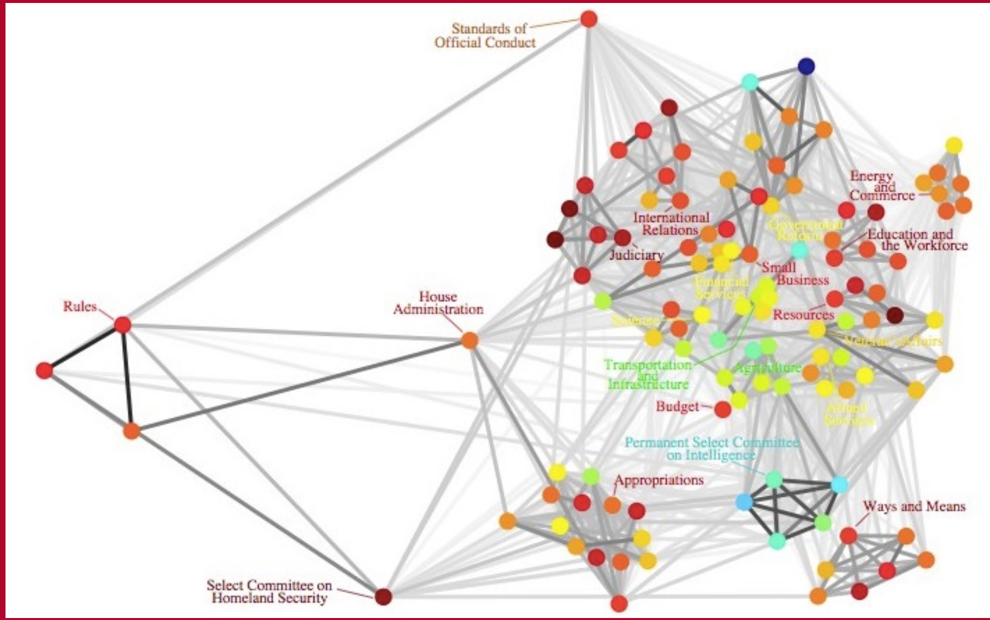
Automatic Control Theory

Chapter 4



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The root locus method

Main contents

1、The root locus concept and root locus equation

2、The root locus procedure

3、General root loci (Zero degree root loci)

The root locus procedure

Review

- **Open loop transfer function**
- **Phase equation**

Continuity   **Symmetry**   **Start and end points**

D(s)=1+G(s)H(s)=0

$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -1$$

what is next

The root locus procedure (Asymptotes, Angles of start and end points, Breakaway point)

Step 5 : Asymptotes of the root loci

渐近线与实轴相交点的坐标为：

$$\sigma_A = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} \longrightarrow$$

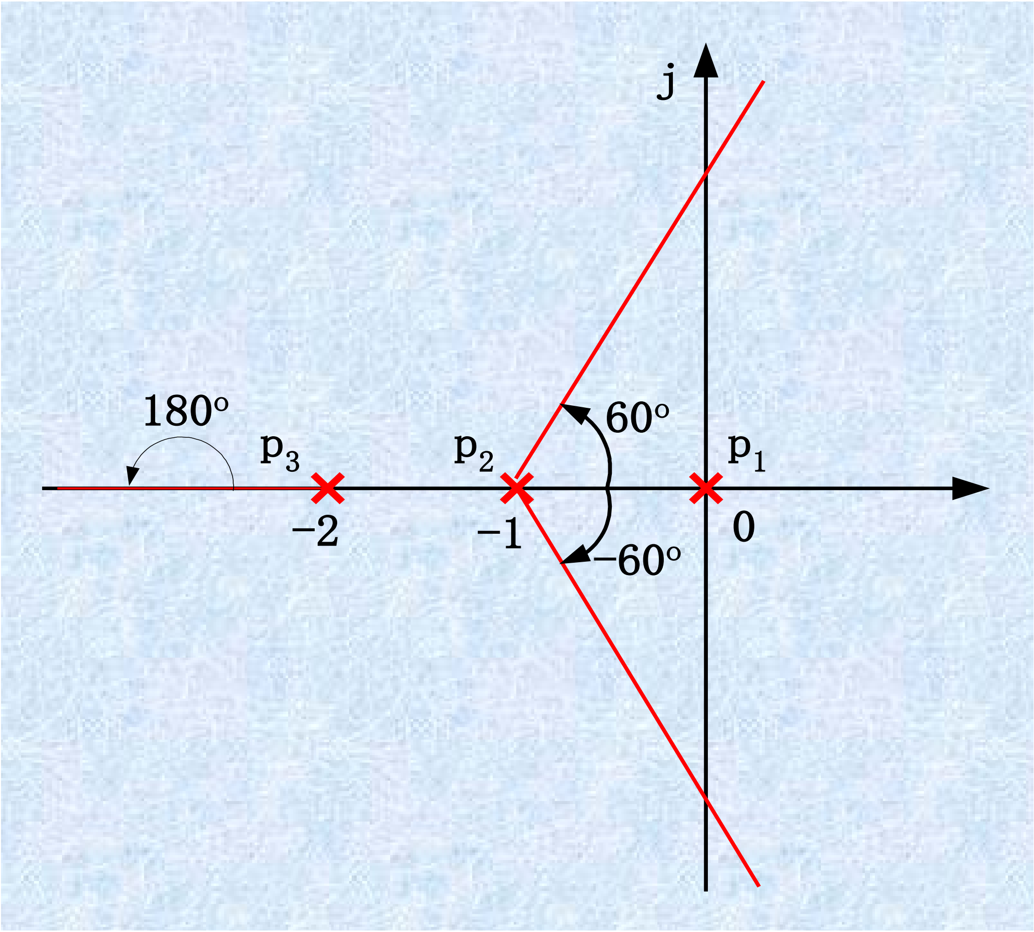
Asymptote centroid

渐近线与实轴正方向的夹角为：

$$\varphi_A = \frac{(2k+1)\pi}{n-m} \longrightarrow$$

Angle of the asymptotes

$$k = 0,1,2,\cdots n-m-1$$



The root locus procedure

$$G(s)H(s)=\frac{K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=K^*\frac{s^m+b_1s^{m-1}+\cdots b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

$$=K^*\frac{s^m}{s^n}\bullet\frac{1+b_1s^{-1}+\cdots+b_{m-1}s^{-m+1}+b_ms^{-m}}{1+a_1s^{-1}+\cdots+a_{n-1}s^{-n+1}+a_ns^{-n}}$$

$$s^{n-m}\frac{1+a_1s^{-1}+\cdots+a_{n-1}s^{-n+1}+a_ns^{-n}}{1+b_1s^{-1}+\cdots+b_{m-1}s^{-m+1}+b_ms^{-m}}=-K^*$$

Root locus equation

$$\frac{1}{x}\left(\frac{1+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n}{1+b_1x+\cdots+b_{m-1}x^{m-1}+b_mx^m}\right)^{\frac{1}{n-m}}=(-K^*)^{\frac{1}{n-m}}$$

$$x=\frac{1}{s}$$

$$\left[\frac{1+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n}{1+b_1x+\cdots+b_{m-1}x^{m-1}+b_mx^m}\right]^{\frac{1}{n-m}}=1+\frac{a_1-b_1}{n-m}x$$

Taylor series at $x=0$

The root locus procedure

$$\frac{1}{x}\left(1+\frac{a_1-b_1}{n-m}x\right)=(-K^*)^{\frac{1}{n-m}}=(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi}\quad (k=0,1,2,\cdots)$$

$$\frac{1}{x}=-\frac{a_1-b_1}{n-m}+(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi}\qquad \frac{1}{x}=s,\ a_1=-\sum_{i=1}^{n}p_i,\ b_1=-\sum_{i=1}^{m}z_i$$

$$s=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}+(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi}$$

set $\varphi_a=\dfrac{(2k+1)\pi}{n-m}$ $\quad\sigma_a=\dfrac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}$

$$s=\sigma_a+(K^*)^{\frac{1}{n-m}}\cdot e^{j\varphi_a}$$

Asymptotes equation

The root locus procedure

Example 1

Open loop transfer function

$$G(s)H(s) = K^*(s+1)/s(s+4)(s^2+2s+2)$$

Get the root locus $K = 0 \rightarrow \infty$

Solution

$$G(s)H(s) = \frac{K^*(s+1)}{s(s+4)(s^2+2s+2)}$$

$p_1 = 0,\ p_a = -4,\ p_3 = -1 + \mathrm{j}1,\ p_4 = -1 - \mathrm{j}1,\ n = 4$

$z_1 = -1,\ m = 1$

The root locus procedure

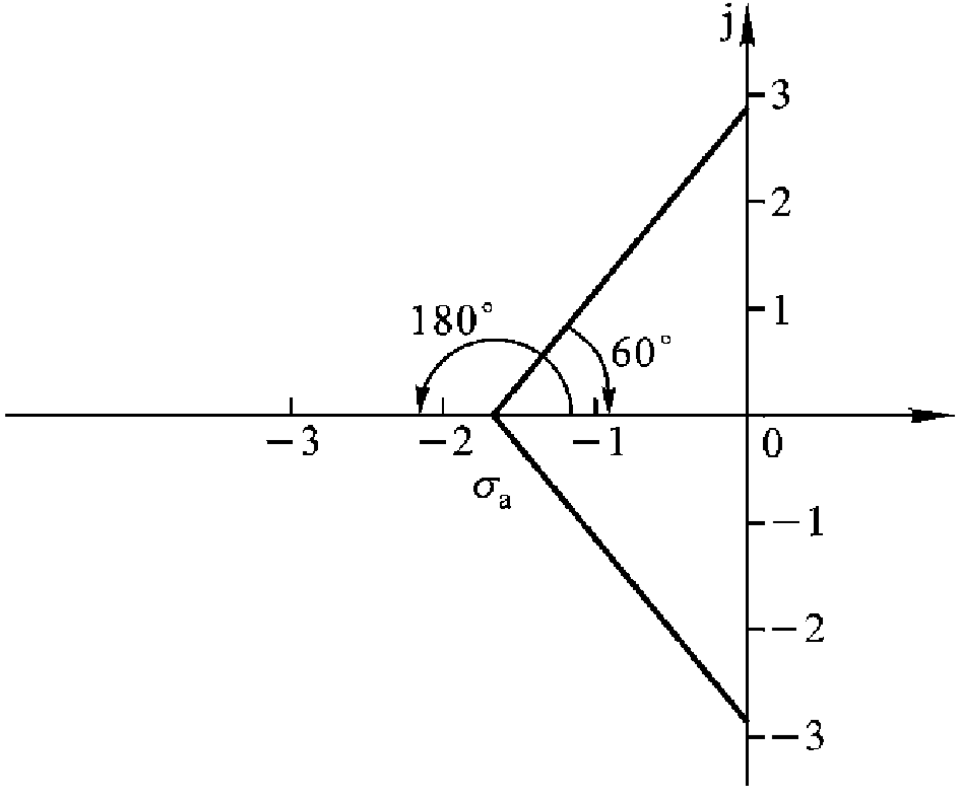
$$\varphi_a = \frac{(2k+1)\pi}{n-m} = \frac{(2k+1)\pi}{4-1} = \frac{(2k+1)\pi}{3}$$

$$\varphi_{a_1} = 60^\circ \qquad (k=0)$$

$$\varphi_{a_2} = 180^\circ \qquad (k=1)$$

$$\varphi_{a_3} = 300^\circ \qquad (k=2)$$

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{0-4-1-j1-1+j1+1}{4-1} = -\frac{5}{3}$$



The root locus procedure

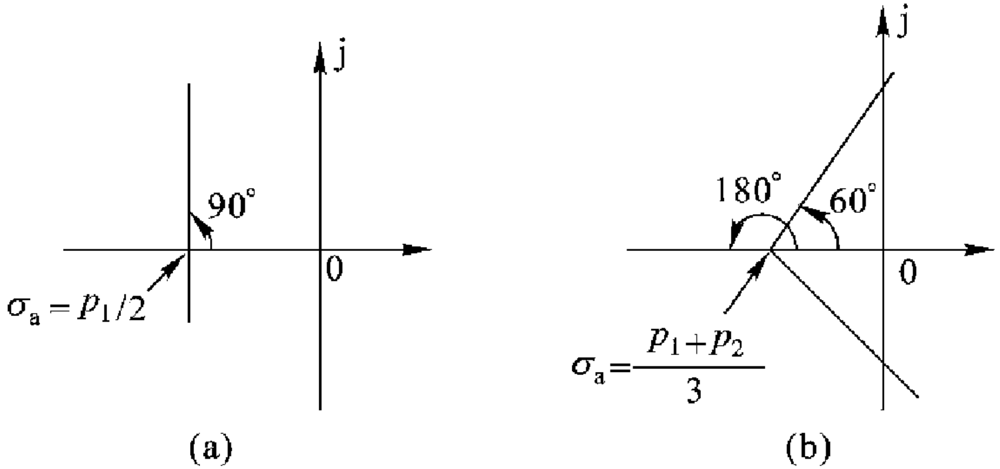
Asymptotes for typical open loop transfer functions

① $G(s)H(s)=\dfrac{K^*}{s(s-p_1)}$。

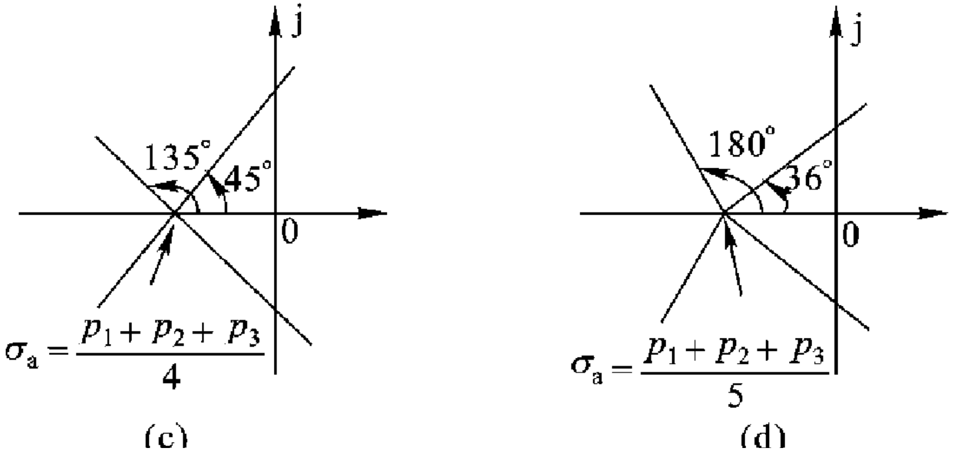
② $G(s)H(s)=\dfrac{K^*}{s(s-p_1)(s-p_2)}$。

③ $G(s)H(s)=\dfrac{K^*}{s(s-p_1)(s-p_2)(s-p_3)}$。

④ $G(s)H(s)=\dfrac{K^*}{s^2(s-p_1)(s-p_2)(s-p_3)}$。



$360°/(n-m)$



The root locus procedure

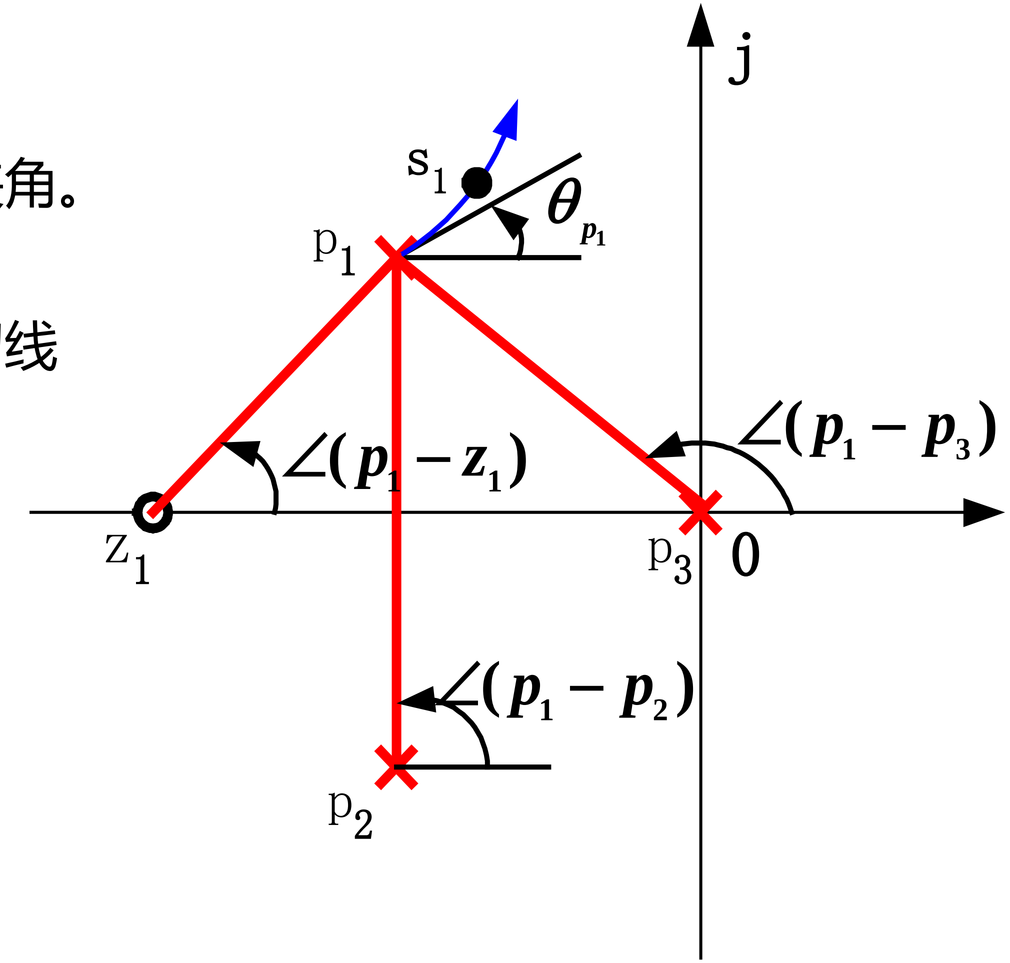
Step 6: Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero.

根轨迹的起始角是指根轨迹在起点处的切线与水平正方向的夹角。

根轨迹的终止角是指终止于某开环零点的根轨迹在该点处的切线与水平正方向的夹角。

$$\theta_{p_i} = (2k+1)\pi + \sum_{j=1}^{m} \angle(p_i - z_j) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \angle(p_i - p_j)$$

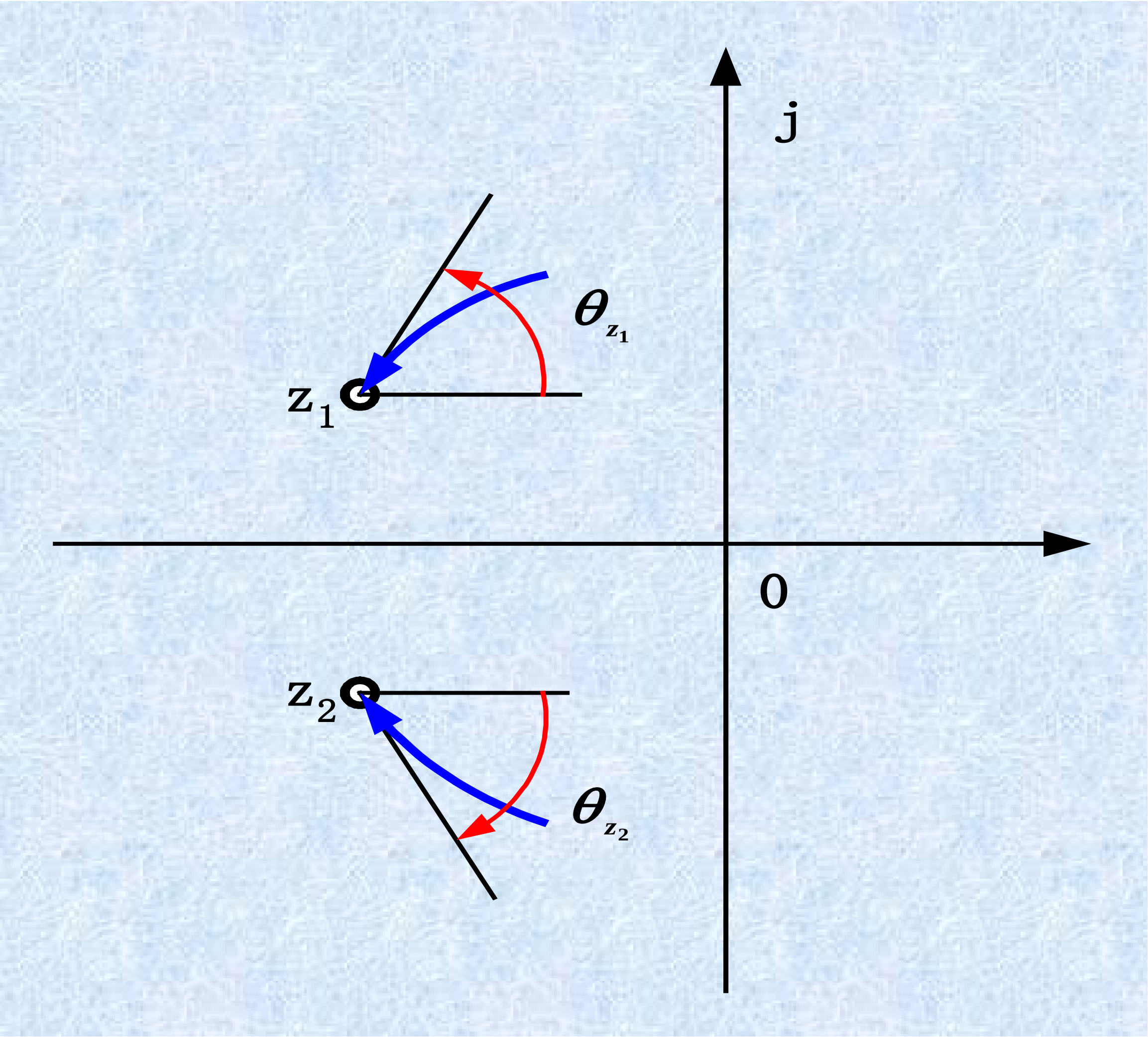
$$= (2k+1)\pi + \sum_{j=1}^{m} \theta_{z_j p_i} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \theta_{p_j p_i}$$



The root locus procedure

$$\theta_{z_i} = (2k+1)\pi + \sum_{j=1}^{n} \angle(z_i - p_j) - \sum_{\substack{j=1 \\ j \neq i}}^{m} \angle(z_i - z_j)$$

$$= (2k+1)\pi + \sum_{j=1}^{m} \theta_{p_j z_i} - \sum_{\substack{j=1 \\ j \neq i}}^{m} \theta_{z_j z_i}$$



The root locus procedure

Phase equation

$$\sum_{j=1}^{m}\angle(s_1-z_j)-\sum_{j=1}^{n}\angle(s_1-p_j)=(2k+1)\pi$$

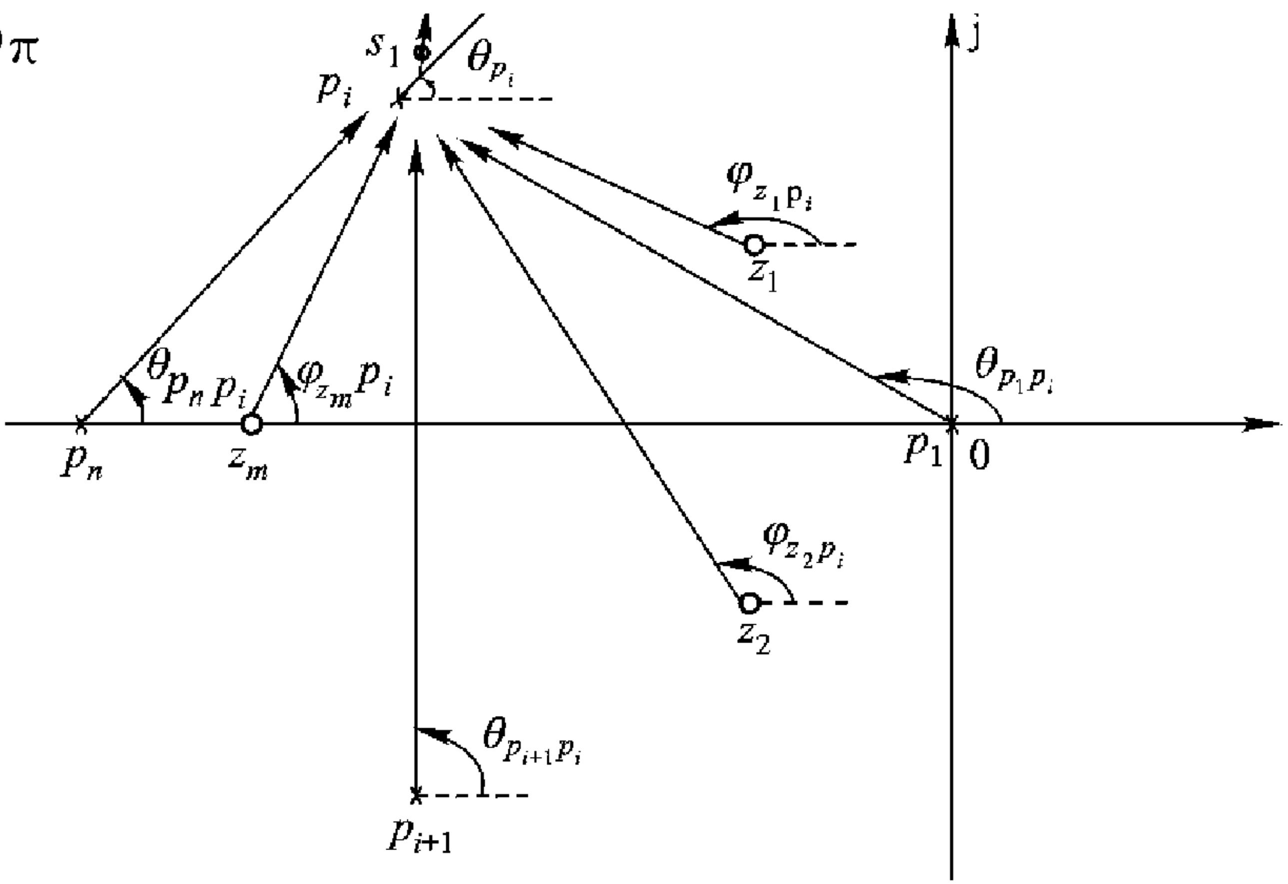
$$\sum_{j=1}^{m}\angle(p_i-z_j)-\sum_{\substack{j=1\\j\neq i}}^{n}\angle(p_i-p_j)-\theta_{p_i}=(2k+1)\pi$$

Set

$$\angle(p_i-z_j)=\varphi_{z_jp_i},\ \angle(p_i-p_j)=\theta_{p_jp_i}$$

$$\theta_{p_i}=(2k+1)\pi+\sum_{j=1}^{m}\varphi_{z_jp_i}-\sum_{\substack{j=1\\j\neq i}}^{n}\theta_{p_jp_i}$$

s_1 close to p_i



The root locus procedure

Example 2

Open loop transfer function

$$G(s)H(s)=\frac{K^*(s+2+\mathrm{j})(s+2-\mathrm{j})}{(s+1+\mathrm{j}2)(s+1-\mathrm{j}2)}$$

Get the root locus

Solution

$$\theta_{p_i}=(2k+1)\pi+\sum_{j=1}^{m}\angle(p_i-z_j)-\sum_{\substack{j=1\\ j\neq i}}^{n}\angle(p_i-p_j)$$

$$\theta_{z_i}=(2k+1)\pi+\sum_{j=1}^{n}\angle(z_i-p_j)-\sum_{\substack{j=1\\ j\neq i}}^{m}\angle(z_i-z_j)$$

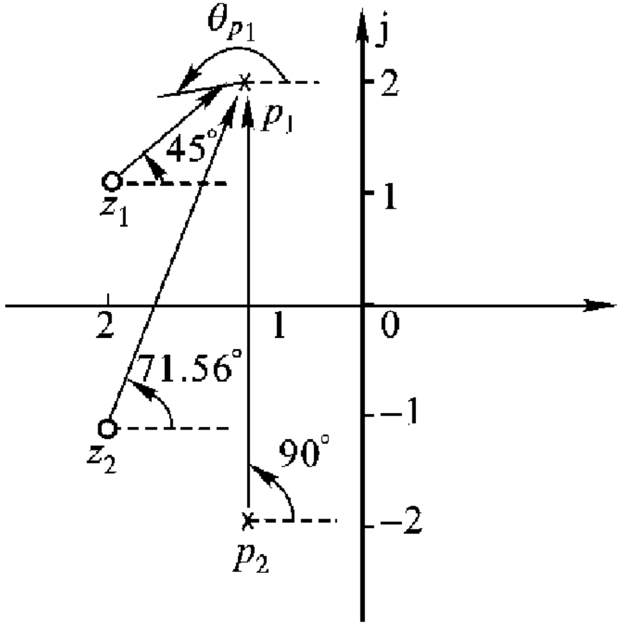
The root locus procedure

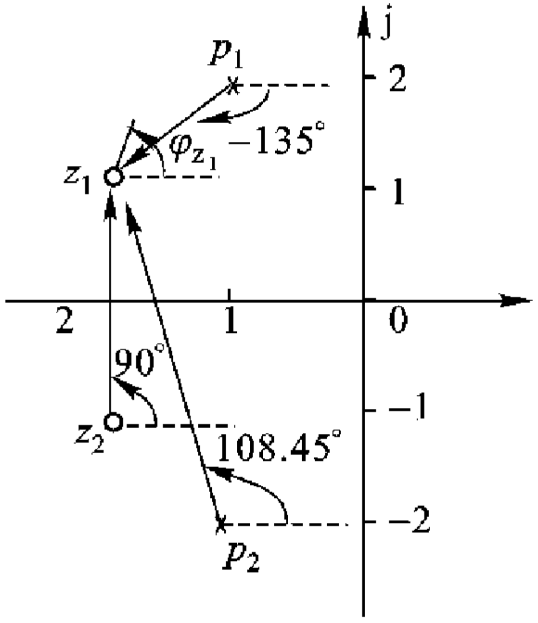
$$\theta_{p_1} = 180^\circ + \varphi_{z_1 p_1} + \varphi_{z_2 p_1} - \varphi_{p_2 p_1}$$
$$= 180^\circ + 45^\circ + 71.56^\circ - 90^\circ = 206.5$$

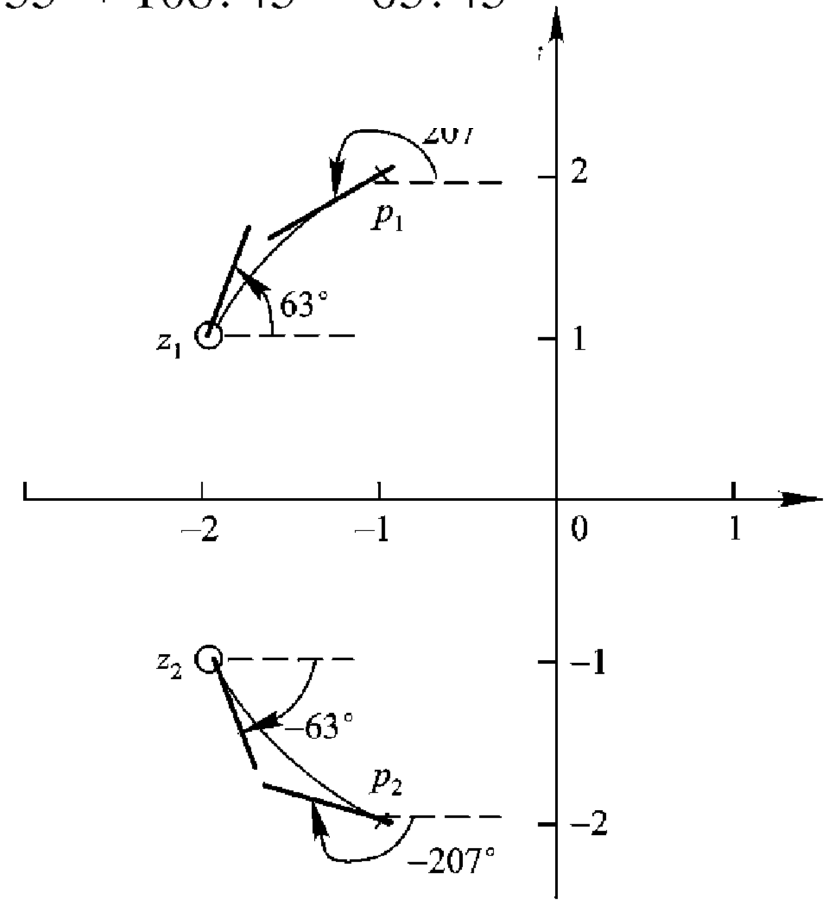
$$\theta_{p_2} = -206.56^\circ$$

$$\varphi_{z_1} = 180^\circ - \varphi_{z_2 z_1} + \theta_{p_1 z_1} + \theta_{p_2 z_1}$$
$$= 180^\circ - 90^\circ - 135^\circ + 108.43^\circ = 63.43^\circ$$

$$\varphi_{z_2} = -63.43^\circ$$





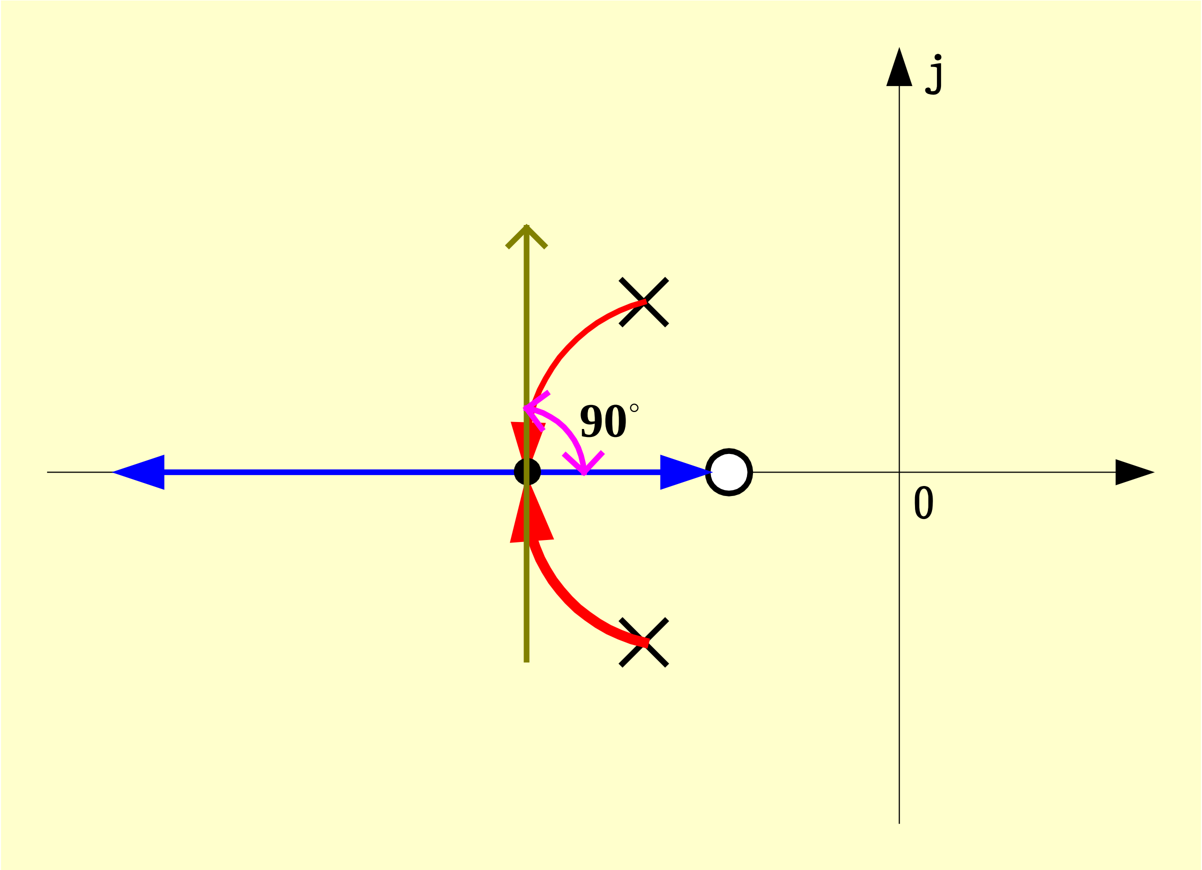


The root locus procedure

Step 7: Determine the breakaway point and the breakaway angle and arrival angle on the real axis.

breakaway point

$$\sum_{i=1}^{m}\frac{1}{d-z_i}=\sum_{i=1}^{n}\frac{1}{d-p_i}$$



The root locus procedure

$$1+\frac{K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0 \qquad \frac{\prod_{i=1}^{n}(s-p_i)+K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0 \qquad D(s)=\prod_{i=1}^{n}(s-p_i)+K^*\prod_{i=1}^{m}(s-z_i)=0$$

$$D(s_1)=\prod_{i=1}^{n}(s_1-p_i)+K^*\prod_{i=1}^{m}(s_1-z_i)=0$$

s_1: multiple root

$$\frac{\mathrm{d}}{\mathrm{d}s_1}D(s_1)=\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)+K^*\prod_{i=1}^{m}(s_i-z_i)\right]=0$$

$$\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)\right]=-K^*\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{m}(s_1-z_i)\right]$$

$$\boxed{\prod_{i=1}^{n}(s_1-p_i)=-K^*\prod_{i=1}^{m}(s_1-z_i)}$$

derived by division

$$\frac{\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)\right]}{\prod_{i=1}^{n}(s_1-p_i)}=\frac{\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{m}(s_1-z_i)\right]}{\prod_{i=1}^{m}(s_1-z_i)}$$

The root locus procedure

$$\boxed{(\ln x)' = \frac{x'}{x}}$$

$$\boxed{\frac{d}{ds_1}\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] = \frac{\frac{d}{ds_1}\prod_{i=1}^{n}(s_1 - p_i)}{\prod_{i=1}^{n}(s_1 - p_i)}}$$

$$\frac{\frac{d}{ds_1}\left[\prod_{i=1}^{n}(s_1 - p_i)\right]}{\prod_{i=1}^{n}(s_1 - p_i)} = \frac{\frac{d}{ds_1}\left[\prod_{i=1}^{m}(s_1 - z_i)\right]}{\prod_{i=1}^{m}(s_1 - z_i)} \Rightarrow \frac{d}{ds_1}\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] = \frac{d}{ds_1}\ln\left[\prod_{i=1}^{m}(s_1 - z_i)\right]$$

$$\sum_{i=1}^{n}\frac{d}{ds_1}\ln(s_1 - p_i) = \sum_{i=1}^{m}\frac{d}{ds_1}\ln(s_1 - z_i)$$

$$\boxed{\begin{aligned}\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] &= \sum_{i=1}^{n}\ln(s_1 - p_i)\\ \ln\left[\prod_{i=1}^{n}(s_1 - z_i)\right] &= \sum_{i=1}^{m}\ln(s_1 - z_i)\end{aligned}}$$

$$\sum_{i=1}^{n}\frac{1}{s_1 - p_i} = \sum_{i=1}^{m}\frac{1}{s_1 - z_i} \Rightarrow \sum_{i=1}^{n}\frac{1}{d - p_i} = \sum_{i=1}^{m}\frac{1}{d - z_i}$$

The root locus procedure

Example 3 $$G(s)H(s)=\frac{K(s+1)}{s^2+3s+3.25}$$ Get the root locus

Solution

$$p_1=-1.5+j1,\ p_2=-1.5-j1 \qquad z_1=-1$$

$$\sigma_a=\frac{-1.5+j1-1.5-j1+1}{2-1}=-2$$

$$\varphi_a=\frac{(2k+1)\pi}{2-1}=\pi$$

Asymptotes equation

The root locus procedure

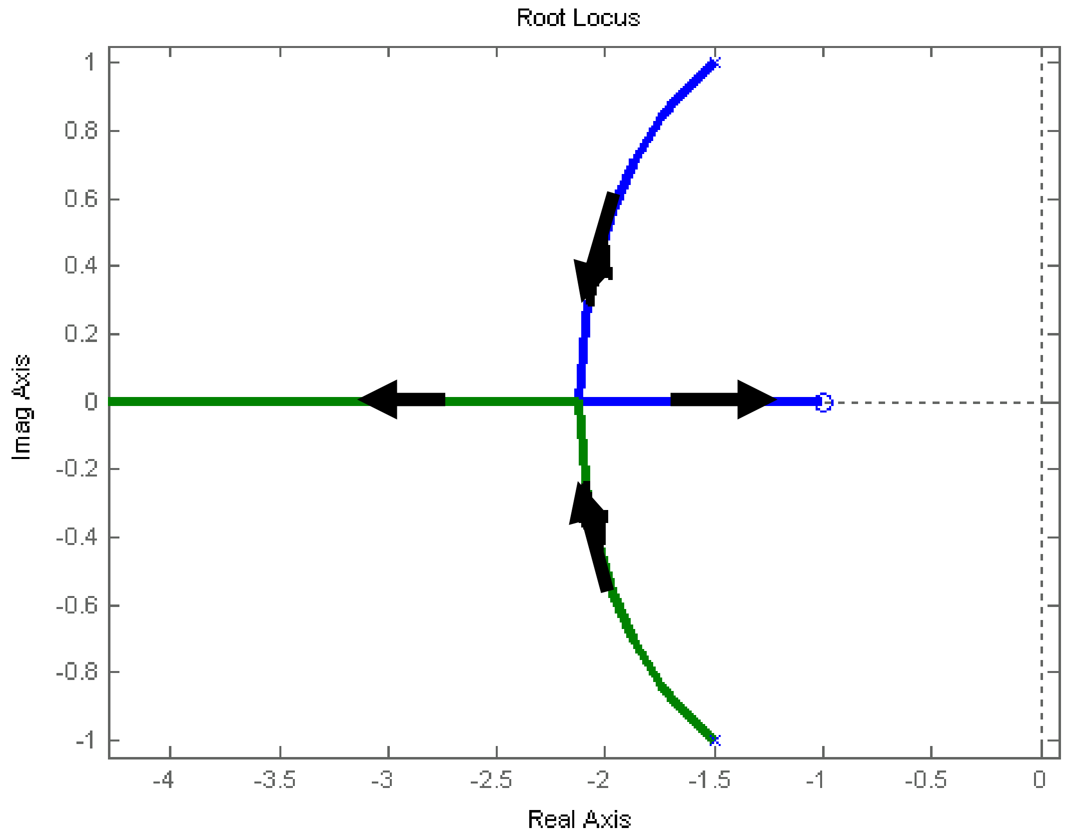
$$\frac{1}{d+1}=\frac{1}{d+1.5+j}+\frac{1}{d+1.5-j}$$

breakaway point

$$d^2+2d-0.25=0$$

$$d=-2.12, \quad d=0.12$$

abandon



Strat point angle

$$\theta_{p_1}=180^\circ+\varphi_{z_1 p_1}-\theta_{p_2 p_1}$$

$$=180^\circ+116.57^\circ-90^\circ=206.57^\circ$$

$$\theta_{p_2}=-206.57^\circ$$

The root locus procedure

核心

- **Definition of root locus**
- **Open loop transfer function**
- **Phase equation**

$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -1$$

Asymptotes, Angles of start and end points, Breakaway point

续

- **Zero degree root loci (Details)**