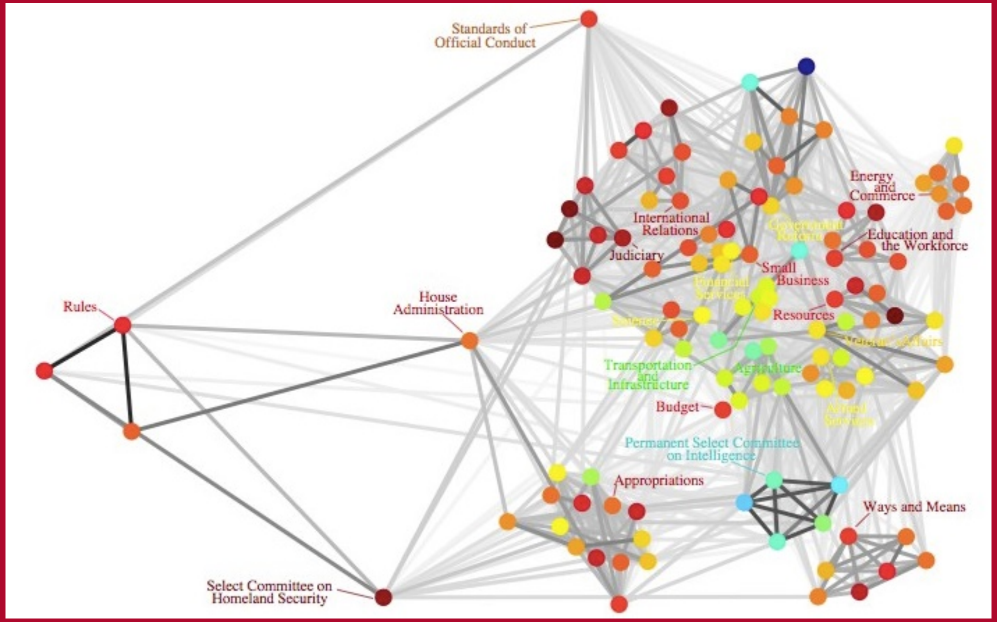
Automatic Control Theory

Chapter 4



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The root locus method

Main contents

1、The root locus concept and root locus equation

2、The root locus procedure

3、General root loci (Zero degree root loci)

The root locus procedure

Review

- **Definition of root locus**
- **Open loop transfer function**
- **Magnitude and Phase equation**

D(s)=1+G(s)H(s)=0

$$G(s)H(s)=\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=-1$$

what is next

The root locus procedure

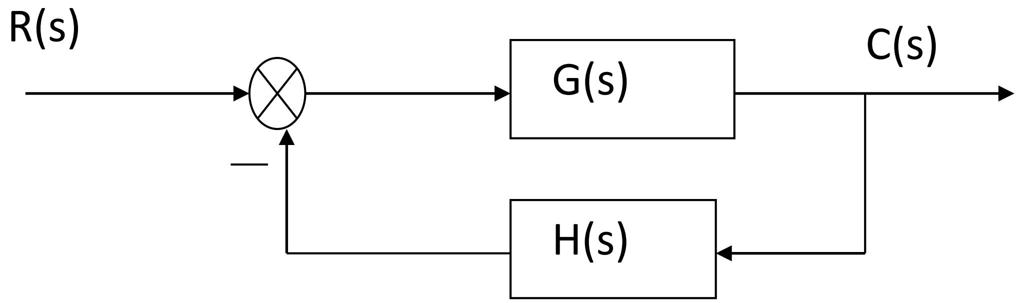
Step 1:

Write the characteristic equation as

$$1 + KP(s) = 0$$

Parameter that is changed

example



The root locus procedure

when $G(s)=\dfrac{K_1(s+5)}{s(s+1)}$ $H(s)=\dfrac{1}{s+4}$

Characteristic equation is

$$1+G(s)H(s)=1+\frac{K_1(s+5)}{s(s+1)(s+4)}=0 \qquad 1+KP(s)=0$$

So

$$K=K_1 \qquad P(s)=\frac{s+5}{s(s+1)(s+2)}$$

The root locus procedure

分子为：s+a

When $$G(s) = \frac{(s+\cancel{5})}{s(s+\cancel{a})} \qquad H(s) = \frac{1}{s+4}$$

分母为：s（s+5）

求标准形式1+KP（s）的characteristic equation

Characteristic equation is $$1+G(s)H(s) = 1+\frac{s+5}{s(s+a)(s+4)} = 0$$

$$s^2(s+4)+as(s+4)+s+5=0 \qquad 1+\frac{as(s+4)}{s^2(s+4)+s+5}=0$$

So $$K = a \qquad P(s) = \frac{s(s+4)}{s^2(s+4)+s+5}$$

Step 2:

Factor P(s), write the polynomial in the form of poles and zeros

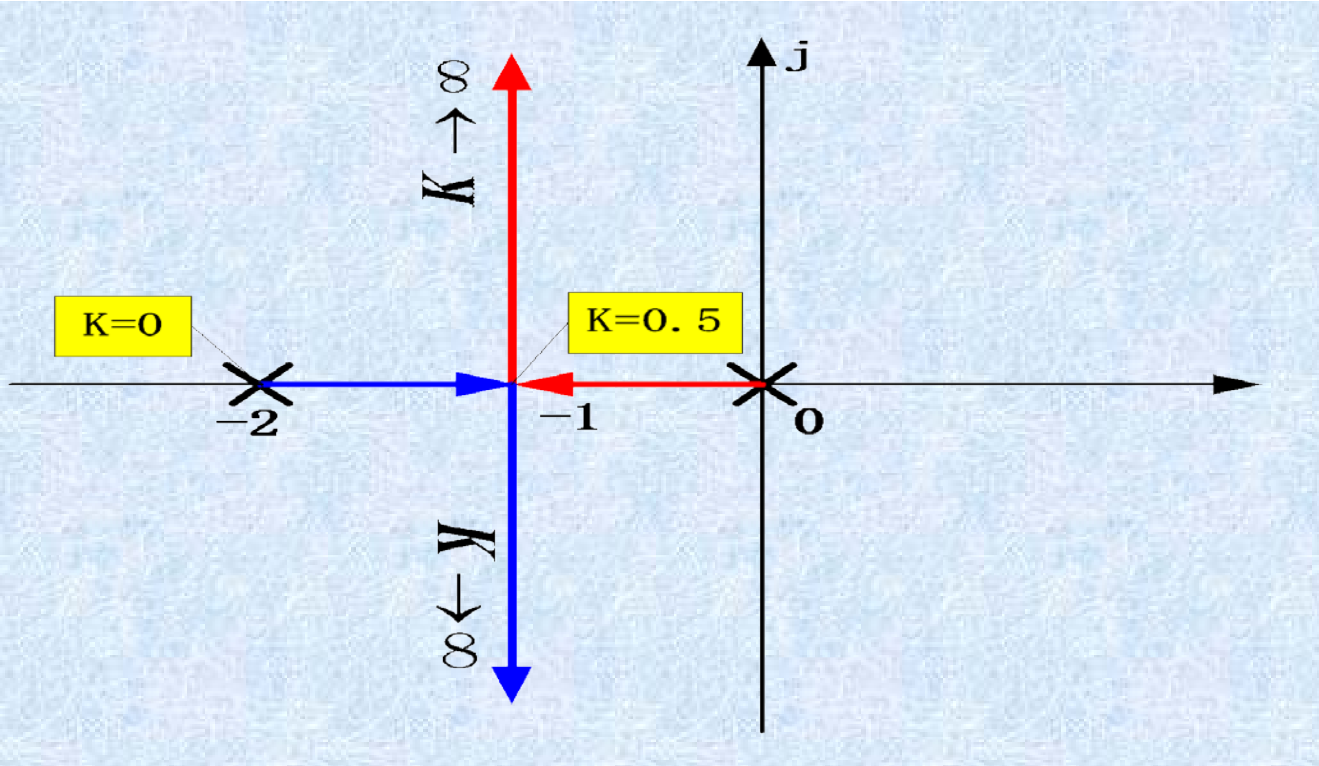
$$1+\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0$$

The number of separate loci is **equal to** the number of **poles**.

n阶系统的特征方程有n个特征根，当K（由0→∞）变动，则n个特征根跟随变化，在s平面上必然出现n条根轨迹。

The root locus procedure

The root loci is continuous, and must be **symmetrical** with respect to the horizontal real axis.



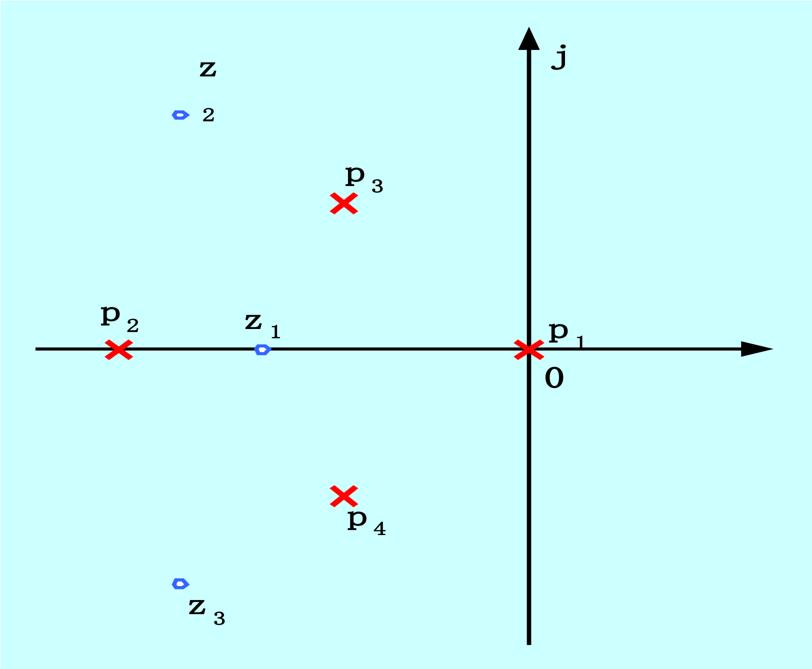
$$1+\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=0$$

Step 3:

Locate the poles and zeros of P(s) on s-plane with selected symbols

$$\prod_{i=1}^{n}(s-p_i)+K\prod_{i=1}^{m}(s-z_i)=0 \qquad 0\le K\le\infty$$

The locus of the roots of the characteristic equation 1+KP(s)=0 begins at the poles of P(s) and end at the zeros of P(s) as **K** increases form 0 to infinity.



The root locus procedure

$$\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -\frac{1}{K^*}$$

$s=p_i$ when $K=0,\ K^*=0$

$s=z_i$ when $K=\infty$

$n>m$

$$\lim_{s\to\infty}\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = \lim_{s\to\infty}\frac{1}{s^{n-m}} = 0$$

The root locus procedure

Step 4: Locate the segments of the **real** axis that are root loci.

The root locus on the **real** axis always lies in a section of the real axis to the left of an **odd** number of poles and zeros.

$$\angle(s_1 - p_2) + \angle(s_1 - p_3) = 0$$

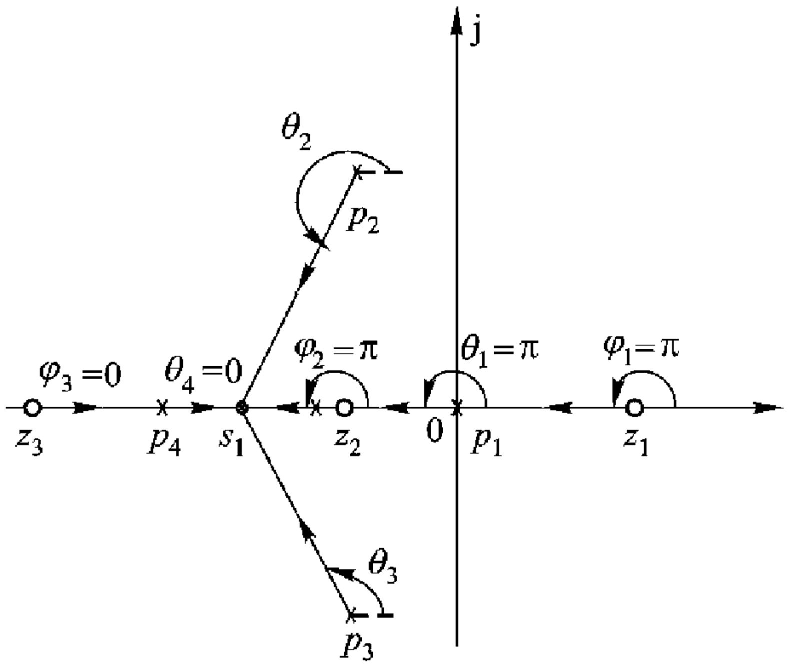
$$\sum_{i=1}^{3} \angle(s_1 - z_i) - \sum_{i=1}^{4} \angle(s_1 - p_i)$$

Phase equation

$$= \angle(s_1 - z_1) + \angle(s_1 - z_2) - \angle(s_1 - p_1)$$

$$= \pi + \pi - \pi \quad = \pi \quad = (2k+1)\pi$$

$$\sum_{i=1}^{l} \angle(s_1 - z_i) - \sum_{i=1}^{h} \angle(s_1 - p_i) = (l-h)\pi \quad = (2k+1)\pi$$



The root locus procedure

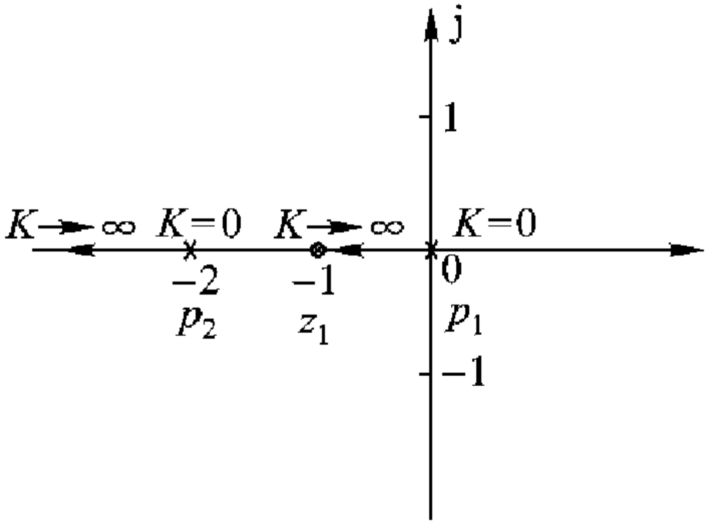
Example 1

Open loop transfer function

$K(s+1)/s(0.5s+1)$ Get the root locus $K=0\to\infty$

Solution

$$G(s)=\frac{2K(s+1)}{s(s+2)}$$



Step 5 : Asymptotes of the root loci

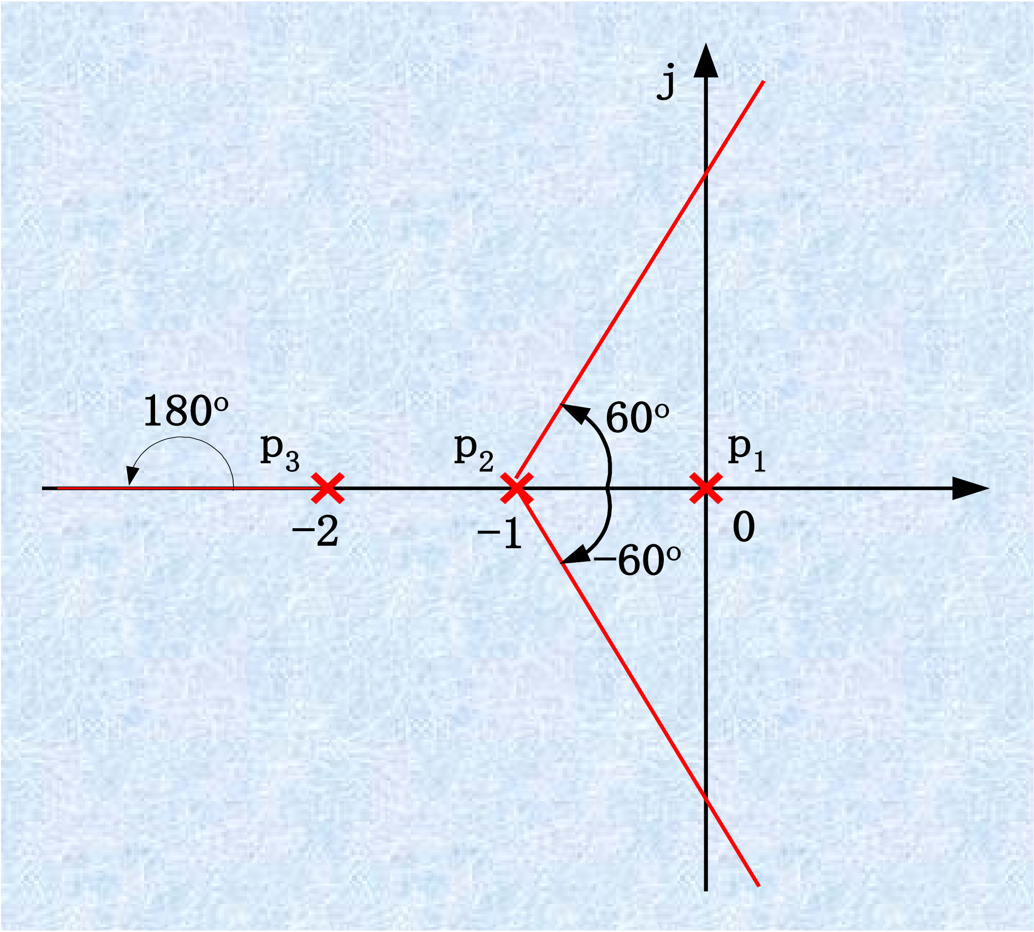
渐近线与实轴相交点的坐标为：

$$\sigma_A = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} \longrightarrow \text{Asymptote centroid}$$

渐近线与实轴正方向的夹角为：

$$\varphi_A = \frac{(2k+1)\pi}{n-m} \longrightarrow \text{Angle of the asymptotes}$$

$$k = 0,1,2,\cdots n-m-1$$



The root locus procedure

$$G(s)H(s)=\frac{K^*\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=K^*\frac{s^m+b_1s^{m-1}+\cdots b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

$$=K^*\frac{s^m}{s^n}\bullet\frac{1+b_1s^{-1}+\cdots+b_{m-1}s^{-m+1}+b_ms^{-m}}{1+a_1s^{-1}+\cdots+a_{n-1}s^{-n+1}+a_ns^{-n}}$$

$$s^{n-m}\frac{1+a_1s^{-1}+\cdots+a_{n-1}s^{-n+1}+a_ns^{-n}}{1+b_1s^{-1}+\cdots+b_{m-1}s^{-m+1}+b_ms^{-m}}=-K^*$$

Root locus equation

$$\frac{1}{x}\left(\frac{1+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n}{1+b_1x+\cdots+b_{m-1}x^{m-1}+b_mx^m}\right)^{\frac{1}{n-m}}=(-K^*)^{\frac{1}{n-m}}\qquad x=\frac{1}{s}$$

$$\left[\frac{1+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n}{1+b_1x+\cdots+b_{m-1}x^{m-1}+b_mx^m}\right]^{\frac{1}{n-m}}=1+\frac{a_1-b_1}{n-m}x$$

Taylor series at $x=0$

The root locus procedure

$$\frac{1}{x}\left(1+\frac{a_1-b_1}{n-m}x\right)=(-K^*)^{\frac{1}{n-m}}=(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi} \qquad (k=0,1,2,\cdots)$$

$$\frac{1}{x}=-\frac{a_1-b_1}{n-m}+(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi} \qquad \frac{1}{x}=s,\ a_1=-\sum_{i=1}^{n}p_i,\ b_1=-\sum_{i=1}^{m}z_i$$

$$s=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}+(K^*)^{\frac{1}{n-m}}e^{j\frac{2k+1}{n-m}\pi}$$

set $\varphi_a=\dfrac{(2k+1)\pi}{n-m}$ $\quad \sigma_a=\dfrac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}$

$$s=\sigma_a+(K^*)^{\frac{1}{n-m}}\cdot e^{j\varphi_a}$$

Asymptotes equation

The root locus procedure

Example 2

Open loop transfer function

$$G(s)H(s) = K^*(s+1)/s(s+4)(s^2+2s+2)$$

Get the root locus $K = 0 \to \infty$

Solution

$$G(s)H(s) = \frac{K^*(s+1)}{s(s+4)(s^2+2s+2)}$$

$p_1 = 0$, $p_a = -4$, $p_3 = -1 + j1$, $p_4 = -1 - j1$, $n = 4$

$z_1 = -1$, $m = 1$

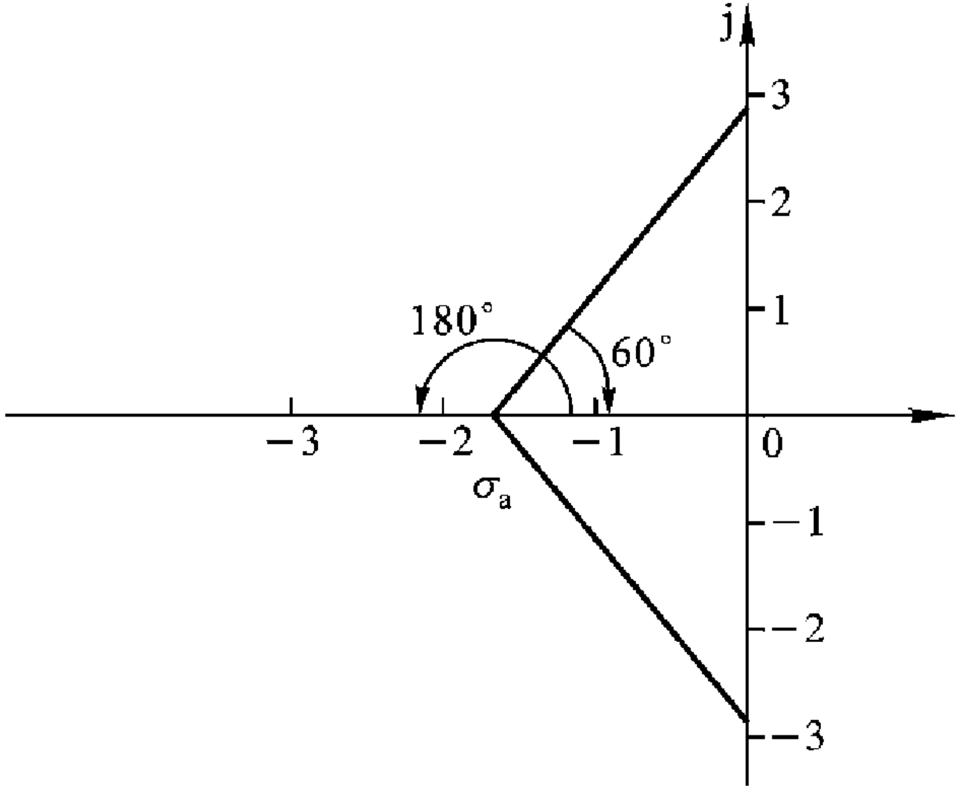
The root locus procedure

$$\varphi_a = \frac{(2k+1)\pi}{n-m} = \frac{(2k+1)\pi}{4-1} = \frac{(2k+1)\pi}{3}$$

$$\varphi_{a_1} = 60^\circ \qquad (k=0)$$

$$\varphi_{a_2} = 180^\circ \qquad (k=1)$$

$$\varphi_{a_3} = 300^\circ \qquad (k=2)$$



$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{0-4-1-j1-1+j1+1}{4-1} = -\frac{5}{3}$$

The root locus procedure

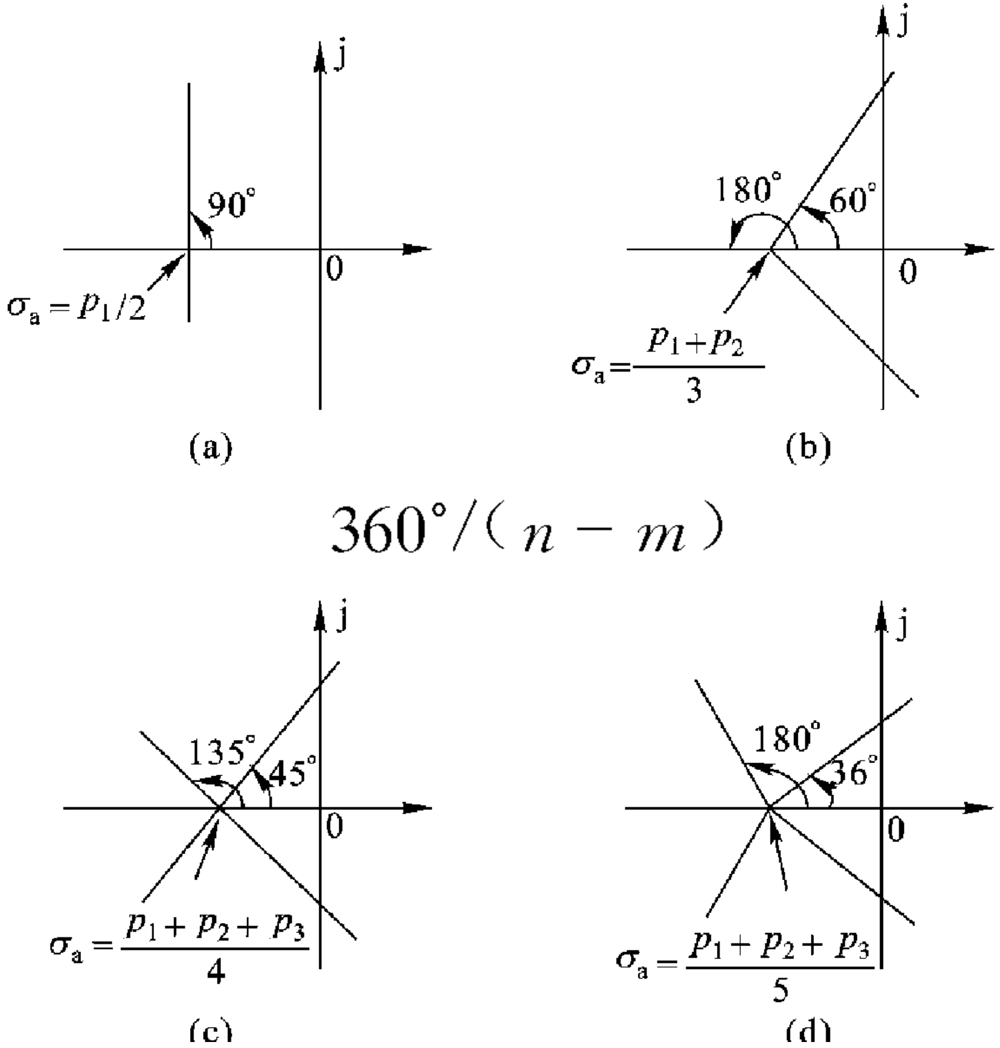
Asymptotes for typical open loop transfer functions

① $G(s)H(s)=\dfrac{K^*}{s(s-p_1)}$。

② $G(s)H(s)=\dfrac{K^*}{s(s-p_1)(s-p_2)}$。

③ $G(s)H(s)=\dfrac{K^*}{s(s-p_1)(s-p_2)(s-p_3)}$。

④ $G(s)H(s)=\dfrac{K^*}{s^2(s-p_1)(s-p_2)(s-p_3)}$。



The root locus procedure

核心

- **Definition of root locus**
- **Open loop transfer function**
- **Phase equation**

$$G(s)H(s)=\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=-1$$

Continuity　**Symmetry**　**Start and end points**　**Asymptotes**

续

- **The root locus procedure（Details）**