Section 9.5: Composition of Functions

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We have previously defined operations on sets such as the integers modulo n. Some sets of functions are no exception. Let A, B', B and C be nonempty sets and consider the functions $f : A \to B'$ and $g : B \to C$. If $B' \subseteq B$, namely, if $\text{range}(f) \subseteq \text{dom}(g)$, then it is possible to create a new function from A to C called the composition of f and g. This composition $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$

Furthermore, it has some useful properties. Consider two functions f and g such that their composition $g \circ f$ is defined, then

(a) If both g and f are injective (surjective), then the composition $g \circ f$ is injective (surjective).

Clearly, one can further conclude that if g and f are bijective, then their composition $g \circ f$ is bijective. Keep in mind that in the beginning of the paragraph we assumed that their composition $g \circ f$ is defined. However, this is not a sufficient condition for $f \circ g$ to be defined. This depends on whether $\text{range}(g) \subseteq \text{dom}(f)$ is true or not.

Also, for nonempty functions f, g, h, if the compositions $g \circ f$ and $h \circ g$ are defined, then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined. Furthermore, $h \circ (g \circ f) = (h \circ g) \circ f$ and so the composition of f, g, h is **associative**.
Lastly, let's prove the following theorem.

Theorem 9.5.1. Let g and f be nonempty functions. If $\text{range}(f) \subseteq \text{dom}(g)$ then $g \circ f$ is a function.

Proof. Assume that $\text{range}(f) \subseteq \text{dom}(g)$. Consider some $(x, y) \in f$. Then, $(y, z) \in g$ and so $(x, z) \in g \circ f$. Hence, for any $x \in \text{dom}(f) = \text{dom}(g \circ f)$, there is an image $g(f(x)) = (g \circ f)(x)$ defined. We now prove that $g \circ f$ is well-defined. Consider two $a, b \in \text{dom}(g \circ f) = \text{dom}(f)$ such that $a = b$. Then, $f(a) = f(b) \in \text{dom}(g)$ and so $g(f(a)) = g(f(b))$. Hence, $(g \circ f)(a) = (g \circ f)(b)$. $\square$