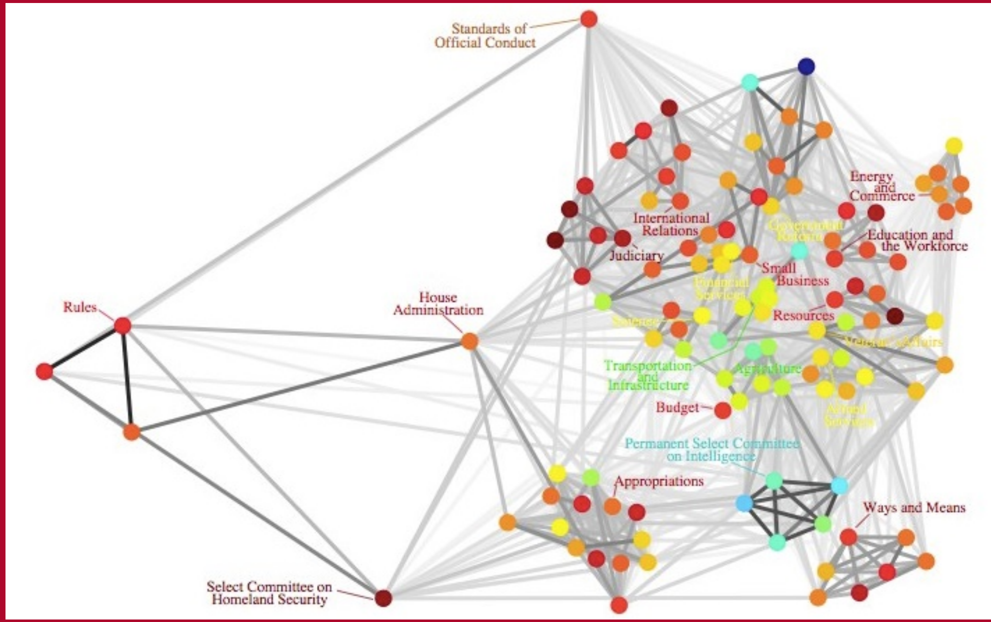
# Automatic Control Theory

## Chapter 2

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# CH2: Mathematical Models of Systems

## Main contents

- Differential Equations of Physical Systems.
- The Transfer function of Linear Systems.
  (The Laplace Transform and Inverse Transform)
- Block Diagram.
- Block Diagram Reduction (Mason's gain formula)

## Review

- Differential Equations of Physical Systems
- Physical laws of the process
- Linear system

## what is next

- The Laplace Transform and Inverse Transform
- The Transfer function of Linear Systems

# The Laplace Transform

## Definition

If a function of time, $f$(t),satisfies

$$\int_0^{\infty} |f(t)| e^{-\sigma_1 t} dt < \infty$$

We have the Laplace transformation for function $f$(t),is

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = L\{f(t)\}$$

# The Laplace Transform

**The Laplace variable *s* can be considered to be differential operator**

so that, we have

复域的乘除法易操作
最好用！
实用的 不是严谨的
⚠️

$$s = \frac{d}{dt}$$ 微分越多

**Useful !**

$$\frac{1}{s} = \int_0^t dt$$

# Important Laplace Transform Pairs 考试的时候不会给

！ 50*1的拉普拉斯变换是50/s

！ 1的拉普拉斯变换是1/s

脉冲函数的拉普拉斯变化为1
单位脉冲函数

$$L[1(t)] = \frac{1}{s}$$

$$L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

$$L[t^n] = \frac{n!}{s^{n+1}}$$

$$L[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

$$L[e^{-at}] = \frac{1}{s+a}$$

# Inverse Laplace Transformation

Inverse Laplace transformation can be denoted

注意表达式写法

$$f(t) = inverse\ Laplace\ transformation\ of\ F(s) = L^{-1}[F(s)]$$

or

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

# Important Theorems of Laplace Transform

(1) linearity

$$L\left[k_1 f_1(t) \pm k_2 f_2(t)\right] = k_1 F_1(s) \pm k_2 F_2(s)$$

(2) differentiation

零初始条件下 useful 公式有用

$$L[\frac{d^k f(t)}{dt^k}] = s^k F(s) - s^{k-1} f(0) - \cdots\cdots f^{(k-1)}(0)$$

# Important Theorems of Laplace Transform

(3) Shife in Time

$$L\left[f(t-T)1(t-T)\right]=e^{-Ts}F(s)$$

(4) Complex Shifting

$$L\left[e^{\mp at}f(t)\right]=F(s\pm a)$$

(5)Initial-Value Theorem

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$$

(6) Final-Value Theorem

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

IF $sF(s)$ does not have poles on or to the right of the imaginary axis in the $s$-plane.
*(Coming soon in Chapter 3 !)*

## Example 1

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy}{dt}+ky(t)=r(t)$$

The Laplace transform of the equation is

$$M[s^2Y(s)-sy(0)-\dot{y}(0)]+b[sY(s)-y(0)]+kY(s)=R(s)$$

when

$$r(t)=0, \quad y(0)=y_0, \quad \dot{y}(0)=0$$

# Solve the differential equations using the Laplace transform

We can get $$Y(s)=\frac{(Ms+b)y_0}{Ms^2+bs+k}=\frac{p(s)}{q(s)}$$ 阶数由分母决定
阶数传递性

when

$$k/M=2 \qquad b/M=3 \qquad y_0=1$$

Then $Y(s)$ becomes

$$Y(s)=\frac{s+3}{(s+1)(s+2)}=\frac{2}{s+1}-\frac{1}{s+2} \tag{1}$$

The inverse Laplace transform of Eq.(1)is

$$y(t)=2e^{-t}-e^{-2t}$$

# Solve the differential equations using the Laplace transform

**Example 2**

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2r(t)$$

When the initial conditions are

$$y(0)=1,\quad \dot{y}(0)=0,\quad r(t)=1$$

The Laplace transform, we obtain

$$Y(s)=\frac{s^2+4s+2}{s(s^2+4s+3)}=\frac{1/2}{s+1}+\frac{-1/6}{s+3}+\frac{2/3}{s}$$

so

$$y(t)=\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}+\frac{2}{3}$$

**Example 3**: Consider the function

$$G(s)=\frac{1}{s(s+1)^3(s+2)}$$

Calculate g(t).

$$G(s)=\frac{k_1}{s}+\frac{k_2}{s+2}+\frac{a_1}{s+1}+\frac{a_2}{(s+1)^2}+\frac{a_3}{(s+1)^3}$$

answer

$$k_1=k_2=\frac{1}{2} \qquad a_1=-1, \quad a_2=0, \quad a_3=-1$$

# The transfer function of linear systems

## Definition

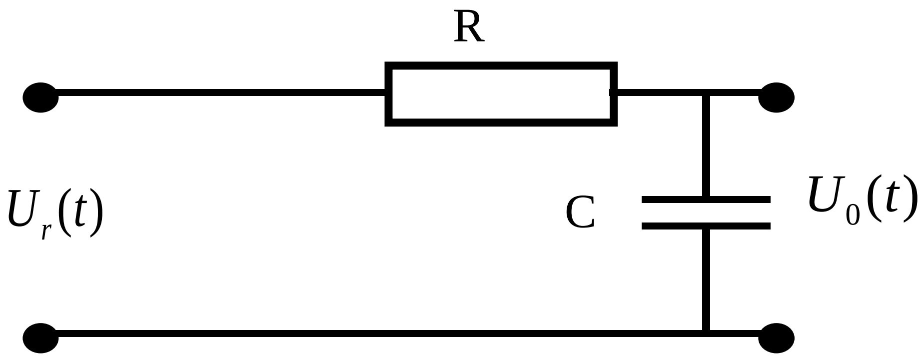
**Transfer function:**

The **ratio** of the Laplace transform of the output variable to the Laplace transform of the input variable, <u>with all initial conditions assumed to be zero</u>.

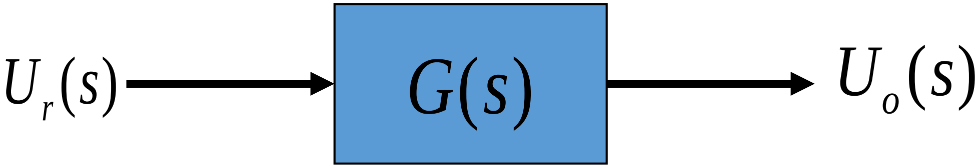
零初始条件下，输出变量的拉氏变换与输入变量的拉氏变换之比。

# The transfer function of linear systems

Example: RC electrical network




$$\frac{U_o(s)}{U_r(s)} = \frac{1/Cs}{R+1/Cs} = \frac{1}{1+RCs}$$



# The transfer function of linear systems

Consider the dynamic system represented by the differential equation

$$\frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_1\frac{dc(t)}{dt} + a_0 c(t)$$

$$= \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} \cdots + b_0 r(t)$$

If the initial conditions are all zero, then the Laplace transform is yielded

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)C(s) \quad = (s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)R(s)$$

The transfer function is

$$\frac{C(s)}{R(s)} = G(s) = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$$

# Some Comments about the Transfer Function

1. The concept of transfer function only applies to  the LTI system.

2. Transfer function is only determined by the structure and parameter of system.

3. Transfer function is a rational proper fraction, and the relationship of the orders of the numerator and denominator is

$$n \geq m$$

n—the order of the denominator
m—the order of the numerator

4. The inverse Laplace transform of transfer function is the impulse response function of the system.

5. The method of the transfer function has some limitation.

(1). It only applies to the SISO system.

(2). It only can reflect the relationship of input and output.

(3). It only can analyze the motion characteristic of zero initial conditions.

# CH2: Mathematical Models of Systems

**核心**

The Laplace Transform and Inverse Transform

The Transfer function of Linear Systems

**续**

**Block Diagram**