Section I: Derivation of relevant equations:

Part 1: Equation for the Orbit path:

In fitting an orbit it is important to understand how one gets from a reference frame which looks at points in the plane of the sky (described by coordinates $\vec{\mathcal{R}} = [X, Y, Z]$ with the origin centered on the Milky Way's Supermassive Black hole (SMBH) Sgr A* and with the X-axis directed due East, and the Y-axis directed due North) to a frame which lies in the plane of an elliptical orbit (one described by $\vec{r} = [x, y, z]$, with the x-axis along the semi-major axis pointed toward periapsis, and the y-axis coplanar with the orbit and pointed parallel to the semi-minor axis[1], and the origin at the focus of the ellipse [which again, lies close enough to the SMBH, that we can set the origin at Sgr A*]). As the origin is not changing, a pure rotation is all that is needed to characterize the transformation between $\vec{\mathcal{R}}$ and $\vec{r}$. The Argument of Periapsis (ω), the Longitude of the Ascending Node (Ω), and the Inclination (i) – angles which form part of the well-known Keplarian Orbital Elements – act as the Euler Angles for this rotation. To get from the sky plane to the orbital plane, one must first rotate around the Z-axis by Ω, then around the X'-axis by i, and finally, around the Z''-axis by ω. The rotation matrices (given by ρ_ω, ρ_Ω, and ρ_i) are:

$$\rho_\omega = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}, \quad \rho_\Omega = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the transformation from $\vec{\mathcal{R}}$ to $\vec{r}$ is given by:

$$\rho_\omega \rho_i \rho_\Omega \vec{\mathcal{R}} = \vec{r}$$

Which can be written as a single transformation, T:

$$T\vec{\mathcal{R}} = \vec{r}$$

where:

$$\begin{aligned} T &= \rho_\omega \rho_i \rho_\Omega \\ &= \begin{bmatrix} \cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega & -\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega & \sin i\sin\omega \\ \sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega & -\sin i\cos\omega \\ \sin i\sin\Omega & \sin i\cos\Omega & \cos i \end{bmatrix} \end{aligned}$$

Applying this transformation to $\vec{\mathcal{R}}$, one obtains three equations:

$$\begin{bmatrix} \cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega & -\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega & \sin i\sin\omega \\ \sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\sin\Omega & -\sin i\cos\omega \\ \sin i\sin\Omega & \sin i\cos\Omega & \cos i \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} &Z(\sin i\sin\omega) + Y(-\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega) + X(\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega) = x \\ &Z(-\sin i\cos\omega) + Y(-\omega\sin\Omega + \cos i\cos\omega\sin\Omega) + X(\sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega) = y \\ &Z(\cos i) + Y(\sin i\cos\Omega) + X(\sin i\sin\Omega) = z = 0 \end{aligned}$$

Because orbits around an ellipse with a short orbital period compared to precession rate are in effectively planar elliptical orbits, one can take $z = 0$ as a good approximation. Taking the last equation and this approximation:

$$Z = -Y(\tan i\cos\Omega) - X(\tan i\sin\Omega)$$

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[1] The orientation of the y and z axes are dependent on the angles used. The x, y, and z axis should, however remain right handed.

Plugging this into the other two equations:

$$x = \left(-Y(\tan i \cos \Omega) - X(\tan i \sin \Omega)\right)(\sin i \sin \omega) + \ Y(-\cos \omega \sin \Omega - \cos i \sin \omega \cos \Omega) \ + \ X(\cos \omega \cos \Omega - \cos i \sin \omega \sin \Omega)$$
$$y = \left(-Y(\tan i \cos \Omega) - X(\tan i \sin \Omega)\right)(-\sin i \cos \omega) \ + \ Y(-\omega \sin \Omega + \cos i \cos \omega \sin \Omega) \ + \ X(\sin \omega \cos \Omega + \cos i \cos \omega \sin \Omega)$$

Or in short:

$$x = \frac{T_{13}}{\cos i}(-T_{32}Y - T_{31}X) + \ T_{12}Y \ + \ T_{11}X$$
$$x = \sec i \left[(T_{11} \cos i - T_{13}T_{31})X + (T_{12} \cos i - T_{13}T_{32})Y\right]$$

$$y = \frac{T_{23}}{\cos i}(-T_{32}Y - T_{31}X) \ + T_{22} \ Y \ + \ T_{21}X$$
$$y = \sec i \left[(T_{21} \cos i - T_{23}T_{31})X + (T_{22} \cos i - T_{23}T_{32})Y\right]$$

In the orbital reference frame, we are trying to fit a 2-D ellipse with the semi-major axis pointed along x-direction:

$$\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 = 1$$

where (x_0, y_0), is the position of the origin relative to the center of the ellipse, a is the length of the semi-major axis, and b is the semi-minor axis. The two axes are related by the eccentricity, e:

$$b = a\sqrt{1 - e^2}$$

Since our origin is taken to be at one of the foci, the coordinates (x_0, y_0) are given by:

$$\left(\sqrt{a^2 - b^2}, 0\right)$$

or, alternatively:

$$(ae, 0)$$

So:

$$x - x_0 = \sec i \left[(T_{11} \cos i - T_{13}T_{31})X + (T_{12} \cos i - T_{13}T_{32})Y\right] - ae$$
$$x - x_0 = \sec i \left[(T_{11} \cos i - T_{13}T_{31})X + (T_{12} \cos i - T_{13}T_{32})Y - ae \cos i\right]$$

and

$$y - y_0 = y$$

Altogether, the equation for the ellipse is:

$$\frac{\left[(T_{11} \cos i - T_{13}T_{31})X + (T_{12} \cos i - T_{13}T_{32})Y - ae \cos i\right]^2}{(a \cos i)^2} + \frac{\left[(T_{21} \cos i - T_{23}T_{31})X + (T_{22} \cos i - T_{23}T_{32})Y\right]^2}{(b \cos i)^2} = 1$$

-or-

$$
\begin{aligned}
& X^2\left(\frac{T_{11}^2}{a^2}+\frac{T_{21}^2}{a^2(1-e^2)}-\frac{2T_{11}T_{13}T_{31}\sec[i]}{a^2}-\frac{2T_{21}T_{23}T_{31}\sec[i]}{a^2(1-e^2)}+\frac{T_{13}^2T_{31}^2\sec^2[i]}{a^2}+\frac{T_{23}^2T_{31}^2\sec^2[i]}{a^2(1-e^2)}\right)\\
& +Y^2\left(\frac{T_{12}^2}{a^2}+\frac{T_{22}^2}{a^2(1-e^2)}-\frac{2T_{12}T_{13}T_{32}\sec[i]}{a^2}-\frac{2T_{22}T_{23}T_{32}\sec[i]}{a^2(1-e^2)}+\frac{T_{13}^2T_{32}^2\sec^2[i]}{a^2}+\frac{T_{23}^2T_{32}^2\sec^2[i]}{a^2(1-e^2)}\right)\\
& +XY\left(\frac{2T_{11}T_{12}}{a^2}+\frac{2T_{21}T_{22}}{a^2(1-e^2)}-\frac{2T_{12}T_{13}T_{31}\sec[i]}{a^2}-\frac{2T_{22}T_{23}T_{31}\sec[i]}{a^2(1-e^2)}-\frac{2T_{11}T_{13}T_{32}\sec[i]}{a^2}\right.\\
& \qquad\qquad \left.-\frac{2T_{21}T_{23}T_{32}\sec[i]}{a^2(1-e^2)}+\frac{2T_{13}^2T_{31}T_{32}\sec^2[i]}{a^2}+\frac{2T_{23}^2T_{31}T_{32}\sec^2[i]}{a^2(1-e^2)}\right)\\
& +Y\left(-\left(\frac{2eT_{12}}{a}\right)+\frac{2eT_{13}T_{32}\sec[i]}{a}\right)\\
& +X\left(-\left(\frac{2eT_{11}}{a}\right)+\frac{2eT_{13}T_{31}\sec[i]}{a}\right)\\
& +(e^2-1)=0
\end{aligned}
$$

Part 2: The LSR Velocity and the Dynamical Mass

Now let's transform from the $\vec{r}$ system back to the $\vec{\mathcal{R}}$ system. The transformation, T, between the two systems is merely the composition of three orthogonal transformation, so the transpose of $T$is also it's inverse:

$$T^{-1} = \begin{bmatrix} \cos\omega\cos\Omega - \cos i \sin\omega\sin\Omega & \sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega & \sin i\sin\Omega \\ -\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\sin\Omega & \sin i\cos\Omega \\ \sin i\sin\omega & -\sin i\cos\omega & \cos i \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\omega\cos\Omega - \cos i \sin\omega\sin\Omega & \sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega & \sin i\sin\Omega \\ -\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\sin\Omega & \sin i\cos\Omega \\ \sin i\sin\omega & -\sin i\cos\omega & \cos i \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$X = (\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega)x + (\sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega)y$$
$$Y = (-\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega)x + (-\omega\sin\Omega + \cos i\cos\omega\sin\Omega)y$$
$$Z = (\sin i\sin\omega)x + (-\sin i\cos\omega)y$$

In our data collection, we have values related to the LSR radial velocities. These radial velocities can be taken to be a close approximation to the velocities in the Z-direction. Since the Euler angles are parameters that change in time based on the rate of precession, we can consider them constant over the time of observation:

$$V_{LSR} \approx \frac{dZ}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}$$
$$V_{LSR} \approx (\sin i\sin\omega)\dot{x} + (-\sin i\cos\omega)\dot{y}$$

We can relate these velocities to the mass of the black hole via the conservation of energy. Recall that Force between two objects in a gravitational field is given by:

$$\vec{F}_{m,M} = -\frac{GMm}{r^2}\hat{r}_{M,m}$$

Where $F_{m,M}$ is the force on m by M, G is the gravitational constant, M is the mass of the black hole, m is the mass of the infalling object, and r is the distance between the two objects. $\hat{r}_{M,m}$ is the unit vector which points from M to m. The gravitational force is conservative, so we expect the potential energy to have a form given by the gradient:

$$-\vec{\nabla}U_g = \vec{F}_{m,M}$$
$$U_g = -\frac{GMm}{r} + \psi$$

where U_g is the potential energy due to gravitation, and ψ is merely the gauge. If we use the classical gauge and let the energy go to zero at infinity – implying that the influence of the black hole has limited range –then the energy is just

$$U_g = -\frac{GMm}{r}$$

The furthest possible place to be (and thus the position with the slowest speed) relative to the ellipse focus for an ellipse size a happens at the radius of $r = a(1+e)$. If the eccentricity is nearly equal to 1, then the absolute farthest place to be from the origin and still be on the ellipse is at $r = 2a$:

$$E_{apoapsis} = E_r$$

$$\frac{1}{2}\cancel{m}v_{2a}^2 - \frac{GM\cancel{m}}{2a} = \frac{1}{2}\cancel{m}v_r^2 - \frac{GM\cancel{m}}{r}$$
$$v_r^2 = v_{2a}^2 + GM\left(\frac{2}{r} - \frac{1}{a}\right)$$
$$v_r = \sqrt{v_{2a}^2 + GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

where v is the velocity, and E is the total energy. By some magic that I, quite frankly do not fully understand $v_a = 0$. I think the idea is that since this is the farthest possible an object can be while still being bound to the SMBH, if it is still moving with any speed at this point it will no longer be travelling on an elliptical orbit, so in order to maintain elliptical orbits, we force the velocity at that point to be zero. In any case, this means:

$$|\overrightarrow{v_r}| = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Because rotations preserve length, the radius, R, in the LSR reference frame has the same length as r:

$$R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}} = r$$
$$(X^2 + Y^2 + Z^2)^{\frac{1}{2}} = (x^2 + y^2)^{\frac{1}{2}}$$

With this length-preserving property, we can also say that:

$$v_r^2 = v_R^2$$
$$\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right)^{\frac{1}{2}} = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}$$

Note that the z-direction velocity stays zero, again, because we are using the assumption that the orbital period is so much shorter that an appreciable precession of the ellipse

Detour: Conservation of Angular Momentum:

The dominant force of attraction between the SMBH and an arbitrary particle of gas is the gravitational force. For a central body of mass, M, being orbited by a test particle mass, m, at a distance, r, the force, F, is given by:

$$F = -\frac{GMm}{r^2}\hat{r}$$

Where G is the constant of universal gravitation. This force is conservative, so it can be derived from a potential, V:

$$F = -\vec{\nabla}V$$
$$V = -\int_{\infty}^{r} \vec{F}\cdot d\vec{s}$$
$$V = -\left(-1\frac{GMm}{r}\right)\Big|_{\infty}^{r} = -\frac{GMm}{r}$$

Constructing the Lagrangian, $\mathcal{L}$, for this system in polar coordinates (r, θ):

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

The Lagrangian's general coordinates, q, obey the Euler-Lagrange Equation:

$$-\frac{d\mathcal{L}}{dq} = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}}\right)$$

Apply the Euler-Lagrange equation to the radial coordinate, we recover the force:

$$-\frac{GMm}{r^2} = \frac{d}{dt}(m\dot{r})$$
$$m\ddot{r} = -\frac{GMm}{r^2}$$

Applying this to the angular coordinate, we obtain a conservation law which relates the angular velocity, ω, to the Angular Momentum, L:

$$0 = \frac{d}{dt}(mr^2\omega)$$
$$mr^2\omega = const = L$$
$$L = mrv_\theta$$
$$\ell \equiv \frac{L}{m} = rv_\theta$$

Where v_θ is the component of the velocity which lies in the theta direction, and ℓ is the *reduced angular momentum* – the angular momentum per unit mass. The total velocity v_t lies tangent to orbit path, and at $r = a - ae$, the velocity will lie completely in the θ direction. Based on this, we can deduce both the angular speed and the radial speeds:

Given:

$$v_t(r,\theta) = v_t(r) = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{GM\left(\frac{2a - r}{ra}\right)}$$

Evaluating $v_t(r)$ at $r_{min} = a - ae$:

$$v_t(r = r_{min}) = v_\theta[r = r_{min}] = \sqrt{GM\left(\frac{2}{r_{min}} - \frac{1}{a}\right)}$$
$$v_\theta[r = r_{min}] = \sqrt{GM\left(\frac{2}{a(1-e)} - \frac{1-e}{a(1-e)}\right)}$$
$$v_\theta[r = r_{min}] = \sqrt{GM\left(\frac{1+e}{a(1-e)}\right)}$$

Given that $\ell = rv_\theta$ and is constant:

$$\ell = (a - ae)\sqrt{GM\left(\frac{1+e}{a(1-e)}\right)}$$
$$v_\theta(r) = \frac{\ell}{r} = \frac{(a-ae)}{r}\sqrt{GM\left(\frac{1+e}{a(1-e)}\right)}$$
$$v_r(r) = \left(v_t^2 - v_\theta^2\right)^{\frac{1}{2}} = \sqrt{GM\left(\frac{2a-r}{ra}\right) - \left(\frac{\ell}{r}\right)^2}$$

Switching to from Polar Back to Cartesian:

Now we turn our attention to transforming between polar coordinates to Cartesian coordinates in the orbit plane:

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$
$$\hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix}$$
$$\hat{x} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$
$$\hat{y} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

The two vectors of interest to us are position and velocity vectors:

Position Vectors:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Velocity Vectors:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix}$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{x} = \sqrt{GM\left(\frac{2a-r}{ra}\right) - \left(\frac{\ell}{r}\right)^2}\cos\theta - \frac{\ell}{r}\sin\theta$$
$$\dot{y} = \sqrt{GM\left(\frac{2a-r}{ra}\right) - \left(\frac{\ell}{r}\right)^2}\sin\theta + \frac{\ell}{r}\cos\theta$$

Cosine and Sine of the true anomaly, of course, can be obtained through the simple relationship between x,y, and r. At some arbitrary point (r, θ), the sine and cosine of the angle is then given simply by:

$$\cos\theta = \frac{x}{r} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}$$
$$\sin\theta = \frac{y}{r} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$
$$\dot{x} = -\frac{y\ell}{x^2+y^2} + \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\sqrt{GM\left(\frac{2a-(x^2+y^2)^{\frac{1}{2}}}{a(x^2+y^2)^{\frac{1}{2}}}\right) - \frac{\ell^2}{x^2+y^2}}$$
$$\dot{y} = \frac{x\ell}{x^2+y^2} + \frac{y}{(x^2+y^2)^{\frac{1}{2}}}\sqrt{GM\left(\frac{2a-(x^2+y^2)^{\frac{1}{2}}}{a(x^2+y^2)^{\frac{1}{2}}}\right) - \frac{\ell^2}{x^2+y^2}}$$

Recall from earlier:

$$V_{LSR} \approx (\sin i \sin\omega)\dot{x} + (-\sin i \cos\omega)\dot{y}$$
$$V_{LSR} - T_{23}\dot{y} - T_{13}\dot{x} = 0$$

and

$$x = \sec i\,[(T_{11}\cos i - T_{13}T_{31})X + (T_{12}\cos i - T_{13}T_{32})Y]$$
$$y = \sec i\,[(T_{21}\cos i - T_{23}T_{31})X + (T_{22}\cos i - T_{23}T_{32})Y]$$