Section 9.5: Composition of Functions

Juan Patricio Carrizales Torres

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We have previously defined operations on sets such as the integers modulo n. Some sets of functions are no exception. Let A, B', B and C be nonempty sets and consider the functions $f : A \to B'$ and $g : B \to C$. If $B' \subseteq B$, namely, if $\text{range}(f) \subseteq \text{dom}(g)$, then it is possible to create a new function from A to C called the composition of f and g. This composition $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$

Furthermore, it has some useful properties. Consider two functions f and g such that their composition $g \circ f$ is defined, then

(a) If both g and f are injective (surjective), then the composition $g \circ f$ is injective (surjective).

Clearly, one can further conclude that if g and f are bijective, then their composition $g \circ f$ is bijective. Keep in mind that in the beginning of the paragraph we assumed that their composition $g \circ f$ is defined. However, this is not a sufficient condition for $f \circ g$ to be defined. This depends on whether $\text{range}(g) \subseteq \text{dom}(f)$ is true or not.

Also, for nonempty functions f, g, h, if the compositions $g \circ f$ and $h \circ g$ are defined, then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined. Furthermore, $h \circ (g \circ f) = (h \circ g) \circ f$ and so the composition of f, g, h is **associative**.
Lastly, let's prove the following theorem.

Theorem 9.5.1. Let g and f be nonempty functions. If $\text{range}(f) \subseteq \text{dom}(g)$ then $g \circ f$ is a function.

Proof. Assume that $\text{range}(f) \subseteq \text{dom}(g)$. Consider some $(x, y) \in f$. Then, $(y, z) \in g$ and so $(x, z) \in g \circ f$. Hence, for any $x \in \text{dom}(f) = \text{dom}(g \circ f)$, there is an image $g(f(x)) = (g \circ f)(x)$ defined. We now prove that $g \circ f$ is well-defined. Consider two $a, b \in \text{dom}(g \circ f) = \text{dom}(f)$ such that $a = b$. Then, $f(a) = f(b) \in \text{dom}(g)$ and so $g(f(a)) = g(f(b))$. Hence, $(g \circ f)(a) = (g \circ f)(b)$. $\square$

Problem 38. Two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 3x^2 + 1$ and $g(x) = 5x - 3$ for all $x \in \mathbb{R}$. Determine $(g \circ f)(1)$ and $(f \circ g)(1)$.

Solution The composition functions $g \circ f : \mathbb{R} \to \mathbb{R}$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ are defined by $g(f(x)) = 5\,(3x^2 + 1) - 3 = 15x^2 + 2$ and $f(g(x)) = 3\,(5x - 3)^2 + 1 = 75x^2 - 90x + 28$ for all $x \in \mathbb{R}$.
Hence, $(g \circ f)(1) = 17$ and $(f \circ g)(1) = 13$.

Problem 39. Two functions $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ and $g : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ are defined by $f([a]) = [3a]$ and $g([a]) = [7a]$.

(a) Determine $g \circ f$ and $f \circ g$.

Solution The composition functions $g \circ f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ and $f \circ g : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ are defined by $g(f([x])) = [21a] = [21][a] = [1][a] = [a]$ and $f(g([x])) = [21a] = [a]$ for every $[a] \in \mathbb{Z}_{10}$. Therefore, $g \circ f = f \circ g$.

(b) What can be concluded as a result of (a)?

Solution Both $g \circ f$ and $f \circ g$ are identity functions on $\mathbb{Z}_{10}$.

Problem 40. Let A and B be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

Proof. Note that $\text{range}(i_A) = A = \text{dom}(f)$ and $\text{range}(f) \subseteq \text{dom}(i_B) = B$. Hence, both functions $f \circ i_A : A \to B$ and $i_B \circ f : A \to B$ are defined by $(f \circ i_A)(x) = f(i_A(x)) = f(x)$ and $(i_B \circ f)(x) = i_B(f(x)) = f(x)$ for every $x \in A$. Both have the same Dominion and rule as f. Hence, $f \circ i_A = i_B \circ f = f$. $\square$