第四节 多元复合函数的求导法则

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$

微分法则 $\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$

本节内容:

一、多元复合函数求导的链式法则

二、多元复合函数的全微分

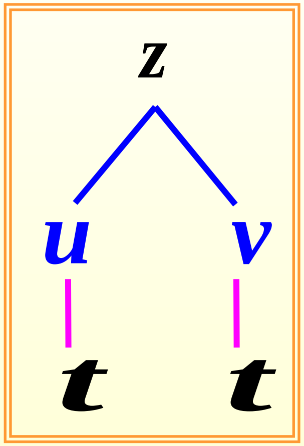
一、多元复合函数求导的链式法则

定理. **若函数** $u=\varphi(t), v=\psi(t)$ 在点 t 可导, $z=f(u,v)$

在对应点 (u,v) **处偏导连续, 则复合函数** $z=f(\varphi(t),\psi(t))$

在点 t 可导, 且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



证 **设 t 取增量 $\triangle t$, 则相应中间变量**

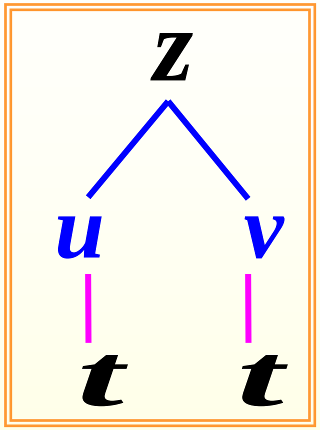
有增量 $\triangle u$, $\triangle v$,

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho) \quad \left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\,)$$

令 $\Delta t\to 0$，则有 $\Delta u\to 0$，$\Delta v\to 0$，

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\qquad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$



$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$

($\triangle t<0$ 时,根式前加 “–”号)

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}$$

（全导数公式）

说明: 若定理中 $f(u,v)$ 在点 (u,v) **偏导数连续**减弱为**偏导数存在**, 则定理结论**不一定成立.**

例如: $z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$

$u = t\,,\quad v = t$

易知: $\left.\dfrac{\partial z}{\partial u}\right|_{\substack{u=0\\v=0}} = f_u(0,0) = 0\,,\quad \left.\dfrac{\partial z}{\partial v}\right|_{\substack{u=0\\v=0}} = f_v(0,0) = 0$

但复合函数 $z = f(t,t) = \dfrac{t}{2}$

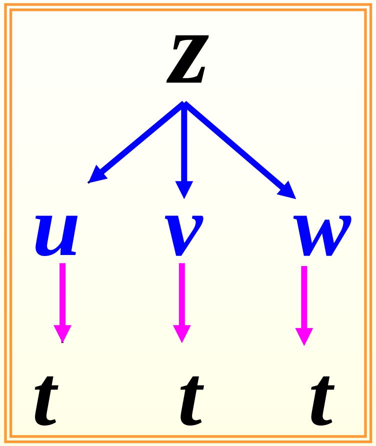
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

推广: 设下面所涉及的函数都可微分.

1) 中间变量多于两个的情形. 例如, $z = f(u,v,w)$,

$u=\varphi(t), v=\psi(t), w=\omega(t)$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

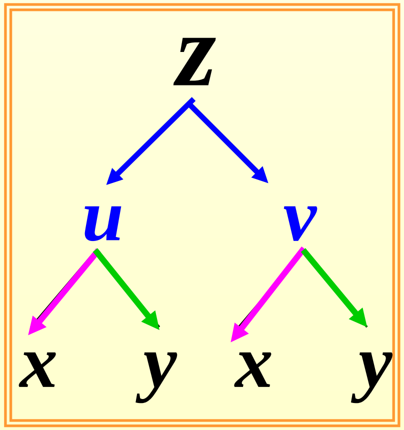


2) 中间变量是多元函数的情形. 例如,

$z = f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$



又如, $z=f(x,v),\ v=\psi(x,y)$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}}=\boxed{\frac{\partial f}{\partial x}}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\begin{tikzcd}[column sep=small] & z=f \arrow[dl] \arrow[dr] & & \\ x & & v \arrow[dl] \arrow[dr] & \\ & x & & y \end{tikzcd}$$

口诀:

分段用乘, 分叉用加, 单路全导, 叉路偏导

注意: 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

$\frac{\partial z}{\partial x}$ 表示 $f(x,\psi(x,y))$ 固定 y 对 x 求导

$\frac{\partial f}{\partial x}$ 表示 $f(x,v)$ 固定 v 对 x 求导

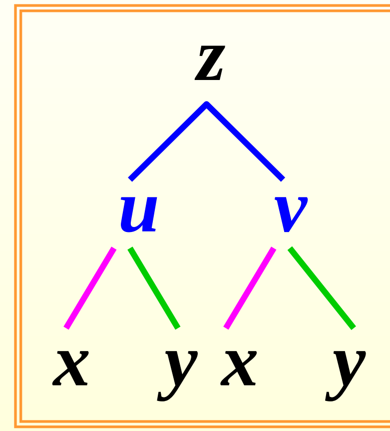
例1 设 $z = e^u \sin v$，$u = xy$，$v = x + y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y \;+ e^u \cos v \cdot 1$$

$$= e^{xy}[y\cdot\sin(x+y) + \cos(x+y)]$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x \;+ e^u \cos v \cdot 1$$

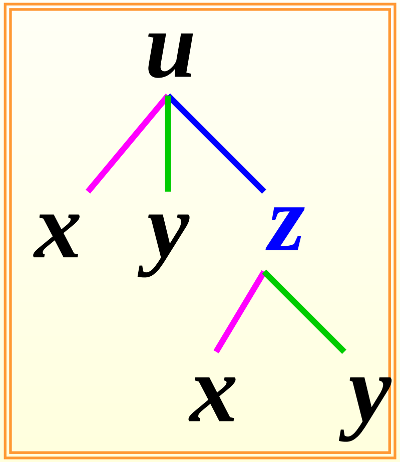
$$= e^{xy}[x\cdot\sin(x+y) + \cos(x+y)]$$

例2 $u = f(x,y,z) = e^{x^2+y^2+z^2}$, $z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解 $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= 2x e^{x^2+y^2+z^2} + 2z e^{x^2+y^2+z^2} \cdot 2x \sin y$$

$$= 2x(1 + 2x^2 \sin^2 y) e^{x^2+y^2+x^4 \sin^2 y}$$



$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2y e^{x^2+y^2+z^2} + 2z e^{x^2+y^2+z^2} \cdot x^2 \cos y$$

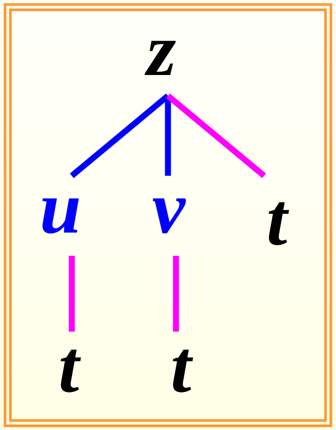
$$= 2(y + x^4 \sin y \cos y) e^{x^2+y^2+x^4 \sin^2 y}$$

例3 设 $z = uv + \sin t$, $u = \mathrm{e}^t$, $v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解 $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

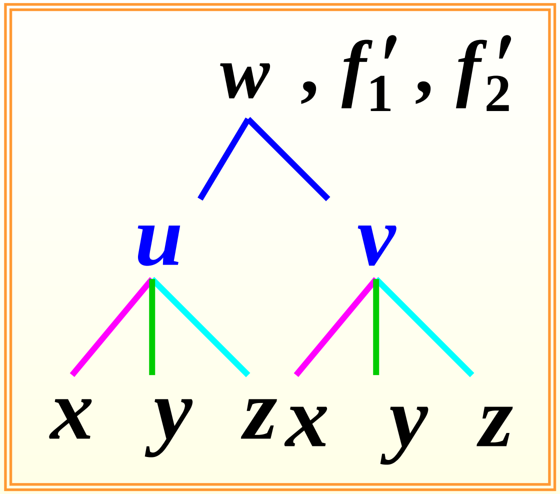
$$= v\,\mathrm{e}^t - u\sin t + \cos t$$

$$= \mathrm{e}^t(\cos t - \sin t) + \cos t$$



例5 设 $w = f(x+y+z, xyz)$, f 具有二阶连续偏导数,

求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.



解 令 $u = x+y+z, v = xyz$, 则

$$w = f(u, v)$$

$$\frac{\partial w}{\partial x} = f'_1 + f'_2 \cdot yz$$

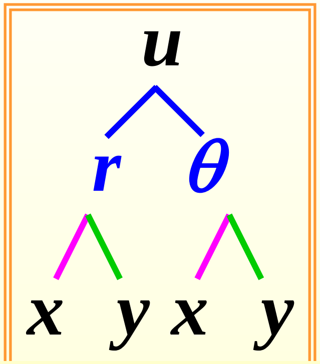
$$\frac{\partial^2 w}{\partial x \partial z} = f''_{11} \cdot 1 + f''_{12} \cdot xy + y f'_2 + yz[f''_{21} \cdot 1 + f''_{22} \cdot xy]$$

$$= f''_{11} + y(x+z) f''_{12} + xy^2 z f''_{22} + y f'_2$$

例8 设 $u=f(x,y)$二阶偏导数连续,求下列表达式在极坐标系下的形式 (1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$, (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解 已知 $x=r\cos\theta$, $y=r\sin\theta$, 则 课本例5，自学

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}$$

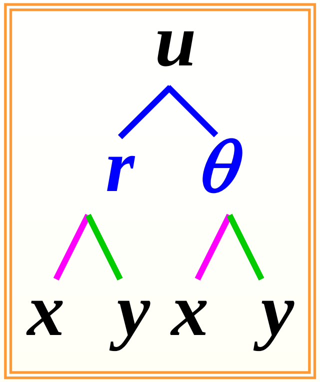


(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-y}{x^2+y^2}$$

$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$



$$\frac{\partial r}{\partial y}=\frac{y}{r}, \quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}$$

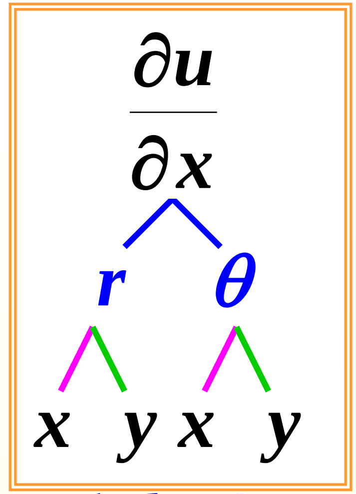
$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\therefore \quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

已知 $\frac{\partial u}{\partial x} = \boxed{\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}}$



(2) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$- \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+ \frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

注意利用已有公式

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta \cancel{- 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$\cancel{+ \frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta \cancel{+ 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos^2\theta}{r^2}$$

$$\cancel{- \frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} \right]$$

二、多元复合函数的全微分

设函数 $z = f(u,v),\ u = \varphi(x,y),\ v = \psi(x,y)$ 都可微，则复合函数 $z = f(\varphi(x,y), \psi(x,y))$ 的全微分为

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$= \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v$$

可见无论 u, v 是自变量还是中间变量，其全微分表达形式都一样，这性质叫做**全微分形式不变性**.

例9 利用全微分形式不变性再解例1.

解 $\mathrm{d}z = \mathrm{d}(\,\mathrm{e}^{u}\sin v\,)$

$$= \mathrm{e}^{u}\sin v\,\mathrm{d}u + \mathrm{e}^{u}\cos v\,\mathrm{d}v$$

$$= \mathrm{e}^{xy}[\sin(x+y)\,\mathrm{d}(xy) + \cos(x+y)\,\mathrm{d}(x+y)]$$

$$= \mathrm{e}^{xy}[\sin(x+y)(y\,\mathrm{d}x + x\,\mathrm{d}y) + \cos(x+y)(\mathrm{d}x+\mathrm{d}y)]$$

$$= \mathrm{e}^{xy}[y\sin(x+y)+\cos(x+y)]\,\mathrm{d}x$$

$$+ \mathrm{e}^{xy}[x\sin(x+y)+\cos(x+y)]\,\mathrm{d}y$$

所以 $$\frac{\partial z}{\partial x} = \mathrm{e}^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$$

$$\frac{\partial z}{\partial y} = \mathrm{e}^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$$

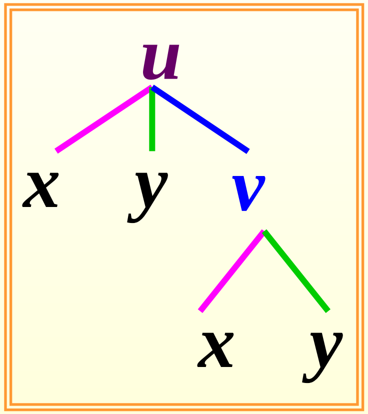
内容小结

1. 复合函数求导的链式法则

“分段用乘,分叉用加，单路全导，叉路偏导”

例如，$u = f(x, y, v),\ v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1' + f_3' \cdot \varphi_1'\,; \qquad \frac{\partial u}{\partial y} = f_2' + f_3' \cdot \varphi_2'$$



2. 全微分形式不变性

对 $z = f(u,v)$, **不论** u , v **是自变量还是中间变量,**

$$\mathrm{d}z = f_u(u,v)\,\mathrm{d}u + f_v(u,v)\,\mathrm{d}v$$

补充题

1. 已知 $f(x,y)\Big|_{y=x^2}=1$, $f_1'(x,y)\Big|_{y=x^2}=2x$, 求 $f_2'(x,y)\Big|_{y=x^2}$.

解 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

2. 设函数 $z = f(x, y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1) = 1, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 3,$$

$\varphi(x) = f(x, f(x,x))$，求 $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. (2001考研)

解 由题设 $\varphi(1) = f(1, f(1,1)) = f(1,1) = 1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1} = 3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$= 3\Big[\underline{f_1'(x, f(x,x))} + \underline{f_2'(x, f(x,x))}\big(\underline{f_1'(x,x) + f_2'(x,x)}\big)\Big]\Big|_{x=1}$$

$$= 3 \cdot [2 + 3 \cdot (2+3)] = 51$$

第五节 隐函数的求导方法

1) 方程在什么条件下才能确定隐函数 .

例如, 方程 $x^2+\sqrt{y}+C=0$ $\begin{cases} C<0 \text{ 时, 能确定隐函数} \\ C>0 \text{ 时, 不能确定隐函数} \end{cases}$

2) 方程能确定隐函数时, 研究其连续性,可微性及求导方法问题.

本节讨论:

一、一个方程所确定的隐函数 及其导数

二、方程组所确定的隐函数组 及其导数

预备知识: $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ (1)

若系数行列式 $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

则方程组(1)有唯一组解. $x_1 = \dfrac{D_1}{D},\ x_2 = \dfrac{D_2}{D}.$

其中, $D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},\ D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$

—克莱姆法则(见高数第一册附录1)

若将行列式的某行或某列数都乘以常数k，相当于原行列式的值乘以常数k。如:

$$\begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

一、一个方程所确定的隐函数及其导数

定理1. 设函数 $F(x,y)$在点 $P(x_0,y_0)$的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0,y_0)=0$;

③ $F_y(x_0,y_0)\neq 0$

则方程 $F(x,y)=0$ 在点 (x_0,y_0) 的某邻域内可唯一确定一个连续函数 $y=f(x)$，满足条件 $y_0=f(x_0)$，并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$ (隐函数求导公式)

定理证明从略，仅就求导公式推导如下：

设 $y = f(x)$ 为方程 $F(x, y) = 0$ 所确定的隐函数，**则**

$$F(x, f(x)) \equiv 0$$

两边对 x 求导

$$\begin{tikzcd}[row sep=small] & & x \\ u = F(x, y) \arrow[urr] \arrow[r] & y \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\mathrm{d} y}{\mathrm{d} x} = 0$$

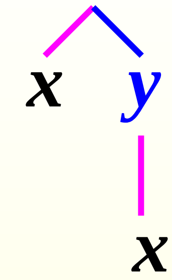
在(x_0, y_0)的某邻域内 $F_y \neq 0$

$$\frac{\mathrm{d} y}{\mathrm{d} x} = -\frac{F_x}{F_y}$$

若$F(x,y)$的二阶偏导数也都连续,

则还可求隐函数的**二阶导数**:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$



$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)+\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=-\frac{F_{xx}F_y-F_{yx}F_x}{F_y^2}-\frac{F_{xy}F_y-F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$=-\frac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}$$

例1 验证方程 $\sin y + e^x - xy - 1 = 0$ 在点(0,0)某邻域可确定一个可导隐函数 $y = f(x)$,并求

$$\left.\frac{dy}{dx}\right|_{x=0},\quad \left.\frac{d^2y}{dx^2}\right|_{x=0}$$

解 令 $F(x,y) = \sin y + e^x - xy - 1$, 则

① $F_x = e^x - y,\ F_y = \cos y - x$ 连续;

② $F(0,0) = 0$;

③ $F_y(0,0) = 1 \neq 0$,

由 定理1 可知, 在 (0,0) 的某邻域内方程存在可导的隐函数 $y = f(x)$, 且

上页 下页

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=-\left.\frac{F_x}{F_y}\right|_{x=0}=-\left.\frac{\mathrm{e}^x-y}{\cos y-x}\right|_{x=0,\,y=0}=-1$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$$

$$=-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x-y}{\cos y-x}\right)\Bigg|_{x=0,\ y=0,\ y'=-1}$$

$$=-\frac{(\mathrm{e}^x-y')(\cos y-x)-(\mathrm{e}^x-y)(-\sin y\cdot y'-1)}{(\cos y-x)^2}\Bigg|_{\substack{x=0\\ y=0\\ y'=-1}}$$

$$=-3$$

上页 下页

导数的另一求法 — 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0, \quad y = y(x)$$

两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

$$y'\Big|_{x=0} = -\left.\frac{e^x - y}{\cos y - x}\right|_{(0,0)} = -1$$

两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

令 $x = 0$，注意此时 $y = 0$，$y' = -1$

$$\left.\frac{d^2 y}{dx^2}\right|_{x=0} = -3$$

定理2 . 若函数 $F(x,y,z)$ 满足:

① 在点 $P(x_0,y_0,z_0)$ 的某邻域内具有**连续偏导数**;

② $F(x_0,y_0,z_0)=0$;

③ $F_z(x_0,y_0,z_0)\neq 0$,

则方程 $F(x,y,z)=0$ 在点 (x_0,y_0,z_0) 某一邻域内可唯一确定一个连续函数 $z=f(x,y)$, 满足 $z_0=f(x_0,y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

定理证明从略, 仅就求导公式推导如下:

设 $z = f(x,y)$ 是方程 $F(x,y,z) = 0$ 所确定的隐函数则

$$F(x, y, f(x, y)) \equiv 0$$

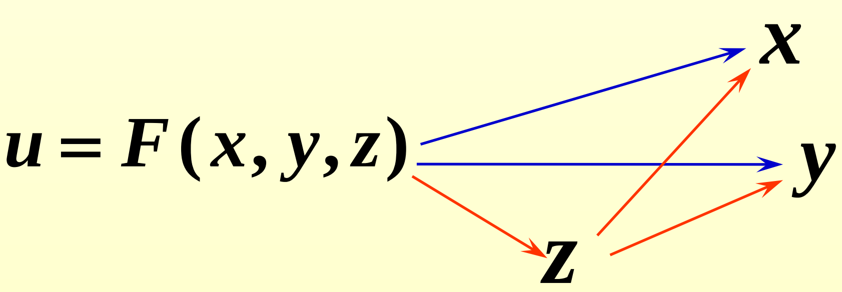
两边对 x 求偏导

$$F_x + F_z \frac{\partial z}{\partial x} = 0$$

在 (x_0, y_0, z_0) 的某邻域内 $F_z \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

同样可得 $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$



例2 设 $x^2+y^2+z^2-4z=0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解法1 **利用隐函数求导**

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解法2　利用公式

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore\quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x}{z-2}=\frac{x}{2-z}$$

两边对 x 求偏导

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

例3 设$F(x^2-y^2,y^2-z^2)=0$，证明$yz\dfrac{\partial z}{\partial x}+zx\dfrac{\partial z}{\partial y}=xy$

证 $\because \dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{F_1'\cdot 2x}{F_2'\cdot(-2z)}=\dfrac{xF_1'}{zF_2'}$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{F_1'\cdot(-2y)+F_2'\cdot 2y}{F_2'\cdot(-2z)}$$

$$=\frac{y(F_2'-F_1')}{zF_2'}$$

$$yz\frac{\partial z}{\partial x}+zx\frac{\partial z}{\partial y}=yx\frac{F_1'}{F_2'}+xy\frac{F_2'-F_1'}{F_2'}=xy$$

设$u=x^2-y^2$

$v=y^2-z^2$

则方程为

$F(u,v)=0$

记$\dfrac{\partial F}{\partial u}=F_1'$

$\dfrac{\partial F}{\partial v}=F_2'$

例4 设$F(x,y)$具有连续偏导数，已知方程 $F(\frac{x}{z},\frac{y}{z})=0$，求 $\mathrm{d}z$.

解法1 利用偏导数公式. 设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数，则

$$\frac{\partial z}{\partial x}=-\frac{F_1'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_1'}{xF_1'+yF_2'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_2'}{xF_1'+yF_2'}$$

故 $$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

解法2 微分法. 对方程两边求微分:

$$F\left(\frac{x}{z},\frac{y}{z}\right)=0$$

$$F_1'\cdot \mathrm{d}\left(\frac{x}{z}\right)+F_2'\cdot \mathrm{d}\left(\frac{y}{z}\right)=0$$

$$F_1'\cdot\left(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2}\right)+F_2'\cdot\left(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2}\right)=0$$

$$\frac{xF_1'+yF_2'}{z^2}\mathrm{d}z=\frac{F_1'\mathrm{d}x+F_2'\mathrm{d}y}{z}$$

$$\mathrm{d}z=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

二、方程组所确定的隐函数组及其导数

隐函数存在定理还可以推广到方程组的情形.

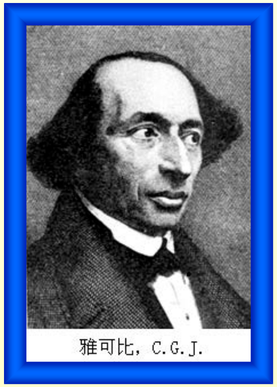
以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的雅可比 行列式.



雅可比，C.G.J.

定理3. 设函数 $F(x,y,u,v),G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0,y_0,u_0,v_0)=0,\ G(x_0,y_0,u_0,v_0)=0$;

③ $J\Big|_P=\dfrac{\partial(F,G)}{\partial(u,v)}\Big|_P\neq 0$,

则方程组 $F(x,y,u,v)=0,\ G(x,y,u,v)=0$ 在点 P 的某一邻域内可**唯一**确定一组满足条件 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$ 的**连续函数** $u=u(x,y),v=v(x,y)$, 且有偏导数公式:

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}$$

(P89)

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

定理证明略.仅推导偏导数公式如下:

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

上页 下页

设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$,则

$$\begin{cases} F(\underline{x},y,u(\underline{x},y),v(\underline{x},y))\equiv 0 \\ G(\underline{x},y,u(\underline{x},y),v(\underline{x},y))\equiv 0 \end{cases}$$

两边对 x 求导得 $\begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$

这是关于 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 的线性方程组，在点P 的某邻域内

系数行列式 $J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$, 故得 $\dfrac{\partial u}{\partial x}=-\dfrac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

同样可得

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

例5 设 $xu-yv=0,\ yu+xv=1$, 求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}$.

解 方程组两边对 x 求导，并移项得

$$\begin{cases} x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=-u \\ y\dfrac{\partial u}{\partial x}+x\dfrac{\partial v}{\partial x}=-v \end{cases}$$

由题设 $J=\begin{vmatrix} x & -y \\ y & x \end{vmatrix}=x^2+y^2\neq 0$

故有
$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}=-\dfrac{xu+yv}{x^2+y^2} \\ \dfrac{\partial v}{\partial x}=\dfrac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}=-\dfrac{xv-yu}{x^2+y^2} \end{cases}$$

练习: 求 $\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}$

答案:
$$\begin{cases} \dfrac{\partial u}{\partial y}=-\dfrac{yu-xv}{x^2+y^2} \\ \dfrac{\partial v}{\partial y}=-\dfrac{xu+yv}{x^2+y^2} \end{cases}$$

上页 下页

P92:10.(3) $\begin{cases} u = f(ux, v+y) \\ v = g(u-x, v^2 y) \end{cases}$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解 此方程组确定了两个二元隐函数 $u = u(x, y)$, $v = v(x, y)$.

对方程组中的每个方程两边同时求对 x 的导数，得

$$\begin{cases} \dfrac{\partial u}{\partial x} = f_1' \cdot \left(u + x \cdot \dfrac{\partial u}{\partial x} \right) + f_2' \cdot \dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial v}{\partial x} = g_1' \cdot \left(\dfrac{\partial u}{\partial x} - 1 \right) + g_2' \cdot 2vy \dfrac{\partial v}{\partial x} \end{cases}$$

设 $s = ux, t = v + y$

$$\begin{tikzcd} & f(s,t) \arrow[dl] \arrow[dr] & \\ s & & t \end{tikzcd}$$

整理得 $$\begin{cases} (xf_1' - 1) \cdot \dfrac{\partial u}{\partial x} + f_2' \cdot \dfrac{\partial v}{\partial x} = -uf_1' \\[2ex] g_1' \cdot \dfrac{\partial u}{\partial x} + (2vyg_2' - 1) \cdot \dfrac{\partial v}{\partial x} = g_2' \end{cases}$$

$$J=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vyg_2'-1 \end{vmatrix}=(xf_1'-1)(2vyg_2'-1)-f_2'g_1'\neq 0$$

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix} -uf_1' & f_2' \\ g_1' & 2vyg_2'-1 \end{vmatrix}}{J}=\frac{-uf_1'(2vyg_2'-1)-f_2'g_1'}{(xf_1'-1)(2vyg_2'-1)-f_2'g_1'}$$

$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix} xf_1'-1 & uf_1' \\ g_1' & g' \end{vmatrix}}{J}=\frac{g_1'(xf_1'+uf_1'-1)}{(xf_1'-1)(2vyg_2'-1)-f_2'g_1'}$$

$$\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}=?$$

练习

例6 设函数 $x = x(u,v),\ y = y(u,v)$ 在点 (u,v) 的某一邻域内有连续的偏导数,且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u, v) 对应的点 (x, y) 的某一邻域内唯一确定一组连续且具有连续偏导数的反函数 $u = u(x,y),\ v = v(x, y)$.

2) 求 $u = u(x,y),\ v = v(x, y)$ 对 x, y 的偏导数.

解 1) 令 $F(x,y,u,v) \equiv x - x(u,v) = 0$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

上页 下页

则有 $$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}\neq 0,$$

由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\begin{cases} x\equiv x(u(x,y),v(x,y)) \\ y\equiv y(u(x,y),v(x,y)) \end{cases} \qquad ①$$

①式两边对 x 求导, 得

$$\begin{cases} 1=\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial x}{\partial v}\cdot\dfrac{\partial v}{\partial x} \\ 0=\dfrac{\partial y}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial y}{\partial v}\cdot\dfrac{\partial v}{\partial x} \end{cases} \qquad ②$$

注意$J\neq 0$，从方程组②解得

$$\frac{\partial u}{\partial x}=\frac{1}{J}\begin{vmatrix} 1 & \dfrac{\partial x}{\partial v} \\ 0 & \dfrac{\partial y}{\partial v} \end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v},$$

$$\frac{\partial v}{\partial x}=\frac{1}{J}\begin{vmatrix} \dfrac{\partial x}{\partial u} & 1 \\ \dfrac{\partial y}{\partial u} & 0 \end{vmatrix}=-\frac{1}{J}\frac{\partial y}{\partial u}$$

同理, ①式两边对y求导, 可得

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial x}{\partial v},\qquad \frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

上页 下页

例6的应用：计算极坐标变换 $x = \underset{u}{r}\cos\underset{v}{\theta},\ y = r\sin\theta$ 的反变换的导数 .

$$\frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r}$$

由于 $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

所以 $\dfrac{\partial r}{\partial x} = \dfrac{1}{J}\dfrac{\partial y}{\partial \theta} = \dfrac{1}{r} r\cos\theta = \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r} = -\frac{1}{r}\sin\theta = -\frac{y}{x^2+y^2}$$

同样有 $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad \dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$

内容小结

1. 隐函数(组) 存在定理

2. 隐函数 (组) 求导方法

方法1. 利用复合函数求导法则直接计算；

方法2. 利用微分形式不变性；

方法3. 代公式 .

思考与练习

设 $z = f(x+y+z, xyz)$,求 $\frac{\partial z}{\partial x}, \frac{\partial x}{\partial z}, \frac{\partial x}{\partial y}$.

提示: $z = f(x+y+z,\ xyz)$

- $$\frac{\partial z}{\partial x} = f_1' \cdot \left(1 + \frac{\partial z}{\partial x}\right) + f_2' \cdot \left(yz + xy\frac{\partial z}{\partial x}\right)$$

$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{f_1' + yz f_2'}{1 - f_1' - xy f_2'}$$

- $$1 = f_1' \cdot \left(\frac{\partial x}{\partial z} + 1\right) + f_2' \cdot \left(yz\frac{\partial x}{\partial z} + xy\right)$$

$$\Longrightarrow \frac{\partial x}{\partial z} = \frac{1 - f_1' - xy f_2'}{f_1' + yz f_2'}$$

- $$0 = f_1' \cdot \left(\frac{\partial x}{\partial y} + 1\right) + f_2' \cdot \left(yz\frac{\partial x}{\partial y} + xz\right)$$

$$\Longrightarrow \frac{\partial x}{\partial y} = -\frac{f_1' + xz f_2'}{f_1' + yz f_2'}$$

解法2. **利用全微分形式不变性同时求出各偏导数.**

$$z = f(x+y+z,\ xyz)$$

$$dz = f_1' \cdot (dx + dy + dz) + f_2' (yz\,dx + xz\,dy + xy\,dz)$$

解出 dx :

$$dx = \frac{-(f_1' + xz f_2')dy + (1 - f_1' - xy f_2')dz}{f_1' + yz f_2'}$$

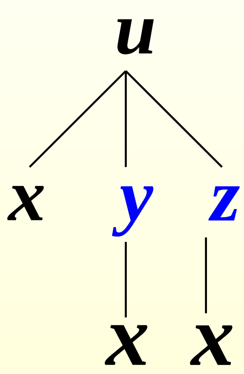
由 $d\,y$, $d\,z$ **的系数即可得** $\dfrac{\partial x}{\partial y}, \dfrac{\partial x}{\partial z}$.

结束 上页 下页

备用题 1. 设 $u=f(x,y,z)$ 有连续的一阶偏导数，又函数 $y=y(x)$ 及 $z=z(x)$ 分别由下列两式确定：

$$e^{xy}-xy=2,\quad e^{x}=\int_0^{x-z}\frac{\sin t}{t}\,\mathrm{d}t,$$ 求 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$. (2001考研)

解 两个隐函数方程两边对 x 求导，得



$$\begin{cases} e^{xy}(y+xy')-(y+xy')=0 \\ e^{x}=\dfrac{\sin(x-z)}{x-z}(1-z') \end{cases}$$

解得 $$y'=-\frac{y}{x},\quad z'=1-\frac{e^{x}(x-z)}{\sin(x-z)}$$

因此 $$\frac{\mathrm{d}u}{\mathrm{d}x}=f_1'-\frac{y}{x}f_2'+\left[1-\frac{e^{x}(x-z)}{\sin(x-z)}\right]f_3'$$

2. 设 $y=y(x)$, $z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. (1999考研)

解法1 分别在各方程两端对 x 求导, 得

$$\begin{cases} z' = f + x\cdot f'\cdot(1+y') \\ F_x + F_y\cdot y' + F_z\cdot z' = 0 \end{cases} \Longrightarrow \begin{cases} -xf'\cdot y' + \underline{z'} = f + xf' \\ F_y\cdot y' + F_z\cdot \underline{z'} = -F_x \end{cases}$$

$$\therefore\ \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\begin{vmatrix} -xf' & f+xf' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -xf' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{(f+xf')F_y - xf'\cdot F_x}{F_y + xf'\cdot F_z}$$

$$(F_y + xf'\cdot F_z \neq 0)$$

上页 下页

解法2 微分法.

$$z = x f(x+y),\ F(x,y,z)=0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + x f' \cdot (\mathrm{d}x + \mathrm{d}y) \\ F_1'\mathrm{d}x + F_2'\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

化简得

$$\begin{cases} (f + x f')\,\mathrm{d}x + x f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1'\,\mathrm{d}x + F_2'\,\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

消去$\mathrm{d}y$可得$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.