Section 9.5: Composition of Functions

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We have previously defined operations on sets such as the integers modulo n. Some sets of functions are no exception. Let A, B', B and C be nonempty sets and consider the functions $f : A \to B'$ and $g : B \to C$. If $B' \subseteq B$, namely, if $\text{range}(f) \subseteq \text{dom}(g)$, then it is possible to create a new function from A to C called the composition of f and g. This composition $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$

Furthermore, it has some useful properties. Consider two functions f and g such that their composition $g \circ f$ is defined, then

(a) If both g and f are injective (surjective), then the composition $g \circ f$ is injective (surjective).

Clearly, one can further conclude that if g and f are bijective, then their composition $g \circ f$ is bijective. Keep in mind that in the beginning of the paragraph we assumed that their composition $g \circ f$ is defined. However, this is not a sufficient condition for $f \circ g$ to be defined. This depends on whether $\text{range}(g) \subseteq \text{dom}(f)$ is true or not.

Also, for nonempty functions f, g, h, if the compositions $g \circ f$ and $h \circ g$ are defined, then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined. Furthermore, $h \circ (g \circ f) = (h \circ g) \circ f$ and so the composition of f, g, h is **associative**.
Lastly, let's prove the following theorem.

Theorem 9.5.1. Let g and f be nonempty functions. If $\text{range}(f) \subseteq \text{dom}(g)$ then $g \circ f$ is a function.

Proof. Assume that $\text{range}(f) \subseteq \text{dom}(g)$. Consider some $(x, y) \in f$. Then, $(y, z) \in g$ and so $(x, z) \in g \circ f$. Hence, for any $x \in \text{dom}(f) = \text{dom}(g \circ f)$, there is an image $g(f(x)) = (g \circ f)(x)$ defined. We now prove that $g \circ f$ is well-defined. Consider two $a, b \in \text{dom}(g \circ f) = \text{dom}(f)$ such that $a = b$. Then, $f(a) = f(b) \in \text{dom}(g)$ and so $g(f(a)) = g(f(b))$. Hence, $(g \circ f)(a) = (g \circ f)(b)$. $\square$

Problem 38. Two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 3x^2 + 1$ and $g(x) = 5x - 3$ for all $x \in \mathbb{R}$. Determine $(g \circ f)(1)$ and $(f \circ g)(1)$.

Solution The composition functions $g \circ f : \mathbb{R} \to \mathbb{R}$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ are defined by $g(f(x)) = 5\,(3x^2+1) - 3 = 15x^2 + 2$ and $f(g(x)) = 3\,(5x-3)^2 + 1 = 75x^2 - 90x + 28$ for all $x \in \mathbb{R}$.
Hence, $(g \circ f)(1) = 17$ and $(f \circ g)(1) = 13$.

Problem 39. Two functions $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ and $g : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ are defined by $f([a]) = [3a]$ and $g([a]) = [7a]$.

(a) Determine $g \circ f$ and $f \circ g$.

Solution The composition functions $g \circ f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ and $f \circ g : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ are defined by $g(f([x])) = [21a] = [21][a] = [1][a] = [a]$ and $f(g([x])) = [21a] = [a]$ for every $[a] \in \mathbb{Z}_{10}$. Therefore, $g \circ f = f \circ g$.

(b) What can be concluded as a result of (a)?

Solution Both $g \circ f$ and $f \circ g$ are identity functions on $\mathbb{Z}_{10}$.

Problem 40. Let A and B be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

Proof. Note that $\text{range}(i_A) = A = \text{dom}(f)$ and $\text{range}(f) \subseteq \text{dom}(i_B) = B$. Hence, both functions $f \circ i_A : A \to B$ and $i_B \circ f : A \to B$ are defined by $(f \circ i_A)(x) = f(i_A(x)) = f(x)$ and $(i_B \circ f)(x) = i_B(f(x)) = f(x)$ for every $x \in A$. Both have the same Dominion and rule as f. Hence, $f \circ i_A = i_B \circ f = f$. $\square$

Problem 41. Let A be a nonempty set and let $f : A \to A$ be a function. Prove that if $f \circ f = i_A$, then f is bijective.

Proof. Assume that $f \circ f = i_A$ for a function $f : A \to A$. First, we show that f is injective. Let $f(a) = f(b)$ for some $a, b \in A$. Then, $(f \circ f)(a) = f(f(a)) = i_A(a) = a$ and $(f \circ f)(b) = f(f(b)) = i_A(b) = b$. Since f is a function and $f(a) = f(b)$, it follows that $f(f(a)) = a = b = f(f(b))$.
We now show that f is surjective. Consider any $c \in A$. Then, there is some $f(c) \in A$ and so $f(f(c)) = (f \circ f)(c) = c$. $\square$

Problem 42. Prove or disprove the following:

(a) If two functions $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.

Proof. Since both f and g are bijective, it follows that f and g are both **injective** and **surjective**. By **Theorem 11**, $g \circ f : A \to C$ is injective and surjective, which, by definition, is a bijective function. $\square$

(b) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is onto, then $g \circ f : A \to C$ is onto.

Solution This is false. Let $A = \{1\}$, $B = \{a, b, c\}$ and $C = \{1, 2, 3\}$. Also, let $f = \{(1, a)\}$ and $g = \{(a, 1), (b, 2), (c, 3)\}$. Hence, g is onto and $g \circ f = \{(1, 1)\}$ is not onto.

(c) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is one-to-one, then $g \circ f : A \to C$ is one-to-one.

Solution This is false. Let $A = \{1, 2\}$, $B = \{a\}$ and $C = \{1\}$. Also, let $f = \{(1, a), (2, a)\}$ and $g = \{(a, 1)\}$. Then, $g \circ f = \{(1, 1), (2, 1)\}$ is not one-to-one.

(d) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.

Solution Such functions exist. Let $A = \{1\}$, $B = \{a, b, c\}$ and $C = \{10\}$. Also, let $f = \{(1, a)\}$ and $g = \{(a, 10), (b, 10), (c, 10)\}$. Then, $g \circ f = \{(1, 10)\}$ is onto and f is not onto.

(e) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.

Proof. We show that this is false. Namely, we prove that if f is not one-to-one, then $g \circ f$ is not one-to-one. Since f is not one-to-one, there are at least two distinct $a, b \in A$ such that $f(a) = f(b)$. Since g is a function and $f(a) = f(b) \in B$, it follows that $g(f(a)) = g(f(b))$. Therefore, there two distinct $a, b \in A$ such that $(g \circ f)(a) = (g \circ f)(b)$ and so $g \circ f$ is not one-to-one. □

Problem 43. For nonempty sets A, B and C, let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove:

If $g \circ f$ is one-to-one, then f is one-to-one.

using as many of the following proof techniques as possible: direct proof, proof by contrapositive, proof by contradiction.

Solution (i) Direct Proof

Proof. Assume that $g \circ f : A \to C$ is one-to-one. Consider some $f(a) = f(b) \in B$ for some $a, b \in A$. Then, $g(f(a)) = (g \circ f)(a) = g(f(b)) = (g \circ f)(b)$. Since $g \circ f$ is one-to-one, it follows that $a = b$. Hence, f is one-to-one. □

(ii) Proof by Contrapositive.

Proof. We show that if f is not one-to-one, then $g \circ f$ is not one-to-one. Since f is not one-to-one, it follows that there are two distinct $a, b \in A$ such that $f(a) = f(b)$. Then, $g(f(a)) = (g \circ f)(a) = g(f(b)) = (g \circ f)(b)$ and $a \neq b$. The function $g \circ f$ is not one-to-one. □

(iii) Proof by Contradiction.

Proof. Suppose that there are functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f$ is one-to-one. We can use the argument made in the **Proof by Contrapositive** to arrive at the conclusion that there are two distinct $a, b \in A$ such that $(g \circ f)(a) = (g \circ f)(b)$, which contradicts our assumption that $g \circ f$ is one-to-one. $\square$

(b) Disprove: If $g \circ f$ is one-to-one, then g is one-to-one.

Solution We disprove this statement by giving functions $f : A \to B$ and $g : B \to C$ such that g is not one-to-one and $g \circ f : A \to C$ is one-to-one. Let $A = \{1\}, B = \{a, b\}$ and $C = \{10\}$, and define $f : A \to B$ and $g : B \to C$ by $f = \{(1, a)\}$ and $g = \{(a, 10), (b, 10)\}$. Then, $g \circ f = \{(1, 10)\}$ is one-to-one and g is not.