Task 3

We have:

$$\begin{aligned}\mathbb{E}[V_{\text{call}}(S_T,0)] =&\frac{1}{\sqrt{2\pi}}\int_{\chi}^{\infty} f_{\text{call}}(s)e^{-\frac{s^2}{2}}\,ds\\ =&\frac{1}{\sqrt{2\pi}}\int_{\chi}^{\infty}\left(S(0)e^{(\mu-\frac{1}{2}\sigma^2)\cdot T+\sigma\sqrt{T}s}-K\right)e^{-\frac{s^2}{2}}\,ds\end{aligned}$$

With:

$$\chi=\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{K}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T\right)$$

It follows:

$$\begin{aligned}\mathbb{E}[V_{\text{call}}(S_T,0)] =&\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\chi} S(0)e^(\mu-\frac{1}{2}\sigma^2)\cdot T-\sigma\sqrt{T}s)e^{-\frac{s^2}{2}}\,ds\\ &-\frac{K}{\sqrt{2\pi}}\int_{\infty}^{-\chi} e^{-\frac{s^2}{2}}\,ds\\ =&S(0)e^{(\mu-\frac{1}{2}\sigma^2)\cdot T}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\chi} e^{-\frac{s^2}{2}-\sigma\sqrt{T}s}ds-K\Phi(-\chi)\\ =&S(0)e^{\mu T}e^{-\frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\chi} e^{-\frac{1}{2}(s+\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T}ds-K\Phi(-\chi)\\ =&S(0)e^{\mu T}\Phi(\sigma\sqrt{T}-\chi)-K\Phi(-\chi)\end{aligned}$$

□