Section 2.3: The Algebraic and Order Limit Theorems

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Now that we have a more formal and clearer definition of convergence for sequences, we can check some properties that come with this meaning. Namely, algebraic and order properties. Before that, we must mention an important theorem, which says that for any convergent sequence (a_n), there is some real number M such that (a_n) is inside $[-M, M]$. Namely, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $(a_n) \to a$ and $(b_n) \to b$. The algebraic property states the following:

(i) $\lim (ca_n) = ca$

(ii) $\lim (a_n + b_n) = a + b$

(iii) $\lim (a_n b_n) = ab$

(iv) $\lim (a_n/b_n) = a/b$, if $b \neq 0$.

THe interesting thing about the arguments given by the author to prove them is that they use the fact that one can make $|a_n - a|$ as small as one wants, namely, for any positive real number ε as small as one can imagine, there is some a_n such that $|a_n - a| < \varepsilon$. Also, these properties helps us interact with combinations of sequences and their limits on a more "familiar" way. On the other hand, the order property states the following:

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.

Noteworthy, if one changes the initial assumption from *for all* for *infinitely many*, the theorem holds. In other words, the property of a finite amount of elements in a sequence is not sufficient to predict the general property of its limit. The first $10^{10^{100}}$ elements can be positive but the rest be negative, which means that the sequence *eventually* aquires the property of negativity.

Problem 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

Proof. Observe that $|x_n - 0| = |x_n|$. Now, consider some positive real number ϵ. Since $(x_n) \to 0$, there is some $N \in \mathbb{N}$ such that $|x_n| = x_n < \epsilon^2$ for all $n \geq N$ $(0 \leq x_n)$. Then, $\sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$. Because $\sqrt{x_n} \geq 0$, it follows that $\sqrt{x_n} = \left|\sqrt{x_n}\right| = \left|\sqrt{x_n} - 0\right| < \epsilon$ for all $n \geq N$. □

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof. Consider some positive real number ϵ. Since $(x_n) \to 0$, it follows that there is some $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon^2$ for all $n \geq N$. Thus, $\sqrt{\epsilon^2} > \sqrt{|x_n - x|} \geq |\sqrt{x_n} - \sqrt{x}|$, and so $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ for all $n \leq N$. Therefore, $(\sqrt{x_n}) \to \sqrt{x}$. □

Problem 2.3.2. Using only Definition 2.2.3 (definition of convergence), prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$

Proof. First, note that $\left|\frac{2x_n-1}{3} - 1\right| = \left|\frac{2x_n-4}{3}\right|$. Now, consider some positive real number ϵ. Then, there is some $N \in \mathbb{N}$ such that $|x_n - 2| < \frac{3}{2}\epsilon$ for all $n \geq N$. Observe that

$$\begin{aligned}\frac{2}{3}|x_n - 2| &= \left|\frac{2}{3}\right||x_n - 2| \\ &= \left|\frac{2}{3}x_n - \frac{4}{3}\right| = \left|\frac{2x_n - 4}{3}\right| \\ &< \epsilon.\end{aligned}$$

Therefore, $\left|\frac{2x_n-1}{3} - 1\right| < \epsilon$ for all $n \geq \mathbb{N}$ and so $\left(\frac{2x_n-1}{3}\right) \to 1$. □

(b) $\left(\frac{1}{x_n}\right) \to 1/2$

Proof. Since $(x_n) \to 2$, it follows that there is an infinity of nonzero x_n that eventually get nearer and nearer to 2. Then, let's consider all $x_n \neq 0$ in the sequence $\left(\frac{1}{x_n}\right)$. Note that $\left|\frac{1}{x_n} - 1/2\right| = \left|\frac{2-x_n}{2x_n}\right| = \frac{|2-x_n|}{2|x_n|}$. Now, let ϵ be any positive real number. Then, there is some $K_1 \in \mathbb{N}$ such that $|x_n - 2| < 1$, which implies that $|x_n| > 1$. Furthermore, there is a positive integer K_2 such that $|x_n - 2| < 2\epsilon$ for all $n \geq K_2$. Thus,

$$|2 - x_n|\frac{1}{2|x_n|} < |2 - x_n|\frac{1}{2} < 2\epsilon\frac{1}{2}$$

for any $n \geq M$, where $M = \max(K_1, K_2)$. Then, $\left|\frac{1}{x_n} - 1/2\right| < \epsilon$ for all $n \geq M$. Hence, $\left(\frac{1}{x_n}\right) \to 1/2$. □

Problem 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof. Since $(x_n) \to l$ and $(z_n) \to l$, it follows that for some positive real numbr ϵ, there is some positive integer K, such that $|x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$ for all $n \geq K$. Hence, $-\epsilon < x_n - l < \epsilon$ and $-\epsilon < z_n - l < \epsilon$, and so $l - \epsilon < x_n, z_n < l + \epsilon$. Since $x_n \leq y_n \leq z_n$, it follows that $l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon$. Thus, $|y_n - l| < \epsilon$ for each $n \geq K$, namely, $(y_n) \to l$. □

Problem 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

Solution In the denominator and numerator, there is a 1 being added. This is constant for all $n \in \mathbb{N}$. Hence, one can define the sequence (1) (all elements are 1), which $(1) \to 1$. Since $(2a_n) \to 2 \cdot 0 = 0$, $(3a_n) \to 3 \cdot 0 = 0$ and $(4a_n a_n) \to 4 \cdot 0 \cdot 0 = 0$, it follows that $(1 + 2a_n) \to 1 + 0 = 1$ and $(1 + 3a_n - 4a_n^4) \to 1 + 0 - 0 = 1$. Thus, $\left(\frac{1}{1+3a_n-4a_n^4} \right) \to \frac{1}{1} = 1$ (recall that all fractions are defined and so $1 + 3a_n - 4a_n^4 \neq 0$ for all $n \in \mathbb{N}$) and

$$\lim \left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2} \right) = 1 * 1 = 1$$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

Solution Note that

$$\begin{aligned} \frac{(a_n + 2)^2 - 4}{a_n} &= \frac{a_n^2 + 4a_n}{a_n} \\ &= a_n + 4 \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $(a_n + 4) \to 0 + 4 = 4$.

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$

Solution Note that

$$\begin{aligned} \frac{\frac{2}{a_n} + 3}{\frac{1}{a_n} + 5} &= \frac{\frac{2+3a_n}{a_n}}{\frac{1+5a_n}{a_n}} \\ &= \frac{2 + 3a_n}{1 + 5a_n} \end{aligned}$$

for all $n \in \mathbb{N}$. Also, $(2 + 3a_n) \to 2 + 3 \cdot 0 = 2$ and $(1 + 5a_n) \to 1 + 5 \cdot 0 = 1$. Thus,

$$\lim \left(\frac{\frac{2}{a_n} + 3}{\frac{1}{a_n} + 5} \right) = \lim \left(\frac{2 + 3a_n}{1 + 5a_n} \right) = 1/1 = 1.$$

Problem 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that z_n is convergent if and only if (x_n) and (y_n) are both convergent with $\lim(x_n) = \lim(y_n)$.

Proof. Assume that (x_n) and (y_n) are both convergent with $\lim(x_n) = \lim(y_n) = M$, where $M \in \mathbb{R}$. We show that $(z_n) \to M$. Consider some positive real number ϵ. By definition of convergence, there is some $K \in \mathbb{N}$ such that $|x_n - M|, |y_n - M| < \epsilon$ for all $n \geq K$. Thus, $|z_n - M| < \epsilon$ for all $n \geq K$.
For the converse, assume that (z_n) is convergent. Specifically, $(z_n) \to M$ for some real number M. Consider some positive real number ϵ. Then, there is some $K \in \mathbb{N}$ such that $|z_n - M| < \epsilon$ for all $n \geq K$. By definition of z_n, it follows that $|x_n - M|, |y_n - M| < \epsilon$ for all $n \geq K$. Thus, $(x_n) \to M$ and $(y_n) \to M$. ☐

Problem 2.3.6. Consider the sequence given $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Excercise 2.3.1, show $\lim(b_n)$ exists and find the value of the limit.

Proof. Note that

$$\begin{aligned}
b_n &= n - \sqrt{n^2 + 2n} \\
&= \left(n - \sqrt{n^2 + 2n}\right) \frac{n + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 + 2n}} \\
&= \frac{(n^2 - (n^2 + 2n))}{n + \sqrt{n^2 + 2n}} \\
&= \frac{-2n}{n + \sqrt{n^2\left(1 + \frac{2}{n}\right)}} \\
&= \frac{-2n}{n\left(1 + \sqrt{1 + \frac{2}{n}}\right)} = \frac{-2}{1 + \sqrt{1 + (2/n)}}
\end{aligned}$$

for each $n \in \mathbb{N}$. Observe that $(1 + (2/n)) \to 1 + 2 \cdot 0 = 1$. Thus, by **Excercise 2.3.1**, $\left(\sqrt{1 + (2/n)}\right) \to \sqrt{1} = 1$ and so $\left(\frac{1}{1+\sqrt{1+(2/n)}}\right) \to 1/(1 + 1) = 1/2$. Therefore, $(b_n) = \left(\frac{-2}{1+\sqrt{1+(2/n)}}\right) \to -2(1/2) = -1$ and so the limit exists. ☐

Problem 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

Solution Such sequences can be constructed. Let (x_n) and (y_n) be defined by $x_n = n$ and $y_n = -n$ for all $n \in \mathbb{N}$. Note that both (x_n) and (y_n) keep approaching objects that are not real numbers, namely, ∞ and $-\infty$ respectively. They diverge. However, $(z_n) = (x_n + y_n) = (n - n) = (0) \to 0$ converges.

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

Proof. We prove that this leads to a contradiction. Let (x_n) be a convergent sequence and (y_n) be a divergent one. Then $(-x_n)$ is convergent. Suppose that $(x_n + y_n)$ converges and so $((x_n + y_n) + (-x_n)) = (y_n)$ is the sum of two convergent sequences. Hence, (y_n) is a convergent sequence. This contradicts our initial assumption. □

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

Solution Let $(b_n) = (1/n)$. Then $(b_n) \to 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$. However, $(1/b_n) = (n)$ diverges to ∞.

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

Proof. We show that this leads to a contradiction. Let (a_n) and (b_n) be unbounded and convergent sequences, respectively. Since (b_n) is convergent, it follows that it is bounded. Also, assume that $(a_n - b_n)$ is bounded. Then, there are real numbers $M, C \geq 0$ such that $|b_n| \leq M$ and $|a_n - b_n| \leq C$ for any $n \in \mathbb{N}$. Thus,

$$\begin{aligned} M + C &\geq |a_n - b_n| + |b_n| \\ &\geq |(a_n - b_n) + b_n| = |a_n| \end{aligned}$$

for every positive integer n. Hence, (a_n) is a bounded sequence. This contradicts our initial assumption. □

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution Let $(a_n) = (0, 0, 0, 0, 0, 0, 0, 0, 0, \dots)$ and $(b_n) = (1, 2, 3, 4, 5, 6, 7, \dots)$. Then $(a_n) \to 0$ and (b_n) does not converge to a real number. However, $(z_n) = (a_n b_n) = (0 \cdot 1, 0 \cdot 2, 0 \cdot 3, \dots) = (0, 0, 0, 0, 0, 0, 0) \to 0$.

Problem 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

Proof. Note that a finite polynomial $p(x)$ of degree k can be expressed as $\sum_{j=0}^{k} a_j x^j$, where a_j are the real coefficients. Hence, each $p(x_n)$ is a member of the sequence of real numbers $\left(\sum_{j=0}^{k} a_j x_n^j\right)$. Since $(x_n) \to x$, it follows, by the algebraic properties of limits of sequences, that $(a_j x_n^j) \to (a_j x^j)$ for all j and so $\left(\sum_{j=0}^{k} a_j x_n^j\right) \to \left(\sum_{j=0}^{k} a_j x^j\right) = p(x)$. Thus, $p(x_n) \to p(x)$. □

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution Let $f : \mathbb{R} \to \mathbb{R}$ be a function be defined by $f(x) = 0$ and $f(a) = 1$ for the rest of $a \in \mathbb{R}$. Then, $f(x_n) = 1$ for all but finitely many $n \in \mathbb{N}$ and so $f(x_n) \to 1 \neq 0 = f(x)$.

Problem 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim(b_n) = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

Proof. Since (a_n) is bounded, we have that $|a_n| \leq M$ for some nonnegative real number M. If $M = 0$, then $a_n = 0$ for all $n \in \mathbb{N}$ and so $(a_n) \to 0$. This implies that $(a_n b_n) \to 0$ since $(b_n) \to 0$. Therefore, we assume that M is nonzero. Now, consider some positive real number ϵ. Then, ϵ/M is positive and so there is some positive integer N such that $|b_n - 0| = |b_n| < \epsilon/M$ for all $n \geq N$. Note that $0 \leq |b_n|$ for all $n \in \mathbb{N}$ and $0 < M$, and so

$$|a_n||b_n| \leq M|b_n| < (\epsilon/M)M = \epsilon.$$

for all $n \geq N$. Therefore, $|a_n||b_n| = |a_n b_n| = |a_n b_n - 0| < \epsilon$ for all $n \geq N$. Hence, $(a_n b_n) \to 0$.

We were not allowed to use the Algebraic Limit Theorem since both (a_n) and (b_n) must converge, but in this case (a_n) is bounded but not necessarily convergent. □

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

Proof. This leads to a contradiction if (a_n) is nonconvergent. In fact, for any sequence (a_n) and convergent sequence $(b_n) \to c \neq 0$, $(a_n b_n)$ diverges. Suppose, to the contrary, that there is some nonconvergent sequence (a_n) and sequence (b_n) that converges to some nonzero c such that $(a_n b_n)$ converges to some b. Since $(b_n) \to c \neq 0$, eventually, all b_n are nonzero, namely, all but finitely many of them are nonzero. Hence, we can create a sequence $(1/b_n)$ defined by $1/b_n$ if $b_n \neq 0$ and 1 if $b_n = 0$. Hence, $(1/b_n) \to 1/c$. Therefore, $\left(\frac{1}{b_n} a_n b_n\right) = (a_n) \to (b/c)$ for all but finitely many of $n \in \mathbb{N}$, and so (a_n) is a convergent sequence. This is a contradiction. □

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Proof. Since (b_n) is convergent, it follows that it is bounded. Furthermore, because $(a_n) \to 0$, by (a), $(a_n b_n) = 0$. □

Problem 2.3.11 (Cesaro Means). (a) Show that if (x_n) is a convergent squence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

Proof. Assume that $(x_n) \to x$. Then, there is some positive number $\varepsilon_{max} > |x_n - x|$ for all $n \in \mathbb{N}$. Consider some positive real numbers ε_1 and a lower one such that $\varepsilon_2 < \varepsilon_{max}$. Thus, $0 < \varepsilon_1 - \varepsilon_2 = \delta$ and so $\varepsilon_1 = \varepsilon_2 + \delta$. Since (x_n) converges to x, we know that there is some $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon_2$ for all $n > N$. Choose some $n > N$ and so

$$\begin{aligned}
|y_n - x| &= \left| \frac{\sum_{i=1}^n x_i}{n} - x \right| \\
&= \left| \frac{\sum_{i=1}^n x_i - xn}{n} \right| = \left| \frac{\sum_{i=1}^n (x_i - x)}{n} \right| \\
&= \frac{1}{n} \left| \sum_{i=1}^n (x_i - x) \right| \leq \frac{1}{n} \sum_{i=1}^n |x_i - x| \\
&< \frac{N\varepsilon_{max} + (n-N)\varepsilon_2}{n} = \varepsilon_2 + \frac{N\varepsilon_{max} - N\varepsilon_2}{n}
\end{aligned}$$

since $|x_n - x| < \varepsilon_2$ for all $n > N$ and $|x_n - x| < \varepsilon_{max}$ for every $n \in \mathbb{N}$, which includes the first N integers. Note that $0 < N\varepsilon_{max} - N\varepsilon_2$ since $\varepsilon_{max} > \varepsilon_2$. By the archimedean property of $\mathbb{R}$, we can choose some positive integer $k > N$ large enough such that $N\varepsilon_{max} - N\varepsilon_2 < k\delta$ and so $N\varepsilon_{max} - N\varepsilon_2 < n\delta$ for all $n \geq k$. Therefore, $\delta > \frac{N\varepsilon_{max} - N\varepsilon_2}{n} > 0$ and

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n |x_i - x| &< \varepsilon_2 + \frac{N\varepsilon_{max} - N\varepsilon_2}{n} \\
&< \varepsilon_2 + \delta = \varepsilon_1.
\end{aligned}$$

for all $n \geq k$. We have shown for any real number ε_1 that there is some $k \in \mathbb{N}$ such that $|y_n - x| < \varepsilon_1$ for all $n \geq k$. Thus, $(y_n) \to x$. □

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.