Week 13

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Section 1: Counterexamples

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A quantified statement of the type $\forall x \in S, R(x)$ can be **disproved** (proved to be false) by showing that $\sim (\forall x \in S, R(x)) \equiv \exists x \in S, \sim R(x)$ is true. If $\forall x \in S, R(x)$ is false, then there is some $x \in S$ for which the open sentence $R(x)$ is false, namely a **counterexample**. Therefore, the truth value of $\forall x \in S, R(x)$ not only depends on the open sentence $R(x)$ but also on the domain S.

Problem 1. Disprove the statement: If a and b are any two real numbers, then $\log(ab) = \log(a) + \log(b)$.

Solution . Let $a \leq 0$ and $b > 0$. Then $\log(ab)$ and $\log(a)$ are not defined (The domain over the open sentence influences on the truth value of the quantified statement).

Problem 2. Disprove the statement: If $n \in \{0, 1, 2, 3, 4\}$, then $2^n + 3^n + n(n-1)(n-2)$ is prime.

Solution . If $n = 4$, then $2^n + 3^n + n(n-1)(n-2) = 121$ is not a prime number. Therefore, $n = 4$ is a counterexample

Problem 3. Disprove the statement: If $n \in \{1, 2, 3, 4, 5\}$, then $3 \mid (2n^2 + 1)$.

Solution . Since $3 \nmid (2(3)^2 + 1)$, it follows that $n = 3$ is a counterexample.

Problem 4. Disprove the statement: Let $n \in \mathbb{N}$. If $\frac{n(n+1)}{2}$ is odd, then $\frac{(n+1)(n+2)}{2}$ is odd.

Solution . Let $n = 2(2k+1)$ where $k \in \mathbb{N}$. Then

$$\begin{aligned}\frac{n(n+1)}{2} &= \frac{2(2k+1)(2(2k+1)+1)}{2} \\ &= (2k+1)(4k+3) = 8k^2 + 10k + 3 = 2(4k^2 + 5k + 1) + 1\end{aligned}$$

Since $4k^2 + 5k + 1 \in \mathbb{N}$, it follows that $\frac{n(n+1)}{2}$ is odd for this values of n. Then,

$$\begin{aligned}\frac{(n+1)(n+2)}{2} &= \frac{(2(2k+1)+1)(2(2k+1)+2)}{2} = \frac{2(2k+2)(2(2k+1)+1)}{2} \\ &= (2k+2)(2(2k+1)+1) = 2(k+1)(2(2k+1)+1)\end{aligned}$$

The positive integer $2(k+1)(2(2k+1)+1)$ is even. Thus, all $n = 2(2k+1)$ where $k \in \mathbb{N}$ are counterexamples.

Problem 5. Disprove the statement: For every two positive integers a and b, $(a+b)^3 = a^3 + 2a^2b + 2ab + 2ab^2 + b^3$.

Solution . Let $a, b \in \mathbb{Z}$ such that $a > 0$ and $b > 0$. Note that

$$\begin{aligned}(a+b)^3 &= (a^2 + 2ab + b^2)(a+b)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Then, let's check for which values of a and b, $a^3+2a^2b+2ab+2ab^2+b^3 = a^3+3a^2b+3ab^2+b^3$ holds.

$$\begin{aligned}a^3 + 2a^2b + 2ab + 2ab^2 + b^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ 2a^2b + 2ab + 2ab^2 &= 3a^2b + 3ab^2\\ 2ab &= a^2b + ab^2\\ ab(2) &= ab(a+b)\end{aligned}$$

Since $a > 0$ and $b > 0$, $ab > 0$ and so we can divide both sides by ab. Then, $2 = a + b$. Therefore, all those positive integers a and b such that $a+b \neq 2$ are counterexamples, namely, $a \neq 1$ or $b \neq 1$.

Problem 6. Let $a, b \in \mathbb{Z}$. Disprove the statement: If ab and $(a+b)^2$ are of opposite parity, then a^2b^2 and $a + ab + b$ are of opposite parity.

Solution . Let a and b be odd integers. Then ab is odd (multiplication of two odd integers) and $a+b$ is even (sum of two odd integers); so $(a+b)^2$ is even. The hypothesis is true. Note that $(ab)^2 = a^2b^2$ is odd (multiplication of two odd integers) and $(a+b)+ab$ is odd (sum of an even and odd integer). They are of the same parity. Therefore, all integers a and b such that both are odd will be counterexamples.

Problem 7. For positive real numbers a and b, it can be shown that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$. If $a = b$, then this inequality is an equality. Consider the following statement:
If a and b are positive real numbers such that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 4$, then $a = b$. Is there a counterexample to this statement?

Solution . If we can show that the previous result is true, then there will be no counterexample. Let $a, b \in \mathbb{R}$ such that $a > 0$, $b > 0$ and $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 4$. Note that,

$$\begin{aligned}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) &= 4\\ 1 + \frac{b}{a} + \frac{a}{b} + 1 &= 4\\ \frac{b}{a} + \frac{a}{b} &= 2\\ a^2 + b^2 &= 2ab\\ a^2 - 2ab + b^2 &= 0\\ (a-b)(a-b) &= 0\end{aligned}$$

Since $(a-b)(a-b)=0$ and $(a-b)=(a-b)$, it follows by *Theorem 4.13* that $a-b=0$; so $a=b$. Therefore, the result is proven to be true and so there are no counterexamples.

Problem 8. In Exercise 7, it is stated that $(a+b)(\frac{1}{a}+\frac{1}{b}) \geq 4$ for every two positive real numbers a and b. Does it therefore follows that $(c^2+d^2)\left(\frac{1}{c^2}+\frac{1}{d^2}\right) \geq 4^2$ for every two positive real numbers c and d?

Solution . Since $c, d \in \mathbb{R}^+$, it follows that $c^2, d^2 \in \mathbb{R}^+$ and so it is true that $(c^2+d^2)\left(\frac{1}{c^2}+\frac{1}{d^2}\right) \geq 4$. However let's check whether $(c^2+d^2)\left(\frac{1}{c^2}+\frac{1}{d^2}\right) \geq 4^2$ holds. Note that,

$$\begin{aligned}
(c^2+d^2)\left(\frac{1}{c^2}+\frac{1}{d^2}\right) &\geq 4^2 \\
1+\frac{d^2}{c^2}+\frac{c^2}{d^2}+1 &\geq 16 \\
\frac{d^2}{c^2}+\frac{c^2}{d^2} &\geq 14 \\
d^4+c^4 &\geq 14c^2d^2 \\
d^4-14c^2d^2+c^4 &\geq 0 \\
(d^2-c^2)^2-12c^2d^2 &\geq 0
\end{aligned}$$

Thus all $c, d \in \mathbb{R}^+$ such that $(d^2-c^2)^2 < 12c^2d^2$ will be counterexamples. If $c=d$, then $c^2=d^2$; so $(d^2-c^2)^2=0$ and $12c^2d^2>0$. Then $(d^2-c^2)^2<12c^2d^2$. Therefore, all $c, d \in \mathbb{R}^+$ such that $c=d$ will be counterexamples. Something that we already knew from the previous problem, namely, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=4$ if and only if $a=b$.

Problem 9. Disprove the statement: For every positive integer x and every integer $n \geq 2$, the equation $x^n+(x+1)^n=(x+2)^n$ has no solution.

Solution . A very famous counterexample is $3^2+4^2=5^2$ ($x=3$ and $n=2$). By Fermat's Last Theorem, all posible counterexamples will only be found in the cases where $n=2$.