Calculus III - MATH 283

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Contents

Seven Handouts

0.1 Handout 1: Domain of Elementary Functions

$$
\begin{aligned}
&f = \frac{u}{v} && \Rightarrow v \neq 0 \\
&f = \sqrt[2k]{u} && \Rightarrow u \geq 0 \quad (k \in \mathbb{N}) \\
&f = \log_v u && \Rightarrow u, v > 0, v \neq 1 \\
&f = \tan(u) && \Rightarrow u \neq \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z}) \\
&f = \cot(u) && \Rightarrow u \neq k\pi \quad (k \in \mathbb{Z}) \\
&f = \arcsin(u) && \Rightarrow -1 \leq u \leq 1 \\
&f = \arccos(u) && \Rightarrow -1 \leq u \leq 1 \\
&f = u^v && \Rightarrow u > 0 \text{ or } v \in \mathbb{N}
\end{aligned}
$$

Important fact: All elementary functions are continuous on their domain except at the isolated points. $v \neq$ constant or $v =$ irrational constant.

0.2 Handout 2: Fundamental Formulas for Integration

$$
\begin{aligned}
&\int u^n u'\,dx &&= \int u^n\,du &&= \frac{u^{n+1}}{n+1} + C && \text{for } n \neq -1\\
&\int \frac{u'}{u}\,dx &&= \int \frac{du}{u} &&= \ln|u| + C &&\\
&\int a^u u'\,dx &&= \int a^u\,du &&= \frac{a^u}{\ln a} + C && \text{for } a > 0, a \neq 1\\
&\int \sin(u)\,dx && &&= -\cos(u) + C &&\\
&\int \cos(u)\,dx && &&= \sin(u) + C &&\\
&\int \frac{1}{\cos^2(u)}\,dx &&= \int \sec^2(u)\,du &&= \tan(u) + C &&\\
&\int \frac{u'dx}{u^2+a^2} &&= \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C_1 &&= -\frac{1}{a} arccot \frac{u}{a} + C_2 &&\\
&\int \frac{u'dx}{a^2-u^2} && &&= \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C_1 &&\\
&\int \frac{u'dx}{\sqrt{a^2-u^2}} &&= \arcsin\left(\frac{u}{a}\right) + C_1 &&= -\arccos\left(\frac{u}{a}\right) + C_2 &&\\
&\int \frac{u'dx}{u\sqrt{u^2\pm a^2}} &&= \frac{1}{a}\ln\left|u+\sqrt{u^2\pm a^2}\right| + C_1 &&= -\frac{1}{a}\ln\left|u+\sqrt{u^2\pm a^2}\right| + C_2 &&\\
&\int \frac{u'dx}{u\sqrt{a^2-u^2}} &&= \frac{1}{a}\arcsin\left(\frac{u}{a}\right) + C_1 &&= -\frac{1}{a}\arccos\left(\frac{u}{a}\right) + C_2 &&
\end{aligned}
$$

0.3 Handout 3: Formulas, definitions, and equations

0.3.1 Basic Trigonometric Equations

$$
\begin{array}{llll}
\sin(x) = a & \text{for } a \in [-1,1] & \Leftrightarrow x = (-1)^n \arcsin(a) + 2n\pi & n \in \mathbb{Z} \\
\sin(x) = 0 & & \Leftrightarrow x = n\pi & n \in \mathbb{Z} \\
\sin(x) = \pm 1 & & \Leftrightarrow x = \pm\dfrac{\pi}{2} + 2n\pi & n \in \mathbb{Z} \\
\cos(x) = a & \text{for } a \in [-1,1] & \Leftrightarrow x = (-1)^n \arccos(a) + 2n\pi & n \in \mathbb{Z} \\
\cos(x) = 0 & & \Leftrightarrow x = \dfrac{\pi}{2} + n\pi & n \in \mathbb{Z} \\
\cos(x) = \pm 1 & & \Leftrightarrow x = 2n\pi & n \in \mathbb{Z} \\
\tan(x) = a & \text{for } a \in \mathbb{R} & \Leftrightarrow x = \arctan(a) + n\pi & n \in \mathbb{Z} \\
\cot(x) = a & \text{for } a \in \mathbb{R} & \Leftrightarrow x = \operatorname{arccot}(a) + n\pi & n \in \mathbb{Z}
\end{array}
$$

0.3.2 Inverse Trigonometric Functions

$$
\begin{array}{llll}
\arcsin(x) = a & \text{for } a \in [-\frac{\pi}{2}, \frac{\pi}{2}] & \Leftrightarrow x = \sin(a) & \text{for } x \in [-1,1] \\
\arccos(x) = a & \text{for } a \in [0, \pi] & \Leftrightarrow x = \cos(a) & \text{for } x \in [-1,1] \\
\arctan(x) = a & \text{for } a \in (-\frac{\pi}{2}, \frac{\pi}{2}) & \Leftrightarrow x = \tan(a) & \text{for } x \in \mathbb{R} \\
\operatorname{arccot}(x) = a & \text{for } a \in (0, \pi) & \Leftrightarrow x = \cot(a) & \text{for } x \in \mathbb{R}
\end{array}
$$

0.3.3 Other Formulas

$$
\sqrt{a^2} = |a|, \qquad e^{a\ln(b)} = b^a, \qquad \log_a(b) = \frac{\ln(b)}{\ln(a)}
$$

0.4 Handout 4: Derivatives

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$c' = 0$$
$$x' = 1$$
$$(u^n)' = nu^{n-1}u'$$
$$(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$$
$$(\sin u)' = \cos u u'$$
$$(\cos u)' = -\sin u u'$$
$$(\tan u)' = \sec^2 u u'$$
$$(\cot u)' = -\csc^2 u u'$$
$$(\sec u)' = \sec u \tan u u'$$
$$(\csc u)' = -\csc u \cot u u'$$
$$(u \pm v)' = u' \pm v'$$
$$(uv)' = u'v + uv'$$
$$(cu)' = cu'$$
$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$
$$\left(\frac{c}{u^n}\right)' = -\frac{ncu'}{u^{n+1}}$$

$$(\frac{u}{c})' = \frac{u'}{c}$$
$$(\frac{c}{u})' = -\frac{cu'}{u^2}$$
$$(e^u)' = e^u u'$$
$$(a^u)' = a^u \ln(a) u'$$
$$(\ln u) = \frac{u'}{u}$$
$$(\log_a u)' = \frac{u'}{u\ln(a)}$$
$$(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$$
$$(\arccos u)' = -\frac{u'}{\sqrt{1-u^2}}$$
$$(\arctan u)' = \frac{u'}{1+u^2}$$
$$(\text{arccot} u)' = -\frac{u'}{1+u^2}$$

0.5 Handout 5: Trigonometry

0.5.1 Trigonometric Identities

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{1}{2}\sin(\alpha-\beta)+\frac{1}{2}\sin(\alpha+\beta)\\
\cos\alpha\cos\beta &= \frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)\\
\sin\alpha\sin\beta &= -\frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta)
\end{aligned}$$

$$\begin{aligned}
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2}\\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta\mp\cos\alpha\sin\beta\\
\cos(\alpha\pm\beta) &= \cos\alpha\cos\beta\mp\sin\alpha\sin\beta\\
\tan(\alpha\pm\beta) &= \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}
\end{aligned}$$

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha\\
\cos(2\alpha) &= \cos^2\alpha-\sin^2\alpha\\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha)\pm\sin(\beta) &= 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)\\
\cos(\alpha)\pm\cos(\beta) &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)
\end{aligned}$$

How to exit trigonometry:

$$\begin{aligned}
\tan\frac{x}{2}=y\Rightarrow\sin x &= \frac{2y}{1+y^2},\quad \cos x=\frac{1-y^2}{1+y^2},\ dx=\frac{2dy}{1+y^2}\ \text{for } x\neq(2k+1)\pi\\
\tan x=y\Rightarrow\sin^2 x &= \frac{y}{1+y^2},\ \cos^2 x=\frac{1}{1+y^2},\ dx=\frac{dy}{1+y^2}\ \text{for } x\neq\frac{\pi}{2}+k\pi
\end{aligned}$$

0.5.2 Trigonometric Chart

Angle	**Sine**	**Cosine**	**Tangent**	**Cotangent**
$-\frac{\pi}{2} - 0$	-1	-0	$+\infty$	$+0$
$-\frac{\pi}{2} + 0$	-1	$+0$	$-\infty$	-0
-0	-0	1	-0	$-\infty$
$+0$	$+0$	1	$+0$	$+\infty$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{2} - 0$	1	$+0$	$+\infty$	$+0$
$\frac{\pi}{2} + 0$	1	-0	$-\infty$	-0
$\pi - 0$	$+0$	-1	-0	$-\infty$
$\pi + 0$	-0	-1	$+0$	$+\infty$

0.6 Handout 6: Power Series for Elementary Functions

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, && x \in (-1,1) \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, && x \in (-\infty,+\infty) \\
\sin x &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, && x \in (-\infty,+\infty) \\
\cos x &= \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}, && x \in (-\infty,+\infty) \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n}}{n}, && x \in (-1,1] \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{2n+1}, && x \in [-1,1] \\
(1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, && x \in (-1,1) \\
\binom{\alpha}{n} &= \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}, && x \in [-1,1]
\end{aligned}
$$

Handout 7.1: Application of Various Types of Integrals in Mechanics

for mass, center of mass, centroid, moments of statics, moments of inertia, length, distance, area, volume, surface area, etc.

of an object O the following types:

- Arbitrary 3D Solid;
- Lamina (2D in 2D);
- Rod (1D in 1D);
- Wire (1D in 3D);
- Shell (2D in 3D);
- Points (0D in 3D);

with respect to Q, as Q could be a point, a straight line, or a plane.

All applications for the general case of inhomogeneous objects and the special case of homogeneous objects are given by only one formula.

Handout 7.1

$$mom_Q^{(k)}(O) = \int_O D^n \delta \, dE$$

where O is the object; Q is a point, straight line, or plane; δ is the density; dE is a portion of O; D is the distance from an arbitrary point P ($P \in O$) to Q.

- $k = 0$: mass, area, length, volume, etc.
- $k = 1$: moment of statics
- $k = 2$: moment of inertia

1. Elementary Functions

1.1 Classifications

All functions belong to specific classifications. Algebraic functions include rational functions, and rational functions consist of polynomial functions.

- **Polynomial Functions**
  - Addition
  - Subtraction
  - Multiplication
- **Rational Functions**
  - Division
- **Algebraic Functions**
  - Rational Powers

All of these operations combined a finite number of times in one formula are known as elementary functions.

1.2 Limits and Continuity

$$\lim_{x\to 0} \frac{1}{x} = \infty \quad \text{(DNE)}$$
$$\lim_{x\to 0^+} \ln(x) = -\infty \quad \text{(DNE)}$$
$$\lim_{x\to 0^-} \ln(x) = \text{DNE}$$

2. Dimensional Analysis

2.1 Introduction

Dimensional analysis is a fundamental tool in understanding the relationships between different physical quantities.

2.2 Classifications

$$\begin{aligned} \text{Circle:} &\quad (x-h)^2 + (y-k)^2 = r^2 \\ \text{Sphere:} &\quad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \\ \text{Disk:} &\quad (x-h)^2 + (y-k)^2 \le r^2 \end{aligned}$$

Here we ask: How many dimensions are needed to visualize these objects? The answer depends on the number of variables in the equation.

- **Plane Lines:** 2 variables, 1 equation
- **Space Lines:** 3 variables, 2 equations

Examples of planes:

$$\text{xy-plane}: z = 0$$

$$\text{xz-plane}: y = 0$$

$$\text{yz-plane}: x = 0$$

Examples of lines:

$$\text{x-axis}: \begin{cases} z = 0 \\ y = 0 \end{cases}$$

$$\text{y-axis}: \begin{cases} x = 0 \\ z = 0 \end{cases}$$

$$\text{z-axis}: \begin{cases} x = 0 \\ y = 0 \end{cases}$$

2.3 Geometric Equations

2.3.1 Distance Between Two Points

The distance d_{PQ} between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in three-dimensional space is given by:

$$d_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2.1}$$

2.3.2 Distance Between a Point and a Plane

$$\text{dist}(P, \Pi) = \frac{|\vec{P} \cdot \vec{n}|}{||\vec{n}||} \tag{2.2}$$

2.3.3 Distance Between Two Parallel Lines

$$\text{dist}(L_1, L_2) = \frac{|\vec{v_1} \cdot \vec{v_2}|}{||\vec{v_1}||} \tag{2.3}$$

2.3.4 Distance Between a Point and a Line

$$\text{dist}(P, L) = \frac{|\vec{P} \times \vec{v}|}{||\vec{v}||} \tag{2.4}$$

2.3.5 Midpoint Formula

The midpoint M_{PQ} between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is calculated as:

$$M_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \tag{2.5}$$

2.3.6 Equation of a Sphere

A sphere centered at $P(p, q, s)$ with radius r has the equation:

$$(x - p)^2 + (y - q)^2 + (z - s)^2 = r^2 \tag{2.6}$$

2.3.7 Equation of a Circle

$$(x-h)^2 + (y-k)^2 = r^2 \tag{2.7}$$

2.3.8 General Quadratic Equation with two unknowns

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \tag{2.8}$$
$$A^2 + B^2 + C^2 \neq 0, \quad A, B, C, D, E, F \in \mathbb{R}$$

2.3.9 Equation of an Ellipsoid

An ellipsoid centered at $P(h, k, l)$ is given by:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1 \tag{2.9}$$

2.3.10 Equation of an Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{2.10}$$

2.3.11 Equation of a Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{2.11}$$

2.3.12 Equation of a Parabola

$$(y-k)^2 = 4p(x-h) \tag{2.12}$$

2.3.13 Equations of a Line

$\vec{a_l} = \langle a, b, c \rangle$ is the vector collinear to the line, and $P(x_0, y_0, z_0)$ is a point on the line.

Parametric Form

$$l = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \tag{2.13}$$

Normal Form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{2.14}$$

2.3.14 Equations of a Plane

$\vec{a_l} = \langle a, b, c\rangle$ is the vector normal to the plane, and $P(x_0, y_0, z_0)$ is a point on the plane.

Point-Normal/Scalar Form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{2.15}$$

2.4 Quadric Surfaces

2.4.1 Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2.16}$$

2.4.2 Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{2.17}$$

2.4.3 Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2.18}$$

2.4.4 Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{2.19}$$

2.4.5 Elliptic Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \tag{2.20}$$

2.4.6 Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \tag{2.21}$$

2.4.7 Describing Quadric Surfaces

To describe a quadric surface, you need to know the cross sections of the surface with the coordinate planes. For example, the cross sections of an ellipsoid with the coordinate planes are ellipses.

Example

Given the equation $x^2 + 4y^2 + 9z^2 = 36$, the cross sections with the coordinate planes are:

- $z = 0$: Ellipse with major axis 6 and minor axis 2 given by the equation:
$$x^2 + 4y^2 = 36 - 9z^2$$
- $y = 0$: Ellipse with major axis 6 and minor axis 3 given by the equation:
$$x^2 + 9z^2 = 36 - 4y^2$$
- $x = 0$: Ellipse with major axis 4 and minor axis 3
$$4y^2 + 9z^2 = 36 - x^2$$

Since all cross sections are ellipses, the surface is an ellipsoid.

Description: This is an ellipsoid centered at the origin with semi-axes of length 6, 3, and 2 along the x, y, and z-axes, respectively.

2.5 Graphing Concepts

Graphing helps visualize functions and equations by showing the set of all points that satisfy them.

2.5.1 Functions of a Single Variable

For a function of a single variable $f(x)$:

- **Domain**: A subset or all of the real number line, often denoted as $\mathbb{R}$ or a specific interval such as $(-\infty, \infty)$.
- **Range**: The set of possible values of $f(x)$; for instance, $[0, \infty)$.
- **Graph**: A curve on the Cartesian plane, representing ordered pairs $(x, f(x))$.

2.5.2 Functions of Multiple Variables

For functions of two or more variables, the domain and range extend to higher dimensions:

- **Domain**: The set of all points in $\mathbb{R}^n$ (e.g., $\mathbb{R}^2$ for a function of two variables or $\mathbb{R}^3$ for three variables) where the function is defined.
  - **Entire Plane**: If the function $f(x, y)$ is defined for all $(x, y) \in \mathbb{R}^2$.
  - **Portion of the Plane**: A subset of $\mathbb{R}^2$, specified by conditions like $x > 0$ or $y \geq 1$ or given by a picture.
- **Range**: The set of output values of the function. For many functions of multiple variables, this is a subset of $\mathbb{R}$.
- **Graph**: For a function $f(x, y)$ in two variables, the graph is a surface in three-dimensional space. For functions of three variables, the graph would exist in four-dimensional space and cannot be directly visualized.

Examples of Functions of Multiple Variables

1: Linear Function of Two Variables

$$f(x, y) = 3x + 5y - 7$$

- **Domain**: $\mathbb{R}^2$ (all real pairs (x, y))
- **Range**: $\mathbb{R}$ (all real values)

- **Graph**: A plane in three-dimensional space.

2: Linear Function of Three Variables

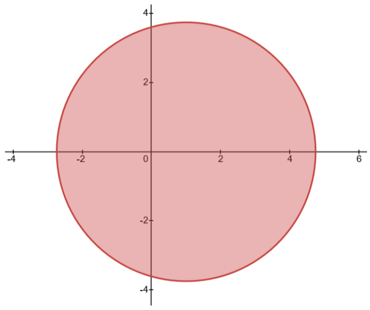
$$f(x, y, z) = 3x - 5y + z - 11$$

- **Domain**: $\mathbb{R}^3$ (all real triples (x, y, z))
- **Range**: $\mathbb{R}$
- **Graph**: Exists in four-dimensional space; it cannot be visualized in three dimensions.

3: Rational Function of Two Variables

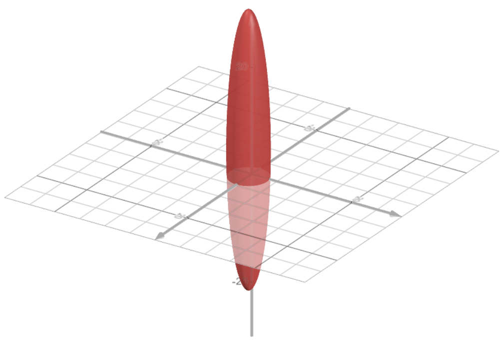
$$f(x, y) = 5 - 7\sqrt{13 - x^2 - y^2 - 2x}$$

- **Domain**: $(x-1)^2 + y^2 \leq 14$, disk with center $(1, 0)$ with radius $\sqrt{14}$.



- **Range**: $\left[5 - 7\sqrt{13}, 5\right]$
- **Graph**: A surface in three-dimensional space. This equation can be rewritten as

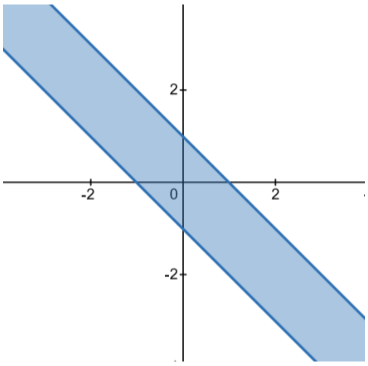
$$\frac{(x-1)^2}{14} + \frac{(y)^2}{14} + \frac{(z-5)^2}{49 \cdot 14} = 1$$



4: Trigonometric Function of Two Variables

$$f(x, y) = 3 - \frac{5}{\pi} \arcsin(x + y)$$

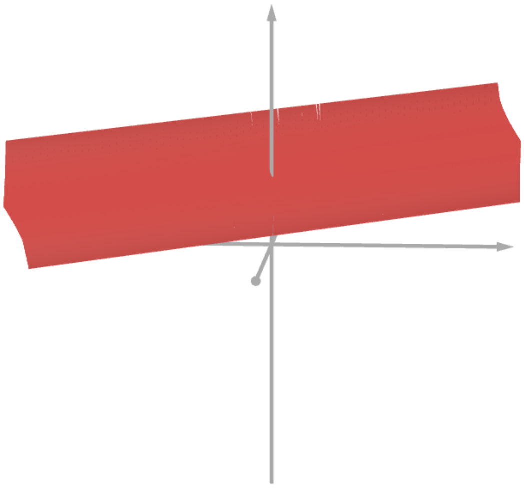
- **Domain**: $-1 \leq x + y \leq 1$



- **Range**:

$$\frac{-\pi}{2} \leq \arcsin(x + y) \leq \frac{\pi}{2},\ z \in \left[\frac{1}{2}, \frac{11}{2}\right]$$

- **Graph**: A surface in three-dimensional space.



2.6 Graphing Dimensions Summary

Graphing dimensions change based on the variables involved:

- **1D**: Interval or union of intervals
- **2D**: Picture

- **3D**: Description
- **4D and Higher**: Equation

2.7 Parametrization

Parametrization is a method to represent a curve or surface in terms of a parameter. For example, a line can be parametrized as:

2.7.1 Parametrization of a Plane Line

$$l = \begin{cases} x = (1-t)x_1 + tx_2 \\ y = (1-t)y_1 + ty_2 \end{cases}$$

where t is the parameter, and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on the line. To parametrize a line segment, t is restricted.

2.7.2 Parametrization of a Circle

$$C = \begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases}$$

where t is the parameter, and (h, k) is the center of the circle with radius r.

2.7.3 Parametrization of a Hyperbola

$$H = \begin{cases} x = h + a\cosh(t) \\ y = k + b\sinh(t) \end{cases}$$

where t is the parameter, and (h, k) is the center of the hyperbola with semi-axes a and b.

2.7.4 Parametrization of a Parabola

$$P = \begin{cases} x = t \\ y = at^2 + bt + c \end{cases}$$

where t is the parameter, and a, b, and c are constants.

If you are able to solve for one variable, there is no need to parametrize the curve.

2.7.5 Parametrization of a Sphere

2.7.6 Examples

1.
$$x^n + y^n = 1$$
$$\begin{cases} x = \cos^{\frac{2}{n}}(t) \\ y = \sin^{\frac{2}{n}}(t) \end{cases}$$

2.
$$x^n - y^n = 1$$
$$\begin{cases} x = \cosh^{\frac{2}{n}}(t) \\ y = \sinh^{\frac{2}{n}}(t) \end{cases}$$

where cosh and sinh are hyperbolic functions given by $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$ and $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$ and satisfy the identity $\cosh^2(t) - \sinh^2(t) = 1$.

3. Functions of Several Variables

Functions of several variables extend the concept of functions to higher dimensions, allowing for more complex mappings and dependencies.

3.1 Continuity

4 conditions for continuity of $f(x, y)$ at a point (x_0, y_0):

- (x_0, y_0) is in the domain of $f(x, y)$
- (x_0, y_0) is not an isolated point
- $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$ exists
- $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = f(x_0, y_0)$

3.2 Equations

3.2.1 Equations of a Tangent Plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{3.1}$$

- Given $P(x_0, y_0, z_0)$ and $f(x, y) = x^2 + y \sin x$

$$\begin{aligned} f(x, y) &= x^2 + y \sin x \\ z &= x^2 + y \sin x \\ F(x, y, z) &= z - x^2 - y \sin x \end{aligned}$$

- $\nabla F = \begin{bmatrix} -2x - y \cos x \\ -\sin x \\ 1 \end{bmatrix}$

3.3 Finding the Tangent Plane

Given a function $f(x, y)$, the equation of the tangent plane at a point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{3.2}$$

Here, f_x and f_y are the partial derivatives of $f(x, y)$ with respect to x and y respectively, evaluated at (x_0, y_0). To find the tangent plane:

1. Find $f_x(x, y)$ and $f_y(x, y)$.
2. Evaluate $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.
3. Substitute into the tangent plane equation.

3.4 Finding Critical Points

To find the critical points of a function $f(x, y)$:

1. Compute the first partial derivatives $f_x(x, y)$ and $f_y(x, y)$.
2. Set the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$.
3. Solve the system of equations to find the points (x_c, y_c) where the gradients are zero.

Once critical points (x_c, y_c) are found, determine their nature (local maxima, local minima, or saddle points) using the second derivative test:

$$D = f_{xx}(x_c, y_c)f_{yy}(x_c, y_c) - [f_{xy}(x_c, y_c)]^2 \tag{3.3}$$

- If $D > 0$ and $f_{xx}(x_c, y_c) > 0$, (x_c, y_c) is a local minimum.
- If $D > 0$ and $f_{xx}(x_c, y_c) < 0$, (x_c, y_c) is a local maximum.
- If $D < 0$, (x_c, y_c) is a saddle point.
- If $D = 0$, the test is inconclusive.

3.5 y-x Convenience Domain

A domain such that when we solve for y or x in all equations of the boundary, we get at most two solutions which are functions of x or y.

3.5.1 Examples

$y = 12 - x^2 \quad , y = x^2 - 4$

y: 2 total solutions.

x: 4 total.

Not a y-x convenience domain, but an x-y.

$$\begin{cases} y = x^2 \\ y = 4x - x^2 \end{cases}$$

x-y convenience domain.

$$A = \int_{x_1}^{x_2} y_2(x) - y_1(x)\, dx$$

$$x^2 = 4x - x^2 \implies x_1 = 0\,, x_2 = 2$$

$$y(1) = \begin{cases} 1 \\ 3 \end{cases}$$

$$\therefore A = \int_0^2 (4x - x^2) - x^2\, dx$$

$$\begin{cases} y = e^x \\ y = xe^x \\ x = 0 \end{cases}$$

x-y convenience domain.

$$A = \int_{x_1}^{x_2} y_2(x) - y_1(x)\, dX$$

$$x = 0 \implies x_1 = 0$$

$$x = xe^x \implies x_2 = 1$$

$$y(.5) = \begin{cases} e^{.5} \\ .5e^{.5} \end{cases}$$

$$\therefore A = \int_0^1 xe^x - e^x\, dx$$

3.5.2 Why?

If we have a y-x convenience domain, we can solve for y or x in all equations of the boundary, and get at most two solutions which are functions of x or y. This makes it easier to set up the double integral.

Steps:

1. Look for equations $x = C$
2. Equate the two solutions and solve for x.
   Get the bounds
   Find the limits
   Use geometric hints/domain of the functions
   Use natural limits. $x, y \in (\infty, -\infty) \ldots$

4. Partial Derivatives

Partial derivatives are used to study functions with multiple variables by differentiating with respect to one variable while keeping the others constant.

4.1 Differentiability

A function of more than two variables is said to be differentiable if the function all of its first partial derivative and all of its second partial derivatives exist and are continuous.

4.2 Introduction

Partial derivatives are used to study how a function changes with respect to one variable while keeping the others constant.

4.2.1 Differentials

The derivative of a function $f(x)$ with respect to x is denoted as: $\frac{dy}{dx}$ or $f'(x)$ where dx is the differential of x and $dy = f'(x)dx$. Differentials are different than derivatives, as they are the change in the function value due to a change in the variable. For example,

$$\text{if } f(x) = x^2, \text{ then } df = 2xdx.$$

$$d(\arctan(u)) = \frac{du}{1+u^2} \text{ but } \arctan(u') = \frac{u'}{1+u^2}$$

Error Estimation

The error in a function $f(x)$ can be estimated by the differential:

$$df = f'(x)dx \tag{4.1}$$

This is useful for approximating the error in a function given a small change in the variable.

Example of Error Estimation

Given the equation $A = lw$, $l = 5m$, $w = 3m$, $dl = 0.001m$, and $dw = 0.001m$, find the error in the area (dA):

$$\begin{aligned} A &= lw \\ dA &= wdl + ldw \\ dA &= (3)(0.001) + (5)(0.001) \\ dA &= 0.008m^2 \end{aligned}$$

4.2.2 Derivatives

$$f(x) \text{is differentiable iff } \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

Linearization

The linearization of a function $f(x)$ at a point x_0 is given by:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \tag{4.2}$$

$$f(x) \approx L(x) \text{ for } x \text{ near } x_0$$

This is useful for approximating the value of a function near a point. x_0 should be an integer close to the point you are trying to approximate.

Example of Linearization

Given the function $f(x) = \sqrt{x}$ and the point $x_0 = 3.9$, we can approximate this by linearizing $f(x)$ at x_0:

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ L(x) &= \sqrt{4} + \frac{1}{2\sqrt{4}}(x - 4) \\ L(x) &= 2 + \frac{1}{4}(x - 4) \\ L(3.9) &= 1.975 \approx \sqrt{3.9} \approx 1.9748 \end{aligned}$$

4.3 Partial Derivatives

4.3.1 Partial Derivatives Definition

The partial derivative of a function $f(x, y)$ with respect to x is denoted as:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

This **not** a fraction. ∂x does not exist, and neither does ∂f.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \tag{4.3}$$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{4.4}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{4.5}$$

Shortcut

The partial derivative of a function $f(x, y, \cdots)$ with respect to x can be calculated by treating all other variables as a constant and differentiating with respect to x:

4.3.2 All Defintions of First and Second Partial Derivatives

You sometimes have to use the definition of a partial derivative to find the actual value of the partial derivative.

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{4.6}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{4.7}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x+h, y) - \frac{\partial f}{\partial y}(x, y)}{h} \tag{4.8}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x, y)}{h} \tag{4.9}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x+h, y) - \frac{\partial f}{\partial x}(x, y)}{h} \tag{4.10}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y \partial y} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x, y+h) - \frac{\partial f}{\partial y}(x, y)}{h} \tag{4.11}$$

4.3.3 Properties of Partial Derivatives

If $f(x, y)$ is continuous at (x_0, y_0), then f_x and f_y exist at (x_0, y_0) and $f_{xy} = f_{yx}$.

Total Differential

The total differential of a function $f(x, y)$ is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If you find df, it is very easy to find both partial derivatives via the coefficients of dx and dy.

4.3.4 Higher-Order Partial Derivatives

The second-order partial derivative of a function $f(x, y)$ with respect to x and y is denoted as:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Order does matter. For example, for some instances:

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

However, these are usually equal for continuous functions. It is read from left to right and can also be notated as f_x, f_{xy}, f_{yx}, etc.

4.4 Examples:

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{2y^3(x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial y} = \frac{x^2(2x^2-y^2)}{(x^2+y^2)^3}$$

For $(x,y) = (0,0)$, we use the limit definition of partial derivatives. The partial derivative of f with respect to x at $(0,0)$ is:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, the partial derivative of f with respect to y at $(0,0)$ is:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Thus, we have:

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

To find the second partial derivatives at $(0,0)$, we proceed similarly:

$$\frac{\partial^2 f}{\partial x \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial^2 f}{\partial y \partial y}(0,0) = \lim_{k\to 0} \frac{\frac{\partial f}{\partial y}(0,k) - \frac{\partial f}{\partial y}(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 0$$

Thus, all second partial derivatives at $(0,0)$ are:

$$\frac{\partial^2 f}{\partial x \partial x}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y \partial y}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0, \frac{\partial^2 f}{\partial y \partial x}(0,0) = DNE$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial x}(x,0) - \frac{\partial f}{\partial x}(0,0)}{x} = \lim_{x \to 0} \frac{1-0}{x} = DNE$$

All partial derviatives combined are:

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{x^4 - x^2y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \begin{cases} \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \begin{cases} \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ DNE, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial x} = \begin{cases} \frac{2y^3(x^2 - y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial y} = \begin{cases} \frac{x^2(2x^2 - y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

4.5 Chain Rule

Given two functions $f(x,y)$ and $f(a,b)$ where a and b are dependent on x and y, the chain rule states:

The partial derivative of the composite function with respect to the first new variable is equal to the partial derivative of the composite function with respect to the first old variable multiplied by partial derivative of the same old variable with respect to the current new variable plus.

4.5.1 Examples:

1. Find $\dfrac{\partial z}{\partial r}$:

$$\begin{aligned} z(r,\theta) &= f(x,y) \\ \text{where } x &= r\cos\theta, \\ \text{and } y &= r\sin\theta \\ \frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \end{aligned}$$

   We do not have $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial f}{\partial y}$, so we cannot find the exact value of $\dfrac{\partial z}{\partial r}$.

2. $g(u,v,\ldots) = f(x,y,\ldots)$

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \ldots \\ \frac{\partial g}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \ldots \\ &\vdots \end{aligned}$$

3. $z = f(x,y)$, $x = s+t$, $y = s-t$

$$\text{Show that:} \quad \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t}$$

$$z(s,t) = z(x,y)$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \\ &= \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t} &= \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \\ &= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \\ \therefore \frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t} &= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \end{aligned}$$

4.6 Second Derivative Test of Multiple Variables

The second derivative test is used to determine the nature of a critical point of a function of multiple variables. The test is as follows:

1. Compute the second partial derivatives of the function $f(x, y)$:

$$f_{xx},\ f_{yy},\ \text{and } f_{xy}.$$

2. Form the **Hessian matrix**:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

3. Calculate the **determinant of the Hessian matrix** (Δf):

$$\delta f = f_{xx} f_{yy} - (f_{xy})^2.$$

4. Use the following rules to classify the critical point:
   - If $\Delta f > 0$ and $f_{xx} > 0$, the critical point is a **local minimum**.
   - If $\Delta f > 0$ and $f_{xx} < 0$, the critical point is a **local maximum**.
   - If $\Delta f < 0$, the critical point is a **saddle point**.
   - If $\Delta f = 0$, the test is **inconclusive**.

4.6.1 Examples

1.

$$f(x, y) = 4x + 6y - x^2 - y^2$$
$$D = \{(x, y),\ x \in [0, 4],\ y \in [0, 5]\}$$
$$df = 4dx + 6dy - 2xdx - 2ydy$$
$$\nabla f = \langle 4 - 2x, 6 - 2y \rangle$$
$$\nabla f = \langle 0, 0 \rangle \implies x = 2, y = 3$$
$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$
$$\Delta f(x, y) = -4 < 0 \implies \text{ local maximum at } (2, 3)$$
$$\text{Vertices: } O = (0, 0),\ A = (4, 0),\ B = (4, 5),\ C = (0, 5)$$

OA:

$$OA,\ x \in (0, 4) \implies f(x, 0) = 4x - x^2$$
$$f'(x) = 4 - 2x = 0 \implies x = 2$$

$$f(2,0) = 8 \implies \text{ local maximum at } (2,0)$$

AB:

$$AB,\, y \in (0,5) \implies f(4,y) = 6y - y^2$$
$$f'(y) = 6 - 2y = 0 \implies y = 3$$
$$f(4,3) = 18 \implies \text{ local maximum at } (4,3)$$

BC:

$$BC,\, x \in (0,4) \implies f(x,5) = 4x - x^2$$
$$f'(x) = 4 - 2x = 0 \implies x = 2$$
$$f(2,5) = 8 \implies \text{ local maximum at } (2,5)$$

CO:

$$CO,\, y \in (0,5) \implies f(0,y) = 6y - y^2$$
$$f'(y) = 6 - 2y = 0 \implies y = 3$$
$$f(0,3) = 18 \implies \text{ local maximum at } (0,3)$$

Boundary	Point	$f(x,y)$
OA	(2, 0)	4
OA	(0, 0)	0
AB	(4, 3)	9
AB	(4, 0)	0
BC	(2, 5)	9
BC	(4, 5)	5
CO	(0, 3)	9
CO	(0, 5)	5
	(2, 3)	13

$$\therefore \text{ global maximum at } (2,3,13)$$

4.7 Lagrange Multipliers

The method of **Lagrange multipliers** is used to find the maximum and minimum values of a function $f(x,y,\dots)$ subject to one or more constraints. The process is as follows:

1. Define the **Lagrangian function**:

$$\mathcal{L}(x,y,\lambda) = f(x,y,\dots) + \lambda \cdot g(x,y,\dots),$$

   where $g(x,y,\dots) = 0$ is the constraint and λ is the **Lagrange multiplier**.

2. Compute the partial derivatives of $\mathcal{L}$ with respect to all variables and λ:

$$\frac{\partial \mathcal{L}}{\partial x},\ \frac{\partial \mathcal{L}}{\partial y},\ \dots,\ \frac{\partial \mathcal{L}}{\partial \lambda}.$$

3. Set these derivatives equal to zero to form a system of equations:

$$\frac{\partial \mathcal{L}}{\partial x} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = 0,$$
$$\vdots$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x, y, \dots) = 0.$$

4. Solve the system of equations to find the critical points.
5. Evaluate $f(x, y, \dots)$ at the critical points to determine the maximum or minimum values, depending on the problem.

Key Notes:

- The method can be extended to functions with multiple constraints. For k constraints $g_1(x, y, \dots) = 0$, $g_2(x, y, \dots) = 0$, $\dots$, $g_k(x, y, \dots) = 0$, introduce a Lagrange multiplier λ_i for each constraint and define:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \dots) = f(x, y, \dots) + \lambda_1 \cdot g_1(x, y, \dots) + \lambda_2 \cdot g_2(x, y, \dots) + \dots.$$

- Always verify that the solutions satisfy the constraints.

4.7.1 Examples

1. Find the maximum and minimum values of $V = xyz$ subject to the constraint $x + y + z = 1$.

$$\mathcal{L}(x, y, z, \lambda) = xyz + \lambda(x + y + z - 1)$$
$$\frac{\partial \mathcal{L}}{\partial x} = yz + \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = xz + \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial z} = xy + \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y + z - 1 = 0.$$

Solving the system of equations, we find $x = y = z = \frac{1}{3}$. Thus, the maximum and minimum values of V are $\frac{1}{27}$.

4.7.2 Shortcut

The Lagrange multiplier method can be simplified by using the following steps:

1. Solve the constraint for each variable

   If it has multiple constraints, solve for the system of the constraints.

2. Substitute the variables into the function

3. Find the critical points

4. Evaluate the function at the critical points

5. Compare the values to find the maximum and minimum

Examples

1. $f(x, y) = x^2 + y^2$, $xy = 1$

$$y = \frac{1}{x} \implies f(x) = x^2 + \frac{1}{x^2}$$

$$f'(x) = 2x - \frac{2}{x^3} = 0 \implies x = 1$$

$$f(1) = 2 \implies \text{ local minimum at } (1, 1)$$

2. $f(x, y) = 3x + y$, $x^2 + y^2 = 10$

$$f(x, y) = \begin{cases} x = \sqrt{10}\cos\theta, \\ y = \sqrt{10}\sin\theta \end{cases}$$

$$f(\sqrt{10}\cos\theta, \sqrt{10}\sin\theta) = 3\sqrt{10}\cos\theta + \sqrt{10}\sin\theta$$

$$f'(\theta) = -3\sqrt{10}\sin\theta + \sqrt{10}\cos\theta = 0 \implies \theta = \frac{5\pi}{6}$$

$$f\left(\sqrt{10}\cos\left(\frac{5\pi}{6}\right), \sqrt{10}\sin\left(\frac{5\pi}{6}\right)\right) = -\sqrt{10}$$

$$\implies \text{ local minimum at } \left(\sqrt{10}\cos\left(\frac{5\pi}{6}\right), \sqrt{10}\sin\left(\frac{5\pi}{6}\right)\right)$$

5. Vector Calculus

Vector calculus explores vector fields and operations such as the gradient, divergence, and curl, which are foundational in physics, engineering, and mathematics.

5.1 Introduction to Vectors and Scalars

In mathematics and physics, **scalars** and **vectors** are fundamental quantities that describe different types of quantities.

Scalars are single numbers that represent quantities that have only magnitude, without any direction. Examples of scalars include:

- Temperature (e.g., 25°C)
- Mass (e.g., 5 kg)
- Distance (e.g., 10 m)

Vectors, on the other hand, are quantities that have both magnitude and direction. Vectors are represented as an ordered list of components in a coordinate system. For example, in three-dimensional space, a vector $\vec{v}$ can be represented as:

$$\vec{v} = \langle v_x, v_y, v_z \rangle.$$

An example of a vector is a displacement of 5 m to the right and 3 m upwards, represented as $\vec{d} = \langle 5, 3, 0 \rangle$.

Vectors are a collection of all directed line segments that have the same length and the same direction. This one and the same length is called the magnitude of the vector and this one and the same direction is called the direction of the vector.

In summary:

- A **scalar** is a quantity that has only magnitude.
- A **vector** is a quantity that has both magnitude and direction.

5.1.1 Vector Addition

You **cannot** add a vector and a scalar. However, you can add two vectors together. Vector addition combines two vectors $\vec{u}$ and $\vec{v}$ to produce a resultant vector $\vec{w}$:

$$\vec{w} = \vec{u} + \vec{v} = \langle u_x + v_x, u_y + v_y, u_z + v_z \rangle \tag{5.1}$$

For example, given these vectors: $u = \langle 1, 2, 3 \rangle$, and $v = \langle 4, 5, 6 \rangle$

$$\vec{u} + \vec{v} = \langle 1 + 4, 2 + 5, 3 + 6 \rangle = \langle 5, 7, 9 \rangle$$

5.1.2 Vector Scalar Multiplication

A vector and scalar can be multiplied, but two vectors cannot be multiplied in the traditional sense.

$$k\vec{u} = \langle ku_x, ku_y, ku_z \rangle \tag{5.2}$$

5.1.3 Scalar Multiples

A scalar multiple of a vector $\vec{u}$ scales its magnitude without changing its direction. If $\vec{v}$ is collinear to $\vec{u}$, then there exists some scalar k where $\vec{v} = k\vec{u}$. For example, given these vectors: $u = \langle 1, 2, 3 \rangle$, and $v = \langle 2, 4, 6 \rangle$

$$2\vec{u} = \vec{v}$$

$$2 \langle 1, 2, 3 \rangle = \langle 2, 4, 6 \rangle$$

For any $\vec{u}$, $\vec{v}$, and scalar k:

$$k\vec{u} = \langle ku_x, ku_y, ku_z \rangle$$

$$\text{iff } \vec{v} = \langle ku_x, ku_y, ku_z \rangle \text{ for some scalar } k, \vec{v} \parallel \vec{u}$$

5.2 Dot Product

The dot product is an operation between two vectors $\vec{u}$ and $\vec{v}$ that produces a scalar, and is calculated as:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + \cdots + u_n v_n = ||\vec{u}||\, ||\vec{v}|| \cos\theta \tag{5.3}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. The dot product is **commutative**:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

5.2.1 Applications of the Dot Product

1. **Magnitude of a Vector**:
$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2$$
$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$$

2. **Determining Perpendicularity**:
$$\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$$
For example, if $\vec{a}_{L_1} \cdot \vec{a}_{L_2} = 0$, then lines L_1 and L_2 are perpendicular.

3. **Finding the Angle Between Two Vectors**:
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}$$

4. **Projection of a Vector**: The projection of $\vec{u}$ onto $\vec{v}$ is:
$$\vec{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

5. **Work Done by a Force**: The work done by a force $\vec{F}$ acting on an object displaced by $\vec{d}$ is:
$$W = \vec{F} \cdot \vec{d}$$

6. **Orthogonal Projections**: The orthogonal projection of $\vec{u}$ onto $\vec{v}$ is:
$$\vec{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$$

7. **Finding the Angle Between Two Planes**: The angle between two planes with normal vectors $\vec{n_1}$ and $\vec{n_2}$ is:
$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{||\vec{n_1}||\,||\vec{n_2}||}$$

8. **Finding the Distance Between a Point and a Plane**: The distance between a point P and a plane with normal vector $\vec{n}$ is:
$$\text{dist}(P, \Pi) = \frac{|\vec{P} \cdot \vec{n}|}{||\vec{n}||}$$

9. **Finding the Distance Between Two Parallel Lines**: The distance between two parallel lines with direction vectors $\vec{v_1}$ and $\vec{v_2}$ is:
$$\text{dist}(L_1, L_2) = \frac{|\vec{v_1} \cdot \vec{v_2}|}{||\vec{v_1}||}$$

5.3 Cross Product

The cross product is an operation on two 3D vectors that yields a vector perpendicular to both:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \langle u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x \rangle \tag{5.4}$$

This resultant vector $\vec{w} = \vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

5.3.1 Shortcut for Cross Product

To find the cross product $\vec{u} \times \vec{v}$, arrange the components as follows:

$$\begin{vmatrix} u_x & u_y & u_z & u_x & u_y & u_z \\ v_x & v_y & v_z & v_x & v_y & v_z \end{vmatrix}$$

Then, calculate each component of the cross product by making crosses between u_y and v_z, u_z and v_y, u_z and v_x, u_x and v_z, u_x and v_y, and u_y and v_x. This yields the components of the cross product vector and is easy to visualize.

5.3.2 Applications of the Cross Product

- **Finding a perpendicular vector to 2 vectors**:

$$\vec{u} \times \vec{v} = \vec{w}$$

$$\vec{w} \perp \vec{u} \text{ and } \vec{v}$$

- **Area of a traingle**:

  The area of a triangle with sides $\vec{u}$ and $\vec{v}$ is:

$$\frac{1}{2}||\vec{u} \times \vec{v}||$$

- **Finding 2D direction in a plane given by 2 vectors or 3 points**:

  The direction of a vector $\vec{w}$ in a plane given by vectors $\vec{u}$ and $\vec{v}$ is:

$$\vec{w} = \vec{u} \times \vec{v}$$

5.4 Triple Scalar Product

The triple scalar product of three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is defined as:

$$\vec{u}\vec{v}\vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}) \tag{5.5}$$

It is often shown as $\vec{u}\vec{v}\vec{w}$ and has a few niche uses.

- **The volume of a parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.**
- **The determinant of a matrix.**

$$\det(M) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \vec{u}\vec{v}\vec{w}$$

- **The volume of a tetrahedron with noncoplanar edges $\vec{u}$, $\vec{v}$, and $\vec{w}$**

$$\left|\frac{\vec{u}\vec{v}\vec{w}}{6}\right|$$

- **The scalar triple product is zero if the vectors are coplanar.**

$$\text{iff } \vec{u}\vec{v}\vec{w} = 0 \text{ then } \vec{u}, \vec{v}, \vec{w} \text{ are coplanar}$$

5.5 Directional Derivatives

5.5.1 Definition

The directional derivative of f at a point (x_0, y_0, z_0) in the direction of a vector $\vec{v}$ is defined as:

$$f'_{\vec{v}}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h\cos\alpha, y_0 + h\cos\beta, z_0 + h\cos\gamma) - f(x_0, y_0, z_0)}{h} \tag{5.6}$$

$$f'_{\vec{v}}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f((x_0, y_0, z_0) + h\frac{\vec{v}}{||\vec{v}||}) - f(x_0, y_0, z_0)}{h} \tag{5.7}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **directional cosines** of $\vec{v}$.

$$\cos\alpha = \frac{v_x}{||\vec{v}||}$$
$$\cos\beta = \frac{v_y}{||\vec{v}||}$$
$$\cos\gamma = \frac{v_z}{||\vec{v}||}$$

5.5.2 Shortcut Formula for Directional Derivative

Using the gradient, the directional derivative can be computed as:

$$f'_{\vec{v}}(x_0, y_0, z_0) = \vec{u} \cdot \nabla f(x_0, y_0, z_0) \tag{5.8}$$

where:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \cdots \right\rangle \tag{5.9}$$

The gradient vector, ∇f, points in the direction of greatest increase of f and has a magnitude equal to the maximum rate of increase.

The shortcut only works when f is differentiable at (x_0, y_0, z_0).

5.5.3 Properties of Directional Derivatives

The maximum value of the directional derivative is the magnitude of the gradient vector.

$$\max f'_{\vec{v}}(x_0, y_0, z_0) = ||\nabla f(x_0, y_0, z_0)|| = \max \left\{ \frac{df}{d\vec{v}} \right\} \tag{5.10}$$

The directional derivative is zero when the input vector is perpendicular to the gradient.

$$\vec{v} \perp \nabla f \implies f'_{\vec{v}}(P_0) = 0 \tag{5.11}$$

5.5.4 Examples

Given the following function f:

$$f(x, y, z) = z \ln(5x + 3y - 1) + x^2 \arctan z + y^3 + 4$$

With the point P_0 and vector $\vec{v}$:

$$P_0(1, -1, 0), \qquad \vec{v} = \langle 2, -1, 2\rangle$$

$$df = dz \ln(5x + 3y - 1) + z\frac{5dx + 3dy}{5x + 3y - 1} + 2x\frac{dz}{1 + z^2} + 3y^2 dy + dx 2x \arctan z$$

Find the directional derivative of f at P_0 in the direction of $\vec{v}$.

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{5z}{5x + 3y - 1} + 2x \arctan z \\[2ex] \dfrac{3z}{5x + 3y - 1} + 3y^2 \\[2ex] \dfrac{2x}{1 + z^2} + \ln(5x + 3y - 1) \end{bmatrix}$$

$$\nabla f(1, -1, 0) = \begin{bmatrix} 0 + 2 \arctan 0 \\ 0 + 3 \\ 0 + 1 \end{bmatrix} = \langle 0, 3, 1\rangle$$

$$\frac{\vec{v}}{\|\vec{v}\|} \cdot \nabla f(1, -1, 0) = \frac{\langle 2, -1, 2\rangle \cdot \langle 0, 3, 1\rangle}{3} = \frac{0 - 3 + 2}{3} = \frac{-1}{3}$$

$$\therefore f'_{\vec{v}}(P_0) = \frac{-1}{3}$$

The directional derivative of f at P_0 in the direction of $\vec{v}$ is $-\dfrac{1}{3}$ while the maximum is $\|\nabla f(P_0)\| = \sqrt{10}$.

6. Multiple Integrals

Multiple integrals extend single-variable integration to functions of several variables, useful in areas such as volume and area calculations.

6.1 Line Integral

6.1.1 Deriving the Line Integral

Given the parametric equation G: $x(t), y(t), t \in [a, b]$ and the function $f(x, y)$, find the area between the parametric curve and the function.

How do we find this? Let's write an integral for this.

$$\int_{t=a}^{t=b} f(x,y)dS$$

dS is the change in the parametric curve's arc length. Given dx and dy, $dS = \sqrt{dx^2 + dy^2}\,dt$, so we can rewrite the earlier equation as

$$\int_a^b f(x,y)\sqrt{dx^2 + dy^2}\,dt$$

This can also be rewritten as

$$\int_a^b f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

6.1.2 Examples

Given the parametric function $x = \cos t$, $y = \sin t$, $t \in [0, \pi/2]$ and $f(x, y) = xy$ find the line integral.

$$\begin{aligned}
\frac{dx}{dt} &= -\sin t \\
\frac{dy}{dt} &= \cos t \\
A &= \int_0^{\pi/2} \sin t \cos t \sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^{\pi/2} \sin t \cos t \cdot 1\, dt \\
&= \int_0^{1} \frac{u}{\sin t} \cdot \sin t\, du \\
&= \left[\frac{u^2}{2}\right]_{u=0}^{u=1} \\
A &= \frac{1}{2}
\end{aligned}$$

6.2 Multiple Integrals

The bounds of the outer integral are always constant.

$$\int_a^b \int_c^d f(x, y)\, dx\, dy$$

a, b are always constants, and c, d are either constants or functions of y.