

## 习题 2.4

1. 用函数极限的  $\varepsilon-\delta$  定义验证下列极限:

$$\begin{aligned} (1) \lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5} = 4; & \quad (2) \lim_{x \rightarrow 2} x^2 = 4; & \quad (3) \lim_{x \rightarrow 0} \cos x = 1; \\ (4) \lim_{x \rightarrow 3} \frac{x-3}{x} = 0; & \quad (5) \lim_{x \rightarrow \frac{\pi}{4}} \sin x = \frac{\sqrt{2}}{2}; & \quad (6) \lim_{x \rightarrow 0} \frac{1}{2^x} = 1. \end{aligned}$$

2. 用函数极限的  $\varepsilon-X$  定义验证下列极限:

$$(1) \lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}; \quad (2) \lim_{x \rightarrow \infty} \frac{3x^2 - 1}{x^2 + 3} = 3; \quad (3) \lim_{x \rightarrow -\infty} a^x = 0 \quad (a > 1).$$

3. 证明  $\lim_{x \rightarrow a} f(x) = A$  的充分必要条件为  $\lim_{x \rightarrow a^+} f(x) = A$  且  $\lim_{x \rightarrow a^-} f(x) = A$ .

4. 设  $\lim_{x \rightarrow \infty} f(x) = A$  ( $A > 0$ ). 试用定义证明  $\lim_{x \rightarrow \infty} \sqrt{f(x)} = \sqrt{A}$ .

5. 求下列函数在指定点处的左、右极限, 并判断函数在该点处是否存在极限:

$$\begin{aligned} (1) f(x) = \frac{\sqrt{(x-1)^2}}{x-1} \text{ 在 } x_0 = 1 \text{ 处;} \\ (2) f(x) = \begin{cases} x+2, & x \leq 2, \\ \frac{1}{x-2}, & x > 2 \end{cases} \text{ 在 } x_0 = 2 \text{ 处;} \\ (3) f(x) = \arctan \frac{1}{x} \text{ 在 } x_0 = 0 \text{ 处;} \\ (4) f(x) = \frac{2^{\frac{1}{x}} - 1}{2^{\frac{1}{x}} + 1} \text{ 在 } x_0 = 0 \text{ 处.} \end{aligned}$$

6. 根据函数极限的定义填写下表:

	$f(x) \rightarrow A$	$f(x) \rightarrow \infty$	$f(x) \rightarrow +\infty$	$f(x) \rightarrow -\infty$
$x \rightarrow x_0$				
$x \rightarrow x_0 + 0$				
$x \rightarrow x_0 - 0$				
$x \rightarrow \infty$				
$x \rightarrow +\infty$				
$x \rightarrow -\infty$				

7. 证明函数  $f(x) = \frac{1}{x} \cos \frac{1}{x}$  在点  $x=0$  的邻域内无界, 但当  $x \rightarrow 0$  时, 并非无穷大.

8. 计算下列极限:

$$\begin{aligned} (1) \lim_{x \rightarrow 4} \frac{x^2 - 6x + 8}{x^2 - 5x + 4}; & \quad (2) \lim_{x \rightarrow a} \frac{x^2 - (a+1)x + a}{x^3 - a^3} \quad (a \neq 0); \\ (3) \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}; & \quad (4) \lim_{x \rightarrow \infty} \frac{(2x-3)^{20} (3x+2)^{30}}{(5x+1)^{50}}; \\ (5) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}; & \quad (6) \lim_{x \rightarrow 1} \frac{\sqrt{3-x} - \sqrt{1+x}}{x^2 - 1}; \end{aligned}$$

$$\begin{aligned}
(7) \lim_{x \rightarrow 4} \frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}; & \quad (8) \lim_{x \rightarrow -8} \frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}; \\
(9) \lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{2}{x^2-1} \right); & \quad (10) \lim_{x \rightarrow \infty} \frac{(x+1)(x^2+1) \cdots (x^n+1)}{[(nx)^n+1]^{\frac{n+1}{2}}}; \\
(11) \lim_{x \rightarrow 1} \frac{x+x^2+\cdots+x^n-n}{x-1}; & \quad (12) \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x}} \sqrt{x^2-a^2} - \sqrt{x}.
\end{aligned}$$

9. 计算下列极限:

$$\begin{aligned}
(1) \lim_{x \rightarrow 0} \frac{\tan 3x}{x}; & \quad (2) \lim_{x \rightarrow 0} \frac{\arcsin x}{x}; \\
(3) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}; & \quad (4) \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}; \\
(5) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}; & \quad (6) \lim_{x \rightarrow 0} \frac{\cos x - \sqrt[3]{\cos x}}{\sin^2 x}; \\
(7) \lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x} - \cos x}{\sin^2 \frac{x}{2}}; & \quad (8) \lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin \left( x - \frac{\pi}{6} \right)}{\frac{\sqrt{3}}{2} - \cos x}; \\
(9) \lim_{x \rightarrow 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2}; & \quad (10) \lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2}; \\
(11) \lim_{n \rightarrow \infty} n \sqrt[n]{n} \left( \tan \frac{x}{\sqrt{n}} - \sin \frac{x}{\sqrt{n}} \right); & \quad (12) \lim_{n \rightarrow \infty} \left( \frac{x}{2} \cos \frac{x}{4} \cdot \cos \frac{x}{2^n} \right).
\end{aligned}$$

10. 计算下列极限:

$$\begin{aligned}
(1) \lim_{x \rightarrow \infty} \left( 1 + \frac{2}{x} \right)^{x+3}; & \quad (2) \lim_{x \rightarrow 0} \sqrt[3]{1-2x}; \\
(3) \lim_{x \rightarrow 0} (1+x^2)^{\frac{1}{1-\cos x}}; & \quad (4) \lim_{x \rightarrow \infty} \left( \frac{2}{x^2} + \cos \frac{1}{x} \right)^{x^2}; \\
(5) \lim_{x \rightarrow 0} (1+\tan x)^{\cot x}; & \quad (6) \lim_{x \rightarrow 0} (1+3 \tan^2 x)^{\cot^2 x}; \\
(7) \lim_{x \rightarrow \infty} \left( \frac{2x-1}{2x+1} \right)^x; & \quad (8) \lim_{x \rightarrow \infty} \left( \frac{x^2}{x^2-1} \right)^x; \\
(9) \lim_{x \rightarrow 0^+} \sqrt[3]{\cos \sqrt{x}}; & \quad (10) \lim_{x \rightarrow a} \left( \frac{\sin x}{\sin a} \right)^{\frac{1}{x-a}}.
\end{aligned}$$

11. 当  $x \rightarrow 0$  时, 试确定下列无穷小对于  $x$  的阶数:

$$\begin{aligned}
(1) x^3 + 1000x^2; & \quad (2) \sqrt[3]{x^2} - \sqrt{x}; \\
(3) \frac{x(x+1)}{1+\sqrt{x}}; & \quad (4) \sqrt{a+x^3} - \sqrt{a} \quad (a > 0); \\
(5) \sqrt{1+x^4} - \sqrt{1-x^4}; & \quad (6) \sqrt{x^2 + \sqrt[3]{x^4}}; \\
(7) \sqrt{1+\tan x} - \sqrt{1-\sin x}; & \quad (8) (\cos x)^x - 1.
\end{aligned}$$

12. 求下列各题中的常数  $a$  :

$$(1) \lim_{n \rightarrow \infty} \left( \frac{n+a}{n-a} \right)^n = \sqrt{e}; \quad (2) \lim_{x \rightarrow \infty} \left( \frac{x+2a}{x-a} \right)^x = 8;$$

$$(3) \lim_{x \rightarrow \infty} \left( 1 + \frac{a}{x} \right)^x = 4; \quad (4) \lim_{x \rightarrow 0} \left( 1 + \frac{x}{a} \right)^{\frac{1}{x}} = 3;$$

$$(5) \text{ 当 } x \rightarrow 0 \text{ 时, } \sqrt[4]{1+ax^2} - 1 \text{ 与 } \cos x - 1 \text{ 是等价无穷小};$$

$$(6) \text{ 当 } x \rightarrow 0^+ \text{ 时, } \sqrt{x+\sqrt{x+\sqrt{x}}} \text{ 与 } \sqrt[4]{x} \text{ 是等价无穷小};$$

$$(7) \text{ 当 } x \rightarrow 1 \text{ 时, } 1-x \text{ 与 } a(1-\sqrt[m]{x}) \text{ } (m \in \mathbb{N}_+) \text{ 是等价无穷小}.$$

13. 计算下列极限:

$$(1) \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{\sin 2x}; \quad (2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{(e^x - 1) \cdot \ln(1+x)};$$

$$(3) \lim_{x \rightarrow 0} \frac{x^2 \tan x}{\sqrt{1-x^2} - 1}; \quad (4) \lim_{x \rightarrow 0} \frac{\ln(\sin^2 x + e^x) - x}{\ln(e^{2x} - x^2) - 2x};$$

$$(5) \lim_{x \rightarrow 0} \frac{\ln \cos ax}{\ln \cos bx} \quad (a, b \neq 0); \quad (6) \lim_{x \rightarrow 0} \frac{\sqrt[n]{1+\alpha x} - \sqrt[m]{1+\beta x}}{x} \quad (m, n \in \mathbb{N}_+);$$

$$(7) \lim_{x \rightarrow 0} \frac{\ln(1+x) + \ln(1-x)}{1 - \cos x + \sin^2 x}; \quad (8) \lim_{x \rightarrow 0} (3e^{\frac{x}{x-1}} - 2)^{\frac{1}{x}};$$

$$(9) \lim_{x \rightarrow 0} \left[ \frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{2}{x}}} + \frac{\sin x}{|x|} \right]; \quad (10) \lim_{x \rightarrow 0} \left( \frac{2^x + 3^x}{2} \right)^{\frac{1}{x}}.$$

14. 求下列各题中的常数  $a, b$  :

$$(1) \lim_{x \rightarrow +\infty} (3x - \sqrt{ax^2 - bx + 1}) = 2;$$

$$(2) \lim_{x \rightarrow \infty} \left( \frac{x^2 + 1}{x + 1} - ax - b \right) = 0;$$

$$(3) \lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{bx^2 - 1}) = 1.$$

15. 已知  $x \rightarrow 0$  时,  $f(x)$  是比  $x$  高阶的无穷小, 且  $\lim_{x \rightarrow 0} \frac{\ln \left( 1 + \frac{f(x)}{\sin 2x} \right)}{3^x - 1} = 5$ , 求  $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$ .