# Task 1

*To prove*:

$$e^{-rT} \cdot \mathbb{E}\left[V\left(S,T\right)\right] = S(0) \cdot A \cdot \Phi\left(d + \sigma\sqrt{T_1}\right) - Ke^{-rT}\Phi(d)$$

with

$$\begin{aligned}
A &= e^{-r(T-T_2)-\sigma^2(T_2-T_1)/2} \\
d &= \frac{\log\left(S(0)/K\right) + \left(r - \frac{1}{2}\sigma^2\right)T_2}{\sigma\sqrt{T_1}} \\
T_1 &= T - \frac{M\left(M-1\right)\left(4M+1\right)}{6M^2}\Delta t = \frac{M(1+M)(1+2M)}{6M^2}\Delta t \\
T_2 &= T - \frac{\left(M-1\right)}{2}\Delta t = \frac{M+1}{2}\Delta t
\end{aligned}$$

*Proof:*
First transform the payoff function so that it only depends on one variable:

$$\begin{aligned}
V\left(S,T\right) &= \max\left\{\left(\prod_{i=1}^{M} S\left(t_i\right)\right)^{1/M} - K, 0\right\} \\
&= \max\left\{\left(\prod_{i=1}^{M} S\left(0\right)e^{\left(r-\frac{1}{2}\sigma^2\right)t_i + \sigma W(t_i)}\right)^{1/M} - K, 0\right\} \\
&= \max\left\{\left(S\left(0\right)^M e^{\sum_{i=1}^{M}\left(r-\frac{1}{2}\sigma^2\right)t_i + \sigma W(t_i)}\right)^{1/M} - K, 0\right\} \\
&= \max\left\{S\left(0\right)\left(e^{\left(r-\frac{1}{2}\sigma^2\right)\sum_{i=1}^{M}\Delta t \cdot i + \sigma\sum_{i=1}^{M}W(t_i)}\right)^{1/M} - K, 0\right\} \\
&= \max\left\{S\left(0\right)e^{\frac{1}{M}\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t\frac{M(M+1)}{2} + \sigma\sum_{i=1}^{M}W(t_i)\right)} - K, 0\right\}
\end{aligned}$$

Since $W(t_i)$ are normal distributed and the increments of the Winer process are independent, it holds:

$$\begin{aligned}
\sum_{i=1}^{M} W\left(t_i\right) =& \sum_{i=1}^{M} \left(M-i+1\right)\left(W\left(t_i\right)-W\left(t_{i-1}\right)\right) \qquad\qquad \text{with } t_0=0 \\
\sim& \mathcal{N}\left(0, \sum_{i=1}^{M} \left(M-i+1\right)^2 \left(t_i-t_{i-1}\right)\right) \\
=& \mathcal{N}\left(0, \Delta t\left(\sum_{i=1}^{M} M^2-2iM+2M+i^2-2i+1\right)\right) \\
=& \mathcal{N}\left(0, \Delta t\left(M^3+2M^2+M-2\left(M+1\right)\sum_{i=1}^{M} i+\sum_{i=1}^{M} i^2\right)\right) \\
=& \mathcal{N}\left(0, \Delta t\left(M^3+2M^2+M-2\left(M+1\right)\frac{M^2+M}{2}+\frac{(2M+1)(M+1)M}{6}\right)\right) \\
=& \mathcal{N}\left(0, \Delta t \cdot \frac{1}{6} M(1+M)(1+2M)\right)
\end{aligned}$$

With this we get the following univariate integrand for $\mathbb{E}\left[V\left(S,T\right)\right]$:

$$\begin{aligned}
f_{geom}^{disc}\left(s\right) :=& \left(S(0)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t\cdot\frac{M+1}{2}+\frac{\sigma}{M}\sqrt{\Delta t\cdot\frac{1}{6}M(1+M)(1+2M)\cdot s}\right)-K\right)^+ \\
=& \left(S(0)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma\sqrt{T_1}\cdot s\right)-K\right)^+
\end{aligned}$$

Now compute:

$$\begin{aligned}
& f\left(\chi\right)=0 \\
\Leftrightarrow& \left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma\sqrt{T_1}\chi=\log\left(\frac{K}{S(0)}\right) \\
\Leftrightarrow& \chi=\frac{-\log\left(\frac{S(0)}{K}\right)-\left(r-\frac{1}{2}\sigma^2\right)T_2}{\sigma\sqrt{T_1}}=-d
\end{aligned}$$

For the expectation we get:

$$
\begin{aligned}
\mathbb{E}\left[V\left(S,T\right)\right]=&\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f_{geom}^{disc}\left(s\right)\cdot e^{-\frac{1}{2}s^2}ds\\
=&\frac{1}{\sqrt{2\pi}}\int_{\infty}^{d}\left(S(0)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma\sqrt{T_1}\cdot s\right)-K\right)\cdot\exp\left(-\frac{1}{2}s^2\right)ds\\
=&\frac{S(0)}{\sqrt{2\pi}}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T_2\right)\int_{\infty}^{d}\exp\left(-\frac{1}{2}s^2+\sigma\sqrt{T_1}\cdot s\right)ds\\
&-\frac{K}{\sqrt{2\pi}}\int_{\infty}^{d}e^{-\frac{1}{2}s^2}ds\\
=&\frac{S(0)}{\sqrt{2\pi}}\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T_2\right)\int_{\infty}^{d}\exp\left(-\frac{1}{2}(s+\sigma\sqrt{T_1})^2+\frac{1}{2}\sigma^2T_1\right)ds\\
&-K\cdot\Phi(d)\\
=&S(0)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T_2+\frac{1}{2}\sigma^2T_1\right)\Phi\left(\sigma\sqrt{T_1}+d\right)-K\cdot\Phi(d)\\
=&S(0)\exp\left(r\cdot T_2-\frac{1}{2}\sigma^2\left(T_2-T_1\right)\right)\Phi\left(\sigma\sqrt{T_1}+d\right)-K\cdot\Phi(d)
\end{aligned}
$$

So the final result is:

$$
\begin{aligned}
V(S,0)=&e^{-rT}\mathbb{E}\left[V\left(S,T\right)\right]\\
=&e^{-rT}\left(S(0)e^{r\cdot T_2-\frac{1}{2}\sigma^2(T_2-T_1)}\Phi\left(\sigma\sqrt{T_1}+d\right)-K\cdot\Phi(d)\right)\\
=&S(0)e^{-r\cdot(T-T_2)-\frac{1}{2}\sigma^2(T_2-T_1)}\Phi\left(\sigma\sqrt{T_1}+d\right)-K\cdot e^{-rT}\cdot\Phi(d)\\
=&S(0)\cdot A\cdot\Phi\left(\sigma\sqrt{T_1}+d\right)-K\cdot e^{-rT}\cdot\Phi(d)
\end{aligned}
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□