 4$.
The implication is true for all $x \in S$.

(E) $Q(x) \Rightarrow P(x)$: If $x^2 = 4$, then $x = -2$.
The implications $Q(-2) \Rightarrow P(-2)$ (both premise and conclusion true) and $Q(0) \Rightarrow P(0)$ (both premise and conclusion false) are true, while $Q(2) \Rightarrow P(2)$ (premise true and conclusion false) is false.

(F) $P(x) \Leftrightarrow Q(x)$: The integer $x = -2$ if and only if $x^2 = 4$.
Both $(P(-2) \Rightarrow Q(-2)) \wedge (Q(-2) \Rightarrow P(-2))$ and $(P(0) \Rightarrow Q(0)) \wedge (Q(0) \Rightarrow P(0))$ are true, while $(P(2) \Rightarrow Q(2)) \wedge (Q(2) \Rightarrow P(2))$ is false since $Q(2) \Rightarrow P(2)$ is false. Therefore, $P(-2) \Leftrightarrow Q(-2)$ and $P(0) \Leftrightarrow Q(0)$ are true, while $P(2) \Leftrightarrow Q(2)$ is false.

Problem 39. For the following open sentences $P(x)$ and $Q(x)$ over a domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.

Before we start with the excercises, it is important to remark that $P(x) \Leftrightarrow Q(x)$ is true only for those values that make both open statements $P(x)$ and $Q(x)$ have the same truth value.

(A) $P(x) : |x| = 4;\ Q(x) : x = 4;\ S = \{-4, -3, 1, 4, 5\}$.

Solution a. The statement $P(-4)$ is true, while $Q(-4)$ is false. Therefore, $P(-4) \Leftrightarrow Q(-4)$ is false.
Both $P(-3)$ and $Q(-3)$ are false. Thus, $P(-3) \Leftrightarrow Q(-3)$ is true.
Both $P(1)$ and $Q(1)$ are false. The biconditional $P(1) \Leftrightarrow Q(1)$ is true.
Since $P(4)$ and $Q(4)$ are true, $P(4) \Leftrightarrow Q(4)$ is true.
Both $P(5)$ and $Q(5)$ are false. Therefore, $P(5) \Leftrightarrow Q(5)$ is true.
The biconditional $P(x) \Leftrightarrow Q(x)$ is true for all $x \in S$ except -4.

(B) $P(x) : x \geq 3;\ Q(x) : 4x - 1 > 12;\ S = \{0, 2, 3, 4, 6\}$.

Solution b. Both $P(0)$ and $Q(0)$ are false, which means that $P(0) \Leftrightarrow Q(0)$ is true.
Both $P(2)$ and $Q(2)$ are false. Therefore, $P(2) \Leftrightarrow Q(2)$ is true.
The statement $P(3)$ is true, but $Q(3)$ is false. Thus, $P(3) \Leftrightarrow Q(3)$ is false.
Since $P(4)$ and $Q(4)$ are true, $P(4) \Leftrightarrow Q(4)$ is true.
Both $P(6)$ and $Q(6)$ are true. Thus, $P(6) \Leftrightarrow Q(6)$ is true.
The biconditional $P(x) \Leftrightarrow Q(x)$ is true for all $x \in S$ except 3.

(C) $P(x) : x^2 = 16;\ Q(x) : x^2 - 4x = 0;\ S = \{-6, -4, 0, 3, 4, 8\}$.

Solution c. Since $P(-6)$ and $Q(-6)$ are false, $P(-6) \Leftrightarrow Q(-6)$ is true.
The statement $P(-4)$ is true, but $Q(-4)$ is false. Therefore, $P(-4) \Leftrightarrow Q(-4)$ is false.
The statement $P(0)$ is false, while $Q(0)$ is true. Thus, $P(0) \Leftrightarrow Q(0)$ is false.
Both $P(3)$ and $Q(3)$ are false, which means that $P(3) \Leftrightarrow Q(3)$ is true.
Both $P(4)$ and $Q(4)$ are true. Therefore, $P(4) \Leftrightarrow Q(4)$ is true. Since $P(8)$ and $Q(8)$ are false, $P(8) \Leftrightarrow Q(8)$ is false.
The biconditional $P(x) \Leftrightarrow Q(x)$ is true just for all $x \in S$ except -4 and 0.

Problem 40. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.

(A) $P(x, y) : x^2 - y^2 = 0$ and; $Q(x, y) : x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.

Solution a. For $(x, y) = (1, -1)$, $P(1, -1) : 0 = 0$; $Q(1, -1) : 1 = -1$.
The statement $P(1, -1)$ is true, but $Q(1, -1)$ is false. Since both have different truth values, $P(1, -1) \Leftrightarrow Q(1, -1)$ is false.

For $(x, y) = (3, 4)$, $P(3, 4) : -7 = 0$; $Q(3, 4) : 3 = 4$.
Both $P(3, 4)$ and $Q(3, 4)$ are false. The biconditional $P(3, 4) \Leftrightarrow Q(3, 4)$ is therefore true.

For $(x, y) = (5, 5)$, $P(5, 5) : 0 = 0$; $Q(5, 5) : 5 = 5$.
Since $P(5, 5)$ and $Q(5, 5)$ are true, $P(5, 5) \Leftrightarrow Q(5, 5)$ is true.

(B) $P(x, y) : |x| = |y|$ and; $Q(x, y) : x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.

Solution b. For $(x, y) = (1, 2)$, $P(1, 2) : 1 = 2$; $Q(1, 2) : 1 = 2$.
Both $P(1, 2)$ and $Q(1, 2)$ are false. Thus, $P(1, 2) \Leftrightarrow Q(1, 2)$ is true.

For $(x, y) = (2, -2)$, $P(2, -2) : 2 = 2$; $Q(2, -2) : 2 = -2$.
Since $P(2, -2)$ is true and $Q(2, -2)$ is false, $P(2, -2) \Leftrightarrow Q(2, -2)$ is false.

For $(x, y) = (6, 6)$, $P(6, 6) : 6 = 6$; $Q(6, 6) : 6 = 6$.
The statements $P(6, 6)$ and $Q(6, 6)$ are true, which means that $P(6, 6) \Leftrightarrow Q(6, 6)$ is true.

(C) $P(x, y) : x^2 + y^2 = 1$ and; $Q(x, y) : x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

Solution c. For $(x, y) = (1, -1)$, $P(1, -1) : 2 = 1$; $Q(1, -1) : 0 = 1$.
Both $P(1, -1)$ and $Q(1, -1)$ are false. The biconditional $P(1, -1) \Leftrightarrow Q(1, -1)$ is therefore true.

For $(x, y) = (-3, 4)$, $P(-3, 4) : 25 = 1$; $Q(-3, 4) : 1 = 1$.
The statements have different truth values, because $P(-3, 4)$ is false and $Q(-3, 4)$ is true. This means that $P(-3, 4) \Leftrightarrow Q(-3, 4)$ is false.

For $(x, y) = (0, -1)$, $P(0, -1) : 1 = 1$; $Q(0, -1) : -1 = 1$.
The statement $P(0, -1)$ is true, but $Q(0, -1)$ is false. Therefore, $P(0, -1) \Leftrightarrow Q(0, -1)$ is false.

For $(x, y) = (1, 0)$, $P(1, 0) : 1 = 1$; $Q(1, 0) : 1 = 1$.
Since $P(1, 0)$ and $Q(1, 0)$ are true, $P(1, 0) \Leftrightarrow Q(1, 0)$ is true.

Problem 41. Determine all values of n in the domain $S = \{1, 2, 3\}$ for which the following is a true statement:
A necessary and sufficient condition for $\frac{n^3+n}{2}$ to be even is that $\frac{n^2+n}{2}$ is odd.

Solution . This open sentence is the biconditional $P(n) \Leftrightarrow Q(n)$ of these two open sentences:

$$P(n) : \tfrac{n^2+n}{2} \text{ is odd.} \quad Q(n) : \tfrac{n^3+n}{2} \text{ is even.}$$

We are now ready to start with the excercise.

For $n = 1$. $P(1) : 1$ is odd; $Q(1) : 1$ is even.
The statement $P(1)$ is true and $Q(1)$ is false. Therefore, $P(1) \Leftrightarrow Q(1)$ is false.

For $n = 2$. $P(2) : 3$ is odd; $Q(2) : 5$ is even.
The statement $P(2)$ is true, but $Q(2)$ is false. The biconditional $P(2) \Leftrightarrow Q(2)$ is false.

For $n = 3$. $P(3) : 6$ is odd; $Q(3) : 15$ is even.
Since $P(3)$ and $Q(3)$ are false, $P(3) \Leftrightarrow Q(3)$ is true.
Of all $n \in S$, only for $n = 3$ the biconditional $P(x) \Leftrightarrow Q(x)$ is true.

Problem 42. Determine all values of n in the domain $S = \{2, 3, 4\}$ for which the following is a true statement:
The integer $\frac{n(n-1)}{2}$ is odd if and only if $\frac{n(n+1)}{2}$ is even.

Solution . This biconditional $P(n) \Leftrightarrow Q(n)$ is composed of the two open statements:

$$P(n) : \tfrac{n(n-1)}{2} \text{ is odd.} \quad Q(n) : \tfrac{n(n+1)}{2} \text{ is even.}$$

For $n = 2$, $P(2) : 1$ is odd; $Q(2) : 3$ is even.
The statement $P(2)$ is true, while $Q(2)$ is false. The biconditional $P(2) \Leftrightarrow Q(2)$ is therefore false.

For $n = 3$, $P(3) : 3$ is odd; $Q(3) : 6$ is even.
Both $P(3)$ and $Q(3)$ are true. Since they have the same truth value, $P(3) \Leftrightarrow Q(3)$ is true.

For $n = 4$, $P(4) : 6$ is odd, $Q(4) : 10$ is even.
Both statements have different truth values, because $P(4)$ is false and $Q(4)$ is true. The biconditional $P(4) \Leftrightarrow Q(4)$ is false.
From the values of $n \in S$, the biconditional $P(n) \Leftrightarrow Q(n)$ is true for $n = 3$.

Problem 43. Let $S = \{1, 2, 3\}$. Consider the following open sentences over the domain S:

$$P(n) : \tfrac{(n+4)(n+5)}{2} \text{ is odd.} \quad Q(n) : 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

Solution . For $n = 1$, $P(1) : 15$ is odd; $Q(1) : 1 > 1$.
The statement $P(1)$ is true, while $Q(1)$ is false. Therefore, the implication $P(1) \Rightarrow Q(1)$ is false.

For $n = 2$, $P(2) : 21$ is odd; $Q(2) : 3 > 2.5$.
Both $P(2)$ and $Q(2)$ are true. Since these two statements have the same truth value, $P(2) \Leftrightarrow Q(2)$ is true.

For $n = 3$, $P(3) : 28$ is odd; $Q(3) : 11 > 6.25$.
Since $P(3)$ is false and $Q(3)$ is true, the implication $Q(3) \Rightarrow P(3)$ is false.
The integer $a = 1$, $b = 3$ and $c = 2$.

Problem 44. Let $S = \{1, 2, 3, 4\}$. Consider the following open sentences over the domain S:

$$P(n) : \frac{n(n-1)}{2} \text{ is even.}$$
$$Q(n) : 2^{n-2} - (-2)^{n-2} \text{ is even.}$$
$$R(n) : 5^{n-1} + 2^n \text{ is prime.}$$

Determine four distinct elements a, b, c, d in S such that
(i) $P(a) \Rightarrow Q(a)$ is false.
(ii) $Q(b) \Rightarrow P(b)$ is true.
(iii) $P(c) \Leftrightarrow R(c)$ is true.
(iv) $Q(d) \Leftrightarrow R(d)$ is false.

Solution . For $n = 1$, $P(1) : 0$ is even; $Q(1) : 1$ is even; $R(1) : 3$ is prime.
The statements $P(1)$ and $R(1)$ are true, while $Q(1)$ is false. Since the statements have these specific truth values, the following statements and their respective truth values resemble the ones we are looking for:
$P(1) \Rightarrow Q(1)$ is false.
$Q(1) \Rightarrow P(1)$ is true.
$P(1) \Leftrightarrow R(1)$ is true.
$Q(1) \Leftrightarrow R(1)$ is false.

For $n = 2$, $P(2) : 1$ is even; $Q(2) : 0$ is even; $R(2) : 9$ is prime.
The statements $P(2)$ and $R(2)$ are false, while $Q(2)$ is true. Therefore:
$P(2) \Leftrightarrow R(2)$ is true.
$Q(2) \Leftrightarrow R(2)$ is false.

For $n = 3$, $P(3) : 3$ is even; $Q(3) : 4$ is even; $R(3) : 33$ is prime.
The statements $P(3)$ and $R(3)$ are false, while $Q(3)$ is true. Therefore:
$P(3) \Leftrightarrow R(3)$ is true.

$Q(3) \Leftrightarrow R(3)$ is false.

For $n = 4$, $P(4) : 6$ is even; $Q(4) : 0$ is even; $R(4) : 141$ is prime.
The statements $P(4)$ and $Q(4)$ are true, while $R(4)$ is false. Therefore:
$Q(4) \Rightarrow P(4)$ is true.
$Q(4) \Leftrightarrow R(4)$ is false.
There are two posible solutions. Either $a = 1, b = 4, c = 2, d = 3$ or
$a = 1, b = 4, c = 3, d = 2$.

Problem 45. Let $P(n) : 2^n - 1$ is a prime. and $Q(n) : n$ is a prime. be open sentences over the domain $S = \{2, 3, 4, 5, 6, 11\}$. Determine all values of $n \in S$ for which $P(n) \Leftrightarrow Q(n)$ is a true statement.

Solution . For $n = 2$, $P(2) : 3$ is a prime; $Q(2) : 2$ is a prime.
Both $P(2)$ and $Q(2)$ are true. Therefore, $P(2) \Leftrightarrow Q(2)$ is true.

For $n = 3$, $P(3) : 7$ is a prime; $Q(3) : 3$ is a prime.
The statements $P(3)$ and $Q(3)$ are true. Thus, $P(3) \Leftrightarrow Q(3)$ is true.

For $n = 4$, $P(4) : 15$ is a prime; $Q(4) : 4$ is a prime.
Both $P(4)$ and $Q(4)$ are false. Therefore, $P(4) \Leftrightarrow Q(4)$ is true.

For $n = 5$, $P(5) : 31$ is a prime; $Q(5) : 5$ is a prime.
Since $P(5)$ and $Q(5)$ are true, $P(5) \Leftrightarrow Q(5)$ is true.

For $n = 6$, $P(6) : 63$ is a prime; $Q(6) : 6$ is a prime.
Both $P(6)$ and $Q(6)$ are false. Therefore, $P(6) \Leftrightarrow Q(6)$ is true.

For $n = 11$, $P(11) : 2047$ is a prime; $Q(11) : 11$ is a prime.
The statement $P(11)$ is false, while $Q(11)$ is true. The biconditional $P(11) \Leftrightarrow Q(11)$ is false.
The biconditional $P(n) \Leftrightarrow Q(n)$ is true for all $n \in S - \{11\}$.