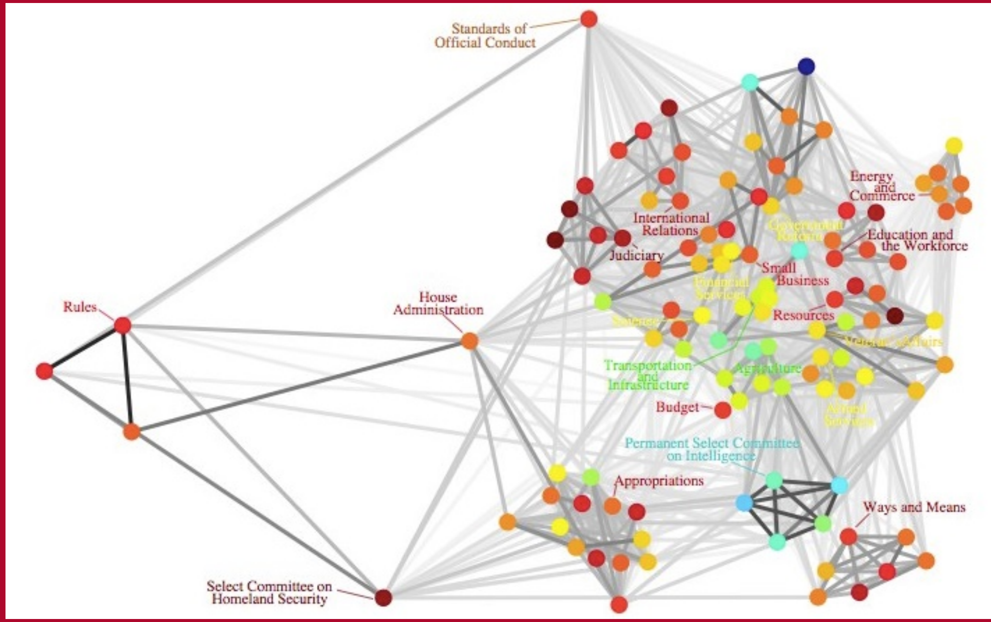
Automatic Control Theory

Chapter 4



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The root locus method

Main contents

1、The root locus concept and root locus equation

2、The root locus procedure

3、General root loci (Zero degree root loci)

The root locus procedure

Review

- **Open loop transfer function**
- **Phase equation**

D(s)=1+G(s)H(s)=0

$$G(s)H(s) = \frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -1$$

Asymptotes, Angles of start and end points, Breakaway point

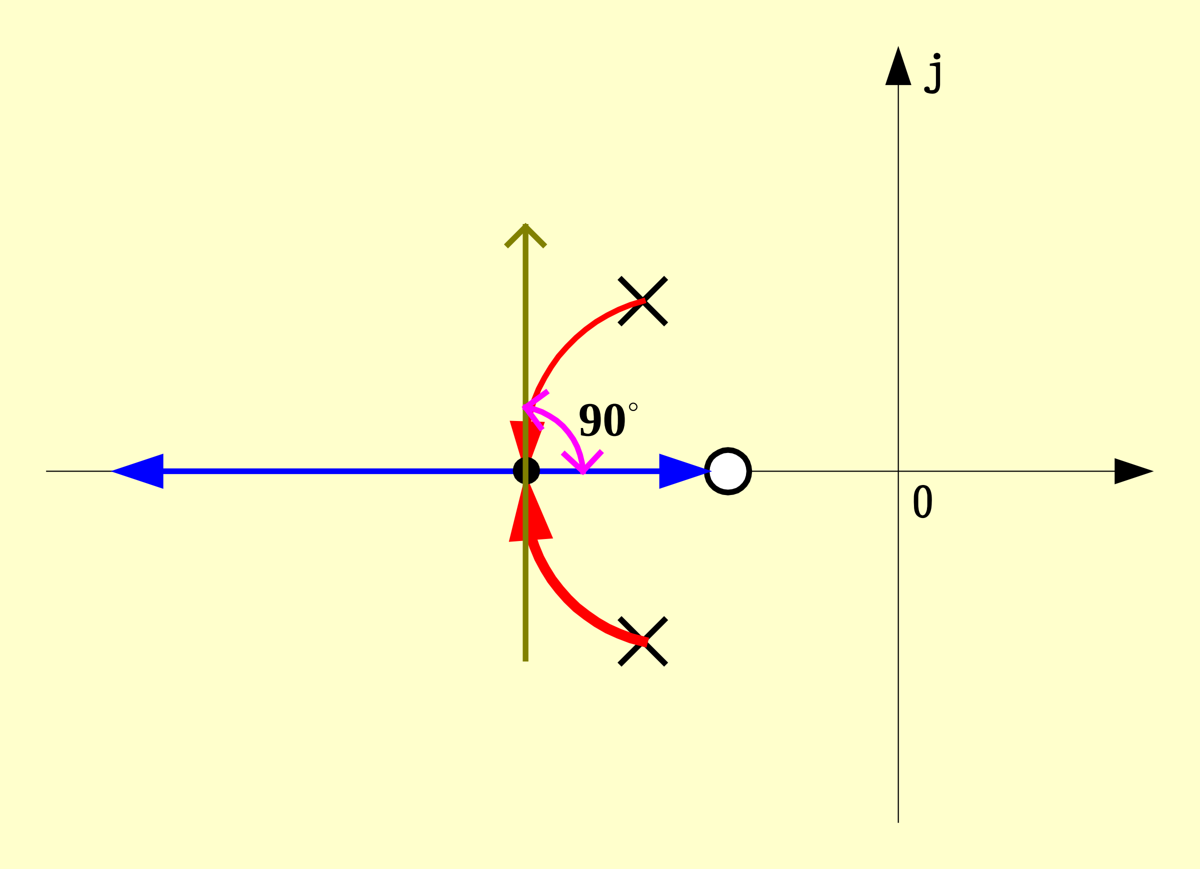
what is next

The root locus procedure (Details)

Step 7: Determine the breakaway point and the breakaway angle and arrival angle on the real axis.

breakaway point

$$\sum_{i=1}^{m}\frac{1}{d-z_i}=\sum_{i=1}^{n}\frac{1}{d-p_i}$$



The root locus procedure

$$1+\frac{K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0 \qquad \frac{\prod_{i=1}^{n}(s-p_i)+K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0 \qquad D(s)=\prod_{i=1}^{n}(s-p_i)+K^*\prod_{i=1}^{m}(s-z_i)=0$$

$$D(s_1)=\prod_{i=1}^{n}(s_1-p_i)+K^*\prod_{i=1}^{m}(s_1-z_i)=0$$

s_1: multiple root

$$\frac{\mathrm{d}}{\mathrm{d}s_1}D(s_1)=\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)+K^*\prod_{i=1}^{m}(s_i-z_i)\right]=0$$

$$\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)\right]=-K^*\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{m}(s_1-z_i)\right]$$

$$\boxed{\prod_{i=1}^{n}(s_1-p_i)=-K^*\prod_{i=1}^{m}(s_1-z_i)}$$

derived by division

$$\frac{\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{n}(s_1-p_i)\right]}{\prod_{i=1}^{n}(s_1-p_i)}=\frac{\frac{\mathrm{d}}{\mathrm{d}s_1}\left[\prod_{i=1}^{m}(s_1-z_i)\right]}{\prod_{i=1}^{m}(s_1-z_i)}$$

The root locus procedure

$$(\ln x)' = \frac{x'}{x}$$

$$\frac{d}{ds_1}\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] = \frac{\frac{d}{ds_1}\prod_{i=1}^{n}(s_1 - p_i)}{\prod_{i=1}^{n}(s_1 - p_i)}$$

$$\frac{\frac{d}{ds_1}\left[\prod_{i=1}^{n}(s_1 - p_i)\right]}{\prod_{i=1}^{n}(s_1 - p_i)} = \frac{\frac{d}{ds_1}\left[\prod_{i=1}^{m}(s_1 - z_i)\right]}{\prod_{i=1}^{m}(s_1 - z_i)} \Rightarrow \frac{d}{ds_1}\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] = \frac{d}{ds_1}\ln\left[\prod_{i=1}^{m}(s_1 - z_i)\right]$$

$$\sum_{i=1}^{n}\frac{d}{ds_1}\ln(s_1 - p_i) = \sum_{i=1}^{m}\frac{d}{ds_1}\ln(s_1 - z_i)$$

$$\ln\left[\prod_{i=1}^{n}(s_1 - p_i)\right] = \sum_{i=1}^{n}\ln(s_1 - p_i)$$

$$\ln\left[\prod_{i=1}^{n}(s_1 - z_i)\right] = \sum_{i=1}^{m}\ln(s_1 - z_i)$$

$$\sum_{i=1}^{n}\frac{1}{s_1 - p_i} = \sum_{i=1}^{m}\frac{1}{s_1 - z_i} \Rightarrow \sum_{i=1}^{n}\frac{1}{d - p_i} = \sum_{i=1}^{m}\frac{1}{d - z_i}$$

The root locus procedure

Example 1

$$G(s)H(s)=\frac{K(s+1)}{s^2+3s+3.25}$$

Get the root locus

Solution

$$p_1=-1.5+j1,\ p_2=-1.5-j1 \qquad z_1=-1$$

$$\sigma_a=\frac{-1.5+j1-1.5-j1+1}{2-1}=-2$$

$$\varphi_a=\frac{(2k+1)\pi}{2-1}=\pi$$

Asymptotes equation

The root locus procedure

$$\frac{1}{d+1}=\frac{1}{d+1.5+j}+\frac{1}{d+1.5-j}$$

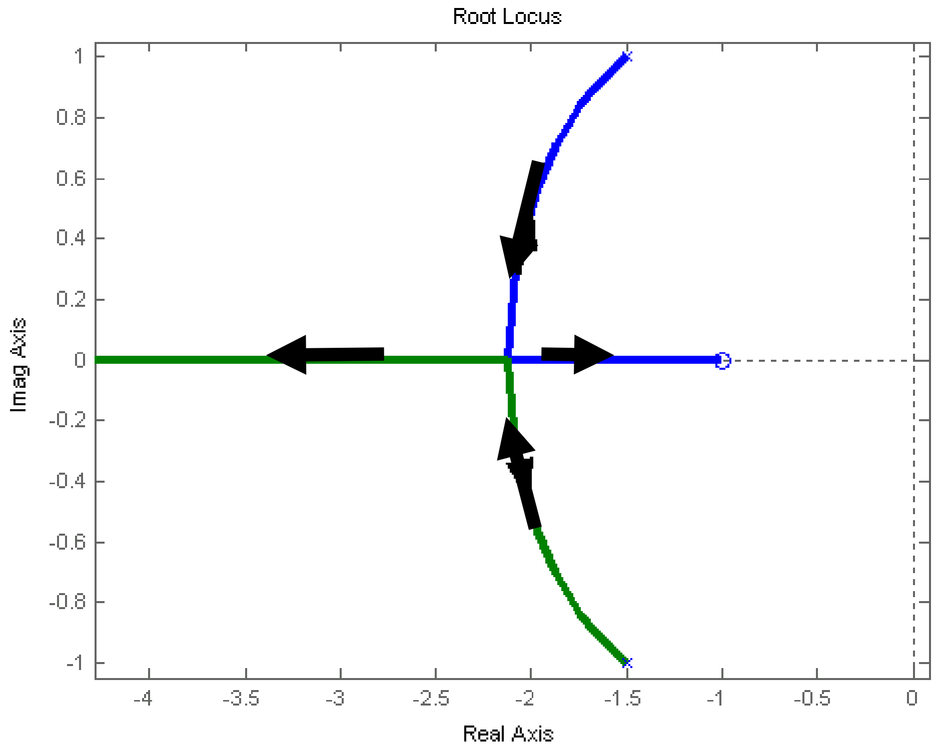
breakaway point

$$d^2+2d-0.25=0 \qquad d=-2.12,\quad d=0.12$$

abandon

Strat point angle

$$\theta_{p_1}=180^\circ+\varphi_{z_1 p_1}-\theta_{p_2 p_1}$$
$$=180^\circ+116.57^\circ-90^\circ=206.57^\circ$$
$$\theta_{p_2}=-206.57^\circ$$



The root locus procedure

分离角是指根轨迹离开重极点处的**切线**与实轴正方向的夹角。

$$\theta_d = \frac{1}{l}[(2k+1)\pi + \sum_{j=1}^{m}\angle(d-z_j) - \sum_{i=l+1}^{n}\angle(d-s_i)]$$

会合角是指根轨迹进入重极点处的**切线**与实轴正方向的夹角。

$$\Psi_d = \frac{1}{l}[(2k+1)\pi + \sum_{i=1}^{n}\angle(d-p_i) - \sum_{i=l+1}^{n}\angle(d-s_i)]$$

d – 为分离点坐标；Z_j – 为P(s)的零点；　p_i – 为P(s)的极点.

s_i – 为当$k=k_d$时，除l个重极点外，

其它$n-l$个非重根。

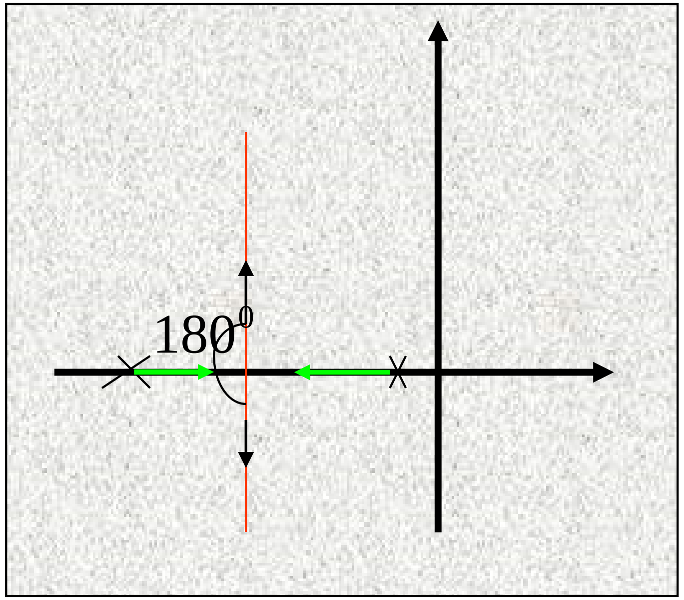
The root locus procedure

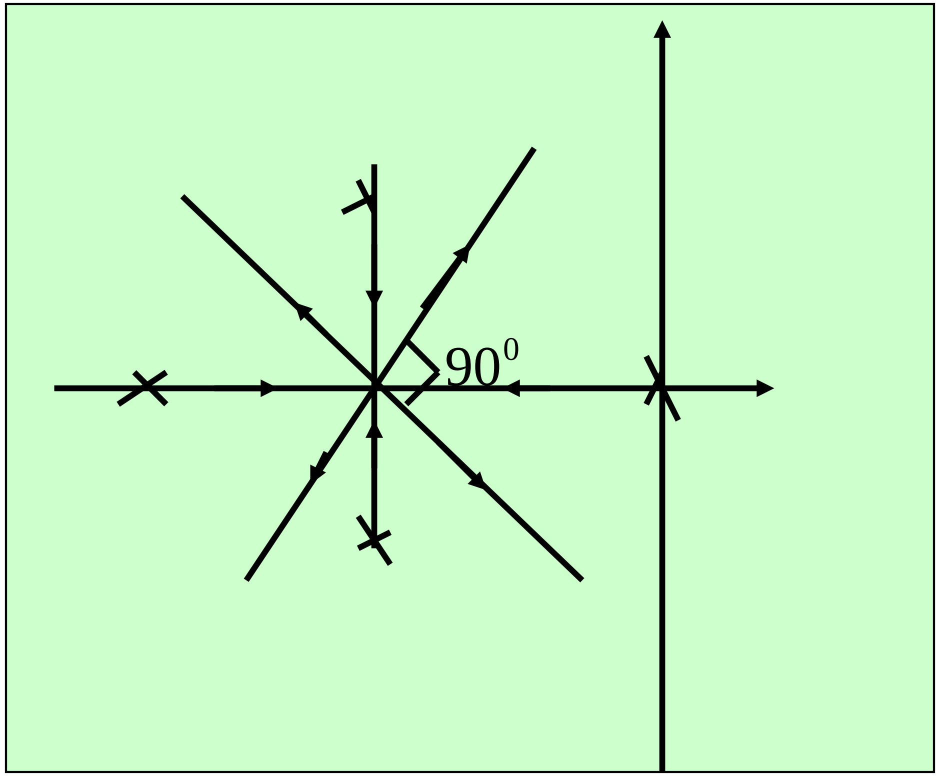
$$若\Psi_d = \frac{1}{l}2k\pi，则\theta_d = \frac{1}{l}(2k+1)\pi$$

$$若\Psi_d = \frac{1}{l}(2k+1)\pi，则\theta_d = \frac{1}{l}2k\pi$$

$$\delta = \frac{\pi}{l}$$

$$k = 0, \pm 1, \pm 2, \cdots$$





Step 8:

Determine the point at which the locus crosses the imaginary axis.

根轨迹与虚轴相交，表明系统闭环特征方程有**纯虚根**，系统处于**临界稳定**状态。

求解方法1：

将S=jw代入闭环特征方程式中1+KP(s)=0中，得： $1+KP(jw)=0$

可分解为： $Re[1+KP(jw)]+jIm[1+KP(jw)]=0$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$Re[1+KP(jw)]=0 \qquad Im[1+KP(jw)]=0$$

求解方法2：

应用Routh-Hurwitz 稳定判据，令Routh表 s所对应的行全为零，从而求得 ω和K 。

The root locus procedure

Example 2 $$G(s)H(s)=\frac{K^*}{s(s+3)(s^2+2s+2)}$$ Get the root locus

Solution

$p_1=0,\ p_2=-3,\ p_3=-1+\mathrm{j}1,\ p_4=-1-\mathrm{j}1$ segments of the real axis from 0 to -3 are root loci

Asymptotes equation

$$\sigma_a=\frac{0-3-1+\mathrm{j}1-1-\mathrm{j}1}{4}=-1.25 \qquad \varphi_a=\frac{(2k+1)\pi}{4}=\pm 45^\circ,\ \pm 135^\circ$$

Breakaway point

$$\sum_{1}^{4}\frac{1}{d-p_i}=0$$

$$\frac{1}{d}+\frac{1}{d+3}+\frac{1}{d+1-\mathrm{j}1}+\frac{1}{d+1+\mathrm{j}1}=0$$

$d_1=-2.3,\ d_{2,3}=-0.92\pm\mathrm{j}0.37$

abandon

The root locus procedure

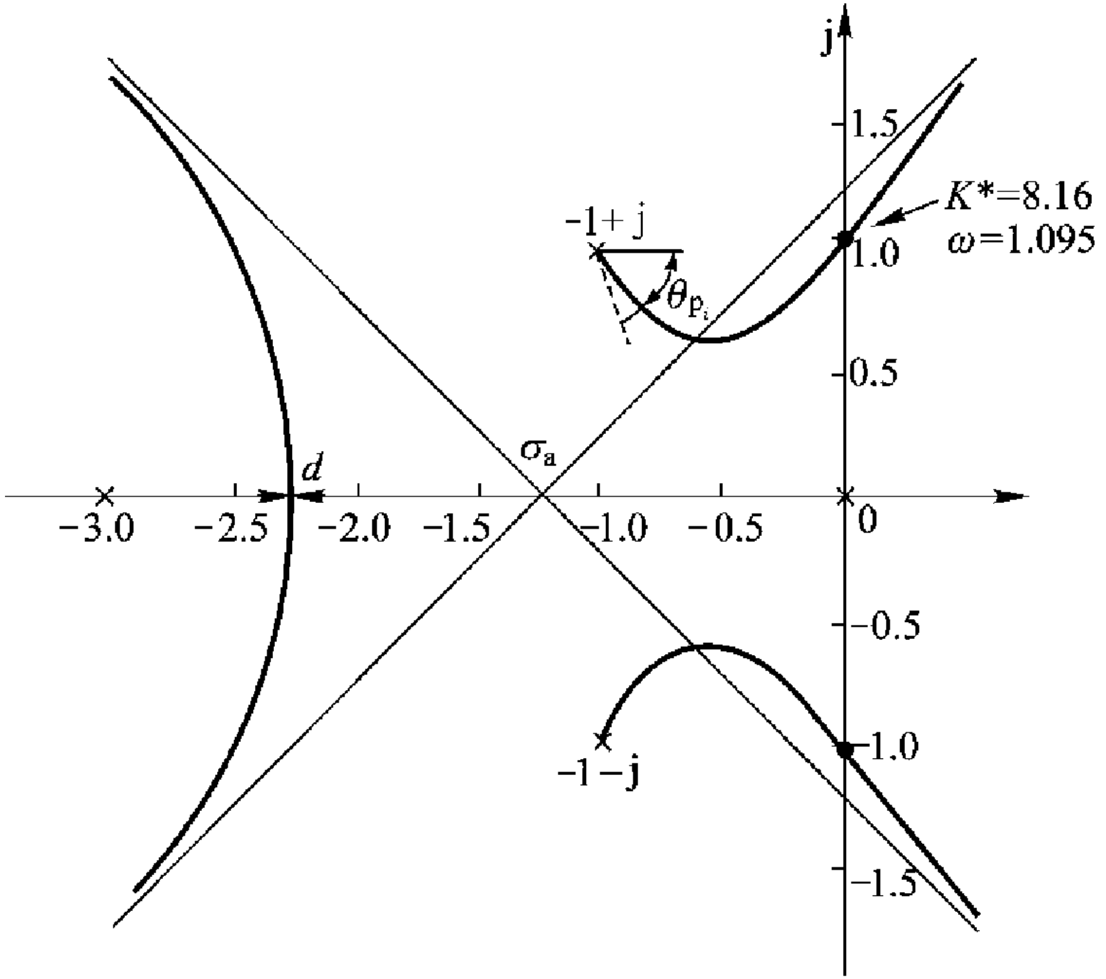
Strat point angle

$$\theta_{p_3} = 180^\circ - \theta_{p_1 p_3} - \theta_{p_2 p_3} - \theta_{p_4 p_3}$$
$$= 180^\circ - 135^\circ - 26.57^\circ - 90^\circ = -71.56^\circ$$
$$\theta_{p_4} = 71.56^\circ$$

Locus crosses the imaginary axis

$$D(s) = s(s+3)(s^2+2s+2) + K^* = 0$$

$$\omega^4 - 8\omega^2 + K^* = 0$$
$$5\omega^3 - 6\omega = 0$$
$$\omega = \pm 1.095, K^* = 8.16$$



The root locus procedure

Sum and Product of all roots of closed-loop characteristic equation

The closed-loop characteristic equation is

$$1 + KP(s) = 1 + \frac{K\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = 0 \qquad \prod_{i=1}^{n}(s-p_i) + K\prod_{j=1}^{m}(s-z_j) = \prod_{i=1}^{n}(s-s_i) = 0$$

$$\begin{cases} -\sum_{i=1}^{n} s_i = a_1 \\ (-1)^n \prod_{i=1}^{n} s_i = a_n \end{cases}$$

When $n - m \geq 2$ $\qquad \boxed{\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} s_i}$ → Sum of roots is constant

The root locus procedure

Conclusion

- The number of separate loci is **equal to** the number of **poles**

- The root loci is continuous, and must be **symmetrical** with respect to the horizontal real axis

- The locus begins at the poles of P(s) and end at the zeros of P(s) as **K** increases form 0 to infinity

- The root locus on the **real** axis always lies in a section of the real axis to the left of an **odd** number of poles and zeros

- Asymptotes of the root loci

$$\sigma_A = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} \qquad \varphi_A = \frac{(2k+1)\pi}{n-m}$$

The root locus procedure

- Angle of departure of the locus from a pole and the angle of arrival of the locus at a zero

$$\theta_{p_i} = (2k+1)\pi + \sum_{j=1}^{m} \angle(p_i - z_j) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \angle(p_i - p_j)$$

$$\theta_{z_i} = (2k+1)\pi + \sum_{j=1}^{n} \angle(z_i - p_j) - \sum_{\substack{j=1 \\ j\neq i}}^{m} \angle(z_i - z_j)$$

- Breakaway point

$$\sum_{i=1}^{m} \frac{1}{d - z_i} = \sum_{i=1}^{n} \frac{1}{d - p_i}$$

- Breakaway angle and arrival angle

$$\boldsymbol{\theta_d = \frac{1}{l}[(2k+1)\pi + \sum_{j=1}^{m} \angle(d - z_j) - \sum_{i=l+1}^{n} \angle(d - s_i)]}$$

$$\boldsymbol{\Psi_d = \frac{1}{l}[(2k+1)\pi + \sum_{i=1}^{n} \angle(d - p_i) - \sum_{i=l+1}^{n} \angle(d - s_i)]}$$

- Locus crosses the imaginary axis

$$1+KP(jw)=0$$

- Sum and Product of all roots of closed-loop characteristic equation

$$-\sum_{i=1}^{n} s_i = a_1 \qquad (-1)^n \prod_{i=1}^{n} s_i = a_n$$

Example 3

Open loop transfer function
$$G(s)=\frac{K}{s(0.05s+1)(0.05s^2+0.2s+1)}$$

Get the root locus, and the root at the critical value of *K*

Solution

n=4 Poles: $0, -20, -2-j4, -2+j4$ segments of the real axis from 0 to -20 are root loci

Asymptotes equation

$$\sigma_a=\frac{\sum_{i=1}^{4}p_i}{n-m}=\frac{-20-2+j4-2-j4}{4}=-6$$

$$\varphi_a=\frac{(2k+1)\pi}{n-m}=\frac{(2k+1)\pi}{4}$$

$k=0$ $\varphi_a=45^\circ$, $k=-1$ $\varphi_a=-45^\circ$

$k=1$ $\varphi_a=135^\circ$, $k=-2$ $\varphi_a=-135^\circ$

The root locus procedure

Strat point angle

$$\theta_{p_3} = 180^\circ - \theta_{p_1 p_3} - \theta_{p_2 p_3} - \theta_{p_4 p_3}$$
$$= 180^\circ - 116.5^\circ - 12.5^\circ - 90^\circ = -39^\circ$$
$$\theta_{p_4} = +39^\circ$$

Breakaway point

$$\frac{1}{d-p_1} + \frac{1}{d-p_3} + \frac{1}{d-p_2} + \frac{1}{d-p_4} = 0$$

$$\frac{1}{d} + \frac{1}{d+20} + \frac{1}{d+2+j4} + \frac{1}{d+2-j4} = 0$$

$$d_1 = -15.1$$

$$d_2 = -1.45 + j2.07, \quad d_3 = -1.45 - j2.07$$

abandon

Locus crosses the imaginary axis

$$D(s) = s(s+20)(s^2+4s+20) + 400K = 0$$

$$\begin{cases} \omega^4 - 100\omega^2 + 400K = 0 \\ -24\omega^3 + 400\omega = 0 \end{cases}$$

$$\omega_1 = 0, \ \omega_2 = 4.1, \ \omega_3 = -4.1$$

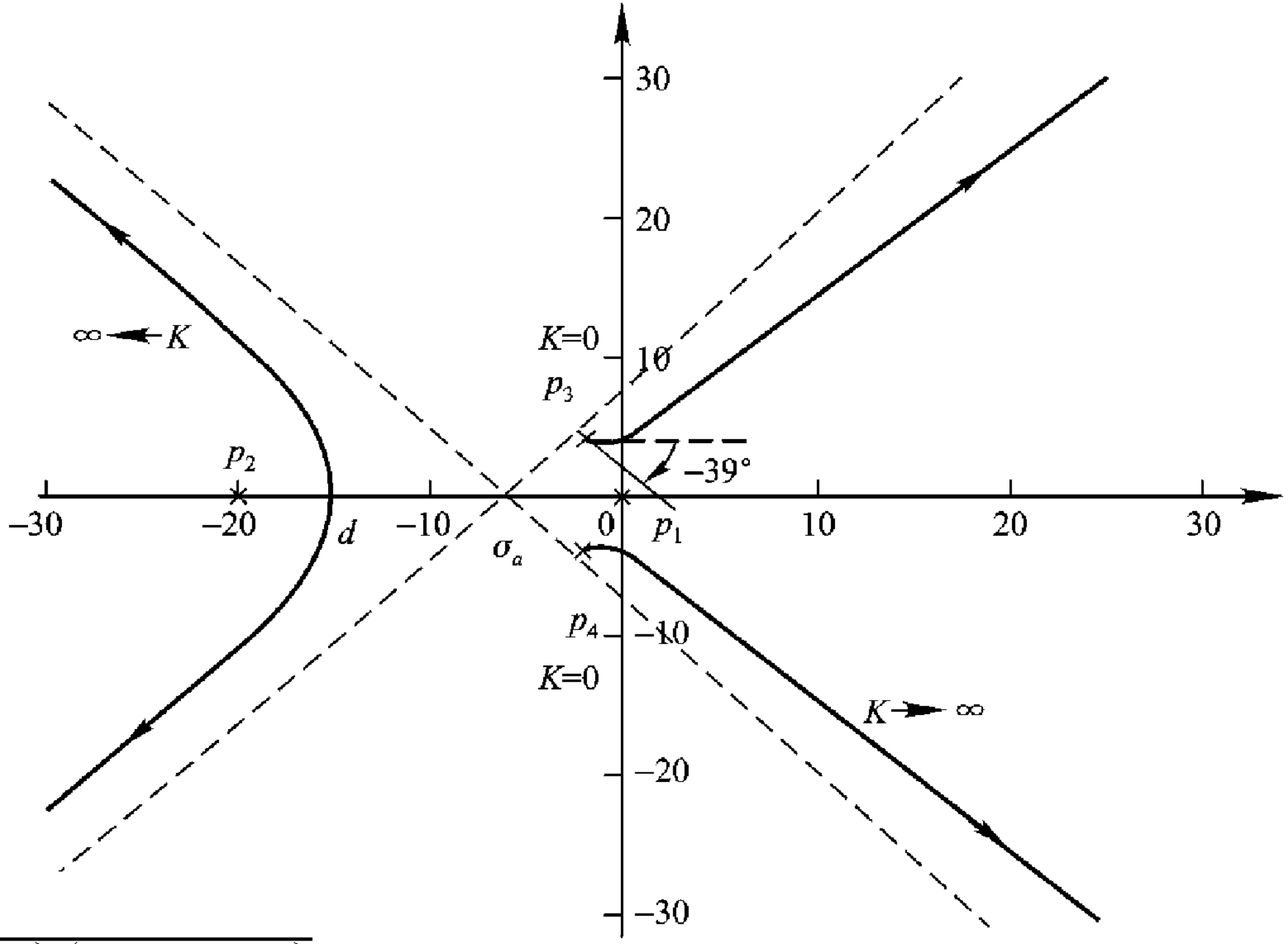
K=3.47

The root locus procedure

K=3.47

$$s_1 = \mathrm{j}4.1,\ s_2 = -\mathrm{j}4.1$$

$$s_3 = -4.2,\ s_4 = -19.8$$



$$\Phi(s) = \frac{1\,388.9}{(s+4.2)(s+19.8)(s+\mathrm{j}4.1)(s-\mathrm{j}4.1)}$$

The root locus procedure

核心

- **Definition of root locus**
- **Open loop transfer function**
- **Phase equation**

$$G(s)H(s)=\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=-1$$

续

- **General root loci (Zero degree root loci)**