Title

Yuchuang [*]

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1 Introduction

Here is an introduction.

2 Formulation

3 Results

4 Conclusions

Aknowledgements

References

[1] (e.g.) Einstein, Albert 1905 *Annalen der Physik* **322**(10) 891-921.

A Priors and Posteriors

These part show prior and conditional posterior $X|\cdot$ for all variables X

A.1 T_0

Prior is multivariate normal

$$\sim N\left(\mu_0, \mathbf{\Sigma_0}\right) \tag{1}$$

Posterior is multivariate normal

$$\sim N(\Psi_0 V_0, \Psi_0) \tag{2}$$

where

$$\mathbf{V}_0 = \mathbf{\Sigma}^{-1}\left[\alpha \mathbf{T}_1 - \alpha(1-\alpha)\mu\mathbf{1}\right] + \mathbf{\Sigma}_0^{-1}\mu_{\mathbf{0}} \tag{3}$$

$$\mathbf{\Psi}_0 = \left(\alpha^2\mathbf{\Sigma}^{-1} + \mathbf{\Sigma}_{\mathbf{0}}^{-1}\right)^{-1} \tag{4}$$

A.2 T_k

Posterior is multivariate normal (here $0 < k < \kappa$)

$$\sim N(\Psi_k V_k, \Psi_k) \tag{5}$$

where

$$\mathbf{V}_k = \mathbf{H}_k^{\mathrm{T}}\mathbf{\Xi}_k^{-1}\left(\mathbf{W}_k - \mathbf{B}_k\right) + \mathbf{\Sigma}^{-1}\left[\alpha\left(\mathbf{T}_{k+1} + \mathbf{T}_{k-1}\right) + (1-\alpha)^2\mu\mathbf{1}\right] \tag{6}$$

$$\mathbf{\Psi}_k = \left[\mathbf{H}_k^{\mathrm{T}}\mathbf{\Xi}_k^{-1}\mathbf{H}_k + \left(1+\alpha^2\right)\mathbf{\Sigma}^{-1}\right]^{-1} \tag{7}$$

A.3 T_κ

Posterior is multivariate normal

$$\sim N(\Psi_k V_\kappa, \Psi_\kappa) \tag{8}$$

where

$$\mathbf{V}_\kappa = \mathbf{H}_\kappa^{\mathrm{T}}\Xi_\kappa^{-1}\left(\mathbf{W}_\kappa - \mathbf{B}_\kappa\right) + \mathbf{\Sigma}^{-1}\left[\alpha\mathbf{T}_{\kappa-1} + (1-\alpha)\mu\mathbf{1}\right] \tag{9}$$

$$\mathbf{\Psi}_\kappa = \left(\mathbf{H}_\kappa^{\mathrm{T}}\mathbf{\Xi}_\kappa^{-1}\mathbf{H}_\kappa + \mathbf{\Sigma}^{-1}\right)^{-1} \tag{10}$$

A.4 α

Prior is uniform

$$\sim U(a_0, a_1) \tag{11}$$

Posterior is truncated normal

$$\sim N_{[a_0,a_1]}\left(\Psi_\alpha V_\alpha, \Psi_\alpha\right) \tag{12}$$

where

$$V_\alpha = \sum_{k=1}^{\kappa} \left(\mathbf{T}_{k-1} - \mu\mathbf{1}\right)^{\mathrm{T}} \mathbf{\Sigma}^{-1} \left(\mathbf{T}_k - \mu\mathbf{1}\right) \tag{13}$$

$$\Psi_\alpha = \left[\sum_{k=1}^{\kappa} \left(\mathbf{T}_{k-1} - \mu\mathbf{1}\right)^{\mathrm{T}} \mathbf{\Sigma}^{-1} \left(\mathbf{T}_{k-1} - \mu\mathbf{1}\right)\right]^{-1} \tag{14}$$

and boundaries a_0, a_1 come from uniform prior

A.5 μ

Prior is normal

$$\sim N\left(\mu_0, \sigma_0^2\right) \tag{15}$$

Posterior is normal

$$\sim N\left(\Psi_\mu V_\mu, \Psi_\mu\right) \tag{16}$$

where

$$V_\mu = \frac{\mu_0}{\sigma_0^2} + (1-\alpha)\mathbf{1}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\left[\sum_{k=1}^{\kappa} \left(\mathbf{T}_k - \alpha\mathbf{T}_{k-1}\right)\right] \tag{17}$$

$$\Psi_\mu = \left(\frac{1}{\sigma_0^2} + \kappa(1-\alpha)^2\mathbf{1}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{1}\right)^{-1} \tag{18}$$

A.6 σ^2

Prior is inverse-gamma

$$\sim InvGamma\left(\lambda_T, \nu_T\right) \tag{19}$$

Posterior is inverse-gamma

$$\sim InvGamma\left(\lambda_T + \frac{N_A\kappa}{2}, \nu_T + \frac{1}{2}\sum_{k=1}^{\kappa} \mathbf{\Delta T}_{k,k-1}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{\Delta T}_{k,k-1}\right) \tag{20}$$

where

$$\mathbf{\Delta T}_{k,k-1} = (\mathbf{T}_k - \alpha\mathbf{T}_{k-1} - (1-\alpha)\mu\mathbf{1}) \tag{21}$$

$$\mathbf{R}_{ij} = \exp\left(-\phi\left|\mathbf{x}_i - \mathbf{x}_j\right|\right) \tag{22}$$

A.7 ϕ

Prior is log normal

$$\sim logNormal\left(\mu_\phi, \sigma_\phi^2\right) \tag{23}$$

Posterior is not analytically distributed. Its probability density function is

$$P(\phi \mid \cdot) \propto P(\phi)\cdot|\mathbf{R}|^{-\kappa/2}\cdot\exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^{\kappa}\mathbf{\Delta T}_{k,k-1}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{\Delta T}_{k,k-1}\right) \tag{24}$$

Since the prior of ϕ is log normal, it is more convenient to sample from $\log(\phi)$. We use transformation $\Phi \equiv \log(\phi)$, and posterior for Φ is

$$P(\Phi \mid \cdot) \propto |\mathbf{R}|^{-\kappa/2}\cdot\exp\left(\frac{-\left(\Phi-\mu_\phi\right)^2}{2\sigma_\phi^2} - \frac{1}{2\sigma^2}\sum_{k=1}^{\kappa}\mathbf{\Delta T}_{k,k-1}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{\Delta T}_{k,k-1}\right) \tag{25}$$

We use Metropolis sampling in this step, and trial values are normal distributions

$$\Phi^* \mid \Phi_{t-1} \sim N\left(\Phi_{t-1}, \sigma_{\Phi,MH}^2\right) \tag{26}$$

A.8 τ_I^2

Prior is inverse-gamma

$$\sim InvGamma\left(\lambda_I, \nu_I\right) \tag{27}$$

Posterior is inverse-gamma

$$\sim InvGamma\left(\lambda_I + \frac{M_I}{2}, \nu_I + \frac{1}{2}\sum_{k=1}^{k}\mathbf{r}_{I,k}^{\mathrm{T}}\mathbf{r}_{I,k}\right) \tag{28}$$

where

$$\mathbf{r}_{I,k} = \mathbf{W}_{I,k} - \left[\mathbf{H}_k\mathbf{T}_k + \mathbf{B}_k\right]_I \tag{29}$$

A.9 τ_P^2

Prior is inverse-gamma

$$\sim InvGamma\left(\lambda_P, \nu_P\right) \tag{30}$$

Posterior is inverse-gamma

$$\sim invGamma\left(\lambda_P + \frac{M_P}{2}, \nu_P + \frac{1}{2}\sum_{k=1}^{\kappa} \mathbf{r}_{P,k}^{\mathrm{T}} \mathbf{r}_{P,k}\right) \tag{31}$$

where

$$\mathbf{r}_{P,k} = \mathbf{W}_{P,k} - \left[\mathbf{H}_k \mathbf{T}_k + \mathbf{B}_k\right]_P \tag{32}$$

A.10 β_1

Prior is normal

$$\sim N\left(\eta_1, \delta_1^2\right) \tag{33}$$

where

$$\eta_1 = \left[\frac{\left(1-\tau_P^2\right)\left(1-\alpha^2\right)}{\sigma^2}\right]^{-1/2} \tag{34}$$

Posterior is normal

$$\sim N\left(\Psi_{\beta_1} V_{\beta_0}, \Psi_{\beta_1}\right) \tag{35}$$

where

$$V_{\beta_1} = \frac{\eta_1}{\delta_1^2} + \frac{1}{\tau_P^2}\sum_{k=1}^{\kappa} \mathbf{T}_k^{\mathrm{T}}\left(\mathbf{W}_{P,k} - \beta_0 \mathbf{1}_{N_{P,k}}\right) \tag{36}$$

$$\Psi_{\beta_1} = \left(\frac{1}{\delta_1^2} + \frac{1}{\tau_P^2}\sum_{k=1}^{\kappa} \mathbf{T}_{P,k}^{\mathrm{T}} \mathbf{T}_{P,k}\right)^{-1} \tag{37}$$

A.11 β_0

Prior is normal

$$\sim N\left(\eta_0, \delta_0^2\right) \tag{38}$$

where $\eta_0 = -\mu_0 \eta_1$ is the negative of the product of prior means for μ and β_1
Posterior is normal

$$\sim N\left(\Psi_{\beta_0} V_{\beta_0}, \Psi_{\beta_0}\right) \tag{39}$$

where

$$V_{\beta_0} = \frac{\eta_0}{\delta_0^2} + \frac{1}{\tau_P^2} \sum_{k=1}^{\kappa} \mathbf{1}_{N_{P,k}}^{\mathrm{T}} \left(\mathbf{W}_{P,k} - \beta_1 \mathbf{H}_{P,k} \mathbf{T}_k \right) \tag{40}$$

$$\Psi_{\beta_0} = \left(\frac{1}{\delta_0^2} + \frac{M_P}{\tau_P^2} \right)^{-1} \tag{41}$$