PU MAT 215 Notes

Aathreya Kadambi

September 2021

Contents

1 Lecture 1: Sets, $\mathbb{Z}$, and $\mathbb{Q}$

Why study analysis if we already took calculus? We need to define and understand things at a more rigorous level in order to answer some questions. For example, is the piece-wise function $f(x) = \frac{1}{p}$ when $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ and $f(x) = 0$ when x is irrational continuous? (It turns out that it is only continuous at irrational points) Is there a function f that is continuous everywhere but differentiable nowhere? (Yes, there are many)

1.1 Sets

- Foundation for everything
- Definition of sets taken for granted
  - Loosely, a collection of elements
  - No order
  - No repeats

There are some operations on sets and some notation like $\cup$, $\cap$, $\setminus$, and A^c.

$A \cup B$	$\{x \mid x \in A \text{ or } x \in B\}$
$A \cap B$	$\{x \mid x \in A, x \in B\}$
$A \setminus B$	$\{x \mid x \in A, x \notin B\}$
A^c	$\{x \mid x \in U, x \notin A\}$ OR $U \setminus A$*

* the set U, the universal set, is usually evident through context. If not, just write $U \setminus A$ directly for clarity

We will use $\varnothing$ to denote the empty set in this document.
Cartesian Product: $A \times B := \{(a, b) \mid a \in A, b \in B\}$.
(a, b) is an ordered pair, but what is an ordered pair in terms of sets?

$$\boxed{(a, b) := \{\{a\}, \{b, \varnothing\}\}}$$

Proposition. Under the definition above, $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Proof.
We will first show the forward direction.

$$(a, b) = (a', b') \Rightarrow \{\{a\}, \{b, \varnothing\}\} = \{\{a'\}, \{b'.\varnothing\}\}$$

This means that either $\{a\} = \{a'\}$ and $\{b, \varnothing\} = \{b', \varnothing\}$ or $\{a\} = \{b', \varnothing\}$ and $\{a'\} = \{b, \varnothing\}$. However, notice that in the second case, a one element set cannot equal a two element set. The only way the two element sets can be one element are if $a = b = a' = b' = \varnothing$, in which case our proposition (and the first case) is true. Thus, the first case must be true, so $\{a\} = \{a'\}$ and $\{b, \varnothing\} = \{b', \varnothing\}$. Then, $a = a'$, and $b = b'$ as desired. ■

Remark. Notice that $b = \varnothing = b'$ doesn't change anything.

1.1.1 Natural Numbers

In this course, we will take everything about natural numbers for granted. However, notice that we can actually define the natural numbers in terms of sets:

$$0 = \{\varnothing\}$$

$$1 = \{\varnothing, \{\varnothing\}\}$$

$$2 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$$
$$\vdots$$

Definition [Map]. If A and B are sets, f is called a **map** from A to B if it assigns a unique element in B for every element in A. Rigorously, a map from A to B, $f : A \to B$, is a subset $f \subseteq A \times B$ such that $\forall\, a \in A\ \exists!\ b \in B$ with $(a, b) \in f$.

We can use our existing notation like $f(n) = n + 1$ to denote a map, but a better more general way to denote a map is as: $n \mapsto n + 1$.

Definition [Equivalence Relation]. An **equivalence relation** on a set A is a relation $\sim$ such that

(1) $x \sim x\ \forall\, x \in A$ (reflexive property)

(2) $x \sim y \Leftrightarrow y \sim x$ (symmetry)

(3) $x \sim y, y \sim z \Rightarrow x \sim z$ (transitive property)

Rigorously, $\sim\ \subseteq A \times A$ such that

(1) $(x, x) \in\ \sim\ \ \forall\, x \in A$ (reflexive property)

(2) $(x, y) \in\ \sim\ \Leftrightarrow (y, x) \in\ \sim$ (symmetry)

(3) $(x, y) \in\ \sim, (y, z) \in\ \sim\ \Rightarrow (x, z) \in\ \sim$ (transitive property)

Remark. Remember 2010 USAJMO Problem 1 hehe. :D

Cooler example is of course 2010 USAJMO Problem 1, but then again what if we make an equivalence relation such that $x \sim y$ iff $x - y$ is divisible by m? These are exactly the residue classes mod m.

Definition [Orbit]. $\mathrm{Orb}(x) :\ \{y \in A \mid x \sim y\}$ (using the notation from the equivalence relation definition). Notice that $\mathrm{Orb}(x) \subseteq A$.

Proposition. $x \sim y$ iff $\mathrm{Orb}(x) = \mathrm{Orb}(y)$.

Proof.
Let us first consider the forward direction. If $\mathrm{Orb}(x) = \mathrm{Orb}(y)$, then $\forall a \in \mathrm{Orb}(x), a \in \mathrm{Orb}(y)$ and vice versa. Thus,
$$\forall a \mid a \sim x, a \sim y$$
Thus, $x \sim y$ by the transitive property as desired. Now, consider the reverse direction. If $x \sim y$, then $\forall a$ such that $a \sim x$, by the transitive property, $a \sim y$. Thus, $\forall a$ such that $a \sim x \Rightarrow a \in \mathrm{Orb}(x)$, $a \sim y \Rightarrow a \in \mathrm{Orb}(y)$. Therefore, $\mathrm{Orb}(x) \subseteq \mathrm{Orb}(y)$. The same argument shows that $\mathrm{Orb}(y) \subseteq \mathrm{Orb}(x)$ (left to the reader hehe) so $\mathrm{Orb}(x) = \mathrm{Orb}(y)$ as desired. ∎

We denote the set of all orbits as $A/\sim$.

1.2 The Integers

We would now like to derive the integers from the natural numbers. In fact, we will do this by creating an equivalence class, which is the integers.

Let $A = \mathbb{N} \times \mathbb{N}$. Define $\sim$ as $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1 + y_2 = x_2 + y_1$. Then $\boxed{\mathbb{Z} := A/\sim}$.

Remark. Essentially, the encoded idea here is that $x_1 - x_2 = y_1 - y_2$ (you can think of it like this but subtraction isn't defined so . . .). The integers here are essentially represented as these ordered pairs (x_1, x_2), so that the integer we are used to is given by $x_1 - x_2$ (a crutch you might use initially to understand the motivation behind this construction but subtraction is of course not defined).

Now we would like to show that $\sim$ is indeed a valid equivalence relation, so we are essentially dividing up all of the ordered pairs in A into these equivalence classes (the orbits), each of which represents an integer.

Proposition. $\sim$ as defined above is a valid equivalence relation.

Proof.
First, we must show reflexivity. This is true, since $(x_1, x_2) \sim (x_1, x_2)$ if and only if $x_1 + x_2 = x_2 + x_1$ (and we assume the commutative property for addition over natural numbers). Next, we must show symmetry. $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1 + y_2 = x_2 + y_1$ if and only if $y_1 + x_2 = y_2 + x_1$ if and only if $(y_1, y_2) \sim (x_1, x_2)$ as desired just by swapping the sides of the equality and using commutative property for natural number addition.

Finally, we must prove transitivity. If $(x_1, x_2) \sim (y_1, y_2)$ and $(y_1, y_2) \sim (z_1, z_2)$, then

$$x_1 + y_2 = x_2 + y_1$$

$$y_1 + z_2 = y_2 + z_1$$

Adding everything up, we get

$$x_1 + z_2 + (y_1 + y_2) = x_2 + z_1 + (y_1 + y_2)$$

after the commutative property. Now recall from our natural number theorems that $a+c = b+c \Rightarrow a = b$. Thus,

$$x_1 + z_2 = x_2 + z_1 \Rightarrow (x_1, x_2) \sim (z_1, z_2)$$

as desired. ■

Thus, we have established our set of integers. Now, we can create a map as follows to redefine $\mathbb{N}$ so that $\mathbb{N} \subseteq \mathbb{Z}$:

$$i : \mathbb{N} \to \mathbb{Z} \mid n \mapsto \text{Orb}((n, 0))$$

where the orbit is with respect to $\sim$ as defined before. Now, we clearly have $\mathbb{N} \subseteq \mathbb{Z}$. Also, from now on, the notation n will automatically mean $\text{Orb}((n, 0))$.

1.2.1 Operations in $\mathbb{Z}$

We can now define operations:

Addition. $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$$(\text{Orb}((x_1, x_2)), \text{Orb}((y_1, y_2))) \mapsto \text{Orb}((x_1 + y_1, x_2 + y_2))$$

Multiplication. $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$$(\text{Orb}((x_1, x_2)), \text{Orb}((y_1, y_2))) \mapsto \text{Orb}((x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1))$$

Remark. We got the motivation for the right side from multiplying $(x_1 - x_2)(y_1 - y_2) = x_1 y_1 + x_2 y_2 - x_1 y_2 - x_2 y_1$.

Negation. $- : \mathbb{Z} \to \mathbb{Z}$

$$\text{Orb}((x_1, x_2)) \mapsto \text{Orb}((x_2, x_1))$$

Remark. From now on though, we will go back to the less rigorous symbols that we use to represent these orbits from this section. For example, the integer 0 really represents $\text{Orb}((0, 0))$.

1.3 The Rationals

We can finally extend one more bit to get to the rationals. Define $A = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$. Let $\sim$ be an equivalence relation such that $(x_1, x_2) = (y_1, y_2)$ if and only if $x_1 y_2 = x_2 y_1$. We can show again that this is indeed an equivalence relation, although I won't include that here. From this equivalence relation on the integers, we get the rational numbers with $\boxed{\mathbb{Q} := A/\sim}$. The rationals are now defined as orbits as well based on this equivalence relation on the integers.

To show the integers as a subset of the rationals, we first redefine the integers (just like how we did the natural numbers) so that the integer n is actually $\text{Orb}((n, 1))$. Basically, we define the map:

$$i : \mathbb{Z} \to \mathbb{Q} \mid n \mapsto \text{Orb}((n, 1))$$

which makes $\mathbb{Z} \subseteq \mathbb{Q}$.

1.4 Induction Axiom

Axiom [Induction Axiom]. If $S \subseteq \mathbb{N}$ such that

(1) $0 \in S$

(2) $\forall n \in S$, $n + 1 \in S$,

then $S = \mathbb{N}$.

We can actually prove directly with the induction axiom. Let $a_n = 2a_{n-1} + 1$ with $a_0 = 0$. Then, show that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Proof.

Let S be the set containing all values of n such that $a_n = 2^n - 1$. Then, since $a_0 = 0 = 2^0 - 1$, $0 \in S$. Also, for all n in S, $a_n = 2^n - 1$. Thus, $a_{n+1} = 2a_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$, so $n + 1 \in S$. From all of this, it follows by the induction axiom that $S = \mathbb{N}$. In other words, the set of values of n such that $a_n = 2^n - 1$ is the natural numbers as desired. ■

2 Precept 1: Modular Arithmetic and Set Theory

2.1 Modular Arithmetic ($\mathbb{Z}/n\mathbb{Z}$)

Fix $n \in \mathbb{N}$. Define an equivalence relation $\sim$ such that

$$a \sim b \Leftrightarrow n \mid (a-b) \Leftrightarrow \exists k \in \mathbb{Z}, a-b = nk$$

Proposition. $\sim$ is an equivalence relation in $\mathbb{Z}$.

Proof.

First, we show reflexivity. Since $a - a = 0$ which is divisible by n, we have $a \sim a$ as desired. Next, we show symmetry. $a \sim b$ implies that $\exists k, nk \mid (a-b) \Rightarrow nk \mid (b-a)$ which implies $b \sim a$ so similarity is true. Finally, we show transitivity. Notice that $a \sim b$ and $b \sim c$ implies that $\exists p, q$ such that $a - b = np$ and $b - c = nq$. Thus, $a - c = n(p+q)$, so $a \sim c$, as desired. ■

Proposition. For $a \in \mathbb{Z}$, define the equivalence class of a

$$[a] = \{b \in \mathbb{Z} : a \sim b\}(\subseteq \mathbb{Z})$$

then $[a] = \{b \in \mathbb{Z} : \exists t \in \mathbb{Z}, b = a + nt\} \subseteq \mathbb{Z}$.

Proof.

We denote the set $\{b \in \mathbb{Z} : \exists t \in \mathbb{Z}, b = a + nt\}$ by S_a. We would like to show that $[a] = S_a$. First, we will show that $[a] \subseteq S_a$. Notice that

$$b \in [a] \Rightarrow a \sim b \Leftrightarrow b \sim a \Rightarrow \exists t \in \mathbb{Z}, b - a = nt \Rightarrow b = a + nt \Rightarrow b \in S_a$$

since $b \in [a] \Rightarrow b \in S_a$, $[a] \subseteq S_a$. In addition,

$$b \in S_a \Rightarrow \exists t \in \mathbb{R}, b = a + nt \Rightarrow b - a = nt \Rightarrow b \sim a \Rightarrow a \sim b \Rightarrow b \in [a]$$

so $S_a \subseteq [a]$. Therefore, $[a] = S_a$ as desired. ■

Definition ($\mathbb{Z}/n\mathbb{Z}$). $\mathbb{Z}/n\mathbb{Z} = \{[a] : a \in \mathbb{Z}\}$.

Proposition. The following are equivalent:

(i) $[a] \cap [b] \neq \varnothing$

(ii) $a \sim b$

(iii) $[a] = [b]$

Proof.

We will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ to show that all three are equivalent.

We may begin by showing that $(i) \Rightarrow (ii)$, or $[a] \cap [b] \neq \varnothing \Rightarrow a \sim b$. Let $k \in [a] \cap [b] \Rightarrow k \in [a], k \in [b]$, so that $a \sim k$ and $b \sim k$. Thus, $a \sim b$, as desired.

Next, we show that $a \sim b \Rightarrow [a] = [b]$. Notice that $c \in [a] \Rightarrow c \sim a \Rightarrow c \sim b \Rightarrow c \in [b]$, so $[a] \subseteq [b]$. Similarly, $d \in [b] \Rightarrow d \sim b \Rightarrow d \sim a \Rightarrow d \in [a]$, so $[b] \subseteq [a]$. Thus, $[a] = [b]$ as desired.

Finally, we show that $[a] = [b] \Rightarrow [a] \cap [b] \neq \varnothing$. Notice that $[a] \cap [b] = [a] \cap [a] \neq \varnothing$ since $a \in [a]$ by reflexivity of a with respect to $\sim$. Therefore, we have shown that all three statements are equivalent. ■

Proposition. $\forall a \in \mathbb{Z}$, $\exists r \in \{0, 1, ..., n-1\}$ such that $[a] = [r]$.

Proof.

By the division algorithm, there exists k in $\mathbb{Z}$ with $a = nk + r$ such that $r \in \{0, 1, ..., n-1\}$. Thus, $a - r = nk$, so $a \sim r$. By our previous proposition, $[a] = [r]$ as desired. ■

Proposition. Let $r, s \in \{0, 1, ..., n-1\}$. Then, $r \sim s \Rightarrow r = s$.

Proof.

Assume for contradiction that $r \neq s$. Then, WLOG $r < s$. Thus, $0 \leq r < s \leq n-1$. This means that $s - r > 0$ and $s - r \leq s \leq n - 1$. However, $r \sim s \Rightarrow s - r = nk$ for some $k \in \mathbb{Z}$. Now, $s - r > 0 \Rightarrow nk > 0 \Rightarrow k > 0$. In other words, $k \geq 1$. Thus, $s - r = nk \geq n$, a contradiction. Thus, $r = s$ as desired. ■

From our propositions, $\mathbb{Z}/n\mathbb{Z} = \{[a] : a \in a \in \mathbb{Z}\} = \{[0], [1], ..., [n-1]\}$.
$\#\mathbb{Z}/n\mathbb{Z} = \#\{[0], [1], ..., [n-1]\} = n$

Proposition. Define $f, g : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ with $f([a], [b]) = [a+b]$ and $g([a], [b]) = [ab]$. Verify that f and g are well-defined (i.e. if $([a], [b]) = ([a'], [b'])$, $f(([a], [b])) = f(([a'], [b']))$ and $g(([a], [b])) = g(([a'], [b']))$).

Remark. By well-defined, we are just checking that $f : X \to Y$ maps $x \in X$ to a *unique* $y \in Y$.

Proof.

Notice that $([a], [b]) = ([a'], [b']) \Leftrightarrow [a] = [a']$ and $[b] = [b']$. This is also equivalent to $a \sim a'$ and $b = b'$ by our previous proposition. In other words, $\exists p, q \in \mathbb{Z}$ such that

$$a = a' + np$$

$$b = b' + nq$$

Thus,

$$a + b = a' + b' + n(p + q)$$

Thus, $(a + b) \sim (a' + b')$, which implies $[a + b] = [a' + b'] \Rightarrow f(([a], [b])) = f(([a'], [b']))$ by our proposition as desired. Now, notice that

$$ab = a'b' + npb' + nqa' + n^2pq = a'b' + n(pb' + qa' + npq)$$

Thus, $ab \sim a'b'$, so $[ab] = [a'b'] \Rightarrow g(([a], [b])) = g(([a'], [b']))$ by our proposition, as desired. ■

Definition (Addition in $\mathbb{Z}/n\mathbb{Z}$). $[a] + [b] = f(([a], [b])) = [a + b]$.

Definition (Multiplication in $\mathbb{Z}/n\mathbb{Z}$). $[a] \times [b] = g(([a], [b])) = [ab]$.

Theorem (Distributive Property on $\mathbb{Z}/n\mathbb{Z}$). $[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c]$.

Proof.

Notice that

$$[a] \cdot ([b] + [c]) = [a] \cdot [b + c] = [ab + ac]$$

from the distributive property on $\mathbb{Z}$ and

$$[a] \cdot [b] + [a] \cdot [c] = [ab] + [ac] = [ab + ac]$$

By the transitive property, we have $[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c]$ as desired. ■

2.2 Set Theory

Remark. Note that rather than U for the universal set, we might want to use X.

Some basic (easy to show so proofs omitted) properties on complements:

(i) $A \cup A^c = X$

(ii) $A \cap A^c = \varnothing$

(iii) $\forall x \in X$, exactly one of $x \in A$ and $x \in A^c$ must be true.

Theorem. $E \subseteq F \Rightarrow F^c \subseteq E^c$.

Proof.

For all $x \in F^c$, notice that either $x \in E \Rightarrow x \in F$ or $x \in E^c$. $x \in F^c \Rightarrow x \notin F$, so $x \notin E$. Thus, $x \in E^c$. In other words, $F^c \subseteq E^c$, as desired. ■

3 Lecture 2: Real Numbers

3.1 Induction

Axiom [Induction Axiom 1]. If $S \subseteq \mathbb{N}$ such that

(1) $0 \in S$

(2) $\forall n \in S$, $n+1 \in S$,

then $S = \mathbb{N}$.

Axiom [Induction Axiom 2]. Let $S \subseteq \mathbb{N}$ be a non-empty set. Then S has a minimum element (namely, $\exists a \in S$ such that $\forall b \in S$ we have $a \leq b$).

These axioms are actually equivalent (the first can be used to show the second and vice versa).

Theorem. Axiom 1 $\Leftrightarrow$ Axiom 2.

Proof.
We start by showing that Axiom 2 $\Rightarrow$ Axiom 1. Assume for contradiction that there exists a set $S \subseteq \mathbb{N}$ following $0 \in S$ and $\forall n \in S, n+1 \in S$, but $S \neq \mathbb{N}$. Consider $S^c = \mathbb{N}\ S$. By the assumption that $S \neq \mathbb{N}$, $S^c \neq \varnothing$. Let $a \in S^c$ be the minimum element. Notice that $0 \in S \Rightarrow 0 \notin S^c \Rightarrow a \neq 0$. Consider $a-1$. Since a is the minimum number in S^c, $a-1 \neq S^c$. Thus, $a-1 \in S$. However, $a-1 \in S \Rightarrow a \in S$, a contradiction. Thus, Axiom 2 implies Axiom 1.

Now we will show that Axiom 1 $\Rightarrow$ Axiom 2. Suppose that $S \subseteq \{0, 1, 2, ..., n\}$ with $S \neq \varnothing$. We claim that S has a minimum element. To show this, we will use induction on n via Axiom 1. With $n = 0$, then $S \in \{0\}$ and $S \neq \varnothing$, so $S = \{0\}$. Thus, $0 \in S$ is clearly the minimum element. Now, we use induction. Suppose that the claim holds for $n = k$. We show that it holds for $n = k+1$. Consider a set $S \in \{0, 1, 2, ..., k+1\}$, and consider $S' = S \cap \{0, 1, 2, ...k\}$. If $S' \neq \varnothing$, then using our induction hypothesis, S' must have a minimum element a. However, since $0 \leq a \leq k$, $a \leq k+1$ as well, so a is also the minimum element of S. Now suppose that $S' = \varnothing$. Then, $S = \{k+1\}$. Then, $k+1 \in S$ is clearly the minimum element in S. Thus, the claim also holds for $k+1$. By induction (Axiom 1), we have shown that $S \in \{0, 1, 2, ..., n\}$ has a minimum element for all $n \in \mathbb{N}$.

Now, consider a set $S \subseteq \mathbb{N}$ with $S \neq \varnothing$. Choose any element $a \in S$. Consider $S' = S \cap \{0, 1, 2, ..., a\} \subseteq \{0, 1, 2, ..., a\}$. Then, by our claim S' must have a minimum element b. By minimality, we have $b \leq a$. Now, consider any element c in S but not in $S \cap \{0, 1, 2, ..., a\}$. Then, $c > a$. Since $b \leq a$, $b < c$. Thus, b is minimal as desired.

Therefore, Axiom 1 and Axiom 2 are equivalent. ∎

3.2 Reals

Some properties of $\mathbb{R}$:

(1) $\mathbb{R}$ is a set, and it is equipped with two operators: $+$ and $\cdot$.

(2) $\mathbb{R}$ is decomposed into the disjoint union of $\mathbb{R} = \{0\} \cup \mathbb{R}^+ \cup \mathbb{R}^-$.

(3) $+$, $\cdot$, and the decomposition $\mathbb{R} = \{0\} \cup \mathbb{R}^+ \cup \mathbb{R}^-$ satisfy the usual set of rules of arithmetic

(4) If $S \subseteq \mathbb{R}$, $S \neq \varnothing$ has an upper bound, then it has a least upper bound.

Remark. This fourth condition is characteristic of the reals. The first 3 properties are also satisfied by the rationals.

Theorem.

(1) $\exists$ a set $\mathbb{R}$ with operators $+$ and $\cdot$ and the decomposition $\mathbb{R} = \{0\} \cup \mathbb{R}^+ \cup \mathbb{R}^-$ such that properties (1)-(4) are satisfied.

(2) If $(\mathbb{R}, +, \cdot, \mathbb{R}^+, \mathbb{R}^-)$ and $(\mathbb{R}', +', \cdot', \mathbb{R}'^+, \mathbb{R}'^-)$ are two sets with additional structures satisfying (1)-(4), then $\exists f : \mathbb{R} \to \mathbb{R}'$ which is both injective and surjective (a bijection).

3.2.1 Operators on $\mathbb{R}$

Definition (Addition).

(1) $x + y = y + x \ \forall \ x, y \in \mathbb{R}$

(2) $x + (y + z) = (x + y) + z) \ \forall \ x, y, z \in \mathbb{R}$

(3) $\exists 0 \in \mathbb{R}$ such that $0 + x = x \ \forall \ x \in \mathbb{R}$

(4) $\forall x \in \mathbb{R}, \exists (-x) \in \mathbb{R}$ such that $x + (-x) = 0$

Definition. $x - y = x + (-y)$

Proposition.

(1) If $x + y = x + z$, then $y = z$

(2) If $x + y = x$, then $y = 0$

(3) If $x + y = 0$, then $y = -x$

(4) $-(-x) = x$

Proof.
(1) If $x + y = x + z$, then $(-x) + (x + y) = (-x) + (x + z)$, so $(-x + x) + y = (-x + x) + z$, so $0 + y = 0 + z$, so $y = z$ as desired.
(2) If $x + y = x$, then $(-x) + (x + y) = (-x) + (x)$, so $(-x + x) + y = (-x + x)$, so $0 + y = 0$, so $y = 0$.
(3) IF $x + y = 0$, then $(-x) + (x + y) = (-x) + 0$, so $(-x + x) + y = -x$, so $0 + y = -x \Rightarrow y = -x$.
(4) $-(-x) + (-x) = 0 = x + (-x)$, so by (1), $-(-x) = x$. ■

Definition (Multiplication). The multiplication operator satisfies:

(1) $xy = yx \ \forall \ x, y \in \mathbb{R}$

(2) $x(yz) = (xy)z) \ \forall \ x, y, z \in \mathbb{R}$

(3) $\exists 1 \in \mathbb{R}$ such that $1 \cdot x = x \ \forall \ x \in \mathbb{R}$

(4) $\forall x \in \mathbb{R}, x \neq 0, \exists \frac{1}{x} \in \mathbb{R}$ such that $x(\frac{1}{x}) = 1$

(5) $1 \neq 0$

The distributive property also holds: $x(y + z) = xy + xz$.

Proposition.

(1) If $xy = xz$, then $y = z$

(2) If $xy = x$ and $x \neq 0$, then $y = 1$

(3) If $xy = x$ and $x \neq 0$, then $y = \frac{1}{x}$

(4) $\frac{1}{1/x} = x$

The proof of this proposition is identical to the above proposition for addition.

Proposition.

(O1) $\forall x, y \in \mathbb{R}^+$, we have $x + y \in \mathbb{R}^+$

(O2) $\forall x, y \in \mathbb{R}^+$, we have $xy \in \mathbb{R}^+$

Definition. We define order on $\mathbb{R}$ as:

(1) $x > y$ iff $x - y \in \mathbb{R}^+$

(2) $x \geq y$ iff $x - y \in \mathbb{R}^+ \cup \{0\}$

(3) $x < y$ iff $y > x$

(4) $y \leq x$ iff $y \geq x$

Proposition.

(1) $x \leq x \; \forall x \in \mathbb{R}$

(2) If $x \leq y$, $y \leq x$, then $x = y$

(3) If $x \geq y$, $y \geq z$, then $x \geq z$

Proof.
(1) $x - x = 0 \in \mathbb{R}^+ \cup \{0\}$ as desired.

Lemma. (1) If $x > 0$, then $(-x) < 0$. (2) If $x < 0$, then $(-x) > 0$.

Proof.
(1) If $x > 0, x \neq 0$, so $-x \neq 0$. Assume for contradiction that $(-x) < 0$ does not hold. Then, since $x \in \mathbb{R}^+$ and $(-x) \in \mathbb{R}^+$, $x + (-x) > 0$ as well. But $x + (-x) = 0$, contradiction.

(2) Similarly for the second part, assume for contradiction that $(-x) < 0$. Then, $x + (-x) < 0$, another contradiction. □

Lemma. If $x > 0$, then $\frac{1}{x} > 0$.

Proof.
Since $x(\frac{1}{x}) = 1 \neq 0$, we have $\frac{1}{x} \neq 0$. If $\frac{1}{x} > 0$ does not hold, then $\frac{1}{x} < 0$. Then, $-\frac{1}{x} > 0$. Thus, $x(-\frac{1}{x}) > 0$. However, $x \cdot (-\frac{1}{x}) = -1$, a contradiction. □

Proposition.

(1) If $z > 0$, then $xz < yz \Leftrightarrow x < y$

(2) If $z < 0$, then $xz < yz \Leftrightarrow x > y$

(3) $x^2 \geq 0 \; \forall \, x \in \mathbb{R}$, $x^2 = 0 \Leftrightarrow x = 0$

(4) If $0 < x < y$, then $\frac{1}{x} > \frac{1}{y} > 0$

Proof.
(1) If $x < y$, then $y - x \in \mathbb{R}^+$. By (O2), $y - x)z \in \mathbb{R}^+$, $yz - xz \in R^+$, $yz > xz$. Now, if $xz < yz$, then $(xz) \cdot \frac{1}{z} < (yz) \cdot \frac{1}{z}$ by the forward direction of the proof. Thus, $x < y$, as desired.

(2) Notice that $xz < yz$

$$\begin{aligned} &\Leftrightarrow yz - xz \in \mathbb{R}^+ \\ &\Leftrightarrow x(-z) - y(-z) \in \mathbb{R}^+ \\ &\Leftrightarrow (-z)x > (-z)y \end{aligned}$$

then apply (1).

(3) If $x > 0$ then by O2, $x^2 > 0$. If $x < 0$, then $-x > 0$ and $(-x)\cdot(-x) = -(x\cdot(-x)) = -(-(x\cdot x)) = x^2$, so $x^2 = (-x)^2 > 0$.

(4) We already proved that $\frac{1}{x} > 0$, $\frac{1}{y} > 0$. Since $y > x$,

$$y(\frac{1}{xy}) > x(\frac{1}{xy})$$

$$\frac{1}{x} > \frac{1}{y}$$

■

3.2.2 Property 4 of $\mathbb{R}$

Recall that the fourth property states that "If $S \subseteq \mathbb{R}$, $S \neq \varnothing$ has an upper bound, then it has a least upper bound." This is also called the **Completeness Axiom**.

If $S \subseteq \mathbb{R}$, define the set of upper bounds of $S = \{x \in \mathbb{R} | \forall y \in S$, we have $x > y\}$. We will denote this set with $U(S)$.

> **Example.** $S = (0, 1)$

Notice that $U(S) = [1, +\infty)$. $U(S)$ has a minimum element, 1. Formally, condition (4) can be written as follows:

Suppose $S \subseteq \mathbb{R}$, $S \neq \varnothing$. If $U(S) \neq \varnothing$, then $\exists a \in U(S)$ such that $\forall\ b \in U(S)$, we have $a \leq b$.

Rationals and Completeness Axiom

An intuitive argument for why $\mathbb{Q}$ does not satisfy the completeness property is as follows: consider $S = (0, \sqrt{2}) \cap \mathbb{Q}$. Then,

$$\begin{aligned} U(S) &= \{x \in \mathbb{Q} | x \geq \sqrt{2}\} \\ &= \{x \in \mathbb{Q} | x > \sqrt{2}\} \end{aligned}$$

and it is plausible to believe that $U(S)$ has no minimum element since $\sqrt{2}$ is not rational (the proof of this is not included in these notes).

4 Lecture 3: Real Numbers and the Archimedian Property

Properties of real numbers:

- $\mathbb{R} = \{0\} \cup \mathbb{R}^+ \cup \mathbb{R}^-$

[O1] $x \in \mathbb{R}^+, y \in \mathbb{R}^+ \Rightarrow x + y \in \mathbb{R}^+$

[O2] $x \in \mathbb{R}^+, y \in \mathbb{R}^+ \Rightarrow xy \in \mathbb{R}^+$

[O3] $x \in \mathbb{R}^-, y \in \mathbb{R}^- \Rightarrow x + y \in \mathbb{R}^-$

- $x > 0 \Rightarrow -x < 0$
- $x < 0 \Rightarrow -x > 0$

Theorem. $1 > 0$

Proof.
Assume for contradiction that $1 < 0$. Then, $(-1) > 0$, but $(-1)(-1) = 1 > 0$, a contradiction. ∎

Definition (Reals). $\mathbb{R}$ is a set equipped with $+$, $\times$, and a definition of positivity, such that the basic rules of arithmetic hold, and satisfies the least upper bound property.

Theorem. $\exists i : \mathbb{Q} \to \mathbb{R}$ an injection such that

(1) $i(1) = 1$, $i(0) = 0$

(2) $i(xy) = i(x)i(y)$

(3) $i(x + y) = i(x) + i(y)$

(4) $i(x) > 0 \Leftrightarrow x > 0$ and $i(x) < 0 \Leftrightarrow x < 0$

Proof.
Let $\frac{p}{q} \in \mathbb{Q}$. Suppose that $q > 0$ and $p, q \in \mathbb{Z}$. Define $i(\frac{p}{q})$ to be the quotient of $|p|$ ones summed divided by q ones summed if $\frac{p}{q} > 0$ and the quotient of $-|p|$ ones summed divided by q ones summed if $\frac{p}{q} < 0$.

To show that i is injective, assume $i(\frac{p}{q}) = i(\frac{p'}{q'})$ with $q, q' > 0$. Then,

$$\frac{p}{q} > 0 \Leftrightarrow \frac{p'}{q'} > 0$$

$$\frac{p}{q} = 0 \Leftrightarrow \frac{p'}{q'} = 0$$

$$\frac{p}{q} < 0 \Leftrightarrow \frac{p'}{q'} < 0$$

By the assumption,

$$\frac{1 + 1 + ... + 1 \ (p \text{ times})}{1 + 1 + ... + 1 \ (q \text{ times})} = \frac{1 + 1 + ... + 1 \ (p' \text{ times})}{1 + 1 + ... + 1 \ (q' \text{ times})}$$

Lemma. Assume $m, n \in \mathbb{R}^{\geq 0}$ such that $1 + 1 + ... + 1$ (m times) $= 1 + 1 + ... + 1$ (n times). Then $m = n$.

Proof. Assume $m \neq n$. Then, WLOG $m > n$. Now, notice that by our assumption

$$1 + 1 + ... + 1 \ (m \text{ times}) \ = 1 + 1 + ... + 1 \ (n \text{ times})$$

$$1{+}1{+}...{+}1 \ (m \text{ times}) \ {+}(-1){+}(-1){+}...{+}(-1) \ (n \text{ times}) \ = 1{+}1{+}...{+}1 \ (n \text{ times}){+}(-1){+}(-1){+}...{+}(-1) \ (n \text{ times})$$

$$1+1+...+1 \ (m-n \text{ times}) \ = 0$$

However, since $1 > 0$, by induction we can see that $1+1+...+1$ $m-n$ times is also greater than zero, a contradiction. Thus, $m = n$ $\square$

Now, cross multiplying and using the distributive property,

$$(1+1+...+1) \ pq' \text{ times } \ = (1+1+...+1) \ p'q \text{ times}$$

Using the lemma,

$$pq' = p'q$$

$$\frac{p}{q} = \frac{p'}{q'}$$

Thus, $i(\frac{p}{q}) = i(\frac{p'}{q'}) \Rightarrow \frac{p}{q} = \frac{p'}{q'}$, so i is injective.

Note that using the theorem, we can actually derive i. If $i(1) = 1$ and $i(0) = 0$, then

$$i(n) = i(1+1+...+1) \ (n \text{ times}) = i(1) + i(1) + ... + i(1) = 1+1+...+1 = n$$

$\blacksquare$

Now, we can identify $\mathbb{Q}$ with its image $i(Q)$ in $\mathbb{R}$ (so $\mathbb{Q} \subseteq \mathbb{R}$).

Theorem. $\mathbb{Q}$ does not satisfy the least upper bound property.

Developing the Proof.

$$S = \{x \in \mathbb{Q} | x > 0, x^2 < 2\}$$

We show that $2 \in \mathbb{Q}$ is an upper bound of S. Notice that $\forall x \in \mathbb{Q}$, if $x > 2$, then $x^2 > 4 \geq 2$, so $x \notin S$. Therefore, 2 is an upper bound for S.

Now assume S has a minimum upper bound a. We deduce a contradiction by showing that $a^2 = 2$ (for which there are no rational numbers a). Notice that since $1 \in S$, $a \geq 1 > 0$.

Suppose $a^2 < 2$. Then, let $x = a + \frac{1}{n}$, where $n \in \mathbb{Z}^+$ such that $n \geq n_0$ where $n_0 \in \mathbb{Q}$ will be found later. We want

$$x^2 = a^2 + \frac{2a}{n} + \frac{1}{n^2} < 2$$

Notice

$$a^2 + \frac{2a}{n} + \frac{1}{n^2} \leq a^2 + \frac{2a}{n} + \frac{1}{n} = a^2 + \frac{2a+1}{n}$$

We want

$$a^2 + \frac{2a+1}{n} < 2 \Rightarrow n > \frac{2a+1}{2-a^2}$$

Thus, we make $a_0 = \frac{2a+1}{2-a^2}$. Now letting $x = a + \frac{1}{n}$,

$$x^2 < a^2 + 2 - a^2 = 2$$

which contradicts that a is the minimum upper bound.

Suppose $a^2 > 2$. Let $x = a - \frac{1}{n}$ where $n \in \mathbb{Z}^+$ and $n \geq n_0$ where we will find n_0 later.

$$x^2 = (a - \frac{1}{n})^2$$

$$= a^2 - 2a \cdot \frac{1}{n} + \frac{1}{n^2}$$

$$\geq a^2 - 2a\frac{1}{n}$$

We need $a^2 - 2a\frac{1}{n} > 2$ so we make $n_0 = \frac{2a}{a^2-2}$. Then, $x^2 > 2$. For all $y \in S$, we have $x^2 > 2 > y^2$, $x, y > 0$ so $x > y$. Therefore x is an upper bound of S which contradicts the minimality of a. Thus, $a^2 = 2$, a contradiction. ■

Proposition. $\exists x \in \mathbb{R}^+$ such that $x^2 = 2$.

Proof.
Let $S = \{x \in \mathbb{R} | x > 0, x^2 < 2\}$. We claim that $2 \in \mathbb{R}$ is an upper bound of S. Suppose we have another element y in $\mathbb{R}$ which is greater than 2. Then,

$$y^2 > 2y > 2 \cdot 2 = 4 > 2$$

so $y \notin S$. Thus $2 \in \mathbb{R}$ is an upper bound of S. Let a be the least upper bound of S. We show that a is positive and $a^2 = 2$.

Since $1^2 < 2$, we have $1 \in S$. Thus, $a \geq 1 > 0$. Suppose $a^2 < 2$. Then let n be a positive integer such that $n > \frac{2a+1}{2-a^2}$. Let $x = a + \frac{1}{n}$, so $x^2 < 2$, so $x \in S$, a contradiction.

Similarly, suppose $a^2 > 2$. Then let n be a positive integer such that $n \geq \max\{\frac{2a}{2-a^2}, \frac{1}{2a}\}$. Let $x = a - \frac{1}{n}$, then $x^2 > 2$. So for all $y \in S$, we have $x^2 > 2 > y^2$, and $x, y > 0$. We have $x^2 - y^2 > 0$, so $(x+y)(x-y) > 0$, but since x and y are both positive, $x - y > 0$. Thus, $x > y$. Therefore, x is an upper bound for S, which contradicts the minimality of a.

Therefore, $a^2 = 2$. ■

Note that we actually have to show that there exists a positive integer n such that $n > \frac{2a+1}{2-a^2}$.

Theorem (Archimedean Property). $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{Z}^+$ such that $n > x$.

Proof.
Assume for contradiction $\exists x$ such that $x \geq n$, for all $n \in \mathbb{Z}^+$. Then, $\mathbb{Z}^+ \subseteq \mathbb{R}$ has an upper bound. Let $a \in \mathbb{R}$ be the least upper bound. Then $a \in \mathbb{R}$ be the least upper bound. Then $a - 1$ is not an upper bound for $\mathbb{Z}^+$, so $\exists n \in \mathbb{Z}^+$ such that $a - 1 < n$. Then, $a < n + 1 \in \mathbb{Z}^+$, and a is not an upper bound of $\mathbb{Z}^+$ which is a contradiction. ■

Notice that $a^2 = 2$ is an element of $\mathbb{R}^+$ but not $\mathbb{Q}$.

Theorem. Suppose $a, b \in \mathbb{R}$, $a < b$. Then $\exists r \in \mathbb{Q}$ such that $a < r < b$. This means that the set of rationals is dense in the reals.

Proof.
Let $n \in \mathbb{Z}^+$ such that

$$n > \frac{1}{b-a}$$

Then $nb > na + 1$. If $a > 1$, consider $S = \{k \in \mathbb{Z}^+ | k > na\}$. Then $S \neq \varnothing$ by the Archimedian property. By induction axiom, Let $m \in S$ be the minimum element. Since $a > 1$, $na \geq a > 1$, so $1 \notin S$. So $m \neq 1$. Hence $m - 1 \in \mathbb{Z}^+$, $m - 1 \notin S$.

$$\Rightarrow m - 1 \leq na$$

$$\Rightarrow m \leq na + 1 < nb$$

$$\Rightarrow na < m < nb$$

$$\Rightarrow a < \frac{m}{n} < b$$

so $r = \frac{m}{n} \in \mathbb{Q}$ satisfies the property. In general, let $n \in \mathbb{Z}^+$ such that $n > 1 - a$. Then $n + a > 1$. By the previous part, there exists a rational r such that

$$n + a < r < n + b$$

$$a < r - n < b$$

as desired. ■

5 Precept 2: Bounds

Problem. Using induction, prove that

$$1^2 + 2^2 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\forall n \in \mathbb{N}$.

Proof.

Let S be the set of all values of n such that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$. Clearly, $1 \in S$ since $1^2 = \frac{1(2)(3)}{6} = 1$. Now, if $k \in S$, then

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

thus,

$$1^2+2^2+...+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2 = (k+1)(\frac{2k^2+k}{6}+k+1) = (k+1)(\frac{2k^2+7k+6}{6})$$

$$= (k+1)(\frac{(k+2)(2k+3)}{6})$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

so $(k+1) \in S$. Thus, by the induction axiom, $S = \mathbb{N}$, so $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all $n \in N$. ■

5.1 Infemum and Supremum

Let $E \subseteq \mathbb{R}$. We say $\alpha = \sup E \Leftrightarrow \alpha$ has the following properties:

(1) α is an upper bound of E.

(2) If $\alpha' < \alpha$, then α' is not an upper bound of E; i.e. $\exists y \in E$ such that $y > \alpha'$.

We say that $\beta = \inf E \Leftrightarrow \beta$ has the following properties:

(1) β is a lower bound of E, i.e. $x \geq \beta \; \forall \; x \in E$,

(2) If $\beta' > \beta$, then β' is not a lower bound of E, i.e. $\exists y \in E$ such that $y < \beta'$.

Problem. Find $\sup S$ and $\inf S$ when $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Solution.

$\forall n \in \mathbb{N}$, $n \geq 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow 1$ is an upper bound of S. Suppose for contradiction that there exists $\alpha' < 1$ such that α' is an upper bound of S. But $1 \in S$, so $1 > \alpha'$, so α' is not an upper bound of S. Thus, $\boxed{\sup S = 1}$.

$\forall n \in \mathbb{N}$, $n > 0 \Rightarrow \frac{1}{n} > 0 \Rightarrow 0$ is a lower bound of S. Suppose there exists $\beta' > 0$ such that β' is a lower bound of S. However, by the Archimedian principle, there exists $k \in \mathbb{N}$ such that $k > \frac{1}{\beta'}$. Thus, $\frac{1}{k} < \beta'$, so β' is not a lower bound. Thus, $\boxed{\inf S = 0}$.

Problem. Find $\sup S$ and $\inf S$ when $S = \{\frac{n-k}{n+k} : n, k \in \mathbb{N}\}$

Solution.
We will start by making some observations.

Notice that $\frac{n-k}{n+k}$ is nonnegative if $n \geq k$ and nonpositive if $n \leq k$.

Also, for $n >> k$, $n - k \approx n \approx n + k$, so $\frac{n-k}{n+k} \approx 1$. If $n = k$, then $\frac{n-k}{n+k} = 0$.

For $n << k$, $n - k \approx -k$ and $n + k \approx k$. Thus, $\frac{n-k}{n+k} \approx -1$.

We might guess that $\sup S = 1$ and $\inf S = -1$.

Notice that $|n-k|$ is either $n-k$ or $k-n$, both of which are less than $n+k$. Thus, $|n-k| < n+k \Rightarrow \frac{|n-k|}{n+k} < 1$. Since $n+k$ is positive, we also have

$$|\frac{n-k}{n+k}| < 1$$

However, this means that

$$-1 < \frac{n-k}{n+k} < 1$$

for all $n, k \in \mathbb{N}$. Thus, 1 is an upper bound of S and -1 is a lower bound of S.

Assume for contradiction that $a < 1$ is an upper bound of S.

We would like to show that $\exists p, q \in \mathbb{N}$ such that $\frac{p-q}{p+q} > a$. Then, $p - q > ap + aq \Rightarrow 0 > (a-1)p + (a+1)q \Rightarrow p > q\frac{1+a}{1-a}$.

Consider $k = 1$ and choose $n \geq \frac{1+a}{1-a} > 1 \in \mathbb{N}$ (since $a < 1$) by the Archimedean Principle. Then,

$$n(1-a) \geq 1 + a \Rightarrow n - na \geq 1 + a \Rightarrow n - 1 \geq na + a \Rightarrow \frac{n-1}{n+1} \geq a$$

so a is not an upper bound of S. Thus, $\sup S = 1$.

Now assume for contradiction that $b > -1$ is a lower bound of S.

We would like to show that $\exists p, q \in \mathbb{N}$ such that $\frac{p-q}{p+1} < b$. Then $p - q < bp + bq \Rightarrow 0 < (b-1)p + (b+1)q \Rightarrow q > p\frac{(1-b)}{(1+b)}$.

Consider $n = 1$ and choose $k \geq \frac{1-b}{1+b} \in \mathbb{N}$ by the Archimedean Principle. Then,

$$k + kb \geq 1 - b \Rightarrow kb + b \geq 1 - k \Rightarrow b \geq \frac{1-k}{1+k}$$

Thus, b is not a lower bound of S, so $\inf S = -1$.

Remark. In class, we just said choose arbitrary k instead of $k = 1$ or arbitrary n instead of $n = 1$. I just chose those because it makes it more tangible.

6 Lecture 4: Cardinality

Corollary. $\forall a, b, \in \mathbb{R}$ such that $a < b$, $\exists r \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r < b$.

Proof.
Suppose for contradiction that r must be rational. Let $a' = a + \sqrt{2}$ and $b' = b + \sqrt{2}$. By the previous theorem, $\exists r' \in \mathbb{Q}$ such that $a' < r' < b'$. Thus, we can let $r = r' - \sqrt{2}$ to get $a' - \sqrt{2} < r' - \sqrt{2} < b' - \sqrt{2} \Rightarrow a < r < b$, but then $\sqrt{2} \in \mathbb{Q}$, a contradiction. Thus, r is irrational and contained in (a, b). ■

The thing is, it turns out that $\mathbb{R}\backslash\mathbb{Q}$ is "much larger" than $\mathbb{Q}$. We will find out what this means specifically later in this lecture.

6.1 Cardinality

Definition (Cardinality). We say that sets A and B have the same **cardinality** if $\exists f : A \to B$ that is both injective and surjective (f is bijective).
Notation (Cardinality). If A and B have the same cardinality, then we write $A \sim B$.

Proposition.

(1) $A \sim A$

(2) $A \sim B \Rightarrow B \sim A$

(3) $A \sim B, B \sim C \Rightarrow A \sim C$

(4) $A \sim A', B \sim B' \Rightarrow A \times B \sim A' \times B'$

Proof.
For part one, simply consider the identity mapping $f : A \to A$, $x \mapsto x$.
For part two, let $f : A \to B$ be the bijective function from A to B. Then consider $f^{-1} : B \to A$ so that $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$. Since $\forall a$ there exists unique b and vice versa, f^{-1} is still bijective.
For part three, let $f : A \to B$ be the bijective function from A to B and $g : B \to C$ be the bijective function from B to C. Then we claim that $g \circ f$ is a bijective function from A to C. Since

$$g(f(a)) = g(f(b)) \Leftrightarrow f(a) = f(b) \Leftrightarrow a = b$$

we must have that $g \circ f$ is bijective, so $A \sim C$.
For part four, let $f : A \to A'$ and $g : B \to B'$ be bijective mappings from A to A' and B to B'. Then, we map

$$(a, b) \mapsto (f(a), g(b))$$

Notice that

$$(f(a), g(b)) = (f(a'), g(b')) \Leftrightarrow f(a) = f(a'), g(b) = g(b') \Leftrightarrow a = a', b = b'$$

so this mapping is indeed bijective. Thus, $A \times B \sim A' \times B'$. ■

Proposition. Two times a certain cardinality is the same cardinality. Having "two times the elements" is the same cardinality. More rigorously,

$$\mathbb{N} \sim \{0, 1\} \times \mathbb{N}$$

Proof.
Simply take the function

$$f : \{0, 1\} \times N \to \mathbb{N}$$

$$(i, n) \to 2n - i$$

■

Proposition. $\{0,1\} \times \mathbb{N} \sim \mathbb{Z}$

Proof. Take the piece-wise function $f : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$

$$f(0,n) = n$$

$$f(1,n) = -n + 1$$

■

And from this, we get $\boxed{\mathbb{N} \sim \mathbb{Z}}$.

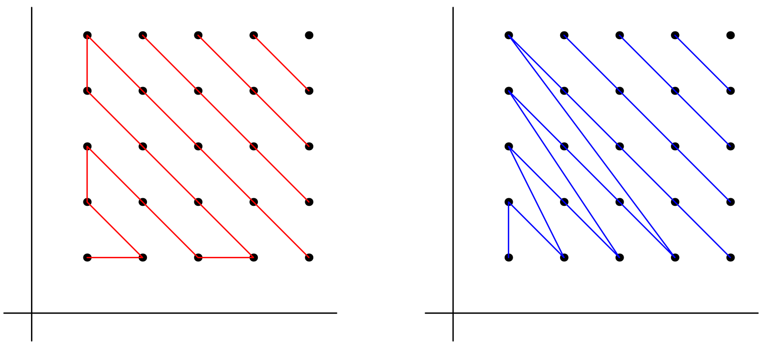
Remark. Hilbert's Hotel!! If we have a full hotel and 50 more people come, we cannot accommodate them. However, if the hotel has an infinite number of rooms, we can rearrange the rooms so that all the people now have rooms to take.

In fact, even if we have a countably infinite number of Hilbert Hotels, we can still fit them into one Hilbert Hotel!

Theorem. $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

Notice that we can still map this!

Remark. We can get geometric intuition on this problem by looking at $\mathbb{N} \times \mathbb{N}$ as a graph and then doing a *walk* through the plane.



The diagram shows two possible walks that demonstrate how $\mathbb{N}^2$ bijects to $\mathbb{N}$. We can actually see that the walk itself is the natural number line, with each visited point on $\mathbb{N}^2$ becoming a point on $\mathbb{N}$.

Proof.

We can take

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$(a,b) \to \frac{(a+b-1)(a+b-2)}{2} + a$$

Remark. This is just our blue path on the plot.

We will now show that f is both injective and surjective.

We will start by showing injectivity.

Lemma. $a + b - 1 = \max\{k \in \mathbb{N} \mid \frac{k(k-1)}{2} < f(a,b)\}$

Proof.

$$\frac{(a+b-1)(a+b-2)}{2} < \frac{(a+b-1)(a+b-2)}{2} + a = f(a,b)$$

so $a+b-1 \in \{k \in \mathbb{N} \mid \dfrac{k(k-1)}{2} < f(a,b)\}$. However,

$$\frac{(a+b)(a+b-1)}{2} = \frac{(a+b-1)(a+b-2)}{2} + (a+b-1) \geq \frac{(a+b-1)(a+b-2)}{2} + a = f(a,b)$$

so $a+b \notin \{k \mid \dfrac{k(k-1)}{2} < f(a,b)\}$. $\square$

By the lemma, since $f(a,b) = f(a',b')$, $a+b-1 = a'+b'-1 \Rightarrow a+b = a'+b'$. Thus,

$$\frac{(a+b-1)(a+b-2)}{2} + a = \frac{(a'+b'-1)(a'+b'-2)}{2} + a' \Rightarrow a = a'$$

so $b = b'$. Thus, $f(a,b) = f(a',b') \Rightarrow a = a', b = b'$. Thus, we have injectivity. Now, we will show surjectivity. For all $n \in \mathbb{N}$, we will show that $\exists (a,b)$ such that $f(a,b) = n$.

Let $k_0 = \max\{k \in \mathbb{N} \mid \dfrac{k(k-1)}{2} < n\}$, $a = n - \dfrac{k_0(k_0-1)}{2}$, and $b = k_0 + 1 - a$. We will now show that $(a,b) \in \mathbb{N} \times \mathbb{N}$ and $f(a,b) = n$.

Remark. Since $\mathbb{Z} \sim \mathbb{N}$, we can also get $\mathbb{Z} \sim \mathbb{Z} \times \mathbb{Z}$.

Theorem. $\mathbb{N} \sim \mathbb{N}^m$ for all $m \in \mathbb{N}$.

Proof.
We use induction. Note that $\mathbb{N} \sim \mathbb{N}$.

Theorem. Suppose $A \subseteq \mathbb{N}$. Then, A is finite or $A \sim \mathbb{N}$.

Proof.
Suppose A is infinite. Define $f : A \to \mathbb{N}$ $n \to \text{card}(\{1,2,...,n\} \cap A)$. Suppose $n, m \in A$, so $n < m$. We show that $f(n) < f(m)$. Notice that

$$m \in \{1,2,...,m\} \cap A$$

but

$$m \notin \{1,2,...,n\} \cap A$$

so $\text{card}(\{1,2,...,n\} \cap A) < \text{card}(\{1,2,...,m\} \cap A)$ which means $f(n) < f(n)$ so f is injective. Now, $\forall k \in \mathbb{N}$, we show that $\exists n \in A$ such that $f(n) = k$. Let S be the set of all k such that $\exists n \in A$ such that $f(n) = k$. Let $n = \min(A)$ by the Well-Ordering principle??. Then, $\{1, ...n\} \cap A = \{n\}$, so $f(n) = 1$. Now, suppose $\exists n \in A$ such that $f(n) = k$. We show that $\exists n' \in A$ such that $f(n') = k+1$. Let $S = \{m \in A \mid m > n\}$. Since A is an infinite set, $S \neq \varnothing$. Let $n' = \min(S)$. Then,

$$f(n') = \text{card}(\{1, ..., n'\} \cap A)$$

so

$$\begin{gathered}
\{1,2,...,n'\} \cap A \\
= (\{1,2,...,n\} \cap A) \cup (\{n+1,...,n'\} \cap A) \\
= (\{1,2,...,n\} \cap A) \cup \{n'\} \\
\text{card}(\{1,2,...,n'\} \cap A) = \text{card}(\{1,2,...,n\} \cap A) + 1 \\
f(n') = f(n) + 1 = k+1
\end{gathered}$$

so by the induction axiom, since $k \in S \Rightarrow (k+1) \in S$, we have $S = \mathbb{N}$, as desired. $\blacksquare$

Corollary. If A is a set $\exists f : A \to \mathbb{N}$ injection. Then either A is finite or $A \sim \mathbb{N}$.

Corollary. $\mathbb{Q} \sim \mathbb{N}$.

Proof.
Define

$$f : \mathbb{Q} \to \mathbb{N} \times \mathbb{Z}$$
$$r \to (q, p)$$

where $r = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q > 0$ such that q is the minimum possible. Then f is an injection and since $\mathbb{N} \times \mathbb{Z} \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, so $\mathbb{Q}$ is finite or $\mathbb{Q} \sim \mathbb{N}$. Thus, $\mathbb{Q} \sim \mathbb{N}$.

Remark. This is a bit unintuitive because $\mathbb{Q}$ is dense while $\mathbb{N}$ is not.

Definition (Countable). If $A \sim \mathbb{N}$, then we say that A is **countable**.
Definition (Uncountable). If A is neither finite nor countable, we say that A is **uncountable**.

Lemma (Nested Interval Property). Suppose $A_n = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\} = [a_n, b_n]$ is a closed interval for each $n \in \mathbb{N}$. Suppose

(1) $A_n \neq \varnothing \ \forall n \in \mathbb{N}$

(2) $A_n \supseteq A_{n+1} \ \forall n \in \mathbb{N}$.

Then $\cap_{n=1}^{\infty} A_n \neq \varnothing$.

Remark. This is only true because of the closed intervals. Otherwise, test $A_n = (0, \frac{1}{n})$ for a counterexample. This property is also not true for $\mathbb{Q}$. Find a counterexample.

Proof.
Since $A_n = [a_n, b_n] \neq \varnothing$, $a_n \leq b_n$. Since $A_n \supseteq A_{n+1}$. Then, $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$. $\forall m, n \in \mathbb{N}$,

$$b_m \geq b_{m+1} \geq ... \geq b_{m+n}$$
$$\geq a_{m+n} \geq a_{m+n-1} \geq ... \geq a_n$$

$\Rightarrow b_m \geq a_n$. So A is bounded from above, and $\sup(A)$ exists. Let $x = \sup(A)$. $\forall b_m$, we have $b_m \geq a_n$ $\forall n$ so b_m is an uper bound of A. Thus, $x \leq b_m$.
In conclusion, $x \geq a_n \forall n$ and $x \leq b_m \forall m$. Then, $x \in \cap_{n=1}^{\infty} A_n$. ■

Theorem. $\mathbb{R}$ is uncountable.

Proof.
Assume that $f : \mathbb{N} \to \mathbb{R}$ is a bijection.

Remark. Since we expect $\mathbb{R}$ to be much "larger" than $\mathbb{N}$, we will try to show that the *surjective* property of f yields a contradiction.

We claim that $\text{Im}(f) = \{f(1), f(2), ..., \}$ be the image of f. We claim that $\exists x \in \mathbb{R}$ such that $x \notin \text{Im}(f)$. Let $A_0 = [0, 1]$. For all $n \in \mathbb{N}$, let A_n be a closed interval $[a_n, b_n]$ such that $a_n < b_n$, $A_n \subseteq A_{n-1}$, and $f(n) \notin A_n$.
For example, let

$$A_n = (a_{n-1}, a_{n-1} + \frac{1}{3}(b_{n-1} - a_{n-1})]$$

if $x \geq \frac{1}{2}(a_{n-1}+b_{n-1})$ and

$$A_n = [b_{n-1} - \frac{1}{3}(b_{n-1} - a_{n-1}), b_{n-1})$$

if $x \leq \frac{1}{2}(a_{n-1}+b_{n-1})$. Let

$$x = \cap_{n=1}^{\infty} A_n$$

Since $f(n) \notin A_n$, $\forall n$, $f(n) \notin \cap_{n=1}^{\infty} A_n$ $\forall n$. So $x \neq f(n) \forall n$. Hence, $x \notin \mathrm{Im}(f)$. ■

Corollary $\mathbb{R} \backslash \mathbb{Q}$ is uncountable.

Continuum Hypothesis. Is there a set whose cardinality is in between that of an uncountable set and a countable set?

This is actually a theorem that has no right answer. Both answers yield valid mathematics. This is one of Hilbert's problems.

7 Lecture 5: Sequences

Definition (Sequence). A **sequence** of real numbers is a map from $\mathbb{N}$ to $\mathbb{R}$.

We denote a sequence with $(a_1, a_2, a_3, ...)$. Alternatively, we can write $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\geq 1}$, (a_n), (b_n), (x_n), or we can explicitely write the function $(\frac{1}{n})$. For example,

$$(\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$$

A visualization of convergence:

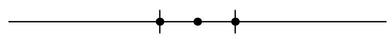
Definition (Absolute Value). We define $|x|$ to be x if $x \geq 0$ and $-x$ if $x < 0$.

We say that "a_n approaches c as $n \to \infty$". Alternatively, notice that we can say that the distance between the sequence term and the limit goes to 0: "$|c - a_n|$ approaches 0 as $n \to \infty$".

Definition (Convergence). We say that (a_n) is **convergent** to c, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - c| < \epsilon$ for all $n \geq N$.

For $\epsilon > 0$, define

$$\begin{aligned} V_\epsilon(c) &= \{x \in \mathbb{R} \mid |x - c| < \epsilon\} \\ &= \{x \in \mathbb{R} \mid -\epsilon < x - c < \epsilon\} \\ &= \{x \in \mathbb{R} \mid c - \epsilon < x < c + \epsilon\} \\ &= (c - \epsilon, c + \epsilon) \end{aligned}$$



We call V_ϵ the "ϵ-neighborhood of c". We can then write a new definition of convergence that is equivalent to the first:

Definition (Convergence). (a_n) is **convergent** to c if and only if $\forall \epsilon > 0$, the set $\{n \in \mathbb{N} \mid a_n \notin V_\epsilon(c)\}$ is finite.

> **Example.** $a_n = \frac{1}{n}$. Then a_n is convergent to 0.

Solution.

Given that $\epsilon > 0$, we want N such that $n \geq N \Rightarrow |a_n - 0| < \epsilon$

$$\Leftrightarrow \frac{1}{n} < \epsilon$$

$$\Leftrightarrow n > \frac{1}{\epsilon}$$

we want $n \geq N \Rightarrow n > \frac{1}{\epsilon}$, so we should take $N \in \mathbb{N} > \frac{1}{n}$.

Suppose $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $\mathbb{N} > \frac{1}{\epsilon}$. The existence of N is from the Archimedean Property). Then for each $n \geq \mathbb{N}$ we have $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ■

We can also prove this using the other definition. Suppose $\epsilon > 0$. Then consider

$$\{n \in \mathbb{N} \mid a_n \notin V_\epsilon\}$$

$$\{n \in \mathbb{N} \mid |a_n - 0| \geq \epsilon\}$$

$$\{n \in \mathbb{N} \mid \frac{1}{n} \geq \epsilon\}$$

$$\{n \in \mathbb{N} \mid n \leq \frac{1}{\epsilon}\}$$

By the Archimedean property, $\exists n_0 \in \mathbb{N}$ such that $n_0 \geq \frac{1}{\epsilon}$. Then,

$$\{n \in \mathbb{N} \mid n \leq \frac{1}{\epsilon}\}$$

$$\{n \in \mathbb{N} \mid n \leq n_0\} = \{1, 2, ..., n_0\}$$

which is finite. ■

Example. $a_n = 0$. Then a_n is convergent to 0.

Proposition (Triangle Inequality). If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$.

Proof.
If $a \geq 0$, $b \geq 0$, then $|a+b| = a+b = |a|+|b|$. If $a < 0, b < 0$, then

$$|a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$$

as desired.

If a, b have different signs, then without loss of generality, assume $a \geq 0$, $b \leq 0$. Let $b' = |b| = -b \geq 0$. Then, $|a| + |b| = a + b'$. On the other hand,

$$|a+b| = \max\{a+b, -(a+b)\}$$

$$= \max\{a-b', b'-a\}$$

Since $a \geq 0$, $b' \geq 0$, we have $a + b' \geq a - b'$ and $a + b' \geq b' - a$, so $|a+b| \leq a + b' = |a| + |b|$. ■

Corollary. If $a, b, c \in \mathbb{R}$, then

$$|a-b| + |b-c| \leq |a-b|$$

Proof.
By the proposition, $|(a-c) + (c-b)| \leq |a-c| + |c-b|$. ■

Theorem. Suppose (a_n) is convergent to both c and c'. Then $c = c'$.

Proof.
$\forall \epsilon_1 > 0, \exists N \in \mathbb{N}$ such that $|a_n - c| < \epsilon_1$ for all $n \geq N$. Similarly, $\forall \epsilon_2 > 0, \exists M \in \mathbb{N}$ such that $|a_n - c'| < \epsilon_2$ for all $n \geq M$.

If $c \neq c'$, take $\epsilon_1 = \epsilon_2 = \frac{|c-c'|}{2}$. Take $n \in \mathbb{N}$ such that $n \geq \max\{N, M\}$. Then we have both

$$|a_n - c| < \epsilon_1 = \frac{|c-c'|}{2}$$

$$|a_n - c'| < \epsilon_2 = \frac{|c-c'|}{2}$$

Since

$$|c-c'| \leq |a_n - c| + |a_n - c'|$$

we have

$$|c-c'| \leq |a_n - c| + |a_n - c'| < \frac{|c-c'|}{2} + \frac{|c-c'|}{2} = |c-c'|$$

which is a contradiction. ■

We can also show this using the other definition.

Suppose that $c \neq c'$. Let $\epsilon_1 = \epsilon_2 = \frac{|c-c'|}{2}$. By the triangle inequality, we have

$$V_{\epsilon_1}(c) \cap V_{\epsilon_2}(c') = \varnothing$$

If $\exists x \in V_{\epsilon_1}(c) \cap V_{\epsilon_2}(c)$, then

$$|x-c| < \epsilon_1, |x-c'| < \epsilon_2$$

implies

$$|c-c'| \leq |x-c| + |x-c'| < \frac{|c-c'|}{2} + \frac{|c-c'|}{2} = |c-c'|$$

which is a contradiction.

We claim that $\exists x \in V_{\epsilon_1}(c) \cap V_{\epsilon_2}(c)$. Then for a_n, either $a_n \notin V_{\epsilon_1}(c)$ or $a_n \notin V_{\epsilon_2}(c')$.

$$\Rightarrow \{n \in \mathbb{N} \mid a_n \notin V_{\epsilon_1}(c)\} \cup \{n \in \mathbb{N} \mid a_n \notin V_{\epsilon_2}(c')\} = \mathbb{N}$$

$$\Rightarrow \text{ at least one of them is infinite}$$

■

Definition. A sequence (a_n) is called bounded if $\exists M > 0$ such that if $\exists M > 0$ such taht $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Proposition. (a_n) is bounded if and only if $\exists M_1, M_2 \in \mathbb{R}$ such that $M_1 \leq a_n \leq M_2$ for all $n \in \mathbb{N}$.

Proof.

(1) If (a_n) is bounded, then $\exists M > 0$ such that

$$|a_n| \leq M$$

so $-M \leq a_n \leq M$ and the statement holds for $M_1 = -M$ and $M_2 = M$.

(2) If $\exists M_1, M_2$ such that $M_1 \leq a_n \leq M_2$ $\forall n$ then take $M > 0$ such that

$$M > \max\{M_2, -M_1\}$$

then $M > M_2$, $-M < M_1$. Then

$$-M < M_1 \leq a_n \leq M_2 < M$$

for all n, so $|a_n < M$ for all n. ■

Theorem. If (a_n) is convergent, then it is bounded.

Proof.

Suppose (a_n) converges to some c. By definition, with $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that $|a_n - c| < 1$ for all $n > \mathbb{N}$. Then for any $n \geq N$,

$$|a_n| \leq |a_n - c| + |c| < 1 + |c|$$

Let $M_1 = \max\{|a_n| \mid n < N\}$, and let $M = \max\{M_1, 1 + |c|\}$. Then, $M \geq |a_n|$ for all n. ■

If (a_n) converges to c, we write

$$\lim_{n \to +\infty} a_n = c$$

or $\lim a_n = c$.

Definition. (a_n) is called **divergent** if it is not convergent to any $c \in \mathbb{R}$.

Theorem. Suppose $\lim a_n = a$ and $\lim b_n = b$. Then

(1) $\lim(a_n + b_n) = a + b$

(2) $\lim(a_n b_n) = ab$

(3) $\lim(ca_n) = ca$, $\forall c \in \mathbb{R}$

(4) If $b \neq 0$, $b_n \neq 0$, then $\lim(\frac{1}{b_n}) = \frac{1}{b}$.

Proof.
By definition,

$$\forall \epsilon_1, \epsilon_2 > 0$$
$$\exists N_1, N_2 \in \mathbb{N}$$

such that $|a_n - a| < \epsilon_1 \ \forall n \geq N_1$ and $|b_n - b| < \epsilon_2 \ \forall n \geq N_2$. Also, $\exists M > 0$ such that $|a_n| < M$.

(1)

Suppose that $\epsilon > 0$. Take $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$ and take N_1, N_2 be the respective lower bounds of n. Take $N = \max\{N_1, N_2\}$. Then $\forall n \geq \mathbb{N}$, we have

$$|a_n - a| < \epsilon_1 = \frac{\epsilon}{2}$$
$$|b_n - b| < \epsilon_2 = \frac{\epsilon}{2}$$
$$\Rightarrow |(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(2)

Suppose $\epsilon > 0$. Take $\epsilon_2 = \frac{\epsilon}{2M}$ and $\epsilon_1 = \frac{\epsilon}{2(|b|+1)}$ and let N_1, N_2 be the respective lower bounds of n. Take $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$,

$$|a_n - a| < \epsilon_1$$
$$|b_n - b| < \epsilon_2$$
$$|a_n| < M$$

Then $|a_n b_n - ab| = |(a_n b_n - a_n b) + (a_n b - ab)|$

$$\begin{aligned}
&\leq |a_n b_n - a_n b| + |a_n b - ab| \\
&\leq |a_n||b_n - b| + |b||a_n - a| \\
&< M\epsilon_2 + |b|\epsilon_1 \\
&= M \cdot \frac{\epsilon}{2M} + |b| \cdot \frac{\epsilon}{2(|b|+1)} \\
&= \frac{1}{2}(1 + \frac{|b|}{|b|+1}) \cdot \epsilon \\
&< \epsilon
\end{aligned}$$

■

Remark. For such short proofs, we can just take the type of argument that we had. However, for much longer proofs where the scratch work is very detailed, we can say take ϵ_1 and ϵ_2 whose values are to be determined later. Then, whe n you get to the step like $M\epsilon_2 + |b|\epsilon_1$, say take ϵ_1 and ϵ_2 to be $\frac{\epsilon}{2M}$ and $\frac{\epsilon}{2(|b|+1)}$ from the beginning.

(3)

Let $b_n = c$. Then, (3) is a special case of (2).

(4)

$\lim b_n = b$, $b \neq 0$. By definition, $\exists N_1$ such that $\forall n \geq \mathbb{N}$, we have

$$|b_n - b| \leq \frac{|b|}{2}$$

Then, $|b| \leq |b_n| + |b - b_n| \leq |b_n| + \frac{|b|}{2}$, so $|b_n| \geq \frac{|b|}{2}$. Hence, $\frac{1}{|b_n|} \leq \frac{2}{|b|}$. Suppose $\epsilon > 0$. Take $\epsilon_2 = \frac{\epsilon |b|^2}{2}$. Then, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \epsilon_2 \ \forall n \geq N_2$$

Let $N = \max\{N_1, N_2\}$, $\forall n \geq \mathbb{N}$, we have

$$|b_n| \geq \frac{|b|}{2}$$

$$|b - b_n| < \epsilon_2$$

Hence

$$|\frac{1}{b} - \frac{1}{b_n}|$$

$$= \frac{|b - b_n|}{|b| \cdot |b_n|}$$

$$< \frac{\epsilon_2}{|b|} \cdot \frac{2}{|b|} = \frac{\epsilon |b|^2}{2} \cdot \frac{1}{|b|} \cdot \frac{1}{2|b|} \epsilon$$

■

8 Precept 3: Rational Exponents

Problem. Consider the following statement for a sequence (a_n):

$$\exists N \in \mathbb{N} \text{ such that } \forall \epsilon > 0 \text{ and } n \geq N, |a_n - a| < \epsilon$$

Describe this statement more explicitly.

Lemma. Let $x, y \in \mathbb{R}$ such that $|x - y| < \epsilon \; \forall \; \epsilon > 0$. Then $x = y$.

Proof.
Suppose for contradiction that $x \neq y$. Then, $|x-y| = \epsilon_0$ for some $\epsilon > 0$. However, we have $|x-y| < \epsilon$ for all $\epsilon > 0$. Since $\epsilon_0 > 0$, let $\epsilon = \epsilon_0$. Then, we get

$$|x - y| < \epsilon_0$$

a contradiction. ■

Consider

$$|a_n - a| < \epsilon \; \forall \; \epsilon > 0 \text{ and } n \geq N$$

Let us fix $k \geq N$. We have that

$$|a_k - a| < \epsilon \; \forall \; \epsilon > 0$$

Using the lemma, $a_k = a$. Thus,

$$\forall n \geq N, a_n = a$$

So the original statement is equivalent to

$$\exists N \in \mathbb{N} \text{ such that } \forall n \geq N, a_n = a$$

Problem. Write down a rigorous definition (using ϵ) of: (a_n) does not converge to a.

Solution.
Recall that (a_n) converges to a if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - a| < \epsilon$$

Thus, we want the negation of the above statement. This is:

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \text{ such that } \exists n \geq N, |a_n - a| \geq \epsilon$$

Problem. Write down the negations of the following statements ($S \subseteq \mathbb{R}$:

(1) $\forall x \in S$, x is positive

(2) $\exists y \in S$, y is positive

(3) $\forall x, A(x)$ ($A(x)$ is some statement involving x)

(4) $\exists x, A(x)$

Solution.

(1) $\exists x \in S$, $x \leq 0$

(2) $\forall y \in S$, $y \leq 0$

(3) $\exists x$, not $A(x)$

(4) $\forall x$, not $A(x)$

Going back to the previous problem, we see

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \text{ such that } \exists n \geq N, |a_n - a| \geq \epsilon$$

Lemma. Let $S \subseteq \mathbb{N}$. Then the following are equivalent:

(1) $\forall M \in \mathbb{N}, \exists m \in M$ such that $m \in S$

(2) $|S| = \infty$

Proof.
We prove this statement by showing that the negation of (1) is equivalent to the negation of (2).

Remark. Interesting strategy :O

The negation of (1) is:

$$(1)' \ \ \exists M \in \mathbb{N}, \forall m \in M \text{ such that } m \in S$$

The negation of (2) is:

$$(2)' \ \ S \text{ is finite}$$

Note that $(1)'$ is equivalent to

$$\exists M \in \mathbb{N} \text{ such that } \forall m \geq M, m \in S^c$$

$$\Leftrightarrow \exists M \in \mathbb{N} \text{ such that } \{m \in \mathbb{N} : m \geq M\} \subseteq S^c$$
$$\Leftrightarrow \exists <\in \mathbb{N} \text{ such that } S \subseteq \{m \in \mathbb{N} : m \geq M\}^c$$
$$= \{m \in \mathbb{N} : m < M\}$$

which is a finite set. Thus, S must also be finite (since it is a subset). Thus, $(1)' \Rightarrow (2)'$. Conversely, let S be a finite set. Define $M_0 = \max S + 1$. The maximum exists since S is finite. Then,

$$m \geq M_0 \Rightarrow m > \max S \Rightarrow m \notin S$$

Thus, $(2)' \Rightarrow (1)'$. Since $(1)'$ and $(2)'$ are equivalent, (1) and (2) are also equivalent. ■

Now, we can simplify our statement:

$$\forall N \in \mathbb{N}, \exists n \geq N \text{ such that } n \in \{k \in \mathbb{N} : |a_k - a| \geq \epsilon\}$$

wihch is equivalent to:

$$\#\{k \in \mathbb{N} : |a_k - a| \geq \epsilon\} = \infty$$

so for infinitely many $n \in \mathbb{N}$, we have $|a_n - a| \geq \epsilon$. Thus, we can write our final definition:

$$\exists \epsilon > 0 \text{ such that for infinitely many } n \in \mathbb{N}, \ a_n - a| \geq \epsilon$$

Proposition (n^{th} root). For an $x \in \mathbb{R}^+$, $\exists$ a unique $y \in \mathbb{R}^+$ such that $y^2 = x$.

$$y = \sup\{t \in \mathbb{R}^+ : t^2 < x\}$$

Notation: $y = x^{1/2}$. For $y \in \mathbb{R}^+$, $y = x^{1/2} \Leftrightarrow y^2 = x$.
For $n \in \mathbb{N}$, $\exists$ a unique $y \in \mathbb{R}^+$ such that $y^n = x$.

$$y = \sup\{t \in \mathbb{R}^+ : t^n < x\}$$

Notation: $y = x^{1/n}$. For $y \in \mathbb{R}^+$, $y = x^{1/n} \Leftrightarrow y^n = x$.

Proposition. If $a, b \in \mathbb{R}^+$ and $n \in \mathbb{N}$, then $(ab)^{1/n} = a^{1/n} \cdot b^{1/n}$.

Proof.
$a^{1/n} = \alpha > 0$ and $b^{1/n} = \beta > 0$ Then $\alpha^n = a$ and $\beta^n = b$. Then,

$$ab = \alpha^n \beta^n = (\alpha\beta)^n, \quad \alpha\beta > 0$$

$$(ab)^{1/n} = \alpha\beta = a^{1/n} b^{1/n}$$

Proposition. Suppose $a > 1$. Then $a^n > 1$ for all $n \in \mathbb{N}$.

Proof.
For $n = 1$, $a^n = a > 1$. Thus, the problem holds for $n = 1$.
Now, we assume that our problem holds for $n = k$ (IH).
Then, $a^{k+1} = a \cdot a^k > a \cdot 1 = a > 1$ (using IH and $a > 0$ and $a > 1$).
Thus, by induction, the problem holds for all $n \in \mathbb{N}$. ■

Proposition. Suppose $0 < a < 1$. Then $a^n < 1 \; \forall \, n \in \mathbb{N}$.

Proof.
$\frac{1}{a} > 1 \Rightarrow (\frac{1}{a})^n > 1$ (by (2))

$$\Rightarrow \frac{1}{a^n} > 1$$

$$\Rightarrow a^n < 1$$

as desired. ■

Proposition. Let $n \in \mathbb{N}$ and $a \in \mathbb{R}^+$. Then $a > 1 \Leftrightarrow a^n > 1$.

Proof.
In a previous proposition we proved that $a > 1 \Rightarrow a^n > 1$. Let $a^n > 1$, and suppose for contradiction that $a \leq 1$. Then either $a < 1$ or $a = 1$. In the first case, by the previous proposition, $a^n < 1$, a contradiction. In the second case, $a = 1 \Rightarrow a^n = 1$, another contradiction. Thus, we must have $a > 1$. Thus, the two statements are equivalent as desired. ■

Proposition. $x, y \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then $x > y \Leftrightarrow x^n > y^n$.

Proof.
Notice that

$$x > y \Leftrightarrow \frac{x}{y} > 1$$

$$\Leftrightarrow \frac{x^n}{y^n} > 1$$

$$\Leftrightarrow x^n > y^n$$

by the previous proposition. ■

Proposition. $n \in \mathbb{N}$, $a, b \in \mathbb{R}^+$. Then $a > b \Leftrightarrow a^{1/n} > b^{1/n}$.

Proof.
$a^{1/n} = \alpha > 0$, $b^{1/n} = \beta > 0$. Then $a = \alpha^n$ and $b = \beta^n$. Thus,

$$a > b \Leftrightarrow \alpha^n > \beta^n$$

$$\Leftrightarrow \alpha > \beta$$
$$\Leftrightarrow a^{1/n} > b^{1/n}$$

■

9 Lecture 6: Convergence

Theorem (Algebraic Limit Theorem). If $\lim a_n = a$, $\lim b_n = b$,

(1) $\lim(a_n + b_n) = a + b$

(2) $\lim(a_n b_n) = ab$

(3) $\lim(ca_n) = ca$

(4) if $b \neq 0$, $b_n \neq 0$m then $\lim \frac{1}{b_n} = \frac{1}{b}$.

Corollary. If $\lim a_n = a$, $\lim b_n = b$, $b \neq 0$, $b_n \neq 0$, then

$$\lim \frac{a_n}{b_n} = \frac{a}{b}$$

Theorem. If $a_n \geq 0$ for all n,

$$\lim a_n = a$$

then $a \geq 0$.

Proof.

Assume $a < 0$. Take $\epsilon = \frac{|a|}{2} > 0$. Since $\lim a_n = a$, there exists $N \in \mathbb{N}$, such that for all $n \geq \mathbb{N}$, we have $|a_n - a| < \epsilon$.

$$|a_n - a| < \epsilon$$
$$-\epsilon < a_n - a < \epsilon$$
$$a_n < +\epsilon = -|a| + \frac{|a|}{2} = -\frac{|a|}{2} < 0$$

which contradicts the assumption that $a_n \geq 0$ for all n. ∎

Theorem (Order Limit Theorem). If $A_n \geq b_n$ for all n, and

$$\lim a_n = a$$
$$\lim b_n = n$$

then $a \geq b$.

Proof.

Since $a_n \geq b_n$ for all n, consider the sequence $c_n = a_n - b_n \geq 0$ for all n. By the algebraic limit theorem, $\lim c_n = \lim a_n + (-b_n) = a + (-b) = a - b$. By the previous theorem, since

$$\lim c_n = a - b$$

and $c_n \geq 0$, $a - b \geq 0 \Rightarrow a \geq b$. ∎

Theorem (Monotone Convergence Theorem).

(a) If (a_n) is bounded and $a_n \leq a_{n+1}$ for all n, then, (a_n) is convergent.

(b) If (a_n) is bounded and $a_n \geq a_{n+1}$ for all n, then (a_n) is convergent.

Proof.

(1)

Let $A = \{a_n \mid n \in \mathbb{N}\}$. Since (a_n) is bounded, A is bounded from above and below. Let $x = \sup A$. We claim that $\lim a_n = x$. Suppose $\epsilon > 0$. Assume for contradiction that there does not exist an a_n such that $a_n > x - \epsilon$. Then, $x - \epsilon \geq a$ for all $a \in A$. However, this contradicts the minimality of $x = \sup A$. Thus, $\forall \epsilon, \exists a_n \mid x - \epsilon < a_n \leq x$. Thus, $\exists N$ with

$$|a_N - x| < \epsilon$$

for all ϵ. Now consider $n > N$. Skipping the induction step, we have $a_n \geq a_N$. Thus,

$$x - \epsilon < a_N \leq a_n$$

for all $n > N$. In addition, $a_n < \sup A = x$. Thus,

$$x - \epsilon < a_n \leq x \Rightarrow |a_n - x| < \epsilon$$

for all $n \geq N$. Thus, we must have $\lim a_n = x$. ■

(2)

For the decreasing sequence, the result follows by the same argument but flipping the inequalities and considering the infimum. We can also use the trick!

Strategy. If we have some result with $\sup$ and some inequalities and we want to use $\inf$ and the opposite inequalities, we can just define $b_n = -a_n$ to prove the result for $\inf$ and the opposite inequalities directly from the result with $\sup$ and those inequalities. See the proof that the infimum exists for a sequence bounded below from the Completeness Axiom.

Definition (Increasing Sequence). If $a_n \leq a_{n+1}$ for all n, then (a_n) is **increasing**.
Definition (Decreasing Sequence). If $a_n \geq a_{n+1}$ for all n, then (a_n) is **decreasing**.
Definition (Monotone). If (a_n) is increasing or decreasing, then (a_n) is **monotone**.

Theorem (Monotone Convergence Theorem). If (a_n) is monotone and bounded, then (a_n) converges.

Definition (Subsequence). If (a_n) is a sequence, and (n_k) is a sequence of natural numbers such that $n_{k+1} > n_k$ for all k, then $(a_{n_k})_{k \in \mathbb{N}}$ is called a **subsequence** of (a_n).

Theorem. If (a_n) is convergent to a and (a_{n_k}) is a subsequence, then

$$\lim a_{n_k} = a$$

Proof.
Suppose $\epsilon > 0$. $\exists N$ such that $\forall n \geq N$, we have

$$|a_n - a| < \epsilon$$

Notice that since (n_k) is a strictly increasing sequence of natural numbers, $n_k \geq k$. This can be shown with induction. Thus, for all $k \geq N$, since $n_k \geq N$,

$$|a_{n_k} - a| < \epsilon$$

must hold for all $k \geq N$. ■

Theorem (Bolzano-Weierstrass). Assume (a_n) is bounded. Then, there exists a subsequence of (a_n) that is convergent.

Proof.

Since (a_n) is bounded, $\exists M > 0$ such that $|a_n| < m$ for all n. Let $I_0 = [-M, M]$. Then $a_n \in I_0$ for all n. Write $I_0 = [-M, 0] \cup [0, M]$. Consider the two sets

$$\{n \in \mathbb{N} \mid a_n \in [-M, 0]\}$$

$$\{n \in \mathbb{N} \mid a_n \in [0, M]\}$$

The union of the two sets above is $\mathbb{N}$ since all a_n are in I_0, and notice that $\mathbb{N}$ is an infinite set. Suppose I_1 is an element in $\{[-M, 0], [0, M]\}$ such that $\{n \in \mathbb{N} \mid a_n \in I_1\}$ is infinite. Now we repeat this process. Suppose I_k is defined. We define I_{k+1} to be a closed subinterval of I_k that has half the length of I_k such that $\{n \in \mathbb{N} \mid a_n \in I_{k+1}\}$ is infinite. This yields a sequence of intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots ...$$

such that

(1) The length of $I_k = \dfrac{2M}{2^k}$

(2) The set $\{n \in \mathbb{N} \mid a_n \in I_k\}$ is infinite for each k.

Let n_1 be an element in

$$\{n \in \mathbb{N} \mid a_n \in I_1\}$$

Suppose $(n_1, ..., n_k)$ are defined for $k \geq 1$. We take n_{k+1} to be an element in

$$\{n \in \mathbb{N} \mid a_n \in I_{k+1}\}$$

to be the minimum element in

$$\{n \in \mathbb{N} \mid a_n \in I_{k+1}\} \backslash \{1, 2, ..., n_k\}$$

We have constructed a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ such that $a_{n_k} \in I_k$ for all k. By the nested interval property,

$$\cap_{k=1}^{\infty} I_k \neq \varnothing$$

Take $x \in \cap_{k=1}^{\infty} I_k$. Then, $a_{n_k} \in I_k$ and $x \in I_k$, so

$$|a_{n_k} - x| \leq \frac{2M}{2^k}$$

Lemma. $\lim_{k \to \infty} \dfrac{1}{2^k} = 0$.

$$|a_{n_k} - x| \leq \frac{2M}{2^k}$$

$$-\frac{2M}{2^k} \leq a_{n_k} - x \leq \frac{2M}{2^k}$$

$$x - \frac{2M}{2^k} \leq a_{n_k} \leq x + \frac{2M}{2^k}$$

By the algebraic limit theorem and the previous lemma,

$$\lim(x + \frac{2M}{2^k}) = \lim(x - \frac{2M}{2^k}) = x$$

so $\lim a_{n_k} = x$ by the Squeeze theorem. ■

Theorem. If (a_n) is bounded, then (a_n) is convergent to a if and only if every convergent subsequence (not including a_n) of (a_n) also converges to a.

Proof.

The forward direction was already proved. Suppose every convergent subsequence of (a_n) converges to a but (a_n) does not converge to a. Since (a_n) does not converge to a, $\exists\epsilon > 0$ such that $\{n \in \mathbb{N} \mid a_n \notin V_\epsilon(a)$ is infinite.

Therefore, there exists a subsequence

$$(a_{n_k})$$

such that $|a_{n_k} - a| \geq \epsilon$ for all k. Since (a_n) is bounded, (a_{n_k}) is also bounded. By the B-W theorem, (a_{n_k}) has a convergent subsequence $(a_{n_{k_i}})_{i\in\mathbb{N}}$. By the assumptions,

$$\lim_{i\to\infty} a_{n_{k_i}} = a$$

so $\exists N$ such that

$$|a_{n_{k_i}} - a| < \epsilon$$

for all $i \geq N$. This is contradicting to the statement that $|a_{n_k} - a| \geq \epsilon$ for all k. ■

10 Lecture 7: Cauchy Sequences and the Axiom of Completeness

10.1 Cauchy Sequences

Definition (Cauchy). A sequence (a_n) is called **Cauchy** if for all $\epsilon > 0$ there exists a natural number N such that for all $n \geq N$, $m \geq \mathbb{N}$, we have $|a_n - a_m| < \epsilon$.

Remark. Wait I acc derived this when doing problem 2 from the last problem set because once you map (a, b) to $\frac{a+b}{2}$ the resulting set of intervals if kind of basically a Cauchy sequence (notice that it doesn't really depend on order).

Theorem (Cauchy Criterion). A sequence is convergent if and only if it is a Cauchy sequence.

Proposition. Every convergent sequence is Cauchy.

Proof.
Take (a_n) is convergent. For all ϵ, let $\epsilon_0 = \frac{\epsilon}{2}$. Suppose $\lim a_n = a$. For $n, m > N \in \mathbb{N}$, then

$$|a_n - a| < \epsilon_0$$

$$|a_m - a| < \epsilon_0$$

where Then,

$$|a_n - a| + |a - a_m| < 2\epsilon_0 = \epsilon$$
$$|a_n - a_m| < |a_n - a| + |a - a_m| < \epsilon$$

so

$$|a_n - a_m| < \epsilon$$

for arbitrary epsilon and $n, m \in \mathbb{N}$. Thus, (a_n) is indeed Cauchy. ■

Proposition. Every Cauchy sequence is bounded.

Proof.

Strategy (Bounding). Sometimes we find multiple bounds for different parts of a sequence. We start by showing that for $n > N$, the sequence is bounded (perhaps because of some limit or other condition) and since the number of terms with $n \leq N$ is finite, it must also be bounded. This seemingly trivial strategy is very useful.

By definition, if (a_n) is a Cauchy sequence, then $\exists N \in \mathbb{N}$ such that for all $n \geq N$ and $m \geq N$, we must have

$$|a_n - a_m| \leq 1$$

When $m = N$, we have

$$|a_n - a_N| \leq 1$$

Using the triangle inequality,

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$$

for all $n \geq N$.

Strategy. Somtimes, when using the triangle inequality on $|a - b|$, if the inequality

$$|a - b| < |a| + |b|$$

is the wrong direction, it is better to take

$$|a| < |a - b| + |b|$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |a_N|\}$. Then if $n < N$,

$$\begin{aligned}|a_n| &\leq \max\{|a_1|, |a_2|, ..., |a_{N-1}|\} \\ &\leq M\end{aligned}$$

If $n \geq N$,

$$|a_n| < 1 + |a_N| \leq M$$

so $|a_n| \leq M$ for all $n \in \mathbb{N}$. ■

Proposition. Every Cauchy sequence is convergent.

Proof.
Scratch Work:
Suppose (a_n) is Cauchy. By the previous proposition, (a_n) is bounded. Thus, by the Bolzano-Weierstrass theorem, the subsequence $(a_{n_k})_{k \in \mathbb{N}}$ of (a_n) such that (a_{n_k}) is convergent. Let

$$a = \lim_{k \to \infty} a_{n_k}$$

We prove that (a_n) is convergent to a. Since (a_n) is Cauchy, $\forall \epsilon_1 > 0$, there exists N_0 such that $|a_m - a_n| < \epsilon_1$ for all $m, n \geq N_0$. Since $\lim_{k \to \infty} a_{n_k} = a$, for all $\epsilon_2 > 0$, there exists N_2 such that $|a_{n_k} - a| < \epsilon_2$ for all $k \geq N_2$.
Let $N = N_1$. Suppose $n \geq N$. Let k be a positive integer such that $k \geq N_2$ and $n_k \geq N_1$. Such k always exists since we have $n_k \geq k$ (shown previously using induction) and we can take $k = \max\{N_1, N_2\}$. Then,

$$|a_{n_k} - a_n| < \epsilon_1$$

$$|a_{n_k} - a| < \epsilon_2$$

Using the triangle inequality,

$$|a_n - a| < |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon_1 + \epsilon_2$$

Proof:
Suppose (a_n) is Cauchy. By the previous proposition, (a_n) is bounded. Thus, by the Bolzano-Weierstrass theorem, the subsequence $(a_{n_k})_{k \in \mathbb{N}}$ of (a_n) such that (a_{n_k}) is convergent. Let

$$a = \lim_{k \to \infty} a_{n_k}$$

We prove that (a_n) is convergent to a. Since (a_n) is Cauchy, $\forall \epsilon > 0$, there exists N_0 such that $|a_m - a_n| < \frac{\epsilon}{2}$ for all $m, n \geq N_0$. Since $\lim_{k \to \infty} a_{n_k} = a$, there exists N_2 such that $|a_{n_k} - a| < \frac{\epsilon}{2}$ for all $k \geq N_2$.
Let $N = N_1$. Suppose $n \geq N$. Let k be a positive integer such that $k \geq N_2$ and $n_k \geq N_1$. Such k always exists since we have $n_k \geq k$ (shown previously using induction) and we can take $k = \max\{N_1, N_2\}$. Then,

$$|a_{n_k} - a_n| < \frac{\epsilon}{2}$$

$$|a_{n_k} - a| < \frac{\epsilon}{2}$$

Using the triangle inequality,

$$|a_n - a| < |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

In conclusion, we have $\lim a_n = a$. ■

Using the above propositions, we have proved the Cauchy Criterion.

10.2 Summary of Sequences

We can break up the propositions and theorems from above into two groups: proofs that used the Axiom of Completeness and the proofs that did not use the Axiom of Completeness.

No Completeness	Completeness
Algebraic Limit Theorem	Nested Interval Property
Order Limit Theorem	Bolzano-Weierstrass Theorem
Squeeze Theorem	Cauchy Criterion
$(a_n) \to a$ means $(a_{n_k}) \to a$ for every subsequence	Monotone Convergence Theorem

Notice that the No Completeness theorems were proved just by playing with ϵ and n, but for the Completeness theorems, we started with the Least Upper Bound Property, which we used to prove the Monotone Convergence Theorem and the Nested Interval Property. We then proved the B-W theorem using $\lim \frac{1}{2^k} = 0$ with the Archimedean property and the Nested Interval Property. Then, we proved the Cauchy Criterion using the B-W theorem.

Essentially, we can actually take the Axiom of Completeness to be any of the theorems in the Completeness category.

Theorem. If $\mathbb{R}$ is an ordered field that satisfies the Archimedean Property, then

(a) Least Upper Bound Property

(b) Nested Interval Property

(c) B-W Theorem

(d) Cauchy Criterion

(e) Monotone Convergence Theorem

(f) Cut Property

are equivalent.

Remark. In fact, different mathematicians started with different starting points in (a), (b), (c), (d), (e), and (f) and it was later found out that the resulting mathematical structures were all the same!! We call the starting point the Axiom of Completeness, and then we can use that to find any of the other theorems in that list. Each of (a)-(f) are "incarnations" of the concept of Completeness which is a property of $\mathbb{R}$.

We will show that theorem in the homework.

Theorem. Cauchy Criterion $\Rightarrow$ Least Upper Bound Property.

Proof.

Remark. Here the Cauchy Criterion is our "Axiom of Completeness".

Suppose $A \subseteq \mathbb{R}$ is a nonempty set that is bounded above. Let u be an upper bound of A. Let $x \in A$, and let $l = x - 1$. Then, $u \geq x > x - 1 = l$. Let $u_1 = u$ and $l_1 = l$. We construct two sequences (u_n) and $(l_n$ as follows: suppose we have defined (u_n, l_n) such that u_n is an upper bound of and l_n is not an upper bound of A. Then, define

$$m_n = \frac{1}{2}(l_n + u_n)$$

and let

$$(l_{n+1}, u_{n+1}) = \begin{cases} (l_n, m_n) & \text{if } m_n \text{ is an upper bound of } A \\ (m_n, u_n) & \text{if } m_n \text{ is not an upper bound of } A \end{cases}$$

Then $|u_{n+1} - l_{n+1}| = \frac{1}{2}|u_n - l_n|$ and u_{n+1} is an upper bound of A, l_{n+1} is not an upper bound. This construction yields two sequences (l_n), (u_n) such that

$$l_n < u_n$$

for all n,

$$|u_n - l_n| = \frac{1}{2^{n-1}}|u_1 - l_1|$$

for all n

$$u_n \text{ is an upper bound of } A$$

$$l_n \text{ is not an upper bound of } A$$

We also have

$$|u_{n+1} - u_n| \leq \frac{1}{2}|u_n - l_n| = \frac{1}{2^n}|u_1 - l_1|$$

$$|l_{n+1} - l_n| \leq \frac{1}{2}|u_n - l_n| = \frac{1}{2^n}|u_1 - l_1|$$

If $m > n$, we have

$$\begin{aligned}
|u_n - u_m| &\leq |u_n - u_{n+1}| + |u_{n+1} - u_{n+2}| + \dots + |u_{m-1} - u_m| \\
&\leq (\frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{m-1}})|u_1 - l_1| \\
&= (\frac{2}{2^n} - \frac{1}{2^{m-1}})(|u_1 - l_1|) \\
&< \frac{1}{2^{n-1}}|u_1 - l_1|
\end{aligned}$$

Remark. Note that we proved the Archimedean Property using the Least Upper Bound Property, but here we are taking the Archimedean Property as an axiom to show that everything is equivalent. Not assuming the Archimedean Property causes some things to fail.

Last time, we proved that $\lim \frac{1}{2^n} = 0$ by the Archimedean Property, so by the algebraic limit theorem,

$$\lim_{n\to\infty} \frac{1}{2^{n-1}}|u_1 - l_1| = 0$$

Then for all $\epsilon > 0$, $\exists N$ such that for all $n \geq N$, we have

$$\frac{1}{2^{n-1}}|u_1 - l_1| < \epsilon$$

Therefore, for any pair m, n with $m > n \geq N$, we have

$$\begin{aligned}
|u_m - u_n| &< \frac{1}{2^{n-1}}|u_1 - l_1| \\
&< \epsilon
\end{aligned}$$

This proves that (u_n) is Cauchy. The same argument proves that (l_n) is Cauchy. By the Cauchy Criterion, both (u_n) and (l_n) are convergent. Let $u = \lim u_n$ and $l = \lim l_n$. We have $(u_n - l_n) = \frac{1}{2^{n-1}}(u_1 - l_1)$. By the algebraic limit theorem,

$$u - l = \lim \frac{1}{2^{n-1}}(n_1 - l_1) = 0$$

We claim that $u = l$ is the least upper bound of A. To show that u is an upper bound, for any $x \in A$, we have $x \leq a_n$ for all n. Now, by the order limit theorem, $x \leq \lim u_n = u$. Now we show that u is the least upper bound.

If $u' < u$ with u' an upper bound, since $\lim l_n = l$, there exists an N such that

$$|l_n - l| < u - u'$$

for all $n \geq N$. Then,

$$u' - u < l_n - l < u - u'$$

(notice that $u - u' > 0$ by the previous assumption). Then,

$$u' - l < l_n - l < l - u'$$

$$u' < l_n$$

Since l_n is not an upper bound of A, we conclude that u' is not an upper bound of A (contradiction). ■

Theorem. The Monotone Convergence Theorem implies the Nested Interval Property.

Incomplete Proof.

Suppose $I_n = [a_n, b_n]$ and $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$. Then one can show that $\lim a_n$ exists and is an element in $\cap_{n=1}^{\infty} A_n$. We will leave the details to the homework.

11 Precept 4: Archimedean Property and Equivalences

Theorem. Assuming the Archimedean Property (AP) and the Nested Interval Property (NIP), we can prove the Axiom of Completeness (Least Upper Bound Theorem or AOC).

Proof.

Lemma. AP implies that $\lim_{n\to\infty} 2^{-n} = 0$.

Proof.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $N \geq \frac{1}{\epsilon}$ (AP). $\forall n \geq N$, $2^n > n > N > \epsilon^{-1}$. Thus, for all $n > N$,

$$0 < 2^{-n} < \epsilon$$

so we must have that $\lim_{n\to\infty} 2^{-n} = 0$. □

Let $A \subseteq \mathbb{R}$ be nonempty and bounded from above. We want to show that $\sup A$ exists. Choose $a \in A$ and c an upper bound of A. Thus, $a \leq c$. Let $I_1 = [a, c]$ be a closed interval. We will introduce a notation:

Definition. We will say that $I = [\alpha, \beta]$ *has property (P)* if $\alpha \in A$ and β is an upper bound of A.

Remark. Only for this proof.

Now, notice that I_1 has property (P). Let us define a nested set of closed intervals

$$\{I_n\}_{n=1}^{\infty}$$

as follows: suppose we have defined $I_k = [a_k, a_c]$ such that I_k has property (P). We will then define $I_{k+1} \subseteq I_k$ as follows:

$$b_k = \frac{a_k + c_k}{2}$$

We will consider two cases:

Case 1: b_k is an upper bound of A

Then, let $I_{k+1} = [a_k, b_k]$. Notice that since $a_k \in A$ and b_k is an upper bound of A, I_{k+1} does indeed satisfy property (P). Also, notice that $|I_{k+1}| = b_k - a_k = \frac{1}{2}(c_k - a_k) = \frac{1}{2}|I_k|$.

Case 2: b_k is not an upper bound of A

Then, $\exists \alpha \in A$ such that $b_k < \alpha \leq c_k$. Then, define $I_{k+1} = [\alpha, c_k]$. Since $\alpha \in A$ and c_k is an upper bound of A, I_{k+1} does indeed satisfy property (P). Similarly to case 1, $|I_{k+1}| = c_k - \alpha < c_k - b_k = \frac{1}{2}(c_k - a_k) = \frac{1}{2}|I_k|$. In other words, $|I_{k+1}| < \frac{1}{2}|I_k|$.

Combining the above cases, $|I_{k+1}| \leq \frac{1}{2}|I_k|$. Define I_1 with property (P). Then if I_k is a closed interval with property (P), we have constructed a closed interval $I_{k+1} \subseteq I_k$ with property P.

Thus, through this construction, we get a sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ such that $\forall n \in \mathbb{N}$, I_n has property (P). Notice that with induction, we can show that $|I_n| \leq 2^{-n+1}|I_1|$. By the lemma, we now know that $|I_n| \to 0$.

Since this is a collection of nested intervals, we can use the nested interval property to see that $\cap_{n=1}^{\infty} I_n \neq \varnothing$ (NIP). We claim that $u = \cap_{n=1}^{\infty} I_n \neq \varnothing$ is the supremum of A.

Suppose for contradiction that u is not an upper bound of A. Then, $\exists x \in A$ with $u < x$. Notice that $u \in \cap_{n=1}^{\infty} I_n$. Thus, $u \in I_n$ for all n, so $a_n \leq u \leq c_n$ for all n. Thus, $c_n - u \leq c_n - a_n = |I_n| = 2^{-n+1}I_1$. By our lemma, $\exists p \in \mathbb{N}$ such that

$$2^{-p+1}|I_1| < x - u$$

Then,

$$c_p - u \leq 2^{-p+1}|I_1| < x - u \Rightarrow c_p < x$$

which is a contradiction as c_p is an upper bound of A. Thus, u is an upper bound of A.

Suppose for contradiction that there exists another upper bound of A, u', with $u' < u$.

$$\forall n \in \mathbb{N}, u \in I_n = [a_n, c_n]$$

$$\forall n \in \mathbb{N}, a_n \leq u \leq c_n$$

$$\forall n \in \mathbb{N}, u - a_n \leq c_n - a_n = |I_n| \leq 2^{-n+1}|I_1|$$

By the lemma, there exists $q \in \mathbb{N}$ such that $2^{-q+1}|I_1| < u - u'$. Thus,

$$u - a_q \leq 2^{-q+1}|I_1| < u - u'$$

Thus, $u' < a_q$, so u' is not an upper bound of A. Thus, $u = \sup A$, which completes the proof. ■

Problem. $\lim_{n\to\infty} n^{1/n} = 1$.

Proof.
We claim that for $n > 1$, $n^{1/n} > 1$. Let $x_n = n^{1/n} - 1$. Then,

$$n = (1 + x_n)^n$$

which we can expand:

$$= (1 + \binom{n}{1} x_n + \binom{n}{2} x_n^2 + ... + \binom{n}{p} x_n^p + ... + x_n^n$$

$$> nchoose2x_n^2$$

by the Binomial theorem. Notice that $n > \dfrac{n(n-1)}{2} x_n^2$. From this, we get that

$$0 < x_n < \sqrt{\frac{2}{n-1}}$$

Notice that 0 converges to 0, and $\sqrt{\dfrac{2}{n-1}}$ also converges to 0. Thus,

$$\lim x_n = 0$$

Since $x_n = n^{1/n} - 1$, $\lim n^{1/n} = 1$. ■

12 Lecture 8. Series

Definition (Series). Suppose (a_n) is a sequence. Define

$$\text{“partial sum”} = \begin{cases} S_1 = a_1 \\ S_2 = a_1 + a_2 \\ \vdots \\ S_n = a_1 + a_2 + \ldots + a_n \end{cases}$$

We call these **series**.

If (s_n) converges to s, we say that $\sum_{k=1}^{\infty} a_k$ converges to s.

Notice that $a_n = s_n - s_{n-1}$ for all $n \geq 2$.

Theorem (Algebraic Limit Theorem for Series). Suppose $\sum_{k=1}^{\infty} a_k = A, \sum_{k=1}^{\infty} b_k = B$. Let $c \in \mathbb{R}$. Then

(1) $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $(A + B)$.

(2) $\sum_{k=1}^{\infty}(ca_k)$ converges to (cA).

Proof.
Define $S_n^a = a_1 + \ldots a_n$ and $S_n^b = b_1 + \ldots b_n$. Then

$$\lim_{n\to\infty} S_n^a = A$$

$$\lim_{n\to\infty} S_n^b = B$$

So $\lim_{n\to\infty}(S_n^a + S_n^b) = A + B$. Since $S_n^a + S_n^b = \sum_{k=1}^{\infty}(a_k + b_k)$, we must have

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B$$

(2) follows from a similar argument. ■

Remark. Notice that in general, $\sum_{k=1}^{\infty} a_k b_k \neq AB$.

Theorem (Cauchy Criterion for Series). If (a_n) is a sequence, then $\sum_{n=1}^{\infty} a_n$ is convergent if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for every (m, n) with $m > n \geq N$, we have

$$|a_{n+1} + a_{n+2} + \ldots + a_n| < \epsilon$$

Proof.
Let $S_n = \sum_{k=1}^{n} a_k$, then for $m > n$, we have $s_m - S_n = a_{n+1} + \ldots + a_m$. By definition, (S_n) is Cauchy if and only if $\forall \epsilon > 0$, $\exists N$ such that for all $m > n \geq N$, we have $|a_{n+1} + \ldots + a_m| < \epsilon$. ■

Theorem (Comparison Test). If $0 \geq a_k \leq b_k$, and $\sum_{k=1} b_k$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent.

Proof.
Since $\sum_{k=1}^{\infty} b_k$ is convergent by the Cauchy criterion, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $m > n \geq N$, we have

$$|b_{n+1} + b_{n+@} + \ldots + b_m| < \epsilon$$

By the assumptions,

$$\begin{aligned} 0 &\leq a_{n+1} + a_{n+2} + \ldots + a_m \\ &\leq b_{n+1} + \ldots + b_m \end{aligned}$$

$$< \epsilon$$

so $|a_{n+1} + ... + a_m| < \epsilon$. By the Cauchy criterion $\sum_{k=1}^{\infty} a_k$ is convergent. ∎

Corollary. If $0 \le a_n \le b_n$, $\forall n$ and $\sum_{k=1}^{\infty}$ is divergent, then $\sum_{k=1}^{\infty} b_n$ is divergent.

Theorem (Absolute Convergence Test). If $\sum_{k=1}^{\infty} |a_k|$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

Proof.
Since $\sigma_{k=1}^{\infty}|a_k|$ is convergent, for all $\epsilon > 0$, $\exists N$ such that for all $m > n \ge N$,

$$||a_{n+1}| + ... + |a_m|| < \epsilon$$

$$|a_{n+1}| + ... + |a_m| < \epsilon$$

By the triangle inequality,

$$|a_{n+1} + ... + a_m| \le |a_{n+1}| + ... + |a_m| < \epsilon$$

By the Cauchy criterion, $\sum_{k=1}^{\infty} a_k$ is convergent. ∎

Theorem. If $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim a_n = 0$.

Proof 1 (Cauchy Criterion).
$\forall \epsilon > 0$, $\exists N$ such that $\forall m > n \ge N$, we have

$$|a_{n+1} + ... + a_m| < \epsilon$$

Take $m = n + 1$. We have

$$|a_{n+1}| < \epsilon$$

so for all $n \ge N + 1$, we have

$$|a_n| < \epsilon$$

Hence $\lim a_n = 0$.

Proof 2 ().
Let $S_n = \sum_{k=1}^{n} a_k$. We have

$$a_n = S_n - S_{n-1}$$

for all $n \ge 2$. Suppose $\lim S_n = S$. We have $\lim S_{n-1} = S$. By the algebraic limit theorem, $\lim a_n = \lim(S_n - S_{n-1}) = 0$. ∎

Theorem (Alternating Series Test). Suppose $a_1 \ge a_2 \ge a_3 \ge ... \ge 0$ and $\lim a_n = 0$. Then

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

is convergent.

Remark. Notice that the inequality condition just means that (a_n) is decreasing and nonnegative.

Proof.
Since $\lim a_n = 0$, $\forall \epsilon > 0$, $\exists N$ such that $|a_n| < \epsilon$ for all $n \ge N$. Suppose $m > n \ge N$. We claim that

$$|(-1)^{n+1} a_{n+1} + ... + (-1)^m a_m| < \epsilon$$

Notice that

$$|(-1)^{n+1} a_{n+1} + ... + (-1)^m a_m| = |a_{n+1} + ... + (-1)^{m-n-1} a_m|$$

We discuss two cases.

Case 1 ($m - n$ is even).

In this case, we have $a_{n+1} - a_{n+2} + ... - a_m$

$$= (a_{n+1} - a_{n+2}) + ... + (a_{m-1} - a_m)$$
$$\geq 0$$

Similarly,

$$a_{n+1} - a_{n+2} + a_{n+3} - ... - a_m$$
$$= a_{n+1} - (a_{n+2} - a_{n+3}) - ... - (a_{m-2} - a_{m-1}) - a_m$$
$$\leq a_{n+1}$$

so $|a_{n+1} - a_{n+2} + ... - a_m| \leq a_{n+1} < \epsilon$.

Case 2 ($m - n$ is odd).

In this case,

$$a_{n+1} - a_{n+2} + ... + a_m$$
$$= (a_{n+1} - a_{n+2}) + ... + (a_{m-2} - a_{m-1}) + a_m$$
$$\geq 0$$

Similarly,

$$a_{n+1} - a_{n+2} + a_{n+3} + ... + a_m$$
$$= a_{n+1} - (a_{n+2} - a_{n+3}) - ... - (a_{m-1} - a_m)$$
$$\leq a_{n+1}$$

so $|a_{n+1} - a_{n+2} + ... + a_m| \leq a_{n+1} < \epsilon$.

Thus, by the Cauchy criterion, the series $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent. ■

Definition (Absolutely Convergent). We say that $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if

$$\sum_{k=1}^{\infty} |a_k|$$

is convergent.

Example. $a_n = b^n$ for $b \neq 0$ and $b \neq 1$. Consider $\sum_{k=1}^{\infty} b^k$. Let $S_n = \sum_{k=1}^{n} b^k = b + b^2 + ... + b^n$. Find a closed form expression for S_n.

Proof.

$$S_n = \sum_{k=1}^{n} b^k$$

$$bS_n = \sum_{k=1}^{n} b^{k+1} = \sum_{k=2}^{n+1} b^k$$

$$S_n - bS_n = \sum_{k=1}^{n} b^k - \sum_{k=2}^{n+1} b^k$$

$$S_n - bS_n = b + \sum_{k=2}^{n} b^k - \sum_{k=2}^{n} b^k - b^{n+1} = b - b^{n+1}$$

Thus, $S_n = \dfrac{b - b^{n+1}}{1 - b}$. ■

Let us take a detour and study the convergence of (b^n).

Lemma. If $b \in (0, 1)$, then $\lim b^n = 0$.

Remark. Notice that here $(0,1)$ is an open interval rather than a set. Also, this proof is not perfect.

Proof.
Since $b \in (0,1)$, we have that

$$0 < b^{n+1} = b^n \cdot b < b^n \cdot 1 = b^n$$

So (b^n) is positive and decreasing. By the monotone convergence theorem, (b^n) is convergent. Let $x = \lim b^n$. Since (b^{2n}) is a subsequence of (b^n) we have

$$\lim b^{2n} = \lim b^n = x$$

By the algebric limit theorem,

$$\lim b^{2n} = \lim b^n \cdot \lim b^n$$

so $x = x^2$. Thus $x(1-x) = 0$ so either $x = 0$ or $x = 1$. Since (b^n) is decreasing, we have $b^n \leq b$ for all $n \in \mathbb{N}$, by the order limit theorem, $x = \lim b^n \leq b < 1$. Therefore, $x = 0$. ■

Lemma. If $b \in (-1,0)$, then $\lim b^n = 0$.

Proof.

$$-|b|^n \leq b^n \leq |b|^n$$

By the previous lemma, we have $\lim |b|^n = 0$. By the algebraic limit theorem, $\lim -|b|^n = 0$. By the Squeeze theorem, $\lim b^n = 0$.

Lemma. If $b > 1$, then (b^n) is unbounded.

Proof.
If $b > 1$, then

$$b^{n+1} = b^n \cdot b > b^n$$

so (b^n) is an increasing sequence. Assume for contradiction that (b^n) is bounded. Then by the monotone convergence theorem, (b^n) is convergent. Let $x = \lim(b^n)$. Then, $\lim(b^{2n}) = \lim(b^n) \cdot \lim(b^n) = x^2$ so $x = x^2$. Thus, $x = 0$ or 1. Since b^n is increasing, $b^n \geq b$ for all $n \in N$. Thus, $\lim(b^n) \geq b > 1$. This is a contradiction. ■

Lemma. If $b < -1$, then (b^n) is unbounded.

Proof.
Since $|b^n| = |b|^n$ and the previous lemma, $(|b|^n)$ is unbounded. By definition, (b^n) is also unbounded. ■

Now back to the discussion of $\sum_{k=1}^{\infty} b^k$. Recall that

$$\sum_{k=1}^{n} b^k = \frac{b - b^{n+1}}{1-b}$$

If $|b| < 1$, then $\lim b^{n+1} = 0$, so $\sum_{k=1}^{\infty} b^k$ converges to $\dfrac{b}{1-b}$. If $|b| > 1$, then

$$S_n = \frac{b - b^{n+1}}{1-b}$$

$$b^n = \frac{b - (1-b)S_n}{b}$$

By the algebraic limit theorem, if (S_n) converges, then (b^n) would also be convergent.

Since (b^n) is unbounded, it has to be divergent, so we conclude that (S_n) is divergent.

Theorem (Harmonic Series).

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent.

Proof.

We show that with $\epsilon = \frac{1}{2}$ such that $\forall N \in \mathbb{N}$, $\exists m > n \geq N$ with

$$|\frac{1}{n+1} + ... + \frac{1}{m}| \geq \epsilon = \frac{1}{2}$$

(so it violates the Cauchy Criterion). Let $n = 2^l$ such that $2^l \geq N$. Let $m = 2^{l+1}$. Then

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{m}$$

$$= \frac{1}{2^l+1} + \frac{1}{2^l+2} + ... + \frac{1}{2^{l+1}}$$

$$\geq \frac{1}{2^{l+1}} + \frac{1}{2^{l+1}} + ... + \frac{1}{2^{l+1}}$$

$$= \frac{2^l}{2^{l+1}} = \frac{1}{2}$$

■

Definition. $x^{p/q} = (\sqrt[q]{x})^p$.

If $r \in \mathbb{R}$, $x > 1$, define $x^r = \sup\{x^p \mid q < r, q \in \mathbb{Q}\}$.

For now we are just going to assume that we have a well defined function for x^r for all real numbers r.

Theorem (p-series). $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is divergent if $p \leq 1$ and is convergent if $p > 1$.

Proof.

Suppose that $p \leq 1$. We have

$$n^p \leq n$$

for all $n \in \mathbb{N}$. So $\frac{1}{n^p} \geq \frac{1}{n} > 0$. By the comparison test, since $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is also divergent.

Suppose that $p > 1$. We show that

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

is convergent. Let $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$. We have (S_n) is increasing. To show that (S_n) is convergent, we only need to show that (S_n) is bounded. Consider the subsequence (S_{2^n}).

$$(S_{2^l} = \frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{(2^l)^p}$$

$$= \frac{1}{1^p} + \sum_{u=0}^{l} [\frac{1}{((2^u)+1)^p} + ... + \frac{1}{(2^u)^p}$$

$$\leq \frac{1}{1^p} + \sum_{u=0}^{l} [\frac{1}{(2^{u-1})^p} + \frac{1}{(2^{u-1})^p} + ... + \frac{1}{(2^{u-1})^p}]$$

$$= \frac{1}{1^p} + \sum_{u=0}^{l} \frac{2^{u-1}}{2^{(u-1)p}}$$

$$= 1 + \sum_{k=0}^{l} (\frac{1}{2^{p-1}})^k \cdot 2^{p-1}$$

13 Lecture 9: More Series and Topology

13.1 Series

Theorem (p-series). $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is divergent if $p \leq 1$ and is convergent if $p > 1$.

Proof.

Let $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$. Then, (S_n) is an increasing sequence so we only need to show that (S_n) is bounded. Consider the subsequence $(S_{2^k})_{k\in\mathbb{N}}$.

$$S_{2^k} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{(2^k)^p}$$

If we groups the first 2^{k-1} terms, we have

$$\begin{aligned}
S_{2^k} &= S_{2^{k-1}} + [\frac{1}{[(2^{k-1}+1]^p} + \frac{1}{[2^{k-1}+2]^p} + \ldots + \frac{1}{(2^k)^p}] \\
&< S_{2^{k-1}} + [\frac{1}{(2^{k-1})^p} + \frac{1}{(2^{k-1})^p} + \ldots + \frac{1}{(2^{k-1})^p}] \\
&= S_{2^{k-1}} + \frac{2^{k-1}}{(2^{k-1})^p} \\
&= S_{2^{k-1}} + \frac{2^p}{2} \cdot (\frac{1}{2^{p-1}})^k
\end{aligned}$$

Let $a = \frac{2^p}{2}$, $b = \frac{1}{2^{p-1}}$. Then

$$S_{2^k} < S_{2^{k-1}} + a \cdot b^k$$

and we have $0 < b < 1$.

$$S_{2^n} < S_1 + ab + ab^2 + \ldots + ab^n$$

$$S_{2^n} < 1 + \sum_{k=1}^{n} ab^k$$

$$S_{2^n} < 1 + a \cdot \frac{b}{1-b}$$

In general, for all n, take $k \in \mathbb{N}$ such that $2^k \geq n$. We have $S_n \leq S_{2^k} < 1 + a \cdot \frac{b}{1-b}$. Then,

$$S_n \leq S_{2^k} < 1 + a\frac{b}{1-b}$$

so (S_n) is bounded from above. ■

Before showing the other part of the theorem, we need a theorem:

Theorem (Rearrangement Theorem). Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=1}^{\infty} a_{f(n)}$ is convergent to the same limit.

Remark. Let $a_n = (-1)^n \frac{1}{n}$ Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Proof.

$$S_n = \sum_{k=1}^{n} a_k$$

$$S'_n = \sum_{k=1}^{n} a_{f(k)}$$

$$S_n - S'_n = \sum_{k=1}^{M} d_k \cdot a_k$$

for $d_k \in \{-1, 0, 1\}$ and $M = \max\{1, 2, ..., n, f(1), f(2), ..., f(n)\}$.

Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, by the Cauchy Criterion, for all positive ϵ, $\exists N$ such that for all $m >\geq N$ such that

$$|a_{n+1}| + |a_{n+2}| + ... + |a_m| < \epsilon$$

Let $N_1 = 1 + \max\{1, 2, 3, ..., N, f^{-1}(1), f^{-1}(2), ..., f^{-1}(N)\}$. Then for $n \geq N$, we have

$$\{1, 2, ..., n\} \supseteq \{1, , 2, ..., N\}$$

$$\{f(1), f(2), ..., f(n)\} \supseteq \{1, 2, ..., N\}$$

So, for each $n \geq N_1$, we have

$$S_n - S'_n = \sigma_{N+1}^{M} d_k a_k \leq \sigma_{N+1}^{M} |a_k|$$

$$|S_n - S'_n| \leq \sigma_{N+1}^{M} |a_k| < \epsilon$$

which follows from the Cauchy Criterion. Thus, the difference sequence converges to zero. Thus,

$$\lim(S_n - S'_n) = 0 \Rightarrow \lim S_n = \lim S'_n$$

■

Remark. If (a_n) is convergent but not absolutely convergent, then

(1) For each $c \in \mathbb{R}$ $\exists f : \mathbb{N} \to \mathbb{N}$ bijection such that $\sum_{n=1}^{\infty} a_{f(n)} = c$.

(2) $\exists f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{(f_n)}$ is divergent.

Try to show this, and then notice that we can see that the set of bijective function from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

Corollary. Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=1}^{\infty} a_{f(n)}$ is also *absolutely* convergent to the same limit.

This follows from the original theorem just by putting in $|a_n|$.

13.2 Topology

Definition (Open Set). If $A \subseteq \mathbb{R}$, we see that A is an open set if for any $x \in A$, there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq A$$

Example. $(0, 1)$ is open. However, $[0, 1]$ is not open.

Example. $\varnothing$ is open.

Proposition. If A is open, then exists a set S of open intervals such that $A = \bigcup_{I \in S} I$.

Proof.

For any $x \in A$, let $\epsilon(x) > 0$ be a positive number such that $(x - \epsilon(x), x + \epsilon(x)) \subseteq A$. Then, we claim that

$$A = \cup_{x \in A}(x - \epsilon(x), x + \epsilon(x))$$

If $x \ inA$, then $x \in (x-\epsilon(x),, x+\epsilon(x))$ so

$$x \ in \cup_{x \in A} (x-\epsilon(x), x+\epsilon(x))$$

$$\Rightarrow A \subseteq \cup_{x \in A}(x-\epsilon(x)+x+\epsilon(x))$$

If $x \in A$ by definition, $(x-\epsilon(x), x+\epsilon(x)) \subseteq A$. So

$$\cup_{x \in A}(x-\epsilon(x), x+\epsilon(x)) \subseteq A$$

■

Theorem. If A_1, A_2 are open, then

$$A_1 \cap A_2$$

is open.

Proof.

$$\forall x \in A_1 \cap A_2$$

we have $x \in A_1$, $x \in A_2$. Thus, there exists $\epsilon_1, \epsilon 2 > 0$ such that

$$(x-\epsilon_1, x+\epsilon_1) \subseteq A_1$$

$$(x-\epsilon_2, x+\epsilon_2) \subseteq A_2$$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then

$$(x-\epsilon, x+\epsilon) \subseteq (x-\epsilon_1, x+\epsilon_1) \subseteq A_1$$
$$(x-\epsilon, x+\epsilon) \subseteq (x-\epsilon_2, x+\epsilon_2) \subseteq A_2$$

so

$$(x-\epsilon, x+\epsilon) \subseteq A_1 \cap A_2$$
$$\Rightarrow A_1 \cap A_2 \text{ is open.}$$

■

Corollary. If A_1, A_2, . . ., A_n is a finite collection of open sets, then $\cap_{i=1}^{n} A_i$ is open.

Theorem. If $\{A_S\}$ is a set of open sets, then

$$\cup_S A_S$$

is also open.

Proof. $\forall x \in \cup A_S$, there exists A_S such that $x \in A_S$. Since A_S is open,

$$\exists \epsilon \text{ s.t. } (x - epsilon, x+\epsilon) \subseteq A_S$$

Then $(x-\epsilon, x+\epsilon) \subseteq \cup A_S$. ■

Definition (Limit Point). Suppose $A \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is called a Limit Point of A if $\forall \epsilon > 0$,

$$(x-\epsilon, x+\epsilon) \cap A = \varnothing$$

Theorem. x is a limit point of A if and only if there exists a sequence (a_n) such that

$$a_n \in A \forall n$$

$$\lim a_n = x$$

Remark. (a_n) does not have to contain distinct elements.

Proof.
If $a_n \in A$ for all n and $\lim a_n = x$, then $\forall \epsilon > 0$, $\exists N$ such that

$$a_n \in (x - \epsilon, x + \epsilon)$$

for all $n \geq N$. Therefore, $(x - \epsilon, x + \epsilon) \cap A \neq \varnothing$. If x is a limit point of A, then $\forall \epsilon > 0$,

$$(x - \epsilon, x + \epsilon) \cap A \neq \varnothing$$

so we can take a_n to be an element of

$$(x - \frac{1}{n}, x + \frac{1}{n}) \cap A$$

We have

$$|a_n - x| < \frac{1}{n}$$
$$a_n \in A$$

for all n. Thus, for all $\epsilon > 0$, $\exists N$ such that $\frac{1}{N} < \epsilon$ by the Archimedean Principle, so

$$|a_n - x| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Definition (Closed). If A contains all the limit points of A, then A is called closed.

Example. $[0, 1]$ is closed, but $(0, 1)$ is not closed.

Example. $\varnothing$ is closed!! :D

Definition (Closure). If $A \subseteq \mathbb{R}$, then $\overline{A}$ is the set of limit points of A. $\overline{A}$ is called the "closure" of A. Notice that $A \subseteq \overline{A}$, and if A is closed, $A = \overline{A}$. Also,

$$A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$$

If $A \subseteq B$ and B is closed, then

$$\overline{A} \subseteq \overline{B} = B$$

so $\overline{A} \subseteq B$.

Theorem. $\overline{A}$ is always closed.

Proof.
Suppose x is a limit point of $\overline{A}$. We need to show that $x \in \overline{A}$. We will show the contrapositive.
If $x \notin \overline{A}$, then $\exists \epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \cap A = \varnothing$$

$\forall y \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2})$, we have

$$(y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2}) \subseteq (x - \epsilon, x + \epsilon)$$

So $(y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2}) \cap A = \varnothing$. Therefore, $y \notin \overline{A}$.

$$(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \cap \overline{A} = \varnothing$$

Therefore x is not a limit point of $\overline{A}$ ■

Corollary.

$$\overline{A} = \cap_{B \text{ closed},\ A \subseteq B} B$$

Theorem. A is open $\Leftrightarrow$ A^c is closed.

Proof.

Suppose A is open, x is a limit point of A^c. we show that $x \in A^c$. Since x is a limit point of A^c, $\forall \epsilon > 0$,

$$(x-\epsilon, x+\epsilon) \cap A^c \neq \varnothing$$

$$\Rightarrow (x-\epsilon, x+\epsilon) \nsubseteq A$$

Since A is open, this implies that $x \notin A$. Thus, $x \in A^c$. Thus, A^c is closed.

Suppose A^c is closed. For $x \in A$, we will show that $\exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq A$. Since $x \in A$, we have $x \notin A^c$. Since A^c is closed, we conclude that x is not a limit point of A^c. Thus,

$$\exists \epsilon > 0$$

such that $(x-\epsilon, x+\epsilon) \cap A^c = \varnothing$. Then,

$$(x-\epsilon, x+\epsilon) \subseteq A$$

so A is open. ■

Corollary.

(1) If A_1, . . ., A_n are closed sets, then

$$\cup_{i=1}^{n} A_n$$

is closed.

(2) If $\{A_S\}$ is a family of closed sets, then $\cap_S A_i$ is closed.

14 Investigation

Interesting. I will try to tackle the theorem from class that we didn't show:

Remark. If (a_n) is convergent but not absolutely convergent, then

(1) For each $c \in \mathbb{R}$ $\exists f : \mathbb{N} \to \mathbb{N}$ bijection such that $\sum_{n=1}^{\infty} a_{f(n)} = c$.

(2) $\exists f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{(f_n)}$ is divergent.

Try to show this, and then notice that we can see that the set of bijective function from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

Proof.
We define sequences x_n and y_n such that $x_1 = \min\{n : a_n \geq 0\}$ and $y_1 = \{n : a_n < 0\}$, and $x_n = \min\{m : a_m \geq 0, m > x_{n-1}\}$ and $y_n = \min\{m : a_m < 0, m > y_{n-1}\}$. Then, by definition $x_n > x_{n-1}$ and $y_n > y_{n-1}$ and the sequences $(a_{x_n})_{n\in\mathbb{N}}$ and $(a_{y_n})_{n\in\mathbb{N}}$ are subsequences of a_n.

Lemma 1. $a_n \geq 0 \Rightarrow$ there exists an x_m with $x_m = n$.

Proof.
Suppose for contradiction that $a_n \geq 0$ and there exists no x_m with $x_m = n$. Then, since (x_n) is strictly increasing, there exists j such that $x_j < n < x_{j+1}$. However, this is a contradiction of the minimality of x_{j+1} since $x_{j+1} = \min\{m : a_m \geq 0, m > x_j\}$, but $n < x_{j+1}$. Thus, there exists x_m with $x_m = n$. ∎

Lemma 2. $a_n < 0 \Rightarrow$ there exists a y_m with $y_m = n$.

Proof.
Suppose for contradiction that $a_n < 0$ and there exists no y_m with $y_m = n$. Then, since (y_n) is strictly increasing, there exists j such that $y_j < n < y_{j+1}$. However, this is a contradiction of the minimality of y_{j+1} since $y_{j+1} = \min\{m : a_m \geq 0, m > y_j\}$, but $n < y_{j+1}$. Thus, there exists y_m with $y_m = n$. ∎

Lemma 3. Let $X = \{x : \exists n, x_n = x\}$ and $Y = \{y : \exists n, y_n = y\}$. Then, $X \cup Y = \mathbb{N}$ and $X \cap Y = \varnothing$.

Proof.
By Lemma 1, $a_x \geq 0 \Leftrightarrow \exists x_n = x$. Thus, $a_x \geq 0 \Leftrightarrow x \in X$. Similarly, by Lemma 2, $a_y < 0 \Leftrightarrow y \in Y$. Since at most one of $a_x \geq 0$ and $a_x < 0$ can be true, for all x, x is in at most one of X and Y. Thus, $X \cap Y = \varnothing$. Also, notice that either $a_x \geq 0$ or $a_x < 0$ for all $x \in \mathbb{N}$, so x must be in at least one of X and Y. Thus, $X \cup Y = \mathbb{N}$. ∎

We claim that $\sum_{n=1}^{\infty} a_{x_n}$ and $\sum_{n=1}^{\infty} a_{y_n}$ are divergent. By Lemma 1 and Lemma 2, x_n are precisely the indices of a_n with nonnegative a_n and y_n are precisely the indices of a_n with negative a_n. Notice that

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |a_{x_n}| + \sum_{n=1}^{\infty} |a_{y_n}| = \sum_{n=1}^{\infty} a_{x_n} - \sum_{n=1}^{\infty} a_{y_n}$$

Assume for contradiction that $\sum a_{x_n}$ and $\sum a_{y_n}$ both converge. Then, by the Algebraic Limit Theorem, $\sum |a_n|$ would converge as well, a contradiction. Thus, at least one of $\sum a_{x_n}$ and $\sum a_{y_n}$ must diverge. Assume $\sum a_{x_n}$ diverges but $\sum a_{y_n}$ converges. Then, since $\sum a_n = \sum a_{x_n} + a_{y_n}$ converges, a_{x_n} must converge as well by the Algebraic Limit Theorem, a contradiction. Similarly, we can show that $\sum a_{x_n}$ converges but $\sum a_{y_n}$ diverges is not possible. Thus, both $\sum_{n=1}^{\infty} a_{x_n}$ and $\sum_{n=1}^{\infty} a_{y_n}$ are divergent. Specifically, since $a_{x_n} \geq 0$ for all n and $a_{y_n} < 0$ for all n, for all M, there exists N_0 and N_1 such that for all $n > N_0$,

$$\sum_{k=1}^{n} a_{x_k} \geq M$$

and for all $n > N_1$,

$$\sum_{k=1}^{n} a_{y_k} \leq -M$$

We now construct the bijection f recursively. Let $s_n = \sum_{k=1}^{n} a_n$ and $s_0 = 0$. Then, if $s_n < c$, let $\alpha_{n+1} = x_{\min\{m : m \in \mathbb{N}, \forall k < m, \alpha_{k+1} \neq x_m\}}$. If $s_n \geq c$, let $\alpha_{n+1} = y_{\min\{m : m \in \mathbb{N}, \forall k < m, \alpha_k \neq y_m\}}$. Notice that by this construction, $\forall k < m$, $\alpha_{n+1} \neq \alpha_k$, and $\forall a \in X$ or $a \in Y$, there is a n such that $\alpha_n = a$. Then, let define f as follows:

$$f : \mathbb{N} \to \mathbb{N}$$

$$n \mapsto \alpha_n$$

We claim that f is a bijection. We start by showing f is an injection. Notice that

$$\alpha_n = \alpha_m \Rightarrow n = m$$

so f is injective. Also, $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$ such that $\alpha_m = n$ since n must be in X or Y by Lemma 3. Thus, f is a bijection and we are done. Now, we must show that this rearrangement indeed converges to c.

15 Precept 5: Quiz

No notes for this precept.

16 Lecture 10: Open, Closed, Compact, Open Covers

Definition (Open). $A \subseteq \mathbb{R}$ is open if and only if for all $x \in A$, there exists $\epsilon > 0$ such that

$$(x-\epsilon, x+\epsilon) \subseteq A$$

Definition (Closed). $A \subseteq \mathbb{R}$ is closed if and only if for every convergent sequence (a_n) such that $a_n \in A$ for all n, we have $\lim a_n \in A$.

Theorem. A is open $\Leftrightarrow$ A^c is closed.

Theorem.

(1) If $A_1, ..., a_n$ is a finite sequence of open sets then $\cap_{i=1}^n A_i$ is open.

(2) If $\{A_s\}$ is a collection of open sets, then $\cup_s A_s$ is open.

Theorem.

(1) If $A_1, ..., a_n$ is a finite sequence of closed sets then $\cup_{i=1}^n A_i$ is closed.

(2) If $\{A_s\}$ is a collection of closed sets, then $\cap_s A_s$ is open.

Definition (Open Cover). Suppose $A \subseteq \mathbb{R}$ and $\{U_s\}$ is a collection of open sets. We say that $\{U_s\}$ is an open cover of A if and only if $A \subseteq \cup_s U_s$. ... definition of subcover and compact

Definition (Compact). A is called compact if every open cover of A has a finite subcover.

Example. $\varnothing$ is compact.

Example. $\{0\} \subseteq \mathbb{R}$ is compact.

Example. If $A \subseteq \mathbb{R}$ is a finite set, then A is compact.

Proof.
Suppose $A = \{x_1, ..., x_n\}$ and $\{U_s\}$ is an open cover of A. Then $x_i \in A \subseteq \cup_S U_s$ for each s. For each, $\exists s_i$ such that $x_i \in U_{s_i}$ and . . . bleh ooof i fell behind

Example. $\mathbb{R}$ is not compact.

Proof.
For $n \in \mathbb{N}$, let $U_n = (-n, n)$. Then $\cup_{n=1}^{\infty} U_n = \mathbb{R}$ so $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$. On the other hand, for any finite subset $S \in \mathbb{N}$, let $M = \max(S)$. We have

$$M + 1 \notin \cup_{n\in S} U_n$$

so $\{U_n\}_{n\in S}$ does not cover $\mathbb{R}$. Therefore, $\mathbb{R}$ is not compact. ■

Remark. Notice that this proof actually applies to any unbounded subset of $\mathbb{R}$.

Definition (Bounded Set). $A \subseteq \mathbb{R}$ is called bounded if $\exists M > 0$ such that $A \subseteq [-M, M]$.

Theorem. If A is compact, then A is bounded.

Proof.

Let $U_n = (-n, n)$ for $n \in \mathbb{N}$. Then $\cup_{n=1}^{\infty} U_n = \mathbb{R}$. So $\cup_{n=1}^{\infty} U_n$ is an open cover of A. Since A is compact, there exists a finite subset of $S \in \mathbb{N}$ such that the collection $\{U_n\}_{n \in S}$ is an open cover of A.

Let $M = \max(S)$. Then

$$A \subseteq \cup_{n \in S} U_n = (-M, M) \subseteq [-M, M]$$

so A is bounded. ■

Theorem. If A is compact, then A is closed.

Proof.
Suppose that $x \in A^c$. we show that there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq A^c$.
For $n \in \mathbb{N}$, let

$$U_n = (-\infty, x - \frac{1}{n}) \cup (x + \frac{1}{n}, +\infty)$$

Consider $\cup_{n=1}^{\infty} U_n$. Then,

$$\cup_{n=1}^{\infty} U_n = \mathbb{R} \backslash \{x\}$$

Since $x \notin A$, we have

$$A \subseteq \cup_{n=1}^{\infty} U_n$$

Since A is compact, there exists a finite subset S of $\mathbb{N}$ such that $A \subseteq \cup_{n \in S} U_n$.
Let $M = \max(S)$. Then

$$\cup_{n \in S} U_n = U_M$$

$$= (-\infty, x - \frac{1}{M}) \cup (x + \frac{1}{M}, +\infty)$$

and we have $A \subseteq (-\infty, x - \frac{!}{M}) \cup (x + \frac{1}{M}, +\infty)$. Then,

$$A^c \supseteq [x - \frac{1}{M}, x + \frac{1}{M}] \subseteq (x - \frac{1}{M}, x + \frac{1}{M})$$

so $(x - \frac{1}{M}, x + \frac{1}{M}) \subseteq A^c$. Hence A^c is open. ■

Theorem. If $A \subseteq \mathbb{R}$ is closed and bounded, then A is compact.

Proof.
The proof is divided into a few steps.

Remark. Although we won't prove it, every open cover has a countable subcover.

Theorem. $[0, 1] \in \mathbb{R}$ is compact.

Proof.
Suppose for contradiction $\{U_s\}$ is an open cover of $[0, 1]$ that has no finite subcover.
Since $[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. $\{U_s\}$ is also an open cover of $[0, \frac{1}{2}]$ and an open cover of $[\frac{1}{2}, 1]$.
Since $\{U_s\}$ has no finite subcover for $[0, 1]$, it has no finite subcover for $[0, \frac{1}{2}]$ or has no finite subcover for $[\frac{1}{2}, 1]$.
Repeating this process, we get a sequence of closed intervals

$$[0, 1] = I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

such that

(1) $\{U_s\}$ has no finite subcover for I_n for all n

(2) length of $I_n = \frac{1}{2^n}$

By the nested interval property, $\cap_{n=0}^{\infty} I_n \neq \varnothing$. Take $x \in \cap_{n=0}^{\infty} I_n$. Since $[0,1] \subseteq U_s U_s$, there exists S_0 such that $x \in U_{s_0}$. Since U_{s_0} is open, $\exists \epsilon > 0$ such that

$$(x-\epsilon, x+\epsilon) \subseteq U_s$$

Take n such that $\frac{1}{2^n} < \epsilon$. Since $x \in I_n$, we have

$$\begin{aligned} I_n &\subseteq [x - \frac{1}{2^n}, x + \frac{1}{2^n}] \\ &\subseteq (x-\epsilon, x+\epsilon) \\ &\subseteq U_s \end{aligned}$$

which is a contradiction since $\{U_s\}$ is a finite subcover. ■

Remark. Note that we can prove that linear functions $f : \mathbb{R} \to \mathbb{R}$ preserve openness and closedness and thus compactness so we can also use that to prove the following theorem.

Theorem. If $a < b$ then $[a, b] \subseteq \mathbb{R}$ is compact.

Theorem. If $A \subseteq \mathbb{R}$ is compact and $B \subseteq \mathbb{R}$ is open. Then $A \backslash B$ is compact.

Proof.
Suppose $\{U_s\}$ is an open cover of $A \backslash B$. Then $\{U_s\} \cup \{B\}$ is an open cover of A. Since A is compact, $\{U_s\} \cup \{B\}$ has a finite subcover $\{U_s\}$ for A.
Then $\{U'\} \backslash \{B\}$ is a finite subcover of $\{U_s\}$ for $A \backslash B$ ■

Theorem. If A is bounded and closed, then A is compact.

To summarize, $A \subseteq \mathbb{R}$ is compact if and only if A is bounded and closed. This theorem is actually an equivalent form of the Axiom of Completeness!
Defintion (Sequentially Compact). $A \subseteq \mathbb{R}$ is called sequentially compact if for any sequence (a_n) with $a_n \in A$ for all n, there exists a subsequence of (a_n) that converges to an element of A.

Theorem. $A \subseteq \mathbb{R}$ is sequentially compact if and only if A is bounded and closed.

Proof.

(1) If A is bounded and closed, we show that A is sequentially compact.

Suppose (a_n) is a sequence such that $a_n \in A$ for all n. Because A is bounded, the sequence (a_n) is also bounded. By the B-W theorem, (a_n) has a convergent subsequence (a_{n_k}). Then, since A is closed,

$$\lim_{k\to\infty} a_{n_k} \in A$$

(2) If A is sequentially compact, then A is bounded.

Assume A is not bounded, then for each $n \in \mathbb{N}$ there exists $a_n \in A$ (we are defining a_n) such that $|a_n| > n$. Consider the sequence (a_n). For every subsetequence $(a_{n_k})_{k\in\mathbb{N}}$, we have

$$|a_{n_k}| > |n_k| = n_k \geq k$$

so $(a_{n_k})_{k\in\mathbb{N}}$ is not bounded, therefore it is not convergent. Therefore A is not sequentially compact.

(3) If A is sequentially compact, then it is closed.

If A is not closed, then there exists a convergent sequence (a_n) such that

$$a_n \in A$$

for all n and

$$\lim a_n \notin A$$

Since (a_n) is convergent, for every subsequence (a_{n_k}) of (a_n) we have

$$\lim_{k\to\infty} a_{n_k} = \lim a_n \notin A$$

Therefore A is not sequentially compact, a contradiction. ■

Definition (Disconnected). $A \subseteq \mathbb{R}$ is called disconnected if $\exists$ open sets $U_1, U_2 \subseteq \mathbb{R}$ such that

(1) $U_1 \cap U_2 = \varnothing$

(2) $U_1 \cap A \neq \varnothing$

(3) $U_2 \cap A \neq \varnothing$

(4) $A \subseteq U_1 \cup U_2$

Definition (Connected). A is connected if and only if it is not disconnected.
Definition (Interval). $A \subseteq \mathbb{R}$ is called an interval if $\forall a < c < b$ such that $a \in A$, $b \in A$, we have $c \in A$.

Theorem. $A \subseteq \mathbb{R}$ is connected if and only if A is an interval.

Example. $\mathbb{Q}$ is not connected.

Proof.
Let $U_1 = (-\infty, \sqrt{2})$, $U_2 = (\sqrt{2}, \infty)$. Then

$$U_1 \cup U_2 = \mathbb{R} \backslash \{\sqrt{2}\} \supseteq \mathbb{Q}$$

■

Proof.

(1) If A is not an interval then it is not connected.

In fact, since A is not an interval, there exits $a < c < b$ such that $a \in A$, $b \in A$, $c \notin A$. Let

$$U_1 = (-\infty, c)$$

$$U_2 = (c, +\infty)$$

Then

$$a \in U_1 \cap A$$

$$b \in U_2 \cap A$$

$$U_1 \cap U_2 = \varnothing$$

$$U_1 \cup U_2 = \mathbb{R} \backslash \{c\} \supseteq A$$

So A is not connected.

(2) If A is an interval, we show that it is connected.

Suppose $\exists U_1, U_2$ such that

$$U_1 \cup U_2 \supseteq A$$

$$U_1 \cap A \neq \varnothing$$

$$U_2 \cap A \neq \varnothing$$

$$U_1 \cap U_2 = \varnothing$$

Let $a \in U_1 \cap A$ and $b \in U_2 \cap A$. Without loss of generality, assume $a < b$. Since A is an interval, we have $[a, b] \subseteq A$. Let

$$S_1 = \{x \in [a, b] \mid x \in U_1\} \subseteq [a, b]$$

$$S_2 = \{x \in [a, b] \mid x \in U_2\} \subseteq [a, b]$$

Then

$$S_1 \cup S_2 = [a, b]$$

$$S_1 \cap S_2 = \varnothing$$

$$a \in S_1$$

$$b \in S_2$$

Let $z = \sup(S_1)$.

17 Lecture 11: FFFF

17.1 Finishing Fundamentals

We will finish up the other definition of the previous theorem:

Proposition. If A is an interval, then A is connected.

Proof.
Suppose for contradiction $\exists U_1, U_2$ such that $U_1 \cap A \neq \varnothing$, $U_2 \cap A \neq \varnothing$, $U_1 \cap U_2 = \varnothing$, and $A \subseteq U_1 \cup U_2$.
Take $a \in U_1 \cap A$ and $b \in U_2 \cap A$. WLOG, we assume that $a < b$.
Define $S_1 = [a, b] \cap U_1 = \{x \in [a, b] \mid x \in U_1\}$ and $S_2 = [a, b] \cap U_2 = \{x \in [a, b] \mid x \in U_2\}$. Then,

$$S_1 \cap S_2 = \varnothing$$

$$S_1 \cap S_2 = [a, b]$$

$$a \in S_1$$

$$b \in S_2$$

Let $x = \sup(S_1)$. Since $a \in S_! \subseteq U_1$ and U_1 is open,

$$\exists \epsilon > 0 \text{ s.t. } (a - \epsilon, a + \epsilon) \subseteq U_1$$

Therefore, $\sup S_1 > a$. Thus, $xneqa$.
Similarly, since $b \in S_2 \subseteq U_2$ and U_2 is open,

$$\exists \epsilon > 0 \text{ s.t. } (b - \epsilon, b + \epsilon) \subseteq U_1$$

Therefore $S_1 \cap (b - \epsilon, b + \epsilon) = \varnothing$, so $\sup S_1 < b$. Thus, $x \neq b$.
If $x \in S_1$, we deduce a contradiction. In fact, if $x \in S_1$, then $x \in U_1$. Since U_1 is open, $\exists \epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq U_1$$

Let $\epsilon_1 = \min\{\epsilon, b - x\}$. Then, $\epsilon_1 > 0$, and

$$[x, x + \epsilon_1) \subseteq (x - \epsilon, x + \epsilon) \subseteq U_1$$

$$[x.x + \epsilon_1) \subseteq [x, b] \subseteq [a, b]$$

$$\Rightarrow [x, x + \epsilon) \subseteq U_1 \cap [a, b] = S_1$$

However this contradicts the assumption that x is the supremum of S_1.
If $x \in S_2$, we deduce a contradiction. Suppose $x \in S_2 \subseteq U_2$, with U_2 open. Then

$$\exists \epsilon > 0, (x - \epsilon, x + \epsilon) \subseteq U_2$$

Since x is an upper bound of S_1 and $S_1 \cup S_2 = [a, b]$, we have

$$(x, b] \subseteq S_2$$

Take $\epsilon_1 = \min\{\epsilon, x - a\}$. Since $x > a$, $\epsilon_1 > 0$, and

$$(x - \epsilon_1, x] \subseteq (x - \epsilon, x + \epsilon) \subseteq U_2$$

$$(x - \epsilon_1, x] \subseteq [a, x] \subseteq [a, b]$$

$$\Rightarrow (x - \epsilon_1, x] \subseteq U_2 \cap [a, b] = S_2$$

Therefore, $(x - \epsilon, b] \subseteq S_2$. Hense,

$$(x - \epsilon_1, b] \cap S_1 = \varnothing$$

so for any $y \in S_1$, we have $y \leq x - \epsilon_1$. This implies that $x - \epsilon_1$ is also an upper bound of S_1, but this contradicts the minimality of x. ■

Theorem. If $A \neq \varnothing \subseteq \mathbb{R}$ is compact, then

(1) $\sup A$ exists

(2) $\sup A \in A$

Reamrk. Specifically, if A is compact, it has a maximum element. By our proof, if $\sup A$ exists and A is closed, then $\sup A \in A$. However, this does not imply that A is compact. We can also use the analogously use the theorem with $\inf$.

Proof.
Since A is compact, it is bounded and closed.
Since $A \neq 0$ and A is bounded, $\sup A$ exists.
Let $x = \sup A$. For each $n \in \mathbb{N}$, $x - \frac{1}{n}$ is not an upper bound of A, so $\exists a \in A$ such that $a_n > x - \frac{1}{n}$. Consider the sequence (a_n). We know that

$$x - \frac{1}{n} < a_n \leq x$$

By the Squeeze Theorem (aka Squeezy Theorem),

$$\lim x - \frac{1}{n} = \lim x = \lim a_n = x$$

Since A is clsoed, this implies $x \in A$. ■

This concludes our discussion of the toplogy of $\mathbb{R}$.

17.2 Funcions Finally

Definition (Function). If $A \subseteq \mathbb{R}$, A function on A is a map from A to $\mathbb{R}$.
We will write $f : A \to \mathbb{R}$ to indicate that f is a function on A.
Definition. Suppose $f : \mathbb{R} \to \mathbb{R}$, and $L \in \mathbb{R}$. Then we say that

$$\lim_{x \to +\infty} f(x) = L$$

if for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

for all $x \geq N$.
Definition. Suppose $f : [a, +\infty) \to \mathbb{R}$ and $L \in \mathbb{R}$. We say that

$$\lim_{x \to +\infty} f(x) = L$$

if for all $\epsilon > 0$ there exists $N \in [a, +\infty)$ such that

$$|f(x) - L| < \epsilon$$

for all $x \geq N$.

Example. Suppose $f : [1, +\infty)$, $f(x) = \frac{1}{x}$. Then $\lim_{x \to +\infty} f(x) = 0$.

Proof.
Suppose $\epsilon > 0$. Take $N = \frac{1}{\epsilon} + 1$.

Remark. Notice that we don't need the Archimedean Principle since N does not need to be a natural number.

Then for all $x \geq N$, we have $x \geq N > 0$ and

$$\frac{1}{x} \leq \frac{1}{N} = \frac{1}{\frac{1}{\epsilon}+1} < \frac{1}{(1/\epsilon)} = \epsilon$$

so $x \in [0, \epsilon) \Rightarrow |x| = x < \epsilon$ as desired. ■

Definition. Suppose $f : (-\infty, a] \to \mathbb{R}$ and $L \in \mathbb{R}$. We say that

$$\lim_{x \to -\infty} f(x) = L$$

if $\forall \epsilon > 0$ $\exists M \in \mathbb{R}$ such that $|f(x) - L| < \epsilon$ for all $x \leq M$.

Definition. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $L \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = L$ if $\epsilon > 0$, $\exists N$ such that $|f(x) - L| < \epsilon$ for all x with $|x| \geq N$.

Proposition. $\lim_{x \to \infty} f(x) = L$ is equivalent to ($\lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$).

We will omit this proof. ■

Theorem. If $\lim_{x \to +\infty} f(x) = A$ and $\lim_{x \to +\infty} g(x) = B$, then

$$\lim_{x \to +\infty} (f(x) + g(x)) = A + B$$

$$\lim_{x \to +\infty} (cf(x)) = cA$$

$$\lim_{x \to +\infty} (f(x)g(x)) = AB$$

and if $B \neq 0$, then $\exists N$ such that $g(x) \neq 0$ for all $x \geq N$, and

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{A}{B}$$

The proof of this is exactly the same proof as the Algebraic Limit Theorem if we replace the analogous things.

Definition (Divergence to $+\infty$). Suppose (a_n) is a sequence. We say that $\lim a_n = +\infty$ if $\forall M > 0$, $\exists N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$. Then we say that (a_n) diverges to $+\infty$.

Theorem. $\lim_{x \to +\infty} f(x)$ converges to L if and only if for every sequence (a_n) such that $\lim a_n = +\infty$,

$$\lim f(a_n) = L$$

Proof.

(1)

Suppose $\lim_{x \to \infty} f(x) = L$ and $\lim a_n = +\infty$. Then $\forall \epsilon > 0$, $\exists N_1$ such that $|f(x) - L| < \epsilon$ for all $x \geq N_1$ and $\exists N_2$ such that $a_n > N_1$ for all $n \geq N_2$. Then for each $n \geq N_2$,

$$a_n > N_1$$

$$|f(a_n) - L| < \epsilon \Rightarrow \lim f(a_n) = L$$

(2)

We will show the contrapositive. Suppose $\lim_{x \to +\infty} f(x)$ does not converge to L. Then, there exists ϵ such that for all N, $\exists x \geq N$ such that

$$|f(x) - L| \geq \epsilon$$

Then for every $n \in \mathbb{N}$, $\exists x_n \geq n$ such that

$$|f(x_n) - L| \geq \epsilon$$

Consider the sequence (x_n) defined by the statement above. Then

$$x_n \geq n$$
$$|f(x_n) - L| \geq \epsilon$$

We claim that $\lim x_n = +\infty$. In fact, for any M, let $N_1 \in \mathbb{N}$ be a natural number such that $N_1 > M$. Then for each $n \geq N_1$, we have

$$x_n \geq n \geq N_1 > M$$

This verifies $\lim x_n = +\infty$. On the other hand, since $|f(x_n) - L| \geq \epsilon$ for all n, by definition $(f(x_n))_{n\in\mathbb{N}}$ does not converge to L. ■

Corollary.

(1) Algebraic Limit Theorem for Functions

(2) Order Limit Theorem for Functions

(3) Squeeze (Squeezy) Theorem for Functions

Corollary. Suppose $\exists (a_n), (b_n)$ such that $\lim a_n = +\infty$, $\lim b_n = +\infty$, $\lim f(a_n) = L_1$, $\lim f(b_n) = L_2$, and $L_1 \neq L_2$. Then $\lim_{x\to+\infty} f(x)$ is not convergent.

Definition (ϵ-δ Definition of Limit for Functions). Suppose $c \in \mathbb{R}$, $r > 0$. $A = (c-r, c) \cup (c, c+r) = (c-r, c+r)\backslash\{c\}$. Suppose $f : A \to \mathbb{R}$, $L \in \mathbb{R}$. We say that $\lim_{x\to c} f(x) = L$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in (c-\delta, c+\delta)\backslash\{c\}$.

Remark. We generally use small Greek letters for small numbers and capitalized English letters for large numbers.

Remark. We actually need to explicitely remove c from $(c-\delta, c+\delta)$ rather than taking the intersection $(c-\delta, c+\delta)$ with the domain of f because even if f is defined at c in a piecewise way so that it is something else, we sitll want the limit to work. For example,

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 100 & x = 1 \end{cases}$$

The reason I was asking for having $(c-\delta, c+\delta) \cap D$ where D is the domain of the function is that what if the function is only defined on the irrationals or something? Then the limit exists nowhere. For example consider the function $f : (\mathbb{R} \to \mathbb{Q}) \to \mathbb{R}$, $x \mapsto 1$.

Theorem. $\lim_{x\to c} f(x) = L$ if and only if for every sequence (a_n) such that $a_n \in (c-r, c+r)\backslash\{c\}$ and $\lim a_n = c$ we have $\lim_{n\to\infty} f(a_n) = L$.

Corollary. We have:

(1) Algebraic Limit Theorem

(2) Order Limit Theorem

(3) Squeeze (Squeezy) Theorem

Corollary. If (a_n), (b_n) are two sequences in $(c-r, c+r)\backslash\{c\}$ such that $\lim a_n = \lim b_n = c$, $\lim f(a_n) = L_1$, $\lim f(b_n) = L_2$, and $L_1 \neq L_2$, then $\lim_{x\to c} f(x)$ is not convergent.

18 Precept 6: Test Review

18.1 Infinite Series

18.1.1 Theorems on the Test

Algebraic Limit Theorem for Series. If $\sum a_n = A$ and $\sum b_n = B$, then

$$\sum a_n + b_n = A + B$$

$$\sum ca_n = cA$$

Cauchy Criterion for Series.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > m \geq N, |a_{m+1} + ... + a_n| < \epsilon$$

$$\Leftrightarrow \sum a_n$$

converges.

Comparison Test. $0 \leq a_n \leq b_n$ for all n means that

$$\sum b_n \text{ conv. } \Rightarrow \exists A, \sum a_n \text{ conv.}$$

$$\sum a_n \text{ div. } \Rightarrow \sum b_n \text{ div.}$$

Remark. On exams, we can use the generalization of the above theorem so that $0 \leq a_n \leq b_n$ only needs to hold true for $n \geq N$.

Absolute Convergence. $\sum a_n$ converges absolutely if and only if $\sum |a_n|$ converges.

Alternating Series. $a_1 \geq a_2 \geq ... \geq a_n \geq ...$, $a_n \to 0$ as $n \to 0$ means that $\sum (-1)^n a_n$ converges.

Remark. On exams, we can use the generalization of the above theorem so that $a_n \geq a_{n+1}$ only needs to hold true for $n \geq N$.

Rearrangement. If $\sum a_n$ converges absolutely, $\sum a_{f(n)}$ converges to the same limit where $f : \mathbb{N} \to \mathbb{N}$ is bijective.

18.1.2 True or False

Exercise. $\sum a_n$ converges absolutely implies that $\sum a_n a_{n+1}$ converges absolutely?

Solution.
True, By the Cauchy Criterion for series, if $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Then, there exists $N \in \mathbb{N}$ such that

$$|a_n| < 1$$

for all $n \geq N$. Thus,

$$|a_n a_{n+1}| \leq |a_{n+1}|$$

and so we use the Comparison Test, we are done (pretty much you should write it up more rigorously though hehe). ■

Exercise. $\sum a_n$ converges then $\sum a_n a_{n+1}$ converges?

Solution.
We have a counterexample: $a_n = \dfrac{(-1)^n}{\sqrt{n}}$. Then,

$$\sum a_n a_{n+1} = \sum \frac{-1}{\sqrt{n(n+1)}}$$

which diverges by Asymptotic comparison with the harmonic series. Obviously write this more rigorously. ■

Remark. We can use the Asymptotic test on the exam (for the previous problem we could have used comparison with $\frac{1}{(n+1)}$.

18.2 Topology of $\mathbb{R}$: Open and Closed Sets

$\mathbb{R}$	Complement	Finite $\cap$	Finite $\cup$	Arbitrary $\cap$	Arbitrary $\cup$
Open	C	O	O	?	O
Closed	O	C	C	C	?

Compactness. For real numbers, all three definitions are equivalent:

(1) Subseq. Conv.

(2) Closed and Bounded

(3) Open Cover

Remark. On the exam, you may use any of the three definitions.

Limit Point. A, x is a limit point if for all $\epsilon > 0$, $\exists a \in A$ such that $|a - x| < \epsilon$.

18.2.1 True or False

Exercise. A, B compact sets then $\{ab : a \in A, b \in B\}$ is compact?

Solution.
This is true. Let $P = \{ab : a \in A, b \in B\}$. Let (p_n) be any sequence in P

Exercise. The intersection of an arbitrary collection of compact sets is compact?

Solution.
This is true. Any arbitrary collection of compact sets is closed, and it is also bounded because it is a subset of any one of the sets which is bounded. Thus, we are done. ■

Exercise. Let A be a compact set, and let $B \neq \varnothing \subset A$. Then the supremum of B exists and $\sup B \in A$.

Solution.
Since A is bounded, B is bounded, so $\sup B$ exists. Also, since there exists (b_n) converging to $\sup B$ in B (we would generally have to prove this), $\sup B$ is a limit point of B, so it must be in A. ■

19 Lecture 12: Limits of Functions

Last time, we defined limits for functions. We will now continue the proof from last class.

Theorem (Sequence Criterion). Let c, r, f be as above (definition of a limit for a function). Then $\lim_{x \to c} f(x) = L$ if and only if the following holds: For every sequence (a_n) such that

$$a_n \in (c - r, c + r) \backslash \{c\} \text{ for all } n$$

$$\lim a_n = c$$

we have $\lim f(a_n) = L$.

Proof.

(1)

If $\lim_{x \to c} f(x) = L$, and (a_n) is a sequence such that

$$a_n \in (c - r, c + r) \backslash \{c\}$$

$$\lim a_n = c$$

we will show that $\lim f(a_n) = L$. For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in (c - r, c + r) \backslash \{c\}$ such that $|x - c| < \delta$, we have

$$|f(x) - L| < \epsilon$$

Also, there exists $N \in \mathbb{N}$ such that $|a_n - c| < \delta$ for all $n \geq N$. By (ii), for $n \geq N$, we have

$$a_n \in (c - r, c + r) \backslash \{c\}$$

$$|a_n - c| < \delta$$

By (i), this implies

$$|f(a) - L| < \epsilon$$

for all $n \geq N$. Thus, $\lim f(a_n) = L$.

(2)

Suppose $\lim_{x \to c} f(x)$ does converge to L. Then $\exists \epsilon > 0$ such that for all $\delta > 0$, $\exists x \in (c - r, c + r) \backslash \{c\}$ such that

$$|x - c| < \delta \text{ and } |f(x) - L| \geq \epsilon$$

For each $n \in \mathbb{N}$, take $\delta = \dfrac{1}{n}$. We conclude that there exists $a_n \in (c - r, c + r) \backslash \{c\}$ such that $|a_n - c| < \dfrac{1}{n}$ and $|f(a_n) - L| \geq \epsilon$. The collection of a_n forms a sequence (a_n) and we have

$$\begin{cases} |a_n - c| < \dfrac{1}{n} \\ a_n \in (c - r, c + r) \backslash \{c\} \\ |f(a_n) - L| \geq \epsilon \end{cases}$$

Since $|a_n - c| < \dfrac{1}{n}$, we have $c - \dfrac{1}{n} < a_n < c + \dfrac{1}{n}$ so $c - \dfrac{1}{n} \leq a_n \leq c + \dfrac{1}{n}$ since $\lim_{n \to +\infty}(c - \dfrac{1}{n}) = \lim_{n \to +\infty}(c + \dfrac{1}{n}) = c$. By the Squeezy theorem,

$$\lim a_n = c$$

since $|f(a) - L| \geq \epsilon$ for all n, by the definition of convergence. Thus, $(f(a_n))_{n \in \mathbb{N}}$ is not convergent to L as desired. ∎

Corollary. The Algebraic Limit Theorem, the Order Limit Theorem, the Squeeze Theorem, and the Uniqueness of limit for functional limits all follow from the above.

Proposition. If $f_1, f_2 : (c-r, c+r)\backslash\{c\} \to \mathbb{R}$ are two functions, suppose $\exists r' \in (0, r)$ such that $f_1 = f_2$ on $(c-r', c+r')\backslash\{c\}$. Then

(1) $\lim_{x\to c} f_1(x)$ converges if and only if $\lim_{x\to c} f_2(x)$ converges

(2) If both $\lim_{x\to c} f_1(x)$ and $\lim_{x\to c} f_2(x)$ converge, we have $\lim_{x\to c} f_1(x) = \lim_{x\to c} f_2(x)$.

Proof.
We first show that

$$\lim_{x\to c}(f_1(x) - f_2(x)) = 0$$

for all $\epsilon > 0$, take $\delta = r'$. Then by the assumption on f_1 and f_2, for all $x \in (c-r, c+r)\backslash\{c\}$ such that $|x - c| < \delta$, we have that

$$f_1(x) = f_2(x)$$

Hence $|(f_1(x) - f_2(x)) - 0| = 0 < \epsilon$. This proves thta

$$\lim_{x\to c}(f_1(x) - f_2(x)) = 0$$

Since $\lim x \to c(f_1(x) - f_2(x)) = 0$, if $\lim_{x\to c} f_1(x)$ is convergent, notice that

$$f_2(x) = f_1(x) - (f_1(x) - f_2(x))$$

so by the algebraic limit theorem, $\lim_{x\to c} f_2(x)$ is convergent and

$$\lim f_2(x) = \lim_{x\to c} f_1(x)$$

Similarly, if $\lim_{x\to c} f_2(x)$ is convergent, then

$$f_1(x) = f_2(x) + (f_1(x) - f_2(x))$$

is also convergent by the algebraic limit theorem so

$$\lim_{x\to c} f_2(x) = \lim_{x\to c} f_1(x)$$

$\blacksquare$

This means we only need to consider a local neighborhood when considering limits for functions. Thus, we can extend the definition of a limit:

Definition. Suppose $f : A \to \mathbb{R}$ is a function and $\exists c \in \mathbb{R}$ with $r > 0$ such that

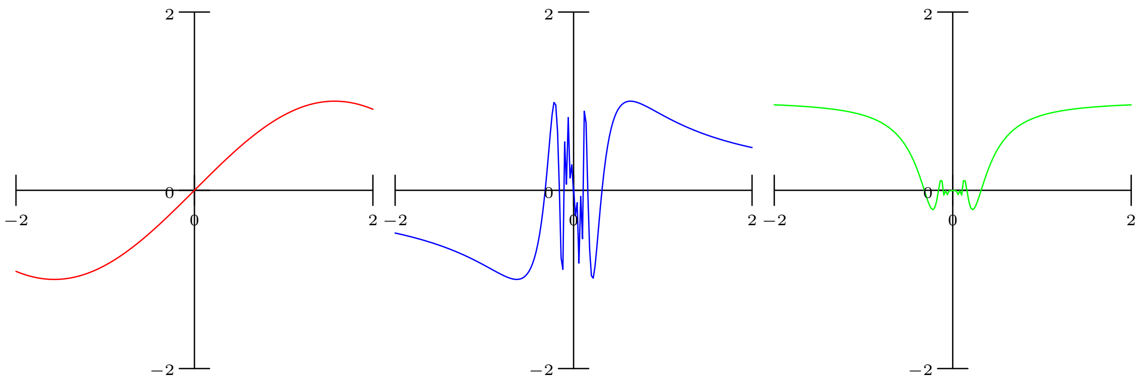
$$(c - r, c + r)\backslash\{c\} \subseteq A$$

Then we say that $\lim_{x\to c} f(x) = L$ if and only if the restriction of f to $(c - r, c + r)\backslash\{c\}$ converges to L as $x \to c$.

Corollary. The definition of $\lim_{x\to c} f(x)$ is independent of the choice of r.

Example. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. We prove that $\lim_{x\to c} f(x) = c$.

Proof.
For all $\epsilon > 0$, take $\delta = \epsilon$. Then for every x such that $|x - c| < \delta$, we have $|f(x) - c| < \epsilon$ so $\lim_{x\to c} f(x) = c$. $\blacksquare$



Since $|\sin(\frac{1}{x})| \leq 1$,

$$|x\sin(\frac{1}{x})| \leq |x|$$

so $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$. We show that $\lim_{x\to 0}|x| = 0$. In fact, for all $\epsilon > 0$, take $\delta = \epsilon$. Then for all x such that $|x-0| < \delta = \epsilon$, this shows that $\lim_{x\to}|x| = 0$. Thus, by the algebraic limit theorem,

$$\lim_{x\to 0}|x| = \lim_{x\to 0} -|x| = 0$$

so by the Squeezy theorem,

$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$$

Example. $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exist.

Proof.

Let $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$. Write $f(x) = \sin(\frac{1}{x})$. Then $f(x_n) = \sin(2\pi n) = 0$ and $f(y_n) = \sin(2\pi n + \frac{\pi}{2}) = 1$ so

$$\begin{cases} \lim x+n = \lim y_n = 0 \\ x_n, y_n \in \mathbb{R}\backslash\{0\} \\ \lim f(x_n) = 0 \\ \lim f(y_n) = 1 \end{cases}$$

By the sequence criterion, $\lim_{x\to 0} f(x)$ is not convergent. ■

Definition. A function $f : A \to \mathbb{R}$ is called *continuous* at $c \in A$ if for all $\epsilon > 0$, $\exists\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Remark. Suppose $A \subseteq (c-r, c+r)$ for some $r > 0$. Then $f : A \to \mathbb{R}$ is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Example. $f(x) = x$ is continuous at all $x \in \mathbb{R}$.

Example.

$$\begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

then $f(x)$ is continuous at $x = 0$.

Theorem. $f : A \to \mathbb{R}$ is continuous at $c \in A$ if and only if the following holds: For every sequence (a_n) such that $a_n \in A$ and $\lim a_n = c$, we have $\lim f(a_n) = f(c)$.

Corollary. If $f, g : A \to \mathbb{R}$ are continuous at $c \in A$, then

(1) $f + g$, kf, $f \cdot g$ are continuous at $x = c$ (where k is a constant)

(2) If $g \neq 0$ on A, then $\frac{f}{g}$ is continuous at $x = c$.

Proposition. Suppose f_1, f_2, $A \to \mathbb{R}$ and $f_1 = f_2$ on $(c - r, c + r) \cap A$ for some $r > 0$. Then, f_1 is continuous at c if and only if f_2 is continuous at c.

Remark. The previous proposition can also be proved using the sequence criterion and the algebraic limit theorem for sequences. It can also be proved using the $\epsilon - \delta$ defintion directly.

Definition. $f : A \to \mathbb{R}$ is called continuous if f is continuous at all $c \in A$.
Definition (Relative Open Set). Suppose $A \subseteq \mathbb{R}$. we say that $U \subseteq A$ is a relative open set (with respect to A) if $\exists$ an open subset $V \subseteq \mathbb{R}$ such that $U = V \cap A$.

Proposition.

(1) If $\{U_s\}$ is a family of relative open sets in A, then $\cup U_s$ is also a relative open set.

(2) If $U_1, ..., U_n$ is a finite family of relative open sets in A, then $\cap_{i=1}^n U_i$ is relatively open.

Proof.

Example. $A = [0, 1]$, $U = (\frac{1}{2}, 1] \subseteq A$ is relatively open (w.r.t. A).

Since U_s is relatively open for each s, there exists $U_S \subseteq \mathbb{R}$ such that U_s is open and

$$U_s = V_s \cap A$$

we have $\cup_s U_s = \cup_s (V_s \cap A) = (\cup_s V_s) \cap A$. Since $\cup V_s$ is open, $\cup U_s$ is relatively open. Similarly, let $V_i \subseteq \mathbb{R}$ be an open subset of $\mathbb{R}$ such that $V_i \cap A = U_i$. Then

$$\begin{aligned} &\cap_{i=1}^n U_i \\ &= \cap_{i=1}^n (V_i \cap A) \\ &= (\cap_{i=1}^n V_i) \cap A \end{aligned}$$

Since $\cap_{i=1}^n V_i$ is open, we have $\cap_{i=1}^n V_i$ is relatively open. ■

Proposition. $U \subseteq A$ is relatively open if and only if for all $x \in U$, $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap A \subseteq U$.

Proof.

(1)

If U is relatively open, then there exists $V \subseteq \mathbb{R}$ an open set such that $U = V \cap A$. Then, for all $x \in U$, $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \leq 1 \Rightarrow (x - \epsilon, x + \epsilon) \cap A \subseteq V \cap A = U$.

(2)

If for each $x \in A$, there exists $\epsilon_x > 0$ such that $(x-\epsilon_x, x+\epsilon_x) \cap A \subseteq U$. Let $V = \cup_{x\to U}(x-\epsilon_x, x+\epsilon_x)$. Then V is open. We claim that $V \cap A = U$. In fact, for each $x \in U$, we have $x \in (x-\epsilon_x, x+\epsilon_x) \subseteq V$ and $x \in U \subseteq A$ so $x \in V \cap A$. This proves that $V \cap A \supseteq U$. On the other hand, for all $x \in U$, by the definition of ϵ_x we have

$$(x-\epsilon_x, x+\epsilon_x) \cap A \subseteq U$$

so

$$\cup_{x\in U}((x-\epsilon_x, x+\epsilon_x) \cap A) \subseteq U$$

$$(\cup_{x\in U}(x-\epsilon_x, x+\epsilon_x)) \cap A = V \cap A \subseteq U$$

we have $V \cap A = U$ ■

Theorem. $f : A \to \mathbb{R}$ is continuous if and only if for every open set $U \subseteq \mathbb{R}$ we have $f^{-1}(U)$ is relatively open in A.

Proof.
Recall that

$$f^{-1}(U) = \{x \in A \mid f(x) \in U\}$$

If for every open set $U \subseteq |R$, $f^{-1}(U)$ is relatively open, we show that f is continuous. Let $c \in A$. For all $\epsilon > 0$, take $U = (f(c)-\epsilon, f(x)+\epsilon)$. By the assumption, $f^{-1}(U)$ is relatively open. Since $f(x) \in U$, $c \in f^{-1}(U)$. By the previous property of relatively open sets, $\exists\delta > 0$ such that

$$(x-\delta, c+\delta) \cap A \subseteq f^{-1}(U)$$

This means that for all $x \in (c-\delta, c+\delta) \cap A$, we have $f(X) \in U$ and hence $|f(x) - f(x)| < \epsilon$. If f is continuous, let $U \subseteq \mathbb{R}$ be an open set, let $c \in f^{-1}(U)$ we have $f(c) \in U$. So $\exists\epsilon > 0$ such that $(f(x)-\epsilon, f(x)+\epsilon) \subseteq U$. Since f is continuous at c, $\exists\delta >$ such that for all $x \in (c-\delta, c+\delta) \cap A$ we have $|f(x) - f(c)| < \epsilon$ so $f(x) \in (f(c)-\epsilon, f(c)+\epsilon) \subseteq U$, so $x \in f^{-1}(U)$ for all $x \in (c-\delta, c+\delta) \cap A$. Since c can be an arbitrary element in $f^{-1}(U)$, by the previous property of relative open sets, we have that $f^{-1}(x)$ is relatively open. ■

Remark. "Definition bashing" is also known as a tautological proof.

20 Lecture 13: Properties of Continuous Functions

Theorem. If $f : A \to \mathbb{R}$ is continuous, and A is compact, then $f(A)$ is compact.

Proof.
We show that every open cover of $f(A)$ has a finite subcover. Let $\{U_s\}$ be an open cover of $f(A)$. Then $A \subseteq \cup U_s$. Then,

$$\forall x \in A, \exists s \text{s.t.} f(x) \in U_s$$

Then for any $a \in A$, there exists U_s such that $a \in f^{-1}(U_s)$. Thus,

$$A \in \cup f^{-1}(U_s)$$

By the previous theorem, $f^{-1}(U_s)$ is relatively open to A for all s. Thus, there exists V_s for each U_s such that $f^{-1}(U_s) = V_s \cap A$. Thus,

$$A \subseteq \cup_s f^{-1}(U_s) \subseteq \cup_s V_s$$

Therefore, $\{V_s\}$ is an open cover of A, which means that $\{V_s\}$ has a finite subcover. Suppose $\{V_1, ..., V_n\}$ is a finite subcover of A. Let $\{U_1, ..., U_n\}$ be the corresponding open sets in $\{U_s\}$. Then we have

$$A \subseteq \cup_{i=1}^n V_i \Rightarrow A = (\cup_{i=1}^n V_i) \cap A = \cup_{i=1}^n (V_i \cap A) = \cup_{i=1}^n f^{-1}(U_i)$$

so $\forall a \in A$, $\exists U_i$ such that $a \in f^{-1}(U_i)$ which means that $\forall a \in A$, there exists U_i such that $f(a) \in U_i$, and that means that $f(A) \subseteq \cup_{i=1}^n U_i$ so $\{U_1, ..., U_n\}$ is a finite subcover of $\{U_s\}$ so $f(A)$ is compact as desired. ■

Remark. This sort of proof that doesn't really depend on much is useful because it is easily generalized. In fact, this actually holds for multivariable functions as well.

Corollary. If $f : A \to \mathbb{R}$ is continuous and A is compact, then $\exists x \in A$ such that

$$f(y) \leq f(x)$$

for all $y \in A$.

Proof.
By the previous theorem, $f(A)$ is compact. By a previous theorem, $\sup f(A)$ exists, and

$$\sup(f(A)) \in f(A)$$

So $\exists x \in A$ such that

$$f(x) = \sup(f(A))$$

Therefore, $f(x)$ is an upper bound of $f(A)$ so $\forall y \in A$, we have $f(y) \leq f(x)$. ■

Remark. Similarly, if $f : A \to \mathbb{R}$ is continuous and A is compact, then there exists $x \in A$ such that $f(x) \leq f(y)$ for all $y \in A$.

Example. Consider $f(x) = x$ on $A = (0, 1)$. Then $f(A) = (0, 1)$, so $f(x)$ has no maximum or minimum value on $(0, 1)$.

Theorem. If $f : A \to \mathbb{R}$ is continuous, and if A is connected, then $f(A)$ is connected.

Recall that by our definition, a set S is called disconnected if there exists open sets U_1, U_2 such that

$$U_1 \cap U_2 = \varnothing$$

$$U_1 \cup U_2 \supseteq S$$
$$U_1 \cap S \neq \varnothing$$
$$U_2 \cap S \neq \varnothing$$

Lemma. A set S is disconnected if there exists relatively open subsets $A, B \subseteq S$ such that

$$A \cap B = \varnothing$$

$$A \cup B = S$$

$$A \neq \varnothing, B \neq \varnothing$$

21 Precept 6: Test Reflection

No notes here!

22 Lecture 14: More Continuity

Recall these very important theorems.

Theorem. $f : A \to \mathbb{R}$ is continuous if and only if for every open set $U \subseteq \mathbb{R}$ we have $f^{-1}(U)$ is relatively open in A.

Theorem. If $f : A \to \mathbb{R}$ is continuous, and A is compact, then $f(A)$ is compact.

Theorem. If $f : A \to \mathbb{R}$ is continuous, and if A is connected, then $f(A)$ is connected.

Corollary. If $f : A \to \mathbb{R}$ is continuous and $A \neq \varnothing$ is compact, then f has a maximum value and a minimum value in A.

A More Direct Proof (Non-topological).

We will only consider the maximum value for now, since they are essentially the same proof. We show that $f(A)$ is bounded from above. Assume for the sake of contradiction that $f(A)$ is not bounded above. Then, for every $M \in \mathbb{R}$, $\exists x \in A$ such that $f(x) > M$. Then, for every $n \in \mathbb{N}$, exists $x_n \in A$ such that $f(x_n) > n$. Since A is compact, it is also sequentially compact, so there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = x \in A$. Recall that we have a sequence criterion for continuity. Applying that, by the definition of (x_n), we have $f(x_{n_k}) > n_k$ so $(f(x_{n_k}))_{k\in\mathbb{N}}$ is an unbounded sequence which yields a contradiction.

Thus, $f(A)$ is bounded from above, so it must have a supremum. Let $\sup f(A) = y$. By the definition of supremum, for each $n \in \mathbb{N}$, $y - \dfrac{1}{n}$ is not an upper bound of $f(A)$, so $\exists x_n \in A$ such that $f(x_n) > y - \dfrac{1}{n}$. Thus,

$$y - \frac{1}{n} < f(x_n) \leq y$$

Using the squeezy theorem, $\lim f(x_n) = y$. Then, by the definition of continuity and sequentially compact, $y = f(\lim x_{n_k})$ for some subsequence. ∎

Remark. We have been discussing infinite sets with topology, but what about finite sets?

Remark. In the definition of continuity, if we let $\delta = g(\epsilon)$, is g continuous? What properties does g satisfy?

Corollary (Intermediate Value Theorem). If $f : [a, b] \to \mathbb{R}$ is continous, $f(a) > y$, $f(b) < y$, then $\exists x \in [a, b]$ such that $f(x) = y$.

Remark. We should actually use $\exists x \in (a, b)$ since it's stronger.

A More Direct Proof (Non-topological).

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < y$, $f(b) > y$. There exists $x \in [a, b]$ such that $f)x_y$.

Let $S = \{x \in [a, b] \mid f(x) < y\}$. Then $S \subseteq [a, b]$ so S is bounded above and $a \in S$. Then, $\sup S$ exists. Let $x_0 = \sup S$.

We show that $x_0 \neq a$, and $x_0 \neq b$. Since $f(a) < y$ and f is continuous at a, take $\epsilon = y - f(a) > 0$, then $\exists \delta > 0$ such that for each $x \in [a, a + \delta)$ we have $|f(x) - f(a)| < \epsilon$. Then,

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

$$f(x) < f(a) + \epsilon = y$$

Thus, for every $x \in [a, a+\delta)$, $f(x) < y$, so $[a, a+\delta) \subseteq S$, so $\sup S \geq a+\delta > a$. A similar argument shows that $\sup S \neq b$.

We deduce a contradiction if $f(x_0) < y$. If $f(x_0) < y$, then let $\epsilon = y - f(x_0) > 0$. $\exists \delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, we have $|f(x) - f(x_0)| < \epsilon$. Then $f(x) < f(x_0) + \epsilon = y$ for all $x \in (x_0 - \delta, x_0 + \delta)$, which is a contradiction since $x_0 + \delta$ would be in S, contradicting the definition that $x_0 = \sup S$. A similar argument shows that $f(x_0) > y$ is not true. Thus, $f(x_0) = y$. ■

Example (Dirichlet Function).

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Then by the density of $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$, f is discontinuous everywhere on $\mathbb{R}$.

Example (Riemann Function). $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{q} & xq \in \mathbb{Z} \text{ and } q \text{ is minimal} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Then, $f(x)$ is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{R} \backslash \mathbb{Q}$.

Proof.

Suppose $c \in \mathbb{Q}$. Then, $f(x) > 0$. For each $n \in \mathbb{N}$, let x_n be an irrational number in $(c - \frac{1}{n}, c + \frac{1}{n})$. Then, by the squeezy theorem, $\lim x_n = c$. Since $f(x_n) = 0$ for all n, $f(c) > 0$. We have $\lim f(x_n) = 0 \neq f(x)$ so f is not continuous at c. Suppose $c \in \mathbb{R} \backslash \mathbb{Q}$. Then $f(x) \geq \frac{1}{n}$ if and only if $x = \frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \{1, 2, ..., n\}$. For each $q \in \{1, 2, ..., n\}$, let $S_q = \{x \in (c-1, c+1) \mid x = \frac{p}{q} \text{ for } p \in \mathbb{Z}\}$.

Remark. This graph is really similar to the super cool graph with the $v_2(x)$ function I have in Desmos!

Then S_q is finite. Let $S = \cup_{q=1}^{n} S_q$. Then $S \subseteq \mathbb{Q}$. So $0 \notin S$. Let $\delta = \min\{|c - x| \mid x \in S\}$. Since S is finite and $c \notin S$, δ exists and is positive. Let $x \in (c - \delta, c + \delta$. Then, $|x - c| < \delta$. This implies that $x \notin S$. So x cannot be written as $\frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \{1, 2, ..., n\}$. This implies that $|f(x)| = f(x) < \frac{1}{n}$.

Remark. You cannot find a function that is continuous at rational points and discontinuous at irrational points. This can be proved by the Baire category theorem.

Definition (Monotone). f is increasing if $x_1 \geq x_2 \Rightarrow f(x_1) \geq f(x_2)$, and f is decreasing if $x_1 \geq x_2 \Rightarrow f(x_1) \leq f(x_2)$. f is monotone if it is either increasing or decreasing.

Example. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then $S = \{c \in \mathbb{R} \mid f \text{ is discontinuous at } c\}$ is countable.

Sketch of proof: Let $c \in S$. WLOG, assume f is increasing. Let $c \in S$. Since f is monotone, $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist (PSET). If f is not continuous at c, we have

$$\lim_{x \to c^+} f(x) > \lim_{x \to c^-} f(x)$$

Let $I_c = (\lim_{x \to c^-} f(c), \lim_{x \to c^+} f(x)) \subseteq \mathbb{R}$. Then $I_c \neq \varnothing$. It is possible to prove that if $c \neq c' \in S$, then $I_c \cap I_{c'} = \varnothing$. By a past PSET problem, $\{I_c \mid c \in S\}$ is countable. Thus S is countable.

Definition (Derivative). Suppose $f : (c - r, c + r) \to \mathbb{R}$ is a function. Define

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Remark. Notice that $\dfrac{f(x)-f(c)}{x-c}$ is only defined on $(c-r,c+r)\setminus\{c\}$, which is why we define the limit of a function on that domain.

Definition (Differentiable). If $f'(c)$ exists, we say that f is differentiable at c.

Theorem. Suppose $f,g:(c-r,c+r)\to\mathbb{R}$ are continuous and differentiable at c. Then $(fg)'=f'g+fg'$ at $x=c$.

Proof.

$$\begin{aligned}\frac{f(x)g(x)-f(c)g(c)}{x-c}&=\frac{f(x)g(x)-f(x)g(c)}{x-c}+\frac{f(x)g(c)-f(c)g(c)}{x-c}\\&=f(x)\frac{g(x)-g(c)}{x-c}+g(c)\frac{f(x)-f(c)}{x-c}\end{aligned}$$

so by the assumptions, the result follows from the algebraic limit theorem. ■

23 Lecture 15: Differentiability

Proposition. Suppose f is differentiable at c. Then, f is continuous at c.

Proof.

Recall that f is continuous of c if and only if $\lim f(x) = f(c)$. We have $f(x) = f(c) + \frac{f(x) - f(c)}{x - c}(x - c)$ when $x \neq c$. Then,

$$\begin{aligned}\lim_{x \to c} f(x) &= \lim_{x \to c}(f(c) + \frac{f(x) - f(c)}{x - c}(x - c)) \\ &= f(c) + f'(x) \cdot 0 = f(c)\end{aligned}$$

■

Proposition. If f, g are differentiable at c and a is constant, then $f - g$, kf, and $f \cdot g$ are all differentiable at c, and

$$(f + g)'(c) = f'(c) + g'(c)$$

$$(kf)'(c) = kf'(c)$$

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

Proof.
Notice that

$$\lim_{x \to c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = f'(c) + g'(c)$$

Also,

$$\lim_{x \to c} \frac{kf(x) - kf(x)}{x - c} = k \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = kf'(x)$$

Finally, we showed the last part last time. ■

Proposition. Suppose $g(x)$ is differentiable at $x = c$ and $g(c) \neq 0$. Then

(1) $\exists \delta > 0$ such that $g(x) \neq 0$ for every $x \in (c - \delta, c + \delta)$.

(2) $\frac{1}{g}$ is differentiable at g.

(3) $(\frac{1}{g})'|_{x-c} = -\frac{g'(c)}{g^2(c)}$

Proof.
Since g is differentiable at $x = c$, it is continuous. Take $\epsilon = |g'(c)| > 0$. Then $\exists \delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, we have

$$|g(x) - g(c)| < \epsilon$$

Then

$$\begin{aligned}|g(c)| &\leq |g(x)| + |g(c) - g(x)| \\ &= |g(x)| + |g(x) - g(c)| \\ &< |g(x)| + \epsilon \\ &= |g(x)| + |g(c)|\end{aligned}$$

on $(c - \delta, c + \delta)$. Thus, $|g(x)| > 0$ on $(c - \delta, c + \delta)$. Therefore, $g(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$.

Then,

$$\frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x - c} = \frac{(\frac{g(c) - g(x)}{g(x)g(c)})}{x - c}$$

$$= \frac{1}{g(x)g(c)} \frac{g(c) - g(x)}{x - c}$$

$$= -\frac{1}{g(x)g(c)} \frac{g(x) - g(c)}{x - c}$$

Thus using the algebraic limit theorem,

$$\lim_{x \to c} \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x - c} = -\frac{1}{g(c)^2} g'(c)$$

$\blacksquare$

Corollary. If f, g are differentiable at c and $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at c and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ at $x = c$.

Proof.

Theorem. Suppose f is differentiable at c and g is differentiable at $f(c)$. Then $(g(f(x)))' = g'(f(c)) \cdot f'(c)$.

A (not completely correct) Proof.

$$\frac{g(f(x)) - g(f(c))}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim_{t \to f(c)} \frac{g(t) - g(f(c))}{t - f(c)} = g'(f(c))$$

so

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = g'(f(c)) \cdot f'(c)$$

End of Fake Proof

Errors with the proof:

- $f(x)$ cannot be $f(c)$ in order for us to put it in the denominator.
- Defining the $t = f(x)$ and doing that is actually ok, but t is actually able to be $f(c)$ which fails for the same reason as above.

Proof.
Define

$$d(t) = \begin{cases} \dfrac{g(t) - g(f(c))}{t - f(c)} & t \neq f(c) \\ g'(f(c)) & t = f(c) \end{cases}$$

By the definition of g', we have

$$\lim_{t \to f(c)} d(t) = g'(f(c)) = d(f(c))$$

so $d(t)$ is continuous at $t = f(c)$.

By one of the homework problems, if $u(c) = \lim_{x \to c} u(x) = a$, and $v(x)$ is continuous at a, then

$$\lim_{x \to c} v(u(x)) = v(a)$$

We show that

$$(*) \frac{g(f(x)) - g(f(c))}{x - c} = d(f(x)) \cdot \frac{f(x) - f(c)}{x - c}$$

We discuss (disusses hehe) two cases. If $f(x) \neq f(x)$, then $d(f(x)) = \dfrac{g(f(x)) - g(f(c))}{f(x) - f(c)}$ and $(*)$ holds. If $f(x) = f(c)$, then the left hand side of $(*)$ is equal to 0 which is equal to the right hand side.

By the homework problem and the continuity of $d(t)$ at $t = f(c)$, we have

$$\lim_{x \to c} d(f(x)) = d(f(c)) = g'(f(c))$$

By the assumption on f,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

so

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = g'(f(c))f'(c)$$

■

Example. $f(x) = k$ for some constant k. Then, $\dfrac{f(x) - f(c)}{x - c} = \dfrac{k - k}{x - c} = 0$ so $f'(c) = 0$ for all c.

Example. $f(x) = x$. Then,

$$\frac{f(x) - f(c)}{x - c} = \frac{x - c}{x - c} = 1$$

so $f'(c) = 1$ for all c.

Example. $f(x) = x^2$, $f(x)' = (x \cdot x)' = (x')x + x(x') = x + x = 2x$.

Example. $f(x) = x^n$ for $n \in \mathbb{N}$. Then, $f'(x) = nx^{n-1}$. This is shown easily with induction (proof omitted here).

This example wasn't stated amazingly, so we can make a proposition:

Proposition. Let $f(x) = x^n$ for $n \in \mathbb{N}$. Then, $f(x)$ is differentiable everywhere on $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

Example. $f(x) = x^{1/n}$ (x >0) for $n \in \mathbb{N}$. In a homework problem this week, we will prove that $f(x)$ is differentiable on $(0, +\infty)$.
By the chain rule and the previous proposition,

$$1 = (x)' = (f(x)^n)' = nf(x)^{n-1} \cdot f'(x) = nx^{\frac{n-1}{n}} f'(x)$$

$$\Rightarrow f'(x) = \frac{1}{n} / x^{1 - \frac{1}{n}} = \frac{1}{n} x^{\frac{1}{n} - 1}$$

Example. Let $f(x) = x^{\frac{m}{n}}$ for $m, n \in \mathbb{N}$. Then, by the chain rule,

$$(f(x))' = ((x^{\frac{1}{n}})^m)'$$

$$= m \cdot (x^{\frac{1}{n}})^{m-1} \cdot (x^{\frac{1}{n}})'$$

$$= m \cdot x^{\frac{m-1}{n}} \cdot \frac{1}{n} \cdot x^{\frac{1}{n} - 1}$$

$$= \frac{m}{n} x^{\frac{m}{n} - 1}$$

Thus, we formulate the following:

Proposition. If $p \in \mathbb{Q}$, then
$$(x^p)' = px^{p-1}$$

Proof.
We only consider $p < 0$ since we proved for $p > 0$. Then,
$$(x^p)' = (\frac{1}{x^{|p|}})'$$
$$= -\frac{(x^{|p|})'}{x^{2|p|}} = -\frac{|p|x^{|p|-1}}{x^{2|p|}}$$
If $p = 0$, then $(x^p)' = 0 = px^{p-1}$ as desired. ■

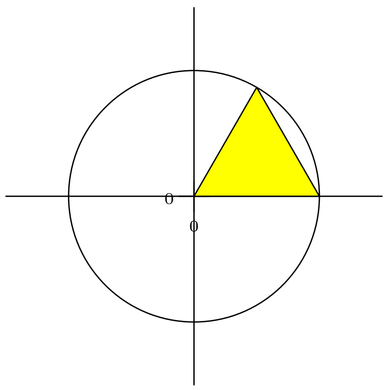
We haven't defined irrational exponents, but this formula actually works.

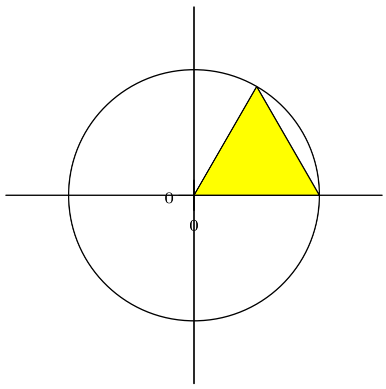
23.1 Semirigorous Discussion of $\sin$ and $\cos$

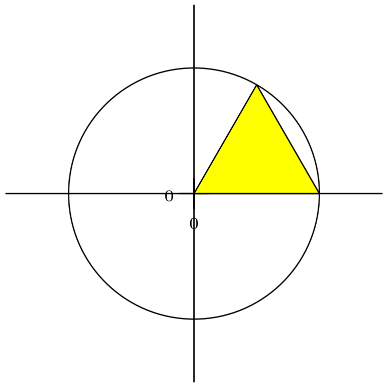
We will take the summation formulas for granted. Specifically,
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

Lemma. If $x \in [0, \frac{\pi}{2}]$, then $\sin x \leq x \leq \tan x$.

Semi-rigorous proof.







Lemma. $\lim_{x\to 0} \sin x = 0$

Proof.
Since $\sin x$ is an odd function, whne $|x| \leq \frac{\pi}{2}$ we have $|\sin x| = \sin|x| \leq |x|$. So $-|x| \leq \sin x \leq |x|$ and $\lim_{x\to 0} \sin x = 0$ by the squeezy theorem. ∎

Lemma. $\lim_{x\to 0} \cos x = 1$

Proof.
When $|x| \leq \frac{\pi}{2}$, we have $\cos x \geq 0$, so $\cos x = \sqrt{1 - \sin^2 x}$ (taken for granted). By the algebraic theorem for square root (shown in homework),

$$\lim_{x\to 0} \cos x = \sqrt{1 - 0^2} = 1$$

∎

Lemma. $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Proof.
If $x \in (0, \frac{\pi}{2})$ we have

$$\sin x \leq x \leq \tan x$$
$$\Rightarrow \frac{\sin x}{x} \leq 1, x \leq \frac{\sin x}{\cos x}$$
$$\Rightarrow \cos x \leq \frac{\sin x}{x} \leq 1$$

By the squeezy theorem,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Since sin is od, $\frac{\sin x}{x}$ is even. Thus,

$$\lim_{x\to 0^-} \frac{\sin x}{x}$$

is also equal to 1. ∎

Proposition. $\sin x$ is differentiable and $\sin' x = \cos x$.

Proof.

We need to study $\dfrac{\sin x - \sin x}{x - c}$. Let $d = x - c$.

$$\frac{\sin x - \sin c}{x - c} = \frac{\sin(c+d) - \sin c}{d}$$

$$= \frac{\sin(c)\cos(d) + \cos(c)\sin(d) - \sin(c)}{d}$$

$$= \cos(c)\frac{\sin(d)}{d} + \sin(c)\frac{\cos(d) - 1}{d}$$

$$\lim_{x\to c} \frac{\sin x - \sin x}{x - c} = \lim_{d\to 0}(\cos c \frac{\sin d}{d} + \sin c \frac{\cos d - 1}{d}$$

$$\cos(d) = \cos(\frac{d}{2} + \frac{d}{2}) = \cos^2\frac{d}{2} - \sin^2\frac{d}{2} = 1 - 2\sin^2\frac{d}{2}$$

so

$$\frac{\cos d - 1}{d} = -\frac{2\sin^2\frac{d}{2}}{d}$$

$$\lim_{d\to 0} \frac{2\sin^2(\frac{d}{2})}{d} = \lim_{d\to 0}(\frac{\sin(d/2)}{d/2}\sin(d/2))1 \cdot 0 = 0$$

Proposition. $\cos'(x) = -\sin x$

24 Precept 7: Example Problems

25 Lecture 16: Applications of Derivatives

Definition (Maximum). Suppose $f : A \to \mathbb{R}$ is a function and $x \in A$. We say that x is a maximum point of f if for all $y \in A$, we have $f(y) \le f(x)$.

Definition (Minimum). Suppose $f : A \to \mathbb{R}$ is a function and $x \in A$. We say that x is a minimum point of f if for all $y \in A$, we have $f(y) \ge f(x)$.

Definition (Maximum). Suppose $f : A \to \mathbb{R}$ is a function and $x \in A$. We say that x is a maximum point of f if there exists $r > 0$ such that for all $y \in (x-r, x+r) \cap A$, we have $f(y) \le f(x)$.

Definition (Maximum). Suppose $f : A \to \mathbb{R}$ is a function and $x \in A$. We say that x is a minimum point of f if there exists $r > 0$ such that for all $y \in (x-r, x+r) \cap A$, we have $f(y) \ge f(x)$.

Theorem. If $f : (c-r, c+r) \to \mathbb{R}$ is differentiable at c and if c is a local minimum or a local maximum, then $f'(c) = 0$.

Remark. I think this combined with a past theorem proves Rolle's theorem.

Proof.
WLOG assume c is a local maximum. Then, $\exists r' \in (0, r]$ such that c is a maximum of f on $(c-r', c+r')$. By the assumption, f is differentiable at c, so $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ converges and

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

We have $\lim_{x\to c^+} \dfrac{f(x)-f(c)}{x-c} = \lim_{x\to c^-} \dfrac{f(x)-f(c)}{x-c} = f'(c)$. Since c is a maximum on $(c-r', c+r')$, we have $f(x) \le f(c)$ for all $x \in (c-r', c+r')$ so $\dfrac{f(x)-f(c)}{x-c} \le 0$ on $(c, c+r')$ and $\dfrac{f(x)-f(c)}{x-c} \ge 0$ on $(c-r', c)$.

Remark. Notice IVT doesnt work here hehehe.

By the order limit theorem,

$$\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0$$

$$\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$$

so $f'(c) \le 0$ and $f'(c) \ge 0$. Therefore, $f'(c) = 0$. ∎

Through the following examples, we can see that f needs to be differentiable at c and c needs to be on the interior of an interval.

Example. $f(x) = |x|$. Then $x = 0$ is a minimum point, but it can be proved that $f(x)$ is not differentiable at $x = 0$.

Example. $f(x) = x$ on $[0, 1]$. Then $x = 1$ is a maximum point but f' is not defined on the boundary (even if we extend the definition of f' to the boundary we would have $f'(1) = 1 \ne 0$).

Theorem (Rolle's Theorem). Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Remark. YOOOO Predicted B-)

Proof.

Since f is continuous on $[a,b]$ and $[a,b]$ is compact. f has a minimum and maximum values. Of $\max_{x\in[a,b]} f(x) = \min_{x\in[a,b]} f(x)$, then f is a constant function so $f'(x) = 0$ for all $x \in [a,b]$. If $\max_{x\in[a,b]} f(x) \neq \min_{x\in[a,b]} f(x)$, then $\max_{x\in[a,b]} f(x) \neq f(a) = f(b)$ or $\min_{x\in[a,b]} f(x) \neq f(a) = f(b)$. WLOG assume $\max_{x\in[a,b]} f(x) \neq f(a) = f(b)$. Then, $\exists c \in (a,b)$ such that $f(c) = \max_{x\in[a,b]} f(x)$. By the previous theorem we have $f'(c) = 0$.

Theorem (Lagrangian Mean Value Theorem). Suppose f is continuous on $[a,b]$ and differentiable on (a,b) (with $a < b$). Then $\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Remark. The Lagrangian MVT recovers Rolle's theorem when $f(a) = f(b)$. HAHAHA I was originally being dumb and thinking $a = b$ hehehehe.

Proof.
Let $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a} \cdot x$. Then, $g(x)$ is continuous on $[a,b]$, and differentiable on (a,b). Also, $g(b) - g(a) = [f(b) - \dfrac{f(b)-f(a)}{b-a} \cdot b] - [f(a) - \dfrac{f(b)-f(a)}{b-a} \cdot a] = (f(b) - f(a)) - \dfrac{f(b)-f(a)}{b-a} \cdot (b-a) = 0$. Therefore $g(a) = g(b)$. By Rolle's theorem, $\exists c \in (a,b)$ such that $g'(c) = 0$, and since $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ as desired. ■

Corollary. If f is differentiable on (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant function on (a,b).

Suppose.
For an arbitrary pair $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Then by the Lagrangian MVT, there exists $t \in (x_1, x_2)$ such that $f'(t) = \dfrac{f(x_2)-f(x_1)}{x_2 - x_1}$. Since $f'(x) = 0$ on (a,b), we have $f(x_2) - f(x-1) = (x_2 - x_1)f'(t) = (x_2 - x_1) \cdot 0 = 0$. ■

Corollary. If f is differentiable on (a,b) and $f'(x) \geq 0$ for all $x \in (a,b)$, then f is an increasing function.

Proof.
Suppose $x_1, x_2 \in (a,b)$ and $x_1 < x_2$. Then $\exists t \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(t)$$

Since $f' \geq 0$ on (a,b), we have

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(t) \geq 0$$

■

Remark. If $f'(x) > 0$ for all $x \in (a,b)$ then for all pairs $x_1, x_2 \in (a,b)$ such thtat $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Example.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Then $f'(x) = 0$ whenever f is differentiable, but f is not a constant function.

,

Example. Use the Lagrangian MVT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$.

Proof.
Let $a_n = n^{1-p} - (n+1)^{1-p}$ (inspired by the integral test). Since $p > 1$, a_n is a positive sequence. We have

$$a_1 + a_2 + ... + a_n = 1^{1-p} - 2^{1-p} + 2^{1-p} - 3^{1-p} + ... + n^{1-p} - (n+1)^{1-p} = 1 - (n+1)^{1-p}$$

Since $p > 1$, $\lim_{n\to\infty}(n+1)^{p-1} = 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent.

Using the Lagrangian MVT, given $f(x) = x^{1-p}$, $f'(x) = (1-p)x^{-p}$, then $\forall n \in \mathbb{N}$, $\exists c_n \in (n, n+1)$ such that $f'(c_n) = \frac{f(n+1) - f(n)}{n+1-n} = f(n+1) - f(n)$. This implies that $f'(c_n) = f(n+1) - f(n) = (n+1)^{1-p} - n^{1-p}$. Thus,

$$(1-p)c_n^{-p} = (n+1)^{1-p} - n^{1-p} \Rightarrow c_n^{-p} = \frac{1}{1-p}((n+1)^{1-p} - n^{1-p})$$

$$= \frac{1}{p-1}(n^{1-p} - (n+1)c_n^{1-p})$$

Since $c_n \in (n, n+1)$, we have $n^{-p} > c_n^{-p} > (n+1)^{-p}$. This implies

$$(n+1)^{-}p < \frac{1}{p-1}(n^{1-p} - (n+1)^{1-p} = \frac{1}{p-1}a_n$$

Recall that $\sum_{n=1}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty}(\frac{1}{p-1}a_n)$ is also convergent. By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^p}$ is convergent, which implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also convergent. ■

Theorem (Cauchy's Mean Value Theorem). Suppose f, g are continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that

$$(f(b) - f(a)g'(c) = (g(b) - g(a))f'(c)$$

This is AKA the Extended Mean Value Theorem.

Remark. If we take $g(x) = x$, then we get Lagrangian Mean Value Theorem because $f'(c)(b-a) = f(b) - f(a)$.

Theorem (L' Hopitol). Suppose f, g are defined on $(c, c+r)$ such that $\lim_{x\to c^+} f(x) = 0$ and $\lim_{x\to c^+} g(x) = 0$ and $g(x) \neq 0$, $g'(x) \neq 0$ on $(c, c+r)$ and $\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$.

Remark. We cannot use L' Hopital cannot be used if $\lim_{x\to c^+} \frac{f'(x)}{g'(x)}$ does not exist.

Definition. We sat that $\lim_{x\to c^+} F(x) = +\infty$ if for every M, $\exists\delta$ such that $f(x) > M$ for all $x \in (c, c+\delta)$.

Theorem. If f, g are differentiable on $(c, c+r)$ and $\lim_{x\to c^+} g(x) = +\infty$ and $g(x) \neq 0$, $g'(x) \neq 0$ on $(c, c+r)$. $\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$.

Remark. Studies show that L'Hopitol might have been rich and interested in math so he paid Bernoulli to get the work.

26 Investigation

Investigation. One thing we discussed in class is how Cauchy's MVT can be shown with Lagrangian MVT.

Theorem.

27 Lecture 17: Sequences of Functions

Definition (Pointwise Convergence). Suppose $A \subseteq \mathbb{R}$, $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions on A, and f is another function on A. We say that f_n pointwise converges to f if for every $x \in A$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.

Example. Let $f_n(x) = \frac{x}{n}$, $f(x) = 0$. Then $f_n(x)$ converges to $f(x)$ pointwise.

Example. Let $A = [0, 1]$. Let $f_n(x) = x^n$. Then

$$\lim_{n\to-\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Example. Define

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Remark. $f_n(x) \to f(x)$ pointwise if and only if for all $x \in A$, for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, we have $|f_n(x) - f(x)| < \epsilon$.

Definition (Uniform Convergence). Suppose $(f_n)(n_\in\mathbb{N}$ is a sequence of functions on A and f is a function on A. We say that (f_n) uniformly converges to f if for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in A$, we have $|f_n(x) - f(x)| < \epsilon$.

Example. Let $f_n(x) = \frac{x}{n} : \mathbb{R} \to \mathbb{R}$, $f(x) = 0 : \mathbb{R} \to \mathbb{R}$. Then $f_n(x)$ converges to $f(x)$ pointwise but not uniformly. On the other hand, if we let $A = [0, 1]$ and consider $f_n(x) = \frac{x}{n} : A \to A$, $f(x) = 0 : A \to A$, then $f_n(x$ converges to $f(x)$ uniformly.

Theorem. Let $f_n : A \to \mathbb{R}$, $f : A \to \mathbb{R}$ be functions. If

- f_n is continuous for every $n \in \mathbb{N}$
- f_n converges to f uniformly

Then f is continuous.

Proof.
For each $c \in A$, $\epsilon > 0$, we show that $\exists\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in A$ with $|x - c| < \delta$.
By the triangle inequality, $|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$. Since $f_n \to f$ uniformly, there exists N such that for all $n \geq N$ and all $x \in A$, we have $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. Take $n = N$. We have $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in A$. Since $f_N(x)$ is continuous, $\exists\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, we have $|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$.
Therefore, for every $x \in A$, with $|x - c| < \delta$ we have

$$|f(x) - f(c)| < |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

■

Remark. By the same proof, if $f_n \to f$ uniformly, and $c \in A$, suppose f_n is continuous at c for all $n \in \mathbb{N}$, then f is continuous at c.

Remark. (f_n) converges pointwise if and only if $(f_n(x))_{n\in\mathbb{N}}$ is convergent for all $x \in A$ which is true if and only if $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy for all $x \in A$ which is true if and only if for each $x \in A$, and every $\epsilon > 0$, there exists N such that for all $m, n > N$, we have $|f_m(x) - f_n(x)| < \epsilon$. It turns out that if we make the N not dependent on x, then we get a Cauchy Criterion for Uniform Convergence.

Theorem (Cauchy Criterion). $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if for every $\epsilon > 0$, $\exists N$ such that for all $m, n \geq \mathbb{N}$ and all $x \in A$, we have $|f_m(x) - f_n(x)| < \epsilon$.

Proof.
Assume (f_n) converges uniformly. Suppose f is the uniform limit of (f_n). Then, for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{\epsilon}{2}$$

for all $n \geq N$. Now, for each pair m, n with $m, n \geq N$, we have

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Syooise that the Cauchy Criterion for Uniform Convergence holds. Then, for each $x \in A$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy so $(f_n(x))_{x\in A}$ is convergent. Therefore, $\exists$ a function f such that $f_n \to f$ pointwise. We show that the convergence is also uniform. For each $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $x \in A$ and $m, n \geq N$, we have

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2}$$

Fix the value of n and let $m \to +\infty$. Since $\lim_{m\to+\infty} f_m(x) = f(x)$, we have $\lim_{m\to+\infty} |f_m(x) - f_n(x)| = |f(x) - f_n(x)|$. If $n, m \geq N$, we have

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \leq \frac{\epsilon}{2}$$

By the Order Limit Theorem, this implies

$$|f(x) - f_n(x)| \leq \frac{\epsilon}{2} < \epsilon$$

for all $n \geq N$, $x \in A$. This implies that $f_n \to f$ uniformly. ■

Example. One application of this is to take limits of differentiable functions that converge to non-differentiable functions. Let $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ and $f(x) = |x|$. Then $f_n(x) > f(x)$, and

$$\begin{aligned} f_n(x) - f(x) &= \sqrt{x^2 + \frac{1}{n^2}} - \sqrt{x^2} \\ &= \frac{1/n^2}{\sqrt{x^2 + \frac{1}{n^2}} + \sqrt{x^2}} \end{aligned}$$

Therefore, $0 \leq f_n(x) - f(x) = \dfrac{1/n^2}{\sqrt{x^2 + \frac{1}{n^2}} + \sqrt{x^2}} \leq \dfrac{1/n^2}{\sqrt{0 + \frac{1}{n^2}} + \sqrt{0^2}} = \dfrac{1}{n}$. For all $\epsilon > 0$, take $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Then, for all $n \geq N$, we have

$$|f_n(x) - f(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Therefore $f_n \to f$ uniformly.

Theorem. Suppose A is an open interval and (f_n) is a sequence of functions on A such that $f_n \to f$ pointwise. Assume

(1) f_n is differentiable for every n

(2) $(f'_n)_{n\in\mathbb{N}}$ converges uniformly to some g.

Then f is differentiable and $f' = g$.

Proof.

For each $c \in A$, we show that $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c} = g(c)$. By definition, for each $\epsilon > 0$ we need to show that $\exists\delta > 0$ such that $|\dfrac{f(x)-f(c)}{x-c} - g(c)| < \epsilon$ for all $x \in (c-\delta, c+\delta)$. By the triangle inequality,

$$|\frac{f(x)-f(c)}{x-c} - g(c)| \le |\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}| + |g(c) - f'_n(c)| + |f'_n(c) - \frac{f_n(x)-f_n(c)}{x-c}|$$

Fix $\epsilon > 0$. Since f'_n converges to g uniformly, there exists N such that $|f'_n(x) - g(x)| < \dfrac{\epsilon}{6}$ for all $n \ge N$ and $x \in A$. Take $n = N$. We have $|g(c) - f'_N(c)| < \dfrac{\epsilon}{6}$. Since f_N is differentiable, $\exists\delta > 0$ such that

$$|f'_N(c) - \frac{f_N(x)-f_N(c)}{x-c}| < \frac{\epsilon}{3}$$

for all $x \in (c-\delta, c+\delta)\backslash\{c\}$. Apply the Lagrangian MVT to $(f_m - f_N)$. We have

$$\frac{f_m(x)-f(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}$$
$$= \frac{(f_m(x)-f_N(x)) - (f_m(c)-f_N(c))}{x-c}$$
$$= f'_m(t) - f'_N(t)$$

for some t between x and c. When $m \ge N$, we have

$$|f'_m(t) - g(t)| < \frac{\epsilon}{6}$$

we have

$$|f'_N(t) - g(t)| < \frac{\epsilon}{6}$$

for all t. So by the triangle inequality, $|f'_m(t) - f'_N(t)| < \dfrac{\epsilon}{3}$. Therefore

$$|\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}| < \frac{\epsilon}{3}$$

for all $m \ge N$. Since $f_m \to f$ pointwise, we have

$$\lim_{m\to+\infty} |\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}|$$
$$= |\frac{f(x)-f(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}|$$

By the order limit theorem, we have

$$|\frac{f(x)-f(c)}{x-c} - \frac{f_N(x)-f_N(c)}{x-c}| \le \frac{\epsilon}{3}$$

In conclusion, for all $x \in (c-\delta, c+\delta)\backslash\{c\}$, $|\dfrac{f(x)-f(c)}{x-c} - g(c)| < \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} + \dfrac{\epsilon}{6} < \epsilon$ ■

28 Precept 8: Mean Value Theorems

28.1 Mean Value Theorems

Rolle's Theorem. Given a function $f : [a, b] \to \mathbb{R}$ with f continuous on $[a, b]$ and differentiable on (a, b), then $f(a) = f(b) \Rightarrow \exists c \in (a, b), f'(c) = 0$.

Lagrangian Mean Value Theorem. Given a function $f : [a, b] \to \mathbb{R}$ with f continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Cauchy Mean Value Theorem. Given a function $f : [a, b] \to \mathbb{R}$ and a function $g : [a, b] \to \mathbb{R}$ with f continuous on $[a, b]$ and differentiable on (a, b) and g continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$, $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.

L'Hopital's Rule.

(1) If f, g are continuous in a neighborhood of a, f and g are differentiable in the neighborhood (possibly not at a), and $f(a) = g(a) = 0$, and $g'(x) \neq 0$ for $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

(2) If f, g are differentiable in a neighborhood of a (possibly not at a), and $\forall x \in (a, b), \lim_{x \to a} g(x) = \infty$, and $g'(x) \neq 0$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Remark. Notice that $g'(x) \neq 0$ only for $x \neq a$. If $x = a$, $g'(a)$ can be zero.

Proof of (2).
For all $\epsilon > 0$, $\exists \delta > 0$ such that if $a < x < a + \delta$,

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$$

Let $t = a + \delta$, and apply Cauchy's Mean Value Theorem: $\forall a < x < t = a + \delta$, $\exists x < c < t \Rightarrow a < c < a + \delta$,

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(t)}{g(x) - g(t)} \Rightarrow L - \frac{\epsilon}{2} < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \frac{\epsilon}{2}$$

Multiply, $\dfrac{g(x) - g(t)}{g(x)} > 0$ (take $a < x < a + \delta_2$ such that $g(x) > g(t)$, $g(x) > 0$.

$$L - \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)}$$

Take $a < x < a + \delta_3$, such that if $a < x < a + \min\{\delta_1, \delta_2, \delta_3\}$,

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

■

Remark. In both parts, if $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = \infty$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \infty$.

True or False. $f : [0,2] \to \mathbb{R}$ continuous and twice differentiable on $(0,2)$. $\exists c \in (0,2)$ such that $f''(c) = f(0) - 2f(1) + f(2)$.

Solution.

Tihs is true. Consider the special case: $g(0) = g(1) = g(2)$. By Rolle's theorem, $\exists 0 < a < 1 < b < 2$ such that $g'(a) = g'(b) = 0$. Thus, there exists $a < c < b$ with $g''(c) = 0$. Let $g(x) = f(x) - Ax^2 - Bx$ with $g(0) = g(1) \Rightarrow f(1) = f(0) = A + B$. $g(2) = g(1) \Rightarrow f(2) - f(1) = 3A + B$. Thus, $A = \dfrac{1}{2}(f(0) - 2f(1) + f(2))$, $B = \dfrac{3}{2}(f(0) - 2f(1) + f(2))$.

29 Investigation

Investigation. Can we generalize the last true or false question from class using Finite Differences (to create a generalized mean value theorem)?

30 Lecture 18: Convergence of Functions and Power Series

30.1 Convergence

Definition (Series Convergence). Suppose $f_n : A \to \mathbb{R}$ is a sequence of functions. Define $S_n = f_1 + f_2 + ... f_n$. We say that $\sum_{n=1}^{\infty} f_n$ is pointwise convergent if (s_n) is pointwise convergent. We say that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent if (s_n) is uniformly convergent.

Theorem. If f_n is continuous for all n and $\sum_{n=1}^{\infty} f_n$ is uniformly convergent to f, then f is also continuous.

Theorem. If $\sum_{n=1}^{\infty} f_n$ is pointwise convergent, every f_n is differentiable, and $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent, then $\sum_{n=1}^{\infty} f_n$ is differentiable and

$$(\sum_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} f_n'$$

Theorem (Cauchy Criterion for the Uniform Convergence of Functions). $\sum_{n=1}^{\infty} f_n$ is uniformly convergent if and only if for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all m, n with $m \geq n \geq N$, we have

$$|\sum_{k=n}^{m} f_k(x)| < \epsilon$$

for all x.

Theorem (Weierstrass M-test) Suppose there exists a sequence of positive numbers (M_n) such that

(1) $|f_n(x)| \leq M_n$ for all x and n

(2) $\sum M_n$ is convergent

Then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

Proof.
Take $\epsilon > 0$. By the Cauchy Criterion, $\sum_{n=1}^{\infty} M_n$, we have that $\exists N \in \mathbb{N}$ such that for all m, n with $m \geq n \geq N$, the inequality

$$|\sum_{k=n}^{m} M_k| < \epsilon$$

holds. By assumption 1,

$$|\sum_{n=1}^{\infty} f_n(x)| \leq \sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} M_k < \epsilon$$

so this series satisfies the Cauchy Criterion for the uniform convergence of functions. Thus, $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent. ∎

30.2 Power Series

Consider a series of functions with $f_n = a_n x^n$. The series is then written as $\sum_{n=0}^{\infty} a_n x^n$. By definition, $\sum_{n=0}^{\infty} a_n x^n$ is the limit of

$$a_0$$

$$a_0 + a_1 x$$

$$a_0 + a_1 x + a_2 x^2$$

$\vdots$

Theorem. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is convergent at x_0. Then for every x such that $|x| < |x_0|$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

Proof.
Since $|x| < |x_0|$, $x_0 \neq 0$. Since

$$|a_n x^n| = |a_n x_0^n|\left|\frac{x}{x_0}\right|^n$$

Since $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, we have $\lim_{n\to+\infty} a_n x_0^n = 0$. So $\exists M > 0$ such that $|a_n x_0^n| \leq M$ for all n. We have

$$- \leq |a_n x^n| \leq M\left|\frac{x}{x_0}^n\right|$$

Since $0 \leq |\frac{x}{x_0}| < 1$, we have $\sum_{n=0}^{\infty} |\frac{x}{x_0}|^n$ is convergent. By comparison test, $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent. ■

Remark. Let $S = \{x \mid \sum a_n x^n\}$ is convergent. Then $x_0 \in S$ implies that $x \in S$ for all x such that $|x| < |x_0|$. This means that $S = (-\infty, +\infty)$ or has the form

$$(-R, R)$$

$$(-R, R]$$

$$[-R, R)$$

$$[-R, R]$$

for some $R \geq 0$. We also have $0 \in S$ trivially.

Definition. If $S = (-R, R), (-R, R], [-R, R)$, or $[-R, R]$, we sa that $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R. If $S = (-\infty, +\infty)$, we say that the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $+\infty$.

Corollary. Suppose R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Then the series is absolutely convergent at every $x \in (-R, R)$.

Theorem. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent at x_0 and let $c = |x_0|$. Then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-c, c]$.

Proof.
Suppose $x \in [-c, c]$, then

$$|a_0 x^n| \leq |a_n| \cdot |c|^n = |a_n x_0^n|$$

by the assumption that $\sum_{n=0}^{\infty} |a_n x_0^n|$ is convergent. Therefore, using the Weierstrass M test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly. ■

Corollary (Uniform Convergence on Compact Subsets). Suppose R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $c \in [0, R)$. Then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-c, c]$.

Theorem. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is convergent at x_0, then $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is absolutely convergent for all x such that $|x| < |x_0|$.

Lemma. Suppose $r > 1$. Then

$$\lim_{n\to+\infty} \frac{n}{r^n} = 0$$

Proof.
Let $u = r - 1 > 0$. We prove that

$$(1+u)^n \geq 1 + nu + \frac{n(n-1)}{2}u^2$$

for all $n \in \mathbb{N}$ (motivated by the Binomial formula) using induction. If $n = 1$, we get

$$1 + u = 1 + u$$

so the statement holds. Suppose that the statement holds for $n = k$. For $n = k + 1$, we have

$$(1+u)^{k+1} = (1+u)^k \cdot (1+u) \geq (1+ku+\frac{k(k-1)}{2}u^2)(1+u) = 1+u+ku+ku^2+\frac{k(k-1)}{2}u^2+\frac{k(k-1)}{2}u^3)$$

$$= 1 + (k+1)u + (k^2 + \frac{k(k-1)}{2})u^2 + \frac{k(k-1)}{2}u^3$$

$$\geq 1 + (k+1)u + \frac{(k+1)k}{2}u^2 = 1 + (k+1)u + \frac{(k+1)k}{2}u^2$$

so the statement holds for $n = k + 1$. By the statement,

$$\frac{n}{r^n} = \frac{n}{(1+u)^n} \leq \frac{n}{1 + nu + \frac{n(n-1)}{2}u^2}$$

When $n \geq 2$,

$$\frac{n}{r^n} \leq \frac{n}{\frac{n(n-1)}{2}u^2} = \frac{2}{n-1}\frac{1}{u^2}$$

Since $\lim_{n\to\infty} \dfrac{1}{n-1} = 0$ and $0 \leq \dfrac{n}{r^n} \leq \dfrac{2}{n-1}\dfrac{1}{u^2}$, the Squeezy theorem implies that

$$\lim_{n\to\infty} \frac{n}{r^n} = 0$$

□

Since $|x| < |x_0|$, we have $x_0 \neq 0$.

$$|na_n x^{n-1}| = |a_n x_0^n| \cdot |\frac{x^{n-1}}{x_0^{n-1}}| \cdot \frac{1}{|x_0|} \cdot n$$

Remark. We can take $|\dfrac{x^{n-1}}{x_0^{n-1}}|n$ goes to 0 by the previous lemma. However, we can also do the following.

Take s such that $|x| < s < |x_0|$. We have

$$|na_{n-1}x^{n-1}| = |a_n x_0^n| \cdot |\frac{x^{n-1}}{x_0^{n-1}}| \cdot \frac{1}{|x_0|} \cdot n$$

$$= \frac{1}{|x_0}|a_n x_0^n| \cdot |\frac{x}{s}|^{n-1}(|\frac{s}{x_0}|^{n-1} \cdot n)$$

Since $\sum -n = 0^\infty a_n x_0^n$ is convergent,

$$\lim_{n\to\infty} a_n x_0^\infty = 0$$

So $\exists M_1$ such that $|a_n x_0^n| \leq M_1$ for all n. By the previous lemma,

$$\lim_{n\to\infty} (|\frac{s}{x_0}|^{n-1} n)$$

$$= \lim_{n\to\infty} (|\frac{s}{x_0}|^{n-1}(n-1)\frac{n}{n-1})$$

$$= 0 \cdot 1 = 0$$

So $\exists M_2$ such that $|\frac{s}{x_0}|^{n-1} \cdot n \le M_2$ for all n. So $|na_nx^{n-1}| \le \frac{1}{|x_0|}M_1M_2|\frac{x}{s}|^{n-1}$. Since $\sum_{n=1}^{\infty}|\frac{x}{s}|^{n-1}$ is convergent, by the comparison test, $\sum_{n=0}^{\infty}|na_nx^{n-1}|$ is also convergent. ■

Remark. We cannot use L'Hopitol's to prove the Lemma because we don't know the derivative of r^x, which actually ends up needing e^x, which we get through power series, sousing L'Hopitol's would be circular logic.

Corollary. If R_1 is the radius of conergence of $\sum a_nx^n$, R_2 is the radius of convergence of $\sum_{n=1}^{\infty} na_nx^{n-1}$, then $R_2 \ge R_1$.

Theorem. Suppose $\sum_{n=1}^{\infty} na_nx^{n-1}$ is convergent at x_0. Then for all x such that $|x| < |x_0|$, the series $\sum_{n=0}^{\infty} a_nx^n$ is absolutely convergent.

Proof.
When $n \ge 1$,

$$a_nx^n = |na_nx_0^{n-1}| \cdot |\frac{x^{n-1}}{x_0^{n-1}}||x||\frac{1}{n}|$$

Since $\sum na_nx_0^{n-1}$ is convergent,

$$\lim_{n\to\infty} na_nx_0^{n-1} = 0$$

So $\exists M$ such that $|na_nx_0^{n-1}| \le M$ for all n, so $|a_nx^n| \le M \cdot |\frac{x^{n-1}}{x_0^{n-1}}|x||\frac{1}{n}| \le |x|M|\frac{x^{n-1}}{x_0^{n-1}}|$. Since $\sum_{n=1}^{\infty}|\frac{x}{x_0}|^{n-1}$ is convergent, we have $\sum_{n=1}^{\infty}|a_nx^n|$ is convergent. ■

Corollary. If R_1 is the radius of conergence of $\sum a_nx^n$, R_2 is the radius of convergence of $\sum_{n=1}^{\infty} na_nx^{n-1}$, then $R_2 = R_1$.

Corollary. Suppose $\sum_{n=0}^{\infty} a_nx^n$ has radius of convergence R. Then for each $c \in [0, R)$, the series

$$\sum_{n=0}^{\infty} na_nx^{n-1}$$

is uniformly convergent on $[-c, c]$. Therefore $\sum_{n=0}^{\infty} a_nx^n$ is differentiable on $(-c, c)$ and $(\sum_{n=0}^{\infty} a_nx^n)' = (\sum_{n=1}^{\infty} na_nx^{n-1})$.

Remark. This implies that $\sum_{n=0}^{\infty} a_nx^n$ is differentiable on $(-c, c)$ for all $c \in [0, R)$ so $\sum_{n=0}^{\infty} aa_nx^n$ is differentiable on $(-R, R)$.

Corollary. For each $k \in \mathbb{N}$, $\sum_{n=0}^{\infty} a_nx^n$ is k-differentiable on $(-R, R)$ and $(\sum_{n=0}^{\infty} a_nx^n)^{(k)} = \sum_{n=0}^{\infty}(a_nx^n)^{(k)}$.

Example. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$. The radius of convergence is 1 and $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ on $(-1, 1)$.

$$\Rightarrow (\sum_{n=0}^{\infty} x^n)' = (\frac{1}{1-x})' = \frac{1}{(1-x)^2}$$

on $(-1, 1)$. At $x = \dfrac{1}{2}$, we have $1 + 2 \cdot \dfrac{1}{2} + 3 \cdot \dfrac{1}{2}^2 + ... = 4$. Replace x with $(-x^2)$ for $x \in (-1, 1)$.

$$1 - x^2 + x^4 - x^6 + x^8 - ... = \frac{1}{1+x^2}$$

Define $f(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots$. Then $f(x)$ has the same raius of convergence as $1 - x^2 + x^4 - x^6 + ...$ and $f'(x) = 1 - x^2 + x^4 - ...$ on $(-R, R)$. We can show that $R = 1$, and hence $f'(x) = \dfrac{1}{1+x^2}$ on $(-1, 1)$. Since $(\arctan x)' = \dfrac{1}{1+x^2}$, $\exists c$ such that $f(x) = \arctan x + c$ on $(-1, 1)$. Plug in $x = 0 \Rightarrow c = 0$. Thus,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$$

on $(-1, 1)$.

31 Lecture 19: Taylor Series

Consider $\sum_{n=0}^{\infty} a_n x^n$. Let R be its radius of convergence. Assume $R > 0$. Define $f(x) = \sum_{n=1}^{\infty} a_n x^n$ on $(-R, R)$. Then f is differentiable infinitely many times and its derivatives are power series.

Corollary. If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are two power series with positive radii of convergence. Suppose $\exists \epsilon > 0$ such that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \text{ on } (-\epsilon, \epsilon)$$

Then $a_n = b_n$ for all n.

Definition (Taylor Series). Suppose $f(x)$ is a function such that $f^{(k)}(0)$ is defined for all k. Define $a_k = \dfrac{f^{(k)}(0)}{k!}$. Then the series $\sum_{n=0}^{\infty} a_n x^n$ is called the "Taylor Series" of f.

Definition. Define

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

(By definition, $x^0 = 1$).

Theorem (Lagrange's Remainder Theorem). Assume $x > 0$ and f is $(n+1)^{\text{th}}$-order differentiable on $(-\epsilon, x+\epsilon)$ for some $\epsilon > 0$. Then $\exists c \in (0, x)$ such that $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$.

Remark. When $n = 0$, $R_0(x) = f(x) - f(0) = f'(c) \cdot x$ which is the same as the Lagrange's Mean Value Theorem.

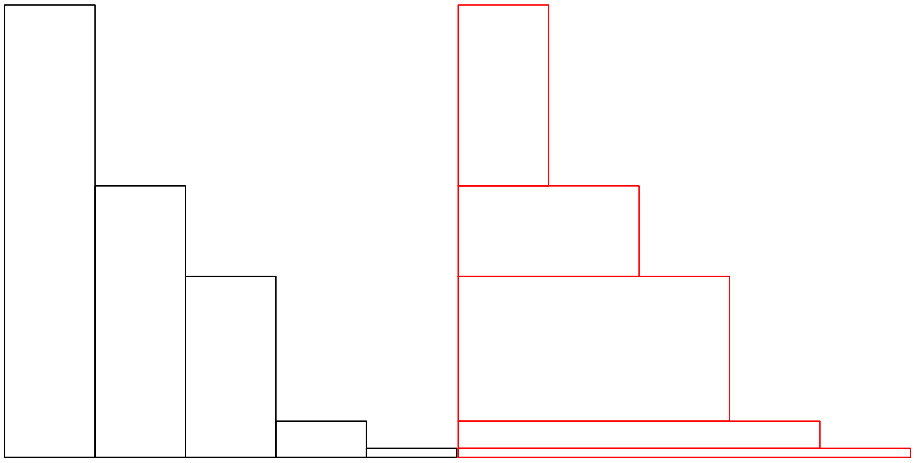
Proof.

$R_n(x) = f(x) - \sum_{k=0}^{n} \dfrac{f^{(k)}(0)}{k!} x^k$. At $x = 0$, $R_n(0) = f(0) - \dfrac{f(0)}{0!} = 0$. Notice that $(x^m)^{(k)} = m(m-1)\ldots(m-k+1)x^{m-k}$, so $(x^m)^{(k)}|_0$ is 0 when $m \neq k$ and $k!$ when $m = k$. For each $k \in \{0, 1, ..., n\}$, $\sum_{k=0}^{n} \dfrac{f^{(m)}(0)}{m!} (x^m)^{(k)}|_{x=0} = (\dfrac{f^{(k)}(0)}{k!} x^k)^{(k)}|_{x=0} = f^{(k)}(0...$

$$R_n(0) = R_n'(0) = ... = R_n^{(k)}(0) = 0$$

Now, we apply the Cauchy Mean Value Theorem . . .

32 Lecture 20:



Corollary. Suppose $\exists A, B > 0$ such that

$$|a_1 + a_2 + ... + a_i| \leq A$$

for all i and

$$B \geq b_1 \geq b_2 \geq b_3 \geq ... \geq 0$$

Then $|\sum_{i=1}^{n} a_i b_i| \leq AB$.

Theorem. Consider $g(x) = \sum_{n=0} \infty a_n x^n$.

(1) If $g(x)$ is convergent at $x = R > 0$, then (x) is uniformly convergent on $[0, R]$.

(2) If $g(x)$ is convergent at $x = -R < 0$, then $g(x)$ is uniformly convergent on $[-R, 0]$.

Proof.
Since $\sum_{k=0}^{\infty} a_k R^k$ is convergent, for all $\epsilon > 0$, $\exists N$ such that for $s \geq t \geq N$,

$$|\sum_{k=t}^{s} a_k R^k| < \frac{\epsilon}{2}$$

It must also follow that

$$|\sum_{k=t}^{s} a_k R^k| \leq \frac{\epsilon}{2}$$

Suppose $m \geq n \geq N$. Then for each $x \in [0, R]$,

$$\sum_{k=n}^{m} a_k x^k) = \sum_{k=n}^{m} (a_k R^k) \cdot (\frac{x}{R})^k$$

Since $x \in [0, R]$, $(\frac{x}{R})^k$ is decreasing and ≤ 1. By a previous statement,

$$|\sum_{k=n}^{s} a_k R^k| \leq \frac{\epsilon}{2}$$

for all $s \in \{n, n+1, ..., m\}$. Apply the previous corollary with $A = \frac{\epsilon}{2}$, $B = 1$. We must have

$$|\sum_{k=n}^{m} (a_k R^k)(\frac{x}{R})^k| \leq \frac{\epsilon}{2}$$

Then $|\sum_{k=n}^{m} a_k x^k| \leq \frac{\epsilon}{2} < \epsilon$ for all $x \in [0, R]$ and all m, n such that $m \geq n \geq N$. Therefore, by the Cauchy Criterion for Uniform Convergence, $g(x)$ is uniformly convergent on $[0, R]$. ■

Remark. Previously we showed that if $g(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent at $R > 0$, then for all $x \in (0, R)$, $g(x)$ is uniformly and absolutely convergent on $[0c, c]$.
This does not imply $g(x)$ is uniformly convergent on $(-R, R)$.

Remark. Using Abel's Theorem, we get

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} + \dots$$

which follows from $\arctan x = \sum \frac{x^{2n+1}}{2n+1}$.

32.1 Integration

Convention. Let $[a, b]$ be a fixed closed interval and let $f : [a, b] \to \mathbb{R}$ be a bounded function.
Definition (Partition). A partition P of $[a, b]$ is a finite subset of $[a, b]$ that includes both a and b.

33 Lecture 21:

Let P be a partition of $[a, b]$. Let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be the elements of P.

$$m_k = \inf_{x \in \{x_1, x_2, \ldots x_n\}} (f(x))$$

$$M_k = \sup_{x \in \{x_1, x_2, \ldots x_n\}} (f(x))$$

We define the upper and lower Riemann sums as:

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

Last, time, we proved that if P_1 and P_2 are two partitions, then

$$U_(f, P_1) \geq L(f, P_2)$$

This was used to show that

$$\inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$
$$\geq \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

Definition. f is called Reimann-integrable if $inf\{U(f, P)\} = \sup\{L(f, P)\}$. In this case, define $\int_a^b f(x)\, dx = \inf\{U(f, P)\} = \sup\{L(f, P)\}$.

Theorem. Suppose f is a bounded function on $[a, b]$. Then f is integrable if and only if for all $\epsilon > 0$, $\exists$ a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Proof.

We start with the first direction. For each $\epsilon > 0$, let P_ϵ be the partition of the interval from $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Let $U(f) = \inf\{U(f, P) \mid P \text{ is a prtition of } [a, b]\}$. Let $L(f) = \sup\{L(f, P) \mid P \text{ is a prtition of } [a, b]\}$. Then

$$U(f) \leq U(f, P_\epsilon)$$
$$L(f) \geq L(f, P_\epsilon)$$

so

$$U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

This shows that

$$U(f) - L(f) < \epsilon$$

for all $\epsilon > 0$, so since $U(f) \geq L(f)$, $U_f = L_f$. Therefore, f is integrable.

We will now suppoes f is integrable. Since f is integrable, $U(f) = L(f)$. Thus, for all $\epsilon > 0$, $\exists$ a partition P_1 such that

$$U(f, P_1) < U(f) + \frac{\epsilon}{2}$$

and there exists a partiion P_2 such that $L(f_1, P_2) > L(f) - \dfrac{\epsilon}{2}$. Let $P = P_1 \cup P_2$. We proved last time that

$$U(f, P) \leq U(f, P_1)$$
$$L(f, P) \geq L(f, P_2)$$

Thus,

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_2)$$

$$< (U(f) + \frac{\epsilon}{2}) - (L(f) - \frac{\epsilon}{2}) = \epsilon$$

■

Remark. Can't we just define a functional sequence or sequence that just converges to the integral by taking systematic partitions now? Or taken the intervals $[L(f), U(f)]$ with the partitions getting more and more refined?

Theorem. If f is continuous on $[a, b]$, then f is integrable.

Proof.
For all $\epsilon > 0$, we need to find a partition P such that $U(f, P) - L(f, P) < \epsilon$. Since f is continuous on $[a, b]$, it is also uniformly continuous. Therefore, $\exists \delta > 0$ such that for all $x, y \in [a, b]$ such that $|x - y| < \delta$, we have $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$. Let P be a partition given by $a = x_0 < x_1 < \ldots < x_n = b$ such that $x_k - x_{k-1} < \delta$ for all $k \in \{1, 2, \ldots n\}$. Recall the defintions of M_k and m_k from above. By the definition of δ, for all $x, y \in [x_{k-1}, x_k]$, we have $f(x) < f(y) + \frac{\epsilon}{2(b-a)}$. Therefore $f(x)$ is a lower bound of $\{f(y) + \frac{\epsilon}{2(b-a)} | y \in [x_{k-1}, x_k]$ so $f(x) \leq \inf\{f(y) + \frac{\epsilon}{2(b-a)} \mid y \in [x_{k-1}, x_k]\}$ so $f(x) \leq \inf\{f(y) + \frac{\epsilon}{2(b-a)} \mid y \in [x_{k-1}, x_k]\} = m_k + \frac{\epsilon}{2(b-a)}$. So $\sup\{f(x) \mid x \in [x_{k-1}, x_k]\} \leq m_k + \frac{\epsilon}{2(b-a)}$. Therefore,

$$M_k \leq m_k + \frac{\epsilon}{2(b-a)}$$

$$M_k - m_k \leq \frac{\epsilon}{2(b-a)}$$

Remark. We could have just used continuity on a compact set to show this.

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

$$= \sum_{k-1}^{n} (M_k - m_k)(x_k - x_{k-1}) \leq \sum_{k=1}^{n} \frac{\epsilon}{2(b-a)}(x_k - x_{k-1})$$

$$= \frac{\epsilon}{2(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1})$$

$$= \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2} < \epsilon$$

■

Theorem. Suppose f is a bounded function on $[a, b]$ such that for all $c \in (a, b)$, the function f is integral on $[c, b]$. Then f is integrable on $[a, b]$.

Proof.
Since f is bounded, $\exists M$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. For each $\epsilon > 0$, we need to find a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $r = \min\{\frac{\epsilon}{4M}, \frac{b-a}{2}\}$ and define $c_0 = a + r$. Since $r > 0$ and $r \leq \frac{b-a}{2}$, we have $c_0 \in (a, b)$. Since $f|_{[a,b]}$ is inteegrable, there exists a partition Q given by

$$c_0 = y_0 < y_1 < y_2 < \ldots < y_n = b$$

such that $U(f \mid_{c_0,b]}, Q) - L(f \mid_{[c_0,b]}, Q) < \frac{\epsilon}{2}$. Since $|f(x)| \leq M$ for all $x \in [a,b]$, we have

$$\sup_{x\in[a,c_0]} f(x) \leq M$$

$$\inf_{x\in[a,c_0]} f(x) \geq -M$$

so $\sup_{x\in[a,c_0]} f(x) - \inf_{x\in[a,c_0]} f(x) \leq 2M$ define P to be the partition of $[a,b]$ given by a, $c_0 = y_0$, y_1, $y_2, \ldots, y_n = b$. Then

$$U(f,P) = [\sup_{x\in[a,c_0]}(f(x))](c_0 - a) + U(f|_{c_0,b}, Q)$$

$$L(f,P) = [\inf_{x\in[a,c_0]}(f(x))](c_0 - a) + L(f|_{c_0,b}, Q)$$

$$\begin{aligned} U(f,P) - L(f,P) &= (\sup_{x\in[a,c_0]} f(x) - \inf_{x\in[a,c_0]}(x)) \cdot (c_0 - a) \\ &+ U(f|_{[a,b]}, Q) - L(f|_{[c_0,b]}, Q)) \\ &< (2M) \cdot (c_0 - a) + \frac{\epsilon}{2} \\ &\leq (2M) \cdot \frac{\epsilon}{4M} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

■

Remark. Can we say that f is Riemann integrable on $[a,b]$ if and only if for all $c \in (a,b)$, $\lim_{x\to c^+} f(x) = f(c)$ or $\lim_{x\to c^-} f(x) = f(c)$?

Theorem. Suppose $< c < b$ and f is integrable on $[a,c]$ and $[c,b]$. Then

(1) f is integrable on $[a,b]$

(2) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Proof.
Suppose $\epsilon > 0$. Since f is integrable on $[a,b]$, $\exists$ a partiiton P_1 given by

$$a = y_0 < y_1 < \ldots < y_n = c$$

such that $U(f|_{[a,c]}, P_1) - L(f|_{[a,c]}, P_2) < \frac{\epsilon}{2}$. Since f is integrable on $[c,b]$, exists a partition P_2 given by $c = z_0 < z_1 < \ldots < z_m = b$ such that $U(f|_{[c,b]}, P_2) - L(f|_{[c,b]}, P_2) < \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2$. Then

$$U(f,P) = U(f|_{[a,c]}, P_1) + U(f|_{[c,b]}, P_2)$$

$$L(f,P) = L(f|_{[a,c]}, P_1) + L(f|_{[c,b]}, P_2)$$

Corollary. If f is a bounded function on $[a,b]$ such that the set of discontinuities is finite, then f is Riemann integrable on $[a,b]$.

Remark. If $[a,b] \subseteq [a',b']$ and f is integrable on $[a',b']$, then f is integrable on $[a,b]$. This is left as an exercise.

Theorem. Suppose f and g are both integrable on $[a,b]$ and c is a constant. Then

(1) $f+g$ is integrable on $[a,b]$.

(2) $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(3) cf is integrable on $[a,b]$

(4) $\int_a^b cf\,dx = c\int_a^b f\,dx$

Proof.
Suppose P is a partition of $[a,b]$ given by

$$a = x_0 < x_1 < \ldots < x_n = b$$

Let $M_k^f = \sup_{x\in[x_{k-1},x_k]}$ and $m_k^f = \inf_{x\in[x_{k-1},x_k]}$ and define others similarly.

Remark. What if instead of defining integration like this, we defined $U(f)$ and $L(f)$ as above and then defined the Integral Bound of f as $[L(f), U(f)]$?

34 Precept 8:

Example. f is a twice differentiable function on $(a, +\infty)$. M_0, M_1, and M_2 are the least upper bounds of $|f|$, $|f'|$, and $|f''|$. Show

$$M_1^2 \leq 4M_0M_2$$

Hint: Use Taylor's Theorem

Solution.

Strategy. Use $a^2 + b^2 \geq 2ab$ to find the infimum of a sum of two things. For example,

$$\frac{2M_0}{h} + \frac{M_2}{2}h \geq 2\sqrt{M_0M_2}$$

Example. f is a twice differentiable function on $(a, +\infty)$. f'' is bounded. $\lim_{x\to+\infty} f(x) = 0$. True or false: $f'(x) \to 0$ as $x \to +\infty$?
Hint: Ues the result from the previous example.

Sketch. True. If $f(x)$ goes to zero, then we just take sufficiently large a and using the first example, $|f(x)| \leq \epsilon$, so $M_1^2 \leq 4\epsilon M_2$ so we can just make epsilon so that M_1 is arbitrarily small.
Proof.

Remark. We can also use the Lagrangian Mean Value Theorem to show that f' is bounded.

From the previous example, for any x, let $a = x - 1$. Then

$$0 \leq |f'(x)| \leq 2\sqrt{\sup_{y>a} |f(y)| \sup_{y>a} |f'(y)|}$$

We have $\exists M$ such that $\sup_{y>a} |f(y)| \leq M$. Also,

$$\lim_{x\to\infty} \sup_{y>a} |f(y)| = 0$$

(lim sup theorem). Thus,

$$\lim_{x\to+\infty} 2\sqrt{\sup_{y>a} |f(y)| \sup_{y>a} |f'(y)|} = 0$$

Using the Squeezy Theorem,

$$\lim_{x\to+\infty} |f'(x)| = 0$$

∎

Example. f is differentiable on $[a, b]$, and $f(a) = 0$. There exists a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. True or false: $f(x) = 0$ on $[a, b]$.
Hint: First assume $b - a \leq \dfrac{1}{2A}$.

Solution.
By the Lagrangian Mean Value Theorem, $f(x) - f(a) = f'(\zeta)(x - a)$. Thus,

$$f(x) = f(\epsilon)(x - a)$$

Since $|f(x)| \leq A|f(\zeta)|||x - a|$, let $M = \sup_{x\in[a,b]} f(x)$

$$M \leq AM|x - a|$$

Thus, $M = 0$ (assuming $b - a \le \frac{1}{2A}$). Thus, $M = 0$, so $f'(x) = 0$. Thus, $f(x) = 0$. If $|b - a| > \frac{1}{2A}$, we can simply break the interval $[a, b]$ into smaller intervals of length less than $\frac{1}{2A}$. ■

Example. f is a real function on $[a, b]$, $n \in \mathbb{N}^+$ $f^{(n-1)}$ exists. For every $x \in [a, b]$, define $Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$. Show the following version of Taylor's theorem:

$$f(\beta) = f(\alpha) + f'(\alpha)(\beta - \alpha) + \ldots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^{n-1} + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$$

35 Lecture 22:

Last time, we saw that

(1) If f is integrable on $[a, b]$ and $[b, c]$, then it is integrable on $[a, c]$.

(2) If f and g are both integrable on $[a, b]$, then $f + g$ is integrable, and

$$\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

(3) If f is integrable on $[a, b]$ and k is a constant, then kf is integrable on $[a, b]$ and

$$\int_a^b kf(x)\ dx = k \int_a^b f(x)\ dx$$

Theorem. If $f(x) \geq 0$ on $[a, b]$ and f in integrable on $[a, b]$, then

$$\int_a^b f(x)\ dx \geq 0$$

Proof.
Let P be any arbitrary partition of $[a, b]$ consisting of elements $a = x_0 < x_1 < \ldots < x_n = b$. Define $M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$. Then $U(f, P) = \sum_{k=1}^n M_k(x_k - x_{k-1})$. Since $f(x) \geq 0$ for all $x \in [a, b]$, we have $M_k \geq 0$ for all k, so $U(f, P) \geq 0$ for all partitions P so

$$\inf\{U(f, P) \mid P \text{ is a partition}\} \geq 0$$

Therefore

$$\int_a^b f(x)\ dx \geq 0$$

■

Corollary.

(1) If f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$$

(2) If f is integrable on $[a, b]$ and $m \leq f(x) \leq M$, on $[a, b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

Proof.
Consider $h(x) = g(x) - f(x)$. Since f and g are both integrable, h is integrable. From the previous theorem, it follows that $\int_a^b g(x) - f(x)\ dx \geq 0$. The result in (1) follows immediately. Applying (1) to the constant functions m and M,

$$\int_a^b mx\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b M\ dx$$

which gives us our desired result. ■

Theorem. If f is integrable on $[a, b]$, then $|f|$ is also integrable.

Proof.
This is a homework problem.

Theorem. If f is integrable on $[a, b]$, then

$$|\int_a^b f(x)\, dx| \le \int_a^b |f(x)|\, dx$$

Proof.
Since $-|f(x)| \le f(x) \le |f(x)|$,

$$\int_a^b -|f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$$
$$-\int_a^b |f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$$
$$|\int_a^b f(x)\, dx| \le \int_a^b |f(x)|\, dx$$

■

This can be viewed an integral version of the Triangle Inequality.

Theorem. If (f_n) is a sequence of functions that is uniformly convergent to f and every f_n is integrable, then f is integrable.

Proof.
This is a homework problem.

Theorem. If (f_n) converges uniformly to f and all f_ns are integrable, then

$$\lim_{n \to +\infty} \int_a^b f_n(x) = \int_a^b f(x)\, dx$$

Proof.

$$|\int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx|$$
$$= |\int_a^b (f_n(x) - f(x))\, dx|$$
$$\le \int_a^b |f_n(x) - f(x)|\, dx$$

Since (f_n) converges to f uniformly, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$$

Then

$$|\int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx| \le \int_a^b \frac{\epsilon}{2(b-a)}\, dx$$
$$= \frac{\epsilon}{2} < \epsilon$$

so $(\int_a^b f_n(x)\, dx)_{n \in \mathbb{N}}$ converges to $\int_a^b f(x)\, dx$. ■

Example. Let $f : [0, 1] \to \mathbb{R}$ be the Dirichlet function on $[0, 1]$. Let $\sigma : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ be a bijection. For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 1 & x = \sigma(k) \text{ for some } k \le n \\ 0 & \text{otherwise} \end{cases}$$

Since f_n is disconinuous only at finitely many points, f_n is integrable on $[0, 1]$.

Claim. For each fixed $x \in [0,1]$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.

Proof. IF $x \in \mathbb{Q} \cap [0,1]$, then $f_n(x) = 0$ for all n, and $f(x) = 0$, so the statement holds. If $x \in \mathbb{Q} \cap [0,1]$, then $\exists$ a unique $m \in \mathbb{N}$ such that $\sigma(m) = x$. Thus

$$f_n(x) = \begin{cases} 1 & n \geq m \\ 0 & n < m \end{cases}$$

and $f(x) = 1$. Therefore $\lim f_n(x) = f(x)$.

Theorem. Suppose $g(x)$ is continuous on $[a,b]$. Define $G(x) = \int_a^x g(s)\ ds$. Then $G(x)$ is differentiable on (a,b) and $G'(c) = g(c)$ for all $c \in (a,b)$.

Proof.
Consider

$$\lim_{x\to c}\left(\frac{G(x)-G(c)}{x-c} - g(c)\right)$$

Since $g(x)$ is continuous for all $\epsilon > 0$ $\exists\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$ we have $|g(x) - g(c)| < \dfrac{\epsilon}{2}$. If $x\ in(c, c+\delta)$ then $G(x) - G(c) = \int_a^x g(s)\ ds - \int_a^c g(s)\ ds = \int_c^x g(s)\ ds$. We have

$$\left|\int_c^x g(s)\ ds - \int_c^s g(c)\ ds\right| = \left|\int_c^x (g(s) - g(c))\ ds\right|$$

$$\leq \int_c^x |g(s) - g(c)|\ ds \leq \int_c^x \frac{\epsilon}{2}\ ds = (x-c)\epsilon/2$$

$$|G(x) - G(c) - (x-c)g(c)| \leq (x-c)\frac{\epsilon}{2}$$

$$\left|\frac{G(x)-G(c)}{x-c} - g(c)\right| \leq \frac{\epsilon}{2} < \epsilon$$

when $x \in (c, c+\delta)$. The theorem can be shown for $x \in (c-\delta, c)$ similarly just taking $G(c) - G(x)$. Then, for all $x \in (c-\delta, c+\delta)\backslash\{c\}$ we have $\left|\dfrac{G(x)-G(c)}{x-c} - g(c)\right| < \epsilon$ so $G'(c) = g(c)$. ■

Theorem. Suppose $F(x)$ is defined on an open interval containing $[a,b]$ and F is differentiable on $[a,b]$. Let $f(x) = F'(x)$. If f is integrable on $[a,b]$,

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Proof.
Let P be a partition of $[a,b]$. Suppose the elements of P are given by $a = x_0 < x_1 < \ldots < x_n = b$. $m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$ and let $M_k = \sup_{x\in[x_{k-1},x_k]} f(x)$. Then $U(f,P) = \sum_{k=1}^n M_k(x_k - x_{k-1})$ and $L(f,P) = \sum_{k=1}^n m_k(x_k - x_{k-1})$. By the Lagrangian Mean Value Theorem applied to F on $[x_{k-1}, x_k]$, for each k there exists $c_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$$

Since

$$m_k \leq f(c_k) \leq M_k$$

We have

$$m_k(x_k - x_{k-1}) \leq F(x_k) - F(x_{k-1}) < M_k(x_k - x_{k-1})$$

Taking a summation of this with respect to k,

$$L(f,P) \leq \sum_{k=1}^{n}(F(x_k) - F(x_{k-1})) \leq U(f,P)$$

$$L(f,P) \leq F(b) - F(a) \leq U(f,P)$$

for all partitions P. Thus,

$$L(f) \leq F(b) - F(a) \leq U(f)$$

so $\int_a^b f(x)\,dx = F(b) - F(a)$ by definition as desired. ■

35.1 Woah the Course Syllabus is Over

Remark. This is like the end of Pokemon SoulSilver after you beat the Champion and after the credits scroll through you get to continue to the next region :D :D :D

Definition (Zero Measure). Suppose $S \in \mathbb{R}$. We say that S has zero measure if for all $\epsilon > 0$, $\exists$ a possibly finite sequence of open intervals (a_n, b_n) such that

(1) $S \in \cup(a_n, b_n)$

(2) $\sum(b_n - a_n) < \epsilon$

Example. If $S \in \mathbb{R}$ is finite suppose

$$S = \{x_1, x_2, \ldots x_n\}$$

Let $a_k = x_k - \dfrac{\epsilon}{4n}$ and $b_k = x_k + \dfrac{\epsilon}{4n}$. Then $S \subseteq \cup_k(a_k, b_k)$ and $\sum_{k=1}^{n}(b_k - a_k) = \dfrac{\epsilon}{2} < \epsilon$.

Remark. We will prove next time that every countable set has zero measure.

Theorem. A bounded function f on $[a,b]$ is integrable if and only if the set of discontinuity of f has zero measure.

Corollary. If f is continuous on $\mathbb{R}$ and g is integrable on $[a,b]$, then $f \circ g$ is also integrable on $[a,b]$.

36 Lecture 23:

Theorem. Suppose f is a bounded function on a closed interval. Then f is integrable if and only if the set of discontinuous points has measure zero.

Definition. Suppose $S \subseteq \mathbb{R}$. Then S is said to have zero measure. If $\forall \epsilon > 0$ $\exists$ a countable collection of open intervals $\{O_i\}$ such that

(1) $S \subseteq \cup O_i$

(2) $\sum_i m(O_i) < \epsilon$

where by definition, if $O_i = (a_i, b_i)$, then $m(O_i) = b_i - a_i$.

Lemma. If S is countable, then it has zero measure.

Proof omitted, fairly simple.

Lemma. If S_1, S_2 are sets with measure zero, then $S_1 \cup S_2$ also has measure zero.

Lemma. If S_1, S_2, $S_3 \dots$ is a countable collection of sets with zero measure, then $\cup_{i\in\mathbb{N}} S_i$ also has zero measure.

Proof.
For all $\epsilon > 0$, since S_k has measure zero for each k, $\exists$ a countable set of open intervals $\{O_i^(k)\}_{i\in\mathbb{N}}$ such that $S_k \subseteq \bigcup_{i\in\mathbb{N}} O_i^{(k)}$.

$$\sum m(O_i^{(k)}) < \frac{\epsilon}{2^{k+1}}$$

Let $\{O_i\} = \cup_{k\in\mathbb{N}}\{O_i^{(k)}\}$. Then

$$\bigcup S_i \subseteq \bigcup O_i$$

$$\sum_{i\in\mathbb{N}} m(O_i) < \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^{k+1}}$$

$$= \frac{\epsilon}{2} < \epsilon$$

■

Example. $\mathbb{Q}$ has measure zero.

Example. $[0, 1]$ does not have measure zero.

We show that any open cover must have a sum of measures greater than one by integrating over functions defined to be 1 on the open intervals so that

$$\int_a^b f_i(x)\, dx = m(O_i)$$

Lemma. If $S_1 \subseteq S_2$ and S_2 has measure zero, then S_1 also has measure zero.

Lemma. If $a < b$, then (a, b) does not have measure zero.

Corollary. If S is a set that contains a non-empty open interval, then S does not have zero measure.

Corollary. If S has zero measure, then S^c is a dense subset of $\mathbb{R}$.

Definition. $D = \{x \in [a, b] \mid f \text{ is not continuous at } x\}$.

Definition (α**-continuity.** Suppose $x \in [a, b]$ and let $\alpha > 0$. We say that f is α-continuous at x if $\exists\delta > 0$ such that for all $y, z \in [a, b] \cap (x - \delta, x + \delta)$, we have

$$|f(y) - f(z)| < \alpha$$

Lemma. If f is continuous at x, then for every $\alpha > 0$, f is α-continuous at x.

Lemma. If f is α-continuous at x for every $\alpha > 0$, then f is continuous at x.

Definition (uniform α**-continuity).** Suppose $A \subseteq [a, b]$ and let $\alpha > 0$. We say that f is uniformly α-continuous on A if $\exists\delta > 0$ such that for all x, y such that $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \alpha$$

Exercise. Suppose $K \subseteq [a, b]$ is a compact set and f is α-continuous on all $x \in K$. Then f is uniformly α-continuous on K.

Define $D_\alpha = \{x \in [a, b] \mid f \text{ is not } \alpha\text{-continuous at } x\}$.

Lemma. D_α is a closed set.

Proof.
Suppose $x \in D^c_a$. If $x \notin [a, b]$ then $\exists\delta > 0$ such that $(x - \delta, x + \delta)$ smth so $(x - \delta, x + \delta) \subseteq D^c_\alpha$.
If $x \in [a, b]$, $x \notin D_\alpha$, then $\exists\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta) \cap [a, b]$ we have

$$|f(y) - f(z)| < \alpha$$

For each $x \in (x - \delta, x + \delta) \cap [a, b]$, take δ' such that $(x' - \delta', x' + \delta') \subseteq (x - \delta, x + \delta)$. Then for all $y, z \in (x' - \delta', x' + \delta')$ ■

To summarize,

(1) Each D_α is closed.

(2) $D = \cup_{\alpha>0} D_\alpha$

Lemma. If $\alpha > \alpha'$ and f is α'-continuous at x. Then f is α-continuous at x.

Corollary. If $\alpha > \alpha'$, then $D_\alpha \subseteq D_{\alpha'}$.

Lemma. D has zero measure if and only if D_α has zero measure for all α.

Lebesgue's Criterion. Suppose f is a bounded function on a closed interval. Then f is integrable if and only if the set of discontinuous points has measure zero.

Proof.
Suppose D has measure zero, we prove that f is integrable. For each $\epsilon > 0$, we show that $\exists$ a partiion P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Take $\alpha = \dfrac{\epsilon}{4M}$ where $M = \sup_{[a,b]}(|f|+1)$. Since D has zero measure, D_α is a bounded closed set with zero measure. So $\exists$ a countable collection of open intervals $\{O_i\}$ such that $D_\alpha \subseteq \bigcup O_i$.

$$\sum m(O_i) < \frac{\alpha\epsilon}{2}$$

Since D_α is compact, we can assume without loss of generality that $\{O_i\}$ is a finite collection. Let P be a partition of $[a,b]$ such that if we write the elements of P as

$$a = x_0 < x_1 < \dots x_n = b$$

then for each k, either $(x_{k-1}, x_k) \subseteq O_i$ for some O_i or $[x_{k-1}, x_k] \cap D_\alpha = \varnothing$.

We also require that if $[x_{k-1}, x_k] \cap D_\alpha = \varnothing$, then $M_k - m_k \subseteq \alpha$. This can be realized by further subdividing $[x_{k-1}, x_k]$ since f is uniformly α-continuous on $[x_{k-1}, x_k]$ when $[x_{k-1}, x_k] \cap D_\alpha] = \varnothing$.

Then $U(f,P) - L(f,P)$

$$\begin{aligned}
&= \sum (M_k - m_k)(x_k - x_{k-1}) \\
&= \sum_{(x_{k-1}, x_k)} (M_k - m_k)(x_k - x_{k-1}) \\
&+ \sum_{[x_{k-}, x_k] \cap D\alpha} (M_k - m_k)(x_k - x_{k-1}) \\
&\leq \sum_{(x_{k-1}, x_k) \subseteq O_i} (2M)(x_k - x_{k-1}) \\
&\leq 2M \sum m(O_i) \\
&< (2M)\frac{\epsilon}{4M} = \frac{\epsilon}{2}
\end{aligned}$$

37 Lecture 24: Lesbesgue, Cantor, and Creating $\mathbb{R}$ from $\mathbb{Q}$

Proposition. Suppose f is integrable on $[a, b]$, then the set of discontinuous points has zero measure.

Proof.

$$D = \{x \in [a, b] \mid f \text{ is discontinuous at } x\}$$
$$D_\alpha = \{x \in [a, b] \mid f \text{ is not } \alpha\text{-continuous at } x$$

Last time, we showed that

$$D = \bigcup_{\alpha > 0} D_\alpha$$

so D has measure zero is equivalent to D_α having zero measure for all $\alpha > 0$. Fix $\alpha, \epsilon > 0$. Since f is integrable, $\exists$ a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \frac{\alpha\epsilon}{2}$$

Let $a = x_0 < x_1 < \ldots < x_n = b$ be the elements of P and recall that $M_k = \sup_{[x_{k-1}, x_k]} f$, $m_k = \inf_{[x_{k-1}, x_k]} f$.

$$\sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\alpha\epsilon}{2}$$

$$\sum_{M_k - m_k \geq \alpha} (M_k - m_k)(x_k - x_{k-1}) + \sum_{M_k - m_k < \alpha} (M_k - m_k)(x_k - x_{k-1}) < \frac{\alpha\epsilon}{2}$$

$$\sum_{M_k - m_k \geq \alpha} (M_k - m_k)(x_k - x_{k-1}) < \frac{\alpha\epsilon}{2}$$

$$\sum_{M_k - m_k \geq \alpha} \alpha(x_k - x_{k-1}) < \frac{\alpha\epsilon}{2}$$

$$\sum_{M_k - m_k \geq \alpha} (x_k - x_{k-1}) < \frac{\epsilon}{2}$$

We have $D_\alpha \subseteq (\bigcup_{M_k - m_k \geq \alpha} [x_{k-1}, x_k]) \cup \{x_1, x_2, \ldots x_{n-1}\}$ Since the total length of the intervals is thus less than $\frac{\epsilon}{2}$, $(\bigcup_{M_k - m_k \geq \alpha} [x_{k-1}, x_k]) \cup \{x_1, x_2, \ldots x_{n-1}\}$ can be covered by infinitely many open intervals with total length less than ϵ. Since $\epsilon > 0$ can be chosen arbitrarily this implies that D_α has measure zero for all $\alpha > 0$. By the previous results, this implies that D has measure zero. ■

37.1 Cantor Sets

What is interesting about Cantor sets is that:

- Measure zero
- Uncountable
- Closed set

Construction of the Cantor Set (me translating construction into math). Let $A_1 = [0, 1]$. Suppose we have A_{n-1} a collection of disjoint closed intervals $\{[a_i, b_i]\}$. Let $A_n = \bigcup_i ([a_i, \frac{b_i + 2a_i}{3}] \cup [\frac{2b_i + a_i}{3}, b_i])$. The Cantor Set is $C = \bigcap_{n=1}^{\infty} A_n$.

Remark. We have not defined a limit of sets, but this intersection is kind of exactly that. I guess we could define the limit of a set $\{A_n\}$ of sets such that $A_n \supset A_{n+1}$ for all n to be $\bigcap_{n \in \mathbb{N}} A_n$.

A_n is a disjoint union of 2^{n-1} intervals and each has length 3^{1-n} and the total length of all the intervals in A_n is thus $(\frac{2}{3})^{n-1}$.

Proposition. The set C has measure zero.

Proof.
For each $\epsilon > 0$, take n such that $(\frac{2}{3})^{n-1} < \epsilon$. Then there is a finite collection of open intervals with total length $< \epsilon$ that covers A_n, and since $C \subseteq A_n$, it also covers C. Thus, by definition, C has zero measure. ■

Proposition. The set C is closed.

Proof.
Since C is an intersection of closed sets, it is also closed. ■

Remark. What is interesting is that we always have a not only countable but also finite number of intervals in the definition of the Cantor Set, but the set C itself is actually uncountable.

Proposition. Let S be the set of sequences consisting of 0s and 1s. Then there is an injection from S to C.

Proof.
Let $(x_n)_{n\in\mathbb{N}}$ be an element of S. Define $I_1 = [0,1]$. Suppose $I_k = [a,b]$. Define

$$I_{k+1} = \begin{cases} [a, a+\frac{1}{3}(b-a)] & x_k = 0 \\ [b-\frac{1}{3}(b-a), b] & x_k = 1 \end{cases}$$

Then $I_k \subseteq A_k$. Define $f : S \to C$ by requiring that $f((x_n)_{n\in\mathbb{N}}) \in \bigcap_{k\in\mathbb{N}} I_k$. By the nested interval property, $\bigcap_{k\in\mathbb{N}} I_k \neq \varnothing$, so such a map f exists. Now we prove that f is injective. Suppose (x_n) and (x'_n) are two different seqeunces. Let $m = \min\{n \in \mathbb{N} \mid x_n \neq x'_n\}$. Let I_k, I'_k be the nested intervals defined by (x_n) and (x'_n) respectively. We have

$$I_k = I'_k \text{ for all } k \leq m$$

Write $I_k = [a,b]$, then I_{m+1} and I'_{m+1} are given by $[a, a+\frac{1}{3}(b-a)]$ and $[b-\frac{1}{3}(b-a), b]$ Since $[a, a+\frac{1}{3}(b-a)] \cap [b-\frac{1}{3}(b-a), b] = \varnothing$. We have $I_{m+1} \cap I'_{m+1} = \varnothing$. Since $f((x_n)) \in \cap_{k\in\mathbb{N}} I_k \subseteq I_{m+1}$ and $f((x'_n)) \in \cap_{k\in\mathbb{N}} I'_k \subseteq I'_{m+1}$. We have $f((x_n)) \neq f((x'_n))$. ■

Remark. It is also possible to show that f is surjective, so it is a bijection.

Proposition. The set S is uncountable.

Remark. We should make the base systems. In other words that for all real numbers r there exists a sequence (a_n) and a base b such that $0 \leq a_n < b$ with $a_n \in \mathbb{N}$ and $\sum a_n b^{-n} = r$?

Proposition. The Cantor set C is uncountable.

Remark. Since C has measure zero, it does not contain any non-empty open interval.

Theorem. Let

$$f(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

Then $f(x)$ is discontinuous on C and continuous on $\mathbb{R} \backslash C$.

Proof.

If $x \neq C$, since C is closed, $\exists \delta > 0$ such that $(x - \delta, x + \delta) \cap C = \varnothing$ so $f = 0$ on $(x - \delta, x + \delta)$. Therefore f is continuous at x.

If $x \in C$, Then for all $\delta > 0$, there exists $y \in (x - \delta x + \delta)$ such that $y \notin C$, so $f(y) = 0$. Since $f(x) = 1$, this implies that f is discontinuous at x. ■

Corollary. $f(x)$ is integrable on $[a, b]$ for all $a < b$.

Remark. If $x \in C$. x is a limit point of C. This is called being a perfect set.

37.2 Construction of $\mathbb{R}$

Definition. Suppose (a_n) is a sequence of rational numbers. We say that (a_n) is Cauchy. If for all $k \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{1}{k}$$

for all $m, n \geq N$.

Definition. Suppose (a_n) is a sequence of rational numbers. We say that $\lim a_n = 0$. If for all $k \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that $|a_n| < \dfrac{1}{k}$ for all $n \geq N$.

Definition. Two Cauchy sequences (a_n) and (b_n) are equivalent if $\lim(a_n - b_n) = 0$.

Remark. We cannot define limits to nonzero numbers because not all limits exist in $\mathbb{Q}$

Definition. Let $\mathbb{R}$ be the set of equivalence classes of Cauchy sequences of rational numbers.

We need to first show that if (a_n) is equivalent to (b_n), and (b_n) is equivalent to (c_n), then (a_n) is equivalent to (c_n). This follows easily from a rational version of the algebraic limit theorem.

Notation. Suppose (a_n) and (b_n) are two Cauchy sequences. We use $\overline{(a_n)}$ to denote the element represented by (a_n) in $\mathbb{R}$.

Definition. Suppose $a = \overline{(a_n)}$ and $b = \overline{(b_n)}$ in $\mathbb{R}$. Define $a + b = \overline{(a_n + b_n)}$ and $ab = \overline{(a_n b_n)}$.

For the above definition to make sense, we need to show that

(1) If (a_n) and (b_n) are Cauchy sequences then $(a_n + b_n)$ and $(a_n b_n)$ are also Cauchy

(2) If (a_n) is equivalent to (a'_n), then $(a_n + b_n)$ is equivalent to $(a'_n + b_n)$ and $(a_n b_n)$ is equivalent to $(a'_n b_n)$.

But we will skip these details for now.

Definition. Define an embedding $i : \mathbb{Q} \to \mathbb{R}$ by taking

$$i(r) = \overline{(r, r, r, \ldots)}$$

Therefore, we can identify $\mathbb{Q}$ as a subset of $\mathbb{R}$ using i. If is not hard now to verify all the classic properties of $\mathbb{R}$.

Ordering: $\overline{(a_n)} > \overline{(b_n)}$ if $\exists k \in \mathbb{N}$ and $N \in N$ such that $a_n > b_n$ for all $n \geq \mathbb{N}$.

We need to show:

(1) If (a_n) is equivalent to (a'_n), then $\overline{(a_n)} > \overline{(b_n)} \Leftrightarrow \overline{(a'_n)} > \overline{(b_n)}$

(2) If $\overline{(a_n)} > \overline{(b_n)}$, $\overline{(b_n)} > \overline{(c_n)}$, then $\overline{(a_n)} > \overline{(c_n)}$.

and we will also just take this for granted.

Archemedian Property on $\mathbb{R}$. Suppose $\overline{(a_n)} \in \mathbb{R}$, then $\exists N \in \mathbb{N}$ such that $\overline{(a_n)} < \overline{(N, N, \ldots)}$.

Proof.
Since (a_n) is Cauchy, $\exists k$ such that $|a_m - a_n| < 1$ for all $m, n \geq k$. Take N such that $N > |a_k| + 2$. Then for all $n \geq k$,

$$|a_n| \leq |a_n - a_k| + |a_k| \leq |a_k| + 1 < N - 1$$

so $a_n + 1 < N$ for all $k \geq N$. By definition, $\overline{(a_n)} < \overline{(N, N, \ldots)}$. ∎

Cauch Criterion. If

$$\begin{aligned} x_1 &= \overline{(a_n^{(1)})} \\ x_2 &= \overline{(a_n^{(2)})} \\ x_3 &= \overline{(a_n^{(3)})} \\ &\cdots \end{aligned}$$

is a Cuachy sequence in $\mathbb{R}$, then (x_n) is convergent in $\mathbb{R}$.

Sketch of Proof.
For all $k \in \mathbb{N}$, $\exists N_k \in \mathbb{N}$ such that for all $m, n \geq N_k$, we have

$$|a_m^{(k)} - a_n^{(k)}| < \frac{1}{k}$$

Let $y_k = a_{N_k}^{(k)}$. Define $y = \overline{(y_n)}_{n \in \mathbb{N}} \in \mathbb{R}$. Then it can be proved that $\lim_{k \to \infty} x_k = y$.

38 Precept 9: The End

Remark. This is the final precept/class of the semester. Damn.

38.1 Reviewing the Riemann Integral

Claim. $L(f, P_1) \leq U(f, P_2)$.

Proof.
Let $Q = P_1 \cup P_2$.

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2)$$

■

We define

$$U(f) = \inf_P \{U(f, P)\}$$

$$L(f) = \sup_P \{L(f, P)\}$$

so $L(f) \leq U(f)$.

We define a function to be Riemann Integrable if f is bounded and $U(f) = L(f)$.

Theorem (Integrability Criterion). A function is Riemann integrable if and only if for all $\epsilon > 0$, $\exists P_\epsilon$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

Theorem. f is continuous on $[a, b]$ implies f is integrable on $[a, b]$.

Proof Sketch.
Continuity on a compact set implies uniform continuity, and then the U and L difference can be bounded using this uniform continuity.

Theorem. If f is bounded and f is integrable on $[c, b]$ for all $a < c < b$, then f is integrable on $[a, b]$.

Theorem. If f is bounded on $[a, b]$ and $a < c < b$, then f is integrable on $[a, b]$ is equivalent to f being integrable on $[a, b]$ and f being integrable on $[b, c]$.

Theorem. If f on $[a, b]$ has finite discontinuities, then f is integrable.

Properties of the Integral. Assume f and g are integrable on $[a, b]$ and $k \in \mathbb{R}$. Then

(i) $f + g$ is integrable, and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

(ii) kf is integrable, and $\int_a^b kf = k \int_a^b f$.

(iii) If $m \leq f \leq M$, then $m(b - a) \leq \int_a^b f \leq M(b - a)$.

(iv) If $f \leq g$, then $\int_a^b f \leq \int_a^b g$.

(v) $|f|$ is integrable, with $\int_a^b |f| \geq \left| \int_a^b f \right|$

Theorem (Integral Limit Theorem). If $f_n \to f$ converges uniformly on $[a,b]$ and each f_n is integrable, then f is integrable, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Proof.

$$|\int_a^b f - \int_a^b f_n| = |\int_a^b f - f_n| \leq \int_a^b |f - f_n|$$

Since (f_n) is uniformly convergent, choose N such that

$$|f(x) - f_n(x)| < \frac{\epsilon}{b-a}$$

for all $x \in [a,b]$ and $n > N$.

True or False. If $f : [a,b] \to \mathbb{R}$ and is increasing then f is integrable.

Proof.
This was a homework problem! :D

True or False. If $f : [a,b] \to \mathbb{R}$ is integrable, then f^2 is also integrable.

Proof.
Let $g = |f|$, so g is integrable. Take $P = \{x_0, x_1, \ldots x_n\}$. Let $M_k = \sup_{[x_{k-1},x_k]} g$ and let $m_k = \inf_{[x_{k-1},x_k]}$. Then

$$U(f^2,P) - L(f^2,P) = \sum_{k=1}^{n} (M_k^2 - m_k^2)(x_k - x_{k-1})$$

Define $M = \sup_{[a,b]} g$. Then

$$U(f^2,P) - L(f^2,P) = \sum_{k=1}^{n} (M_k^2 - m_k^2)(x_k - x_{k-1}) \leq 2M \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = 2M \in_a^b |f|$$

so $g^2 = f^2$ is integrable as desired. ■

True or False. If $f, g : [a,b] \to \mathbb{R}$ are integrable, then fg is integrable.

Proof.
Note that f^2 and g^2 are integrable. Also, $f+g$ is integrable, so $(f+g)^2 = f^2 + 2fg + g^2$ is integrable. Thus, fg is integrable. ■

True or False. If each $f_n : [a,b] \to \mathbb{R}$ is integrable and positive, and $f_n(x) \to 0$ as $n \to \infty$ for any $x \in [a,b]$, then $\lim_{n\to\infty} \int_a^b f_n = 0$.

Counterexample. Let

$$f_n = \begin{cases} n^2 & \frac{1}{n^2} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

This decays to zero but the limit is not zero.

38.2 Fundamental Theorem Calculus

Fundamental Theorem of Calculus.

(i) f is integrable on $[a,b]$, $F'(x) = f(x)$ on $[a,b]$, implies $\int_a^b f = F(b) - F(a)$

(ii) g is integrable on $[a,b]$, $G(x) = \int_a^x g$ on $[a,b]$ implies G is continuous, and if g is continuous at $c \in [a,b]$, then G is differentiable at c with $G'(c) = g(c)$.

True or False. If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists some $g : [a,b] \to \mathbb{R}$ such that $f = g'$.

True.

True or False. $f : [a,b] \to \mathbb{R}$. If $\int_a^x f = 0$ for any x then $f(x) = 0$ for all x.