Probabilistic Robotics Course

Particle Distributions
Particle Filters

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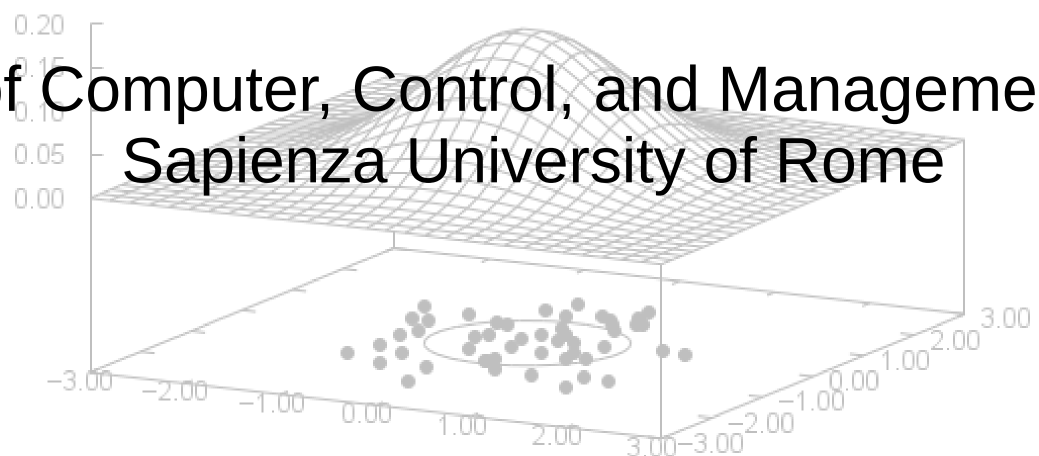


Sampling from a Distribution

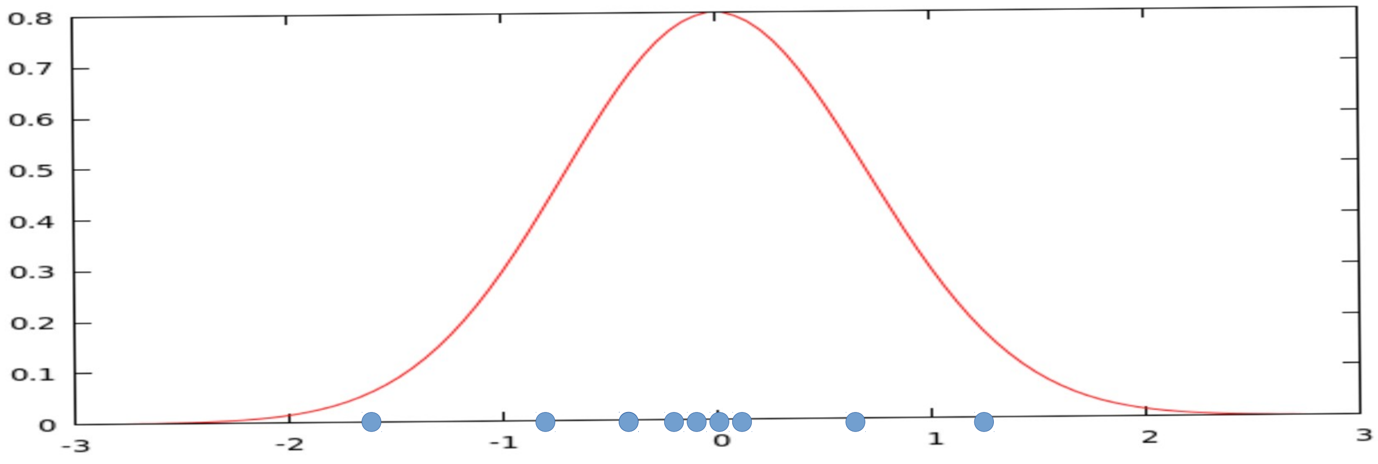
Sampling means generating a set of samples, given we know a distribution:

$$x^{(i)} \sim p(x)$$

Most of the random number generators produce samples from the uniform distribution:

$$y^{(i)} \sim U(0,1)$$

How can we generate samples from *p(x)?*



Generating Samples

We assume to have p(x) in closed form

1.compute the cumulative distribution

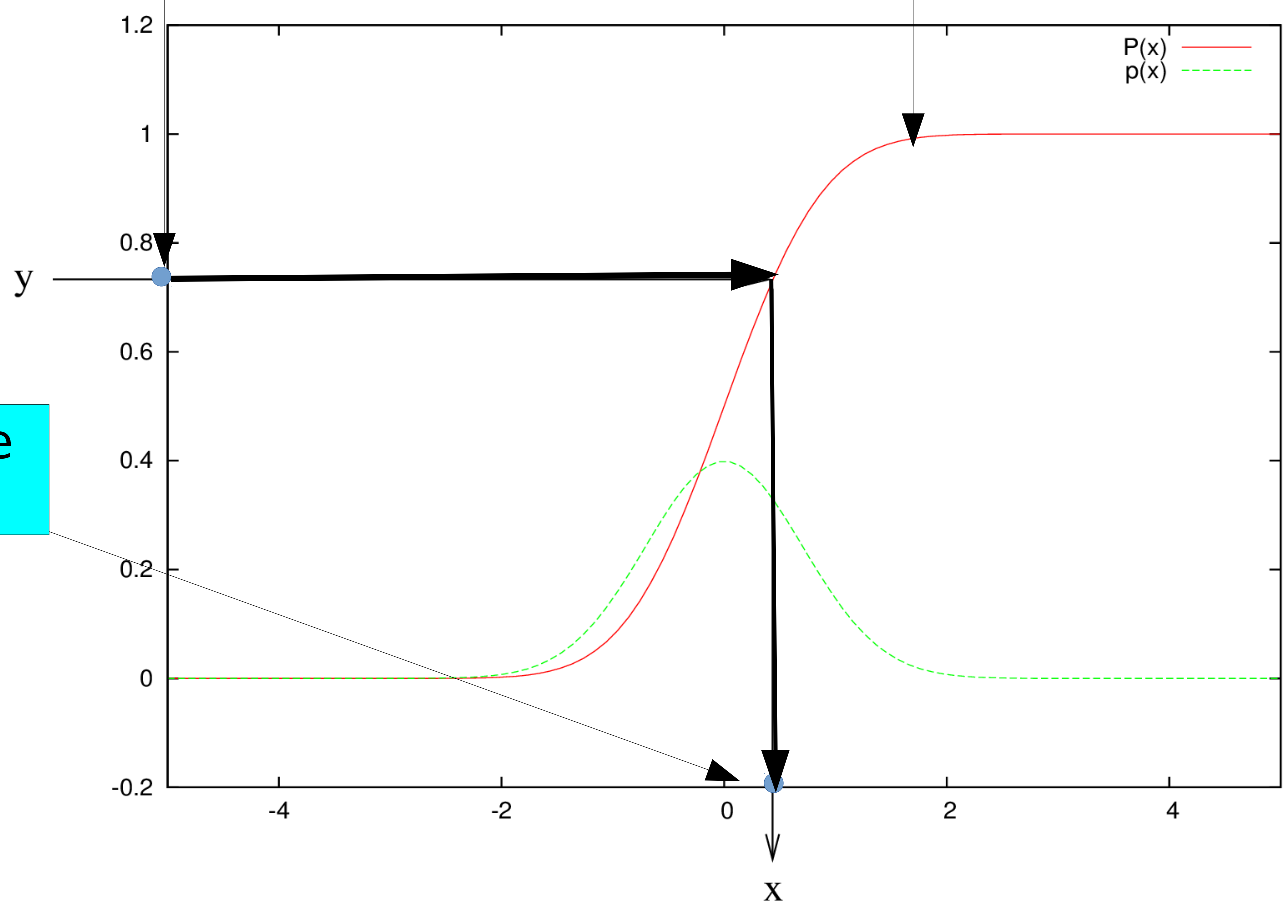
$$P(x) = \int_{-\infty}^{x} p(x')dx'$$

2. draw a sample from U(0,1)

$$y^{(i)} \sim U(0,1)$$

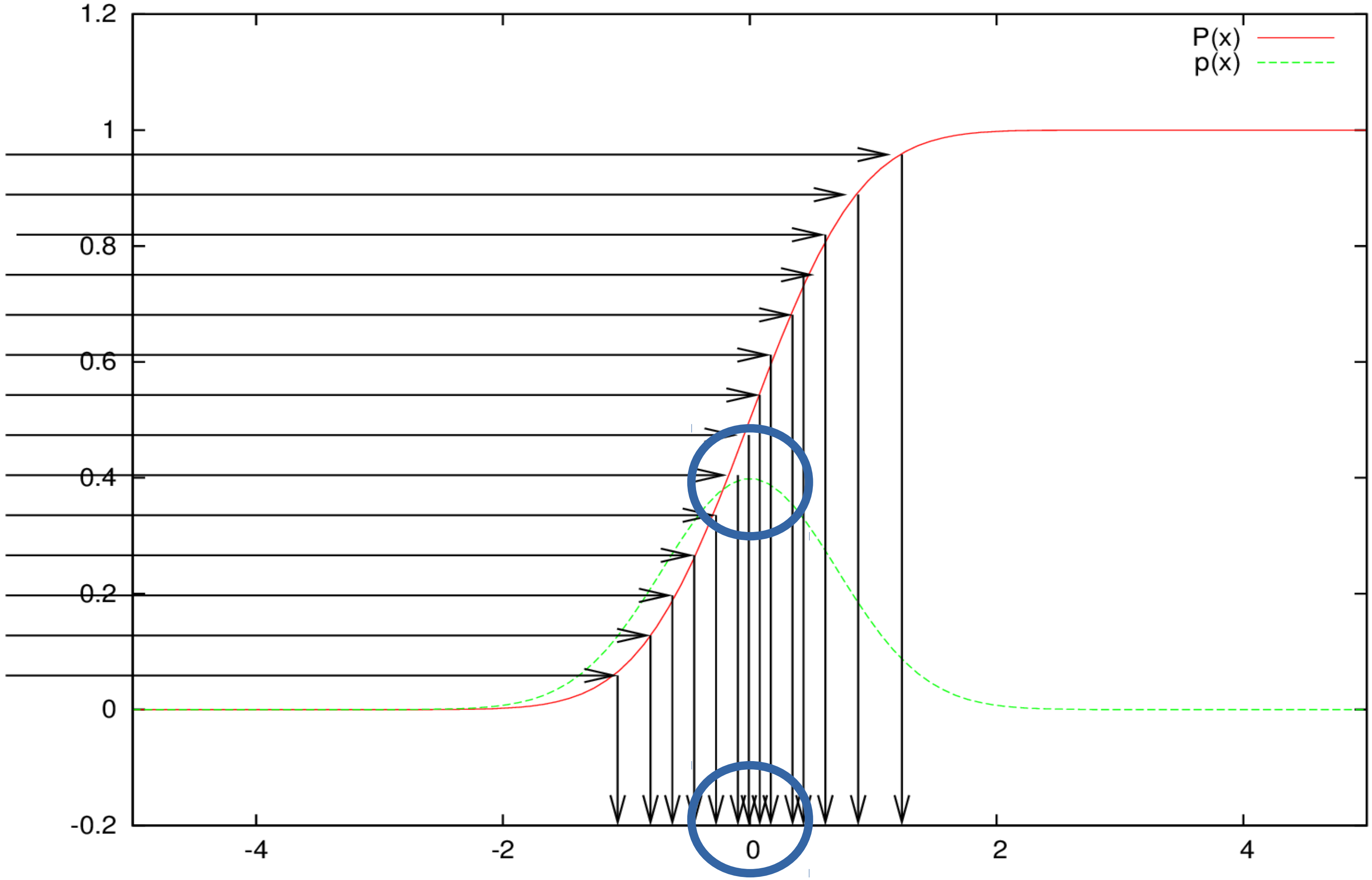
3. compute the inverse of the cumulative at $y^{(i)}$

$$x^{(i)} = P^{-1}(y^{(i)})$$



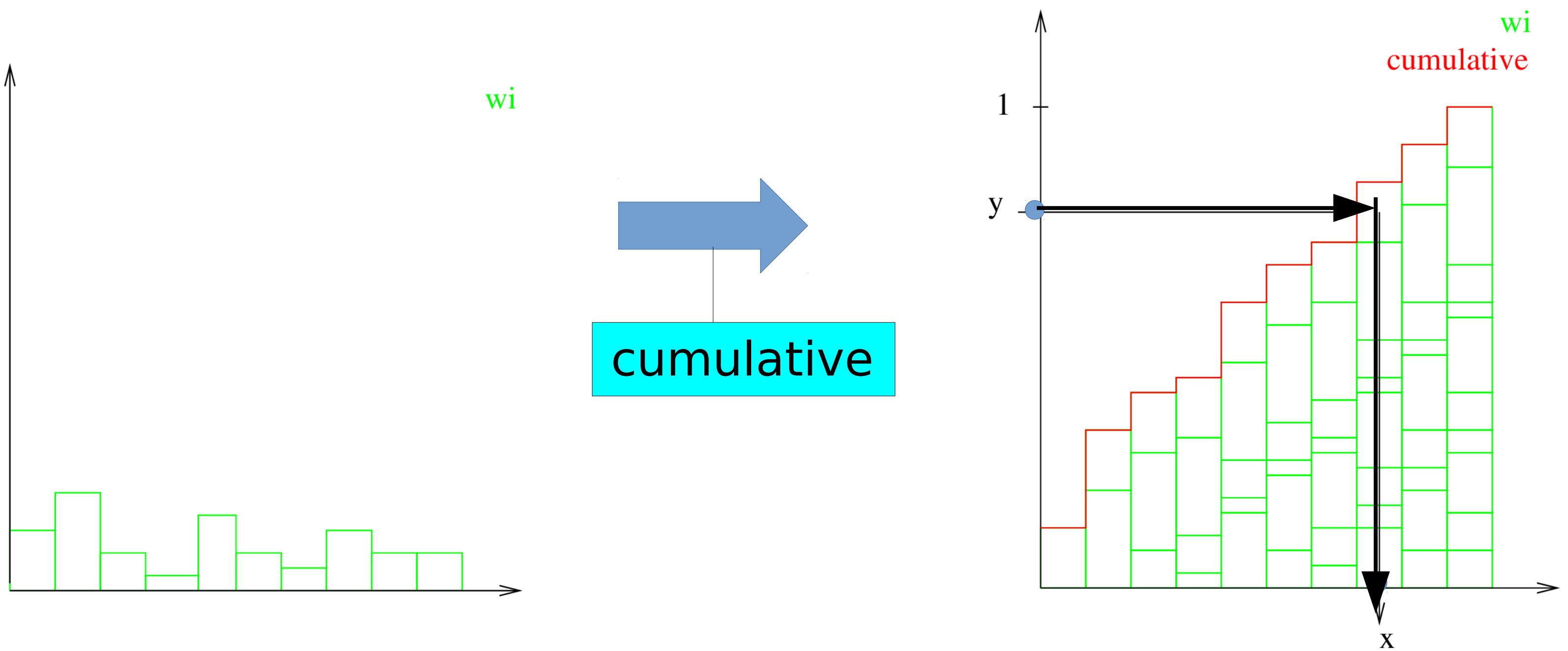
Generating Samples

Iterating this process generates denser samples where *p(x)* is higher



Discrete Case

If the distribution is discrete, we can do a similar process. The cumulative will look like a stair with uneven steps



Uniform Sampling

We will encounter the task of generating N samples from a discrete distribution.

Calling the random number generator N times might be expensive.

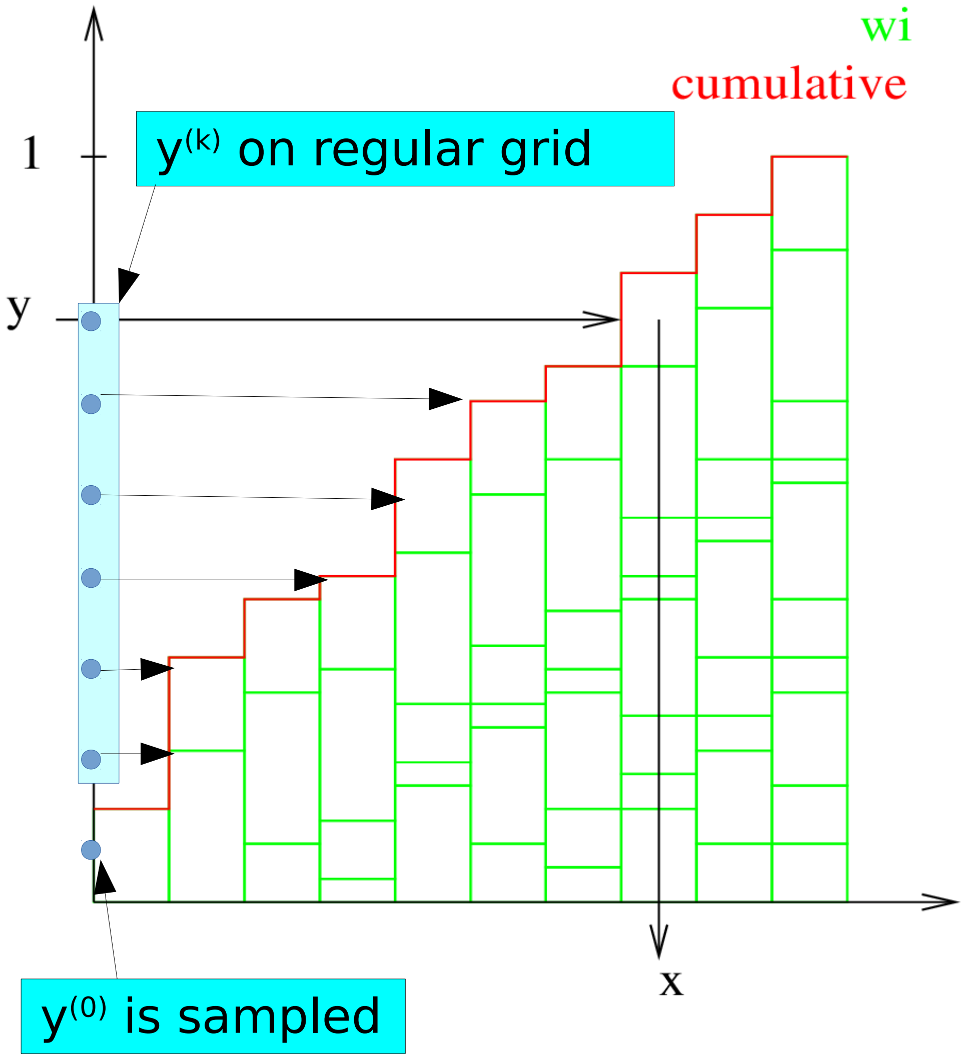
An alternative approach is uniform sampling

- sample a value *y(1)* between 1 and 1/N

$$y^{(0)} \sim U(0, \frac{1}{N})$$

- pick the remaining $y^{(i)}$ samples in a regular grid

$$y^{(k)} = y^{(0)} + \frac{k}{N}$$



Uniform Sampling

Octave function

```
function sampled_indices=uniformSample(weights, num_desired_samples)
  %normalize the weights (if they are not normalized)
  normalizer=1./sum(weights);
  %resize the indices
  sampled_indices=zeros(num_desired_samples,1);
  step=1./num_desired_samples;

  y0=rand()*step;      %sample between 0 and 1/num_desired_samples
  yi=y0;               %value of the sample on the y space
  cumulative =0;       %this is our running cumulative distribution
  sample_index=1;      %the index of output where we write the sampled idx
  for (weight_index=1:size(weights,1))
      cumulative += normalizer*weights(weight_index); %update cumulative
      % fill with current_weight_index
      % until the cumulative does not become larger than yi
      while (cumulative>yi)
            sampled_indices(sample_index)=weight_index;
            sample_index++;
            yi+=step;
      endwhile
  endfor
endfunction
```

Importance Sampling

Sometimes we do not know the sampling distribution, so we cannot compute the inverse cumulative. In this case, we can generate **weighted** samples

1.sample from a known distribution $\pi(x)$ possibly close to $p(x)$

$$x^{(i)} \sim \pi(x)$$

2.compute a weight by evaluating $\pi(x)$ and $p(x)$ at the sampled point

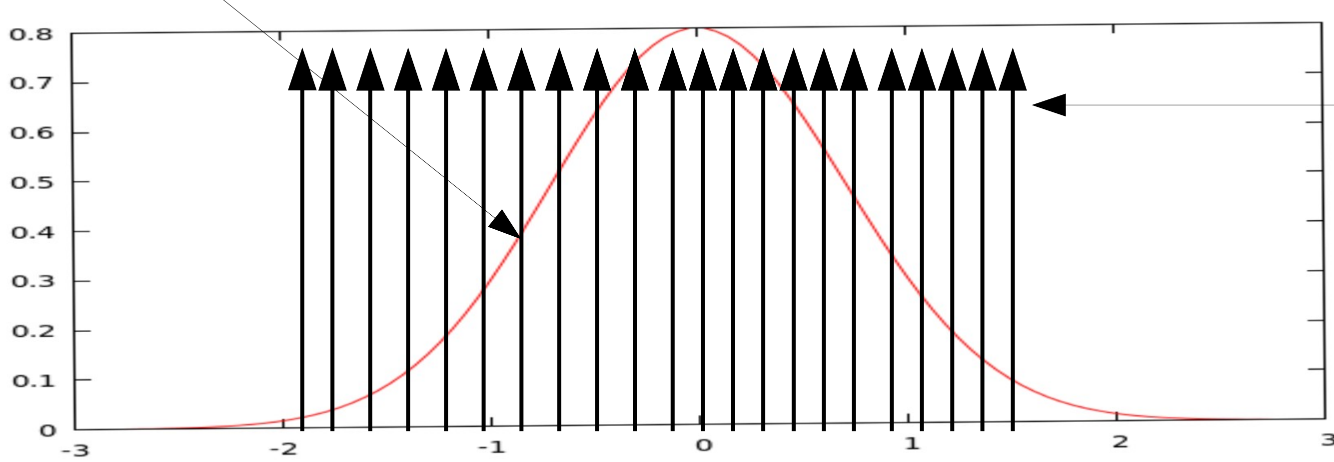
$$w^{(i)} = \frac{p(x^{(i)})}{\pi(x^{(i)})}$$

target distribution

proposal distribution

Gaussian (target)

samples generated by e.g. by a uniform (proposal)



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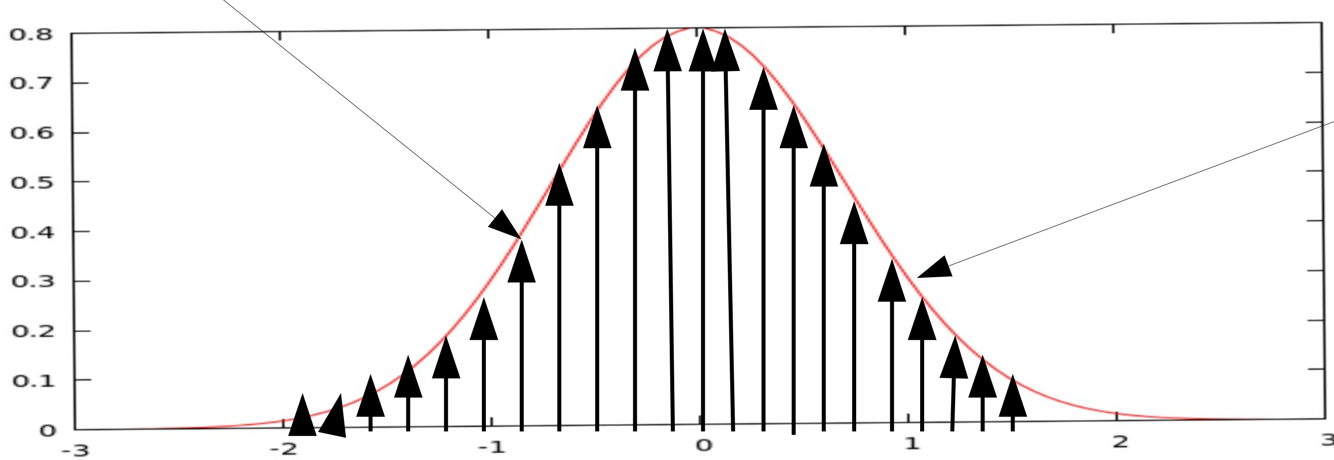


weights recover the difference between target and proposal

Choice of Proposal

Care must be taken when choosing the proposal

- The proposal $\pi(x)$ should cover all the relevant portion of the target $p(x)$ otherwise some feasible samples might not be generated

$$p(x) > 0 \Rightarrow \pi(x) > 0$$

In the ideal case of sampling from the target distribution, the weights would be uniform

Resampling

If we want to turn a weighed sample set into an unweighted one (uniform), we need:

- to repeat samples having high weights
- suppress samples with low weight

This can be done

- by drawing a set of *indices* j from the normalized weights distribution such that

$$p(j) = \boxed{\tilde{w}^{(j)}} = \frac{w^{(j)}}{\sum_i w^{(i)}}$$

normalize
d weights

Repeating the samples according to the indices generated through the sampling procedure

Resampling

How to proceed?

1. sample a set of indices with probability proportional to the weights

input weighed sample set

$$\left\{ \left\langle w^{(i)}, x^{(i)} \right\rangle \right\}$$

array of sample indices

$$\{j^{(i)}\} = \mathrm{uniformSample}(\{w^{(i)}\})$$

resulting samples

$$\tilde{x}^{(i)} = x^{(j(i))}$$

2. the new sample in position *i* is the old sample in position *j[i]*

Particle Densities

We can represent an approximation of a density function by a set of weighed samples

The "denser" the samples in a region, the higher will be the probability of that region

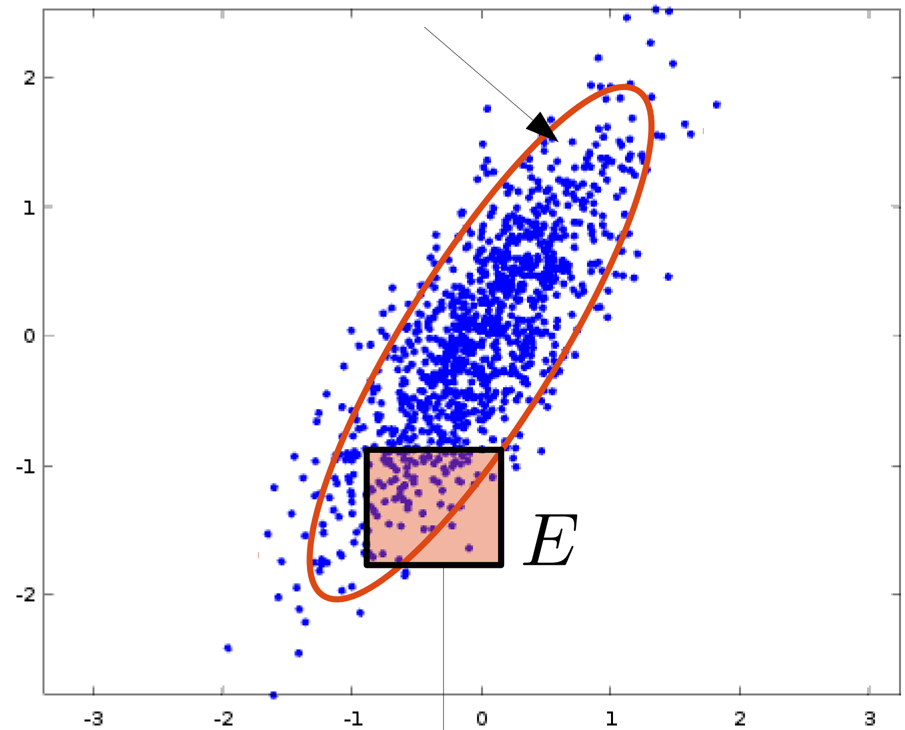
$$\mathbf{x}^{[i]} \sim p(\mathbf{x})$$

Dirac centered in $x^{(i)}$

$$p(\mathbf{x}) \simeq \sum_i w^{(i)} \delta(\mathbf{x} - \mathbf{x}^{(i)})$$

$$\int_E p(\mathbf{x}) d\mathbf{x} \simeq \sum_{\mathbf{x}^{[i]} \in E} w^{(i)}$$

Gaussian



The probability that x falls in a region E can be obtained by summing the *weights* in the region

Why Particles are Cool

Can represent arbitrary distributions

Easy to “visualize”

Easy to manipulate

Good for small state spaces

Transformation

Transformation is easy:

Sampled density

$$p(\mathbf{x}_a) \simeq \sum_i w^{(i)} \delta(\mathbf{x}_a - \mathbf{x}_a^{(i)})$$

$$\mathbf{x}_b = \mathbf{f}(\mathbf{x}_a)$$

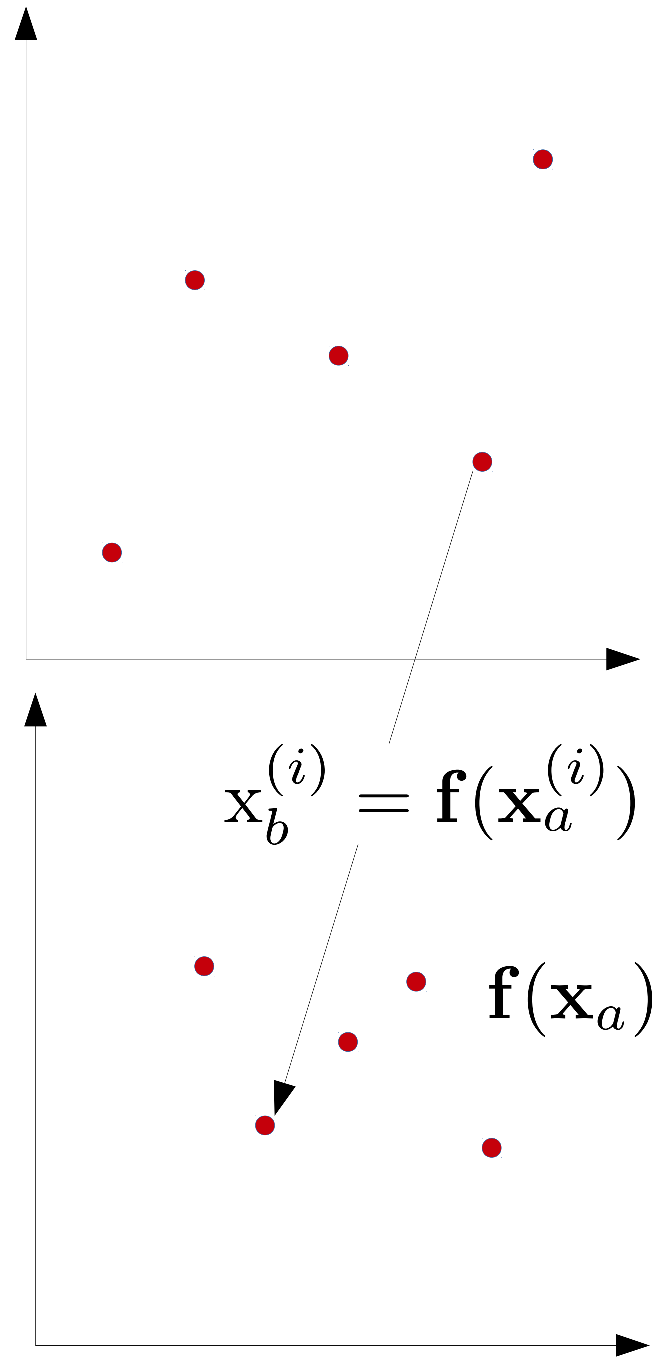
function of random variable

sampled density on x_b

$$p(\mathbf{x}_b) = \simeq \sum_i w^{(i)} \delta\left(\mathbf{x}_b - \mathbf{f}(\mathbf{x}_a^{(i)})\right)$$

$$\mathbf{x}_b^{(i)} = \mathbf{f}(\mathbf{x}_a^{(i)})$$

can be implemented by transforming each sample with f



Marginalization

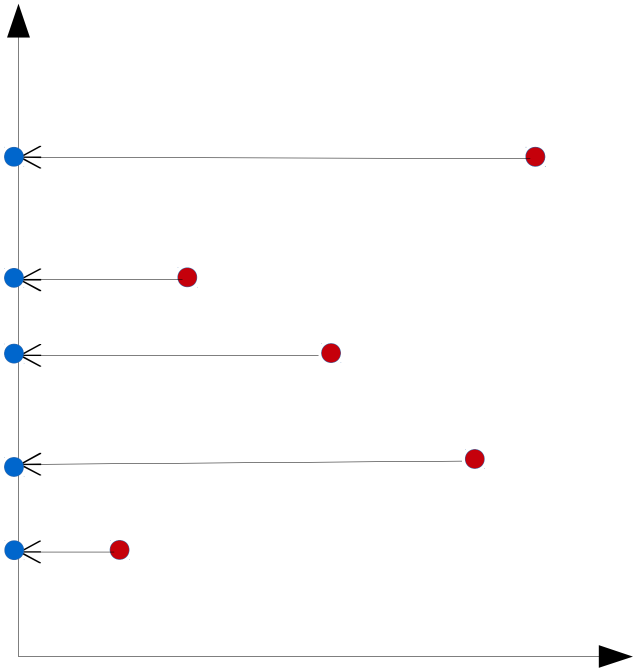
Marginalization just deletes from the sample set the coordinates of the marginalized component:

Sampled density on x_a,x_b

$$p(\mathbf{x}_a, \mathbf{x}_b) \simeq \sum_i w^{(i)} \delta \left(\begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} - \begin{pmatrix} \mathbf{x}_a^{(i)} \\ \mathbf{x}_b^{(i)} \end{pmatrix} \right)$$

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_b$$

$$= \simeq \sum_i w^{(i)} \delta \left(\mathbf{x}_a - \mathbf{x}_a^{(i)} \right)$$



Chain Rule

$$p(\mathbf{x}_a) \simeq \sum_i w^{(i)} \delta(\mathbf{x}_a - \mathbf{x}_a^{(i)})$$

Sampled density on x_a

$$p(\mathbf{x}_b | \mathbf{x}_a)$$

Conditional on $x_b|x_a$

$$\mathbf{x}_b^{(i)} \sim p(\mathbf{x}_b | \mathbf{x}_a^{(i)})$$

1. Generate a sample from the conditional, for each sample in the conditioning

$$p(\mathbf{x}_a, \mathbf{x}_b) \simeq \sum_i w^{(i)} \delta\left(\begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} - \begin{pmatrix} \mathbf{x}_a^{(i)} \\ \mathbf{x}_b^{(i)} \end{pmatrix}\right)$$

2. Stack the samples to get a **Particle** from the joint distribution

Conditioning

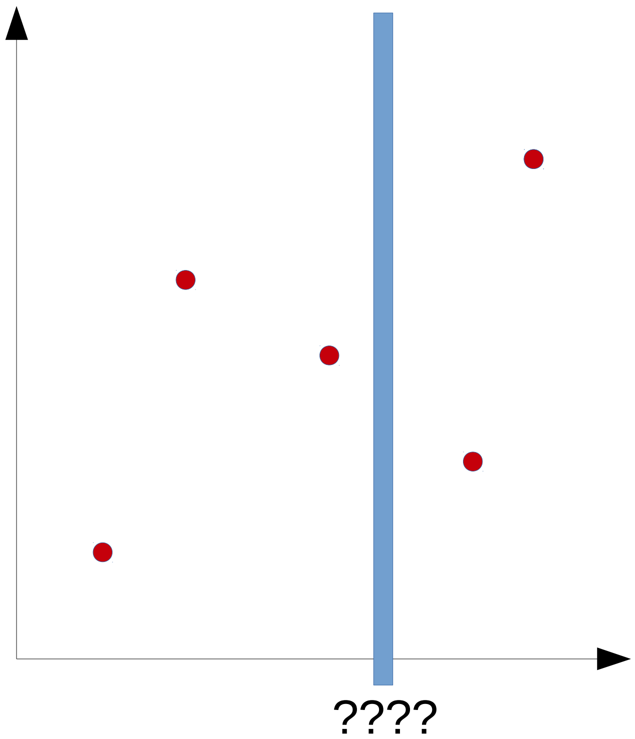
$$p(\mathbf{x}_a, \mathbf{x}_b) \simeq \sum_i w^{(i)} \delta\left(\begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} - \begin{pmatrix} \mathbf{x}_a^{(i)} \\ \mathbf{x}_b^{(i)} \end{pmatrix}\right)$$

$$p(\mathbf{x}_a|\mathbf{x}_b) = \frac{p(\mathbf{x}_a, \mathbf{x}_b)}{p(\mathbf{x}_b)}$$

Not easy

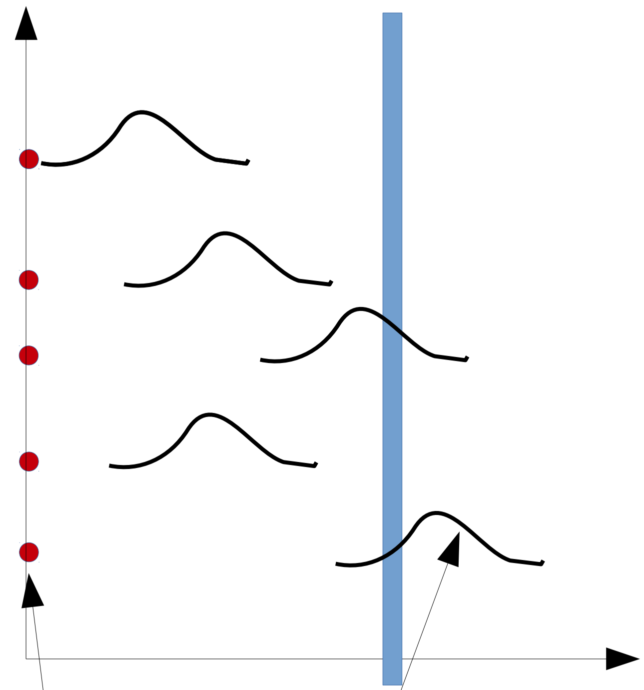
Reason:

- Samples do not like to be sliced



Conditioning

Things would be different if we would have for each sample a conditional distribution on x_b



$$p(\mathbf{x}_a, \mathbf{x}_b) \simeq \sum_i w^{(i)} \delta\left(\mathbf{x}_a - \underbrace{\mathbf{x}_a^{(i)}}_{\text{sample}}\right) \underbrace{p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})}_{\text{conditional given the sample}}$$

Conditioning

$$p(\mathbf{x}_a, \mathbf{x}_b) \simeq \sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})$$

mixture of conditional distributions for each sample

$$p(\mathbf{x}_a | \mathbf{x}_b) = \frac{p(\mathbf{x}_a, \mathbf{x}_b)}{p(\mathbf{x}_b)} = \frac{p(\mathbf{x}_a, \mathbf{x}_b)}{\int p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_a}$$

expand the conditioning through chain rule and marginalization

$$\simeq \frac{\sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})}{\int \left[\sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})\right] d\mathbf{x}_a}$$

apply the mixture approximation

Flip sum and integral

$$= \frac{\sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})}{\sum_i w^{(i)} p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)}) \int \left[\delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right)\right] d\mathbf{x}_a}$$

This is 1

Normalizer

$$= \frac{1}{\sum_i w^{(i)} p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})} \sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})$$

Conditioning

Note that conditioning only affects the weights:

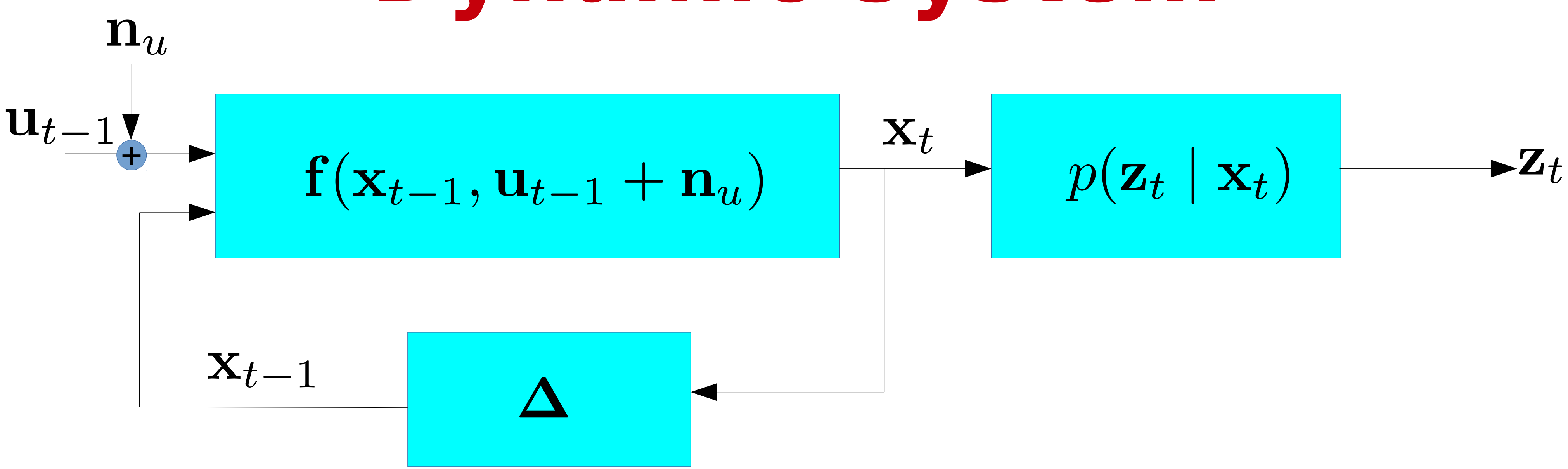
$$p(\mathbf{x}_a|\mathbf{x}_b) \simeq \frac{1}{\sum_i w^{(i)} p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})} \sum_i w^{(i)} \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right) p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})$$

$$= \eta \sum_i w^{(i)} p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)}) \delta\left(\mathbf{x}_a - \mathbf{x}_a^{(i)}\right)$$

$$w_{a|b}^{(i)} \propto w_a^{(i)} p(\mathbf{x}_b \mid \mathbf{x}_a^{(i)})$$

To implement conditioning we need to multiply each weight by the conditional of the sample evaluated at the conditioning variable

Dynamic System



$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$

we know the transition function

$$p(\mathbf{x}) \simeq \sum_i w^{(i)} \delta(\mathbf{x} - \mathbf{x}^{(i)})$$

state PDF is a set of weighed samples

$$\mathbf{n}_u \sim p(\mathbf{n}_u)$$

additive noise distributed according to a $p(n_u)$ we can sample from

$$p(\mathbf{z}_t \mid \mathbf{x}_t)$$

we can evaluate pointwise the observation model

Prediction

$$p(\mathbf{x}_{t-1|t-1}) \simeq \sum_i w^{(i)}_{t-1|t-1} \delta(\mathbf{x}_{t-1|t-1} - \mathbf{x}^{(i)}_{t-1|t-1})$$

prior

$$\mathbf{n}^{(i)}_u \sim p(\mathbf{n}_u)$$

1. generate I noise samples.
<x(i),n(i)> are samples from the joint distribution

$$p(\mathbf{x}_{t|t-1}) \simeq \sum_i w^{(i)}_{t-1|t-1} \delta(\mathbf{x}_{t|t-1} - \mathbf{f}(\mathbf{x}^{(i)}_{t-1|t-1}, \mathbf{u}_{t-1} + \mathbf{n}^{(i)}_u))$$

2. transform each sample with its noise trough f

$$\mathbf{x}^{(i)}_{t|t-1} = \mathbf{f}(\mathbf{x}^{(i)}_{t-1|t-1}, \mathbf{u}_{t-1} + \mathbf{n}^{(i)}_u)$$

samples of PDF after prediction

$$w^{(i)}_{t|t-1} = w^{(i)}_{t-1|t-1}$$

weights of PDF after prediction (unchanged)

Update

$$p(\mathbf{x}_{t|t-1}) \simeq \sum_i w^{(i)}_{t|t-1} \delta(\mathbf{x}_{t|t-1} - \mathbf{x}^{(i)}_{t|t-1})$$

prediction

$$p(\mathbf{x}_{t|t}) \simeq \sum_i w^{(i)}_{t|t-1} p(\mathbf{z}_t \mid \mathbf{x}^{(i)}_t) \delta(\mathbf{x}_{t|t-1} - \mathbf{x}^{(i)}_{t|t-1})$$

$$w^{(i)}_{t|t} = w^{(i)}_{t|t-1} p(\mathbf{z}_t \mid \mathbf{x}^{(i)}_{t|t-1})$$

conditioned

$$\mathbf{x}^{(i)}_{t|t} = \mathbf{x}^{(i)}_{t|t-1}$$

Weights after update. Multiply each weight of prediction by the likelihood of the measurement

Samples after update. Unchanged.

Resample a new generation to focus computation on likely regions of the state space

Particle Filter (Wrapup)

Predict

$$\mathbf{n}_u^{(i)} \sim p(\mathbf{n}_u)$$
$$\mathbf{x}_{t|t-1}^{(i)} = \mathbf{f}(\mathbf{x}_{t-1|t-1}^{(i)}, \mathbf{u}_{t-1} + \mathbf{n}_u^{(i)})$$
$$w_{t|t-1}^{(i)} = w_{t-1|t-1}^{(i)}$$

1. generate I noise samples.

2. apply f to each sample+noise

Update

$$w_{t|t}^{(i)} = w_{t|t-1}^{(i)} p(\mathbf{z}_t \mid \mathbf{x}_{t|t-1}^{(i)})$$
$$\mathbf{x}_{t|t}^{(i)} = \mathbf{x}_{t|t-1}^{(i)}$$

3. multiply each weight by the conditional of the measurement evaluated at the weight

4. Resample a new generation to focus computation on likely regions of the state space