Section 8.3: Equivalence Relations

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This chapter reviews some properties that we realized and proved in the problems of **Section 8.3**. However, there's something worth noting. Let R be some relation on some nonempty set A. I previously showed that the union of the equivalence classes by R is A and they all are pairwise disjoint. Nevertheless, I didn't ponder on it much to realize what this meant, namely, that the set of these distinct equivalence classes is a partition of A!!!! This was proven by the authors by just showing that each $x \in A$ belongs to exaclty one equivalence class by R.

Problem 36. Give an example of an equivalence relation R on the set $A = \{v, w, x, y, z\}$ such that there are exactly three distinct equivalence classes. What are the equivalence classes for your example?

Solution 36. Consider the parition $P = \{\{v\}, \{w\}, \{x, y, z\}\}$ of A. By **Theorem 4**, the relation R definded by $a \; R \; b$ if $a, b \in X$ for some $X \in P$ is an equivalence relation. Hence, the distinct equivalance classe are

$$\begin{aligned} a_1 &= \{x, y, z\} \\ a_2 &= \{w\} \\ a_3 &= \{v\} \end{aligned}$$

Problem 37. A relation R is defined on $\mathbb{N}$ by $a \; R \; b$ if $a^2 + b^2$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Proof. We first prove that R is an equivalence relation. Consider some positive integer c. Then, $c^2 + c^2 = 2c^2$. Since c^2 is an integer, it follows that $2c^2$ is even and so $c \; R \; c$. Hence, R is reflexive. Let $a, b \in \mathbb{N}$. By the commutative property of sums on real numbers, it follows that if $a^2 + b^2$ is even, then $b^2 + a^2$ is equal to the same even number. Therefore, $a \; R \; b$ implies $b \; R \; a$ and so R is symmetric. Consider $x, y, z \in \mathbb{Z}$ such that $x \; R \; y$ and $y \; R \; z$. Hence, $x^2 + y^2 = 2m$ and $y^2 + z^2 = 2n$ for $m, n \in \mathbb{Z}$. Thus, $x^2 = 2m - y^2$ and $z^2 = 2n - y^2$. Therefore,

$$\begin{aligned} x^2 + z^2 &= (2m - y^2) + (2n - y^2) \\ &= 2m + 2n - 2y^2 = 2(m + n - y^2). \end{aligned}$$

Because $m+n-y^2 \in \mathbb{Z}$, it follows that x^2+z^2 is even and so $x\ R\ z$, which implies that R is transitive.

Once R is shown to be an equivalence relation, we now determine the dsitinct equivalence classes. Let x be an even positive integer. Then x^2 is even. Consider some $y \in \mathbb{N}$. Note that y^2+x^2 is even if and only if y^2 is even. We also know that y^2 is even if and only if y is even. Therefore,

$$[x] = \{n \in \mathbb{N} : n \text{ is even}\}.$$

Consider positive integers y and z. If y is and odd positive integer, then z^2+y^2 is odd if and only if z^2 is odd. Hence, z must be odd.

$$[y] = \{n \in \mathbb{N} : n \text{ is odd}\}.$$

Since the set of even and odd positive integers is a partition of $\mathbb{N}$, it follows that there only two distinct equivalence classes. $\square$

Problem 38. Let R be a relation defined on the set $\mathbb{N}$ by $a\ R\ b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is and equivalence relation.

Solution 38. The relation R on $\mathbb{N}$ is not an equivalence relation. Consider the positive integers 2, 3 and 5. Since $2 \mid (2 \cdot 3)$ and $2 \mid (2 \cdot 5)$, it follows that $3\ R\ 2$ and $2\ R\ 5$. However, $3 \nmid (2 \cdot 5)$ and $5 \nmid (2 \cdot 3)$. Hence, $3 \not\!R\ 5$ and so R is not transitive. This implies that R is not an equivalence relation.

Problem 39. Let S be a nonempty subset of $\mathbb{Z}$ and let R be a relation defined on S by $x\ R\ y$ if $3 \mid (x+2y)$.

(a) Prove that R is an equivalence relation.

Proof. Let S be some nonempty subset of $\mathbb{Z}$ and R some relation on S defined by $x\ R\ y$ if $3 \mid (x+2y)$. For some integer $x \in S$, $x+2x = 3x$ and so $3 \mid 3x$. Hence, $x\ R\ x$ is reflexive.
Let $x, y \in S$ such that $x\ R\ y$. Hence, $x+2y = 3c$ for some integer c. Then, $x = 3c-2y$ and so

$$\begin{aligned} y+2x &= y+2(3c-2y) \\ &= y+6c-4y \\ &= 3(2c-y). \end{aligned}$$

Since $2c-y \in \mathbb{Z}$, it follows that $3 \mid (y+2x)$ and so $y\ R\ x$ (R is symmetric).
Consider some $x, y, z \in S$ sucht that $x\ R\ y$ and $y\ R\ z$. Therefore, $x+2y = 3a$ and $y+2z = 3b$ for $a, b \in \mathbb{Z}$. Then, $x = 3a-2y$ and $2z = 3b-y$. Note that

$$\begin{aligned} x+2z &= 3a-2y+3b-y \\ &= 3(a-y+b). \end{aligned}$$

Since $a-y+b \in \mathbb{Z}$, it follows that $3 \mid (x+2z)$ and so $x\ R\ z$ (R is transitive). $\square$

(b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

Solution (b). The distinct equivalence classes are:

$$\begin{aligned}
A_1 &= \{-6, 0\} = [-6] = [0] \\
A_2 &= \{5, -7\} = [-7] = [5] \\
A_3 &= \{-2, 1, 4, 7\} = [-2] = [1] = [4] = [7]
\end{aligned}$$

Problem 40. A relation R is defined on $\mathbb{Z}$ by $x\ R\ y$ if $3x - 7y$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Solution 40. First, we show that R is an equivalence relation.

Proof. We show that R is reflexive. Consider some integer x. Then, $3x - 7x = -4(x) = 2(-2x)$, where $-2x \in \mathbb{Z}$ and so it is even. Hence, $x\ R\ x$.
We prove that R is symmetric. Consider two integers x and y such that $x\ R\ y$. Hence, $3x - 7y = 2c$ for some integer c. Then, $3y - 7x = 2c + 10y - 10x = 2(c + 5y - 5x)$. Since $c + 5y - 5x \in \mathbb{Z}$, it follows that $3y - 7x$ is even and so $y\ R\ x$.
Now, consider three integers x, y, z such that $x\ R\ y$ and $y\ R\ z$. Thus, $3x - 7y = 2a$ and $3y - 7z = 2b$ for some $a, b \in \mathbb{Z}$. Note that $(3x - 7y) + (3y - 7z) = 2a + 2b$ and so $3x - 7z = 2a + 2b + 4y = 2(a + b + y)$. Since $a + b + y \in \mathbb{Z}$, it follows that $3x - 7y$ is even and so $x\ R\ z$. $\square$

Now that it has been proven that R is an equivalence relation. We proceed to determine its equivalence classes. We first determine the equivalence class for some even integer, say 0. Then

$$\begin{aligned}
[0] &= \{x \in \mathbb{Z} : x\ R\ 0\} \\
&= \{x \in \mathbb{Z} : 3x - 70 \text{ is even}\} \\
&= \{x \in \mathbb{Z} : 3x \text{ is even}\} \\
&= \{x \in \mathbb{Z} : x \text{ is even}\}.
\end{aligned}$$

Now, consider some odd integer, say 1. Then

$$\begin{aligned}
[1] &= \{x \in \mathbb{Z} : x\ R\ 1\} \\
&= \{x \in \mathbb{Z} : 3x - 7 \text{ is even}\} \\
&= \{x \in \mathbb{Z} : 3x \text{ is odd}\} \\
&= \{x \in \mathbb{Z} : x \text{ is odd}\}.
\end{aligned}$$

Therefore, there are two distinct equivalence classes, namley, the set of even integers and the set of odd ones.

Problem 41. (a) Prove that the intersection of two equivalence relations on a nonempty set is an equivalence relation.

Proof. Let R_1 and R_2 be two equivalence relations on some nonempty set A. Let their intersection be the set K. Since both R_1 and R_2 are reflexive, it follows that if $x \in A$, then $(x, x) \in R_1, R_2$, and so $(x, x) \in K$. Hence, K is reflexive. Consider some $a, b \in A$ such that $a\ K\ b$ (Recall that $a\ K\ b$ is the same as saying $(a, b) \in K$). Then, $a\ R_1\ b$ and $a\ R_2\ b$. Since both are symmetric, $b\ [R_1, R_2]\ a$ (b is related to a by both R_1 and R_2) and so $b\ K\ a$, which implies that K is symmetric.
Now consider some $a, b, c \in A$ such that $a\ K\ b$ and $b\ K\ c$. Therefore, $a\ [R_1, R_2]\ b$ and $b\ [R_1, R_2]\ c$. Since both relations are transitive, it follows that $a\ [R_1, R_2]\ c$. Therefore, $a\ K\ c$, which implies that K is transitive. Thus, K, namely, the intersection of two equivalence relations on a nonempty set, is an equivalence relation. □

Lemma 8.4.1. Let a, b be integers. $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$ is a necessary and sufficiente condition for $a \equiv b \pmod 6$

Proof. Assume that $a \equiv b \pmod 6$, then $a = 6(c) + b = 2(3c) + b = 3(2c) + b$ for some integer c. Since $2c, 3c \in \mathbb{Z}$, it follows that $a \equiv b \pmod 3$ and $a \equiv b \pmod 2$.
Suppose that $a \equiv b \pmod 3$ and $a \equiv b \pmod 2$. Hence, $a = 3x + b = 2y + b$ and so $3x = 2y$ for some $x, y \in \mathbb{Z}$. Hence, $3x$ is even and so $2 \mid x$. Hence $3x = 3 \cdot 2(c)$ for some $c \in \mathbb{Z}$. Therefore, $a = 6(c) + b$, which implies that $a \equiv b \pmod 6$. □

(b) Consider the equivalence relations R_2 and R_3 defined on $\mathbb{Z}$ by $a\ R_2\ b$ if $a \equiv b \pmod 2$ and $a\ R_3\ b$ if $a \equiv b \pmod 3$. By (a), $R_1 = R_2 \cap R_3$ is an equivalence relation on $\mathbb{Z}$. Determine the distinct equivalence classes in R_1.

Solution b. Note that R_1 is defined by $a\ R_1\ b$ if $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$. Since both 2 and 3 are prime, by the prevoius Lemma, $a \equiv b \pmod 2$ and $a \equiv b \pmod 3 \iff a \equiv b \pmod 6$. Thus,

$$\begin{aligned}
[a] &= \{x \in \mathbb{Z} : x\ R\ a\} \\
&= \{x \in \mathbb{Z} : x \equiv a \pmod 6\} \\
&= \{x \in \mathbb{Z} : x = 6m + a, m \in \mathbb{Z}\}.
\end{aligned}$$

Recall that any integer can be expressed as $6c + b$ for exaclty one (c, b), where $c \in \mathbb{Z}$ and $b \in \{0, 1, 2, 3, 4, 5\}$ by the **Divition Algorithm**. Hence,

$$\begin{aligned}
[0] &= \{6x + 0 : x \in \mathbb{Z}\} \\
[1] &= \{6x + 1 : x \in \mathbb{Z}\} \\
[2] &= \{6x + 2 : x \in \mathbb{Z}\} \\
[3] &= \{6x + 3 : x \in \mathbb{Z}\} \\
[4] &= \{6x + 4 : x \in \mathbb{Z}\} \\
[5] &= \{6x + 5 : x \in \mathbb{Z}\}
\end{aligned}$$

Problem 42. Prove or disprove: The union of two equivalence relations on a nonempty set is an equivalence relation.

Solution 42. This is false. Consider the set $A = \{a, b, c\}$ and relations

$$R_1 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\} \quad \text{and}$$
$$R_2 = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}.$$

Hence, both R_1 and R_2 are equivalence relations and their union is

$$K = R_1 \cup R_2 = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b),(a,b),(b,a)\}.$$

Note that $(a,b),(b,c) \in K$, however, $(a,c) \notin K$. Thus, K is not transitive and so it is not an equivalence relation.

Problem 43. Let $A = \{u, v, w, x, y, z\}$. The relation

$$R = \{(u,u),(u,v),(u,w),(v,u),(v,v),(v,w),(w,u),(w,v),$$
$$(w,w),(x,x),(x,y),(y,x),(y,y),(z,z)\}$$

defined on A is an equivalence relation. In particular, $[u] = [v] = [w] = \{u, v, w\}$, $[x] = [y] = \{x, y\}$ and $[z] = \{z\}$; so $|[u]| = |[v]| = |[w]| = 3$ and $|[x]| = |[y]| = 2$, while $|[z]| = 1$. Therefore, $|[u]|+|[v]|+|[w]|+|[x]|+|[y]|+|[z]| = 14$. Let $A = \{a_1, a_2, \ldots, a_n\}$ be an n-element set and let R be an equivalence relation defined on A. Prove that $\sum_{i=1}^{n} |[a_i]|$ is even if and only if n is even.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be an n-element set and let R be an equivalence relation defined on A. Also, let $r_1, r_2, \ldots, r_k$ be the distinct equivalence classes by R.

Recall that $\{r_1, r_2, \ldots, r_k\}$ is a partition of A, and so

$$\sum_{i=1}^{k} |r_i| = |A|.$$

We know that $[x] = [w] \iff x \in [w]$ for any elements $x, w \in A$. Hence, for any positive integer $i \leq k$, r_i contains $|r_i|$ unique elements of A and $|[x]| = |r_i|$ for every $x \in [r_i]$. Hence,

$$\sum_{i=1}^{n} |[a_i]| = \sum_{i=1}^{k} |r_i|^2$$

(check **Corollary 8.3.3**). Since x^2 is even if and only if x is even, the sum $\sum_{i=1}^{k} |r_i|^2 = \sum_{i=1}^{n} |[a_i]|$ has the same parity as $\sum_{i=1}^{k} |r_i| = |A|$ (same quantitiy of even and odd numbers being added).

$\square$