Task 1

To prove:

$$e^{-rT} \cdot \mathbb{E}\left[V\left(S,T\right)\right] = S(0) \cdot A \cdot \Phi\left(d + \sigma\sqrt{T_1}\right) - Ke^{-rT}\Phi(d)$$

with

$$\begin{aligned} A &= e^{-r(T-T_2)-\sigma^2(T_2-T_1)/2} \\ d &= \frac{\log\left(S(0)/K\right) + \left(r - \frac{1}{2}\sigma^2\right)T_2}{\sigma\sqrt{T_1}} \\ T_1 &= T - \frac{M\left(M-1\right)\left(4M+1\right)}{6M^2}\Delta t \\ T_2 &= T - \frac{(M-1)}{2}\Delta t = \frac{M+1}{2}\Delta t \end{aligned}$$

Proof:
First transform the payoff function so that it only depends on one variable:

$$\begin{aligned} V\left(S,T\right) =& \max\left\{\left(\prod_{i=1}^{M} S\left(t_i\right)\right)^{1/M} - K, 0\right\} \\ =& \max\left\{\left(\prod_{i=1}^{M} S\left(0\right)e^{\left(r-\frac{1}{2}\sigma^2\right)t_i+\sigma W(t_i)}\right)^{1/M} - K, 0\right\} \\ =& \max\left\{\left(S\left(0\right)^M e^{\sum_{i=1}^{M}\left(r-\frac{1}{2}\sigma^2\right)t_i+\sigma W(t_i)}\right)^{1/M} - K, 0\right\} \\ =& \max\left\{S\left(0\right)\left(e^{\left(r-\frac{1}{2}\sigma^2\right)\sum_{i=1}^{M}\Delta t\cdot i+\sigma\sum_{i=1}^{M}W(t_i)}\right)^{1/M} - K, 0\right\} \\ =& \max\left\{S\left(0\right)e^{\frac{1}{M}\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t\frac{M(M+1)}{2}+\sigma\sum_{i=1}^{M}W(t_i)\right)} - K, 0\right\} \end{aligned}$$

Since $W(t_i)$ are normal distributed and the increments of the Winer process are independent, it holds:

$$\begin{aligned} \sum_{i=1}^{M} W\left(t_i\right) =& \sum_{i=1}^{M}\left(M-i+1\right)\left(W\left(t_i\right) - W\left(t_{i-1}\right)\right) && \text{with } t_0 = 0 \\ \sim& \mathcal{N}\left(0, \sum_{i=1}^{M}\left(M-i+1\right)\left(t_i - t_{i-1}\right)\right) && = \mathcal{N}\left(0, \sum_{i=1}^{M}\left(M-i+1\right)\Delta t\right) \\ =& \mathcal{N}\left(0, \Delta t\left(M^2 + M - \frac{M^2+M}{2}\right)\right) && = \mathcal{N}\left(0, \Delta t \cdot \frac{M^2+M}{2}\right) \end{aligned}$$

With this we get the following univariate integrand for $\mathbb{E}\left[V\left(S,T\right)\right]$:

$$f_{geom}^{disc}\left(s\right) := \left(S(0)exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t \cdot \frac{M+1}{2} + \frac{\sigma}{M}\sqrt{\Delta t \cdot \frac{M^2+M}{2}}s\right) - K\right)^+$$

Now compute:

$$\begin{aligned}
&f\left(\chi\right)=0\\
\Leftrightarrow&\left(r-\frac{1}{2}\sigma^2\right)\Delta t\cdot\frac{M+1}{2}+\frac{\sigma}{M}\sqrt{\Delta t\cdot\frac{M^2+M}{2}}\chi=\log\left(\frac{K}{S(0)}\right)\\
\Leftrightarrow&\sigma\sqrt{\Delta t\cdot\frac{M^2+M}{2M^2}}\chi=\log\left(\frac{K}{S(0)}\right)-\left(r-\frac{1}{2}\sigma^2\right)T_1
\end{aligned}$$

...

$\square$