Week ??

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Section 2: A More General Principle of Mathematical Induction

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Previously, we were able to prove statements of the form $\forall n \in \mathbb{N}, P(n)$ thanks to the Well-ordering Principle and Principle of Mathematical Induction, both concerning the set $\mathbb{N}$. However, consider the case where we want to prove that some open sentence $P(n)$ is true for all $n \in S$, where $S = \{i \in \mathbb{Z} : i \geq m \in \mathbb{Z}\}$ (note that m is some **fixed** integer). Luckily, we can show that this arbitrary S is well-ordered (It will be shown in **Problem 17**). This fact will be usefull to prove the following general Pinciple of Mathematical Induction

Theorem 1. Let $S = \{i \in \mathbb{Z} : i \geq m\}$ where m is some fixed integer. Also, let $P(n)$ be some open sentence over S. If

1. $P(m)$ and
2. $\forall k \in S, P(k) \implies P(k+1)$,

then $\forall n \in S, P(n)$.

Proof. Let $P(n)$ be an open sentence over S, where $S = \{i \in \mathbb{Z} : i \geq m\}$ for some fixed integer m. Then, assume, to the contrary, that there is some nonempty set $T \subseteq S$ of contradictions for the statement $\forall n \in S, P(n)$ and that both $P(m)$ and $\forall k \in S, P(k) \implies P(k+1)$ are true. Then, there is at least some $x \in T$ and $x > m$ since $P(m)$ is true and m is the lowest element of S. Since $T \subseteq S$ and S is well-ordered, it follows that T has some lowest element y. Because y is the lowest element in T, the integer $y - 1 \notin T$ and so $y - 1 \geq m$. Therefore, $y - 1 \in S$ and so $P(y-1)$ is true. However, $P(y)$ must also be true since $\forall k \in S, P(k) \implies P(k+1)$ is true. This contradicts our initial assumption that y was the lowset element of T. $\square$

Of course, if $m = 1$, then $S = \mathbb{N}$ and we have our loved **Principle of Mathematical Induction** for the positive integers.

Problem 17. Prove Theorem 7: For each integer m, the set $S = \{i \in \mathbb{Z} : i \geq m\}$ is well-ordered.

Proof. Let $S = \{i \in \mathbb{Z} : i \geq m\}$ for some fixed integer m. Let's consider some nonempty subset T of S. We must show that every nonempty subset of S has a least element. If $T \subseteq \mathbb{N}$, then T has a least element since $\mathbb{N}$ is well-ordered. On the other hand, we may assume that

$T \not\subseteq \mathbb{N}$. Thus, $T - \mathbb{N}$ is a finite nonempty set and so it has a least element t. Therefore, $t \leq 0$ and so $x \geq t$ for every $x \in T$; so t is the least element of T. We may conclude that any nonempty subset T of S has a least element and so S is well-ordered. $\square$

Problem 18. Prove that $2^n > n^3$ for every integer $n \geq 10$.

Proof. We proceed by induction. Note that $2^{10} = 2^3 \cdot (2^7) = 2^3 \cdot (128) > 2^3 \cdot (125) = 2^3 \cdot 5^3 = 10^3$. Therefore, the statement is true for $n = 10$. Assume that $2^k > k^3$ for some integer $k \geq 10$. We now show that $2^{k+1} > (k+1)^3$. Note that

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &> k^3 \cdot 2 = k^3 + k^3 \\ &\geq k^3 + 10k^2 = k^3 + 3k^2 + 7k^2 \\ &\geq k^3 + 3k^2 + 70k = k^3 + 3k^2 + 3k + 67 \\ &\geq k^3 + 3k^2 + 3k + 1 = (k+1)^3 \end{aligned}$$

By the principle of mathematical induction, $2^n > n^3$ for every integer $n \geq 10$. $\square$

Problem 19. Prove the following implication for every integer $n \geq 2$: If $x_1, x_2, \ldots, x_n$ are any n real numbers such that $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 0$, then at least one of the numbers $x_1, x_2, \ldots, x_n$ is 0. (Use the fact that if the product of two real numbers is 0, then at least one of the numbers is 0).

Proof. We use induction. Consider two arbitrary real numbers m, n such that $m \cdot n = 0$. By **Theorem 4.13**, either $m = 0$ or $n = 0$ and so the statement is true for 2 real numbers. Let $k \geq 2$ be some integer. Assume that if $a_1, a_2, \ldots, a_k$ are some k arbitrary real numbers such that $a_1 \cdot a_2 \cdot \ldots \cdot a_k = 0$, then at least one of them is 0. We show that if $b_1 \cdot b_2 \cdot \ldots \cdot b_{k+1} = 0$ for some arbitrary real numbers $b_1, b_2, \ldots, b_{k+1}$, then at least one of the $k+1$ real numbers is 0.

First, let $b_1, b_2, \ldots, b_{k+1}$ be some arbitrary $k+1$ real numbers such that $b_1 \cdot b_2 \cdot \ldots \cdot b_{k+1} = 0$. Note that

$$b_1 \cdot b_2 \cdot \ldots \cdot b_{k+1} = [b_1 \cdot b_2 \cdot \ldots \cdot b_k] \cdot b_{k+1} = 0$$

By **Theorem 4.13**, either $b_1 \cdot b_2 \cdot \ldots \cdot b_k = 0$ or $b_{k+1} = 0$. If $b_{k+1} = 0$, then at least one the $k+1$ real numbers is 0. On the other hand, if $b_1 \cdot b_2 \cdot \ldots \cdot b_k = 0$, then, by our inductive hypothesis, at least one of the k real numbers is 0 and so at least one of the $k+1$ real numbers is 0.

By the principle of mathematical induction, this result is true. $\square$

Problem 20. Use mathematical induction to prove that every finite nonempty set of real numbers has a largest element.

Proof. We proceed by induction. Let $A = \{a\}$ for some $a \in \mathbb{R}$. Then $a \geq x$ for all $x \in A$ and so a is the largest element of A. Thus, the result is true for some set A of real numbers with cardinality 1.
Now, assume that every set B of real numbers with cardinality k, where the integer $k \geq 1$, has some element b such that $b \geq x$ for all $x \in B$. We show that the set C of real numbers with cardinality $k+1$ has a largest element.
Let C be a set of real numbers with cardinality $k+1$. Consider the subset $D = (C - \{m\})$ for some $m \in C$. Since $|D| = k$, it follows, by the inductive hypothesis, that there is some $c \in D$ such that $c \geq x$ for all $x \in D$. If $c \geq m$, then c is the largest element of C. On the other hand, if $m > c$, then m is the largest element of C. Therefore, the set C has a largest element in both cases.
By the principle of mathematical induction, every finite nonempty set of real numbers has a largest element. □

(b) Use (a) to prove that every finite nonempty set of real numbers has a smallest element.

Proof. Let A be a set of real numbers with cardinality k ($k \in \mathbb{N}$). Consider the set $B = \{a : -a \in A\}$. Since B is a finite nonempty set of real numbers, it follows, by the previous result, that there is some $m \in B$ such that $m \geq x$ for every $x \in B$. Then $-m \in A$ and so $-m \leq -x$ for all $-x \in A$. Thus, the number $-m$ is the smallest element of A. □

Problem 21. Prove that $4 \mid (5^n - 1)$ for every nonnegative integer n.

Proof. We proceed by induction. Since $5^0 - 1 = 0 = 4 \cdot 0$, it follows that $4 \mid (5^0 - 1)$ and so the statement is true for $n = 0$.
Assume that $4 \mid (5^k - 1)$ for some integer $k \geq 0$. Thus, $5^k - 1 = 4c$, where $c \in \mathbb{Z}$, and so $5^k = 4c + 1$. We then show that $4 \mid (5^{k+1} - 1)$. Note that

$$\begin{aligned} 5^{k+1} - 1 &= 5^k \cdot 5 - 1 \\ &= (4c+1) \cdot 5 - 1 = 20c + 5 - 1 \\ &= 4(5c+1) \end{aligned}$$

Since $(5c+1) \in \mathbb{Z}$, it follows that $4 \mid (5^{k+1} - 1)$.
The result follows by the Principle of Mathematical Induction. □

Problem 22. Prove that $3^n > n^2$ for every positive integer n.

Proof. We use induction. Since $3^1 = 3 > 1 = 1^2$, the result is true for $n = 1$. Assume that $3^k > k^2$ where $k \in \mathbb{N}$. We then show that $3^{k+1} > (k+1)^2$. If $k = 1$, then $3^{k+1} = 9 > 4 = (k+1)^2$. Therefore, we may assume that $k \geq 2$. Observe that

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3 \\ &> k^2 \cdot 3 = k^2 + 2k^2 \\ &\geq k^2 + 4k = k^2 + 2k + 2k \geq k^2 + 2k + 4 \\ &> k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

The result follows by the Principle of Mathematical Induction. □

Problem 23. Prove that $7 \mid (3^{2n} - 2^n)$ for every nonnegative integer n.

Proof. We proceed by induction. Since $3^{2\cdot 0} - 2^0 = 0 = 7 \cdot 0$, it follows that $7 \mid (3^{2n} - 2^n)$ for $n = 0$.
Assume that $7 \mid (3^{2k} - 2^k)$ where k is a nonnegative integer. Thus, $3^{2k} - 2^k = 7c$ for some integer c and so $3^{2k} = 7c + 2^k$. We then show that $7 \mid (3^{2(k+1)} - 2^{k+1})$. Observe that

$$\begin{aligned}
3^{2(k+1)} - 2^{k+1} &= 3^{2k} \cdot 3^2 - 2^{k+1} \\
&= 3^2(7c + 2^k) - 2^k \cdot 2 \\
&= 3^2 \cdot 7c + 3^2 \cdot 2^k - 2^k \cdot 2 \\
&= 3^2 \cdot 7c + 2^k(3^2 - 2) = 3^2 \cdot 7c + 2^k \cdot 7 \\
&= 7(3^2 c + 2^k)
\end{aligned}$$

Because $(3^2c + 2^k) \in \mathbb{Z}$, it follows that $7 \mid (3^{2(k+1)} - 2^{k+1})$. By the Principle of Mathematical Induction, $7 \mid (3^{2n} - 2^n)$ for any nonnegative integer n. □

Problem 24. Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer n,

$$(1 + x)^n \geq 1 + nx.$$

Proof. We proceed by induction. Let $x > -1$ be some real number. Since $(1+x)^1 = 1+x \geq 1 + 1x$, it follows that the statement is true for $n = 1$. Assume that $(1 + x)^k \geq 1 + kx$ for some $k \in \mathbb{N}$. We then prove that $(1 + x)^{k+1} \geq 1 + (k + 1)x$. Observe that

$$(1 + x)^{k+1} = (1 + x)^k(1 + x).$$

Because $x + 1 > 0$, it follows that

$$\begin{aligned}
(1 + x)^k(1 + x) &\geq (1 + kx)(1 + x) \\
&= 1 + kx + x + kx^2 = 1 + (k + 1)x + kx^2.
\end{aligned}$$

Note that $x^2 \geq 0$ and so $kx^2 \geq 0$. Therefore,

$$(1 + x)^{k+1} \geq 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x.$$

By the principle of mathematical induction, the Bernoulli's Identity holds. □

Problem 25. Prove that $n! > 2^n$ for every integer $n \geq 4$.

Proof. We use induction to prove this. Observe that $4! = 24 > 16 = 2^4$ and so $n! > 2^n$ for $n = 4$. Suppose that $k! > 2^k$ for some integer $k \geq 4$. We then show that

$$(k + 1)! > 2^{k+1}.$$

Note that

$$\begin{aligned}
(k + 1)! &= (k + 1)k! \\
&> 2^k(k + 1)
\end{aligned}$$

since $k+1>0$. Because $k+1 \geq 5$, it follows that

$$\begin{aligned}(k+1)! &> 2^k(k+1)\\ &\geq 2^k \cdot 5\\ &> 2^k \cdot 2 = 2^{k+1}\end{aligned}$$

By the principle of mathematical induction, $n! > 2^n$ holds for any integer $n \geq 4$. $\square$

Problem 26. Prove that $81 \mid (10^{n+1} - 9n - 10)$ for every nonnegative integer n.

Proof. We use induction to prove this statement. Since $10^{0+1} - 9 \cdot 0 - 10 = 0 = 0 \cdot 81$, it follows that $81 \mid (10^{0+1} - 9 \cdot 0 - 1)$ and so the statement holds for $n = 0$. We then assume that $81 \mid (10^{k+1} - 9k - 10)$ for any nonnegative integer k. Thus, $10^{k+1} - 9k - 10 = 81c$, where $c \in \mathbb{Z}$, and so $10^{k+1} = 81c + 9k + 10$. We proceed to show that $81 \mid (10^{(k+1)+1} - 9(k+1) - 10)$. Note that

$$\begin{aligned}10^{(k+1)+1} - 9(k+1) - 10 &= 10^{k+1} \cdot 10 - 9k - 9 - 10\\ &= 10(81c + 9k + 10) - 9k - 19\\ &= 81(10c) + 90k - 9k + 100 - 19\\ &= 81(10c) + 81k + 81\\ &= 81(10c + k + 1)\end{aligned}$$

Because $(10c + k + 1) \in \mathbb{Z}$, it follows that $81 \mid (10^{(k+1)+1} - 9(k+1) - 10)$.
By the principle of mathematical induction, $81 \mid (10^{n+1} - 9n - 10)$ for every nonnegative integer n. $\square$

Problem 27. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every positive integer n.

Proof. We proceed by induction. Since $1 \leq 1 = 2 - 1$, the inequality holds for $n = 1$. We assume that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

for any $k \in \mathbb{N}$. We then show that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}.$$

Observe that

$$
\begin{aligned}
1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{(k+1)^2} &= 1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \\
&\leq 2-\frac{1}{k}+\frac{1}{(k+1)^2} \\
&= 2-\frac{(k+1)^2-k}{k(k+1)^2} \\
&= 2-\frac{k^2+2k+1-k}{k(k+1)^2} \\
&= 2-\frac{k^2+k+1}{k(k+1)^2} = 2-\frac{k(k+1)+1}{k(k+1)^2} \\
&= 2-\frac{1}{k+1}-\frac{1}{k(k+1)^2} \\
&< 2-\frac{1}{k+1}
\end{aligned}
$$

since $\frac{1}{k(k+1)^2} > 0$. Therefore,

$$1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{(k+1)^2} \leq 2-\frac{1}{k+1}.$$

By the principle of mathematical induction, $1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for every $n \in \mathbb{N}$. $\square$

Problem 28. Assume that if $3 \mid 2a$, where $a \in \mathbb{Z}$, then $3 \mid a$ is true. Prove the following generalization: Let $a \in \mathbb{Z}$. For every positive integer n, if $3 \mid 2^n a$, then $3 \mid a$.

Proof. We proceed by induction. Let $P(n) : 3 \mid 2^n a \implies 3 \mid a$. By the assumption given in the problem, the statement is true for $n = 1$. Let $k \in \mathbb{N}$ and suppose that $P(k)$ is true. We show that $P(k+1)$ is true. Assume that $3 \mid (2^{k+1}a)$. Then, $2^{k+1}a = 3m$ for some $m \in \mathbb{Z}$. Observe that

$$2^{k+1}a = 2(2^k a) = 3m.$$

Therefore, $3 \mid (2(2^k a))$. Since $2^k a \in \mathbb{Z}$, it follows, by the assumption from the problem, that $3 \mid (2^k a)$. By the induction hypothesis, it follows that $3 \mid a$. Thus, $P(k+1) : 3 \mid (2^{k+1}a) \implies 3 \mid a$ is true. By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$ $\square$

Problem 29. Prove that if $A_1, A_2, \ldots, A_n$ are any $n \geq 2$ sets, then

$$\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$$

Proof. We proceed by induction. Let A_1, A_2 be arbitrary sets. By the Morgan's Laws, $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ and so the statement is true for $n = 2$. Assume that if $B_1, B_2, \ldots, B_k$ are $k \geq 2$ arbitrary sets, then

$$\overline{B_1 \cap B_2 \cap \ldots \cap B_k} = \overline{B_1} \cup \overline{B_2} \cup \ldots \cup \overline{B_k}.$$

We show that for the arbitrary sets $C_1, C_2, \ldots, C_{k+1}$, the equality

$$\overline{C_1 \cap C_2 \cap \ldots \cap C_{k+1}} = \overline{C_1} \cup \overline{C_2} \cup \ldots \cup \overline{C_{k+1}}$$

holds. Note that

$$\begin{aligned}\overline{C_1 \cap C_2 \cap \ldots \cap C_{k+1}} &= \overline{C_1 \cap C_2 \cap \ldots \cap C_k \cap C_{k+1}} \\ &= \overline{\bigcap_{x=1}^{k} C_x \cap C_{k+1}}.\end{aligned}$$

By the Morgan's Laws,

$$\begin{aligned}\overline{\bigcap_{x=1}^{k} C_x \cap C_{k+1}} &= \overline{\bigcap_{x=1}^{k} C_x} \cup \overline{C_{k+1}} \\ &= \overline{C_1 \cap C_2 \cap \ldots \cap C_k} \cup \overline{C_{k+1}} \\ &= \overline{C_1} \cup \overline{C_2} \cup \ldots \cup \overline{C_k} \cup \overline{C_{k+1}}.\end{aligned}$$

By the principle of mathematical induction, this result is true. □

Problem 30. Recall...

Lemma 1. For integers $n \geq 2, a, b, c, d$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then both $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$

Use this lemma and mathematical induction to prove the following: For any $2m$ integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ for which $a_i \equiv b_i \pmod{n}$ for $1 \leq i \leq m$,

1. $a_1 + a_2 + \ldots + a_m \equiv b_1 + b_2 + \ldots + b_m \pmod{n}$ and

   Proof. We proceed by induction. Let $x_1, x_2, y_1, y_2, n \in \mathbb{Z}$ such that $n \geq 2$ and $x_i \equiv y_i \pmod{n}$ for $1 \leq i \leq 2$. By **Lemma 1**, $x_1 + x_2 \equiv y_1 + y_2 \pmod{n}$ and so the result is true for $m = 2$. Now, for any $2k$ integers, where $k \geq 2$, $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ for which $a_i \equiv b_i \pmod{n}$ for $1 \leq i \leq k$, assume that

   $$a_1 + a_2 + \ldots + a_k \equiv b_1 + b_2 + \ldots + b_k \pmod{n}.$$

   We show that for any $2(k + 1)$ integers $c_1, c_2, \ldots, c_{k+1}$ and $d_1, d_2, \ldots, d_{k+1}$ for which $c_i \equiv d_i \pmod{n}$ for $1 \leq i \leq k + 1$, the equality

   $$c_1 + c_2 + \ldots + c_{k+1} \equiv d_1 + d_2 + \ldots + d_{k+1} \pmod{n}$$

   holds. By the inductive hypothesis,

   $$c_1 + c_2 + \ldots + c_k \equiv d_1 + d_2 + \ldots + d_k \pmod{n}.$$

   Also, we know that

   $$c_{k+1} \equiv d_{k+1} \pmod{n}.$$

Since both $\sum_{x=1}^{k} c_x$ and $\sum_{x=1}^{k} d_x$ are also integers, it follows, by **Lemma 1**, that

$$\sum_{x=1}^{k} c_x + c_{k+1} \equiv \sum_{x=1}^{k} d_x + d_{k+1} \pmod{n} \text{ and so}$$
$$c_1 + c_2 + \ldots + c_{k+1} \equiv d_1 + d_2 + \ldots + d_{k+1} \pmod{n}.$$

By the principle of mathematical induction, this result is true. □

2. $a_1 a_2 \cdots a_m \equiv b_1 b_2 \cdots b_m \pmod{n}$.

Proof. We proceed by induction. Let $x_1, x_2, y_1, y_2, n \in \mathbb{Z}$ such that $n \geq 2$ and $x_i \equiv y_i \pmod{n}$ for $1 \leq i \leq 2$. By **Lemma 1**, $x_1 x_2 \equiv y_1 y_2 \pmod{n}$ and so the result is true for $m = 2$. Now, for any $2k$ integers, where $k \geq 2$, $a_1, a_2 \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ for which $a_i \equiv b_i \pmod{n}$ for $1 \leq i \leq k$, assume that

$$a_1 a_2 \cdots a_k \equiv b_1 b_2 \cdots b_k \pmod{n}.$$

We show that for any $2(k+1)$ integers $c_1, c_2, \ldots, c_{k+1}$ and $d_1, d_2, \ldots, d_{k+1}$ for which $c_i \equiv d_i \pmod{n}$ for $1 \leq i \leq k+1$, the equality

$$c_1 c_2 \cdots c_{k+1} \equiv d_1 d_2 \cdots d_{k+1} \pmod{n}.$$

holds. By the inductive hypothesis,

$$c_1 c_2 \cdots c_k \equiv d_1 d_2 \cdots d_k \pmod{n}.$$

Also we know that

$$c_{k+1} \equiv d_{k+1} \pmod{n}.$$

By **Lemma 1**,

$$\left(\prod_{x=1}^{k} c_x\right) c_{k+1} \equiv \left(\prod_{x=1}^{k} d_x\right) d_{k+1} \pmod{n} \text{ and so}$$
$$c_1 c_2 \cdots c_{k+1} \equiv d_1 d_2 \cdots d_{k+1} \pmod{n}.$$

The result follows by the Principle of Mathematical Induction. □

Problem 31. Lemma 2. Let x and y be positive real numbers. Then,

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof. Let x and y be positive real numbers. We consider two cases.
Case 1. $x = y$. Therefore, $\frac{x}{y} + \frac{y}{x} = 1 + 1 = 2$.

Case 2. Either $x > y$ or $y > x$. Without loss of generality, assume the later. Then, $y = x + c$, where c is a positive real number. Therefore,

$$\begin{aligned}
\frac{x}{y} + \frac{y}{x} &= \frac{x}{x+c} + \frac{x+c}{x} \\
&= \frac{x}{x+c} + \frac{x}{x} + \frac{c}{x} \\
&= 1 + \frac{x^2 + c(x+c)}{x(x+c)} \\
&= 1 + \frac{x^2 + xc + c^2}{x^2 + xc} > 2
\end{aligned}$$

since

$$\frac{x^2 + xc + c^2}{x^2 + xc} > 1.$$

Therefore, the result is true. $\square$

Result 31. Prove for every $n \geq 1$ positive real numbers $a_1, a_2, \ldots, a_n$ that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \geq n^2.$$

Proof. We use induction. Let x be some positive real number. Because $\frac{x}{x} = 1 \geq 1^2$, the statement is true for $n = 1$. Suppose for every $k \geq 1$ positive real numbers $a_1, a_2, \ldots, a_k$ that

$$\left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \geq k^2.$$

We prove for any $k+1$ positive real numbers $b_1, b_2, \ldots, b_{k+1}$ that

$$\left(\sum_{i=1}^{k+1} b_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \geq (k+1)^2.$$

Note that

$$
\begin{aligned}
\left(\sum_{i=1}^{k+1} b_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) &= b_1\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) + b_2\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) + \ldots + b_{k+1}\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \\
&= \left(\sum_{i=1}^{k} b_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) + b_{k+1}\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \\
&= \frac{1}{b_1}\left(\sum_{i=1}^{k} b_i\right) + \frac{1}{b_2}\left(\sum_{i=1}^{k} b_i\right) + \ldots + \frac{1}{b_{k+1}}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \\
&= \left(\sum_{i=1}^{k} b_i\right)\left(\sum_{i=1}^{k} \frac{1}{b_i}\right) + \frac{1}{b_{k+1}}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \\
&\geq k^2 + \frac{1}{b_{k+1}}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) \\
&= k^2 + \frac{1}{b_{k+1}}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k} \frac{1}{b_i}\right) + \frac{b_{k+1}}{b_{k+1}} \\
&= k^2 + \sum_{i=1}^{k}\left(\frac{b_i}{b_{k+1}} + \frac{b_{k+1}}{b_i}\right) + 1
\end{aligned}
$$

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By **Lemma 2**, $\frac{b_i}{b_{k+1}} + \frac{b_{k+1}}{b_i} \geq 2$ for $1 \leq i \leq k$ and so

$$
\begin{aligned}
\left(\sum_{i=1}^{k+1} b_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{b_i}\right) &\geq k^2 + \sum_{i=1}^{k}\left(\frac{b_i}{b_{k+1}} + \frac{b_{k+1}}{b_i}\right) + 1 \\
&\geq k^2 + 2k + 1 = (k+1)^2
\end{aligned}
$$

Therefore, the result follows by the Principle of Mathematical Induction. □

Problem 32. Prove for every $n \geq 2$ positive real numbers $a_1, a_2, \ldots, a_n$ that

$$(n-1)\sum_{i=1}^{n} a_i^2 \geq 2 \sum_{1 \leq i < j \leq n} a_i a_j.$$

Proof. We proceed by induction. Let x and y be two positive real numbers. Note that, according to the result,

$$
\begin{aligned}
(2-1)(x^2+y^2) &= x^2 + y^2 \\
&\geq 2xy
\end{aligned}
$$

and so

$$
\begin{aligned}
x^2 - 2xy + y^2 &= (x-y)^2 \\
&\geq 0.
\end{aligned}
$$

Thus, the result is true for $n = 2$. Assume for $k \geq 2$ positive real numbers $a_1, a_2, \ldots, a_k$ that

$$(k-1)\sum_{i=1}^{k} a_i^2 \geq 2 \sum_{1 \leq i < j \leq k} a_i a_j.$$

We now show for $k+1$ positive real numbers $b_1, b_2, \ldots, b_{k+1}$ that

$$((k+1)-1)\sum_{i=1}^{k+1} b_i^2 \geq 2 \sum_{1 \leq i < j \leq k+1} b_i b_j.$$

Observe that

$$\begin{aligned}
((k+1)-1)\sum_{i=1}^{k+1} b_i^2 &= (k-1)\sum_{i=1}^{k} b_i^2 + (k-1)b_{k+1}^2 + \sum_{i=1}^{k} b_i^2 \\
&= (k-1)\sum_{i=1}^{k} b_i^2 + \sum_{i=1}^{k} b_i^2 + k b_{k+1}^2 \\
&\geq 2 \sum_{1 \leq i < j \leq k} b_i b_j + \sum_{i=1}^{k} b_i^2 + k b_{k+1}^2 \\
&= 2 \sum_{1 \leq i < j \leq k} b_i b_j + \sum_{i=1}^{k} (b_i^2 + b_{k+1}^2).
\end{aligned}$$

Since

$$\begin{aligned}
b_i^2 + b_{k+1}^2 &= (b_i - b_{k+1})^2 + 2b_i b_{k+1} \\
&\geq 2b_i b_{k+1}
\end{aligned}$$

for $1 \leq i \leq k$, it follows that

$$\begin{aligned}
((k+1)-1)\sum_{i=1}^{k+1} b_i^2 &\geq 2 \sum_{1 \leq i < j \leq k} b_i b_j + \sum_{i=1}^{k} (b_i^2 + b_{k+1}^2) \\
&\geq 2 \sum_{1 \leq i < j \leq k} b_i b_j + 2\sum_{i=1}^{k} b_i b_{k+1} \\
&= 2 \sum_{1 \leq i < j \leq k+1} b_i b_j.
\end{aligned}$$

By the Principle of Mathematical Induction, the result is true. $\square$