Section 8.2: Properties of relations

Juan Patricio Carrizales Torres

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This chapter mentioned three properties of interested for some relation R on a single set A. Since most of these properties involve implications with universal quantifiers, the easiest way to check wether a relation has certain property is by looking for specific examples for which the implication in question is false.

(a) **Reflexive Property:** if $x \in A$, then $(x, x) \in R$. (x is related to itself)

(b) **Symmetric Property:** $\forall x, y \in A$, if $x \; R \; y$, then $y \; R \; x$ (x is related to y and viceversa). Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$ and $y \not{R} \; x$. For this to happen, it is necessary that $x \neq y$.

(c) **Transitive Property:** $\forall x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$. Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$, $y \; R \; z$ and $x \not{R} \; z$. For this to happen, it is necessary that $x \neq y$ and $z \neq y$.

Problem 11. Let $A = \{a, b, c, d\}$ and let

$$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$$

be a relation on A. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 11. The relation is reflexive since $\{(a, a), (b, b), (c, c), (d, d)\} \subset R$. Also, it is transitive since $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in A$ is fulfilled. However, the relation is not symmetric since $(a, b) \in R$ and $(b, a) \notin R$.

Problem 13. Let $S = \{a, b, c\}$. Then $R = \{(a, b)\}$ is a relation on S. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 13. The relation S is transitive since the implication $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in S$ is fulfilled vacuously. However, it is neither reflexive because $(a, a) \notin R$ nor symmetrice since $(a, b) \in R$ but $(b, a) \notin R$.

Problem 14. Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

Solution 14. Let $R = \{(a,b),(b,c)\}$. The relation R is not reflexive since $(a,a) \notin R$, it is not symmetric because $(a,b) \in R$ and $(b,a) \notin R$ and it is not transitive since $(a,b),(b,c) \in R$ but $(a,c) \notin R$.

Problem 15. A relation R is defined on $\mathbb{Z}$ by $a\ R\ b$ if $|a-b| \leq 2$. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 15. The relation R is reflexive since $|a-a| = 0 \leq 2$ for any $a \in \mathbb{Z}$ and so $a\ R\ a$. It is symmetric since for any $a, b \in \mathbb{Z}$, if $|a-b| \leq 2$, then $|b-a| = |a-b| \leq 2$. However, it is not transitive since $|3-1| = 2$ and $|1-0| = 1$ but $|3-0| = 3 > 2$.

Problem 16. Let $A = \{a,b,c,d\}$. How many relations defined on A are reflexive, symmetric and transitive and contain the ordered pairs $(a,b),(b,c),(c,d)$?

Solution 16. contenidos...