Section 2.3: The Algebraic and Order Limit Theorems

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Now that we have a more formal and clearer definition of convergence for sequences, we can check some properties that come with this meaning. Namely, algebraic and order properties. Before that, we must mention an important theorem, which says that for any convergent sequence (a_n), there is some real number M such that (a_n) is inside $[-M, M]$. Namely, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $(a_n) \to a$ and $(b_n) \to b$. The algebraic property states the following:

(i) $\lim (ca_n) = ca$

(ii) $\lim (a_n + b_n) = a + b$

(iii) $\lim (a_n b_n) = ab$

(iv) $\lim (a_n/b_n) = a/b$, if $b \neq 0$.

THe interesting thing about the arguments given by the author to prove them is that they use the fact that one can make $|a_n - a|$ as small as one wants, namely, for any positive real number ε as small as one can imagine, there is some a_n such that $|a_n - a| < \varepsilon$. Also, these properties helps us interact with combinations of sequences and their limits on a more "familiar" way. On the other hand, the order property states the following:

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.

Noteworthy, if one changes the initial assumption from *for all* for *infinitely many*, the theorem holds. In other words, the property of a finite amount of elements in a sequence is not sufficient to predict the general property of its limit. The first $10^{10^{100}}$ elements can be positive but the rest be negative, which means that the sequence *eventually* aquires the property of negativity.

Problem 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

Proof. Observe that $|x_n - 0| = |x_n|$. Now, consider some positive real number ϵ. Since $(x_n) \to 0$, there is some $N \in \mathbb{N}$ such that $|x_n| = x_n < \epsilon^2$ for all $n \geq N$ $(0 \leq x_n)$. Then, $\sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$. Because $\sqrt{x_n} \geq 0$, it follows that $\sqrt{x_n} = |\sqrt{x_n}| = |\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$. $\qquad \square$

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof. Consider some positive real number ϵ. Since $(x_n) \to 0$, it follows that there is some $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon^2$ for all $n \geq N$. Thus, $\sqrt{\epsilon^2} > \sqrt{|x_n - x|} \geq |\sqrt{x_n} - \sqrt{x}|$, and so $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ for all $n \leq N$. Therefore, $(\sqrt{x_n}) \to \sqrt{x}$. $\qquad \square$

Problem 2.3.2. Using only Definition 2.2.3 (definition of convergence), prove that if $(x_n) \to 2$, then

1. $\left(\frac{2x_n - 1}{3}\right) \to 1$

   Proof. First, note that $\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right|$. Now, consider some positive real number ϵ. Then, there is some $N \in \mathbb{N}$ such that $|x_n - 2| < \frac{3}{2}\epsilon$ for all $n \geq N$. Observe that

   $$\frac{2}{3}|x_n - 2| = \left|\frac{2}{3}\right| |x_n - 2|$$
   $$= \left|\frac{2}{3}x_n - \frac{4}{3}\right| = \left|\frac{2x_n - 4}{3}\right|$$
   $$< \epsilon.$$

   Therefore, $\left|\frac{2x_n - 1}{3} - 1\right| < \epsilon$ for all $n \geq \mathbb{N}$ and so $\left(\frac{2x_n - 1}{3}\right) \to 1$. $\qquad \square$

2. $\left(\frac{1}{x_n}\right) \to 1/2$

   Proof. Since $(x_n) \to 2$, it follows that there is an infinity of nonzero x_n that eventually get nearer and nearer to 2. Then, let's consider all $x_n \neq 0$ in the sequence $\left(\frac{1}{x_n}\right)$.

   Note that $\left|\frac{1}{x_n} - 1/2\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|2 - x_n|}{2|x_n|}$. Now, let ϵ be any positive real number. Then, there is some $K_1 \in \mathbb{N}$ such that $|x_n - 2| < 1$, which implies that $|x_n| > 1$. Furthermore, there is a positive integer K_2 such that $|x_n - 2| < 2\epsilon$ for all $n \geq K_2$. Thus,

   $$|2 - x_n| \frac{1}{2|x_n|} < |2 - x_n| \frac{1}{2} < 2\epsilon \frac{1}{2}$$

   for any $n \geq M$, where $M = \max(K_1, K_2)$. Then, $\left|\frac{1}{x_n} - 1/2\right| < \epsilon$ for all $n \geq M$. Hence, $\left(\frac{1}{x_n}\right) \to 1/2$. $\qquad \square$

Problem 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof. Since $(x_n) \to l$ and $(z_n) \to l$, it follows that for some positive real numbr ϵ, there is some positive integer K, such that $|x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$ for all $n \geq K$. Hence, $-\epsilon < x_n - l < \epsilon$ and $-\epsilon < z_n - l < \epsilon$, and so $l - \epsilon < x_n, z_n < l + \epsilon$. Since $x_n \leq y_n \leq z_n$, it follows that $l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon$. Thus, $|y_n - l| < \epsilon$ for each $n \geq K$, namely, $(y_n) \to l$. $\qquad\square$

Problem 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

1. $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

2. $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

3. $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$