Section 8.1: Equivalence Relations

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May 9, 2022

Consider two sets A and B. A **relation** R **from** A **to** B is a subset of the cartesian product $A \times B$. Therefore, $(a, b) \in R$ means that $a \in A$ is related to $b \in B$ by R, namely, $a\, R\, b$. On the other hand, if $(a, b) \notin R$, then $a \in A$ is not related to $b \in B$ by R, namely, $a \not R\, b$.

Also, if $|A \times B| = n$ for some positive integer n, then there are 2^n possible relations from A to B. Note that R can be seen as a function with some domain and range. Hence

$$\mathrm{dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

and

$$\mathrm{range}(R) = \{b \in B : (a, b) \in R \text{ for some } a \in B\}.$$

The relation R has too some type of inverse called the **inverse relation** R^{-1}, which is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

However, functions are just a subset of relations. All functions are relations, but not all relations are functions. Finally, a **relation on a set** A is just a relation from A to A. Therefore, it is a subset of $A \times A$.

Problem 1. Let $A = \{a, b, c\}$ and $B = \{r, s, t, u\}$. Furthermore, let $R = \{(a, s), (a, t), (b, t)\}$ be a relation from A to B. Determine $\mathrm{dom}(R)$ and $\mathrm{range}(R)$.

Solution 1. The domain and range of the relation R are as follows:

$$\begin{aligned} \mathrm{dom}(R) &= \{a, b\} \quad \text{and} \\ \mathrm{range}(R) &= \{s, t\} \end{aligned}$$

Problem 2. Let A be a nonempty set and $B \subseteq \mathcal{P}(A)$. Define a relation R from A to B by $x\, R\, Y$ if $x \in Y$. Give an example of two sets A and B that illustrate this. What is R for these two sets?

Solution 2. Let $A = \{a, b, 3, 4\}$ and $B = \{\{a, b\}, \{3, 4\}, \emptyset\}$, and so $B \subseteq \mathcal{P}(A)$. Hence,

$$\begin{aligned} R &= \{(x, Y) : x \in Y\} \\ &= \{(a, \{a, b\}), (b, \{a, b\}), (3, \{3, 4\}), (4, \{3, 4\})\} \end{aligned}$$

Problem 3. Let $A = \{0, 1\}$. Determine all the relations on A.

Solution 3. Since $|A \times A| = 4$, it follows that there are $|\mathcal{P}(A \times A)| = 2^4 = 16$ possible relations on A. These are the following

$$\begin{aligned}
R_1 &= \emptyset \\
R_2 &= \{(0,0)\} \\
R_3 &= \{(0,1)\} \\
R_4 &= \{(0,0),(0,1)\} \\
R_5 &= \{(1,0)\} \\
R_6 &= \{(1,1)\} \\
R_7 &= \{(1,0),(1,1)\} \\
R_8 &= \{(0,0),(1,0)\} \\
R_9 &= \{(0,0),(1,1)\} \\
R_{10} &= \{(0,1),(1,0)\} \\
R_{11} &= \{(0,1),(1,1)\} \\
R_{12} &= \{(0,0),(0,1),(1,0)\} \\
R_{13} &= \{(0,0),(0,1),(1,1)\} \\
R_{14} &= \{(1,0),(1,1),(0,0)\} \\
R_{15} &= \{(1,0),(1,1),(0,1)\} \\
R^{16} &= A \times A
\end{aligned}$$

Problem 6. A relation R is defined on $\mathbb{N}$ by $a\ R\ b$ if $a/b \in \mathbb{N}$. For $c, d \in \mathbb{N}$, under what conditions is $c\ R^{-1}\ d$?

Solution 6. Note that $d/c \in \mathbb{N} \iff d\ R\ c$. Also, we know that $d\ R\ c \iff c\ R^{-1}\ d$. Hence, $d/c \in \mathbb{N} \iff c\ R^{-1}\ d$.

From this colloraly we can derived some interesting and useful theorem:

Theorem inverse. Let R be a relation with condition $P()$. Then, the condition for R^{-1} is also $P()$.

Proof. We know that $P(x,y) \iff (x,y) \in R$ and $(x,y) \in R \iff (y,x) \in R^{-1}$. Therefore, $P(x,y) \iff (y,x) \in R^{-1}$. □

Problem 7. For the relation $R = \{(x,y) : x + 4y \text{ is odd}\}$ defined on $\mathbb{N}$, what is R^{-1}?

Solution 7. Note that $4y = 2(2y)$ is even for all $y \in \mathbb{N}$. Therefore, x must be an odd integer in order for $x + 4y$ to be odd. Using **theorem inverse**,

$$\begin{aligned}
R^{-1} &= \{(y,x) : x + 4y \text{ is odd}\} \\
&= \{(y,x) : x \text{ is odd}\} \\
&= \mathbb{N} \times \mathbb{I},
\end{aligned}$$

where $\mathbb{I}$ is the set of odd positive integers.

Problem 8. For the relation $R = \{(x, y) : x \leq y\}$ defined on $\mathbb{N}$, what is R^{-1}?

Solution 8. Using **theorem inverse**.

$$R^{-1} = \{(y, x) : x \leq y\}$$

We can even change variables:

$$R^{-1} = \{(a, b) : b \leq a\}$$

Problem 9. Let A and B be sets such that $|A| = |B| = 4$.

(a) Prove or disprove: If R is a relation from A to B where $|R| = 9$ and $R = R^{-1}$, then $A = B$.

Solution 9. This statement is false. Note that $|R| = 9 = 3!$. Let's give a counterexmaple. Let $A = \{2, 4, 6, 9\}$ and $B = \{2, 4, 6, 11\}$. Consider some relation R from A to B such that

$$\begin{aligned} R &= \{(a, b) : a \text{ and } b \text{ are even}\} \\ &= \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\} \end{aligned}$$

and so

$$R^{-1} = \{(2, 2), (4, 2), (6, 2), (2, 4), (4, 4), (6, 4), (2, 6), (4, 6), (6, 6)\}.$$

Hence, $|R| = 9$ and $R = R^{-1}$. However, $A \neq B$.

(b) Show that by making a small change in the statement in (a), a different a different response to the resulting statement can be obtained.

Solution b. **Lemma 9.b.l.** Let A and B be two nonempty sets. Then, $A \times B = B \times A$ is a necessary and sufficient condition for $A = B$.

Proof. Let $A = B$. Then $A \times B = A \times A = B \times A$.
For the converse, assume that $A \neq B$. Then, either $A \not\subseteq B$ or $B \not\subseteq A$. Without loss of generality, let $A \not\subseteq B$ and so there is some $x \in A$ such that $x \notin B$. Then, there is some $(x, b) \in A \times B$. Since $x \notin B$, $(x, b) \notin B \times A$. Hence, $A \times B \neq B \times A$. $\square$

Theorem 9.b. If R is a relation from A to B where $|R| = 16$ and $R = R^{-1}$, then $A = B$.

Proof. Consider some sets A and B with $|A| = |B| = 4$. Sine $|R| = 16$ and $|A \times B| = 4 \cdot 4 = 16$, it follows, by definition of $R \subseteq A \times B$, that $R = A \times B$. Hence, $R^{-1} = \{(b, a) : (a, b) \in R = A \times B\} = B \times A$. However, we assumed that $R = R^{-1}$, which implies that $A \times B = B \times A$. By **Lemma 9.b.l**, $A = B$. $\square$

However, we are interested in the minimum amount of $|R|$ to imply $A = B$. Since $3! = 9$, adding another element would need for a fourth element of A and B to be equal.