Introduction to Bandits: Algorithms and Theory
Part 2: Bandits with large sets of actions

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K-armed bandit, with $K = 4$

At each round t, select a tap. Optimize quality of n selected beers.

Bandit with a large number of arms

Goal: optimize the beer you drink before you get drunk...

Part 2: Bandits with a large set of actions

The number of arms is larger then the number of rounds.

- Unstructured set of actions:
  1. Many-armed bandits
- Structured set of actions:
  2. Linear bandits
  3. Lipschitz bandits
  4. Bandits in trees
- Extensions

We consider the “optimism in the face of uncertainty” principle in stochastic environments.

A few references on bandits since 2005...

[Abbasi-Yadkori, 2009] [Abernethy, Hazan, Rakhlin, 2008] [Abernethy, Bartlett, Rakhlin, Tewari, 2008] [Abernethy, Agarwal, Bartlett, Rakhlin, 2009] [Audibert, Bubeck, 2010] [Audibert, Munos, Szepesvári, 2009] [Audibert, Bubeck, Lugosi, 2011] [Auer, Ortner, Szepesvári, 2007] [Auer, Ortner, 2010] [Awerbuch, Kleinberg, 2008] [Bartlett, Hazan, Rakhlin, 2007] [Bartlett, Dani, Hayes, Kakade, Rakhlin, Tewari, 2008] [Bartlett, Tewari, 2009] [Ben-David, Pal, Shalev-Shwartz, 2009] [Blum, Mansour, 2007] [Bubeck, 2010] [Bubeck, Munos, 2010] [Bubeck, Munos, Stoltz, 2009] [Bubeck, Munos, Stoltz, Szepesvári, 2008] [Cesa-Bianchi, Lugosi, 2006] [Cesa-Bianchi, Lugosi, 2009] [Chakrabarti, Kumar, Radlinski, Upfal, 2008] [Chu, Li, Reyzin, Schapire, 2011] [Coquelin, Munos, 2007] [Dani, Hayes, Kakade, 2008] [Dorard, Glowacka, Shawe-Taylor, 2009] [Filippi, 2010] [Filippi, Cappé, Garivier, Szepesvári, 2010] [Flaxman, Kalai, McMahan, 2005] [Garivier, Cappé, 2011] [Grünewälder, Audibert, Opper, Shawe-Taylor, 2010] [Guha, Munagala, Shi, 2007] [Hazan, Agarwal, Kale, 2006] [Hazan, Kale, 2009] [Hazan, Megiddo, 2007] [Honda, Takemura, 2010] [Jaksch, Ortner, Auer, 2010] [Kakade, Shalev-Shwartz, Tewari, 2008] [Kakade, Kalai, 2005] [Kale, Reyzin, Schapire, 2010] [Kanade, McMahan, Bryan, 2009] [Kleinberg, 2005] [Kleinberg, Slivkins, 2010] [Kleinberg, Niculescu-Mizil, Sharma, 2008] [Kleinberg, Slivkins, Upfal, 2008] [Kocsis, Szepesvári, 2006] [Langford, Zhang, 2007] [Lazaric, Munos, 2009] [Li, Chu, Langford, Schapire, 2010] [Li, Chu, Langford, Wang, 2011] [Lu, Pàl, Pàl, 2010] [Maillard, 2011] [Maillard, Munos, 2010] [Maillard, Munos, Stoltz, 2011] [McMahan, Streeter, 2009] [Narayanan, Rakhlin, 2010] [Ortner, 2008] [Pandey, Agarwal, Chakrabarti, Josifovski, 2007] [Poland, 2008] [Radlinski, Kleinberg, Joachims, 2008] [Rakhlin, Sridharan, Tewari, 2010] [Rigollet, Zeevi, 2010] [Rusmevichientong, Tsitsiklis, 2010] [Shalev-Shwartz, 2007] [Slivkins, Upfal, 2008] [Slivkins, 2011] [Srinivas, Krause, Kakade, Seeger, 2010] [Stoltz, 2005] [Sundaram, 2005] [Wang, Kulkarni, Poor, 2005] [Wang, Audibert, Munos, 2008]

and I surely missed many relevant references...

Unstructured set of actions: Examples

There is an infinite number of arms. The rewards received so far do not tell us anything about the value of unobserved arms.
Example: Enjoy Parisian restaurants.
Each day, select a restaurant:

- among the ones where you have already been
  - because it is good (Exploitation)
  - or not well known (Exploration)
- or choose a new one randomly (Discovery)

Other examples:

- Mining for valuable resources (such as gold or oil): exploit good wells, explore unknown wells, or start diging at a new location.
- Marketing (e.g. send catalogues to good customers, uncertain customers, or random people).

Many-armed bandits: Assumptions

We make a (probabilistic) assumption about the mean-value of any new arm.

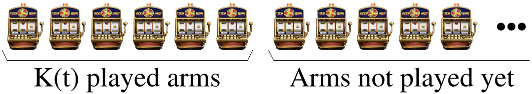
- **Usual assumption**: the distribution of the mean-reward of a new arm is known [Banks, Sundaram, 1992], [Berry, Chen, Zame, Heath, Shepp, 1997].
- **Weaker assumption:** [Wang, Audibert, Munos, 2008] We know $\beta > 0$ such that

$$\mathbb{P}(\mu(\text{new arm}) > \mu^* - \varepsilon) = \Theta(\varepsilon^\beta),$$

β characterizes the probability of selecting near-optimal arms

Large $\beta \implies$ small chance of pulling good arm, thus one needs to pull many arms. And vice-versa.

The UCB-AIR strategy



UCB with Arm Increasing Rule [Wang, Audibert, Munos, 2008]:

- $K(0) = 0$. At time $t+1$, pull a new arm if

$$K(t) < \begin{cases} t^{\frac{\beta}{2}} & \text{if } \beta < 1 \text{ and } \mu^* < 1 \\ t^{\frac{\beta}{\beta+1}} & \text{if } \beta \geq 1 \text{ or } \mu^* = 1 \end{cases}$$

- Otherwise, apply UCB–V [Audibert, Munos, Szepesvári, 2009] on the $K(t)$ drawn arms, i.e., play

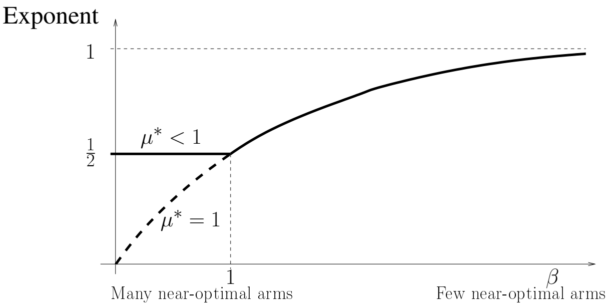
$$\underset{1\leq k\leq K(t)}{\operatorname{argmax}} \underbrace{\widehat{\mu}_{k,t}}_{\text{empirical rewards}} + \underbrace{\sqrt{\frac{2\widehat{V}_{k,t}\mathcal{E}_t}{T_k(t)}} + \frac{3\mathcal{E}_t}{T_k(t)}}_{\text{Confidence interval}},$$

with exploration sequence: $c\log(\log t) \leq \mathcal{E}_t \leq \log t$.

Regret analysis of UCB-AIR

Upper bound on the regret of UCB-AIR:

$$\mathbb{E}R_n = \begin{cases} \widetilde{O}(\sqrt{n}) & \text{if } \beta < 1 \text{ and } \mu^* < 1 \\ \widetilde{O}\left(n^{\frac{\beta}{1+\beta}}\right) & \text{if } \mu^* = 1 \text{ or } \beta \geq 1 \end{cases}$$



Lower bound: $\forall \beta > 0, \mu^* \leq 1$, for any algorithm $\mathbb{E}R_n = \Omega(n^{\frac{\beta}{1+\beta}})$.

Remarks and possible extensions

Remarks

- When $\beta > 1$ or $\mu^* = 1$ the upper and lower bounds match (up to logarithmic factor).
- Exploration-Exploitation-Discovery tradeoff:
  - **Exploitation**: Pull a good arm
  - **Exploration**: Pull an uncertain arm
  - **Discovery**: Pull a new arm
- The exploration sequence $\mathcal{E}_t$ can be of order $\log \log t$ (instead of $\log t$): discovery replaces exploration
- **Open question**: similar performance when β is unknown? (i.e. adaptive strategy that estimates β while minimizing regret).

Structured set of actions or rewards

The mapping *Arms* $\rightarrow$ *Reward* possesses some known structure:

- Linear
- Lipschitz
- Tree structure

Reward samples from observed arms provides information about unseen arms.

Linear bandits

Outline of this section:

- Linear reward function
- UCB type of algorithms: Confidence Ellipsoid
- Extensions

References: [Auer, 2002], [Dani, Hayes, Kakade, 2008], [Abbasi-Yadkori, 2009], [Rusmevichientong, Tsitsiklis, 2010], [Filippi, Cappé, Garivier, Szepesvári, 2010].

Linear mean-reward function

The set of arms $\mathcal{X}$ is a subset of $\mathbb{R}^D$.
The mean-reward function is linear: $x \in \mathcal{X} \mapsto \langle x, \alpha^* \rangle$, where $\alpha^* \in \mathbb{R}^D$ is an unknown parameter.
At each time step t,

- Select $x_t \in \mathcal{X}$,
- Observe $y_t = \langle x_t, \alpha^* \rangle + \eta_t$, where $\mathbb{E}[\eta_t | x_t] = 0$.
  (we assume the noise is bounded or sub-Gaussian).

Let $x^* = \mathrm{argmax}_{x \in \mathcal{X}} \langle x, \alpha^* \rangle$ be the best arm in $\mathcal{X}$.
Define the regret:

$$R_n = n \langle x^*, \alpha^* \rangle - \sum_{t=1}^{n} y_t.$$

No need to estimate the mean-reward of all arms, estimating α^* is enough. So the regret will scale with D and not with the number of arms (which may be infinite).

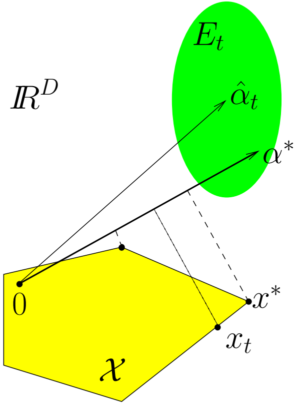
Geometric intuition

Choose $x_t \in \mathcal{X}$ and get:

$$y_t = \langle x_t, \alpha^* \rangle + \eta_t$$

(provides information about α^* along the direction x_t).

Idea: Build a high probability confidence set E_t s.t. $\alpha^* \in E_t$ w.h.p.



Play the arm $x \in \mathcal{X}$ that maximizes $\langle x, \alpha \rangle$ for some $\alpha \in E_t$.

Confidence Ball algorithms

[Dani, Hayes, Kakade, 2008]
UCB idea: Define a least-squares estimate $\widehat{\alpha}_t$ of α^*:

$$\widehat{\alpha}_t = A_t^{-1} \sum_{s=1}^{t} y_s x_s, \text{ where } A_t = \Big(\sum_{s=1}^{t} x_s x_s^T + A_0 \Big),$$

and a confidence ellipsoid E_t around $\widehat{\alpha}_t$:

$$E_t = \big\{ \alpha \in \mathbb{R}^D, \big\| \alpha - \widehat{\alpha}_t \big\|_{2,A_t} \leq \rho(t) \big\}, \text{ where } \rho(t) = c\sqrt{D} \log(t/\delta).$$

Property: w.p. $1-\delta$, $\alpha^* \in E_t$ for all $t \geq 1$.

Algorithm: At round $t+1$, select arm

$$x_{t+1} = \operatorname*{argmax}_{x \in \mathcal{X}} \max_{\alpha \in E_t} \langle x, \alpha \rangle .$$

Regret analysis

[Dani, Hayes, Kakade, 2008], [Rusmevichientong, Tsitsiklis, 2010]

Upper bounds:

- *Problem independent:* With probability $1-\delta$,

$$R_n = O\big(D\sqrt{n}(\log n/\delta)^{3/2}\big)$$

- *Problem dependent:* With probability $1-\delta$,

$$R_n = O\big(\frac{D^2}{\Delta}(\log n/\delta)^3\big),$$

  where Δ is the gap (mean reward difference between best and second best extremal points). Useful when $\mathcal{X}$ is finite or a polytope.

Lower bound: there exists a set $\mathcal{X}$ such that for any algorithm,

$$R_n = \Omega(D\sqrt{n}).$$

Possible extensions [1]

- One may consider ℓ_1-**ellipsoid** instead of ℓ_2, which yield a slightly poorer regret $\tilde{O}(D^{3/2}\sqrt{n})$ but which is more computationally efficient (computation of $\max_{\alpha \in E_t} \langle x, \alpha \rangle$ is $O(D)$).
- **Generalized Linear models** [Filippi, Cappé, Garivier, Szepesvári, 2010]:

$$y_t = \mu(\langle x_t, \alpha^*, \rangle) + \eta_t,$$

where μ is a real-valued function (such as logistic regression function, in order to deal with binary rewards). GLM-UCB selects the arm:

$$\operatorname*{argmax}_{x \in \mathcal{X}} \Big(\mu(\langle x, \widehat{\alpha}_t, \rangle) + \rho(t) \|x\|_{2, A_t^{-1}} \Big),$$

with enjoys similar performance guarantees.

Possible extensions [2]

- **Linear combination of features**:

$$y_t = \langle \varphi(x_t), \alpha^* \rangle + \eta_t,$$

  and apply previous analysis with the set of arms $\varphi(\mathcal{X})$.
- **Sparse linear bandits**: α^* is sparse. Derive algorithms that scale with $\|\alpha^*\|_0$ instead of D.
- **Open question**: is it possible to improve the upper- and lower- bounds in terms of a measure of the quantity of near-optimal states?

$$\mathcal{X}_\varepsilon = \{x \in \mathcal{X}, \langle x, \alpha^* \rangle \geq \langle x^*, \alpha^* \rangle - \varepsilon\}.$$

We now consider more general reward functions f.

$\mathcal{X}$-armed bandits

Outline of this Section:

- Gentle start: Optimization of a deterministic Lipschitz function
- Adding noise
- Relaxing Lipschitz assumption
- Hierarchical Optimistic Optimization (HOO)

References: [Agrawal, 1995], [Kleinberg, 2004], [Auer, Ortner, Szepesvári, 2007], [Kleinberg, Slivkins, Upfall, 2008], [Bubeck, Munos, Stoltz, Szepesvári, 2011].

Optimization of a deterministic Lipschitz function

Problem: Find online the maximum of $f : \mathcal{X} \to \mathbb{R}$, assumed to be Lipschitz: $|f(x) - f(y)| \leq \ell(x, y)$.

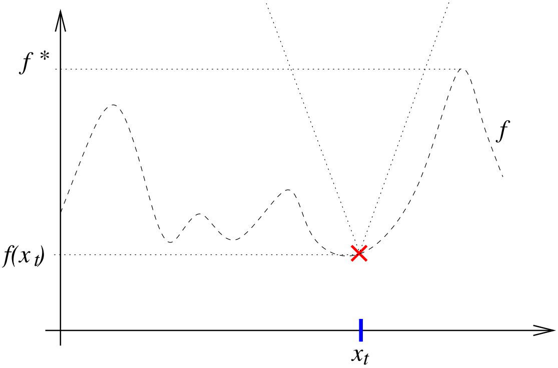
- At each time step t, select $x_t \in \mathcal{X}$
- Observe $f(x_t)$
- Goal: maximize the sum of rewards.

Define the cumulative regret

$$R_n = \sum_{t=1}^{n} \left[f^* - f(x_t) \right],$$

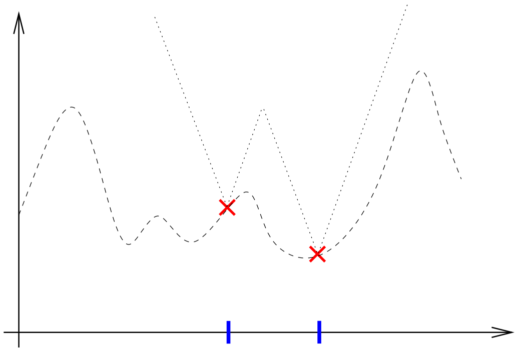
where $f^* = \sup_{x \in \mathcal{X}} f(x)$

Example in 1d



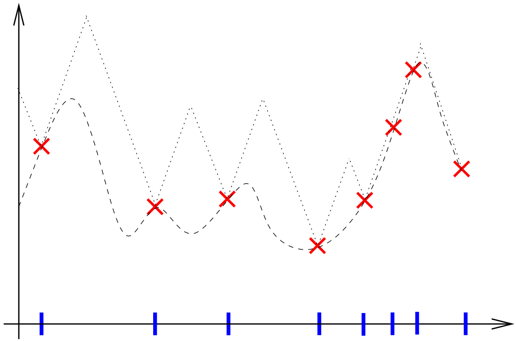
Lipschitz property $\rightarrow$ the evaluation of f at x_t provides a first upper-bound on f.

Example in 1d (continued)



New point $\rightarrow$ refined upper-bound on f.

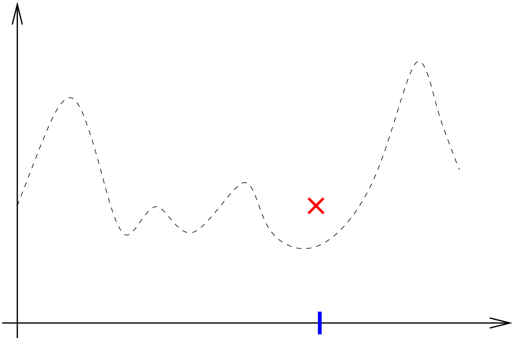
Example in 1d (continued)



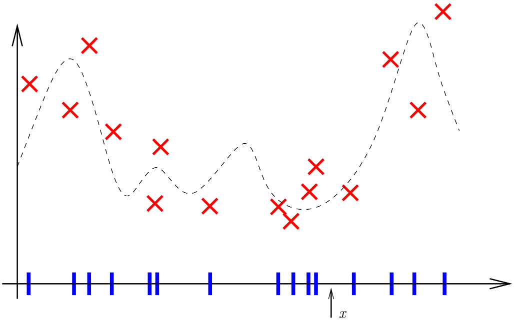
Question: where should one sample the next point?
Answer: select the point with highest upper bound!
"Optimism in the face of (partial observation) uncertainty"

Lipschitz optimization with noisy evaluations

f is still Lipschitz, but now, the evaluation of f at x_t returns a noisy evaluation r_t of $f(x_t)$, i.e. such that $\mathbb{E}[r_t|x_t] = f(x_t)$.

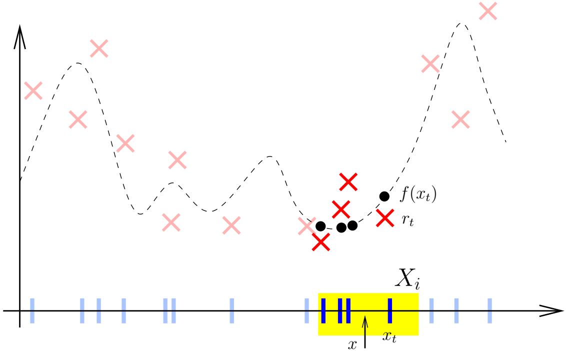


Where should one sample next?



How to define a high probability upper bound at any state x?

UCB in a given domain

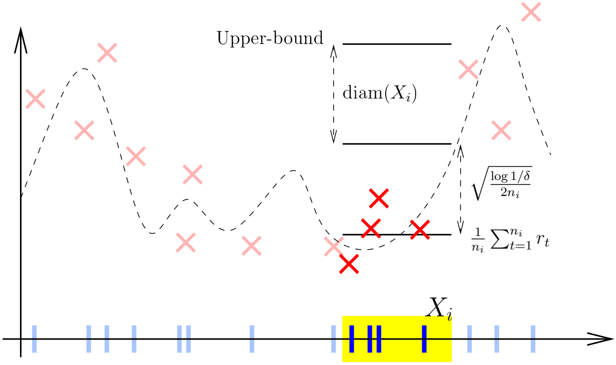


For a fixed domain $X_i \ni x$ containing n_i points $\{x_t\} \in X_i$, we have that $\sum_{t=1}^{n_i} r_t - f(x_t)$ is a Martingale. Thus by Azuma's inequality,

$$\frac{1}{n_i}\sum_{t=1}^{n_i} r_t + \sqrt{\frac{\log 1/\delta}{2n_i}} \geq \frac{1}{n_i}\sum_{t=1}^{n_i} f(x_t) \geq f(x) - diam(X_i),$$

since f is Lipschitz (where $diam(X_i) = \sup_{x,y\in X_i} \ell(x,y)$).

High probability upper bound



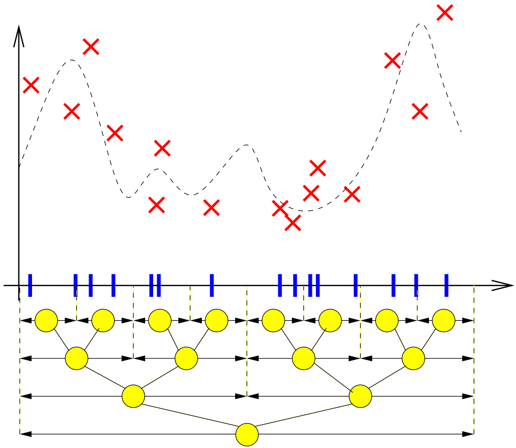
$$\text{w.p. } 1-\delta, \qquad \frac{1}{n_i}\sum_{t=1}^{n_i} r_t + \sqrt{\frac{\log 1/\delta}{2n_i}} + diam(X_i) \geq \sup_{x\in X_i} f(x).$$

Tradeoff between size of the confidence interval and diameter.

By considering several domains we can derive a tigher upper bound.

A hierarchical decomposition

Use a tree of partitions at all scales:

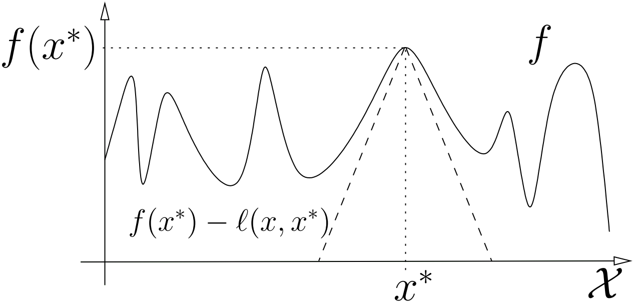


$$B_i(t) \stackrel{\mathrm{def}}{=} \min\left\{ \widehat{\mu}_i(t) + \sqrt{\frac{2\log(t)}{T_i(t)}} + \mathrm{diam}(i), \max_{j \in \mathcal{C}(i)} B_j(t) \right\}$$

$\mathcal{X}$-armed bandits

Let $\mathcal{X}$ be a space equipped with a semi-metric $\ell(x, y)$.
Let $f(x)$ be a function such that:

$$f(x^*) - f(x) \leq \ell(x, x^*),$$



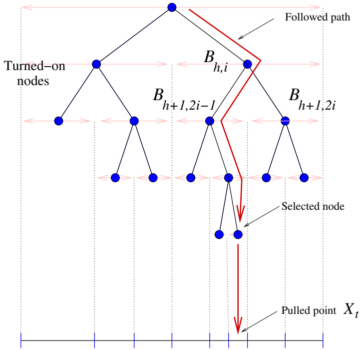
Hierarchical Optimistic Optimization (HOO)

[Bubeck, Munos, Stoltz, Szepesvári, 2008]: Consider a tree of partitions of $\mathcal{X}$, each node i corresponds to a subdomain X_i.

HOO Algorithm:

Let $\mathcal{T}_t$ be the set of expanded nodes at round t.

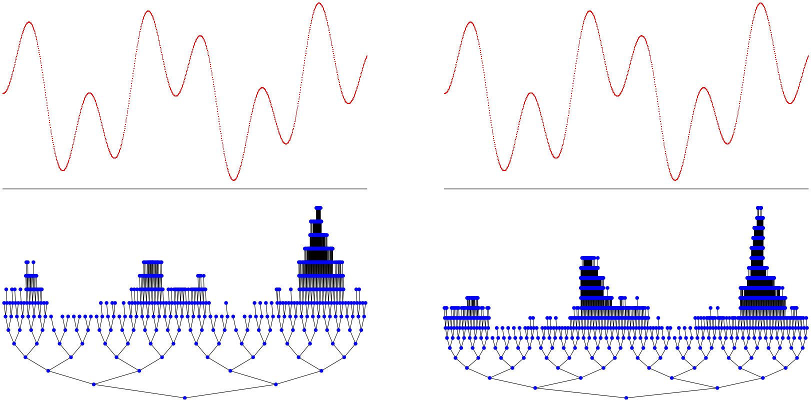
- $\mathcal{T}_1 = \{\text{root}\}$ (space $\mathcal{X}$)
- At t, select a leaf I_t of $\mathcal{T}_t$ by maximizing the B-values,
- $\mathcal{T}_{t+1} = \mathcal{T}_t \cup \{I_t\}$
- Select $x_t \in X_{I_t}$
- Observe reward r_t and update the B-values:



$$B_i(t) \stackrel{\text{def}}{=} \min\left[\widehat{\mu}_i(t) + \sqrt{\frac{2\log(t)}{T_i(t)}} + diam(i), \max_{j\in\mathcal{C}(i)} B_j(t)\right]$$

Example in 1d

$r_t \sim \mathcal{B}(f(x_t))$ a Bernoulli distribution with parameter $f(x_t)$



Resulting tree at time $n = 1000$ and at $n = 10000$.

Analysis of HOO

Let d be the **near-optimality dimension** of f in $\mathcal{X}$: i.e. such that the set of ε-optimal states

$$X_\varepsilon \stackrel{\text{def}}{=} \{x \in \mathcal{X}, f(x) \geq f^* - \varepsilon\}$$

can be covered by $O(\varepsilon^{-d})$ balls of radius ε.
Then

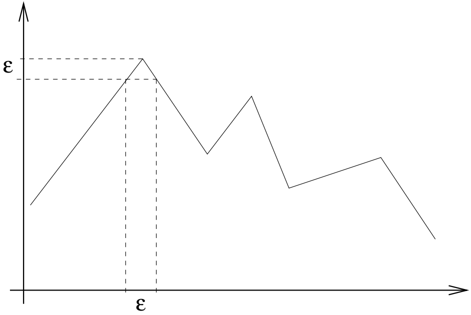
$$\mathbb{E}R_n = \widetilde{O}(n^{\frac{d+1}{d+2}}).$$

(Similar to Zooming algorithm of [Kleinberg, Slivkins, Upfall, 2008], but HOO requires a tree of partitions whereas Zooming requires a sampling oracle)

Example 1:

Assume the function is locally peaky around its maximum:

$$f(x^*) - f(x) = \Theta(||x^* - x||).$$

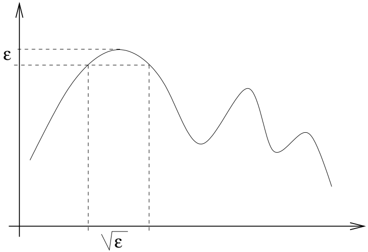


It takes $O(\varepsilon^0)$ balls of radius ε to cover X_ε. Thus $d = 0$ and the regret is $\widetilde{O}(\sqrt{n})$.

Example 2:

Assume the function is locally quadratic around its maximum:

$$f(x^*) - f(x) = \Theta(||x^* - x||^\alpha), \text{ with } \alpha = 2.$$



- For $\ell(x, y) = ||x - y||$, it takes $O(\varepsilon^{-D/2})$ balls of radius ε to cover X_ε. Thus $d = D/2$ and $R_n = \widetilde{O}(n^{\frac{D+2}{D+4}})$.
- For $\ell(x, y) = ||x - y||^2$, it takes $O(\varepsilon^0)$ ℓ-balls of radius ε to cover X_ε. Thus $d = 0$ and $R_n = \widetilde{O}(\sqrt{n})$.

Known smoothness around the maximum

Consider $\mathcal{X} = [0,1]^d$. Assume that f has a finite number of global maxima and is locally α-smooth around each maximum x^*, i.e.

$$f(x^*) - f(x) = \Theta(||x^* - x||^\alpha).$$

Then, by choosing $\ell(x,y) = ||x-y||^\alpha$, X_ε is covered by $O(1)$ ℓ-balls of "radius" ε. Thus the near-optimality dimension $d = 0$, and the regret of HOO is:

$$\mathbb{E}R_n = \widetilde{O}(\sqrt{n}),$$

The rate of growth is **independent of the ambient dimension** D.

Conclusions on $\mathcal{X}$-armed bandits

The near-optimality dimension may be seen as an excess order of smoothness of f (around its maxima) compared to what is known:

- **If the smoothness order of the function is known** then the regret of HOO is $\widetilde{O}(\sqrt{n})$
- **If the smoothness is underestimated**, for example f is α-smooth but we only use $\ell(x, y) = ||x - y||^\beta$, with $\beta < \alpha$, then the near-optimality dimension is $d = D(1/\beta - 1/\alpha)$ and the regret is $\widetilde{O}(n^{(d+1)/(d+2)})$
- **If the smoothness is overestimated**, the local-Lipschitz assumption is violated, thus there is no guarantee.
  For example UCT [Kocsis, Szepesvári, 2006] can be arbitrarily poor [Coquelin, Munos, 2007].

Bandits in trees

Outline of this Section:

- A more structured problem: finding a path in a tree
- An algorithm that does not fully use the reward structure
- An algorithm that does!

References: [Kocsis, Szepesvári, 2006], [Coquelin, Munos, 2007], [Bubeck, Munos, 2010]

A more structured problem

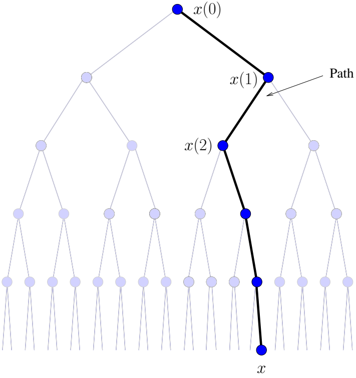
Finding an path in a tree: each arm is a path (in a graph or tree) and the *value of the path is the sum of rewards along the path*.

Example:

- Infinite horizon with γ-discounted rewards. K actions.
- Space of arms $\mathcal{X}$ = set of paths (infinite sequence of actions).
- Reward along a path x_t:

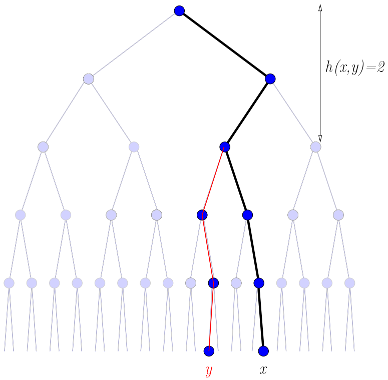
$$y_t = \sum_{i \geq 0} \gamma^i y_t(i),$$

  where $y_t(i) \sim \nu(x_t(i)) \in [0,1]$.
- Write $\mu(x(i)) = \mathbb{E}[\nu(x(i))]$, and $f(x) = \sum_{i \geq 0} \gamma^i \mu(x(i))$



Using HOO

- **Prop:** The mean-reward function $f(x) = \sum_{i\geq 1} \gamma^i \mu(x(i))$ is Lipschitz w.r.t. the metric: $\ell(x,y) = \frac{\gamma^{h(x,y)}}{1-\gamma}$.
- **Use HOO**: At round t, play the path x_t maximizing the B-value.
- Observe sample reward $y_t = \sum_{i\geq 1} \gamma^i y_t(i)$ of the path and use it to update the B-values.



Problem: HOO does not make full use of the tree structure: It uses the sample reward y_t of a path x_t but not the sample rewards $y_t(i)$ of all nodes $x_t(i)$ of the path x_t.

Optimistic sampling using the tree structure

OLOP algorithm [Bubeck, Munos, 2010]:

- At round t, play path x_t (up to depth $h = \frac{1}{2}\frac{\log n}{\log 1/\gamma}$)
- Observe sample rewards $y_t(i)$ of each node along the path x_t
- Compute empirical rewards for each node $x(i)$ of depth $i \leq h$

$$\widehat{\mu}_t(x(i)) = \frac{1}{T_{x(i)}(t)} \sum_{s=1}^{t} y_t(i)\mathbb{I}\{x(i) \in x_t\} \text{ where } T_{x(i)}(t) = \sum_{s=1}^{t} \mathbb{I}\{x(i) \in x_t\}$$

- Define bound for each path x:

$$B_t(x) = \min_{1\leq j\leq h} \Big[\sum_{i=1}^{j} \gamma^i \Big(\widehat{\mu}(x(i)) + \sqrt{\frac{2\log n}{T_{x(i)}(t)}}\Big) + \frac{\gamma^{j+1}}{1-\gamma}\Big]$$

- Select path $x_{t+1} = \mathrm{argmax}_x B_t(x)$

This algorithm fully uses the tree structure of the rewards.

Performance guarantee of OLOP

Consider the near-optimality dimension d, i.e., such that

$$\mathcal{X}_\varepsilon = \{x \in \mathcal{X}, f(x) \geq f^* - \varepsilon\}$$

is covered by $O(\varepsilon^{-d})$ ℓ-balls of size ε.
Regret of OLOP: (Open Loop Optimistic Planning) after n calls to the generative model,

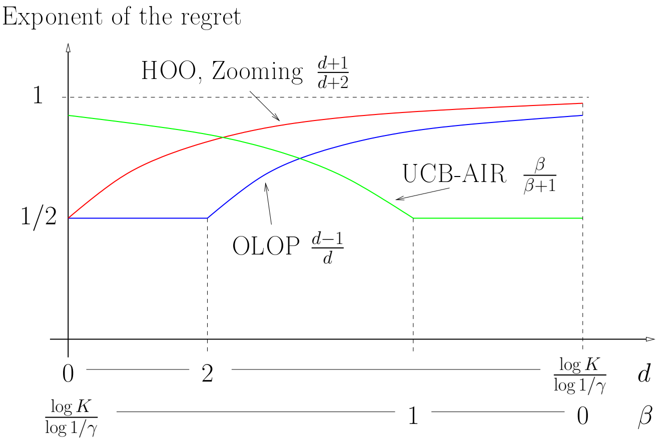
$$R_n = nf^* - \mathbb{E}\Big[\sum_{t=1}^{n} f(x_t)\Big] = \begin{cases} \widetilde{O}\Big(n^{\frac{d-1}{d}}\Big) & \text{if } d > 2 \\ \widetilde{O}(\sqrt{n}) & \text{if } d \leq 2 \end{cases}$$

Another measure of the set of near-optimal paths is $\beta \geq 0$:

$$\mathbb{P}(\text{Random path is } \varepsilon\text{-optimal}) = O(\varepsilon^\beta).$$

Note that we have $d = \frac{\log K}{\log 1/\gamma} - \beta \in [0, \frac{\log K}{\log 1/\gamma}]$.

Comparison: OLOP, HOO, Zooming, UCB-AIR



Conclusion on stochastic bandits

- Success of "Optimism in the face of uncertainty" principle
- Use reward structure as much as possible
- Better concentration inequalities $\Longrightarrow$ better bounds
- Regret bounds expressed in terms of a measure of near-optimal solutions

Other topics in stochastic bandits

A few pointers:

- **Contextual bandits** [Woodroofe, 1979], [Auer, 2002], [Wang, Kulkarni, Poor, 2005], [Pandey, Agarwal, Chakrabarti, Josifovski, 2007], [Langford, Zhang, 2007], [Hazan, Megiddo, 2007], [Rigollet, Zeevi, 2010], [Chu, Li, Reyzin, Schapire, 2011], [Slivkins, 2011].
- **Restless bandits** [Whittle, 1988], [Bertsimas, Nino-Mora, 1994], [Guha, Munagala, Shi, 2007], [Filippi, 2010].
- **Markov decision processes** [Burnetas, Katehakis, 1997], [Auer, Ortner, 2007], [Jaksch, Ortner, Auer, 2010], [Bartlett, Tewari, 2009].
- **Gaussian bandits** [Dorard, Glowacka, Shawe-Taylor, 2009], [Grünewälder, Audibert, Opper, Shawe-Taylor, 2010], [Srinivas, Krause, Kakade, Seeger, 2010]
- **Sleeping bandits** [Kleinberg, Niculescu-mizil, Sharma, 2008], [Kanade, McMahan, Bryan, 2009], **mortal bandits** [Chakrabarti, Kumar, Radlinski, Upfal, 2008], ...

Topics in adversarial bandits

At each round t,

- Simultaneously, the adversary selects a function $f_t : \mathcal{X} \mapsto \mathbb{R}$, and the player chooses $x_t \in \mathcal{X}$
- The reward $f_t(x_t)$ is revealed.

The performance of the player is compared to the best constant strategy:

$$R_n = \max_{x \in \mathcal{X}} \sum_{t=1}^{n} f_t(x) - \sum_{t=1}^{n} f_t(x_t).$$

Performance depends on

- Full versus bandit information
- Class of functions f_t
- Shape of the action space $\mathcal{X}$

[Cesa-Bianchi, Lugosi, 2006]

A few pointers

- **Linear bandits** [Dani, Hayes, Kakade, 2008], [Abernethy, Hazan, Rakhlin, 2008], [Awerbuch, Kleinberg, 2008],
- **Convex bandits** [Zinkevich, 2003], [Flaxman, Kalai, McMahan, 2005], [Hazan, Agarwal, Kale, 2006], [Bartlett, Hazan, Rakhlin, 2007], [Shalev-Shwartz, 2007], [Abernethy, Bartlett, Rakhlin, Tewari, 2008], [Narayanan, Rakhlin, 2010]
- **Lipschitz bandits** [Maillard, Munos, 2010]
- **Countable bandits** [Poland, 2008]
- **Combinatorial bandits** [Cesa-Bianchi, Lugosi, 2009], [Audibert, Bubeck, Lugosi, 2011]
- **Online learning** in stochastic/adversarial environments [Auer, Cesa-Bianchi, Freund, Schapire, 2002], [Kakade, Kalai, 2005], [Cesa-Bianchi et al. 2009], [Abernethy, Agarwal, Bartlett, Rakhlin, 2009], [Ben-David, Pal, Shalev-Shwartz, 2009], [Lazaric, Munos, 2009], [Rakhlin, Sridharan, Tewari, 2011].

Thank you

Material available on the Tutorial web page:

`https://sites.google.com/site/banditstutorial`

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