Probabilistic Robotics Course

Differentiation

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Outline

- Computing Derivatives
- Analytical
- Numerical Differentiation
- Automatic Differentiation

Approaching the problem

We want to develop a KF based algorithm to track the position of Orazio as it moves

The inputs of our algorithms will be

- velocity measurements
- landmark measurements

The prior knowledge about the map is represented by the location of each landmark in the world

Derivatives

In many cases one has to compute derivatives of complicated multivariate functions

Computing them by hand is

- tedious
- error prone
- requires a lot of time

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

Computers like doing boring stuff

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Numerical Differentiation

Use the definition of derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Given a set of perturbation vectors

$$\epsilon_1 = \begin{pmatrix} \epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} 0 \\ \epsilon \\ 0 \\ \vdots \end{pmatrix} \cdots, \epsilon_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \end{pmatrix}$$

Symmetry around linearization point leads to lower numerical errors

we compute the i[th] column by the following

$$\frac{\partial \mathbf{f}}{\partial x_i} \simeq \frac{\mathbf{f}(\mathbf{x}+\epsilon_i) - \mathbf{f}(\mathbf{x}-\epsilon_i)}{2\epsilon}$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Each column requires evaluating ***f*** 2 times

Numerical Differentiation

Choosing epsilon might be non trivial

- too small leads to machine precision errors
- too large poor derivative
- computation might be lowered

However

- easy to implement
- most of the times it works well
- can be used to check your hand-computed derivatives

Automatic Differentiation

Can we get the computer giving us the exact value of the derivative at a point, **without** computing the derivatives analytically?

Derivatives are mechanic!

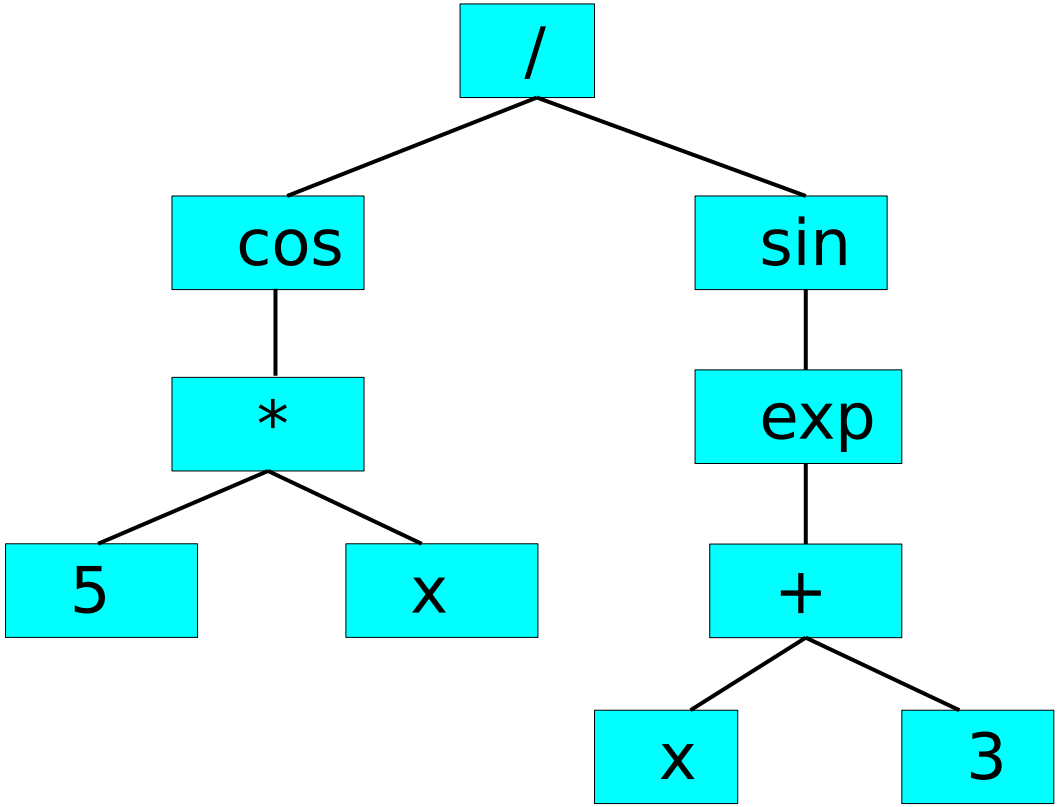
Can't we get the computer **evaluating** them for us?

Parsing Tree

Consider the following expression

```
cos(5*x)/(sin(exp(x+3)))
```

Its parsing tree looks like that



Chain Rule

To **evaluate** the derivative of a nested function

$$f(g(x))$$

in a point $\breve{x}$ we need to know

- The **formula** of the derivative of *f*

$$\frac{\partial f(y)}{\partial y} = f'(y)$$

- The **value** of the argument of *f*

$$u = g(\breve{x})$$

- The **value** of the derivative of the argument

$$u' = g'(\breve{x})$$

Chain rule tells us that

$$\frac{\partial f(g(x))}{\partial x} |_{x=\breve{x}} = f'(u)u'$$

Parsing Tree for Derivatives

For a generic function, we can use the parsing tree to compute

- the value of a function
- the value of the derivative of the function

We need to replace the basic type (float/double) with a pair (u, u'), and redefine the operators consistently

Atoms and Unary Functions

Atoms

- The variable used for differentiation becomes a pair [x, 1]
- All constants become a pair [c_i, 0]

Transcendental functions

- sin([u,u'])=[sin(u),cos(u)*u']
- cos([u,u'])=[cos(u),-sin(u)*u']
- exp([u,u'])=[exp(u),exp(u)*u']
- ....

Binary Functions (operators)

Sum, Subtraction, Multiplications and Division are implemented by applying the derivative rules on their arguments

- [u,u']+[v,v']=[u+v,u'+v']
- [u,u']-[v,v']=[u-v,u'-v']
- [u,u']*[v,v']=[uv,u'v+v'u]
- [u,u']/[v,v']=[uv,(u'v-v'u)/v^2]

Template will do

Have a look at the **autodiff** example on the course page.

It's a minimal c++ implementation, that can embedded in Eigen.