Section 8.2: Properties of relations

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May 13, 2022

This chapter mentioned three properties of interested for some relation R on a single set A. Since most of these properties involve implications with universal quantifiers, the easiest way to check wether a relation has certain property is by looking for specific examples for which the implication in question is false.

(a) **Reflexive Property:** if $x \in A$, then $(x, x) \in R$. (x is related to itself)

(b) **Symmetric Property:** $\forall x, y \in A$, if $x \; R \; y$, then $y \; R \; x$ (x is related to y and viceversa). Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$ and $y \not R \; x$. For this to happen, it is necessary that $x \neq y$.

(c) **Transitive Property:** $\forall x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$. Note that for the relation R to not be symmetric, it must be true that $x \; R \; y$, $y \; R \; z$ and $x \not R \; z$. For this to happen, it is necessary that $x \neq y$ and $z \neq y$.

Problem 11. Let $A = \{a, b, c, d\}$ and let

$$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$$

be a relation on A. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 11. The relation is reflexive since $\{(a, a), (b, b), (c, c), (d, d)\} \subset R$. Also, it is transitive since $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in A$ is fulfilled. However, the relation is not symmetric since $(a, b) \in R$ and $(b, a) \notin R$.

Problem 13. Let $S = \{a, b, c\}$. Then $R = \{(a, b)\}$ is a relation on S. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 13. The relation S is transitive since the implication $(x, y), (y, z) \in R \implies (x, z) \in R$ for any $x, y, z \in S$ is fulfilled vacuously. However, it is neither reflexive because $(a, a) \notin R$ nor symmetrice since $(a, b) \in R$ but $(b, a) \notin R$.

Problem 14. Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

Solution 14. Let $R = \{(a, b), (b, c)\}$. The relation R is not reflexive since $(a, a) \notin R$, it is not symmetric because $(a, b) \in R$ and $(b, a) \notin R$ and it is not transitive since $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Problem 15. A relation R is defined on $\mathbb{Z}$ by $a\ R\ b$ if $|a - b| \leq 2$. Which of the properties reflexive, symmetric and transitive does the relation R possess? Justify your answers.

Solution 15. The relation R is reflexive since $|a - a| = 0 \leq 2$ for any $a \in \mathbb{Z}$ and so $a\ R\ a$. It is symmetric since for any $a, b \in \mathbb{Z}$, if $|a - b| \leq 2$, then $|b - a| = |a - b| \leq 2$. However, it is not transitive since $|3 - 1| = 2$ and $|1 - 0| = 1$ but $|3 - 0| = 3 > 2$.

Problem 16. Let $A = \{a, b, c, d\}$. How many relations defined on A are reflexive, symmetric and transitive and contain the ordered pairs $(a, b), (b, c), (c, d)$?

Solution 16. In order for a relation R on A to be reflexive it must be true that $\{(a, a), (b, b), (c, c), (d, d)\} \subseteq R$. Since $(a, b), (b, c), (c, d) \in R$, it follows that $(b, a), (c, b), (d, c) \in R$ so that R is symmetric. Because, so far

$$\{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c)\} \subseteq R$$

, it follows that $(a, c), (c, a), (b, d) \in R$ for R to be transitive. Since $(b, d) \in R$, it follows that $(d, b) \in R$ so that the symmetric property is mantained. However, $(d, b), (b, a) \in R$ and so $(d, a) \in R$ so that it is transitive. This implies $(a, d) \in R$ since R must be symmetric. Hence,

$$\begin{aligned} R &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (c, d), (b, a), (c, b), (d, c), (a, c), (c, a), (b, d), (d, b), (d, a), (a, d)\} \\ &= A \times A \end{aligned}$$

Since $R \subseteq A \times A$, it follows that there is only one possible relation R on A that fulfills the conditions.

Problem 18. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric but not transitive.

Solution (a). contenidos...

(b) reflexive and transitive but not symmetric.

Solution (b). contenidos...

(c) symmetric and transitive but not reflexive.

Solution (c). contenidos...

(d) reflexive but neither symmetric nor transitive.

Solution (d). contenidos...

(e) symmetric but neither reflexive nor transitive.

Solution (e). contenidos...

(f) transitive but neither reflexive nor symmetric.

Solution (f). contenidos...

Problem 19. A relation R is defined on $\mathbb{Z}$ by $x \ R \ y$ if $x \cdot y \geq 0$. Prove or disprove the following:

(a) R is reflexive.

(b) R is symmetric.

(c) R is transitive.

Problem 20. Determine the maximum number of elements in a relation R on a $3-element$ set such that R has none of the properties reflexive, symmetric and transitive.

Solution 20. Let R be a relation on a 3-element set B that has none of the properties reflexive, symmetric and transitive. Let's check the maximum number of elements R can contain. Since $R \subseteq B \times B$, it follows that $|R| \leq 9$. However, since R is not reflexive, it follows that $(b,b) \notin R$ for some $b \in B$ and so $|R| \leq 8$.

Because R is not symmetric, it follows that $(b,a) \in R$ and $(a,b) \notin R$ for some different $a, b \in B$ and so $|R| \leq 7$. Also, since R is not transitive, it follows that $(a,b), (b,c) \in R$ and $(a,c) \notin R$ for some $a, b, c \in B$ such that $a \neq b$ and $b \neq c$. Thus, either $c \neq a$ or $c = a$, however note that we already got rid of those two such ordered pairs and so the maximum number of elements in R is 7.

Problem 21. Let S be the set of all polynomials of degree at most 3. An element $s(x)$ of S can then be expressed as $s(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. A relation

Solution 21. contenidos...

Problem 22. A relation R is defined on $\mathbb{N}$ by $a \ R \ b$ if either $a \mid b$ or $b \mid a$. Determine which of the properties reflexive, symmetric and transitive are possessed by R.