# Week 9

Juan Patricio Carrizales Torres
Section 2: Direct Proofs

September 17, 2021

Let $P(x)$ and $Q(x)$ be open sentences over a domain $S$ such that they have some "connection". If one wants to prove that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$, one may assume that $P(x)$ is true for an arbitrary $x \in S$ and show that $Q(x)$ is true for that same $x$. This is known as a direct proof and it uses the fact that an implication statement can only be false when the hypothesis is true and the conclusion is false.

**Problem 8.** Prove that if $x$ is an odd integer, then $9x + 5$ is even.

*Proof.* Assume that $x$ is an odd integer. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. Hence,

$$9(2k + 1) + 5 = 18x + 14 = 2(9x + 7)$$

Since $9x + 7$ is an integer, it follows that $9x + 5$ is even. $\square$

**Problem 9.** Prove that if $x$ is an even integer, then $5x - 3$ is an odd integer.

*Proof.* Since $x$ is an even integer, we can write $x = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$5(2k) - 3 = 2(5k) - 4 + 1 = 2(5k - 2) + 1$$

Because $5k - 2$ is an integer, $5x - 3$ is odd. $\square$

**Problem 10.** Prove that if $a$ and $c$ are odd integers, then $ab + bc$ is even for every integer $b$.

*Proof.* Let $a$ and $c$ be odd integers. Then $a = 2m + 1$ and $c = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore,

$$(2m + 1)b + b(2n + 1) = b(2m + 2n + 2) = 2b(m + n + 1) \tag{1}$$

Since $b(m + n + 1)$ is an integer, $ab + bc$ is even $\square$

**Problem 11.** Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

*Proof.* Let $1 - n^2 > 0$. Then $0 \leq n^2 < 1$ and so $n = 0$. Hence, $3(0) - 2 = -2 = 2(-1)$. Thus, $3n - 2$ is even. $\square$

**Problem 12.** Let $x \in \mathbb{Z}$. Prove that if $2^{2x}$ is an odd integer, then $2^{-2x}$ is an odd integer.

*Proof.* Let $2^{2x}$ be an odd integer. If $x < 0$ then $2^{2x}$ is not an integer; while if $x > 0$, then $2^{2x}$ is even (2 multiplies itself $2x$ times). Since $2^{2(0)} = 1$, it follows that $x = 0$. Therefore, $2^{-2(0)} = 1$ is odd. $\square$

**Problem 13.** Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n+1)^2(n+2)^2/4$ is even, then $(n+2)^2(n+3)^2/4$ is even.

*Proof.* Let $n \in S$ such that $(n+1)^2(n+2)^2/4$ is even. Since $(n+1)^2(n+2)^2/4 = 1$ when $n = 0$, $(n+1)^2(n+2)^2/4 = 9$ when $n = 1$, and $(n+1)^2(n+2)^2/4 = 36$ when $n = 2$, it follows that $n = 2$. Therefore, when $n = 2$, $(n+2)^2(n+3)^2/4 = 100$, which is even. $\square$

**Problem 14.** Let $S = \{1, 5, 9\}$. Prove that if $n \in S$ and $\frac{n^2+n-6}{2}$ is odd, then $\frac{2n^3+3n^2+n}{6}$ is even.

*Proof.* Note that for all $n \in S$, $\frac{n^2+n-6}{2}$ is even. Therefore, this implication is true vacuously. $\square$

**Problem 15.** Let $A = \{n \in \mathbb{Z} : n > 2 \text{ and } n \text{ is odd}\}$ and $B = \{n \in \mathbb{Z} : n < 11\}$. Prove that if $n \in A \cap B$, then $n^2 - 2$ is prime.

*Proof.* Assume that $n \in A \cap B$. Then $2 < n < 11$ and $n$ is odd, and so $n \in \{3, 5, 7, 9\} = A \cap B$. Note that $3^2 - 2 = 7$, $5^2 - 2 = 23$, $7^2 - 2 = 47$ and $9^2 - 2 = 79$ are all prime numbers. Thus, this implication is true. $\square$