Notes on Analysis

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1 Meta

The goal of this PDF is to put together a collection of some of the most nuanced/interesting concepts I have learned in analysis classes in my life.

These notes will aim to write more clearly and concisely what I could not find written cleanly in other places, or took me too long to understand. In that sense, it is more of a reference than true exposition.

There is not much added value from simply rewriting what already has been written, so in places where great existing literature exists, I will choose to cite it. However, I recognize that this may create accessibility issues. If you are interested but cannot find a reference, please contact me at `aathreyakadambi@gmail.com` and I will help look it up for you.

2 Examples We Shouldn't Need to Recite in The Afterlife

> "Did you know: In one's afterlife, one is condemned to finding counter-examples to all false statements made in life?
> Hence the advice: Start early!"
>
> — Alexander Givental's Personal/Professional Homepage

I thought it would be best to start these notes with many examples of objects I wish to study. For the purposes of these notes, most things studied will be *linear*. That is because the study of linear objects is itself an extremely vast and interesting subject. Importantly, spaces of functions tend to have linear properties. In future notes I hope to also study other types of curved spaces which require a somewhat different mindset. In other future notes, I would love to also consider things with more heavy algebraic machinery.

2.1 Banach Spaces

Take Ω to be a σ-finite measure space.

Space	Dual space	Key properties
$\ell_p,\ 1 < p < \infty$	$\ell_q,\ \frac{1}{p} + \frac{1}{q} = 1$	reflexive, separable
ℓ_1	ℓ_∞	separable, not reflexive
ℓ_∞	$ba(\mathbb{N})$	not separable, not reflexive
ℓ_2	ℓ_2	Hilbert, self-dual
c_0	ℓ_1	separable, not reflexive
c	$\ell_1 \oplus \mathbb{R}$	separable, not reflexive
$L^p(\Omega),\ 1 < p < \infty$	$L^q(\Omega)$	reflexive (if σ-finite)
$L^1(\Omega)$	$L^\infty(\Omega)$	not reflexive
$L^\infty(\Omega)$	$ba(\Omega)$	dual larger than L^1
$C(K)$	$M(K)$	Radon measures (Riesz)

2.2 Hilbert Spaces

2.3 Probability Spaces

2.4 Spaces of Distributions

2.5 Basic Operators

- Shift and Cut Operators

2.6 Differential Operators

3 Different Notions of Convergence

In some sense, analysis begins with limits. And taking "limits" requires a notion of convergence, which comes from having a topology. In certain spaces, such as probability spaces and vector spaces, there can be many useful notions of convergence. I'll aim to start with a birds-eye view of convergence via a series of definitions, then discuss various questions relating notions of convergence.

Beware: it may be useful to have seen these definitions before (as I have when I'm writing this). But if it's your first time seeing these, feel free to send me feedback at `aathreyakadambi@gmail.com` about how readable this is and for what words you would have appreciated more explanation. This note is meant to serve more as a reference than a thorough introduction.

3.1 Topologies on Functions

Functions map from one space to another, denoted $f : A \to B$. When B has a topology on it, we define what is called pointwise convergence:

Pointwise Convergence. $f_\alpha \to f$ pointwise when $f_\alpha(x) \to f(x)$ for every $x \in A$.

- Note the dependence on the topology of A: this is where the notion of $f_\alpha(x) \to f(x)$ comes from.

With a bit more structure on B, it is possible to define other types of convergence, such as uniform convergence. Though for full generality B could simply be a "uniform space",[1] we allow ourselves to indulge and for simplicity and take B with a metric $d : B \times B \to \mathbb{R}_{\geq 0}$. Then uniform convergence can be defined as:

Uniform Convergence. $f_\alpha \to f$ uniformly when $\sup_{x \in A} d(f_\alpha(x), f(x)) \to 0$.

- This is "uniform" because the function gets closer everywhere evenly, rather than converging much slower in one place compared to another.

3.2 Topologies on Banach Spaces

Consider a Banach space X. Here are some important topologies:

Norm Topology. This is simply the topology from the norm, namely: $x_\alpha \to x$ when $\|x_\alpha - x\| \to 0$.

- Strong convergence is convergence in the norm topology.
- Useful strat: by the reverse triangle inequality, $|\|x_\alpha\| - \|x\|| \leq \|x_\alpha - x\|$, so a side effect is that if x_α converges in norm to x, $\|x_\alpha\| \to \|x\|$ (of course, the converse isn't true).

Weak Topology. This is the topology such that all linear functionals on X are continuous.

- $x_n \xrightarrow{w} x$ if $l(x_n) \to l(x)$ for every linear functional l.

3.3 Topologies on Bounded Operators

Consider Banach spaces X and Y, and the space of bounded operators $\mathcal{L}(X, Y)$. These spaces are particularly rich in notions of convergence because they inherit many topologies from being both function spaces and Banach spaces. Here I list a few important topologies on $\mathcal{L}(X, Y)$.

Uniform/Norm Topology. This topology is the norm topology on $\mathcal{L}(X, Y)$ with the norm $\|T\| = \sup_{\|x\| \neq 0} \frac{\|Tx\|}{\|x\|}$.

Strong Operator Topology. This topology is the weakest toplogy on $\mathcal{L}(X, Y)$ such that all evaluation maps $E_x : \mathcal{L}(X, Y) \to Y$, $T \mapsto Tx$ are continuous.

- Strong convergence $T_\alpha \xrightarrow{s} T$ is equivalent to $\|T_\alpha x - Tx\| \to 0$ for all x (pointwise convergence w.r.t. norm topology on Y).

[1] This notion gives meaning to "how close" via entourages instead of just "close" as a topology does via neighborhoods. I leave it to the interested reader to investigate further and generalize results sufficiently for their interests. This investigation is particularly insightful when considering: when does convergence imply existence of a limit?

Weak Operator Topology (WOT). This topology is the weakest toplogy on $\mathcal{L}(X,Y)$ such that all evaluation maps $E_{x,l} : \mathcal{L}(X,Y) \to \mathbb{C}$, $T \mapsto l(Tx)$ are continuous.

- Weak convergence $T_\alpha \xrightarrow{w} T$ is equivalent to $|l(T_\alpha x) - l(Tx)| \to 0$ for all l, x (pointwise convergence w.r.t. weak topology on Y).

Weak (Banach Space) Topology. This is the weak topology on $\mathcal{L}(X,Y)$, treating it as a Banach space.

- Convergence would be equivalent to $|l(T_\alpha) - l(T)| \to 0$ for all $l \in \mathcal{L}(X,Y)^*$

3.4 Topologies on Measurable Functions

3.5 Caution with Cauchy-ness

Theorem. In a Banach space B, norm-Cauchy is equivalent to uniformly weak-Cauchy over the unit ball in B^*.[a]

[a] "Uniformly" requires restricting to the unit ball in B^*. Recall meaning from the topologies on function section.

3.6 When Does Convergence Imply Existence of A Limit?

This is a very common question in analysis that we hear time and time again, ever since the first consideration of whether Cauchy sequences converge.

Theorem. Take H a separable Hilbert space. Then $\mathcal{L}(H)$ WOT-Cauchy sequentially complete.[a]

[a] I say "sequentially complete" since the question is more nuanced for nets.

Proof Sketch. Uniform boundedness principle allows us to lift the property that $\sup_n |(T_n x, y)|$ is bounded to the property that $\sup_n \|T_n x\|$ is bounded, from which we construct a bounded sesquilinear form and use Riesz lemma. See Reed & Simon Theorem 6.1 for a full proof. ∎

4 Hahn-Banach Theorem: Magic to Make Functionals Appear From Thin Air

5 Uniform Boundedness Principle: The Principle of Bound Lifting