Chapter 1: Vector Spaces

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Problem 1.1. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. Show that if $\alpha, \beta \in \mathbb{F}$ and if $\mathbf{v}$ is a nonzero vector in $\mathcal{V}$, then $\alpha\mathbf{v} = \beta\mathbf{v} \implies \alpha = \beta$. [HINT: $\alpha - \beta \neq 0 \implies \mathbf{v} = (\alpha - \beta)^{-1}(\alpha - \beta)\mathbf{v}$.]

Proof. Suppose, to the contrary, that there are distinct $\alpha, \beta \in \mathbb{F}$ such that for some nonzero $\mathbf{v} \in \mathcal{V}$ we have $\alpha\mathbf{v} = \beta\mathbf{v}$. Then, $\alpha - \beta \neq 0$ and so $\mathbf{v} = (\alpha - \beta)^{-1}(\alpha - \beta)\mathbf{v}$. Hence,

$$\mathbf{v} = (\alpha - \beta)^{-1}\alpha\mathbf{v} - (\alpha - \beta)^{-1}\beta\mathbf{v} = (\alpha - \beta)^{-1}(\alpha\mathbf{v} - \beta\mathbf{v}).$$

Since $\alpha\mathbf{v} = \beta\mathbf{v}$, it follows that $\alpha\mathbf{v} - \beta\mathbf{v} = \beta\mathbf{v} - \beta\mathbf{v} = \mathbf{0}$. This implies that $\mathbf{v} = (\alpha - \beta)^{-1}\mathbf{0} = \mathbf{0}$. This is a contradiction to our assumption that $\mathbf{v}$ was nonzero.

Another way to prove this directly is by using the fact, for some $\alpha \in \mathbb{F}$ and nonzero vector $\mathbf{v}$, that $\alpha\mathbf{v} = \mathbf{0} \implies \alpha = 0$. A proof reads as follows:

Let $\alpha, \beta \in \mathbb{F}$ and $\mathbf{v} \in \mathcal{V}$ be some nonzero vector such that $\alpha\mathbf{v} = \beta\mathbf{v}$. Then, $\alpha\mathbf{v} - \beta\mathbf{v} = \beta\mathbf{v} - \beta\mathbf{v} = \mathbf{0}$ and so $(\alpha - \beta)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v}$ is nonzero, it follows that $\alpha - \beta = 0$ and so $\alpha = \beta$. $\square$

Problem 1.2. Show that the space $\mathbb{R}^3$ endowed with the rule

$$\mathbf{x} \mathbin{\square} \mathbf{y} = \begin{bmatrix} \max(x_1, y_1) \\ \max(x_2, y_2) \\ \max(x_3, y_3) \end{bmatrix}$$

for vector addition and the usual rule for scalar multiplication is not a vector space over $\mathbb{R}$.

Proof. We show that this space has no unique additive identity. Consider some $\mathbf{x} = (x_1, x_2, x_3)$. Then, both $\mathbf{y} = (x_1 - 1, x_2 - 1, x_3 - 1)$ and $\mathbf{z} = (x_1 - 2, x_2 - 2, x_3 - 2)$ are in $\mathbb{R}^3$ and they are distinct. Note that $\mathbf{x} \mathbin{\square} \mathbf{y} = \mathbf{x}$ and $\mathbf{x} \mathbin{\square} \mathbf{z} = \mathbf{x}$.

In fact, one can easily show that there is no vector that is an additive inverse of every vector (the zero vector $\mathbf{0}$) since one can easily construct a vector with elements lower than the ones from any other vector. $\square$

Problem 1.3. Let $\mathcal{C} \subset \mathbb{R}^3$ denote the set of vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that th polynomial $a_1 + a_2 t + a_3 t^2 \geq 0$ for every $t \in \mathbb{R}$. Show that it is closed under vector addition (i.e., $\mathbf{a}, \mathbf{b} \in \mathcal{C} \implies \mathbf{a} + \mathbf{b} \in \mathcal{C}$), but that $\mathcal{C}$ is not a vector space over $\mathbb{R}$. [REMARK: A set $\mathcal{C}$ with the indicated two properties is called a **cone**.]

Proof. We first show that $\mathcal{C}$ is closed under addition. Consider any $\mathbf{a}, \mathbf{b} \in \mathcal{C}$. Then, for every $t \in \mathbb{R}$ we have $a_1 + a_2 t + a_3 t^2 \geq 0$ and $b_1 + b_2 t + b_3 t^2 \geq 0$. Then,

$$a_1 + a_2 t + a_3 t^2 + b_1 + b_2 t + b_3 t^2 = (a_1 + b_1) + (a_2 + b_2)t + (a_3 + b_3)t^2 \geq 0$$

for every $t \in \mathbb{R}$. Thus, $\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \in \mathcal{C}$. However, it is not close under scalar multiplication. Consider some nonzero $\mathbf{v} \in \mathcal{C}$ and let $\alpha = -1$. Since $v_1 + v_2 t + v_3 t^2 \geq 0$ for every $t \in \mathbb{R}$, it follows that $(-v_1) + (-v_2)t + (-v_3)t^2 < 0$ for every positive t. Hence, $(-1)\mathbf{v} \notin \mathcal{C}$ and so it is not a vector space over $\mathbb{R}$. □

Problem 1.5. Let $\mathcal{F}$ denote the set of continuous real-valued functions $f(x)$ on the interval $0 \leq x \leq 1$. Show that $\mathcal{F}$ is a vector space over $\mathbb{R}$ with respect to the natural rules of vector addition $((f_1 + f_2)(x) = f_1(x) + f_2(x))$ and scalar multiplication $((\alpha f)(x) = \alpha f(x))$.

Proof. (a) **Closed under vector addition**

Consider two functions $f, g \in \mathcal{F}$. Let $x \in [0,1]$. Then, $f(x), g(x) \in \mathbb{R}$ and so $(f + g)(x) = f(x) + g(x) \in \mathbb{R}$ since $\mathbb{R}$ is closed under addition. Therefore, $f + g$ is a real-valued function on the interval $[0,1]$ and so $(f + g) \in \mathcal{F}$.

(b) **Closed under scalar multiplication**

Consider some function $f \in \mathcal{F}$ and real number α. Let $x \in [0,1]$. Then, $f(x) \in \mathbb{R}$ and so $(\alpha f)(x) = \alpha f(x) \in \mathbb{R}$ since $\mathbb{R}$ is closed under multiplication. Thus, αf is a real-valued function on the interval $[0,1]$ and so $\alpha f \in \mathcal{F}$.

(c) **Vector addition is commutative**

Let $f, g \in \mathcal{F}$ and $x \in [0,1]$. Then, $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$ since addition in the set of real numbers is commutative.

(d) **Vector addition is associative**

Let $f, g, h \in \mathcal{F}$ and $x \in [0,1]$. Then, $((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g+h)(x) = (f + (g+h))(x)$ since addition in $\mathbb{R}$ is associative (the order of addition does not matter).

(e) **Existence of additive identity**

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$ for all $x \in [0,1]$. Then, f is a continous real-valued function and so $f \in \mathcal{F}$. Consider any $g \in \mathcal{F}$ and let $a \in [0,1]$. Then, $(f+g)(a) = f(a) + g(a) = 0 + g(a) = g(a)$ since 0 is the additive identity of real numbers. Thus, f is an additive identitive in $\mathcal{F}$.

(f) **Existence of additive inverse**

Consider some $f \in \mathcal{F}$. Let $g : [0,1] \to \mathbb{R}$ be defined by $g(x) = -f(x)$ for all $x \in [0,1]$. Consider some $x \in [0,1]$ and so $(f+g)(x) = f(x) + g(x) = f(x) - f(x) = 0$. Hence, g is the additive inverse of f.

(g) $f \in \mathcal{F} \implies (1)f = f$

Let $f \in \mathcal{F}$. Consider any $x \in [0,1]$ and so $f(x) = (1)f(x)$. Thus, $f = (1)f$.

(h) For any $\alpha, \beta \in \mathbb{R}$ and vector $f \in \mathcal{F}$, $\alpha(\beta f) = (\alpha\beta)f$

Let $f \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{R}$. Consider any $x \in [0,1]$ and so $\alpha(\beta f)(x) = \alpha(\beta f(x)) = (\alpha\beta)f(x)$ since multiplication in $\mathbb{R}$ is associative. Thus, $\alpha(\beta f) = (\alpha\beta)f$

(i) For any $\alpha, \beta \in \mathbb{R}$ and vector $f \in \mathcal{F}$, $(\alpha + \beta)f = \alpha f + \beta f$

Let $f \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{R}$. Consider any $x \in [0,1]$ and so $(\alpha+\beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$ since multiplication over addition is distributive for real numbers.

□

Lemma 1. Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{M}$ over $\mathbb{F}$. Then, $\mathcal{S}$ is a vector space if and only if for every pair of vectors $\mathbf{v}, \mathbf{a} \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{F}$, $\alpha\mathbf{v} + \beta\mathbf{a} \in \mathcal{S}$.

Proof. Assume that $\mathcal{S}$ is a vector space and so it is closed under addition and scalar multiplication. Let $\mathbf{v}, \mathbf{a} \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{F}$, then $\alpha\mathbf{v}, \beta\mathbf{a} \in \mathcal{S}$ and so $\alpha\mathbf{v} + \beta\mathbf{a} \in \mathcal{S}$.
Suppose, for every pair of vectors $\mathbf{v}, \mathbf{a} \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{F}$, that $\alpha\mathbf{v} + \beta\mathbf{a} \in \mathcal{S}$. Let $\alpha = 0$ and $\beta \in \mathbb{F}$. Consider any vectors $\mathbf{v}, \mathbf{a} \in \mathcal{S}$. Then, $\alpha\mathbf{v} = 0$ is the additive identity of $\mathcal{M}$ and so $\beta\mathbf{a} = \alpha\mathbf{v} + \beta\mathbf{a} \in \mathcal{S}$. Thus, $\mathcal{S}$ is closed under scalar multiplication.
Consider some vectors $\mathbf{v}, \mathbf{a} \in \mathcal{S}$ and let $\alpha = \beta = 1$. Then, $\mathbf{v} + \mathbf{a} = (1)\mathbf{v} + (1)\mathbf{a} = \alpha\mathbf{v} + \beta\mathbf{a} \in \mathcal{S}$ since $\mathbf{v}, \mathbf{a} \in \mathcal{M}$. Therefore, $\mathcal{S}$ is closed under addition and so it is a vector space. □

Problem 1.6. Let F_0 denote the set of continuous real-valued functions $f(x)$ on the interval $0 \le x \le 1$ that met the auxiliary constraints $f(0) = 0$ and $f(1) = 0$. Show that F_0 is a vector space over $\mathbb{R}$ with respect to the natural rules of vector addition and scalar multiplication that were introducd in **Excercise 1.5** and that F_0 is a subspace of the vector space $\mathcal{F}$ that was considered there.

Proof. By definition, $F_0 \subseteq \mathcal{F}$. Let's prove that it is closd under addition and scalar multiplication. Consider some $f, g \in F_0$ and $\alpha, \beta \in \mathbb{R}$. Then, $\alpha f + \beta g$ is a real-valued function since $f, g \in \mathcal{F}$. Particularly, $(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = 0 + 0 = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$ and so, by condition, it is a vector in F_0. Therefore, F_0 is a subspace of $\mathcal{F}$. □

Problem 1.7. Let F_1 denote the set of continuous real-valued functions $f(x)$ on the interval $0 \le x \le 1$ that meet the auxiliary constraints $f(0) = 0$ and $f(1) = 1$. Show that F_1 is not a vector space over $\mathbb{R}$ with respect to the natural rules of vector addition and scalar multiplication that were introduced in **Exercise 1.5**.

Proof. We know that $F_1 \subseteq \mathcal{F}$. Consider some $f \in F_1$. Then, $(2)f$ is a continuous real-valued function since $f \in \mathcal{F}$. However, note that $(2f)(1) = (2)f(1) = 2 \neq 1$ and so $(2)f \notin F_1$. Hence, F_1 is not closed under scalar multiplication and so F_1 is not a subspace of $\mathcal{F}$. □