Section 8.3: Equivalence Relations

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In this section, the concept of **Equivalence Relation** on some set A is introduced. In short words, an **Equivalence Relation** on some set A is one that has is reflexive, symmetric and transitive. One of the best examples is the relation R defined by $x\ R\ y$ if $x = y$. Also, an important subset to understand the behavior of these type of relations is the **equivalence class**. Basically, an **equivalence class** $[a]$ contains all elements $x \in A$ that are related to some specific $a \in A$, namely,

$$[a] = \{x \in A : x\ R\ a\}$$

Note that if $b \in [a]$ (b is related to a), then b and a are "equivalent". Note that $a \in [b]$ and $[b] = [a]$ due to the symmetric and transitive properties of R. Quite interesting!!!

Lemma 8.3.1. Let R be an equivalence relation on an nonempty set A. Then, $a\ R\ b$ for some $a, b \in A$ is a necessary and sufficiente condition for $[a] = [b]$.

Proof. Because R is reflexive, $a \in [a]$ and $b \in [b]$ and so they are nonempty. Consider some $x \in [a]$, then $x\ R\ a$. Note that $a\ R\ b$ and so, by the transtitive property of R, $x\ R\ b$. Hence, $x \in [b]$ which implies that $[a] \subseteq [b]$.

Now consider some $y \in [b]$ and so $y\ R\ b$. Since R is symmetric and $a\ R\ b$, it follows that $b\ R\ a$. Thus, by the transitive property, $y\ R\ a$ and so $y \in [a]$. Therefore, $[b] \subseteq [a]$ and so $[a] = [b]$.

For the converse, assume that $[a] = [b]$. Since R is reflexive, it follows that $a \in [a]$ and so $a \in [b]$. Hence, $a\ R\ b$. □

Note that this implies that the union of all equivalence classes of A is A itself!!!

Corollary 8.3.1. Let R be an equivalence relation on an nonempty set A and consider some $a, b \in A$. Then, $[a] = [b]$ if and only if $a, b \in [a]$.

Proof. Assume that $[a] = [b]$. By **Lemma 8.3.1** and the symmetric property of R, $a\ R\ b$ and $b\ R\ a$. Therefore, $a \in [b]$ and $b \in [a]$, and so $a, b \in [a] = [b]$.

For the converse, suppose that $a, b \in [a]$. Then, $a\ R\ b$. By **Lemma 8.3.1**, $[a] = [b]$. □

Lemma 8.3.2. Let R be an equivalence relation on an nonempty set A such that $\{[a]_i : i \leq n\}$ is the set of all equivalence classes for some $n \in \mathbb{N}$ (finite quantity of equivalence classes). Then,

$$\bigcup_{i=1}^{n} [a_i] = A$$

Proof. Suppose, to the contrary, that

$$\bigcup_{i=1}^{n}[a_i] \neq A.$$

Hence, either

$$\bigcup_{i=1}^{n}[a_i] \not\subseteq A \quad \text{or} \quad \bigcup_{i=1}^{n}[a_i] \not\supseteq A.$$

Suppose the first. Then, there exists some $x \in \bigcup_{i=1}^{n}[a_i]$ such that $x \notin A$. This implies that $x \in [a_k]$ for some positive integer k. However, $x \notin A$ and this contradicts the fact that $[a_k] = \{x \in A : x \, R \, a_k\}$.

Thus, we can assume that $\bigcup_{i=1}^{n}[a_i] \not\supseteq A$. Then, there is some $y \in A$ such that $y \notin \bigcup_{i=1}^{n}[a_i]$. Because $\{[a]_i : i \leq n\}$ is the set of all equivalence classes by R, it follows that $y \not R \, a$ for any $a \in A$. Hence $(y, y) \notin R$. However, this contradicts the fact that R is reflexive.

Thus,

$$\bigcup_{i=1}^{n}[a_i] = A.$$

$\square$

Problem 24. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a \, R \, c$, $c \, R \, d$, $d \, R \, g$ and $b \, R \, f$. If there are three distinct equivalence classes resulting from R, then determine these equivalence classes and determine all elements of R.

Solution 24. By repetitive use of **Lemma 8.3.1**, we conclude that $[a] = [c] = [d] = [g]$ and $[b] = [f]$. Also, since e is not related to any element of A, it follows that the remaining equivalence class is $[e]$. Note that the reflexive property of R implies that $g \, R \, g$ and $f \, R \, f$. Therefore, by the transitive property,

$$\begin{aligned}
&[g] = \{a, g, d, c\} = [a] = [c] = [d] \\
&[f] = \{b, f\} = [b] \\
&[e] = \{e\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
R = \{&(a, a), (g, a), (d, a), (c, a), (a, c), (g, c), (d, c), \\
&(c, c), (a, d), (g, d), (d, d), (c, d), (a, g), (g, g), \\
&(d, g), (c, g), (b, b), (f, b), (b, f), (f, f), (e, e)\}.
\end{aligned}$$

This is a taste of how useful equivalence classes can be. Wow!!!

Problem 25. Let $A = \{1, 2, 3, 4, 5, 6\}$. The relation

$$\begin{aligned}
R = \{&(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), \\
&(3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}
\end{aligned}$$

is an equivalence relation on A. Determine the distinct equivalence classes.

Solution 25. Since R is an equivalence relation on A, then we can use **Lemma 1.8.3** to determine the equivalence classes. Note that $(1,1),(5,1),(2,2),(6,2),(3,2),(4,4) \in R$. Hence,

$$\begin{aligned}
[1] &= \{1,5\} = [5] \\
[2] &= \{2,6,3\} = [3] = [6] \\
[4] &= \{4\}
\end{aligned}$$

Problem 26. Let $A = \{1,2,3,4,5,6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1,4,5\},\{2,6\}$ and $\{3\}$. What is R?

Solution 26. We now that

$$\{1,4,5\} = [1] = [4] = [5]\{2,6\} \qquad = [2] = [6]\{3\} = [3]$$

The relation

$$R = \{(1,1),(4,1),(5,1),(1,4),(4,4),(5,4)\}$$