Week 16

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Section 1: The Principle of Mathematical Induction

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Thus far, we have explored three types of mathematical proofs for the quantified statement $\forall x \in S, R(x)$, namely, direct proof, proof by contrapositive and proof by contradiction. However, for some sets S, it is posible to prove $\forall x \in S, R(x)$ by using the Principle of Mathematical Induction.

Before delving into it, we must define what is understood for a **least element** of an arbitrary set X of real numbers. The **least element** of an arbitrary set X of real numbers is some $m \in X$ such that $\forall n \in X, n \geq m$ is true. This m is unique. Thus, some sets can have **least elements** and others don't. For example, the least element of $\mathbb{N}$ is 1, but for the open interval $(0, 1)$ there is no least element. Now, if all nonempty subsets of an arbitrary nonempty set X of real numbers have a least element, then we say that X is **well-ordered**. Note that having a least element is a necessary condition for a nonempty set to be well-ordered, but it is not sufficient (i.e., $[0, 1]$ has 0 as a least element, but $(0, 1) \subset [0, 1]$ has no least element).

In number theory, the **Well-ordering principle** states that the set $\mathbb{N}$ is well-ordered. We don't prove it here and so we take it as an axiom. From the **Well-ordering principle** we get the **Principle of Mathematical Induction** (Theorem 1).

Theorem 1. For each positive integer n, let $P(n)$ be a statement. If

$$P(1) \text{ is true and}$$
$$\forall k \in \mathbb{N}, P(k) \implies P(k+1) \text{ is true}$$

$$\text{then}$$
$$\forall n \in \mathbb{N}, P(n) \text{ is true.}$$

Proof. Assume, to the contrary, that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$ are true, and that there are $n \in \mathbb{N}$ such that $P(n)$ is false. Let S be the set of all counterexamples for $\forall n \in \mathbb{N}, P(n)$. Since $S \neq \emptyset$ and $S \subseteq \mathbb{N}$, it follows by the **Well-ordering principle** that S has a least element m; so $m \in S$. Because $m \in \mathbb{N}$ and $P(1)$ is true, $m \geq 2$ and so $m - 1 \in \mathbb{N}$ and $m - 1 \notin S$ (m is the least element of S). Thus, $P(m-1)$ is true and, by the second condition $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$, $P(m)$ must be true. However, this implies that $m \notin S$, which leads to a contradiction. □

Therefore, a proof by induction uses the **Principle of Mathematical Induction**. In such proof, for an open sentence $P(x)$ over $\mathbb{N}$, it suffices to show that $P(1)$ is true (basis step) and that $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$ is true (inductive step) so that one can conclude that $\forall n \in \mathbb{N}, P(n)$ is true by **Modus Pollens** $(([p \wedge (p \implies q)] \implies q) \equiv T)$

Problem 1. Which of the following sets are well-ordered?

(a) $S = \{x \in \mathbb{Q} : x \geq -10\}$

Solution a. Consider the set $M = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Since $M \subset S$ and M has no least element, it follows that S is not well-ordered.

(b) $S = \{-2, -1, 0, 1, 2\}$

Solution b. Since S is a nonempty finite set of real numbers, it follows that it is well-ordered.

(c) $S = \{x \in \mathbb{Q} : -1 \leq x \leq 1\}$

Solution c. Consider the set $M = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Just as in example (a), $M \subset S$ and M has no least element. Therefore, S is not well-ordered.

(d) $S = \{p : p \text{ is a prime}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$.

Solution d. It can be seen that $S \subset \mathbb{N}$. Since $\mathbb{N}$ is well ordered, all nonempty subsets of $\mathbb{N}$ have a least element. Therefore, all nonempty subsets of S have a least element (since all subsets of S are subsets of $\mathbb{N}$) and so S is well-ordered.

Problem 2. Prove that if A is any well-ordered set of real numbers and B is a nonempty subset of A, then B is also well-ordered.

Proof. Let A and B be two arbitrary nonempty sets of real numbers such that A is well-ordered and $B \subseteq A$. Since A is well-ordered, it follows that each nonempty subset of A has a least element. Because $B \subseteq A$, it follows that every subset of B is also a subset of A and so every nonempty subset of B has a least element. Thus, B is well-ordered. □

Problem 3. Prove that every nonempty set of negative integers has a largest element.

Proof. Let S be a nonempty subset of $\mathbb{N}$. Since $\mathbb{N}$ is well ordered, it follows that there is some $m \in S$ such that $x \geq m$ for every $x \in S$. Now, consider the set $M = \{-n : n \in S\}$. Then $-m \in M$ and so $-x \leq -m$ for every $-x \in M$. Thus, $-m$ is the largest element in M. □

Problem 4. Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for every positive integer n

(1) by mathematical induction

Proof. We proceed by induction. Since $2 \cdot 1 - 1 = 1 = 1^2$, the statement is true for $n = 1$. Then, assume that the statement is true for an arbitrary positive integer k, namely,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2.$$

We now show that the statement is true for $k + 1$, namely,

$$1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$$

Note that

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k + 1) &= [1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) \\ &= k^2 + 2k + 1 = (k + 1)^2 \end{aligned}$$

Thus, by the principles of mathematical induction, for every positive integer n,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

□

(b) by adding $1 + 3 + 5 + \ldots + (2n - 1)$ and $(2n - 1) + (2n - 3) + \ldots + 1$.

Proof. Let $S = 1 + 3 + 5 + \ldots + (2n - 1)$ for any positive integer n. Note that, by inverting the orther of the terms, $S = (2n - 1) + (2n - 3) + (2n - 5) + \ldots + 1$. Adding them we get

$$2S = [(2n-1)+1]+[(2n-3)+3]+[(2n-5)+5]+\ldots+[1+(2n-1)] = 2n+2n+2n+\ldots+2n.$$

Since there are n terms, it follows that $2S = 2n^2$ or $S = n^2$. Therefore,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

for every positive integer n. □

Problem 5. Use mathematical induction to prove that

$$1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$$

for every positive integer n.

Proof. We proceed by induction. Since $1 = 2 - 1$, the statement is true when $n = 1$. Now, assume that $1 + 5 + 9 + \ldots + (4k - 3) = 2k^2 - k$ where k is a positive integer. We then show that $1 + 5 + 9 + \ldots + (4k + 1) = 2(k + 1)^2 - (k + 1)$. Observe that

$$\begin{aligned} 1 + 5 + 9 + \ldots + (4k + 1) &= [1 + 5 + 9 + \ldots + (4k - 3)] + (4k + 1) \\ &= (2k^2 - k) + 4k + 1 = 2k^2 + 4k + 2 - k - 1 \\ &= 2(k^2 + 2k + 1) - (k + 1) = 2(k + 1)^2 - (k + 1) \end{aligned}$$

By the principle of mathematical induction, $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$ for every positive integer n. □

Problem 6. (a) We have seen that $1^2 + 2^2 + \ldots + n^2$ is the number of squares in an $n \times n$ square composed of n^2 1×1 squares. What does $1^3+2^3+3^3+\ldots+n^3$ represent geometrically?

Solution . Let $n \geq 1$. In $\mathbb{R}^3$, for a $k \times k \times k$ cube, where $1 \leq k \leq n$, each of the variables in the ordered triple (x, y, z) (lower corner of a cube) can have values $0 \leq x, y, z \leq n-k$ and so there are $(n-k+1)^3$ possible ordered triples (possible different positions for a $k \times k \times k$ cube inside a $n \times n \times n$ cube). Thus, the number of different cubes (different proportions or position) in an $n \times n \times n$ cube composed of n^3 $1 \times 1 \times 1$ cubes is

$$\sum_{k=1}^{n}(n-k+1)^3 = n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3$$
$$= 1^3 + 2^3 + \ldots + (n-1)^3 + n^3 = \sum_{k=1}^{n} k^3$$

(b) Use mathematical induction to prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

Proof. We proceed by induction. Since $1^3 = 1 = \frac{1^2\cdot(1+1)^2}{4}$, the statement holds for $n = 1$. Assume that $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$ for some positive integer k. We then show that the statement holds for $n = k + 1$, that is, $1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$. Note that

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 &= [1^3 + 2^3 + 3^3 + \ldots + k^3] + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

By the principle of mathematical induction, the equality holds for any positive integer n. $\square$

Problem 7. Find another formula suggested by Exercises 4 and 5, and verify your formula by mathematical induction.

Result 1. Let $n \in \mathbb{N}$. Then $1 + 7 + 13 + \ldots + (6n-5) = 3n^2 - 2n$

Proof. We prove this by induction. Since $6 - 5 = 1 = 3 \cdot 1^2 - 2$, the statement is true for $n = 1$. Assume that $1 + 7 + 13 + \ldots + (6k-5) = 3k^2 + 2k$ for an arbitrary positive integer k. We then show that $1 + 7 + 13 + \ldots + (6(k+1)-5) = 3(k+1)^2 - 2(k+1)$. Note that

$$\begin{aligned} 1 + 7 + 13 + \ldots + (6(k+1)-5) &= [1 + 7 + 13 + \ldots + (6k-5)] + (6(k+1)-5) \\ &= (3k^2 - 2k) + 6k + 6 - 5 = (3k^2 + 6k + 3) - 2k - 2 \\ &= 3(k^2 + 2k + 1) - 2(k+1) = 3(k+1)^2 - 2(k+1) \end{aligned}$$

By the principle of mathematical induction,

$$1 + 7 + 13 + \ldots + (6n-5) = 3n^2 - 2n$$

for every positive integer n. (Lol, what a lovely coincidence :D) $\square$

Problem 8. Find a formula for $1+4+7+\ldots+(3n-2)$ for positive integers n, and then verify your formula by induction.

Solution . Let $S=1+4+7+\ldots+(3n-2)$ for some positive integer n. By inverting the order of the terms, we conclude that $S=(3n-2)+(3(n-1)-2)+(3(n-2)-2)\ldots+1$. Therefore,

$$\begin{aligned}2S &= [(3n-2)+1]+[(3(n-1)-2)+4]+\ldots+[1+(3n-2)] \\ &= (3n-1)+(3n-1)+\ldots+(3n-1)\end{aligned}$$

Thus, $2S=n(3n-1)$ or $S=\frac{3n^2-n}{2}$ for any positive integer n.

Result 2. Let n be some positive integer. Then $1+4+7+\ldots+(3n-2)=\frac{3n^2-n}{2}$.

Proof. We proceed by induction. Since $3-2=1=\frac{3\cdot 1^2-1}{2}$, it follows that the statement is true for $n=1$. Now, suppose that $1+4+7+\ldots+(3k-2)=\frac{3k^2-k}{2}$ for an arbitrary positive integer k. We then show that $1+4+7+\ldots+(3(k+1)-2)=\frac{3(k+1)^2-(k+1)}{2}$. Observe that

$$\begin{aligned}1+4+7+\ldots+(3(k+1)-2) &= [1+4+7+\ldots+(3k-2)]+(3(k+1)-2) \\ &= \frac{3k^2-k}{2}+3(k+1)-2=\frac{3k^2-k+6(k+1)-4}{2} \\ &= \frac{(3k^2+6k+3)-k+3-4}{2}=\frac{3(k+1)^2-(k+1)}{2}\end{aligned}$$

By the principle of mathematical induction,

$$1+4+7+\ldots+(3n-2)=\frac{3n^2-n}{2}$$

for every positive integer n. $\square$

Problem 9. Prove that $1\cdot 3+2\cdot 4+3\cdot 5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6}$ for every positive integer n.

Proof. We proceed by induction. Since $1(1+2)=3=\frac{1(1+1)(2+7)}{6}$, it follows that the statement is true for $n=1$. Assume that $1\cdot 3+2\cdot 4+3\cdot 5+\ldots+k(k+2)=\frac{k(k+1)(2k+7)}{6}$ for some positive integer k. We then show that $1\cdot 3+2\cdot 4+3\cdot 5+\ldots+(k+1)(k+3)=\frac{(k+1)(k+2)(2(k+1)+7)}{6}$. Note that

$$\begin{aligned}1\cdot 3+2\cdot 4+\ldots+(k+1)(k+3) &= [1\cdot 3+2\cdot 4+\ldots+k(k+2)]+(k+1)(k+3) \\ &= \frac{k(k+1)(2k+7)}{6}+(k+1)(k+3) \\ &= \frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6} \\ &= \frac{(k+1)(k(2k+7)+6(k+3))}{6}=\frac{(k+1)(2k^2+7k+6k+18)}{6} \\ &= \frac{(k+1)(k+2)(2k+9)}{6}=\frac{(k+1)(k+2)(2(k+1)+7)}{6}\end{aligned}$$

By the principle of mathematical induction, this statement is true for every positive integer n. $\square$

Problem 10. Let $r \neq 1$ be a real number. Use induction to prove that $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for every positive integer n.

Proof. We prove this by induction. For $n = 1$ we have $ar^{1-1} = a = \frac{a(1-r^1)}{1-r}$, which is true. Assume that $a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$ where $k \in \mathbb{N}$. We then show that $a + ar + ar^2 + \ldots + ar^k = \frac{a(1-r^{k+1})}{1-r}$. Observe that

$$\begin{aligned} a + ar + ar^2 + \ldots + ar^k &= [a + ar + ar^2 + \ldots + ar^{k-1}] + ar^k \\ &= \frac{a(1-r^k)}{1-r} + ar^k = \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\ &= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r} = \frac{a(1-r^{k+1})}{1-r} \end{aligned}$$

By the principle of mathematical induction,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

where $n \in \mathbb{N}$. ☐

Problem 11. Prove that $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(n+2)(n+3)} = \frac{n}{3n+9}$ for every positive integer n.

Proof. We proceed by induction. Since $\frac{1}{(1+2)(1+3)} = \frac{1}{12} = \frac{1}{3+9}$, the statement is true for $n = 1$. Assume that $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(k+2)(k+3)} = \frac{k}{3k+9}$ for an arbitrary positive integer k. We then show that $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(k+3)(k+4)} = \frac{k+1}{3k+12}$. Observe that

$$\begin{aligned} \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(k+3)(k+4)} &= \left[\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(k+2)(k+3)}\right] + \frac{1}{(k+3)(k+4)} \\ &= \frac{k}{3k+9} + \frac{1}{(k+3)(k+4)} = \frac{k(k+4)+3}{3(k+3)(k+4)} \\ &= \frac{(k+3)(k+1)}{3(k+3)(k+4)} = \frac{k+1}{3k+12} \end{aligned}$$

By the principle of mathematical induction,

$$\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{(n+2)(n+3)} = \frac{n}{3n+9}$$

for every positive integer n. ☐

Problem 12. Consider the open sentence $P(n) : 9 + 13 + \ldots + (4n+5) = \frac{4n^2+14n+1}{2}$, where $n \in \mathbb{N}$.

(a) Verify the implication $P(k) \implies P(k+1)$ for an arbitrary positive integer k.

Solution . Assume that $P(k)$ is true for some $k \in \mathbb{N}$, namely, $9 + 13 + \ldots + (4k+5) = \frac{4k^2+14k+1}{2}$. We then show that $P(k+1) : 9+13+\ldots+(4k+9) = \frac{4(k+1)^2+14(k+1)+1}{2} = \frac{4k^2+22k+19}{2}$. Note that

$$\begin{aligned}9 + 13 + \ldots + (4k+9) &= [9 + 13 + \ldots + (4k+5)] + (4k+9) \\ &= \frac{4k^2+14k+1}{2} + 4k + 9 = \frac{4k^2+14k+1+8k+18}{2} \\ &= \frac{4k^2+22k+19}{2}.\end{aligned}$$

(b) Is $\forall n \in \mathbb{N}, P(n)$ true?

Solution . Proving the truth of $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$ is not enough to determine whether $\forall n \in \mathbb{N}, P(n)$ is true or false. This is so, since $P(k) \implies P(k+1)$ is also considered to be true when $P(k)$ is false (a fact harnessed by the direct proof). For example, for $n = 1$ we have that $P(1) : 4 + 5 = 9 = \frac{19}{2} = \frac{4 \cdot 1^2+14+1}{2}$ (basis step), which is clearly false.

Problem 13. Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ for every positive integer n.

Proof. We proceed by induction. Since $1 \cdot 1! = 1 = (1+1)! - 1$, the equation holds for $n = 1$. Assume that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$$

for some positive integer k. We now show that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + (k+1) \cdot (k+1)! = (k+2)! - 1.$$

Note that

$$\begin{aligned}1 \cdot 1! + 2 \cdot 2! + \ldots + (k+1) \cdot (k+1)! &= [1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k!] + (k+1) \cdot (k+1)! \\ &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)! \cdot [(k+1) + 1] - 1 = ((k+2) - 1)! \cdot (k+2) - 1 \\ &= (k+2)! - 1\end{aligned}$$

By the principle of mathematical induction,

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$$

for any positive integer n. □

Problem 14. Prove that $2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! \geq [(n+1)!]^n$ for every positive integer n.

Proof. We proceed by induction. Since $(2 \cdot 1)! = 2! = [(1+1)!]^1$, the statement is true for $n = 1$. Assume that

$$2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k)! \geq [(k+1)!]^k$$

for some $k \in \mathbb{N}$. We then show that

$$2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k+2)! > [(k+2)!]^{k+1}.$$

Observe that

$$\begin{aligned}
2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k+2)! &= [2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k)!] \cdot (2k+2)! \\
&\geq [(k+1)!]^k (2k+2)! \qquad \text{since } (2k+2)! \in \mathbb{N}. \\
&= [(k+1)!]^k \cdot 1 \cdot 2 \cdot \ldots \cdot (2(k+1)-(k+1)) \cdot \ldots \cdot (2(k+1)-1) \cdot (2(k+1)) \\
&= [(k+1)!]^{k+1} (2(k+1)-k) \cdot \ldots \cdot (2(k+1)-1) \cdot (2(k+1)) \\
&= [(k+1)!]^{k+1} (2(k+1)-k)[m]
\end{aligned}$$

where the positive integer $m = (2(k+1)-(k-1)) \ldots (2(k+1)-1)(2(k+1)-0)$. Note that m is a multiplication of k positive terms and each of them are greater than $k+2 > 0$. Therefore,

$$[(k+1)!]^{k+1}(k+2)[m] > [(k+1)!]^{k+1}(k+2)\left[(k+2)^k\right]$$

and so

$$\begin{aligned}
2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k+2)! &> [(k+1)!]^{k+1}(k+2)\left[(k+2)^k\right] \\
&> [(k+1)!]^{k+1}(k+2)^{k+1} = [(k+2)!]^{k+1}
\end{aligned}$$

Remember that for $n=1$, it is true that $2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! = [(n+1)!]^n$. Since we have proven that

$$2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2k)! \geq [(k+1)!]^k \implies 2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2(k+1))! > [(k+2)!]^{k+1}$$

for every positive integer k, it follows that $2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! = [(n+1)!]^n$ if and only if $n=1$ and for $n>1$ we have that $2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! > [(n+1)!]^n$. Thus, by the principle of mathematical induction,

$$2! \cdot 4! \cdot 6! \cdot \ldots \cdot (2n)! \geq [(n+1)!]^n$$

for every positive integer n. □

Problem 15. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$ for every positive integer n.

Proof. We proceed by induction. Since $\frac{1}{\sqrt{1}} = 1 = 2\sqrt{1} - 1$, the statement is true for $n = 1$. Assume that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k} - 1$$

for some arbitrary positive integer k. Now, we show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1.$$

Observe that

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} &= \left[\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}}\right] + \frac{1}{\sqrt{k+1}} \\
&\leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \\
&= \frac{2\sqrt{k^2+k}+1}{\sqrt{k+1}} - 1
\end{aligned}$$

Since $4(k^2+k) < (2k+1)^2$, it follows that $2\sqrt{k^2+k} < |2k+1| = 2k+1$ and so $2\sqrt{k^2+k}+1 < 2k+2$. Thus,

$$\begin{aligned} \frac{2\sqrt{k^2+k}+1}{\sqrt{k+1}} - 1 &< \frac{2(k+1)}{\sqrt{k+1}} - 1 \\ &< 2\sqrt{k+1} - 1 \end{aligned}$$

and so

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 1$$

or

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$

By the principle of mathematical induction,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$$

for any positive integer n. $\square$

Problem 16. Prove that $7 \mid [3^{4n+1} - 5^{2n-1}]$ for every positive integer n.

Proof. We proceed by induction. Since $3^{4+1} - 5^{2-1} = 238 = 7(34)$, it follows that $7 \mid [3^{4+1} - 5^{2-1}]$ and so the statement is true for $n = 1$. Assume that $7 \mid [3^{4k+1} - 5^{2k-1}]$ for some positive integer k. We then show that $7 \mid [3^{4k+5} - 5^{2k+1}]$. Since $7 \mid [3^{4k+1} - 5^{2k-1}]$, it follows that $3^{4k+1} - 5^{2k-1} = 7c$, where $c \in \mathbb{Z}$. Note that

$$\begin{aligned} 3^{4k+5} - 5^{2k+1} &= 3^{4k+1+4} - 5^{2k-1+2} \\ &= 3^{4k+1} \cdot 3^4 - 5^{2k-1} \cdot 5^2 \\ &= (7c + 5^{2k-1}) \cdot 3^4 - 5^{2k-1} \cdot 5^2 \\ &= 7c \cdot 3^4 + 5^{2k-1}(3^4 - 5^2) \\ &= 7c \cdot 3^4 + 5^{2k-1}(56) \\ &= 7(3^4 c + 5^{2k-1} \cdot 8) \end{aligned}$$

Because $3^4c + 5^{2k-1} \cdot 8 \in \mathbb{Z}$, it follows that $7 \mid [3^{4k+5} - 5^{2k+1}]$. By the principle of mathematical induction,

$$7 \mid [3^{4n+1} - 5^{2n-1}]$$

for every integer n. $\square$