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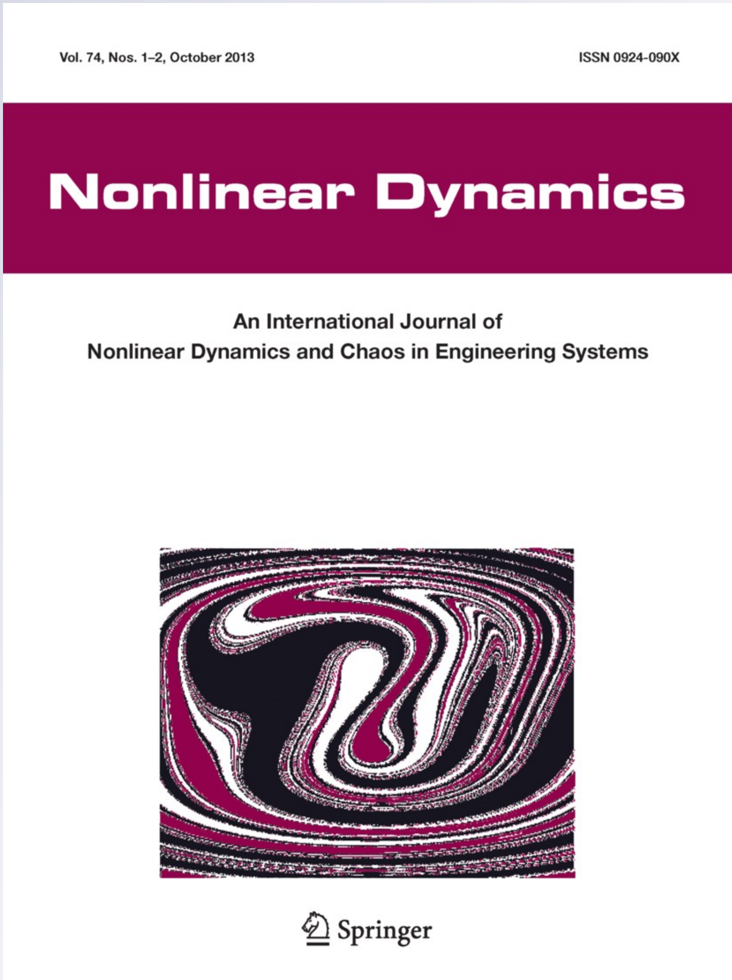

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ORIGINAL PAPER

On the stability analysis of microscopic traffic car-following model: a case study

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Abstract The stability analysis of the microscopic traffic car-following model is an important issue. The present paper systematically discusses the local stability and asymptotic stability of the car-following model; meanwhile, the corresponding Lyapunov stability is also proposed from the viewpoint of control. Moreover, taking the full velocity difference (FVD) car-following model as a case, the difference among the three stability analysis approaches and the simulation are conducted. Finally, the results reveal that it can improve the dynamic performance when keep the value of the gain factor k constant and increase the value of sensitivity coefficient of velocity difference λ; and so is it when it keeps the value of a sensitivity coefficient of velocity difference λ constant and increases the value of the gain factor k, while the value of the gain factor k is dominant in this process.

Y. Li (✉) · H. Zhu · M. Cen · Y. Li · R. Li
College of Automation, Chongqing University of Posts and Telecommunications (CQUPT), Chongqing, 400065, P.R. China
e-mail: laf1212@163.com

Y. Li · H. Zhu · M. Cen · Y. Li · R. Li
Automotive Electronics and Embedded System Engineering Research Center, CQUPT, 400065, Chongqing, P.R. China

D. Sun
College of Automation, Chongqing University, Chongqing, 400044, P.R. China

Keywords Car-following model · Local stability · Asymptotic stability · Lyapunov stability

1 Introduction

Nowadays, traffic problems have received considerable attention in academia and society. To exactly describe various complex traffic phenomena, researchers propose a lot of traffic models, which mainly include microscopic traffic models [1–12] and macroscopic traffic models [13–17]. In the microscopic level, an individual vehicle is represented by a particle and the vehicle traffic is treated as a system of interacting particles driven far from equilibrium, and the car-following model becomes a highly active research area with great progress [1]. Undoubtedly, one appealing issue of car-following model is the stability analysis, which reflects the dynamic performance of the model.

However, the existing stability analysis mainly focuses on the stability condition, e.g., [2–12]. On the other hand, the idea of stability analysis derives from the viewpoint of fluid mechanics. Clearly, the traffic flow system is a nonlinear and strong coupling complex system, and the stability of traffic flow means that the ability of vehicles in the traffic system move freely with safe velocity and space distance. In other words, if the stability region is larger, the opportunity of traffic congestion or collision is lower. Therefore, the motivation of this present paper relies on the following consideration: (1) Considering the traffic flow system from the viewpoint of system and control, how about the Lyapunov stability of the car-following model? (2) Although the stability condition of car-following model has been obtained, and how about the change between the stability condition and the corresponding parameters?

The organization of this paper is as follows: in Sect. 2, we briefly give a review of car-following models; in Sect. 3, taking the FVD model as a study case, the stability analysis, which includes local stability and asymptotic stability, is discussed, while Lyapunov analysis of the FVD model is also conducted; in Sect. 4, we give some simulations and results; and then, we give the conclusion in the final section.

2 Car-following model

In the car-following theory, the relation between the preceding vehicle and the following vehicle is described in which each individual vehicle always decelerates or accelerates as a response of its surrounding stimulus. Thus, the motion equation of the nth vehicle can be presented in the following way [1]:

$$[\text{Response}]_n \propto [\text{Stimulus}]_n \tag{1}$$

Those proposed models vary according to the definitions of the stimulus. Generally speaking, the stimulus may include the speed of the vehicle, the acceleration of the vehicle, the relative speed, and spacing between the nth and $n+1$th vehicle (the nth vehicle follows the $n+1$th vehicle).

In 1995, Bando et al. [2] argued that the driver seeks a safe velocity determined by the distance from the leading vehicle in the car-following process and they proposed a car-following model, by which an optimal velocity was brought forward. The model is known as the optimal velocity (OV) model, and the mathematical formula is given by [2]:

$$a_n(t) = k\big[V\big(\Delta x_n(t)\big) - v_n(t)\big] \quad (n = 1, 2, \ldots, N) \tag{2}$$

where $x_n(t) > 0$, $v_n(t) > 0$ and $a_n(t)$ represent position (m), velocity (m/s), and acceleration (m/s^2) of the nth vehicle, respectively, $t \in \mathbb{R}$ is the time (s); $N \in \mathbb{N}$ is the number of preceding vehicles in consideration. $k \in \mathbb{R}$ and $k > 0$ is a coefficient of proportionality of the driver reaction to the stimulus, variously referred to as the sensitivity coefficient or the gain factor. And $\Delta x_n(t) = x_{n+1}(t) - x_n(t)$ is the spatial headway distance between the preceding vehicle $n+1$ and the following vehicle n at time t. And $V(\bullet)$ is the optimal velocity function, which is a function of $\Delta x_n(t)$ and its general form is as follows [2]:

$$\begin{aligned} &V\big(\Delta x_n(t)\big) \\ &\quad = \big[\tanh\big(\Delta x_n(t) - h_c\big) + \tanh(h_c)\big]v_{\max}/2 \end{aligned} \tag{3}$$

Here, $v_{\max}$ is the maximal speed of the vehicle, h_c is the safe distance, $\tanh(\bullet)$ is the hyperbolic tangent function, and (3) is a monotonically increasing function with $\Delta x_n(t)$, and it has an upper bound. Generally speaking, when $\Delta x_n(t) \to 0$, and $V(\Delta x_n(t)) \to 0$ avoid the collision, while $\Delta x_n(t) \to \infty$ and $V(\Delta x_n(t)) \to v_{\max}$. This means that the vehicles are free without interaction.

Although the OV model was in good agreement with the empirical data to some extent, however, the OV model could not explain the traffic phenomena described by Treiber et al. [3]. That is, if the preceding cars are much faster than the following ones, then the vehicle would not brake, even if its headway is smaller than the safe distance, because the headway between the two vehicles will become larger. Given the observed car-following phenomena, Jiang et al. [4] argued that the relative speed between the leading and the following vehicles had an impact on the behavior of the following driver. Consequently, they proposed a full velocity difference (FVD) model as follows by taking both positive and negative velocity difference into account [4]:

$$a_n(t) = k\big[V\big(\Delta x_n(t)\big) - v_n(t)\big] + \lambda \Delta v_n(t) \tag{4}$$

where $\Delta v_n(t) = v_{n+1}(t) - v_n(t)$ is the velocity difference between the preceding vehicle $n+1$ and the following vehicle n at time t. And $\lambda \in \mathbb{R}$ and $\lambda \geq 0$ is the sensitivity coefficient of the velocity difference.

The FVD model considers the effects of both the headway and the velocity difference; theoretically, it is more realistic and exact than the OV model. And the numerical investigations indicated that the FVD model could describe the phase transition of traffic flow and estimate the evolution of traffic congestion. Based on the OV model and FVD model, some extended car-following models have sprung up over the last few years [5–12].

3 Stability analysis

For all types of the car-following models, stability analysis is definitely of an importance issue [1–12]. In regards to car-following models, one fundamental purpose is to learn about the dynamic characteristics of traffic flow evolution in the single lane via the car-following behavior between the individual vehicles.

Taking the FVD model as a study case, the stability of the car-following model is systematically and concretely addressed in the present paper, specifically including the local stability and asymptotic stability, and the local stability is concerned about a vehicle's response to the disturbances of its proceeding vehicle while the asymptotic stability cares about the propagation of the disturbances among a platoon of vehicles. Moreover, the Lyapunov stability analysis is also conducted for comparison. Well, some assumptions are predefined as follows before detailed discussion:

Assumption 1 All the vehicles move in the single lane and no overtaking.

Assumption 2 When $t \leq 0$, all the vehicles move at a constant initial velocity; when $t = 0+$, the leading vehicle will deviate the initial position and result in the fluctuation of the steady traffic flow, therefore, the driver will coordinate the fluctuation through adjusting the vehicle speed.

The car-following phenomenon in the car-following theory is simply described as in Fig. 1.

In Fig. 1, b is the headway distance of two successive vehicles, and L is the length of the considering road.

3.1 Local stability

Local stability of traffic flow refers that in what condition the velocities of consecutive two cars roughly equal and the relative spacing distance generally maintain a certain constant value. And local stability focuses on the characteristics of the following vehicle responding to the disturbance of the preceding vehicle.

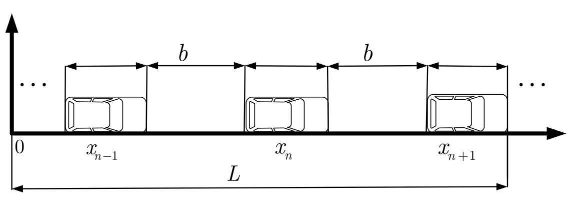

Fig. 1 Sketch of car-following phenomenon

According to [18, 19], we choose the variables $v_n(t)$ and $\Delta x_n(t)$ as the state-variables based on the dynamical equation (4), and the FVD model is rewritten as

$$\begin{cases} \dot{v}_n(t) = k[V(\Delta x_n(t)) - v_n(t)] + \lambda \Delta v_n(t) \\ \Delta \dot{x}_n(t) = v_{n+1}(t) - v_n(t) \end{cases} \tag{5}$$

supposing the lead vehicle runs constantly with speed v_0, and then the following vehicles have the following steady states [19]:

$$\begin{bmatrix} v_n^*(t) & \Delta x_n^*(t) \end{bmatrix} = \begin{bmatrix} v_0 & V^{-1}(v_0) \end{bmatrix}^{\mathrm{T}} \tag{6}$$

and the state-space expression of the FVD model will be obtained when the system (5) is linearized at the steady state (6):

$$\begin{cases} \begin{bmatrix} \delta \dot{v}_n(t) \\ \delta \Delta \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -k-\lambda & k\Gamma \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta v_n(t) \\ \delta \Delta x_n(t) \end{bmatrix} + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \delta v_{n+1}(t) \\ \delta v_n(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \delta v_n(t) \\ \delta \Delta x_n(t) \end{bmatrix} \end{cases} \tag{7}$$

where $\Delta \delta v_n(t) = \delta v_{n+1}(t) - \delta v_n(t)$. And Γ is the slope of OV function at $\Delta x_n(t) = V^{-1}(v_0)$:

$$\Gamma = \left. \frac{\mathrm{d}V(\Delta x_n(t))}{\mathrm{d}\Delta x_n(t)} \right|_{\Delta x_n(t) = V^{-1}(v_0)}$$

from frequency domain viewpoint, this linearized system can be written as

$$V_n(s) = G_{11}(s)\Delta X_n(s) + G_{12}(s)V_{n+1}(s) \tag{8}$$

where $\Delta X_n(s) = L(\delta \Delta x_n(t))$, $V_n(s) = L(\delta v_n(t))$, $V_{n+1}(s) = L(\delta v_{n+1}(t))$, $G_{11}(s) = k\Gamma/(s + k + \lambda)$, $G_{12}(s) = \lambda/(s + k + \lambda)$. $L(\bullet)$ denotes the Laplace transform.

The relation between the $n+1$th vehicle velocity disturbance and the nth vehicle velocity disturbance is described as in Fig. 2.

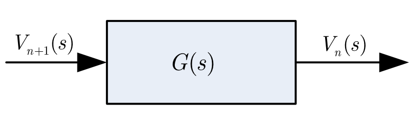

Fig. 2 Block diagram of the two successive vehicles

From Fig. 2, we can get its mathematic description [19]:

$$V_n(s) = G(s) \bullet V_{n+1}(s) \tag{9}$$

substitute (8) into (9), and the transform function $G(s)$is given by

$$G(s) = \frac{V_n(s)}{V_{n+1}(s)} = \frac{\lambda s + k\Gamma}{s^2 + (k+\lambda)s + k\Gamma} = \frac{\lambda s + k\Gamma}{d(s)} \tag{10}$$

Here, the characteristic polynomial $d(s)$ is

$$d(s) = s^2 + (k+\lambda)s + k\Gamma \tag{11}$$

Definition 1 [19] Assume that characteristic polynomial $d(s)$ is stable. If H_∞-norm of $G(s)$ is greater than 1, that is,

$$\left\|G(s)\right\|_\infty = \sup_{\omega\in[0,\infty)} \left|G(\mathrm{i}\omega)\right| > 1 \tag{12}$$

then traffic jam occurs in the OV model.

Theorem 1 *If the following condition is satisfied,*

$$\Gamma \le \lambda + k/2 \tag{13}$$

then the FVD model is local stable.

Proof Firstly, according to Hurwitz stability criterion in control theories $d(s)$ is stable if

$$\begin{cases} \lambda + k > 0 \\ k\Gamma > 0 \end{cases} \tag{14}$$

it is known that $\lambda \ge 0$, $k > 0$ and $\Gamma > 0$ during the procedure of modeling, so this inequation is held on.

Secondly, consider $\|G(s)\|_\infty \le 1$, that is,

$$\left\|\bar{G}(s)\right\|_\infty = \sup_{\omega\in[0,\infty)} \left|\bar{G}(\mathrm{i}\omega)\right| = \sup_{\omega\in[0,\infty)} \sqrt{\left(k^2\Gamma^2 + \lambda^2\omega^2\right)/\left[\left(k\Gamma - \omega^2\right)^2 + (k+\lambda)^2\omega^2\right]} \le 1 \tag{15}$$

Equation (15) is tenable if the following equation is fulfilled:

$$\sqrt{\left(k^2\Gamma^2 + \lambda^2\omega^2\right)/\left[\left(k\Gamma - \omega^2\right)^2 + (k+\lambda)^2\omega^2\right]} \le 1 \tag{16}$$

it means that

$$\Gamma \le \lambda + k/2 \tag{17}$$

So the proof is completed. □

3.2 Asymptotic stability

Asymptotic stability of traffic flow refers to the attenuation or increase of the amplitude of perturbation as it moves down a platoon of vehicles upstream of the perturbation. Although a chain of vehicles may be locally stable, it may result in an asymptotically unstable situation when each vehicle amplifies the signal and passes it on to the vehicles upstream. Asymptotic stability is concerned about the disturbance propagation characteristics in the platoon of vehicles.

Assumption 3 Supposing the initial state of the traffic flow is the steady state, and the headway is b, while the corresponding optimal velocity is $V(b)$.

Theorem 2 *If the following condition is satisfied,*

$$V'(b) \le \lambda + k/2 \tag{18}$$

then the FVD model is asymptotic stable.

Proof According to the above assumption, the position solution to the stability flow is [4]:

$$x_n^0(t) = bn + V(b)t \tag{19}$$

adding a disturbance $y_n(t)$ to (19), it will become

$$y_n(t) = x_n(t) - x_n^0(t) \tag{20}$$

where b in (18) means the space headway of two successional vehicle, while b in (19) represents by a constant. For convenience and comparison, we let b be the constant in the initial proof stage, and it makes actually no difference to the result at all. For n, the reason is the same as the foregoing.

When substituting (20) into (4) and using the Taylor expansion, it will deduce

$$\ddot{y}_n(t) = k\left[V'(b)\Delta y_n(t) - \dot{y}_n(t)\right] + \lambda\Delta\dot{y}_n(t) \tag{21}$$

set $y_n(t) = \exp(\mathrm{i}\alpha_\eta n + zt)$, where iis an imaginary unit, z is a complex number, and α_η means the frequency, and substituting it in (21) and according to the Fourier transform [1, 4–7, 9–12]:

$$z^2 + z\big[k - \lambda\big(\exp(\mathrm{i}\alpha_\eta) - 1\big)\big] - kV'(b)\big(\exp(\mathrm{i}\alpha_\eta) - 1\big) = 0 \tag{22}$$

set $z = r + w\mathrm{i}$, using the Euler's formula $\exp(\mathrm{i}\alpha_\eta) = \cos\alpha_\eta + \mathrm{i}\sin\alpha_\eta$. Denote $\sigma_c = k - \lambda(\cos\alpha_\eta - 1)$, $\sigma_s = \lambda\sin\alpha_\eta$. Substitute them into (22) and let $r = 0$, in which

$$-w^2 + w\sigma_s - kV'(b)(\cos\alpha_\eta - 1) + \mathrm{i}\big[w\sigma_c - kV'(b)\sin\alpha_\eta\big] = 0 \tag{23}$$

and supposing the real and imagine part of (23) is zero, and we will get

$$V'(b) = \frac{\sigma_c\sigma_s\sin\alpha_\eta - \sigma_c^2(\cos\alpha_\eta - 1)}{k(\sin\alpha_\eta)^2} \tag{24}$$

where $\alpha_\eta \to 0$, the critical stability curve is obtained according to the L'Hospital principle:

$$V'(b) = \lambda + k/2 \tag{25}$$

then the asymptotic stability condition is

$$V'(b) \le \lambda + k/2 \tag{26}$$

So the proof is over. □

Remark 1 According to (17) and (26), we get to know that the asymptotic stability is equivalent to the local stability to some extent. General speaking, for a time-invariant (TI) system, asymptotic stability of nonoscillation can insure local stability.

In the following Fig. 3, the curves represent the critical curve of the OV model ($\lambda = 0$) and the FVD model $\lambda = 0.2\ \mathrm{s}^{-1}$, respectively. The above region of the critical curve is the stability region, while the below region is the unstability region. Figure 3 shows that the FVD model improves the stability through introducing the velocity difference.

3.3 Lyapunov stability

Lyapunov stability of traffic flow mainly focus on the system performance of traffic flow from the control and system viewpoint, where the traffic flow is treated as a system and the initial perturbation of the leading vehicle is viewed as a stimulus. Lyapunov stability is

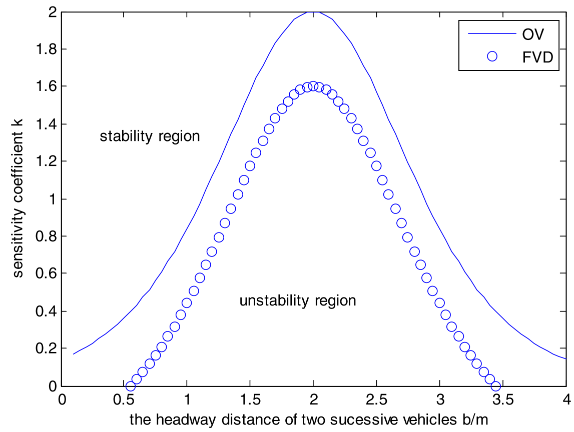

Fig. 3 The critical curve between sensitivity coefficient and distance of the successive vehicles

concerned about the characteristics of the disturbance response of the traffic flow system including the following vehicle but not only the following vehicle.

As mentioned in the previous Introduction, the traffic flow system is a nonlinear and strong coupling complex system. In order to conduct the Lyapunov stability analysis of the FVD model from the viewpoint of control, a theorem is given as follows.

Theorem 3 *In regards to the FVD model, if* $k > 0$, $\lambda \ge 0$, *the FVD model is Lyapunov stable.*

Proof According to [18, 19], and when (7) is obtained as follows:

$$\begin{cases} \begin{bmatrix} \delta\dot{v}_n(t) \\ \delta\Delta\dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -k-\lambda & k\Gamma \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta v_n(t) \\ \delta\Delta x_n(t) \end{bmatrix} + \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \delta v_{n+1}(t) \\ \delta v_n(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \delta v_n(t) \\ \delta\Delta x_n(t) \end{bmatrix} \end{cases} \tag{27}$$

Let $P = [\delta v_n(t)\ \delta\Delta x_n(t)]^{\mathrm{T}}$, $A = \left[\begin{smallmatrix} -k-\lambda & k\Gamma \\ -1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\mathrm{T}}$, then Eq. (27) becomes

$$\begin{cases} \dot{P} = AP + B\delta v_{n+1}(t) \\ \delta v_n(t) = CP \end{cases} \tag{28}$$

Regarding to (28), the state feedback law is generally designed as follows:

$$\delta v_{n+1}(t) = KP = \begin{bmatrix} k_1 & k_2 \end{bmatrix} P \tag{29}$$

So, the following main task is how to design k_1, k_2 to ensure the stability of the system. In order to achieve the goal, it needs to make the roots of the characteristic equation of the closed-loop system to locate in the left half-complex plane. Thus, we substitute (29) into the first equation of (28) and the closed-loop equation will be obtained

$$\dot{P} = MP \tag{30}$$

where

$$M = A + BK = \begin{bmatrix} -k - \lambda + \lambda k_1 & k\Gamma + \lambda k_2 \\ -1 + k_1 & k_2 \end{bmatrix}$$

To obtain the characteristic roots, the matrix M can be decomposed as

$$M = QDQ^{-1}$$

where

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$$\Theta = \sqrt{k^2 - 2k\lambda k_1 - 4\Gamma k k_1 + 2kk_2 + 2k\lambda + 4\Gamma k + \lambda^2 k_1^2 - 6\lambda k_1 k_2 - 2\lambda^2 k_1^2 + k_2^2 + 6\lambda k_2 + \lambda^2}$$

$$d_1 = \frac{1}{2}(k_2 - k - \lambda + k_1\lambda - \Theta), \quad d_2 = \frac{1}{2}(k_2 - k - \lambda + k_1\lambda + \Theta)$$

$$q_{11} = \frac{(k + k_2 + \Theta)}{2(k_1 - 1)} - \frac{\lambda}{2}, \quad q_{12} = \frac{(k + k_2 - \Theta)}{2(k_1 - 1)} - \frac{\lambda}{2}, \qquad q_{21} = 1, \quad q_{22} = 1$$

If the new state of the system (30) is supposed as

$$\xi = Q^{-1}P$$

The closed-loop state equation of (30) $\dot{P} = MP$ is

$$\begin{aligned} \dot{\xi} &= Q^{-1}\dot{P} = Q^{-1}MP = Q^{-1}MQ\xi \\ &= Q^{-1}QDQ^{-1}Q\xi = D\xi \end{aligned}$$

Namely,

$$\dot{\xi} = D\xi$$

If we choose the Lyapunov candidate function as

$$E = \frac{1}{2}\xi^T\xi$$

then

$$\dot{E} = \frac{1}{2}\left(\xi^T\dot{\xi} + \dot{\xi}^T\xi\right) = \frac{1}{2}\left(\xi^T D\xi + \xi^T D^T\xi\right)$$

Because D is diagonal matrix, we have

$$\dot{E} = \frac{1}{2}\left(\xi^T D\xi + \xi^T D\xi\right) = \xi^T D\xi \tag{31}$$

If D is a negative definite matrix, we have $\dot{E} < 0$ and the system is stable. In other words, if the system is stable, the matrix D should be negative definite, namely

$$\begin{aligned} d_1 = \frac{1}{2}\Big(&k_2 - k - \lambda + k_1\lambda \\ &- \sqrt{k^2 - 2k\lambda k_1 - 4\Gamma kk_1 + 2kk_2 + 2k\lambda + 4\Gamma k + \lambda^2 k_1^2 - 6\lambda k_1 k_2 - 2\lambda^2 k_1^2 + k_2^2 + 6\lambda k_2 + \lambda^2}\Big) < 0 \\ d_2 = \frac{1}{2}\Big(&k_2 - k - \lambda + k_1\lambda \\ &+ \sqrt{k^2 - 2k\lambda k_1 - 4\Gamma kk_1 + 2kk_2 + 2k\lambda + 4\Gamma k + \lambda^2 k_1^2 - 6\lambda k_1 k_2 - 2\lambda^2 k_1^2 + k_2^2 + 6\lambda k_2 + \lambda^2}\Big) < 0 \end{aligned}$$

Considering $k > 0$ and $\lambda \geq 0$, and the above conditions can be definitely satisfied through proper selection of control gains k_1 and k_2, it is easy to verify the controllability of the linear system (28).

Therefore, $\dot{E} < 0$ can be guaranteed, and the system is Lyapunov stable. □

Remark 2 In terms of (17), (26), and (31), we can get to know that the local stability and asymptotic stability of the FVD model are more strict than Lyapunov stability.

4 Simulation

The simulations are carried out under the same conditions as in [2, 19], and there are 100 vehicles running on a single lane under the periodic boundary condition. The road length is 200 m. The initial parameters are set as below [2, 19]:

$$x_1(0) = 10\ \mathrm{m}, \qquad x_n(0) = (n-1)L/N$$
$$(n = 2, 3, \ldots, N)$$
$$\dot{x}_n(0) = V(L/N) \quad (n = 1, 2, \ldots, N)$$
$$v_{\max} = 2\ \mathrm{m/s}, \qquad x_c = 2\ \mathrm{m}, \qquad v_0 = 0.964\ \mathrm{m/s}$$

For comparison, we choose three pairs of parameters $k = 1\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$; $k = 1\ \mathrm{s}^{-1}$, $\lambda = 1\ \mathrm{s}^{-1}$; $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, respectively. Figures 4 and 5 show the space headway distribution and velocity distribution of all vehicles at $t = 100$ s, and from Figs. 4 and 5 we can get to know that when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, there is obviously a large oscillation in the space headway distribution and velocity distribution, and finally tends to be stable, while when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 1\ \mathrm{s}^{-1}$ or $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, the distributions of the space headway and velocity are tending to stationary distribution although the fluctuation of the space headway is within 2 m narrow range and the fluctuation of velocity is also within 0.964 m/s narrow range. Moreover, the corresponding amplitude of fluctuation is smaller when $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$. The cause of the difference is that when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, the FVD model is just Lyapunov stable, neither the local stable nor the asymptotic stable; while when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 1\ \mathrm{s}^{-1}$ or $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, the FVD model is Lyapunov stable, as well as asymptotic stable. Further, the FVD model has better dynamic performance when $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$.

Figure 6(a) reveals the gain curve $|G(jw)|$ of the transfer function $G(s)$ and from Fig. 6(a) we can see that when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, a peak occurs in the gain curve and damps immediately, and the steady state error is very small, while when $k = 1\ \mathrm{s}^{-1}$, $\lambda = 1\ \mathrm{s}^{-1}$, there is no peak but the decay is slow and the steady state error is larger. The change tells us that if we keep the value of the gain factor k constant and increase the value of sensitivity coefficient of velocity difference λ, the dynamic performance of the traffic flow system will be improved. On the other hand, when $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$, the peak is also disappearing and the decay is fast; meanwhile, the steady state error is very small. From the fact that we get to know that when we keep the value of sensitivity coefficient of velocity difference λ constant and increase the value of the gain factor k, the dynamic performance of the traffic flow system will also be enhanced, and the value of the gain factor k is dominant in this process. Figure 6(b) shows the step response, which indicates that the traffic flow system is Lyapunov stable under the three pairs of parameters according to Theorem 3. What is more, the dynamic performance of the system

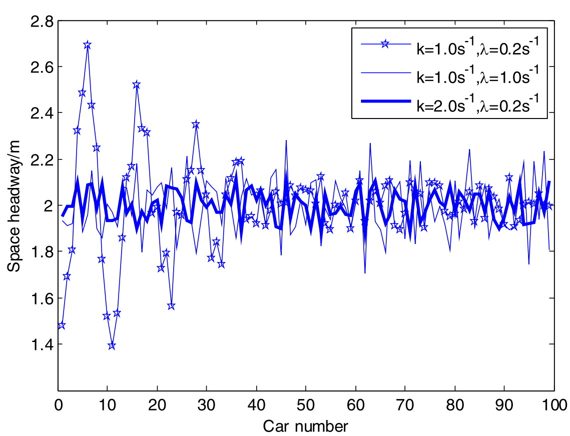

Fig. 4 The space headway of different vehicles

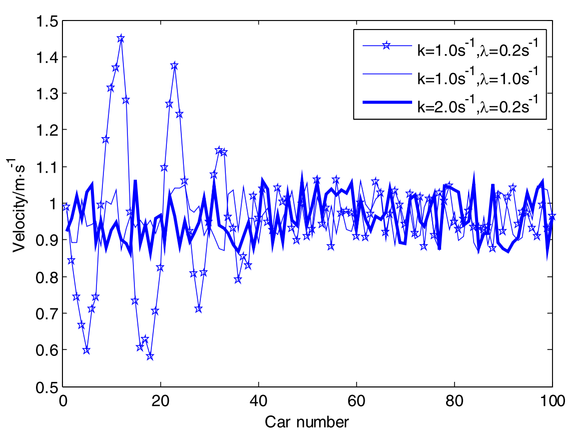

Fig. 5 The velocity of different vehicles

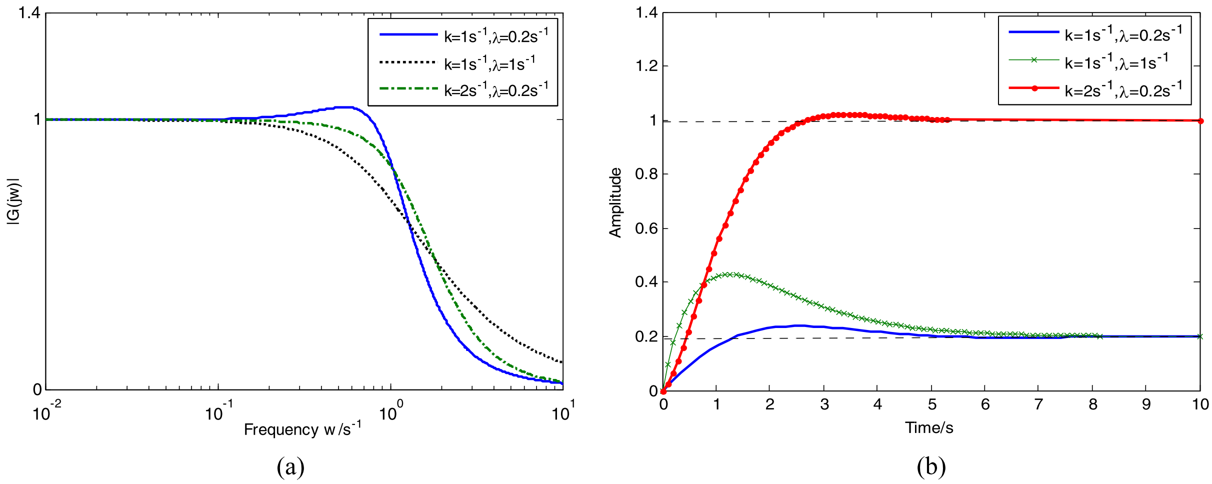

Fig. 6 The character of the $G(s)$: (**a**) the gain curve; (**b**) the step response

is better when $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$ in terms of overshoot, rising velocity, and the steady state error.

Therefore, we can draw a conclusion according to Figs. 4, 5, and 6 and Theorem 3: The FVD mode is Lyapunov stable under the three pairs of parameters, which is in agreement with the previous theoretical analysis and the dynamic performance of the traffic flow system is better under $k = 2\ \mathrm{s}^{-1}$, $\lambda = 0.2\ \mathrm{s}^{-1}$.

And based on these results, it can help to improve the traffic control by designing reasonable parameters k and λ. And k means the driver should be concentrating and sensitive and, therefore, make a quick response, which is more important than adjusting the sensitivity coefficient of the velocity difference represented by λ.

5 Conclusion

Taking the FVD model as the study case, the present paper systematically discusses the local stability and asymptotic stability; what is more, different with the traditional analysis method, the Lyapunov stability is conducted from a control point of view. And related simulations are carried out to reveal that it can improve the dynamic performance of the traffic flow system when keeping the value of the gain factor $k$constant and increasing the value of the sensitivity coefficient of velocity difference λ; and so is it when keeping the value of the sensitivity coefficient of velocity difference λ constant and increasing the value of the gain factor k, while the value of the gain factor $k$is dominant in this process.

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