Calculus III - MATH 283

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Fall 2024

Contents

1. Elementary Functions

1.1 Classifications

All functions belong to specific classifications. Algebraic functions include rational functions, and rational functions consist of polynomial functions.

- **Polynomial Functions**
  - Addition
  - Subtraction
  - Multiplication
- **Rational Functions**
  - Division
- **Algebraic Functions**
  - Rational Powers

All of these operations combined a finite number of times in one formula are known as elementary functions.

1.2 Limits and Continuity

$$\lim_{x\to 0} \frac{1}{x} = \infty \quad \text{(DNE)}$$
$$\lim_{x\to 0^+} \ln(x) = -\infty \quad \text{(DNE)}$$
$$\lim_{x\to 0^-} \ln(x) = \text{DNE}$$

2. Dimensional Analysis

2.1 Introduction

Dimensional analysis is a fundamental tool in understanding the relationships between different physical quantities.

2.2 Classifications

$$\begin{aligned} \text{Circle:} \quad & (x-h)^2 + (y-k)^2 = r^2 \\ \text{Sphere:} \quad & (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \\ \text{Disk:} \quad & (x-h)^2 + (y-k)^2 \leq r^2 \end{aligned}$$

Here we ask: How many dimensions are needed to visualize these objects? The answer depends on the number of variables in the equation.

- **Plane Lines:** 2 variables, 1 equation
- **Space Lines:** 3 variables, 2 equations

Examples of planes:

$$\text{xy-plane}: \; z = 0$$
$$\text{xz-plane}: \; y = 0$$
$$\text{yz-plane}: \; x = 0$$

Examples of lines:

$$\text{x-axis}: \begin{cases} z = 0 \\ y = 0 \end{cases}$$

$$\text{y-axis}: \begin{cases} x = 0 \\ z = 0 \end{cases}$$

$$\text{z-axis}: \begin{cases} x = 0 \\ y = 0 \end{cases}$$

2.3 Geometric Equations

2.3.1 Distance Between Two Points

The distance d_{PQ} between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in three-dimensional space is given by:

$$d_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2.1}$$

2.3.2 Midpoint Formula

The midpoint M_{PQ} between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is calculated as:

$$M_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \tag{2.2}$$

2.3.3 Equation of a Sphere

A sphere centered at $P(p, q, s)$ with radius r has the equation:

$$(x - p)^2 + (y - q)^2 + (z - s)^2 = r^2 \tag{2.3}$$

2.3.4 Equation of an Ellipsoid

An ellipsoid centered at $P(h, k, l)$ is given by:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} + \frac{(z - l)^2}{c^2} = 1 \tag{2.4}$$

2.3.5 Equations of a Line

$\vec{a_l} = \langle a, b, c \rangle$ is the vector collinear to the line, and $P(x_0, y_0, z_0)$ is a point on the line.

Parametric Form

$$l = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \tag{2.5}$$

Normal Form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{2.6}$$

2.3.6 Equations of a Plane

$\vec{a_l} = \langle a, b, c \rangle$ is the vector normal to the plane, and $P(x_0, y_0, z_0)$ is a point on the plane.

Point-Normal/Scalar Form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{2.7}$$

2.4 Graphing Concepts

Graphing helps visualize functions and equations by showing the set of all points that satisfy them.

2.4.1 Functions of a Single Variable

For a function of a single variable $f(x)$:

- **Domain**: A subset or all of the real number line, often denoted as $\mathbb{R}$ or a specific interval such as $(-\infty, \infty)$.
- **Range**: The set of possible values of $f(x)$; for instance, $[0, \infty)$.
- **Graph**: A curve on the Cartesian plane, representing ordered pairs $(x, f(x))$.

2.4.2 Functions of Multiple Variables

For functions of two or more variables, the domain and range extend to higher dimensions:

- **Domain**: The set of all points in $\mathbb{R}^n$ (e.g., $\mathbb{R}^2$ for a function of two variables or $\mathbb{R}^3$ for three variables) where the function is defined.
  - **Entire Plane**: If the function $f(x, y)$ is defined for all $(x, y) \in \mathbb{R}^2$.
  - **Portion of the Plane**: A subset of $\mathbb{R}^2$, specified by conditions like $x > 0$ or $y \geq 1$ or given by a picture.

- **Range**: The set of output values of the function. For many functions of multiple variables, this is a subset of $\mathbb{R}$.
- **Graph**: For a function $f(x, y)$ in two variables, the graph is a surface in three-dimensional space. For functions of three variables, the graph would exist in four-dimensional space and cannot be directly visualized.

Examples of Functions of Multiple Variables

1: Linear Function of Two Variables

$$f(x, y) = 3x + 5y - 7$$

- **Domain**: $\mathbb{R}^2$ (all real pairs (x, y))
- **Range**: $\mathbb{R}$ (all real values)
- **Graph**: A plane in three-dimensional space.

2: Linear Function of Three Variables

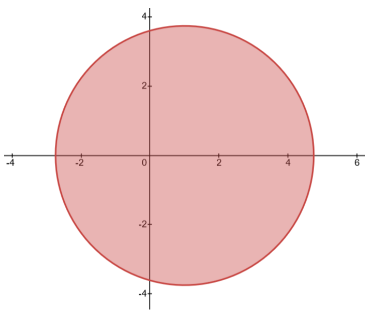
$$f(x, y, z) = 3x - 5y + z - 11$$

- **Domain**: $\mathbb{R}^3$ (all real triples (x, y, z))
- **Range**: $\mathbb{R}$
- **Graph**: Exists in four-dimensional space; it cannot be visualized in three dimensions.

3: Rational Function of Two Variables

$$f(x, y) = 5 - 7\sqrt{13 - x^2 - y^2 - 2x}$$

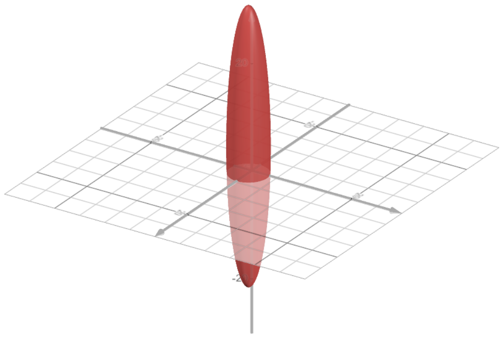
- **Domain**: $(x - 1)^2 + y^2 \leq 14$, disk with center $(1, 0)$ with radius $\sqrt{14}$.



- **Range**: $\left[5 - 7\sqrt{13}, 5\right]$

- **Graph**: A surface in three-dimensional space. This equation can be rewritten as

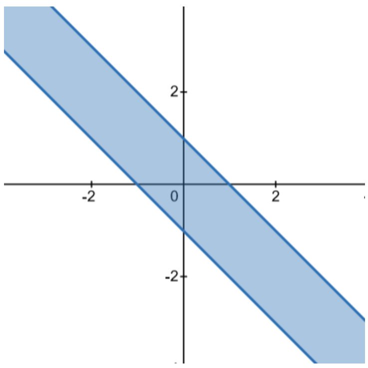
$$\frac{(x-1)^2}{14} + \frac{(y)^2}{14} + \frac{(z-5)^2}{49 \cdot 14} = 1$$



4: Trigonometric Function of Two Variables

$$f(x, y) = 3 - \frac{5}{\pi} \arcsin(x + y)$$

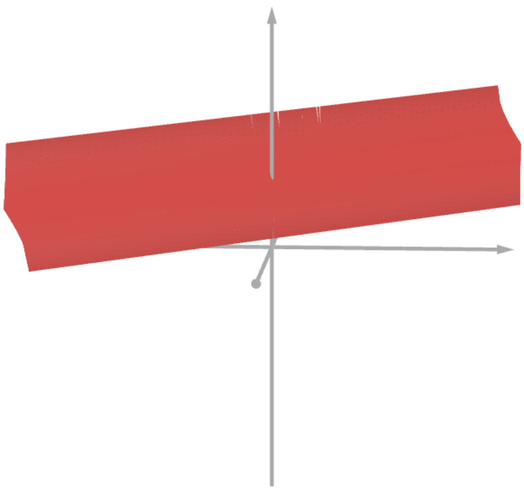
- **Domain**: $-1 \leq x + y \leq 1$



- **Range**:

$$\frac{-\pi}{2} \leq \arcsin(x + y) \leq \frac{\pi}{2},\ z \in \left[\frac{1}{2}, \frac{11}{2}\right]$$

- **Graph**: A surface in three-dimensional space.



2.5 Graphing Dimensions Summary

Graphing dimensions change based on the variables involved:

- **1D**: Interval or union of intervals
- **2D**: Picture
- **3D**: Description
- **4D and Higher**: Equation

3. Functions of Several Variables

Functions of several variables extend the concept of functions to higher dimensions, allowing for more complex mappings and dependencies.

4. Partial Derivatives

Partial derivatives are used to study functions with multiple variables by differentiating with respect to one variable while keeping the others constant.

4.1 Differentiability

A function of more than two variables is said to be differentiable if the function all of its first partial derivative and all of its second partial derivatives exist and are continuous.

4.2 Introduction

Partial derivatives are used to study how a function changes with respect to one variable while keeping the others constant.

4.2.1 Differentials

The derivative of a function $f(x)$ with respect to x is denoted as: $\frac{dy}{dx}$ or $f'(x)$ where dx is the differential of x and $dy = f'(x)dx$. Differentials are different than derivatives, as they are the change in the function value due to a change in the variable. For example,

$$\text{if } f(x) = x^2, \text{ then } df = 2xdx.$$

$$d(\arctan(u)) = \frac{du}{1+u^2} \text{ but } \arctan(u') = \frac{u'}{1+u^2}$$

4.2.2 Derivatives

$$f(x)\text{is differentiable iff } \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists}$$

4.3 Partial Derivatives

4.3.1 Partial Derivatives Definition

The partial derivative of a function $f(x, y)$ with respect to x is denoted as:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

This **not** a fraction. ∂x does not exist, and neither does ∂f.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \tag{4.1}$$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{4.2}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{4.3}$$

Shortcut

The partial derivative of a function $f(x, y, \cdots)$ with respect to x can be calculated by treating all other variables as a constant and differentiating with respect to x:

4.3.2 All Defintions of First and Second Partial Derivatives

You sometimes have to use the definition of a partial derivative to find the actual value of the partial derivative.

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{4.4}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{4.5}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(x+h, y) - \frac{\partial f}{\partial y}(x, y)}{h} \tag{4.6}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x, y)}{h} \tag{4.7}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(x+h, y) - \frac{\partial f}{\partial x}(x, y)}{h} \tag{4.8}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y \partial y} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(x, y+h) - \frac{\partial f}{\partial y}(x, y)}{h} \tag{4.9}$$

4.3.3 Properties of Partial Derivatives

If $f(x, y)$ is continuous at (x_0, y_0), then f_x and f_y exist at (x_0, y_0) and $f_{xy} = f_{yx}$.

Total Differential

The total differential of a function $f(x, y)$ is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If you find df, it is very easy to find both partial derivatives via the coefficients of dx and dy.

4.3.4 Higher-Order Partial Derivatives

The second-order partial derivative of a function $f(x, y)$ with respect to x and y is denoted as:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Order does matter. For example, for some instances:

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

However, these are usually equal for continuous functions. It is read from left to right and can also be notated as f_{xy} or f_{xy}.

4.4 Examples:

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{2y^3(x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial y} = \frac{x^2(2x^2-y^2)}{(x^2+y^2)^3}$$

For $(x,y) = (0,0)$, we use the limit definition of partial derivatives. The partial derivative of f with respect to x at $(0,0)$ is:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Similarly, the partial derivative of f with respect to y at $(0,0)$ is:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

Thus, we have:

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

To find the second partial derivatives at $(0,0)$, we proceed similarly:

$$\frac{\partial^2 f}{\partial x \partial x}(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial^2 f}{\partial y \partial y}(0,0) = \lim_{k \to 0} \frac{\frac{\partial f}{\partial y}(0,k) - \frac{\partial f}{\partial y}(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 0$$

Thus, all second partial derivatives at $(0,0)$ are:

$$\frac{\partial^2 f}{\partial x \partial x}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y \partial y}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0, \frac{\partial^2 f}{\partial y \partial x}(0,0) = DNE$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial x}(x,0) - \frac{\partial f}{\partial x}(0,0)}{x} = \lim_{x\to 0} \frac{1-0}{x} = DNE$$

All partial derviatives combined are:

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{x^4 - x^2y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \begin{cases} \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \begin{cases} \frac{2x^3y^2 - 2xy^4}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ DNE, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial x} = \begin{cases} \frac{2y^3(x^2 - y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial y} = \begin{cases} \frac{x^2(2x^2 - y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

5. Multiple Integrals

Multiple integrals extend single-variable integration to functions of several variables, useful in areas such as volume and area calculations.

6. Vector Calculus

Vector calculus explores vector fields and operations such as the gradient, divergence, and curl, which are foundational in physics, engineering, and mathematics.

6.1 Introduction to Vectors and Scalars

In mathematics and physics, **scalars** and **vectors** are fundamental quantities that describe different types of quantities.

Scalars are single numbers that represent quantities that have only magnitude, without any direction. Examples of scalars include:

- Temperature (e.g., 25°C)
- Mass (e.g., 5 kg)
- Distance (e.g., 10 m)

Vectors, on the other hand, are quantities that have both magnitude and direction. Vectors are represented as an ordered list of components in a coordinate system. For example, in three-dimensional space, a vector $\vec{v}$ can be represented as:

$$\vec{v} = \langle v_x, v_y, v_z \rangle.$$

An example of a vector is a displacement of 5 m to the right and 3 m upwards, represented as $\vec{d} = \langle 5, 3, 0 \rangle$.

In summary:

- A **scalar** is a quantity that has only magnitude.
- A **vector** is a quantity that has both magnitude and direction.

6.1.1 Vector Addition

You **cannot** add a vector and a scalar. However, you can add two vectors together. Vector addition combines two vectors $\vec{u}$ and $\vec{v}$ to produce a resultant vector $\vec{w}$:

$$\vec{w} = \vec{u} + \vec{v} = \langle u_x + v_x, u_y + v_y, u_z + v_z \rangle$$

For example, given these vectors: $u = \langle 1, 2, 3 \rangle$, and $v = \langle 4, 5, 6 \rangle$

$$\vec{u} + \vec{v} = \langle 1 + 4, 2 + 5, 3 + 6 \rangle = \langle 5, 7, 9 \rangle$$

6.1.2 Vector Scalar Multiplication

A vector and scalar can be multiplied, but two vectors cannot be multiplied in the traditional sense.

$$k\vec{u} = \langle ku_x, ku_y, ku_z \rangle$$

6.1.3 Scalar Multiples

A scalar multiple of a vector $\vec{u}$ scales its magnitude without changing its direction. If $\vec{v}$ is collinear to $\vec{u}$, then there exists some scalar k where $\vec{v} = k\vec{u}$. For example, given these vectors: $u = \langle 1, 2, 3 \rangle$, and $v = \langle 2, 4, 6 \rangle$

$$2\vec{u} = \vec{v}$$

$$2 \langle 1, 2, 3 \rangle = \langle 2, 4, 6 \rangle$$

For any $\vec{u}$, $\vec{v}$, and scalar k:

$$k\vec{u} = \langle ku_x, ku_y, ku_z \rangle$$

$$\text{iff } \vec{v} = \langle ku_x, ku_y, ku_z \rangle \text{ for some scalar } k, \vec{v} \parallel \vec{u}$$

6.2 Dot Product

The dot product is an operation between two vectors $\vec{u}$ and $\vec{v}$ that produces a scalar, and is calculated as:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + \cdots + u_n v_n = ||\vec{u}||\, ||\vec{v}|| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. The dot product is **commutative**:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

6.2.1 Applications of the Dot Product

1. **Magnitude of a Vector**:

$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2$$

$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$$

2. **Determining Perpendicularity**:

$$\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$$

For example, if $\vec{a}_{L_1} \cdot \vec{a}_{L_2} = 0$, then lines L_1 and L_2 are perpendicular.

6.3 Cross Product

The cross product is an operation on two 3D vectors that yields a vector perpendicular to both:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \langle u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x \rangle$$

This resultant vector $\vec{w} = \vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

6.3.1 Shortcut for Cross Product

To find the cross product $\vec{u} \times \vec{v}$, arrange the components as follows:

$$\begin{vmatrix} u_x & u_y & u_z & u_x & u_y & u_z \\ v_x & v_y & v_z & v_x & v_y & v_z \end{vmatrix}$$

Then, calculate each component of the cross product by making crosses between u_y and v_z, u_z and v_y, u_z and v_x, u_x and v_z, u_x and v_y, and u_y and v_x. This yields the components of the cross product vector and is easy to visualize.

6.4 Triple Scalar Product

The triple scalar product of three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is defined as:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$$

It is often shown as $\vec{u}\vec{v}\vec{w}$ and has a few niche uses.

- The volume of a parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.
- The determinant of a matrix.

$$\det(M) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \vec{u}\vec{v}\vec{w}$$

- The volume of a tetrahedron with noncoplanar edges $\vec{u}$, $\vec{v}$, and $\vec{w}$

$$\left| \frac{\vec{u}\vec{v}\vec{w}}{6} \right|$$

- The scalar triple product is zero if the vectors are coplanar.

$$\text{iff } \vec{u}\vec{v}\vec{w} = 0 \text{ then } \vec{u}, \vec{v}, \vec{w} \text{ are coplanar}$$

6.5 Directional Derivatives

6.5.1 Definition

The directional derivative of f at a point (x_0, y_0, z_0) in the direction of a vector $\vec{v}$ is defined as:

$$f'_{\vec{v}}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h\cos\alpha, y_0 + h\cos\beta, z_0 + h\cos\gamma) - f(x_0, y_0, z_0)}{h}$$

$$f'_{\vec{v}}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f((x_0, y_0, z_0) + h\frac{\vec{v}}{\|\vec{v}\|}) - f(x_0, y_0, z_0)}{h}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **directional cosines** of $\vec{v}$.

6.5.2 Shortcut Formula for Directional Derivative

Using the gradient, the directional derivative can be computed as:

$$f'_{\vec{v}}(x_0, y_0, z_0) = \vec{u} \cdot \nabla f(x_0, y_0, z_0)$$

where:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The gradient vector, ∇f, points in the direction of greatest increase of f and has a magnitude equal to the maximum rate of increase.