高等数学

积 分 表

公 式 推 导

目　　录

（一）含有 $ax+b$ 的积分（1~9）

1. $\int \frac{dx}{ax+b} = \frac{1}{a} \cdot ln\left| ax+b \right| + C$

证明：被积函数 $f(x) = \frac{1}{ax+b}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

令 $ax+b=t \ (t \neq 0)$，则 $dt = adx$，$\therefore\ dx = \frac{1}{a}dt$

$$\therefore \int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{1}{t}dt$$

$$= \frac{1}{a} \cdot ln \left| t \right| + C$$

将 $t = ax+b$ 代入上式得：$\int \frac{dx}{ax+b} = \frac{1}{a} \cdot ln\left| ax+b \right| + C$

2. $\int (ax+b)^{\mu} dx = \frac{1}{a(\mu+1)} \cdot (ax+b)^{\mu+1} + C \qquad (\mu \neq -1)$

证明：令 $ax+b=t$，则 $dt = adx$，$\therefore\ dx = \frac{1}{a}dt$

$$\therefore \int (ax+b)^{\mu} dx = \frac{1}{a}\int t^{\mu} dt$$

$$= \frac{1}{a(\mu+1)} \cdot t^{\mu+1} + C$$

将 $t = ax+b$ 代入上式得：$\int (ax+b)^{\mu} dx = \frac{1}{a(\mu+1)} \cdot (ax+b)^{\mu+1} + C$

3. $\int \frac{x}{ax+b} dx = \frac{1}{a^2}\left(ax+b - b \cdot ln\left| ax+b \right|\right) + C$

证明：被积函数 $f(x) = \frac{x}{ax+b}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

令 $ax+b=t \ (t \neq 0)$，则 $x = \frac{1}{a}(t-b)$，$dx = \frac{1}{a}dt$

$$\therefore \int \frac{x}{ax+b} dx = \int \frac{\frac{1}{a}(t-b)}{t} \cdot \frac{1}{a} dt = \frac{1}{a^2}\int \left(1 - \frac{b}{t}\right) dt$$

$$= \frac{1}{a^2}\int dt - \frac{1}{a^2}\int \frac{b}{t} dt$$

$$= \frac{t}{a^2} - \frac{b}{a^2} \cdot ln\left| t \right| + C$$

$$= \frac{1}{a^2}\left(t - b \cdot ln\left| t \right|\right) + C$$

将 $t = ax+b$ 代入上式得：$\int \frac{x}{ax+b} dx = \frac{1}{a^2}\left(ax+b - b \cdot ln\left| ax+b \right|\right) + C$

4. $\int \frac{x^2}{ax+b}dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b\,(ax+b) + b^2 \cdot ln\,|\,ax+b\,|\right] + C$

证明:$\int \frac{x^2}{ax+b}dx = \frac{1}{a^2}\int \frac{(ax+b)^2 - 2abx - b^2)}{ax+b}dx$

$$= \frac{1}{a^2}\int (ax+b)dx - \frac{1}{a^2}\int \frac{2abx}{ax+b}dx - \frac{1}{a^2}\int \frac{b^2}{ax+b}dx$$

$\because\ \frac{1}{a^2}\int (ax+b)dx = \frac{1}{2a^3}(ax+b)^2 + C_1$

$$\frac{1}{a^2}\int \frac{2abx}{ax+b}dx = \frac{2b}{a^3}\int \frac{ax+b-b}{ax+b}d(ax)$$

$$= \frac{2b}{a^3}\int dx - \frac{2b^2}{a^3}\int \frac{1}{ax+b}d(ax+b)$$

$$= \frac{2b}{a^3}x - \frac{2b^2}{a^3}ln\,|\,ax+b\,| + C_2$$

$$\frac{1}{a^2}\int \frac{b^2}{ax+b}dx = \frac{b^2}{a^3}\int \frac{1}{ax+b}d(ax+b) = \frac{b^2}{a^3}ln\,|\,ax+b\,| + C_3$$

由以上各式整理得:$\int \frac{x^2}{ax+b}dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b\,(ax+b) + b^2 \cdot ln\,|\,ax+b\,|\right] + C$

5. $\int \frac{dx}{x\,(ax+b)} = -\frac{1}{b}\cdot ln\left|\frac{ax+b}{x}\right| + C$

证明:被积函数 $f(x) = \frac{1}{x\cdot(ax+b)}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{1}{x\cdot(ax+b)} = \frac{A}{x} + \frac{B}{ax+b}$,则 $1 = A(ax+b) + Bx = (Aa+B)x + Ab$

$\therefore$ 有 $\begin{cases} Aa + B = 0 \\ Ab = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b} \\ B = -\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x\,(ax+b)} = \int [\frac{1}{bx} - \frac{a}{b\cdot(ax+b)}]dx = \frac{1}{b}\int \frac{1}{x}dx - \frac{a}{b}\int \frac{1}{ax+b}dx$

$$= \frac{1}{b}\int \frac{1}{x}dx - \frac{1}{b}\int \frac{1}{ax+b}d(ax+b)$$

$$= \frac{1}{b}\cdot ln\,|\,x\,| - \frac{1}{b}\cdot ln\,|\,ax+b\,| + C$$

$$= \frac{1}{b}\cdot ln\left|\frac{x}{ax+b}\right| + C$$

$\boxed{\text{提示:} log_a b^{-1} = -log_a b}$

$$= -\frac{1}{b}\cdot ln\left|\frac{ax+b}{x}\right| + C$$

6. $\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \cdot ln\left|\frac{ax+b}{x}\right| + C$

证明：被积函数 $f(x) = \frac{1}{x^2 \cdot (ax+b)}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{1}{x^2 \cdot (ax+b)} = \frac{\mathrm{A}}{x} + \frac{B}{x^2} + \frac{C}{ax+b}$，则 $1 = \mathrm{A}x(ax+b) + \mathrm{B}(ax+b) + Cx^2$

即 $x^2(\mathrm{A}a + \mathrm{C}) + x(Ab + aB) + \mathrm{B}b = 1$

$$\therefore \text{ 有 } \begin{cases} Aa + C = 0 \\ Ab + aB = 0 \\ Bb = 1 \end{cases} \Rightarrow \begin{cases} \mathrm{A} = -\frac{a}{b^2} \\ \mathrm{B} = \frac{1}{b} \\ C = \frac{a^2}{b^2} \end{cases}$$

$$\text{于是 } \int \frac{dx}{x^2(ax+b)} = -\frac{a}{b^2}\int \frac{1}{x}dx + \frac{1}{b}\int \frac{1}{x^2}dx + \frac{a^2}{b^2}\int \frac{1}{ax+b}dx$$

$$= -\frac{a}{b^2}\int \frac{1}{x}dx + \frac{1}{b}\int \frac{1}{x^2}dx + \frac{a}{b^2}\int \frac{1}{ax+b}d(ax+b)$$

$$= -\frac{a}{b^2} \cdot ln|x| - \frac{1}{bx} + \frac{a}{b^2} \cdot ln|ax+b| + C$$

$$= -\frac{1}{bx} + \frac{a}{b^2} \cdot ln\left|\frac{ax+b}{x}\right| + C$$

7. $\int \frac{x}{(ax+b)^2}dx = \frac{1}{a^2}\left(ln|ax+b| + \frac{b}{ax+b}\right) + C$

证明：被积函数 $f(x) = \frac{x}{(ax+b)^2}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

设 $\frac{x}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$，则 $x = \mathrm{A}(ax+b) + \mathrm{B}$

即 $x \cdot \mathrm{A}a + (Ab + B) = x$

$$\therefore \text{ 有 } \begin{cases} Aa = 1 \\ Ab + B = 0 \end{cases} \Rightarrow \begin{cases} \mathrm{A} = \frac{1}{a} \\ B = -\frac{b}{a} \end{cases}$$

$$\text{于是 } \int \frac{x}{(ax+b)^2}dx = \frac{1}{a}\int \frac{1}{ax+b}dx - \frac{b}{a}\int \frac{1}{(ax+b)^2}dx$$

$$= \frac{1}{a^2}\int \frac{1}{ax+b}d(ax+b) - \frac{b}{a^2}\int \frac{1}{(ax+b)^2}d(ax+b)$$

$$= \frac{1}{a^2} \cdot ln|ax+b| + \frac{b}{a^2(ax+b)} + C$$

$$= \frac{1}{a^2}\left(ln|ax+b| + \frac{b}{ax+b}\right) + C$$

8. $\int \frac{x^2}{(ax+b)^2}dx = \frac{1}{a^3}\left(ax+b-2b\cdot ln\,|\,ax+b\,|-\frac{b^2}{ax+b}\right)+C$

证明：被积函数 $f(x)=\frac{x^2}{(ax+b)^2}$ 的定义域为$\{x\,|\,x\neq -\frac{b}{a}\}$

令 $ax+b=t\quad (t\neq 0)$, 则 $x=\frac{1}{a}(t-b)$, $dx=\frac{1}{a}dt$

$$\therefore\quad \frac{x^2}{(ax+b)^2}=\frac{(b-t)^2}{a^2t^2}=\frac{b^2+t^2-2bt}{a^2t^2}$$

$$\therefore\quad \int \frac{x^2}{(ax+b)^2}dx=\int \frac{b^2+t^2-2bt}{a^3t^2}dt=\frac{b^2}{a^3}\int \frac{1}{t^2}dt+\frac{1}{a^3}\int dt-\frac{2b}{a^3}\int \frac{1}{t}dt$$

$$=-\frac{b^2}{a^3t}+\frac{1}{a^3}\cdot t-\frac{2b}{a^3}\cdot ln\,|\,t\,|+C$$

$$=\frac{1}{a^3}(t-2b\cdot ln\,|\,t\,|-\frac{b^2}{t})+C$$

将$t=ax+b$代入上式得：$\int \frac{x^2}{(ax+b)^2}dx=\frac{1}{a^3}\left(ax+b-2b\cdot ln\,|\,ax+b\,|-\frac{b^2}{ax+b}\right)+C$

9. $\int \frac{dx}{x(ax+b)^2}=\frac{1}{b(ax+b)}-\frac{1}{b^2}\cdot ln\,|\,\frac{ax+b}{x}\,|+C$

证明：被积函数 $f(x)=\frac{1}{x(ax+b)^2}$ 的定义域为 $\{x\,|\,x\neq -\frac{b}{a}\}$

设：$\frac{1}{x(ax+b)^2}=\frac{A}{x}+\frac{B}{ax+b}+\frac{D}{(ax+b)^2}$

则 $1=A(ax+b)^2+Bx(ax+b)+Dx$

$=Aa^2x^2+Ab^2+2Aabx+Bax^2+Bbx+Dx$

$=x^2(Aa^2+Ba)+x(2Aab+Bb+D)+Ab^2$

$$\therefore\ 有\ \begin{cases} Aa^2+Ba=0 \\ 2Aab+Bb+D=0 \\ Ab^2=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b^2} \\ B=-\frac{a}{b^2} \\ D=-\frac{a}{b} \end{cases}$$

于是 $\int \frac{dx}{x(ax+b)}=\frac{1}{b^2}\int \frac{1}{x}dx-\frac{a}{b^2}\int \frac{1}{ax+b}dx-\frac{a}{b}\int \frac{1}{(ax+b)^2}dx$

$$=\frac{1}{b^2}\cdot ln|x|-\frac{1}{b^2}\cdot ln|ax+b|+\frac{1}{b}\cdot\frac{1}{ax+b}+C$$

$$=\frac{1}{b(ax+b)}-\frac{1}{b^2}\cdot ln|\frac{ax+b}{x}|+C$$

（二）含有 $\sqrt{ax+b}$ 的积分（10~18）

10. $\int\sqrt{ax+b}\,dx=\frac{2}{3a}\cdot\sqrt{(ax+b)^3}+C$

证明：$\int\sqrt{ax+b}\,dx=\frac{1}{a}\int(ax+b)^{\frac{1}{2}}d(ax+b)=\frac{1}{a}\cdot\frac{1}{1+\frac{1}{2}}\cdot(ax+b)^{\frac{1}{2}+1}+C$

$$=\frac{2}{3a}\cdot\sqrt{(ax+b)^3}+C$$

11. $\int x\sqrt{ax+b}\,dx=\frac{2}{15a^2}\cdot(3ax-2b)\cdot\sqrt{(ax+b)^3}+C$

证明：令$\sqrt{ax+b}=t\quad(t\geq 0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，$x\sqrt{ax+b}=\frac{t^2-b}{a}\cdot t$

$$\therefore\ \int x\sqrt{ax+b}\,dx=\int\frac{t^2-b}{a}\cdot t\cdot\frac{2t}{a}dt=\frac{2}{a^2}\int(t^4-bt^2)dt$$

$$=\frac{2}{5a^2}\int dt^5-\frac{2b}{3a^2}\int dt^3=\frac{2}{5a^2}\cdot t^5-\frac{2b}{3a^2}\cdot t^3+C$$

$$=\frac{2t^3}{15a^2}(3t^2-5b)+C$$

将$t=\sqrt{ax+b}$代入上式得：$\int x\sqrt{ax+b}\,dx=\frac{2}{15a^2}[3(ax+b)-5b]\cdot\sqrt{(ax+b)^3}+C$

$$=\frac{2}{15a^2}\cdot(3ax-2b)\cdot\sqrt{(ax+b)^3}+C$$

12. $\int x^2\sqrt{ax+b}\,dx=\frac{2}{105a^3}\cdot(15a^2x^2-12abx+8b^2)\cdot\sqrt{(ax+b)^3}+C$

证明：令$\sqrt{ax+b}=t\quad(t\geq 0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$x^2\sqrt{ax+b}=\frac{(t^2-b)^2}{a^2}\cdot t=\frac{t^5+b^2t-2bt^3}{a^2}$$

$$\therefore\ \int x^2\sqrt{ax+b}\,dx=\frac{2}{a^3}\int t\cdot(t^5+b^2t-2bt^3)dt$$

$$=\frac{2}{a^3}\int t^6dt-\frac{2b^2}{a^3}\int t^2dt-\frac{4b}{a^3}\int t^4dt$$

$$=\frac{2}{a^3}\cdot\frac{1}{1+6}\cdot t^{6+1}+\frac{2b^2}{a^3}\cdot\frac{1}{1+2}\cdot t^{1+2}-\frac{4b}{a^3}\cdot\frac{1}{1+4}\cdot t^{4+1}+C$$

$$=\frac{2}{7a^3}\cdot t^7+\frac{2b^2}{3a^3}\cdot t^3-\frac{4b}{5a^3}\cdot t^5+C$$

$$=\frac{2t^3}{105a^3}\cdot(15t^4+35b^2-42bt^2)+C$$

将$t=\sqrt{ax+b}$代入上式得：

$$\int x^2\sqrt{ax+b}\,dx=\frac{2}{105a^3}\cdot\sqrt{(ax+b)^3}\left[15a^2x^2+15b^2+30abx+35b^2-42b\cdot(ax+b)\right]$$

$$=\frac{2}{105a^3}\cdot(15a^2x^2-12abx+8b^2)\cdot\sqrt{(ax+b)^3}+C$$

13. $\int \frac{x}{\sqrt{ax+b}}dx = \frac{2}{3a^2}\cdot(ax-2b)\cdot\sqrt{(ax+b)}+C$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{x}{\sqrt{ax+b}}dx = \int \frac{t^2-b}{at}\cdot\frac{2t}{a}dt$$

$$= \frac{2}{a^2}\int t^2dt - \frac{2}{a^2}\int bdt$$

$$= \frac{2}{a^2}\cdot\frac{1}{1+2}\cdot t^{2+1} - \frac{2b}{a^2}\cdot t + C$$

$$= \frac{2}{3a^2}\cdot t^3 - \frac{2b}{a^2}\cdot t + C$$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2}{3a^2}\cdot(ax+b)\cdot\sqrt{(ax+b)} - \frac{2b}{a^2}\cdot\sqrt{(ax+b)}+C$

$$= \frac{2}{3a^2}\cdot(ax-2b)\cdot\sqrt{(ax+b)}+C$$

14. $\int \frac{x^2}{\sqrt{ax+b}}dx = \frac{2}{15a^3}\cdot(3a^2x^2-4abx+8b^2)\cdot\sqrt{(ax+b)}+C$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{x^2}{\sqrt{ax+b}}dx = \int (\frac{t^2-b}{a})^2\cdot\frac{1}{t}\cdot\frac{2t}{a}dt$$

$$= \frac{2}{a^3}\int (t^4+b^2-2bt^2)dt$$

$$= \frac{2}{a^3}\int t^4dt + \frac{2}{a^3}\int b^2dt - \frac{4b}{a^3}\int t^2dt$$

$$= \frac{2}{a^3}(\frac{1}{5}t^5 + b^2t - \frac{2b}{3}t^3) + C$$

$$= \frac{2t}{15a^3}\cdot(3t^4+15b^2-10bt^2)+C$$

将$t=\sqrt{ax+b}$代入上式得：

$$\int \frac{x^2}{\sqrt{ax+b}}dx = \frac{2}{15a^3}\cdot\sqrt{(ax+b)}\cdot\left[3(a^2x^2+b^2+2abx)+15b^2-10b\cdot(ax+b)\right]\cdot\sqrt{(ax+b)}+C$$

$$= \frac{2}{15a^3}\cdot(3a^2x^2-4abx+8b^2)\cdot\sqrt{(ax+b)}+C$$

15. $$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & (b>0) \\ \frac{2}{\sqrt{-b}} \cdot arctan\sqrt{\frac{ax+b}{-b}} + C & (b<0) \end{cases}$$

证明：令$\sqrt{ax+b}=t \quad (t>0)$，则$x=\frac{t^2-b}{a}$ ，$dx=\frac{2t}{a}dt$ ，

$$\therefore \int \frac{dx}{x\sqrt{ax+b}} = \int \frac{1}{\frac{t^2-b}{a} \cdot t} \cdot \frac{2t}{a}dt$$

$$= \int \frac{2}{t^2-b}dt$$

1.当b > 0时，$\int \frac{2}{t^2-b}dt = 2\int \frac{1}{t^2-(\sqrt{b})^2}dt$

$$\boxed{\text{公式 } 21: \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot ln\left|\frac{x-a}{x+a}\right| + C}$$

$$= \frac{1}{\sqrt{b}} \cdot ln\left|\frac{t-\sqrt{b}}{t+\sqrt{b}}\right| + C$$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

2.当$b<0$时，$\int \frac{2}{t^2-b}dt = 2\int \frac{1}{t^2+(\sqrt{-b})^2}dt$

$$\boxed{\text{公式 } 19: \int \frac{dx}{x^2+a^2} = \frac{1}{a} \cdot arctan\frac{x}{a} + C}$$

$$= \frac{2}{\sqrt{-b}} \cdot arctan\frac{\text{t}}{\sqrt{-b}} + C$$

将$t=\sqrt{ax+b}$代入上式得：$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \cdot arctan\sqrt{\frac{ax+b}{-b}} + C$

综合讨论1，2得：$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & (b>0) \\ \frac{2}{\sqrt{-b}} \cdot arctan\sqrt{\frac{ax+b}{-b}} + C & (b>0) \end{cases}$$

16. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$

证明：设 $\frac{1}{x^2\cdot\sqrt{ax+b}} = \frac{\mathrm{A}}{x\sqrt{ax+b}} + \frac{B\sqrt{ax+b}}{x^2}$，则 $1 = \mathrm{A}x + \mathrm{B}(ax+b)$

$$\therefore \text{ 有 } \begin{cases} A + Ba = 0 \\ Bb = 1 \end{cases} \Rightarrow \begin{cases} \mathrm{A} = -\frac{a}{b} \\ \mathrm{B} = \frac{1}{b} \end{cases}$$

于是 $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx + \frac{1}{b}\int \frac{\sqrt{ax+b}}{x^2}dx$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{1}{b}\int \sqrt{ax+b}\, d\frac{1}{x}$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b}\int \frac{1}{x} d\sqrt{ax+b}$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b}\int \frac{1}{x}\cdot\frac{a}{2}(ax+b)^{-\frac{1}{2}}dx$$

$$= -\frac{a}{b}\int \frac{1}{x\sqrt{ax+b}}dx - \frac{\sqrt{ax+b}}{bx} + \frac{a}{2b}\int \frac{1}{x\sqrt{ax+b}}dx$$

$$= -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$

17. $\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

证明：令$\sqrt{ax+b} = t \quad (t \ge 0)$，则 $x = \frac{t^2-b}{a}$ ，$dx = \frac{2t}{a}dt$

$$\therefore \int \frac{\sqrt{ax+b}}{x}dx = \int \frac{at}{t^2-b}\cdot\frac{2t}{a}dt = 2\int \frac{t^2}{t^2-b}dt$$

$$= 2\int \frac{t^2-b^2+b^2}{t^2-b}dt = 2\int dt + 2b\int \frac{1}{t^2-b}dt$$

$$= 2t + 2b\int \frac{1}{t^2-b}dt$$

$\because b$取值为R，符号可正可负 $\therefore \int \frac{1}{t^2-b}dt$ 不能明确积分

$$\therefore \int \frac{\sqrt{ax+b}}{x}dx = 2t + 2b\int \frac{1}{t^2-b}dt$$

$$= 2t + 2b\int \frac{1}{t^2-b}\cdot\frac{a}{2t}dx$$

将$t = \sqrt{ax+b}$代入上式得：$\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{(ax+b)} + 2b\int \frac{1}{ax+b-b}\cdot\frac{a}{2\sqrt{ax+b}}dx$

$$= 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$$

18. $\int \frac{\sqrt{ax+b}}{x^2}dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$

证明：$\int \frac{\sqrt{ax+b}}{x^2}dx = -\int \sqrt{ax+b}\, d\frac{1}{x}$

$$= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x}\, d\sqrt{ax+b}$$

$$= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x}\cdot (ax+b)^{-\frac{1}{2}} \cdot \frac{a}{2}dx$$

$$= -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$$

（三）含有 $x^2 \pm a^2$ 的积分（19~21）

19. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot arctan\frac{x}{a} + C$

证明：令 $x = a\cdot tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$，则 $dx = d(a\cdot tant) = a\cdot sec^2 t\, dt$

$$\frac{1}{x^2+a^2} = \frac{dx}{a^2\cdot(1+tan^2 t)} = \frac{1}{a^2 sec^2 t}$$

$$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{1}{a^2 sec^2 t}\cdot a\cdot sec^2 t\, dt$$

$$= \frac{1}{a}\int dt$$

$$= \frac{1}{a}\cdot t + C$$

$\because\ x = a\cdot tant \qquad \therefore\ t = arctan\frac{x}{a}$

将$t = arctan\frac{x}{a}$代入上式得：$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot arctan\frac{x}{a} + C$

20. $\int\frac{dx}{(x^2+a^2)^n}=\frac{x}{2(n-1)\cdot a^2\cdot(x^2+a^2)^{n-1}}+\frac{2n-3}{2(n-1)\cdot a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}$

证明：$\int\frac{dx}{(x^2+a^2)^n}=\frac{x}{(x^2+a^2)^n}-\int x\,d\,\frac{1}{(x^2+a^2)^n}$

$$=\frac{x}{(x^2+a^2)^n}-\int x\cdot(-n)\cdot(x^2+a^2)^{-n-1}\cdot 2x\,dx$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{1}{(x^2+a^2)^n}dx-2na^2\int\frac{1}{(x^2+a^2)^{n+1}}dx$$

移项并整理得：$(1-2n)\int\frac{dx}{(x^2+a^2)^n}=\frac{x}{(x^2+a^2)^n}-2na^2\int\frac{1}{(x^2+a^2)^{n+1}}dx$

$$\therefore\ \int\frac{1}{(x^2+a^2)^{n+1}}dx=\frac{1}{2na^2}\left[\frac{x}{(x^2+a^2)^n}+(2n-1)\int\frac{dx}{(x^2+a^2)^n}\right]$$

令$n+1=n$，则$\int\frac{dx}{(x^2+a^2)^n}=\frac{1}{2(n-1)\cdot a^2}\left[\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)\int\frac{dx}{(x^2+a^2)^{n-1}}\right]$

$$=\frac{x}{2(n-1)\cdot a^2\cdot(x^2+a^2)^{n-1}}+\frac{2n-3}{2(n-1)\cdot a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}$$

21. $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+C$

证明：$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int[\frac{1}{x-a}-\frac{1}{x+a}]\,dx$

$$=\frac{1}{2a}\int\frac{1}{x-a}dx-\frac{1}{2a}\int\frac{1}{x+a}dx$$

$$=\frac{1}{2a}\cdot ln\left|x-a\right|-\frac{1}{2a}\cdot ln\left|x+a\right|+C$$

$$=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+C$$

（四）含有 $ax^2+b\ (a>0)$ 的积分（22~28）

22. $$\int\frac{dx}{ax^2+b}=\begin{cases}\dfrac{1}{\sqrt{ab}}\cdot arctan\sqrt{\dfrac{a}{b}}\cdot x+C & (b>0)\\[2ex] \dfrac{1}{2\sqrt{-ab}}\cdot ln\left|\dfrac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C & (b<0)\end{cases}\qquad (a>0)$$

证明：

1.当b > 0时，$\dfrac{1}{ax^2+b}=\dfrac{1}{x^2+\dfrac{b}{a}}\cdot\dfrac{1}{a}=\dfrac{1}{x^2+(\sqrt{\dfrac{b}{a}})^2}\cdot\dfrac{1}{a}$

$$\therefore\ \int\frac{dx}{ax^2+b}=\frac{1}{a}\int\frac{1}{x^2+(\sqrt{\frac{b}{a}})^2}dx$$

$$=\frac{1}{a}\cdot\sqrt{\frac{a}{b}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+\mathrm{C}$$

$$=\frac{1}{\sqrt{ab}}\cdot arctan\sqrt{\frac{a}{b}}\cdot x+C$$

2.当$b<0$时，$\dfrac{1}{ax^2+b}=\dfrac{1}{x^2-(-\dfrac{b}{a})}\cdot\dfrac{1}{a}=\dfrac{1}{x^2-(\sqrt{-\dfrac{b}{a}})^2}\cdot\dfrac{1}{a}$

$$\therefore\ \int\frac{dx}{ax^2+b}=\frac{1}{a}\int\frac{1}{x^2-(\sqrt{-\frac{b}{a}})^2}dx$$

$$=\frac{1}{2\sqrt{\frac{-b}{a}}}\cdot\frac{1}{a}\cdot ln\left|\frac{x-\sqrt{\frac{-b}{a}}}{x+\sqrt{\frac{-b}{a}}}\right|+C$$

$$=\frac{1}{2\sqrt{-ab}}\cdot ln\left|\frac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C$$

综合讨论1,2得：$$\int\frac{dx}{ax^2+b}=\begin{cases}\dfrac{1}{\sqrt{ab}}\cdot arctan\sqrt{\dfrac{a}{b}}\cdot x+C & (b>0)\\[2ex] \dfrac{1}{2\sqrt{-ab}}\cdot ln\left|\dfrac{\sqrt{a}\cdot x-\sqrt{-b}}{\sqrt{a}\cdot x+\sqrt{-b}}\right|+C & (b<0)\end{cases}$$

23. $$\int\frac{x}{ax^2+b}dx=\frac{1}{2a}\cdot ln\left|ax^2+b\right|+C\qquad (a>0)$$

证明：$$\int\frac{x}{ax^2+b}dx=\frac{1}{2}\int\frac{1}{ax^2+b}dx^2$$

$$=\frac{1}{2a}\int\frac{1}{ax^2+b}d(ax^2+b)$$

$$=\frac{1}{2a}\cdot ln\left|ax^2+b\right|+C$$

24. $\int \frac{x^2}{ax^2+b}dx = \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明：$\int \frac{x^2}{ax^2+b}dx = \frac{b}{a}\int \frac{ax^2}{ax^2+b}\cdot\frac{1}{b}dx$

$$= \frac{b}{a}\int (\frac{1}{b} - \frac{1}{ax^2+b})dx$$

$$= \frac{b}{a}\int \frac{1}{b}dx - \frac{b}{a}\int \frac{1}{ax^2+b}dx$$

$$= \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^2+b}$$

25. $\int \frac{dx}{x(ax^2+b)} = \frac{1}{2b}\cdot ln\frac{x^2}{|ax^2+b|} + C \qquad (a>0)$

证明：$\int \frac{dx}{x(ax^2+b)} = \int \frac{x}{x^2(ax^2+b)}dx$

$$= \frac{1}{2}\int \frac{1}{x^2(ax^2+b)}dx^2$$

设：$\frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$

则 $1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$

$\therefore$ 有 $\begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x(ax^2+b)} = \frac{1}{2}\int [\frac{1}{bx^2} - \frac{a}{b(ax^2+b)}]dx^2$

$$= \frac{1}{2b}\int \frac{1}{x^2}dx^2 - \frac{a}{2b}\int \frac{1}{ax^2+b}dx^2$$

$$= \frac{1}{2b}\int \frac{1}{x^2}dx^2 - \frac{1}{2b}\int \frac{1}{ax^2+b}d(ax^2+b)$$

$$= \frac{1}{2b}\cdot ln\left|x^2\right| - \frac{1}{2b}\cdot ln\left|ax^2+b\right| + C$$

$$= \frac{1}{2b}\cdot ln\frac{x^2}{|ax^2+b|} + C$$

26. $\int \frac{dx}{x^2(ax^2+b)} = -\frac{1}{bx} - \frac{a}{b}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明：设：$\frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$

则 $1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$

$\therefore$ 有 $\begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b} \\ B = -\frac{a}{b} \end{cases}$

于是 $\int \frac{dx}{x^2(ax^2+b)} = \int [\frac{1}{bx^2} - \frac{a}{b(ax^2+b)}]dx$

$= \frac{1}{b}\int \frac{1}{x^2}dx - \frac{a}{b}\int \frac{1}{ax^2+b}dx$

$= -\frac{1}{bx} - \frac{a}{b}\int \frac{dx}{ax^2+b}$

27. $\int \frac{dx}{x^3(ax^2+b)} = \frac{a}{2b^2} ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C \qquad (a>0)$

证明：$\int \frac{dx}{x^3(ax^2+b)} = \int \frac{x}{x^4(ax^2+b)}dx$

$= \frac{1}{2}\int \frac{1}{x^4(ax^2+b)}dx^2$

设：$\frac{1}{x^4(ax^2+b)} = \frac{A}{x^2} + \frac{B}{x^4} + \frac{C}{ax^2+b}$

则 $1 = Ax^2(ax^2+b) + B(ax^2+b) + Cx^4$

$= (Aa+C)x^4 + (Ab+Ba)x^2 + Bb$

$\therefore$ 有 $\begin{cases} Aa+C=0 \\ Ab+Ba=0 \\ Bb=1 \end{cases} \Rightarrow \begin{cases} B = \frac{1}{b} \\ A = -\frac{a}{b^2} \\ C = \frac{a^2}{b^2} \end{cases}$

于是 $\int \frac{dx}{x^3(ax^2+b)} = -\frac{a}{2b^2}\int \frac{1}{x^2}dx^2 + \frac{1}{2b}\int \frac{1}{x^4}dx^2 + \frac{a^2}{2b^2}\int \frac{1}{ax^2+b}dx^2$

$= -\frac{a}{2b^2}\cdot ln|x^2| - \frac{1}{2bx^2} + \frac{a}{2b^2}\cdot ln|ax^2+b| + C$

$= \frac{a}{2b^2} ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C$

28. $\int \frac{dx}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{dx}{ax^2+b} \qquad (a>0)$

证明：$\int \frac{dx}{(ax^2+b)^2} = -\int \frac{1}{2ax} d\frac{1}{ax^2+b} = -\frac{1}{2ax}\cdot\frac{1}{ax^2+b} + \int \frac{1}{ax^2+b} d\frac{1}{2ax}$

$$= -\frac{1}{2ax}\cdot\frac{1}{ax^2+b} - \int \frac{1}{ax^2+b}\cdot\frac{1}{2ax^2}dx$$

设：$\frac{1}{2ax^2(ax^2+b)} = \frac{A}{2ax^2} + \frac{B}{ax^2+b}$，则 $1 = A(ax^2+b) + 2Bax^2 = (Aa+2Ba)x^2 + Ab$

$$\therefore \text{有} \begin{cases} Aa+2Ba=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{1}{2b} \end{cases}$$

于是 上式 $= -\frac{1}{2ax(ax^2+b)} - \int (\frac{1}{2abx^2} - \frac{1}{2b(ax^2+b)})dx$

$$= -\frac{1}{2ax(ax^2+b)} - \frac{1}{2ab}\int \frac{1}{x^2}dx + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx$$

$$= -\frac{1}{2ax(ax^2+b)} + \frac{1}{2abx} + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx = \frac{ax^2+b-b}{2abx(ax^2+b)} + \frac{1}{2b}\int \frac{1}{2b(ax^2+b)}dx$$

$$= \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{dx}{ax^2+b}$$

（五）含有 $ax^2+bx+c\ (a>0)$ 的积分（29~30）

29. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\cdot arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2<4ac) \\ \frac{1}{\sqrt{b^2-4ac}}\cdot ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (b^2>4ac) \end{cases} \qquad (a>0)$$

证明：$\because\ ax^2+bx+c = \frac{1}{4a}\left[(2ax+b)^2 + (4ac-b^2)\right]$

$$\therefore \int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(4ac-b^2)}dx$$

1.当 $b^2<4ac$ 时，$\int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(\sqrt{4ac-b^2})^2}dx$

$\boxed{\text{公式 19：} \int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot arctan\frac{x}{a} + C}$

$$= \frac{4a}{2a}\int \frac{1}{(2ax+b)^2+(\sqrt{4ac-b^2})^2}d(2ax+b)$$

$$= \frac{2}{\sqrt{4ac-b^2}}\cdot arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C$$

2.当 $b^2>4ac$ 时，$\int \frac{dx}{ax^2+bx+c} = 4a\int \frac{1}{(2ax+b)^2+(4ac-b^2)}dx$

$$= 4a\int \frac{1}{(2ax+b)^2-(b^2-4ac)}dx$$

$\boxed{\text{公式 21：} \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right| + C}$

$$= \frac{4a}{2a}\int \frac{1}{(2ax+b)^2-(\sqrt{b^2-4ac})^2}d(2ax+b)$$

$$= \frac{1}{\sqrt{b^2-4ac}}\cdot ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C$$

综合讨论1,2得：$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\cdot arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2<4ac) \\ \frac{1}{\sqrt{b^2-4ac}}\cdot ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (b^2>4ac) \end{cases}$$

30. $\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\cdot ln\left| ax^2+bx+c \right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c} \qquad (a>0)$

证明：$\int \frac{x}{ax^2+bx+c}dx = \int \frac{1}{2a}\cdot\frac{2ax+b-b}{ax^2+bx+c}dx$

$$= \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \frac{1}{2a}\int \frac{-b}{ax^2+bx+c}dx$$

$$= \frac{1}{2a}\int \frac{1}{ax^2+bx+c}d(ax^2+bx+c) - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$$

$$= \frac{1}{2a}\cdot ln\left| ax^2+bx+c \right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

（六）含有 $\sqrt{x^2+a^2}$ $(a>0)$ 的积分（31~44）

31. $\int \frac{dx}{\sqrt{x^2+a^2}} = arsh\frac{x}{a} + C_1 = ln\,(x+\sqrt{x^2+a^2}) + \mathrm{C} \qquad (a>0)$

证明：被积函数 $f(x) = \frac{1}{\sqrt{x^2+a^2}}$ 的定义域为 $\{x \mid x\in R\}$

可令 $x = a\,tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$， 则 $dx = d(a\,tant) = a\,sec^2 tdt$，$\sqrt{x^2+a^2} = \mid a\,sect \mid$

$\because\ -\frac{\pi}{2}<t<\frac{\pi}{2}, sect = \frac{1}{cost} > 0$，$\therefore\ \sqrt{x^2+a^2} = a\,sect$

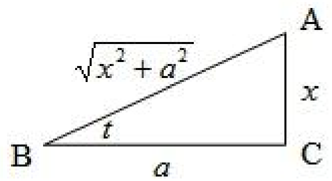
$\therefore\ \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{1}{a\,sect}\cdot a\,sec^2 t\,dt$ 　　$\boxed{\text{公式87:} \int sectdt = ln\mid sect+tant\mid + C}$

$$= \int sect\,dt$$

$$= ln\left| sect+tant \right| + C_2$$

在 $\mathrm{Rt}\Delta ABC$ 中，设 $\angle \mathrm{B} = t$，$\mid BC \mid = a$，则 $\mid \mathrm{AC} \mid = x$，$\mid \mathrm{AB} \mid = \sqrt{x^2+a^2}$

$\therefore\ sect = \frac{1}{cost} = \frac{\sqrt{x^2+a^2}}{a}$ ，$tant = \frac{x}{a}$



$\therefore\ \int \frac{dx}{\sqrt{x^2+a^2}} = ln\left| sect+tant \right| + C_2$

$$= ln\left| \frac{\sqrt{x^2+a^2}+x}{a} \right| + C_2$$

$$= ln\left| \sqrt{x^2+a^2}+x \right| - lna + C_2$$

$$= ln\left| \sqrt{x^2+a^2}+x \right| + C_3$$

$\because\ \sqrt{x^2+a^2}+x>0$

$\therefore\ \int \frac{dx}{\sqrt{x^2+a^2}} = ln\,(x+\sqrt{x^2+a^2}) + \mathrm{C}$

32. $\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{1}{\sqrt{(x^2+a^2)^3}}$的定义域为$\{x \mid x\in R\}$

可令$x=a\,tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，　则$dx=d(a\,tant)=a\,sec^2 tdt$，$\sqrt{(x^2+a^2)^3}=|a^3\,sec^3 t|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}, sect=\frac{1}{cost}>0$，$\therefore \sqrt{(x^2+a^2)^3}=a^3\,sec^3 t$

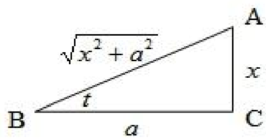
$$\therefore \int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \int \frac{1}{a^3\,sec^3 t}\cdot a\,sec^2 t\,dt = \frac{1}{a^2}\int \frac{1}{sect}dt$$

$$= \frac{1}{a^2}\int \cos t dt = \frac{1}{a^2} sint + C$$

在RtΔABC中，设$\angle B=t$，$|BC|=a$，则$|AC|=x$，$|AB|=\sqrt{x^2+a^2}$

$$\therefore \sin t = \frac{|AC|}{|AB|} = \frac{x}{\sqrt{x^2+a^2}}$$

$$\therefore \int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \frac{1}{a^2}\cdot sint + C = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$



33. $\int \frac{x}{\sqrt{x^2+a^2}}dx = \sqrt{x^2+a^2} + C \qquad (a>0)$

证明：令$\sqrt{x^2+a^2}=t \quad (t>0)$，则$x=\sqrt{t^2-a^2}$

$$\therefore dx = \frac{1}{2}(t^2-a^2)^{-\frac{1}{2}}\cdot 2tdt = \frac{t}{\sqrt{t^2-a^2}}dt$$

$$\therefore \int \frac{x}{\sqrt{x^2+a^2}}dx = \int \frac{\sqrt{t^2-a^2}}{t}\cdot \frac{t}{\sqrt{t^2-a^2}}dt$$

$$= \int dt = t + C$$

将$t=\sqrt{x^2+a^2}$代入上式得：$\int \frac{x}{\sqrt{x^2+a^2}}dx = \sqrt{x^2+a^2} + C$

34. $\int \frac{x}{\sqrt{(x^2+a^2)^3}}dx = -\frac{1}{\sqrt{x^2+a^2}} + C \qquad (a>0)$

证明：$\int \frac{x}{\sqrt{(x^2+a^2)^3}}dx = \int x\cdot(x^2+a^2)^{-\frac{3}{2}}dx = \frac{1}{2}\int (x^2+a^2)^{-\frac{3}{2}}dx^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-\frac{3}{2}}d(x^2+a^2)$$

$$= \frac{1}{2}\times\frac{1}{1-\frac{3}{2}}\cdot(x^2+a^2)^{1-\frac{3}{2}} + C$$

$$= -\frac{1}{\sqrt{x^2+a^2}} + C$$

35. $\int \frac{x^2}{\sqrt{x^2+a^2}}dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} - \frac{a^2}{2}ln(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证明: $\int \frac{x^2}{\sqrt{x^2+a^2}}dx = \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$

$$= \int \sqrt{x^2+a^2}dx - a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$\because \int \sqrt{x^2+a^2}dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot ln(x+\sqrt{x^2+a^2}) + C$ (公式39)

$\int \frac{1}{\sqrt{x^2+a^2}}dx = ln(x+\sqrt{x^2+a^2}) + C$ (公式31)

$\therefore \int \frac{x^2}{\sqrt{x^2+a^2}}dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}ln(x+\sqrt{x^2+a^2}) - a^2\cdot ln(x+\sqrt{x^2+a^2}) + C$

$$= \frac{x}{2}\cdot\sqrt{x^2+a^2} - \frac{a^2}{2}\cdot ln(x+\sqrt{x^2+a^2}) + C$$

36. $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = -\frac{x}{\sqrt{x^2+a^2}} + ln(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证明: 被积函数 $f(x) = \frac{x^2}{\sqrt{(x^2+a^2)^3}}$ 的定义域为 $\{x \mid x\in R\}$

可令 $x = a\,tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$, 则 $dx = d(a\,tant) = a\,sec^2tdt$, $\frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{a^2tan^2t}{|a^3\,sec^3t|}$

$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, sect = \frac{1}{cost} > 0$, $\therefore \frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{tan^2t}{a\,sec^3t}$

$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{tan^2t}{a\,sec^3t}\cdot a\,sec^2tdt = \int \frac{tan^2t}{sect}dt = \int \frac{sec^2t-1}{sect}dt$

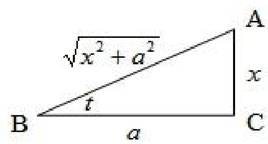
$$= \int sectdt - \int \frac{1}{sect}dt = \int sectdt - \int \cos tdt$$

$$= ln\left|sect+tant\right| - sint + C_1$$

$\boxed{\text{公式 87: } \int \sec t\,dt = ln|sect+tant| + C}$

在Rt$\varDelta ABC$中，设$\angle B = t$, $|BC| = a$,则 $|AC| = x$, $|AB| = \sqrt{x^2+a^2}$

$\therefore sint = \frac{x}{\sqrt{x^2+a^2}}, tant = \frac{x}{a}, sect = \frac{1}{cost} = \frac{\sqrt{x^2+a^2}}{a}$



$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = ln\left|sect+tant\right| - sint + C_1$

$$= ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

$$= ln\left|\sqrt{x^2+a^2}+x\right| - \frac{x}{\sqrt{x^2+a^2}} - lna + C_1$$

$\because \sqrt{x^2+a^2} + x > 0$

$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = -\frac{x}{\sqrt{x^2+a^2}} + ln(x+\sqrt{x^2+a^2}) + C$

37. $\int \frac{dx}{x\cdot\sqrt{x^2+a^2}} = \frac{1}{a}\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C \qquad (a>0)$

证明：令$\sqrt{x^2+a^2}=t \quad (t>0)$，则$x=\sqrt{t^2-a^2}$

$$\therefore\ dx = \frac{1}{2}(t^2-a^2)^{-\frac{1}{2}}\cdot 2tdt = \frac{t}{\sqrt{t^2-a^2}}dt$$

$$\therefore\ \int\frac{dx}{x\cdot\sqrt{x^2+a^2}} = \int\frac{1}{t\cdot\sqrt{t^2-a^2}}\cdot\frac{t}{\sqrt{t^2-a^2}}dt$$

$$= \int\frac{1}{t^2-a^2}dt$$

$\boxed{\text{公式 21：}\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+\mathrm{C}}$

$$= \frac{1}{2a}\cdot ln\left|\frac{t-a}{t+a}\right|+C$$

$$= \frac{1}{2a}\cdot ln\left|\frac{(t-a)^2}{t^2-a^2}\right|+C$$

将$t=\sqrt{x^2+a^2}$代入上式得：$\int\frac{dx}{x\cdot\sqrt{x^2+a^2}} = \frac{1}{2a}\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2+a^2-a^2}\right|+C$

$\boxed{\text{提示：}log_a b^n = nlog_a b}$

$$= \frac{1}{2a}\cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2}\right|+C$$

$$= \frac{1}{a}\cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$$

38. $\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}+C \qquad (a>0)$

证明：$\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}} = -\int\frac{1}{\sqrt{x^2+a^2}}d\frac{1}{x}$

令$t=\frac{1}{x} \quad (t\neq 0)$，则$x=\frac{1}{t}$

$$\therefore\ -\int\frac{1}{\sqrt{x^2+a^2}}d\frac{1}{x} = -\int\frac{1}{\sqrt{\frac{1}{t^2}+a^2}}dt = -\int\frac{t}{\sqrt{1+a^2t^2}}dt$$

$$= -\frac{1}{2a^2}\int\frac{2a^2t}{\sqrt{1+a^2t^2}}dt$$

$$= -\frac{1}{2a^2}\int\frac{1}{\sqrt{1+a^2t^2}}d(1+a^2t^2)$$

$$= -\frac{1}{2a^2}\cdot\frac{1}{1-\frac{1}{2}}(1+a^2t^2)^{1-\frac{1}{2}}+C$$

$$= -\frac{1}{a^2}\cdot\sqrt{1+a^2t^2}+C$$

将$t=\frac{1}{x}$代入上式得：$\int\frac{dx}{x^2\cdot\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}+C$

39. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证法 1: $\because \int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int x\,d\sqrt{x^2+a^2}$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$\therefore \int \sqrt{x^2+a^2}\,dx + \int \frac{x^2}{\sqrt{x^2+a^2}}dx = x\sqrt{x^2+a^2} \qquad ①$$

$$又 \int \sqrt{x^2+a^2}\,dx - \int \frac{x^2}{\sqrt{x^2+a^2}}dx = \int \frac{a^2}{\sqrt{x^2+a^2}}dx \qquad ②$$

$\boxed{公式 31: \int \frac{1}{\sqrt{x^2+a^2}}dx = \ln(x+\sqrt{x^2+a^2}) + C \quad (a>0)}$

$$= a^2\cdot \ln(x+\sqrt{x^2+a^2}) + C_1$$

由① +②得，$2\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} + a^2\cdot \ln(x+\sqrt{x^2+a^2})$

即 $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln(x+\sqrt{x^2+a^2}) + C$

39. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证法 2: 令 $x = a\cdot \tan t \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$ ，则 $\sqrt{x^2+a^2} = a\sqrt{1+\tan^2 t} = |a\cdot \sec t|$,

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \sqrt{x^2+a^2} = a\cdot \sec t$$

$\boxed{提示: 1+\tan^2 t = \sec^2 t}$

$$\therefore \int \sqrt{x^2+a^2}\,dx = \int a\cdot \sec t\,d(a\cdot \tan t) = a^2\int \sec t\,d\tan t$$

$$= a^2\sec t\cdot \tan t - a^2\int \tan t\,d\sec t \qquad ①$$

$$又 \int \tan t\,d\sec t = \int \tan t\cdot \sec t\cdot \tan t\,dt = \int \frac{\sin^2 t}{\cos^3 t}dt$$

$$= \int \frac{1-\cos^2 t}{\cos^3 t}dt = \int \frac{1}{\cos t}\cdot\frac{1}{\cos^2 t}dt - \int \frac{1}{\cos t}dt$$

$$= \int \sec t\,d\tan t - \int \sec t\,dt \qquad ②$$

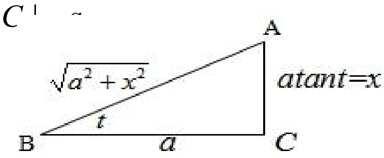
联立①②有 $a^2\int \sec t\,d\tan t = \frac{1}{2}(a^2\sec t\cdot \tan t + a^2\int \sec t\,dt) \qquad ③$

又 $\int \sec t\,dt = \ln|\sec t + \tan t| + C_1$ （公式 87） $\qquad ④$

联立③④有 $a^2\int \sec t\,d\tan t = \frac{1}{2}a^2\sec t\cdot \tan t + \frac{1}{2}a^2\ln|\sec t+\tan t| + C_2 \qquad ⑤$

$\because x = a\cdot \tan t, \therefore$ 在RtΔABC中，可设 $\angle B = t, |BC$

则 $|AC| = a\cdot \tan t = x, |AB| = \sqrt{a^2+x^2}$



$$\therefore \sec t = \frac{1}{\cos t} = \frac{\sqrt{a^2+x^2}}{a}, \tan t = \frac{x}{a}$$

$$\therefore \frac{1}{2}a^2\sec t\cdot \tan t + \frac{1}{2}a^2\ln|\sec t+\tan t| = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|$$

$$= \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln(x+\sqrt{x^2+a^2}) - \frac{a^2}{2}\cdot \ln a$$

综合①②③④⑤得 $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot \ln(x+\sqrt{x^2+a^2}) + C$

40. $\int\sqrt{(x^2+a^2)^3}\,dx=\frac{x}{8}\cdot(2x^2+5a^2)\sqrt{x^2+a^2}+\frac{3}{8}\cdot a^4\cdot ln\,(x+\sqrt{x^2+a^2})+C \qquad (a>0)$

证明：被积函数 $f(x)=\sqrt{(x^2+a^2)^3}$的定义域为$\{x\mid x\in R\}$

可令$x=a\,tant \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$， 则$\sqrt{(x^2+a^2)^3}=|a^3\,sec^3t\,|$

$\because\ -\frac{\pi}{2}<t<\frac{\pi}{2}, sect=\frac{1}{cost}>0$， $\therefore\ \sqrt{(x^2+a^2)^3}=a^3\cdot sec^3t$

$$\begin{aligned}\therefore\ \int\sqrt{(x^2+a^2)^3}\,dx&=\int a^3\cdot sec^3t\,d\,(a\,tant)=a^4\int sec^3t\,d\,tant\\&=a^4sec^3t\cdot tant-a^4\int tant\,d\,sec^3t\\&=a^4sec^3t\cdot tant-a^4\int tant\cdot 3\cdot sec^2t\cdot sect\,tant\,dt\\&=a^4sec^3t\cdot tant-3a^4\int tan^2t\cdot sec^3t\,dt\\&=a^4sec^3t\cdot tant-3a^4\int tan^2t\cdot sect\,d\,tant\\&=a^4sec^3t\cdot tant-3a^4\int (sec^2t-1)\cdot sect\,d\,tant\\&=a^4sec^3t\cdot tant-3a^4\int sec^3t\,d\,tant+3a^4\int sect\,d\,tant\end{aligned}$$

移项并整理的：$a^4\int sec^3t\,d\,tant=\frac{1}{4}(a^4sec^3t\,tant+3a^4\int sect\,d\,tant)$ ①

$$\begin{aligned}\because\ \int sect\,d\,tant&=sect\cdot tant-\int tant\,d\,sect\\&=sect\cdot tant-\int tan^2t\cdot sectdt\\&=sect\cdot tant-\int (sec^2t-1)\cdot sectdt\\&=sect\cdot tant-\int sec^3t\,dt+\int sect\,dt\end{aligned}$$ ②

又$\because\ \int sect\,d\,tant=\int sec^3t\,dt$ ③

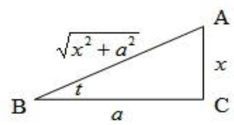
公式 87：$\int \sec t\,dt=ln\,|\,sect+tant\,|+C$

联立②③得：$a^4\int sec^3t\,d\,tant=\frac{1}{2}\cdot sect\cdot tant+\frac{1}{2}\int sect\,dt$

$=\frac{1}{2}\cdot sect\cdot tant+\frac{1}{2}ln\,|\,sect+tant\,|+C_1$ ④

联立①④得 $a^4\int sec^3t\,d\,tant=\frac{1}{4}a^4sec^3t\cdot tant+\frac{3}{8}a^4sect\cdot tant+\frac{3}{8}a^4\cdot ln\,|\,sect+tant\,|+C_1$

在RtΔABC中，设$\angle B=t$，$|BC|=a$，则$|AC|=x$，$|AB|=\sqrt{x^2+a^2}$



$\therefore\ tant=\frac{x}{a}, sect=\frac{1}{\cos t}=\frac{\sqrt{x^2+a^2}}{a}$

$$\begin{aligned}\therefore\ a^4\int sec^3t\,d\,tant&=\frac{a^4}{4}\cdot\frac{x}{a}\cdot\frac{x^2+a^2}{a^3}\cdot\sqrt{x^2+a^2}+\frac{3a^4}{8}\cdot\frac{\sqrt{x^2+a^2}}{a}\cdot\frac{x}{a}+\frac{3}{8}a^4\cdot ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+C_1\\&=\frac{x}{4}(x^2+a^2)\sqrt{x^2+a^2}+\frac{3a^2\cdot x}{8}\cdot\sqrt{x^2+a^2}+\frac{3a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right|+C\end{aligned}$$

$\therefore\ \int\sqrt{(x^2+a^2)^3}\,dx=\frac{x}{8}\cdot(2x^2+5a^2)\sqrt{x^2+a^2}+\frac{3}{8}\cdot a^4\cdot ln\,(x+\sqrt{x^2+a^2})+C$

41. $\int x \cdot \sqrt{x^2+a^2}dx = \frac{1}{3}\sqrt{(x^2+a^2)^3}+C \qquad (a>0)$

证明：$\int x \cdot \sqrt{x^2+a^2}dx = \frac{1}{2}\int (x^2+a^2)^{\frac{1}{2}}dx^2$

$$= \frac{1}{2}\int (x^2+a^2)^{\frac{1}{2}}d(x^2+a^2)$$

$$= \frac{1}{2}\times\frac{1}{1+\frac{1}{2}}\cdot(x^2+a^2)^{1+\frac{1}{2}}+C$$

$$= \frac{1}{3}\sqrt{(x^2+a^2)^3}+C$$

42. $\int x^2 \cdot \sqrt{x^2+a^2}\, dx = \frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2}) + C \qquad (a>0)$

证明：被积函数 $f(x) = x^2 \cdot \sqrt{x^2+a^2}$的定义域为$\{x \mid x \in R\}$

可令$x = a\,tant \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$， 则$x^2 \cdot \sqrt{(x^2+a^2)} = a^2 tan^2 t \mid a\,sect \mid$

$\because\ -\frac{\pi}{2} < t < \frac{\pi}{2}, sect = \frac{1}{cost} > 0$， $\therefore\ x^2 \cdot \sqrt{(x^2+a^2)} = a^3 tan^2 t \cdot sect$

$$\therefore\ \int x^2 \cdot \sqrt{(x^2+a^2)}\, dx = \int a^3 tan^2 t \cdot sect\ d\,(a\,tant) = a^4 \int tan^2 t \cdot sect\ d\ tant = a^4 \int tan^2 t \cdot sec^3 t\ dt$$

$$= a^4 \int tant \cdot sec^2 t\ dsect$$

$$= a^4 \int tant \cdot (1+tan^2 t)\ dsect = a^4 \int tant\ dsect + a^4 \int tan^3 t\ dsect$$

$$= a^4 \int tant\ dsect + a^4 \cdot tan^3 t \cdot sect - a^4 \int sect\ dtan^3 t$$

$$= a^4 \int tant\ dsect + a^4 \cdot tan^3 t \cdot sect - 3a^4 \int sec^3 t\ tan^2 t\ dt$$

$$= a^4 \int tant\ dsect + a^4 \cdot tan^3 t \cdot sect - 3a^4 \int sec^2 t\ tant\ dsect$$

移项并整理的：$a^4 \int tant \cdot sec^2 t\ dsect = \frac{1}{4}(a^4 \int tant\ dsect + a^4 \cdot tan^3 t \cdot sect)$

$$= \frac{a^4}{4}\int tant\ dsect + \frac{a^4}{4}\cdot tan^3 t \cdot sect \qquad ①$$

$\because\ \int tant\ d\ sect = sect \cdot tant - \int sect\ dtant \ = sect \cdot tant - \int sec^3 t dt$

$$= sect \cdot tant - \int (1+tan^2 t)\cdot sectdt$$

$$= sect \cdot tant - \int sect\ dt - \int tan^2 t\ sect\ dt$$

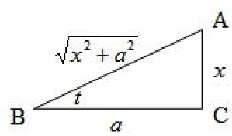
$$= sect \cdot tant - \int sect\ dt - \int tant\ d\ sect$$

移项并整理得：$\int tant\ d\ sect = \frac{1}{2}\cdot sect \cdot tant - \frac{1}{2}\int sect\ dt$ $\boxed{\text{公式87:}\int sect\ dt = ln\,|\,sect+tant\,| + C}$

$$= \frac{1}{2}\cdot sect \cdot tant - \frac{1}{2} ln\,|\ sect + tant\ | + C_1 \qquad ②$$

联立①②得：$a^4 \int tant \cdot sec^2 t\ dsect = \frac{a^4}{8} sect \cdot tant - \frac{a^4}{8} ln\,|\ sect + tant\ | + \frac{a^4}{4} tan^3 t \cdot sect + C_1$

在RtΔABC中，设$\angle B = t$，$|\,BC\,| = a$ ，则$|\,AC\,| = x$，$|\,AB\,| = \sqrt{x^2+a^2}$



$\therefore\ tant = \frac{x}{a}, sect = \frac{1}{\cos t} = \frac{\sqrt{x^2+a^2}}{a}$

$$\therefore\ a^4 \int tant \cdot sec^2 t\ dsect = \frac{a^4}{8}\cdot\frac{x}{a}\cdot\frac{\sqrt{x^2+a^2}}{a} - \frac{a^4}{8}\cdot ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + \frac{a^4}{4}\cdot\frac{x^3}{a^3}\cdot\frac{\sqrt{x^2+a^2}}{a} + C_1$$

$$= \frac{a^4 x}{8}\cdot\sqrt{x^2+a^2} - \frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right| + \frac{x^3}{4}\cdot\sqrt{x^2+a^2} + C_2$$

$$= \frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right| + C$$

$\because\ \sqrt{x^2+a^2} + x > 0$ ， $\therefore\ \frac{a^4}{8}\cdot ln\left|\sqrt{x^2+a^2}+x\right| = \frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2})$

$$\therefore\ \int x^2 \cdot \sqrt{x^2+a^2}\ dx = \frac{x}{8}\cdot(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\cdot ln\,(x+\sqrt{x^2+a^2}) + C$$

43. $\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} + a \cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{\sqrt{x^2+a^2}}{x}$的定义域为 $\{x \mid x \neq 0\}$

令$\sqrt{x^2+a^2}=t \quad (t \geq 0$且$t \neq a)$，则$x=\sqrt{t^2-a^2}$

$$\therefore\ dx = \frac{1}{2}(t^2-a^2)^{-\frac{1}{2}} \cdot 2tdt = \frac{t}{\sqrt{t^2-a^2}}dt$$

$$\therefore \int \frac{\sqrt{x^2+a^2}}{x}dx = \int \frac{t}{\sqrt{t^2-a^2}} \cdot \frac{t}{\sqrt{t^2-a^2}}dt = \int \frac{t^2}{t^2-a^2}dt$$

$$= \int \frac{t^2-a^2+a^2}{t^2-a^2}dt = \int dt + a^2 \int \frac{1}{t^2-a^2}dt$$

$\boxed{公式\ 21: \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot ln\left|\frac{x-a}{x+a}\right| + C}$

$$= t + a^2 \cdot \frac{1}{2a} \cdot ln\left|\frac{t-a}{t+a}\right| + C = t + \frac{a}{2} \cdot ln\left|\frac{(t-a)^2}{t^2-a^2}\right| + C$$

将$t=\sqrt{x^2+a^2}$代入上式得：$\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} + \frac{a}{2} \cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)^2}{x^2+a^2-a^2}\right| + C$

$$= \sqrt{x^2+a^2} + a \cdot ln\left|\frac{(\sqrt{x^2+a^2}-a)}{x}\right| + C$$

$$= \sqrt{x^2+a^2} + a \cdot ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C$$

44. $\int \frac{\sqrt{x^2+a^2}}{x^2}dx = -\frac{\sqrt{x^2+a^2}}{x} + ln\,(x+\sqrt{x^2+a^2}) + \mathrm{C} \qquad (a>0)$

证明：被积函数 $f(x)=\frac{\sqrt{x^2+a^2}}{x^2}$的定义域为 $\{x \mid x \neq 0\}$

1. 当$x>0$时，可令 $x = a\,tant \quad (0<t<\frac{\pi}{2})$， 则$dx = d(a\,tant) = a\,sec^2 tdt$ ，

$\frac{\sqrt{x^2+a^2}}{x^2} = \frac{|a\,sect|}{a^2\,tan^2 t}$ ， $\because\ 0<t<\frac{\pi}{2}$，$sect = \frac{1}{cost} > 0$， $\therefore\ \frac{\sqrt{x^2+a^2}}{x^2} = \frac{sect}{a\,tan^2 t}$

$$\therefore \int \frac{\sqrt{x^2+a^2}}{x^2}dx = \int \frac{sect}{a\,tan^2 t} \cdot a\,sec^2 tdt = \int \frac{sect}{tan^2 t} \cdot (1+tan^2 t)dt$$

$$= \int sect\,dt + \int \frac{sect}{tan^2 t}dt = \int sect\,dt + \int \frac{1}{cost} \cdot \frac{cos^2 t}{sin^2 t}dt$$

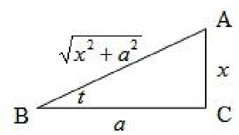
$$= \int sect\,dt + \int \frac{cost}{sin^2 t}dt = \int sect\,dt + \int \frac{1}{sin^2 t}dsint$$

$\boxed{公式\ 87: \int \sec t\,dt = ln|sect + tant| + C}$

$$= ln|sect + tant| - \frac{1}{sint} + C_1$$

在$\mathrm{Rt}\Delta ABC$中，设 $\angle \mathrm{B}=t$，$|BC|=a$，则$|\mathrm{AC}|=x$，$|\mathrm{AB}|=\sqrt{x^2+a^2}$

$$\therefore\ sint = \frac{x}{\sqrt{x^2+a^2}},\ tant = \frac{x}{a},\ sect = \frac{1}{cost} = \frac{\sqrt{x^2+a^2}}{a}$$



$$\therefore \int \frac{\sqrt{x^2+a^2}}{x^2}dx = ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| - \frac{\sqrt{x^2+a^2}}{x} + C_1$$

$$= -\frac{\sqrt{x^2+a^2}}{x} + ln\left|\sqrt{x^2+a^2}+x\right| - lna + C_1$$

$\because\ \sqrt{x^2+a^2}+x>0 \qquad \therefore \int \frac{\sqrt{x^2+a^2}}{x^2}dx = -\frac{\sqrt{x^2+a^2}}{x} + ln\,(\sqrt{x^2+a^2}+x) + \mathrm{C}$

2. 当$x<0$时，同理可证得： $\int \frac{\sqrt{x^2+a^2}}{x^2}dx = -\frac{\sqrt{x^2+a^2}}{x} + ln\,(\sqrt{x^2+a^2}+x) + \mathrm{C}$

综合讨论 1，2 得：$\int \frac{\sqrt{x^2+a^2}}{x^2}dx = -\frac{\sqrt{x^2+a^2}}{x} + ln\,(\sqrt{x^2+a^2}+x) + \mathrm{C}$

（七）含有 $\sqrt{x^2-a^2}$ $(a>0)$ 的积分（45~58）

45. $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C \qquad (a>0)$

证法1：被积函数 $f(x)=\frac{1}{\sqrt{x^2-a^2}}$ 的定义域为 $\{x\,|\,x>a或x<-a\}$

1. 当 $x>a$ 时，可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$，则 $dx=a\cdot sect\cdot tantdt$

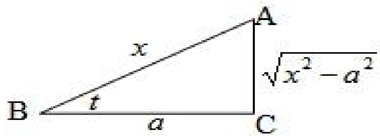
$\sqrt{x^2-a^2}=a\sqrt{sec^2t-1}=a\cdot|tant|$ $\because 0<t<\frac{\pi}{2}$，$\sqrt{x^2-a^2}=a\cdot tant$

$$\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\cdot sect\cdot tant}{a\cdot tant}dt=\int sectdt \qquad \boxed{公式\ 87：\int sec\,tdt=ln|sec\,t+tan\,t|+C}$$

$$=ln|sect+tant|+C_2$$

在Rt ΔABC中，可设 $\angle B=t$，$|BC|=a$，则 $|AB|=x$，$|AC|=\sqrt{x^2-a^2}$

$$\therefore\ sec\,t=\frac{1}{cos\,t}=\frac{x}{a},\ tan\,t=\frac{|AC|}{|BC|}=\frac{\sqrt{x^2-a^2}}{a}$$



$$\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=ln|sec\,t+tan\,t|=ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|$$

$$=ln|x+\sqrt{x^2-a^2}|+C_3$$

2 .当 $x<-a$，即 $-x>a$ 时，令 $\mu=-x$，即 $x=-\mu$

由讨论 1可知 $\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{d\mu}{\sqrt{\mu^2-a^2}}=-ln|\mu+\sqrt{\mu^2-a^2}|+C_4$

$$=-ln|-x+\sqrt{x^2-a^2}|+C_4=ln\frac{1}{|-x+\sqrt{x^2-a^2}|}+C_4$$

$$=ln\frac{|-x+\sqrt{x^2-a^2}|}{a^2}+C_4$$

$$=ln|-x-\sqrt{x^2-a^2}|+C_5$$

综合讨论 1,2，可写成 $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C$

45. $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C \qquad (a>0)$

证法2: 被积函数 $f(x)=\frac{1}{\sqrt{x^2-a^2}}$的定义域为 $\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时，可设$x=a\cdot cht\ (t>0)$，则$t=arch\frac{x}{a}$

$\sqrt{x^2-a^2}=\sqrt{a^2ch^2t-a^2}=a\cdot sht\ ,dx=a\cdot shtdt$

$$\therefore\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\cdot sht}{a\cdot sht}dt=\int dt=t+C_1$$

$$=arch\frac{x}{a}+C=ln\left[\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right]+C_2$$

$$=ln|x+\sqrt{x^2-a^2}|+C_3$$

2.当$x<-a$,即$-x>a$时，令 $\mu=-x$,即$x=-\mu$

由讨论 1可知 $\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{d\mu}{\sqrt{\mu^2-a^2}}=-ln|\mu+\sqrt{\mu^2-a^2}|+C_4$

$$=-ln(-x+\sqrt{x^2-a^2})+C_4=ln\frac{1}{|-x+\sqrt{x^2-a^2}|}+C_4$$

$$=ln\frac{|-x+\sqrt{x^2-a^2}|}{a^2}+C_4$$

$$=ln|-x-\sqrt{x^2-a^2}|+C_5$$

综合讨论 1,2，可写成 $\int\frac{dx}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\cdot arsh\frac{|x|}{a}+C_1=ln|x+\sqrt{x^2-a^2}|+C$

46. $\int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\cdot\sqrt{x^2-a^2}}+C \qquad (a>0)$

证明:被积函数 $f(x)=\frac{1}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x\mid x>a或x<-a\}$

1. 当 $x>a$ 时,可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,则 $dx=a\cdot sect\cdot tantdt$

$\sqrt{(x^2-a^2)^3}=\left|a^3\cdot tan^3t\right| \quad \because 0<t<\frac{\pi}{2}$,$tant>0$,$\sqrt{(x^2-a^2)^3}=a^3\cdot tan^3t$

$$\therefore \int\frac{dx}{\sqrt{(x^2-a^2)^3}}=\int\frac{a\cdot sect\cdot tant}{a^3\cdot tan^3t}dt=\frac{1}{a^2}\int\frac{sect}{tan^3t}dt$$

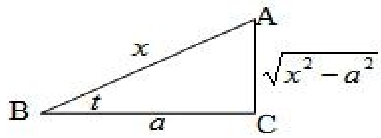
$$=\frac{1}{a^2}\int\frac{1}{cos\,t}\cdot\frac{cos^2t}{sin^2t}dt=\frac{1}{a^2}\int\frac{cos\,t}{sin^2t}dt$$

$$=\frac{1}{a^2}\int\frac{1}{sin^2t}d\,sint$$

$$=-\frac{1}{a^2\,sin\,t}+C$$

在Rt ΔABC中,可设 $\angle B=t$,$|BC|=a$,则$|AB|=x$,$|AC|=\sqrt{x^2-a^2}$

$$\therefore sin\,t=\frac{\sqrt{x^2-a^2}}{x}$$



$$\therefore \int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\cdot\sqrt{x^2-a^2}}+C$$

2 .当 $x<-a$,即 $-x>a$ 时,令 $\mu=-x$,即 $x=-\mu$

$$\therefore \int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\int\frac{d\mu}{\sqrt{(\mu^2-a^2)^3}}$$

由讨论 1可知 $-\int\frac{d\mu}{\sqrt{(\mu^2-a^2)^3}}=\frac{\mu}{a^2\cdot\sqrt{(\mu^2-a^2)}}+C$

将 $\mu=-x$ 代入得:$\int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\cdot\sqrt{x^2-a^2}}+C$

综合讨论 1,2 得:$\int\frac{dx}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\cdot\sqrt{x^2-a^2}}+C$

47. $\int\frac{x}{\sqrt{x^2-a^2}}dx=\sqrt{x^2-a^2}+C \qquad (a>0)$

证明:$\int\frac{x}{\sqrt{x^2-a^2}}dx=\frac{1}{2}\int(x^2-a^2)^{-\frac{1}{2}}dx^2$

$$=\frac{1}{2}\int(x^2-a^2)^{-\frac{1}{2}}d(x^2-a^2)$$

$$=\frac{1}{2}\times\frac{1}{1-\frac{1}{2}}(x^2-a^2)^{1-\frac{1}{2}}+C$$

$$=\sqrt{x^2-a^2}+C$$

48. $\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = -\frac{1}{\sqrt{x^2-a^2}} + C \qquad (a>0)$

证明：被积函数 $f(x) = \frac{x}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x \mid x > a 或 x < -a\}$

1. 当 $x > a$ 时，可设 $x = a \cdot sect \quad (0 < t < \frac{\pi}{2})$，则 $dx = a \cdot sect \cdot tantdt$

$$\frac{x}{\sqrt{(x^2-a^2)^3}} = \frac{a \cdot sect}{\left|a^3 \cdot tan^3 t\right|} \quad \because 0 < t < \frac{\pi}{2}, \quad \frac{x}{\sqrt{(x^2-a^2)^3}} = \frac{sect}{a^2 \cdot tan^3 t}$$

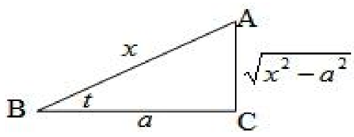
$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = \int \frac{sect}{a^2 \cdot tan^3 t} \cdot a \cdot sect \cdot tant\, dt$$

$$= \frac{1}{a}\int \frac{sec^2 t}{tan^2 t}dt = \frac{1}{a}\int \frac{1}{sin^2 t}dt$$

$$= -\frac{1}{a}\int -csc^2 tdt = -\frac{1}{a} \cdot cot\, t + C$$

在Rt ΔABC中，可设 $\angle B = t$，$|BC| = a$，则 $|AB| = x$，$|AC| = \sqrt{x^2-a^2}$

$$\therefore \ cot\, t = \frac{a}{\sqrt{x^2-a^2}}$$



$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = -\frac{1}{a} \cdot \frac{a}{\sqrt{x^2-a^2}} + C = -\frac{1}{\sqrt{x^2-a^2}} + C$$

2. 当 $x < -a$，即 $-x > a$ 时，令 $\mu = -x$，即 $x = -\mu$

$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = \int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}}d\mu$$

由讨论 1可知 $\int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}}d\mu = -\frac{1}{\sqrt{\mu^2-a^2}} + C$

将 $\mu = -x$ 代入得：$\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = -\frac{1}{\sqrt{x^2-a^2}} + C$

综合讨论 1,2 得：$\int \frac{x}{\sqrt{(x^2-a^2)^3}}dx = -\frac{1}{\sqrt{x^2-a^2}} + C$

49. $\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2} \cdot ln\left| x + \sqrt{x^2-a^2} \right| + C \qquad (a>0)$

证明：$\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$

$$= \int (\sqrt{x^2-a^2} + \frac{a^2}{\sqrt{x^2-a^2}})dx$$

$$= \int \sqrt{x^2-a^2}dx + a^2\int \frac{1}{\sqrt{x^2-a^2}}dx$$

$\because \int \sqrt{x^2-a^2}\, dx = \frac{x}{2} \cdot \sqrt{x^2-a^2} - \frac{a^2}{2} \cdot ln\left| x + \sqrt{x^2-a^2} \right| + C$ ① (公式53)

$a^2\int \frac{dx}{\sqrt{x^2-a^2}} = a^2 \cdot ln\left| x + \sqrt{x^2-a^2} \right| + C$ ② (公式45)

$\therefore$ 由①+②得：$\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2} \cdot ln\left| x + \sqrt{x^2-a^2} \right| + C$

50. $\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx = -\frac{x}{\sqrt{x^2-a^2}} + ln\left| x+\sqrt{x^2-a^2} \right| + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{x^2}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x \mid x>a 或 x<-a\}$

1. 当 $x>a$ 时，可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$，则 $dx=a\cdot sect\cdot tantdt$

$$\frac{x^2}{\sqrt{(x^2-a^2)^3}}=\frac{a^2\cdot sec^2t}{\left|a^3\cdot tan^3 t\right|} \because 0<t<\frac{\pi}{2}, \therefore \frac{x^2}{\sqrt{(x^2-a^2)^3}}=\frac{sec^2t}{a\cdot tan^3 t}$$

$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=\int \frac{sec^2 t}{a\cdot tan^3 t}\cdot a\cdot sect\cdot tant\, dt=\int\frac{sec^3 t}{tan^2 t}dt=\int\frac{1}{cos^3 t}\cdot\frac{cos^2 t}{sin^2 t}dt=\int\frac{1}{sin^2 t\cdot cos\, t}dt$$

$$=\int\frac{cos\, t}{sin^2 t\cdot cos^2 t}dt=\int\frac{1}{sin^2 t\,(1-sin^2 t)}d\, sin\, t=\int(\frac{1}{sin^2 t}+\frac{1}{1-sin^2 t})d\, sin\, t$$

$$=\int\frac{1}{sin^2 t}d\, sin\, t+\int\frac{1}{1-sin^2 t}d\, sin\, t=\int\frac{1}{sin^2 t}d\, sin\, t-\int\frac{1}{sin^2 t-1}d\, sin\, t$$

$$=\int\frac{1}{sin^2 t}d\, sin\, t+\frac{1}{2}\int(\frac{1}{sin\, t+1}-\frac{1}{sin\, t-1})d\, sin\, t$$

$$=\int\frac{1}{sin^2 t}d\, sin\, t+\frac{1}{2}\int\frac{1}{sin\, t+1}d(sin\, t+1)-\frac{1}{2}\int\frac{1}{sin\, t-1}d(sin\, t-1)$$

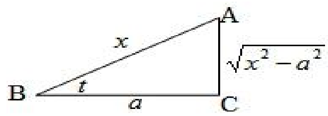
$$=-\frac{1}{sin\, t}+\frac{1}{2}ln\left|sin\, t+1\right|-\frac{1}{2}ln\left|sin\, t-1\right|+C_1$$

$$=-\frac{1}{sin\, t}+\frac{1}{2}ln\left|\frac{sin\, t+1}{sin\, t-1}\right|+C_1=-\frac{1}{sin\, t}+\frac{1}{2}ln\left|\frac{(sin\, t+1)^2}{sin^2 t-1}\right|+C_1$$

$$=-\frac{1}{sin\, t}+\frac{1}{2}ln\left|\frac{(sin\, t+1)^2}{cos^2 t}\cdot(-1)\right|+C_1=-\frac{1}{sin\, t}+ln\left|tan\, t+sec\, t\right|+C_2$$

在Rt ΔABC 中，可设 $\angle B=t, |BC|=a$，则 $|AB|=x, |AC|=\sqrt{x^2-a^2}$

$$\therefore \quad sin\, t=\frac{\sqrt{x^2-a^2}}{x}, \; tan\, t=\frac{\sqrt{x^2-a^2}}{a}, \; sec\, t=\frac{x}{a}$$



$$\therefore \int\frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C_2=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$$

2. 当 $x<-a$，即 $-x>a$ 时，令 $\mu=-x$，即 $x=-\mu$

$$\therefore \int\frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\int\frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}}d\mu$$

由讨论 1可知 $-\int\frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}}d\mu=\frac{\mu}{\sqrt{\mu^2-a^2}}-ln\left|\mu+\sqrt{\mu^2-a^2}\right|+C$

将 $\mu=-x$ 代入得：$\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|-x+\sqrt{x^2-a^2}\right|+C$

$$=\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|\frac{(\sqrt{x^2-a^2}-x)(\sqrt{x^2-a^2}+x)}{\sqrt{x^2-a^2}+x}\right|+C$$

$$=-\frac{x}{\sqrt{x^2-a^2}}-ln\left|\frac{x^2-a^2-x^2}{(\sqrt{x^2-a^2}+x)}\right|+C=-\frac{x}{\sqrt{x^2-a^2}}+2\,ln\, a-ln\left|\frac{1}{\sqrt{x^2-a^2}+x}\right|+C$$

提示：$log_a b^n = n log_a b$

$$=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$$

综合讨论 1,2 得：$\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}dx=-\frac{x}{\sqrt{x^2-a^2}}+ln\left|x+\sqrt{x^2-a^2}\right|+C$

51. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\,\frac{a}{|x|} + C \qquad (a>0)$

证法1:被积函数$f(x)=\frac{1}{x\sqrt{x^2-a^2}}$的定义域为$\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时,可设$x=a\cdot sect \quad (0<t<\frac{\pi}{2})$,则

$$x\sqrt{x^2-a^2} = a^2\cdot sect\sqrt{sec^2t-1} = a^2 sect\cdot tant \;, dx = a\cdot sect\cdot tant\,d\,t$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\cdot sect\cdot tant}{a^2 sect\cdot tant}dt = \int \frac{1}{a}dt$$

$$= \frac{1}{a}t + C_1$$

$$\because x = a\cdot sect, \therefore cost = \frac{a}{x}, \therefore \; t = arccos\frac{a}{x}$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\frac{a}{x} + C$$

2 .当$x<-a$,即$-x>a$时，令 $\mu=-x$,即$x=-\mu$

由讨论1可知$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{d\mu}{\mu\sqrt{\mu^2-a^2}} = \frac{1}{a}\cdot arccos\frac{a}{\mu} + C_2$

$$= \frac{1}{a}\cdot arccos\,\frac{a}{-x} + C$$

综合讨论1,2，可写成$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cdot arccos\,\frac{a}{|x|} + C$

51. $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\cdot arccos\ \frac{a}{|x|}+C \qquad (a>0)$

证法2：被积函数 $f(x)=\dfrac{1}{x\sqrt{x^2-a^2}}$ 的定义域为 $\{x\,|\,x>a或x<-a\}$

1. 当$x>a$时，可设$x=a\cdot cht \quad (0<t)$，则

$x\sqrt{x^2-a^2}=a\cdot cht\cdot a\cdot sht=a^2cht\cdot sht\ , dx=a\cdot sht\,dt$

$$\therefore \int\frac{dx}{x\sqrt{x^2-a^2}}=\int\frac{a\cdot sht}{a\cdot cht\cdot sht}dt=\int\frac{1}{a}\cdot\frac{1}{cht}dt$$

$$=\frac{1}{a}\int\frac{cht}{ch^2t}dt=\frac{1}{a}\int\frac{1}{1+sh^2t}dsht$$

$$=\frac{1}{a}\cdot arctan(sht)+C$$

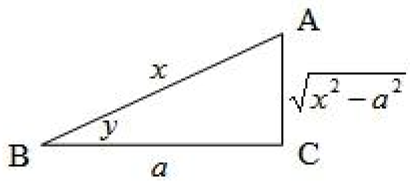
$\boxed{公式\ 19：\int\frac{dx}{x^2+a^2}=\frac{1}{a}arctan\ \frac{x}{a}+C}$

$\because\ x=a\cdot cht,\ \therefore\ cht=\frac{x}{a},\ \therefore\ \ sht=\sqrt{1-ch^2t}=\frac{\sqrt{x^2-a^2}}{a}$

在$Rt\Delta ABC$中，设 $tany=sht=\dfrac{\sqrt{x^2-a^2}}{a},\angle B=y,|BC|=a$

$\therefore\ y=arctan(sht)\ ,|AC|=\sqrt{x^2-a^2}\ ,|AB|=\sqrt{|AC|^2+|BC|^2}=x$

$\therefore\ cosy=\dfrac{|BC|}{|AB|}=\dfrac{a}{x}$



即$cosy=cos\,arctan(sht)=\dfrac{a}{x}$

$\therefore\ arctan(sht)=arccos\dfrac{a}{x}+C$

$$\therefore\ \int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\cdot arctan(sht)+C=\frac{1}{a}\cdot arccos\ \frac{a}{x}+C$$

2 .当$x<-a$，即$-x>a$时，令 $\mu=-x$，即$x=-\mu$

由讨论 1可知 $\int\dfrac{dx}{x\sqrt{x^2-a^2}}=\int\dfrac{d\mu}{\mu\sqrt{\mu^2-a^2}}=\dfrac{1}{a}\cdot arccos\dfrac{a}{\mu}+C_2$

$$=\frac{1}{a}\cdot arccos\frac{a}{-x}+C$$

综合讨论1,2，可写成 $\int\dfrac{dx}{x\sqrt{x^2-a^2}}=\dfrac{1}{a}\cdot arccos\ \dfrac{a}{|x|}+C$

52. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} + C \qquad (a>0)$

证明:被积函数 $f(x) = \frac{1}{x^2\sqrt{x^2-a^2}}$的定义域为 $\{x \mid x>a或x<-a\}$

1. 当$x>a$时,可设$x=\frac{1}{t} \quad (0<t<\frac{1}{a})$,则$dx = -\frac{1}{t^2}dt$, $\frac{1}{x^2\sqrt{x^2-a^2}} = \frac{t^3}{\sqrt{1-a^2t^2}}$

$$\therefore \int \frac{dx}{x^2\sqrt{x^2-a^2}} = \int \frac{t^3}{\sqrt{1-a^2t^2}}\cdot(-\frac{1}{t^2})dt$$

$$= -\int \frac{t}{\sqrt{1-a^2t^2}}dt \quad = -\frac{1}{2}\int (1-a^2t^2)^{-\frac{1}{2}}dt^2$$

$$= \frac{1}{2a^2}\int (1-a^2t^2)^{-\frac{1}{2}}d(1-a^2t^2) \quad = \frac{1}{2a^2}\cdot\frac{1}{1-\frac{1}{2}}\cdot(1-a^2t^2)^{1-\frac{1}{2}} + C$$

$$= \frac{\sqrt{1-a^2t^2}}{a^2} + C$$

将$x=\frac{1}{t}$,即$t=\frac{1}{x}$代入上式得: $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{1}{a^2}\cdot\sqrt{1-a^2(\frac{1}{x})^2} + C = \frac{1}{a^2}\cdot\sqrt{\frac{x^2-a^2}{x^2}} + C$

$$= \frac{1}{a^2}\cdot\frac{\sqrt{x^2-a^2}}{|x|} + C$$

$$\because x>a>0 \quad \therefore \int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} + C$$

2 .当$x<-a$,即$-x>a$时，令 $\mu=-x$,即$x=-\mu$

由讨论 1可知$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = -\int \frac{d\mu}{\mu^2\sqrt{\mu^2-a^2}} = -\frac{\sqrt{\mu^2-a^2}}{a^2\mu} + C$

将$\mu=-x$代入上式得:$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} + C$

综合讨论 1,2 得:$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} + C$

53. $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C \qquad (a>0)$

证明:被积函数 $f(x)=\sqrt{x^2-a^2}$的定义域为 $\{x\mid x>a或x<-a\}$

1. 当 $x>a$时，可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$，则$\sqrt{x^2-a^2}=\left|a\cdot tan\,t\right|$

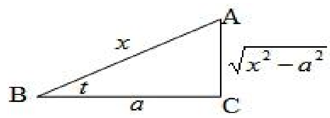
$\because 0<t<\frac{\pi}{2}$，$\therefore \sqrt{x^2-a^2}=a\cdot tan\,t$

$$\begin{aligned}\therefore \int\sqrt{x^2-a^2}\,dx &=\int a\cdot tan\,t\,d\,(a\cdot sect)=a^2\int tan\,t\,d\,sec\,t\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,d\,tan\,t\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec^3\,t\,dt\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,(1+tan^2\,t)\,dt\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,dt-a^2\int sec\,t\,tan^2\,t\,dt\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\int sec\,t\,dt-a^2\int tan\,t\,d\,sec\,t\\ &=a^2\cdot tan\,t\cdot sec\,t-a^2\cdot ln\left|sec\,t+tan\,t\right|-a^2\int tan\,t\,d\,sec\,t\end{aligned}$$

移项并整理得：$a^2\int tan\,t\,d\,sec\,t=\frac{a^2}{2}\cdot tan\,t\cdot sec\,t-\frac{a^2}{2}\cdot ln\left|sec\,t+tan\,t\right|+C_1$

在Rt ΔABC中，可设 $\angle B=t$，$|BC|=a$，则$|AB|=x$，$|AC|=\sqrt{x^2-a^2}$

$\therefore\ tan\,t=\frac{\sqrt{x^2-a^2}}{a}$，$sec\,t=\frac{x}{a}$



$$\begin{aligned}\therefore \int\sqrt{x^2-a^2}\,dx &=a^2\int tan\,t\,d\,sec\,t\\ &=\frac{a^2}{2}\cdot\frac{\sqrt{x^2-a^2}}{a}\cdot\frac{x}{a}-\frac{a^2}{2}\cdot ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|+C_1\\ &=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C\end{aligned}$$

2 .当 $x<-a$时，可设 $x=a\cdot sect \quad (-\frac{\pi}{2}<t<0)$ 同理可证

综合讨论 1,2 得：$\int\sqrt{x^2-a^2}\,dx==\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$

54. $\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{8}\cdot(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C\qquad(a>0)$

证明: $\int\sqrt{(x^2-a^2)^3}\,dx=x\cdot(x^2-a^2)^{\frac{3}{2}}-\int xd\,(x^2-a^2)^{\frac{3}{2}}$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-\int x\cdot\frac{3}{2}\cdot(2x)\cdot(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int x^2(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int(x^2-a^2+a^2)(x^2-a^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(x^2-a^2)^{\frac{3}{2}}-3\int(x^2-a^2)^{\frac{3}{2}}d\,x-3a^2\int(x^2-a^2)^{\frac{1}{2}}d\,x$$

移项并整理得: $\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{4}\cdot(x^2-a^2)^{\frac{3}{2}}-\frac{3a^2}{4}\int(x^2-a^2)^{\frac{1}{2}}d\,x$ ①

又 $\int(x^2-a^2)^{\frac{1}{2}}d\,x=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$ (公式 53) ②

联立①②得:

$$\int\sqrt{(x^2-a^2)^3}\,dx=\frac{x}{4}(x^2-a^2)^{\frac{3}{2}}-\frac{3x}{8}\cdot a^2\cdot\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$=(\frac{x^3}{4}-\frac{a^2x}{4})\sqrt{x^2-a^2}-\frac{3x}{8}\cdot a^2\cdot\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$=\frac{x}{8}\cdot(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3}{8}\cdot a^4\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

55. $\int x\sqrt{x^2-a^2}\,dx=\frac{1}{3}\sqrt{(x^2-a^2)^3}+\mathrm{C}\qquad(a>0)$

证明: $\int x\sqrt{x^2-a^2}\,dx=\frac{1}{2}\int\sqrt{x^2-a^2}\,dx^2$

$$=\frac{1}{2}\int(x^2-a^2)^{\frac{1}{2}}\,d\,(x^2-a^2)$$

$$=\frac{1}{2}\times\frac{1}{1+\frac{1}{2}}\cdot(x^2-a^2)^{1+\frac{1}{2}}+C$$

$$=\frac{1}{3}\sqrt{(x^2-a^2)^3}+\mathrm{C}$$

56. $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\cdot ln\left|x+\sqrt{x^2-a^2}\right| + C \qquad (a>0)$

证明：被积函数 $f(x)=x^2\sqrt{x^2-a^2}$的定义域为 $\{x\mid x>a或x<-a\}$

1.当$x>a$时，可令$x=a\cdot sect \quad (0<t<\frac{\pi}{2})$， 则$x^2\sqrt{x^2-a^2}=a^2sec^2t\,|\,a\,tant\,|$

$\because\ 0<t<\frac{\pi}{2}, tan\,t>0$， $\therefore\ x^2\sqrt{x^2-a^2}=a^3sec^2t\cdot tant$

$$\therefore\ \int x^2\sqrt{x^2-a^2}\,dx=\int a^3sec^2t\cdot tant\ d\,(a\ sect)=a^4\int sec^3t\ d\ tan^2t\ dt \qquad ①$$

$$=\frac{a^4}{3}\int sect\cdot 3\cdot sec^2t\cdot tan^2t\ dt=\frac{a^4}{3}\int sect\ d\ tan^3t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3t-\frac{a^4}{3}\int tan^3t\ d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3t-\frac{a^4}{3}\int tan\,t\,(sec^2t-1)d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3t-\frac{a^4}{3}\int tan\,t\cdot sec^2td\ sec\,t+\frac{a^4}{3}\int tan\,t\ d\ sec\,t$$

$$=\frac{a^4}{3}\cdot sec\,t\cdot tan^3t-\frac{a^4}{3}\int tan^2t\cdot sec^3t\ dt+\frac{a^4}{3}\int tan\,t\ d\ sec\,t$$

移项并整理得： $a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}\cdot sec\,t\cdot tan^3t+\frac{a^4}{4}\int tan\,t\ d\ sec\,t \qquad ②$

又 $\int tant\ d\ sect=sect\cdot tant-\int sect\ dtant\ =sect\cdot tant-\int sec^3tdt$

$$=sect\cdot tant-\int(1+tan^2t)\cdot sectdt$$

$$=sect\cdot tant-\int sect\ dt-\int tan^2t\ sect\ dt$$

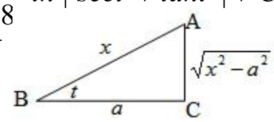
$$=sect\cdot tant-\int sect\ dt-\int tant\ d\ sect$$

移项并整理得： $\int tant\ d\ sect=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}\int sect\ dt$

$$=\frac{1}{2}\cdot sect\cdot tant-\frac{1}{2}ln\left|sect+tant\right|+C_1 \qquad ③$$

将③式代入②式得： $a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}tan^3t\cdot sect+\frac{a^4}{8}sect\cdot tant-\frac{a^4}{8}ln\left|sect+tant\right|+C_1$

在RtΔABC中，设 $\angle B=t$，$|BC|=a$，则$|AB|=x$，$|AC|=\sqrt{x^2-a^2}$



$\therefore\ tant=\frac{\sqrt{x^2-a^2}}{a}, sect=\frac{1}{cos\,t}=\frac{x}{a}$

$$\therefore\ a^4\int sec^3t\ d\ tan^2t\ dt=\frac{a^4}{4}\cdot\frac{x}{a}\cdot\frac{x^2-a^2}{a^3}\cdot\sqrt{x^2-a^2}+\frac{a^4}{8}\cdot\frac{\sqrt{x^2-a^2}}{a}\cdot\frac{x}{a}-\frac{a^4}{8}\cdot ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\frac{x}{4}\cdot(x^2-a^2)\cdot\sqrt{x^2-a^2}+\frac{a^2x}{8}\cdot\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C_2$$

$$=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$$

2. 当$x<-a$,即$-x>a$时， 令$\mu=-x$,则$x=-\mu$

由讨论 1得：$\int x^2\sqrt{x^2-a^2}\,dx=-\int\mu^2\sqrt{\mu^2-a^2}\,d\mu$

$$=\frac{-\mu}{8}\cdot(2\mu^2-a^2)\sqrt{\mu^2-a^2}+\frac{a^4}{8}\cdot ln\left|-\mu+\sqrt{\mu^2-a^2}\right|+C$$

将$\mu=-x$代入上式得： $\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$

综合讨论 1,2得：$\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C$

57. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C \qquad (a>0)$

证法1：被积函数 $f(x)=\frac{\sqrt{x^2-a^2}}{x}$ 的定义域为 $\{x \mid x>a或x<-a\}$

1. 当 $x>a$ 时，可设 $x=a\cdot sect \quad (0<t<\frac{\pi}{2})$，

则 $\frac{\sqrt{x^2-a^2}}{x}=\frac{a\cdot tant}{a\cdot sect}$， $\quad dx = a\cdot sect\cdot tant\ d\ t$

$$\therefore \int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{a\cdot tant\cdot a\cdot sect\cdot tant}{a\cdot sect}dt = \int a\cdot tan^2 tdt$$

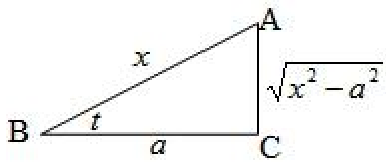
$$= a\int \frac{sin^2 t}{cos^2 t}dt = a\int \frac{1-cos^2 t}{cos^2 t}dt = a\int \frac{1}{cos^2 t}dt - \int dt$$

$$= a\cdot tant - a\cdot t + C$$

$\because \ x=a\cdot sect, \ \therefore \ cost=\frac{a}{x}, \ \therefore \ t = arccos\ \frac{a}{x}$

在 $Rt\Delta ABC$ 中，设 $\angle B=t, |BC|=a$，则 $|AB|=x$，$|AC|=\sqrt{x^2-a^2}$

$\therefore \ tant = \frac{|AC|}{|BC|}=\frac{\sqrt{x^2-a^2}}{a}$



$$\therefore \int \frac{\sqrt{x^2-a^2}}{x}dx = a\cdot tant - a\cdot t + C$$

$$= \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{x} + C$$

2. 当 $x<-a$，即 $-x>a$ 时，令 $\mu=-x$，即 $x=-\mu$

由讨论 1可知 $\int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{\sqrt{\mu^2-a^2}}{\mu}d\ \mu = \sqrt{\mu^2-a^2} - a\cdot arccos\ \frac{a}{\mu} + C$

$$= \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{-x} + C$$

综合讨论 1,2，可写成：$\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C$

57. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C \qquad (a>0)$

证法 2：被积函数 $f(x)=\frac{\sqrt{x^2-a^2}}{x}$ 的定义域为 $\{x \mid x>a 或 x<-a\}$

1．当 $x>a$ 时，可设 $x=a\cdot ch\,t \quad (0<t)$，

则 $\frac{\sqrt{x^2-a^2}}{x}=\frac{a\cdot sht}{a\cdot cht}=\frac{sht}{cht}$，$dx=a\cdot sht\ dt$

$$\therefore \int \frac{\sqrt{x^2-a^2}}{x}dx=\int \frac{sht}{cht}\cdot a\cdot sht\ dt = a\int \frac{sh^2t}{cht}dt$$

$$= a\int \frac{ch^2t-1}{cht}dt = a\int chtdt - a\int \frac{cht}{ch^2t}dt$$

$$= a\int chtdt - a\int \frac{1}{1+sh^2t}dsht$$

$$= a\cdot sht - a\cdot arctan\,(sht) + C$$

提示：$ch^2t - sh^2t = 1$
$(cht)' = sht$
$(sh\,t)' = cht$

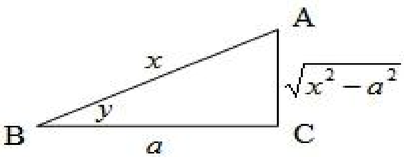
公式 19：$\int \frac{dx}{x^2+a^2}=\frac{1}{a}\cdot arctan\ \frac{x}{a}+C$

$\because\ x=a\cdot cht,\ \therefore\ cht=\frac{x}{a},\ \therefore\ sht=\sqrt{1-ch^2t}=\frac{\sqrt{x^2-a^2}}{a}$

在 RtΔABC 中，设 $tany = sht = \frac{\sqrt{x^2-a^2}}{a}, \angle B = y, |BC| = a$

$\therefore\ y = arctan\ (sht), |AC| = \sqrt{x^2-a^2}, |AB| = \sqrt{|AC|^2+|BC|^2} = x$

$\therefore\ cosy = \frac{|BC|}{|AB|} = \frac{a}{x}$



即 $cosy = cos\ arctan\,(sht) = \frac{a}{x}$

$\therefore\ arctan\,(sht) = arccos\ \frac{a}{x}$

$$\therefore\ \int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{x} + C$$

2．当 $x<-a$，即 $-x>a$ 时，令 $\mu=-x$，即 $x=-\mu$

由讨论 1 可知 $\int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{\sqrt{\mu^2-a^2}}{\mu}d\mu = \sqrt{\mu^2-a^2} - a\cdot arccos\ \frac{a}{\mu} + C$

$$= \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{-x} + C$$

综合讨论 1，2，可写成：$\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\cdot arccos\ \frac{a}{|x|} + C$

58. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x} + ln\left|x+\sqrt{x^2-a^2}\right| + C \qquad (a>0)$

证明：$\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\int \sqrt{x^2-a^2}\,d\frac{1}{x}$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{x} d\sqrt{x^2-a^2}$$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{x}\cdot\frac{1}{2}\cdot 2x\cdot(x^2-a^2)^{-\frac{1}{2}}dx$$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \int \frac{1}{\sqrt{x^2-a^2}}dx$$

$\boxed{\text{公式 45:} \int \frac{dx}{\sqrt{x^2-a^2}} = ln\left|x+\sqrt{x^2-a^2}\right| + C}$

$$= -\frac{\sqrt{x^2-a^2}}{x} + ln\left|x+\sqrt{x^2-a^2}\right| + C$$

（八）含有 $\sqrt{a^2-x^2}$ $(a>0)$ 的积分（59~72）

59. $\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{1}{\sqrt{a^2-x^2}}$ 的定义域为 $\{x\,|-a<x<a\}$

$\therefore$ 可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx=a\cdot cost\,dt$ ，$\frac{1}{\sqrt{a^2-x^2}}=\frac{1}{|a\cdot cost|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$cost>0$ $\therefore$ $\frac{1}{\sqrt{a^2-x^2}}=\frac{1}{a\cdot cost}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{a\cdot cost}\cdot a\cdot cost\,dt$

$$= \int dt$$

$$= t + C$$

$\because x=a\cdot sint \quad \therefore t=arcsin\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{a} + C$

60. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2 \cdot \sqrt{a^2-x^2}} + C \qquad (a>0)$

证明:被积函数 $f(x) = \frac{1}{\sqrt{(a^2-x^2)^3}}$ 的定义域为 $\{x \mid -a<x<a\}$

$\therefore$ 可设 $x = a \cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx = a \cdot cost\,dt$ ，$\frac{1}{\sqrt{(a^2-x^2)^3}} = \frac{1}{|a^3 \cdot cos^3 t|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$cost>0$ $\therefore$ $\frac{1}{\sqrt{(a^2-x^2)^3}} = \frac{1}{a^3 \cdot cos^3 t}$

$$\therefore \int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \int \frac{1}{a^3 \cdot cos^3 t} \cdot a \cdot cost\,dt$$

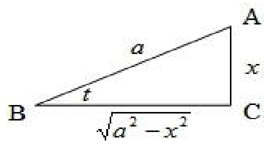
$$= \int \frac{1}{a^2 \cdot cos^2 t} dt$$

$$= \int \frac{1}{a^2} \cdot sec^2 t\,dt$$

$$= \frac{1}{a^2} \cdot tant + C$$

在Rt ΔABC中，设 $\angle B = t$,$|AB| = a$,则 $|AC| = x$,$|BC| = \sqrt{a^2-x^2}$

$\therefore$ $tant = \frac{x}{\sqrt{a^2-x^2}}$



$$\therefore \int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2 \cdot \sqrt{a^2-x^2}} + C$$

61. $\int \frac{x}{\sqrt{a^2-x^2}} dx = -\sqrt{a^2-x^2} + \mathrm{C} \qquad (a>0)$

证明:$\int \frac{x}{\sqrt{a^2-x^2}} dx = \frac{1}{2} \int (a^2-x^2)^{-\frac{1}{2}} dx^2$

$$= -\frac{1}{2} \int (a^2-x^2)^{-\frac{1}{2}} d(a^2-x^2)$$

$$= -\frac{1}{2} \times \frac{1}{1-\frac{1}{2}} \cdot (a^2-x)^{1-\frac{1}{2}} + C$$

$$= -\sqrt{a^2-x^2} + \mathrm{C}$$

62. $\int \frac{x}{\sqrt{(a^2-x^2)^3}}dx = \frac{1}{\sqrt{a^2-x^2}} + \mathrm{C} \qquad (a>0)$

证明：$\int \frac{x}{\sqrt{(a^2-x^2)^3}}dx = \frac{1}{2}\int (a^2-x^2)^{-\frac{3}{2}}dx^2$

$$= -\frac{1}{2}\int (a^2-x^2)^{-\frac{3}{2}}d(a^2-x^2)$$

$$= -\frac{1}{2}\times\frac{1}{1-\frac{3}{2}}\cdot(a^2-x^2)^{1-\frac{3}{2}} + C$$

$$= \frac{1}{\sqrt{a^2-x^2}} + \mathrm{C}$$

63. $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot \arcsin\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x) = \frac{x^2}{\sqrt{a^2-x^2}}$ 的定义域为 $\{x \mid -a<x<a\}$

$\therefore$ 可设 $x = a\cdot \sin t \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx = a\cdot\cos t\,dt$ ，$\frac{x^2}{\sqrt{a^2-x^2}} = \frac{a^2\cdot\sin^2 t}{|a\cdot\cos t|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0 \quad \therefore \quad \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a\cdot\sin^2 t}{\cos t}$

$\therefore \int \frac{x^2}{\sqrt{a^2-x^2}}dx = \int \frac{a\cdot\sin^2 t}{\cos t}\cdot a\cdot\cos t\,dt$

$$= a^2\int \sin^2 t\,dt$$

$$= a^2\int \frac{1-\cos 2t}{2}dt$$

$$= \frac{a^2}{2}\int dt - \frac{a^2}{4}\int \cos 2t\,d(2t)$$

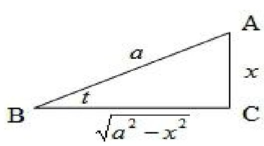
$$= \frac{a^2}{2}\cdot t - \frac{a^2}{4}\cdot\sin 2t + C$$

$$= \frac{a^2}{2}\cdot t - \frac{a^2}{2}\cdot\sin t\cdot\cos t + C$$

提示：$\cos 2t = \cos^2 t - \sin^2 t$
$= 1 - 2\sin^2 t$
$\sin 2t = 2\cdot\sin t\cdot\cos t$

在Rt ΔABC中，设 $\angle B = t$,$|AB| = a$,则 $|AC| = x$,$|BC| = \sqrt{a^2-x^2}$

$\therefore \sin t = \frac{x}{a}$ ，$\cos t = \frac{\sqrt{a^2-x^2}}{a}$



$\therefore \int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\arcsin\frac{x}{a} + C$

64. $\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\,dx = \frac{x}{\sqrt{a^2-x^2}} - arcsin\,\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{x^2}{\sqrt{(a^2-x^2)^3}}$ 的定义域为 $\{x\,|-a<x<a\}$

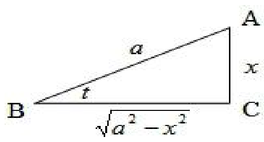
$\therefore$ 可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx=a\cdot\cos t\,dt$ ，$\frac{x^2}{\sqrt{(a^2-x^2)^3}}=\frac{a^2\cdot sin^2t}{|a^3\cdot cos^3t|}$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0$ $\therefore$ $\frac{x^2}{\sqrt{(a^2-x^2)^3}}=\frac{sin^2t}{a\cdot cos^3t}$

$$\begin{aligned}
\therefore \int\frac{x^2}{\sqrt{a^2-x^2}}\,dx &= \int\frac{sin^2t}{a\cdot cos^3t}\cdot a\cdot\cos t\,dt \\
&= \int\frac{sin^2t}{cos^2t}\,dt \\
&= \int\frac{1-cos^2t}{cos^2t}\,dt \\
&= \int\frac{1}{cos^2t}\,dt-\int dt \\
&= \int d\,tant-\int dt \\
&= tant-t+C
\end{aligned}$$

在Rt ΔABC中，设 $\angle B=t\,,|AB|=a$，则 $|AC|=x\,,|BC|=\sqrt{a^2-x^2}$

$\therefore\ tant=\frac{x}{\sqrt{a^2-x^2}}$



$\therefore\ \int\frac{x^2}{\sqrt{(a^2-x^2)^3}}\,dx=\frac{x}{\sqrt{a^2-x^2}}-arcsin\,\frac{x}{a}+C$

65. $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C \qquad (a>0)$

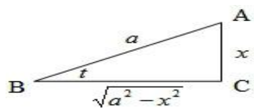
证明：被积函数 $f(x) = \frac{1}{x\sqrt{a^2-x^2}}$ 的定义域为 $\{x \mid -a < x < a 且 x \neq 0\}$

1.当 $-a < x < 0$ 时，可设 $x = a \cdot sint \quad (-\frac{\pi}{2} < t < 0)$，则 $dx = a \cdot cos\, t\, dt$

$x\sqrt{a^2-x^2} = a \cdot sint \cdot |a \cdot cos\, t| \quad \because -\frac{\pi}{2} < t < 0,\ cos\, t > 0 \ \therefore\ x\sqrt{a^2-x^2} = a^2 \cdot sint \cdot cos\, t$

$$\begin{aligned}
\therefore \int \frac{dx}{x\sqrt{a^2-x^2}} &= \int \frac{1}{a^2 \cdot sint \cdot cos\, t} \cdot a \cdot cos\, t\, dt \\
&= \frac{1}{a}\int \frac{1}{sint} dt \\
&= \frac{1}{a}\int \frac{sint}{sin^2 t} dt \\
&= -\frac{1}{a}\int \frac{1}{1-cos^2 t} d\, cos\, t \\
&= -\frac{1}{2a}\int \left(\frac{1}{1+cost} + \frac{1}{1-cost}\right) d\, cos\, t \\
&= -\frac{1}{2a}\int \frac{1}{1+cost} d(cos\, t+1) + \frac{1}{2a}\int \frac{1}{1-cost} d(1-cos\, t) \\
&= -\frac{1}{2a} \cdot ln|1+cost| + \frac{1}{2a} \cdot ln|cost-1| + C_1 \\
&= \frac{1}{2a} \cdot ln\left|\frac{cost-1}{1+cost}\right| + C_1 \\
&= \frac{1}{2a} \cdot ln\left|\frac{(cost-1)^2}{1-cos^2 t} \cdot (-1)\right| + C_1 \\
&= \frac{1}{2a} \cdot ln\left|\frac{(cost-1)^2}{sin^2 t}\right| + C_2 \\
&= \frac{1}{a} \cdot ln\left|\frac{cost-1}{sint}\right| + C_2 \\
&= \frac{1}{a} \cdot ln|cott - csct| + C_2
\end{aligned}$$

在Rt ΔABC 中，设 $\angle B = t, |AB| = a$，则 $|AC| = x, |BC| = \sqrt{a^2-x^2}$



$\therefore\ cott = \frac{\sqrt{a^2-x^2}}{x},\ csct = \frac{1}{sint} = \frac{a}{x}$

$$\begin{aligned}
\therefore \int \frac{dx}{x\sqrt{a^2-x^2}} &= \frac{1}{a} \cdot ln\left|\frac{\sqrt{a^2-x^2}-a}{x}\right| + C_2 = \frac{1}{a} \cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x} \cdot (-1)\right| + C_2 \\
&= \frac{1}{a} \cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + C_3
\end{aligned}$$

$\because\ a-\sqrt{a^2-x^2} > 0$

$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C$

2.当 $0 < x < a$ 时，可设 $x = a \cdot sint \quad (0 < t < \frac{\pi}{2})$，同理可证

综合讨论 1, 2 得：$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C$

66. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{1}{x^2\sqrt{a^2-x^2}}$ 的定义域为 $\{x \mid -a<x<a 且 x\neq 0\}$

1.当 $-a<x<0$ 时，可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$，则 $dx=a\cdot\cos t\,dt$ ，

$$\frac{1}{x^2\sqrt{a^2-x^2}} = \frac{1}{a^2\cdot sin^2 t}\cdot\frac{1}{|a\cdot\cos t|}$$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ， $\cos t>0$ $\therefore$ $\frac{1}{x^2\sqrt{a^2-x^2}} = \frac{1}{a^3\cdot sin^2 t\cdot cost}$

$$\therefore \int \frac{dx}{x^2\sqrt{a^2-x^2}} = \int \frac{1}{a^3\cdot sin^2 t\cdot cost}\cdot a\cdot\cos t\,dt$$

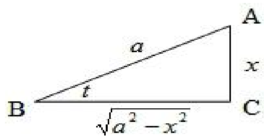
$$= \frac{1}{a^2}\int\frac{1}{sin^2 t}dt$$

$$= -\frac{1}{a^2}\int - csc^2 t\,dt$$

$$= -\frac{1}{a^2}\cdot cott + C$$

在Rt ΔABC中，设 $\angle B=t$,$|AB|=a$,则$|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$$\therefore \quad cott = \frac{\sqrt{a^2-x^2}}{x}$$



$$\therefore \int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$$

2.当 $0<x<a$ 时，可设 $x=a\cdot sint \quad (0<t<\frac{\pi}{2})$,同理可证

综合讨论 1,2 得：$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

67. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot arcsin\,\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x)=\sqrt{a^2-x^2}$ 的定义域为 $\{x\,|\,-a<x<a\}$

$\therefore$ 可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $dx = a\cdot cos\,t\,dt$ ，$\sqrt{a^2-x^2} = |\,a\cdot cos\,t\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$cos\,t>0$ $\therefore$ $\sqrt{a^2-x^2} = a\cdot cos\,t$

$$\begin{aligned}\therefore \int\sqrt{a^2-x^2}\,dx &= \int a\cdot cos\,t\cdot a\cdot cos\,t\,dt\\ &= a^2\int cos^2 t\,dt\\ &= a^2\int(1-sin^2 t)\,dt\\ &= a^2\int dt - a^2\int sin^2 t\,dt \qquad ①\end{aligned}$$

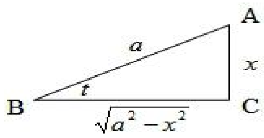
$$\begin{aligned}\text{又}\int\sqrt{a^2-x^2}\,dx &= a^2\int cos^2 t\,dt\\ &= a^2\int cost\ d\ sint\\ &= a^2\cdot sint\cdot cost - a^2\int sint\ d\,cost\\ &= a^2\cdot sint\cdot cost + a^2\int sin^2 t\,dt \qquad ②\end{aligned}$$

由① + ②得：$2\int\sqrt{a^2-x^2}\,dx = a^2\int dt + a^2\cdot sint\cdot cost = a^2 t + a^2\cdot sint\cdot cost$

$\therefore \int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}t + \frac{a^2}{2}\cdot sint\cdot cost + C$

在Rt ΔABC中，设 $\angle B = t$,$|AB|=a$,则 $|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$\therefore \ sint = \frac{x}{a}$ ，$cost = \frac{\sqrt{a^2-x^2}}{a}$



$$\begin{aligned}\therefore \int\sqrt{a^2-x^2}\,dx &= \frac{a^2}{2}\cdot arcsin\,\frac{x}{a} + \frac{a^2}{2}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a} + C\\ &= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot arcsin\,\frac{x}{a} + C\end{aligned}$$

68. $\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{8}\cdot(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C \qquad (a>0)$

证明：$\int\sqrt{(a^2-x^2)^3}\,dx=x\cdot(a^2-x^2)^{\frac{3}{2}}-\int xd\,(a^2-x^2)^{\frac{3}{2}}$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}-\int x\cdot\frac{3}{2}\cdot(-2x)\cdot(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}+3\int x^2(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}+3\int(x^2-a^2+a^2)(a^2-x^2)^{\frac{1}{2}}d\,x$$

$$=x\cdot(a^2-x^2)^{\frac{3}{2}}-3\int(a^2-x^2)^{\frac{3}{2}}d\,x+3a^2\int(a^2-x^2)^{\frac{1}{2}}d\,x$$

移项并整理得：$\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{4}\cdot(a^2-x^2)^{\frac{3}{2}}+\frac{3a^2}{4}\int(a^2-x^2)^{\frac{1}{2}}d\,x$ ①

又$\int(a^2-x^2)^{\frac{1}{2}}d\,x=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C$ (公式 67) ②

联立①②得：

$$\int\sqrt{(a^2-x^2)^3}\,dx=\frac{x}{4}(a^2-x^2)^{\frac{3}{2}}+\frac{3x}{8}\cdot a^2\cdot\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

$$=(\frac{a^2x}{4}-\frac{x^3}{4})\sqrt{a^2-x^2}+\frac{3x}{8}\cdot a^2\cdot\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

$$=\frac{x}{8}\cdot(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}\cdot a^4\cdot arcsin\frac{x}{a}+C$$

69. $\int x\sqrt{a^2-x^2}dx=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C \qquad (a>0)$

证明：被积函数 $f(x)=x\sqrt{a^2-x^2}$ 的定义域为 $\{x\,|-a<x<a\}$

$\therefore$ 可设$x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则$dx=a\cdot\cos t\,dt$ ，$x\sqrt{a^2-x^2}=a\cdot\sin t\cdot|\,a\cdot\cos t\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$ ，$\cos t>0$ $\therefore$ $x\sqrt{a^2-x^2}=a^2\cdot sint\cdot cost$

$\therefore$ $\int x\sqrt{a^2-x^2}dx=\int a^2\cdot sint\cdot cost\cdot a\cdot\cos t\,dt=a^3\int\cos^2t\cdot sint\,dt$

$$=-a^3\int cos^2t\,dcost=-\frac{a^3}{3}cos^3t+C$$

$$=-\frac{a^3}{3}(1-sin^2t)^{\frac{3}{2}}+C$$

$\because$ $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$ ，$\therefore$ $sint=\frac{x}{a}$

$\therefore$ $(1-sin^2t)^{\frac{3}{2}}=(\frac{a^2-x^2}{a^2})^{\frac{3}{2}}=\frac{\sqrt{(a^2-x^2)^3}}{a^3}$

$\therefore$ $\int x\sqrt{a^2-x^2}dx=-\frac{a^3}{3}(1-sin^2t)^{\frac{3}{2}}+C$

$$=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C$$

70. $\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\cdot arcsin\,\frac{x}{a}+C \qquad (a>0)$

证明：被积函数 $f(x)=x^2\sqrt{a^2-x^2}$的定义域为 $\{x\,|\,-a<x<a\}$

$\therefore$ 可令 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<\frac{\pi}{2})$，则 $x^2\sqrt{a^2-x^2}=a^2\cdot sin^2t\,|\,a\cdot cost\,|$

$\because -\frac{\pi}{2}<t<\frac{\pi}{2}$，$\cos t>0$，$\therefore\ x^2\sqrt{a^2-x^2}=a^3\cdot sin^2t\cdot cost$

$$\begin{aligned}\therefore \int x^2\sqrt{a^2-x^2}\,dx &=\int a^3 sin^2t\cdot cost\ d\,(a\cdot sint)=a^4\int sin^2t\cdot cos^2t\,dt\\&=\frac{a^4}{3}\int 3\cdot sin^2t\cdot cos\,t\cdot cos\,t\,dt\\&=\frac{a^4}{3}\int cos\,t\ d\ sin^3t\\&=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sin^3t\,d\,cost\\&=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\cdot(1-cos^2t)\,d\,cost\\&=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\ d\,cost+\frac{a^4}{3}\int sint\cdot cos^2t\,d\,cost\\&=\frac{a^4}{3}\cdot\cos t\cdot sin^3t-\frac{a^4}{3}\int sint\ d\,cost+\frac{a^4}{3}\int sin^2t\cdot cos^2t\,dt\end{aligned}$$

移项并整理得：$\int x^2\sqrt{a^2-x^2}\,dx=a^4\int sin^2t\cdot cos^2t\,dt$

$$=\frac{a^4}{4}\cdot\cos t\cdot sin^3t-\frac{a^4}{4}\int sint\ d\,cost \qquad ①$$

$$\begin{aligned}\because \int sint\ d\ cost &=sint\cdot cost-\int cost\ d\ sint\\&=sint\cdot cost-\int\cos^2t\,dt\\&=sint\cdot cost-\int(1-sin^2t)dt\\&=sint\cdot cost-\int dt+\int sin^2t\,dt \qquad ②\end{aligned}$$

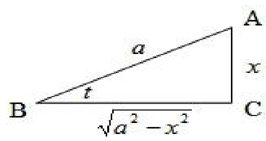
又 $\because \int sint\ d\ cost=-\int sin^2t\,dt \qquad ③$

联立②③得：$\int sint\ d\ cost=\frac{1}{2}\cdot sint\cdot cost-\frac{1}{2}\int dt$

$$=\frac{1}{2}\cdot sint\cdot cost-\frac{1}{2}\cdot t+C_1 \qquad ④$$

联立①④得：$\int x^2\sqrt{a^2-x^2}\,dx=\frac{a^4}{4}\cdot\cos t\cdot sin^3t-\frac{a^4}{8}\cdot sint\cdot cost-\frac{a^4}{8}\cdot t+C$

在Rt$\varDelta ABC$中，设 $\angle B=t$，$|AB|=a$，则 $|AC|=x$，$|BC|=\sqrt{a^2-x^2}$



$\therefore\ cost=\frac{\sqrt{a^2-x^2}}{a}$，$sint=\frac{x}{a}$

$$\begin{aligned}\therefore \int x^2\sqrt{a^2-x^2}\,dx &=\frac{a^4}{4}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x^3}{a^3}-\frac{a^4}{8}\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a}+\frac{a^4}{8}\cdot arcsin\,\frac{x}{a}+C\\&=\frac{x}{8}\cdot(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\cdot arcsin\,\frac{x}{a}+C\end{aligned}$$

71. $\int \frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{1}{x\sqrt{a^2-x^2}}$ 的定义域为 $\{x\,|-a<x<a且x\neq 0\}$

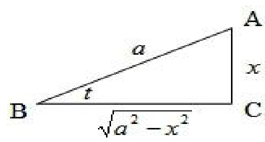
1.当 $-a<x<0$ 时，可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$，则 $dx=a\cdot\cos t\,dt$

$\frac{\sqrt{a^2-x^2}}{x}=\frac{|a\cdot\cos t|}{a\cdot sint}$ $\because -\frac{\pi}{2}<t<0$ ，$\cos t>0$ $\therefore$ $\frac{\sqrt{a^2-x^2}}{x}=\frac{\cos t}{sint}$

$$\begin{aligned}
\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx &= \int\frac{\cos t}{sint}\cdot a\cdot\cos t\,dt=a\int\frac{\cos^2 t}{sint}dt \\
&=a\int\frac{1-sin^2t}{sint}dt=a\int\frac{1}{sint}dt-a\int sint\,dt \\
&=a\int\frac{sint}{sin^2t}dt-a\int sint\,dt=-a\int\frac{1}{1-cos^2t}dcost-a\int sint\,dt \\
&=-\frac{a}{2}\int(\frac{1}{1+cost}+\frac{1}{1-cost})d\cos t-a\int sint\,dt \\
&=-\frac{a}{2}\int\frac{1}{1+cost}d(\cos t+1)+\frac{a}{2}\int\frac{1}{cost-1}d(cost-1)-a\int sint\,dt \\
&=-\frac{a}{2}\cdot ln|1+cost|+\frac{a}{2}\cdot ln|cost-1|+a\cdot\cos t+C_1 \\
&=\frac{a}{2}\cdot ln\left|\frac{cost-1}{1+cost}\right|+a\cdot\cos t+C_1 \\
&=\frac{a}{2}\cdot ln\left|\frac{(cost-1)^2}{1-cos^2t}\cdot(-1)\right|+a\cdot\cos t+C_1 \\
&=\frac{a}{2}\cdot ln\left|\frac{(cost-1)^2}{sin^2t}\right|+a\cdot\cos t+C_2 \\
&=a\cdot ln\left|\frac{cost-1}{sint}\right|+a\cdot\cos t+C_2 \\
&=a\cdot ln|cott-csct|+a\cdot\cos t+C_2
\end{aligned}$$

在Rt ΔABC中，设 $\angle B=t$,$|AB|=a$,则 $|AC|=x$,$|BC|=\sqrt{a^2-x^2}$

$\therefore$ $cott=\frac{\sqrt{a^2-x^2}}{x}$ ，$csct=\frac{1}{sint}=\frac{a}{x}$ ，$cost=\frac{\sqrt{a^2-x^2}}{a}$



$$\begin{aligned}
\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx &= a\cdot ln\left|\frac{\sqrt{a^2-x^2}-a}{x}\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_2 \\
&=a\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\cdot(-1)\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_2 \\
&=\frac{1}{a}\cdot ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+a\cdot\frac{\sqrt{a^2-x^2}}{a}+C_3
\end{aligned}$$

$\because a-\sqrt{a^2-x^2}>0$

$\therefore \int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

2.当 $0<x<a$ 时，可设 $x=a\cdot sint \quad (0<t<\frac{\pi}{2})$,同理可证

综合讨论 1,2 得：$\int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\cdot ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

72. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C \qquad (a>0)$

证明：被积函数 $f(x)=\frac{\sqrt{a^2-x^2}}{x^2}$ 的定义域为 $\{x\,|\,-a<x<a 且 x\neq 0\}$

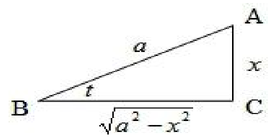
1.当 $-a<x<0$ 时，可设 $x=a\cdot sint \quad (-\frac{\pi}{2}<t<0)$，则 $dx=a\cdot\cos t\,dt$，$\frac{\sqrt{a^2-x^2}}{x^2}=\frac{|\,a\cdot\cos t\,|}{a^2\cdot sin^2 t}$

$\because -\frac{\pi}{2}<t<0$，$\cos t>0$ $\therefore \frac{\sqrt{a^2-x^2}}{x^2}=\frac{\cos t}{a\cdot sin^2 t}$

$$\begin{aligned}\therefore \int \frac{\sqrt{a^2-x^2}}{x^2}dx &= \int \frac{\cos t}{a\cdot sin^2 t}\cdot a\cdot\cos t\,dt \\ &= \int \frac{cos^2 t}{sin^2 t}dt \\ &= \int \frac{1-sin^2 t}{sin^2 t}dt \\ &= \int csc^2 t dt - \int dt \\ &= -cott - t + C\end{aligned}$$

在Rt ΔABC 中，设 $\angle B=t$，$|AB|=a$，则 $|AC|=x$，$|BC|=\sqrt{a^2-x^2}$

$\therefore \ cott = \frac{\sqrt{a^2-x^2}}{x}$



$\therefore \int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C$

2.当 $0<x<a$ 时，可设 $x=a\cdot sint \quad (0<t<\frac{\pi}{2})$，同理可证

综合讨论 1，2 得：$\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - arcsin\frac{x}{a} + C$

（九）含有 $\sqrt{\pm a^2+bx+c}\ (a>0)$ 的积分（73~78）

73． $\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+\text{C}\qquad(a>0)$

证明：若被积函数 $f(x)=\frac{1}{\sqrt{ax^2+bx+c}}$ 成立，则 $ax^2+bx+c>0$ 恒成立

$$\because\ a>0\quad\therefore\ \Delta=b^2-4ac>0$$

$$\because ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+4ac-b^2]$$

$$=\frac{1}{4a}[(2ax+b)^2-(\sqrt{b^2-4ac})^2]$$

$$\therefore\ \int\frac{dx}{\sqrt{ax^2+bx+c}}=2\sqrt{a}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}dx$$

$$=\frac{2\sqrt{a}}{2a}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}d(2ax+b)$$

$$=\frac{1}{\sqrt{a}}\int\frac{1}{\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}}d(2ax+b)$$

$\boxed{公式45:\int\frac{dx}{\sqrt{x^2-a^2}}=ln\,|\,x+\sqrt{x^2-a^2}\,|+C}$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|+C$$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+\sqrt{4a\cdot(ax^2+bx+c)}\right|+C$$

$$=\frac{1}{\sqrt{a}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+\text{C}$$

74. $\int\sqrt{ax^2+bx+c}\,dx=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+\text{C}\qquad(a>0)$

证明：若被积函数 $f(x)=\sqrt{ax^2+bx+c}$ 成立，则 $ax^2+bx+c>0$ 恒成立

$$\because\ a>0\quad\therefore\ \Delta=b^2-4ac>0$$

$$\because ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+4ac-b^2]$$

$$=\frac{1}{4a}[(2ax+b)^2-(\sqrt{b^2-4ac})^2]$$

$\boxed{公式53:\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cdot ln\left|x+\sqrt{x^2-a^2}\right|+C}$

$$\therefore\ \int\sqrt{ax^2+bx+c}\,dx=\frac{1}{2\sqrt{a}}\int\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\,dx$$

$$=\frac{1}{2a\cdot2\sqrt{a}}\int\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\,d(2ax+b)$$

$$=\frac{1}{4a\cdot\sqrt{a}}\cdot\left[\frac{2ax+b}{2}\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}-\frac{b^2-4ac}{2}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|\right]$$

$$=\frac{1}{4\sqrt{a^3}}\cdot\frac{2ax+b}{2}\cdot2\sqrt{a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+\sqrt{(2ax+b)^2-(\sqrt{b^2-4ac})^2}\right|+C$$

$$=\frac{1}{4\sqrt{a^3}}\cdot\frac{2ax+b}{2}\cdot2\sqrt{a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+\sqrt{4a\cdot(ax^2+bx+c)}\right|+C$$

$$=\frac{2ax+b}{4a}\cdot\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8\sqrt{a^3}}\cdot ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

75. $\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2\sqrt{a^3}} \cdot ln\left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + \mathrm{C} \qquad (a>0)$

证明：$\because d(ax^2+bx+c)=(2ax+b)dx$

$\therefore$ 可将$\int \frac{x}{\sqrt{ax^2+bx+c}}dx$ 变换成$\int \left[\frac{1}{\sqrt{ax^2+bx+c}} \cdot \left(\frac{2ax+b}{2a} - \frac{b}{2a} \right) \right] dx$

$\therefore$ 上式$=\frac{1}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}} \cdot (2ax+b)dx - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

$$=\frac{1}{2a}\int (ax^2+bx+c)^{-\frac{1}{2}}d(ax^2+bx+c) - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

$$=\frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

又$\frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{b}{2a} \cdot \frac{1}{\sqrt{a}} \cdot ln\left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C_1 \qquad$ (公式73)

$$=\frac{b}{2\sqrt{a^3}} \cdot ln\left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C_1$$

$\therefore \int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2\sqrt{a^3}} \cdot ln\left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + \mathrm{C}$

76. $\int \frac{dx}{\sqrt{c+bx-ax^2}} = \frac{1}{\sqrt{a}} \cdot arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + \mathrm{C} \qquad (a>0)$

证明：若被积函数 $f(x)=\frac{1}{\sqrt{c+bx-ax^2}}$ 成立，则$c+bx-ax^2>0$有解

$\because a>0 \quad \therefore \Delta = b^2+4ac>0$

$\because c+bx-ax^2 = \frac{1}{4a}[b^2-(2ax-b)^2]+c$

$$=\frac{b^2+4ac}{4a} - \frac{(2ax-b)^2}{4a}$$

$\therefore \int \frac{dx}{\sqrt{c+bx-ax^2}} = 2\sqrt{a}\int \frac{1}{\sqrt{(b^2+4ac)^2-(2ax-b)^2}}dx$

$$=\frac{1}{\sqrt{a}} \cdot arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + \mathrm{C}$$

$\boxed{\text{原题：}\int \frac{dx}{\sqrt{c+bx-ax^2}} = -\frac{1}{\sqrt{a}} \cdot arcsn \frac{2ax-b}{\sqrt{b^2+4ac}} + \mathrm{C}\text{ 有误}}$

77. $\int\sqrt{c+bx-ax^2}dx=\frac{2ax-b}{8a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+\mathrm{C}\qquad(a>0)$

证明:若被积函数 $f(x)=\sqrt{c+bx-ax^2}$ 成立，则 $c+bx-ax^2\geq0$ 有解

$$\because\ a>0\quad\therefore\ \Delta=b^2+4ac\geq0$$

$$\because c+bx-ax^2=\frac{1}{4a}[b^2-(2ax-b)^2]+c$$

$$=\frac{b^2+4ac}{4a}-\frac{(2ax-b)^2}{4a}$$

公式67: $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot arcsin\frac{x}{a}+C$

$$\therefore\ \int\sqrt{c+bx-ax^2}\,dx=\frac{1}{2\sqrt{a}}\int\sqrt{(b^2+4ac)^2-(2ax-b)^2}\,dx$$

$$=\frac{1}{2\sqrt{a}\cdot2a}\int\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}\,d(2ax-b)$$

$$=\frac{1}{4\sqrt{a^3}}\left[\frac{2ax-b}{2}\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}+\frac{b^2+4ac}{2}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}\right]+C$$

$$=\frac{2ax-b}{8\sqrt{a^3}}\sqrt{4a\cdot(c+bx-ax^2)}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

$$=\frac{2ax-b}{8a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+\mathrm{C}$$

78. $\int\frac{x}{\sqrt{c+bx-ax^2}}dx=-\frac{1}{a}\sqrt{c+bx-ax^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+\mathrm{C}\qquad(a>0)$

证明:若被积函数 $f(x)=\frac{x}{\sqrt{c+bx-ax^2}}$ 成立，则 $c+bx-ax^2>0$ 有解

$$\because\ a>0\quad\therefore\ \Delta=b^2+4ac>0$$

$$\because c+bx-ax^2=\frac{1}{4a}[b^2-(2ax-b)^2]+c$$

$$=\frac{1}{4a}\left[b^2+4ac-(2ax-b)^2\right]$$

$$\therefore\ \int\frac{x}{\sqrt{c+bx-ax^2}}dx=2\sqrt{a}\int\frac{x}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,dx$$

公式 61 : $\int\frac{x}{\sqrt{a^2-x^2}}\,dx=-\sqrt{a^2-x^2}+\mathrm{C}$

$$=2\sqrt{a}\cdot\frac{1}{2a}\cdot\frac{1}{2a}\int\frac{2ax-b+b}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)$$

$$=\frac{1}{2\sqrt{a^3}}\int\frac{2ax-b}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)+\frac{b}{2\sqrt{a^3}}\int\frac{1}{\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}}\,d(2ax-b)$$

$$=-\frac{1}{2\sqrt{a^3}}\sqrt{(\sqrt{b^2+4ac})^2-(2ax-b)^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

公式59: $\int\frac{dx}{\sqrt{a^2-x^2}}=arcsin\frac{x}{a}+C$

$$=-\frac{1}{2\sqrt{a^3}}\sqrt{4a\cdot(c+bx-ax^2)}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$$

$$=-\frac{1}{a}\sqrt{c+bx-ax^2}+\frac{b}{2\sqrt{a^3}}\cdot arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+\mathrm{C}$$

（十）含有 $\sqrt{\pm\frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)(b-x)}$ 的积分（79~82）

79. $\int\sqrt{\frac{x-a}{x-b}}dx=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\cdot ln\,(\sqrt{|x-a|}+\sqrt{|x-b|})+C$

证明：$\because\sqrt{\frac{x-a}{x-b}}>0$ 可令 $t=\sqrt{\frac{x-a}{x-b}}\quad(t>0)$，则$x=\frac{a-bt^2}{1-t^2}$ ，$dx=\frac{2t\cdot(a-b)}{(1-t^2)^2}dt$

$$\therefore\int\sqrt{\frac{x-a}{x-b}}dx=\int t\cdot\frac{2t\cdot(a-b)}{(1-t^2)^2}dt=2(a-b)\int\frac{t^2}{(1-t^2)^2}dt$$

$$=2(b-a)\int\frac{1-t^2+1}{(1-t^2)^2}dt=2(b-a)\int[\frac{1}{1-t^2}-\frac{1}{(1-t^2)^2}]dt$$

$$=2(b-a)\int\frac{1}{1-t^2}dt-2(b-a)\int\frac{1}{(1-t^2)^2}dt=2(a-b)\int\frac{1}{t^2-1}dt+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

$$=2(a-b)\cdot\frac{1}{2}\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

对于$\int\frac{1}{(1-t^2)^2}dt=\int\frac{1}{(t^2-1)^2}dt\qquad(t>0)$

$\therefore$ 可令 $t=sec\,k\qquad(0<k<\frac{\pi}{2})$，则 $(t^2-1)^2=tan^4k$，$d\,sec\,k=sec\,k\cdot tan\,kdk$

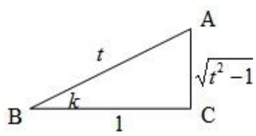
$$\therefore\int\frac{1}{(t^2-1)^2}dt=\int\frac{1}{tan^4k}\cdot sec\,k\cdot tan\,kdk=\int\frac{sec\,k}{tan^4k}dk=\int\frac{cos^2k}{sin^3k}dk$$

$$=\int\frac{1-sin^2k}{sin^3k}dk=\int\frac{1}{sin^3k}dk-\int\frac{1}{sin\,k}dk=-\frac{1}{2}\cdot\frac{cos\,k}{sin^2k}+\frac{1}{2}\int\frac{1}{sin\,k}dk-\int\frac{1}{sin\,k}dk$$

$$=-\frac{1}{2}\cdot\frac{cos\,k}{sin^2k}-\frac{1}{2}\int\frac{1}{sin\,k}dk=-\frac{1}{2}\cdot ln\,|\,csck-cotk\,|-\frac{1}{2}\cdot\frac{cos\,k}{sin^2k}$$

在 $Rt\Delta ABC$中，$\angle B=k$ ，$|BC|=1$ 则$|AC|=\sqrt{t^2-1}$ ，$|AB|=t$

$\therefore\ csc\,k=\frac{1}{sin\,k}=\frac{t}{\sqrt{t^2-1}}$ ，$cot\,k=\frac{1}{\sqrt{t^2-1}}$ ，$cos\,k=\frac{1}{t}$ ，$sin\,k=\frac{\sqrt{t^2-1}}{t}$



$$\therefore\ \int\sqrt{\frac{x-a}{x-b}}dx=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)[-\frac{1}{2}\cdot ln\left|\frac{t-1}{\sqrt{t^2-1}}\right|-\frac{t}{2(t^2-1)}]+C_1$$

$$=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|-(a-b)\cdot ln\left|\frac{t-1}{\sqrt{t^2-1}}\right|-\frac{(a-b)\cdot t}{t^2-1}+C_1$$

$$=(a-b)\cdot ln\left|\frac{\sqrt{t^2-1}}{t+1}\right|-\frac{(a-b)\cdot t}{(t^2-1)}+C_1$$

将$t=\sqrt{\frac{x-a}{x-b}}$代入上式得：$\therefore\ \int\sqrt{\frac{x-a}{x-b}}dx=(a-b)\cdot ln\left|\frac{\sqrt{\frac{b-a}{|x-b|}}}{\frac{\sqrt{|x-a|}+\sqrt{|x-b|}}{\sqrt{|x-b|}}}\right|-(a-b)\sqrt{\frac{x-a}{x-b}}\cdot\frac{x-b}{b-a}+C_1$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\,ln\left|\frac{\sqrt{b-a}}{\sqrt{|x-a|}+\sqrt{|x-b|}}\right|+C_1$$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\,ln\left|\sqrt{b-a}\right|+(b-a)\,ln\left|\sqrt{|x-a|}+\sqrt{|x-b|}\right|+C_1$$

$$=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\cdot ln\,(\sqrt{|x-a|}+\sqrt{|x-b|})+C$$

80. $\int\sqrt{\frac{x-a}{b-x}}dx=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\cdot arcsin\sqrt{\frac{x-a}{b-a}}+C$

证明：$\because\sqrt{\frac{x-a}{b-x}}>0$ 可令 $t=\sqrt{\frac{x-a}{b-x}}$ $(t>0)$，则$x=\frac{a+bt^2}{1+t^2}$ ，$dx=\frac{2t\cdot(b-a)}{(1+t^2)^2}dt$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=\int t\cdot\frac{2t\cdot(b-a)}{(1+t^2)^2}dt=2(b-a)\int\frac{t^2}{(1+t^2)^2}dt$$

$$=2(b-a)\int\frac{1+t^2-1}{(1+t^2)^2}dt=2(b-a)\int[\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}]dt$$

$$=2(b-a)\int\frac{1}{1+t^2}dt-2(b-a)\int\frac{1}{(1+t^2)^2}dt=2(b-a)arcsint-2(a-b)\int\frac{1}{(1+t^2)^2}dt$$

$$=2(a-b)\cdot\frac{1}{2}\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt=(a-b)\cdot ln\left|\frac{t-1}{t+1}\right|+2(a-b)\int\frac{1}{(1-t^2)^2}dt$$

对于$\int\frac{1}{(1+t^2)^2}dt$ $(t>0)$

$\therefore$ 可令 $t=tan\,k$ $(0<k<\frac{\pi}{2})$，则$(t^2+1)^2=sec^4k$，$dt=sec^2kdk$

$$\therefore\int\frac{1}{(1+t^2)^2}dt=\int\frac{1}{sec^4k}\cdot sec^2kdk=\int\frac{1}{sec^2k}dk=\int cos^2kdk$$

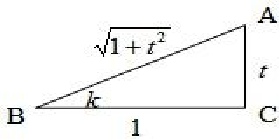
$$=\frac{1}{2}\int(1+cos\,2k)dk=\frac{1}{2}\int dk+\frac{1}{2}\int cos\,2kdk$$

$$=\frac{k}{2}+\frac{1}{4}\cdot sin\,2k+C_1$$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=2(b-a)k-2(b-a)[\frac{k}{2}+\frac{1}{4}\cdot sin\,2k]+C_1$$

$$=(b-a)k-(b-a)sin\,k\cdot cos\,k+C_1$$

在$\mathrm{Rt}\Delta ABC$中，$\angle\mathrm{B}=k$ ，$|\mathrm{BC}|=1$ 则$|\mathrm{AC}|=1$ ，$|\mathrm{AB}|=\sqrt{t^2+1}$



$\therefore$ $cos\,k=\frac{1}{\sqrt{t^2+1}}$ ，$sin\,k=\frac{t}{\sqrt{t^2+1}}$

$$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=(b-a)arcsin\frac{t}{\sqrt{t^2+1}}-(b-a)\cdot\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}+C_1$$

$$=(b-a)arcsin\frac{t}{\sqrt{t^2+1}}-(b-a)\cdot\frac{t}{t^2+1}+C_1$$

将$t=\sqrt{\frac{x-a}{b-x}}$代入上式得：$\therefore\int\sqrt{\frac{x-a}{b-x}}dx=(b-a)arcsin\left(\sqrt{\frac{b-x}{b-a}}\sqrt{\frac{x-a}{b-x}}\right)-(b-a)\cdot\sqrt{\frac{x-a}{b-x}}\sqrt{\frac{b-x}{b-a}}+C_1$

$$=(b-a)arcsin\sqrt{\frac{x-a}{b-a}}-(b-x)\cdot\sqrt{\frac{x-a}{b-x}}+C_1$$

$$=(b-a)arcsin\sqrt{\frac{x-a}{b-a}}+(x-b)\cdot\sqrt{\frac{x-a}{b-x}}+C$$

81. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C \qquad (a<b)$

证明：$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{1}{|x-a|} \cdot \sqrt{\frac{x-a}{b-x}}\ dx$

令 $t = \sqrt{\frac{x-a}{b-x}}$，则$x = \frac{a+bt^2}{1+t^2}$，$|x-a| = \left|\frac{(b-a)t^2}{1+t^2}\right|$，$dx = \frac{2t(b-a)}{(1+t^2)^2}dt$

$\because b>a$，$\therefore$ $|x-a| = (b-a)\cdot\frac{t^2}{1+t^2}$

于是 $\int \frac{1}{|x-a|}\cdot\sqrt{\frac{x-a}{b-x}}\ dx = \int \frac{1}{b-a}\cdot\frac{1+t^2}{t^2}\cdot t\cdot\frac{2t\cdot(b-a)}{(1+t^2)^2}dt$

$= 2\int \frac{1}{1+t^2}dt = 2\arctan t + C$ （公式19）

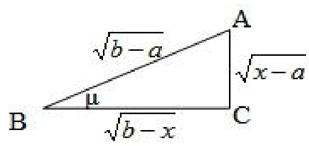
$= 2\arctan\sqrt{\frac{x-a}{b-x}} + C$

令 $\tan\mu = \sqrt{\frac{x-a}{b-x}}$，则 $\mu = \arctan\sqrt{\frac{x-a}{b-x}}$

在$\mathrm{Rt}\Delta ABC$中，$\angle \mathrm{B} = \mu$，$|\mathrm{AC}| = \sqrt{x-a}$

$\therefore$ $|\mathrm{BC}| = \sqrt{b-x}$，$|\mathrm{AB}| = \sqrt{|\mathrm{AC}|^2 + |\mathrm{BC}|^2} = \sqrt{b-a}$

$\therefore$ $\sin\mu = \sqrt{\frac{x-a}{b-a}}$，$\therefore$ $\mu = \arcsin\sqrt{\frac{x-a}{b-a}}$



$\therefore$ $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C$

82. $\int\sqrt{(x-a)(b-x)}dx=\frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\cdot arcsin\sqrt{\frac{x-a}{b-x}}+\mathrm{C}\quad(a<b)$

证明：$\int\sqrt{(x-a)(b-x)}dx=\int|x-a|\sqrt{\frac{b-x}{x-a}}dx$

$\because\sqrt{\frac{b-x}{x-a}}>0$ 可令 $t=\sqrt{\frac{b-x}{x-a}}\quad(t>0)$，则$x=\frac{b+at^2}{1+t^2}$ ，$dx=\frac{2at\cdot(1+t^2)-2t(at^2+b)}{(1+t^2)^2}dt=\frac{2t(a-b)}{(1+t^2)^2}dt$

$$|x-a|=\left|\frac{at^2+b-a-at^2}{1+t^2}\right|=\left|\frac{b-a}{1+t^2}\right|$$

$\because\ a<b\quad\therefore\ |x-a|=\frac{b-a}{1+t^2}$

$$\therefore\ \int\sqrt{(x-a)(b-x)}dx=\int\frac{b-a}{1+t^2}\cdot t\cdot\frac{2t(a-b)}{(1+t^2)^2}dt$$

$$=-2(a-b)^2\int\frac{t^2}{(1+t^2)^3}dt$$

对于$\int\frac{t^2}{(1+t^2)^3}dt\quad(t>0)$ $\therefore$ 可令 $t=tan\,k\quad(0<k<\frac{\pi}{2})$，则$(t^2+1)^3=sec^6k$，$dt=sec^2kdk$

$$\therefore\ \int\frac{t^2}{(1+t^2)^3}dt=\int\frac{tan^2k}{sec^6k}\cdot sec^2kdk=\int\frac{tan^2k}{sec^4k}dk=\int sin^2k\cdot cos^2kdk$$

$$=\frac{1}{4}\int(2\sin k\cdot cos\,k)^2dk=\frac{1}{4}\int sin^2 2k\cdot dk$$

$$=\frac{1}{8}\left[\frac{2k}{2}-\frac{1}{4}\cdot sin\,4k\right]+C$$

$$=\frac{k}{8}-\frac{1}{32}\cdot sin\,4k+C$$

$$=\frac{k}{8}-\frac{1}{32}\cdot(4\sin k\cdot cos^3k-4\sin^3k\cdot cos\,k)+C$$

$$=\frac{k}{8}-\frac{1}{8}\cdot sin\,k\cdot cos^3k+\frac{1}{8}sin^3k\cdot cos\,k+C$$

联立以上两式得：$\int\sqrt{(x-a)(b-x)}dx=-2(b-a)^2\cdot\frac{1}{8}\cdot(k-sin\,k\cdot cos^3k+sin^3k\cdot cos\,k)+\mathrm{C}$

$$=-\frac{(b-a)^2}{4}\cdot(k-sin\,k\cdot cos^3k+sin^3k\cdot cos\,k)+\mathrm{C}$$

在RtΔABC中，$\angle\mathrm{B}=k$ ，$|\mathrm{BC}|=1$ 则$|\mathrm{AC}|=t$ ，$|\mathrm{AB}|=\sqrt{t^2+1}$

$\therefore\ \ cos\,k=\frac{t}{\sqrt{t^2+1}}$ ，$sin\,k=\frac{1}{\sqrt{t^2+1}}$

$$\therefore\ \int\sqrt{(x-a)(b-x)}dx=-\frac{(b-a)^2}{4}\cdot(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{1}{\sqrt{t^2+1}}\cdot\frac{t^2}{t^2+1}\cdot\frac{t}{\sqrt{t^2+1}}$$
$$+\frac{1}{t^2+1}\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}})$$

$$=-\frac{(b-a)^2}{4}\cdot(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{t^3}{(t^2+1)^2}+\frac{t}{(t^2+1)^2})+\mathrm{C}$$

$$=-\frac{(b-a)^2}{4}\cdot(arcsin\frac{t}{\sqrt{t^2+1}}-\frac{t(t^2-1)}{(t^2+1)^2})+\mathrm{C}$$

将$t=\sqrt{\frac{b-x}{x-a}}$代入上式得：$\int\sqrt{(x-a)(b-x)}dx=\frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\cdot arcsin\sqrt{\frac{x-a}{b-x}}+\mathrm{C}$

（十一）含有三角函数的积分（83~112）

83. $\int sinx\,dx = -cosx + C$

证明：$\int sinx\,dx = -\int (-sinx)\,dx$

$\because\ (cosx)' = -sinx$ 即 $cosx$为$-sinx$的原函数

$\therefore\ \int sinx\,dx = -\int dcosx$

$= -cosx + C$

84. $\int cos\,x\,dx = sin\,x + C$

证明：$\because\ (sin\,x)' = cos\,x$即 $sin\,x$为$cos\,x$的原函数

$\therefore\ \int cos\,x\,dx = \int d\,sin\,x$

$= sin\,x + C$

85. $\int tan\,x\,dx = -ln\left|\,cosx\,\right| + C$

证明：$\int tan\,x\,dx = \int \frac{sinx}{cos\,x}\,dx$

$= -\int \frac{1}{cos\,x}\,d\,cos\,x$

$= -ln\left|\,cosx\,\right| + C$

86. $\int cot\,x\,dx = ln\left|\,sin\,x\,\right| + C$

证明：$\int cot\,x\,dx = \int \frac{cos\,x}{sin\,x}\,dx$

$= \int \frac{1}{sin\,x}\,d\,sin\,x$

$= ln\left|\,sin\,x\,\right| + C$

87. $\int sec\,xdx = ln\,|\,tan\,(\frac{\pi}{4} + \frac{x}{2})\,| + \mathrm{C} = ln|sec\,x + tan\,x| + \mathrm{C}$

证明：$\int sec\,xdx = \int \frac{1}{cos\,x}dx = \int \frac{cos\,x}{cos^2\,x}dx$

$= \int \frac{1}{1 - sin^2\,x}d\,sin\,x = \frac{1}{2}\int \frac{1}{1 + sin\,x}d\,sin\,x + \frac{1}{2}\int \frac{1}{1 - sin\,x}d\,sin\,x$

$= \frac{1}{2}\cdot ln\,|1 + sin\,x\,| - \frac{1}{2}\cdot ln\,|1 - sin\,x\,| + \mathrm{C}$

$= \frac{1}{2}\cdot ln\,|\,\frac{1 + sin\,x}{1 - sin\,x}\,| + \mathrm{C} = \frac{1}{2}\cdot ln\,\left|\frac{(1 + sin\,x)^2}{1 - sin^2\,x}\right| + C$

$= \frac{1}{2}\cdot ln\left|\frac{(1 + sin\,x)^2}{cos^2\,x}\right| + C = ln\,|\,\frac{1 + sin\,x}{cos\,x}\,| + \mathrm{C}$

$= ln\,\left|\frac{1}{cos\,x} - \frac{sin\,x}{cox}\right| + C$

$= ln\,|\,sec\,x + tan\,x\,| + \mathrm{C}$

88. $\int csc\,x\,dx = ln\left|tan\frac{x}{2}\right| + C = ln\left|csc\,x - cot\,x\right| + C$

证法1：$\because csc\,x = \frac{1}{sin\,x} = \frac{1}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{sin^2\frac{x}{2}+cos^2\frac{x}{2}}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{1+tan^2\frac{x}{2}}{2\,tan\frac{x}{2}}$

又$\because\ d\,tan\frac{x}{2} = \frac{1}{2}\cdot\frac{1}{cos^2\frac{x}{2}}dx$

$\therefore\ dx = 2\cdot cos^2\frac{x}{2}d\,tan\frac{x}{2}$

$$\begin{aligned}\therefore\ \int csc\,x\,dx &= \int\frac{1}{2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}}\cdot 2\cdot cos^2\frac{x}{2}d\,tan\frac{x}{2}\\ &= \int\frac{1}{tan\frac{x}{2}}d\,tan\frac{x}{2}\\ &= ln\left|tan\frac{x}{2}\right| + C\end{aligned}$$

$\because\ tan\frac{x}{2} = \frac{sin\frac{x}{2}}{cos\frac{x}{2}} = \frac{sin^2\frac{x}{2}}{sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{2\,sin^2\frac{x}{2}}{2\,sin\frac{x}{2}\cdot cos\frac{x}{2}} = \frac{1-cos\,x}{sin\,x} = csc\,x - cot\,x$

$\therefore\ \int csc\,x\,dx = ln\left|tan\frac{x}{2}\right| + C = ln\left|csc\,x - cot\,x\right| + C$

证法2：$\int csc\,x\,dx = \int\frac{1}{sint}dt$

$$\begin{aligned}&= \int\frac{sint}{sin^2t}dt\\ &= -\int\frac{1}{1-cos^2t}d\,cos\,t\\ &= -\frac{1}{2}\int(\frac{1}{1+cost}+\frac{1}{1-cost})d\,cos\,t\\ &= -\frac{1}{2}\int\frac{1}{1+cost}d(cos\,t+1) + \frac{1}{2}\int\frac{1}{1-cost}d(1-cos\,t)\\ &= -\frac{1}{2}\cdot ln\left|1+cost\right| + \frac{1}{2}\cdot ln\left|cost-1\right| + C_1\\ &= \frac{1}{2}\cdot ln\left|\frac{cost-1}{1+cost}\right| + C_1\\ &= \frac{1}{2}\cdot ln\left|\frac{(1-cos\,t)^2}{1-cos^2t}\cdot(-1)\right| + C_1\\ &= \frac{1}{2}\cdot ln\left|\frac{(1-cos\,t)^2}{sin^2t}\right| + C_2\\ &= ln\left|\frac{1-cost}{sint}\right| + C_2\\ &= ln\left|csc\,x - cot\,x\right| + C\end{aligned}$$

89. $\int sec^2 x\,dx = tan\,x + C$

证明：$\because\ (tan\,x)' = sec^2 x$ 即 $tan\,x$ 为 $sec^2 x$ 的原函数

$$\therefore \int sec^2 x\,dx = \int d\,tan\,t$$
$$= tan\,x + C$$

90. $\int csc^2 x\,dx = -cotx + C$

证明：$\int csc^2 x\,dx = -\int (-csc^2 x)\,dx$

$\because\ (cotx)' = -csc^2 x$ 即 $cot\,x$ 为 $-csc^2 x$ 的原函数

$$\therefore \int csc^2 x\,dx = -\int dcotx$$
$$= -cotx + C$$

91. $\int sec\,x \cdot tan\,x\,dx = sec\,x + C$

证明：$\because\ (sec\,x)' = sec\,x \cdot tan\,x$ 即 $sec\,x$ 为 $sec\,x \cdot tan\,x$ 的原函数

$$\therefore \int sec\,x \cdot tan\,x\,dx = \int d\,sec\,x$$
$$= sec\,x + C$$

92. $\int cscx \cdot cot\,x\,dx = -csc\,x + C$

证明：$\int cscx \cdot cot\,x\,dx = -\int (-cscx \cdot cot\,x)\,dx$

$\because\ (csc\,x)' = -cscx \cdot cot\,x$ 即 $csc\,x$ 为 $-cscx \cdot cot\,x$ 的原函数

$$\therefore \int cscx \cdot cot\,x\,dx = -\int d\,csc\,x$$
$$= -csc\,x + C$$

93. $\int sin^2 x\,dx = \frac{x}{2} - \frac{1}{4} \cdot sin\,2x + C$

证明：$\int sin^2 x\,dx = \int (\frac{1}{2} - \frac{1}{2} \cdot cos2x)\,dx$

$$= \frac{1}{2}\int dx - \frac{1}{4}\int cos2x\,d2x$$
$$= \frac{x}{2} - \frac{1}{4} sin\,2x + C$$

提示：$sin^2 x = \frac{1 - cos\,2x}{2}$

94. $\int cos^2 x\,dx = \frac{x}{2} + \frac{1}{4} \cdot sin\,2x + C$

证明：$\int cos^2 x\,dx = \int (\frac{1}{2} + \frac{1}{2} \cdot cos2x)\,dx$

$$= \frac{1}{2}\int dx + \frac{1}{4}\int cos2x\,d2x$$
$$= \frac{x}{2} + \frac{1}{4} sin\,2x + C$$

提示：$cos^2 x = \frac{1 + cos\,2x}{2}$

95. $\int sin^n x\, dx = -\frac{1}{n}\cdot sin^{n-1} x\cdot cos\, x + \frac{n-1}{n}\int sin^{n-2} x\, dx$

证明：$\int sin^n x\, dx = \int sin^{n-1} x\cdot sin\, x\, dx$

$$
\begin{aligned}
&= -\int sin^{n-1} x\, d\, cos\, x \\
&= -cos\, x\cdot sin^{n-1} x + \int cos\, x\, d\, sin^{n-1} x \\
&= -cos\, x\cdot sin^{n-1} x + \int cos\, x\cdot (n-1)\cdot sin^{n-2} x\cdot cos\, x\, dx \\
&= -cos\, x\cdot sin^{n-1} x + (n-1)\int cos^2 x\cdot sin^{n-2} x\, dx \\
&= -cos\, x\cdot sin^{n-1} x + (n-1)\int (1-sin^2 x)\cdot sin^{n-2} x\, dx \\
&= -cos\, x\cdot sin^{n-1} x + (n-1)\int sin^{n-2} x\, dx - (n-1)\int sin^n x\, dx
\end{aligned}
$$

移项并整理得：$n\int sin^n x\, dx = -cos\, x\cdot sin^{n-1} x + (n-1)\int sin^{n-2} x\, dx$

$\therefore \int sin^n x\, dx = -\frac{1}{n}\cdot sin^{n-1} x\cdot cos\, x + \frac{n-1}{n}\int sin^{n-2} x\, dx$

96. $\int cos^n x\, dx = \frac{1}{n}\cdot cos^{n-1} x\cdot sin\, x + \frac{n-1}{n}\int cos^{n-2} x\, dx$

证明：$\int cos^n x\, dx = \int cos^{n-1} x\cdot cos\, x\, dx$

$$
\begin{aligned}
&= \int cos^{n-1} x\, d\, sin\, x \\
&= sin\, x\cdot cos^{n-1} x - \int sin\, x\, d\, cos^{n-1} x \\
&= sin\, x\cdot cos^{n-1} x + \int sin\, x\cdot (n-1)\cdot cos^{n-2} x\cdot sin\, x\, dx \\
&= sin\, x\cdot cos^{n-1} x + (n-1)\int sin^2 x \cdot s\, cos^{n-2} x\, dx \\
&= sin\, x\cdot cos^{n-1} x + (n-1)\int (1-cos^2 x)\cdot cos^{n-2} x\, dx \\
&= sin\, x\cdot cos^{n-1} x + (n-1)\int cos^{n-2} x\, dx - (n-1)\int cos^n x\, dx
\end{aligned}
$$

移项并整理得：$n\int cos^n x\, dx = sin\, x\cdot cos^{n-1} x + (n-1)\int cos^{n-2} x\, dx$

$\therefore \int sin^n x\, dx = \frac{1}{n}\cdot sin\, x\cdot cos^{n-1} x + \frac{n-1}{n}\int cos^{n-2} x\, dx$

97. $\int \frac{dx}{\sin^n x} dx = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$

证明: $\int \frac{dx}{\sin^n x} dx = -\int \frac{1}{\sin^{n-2} x} \cdot \frac{1}{-\sin^2 x} dx$

$= -\int \frac{1}{\sin^{n-2} x} d\cot x$

$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x \, d\frac{1}{\sin^{n-2} x}$

$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x \cdot (2-n) \cdot \sin^{1-n} x \cdot \cos x \, dx$

$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{\cos^2 x}{\sin^n x} dx$

$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{1-\sin^2 x}{\sin^n x} dx$

$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{dx}{\sin^n x} dx - (2-n) \int \frac{1}{\sin^{n-2} x} dx$

移项并整理得: $(n-1) \int \frac{dx}{\sin^n x} dx = -\frac{\cot x}{\sin^{n-2} x} - (2-n) \int \frac{1}{\sin^{n-2} x} dx$

$= -\frac{\cos x}{\sin^{n-1} x} + (n-2) \int \frac{1}{\sin^{n-2} x} dx$

$\therefore \int \frac{dx}{\sin^n x} dx = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$

98. $\int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$

证明: $\int \frac{dx}{\cos^n x} = \int \frac{1}{\cos^{n-2} x} \cdot \frac{1}{\cos^2 x} dx$

$= \int \frac{1}{\cos^{n-2} x} d\tan x$

$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x \, d\frac{1}{\cos^{n-2} x}$

$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x \cdot (2-n) \cdot \cos^{1-n} x \cdot \sin x \, dx$

$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{\sin^2 x}{\cos^n x} dx$

$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{1-\cos^2 x}{\cos^n x} dx$

$= \frac{\sin x}{\cos^{n-1} x} - (n-2) \int \frac{dx}{\cos^n x} dx + (n-2) \int \frac{1}{\cos^{n-2} x} dx$

移项并整理得: $(n-1) \int \frac{dx}{\cos^n x} = \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$

$= \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$

$\therefore \int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$

99. $\int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x+\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$ ①

$$=-\frac{1}{m+n}\cdot cos^{m+1}x\cdot sin^{n-1}x+\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx \quad ②$$

证明①：$\because d\,sin^{m+n}xdx=(m+n)\cdot sin^{m+n-1}x\cdot cos\,xdx$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\int cos^{m-1}x\cdot sin^{1-m}xd\,sin^{m+n}x$$

$$=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x-\frac{1}{m+n}\int sin^{m+n}xd(cos^{m-1}x\cdot sin^{1-m}x)$$

$\because d(cos^{m-1}x\cdot sin^{1-m}x)=[-(m-1)\cdot cos^{m-2}x\cdot sin\,x\cdot sin^{1-m}x+(1-m)\cdot sin^{1-m-1}x\cdot cos\,x\cdot cos^{m-1}x]dx$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^m x\cdot(sin^2x\cdot cos^{-2}x+1)]dx$$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^m x\cdot(\frac{sin^2x+cos^2x}{cos^2x})]dx$$

$$=[(1-m)\cdot sin^{-m}x\cdot cos^{m-2}x]dx$$

$$\therefore -\frac{1}{m+n}\int sin^{m+n}xd(cos^{m-1}x\cdot sin^{1-m}x)=\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{1}{m+n}\cdot cos^{m-1}x\cdot sin^{n+1}x+\frac{m-1}{m+n}\int cos^{m-2}x\cdot sin^n xdx$$

证明②：$\because d\,cos^{m+n}x=-(m+n)\cdot cos^{m+n-1}x\cdot sin\,xdx$

$$\therefore \int cos^m x\cdot sin^n xdx=\frac{-1}{m+n}\int cos^{1-n}x\cdot sin^{n-1}xd\,cos^{m+n}x$$

$$=\frac{-1}{m+n}\cdot sin^{n-1}x\cdot cos^{m+1}x+\frac{1}{m+n}\int cos^{m+n}xd(sin^{n-1}x\cdot cos^{1-n}x)$$

$\because d(sin^{n-1}x\cdot cos^{1-n}x)=[(n-1)\cdot sin^{n-2}x\cdot cos\,x\cdot cos^{1-n}x-(1-n)\cdot cos^{1-n-1}x\cdot sin\,x\cdot sin^{n-1}x]dx$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^n x\cdot(sin^{-2}x\cdot cos^2x+1)]dx$$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^n x\cdot(\frac{sin^2x+cos^2x}{sin^2x})]dx$$

$$=[(n-1)\cdot cos^{-n}x\cdot sin^{n-2}x]dx$$

$$\therefore \frac{1}{m+n}\int cos^{m+n}xd(sin^{n-1}x\cdot cos^{1-n}x)=\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx$$

$$\therefore \int cos^m x\cdot sin^n xdx=-\frac{1}{m+n}\cdot cos^{m+1}x\cdot sin^{n-1}x+\frac{n-1}{m+n}\int cos^m x\cdot sin^{n-2}xdx$$

100. $\int sin\,ax \cdot cos\,bx\,dx = -\frac{1}{2(a+b)} \cdot cos(a+b)x - \frac{1}{2(a-b)} \cdot cos(a-b)x + \mathrm{C}$

证明：$\int sin\,ax \cdot cos\,bx\,dx = \int \frac{1}{2}[sin\,(a+b)x + sin(a-b)x]dx$

提示： $sin\ \alpha\ cos\ \beta = \frac{1}{2}[sin\ (\alpha+\beta) + sin\ (\alpha-\beta)]$

$$= \frac{1}{2}\int sin\,(a+b)x\,dx \frac{1}{2}\int sin(a-b)xdx$$

$$= \frac{1}{2(a+b)}\int sin\,(a+b)x\,d(a+b)x + \frac{1}{2(a-b)}\int sin\,(a-b)x\,d(a-b)x$$

$$= -\frac{1}{2(a+b)} \cdot cos(a+b)x - \frac{1}{2(a-b)} \cdot cos(a-b)x$$

101. $\int sin\,ax \cdot sin\,bx\,dx = -\frac{1}{2(a+b)} \cdot sin\,(a+b)x + \frac{1}{2(a-b)} \cdot sin\,(a-b)x + \mathrm{C}$

证明：$\int sin\,ax \cdot sin\,bx\,dx = \int \frac{1}{2}[cos\,(a-b)x - cos(a+b)x]dx$

提示： $sin\ \alpha\ sin\ \beta = -\frac{1}{2}[cos\ (\alpha+\beta) + cos\ (\alpha-\beta)]$

$$= \frac{1}{2}\int cos\,(a-b)x\,dx - \frac{1}{2}\int cos\,(a+b)x\,dx$$

$$= \frac{1}{2(a-b)}\int cos\,(a-b)x\,d(a-b)x - \frac{1}{2(a+b)}\int cos\,(a+b)x\,d(a+b)x$$

$$= \frac{1}{2(a-b)} \cdot sin\,(a-b)x - \frac{1}{2(a+b)} \cdot sin\,(a+b)x + \mathrm{C}$$

102. $\int cos\,ax \cdot cos\,bx\,dx = \frac{1}{2(a+b)} \cdot sin\,(a+b)x + \frac{1}{2(a-b)} \cdot sin\,(a-b)x + \mathrm{C}$

证明：$\int cos\,ax \cdot cos\,bx\,dx = \int \frac{1}{2}[cos\,(a+b)x + cos(a-b)x]dx$

提示：$cos\,\alpha\,cos\,\beta = \frac{1}{2}[cos\,(\alpha+\beta) + cos\,(\alpha-\beta)]$

$$= \frac{1}{2}\int cos\,(a+b)x\,dx + \frac{1}{2}\int cos\,(a-b)x\,dx$$

$$= \frac{1}{2(a+b)}\int cos\,(a+b)x\,d(a+b)x + \frac{1}{2(a-b)}\int cos\,(a-b)x\,d(a-b)x$$

$$= \frac{1}{2(a+b)} \cdot sin\,(a+b)x + \frac{1}{2(a-b)} \cdot sin\,(a-b)x + \mathrm{C}$$

103. $\int \frac{dx}{a+b\cdot \sin x}=\frac{2}{\sqrt{a^2-b^2}}\cdot \arctan\frac{a\cdot \tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C \qquad (a^2>b^2)$

证明：令$t=\tan\frac{x}{2}$，则 $\sin x=2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}=\frac{2\cdot\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$

$$dt=(\tan\frac{x}{2})dx=\frac{1}{2}\cdot\sec^2\frac{x}{2}dx=\frac{1}{2}(1+\tan^2\frac{x}{2})\,dx=\frac{1}{2}(1+t^2)\,dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt\ ,\ a+b\cdot\sin x=a+\frac{2bt}{1+t^2}=\frac{a\,(1+t^2)+2bt}{1+t^2}$$

$$\therefore\ \int\frac{dx}{a+b\cdot\sin x}=\int\frac{1+t^2}{a\,(1+t^2)+2bt}\cdot\frac{2}{1+t^2}dt$$

$$=2\int\frac{1}{a\,t^2+2bt+a}dt$$

$$=2\int\frac{1}{a\,(t+\frac{b}{a})^2-\frac{b^2}{a}+a}dt$$

$$=2a\int\frac{1}{(at+b)^2+(a^2-b^2)}dt$$

$$=2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)$$

当$a^2>b^2$，即$a^2-b^2>0$时

$$2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)=2\int\frac{1}{(at+b)^2+(\sqrt{a^2-b^2})^2}d(at+b)$$

$\boxed{公式19：\int\frac{dx}{x^2+a^2}=\frac{1}{a}\cdot\arctan\frac{x}{a}+C}$

$$=\frac{2}{\sqrt{a^2-b^2}}\cdot\arctan\frac{at+b}{\sqrt{a^2-b^2}}+C$$

将$t=\tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\cdot\arctan\frac{a\cdot\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C$

104. $\int \frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\cdot ln\left|\frac{a\cdot tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\cdot tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+\mathrm{C} \qquad (a^2<b^2)$

证明：令$t=tan\frac{x}{2}$，则 $\sin x=2\cdot sin\frac{x}{2}\cdot cos\frac{x}{2}=\frac{2\cdot tan\frac{x}{2}}{1+tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$

$$dt=(tan\frac{x}{2})dx=\frac{1}{2}\cdot sec^2\frac{x}{2}dx=\frac{1}{2}(1+tan^2\frac{x}{2})\,dx=\frac{1}{2}(1+t^2)\,dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt\ ,\ a+b\sin x=a+\frac{2bt}{1+t^2}=\frac{a\,(1+t^2)+2bt}{1+t^2}$$

$$\begin{aligned}\therefore\ \int\frac{dx}{a+b\sin x}&=\int\frac{1+t^2}{a\,(1+t^2)+2bt}\cdot\frac{2}{1+t^2}dt\\&=2\int\frac{1}{a\,t^2+2bt+a}dt\\&=2\int\frac{1}{a\,(t+\frac{b}{a})^2-\frac{b^2}{a}+a}dt\\&=2a\int\frac{1}{(at+b)^2+(a^2-b^2)}dt\\&=2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)\end{aligned}$$

当$a^2<b^2$，即$a^2-b^2<0$时

$$\begin{aligned}2\int\frac{1}{(at+b)^2+(a^2-b^2)}d(at+b)&=2\int\frac{1}{(at+b)^2-(b^2-a^2)}d(at+b)\\&=2\int\frac{1}{(at+b)^2-(\sqrt{b^2-a^2})^2}d(at+b)\\&=2\times\frac{1}{2\sqrt{b^2-a^2}}\cdot ln\left|\frac{at+b-\sqrt{b^2-a^2}}{at+b+\sqrt{b^2-a^2}}\right|+C\end{aligned}$$

公式 21：$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot ln\left|\frac{x-a}{x+a}\right|+\mathrm{C}$

将$t=tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\cdot ln\left|\frac{a\cdot tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\cdot tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+\mathrm{C}$

105. $\int\frac{dx}{a+b\cdot\cos x}=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot\tan\frac{x}{2}\right)+C\qquad(a^2>b^2)$

证明：令$t=\tan\frac{x}{2}$，则 $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$

$$\therefore\ a+b\cdot\cos x=a+b\cdot\frac{1-t^2}{1+t^2}=\frac{(a+b)+t^2(a-b)}{1+t^2}$$

$$\because\ dt=d\tan\frac{x}{2}=\frac{1}{2}\cdot\sec^2\frac{x}{2}dx=\frac{1}{2\cos^2\frac{x}{2}}dx=\frac{1}{1+\cos x}dx=\frac{1+t^2}{2}dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt$$

提示：$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$$\therefore\ \int\frac{dx}{a+b\cdot\cos x}=\int\frac{2}{(a+b)+t^2(a-b)}dt$$

当$|a|>|b|$，即$a^2>b^2$时

$$\int\frac{2}{(a+b)+t^2(a-b)}dt=\frac{2}{a-b}\int\frac{1}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2+t^2}dt$$

公式19：$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\cdot\arctan\frac{x}{a}+C$

$$=\frac{2}{a-b}\cdot\sqrt{\frac{a-b}{a+b}}\cdot\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=\frac{2}{a-b}\cdot\sqrt{\frac{a-b}{a+b}}\cdot\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=2\sqrt{\frac{1}{(a+b)\cdot(a-b)}}\cdot\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

$$=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\cdot\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot t\right)+C$$

将$t=\tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\cdot\cos x}=\frac{2}{a+b}\cdot\sqrt{\frac{a+b}{a-b}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\cdot\tan\frac{x}{2}\right)+C$

106. $\int \frac{dx}{a+b\cdot\cos x}=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right|+C \qquad (a^2<b^2)$

证明：令$t=\tan\frac{x}{2}$，则 $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$

$$\therefore\ a+b\cdot\cos x=a+b\cdot\frac{1-t^2}{1+t^2}=\frac{(a+b)+t^2(a-b)}{1+t^2}$$

$$\because\ dt=d\tan\frac{x}{2}=\frac{1}{2}\cdot\sec^2\frac{x}{2}dx=\frac{1}{2\cos^2\frac{x}{2}}dx=\frac{1}{1+\cos x}dx=\frac{1+t^2}{2}dx$$

$$\therefore\ dx=\frac{2}{1+t^2}dt$$

提示：$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$$\therefore\ \int\frac{dx}{a+b\cdot\cos x}=\int\frac{2}{(a+b)+t^2(a-b)}dt$$

当 $a^2<b^2$，即$|a|<|b|$，$\therefore\ b-a>0$

$$\int\frac{2}{(a+b)+t^2(a-b)}dt=\int\frac{2}{(a+b)-t^2(b-a)}dt$$

$$=\frac{2}{b-a}\int\frac{1}{\left(\sqrt{\frac{a+b}{b-a}}\right)^2-t^2}dt=\frac{2}{a-b}\int\frac{1}{t^2-\left(\sqrt{\frac{a+b}{b-a}}\right)^2}dt$$

$$=\frac{2}{a-b}\cdot\frac{1}{2}\cdot\sqrt{\frac{b-a}{a+b}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C=\frac{1}{a-b}\cdot\sqrt{\frac{b-a}{a+b}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C$$

公式 21：$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right|+C$

$$=(-1)\sqrt{\frac{1}{(a+b)\cdot(b-a)}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C=-\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{t-\sqrt{\frac{a+b}{b-a}}}{t+\sqrt{\frac{a+b}{b-a}}}\right|+C$$

$$=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{t+\sqrt{\frac{a+b}{b-a}}}{t-\sqrt{\frac{a+b}{b-a}}}\right|+C$$

将$t=\tan\frac{x}{2}$代入上式得：$\int\frac{dx}{a+b\cdot\cos x}=\frac{1}{a+b}\cdot\sqrt{\frac{a+b}{b-a}}\cdot\ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right|+C$

107. $\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\cdot \arctan\left(\frac{b}{a}\cdot \tan x\right) + C$

证明：$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{1}{\cos^2 x}\cdot \frac{1}{a^2 + b^2\tan^2 x}dx$

$$= \int \frac{1}{a^2 + b^2\tan^2 x}\, d\tan x$$

$$= \frac{1}{b^2}\int \frac{1}{(\frac{a^2}{b^2} + \tan^2 x)}\, d\tan x$$

$$= \frac{1}{b^2}\int \frac{1}{(\frac{a}{b})^2 + \tan^2 x)}\, d\tan x$$

$\boxed{公式19：\int \frac{dx}{x^2+a^2} = \frac{1}{a}\cdot \arctan\frac{x}{a} + C}$

$$= \frac{1}{b^2}\cdot \frac{b}{a}\cdot \arctan\left(\frac{b}{a}\cdot \tan x\right) + C$$

$$= \frac{1}{ab}\cdot \arctan\left(\frac{b}{a}\cdot \tan x\right) + C$$

108. $\int \frac{dx}{a^2\cos^2 x - b^2\sin^2 x} = \frac{1}{2ab}\cdot \ln\left|\frac{b\cdot \tan x + a}{b\cdot \tan x - a}\right| + C$

证明：$\int \frac{dx}{a^2\cos^2 x - b^2\sin^2 x} = \int \frac{1}{\cos^2 x}\cdot \frac{1}{a^2 - b^2\tan^2 x}dx$

$$= \int \frac{1}{a^2 - b^2\tan^2 x}\, d\tan x$$

$$= \frac{1}{b}\int \frac{1}{a^2 - (b\cdot \tan x)^2}\, d(b\cdot \tan x)$$

$$= -\frac{1}{b}\int \frac{1}{(b\cdot \tan x)^2 - a^2}\, d(b\cdot \tan x)$$

$$= -\frac{1}{b}\cdot \frac{1}{2a}\cdot \ln\left|\frac{b\cdot \tan x - a}{b\cdot \tan x + a}\right| + C$$

$\boxed{公式\ 21：\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\cdot \ln\left|\frac{x-a}{x+a}\right| + C}$

$$= -\frac{1}{2ab}\cdot \ln\left|\frac{b\cdot \tan x - a}{b\cdot \tan x + a}\right| + C$$

$\boxed{提示：\log_a b^{-1} = -\log_a b}$

$$= \frac{1}{2ab}\cdot \ln\left|\frac{b\cdot \tan x + a}{b\cdot \tan x - a}\right| + C$$

109. $\int x\cdot \sin ax\,dx=\frac{1}{a^2}\cdot \sin ax-\frac{1}{a}\cdot x\cdot \cos ax+C$

证明：$\int x\cdot \sin ax\,dx=-\frac{1}{a}\int x\,d\cos ax$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a}\int \cos ax\,dx$$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a^2}\int \cos ax\,dax$$

$$=-\frac{1}{a}\cdot x\cdot \cos ax+\frac{1}{a^2}\cdot \sin ax+C$$

110. $\int x^2\cdot \sin ax\,dx=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax+\frac{2}{a^3}\cdot \cos ax+C$

证明：$\int x^2\cdot \sin ax\,dx=-\frac{1}{a}\int x^2\,d\cos ax$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{1}{a}\int \cos ax\,dx^2$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a}\int x\cdot \cos ax\,dx$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot \int x\,d\sin ax$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax-\frac{2}{a^3}\cdot \int \sin ax\,dax$$

$$=-\frac{1}{a}\cdot x^2\cdot \cos ax+\frac{2}{a^2}\cdot x\cdot \sin ax+\frac{2}{a^3}\cdot \cos ax$$

111. $\int x\cdot \cos ax\,dx=\frac{1}{a^2}\cdot \cos ax-\frac{1}{a}\cdot x\cdot \sin ax+C$

证明：$\int x\cdot \cos ax\,dx=\frac{1}{a}\int x\,d\sin ax$

$$=\frac{1}{a}\cdot x\cdot \sin ax-\frac{1}{a}\int \sin ax\,dx$$

$$=\frac{1}{a}\cdot x\cdot \sin ax-\frac{1}{a^2}\int \sin ax\,dax$$

$$=\frac{1}{a}\cdot x\cdot \sin ax+\frac{1}{a^2}\cdot \cos ax+C$$

112. $\int x^2\cdot \cos ax\,dx=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \sin ax+C$

证明：$\int x^2\cdot \cos ax\,dx=\frac{1}{a}\int x^2\,d\sin ax$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax-\frac{1}{a}\int \sin ax\,dx^2$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a}\int x\cdot \sin ax\,dx$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax-\frac{2}{a^2}\cdot \int x\,d\cos ax$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \int \cos ax\,dax$$

$$=\frac{1}{a}\cdot x^2\cdot \sin ax+\frac{2}{a^2}\cdot x\cdot \cos ax-\frac{2}{a^3}\cdot \sin ax+C$$

（十二）含有反三角函数的积分（其中$a>0$）（113~121）

113. $\int arcsin\frac{x}{a}dx = x\cdot arcsin\frac{x}{a}+\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：$\int arcsin\frac{x}{a}dx = x\cdot arcsin\frac{x}{a}-\int x\,d\ \ arcsin\frac{x}{a}$

$$= x\cdot arcsin\frac{x}{a}-\int x\cdot\frac{1}{\sqrt{1-(\frac{x}{a})^2}}\cdot\frac{1}{a}dx$$

$$= x\cdot arcsin\frac{x}{a}-\int\frac{x}{\sqrt{a^2-x^2}}dx$$

$$= x\cdot arcsin\frac{x}{a}-\frac{1}{2}\int\frac{1}{\sqrt{a^2-x^2}}dx^2$$

$$= x\cdot arcsin\frac{x}{a}+\frac{1}{2}\int(a^2-x^2)^{-\frac{1}{2}}d(a^2-x^2)$$

$$= x\cdot arcsin\frac{x}{a}+\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}\cdot(a^2-x^2)^{1-\frac{1}{2}}+C$$

$$= x\cdot arcsin\frac{x}{a}+\sqrt{a^2-x^2}+C$$

114. $\int x\cdot arcsin\frac{x}{a}dx = (\frac{x^2}{2}-\frac{a^2}{4})\cdot arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C \qquad (a>0)$

证明：令$t = arcsin\frac{x}{a}$，则 $x = a\cdot sin\,t$

$$\therefore\ \int x\cdot arcsin\frac{x}{a}dx = \int a\cdot sin\,t\cdot t\,d(a\cdot sin\,t) = a^2\int t\cdot sin\,t\cdot cos\,t\,dt$$

$$= \frac{a^2}{2}\int t\cdot sin\,2t\,dt = -\frac{a^2}{4}\int t\,d\,cos\,2t$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{4}\int cos\,2t\,dt$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{8}\int cos\,2t\,d2t$$

$$= -\frac{a^2}{4}\cdot t\cdot cos\,2t+\frac{a^2}{8}\cdot sin\,2t+C$$

$$= -\frac{a^2}{4}\cdot t\cdot(2cos^2t-1)+\frac{a^2}{4}\cdot sin\,t\cdot cos\,t+C$$

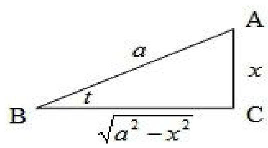
$$= -\frac{a^2}{2}\cdot t\cdot cos^2t+\frac{a^2}{4}\cdot t+\frac{a^2}{4}\cdot sin\,t\cdot cos\,t+C$$

提示：$sin\,2x = 2\cdot sin\,x\cdot cos\,x$

$cos\,2x = cos^2x-sin^2x$

$= 2cos^2x-1$

在Rt ΔABC中，可设$\angle B = t$，$|AB| = a$，则$|AC| = x$，$|BC| = \sqrt{a^2-x^2}$



$$\therefore\ \ cos\,t = \frac{\sqrt{a^2-x^2}}{a}\ ,\ \ sin\,t = \frac{x}{a}$$

$$\therefore\ \int x\cdot arcsin\frac{x}{a}dx = -\frac{a^2}{2}\cdot arcsin\frac{x}{a}\cdot\frac{a^2-x^2}{a^2}+\frac{a^2}{4}\cdot arcsin\frac{x}{a}+\frac{a^2}{4}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}+C$$

$$= \frac{x^2-a^2}{2}\cdot arcsin\frac{x}{a}+\frac{a^2}{4}\cdot arcsin\frac{x}{a}+\frac{x}{4}\cdot\sqrt{a^2-x^2}+C$$

$$= (\frac{x^2}{2}-\frac{a^2}{4})\cdot arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C$$

115. $\int x^2 \cdot arcsin\frac{x}{a}dx = \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C \qquad (a>0)$

证明：令$t = arcsin\frac{x}{a}$，则 $x = a\cdot sin\,t$

$$\therefore \int x^2\cdot arcsin\frac{x}{a}dx = \int a^2\cdot sin^2 t\cdot t\,d(a\cdot sin\,t) = a^3\int t\cdot sin^2 t\cdot cost\,dt$$

$$= \frac{a^3}{3}\int t\,d\,sin^3 t$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin^3 t\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin\,t\,(1-cos^2 t)\,dt$$

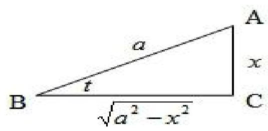
$$= \frac{a^3}{3}\cdot t\cdot sin^3 t - \frac{a^3}{3}\int sin\,t\,dt + \frac{a^3}{3}\int sin\,t\cdot cos^2 t\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos\,t - \frac{a^3}{3}\int cos^2 t\,d\,cos\,t$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos\,t - \frac{a^3}{3}\cdot\frac{1}{1+2}\cdot cos^3 t + C$$

$$= \frac{a^3}{3}\cdot t\cdot sin^3 t + \frac{a^3}{3}\cdot cos\,t - \frac{a^3}{9}\cdot cos^3 t + C$$

在Rt ΔABC中，可设 $\angle B = t$，$|AB| = a$，则 $|AC| = x$，$|BC| = \sqrt{a^2-x^2}$



$$\therefore\ cos\,t = \frac{\sqrt{a^2-x^2}}{a}\ ,\ \ sin\,t = \frac{x}{a}$$

$$\therefore \int x^2\cdot arcsin\frac{x}{a}dx = \frac{a^3}{3}\cdot arcsin\frac{x}{a}\cdot\frac{x^3}{a^3} + \frac{a^3}{3}\cdot\frac{\sqrt{a^2-x^2}}{a} - \frac{a^3}{9}\cdot\frac{a^2-x^2}{a^3}\cdot\sqrt{a^2-x^2} + C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{a^2}{3}\cdot\sqrt{a^2-x^2} - \frac{a^2-x^2}{9}\cdot\sqrt{a^2-x^2} + C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

116. $\int arccos\,\frac{x}{a}dx = x\cdot arccos\,\frac{x}{a} - \sqrt{a^2 - x^2} + C \qquad (a > 0)$

证明：$\int arccos\,\frac{x}{a}dx = x\cdot arccos\,\frac{x}{a} - \int x\,d\ arccos\,\frac{x}{a}$

$$= x\cdot arccos\,\frac{x}{a} + \int x\cdot\frac{1}{\sqrt{1-(\frac{x}{a})^2}}\cdot\frac{1}{a}dx$$

$$= x\cdot arccos\,\frac{x}{a} + \int \frac{x}{\sqrt{a^2 - x^2}}dx$$

$$= x\cdot arccos\,\frac{x}{a} + \frac{1}{2}\int \frac{1}{\sqrt{a^2 - x^2}}dx^2$$

$$= x\cdot arccos\,\frac{x}{a} - \frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}d(a^2 - x^2)$$

$$= x\cdot arccos\,\frac{x}{a} - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}\cdot(a^2 - x^2)^{1-\frac{1}{2}} + C$$

$$= x\cdot arccos\,\frac{x}{a} - \sqrt{a^2 - x^2} + C$$

117. $\int x\cdot arccos\ \frac{x}{a}dx = (\frac{x^2}{2} - \frac{a^2}{4})\cdot arccos\ \frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2} + C \qquad (a > 0)$

证明：令 $t = arccos\ \frac{x}{a}$ ，则 $x = a\cdot cos\,t$

$$\therefore\ \int x\cdot arccos\ \frac{x}{a}dx = \int a\cdot cos\,t\cdot t\,d(a\cdot cos\,t) = -a^2\int t\cdot cos\,t\cdot\ sin\,t\,dt$$

$$= -\frac{a^2}{2}\int t\cdot sin\,2t\,dt = \frac{a^2}{4}\int t\,d\,cos\,2t$$

$$= \frac{a^2}{4}\cdot t\cdot cos\,2t - \frac{a^2}{4}\int cos\,2t\,dt$$

$$= \frac{a^2}{4}\cdot t\cdot cos\,2t - \frac{a^2}{8}\int cos\,2t\,d\,2t$$

$$= \frac{a^2}{4}\cdot t\cdot cos\,2t - \frac{a^2}{8}\cdot sin\,2t + C$$

提示：$sin\,2x = 2\cdot sin\,x\cdot cos\,x$

$cos\,2x = cos^2 x - sin^2 x$

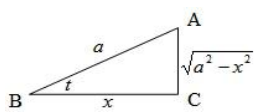
$= 2\,cos^2 x - 1$

$$= \frac{a^2}{4}\cdot t\cdot(2\,cos^2 t - 1) - \frac{a^2}{4}\cdot sin\,t\cdot cos\,t + C$$

$$= \frac{a^2}{2}\cdot t\cdot cos^2 t - \frac{a^2}{4}\cdot t - \frac{a^2}{4}\cdot sin\,t\cdot cos\,t + C$$

在Rt $\varDelta ABC$中，可设 $\angle B = t$,$|AB| = a$,则 $|BC| = x$,$|AC| = \sqrt{a^2 - x^2}$

$$\therefore\ \ sin\,t = \frac{\sqrt{a^2 - x^2}}{a}\ ,\ \ cos\,t = \frac{x}{a}$$



$$\therefore\ \int x\cdot arccos\ \frac{x}{a}dx = \frac{a^2}{2}\cdot arcsin\ \frac{x}{a}\cdot\frac{x^2}{a^2} - \frac{a^2}{4}\cdot arcsin\ \frac{x}{a} - \frac{a^2}{4}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{x^2}{2}\cdot arcsin\ \frac{x}{a} - \frac{a^2}{4}\cdot arcsin\ \frac{x}{a} - \frac{x}{4}\cdot\sqrt{a^2 - x^2} + C$$

$$= (\frac{x^2}{2} - \frac{a^2}{4})\cdot arcsin\ \frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2} + C$$

118. $\int x^2 \cdot arccos\frac{x}{a}dx = \frac{x^3}{3}\cdot arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C \qquad (a>0)$

证明：令$t = arccos\frac{x}{a}$ ，则 $x = a\cdot cos\,t$

$$\therefore \int x^2\cdot arccos\frac{x}{a}dx = \int a^2\cdot cos^2 t\cdot t\, d(a\cdot cos\,t) = -a^3\int t\cdot cos^2 t\cdot sin\,t\,dt$$

$$= \frac{a^3}{3}\int t\, d\,cos^3 t$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos^3 t\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos\,t\,(1-sin^2 t)\,dt$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\int cos\,t\,dt + \frac{a^3}{3}\int cos\,t\cdot sin^2 t\,dt$$

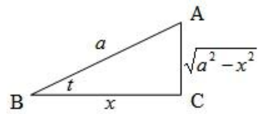
$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{3}\int sin^2 t\,d\,sin\,t$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{3}\cdot\frac{1}{1+2}\cdot sin^3 t + C$$

$$= \frac{a^3}{3}\cdot t\cdot cos^3 t - \frac{a^3}{3}\cdot sin\,t + \frac{a^3}{9}\cdot sin^3 t + C$$

在Rt ΔABC中， 可设 $\angle B = t$,$|AB| = a$,则$|BC| = x$,$|AC| = \sqrt{a^2-x^2}$

$\therefore\ sin\,t = \frac{\sqrt{a^2-x^2}}{a}$ ， $cos\,t = \frac{x}{a}$



$$\therefore \int x^2\cdot arccos\frac{x}{a}dx = \frac{a^3}{3}\cdot arcsin\frac{x}{a}\cdot\frac{x^3}{a^3} - \frac{a^3}{3}\cdot\frac{\sqrt{a^2-x^2}}{a} + \frac{a^3}{9}\cdot\frac{a^2-x^2}{a^3}\cdot\sqrt{a^2-x^2} + C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} - \frac{a^2}{3}\cdot\sqrt{a^2-x^2} + \frac{a^2-x^2}{9}\cdot\sqrt{a^2-x^2} + C$$

$$= \frac{x^3}{3}\cdot arcsin\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

119. $\int arctan\frac{x}{a}dx = x\cdot arctan\frac{x}{a} - \frac{a}{2}\cdot ln\,(a^2+x^2) + C \qquad (a>0)$

证明：$\int arctan\frac{x}{a}dx = x\cdot arctan\frac{x}{a} - \int x\,d\,x\cdot arctan\frac{x}{a}$

$$= x\cdot arctan\frac{x}{a} - \int x\cdot\frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}dx$$

$$= x\cdot arctan\frac{x}{a} - a\int\frac{x}{a^2+x^2}dx$$

$$= x\cdot arctan\frac{x}{a} - \frac{a}{2}\int\frac{1}{a^2+x^2}dx^2$$

$$= x\cdot arctan\frac{x}{a} - \frac{a}{2}\int\frac{1}{a^2+x^2}d(a^2+x^2)$$

$$= x\cdot arctan\frac{x}{a} - \frac{a}{2}\cdot ln\left|\,a^2+x^2\,\right| + C$$

$\because\ a^2+x^2>0$

$$\therefore\ \int arctan\frac{x}{a}dx = x\cdot arctan\frac{x}{a} - \frac{a}{2}\cdot ln\,(a^2+x^2) + C$$

120. $\int x\cdot arctan\frac{x}{a}dx=\frac{1}{2}(a^2+x^2)\cdot arctan\frac{x}{a}-\frac{a}{2}\cdot x+C \qquad (a>0)$

证明：令$t=arctan\frac{x}{a}$，则 $x=a\cdot tan\,t$

$$\therefore \int x\cdot arctan\frac{x}{a}dx=\int a\cdot tan\,t\cdot t\,d(a\cdot tan\,t)=a^2\int t\cdot sec^2 t\cdot tan\,t\,dt$$

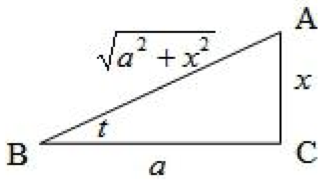
$$=\frac{a^2}{2}\int t\,d\,sec^2 t$$

$$=\frac{a^2}{2}\cdot t\cdot sec^2 t-\frac{a^2}{2}\int sec^2 t\,dt$$

$$=\frac{a^2}{2}\cdot t\cdot sec^2 t-\frac{a^2}{2}\cdot tan\,t+C$$

在Rt ΔABC中，可设$\angle B=t$，$|BC|=a$，则$|AC|=x$，$|AB|=\sqrt{a^2+x^2}$

$$\therefore\ sec\,t=\frac{1}{cos\,t}=\frac{\sqrt{a^2+x^2}}{a}\ ,\ tan\,t=\frac{x}{a}$$



$$\therefore \int x\cdot arctan\frac{x}{a}dx=\frac{a^2}{2}\cdot arctan\frac{x}{a}\cdot\frac{a^2+x^2}{a^2}-\frac{a^2}{2}\cdot\frac{x}{a}+C$$

$$=\frac{1}{2}(a^2+x^2)\cdot arctan\frac{x}{a}-\frac{a}{2}\cdot x+C$$

121. $\int x^2\cdot arctan\frac{x}{a}dx=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln(a^2+x^2)+C \qquad (a>0)$

证明：$\therefore \int x^2\cdot arctan\frac{x}{a}dx=\frac{1}{3}\int arctan\frac{x}{a}dx^3$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{1}{3}\int x^3\cdot\frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}dx$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{3}\int\frac{x^3}{a^2+x^2}dx$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\int\frac{x^2}{a^2+x^2}dx^2$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\int\frac{x^2+a^2-a^2}{a^2+x^2}dx^2$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\int dx^2+\frac{a}{6}\int\frac{a^2}{a^2+x^2}dx^2$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\int dx^2+\frac{a^3}{6}\int\frac{1}{a^2+x^2}d(x^2+a^2)$$

$$=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln\left|a^2+x^2\right|+C$$

$\because\ a^2+x^2>0$

$$\therefore \int x^2\cdot arctan\frac{x}{a}dx=\frac{x^3}{3}\cdot arctan\frac{x}{a}-\frac{a}{6}\cdot x^2+\frac{a^3}{6}ln(a^2+x^2)+C$$

（十三）含有指数函数的积分（122~131）

122. $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$

证明：$\int a^x dx = \frac{1}{\ln a} \int \ln a \cdot a^x dx$

$\because (a^x)' = a^x \ln a$，即$a^x \ln a$的原函数为$a^x$

$\therefore \int a^x dx = \frac{1}{\ln a} \int da^x$

$= \frac{1}{\ln a} \cdot a^x + C$

123. $\int e^{ax} dx = \frac{1}{a} \cdot e^{ax} + C$

证明：令 $ax = \mu$，则 $x = \frac{\mu}{a}$，$dx = \frac{1}{a} d\mu$

$\therefore \int e^{ax} dx = \frac{1}{a} \int e^{\mu} d\mu = \frac{1}{a} \cdot e^{\mu} + C$

$= \frac{1}{a} \cdot e^{ax} + C$

124. $\int x \cdot e^{ax} dx = \frac{1}{a^2}(ax-1)e^{ax} + C$

证明：$\int x \cdot e^{ax} dx = \frac{1}{a} \int x\, de^{ax}$

$= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a} \int e^{ax} dx$

$= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a^2} \int e^{ax} dax$

$= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a^2} e^{ax} + C$

$= \frac{1}{a^2}(ax-1)e^{ax} + C$

125. $\int x^n \cdot e^{ax} dx = \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx$

证明：$\int x^n \cdot e^{ax} dx = \frac{1}{a} \int x^n \, de^{ax}$

$= \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{1}{a} \int e^{ax} dx^n$

$= \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx$

126. $\int x \cdot a^x dx = \frac{x}{ln\, a} \cdot a^x - \frac{1}{(ln\, a)^2} \cdot a^x + C$

证明：$\int x \cdot a^x dx = \frac{1}{ln\, a}\int x\, da^x$

$$= \frac{1}{ln\, a} \cdot x \cdot a^x - \frac{1}{ln\, a}\int a^x dx \qquad \boxed{\text{公式}122: \int a^x dx = \frac{1}{ln\, a} \cdot a^x + C}$$

$$= \frac{1}{ln\, a} \cdot x \cdot a^x - \frac{1}{(ln\, a)^2} \cdot a^x + C$$

127. $\int x^n \cdot a^x dx = \frac{1}{ln\, a} \cdot x^n \cdot a^x - \frac{n}{ln\, a}\int x^{n-1} \cdot a^x dx$

证明：$\int x^n \cdot a^x dx = \frac{1}{ln\, a}\int x^n\, da^x$

$$= \frac{1}{ln\, a} \cdot x^n \cdot a^x - \frac{1}{ln\, a}\int a^x dx^n$$

$$= \frac{1}{ln\, a} \cdot x^n \cdot a^x - \frac{n}{ln\, a}\int x^{n-1} \cdot a^x dx$$

128. $\int e^{ax} \cdot sin\, bx\, dx = \frac{1}{a^2+b^2} \cdot e^{ax}(a \cdot sin\, bx - b \cdot cos\, bx) + C$

证明：$\int e^{ax} \cdot sin\, bx\, dx = -\frac{1}{b}\int e^{ax} d\, cos\, bx$

$$= -\frac{1}{b} \cdot e^{ax} \cdot cos\, bx + \frac{1}{b}\int cos\, bx de^{ax}$$

$$= -\frac{1}{b} \cdot e^{ax} \cdot cos\, bx + \frac{a}{b^2} \cdot e^{ax} \cdot sin\, bx - \frac{a}{b^2}\int sin\, bx\, de^{ax}$$

$$= -\frac{1}{b} \cdot e^{ax} \cdot cos\, bx + \frac{a}{b^2} \cdot e^{ax} \cdot sin\, bx - \frac{a}{b^2}\int sin\, bx\, de^{ax}$$

移项并整理得：$\frac{a^2+b^2}{b^2}\int e^{ax} \cdot sin\, bx\, dx = -\frac{1}{b} \cdot e^{ax} \cdot cos\, bx + \frac{a}{b^2} \cdot e^{ax} \cdot sin\, bx + C$

$\therefore \int e^{ax} \cdot sin\, bx\, dx = -\frac{b}{a^2+b^2} \cdot e^{ax} \cdot cos\, bx + \frac{a}{a^2+b^2} \cdot e^{ax} \cdot sin\, bx + C$

$$= \frac{1}{a^2+b^2} \cdot e^{ax}(a \cdot sin\, bx - b \cdot cos\, bx) + C$$

129. $\int e^{ax}\cdot \cos bxdx=\dfrac{1}{a^2+b^2}\cdot e^{ax}(b\cdot \sin bx+a\cdot \cos bx)+C$

证明：$\int e^{ax}\cdot \cos bxdx=\dfrac{1}{b}\int e^{ax}d\sin bx$

$$
\begin{aligned}
&=\frac{1}{b}\cdot e^{ax}\cdot \sin bx-\frac{1}{b}\int \sin bxde^{ax}\\
&=\frac{1}{b}\cdot e^{ax}\cdot \sin bx-\frac{a}{b}\int \sin bx\cdot e^{ax}dx\\
&=\frac{1}{b}\cdot e^{ax}\cdot \sin bx+\frac{a}{b^2}\int e^{ax}d\cos bx\\
&=\frac{1}{b}\cdot e^{ax}\cdot \sin bx+\frac{a}{b^2}\cdot e^{ax}\cdot \cos bx-\frac{a}{b^2}\int \cos bxde^{ax}\\
&=\frac{1}{b}\cdot e^{ax}\cdot \sin bx+\frac{a}{b^2}\cdot e^{ax}\cdot \cos bx-\frac{a^2}{b^2}\int e^{ax}\cdot \cos bxdx
\end{aligned}
$$

$\therefore\ (1+\dfrac{a^2}{b^2})\int e^{ax}\cdot \cos bxdx=\dfrac{a^2+b^2}{b^2}\int e^{ax}\cdot \cos bxdx=\dfrac{1}{b}\cdot e^{ax}\cdot \sin bx+\dfrac{a}{b^2}\cdot e^{ax}\cdot \cos bx$

$\therefore\ \int e^{ax}\cdot \cos bxdx=\dfrac{1}{a^2+b^2}\cdot e^{ax}(b\cdot \sin bx+a\cdot \cos bx)+C$

130. $\int e^{ax}\cdot sin^n bx\,dx=\dfrac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot sin^{n-1}bx(a\cdot sin\,bx-nb\cdot cos\,bx)$

$$+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot sin^{n-2}bx\,dx$$

证明：$\int e^{ax}\cdot sin^n bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\cdot sin^2bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\cdot(1-cos^2bx)\,dx$

$$=\int e^{ax}\cdot sin^{n-2}bx\,dx-\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2bx\,dx \qquad ①$$

又 $\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2bx\,dx=\dfrac{1}{b\cdot(n-1)}\int e^{ax}\cdot cos\,bx\,d\,sin^{n-1}bx$

$$=\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos\,bx\cdot sin^{n-1}bx-\frac{1}{b\cdot(n-1)}\int sin^{n-1}bx\,d(e^{ax}\cdot cos\,bx) \qquad ②$$

又 $\int sin^{n-1}bx\,d(e^{ax}\cdot cos\,bx)=\int sin^{n-1}bx\,(a\cdot e^{ax}\cdot cos\,bx-b\cdot sin\,bx\cdot e^{ax})dx$

$$=a\int e^{ax}\cdot sin^{n-1}bx\cdot cos\,bx\,dx-b\int sin^n bx\cdot e^{ax}dx \qquad ③$$

又 $\int e^{ax}\cdot sin^{n-1}bx\cdot cos\,bx\,dx=\dfrac{1}{b}\int e^{ax}\cdot sin^{n-1}bx\,d\,sin\,bx$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{1}{b}\int sin\,bx\,d(e^{ax}\cdot sin^{n-1}bx)$$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{1}{b}\int sin\,bx[a\cdot e^{ax}\cdot sin^{n-1}bx+b\cdot(n-1)sin^{n-2}bx\cdot cos\,bx\cdot e^{ax}]dx$$

$$=\frac{1}{b}\cdot e^{ax}\cdot sin^n bx-\frac{a}{b}\int sin^n bx\cdot e^{ax}dx-(n-1)\int e^{ax}\cdot sin^{n-1}bx\cdot cos\,bx\,dx$$

移项并整理得：$\int e^{ax}\cdot sin^{n-1}bx\cdot cos\,bx\,dx=\dfrac{1}{bn}\cdot e^{ax}\cdot sin^n bx-\dfrac{a}{bn}\int sin^n bx\cdot e^{ax}dx$ ④

将④式代入③式的得：$\int sin^{n-1}bx\,d(e^{ax}\cdot cos\,bx)$

$$=\frac{a}{bn}\cdot e^{ax}\cdot sin^n bx-\frac{a^2}{bn}\int sin^n bx\cdot e^{ax}dx-b\int sin^n bx\cdot e^{ax}dx$$

$$=\frac{a}{bn}\cdot e^{ax}\cdot sin^n bx-\frac{a^2+b^2n}{bn}\int sin^n bx\cdot e^{ax}dx \qquad ⑤$$

将⑤式代入②式得：$\int e^{ax}\cdot sin^{n-2}bx\cdot cos^2bx\,dx=\dfrac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos\,bx\cdot sin^{n-1}bx$

$$-\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot sin^n bx+\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int sin^n bx\cdot e^{ax}dx \qquad ⑥$$

将⑥式代入①式得：$\int e^{ax}\cdot sin^n bx\,dx=\int e^{ax}\cdot sin^{n-2}bx\,dx-\dfrac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos\,bx\cdot sin^{n-1}bx$

$$+\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot sin^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int sin^n bx\cdot e^{ax}dx$$

移项并整理得：$\int e^{ax}\cdot sin^n bx\,dx$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\left[\int e^{ax}\cdot sin^{n-2}bx\,dx-\frac{1}{b\cdot(n-1)}\cdot e^{ax}\cdot cos\,bx\cdot sin^{n-1}bx+\frac{1}{n\cdot(n-1)b^2}\cdot e^{ax}\cdot sin^n bx\right]$$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\cdot\int e^{ax}\cdot sin^{n-2}bx\,dx-\frac{bn}{a^2+b^2n^2}\cdot e^{ax}\cdot cos\,bx\cdot sin^{n-1}bx+\frac{a}{a^2+b^2n^2}\cdot e^{ax}\cdot sin^n bx$$

$$=\frac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot sin^{n-1}bx(a\cdot sin\,bx-nb\cdot cos\,bx)$$

$$+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot sin^{n-2}bx\,dx$$

131. $\int e^{ax}\cdot cos^n bx\,dx=\dfrac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot cos^{n-1}bx(a\cdot cos\,bx+nb\cdot sin\,bx)$

$$+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot cos^{n-2}bx\,dx$$

证明：$\int e^{ax}\cdot cos^n bx\,dx=\int e^{ax}\cdot cos^{n-2}bx\cdot cos^2 bx\,dx=\int e^{ax}\cdot cos^{n-2}bx\cdot(1-sin^2 bx)\,dx$

$$=\int e^{ax}\cdot cos^{n-2}bx\,dx-\int e^{ax}\cdot cos^{n-2}bx\cdot sin^2 bx\,dx \qquad ①$$

又 $\int e^{ax}\cdot cos^{n-2}bx\cdot sin^2 bx\,dx=\dfrac{1}{b\cdot(1-n)}\int e^{ax}\cdot sin\,bx\,d\,cos^{n-1}bx$

$$=\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot sin\,bx\cdot cos^{n-1}bx-\frac{1}{b\cdot(1-n)}\int cos^{n-1}bx\,d(e^{ax}\cdot sin\,bx) \qquad ②$$

又 $\int cos^{n-1}bx\,d(e^{ax}\cdot sin\,bx)=\int cos^{n-1}bx\,(a\cdot e^{ax}\cdot sin\,bx+b\cdot cos\,bx\cdot e^{ax})dx$

$$=a\int e^{ax}\cdot cos^{n-1}bx\cdot sin\,bx\,dx+b\int cos^n bx\cdot e^{ax}dx \qquad ③$$

又 $\int e^{ax}\cdot cos^{n-1}bx\cdot sin\,bx\,dx=-\dfrac{1}{b}\int e^{ax}\cdot cos^{n-1}bx\,d\,cos\,bx$

$$=-\frac{1}{b}\cdot e^{ax}\cdot cos^n bx+\frac{1}{b}\int cos\,bx\,d(e^{ax}\cdot cos^{n-1}bx)$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot cos^n bx+\frac{1}{b}\int cos\,bx[a\cdot e^{ax}\cdot cos^{n-1}bx-b\cdot(n-1)cos^{n-2}bx\cdot sin\,bx\cdot e^{ax}]dx$$

$$=-\frac{1}{b}\cdot e^{ax}\cdot cos^n bx+\frac{a}{b}\int cos^n bx\cdot e^{ax}dx-(n-1)\int e^{ax}\cdot cos^{n-1}bx\cdot sin\,bx\,dx$$

移项并整理得：$\int e^{ax}\cdot cos^{n-1}bx\cdot sin\,bx\,dx=-\dfrac{1}{bn}\cdot e^{ax}\cdot cos^n bx+\dfrac{a}{bn}\int cos^n bx\cdot e^{ax}dx \qquad ④$

将④式代入③式的得：$\int cos^{n-1}bx\,d(e^{ax}\cdot sin\,bx)$

$$=-\frac{a}{bn}\cdot e^{ax}\cdot cos^n bx+\frac{a^2}{bn}\int cos^n bx\cdot e^{ax}dx+b\int cos^n bx\cdot e^{ax}dx$$

$$=-\frac{a}{bn}\cdot e^{ax}\cdot cos^n bx+\frac{a^2+b^2n}{bn}\int cos^n bx\cdot e^{ax}dx \qquad ⑤$$

将⑤式代入②式得：$\int e^{ax}\cdot cos^{n-2}bx\cdot sin^2 bx\,dx=\dfrac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot sin\,bx\cdot cos^{n-1}bx$

$$+\frac{a}{b^2\cdot n\cdot(1-n)}\cdot e^{ax}\cdot cos^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(1-n)}\int cos^n bx\cdot e^{ax}dx \qquad ⑥$$

将⑥式代入①式得：$\int e^{ax}\cdot cos^n bx\,dx=\int e^{ax}\cdot cos^{n-2}bx\,dx-\dfrac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot sin\,bx\cdot cos^{n-1}bx$

$$+\frac{a}{b^2\cdot n\cdot(n-1)}\cdot e^{ax}\cdot cos^n bx-\frac{a^2+b^2n}{b^2\cdot n\cdot(n-1)}\int cos^n bx\cdot e^{ax}dx$$

移项并整理得：$\int e^{ax}\cdot cos^n bx\,dx$

$$=\frac{n\cdot(1-n)b^2}{-a^2-b^2n^2}\left[\int e^{ax}\cdot cos^{n-2}bx\,dx-\frac{1}{b\cdot(1-n)}\cdot e^{ax}\cdot sin\,bx\cdot cos^{n-1}bx-\frac{a}{n\cdot(1-n)b^2}\cdot e^{ax}\cdot cos^n bx\right]$$

$$=\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\cdot\int e^{ax}\cdot cos^{n-2}bx\,dx+\frac{bn}{a^2+b^2n^2}\cdot e^{ax}\cdot sin\,bx\cdot cos^{n-1}bx+\frac{a}{a^2+b^2n^2}\cdot e^{ax}\cdot cos^n bx$$

$$=\frac{1}{a^2+b^2n^2}\cdot e^{ax}\cdot cos^{n-1}bx(a\cdot cos\,bx+nb\cdot sin\,bx)+\frac{n\cdot(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cdot cos^{n-2}bx\,dx$$

（十四）含有对数函数的积分（132~136）

132. $\int ln\,x dx = x\cdot ln\,x - x + C$

证明：$\int ln\,x dx = x\cdot ln\,x - \int x\,d\,ln\,x$

$$= x\cdot ln\,x - \int x\cdot\frac{1}{x}dx$$

$$= x\cdot ln\,x - \int dx$$

$$= x\cdot ln\,x - x + C$$

133. $\int\frac{dx}{x\cdot ln\,x}dx = ln\left|\,ln\,x\,\right| + C$

证明：$\int\frac{dx}{x\cdot ln\,x}dx = \int\frac{1}{ln\,x}d\,ln\,x$

$$= ln\left|\,ln\,x\,\right| + C$$

$\boxed{\text{提示：}(ln\,x)' = \frac{1}{x}}$

134. $\int x^n\cdot ln\,x\,dx = \frac{1}{n+1}\cdot x^{n+1}(ln\,x - \frac{1}{n+1}) + C$

证明：$\int x^n\cdot ln\,x\,dx = \int\frac{ln\,x}{n+1}\cdot(n+1)\cdot x^n dx$

$$= \int\frac{ln\,x}{n+1}dx^{n+!}$$

$$= \frac{ln\,x}{n+1}\cdot x^{n+1} - \frac{1}{n+1}\int x^{n+1}d\,ln\,x$$

$$= \frac{ln\,x}{n+1}\cdot x^{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{ln\,x}{n+1}\cdot x^{n+1} - (\frac{1}{n+1})^2\cdot x^{n+1} + C$$

$$= \frac{1}{n+1}\cdot x^{n+1}(ln\,x - \frac{1}{n+1}) + C$$

135. $\int (lnx)^n dx = x\cdot(lnx)^n - n\int (lnx)^{n-1}dx$

$$= x\sum_{k=0}^{n}(-1)^{n-k}\cdot\frac{n!}{k!}\cdot(lnx)^k$$

证明：$\int (lnx)^n dx = x\cdot(lnx)^n - \int xd(\ln x)^n$

$$= x\cdot(lnx)^n - \int x\cdot n\cdot(\ln x)^{n-1}\cdot\frac{1}{x}dx$$

$$= x\cdot(lnx)^n - n\int (lnx)^{n-1}dx$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\int xd(\ln x)^{n-1}$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\int (lnx)^{n-2}dx$$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\cdot x\cdot(lnx)^{n-2} - n\cdot(n-1)\cdot(n-2)\int (lnx)^{n-3}dx$$

$\cdots\cdots\cdot$

$$= x\cdot(lnx)^n - n\cdot x\cdot(lnx)^{n-1} + n\cdot(n-1)\cdot x\cdot(lnx)^{n-2} - n\cdot(n-1)\cdot(n-2)(lnx)^{n-3}$$

$$+\cdots\cdots+(-1)^{n-k}\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)\cdot(lnx)^{n-k}+\cdots\cdots$$

$$+(-1)^2\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots 5\times4\times3\cdot(lnx)^{3-1}\cdot x$$

$$+(-1)^1\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots 4\times3\times2\cdot(lnx)^{2-1}\cdot x$$

$$+(-1)^0\cdot n\cdot(n-1)\cdot(n-2)\cdots\cdots 3\times2\times1\cdot(lnx)^{1-1}\cdot x$$

$$= x\sum_{k=0}^{n}(-1)^{n-k}\cdot\frac{n!}{k!}\cdot(lnx)^k$$

136. $\int x^m\cdot(\ln x)^{\mathrm{n}}\,dx = \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^m\cdot(\ln x)^{\mathrm{n}-1}\,dx$

证明：$\int x^m\cdot(\ln x)^{\mathrm{n}}\,dx = \frac{1}{m+1}\int(\ln x)^{\mathrm{n}}dx^{m+1}$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{1}{m+1}\int x^{m+1}\,d(\ln x)^{\mathrm{n}}$$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^{m+1}\cdot(\ln x)^{\mathrm{n}-1}\cdot\frac{1}{x}dx$$

$$= \frac{1}{m+1}\cdot x^{m+1}\cdot(\ln x)^{\mathrm{n}} - \frac{\mathrm{n}}{m+1}\int x^m\cdot(\ln x)^{\mathrm{n}-1}\,dx$$

（十五）含有双曲函数的积分（137~141）

137. $\int shx\, dx = chx + C$

证明：$\because\ (chx)' = shx$，即chx为shx的原函数

$$\therefore \int shx\, dx = \int d\, chx = chx + C$$

138. $\int ch\, x\, dx = shx + C$

证明：$\because\ (shx)' = chx$，即shx为chx的原函数

$$\therefore \int ch\, x\, dx = \int d\, shx = shx + C$$

139. $\int th\, x\, dx = ln\, chx + C$

证明：
$$\int th\, x\, dx = \int \frac{shx}{chx} dx = \int \frac{1}{chx} d\, chx = ln\, chx + C$$

140. $\int sh^2 x\, dx = -\frac{x}{2} + \frac{1}{4} sh\, 2x + C$

证明：
$$\int sh^2 x\, dx = \int \left(\frac{e^x - e^{-x}}{2} \right)^2 dx = \frac{1}{4} \int (e^{2x} + e^{-2x} - 2) dx = \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} - \frac{x}{2} + C = -\frac{x}{2} + \frac{1}{4} \cdot \frac{e^{2x} - e^{-2x}}{2} + C = -\frac{x}{2} + \frac{1}{4} \cdot sh\, 2x + C$$

提示：$chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)

$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

141. $\int ch^2 x\, dx = \frac{x}{2} + \frac{1}{4} \cdot sh\, 2x + C$

证明：
$$\int ch^2 x\, dx = \int \left(\frac{e^x + e^{-x}}{2} \right)^2 dx = \frac{1}{4} \int (e^{2x} + e^{-2x} + 2) dx = \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} + \frac{x}{2} + C = \frac{x}{2} + \frac{1}{4} \cdot \frac{e^{2x} - e^{-2x}}{2} + C = \frac{x}{2} + \frac{1}{4} \cdot sh\, 2x + C$$

提示：$chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)

$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

（十六）定积分（142~147）

142. $\int_{-\pi}^{\pi} cos\, nx\, dx = \int_{-\pi}^{\pi} sin\, nx\, dx = 0$

证明①：$\int_{-\pi}^{\pi} cos\, nx\, dx = \frac{1}{n}\int_{-\pi}^{\pi} cos\, nx\, dnx$

$$= \frac{1}{n}\cdot(sin\, nx\Big|_{-\pi}^{\pi})$$

$$= \frac{1}{n}\cdot sin\,(n\pi) - \frac{1}{n}\cdot sin\,(-n\pi)$$

$$= \frac{2}{n}\cdot sin\,(n\pi)$$

$$= 0$$

证明②：$\int_{-\pi}^{\pi} sin\, nx\, dx = \frac{1}{n}\int_{-\pi}^{\pi} sin\, nx\, dnx$

$$= -\frac{1}{n}\cdot(cos\, nx\Big|_{-\pi}^{\pi})$$

$$= -\frac{1}{n}\cdot cos\,(n\pi) + \frac{1}{n}\cdot cos\,(-n\pi)$$

$$= 0$$

综合证明①②得：$\int_{-\pi}^{\pi} cos\, nx\, dx = \int_{-\pi}^{\pi} sin\, nx\, dx = 0$

143. $\int_{-\pi}^{\pi} cos\, mx\cdot sin\, nx\, dx = 0$

$\boxed{\text{公式 100：}\int sin\ ax\cdot cos\ bx\ dx = -\frac{1}{2(a+b)}\cdot cos\,(a+b)x - \frac{1}{2(a-b)}\cdot cos\,(a-b)x + C}$

证明：1.当$m \neq n$时

$$\int_{-\pi}^{\pi} cos\, mx\cdot sin\, nx\, dx = -\frac{1}{2(m+n)}\cdot cos(m+n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(n-m)}cos(n-m)x\Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{2(m+n)}[cos(m+n)\pi - cos(m+n)\pi] - \frac{1}{2(n-m)}[cos(n-m)\pi - cos(n-m)(-\pi)]$$

$$= 0 + 0 = 0$$

2.当$m = n$时

$$\int_{-\pi}^{\pi} cos\, mx\cdot sin\, nx\, dx = \int_{-\pi}^{\pi} cos\, mx\cdot sin\, mx\, dx$$

$\boxed{\text{提示：}sin\, 2x = 2\cdot sin\, x\cdot cos\, x}$

$$= \frac{1}{2m}\int_{-\pi}^{\pi} sin\, 2mx\ d\ mx$$

$$= \frac{1}{4m}\int_{-\pi}^{\pi} sin\, 2mx\ d\ 2mx$$

$$= -\frac{1}{4m}\cdot cos\, 2mx\Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{4m}\cdot[cos\, 2m\pi - cos(-2m\pi)]$$

$$= 0$$

综合讨论1,2得：$\int_{-\pi}^{\pi} cos\, nx\, dx = \int_{-\pi}^{\pi} cos\, mx\cdot sin\, nx\, dx = 0$

144. $\int_{-\pi}^{\pi} cos\, mx \cdot cos\, nx\, dx = \begin{cases} 0\,, & m \neq n \\ \pi, & m = n \end{cases}$

证明：1.当$m \neq n$时

$$\int_{-\pi}^{\pi} cos\, mx \cdot cos\, nx\, dx = \frac{1}{2(m+n)} \cdot sin\,(m+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)} sin\,(m-n)x \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2(m+n)}[sin\,(m+n)\pi - sin\,(m+n)(-\pi)] - \frac{1}{2(m-n)}[sin\,(m-n)\pi + sin\,(m-n)(-\pi)]$$

$$= 0 - 0 = 0$$

公式102：$\int cos\, ax \cdot cos\, bx\, dx = \frac{1}{2(a+b)} \cdot sin\,(a+b)x + \frac{1}{2(a-b)} \cdot sin\,(a-b)x + C$

2.当$m = n$时

$$\int_{-\pi}^{\pi} cos\, mx \cdot cos\, nx\, dx = \int_{-\pi}^{\pi} cos\, mx \cdot cos\, mx\, dx$$

$$= \frac{1}{m}\int_{-\pi}^{\pi} cos^2\, mx\, d\, mx$$

公式 94：$\int cos^2 x\, dx = \frac{x}{2} + \frac{1}{4} \cdot sin\, 2x + C$

$$= \frac{1}{4m} \cdot sin\, 2mx \Big|_{-\pi}^{\pi} + \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{4m} \cdot [sin\, 2m\pi - sin\,(-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

综合讨论1,2得：$\int_{-\pi}^{\pi} cos\, mx \cdot cos\, nx\, dx = \begin{cases} 0\,, & m \neq n \\ \pi, & m = n \end{cases}$

145. $\int_{-\pi}^{\pi} sin\, mx \cdot sin\, nx\, dx = \begin{cases} 0\,, & m \neq n \\ \pi, & m = n \end{cases}$

证明：1.当$m \neq n$时

$$\int_{-\pi}^{\pi} sin\, mx \cdot sin\, nx\, dx = -\frac{1}{2(m+n)} \cdot sin\,(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)} sin\,(m-n)x \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{2(m+n)}[sin\,(m+n)\pi - sin\,(m+n)(-\pi)] + \frac{1}{2(m-n)}[sin\,(m-n)\pi - sin\,(m-n)(-\pi)]$$

$$= 0 + 0 = 0$$

公式101：$\int sin\, ax \cdot sin\, bx\, dx = -\frac{1}{2(a+b)} \cdot sin\,(a+b)x + \frac{1}{2(a-b)} \cdot sin\,(a-b)x + C$

2.当$m = n$时

$$\int_{-\pi}^{\pi} sin\, mx \cdot sin\, nx\, dx = \int_{-\pi}^{\pi} sin^2\, mx\, dx$$

公式 93：$\int sin^2 x\, dx = \frac{x}{2} - \frac{1}{4} \cdot sin\, 2x + C$

$$= \frac{1}{m}\int_{-\pi}^{\pi} sin^2\, mx\, d\, mx$$

$$= \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi} - \frac{1}{4m} \cdot sin\, 2mx \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{4m} \cdot [sin\, 2m\pi - sin\,(-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

综合讨论1,2得：$\int_{-\pi}^{\pi} sin\, mx \cdot sin\, nx\, dx = \begin{cases} 0\,, & m \neq n \\ \pi, & m = n \end{cases}$

146. $\int_0^{\pi} sin\,mx \cdot sin\,nx\,dx = \int_0^{\pi} cos\,mx \cdot cos\,nx\,dx \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \end{cases}$

证明：1.当$m \neq n$时

$$\int_0^{\pi} sin\,mx \cdot sin\,nx\,dx = -\frac{1}{2(m+n)} \cdot sin\,(m+n)x\Big|_0^{\pi} + \frac{1}{2(m-n)} sin\,(m-n)x\Big|_0^{\pi}$$

$$= -\frac{1}{2(m+n)}[sin\,(m+n)\pi - sin\,0] + \frac{1}{2(m-n)}[sin\,(m-n)\pi - sin\,0]$$

$$= 0 + 0 = 0$$

$$\int_0^{\pi} cos\,mx \cdot cos\,nx\,dx = \frac{1}{2(m+n)} \cdot sin\,(m+n)x\Big|_0^{\pi} + \frac{1}{2(m-n)} sin\,(m-n)x\Big|_0^{\pi}$$

$$= \frac{1}{2(m+n)}[sin\,(m+n)\pi - sin\,0] + \frac{1}{2(m-n)}[sin\,(m-n)\pi + sin\,0]$$

$$= 0 + 0 = 0$$

2.当$m = n$时

$$\int_0^{\pi} sin\,mx \cdot sin\,nx\,dx = \int_0^{\pi} sin^2\,mx\,dx$$

$$= \frac{1}{m}\int_0^{\pi} sin^2\,mx\,d\,mx$$

$$= \frac{1}{2m} \cdot mx\Big|_0^{\pi} - \frac{1}{4m} \cdot sin\,2mx\Big|_0^{\pi}$$

$$= -\frac{1}{4m} \cdot [sin\,2m\pi - sin\,0] + \frac{\pi}{2} + 0$$

$$= \frac{\pi}{2}$$

$$\int_0^{\pi} cos\,mx \cdot cos\,nx\,dx = \int_0^{\pi} cos\,mx \cdot cos\,mx\,dx$$

$$= \frac{1}{m}\int_0^{\pi} cos^2\,mx\,d\,mx$$

$$= \frac{1}{4m} \cdot sin\,2mx\Big|_0^{\pi} + \frac{1}{2m} \cdot mx\Big|_0^{\pi}$$

$$= \frac{1}{4m} \cdot [sin\,2m\pi - sin\,0] + \frac{\pi}{2} + 0$$

$$= \frac{\pi}{2}$$

综合讨论1,2得：$\int_0^{\pi} sin\,mx \cdot sin\,nx\,dx = \int_0^{\pi} cos\,mx \cdot cos\,nx\,dx \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \end{cases}$

以上所用公式：

公式 101：$\int sin\ ax \cdot sin\ bx\ dx = -\frac{1}{2(a+b)} \cdot sin\ (a+b)x + \frac{1}{2(a-b)} \cdot sin\ (a-b)x + C$

公式 102：$\int cos\ ax \cdot cos\ bx\ dx = \frac{1}{2(a+b)} \cdot sin\ (a+b)x + \frac{1}{2(a-b)} \cdot sin\ (a-b)x + C$

公式 93：$\int sin^2 x\ dx = \frac{x}{2} - \frac{1}{4} \cdot sin\ 2x + C$

公式 94：$\int cos^2\ x\ dx = \frac{x}{2} + \frac{1}{4} \cdot sin\ 2x + C$

147. $I_n=\int_0^{\frac{\pi}{2}} sin^n x\,dx=\int_0^{\frac{\pi}{2}} cos^n x\,dx$

$$I_n=\frac{n-1}{n}I_{n-2}$$

$$=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\ (n\text{为大于1的正奇数}),\ I_1=1\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\ (n\text{为正偶数}),\ I_0=\frac{\pi}{2}\end{cases}$$

证明①：$I_n=\int_0^{\frac{\pi}{2}} sin^n x\,dx=-\frac{1}{n}\cdot sin^{n-1}x\cdot cos\,x\Big|_0^{\frac{\pi}{2}}+\frac{n-1}{n}\int_0^{\frac{\pi}{2}} sin^{n-2}x\,dx$

$$=-\frac{1}{n}(sin^{n-1}\frac{\pi}{2}\cdot cos\frac{\pi}{2}-sin^{n-1}0\cdot cos0)+\frac{n-1}{n}\int_0^{\frac{\pi}{2}} sin^{n-2}x\,dx$$

$$=\frac{n-1}{n}\int_0^{\frac{\pi}{2}} sin^{n-2}x\,dx=\frac{n-1}{n}I_{n-2}$$

当n为正奇数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\int_0^{\frac{\pi}{2}} sinx\,dx$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot(-cos\,x)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

特别的，当$n=1$时，$I_n=\int_0^{\frac{\pi}{2}} sinx\,dx=(-cos\,x)\Big|_0^{\frac{\pi}{2}}=1$

当n为正偶数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} sin^0 x\,dx$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot(x)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

特别的，当$n=0$时，$I_n=\int_0^{\frac{\pi}{2}} sin^0 x\,dx=(x)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}$

证明②：$I_n=\int_0^{\frac{\pi}{2}} cos^n x\,dx\cdots\cdots$亦同理可证

附录：常数和基本初等函数导数公式

1. $(C)' = 0$ $\quad$ (C为常数)

2. $(x^{\mu})' = \mu \cdot x^{\mu-1}$ $\quad$ $(x \neq 0)$

3. $(\sin x)' = \cos x$

4. $(\cos x)' = -\sin x$

5. $(\tan x)' = \sec^2 x$

6. $(\cot x)' = -\csc^2 x$

7. $(\sec x)' = \sec x \cdot \tan x$

8. $(\csc x)' = -\csc x \cdot \cot x$

9. $(a^x)' = a^x \cdot \ln a$ $\quad$ (a为常数)

10. $(e^x)' = e^x$

11. $(\log_a x)' = \dfrac{1}{x \cdot \ln a}$ $\quad$ $(a > 0)$

12. $(\ln x)' = \dfrac{1}{x}$

13. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

14. $(\arccos x)' = \dfrac{1}{-\sqrt{1-x^2}}$

15. $(\arctan x)' = \dfrac{1}{1+x^2}$

16. $(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$

说 明

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2.本讲义为方便各位学友阅读，排版采用每一单面都是一个或几个完整证明过程的原则;

3.由于本讲义编辑的比较匆忙，难免有些推导和输入错误，还望广大学友给予批评和指正。反馈邮箱 L0VE1193345021@qq.com

2013 年 5 月

献给

我们的

大一