Section 1.2: Some Preliminaries

Juan Patricio Carrizales Torres

April 26, 2022

Problem 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof. Suppose, to the contrary, that $\sqrt{3}$ is rational. Thus, $\sqrt{3} = \frac{a}{b}$ for some integers a and $b \neq 0$. We may further assume that $\frac{a}{b}$ is totally simplified and so both integers have no common factors. Therefore, $a^2 = 3b^2$ and so $3 \mid a^2$ and $3 \mid a$. Thus, $a = 3r$ for some $r \in \mathbb{Z}$ and so $3r = b^2$, which implies that $3 \mid b$. Note that a and b have a factor in common, which leads to a contradiction.

Note that $\frac{a^2}{b^2} = 6 = 3(2)$ and $a^2 = 3(2)b^2$. Therefore, we have the starting "ingredients" to use the same technique to prove that $\sqrt{6}$ is irrational with a similar argument. $\square$

(b) Where does the proof of **Theorem 1.1.1** break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution (b). The proof breaks because $2^2 = 4$, and so $4r^2 = 2(2)b^2$ implies that $r^2 = b^2$ and so we are devoid of the chance of showing that a and b have 2 as a common factor with this specific argument.

Problem 1.2.2. Show that there is no rational number r satysfying $2^r = 3$

Proof. Suppose, to the contrary, that there is some rational number r such that $2^r = 3$. Thus,

$$2^{\frac{a}{b}} = 3$$

for some integers a and $b \neq 0$. We may further assume that $a \neq 0$ since $1 \neq 3$. Also note that $\frac{a}{b} > 0$ because 3 is an integer. Thus,

$$2^{\frac{a}{b}} = (2^a)^{\frac{1}{b}} = 3$$

which implies that $2^a = 3^b$. If $a, b > 0$, then 2^a and 3^b are an even and an odd number, respectively, which is a contradiction. On the other hand, if $a, b < 0$, then $2^a \cdot 3^{-b} = 3^b \cdot 3^{-b} = 1$. Note that $2^a \cdot 3^{-b} = 1$ implies that $3^{-b} = 2^{-a}$. Since $-a, -b > 0$, it follows that 2^{-a} and 3^{-b} are even and odd numbers, respectively. This is a contradiciton. $\square$

Problem 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

Solution a. This statement is false. Let $A_n = \{x \in \mathbb{N} : x \geq n\}$. Then, for every $n \in \mathbb{N}$, $A_n \supseteq A_{n+1}$ and A_n is infinite. Assume, to the contrary, that $\bigcap_{n=1}^{\infty} A_n$ contains some element $m \in \mathbb{N}$. However, $m \notin A_{m+1}$, which leads to a contradiciton.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

Solution b. This seems to be true. Since they are nested and finite sets, it follows that $|A_1| \geq A_n$ for all $n \in \mathbb{N}$. However, we only have a finite quantity number of elements for an infinite sequence of nonempty finite nested sets. Therefore, for some $k \in \mathbb{N}$, all sets $n \geq k$ must be equal. Hence, $\bigcap_{n=1}^{\infty} A_n = A_k$.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

Solution c. This is false. Let $A = \{1, 2, 3\}$, $B = \{3\}$ and $C = \{8, 2\}$. Then $A \cap (B \cup C) = A \cap \{8, 2, 3\} = \{2, 3\}$ and $(A \cap B) \cup C = \{3\} \cup \{8, 2\} = \{8, 2, 3\}$. Hence, $A \cap (B \cup C) \neq (A \cap B) \cup C$ and this represents a counterexample.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

Solution d. This is true.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution e. This is true.

Problem 1.2.5. (De Morgan's Laws). Let A and B b subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cup B)^c \subseteq A^c \cup B^c$.

Proof. Assume that $x \in (A \cap B)^c$. Then, $x \in \mathbb{R}$ such that $x \notin A \cap B$ and so $x \notin A$ or $x \notin B$. Therefore, $x \in A^c$ or $x \in B^c$, namely, $x \in A^c \cup B^c$. □

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

Proof. Let $x \in A^c \cup B^c$. Then, $x \in \mathbb{R}$ and either $x \notin A$ or $x \notin B$. Thus, $x \in \mathbb{R}$ and $x \notin A \cap B$, namely, $x \in (A \cap B)^c$. □

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof. With arguments **(a)** and **(b)** one can conlcude that $(A \cup B)^c = A^c \cap B^c$ (inclusion both ways). $\square$

Problem 1.2.6. (a) Verify the triangle inequality in the special case where a and b have the same sign.

Proof. Let $a, b \geq 0$. Hence, $a + b \geq 0$ and so $|a + b| = a + b = |a| + |b|$. On the other hand, consider some real numbers $a, b < 0$. Thus, $|a + b| < 0$ and so $|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$. $\square$

Lemma 1.2.6.b. For any $a, b \in \mathbb{R}$, $(a + b)^2 \leq (|a| + |b|)^2$.

Proof. Let $a, b \in \mathbb{R}$. Consider $(a + b)^2 = a^2 + 2ab + b^2$. Note that $a^2 + 2ab + b^2 = |a|^2 + 2ab + |b|^2 \leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$. $\square$

(b) Find an efficient proof for all the cases at once by first demonstrating $(a + b)^2 \leq (|a| + |b|)^2$.

Proof. Let a and b be some real numbers. Since either $|a+b| = a+b$ or $|a+b| = -(a+b)$, it follows that $(|a + b|)^2 = (a + b)^2 \leq (|a| + |b|)^2$, by **Lemma 1.2.6.b**. Because $|a + b|, |a| + |b| \geq 0$, it follows that $|a + b| \leq |a| + |b|$. $\square$

(c) Prove $|a - b| \leq |a - c| + |c - d| + |d - b|$ for all a, b, c, and d.

Proof. Consider some real numbers a, b, c and d. Note that

$$\begin{aligned} |[(a - c) + (c - d)] + (d - b)| &\leq |(a - c) + (c - d)| + |d - b| \\ &\leq |a - c| + |c - d| + |d - b|. \end{aligned}$$

Since $|[(a - c) + (c - d)] + (d - b)| = |a - b|$, it follows that $|a - b| \leq |a - c| + |c - d| + |d - b|$, as desired.

This seems to be a particular case of a more general statement, namely, let $\{a_1, a_2, \ldots, a_n\}$ be a set of n real numbers, then $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$. $\square$

(d) Prove $||a| - |b|| \leq |a - b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Proof. Consider some real numbers a and b. Note that

$$\begin{aligned} ||a| - |b|| &= ||(a - b) + b| - |b|| \\ &\leq ||a - b| + |b| - |b|| = ||a - b|| \\ &= |a - b|. \end{aligned}$$

Therefore, $||a| - |b|| \leq |a - b|$, as desired. $\square$

Problem G. iven a function f and s subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

Solution (a). Since for any $x \in A$, $x \geq 0$, it follows that $a < b$ implies $a^2 < b^2$ for some $a, b \in A$. Hence, $f(A) = [0, 4]$. The same argument can be used to conclude that $f(B) = [1, 16]$.

Note that

$$\begin{aligned} f(A \cap B) &= f([0,2] \cap [1,4]) = f([1,2]) \\ &= [1,4] \\ &= [0,4] \cap [1,16] = f(A) \cap f(B). \end{aligned}$$

Also,

$$\begin{aligned} f(A \cup B) &= f([0,2] \cup [1,4]) = f([0,4]) \\ &= [0,16] \\ &= [0,4] \cup [1,16] = f(A) \cup f(B). \end{aligned}$$

Therefore, for this particular case, both $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$ hold.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

Solution b. Let $A = [-4, -1]$ and $B = [-5, -1]$. Then $f(A \cap B) = f([-4, -1]) = [1, 16]$ and $f(A) \cap f(B) = [1, 16] \cap [1, 25] = [1, 25]$. Therefore, for this particular case, $f(A \cap B) \subset f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Proof. Consider some arbitrary function $g : \mathbb{R} \to \mathbb{R}$ and sets $A, B \subseteq \mathbb{R}$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, if follows that $g(A \cap B) \subseteq g(A)$ and $g(A \cap B) \subseteq g(B)$. Therefore, $g(A \cap B) \subseteq g(A) \cap g(B)$. $\square$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary funciton g.

Result d. For an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cup B) = g(A) \cup g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Proof. Consider some arbitrary function $g : \mathbb{R} \to \mathbb{R}$ and sets $A, B \subseteq \mathbb{R}$. Let $y \in g(A \cup B)$. This implies that there is some $x \in A \cup B$ such that $g(x) = y$. However, $x \in A$ or $x \in B$ and so $g(x) \in g(A)$ or $g(x) \in g(B)$. Hence, $g(A \cup B) \subseteq g(A) \cup g(B)$.

For the converse, let $y \in g(A) \cup g(B)$. Then, either $y \in g(A)$ or $y \in g(B)$. This means that there is some $x \in A$ or $x \in B$ such that $g(x) = y$. Then, $x \in A \cup B$ and so $g(x) = y \in g(A \cup B)$. Thus, $g(A \cup B) = g(A) \cup g(B)$.

□

Problem 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1,1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

Solution a. Let $f(x) = x^2$. Then, $f : \mathbb{N} \to \mathbb{N}$ is 1-1 since every natural number has one unique square and it is not onto since not all natural numbers are perfect squares.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

Solution b. Let

$$f(x) = \begin{cases} 1 & \text{if } x \leq 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Solution c. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ n & \text{if } \exists n \in \mathbb{N}, x = 2n \\ -n & \text{if } \exists n \in \mathbb{N}, x = 2n + 1 \end{cases}$$

Problem 1.2.9. Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is calles the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

Solution a. We know that $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. Hence,

$$\begin{aligned} f^{-1}(A \cap B) &= f^{-1}([-1, 1]) \\ &= [-1, 1] = [-2, 2] \cap [-1, 1] \\ &= f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

Also,

$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}([-1,4]) \\ &= [-2,2] = [-2,2] \cup [-1,1] \\ &= f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

(b) The good behavior of preimages demonstarted in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Proof. Let $A, B \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We divide this proof in two parts.

(a) We show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$. First, consider some $x \in g^{-1}(A \cap B)$. Then, there is some $y = g(x) \in A \cap B$ and so $y \in A$ and $y \in B$. Thus, $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ and so $x \in g^{-1}(A) \cap g^{-1}(B)$. Therefore, $g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$.

For the converse, consider some $x \in g^{-1}(A) \cap g^{-1}(B)$. Then, $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, and so there is some $y = g(x) \in A$ and $y = g(x) \in B$. Hence, $y \in A \cap B$, which implies that $x \in g^{-1}(A \cap B)$. Therefore, $g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$.

(b) We show that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$. Consider some $x \in g^{-1}(A \cup B)$. Then, there is some $y = g(x) \in A \cup B$ and so either $y \in A$ or $y \in B$. This implies that $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. Hence, $x \in g^{-1}(A) \cup g^{-1}(B)$ and so $g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B)$.

For the converse, let $x \in g^{-1}(A) \cup g^{-1}(B)$. Then, there is some $y = f(x)$ such that either $y \in A$ or $y \in B$. Hence, $y \in A \cup B$, which implies that $x \in g^{-1}(A \cup B)$. Therefore, $g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B)$.

$\square$

Problem 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

Proof. Since $y_1 = 6$, it follows that $y_2 = (2 \cdot 6 - 6)/3 = 2$ and so $y_1, y_2 > -6$. Assume that $y_k > -6$ for any $k \geq 2$. We show that $y_{k+1} > -6$. Note that

$$\begin{aligned} y_{k+1} &= \frac{2y_k - 6}{3} = \frac{2}{3}y_k - 2 \\ &> \frac{2}{3}(-6) - 2 \\ &= -6. \end{aligned}$$

By the Principle of Mathematical Induction, $y_n > -6$ for all $n \in \mathbb{N}$.

The number -6 seems to be some type of *limit* the sequence approaches. $\square$

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Problem 1.2.13. (a) Show how induction can be used to conclude that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$$

Proof. We proceed by induction. By **De Morgan's Laws**, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ for any sets A_1 and A_2. Hence, the statement is true for $n = 2$. Assume for any k sets $B_1, B_2, B_3, \ldots, B_k$ that

$$\left(\bigcup_{i=1}^{k} B_i\right)^c = \bigcap_{i=1}^{k} B_i^c.$$

We show fro any $k+1$ sets $C_1, C_2, C_3, \ldots, C_{k+1}$ that

$$\left(\bigcup_{i=1}^{k+1} C_i\right)^c = \bigcap_{i=1}^{k+1} C_i^c.$$

Note that

$$\left(\bigcup_{i=1}^{k+1} C_i\right)^c = \left(\left(\bigcup_{i=1}^{k} C_i\right) \cup C_{k+1}\right)^c.$$

Since $\bigcup_{i=1}^{k} C_i$ is a set, it follows, by **De Morgan's Laws**, that

$$\begin{aligned}\left(\left(\bigcup_{i=1}^{k} C_i\right) \cup C_{k+1}\right)^c &= \left(\bigcup_{i=1}^{k} C_i\right)^c \cap C_{k+1}^c \\ &= \bigcap_{i=1}^{k} C_i^c \cap C_{k+1}^c = \bigcap_{i=1}^{k+1} C_i^c\end{aligned}$$

according to our inductive hypothesis. By the Theorem of Mathematical Induction, for any $n \geq 2$ sets $A_1, A_2, \ldots, A_n$, it is true that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$$

□