# LINEAR ALGEBRA

4th STEPHEN H.FRIEDBERG, ARNOLD J.INSEL, LAWRENCE E.SPENCE

# Exercises Of Chapter 1-4

$http : //math.pusan.ac.kr/cafe\_home/chuh/$

# Contents

# §1. Vector Spaces

## 1.1. Introduction

1. Only the pairs in (b) and (c) are parallel

(a) $x = (3, 1, 2)$ and $y = (6, 4, 2)$

$\nexists\ 0 \neq t \in \mathbb{R}$ s.t. $y = tx$

(b) $(9, -3, -21) = 3(-3, 1, 7)$

(c) $(5, -6, 7) = -1(-5, 6, -7)$

(d) $x = (2, 0, -5)$ and $y = (5, 0, -2)$

$\nexists\ 0 \neq t \in \mathbb{R}$ s.t. $y = tx$

2. (a) $x = (3, -2, 4) + t(-8, 9, -3)$

(b) $x = (2, 4, 0) + t(-5, -10, 0)$

(c) $x = (3, 7, 2) + t(0, 0, -10)$

(d) $x = (-2, -1, 5) + t(5, 10, 2)$

3. (a) $x = (2, -5, -1) + s(-2, 9, 7) + t(-5, 12, 2)$

(b) $x = (-8, 2, 0) + s(9, 1, 0) + t(14, -7, 0)$

(c) $x = (3, -6, 7) + s(-5, 6, -11) + t(2, -3, -9)$

(d) $x = (1, 1, 1) + s(4, 4, 4) + t(-7, 3, 1)$

4. $x = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n,\ i = 1, 2, \cdots, n$

$0 = (0, 0, \cdots, 0) \in \mathbb{R}^n$ s.t. $x + 0 = x, \forall x \in \mathbb{R}^n$

5. $x = (a_1, a_2) \Rightarrow tx = t(a_1, a_2) = (ta_1, ta_2)$

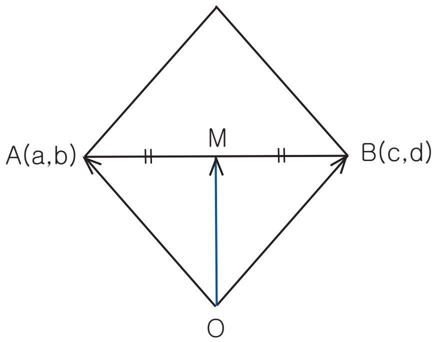
6. $A + B = (a + c, b + d),\ M = (\frac{a+c}{2}, \frac{b+d}{2})$

7.

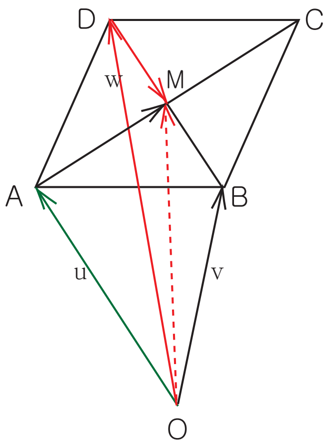
$C = (v - u) + (w - u) + u = v + w - u$

$\overrightarrow{OD} + \frac{1}{2}\overrightarrow{DB} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$i.e.\ w + \frac{1}{2}(v - w) = \frac{1}{2}(v + w) = u + \frac{1}{2}(v + w - u - u)$



1−6



1−7

## 1.2. Vector Spaces

1.

(a) T

(b) F (If $\exists 0'$ s.t. $x + 0' = x, \forall x \in V$, then $0' = 0 + 0' = 0' + 0 = 0, \ \therefore \ 0' = 0$)

(c) F (If $x = 0, a \neq b$, then $a \cdot 0 = b \cdot 0$ but $a \neq b$)

(d) F (If $a = 0, x \neq y$, then $a \cdot x = 0 = a \cdot y$ but $x \neq y$)

(e) T

(f) F (An $m \times n$ matrix has $m$ rows and $n$ cilumns)

(g) F

(h) F (If $f(x) = ax + b, g(x) = -ax + b$, then deg$f$=deg$g$=1, deg($f + g$)=0)

(i) T (p.10 Example4)

(j) T

(k) T (p.9 Example3)

2. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3. $M_{13} = 3, \ M_{21} = 4, \ M_{22} = 5$

4.
(a) $\begin{pmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{pmatrix}$

(d) $\begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & 40 \end{pmatrix}$

(e) $2x^4 + x^3 + 2x^2 - 2x + 10$

(f) $-x^3 + 7x^2 + 16$

(g) $10x^7 - 30x^4 + 40x^2 - 15x$

(h) $3x^5 - 6x^3 + 12x + 6$

5.
$$U = \begin{pmatrix} 8 & 3 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 9 & 1 & 4 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 17 & 4 & 5 \\ 6 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix}$$

(∗) The total number of crossings

| | Fall | Spring | Winter |
|---|---|---|---|
| Brook trout | 17 | 4 | 5 |
| Rainbow trout | 6 | 0 | 0 |
| Brown trout | 4 | 1 | 0 |

6.
$$M = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 5 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \end{pmatrix}, \ 2M - A = \begin{pmatrix} 3 & 1 & 1 & 4 \\ 4 & 0 & 1 & 3 \\ 5 & 2 & 1 & 9 \end{pmatrix}$$

35 suites were sold during the June sale

7. We are going to show that $f(t) = g(t)$ and $(f+g)(t) = h(t),\ \forall t \in S = \{0,1\}$

(i) $f(0) = 1 = g(0), f(1) = 3 = g(1)$

(ii) $(f+g)(0) = 2 = h(0), (f+g)(1) = 6 = h(1)$

$\therefore$ In $\mathcal{F}(S,R), f = g$ and $f + g = h$

8. VS 7, 8

9. (a) Exercise 1(b)

(b) If $\exists y'$ s.t. $x + y' = 0$, then $x + y = 0x + y'$

By the theorem 1.1, $y = y'$

(c) $a \cdot 0 + a \cdot 0 = a(0+0) = a \cdot 0 = a \cdot 0 + 0$

By the theorem 1.1, $a \cdot 0 = 0$

10. $V = D(\mathbb{R}),\ \forall s \in \mathbb{R}$

(1) $\forall f, g \in V,\ f + g = g + f$

$(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$

$\therefore\ f + g = g + f$

(2) $\forall f, g, h \in V,\ (f+g) + h = f + (g+h)$

$((f+g)+h)(s) = (f+g)(s) + h(s) = f(s) + g(s) + h(s)$

$= f(s) + (g+h)(s) = (f + (g+h))(s)$

$\therefore\ (f+g) + h = f + (g+h)$

(3) $\exists 0 \in V$ s.t. $f+0=f,\ \forall f \in V$

$(f+f')(s)=f(s)+f'(s)=0(s)$

$\therefore\ f'(s)=0(s)-f(s)=(0-f)(s)=(-f)(s)$

$\therefore\ f'=-f$

(5) $\forall f \in V,\ 1\cdot f=f$

$(1\cdot f)(s)=1(f(s))=f(s)$

$\therefore\ 1\cdot f=f$

(6) $\forall a,b \in F, (ab)f=a(bf)$

$((ab)f)(s)=(ab)f(s)=a(bf(s))=a(bf)(s)$

$\therefore\ (ab)f=a(bf)$

(7) $\forall a \in F,\ a(f+g)=af+ag$

$a(f+g)(s)=a(f(s)+g(s))=af(s)+ag(s)=(af+ag)(s)$

$\therefore\ a(f+g)=af+ag$

(8) $\forall a,b \in F,\ (a+b)f=af+bf$

$(a+b)f(s)=af(s)+bf(s)=(af+bf)(s)$

$\therefore\ (a+b)f=af+bf$

11. $V=\{0\},\ \forall a,b \in F$

12.

(1) $\forall f,g \in V, t \in \mathbb{R}$

$(f+g)(-t)=f(-t)+g(-t)=f(t)+g(t)=(f+g)(t)\ \therefore\ f+g \in V$

$(g+f)(-t) = (g+f)(t) \ \therefore \ g+f \in V$

$\therefore \ f+g = g+f$

(2) $\forall f, g, h \in V, \ (f+g)+h = f+(g+h)$

(3) $\exists 0 \in V$ s.t. $f+0 = f, \ \forall f \in V$

(4) $\exists f' \in V$ s.t. $f+f' = 0, \ \forall f \in V, \ f' = -f$

(5) $\forall f \in V, 1f = f$

(6) $\forall a, b \in F, \ (ab)f = a(bf)$

$((ab)f(-t) = (ab)f(t), \ \therefore abf \in V)$

(7) $\forall a \in F, \ \forall f, g \in V, \ a(f+g) = af + ag$

(8) $\forall a, b \in F, \ \forall f \in V, \ (a+b)f = af + bf$

13. No, (VS 4) fails

(VS 3) $\exists \ (0,1) \in V$ s.t. $(a_1, a_2) + (0,1) = (a_1, a_2), \ \forall (a_1, a_2) \in V$

(VS 4) If $a_2 = 0$, then $\nexists (b_1, b_2) \in V$ s.t. $(a_1, 0) + (b_1, b_2) = (0, 1)$

14. Yes ($\because \ \mathbb{R} \subseteq \mathbb{C}$)

15. No ($\because \ \mathbb{C} \subsetneq \mathbb{R}$)

$\alpha \in F = \mathbb{C}, \ \alpha x \notin V = \mathbb{R}^n, \ \forall x \in V$

16. Yes ($\because \ \mathbb{Q} \subseteq \mathbb{R}$)

17. No, (VS 5) fails

(VS 5) If $a_2 \neq 0$, then $1(a_1, a_2) = (a_1, 0) \neq (a_1, a_2)$

18. No, (VS 1) fails

(VS 5) If $a_1 \neq b_1$, then it fails to hold (VS 1)

19. No, (VS 8) fails

(VS 8) If $c_1 + c_2 \neq 0,\ c_1 \neq 0,\ c_2 \neq 0$, then it fails to hold (VS 8)

20. (VS 1) $\forall \{a_n\}, \{b_n\} \in V$

$\{a_n\} + \{b_n\} = \{a_1 + b_1, a_2 + b_2, \cdots\} = \{b_1 + a_1, b_2 + a_2, \cdots\} = \{b_n\} + \{a_n\}$

(VS 2) $(\{a_n\} + \{b_n\}) + \{c_n\} = \{a_n\} + (\{b_n\} + \{c_n\})$

(VS 3) $\exists \{0\}$ s.t. $\{a_n\} + \{0\} = \{a_n\}$

(VS 4) $\exists \{-a_n\}$ s.t. $\{a_n\} + \{-a_n\} = \{a_n\} - \{a_n\} = \{0\}, \forall \{a_n\} \in V$

(VS 5) $\exists \{1\}$ s.t. $\{1\}\{a_n\} = \{a_n\}, \forall \{a_n\} \in V$

(VS 6) $\forall \alpha, \beta \in F,\ (\alpha\beta)\{a_n\} = \alpha(\beta\{a_n\})$

(VS 7) $\forall \alpha \in F, \forall \{a_n\}, \{b_n\} \in V, \alpha(\{a_n\} + \{b_n\}) = \alpha\{a_n\} + \alpha\{b_n\}$

(VS 8) $\forall \alpha, \beta \in F,\ (\alpha + \beta)\{a_n\} = \alpha\{a_n\} + \beta\{a_n\}$

22. $2^{mn}$

## 1.3. Subspaces

1. (a) F (p.1 Definition of subspace)

(b) F ($0 \notin \varnothing$)

(c) T ($V$ and $\{\varnothing\}$ are subspaces of $V$)

(d) F (p.19 Theorem 1.4)

(e) F

(f) F (p.18 Example 4)

(g) F ($(0,0,0) \in W$, but $(0,0,0) \notin R^2$)

2. (b), (c), (e), (f), (g) are not square matrices

(a) -5, (d) 12, (h) -6

3. $\forall A, B \in M_{m\times n}(F), a, b \in F (1 \leq i \leq m, 1 \leq j \leq n)$

$(aA+bB)^t_{ij} = (aA+bB)_{ji} = (aA)_{ji} + (bB)_{ji}$

$= a(A)_{ji} + b(B)_{ji} = aA^t_{ij} + bB^t_{ij} = (aA^t + bB^t)_{ij}$

$\therefore \ (aA+bB)^t = aA^t + bB^t$

4. $(A^t)^t_{ij} = (A^t)_{ji} = A_{ij}$

5. $(A+A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$

$\therefore \ A + A^t$ is symmetric

6. $tr(aA+bB) = \sum_{i=1}^{n}(aA+bB)_{ii} = \sum_{i=1}^{n}(aA)_{ii} + \sum_{i=1}^{n}(bB)_{ii} = atr(A) + btr(B)$

7. $A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & O & \\ & O & a_{33} & \\ & & & a_{44} \end{pmatrix} \Rightarrow \quad A^t = A$

$\therefore$ $A$ is symmetric

8. (a) Yes

(b) No ($(0,0,0) \notin W_2$)

(c) Yes

(d) Yes

(e) No ($(0,0,0) \notin W_5$)

(f) No $x+y \notin W_6$), $\forall x, y \in W_6$

9. (1) $W_1 \cap W_3 = \{0\}$ is a subspace of $R^3$

(2) $W_1 \cap W_4 = W_1$ is a subspace of $R^3$

(3) $W_3 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 \mid 3a_1 = 11a_2, 3a_3 = 23a_2\}$ is a subspace of $R^3$

10. (i) $W_1$ is a subspace of $F^n$

(ii) $W_2$ is not a subspace of $F^n$

($\because (0,0,0) \notin W_2$)

11. No (The given set is not closed under addition)

(Example) $\forall f, g \in W$

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,\ degf = n$

$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0,\ degg = n$

If $b_n = -a_n$, then $deg(f+g) = n-1$

$\therefore\ f + g \notin W$

$\therefore\ W$ is not a subspace of $V$

15. Yes

17. ($\Rightarrow$) Theorem 1.3

($\Leftarrow$) $W$ is a subspace of $V$

(i) $0 \in W$ ($a = 0 \in F$: field )

(ii) $ax \in W$

(iii) $x + y \in W$

18. ($\Rightarrow$) Theorem 1.3

($\Leftarrow$) $W$ is a subspace of $V$

(i) $0 \in W$

(ii) $ax \in W$ ($y = 0 \in W$, by (i) )

(iii) $x + y \in W$ ($a = 1 \in F$: field)

19. ($\Leftarrow$) If $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$ is a subspace of $V$

If $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$ is a subspace of $V$

$\therefore\ W_1 \cup W_2$ is a subspace of $V$

($\Rightarrow$) If $\exists\ a \notin W_1, a \in W_2\ \exists\ b \notin W_2, b \in W_1$,

then $ab \in W_1 \cup W_2$

But if $ab \in W_1$, then $a = (ab)b^{-1} in W_1$

if $ab \in W_2$, then $b = (ba)a^{-1} in W_2$

It's a contrdiction

$\therefore\ W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

20. Induction on $n$

In case of $n = 2$, it's clear

Assume that this holds for $n = k - 1$ $(k > 2)$

By the induction hypothesis,

$\sum_{i=1}^{k-1} a_i w_i + a_k w_k \in W,\ w_k \in W,\ a_k \in F$

$\therefore\ \forall w_i \in W,\ \sum_{i=1}^{n} a_i w_i \in W$, where $\forall a_i \in F,\ i = 1, 2, \cdots, n$

21. (i) $\{0\} \to 0,\ \therefore\ \{0\} \in W$

(ii) $\lim\{a_n\} = a, \lim\{b_n\} = b$

$\lim(\{a_n\} + \{b_n\}) = \lim\{a_n\} + \lim\{b_n\} = a + b$

$\therefore\ \{a_n\} + \{b_n\} \in W$

(iii) $\lim c\{a_n\} = c\lim\{a_n\}$

$\therefore\ c\{a_n\} \in W$

$\therefore\ W$ is a subspace of $V$

22.

Let $W_1 = \{g \in F(F_1, F_2) \mid g(-t) = g(t)$, for each $t \in F_1\}$

$W_2 = \{g \in F(F_1, F_2) \mid g(-t) = -g(t)$, for each $t \in F_1\}$

(1) $\forall g_1, g_2 \in W_1,\ \forall c \in F$

(i) $0 \in W_1$

(ii) $(g_1 + g_2)(-t) = g_1(-t) + g_2(-t) = g_1(t) + g_2(t) = (g_1 + g_2)(t)$

$\therefore\ g_1 + g_2 \in W_1$

(iii) $cg_1(-t) = c(g_1(t)) = cg_1(t)$

$\therefore\ cg \in W_1$

$\therefore\ W_1$ is a subspace of $V$

(2) $\forall g_1, g_2 \in W_2,\ \forall c \in F$

(i) $0 \in W_2$

(ii) $(g_1 + g_2)(-t) = g_1(-t) + g_2(-t) = -g_1(t) - g_2(t) = -(g_1 + g_2)(t)$

$\therefore\ g_1 + g_2 \in W_2$

(iii) $cg_1(-t) = c(-g_1(t)) = -cg_1(t)$

$\therefore\ cg \in W_2$

$\therefore\ W_2$ is a subspace of $V$

23. (a) $W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in_2\}$

(i) $0 = 0 + 0 \in W_1 + W_2$

(ii) $\forall x_1 + y_1,\ x_2 + y_2 \in W_1 + W_2$

$(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$

(iii) $\forall x_1 + y_1 \in W_1 + W_2,\ \forall c \in F$

$c(x_1 + y_1) = cx_1 + cy_1 \in W_1 + W_2$

(b) $\forall\ W$ as a subspace of $V$ s.t. $W_1 \subseteq W,\ W_2 \subseteq W$

$\forall x \in W_1 \subseteq W,\ \forall y \in W_2 \subseteq W \ \Rightarrow\ x + y \in W_1 + W_2 \subseteq W$

24. $V = F^n$

(i) $W_1 \cap W_2 = \{(0, 0, \cdots, 0)\}$

(ii) $W_1 + W_2 \subseteq V$ is clear

$\forall v = (a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_{n-1}, 0) + (0, 0, \cdots, 0, a_n) \in W_1 + W_2$

$\therefore\ V = W_1 + W_2$

25. (a) $W_1 \cap W_2 = \{0\}$

(b) $V = \{f(x) \in P(F) \mid f(x) = a_0 + a_1 x + \cdots\}$

$(\because)$ Since $W_1 \subseteq V$ and $W_2 \subseteq V$, $W_1 + W_2 \subseteq V$ is clear

$\forall f \in V,\ f = a_0 + a_2 x + \cdots + a_1 x + a_3 x^3 + \cdots \in W_1 + W_2$

$\therefore\ V = P(F) = W_1 \oplus W_2$

26. (a) $W_1 \cap W_2 = \{A \in M_{m\times n} \mid A_{ij} = 0\ \forall i, j\} = \{0\}$

(b) $W_1 + W_2 \subseteq V$ is clear

$$\forall A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n},$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & & \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(n-1)} & 0 \end{pmatrix} \in W_1 + W_2$$

$\therefore\ V = P(F) = W_1 \oplus W_2$

27. (a) $W_1 \cap W_2 = \{A \in M_{m\times n} \mid A_{ij} = 0\ \forall i, j\} = \{0\}$

(b) $W_1 + W_2 \subseteq V$ is clear

$$\forall A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n},$$

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & a_{mn} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{(n-1)n} \end{pmatrix} \in W_1 + W_2$$

$\therefore\ V = P(F) = W_1 \oplus W_2$

28. (a) $W_1 \cap W_2 = \{0\}$

(b) $W_1 + W_2 \subseteq V$ is clear

$\forall A \in V,\ A = (\frac{A-A^t}{2}) + (\frac{A+A^t}{2}) \in W_1 + W_2$

$\therefore\ V \in W_1 + W_2$

(cf) $char(F) \neq 2$

$\{\frac{1}{2}(A - A^t)\}^t = \frac{1}{2}(A^t - A) = -\{\frac{1}{2}(A - A^t)\}\ \therefore \frac{1}{2}(A - A^t) \in W_1$

$\{\frac{1}{2}(A - A^t)\}^t = \frac{1}{2}(A + A^t)\ \therefore \frac{1}{2}(A + A^t) \in W_2$

29. $W_1$ : strictly lower triangular matrices

$W_2$ : all symmetric matrices

(a) $W_1 \cap W_2 = \{0\}$

(b) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in V$ and $A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \in W_2$

then $\forall A = (A - A') + A' \in W_1 + W_2$

$\therefore\ V \subseteq W_1 + W_2$

$\therefore\ V = W_1 + W_2$

30.

$(\Rightarrow)$ If $x = x_1 + x_2 = x_3 + x_4,\ x_1, x_3 \in W_1$ and $x_2, x_4 \in W_2$

then $x_1 - x_3 = x_4 - x_2 \in W_1 \cap W_2 = \{0\}$

$\therefore\ x_1 = x_3,\ x_2 = x_4$

$(\Leftarrow)$ (i) $V = W_1 + W_2$ is clear

(ii) If $w \in W_1 \cap W_2$, then $w = w + 0 = 0 + 0$

$\therefore\ w = 0$

$\therefore\ W_1 \cap W_2 = \{0\}$

31. (a) ($\Leftarrow$) $v \in W \;\Rightarrow\; v + W = W$

$\therefore\; v + W$ is a subspace of $V$

($\Rightarrow$) $v + w_1, v + w_2 \in W$

$v + w_1 + v + w_2 = v + (w_1 + w_2 + v) \in v + W$

$\therefore\; w_1 + w_2 + v \in W$

$\therefore\; v \in W$

(b) $v_1 - v_2 \in W \Leftrightarrow v_1 - v_2 + W = W \;\Leftrightarrow\; v_1 + W = v_2 + W$

(c) Since $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$,

$(v_1 - v_1') + (v_2 - v_2') = (v_1 + v_2) - (v_1' + v_2') \in W$

$\therefore\; (v_1 + v_2) + W = (v_1' + v_2') + W$

$\therefore\; (v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$

Since $a(v_1 - v_1') \in W,\; a(v_1 + W) = a(v_1' + W)$

(d) $S = V/W = \{v + W \mid v \in V\}$ is a vector space

(VS 1) $(v_1+W)+(v_2+W) = (v_1+v_2)+W = (v_2+v_1)+W = (v_2+W)+(v_1+W)$

(VS 2) $\{(v_1 + W) + (v_2 + W)\} + (v_3 + W) = (v_1 + v_2) + W + v_3 + W$

$= (v_1 + v_2 + v_3) + W = v_1 + (v_2 + v_3) + W$

$= \{v_1 + W\} + \{(v_2 + v_3) + W\}$

$= (v_1 + W) + \{(v_2 + W) + (v_3 + W)\}$

(VS 3) $\exists 0 + W$ s.t. $(v + W) + (0 + W) = (v + 0) + W = v + W$

(VS 4) $\forall v + W,\; (1 + W)(v + W) = v + W$

(VS 5) $\exists\; -v + W$ s.t. $(v + W) + (-v + W) = 0 + W$

(VS 6) $\forall a, b \in F,\ (ab)(v+W) = abv+W = a(bv)+W = a(bv+W) = a(b(v+W))$

(VS 7) $\forall a \in F,\ a(v_1 + W + v_2 + W) = av_1 + av_2 + W = av_1 + W + av_2 + W = a(v_1 + W) + a(v_2 + W)$

(VS 8) $(a + b)(v + W) = (a + b)v + W = av + bv + W = a(v + W) + b(v + W)$

## 1.4. Linear Combinations and Systems of Linear Equations

1. (a) T ($0v = 0,\ \forall v \in V$)

(b) F (p.30 $span(\varnothing) = \{0\}$)

(c) T (p.30 Theorem 1.5)

(d) F (p.27)

(e) T (p.27)

(f) F (p.33 2(b))

2. (a) $\{r(1,1,0,0) + s(-3,0,-2,1) + (5,0,4,0) \mid r, s \in R\}$

(b) $(-2,-4,-3)$

(c) There are no solutions

(d) $\{r(-8,3,1,0) + (-16,9,0,2) \mid r \in R\}$

(e) $\{r(0,-3,1,0,0) + s(-3,-2,0,1,0) + (-4,3,0,0,5) \mid r, s \in R\}$

(f) $(3,4,-2)$

3. (a) yes $(-2,0,3) = 4(1,3,0) + (-3)(2,4,-1)$

(b) Yes $(1,2,-3) = 5(-3,2,1) + 8(2,-1,-1)$

(c) No

(d) Yes $(2,-1,0) = \frac{4}{5}(1,2,-3) + \frac{6}{5}(1,-3,2)$

(e) No

(f) Yes $(-2,2,2) = 4(1,2,-1) + 2(-3,-3,3)$

4. (a) Yes $(x^3 - 3x + 5) = 3(x^3 + 2x^2 - x + 1) + (-2)(x^3 + 3x^2 - 1)$

(b) No

(c) Yes $4, -3$

(d) Yes $-2, 5$

(e) No

(f) No

5. (a) Yes $(2, -1, 1) = 1(1, 0, 2) + (-1)(-1, 1, 1) \in span(S)$

(b) No

(c) No

(d) Yes $2, -1$

(e) Yes $-1, 3, 1$

(f) No

(g) Yes $3, 4, -2$

(h) No

6. Let $span\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} = W$

$\forall v = (a_1, a_2, a_3) \in F^3$,

$v = r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1)$ s.t. $r = \frac{1}{2}(a_1 + a_2 - a_3),\ s = \frac{1}{2}(a_1 - a_2 + a_3),\ t = \frac{1}{2}(-a_1 + a_2 + a_3) \in W$

$\therefore\ V \subseteq W$ Since $W \subseteq F^3$ is clear, $W = F^3$

7. $\forall v = (a_1, a_2, \cdots, a_n) \in F^n$

$v = a_1 e_1 + a_1 e_2 + \cdots + a_n e_n,\ \forall a_i \in F,\ i = 0, 1, \cdots, n$

8. $\forall f(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n(F)$

$f(x) = a_0 \cdot 1 + a_1 \cdot x + \cdots + a_n \cdot x^n, \forall a_i F, i = 0, 1, \cdots, n$

9. $\forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(F)$

$A = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

10. $\forall A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \in M_{2\times 2}(F)$

$A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in span\{M_1, M_2, M_3\}$

11. If $x \neq 0$, then $span(\{x\}) = \{ax \mid a \in F\}$ is the line through the origin in $R^3$ Otherwise $span(\{x\}) = \{ax \mid a \in F\}$ is the origin

12. $(\Leftarrow)$ By the theorem 1.5, $span(W)$ is a subspace of $V$

$\therefore\ W$ is a subspace of $V$

$(\Rightarrow)\ \forall v \in W,\ v = 1 \cdot v \in span(W)$

$\therefore\ W \subseteq span(W)$

$\forall v \in span(W),\ v = a_1 v_1 + \cdots + a_n v_n$

Since $\forall a_i v_i \in W,\ v = \sum_{i=1}^{n} \in W$

$\therefore\ span(W) \subseteq W$

13. (i) $S_1 \subseteq S_2 \subseteq span(S_2)\ \Rightarrow\ span(S_1) \subseteq span(S_2)$

(ii) By (i) $span(S_1) = V \subseteq span(S_2) \subseteq V$

$\therefore\ span(S_2) = V$

14. $S_1, S_2 \subseteq V,\ S_1 = \{x_1, x_2, \cdots, x_m\}, S_2 = \{x_{m+1}, x_{m+2}, \cdots, x_n\}$

(i) Since $S_1, S_2 \subseteq S_1 \cup S_2,\ span(S_1), span(S_2) \subseteq span(S_1 \cup S_2)$

If $v = \sum_{i=1}^{m} a_i x_i + \sum_{i=m+1}^{n} a_i x_i \in span(S_1) + span(S_2),\ \forall a_i \in F$

then $v \in span(S_1 \cup S_2)$

(ii) If $v = \sum_{i=1}^{n} a_i x_i \in span(S_1 \cup S_2,\ \forall a_i \in F$

then $v = \sum_{i=1}^{m} a_i x_i + \sum_{i=m+1}^{n} a_i x_i \in span(S_1) + span(S_2)$

15. Since $S_1 \cap S_2 \subseteq S_1 \subseteq span(S_1)$ and $S_1 \cap S_2 \subseteq S_2 \subseteq span(S_2)$,

$span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$

16. $\forall v \in span(S)$, suppose $v = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$

then $(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = 0$

$\therefore\ (a_1 - b_1) = \cdots = (a_n - b_n) = 0$

$\therefore\ a_1 = b_1, \cdots, a_n = b_n$

17. $W$ must be a finite set

(i) $F$ is an infinite field

If $\exists 0 \neq w \in W$, then $\{aw \mid a \in F\} \Rightarrow W = \{0\}$

(ii) $F$ is a finite field

If $\beta = \{w_1, \cdots, w_n\}$, then $\mid W \mid = \mid F \mid^{|\beta|}$

$\therefore \ \dim W < \infty$

## 1.5. Linear Dependence and Linear Independence

1. (a) If $S$ is a linearly dependent set, then each vector in $S$ is a linear combination of other vector in $S$.

*Ans* : F

(Example) $V = R^3$, $S$=$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1), e_4 = (1,1,0)\}$

$\{e_1, e_2, e_3\}$ : linearly independent

$\{e_1, e_2, e_4\}$ : linearly dependent

(b) Any set containing the zero vector is linearly dependent.

*Ans* : T ($\because$) $\forall a \in F, 0 = a \cdot 0, \quad a \neq 0$

(c) The empty set is linearly dependent.

*Ans* : F ($\because$) linearly dependent set must be non-empty.

(d) Subsets of linearly dependent sets are linearly dependent.

*Ans* : F ($\because$) theorem 1.6

(e) Subsets of linearly independent sets are linearly independent.

*Ans* : T ($\because$) the corollarly from theorem 1.6

(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \cdots, x_n$ are linearly independent, then all the scalars $a_i$ are zero.

*Ans* : T ($\because$) from the definition.

2. (a), (d), (e), (g), (h), (j) : linearly independent

(b), (c), (f), (i) : linearly dependent

3. In $M_{2\times 3}(F)$, prove that the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is linearly dependent.

($\because$) $a+d=0, b+d=0, c+d=0,$

$a+e=0, b+e=0, c+e=0$

$\Rightarrow a=-d, b=-d, c=-d, d, e=d$

$\therefore$ the given set is linearly dependent.

4. In $F^n$, let $e_j$ denote the vector whose $j$th coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \cdots, e_n\}$ is linearly independent.

($\because$) $(a_1, a_2, \cdots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n = (0, \cdots, 0)$

$\therefore a_1 = a_2 = \cdots = a_n = 0$

$\therefore \{e_1, e_2, \cdots, e_n\}$ is linearly independent.

5. Show that the set $\{1, x, x^2, \cdots, x^n\}$ is linearly independent in $P_n(F)$.

($\because$) If $a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\cdot 1 + 0\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_n$,

then $\forall a_i = 0$, $1 \le i \le n$.

6. In $M_{n\times n}(F)$, let $E^{ij}$ denote the matrix whose only nonzero entry is 1 in the $i$th row and $i$th column.

Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

$(\because)$ $\forall A \in M_{n\times n}(F)$,

If $A = a_{11}E^{11} + a_{12}E^{12} + \cdots + a_{nn}E^{nn} = 0$

then,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\therefore \forall a_{ij} = 0, \quad \text{where } 1 \leq i \leq m, 1 \leq j \leq n$

7. Recall from Example 3 in section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace.

Find a linearly independent set that generates this subspace.

$(\because)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\forall \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a, b \in F$

8. Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space $F^3$.

(a) Prove that if $F = R$, then $S$ is linearly independent.

(b) Prove that if $F$ has characteristic 2, then $S$ is linearly independent.

$(\because)$ (a) $a(1,1,0) + b(1,0,1) + c(0,1,1) = (0,0,0), \quad \forall a, b, c \in F$

then $a+b=0, a+c=0, b+c=0$

$\therefore a=b=c=0$

(b) If $a=b=c=1$, then $a(1,1,0)+b(1,0,1)+c(0,1,1)=(2,2,2)=(0,0,0)$

$\therefore$ $\{(1,1,0),(1,0,1),(0,1,1)\}$ is linearly dependent

9. Let $u$ and $v$ be distinct vectors in a vector space $V$.

Show that $\{u,v\}$ is linearly dependent if and only if $u$ or $v$ is a multiple of the other.

($\because$) $u \neq v, \{u,v\}$ : linearly independent

($\Rightarrow$) If au+bv= 0,

$(1) a \neq 0 \Rightarrow u = \frac{-a}{b} v$

$(2) a = 0 \Rightarrow b \neq 0 \Rightarrow v = \frac{-a}{b} u$

$(\Leftarrow) u = kv \Rightarrow 1u - kv = 0$

10. Give an example of three linearly dependent vectors in $R^3$ such that none of the three is a multiple of another.

(Example) $\{(-1,0,-1),(1,-1,0),(0,1,1)\}$

11. Let $S=\{u_1,u_2,\cdots,u_n\}$ be a linearly independent subset of a vector space $V$ over the field $Z_2$. How many vectors are there in span($S$)?

$\mid span(\text{S}) \mid = \mid F \mid^{|n|}$, when the set is linearly independent.

* what if the given set is not linearly independent?

then the number of vectors in the set is smaller than $\mid F \mid^{|n|}$.

(Example) $a_i = \{0, 1\}$,

$v_1 = (-1, 0, 1), v_2 = (1, -1, 0), v_3 = (0, 1, 1)$

span($S$)=$\{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1)\}$

12. Prove Theorem 1.6 and its corollary.

Let $S_1 = \{u_1, \cdots, u_n\},\ S_2 = \{u_1, \cdots, u_n, v\}$

(i) $a_1u_1 + a_2u_2 + \cdots + a_nu_n + a_{n+1}v = 0$, not all $a_i \neq 0$ $(n \geq 1)$ then $v = b_1u_1 + \cdots + b_nu_n$, where $b_i = -\frac{a_i}{a_{n+1}} \in F$

$\therefore$ $S_2$ is linearly independent

(ii) $a_1u_1 + \cdots + a_nu_n + 0v = 0 \ \Rightarrow \ a_1 = \cdots = a_n = 0$

$\therefore$ $S_1$ is linearly independent

13. Let $V$ be a vector space over a field of characteristic not equal to two.

(a) Let $u$ and $v$ be distinct vectors in $V$.

Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

$(\because)(\Rightarrow)$ Suppose $a(u + v) + b(u - v) = 0,\ a, b \in F$

$(a + b)u + (a - b)v = 0$

$\therefore a = b = 0$

$(\Leftarrow)$ $\{u + v, u - v\}$ is linearly independent.

$au + bv = (\frac{a+b}{2})(u + v) + (\frac{a-b}{2})(u - v) = 0$

$(\frac{a+b}{2}) = 0$ and $(\frac{a-b}{2}) = 0$

$\therefore a = b = 0$

(b)

$(\Rightarrow)$ $a(u+v) + b(u+w) + c(v+w) = 0$

$\Rightarrow$ $(a+b)u + (a+c)v + (b+c)w = 0$

$\therefore$ $a+b = a+c = b+c = 0$

$\therefore$ $a = b = c = 0$

$(\Leftarrow)$ $au + bv + cw = (\frac{a+b}{2})(u+v) + (\frac{a+c}{2})(u+w) + (\frac{b+c}{2})(v+w)$

$(\frac{a+b}{2}) = (\frac{a+c}{2}) = (\frac{b+c}{2}) = 0$

$\therefore$ $a = b = c = 0$

14. $(\Leftarrow)$ By the exercise 1(b), if $S = \{0\}$, then $S$ is linearly dependent

If $v = a_1u_1 + \cdots + a_nu_n$, then $a_1u_1 + \cdots + a_nu_n - 1 \cdot v = 0$

$\therefore$ $S = \{u_1, \cdots, u_n, v\}$ is linearly dependent

$(\Rightarrow)$ If $S \neq \{0\}$, $\exists a_i \neq 0$ s.t. $a_1u_1 + \cdots + a_nu_n + a_{n+1}v = 0$

Let $a_{n+1} \neq 0$

$v = b_1u_1 + \cdots b_nv_n \in span(\{u_1, \cdots, u_n\})$, where $b_i = -\frac{a_i}{a_{n+1}} \in F$

15. $(\Leftarrow)$ By the theorem 1.6

$(\Rightarrow)$

If $u_1 = 0$, then it's clear

So we may assume $u_1 \neq 0$

Let $k \geq 0$ be the first integer s.t. $u_1, \cdots, u_k$ linearly independent and $\{u_1, \cdots, u_k, u_{k+1}\}$ linearly dependent

So $a_1u_1 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0$ for some scalar $a_1, a_2, \cdots, a_{k+1}$ (not all zero)

If $a_{k+1} = 0$, then $a_1u_1 + \cdots + a_ku_k + a_{k+1}u_{k+1} = a_1u_1 + \cdots + a_ku_k = 0$

$\therefore\ a_1 = \cdots = a_k = a_{k+1} = 0$

It's a contradiction

Thus $u_{k+1} = b_1u_1 + \cdots + b_ku_k \in span(u_1, \cdots, u_k)$, where $b_i = -\frac{a_i}{a_{k+1}}$

16. $(\Rightarrow)$ By the corollary of theorem 1.6

$(\Leftarrow)$ $S \subseteq S$ is linearly independent

17. Let $M^{(1)} = (a_{11}, 0, \cdots, 0)^t, M^{(2)} = (a_{12}, a_{22}, 0, \cdots, 0)^t, \cdots, M^{(n)} = (a_{1n}, a_{2n}, \cdots, a_{nn})^t$

$M \in span\{M^{(1)}, M^{(2)}, \cdots, M^{(n)} \mid a_{ii} \neq 0\}$

Suppose $k_1M^{(1)} + \cdots + k_nM^{(n)} = 0$

$k_1 = k_2 = \cdots = k_n = 0,\ \forall a_{ii} \neq 0$

$\therefore\ S$ is linearly independent

18. $f_0(x) = a_0$

$f_1(x) = a_0 + a_1x$

$\vdots$

$f_n(x) = a_0 + a_1x + \cdots + a_nx^n$

$\Rightarrow\ k_0f_0(x) + k_1f_1(x) + \cdots + k_nf_n(x) = 0$

$\Rightarrow \;\; k_n a_n = 0$

$(k_{n-1} + k_n)a_{n-1} = 0$

$\vdots$

$(k_0 + \cdots + k_n)a_0 = 0$

$\therefore \;\; \forall k_i = 0$

19.

$(a_1 A_1 + \cdots + a_k A_k)^t = (0)^t \;\; \Rightarrow \;\; a_1 A_1^t + \cdots + a_k A_k^t = 0$

$\therefore \;\; a_1 = \cdots = a_k = 0$

$\therefore \;\; \{A_1^t, \cdots, A_k^t\}$ is linearly independent

20.

$ae^{rt} + be^{st} = 0, \;\; r \neq s$

$\Rightarrow \;\; a + be^{(s-r)t} = 0$

Since $e^{(s-r)t} \neq 0, \;\; a = b = 0$

$\therefore \;\; \{e^{rt}, e^{st}\}$ is linearly independent

## 1.6. Bases and Dimension

1. (a) F ($\because$)$\varnothing$ is a basis for the zero vector space.

* $span\{\varnothing\} = \{0\}$ and $\varnothing$ is linearly independent.

(b) T ($\because$) Theorem 1.9 ; If a vector space $V$ is generated by a finite set $S$, then some subset of $S$ is a basis for $V$.

(c) F (Counterexample) $\{1, x, x^2, \cdots\}$ is a basis for $P(F)$

(d) F ($\because$) Corollary 2 (c) from Theorem 1.10

Every linearly independent subset of $V$ can be extended to a basis for $V$.

(e) T ($\because$) Corollary 1 from Theorem 1.10

Let $V$ be a vector space having a finite basis. Then every basis for $V$ contains the same number of vectors.

(f) F ($\because$) $\{1, x, x^2, \cdots, x^n\}$ is a basis for $P_n(F)$

(g) F ($\because$) The dimension of $M_{m\times n}(F)$ is $m \times n$

(h) T ($\because$) Replacement theorem.

(i) F

($\because$) Theorem 1.8. Let $V$ be a vector space and $\beta = \{u_1, u_2, \cdots, u_n\}$ be a subset of $V$. Then $\beta$ is a basis for $V$ if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$.

(Example) $V = R^2$, $S = \{v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, 1)\}$

$(a, b) = av_1 + bv_2 + 0v_3 = 0v_1 + (b - a)v_2 + av_3$

(j) T

Theorem 1.11. Let $W$ be a subspace of a finite-dimensional vector space $V$. Then $W$ is finite-dimensional and dim(W)$\leq$ dim(V). Moreover, if dim(W)=dim(V), then $V = W$.

(k) T

(i) The vector space $\{0\}$ has dimension zero and 0 is unique element in $V$.

So $V$ has exactly one subspace with dimension 0.

(ii) From the theorem 1.11, $W \leq V$ as a subspace

If dim($W$)=dim($V$), then $V = W$

So $V$ has exactly one subspace with dimension $n$.

(l) T

($\Rightarrow$) If $S$ is linearly independent,

Let $W$ be a space spanned by $S$.

Then $S$ is a basis for $W$ (the corollary 2 from 1.10)

And dim$W = n$

$\therefore W = V$

$\therefore S$ is a basis for $V$

($\Leftarrow$) If $S$ is a generating set for $V$ that contains $n$ vectors,

then by the corollary 2 from 1.10, $S$ is linearly independent.

2. (a) $\{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$ : a basis for $R^3$

($\because$) $0 = a \cdot (1, 0, -1) + b \cdot (2, 5, 1) + c \cdot (0, -4, 3) = (a + 2b, 5b - 4c, -a + b + 3c)$

$\therefore a = b = c = 0$

(*)

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & -4 & 3 \end{vmatrix} = 27 \neq 0$$

$\therefore \{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$ is linearly independent.

(b) $\{(2, -4, 1), (0, 3, -1), (6, 0, -1)\}$ : a basis for $R^3$

$(\because)\ 0 = a \cdot (2, -4, 1) + b \cdot (0, 3, -1) + c \cdot (6, 0, -1)$

$= (2a + 6c, -4a, a - b - c)$

$\therefore a = b = c = 0$

(c) $\{(1, 2, -1), (1, 0, 2), (2, 1, 1)\}$ : a basis for $R^3$

(d) $\{(-1, 3, 1), (2, -4, -3), (-3, 8, 2)\}$ : a basis for $R^3$

(e) $\{(1, -3, -2), (-3, 1, 3), (-2, -10, -2)\}$

$(\because)\ 0 = a \cdot (1, -3, -2) + b \cdot (-3, 1, 3) + c \cdot (-2, -10, -2)$

$= (a - 3b - 2c, -3a + b - 10c, -2a + 3b - 2c)$

$\therefore a = 2b, c = \frac{-1}{2} b$

$\therefore \exists (4, 2, 1) \neq (0, 0, 0)$

(*)

$$\begin{vmatrix} 1 & -3 & -2 \\ -3 & 1 & 3 \\ -2 & 10 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & -2 \\ 0 & -8 & -3 \\ 0 & -16 & -6 \end{vmatrix} = 0$$

$\therefore \{(1, -3, -2), (-3, 1, 3), (-2, -10, -2)\}$ is linearly dependent.

3. (a) $\{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$

$(\because)\ 0 = a\cdot(-1-x+2x^2)+b\cdot(2+x-2x^2)+c\cdot(1-2x+4x^2)$

$=(1+2a-2b+4c)x^2)+(-a+b-2c)x+(-a+2b+c)$

$\therefore a=-5c, b=-3c$

$\therefore \exists(-5,3,1)\neq(0,0,0)$

(*)

$$\begin{vmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 1 & -3 & 6 \end{vmatrix} = 0$$

$\therefore \{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$ is linearly dependent.

(b) $\{1+2x+x^2, 3+x^2, x+x^2\}$ : a basis for $P^2(R)$

(c) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$ : a basis for $P^2(R)$

(d) $\{-1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2\}$: a basis for $P^2(R)$

(e) $\{1+2x-x^2, 4-2x+x^2, -1+18x-9x^2\}$

$(\because)\ 0 = a\cdot(-1-x+2x^2)+b\cdot(4-2x+x^2)+c\cdot(-1+18x-9x^2)$

$=(-a+b-9c)x^2)+(2a-2b+18c)x+(a+4b-c)$

$\therefore a=-11c, b=-2c$

$\therefore \exists(-11,-2,1)\neq(0,0,0)$

4. No.

$|\{x^3-2x^2+1, 4x^2-x+3, 3x-2\}| = 3$ and $\dim(P_3(R))=4$

The generating set for $V$ contains at least 4 vectors.

5. No.

Any $n+1$ or more vectors in $V$ are linearly dependent.

Since $\dim(R^3) = 3$, every linearly independent set contains at most 3 vectors.

6.

$$\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$$

7.

Select any nonzero vector in the given set, say $u_1$, to be a vector in the basis.

Since $u_3 = -4u_1$, the set $\{u_1, u_3\}$ is linearly dependent.

Hence we don't include $u_3$

On the other hand, the set $\{u_1, u_2\}$ is linearly independent. Thus we include $u_2$.

And the set $\{u_1, u_2, u_4\}$ is linearly dependent. So we exclude $u_4$.

$\therefore$ $\{u_1, u_2, u_5\}$ is a basis for $R^3$.

(*)

$$\begin{vmatrix}2 & -3 & 1\\1 & 4 & -2\\1 & 37 & -17\end{vmatrix} = \begin{vmatrix}0 & -11 & 5\\1 & 4 & -2\\0 & 33 & -15\end{vmatrix} = -\begin{vmatrix}-11 & 5\\33 & -15\end{vmatrix} = 0$$

$\therefore \{u_1, u_2, u_4\}$ is linearly dependent.

8. $\{u_1, u_3, u_4, u_8\}$ is a basis for $W$.

9. $a, b, c, d \in F$, $(a_1, a_2, a_3, a_4) = au_1 + bu_2 + cu_3 + du_4$

$= (a, a+b, a+b+c, a+b+c+d)$

$\therefore a = a_1, b = a_2 - a_1, c = a_3 - a_2, d = a_4 - a_3$

$\therefore (a_1, a_2, a_3, a_4) = a_1u_1 + (a_2 - a_1)u_2 + (a_3 - a_2)u_3 + (a_4 - a_3)u_4$

10. (a)

$f_0(x) = \frac{1}{3}(x^2 - 1),\ f_1(x) = -\frac{1}{2}(x^2 + x - 2),\ f_2(x) = \frac{1}{6}(x^2 + 3x + 2)$

$\therefore\ g(x) = \sum_{i=0}^{2} b_i f_i(x) = -4x^2 - x + 8$

(b)

$f_0(x) = \frac{1}{35}(x^2 - 4x + 3),\ f_1(x) = -\frac{1}{10}(x^2 + x - 12),\ f_2(x) = \frac{1}{14}(x^2 + 3x - 4)$

$\therefore\ g(x) = \sum_{i=0}^{2} b_i f_i(x) = -3x + 12$

(c)

$f_0(x) = -\frac{1}{15}(x^3 - 3x^2 - x + 3),\ f_1(x) = -\frac{1}{8}(x^3 - 2x^2 - 5x + 6),\ f_2(x) = -\frac{1}{12}(x^3 - 7x^2 - 6),\ f_3(x) = \frac{1}{40}(x^3 + 2x^2 - x - 2)$

$\therefore\ g(x) = \sum_{i=0}^{3} b_i f_i(x) = -x^3 + 2x^2 + 4x - 5$

(d)

$f_0(x) = -\frac{1}{12}(x^3 + x^2 - 2x),\ f_1(x) = \frac{1}{6}(x^3 + 2x^2 - 3x),\ f_2(x) = -\frac{1}{6}(x^3 + 4x^2 + x -$

6), $f_3(x) = \frac{1}{12}(x^3 + 5x^2 + 6x)$

$\therefore\ g(x) = \sum_{i=0}^{3} b_i f_i(x) = -3x^3 - 6x^2 + 4x + 15$

11. (i)

We need to show that $\{u+v, au\}$ is linearly independent

$k_1, k_2$ are scalars,

If $k_1(u+v) + k_2(au) = (k_1 + ak_2)u + (k_1 v) = 0$

Since $\{u, v\}$ is a basis for $V$,

$\therefore k_1 + ak_2 = 0, k_1 = 0$ ($a$ is a nonzero scalar)

$\therefore k_1 = k_2 = 0$

(ii)

If $k_1(au) + k_2(bv) = (ak_1)u + (bk_2)v = 0$

Since $a \neq 0, b \neq 0$ and $\{u, v\}$ is a basis for $V$.

$\therefore k_1 = k_2 = 0$

$\therefore \{au, bv\}$ is a basis for $V$.

12. $k_1(u + v + W) + k_2(v + w) + k_3(w) = 0,\ k_1, k_2, k_3 \in F$

$\Rightarrow\ k_1 u + (k_1 + k_2)v + (k_1 + k_2 + k_3)w = 0$

$\Rightarrow\ k_1 = k_2 = k_3 = 0$

13. $\{a(1,1,1) | a \in R\}$ is a solution set

$\therefore\ \{(1,1,1)\}$ is a basis for the given system

14. $\{(0,0,1,0,0),(0,0,0,1,0),(0,1,0,0,0),(0,0,0,0,1)\}$ : a basis for $W_1$

$\{(1,0,0,0,-1),(0,1,1,1,0)\}$ : a basis for $W_2$

$\therefore$ dim($W_1$)=4, dim($W_2$)=2

(*) The dimension of the solution space $AX=0$ is equal to $n-$rank$A$

($n$ is the number of rows of $A$)

(i) $a_1 - a_3 - a_4 = 0$ i.e. $a_1 + 0a_2 - a_3 - a_4 + 0a_5 = 0$—(*)

$$A = (1,0,-1,-1,0)_{1\times 5}, \quad X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

(*) is equal to $AX = 0$

The dimension of the solution space $AX=0$ is equal to $n-$rank$A = 5-1 = 4$

(ii) $a_2 = a_3, a_2 = a_4, a_1 + a_5 = 0$

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}_{3\times 5}$$

$\therefore$ rank($A$)=3

The dimension of the solution space $AX=0$ is equal to $n-$rank$A = 5-3 = 2$

15.

(i) $\{(\sum_{i,j=1}^{n} E_{ij}$, where $i \neq j)$ and $(**)\}$ : a basis for $W$.

$(**) = \{(1,-1,0,\cdots,0),(0,1,-1,\cdots,0),\cdots,(0,\cdots,1,-1,0),(0,\cdots,1,-1)\}$

that is, $i$th component of the element in $(**)$ is $-1$

(ii) Dim$(W)$=$n^2 - n + (n-1) = n^2 - 1$

$(\because)$ $n^2$ : the dim of $M_{n\times n}(F)$, $n$ : the number of vectors consist of diagonal,

$(n-1)$ : the number of vectors consist of $(**)$

* When char$F$=2, if $A^t = -A$, then $a_{ij} = -a_{ji}$ and $a_{ii} = -a_{ii}$

$\therefore a_{ii} = 0$

(Example)

$W = \{(a_1,a_2,a_3,a_4,a_5) | \sum_{i=1}^{n} a_1 = 0\} \leq R^5$ is a subspace of $V$, $\dim(W) = 4$

$\beta = \{v_1 = (1,-1,0,0,0), v_2 = (1,0,-1,0,0), v_3 = (1,0,0,-1,0), v_4 = (1,0,0,0,-1)\}$

(or $\beta = \{(1,-1,0,0,0),(0,1,-1,0,0),(0,0,1,-1,0),(0,0,0,1,-1)\}$)

From $a_1 + a_2 + a_3 + a_4 + a_5 = 0$, $a_5 = -(a_1 + a_2 + a_3 + a_4)$

$(a_1,a_2,a_3,a_4,a_5) = (a_1,a_2,a_3,a_4,-(a_1+a_2+a_3+a_4)$

$= (a_1,0,0,0,-a_1) + (0,a_2,0,0,-a_2) + (0,0,a_3,0,-a_3) + (0,0,0,a_4,-a_4)$

16.

$\forall A \in W$ : the set of all upper triangular $n \times n$ matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

(i) $\{E_{ij} | E_{ij} = 0, i > j\}$ is a basis for $W$

(ii) $\dim(W) = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$

17.

$\forall A \in W$ : the set of all skew-symmetric $n \times n$ matrices

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & & \ddots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{pmatrix}, \quad a_{ij} = -a_{ji}$$

(i) A basis for $W$

$$\left\{ \left( \begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline \text{—} & & \\ & & \end{array} \right), \left( \begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ \hline \text{—} & & & \end{array} \right), \cdots, \left( \begin{array}{c|cc} & & \\ \hline \text{—} & 0 & 1 \\ & 1 & 0 \end{array} \right) \right\}$$

(ii) $\dim(W) = 1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)$

(iii) In case of char$F$=2

if $A^t = -A$, then $a_{ij} \neq -a_{ji}$

$(\because)$ if $a_{ii} = -a_{ii} \not\Rightarrow a_{ii} = 0$ in char$(F)$=2

18. $V$ consists of all sequences $\{a_n\}$ in $F$ that have only a finite number of nonzero terms $a_n$

Let $\{e_i\} = \{0, \cdots, 0, 1, 0 \cdots, 0\}$ *i.e.* the $i$th term is 1

then $\{e_i \mid i = 1, \cdots, n\}$ is a basis for $V$

19. $(\Leftarrow)$ Suppose each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$, then clearly $\beta$ spans $V$.

If $0 = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

And we also have $0 = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n$

By hypothesis, the representation of zero as a linear combination of the $u_i$ is unique. Hence each $a_i = 0$, and the $u_i$ are linearly independent.

20. (a) Prove that there is a subset of $S$ that is a basis for $V$. (Be careful not to assume that $S$ is finite.)

If $V = (0) \Rightarrow \varnothing(\subseteq S)$ a basis. So we may assume that $V \neq (0)$

Let $\mathcal{C} = \{B | B \subseteq S, B \text{ is linearly independent}\}$

then $\forall B \in \mathcal{C}$, $|B| \leq n(= \dim V)$

(i) Choose $B' \in \mathcal{C}$ with maximal element

*i.e.* $B' \in \mathcal{C}$ and $\forall B \in \mathcal{C}$, $|B| \leq |B'|$

(ii)

Claim $S \subseteq span(B')$

($\because$) If not, $S \nsubseteq span(B')$, then $\exists v \in S, v \notin span(B')$

By the theorem 1.7(p.39), $B' \cup \{v\}$ is linearly independent

Since $B' \subsetneq B' \cup \{v\} \subseteq S$, this contradicts to the maximality of $B'$

And $V = span(S) \subseteq span(span(B')) = span(B')$

$\therefore\ V = span(B')$

Therefore $B'$ is a basis for $V$

(b)

Let $Q \subseteq V$ s.t. $span(Q) = V$ and $\mid Q \mid < n$

From $(a)$, we can find a subset $Q'$ of $Q$ is a basis for $V$.

This contradicts to the following fact :

If $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for $V$, then any set of $n+1$ vectors in $V$ is linearly dependent

$\therefore$ Any spanning set $S$ for $V$ must contain at least $n$ vectors.

21. (i) Suppose $\dim V = \infty$

$\Rightarrow\ V$ has an infinite set of linearly independent

(ii) By the Replacement theorem

22. $\dim(W_1 \cap W_2) = \dim(W_1)\ \Leftrightarrow\ W_1 \cap W_2 = W_1\ \Leftrightarrow\ W_1 \subseteq W_2$

23. (a) $\dim(W_1) = \dim(W_2)$ if and only if $v \in span\{v_1, v_2, \cdots, v_k\} = W_1$

(b) If $\dim(W_1) \neq \dim(W_2)$, then $\dim(W_1) < \dim(W_2)$

($\because$) By (a), $v \in W_1$ and $v \notin W_2$

By the exercise 20, $\dim(W_1) \leq k$ and $\dim(W_2) \leq k+1$

$\therefore$ $\dim(W_1) \leq \dim(W_2)$

$\therefore$ $\dim(W_1) < \dim(W_2)$

24. $f(x) = k_n x^n + \cdots + k_1 x + k_0,\ k_n \neq 0, \forall k_i \in R$

Let $a_0 f(x) + a_1 f'(x) + \cdots + a_n f^{(n)}(x) = 0$, for some scalars $a_0, \cdots, a_n \in R$

then $(a_0 k_n)x^n + (a_0 k_{n-1} + a_1 k_n)x^{n-1} + \cdots + (a_0 k_1 + 2!a_1 k_2 + 3!a_2 k_3 + \cdots + n!a_{n-1}k_n)x + (\sum_{m=0}^{n} m!a_m k_m) = 0$

By equating the coefficient of $x^k$ on both sides of this equation for $k = 0, 1, 2, \cdots, n$, we obtain $a_0 = a_1 = \cdots = a_n = 0$ (since char$R$=0)

It follows from (b) of corollary 2 (p.48) that $\{f(x), f'(x), \cdots, f^{(n)}(x)\}$ is a basis for $P_n(R)$.

$\therefore$ $\forall g(x) \in P_n(R),\ \exists\ c_0, \cdots, c_n \in R$ s.t. $g(x) = c_0 f(x) + c_1 f'(x) + \cdots + c_n f^{(n)}(x)$

25. $Z = \{(v, w) \mid v \in V \text{ and } w \in W\} = V \times W$

$\dim(Z) = \dim(Z) \times \dim(W) = mn$

$\beta = \{v_1, \cdots, v_m\}$ a basis for $V$

$\gamma = \{w_1, \cdots, w_n\}$ a basis for $W$

then $\alpha = \{(v_1, 0), \cdots, (v_m, 0), \cdots, (0, w_1), \cdots, (0, w_n)\}$ a basis for $V \times W$

26. $W = \{f \in P_n(R) \mid f(a) = 0\}$ is a subspace of $V = P_n(R)$

$i.e.$ $\forall f \in W$ forms $(x-a)(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$

$\therefore\ \dim(W) = n$

27.

$\dim(W_1 \cap P_n(F)) = \begin{matrix} \frac{n}{2} & \text{if n : even} \\ \frac{n+1}{2} & \text{if n : odd} \end{matrix}$ , $\dim(W_2 \cap P_n(F)) = \begin{matrix} \frac{n}{2}+1 & \text{if n : even} \\ \frac{n+1}{2} & \text{if n : odd} \end{matrix}$

28.

$\dim V = 2n$

$\beta = \{v_1, v_2, \cdots, v_n\}$ a basis for $V$ over $\mathbb{C}$, $\dim V = n$

$\beta' = \{v_1, v_2, \cdots, v_n, v_1 i, v_2 i, \cdots, v_n i\}$ a basis for $V$ over $\mathbb{R}$

$a_1 v_1 + \cdots + a_n v_n + b_1 v_v i + \cdots + b_n v_n i = 0$

$\Rightarrow\ (a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n = 0$

29.

Let $\beta = \{u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_p\}$ (i) $\beta$ spans $W_1 + W_2$
$\forall v \in W_1 + W_2$, $v = (\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i) + (\sum_{i=1}^{k} c_i u_i + \sum_{i=1}^{p} d_i w_i) \in span(\beta)$, $w_1 \in W_1$, $w_2 \in W_2$ (ii) $\beta$ is linearly independent

Suppose that $a_1u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_m v_m + c_1 w_1 + \cdots + c_p w_p = 0 ----(*)$

, where $a_1, \cdots, a_k, b_1, \cdots, b_m, c_1, \cdots, c_p \in F$

Let $v = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i = -\sum_{i=1}^{p} c_i w_i \in W_1 \cap W_2$

Since $\{u_1, \cdots, u_k\}$ is a basis for $W_1 \cap W_2$, $\exists d_i \in F$ s.t. $v = \sum_{i=1}^{k} d_i u_i$

$\therefore\ \sum_{i=1}^{k} d_i u_i + \sum_{i=1}^{p} c_i w_i = 0 \ \Rightarrow\ c_1 = c_2 = \cdots = c_p = 0$

By $(*)$, $a_1 = \cdots = a_k = b_1 = \cdots = b_m = 0$

$\therefore\ \beta$ is linearly independent

$\therefore\ \beta$ is a basis for $W_1 + W_2$

$\therefore\ \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

(b) $\dim(W_1 + W_2) = 0 \ \Leftrightarrow\ W_1 \cap W_2 = \varnothing$

30.

(i) $W_1$ and $W_2$ are subspaces of $V$

$\forall A = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{a} \end{pmatrix}, B = \begin{pmatrix} \mathrm{d} & \mathrm{e} \\ \mathrm{f} & \mathrm{d} \end{pmatrix} \in W_1,\ \alpha \in F$

$\alpha A + B = \begin{pmatrix} \alpha\mathrm{a+d} & \alpha\mathrm{b+e} \\ \alpha\mathrm{c+f} & \alpha\mathrm{a+d} \end{pmatrix} \in W_1$

$\forall A = \begin{pmatrix} 0 & \mathrm{a} \\ -\mathrm{a} & \mathrm{b} \end{pmatrix}, B = \begin{pmatrix} 0 & \mathrm{c} \\ -\mathrm{c} & \mathrm{d} \end{pmatrix} \in W_2,\ \alpha \in F$

$\alpha A + B = \begin{pmatrix} 0 & \alpha\mathrm{a+c} \\ -\alpha\mathrm{a\text{-}c} & \mathrm{b+d} \end{pmatrix} \in W_2$

(ii) $\dim(W_1) = 3$, $\dim(W_2) = 2$, $\dim(W_1 \cap W_2) = 1$ and $\dim(W_1 + W_2) = 4$

31. (a)

$(\because)$ $W_1 \cap W_2$ is a subspace of $W_1$,

By the theorem 1.11, $\dim(W_1 \cap W_2) \leq \dim(W_1) \leq n$

$\therefore\ dim(W_1 \cap W_2) \leq n$

(b)

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \leq m + n - \dim(W_1 + W_2)$

$\therefore\ \dim(W_1 + W_2) \leq m + n$

32.

(a) $\dim(W_1 \cap W_2) = \dim(W_2)$

$W_1 = \{(a_1, a_2, 0) | a_1, a_2 \in F\}$, dim($W_1$)=2

$W_2 = \{(a_1, 0, 0) | a_1 \in F\}$, dim($W_2$)=1

(b) dim($W_1 \cap W_2$)=0

$W_1 = \{(a_1, 0, 0) | a_1 \in F\}$, dim($W_1$)=1

$W_2 = \{(0, a_2, a_3) | a_2, a_3 \in F\}$, dim($W_2$)=2

(c) $W_1 = \{(a_1, 0, a_3) | a_1, a_3 \in F\}$, dim($W_1$)=2

$W_2 = \{(a_1, a_2, 0) | a_1, a_2 \in F\}$, dim($W_2$)=2

33. $V = W_1 \oplus W_2 \ \Leftrightarrow\ \beta_1 \cap \beta_2 = \varnothing,\ \beta_1 \cup \beta_2 :$ a basis for $V$

($\Rightarrow$)

(i) Suppose that $a_1u_1 + a_2u_2 + \cdots + b_1w_1 + b_2w_2 + \cdots = 0\ a_i, b_j \in F$

Then $a_1u_1 + a_2u_2 + \cdots = -(b_1w_1 + b_2w_2 + \cdots) \in W_1 \cap W_2 = \{0\}$

$\therefore\ \forall a_i = b_j = 0$

$\therefore\ \beta_1 \cup \beta_2$ is linearly independent

Let $v = u + w \in W_1 + W_2$

Since $\beta_1, \beta_2$ spans $W_1, W_2$, respectively

$\exists\ a_i, b_j \in F$ s.t. $v = a_1u_1 + a_2u_2 + \cdots + b_1w_1 + b_2w_2 + \cdots$

$\therefore\ \beta_1 \cup \beta_2$ spans $V$

(ii) If $\exists 0 \neq u \in \beta_1 \cap \beta_2$, then $W_1 \cap W_2 \neq \{0\}$

$\therefore\ \beta_1 \cap \beta_2 = \varnothing$

($\Leftarrow$)

(i) Since $\beta_1 \cup \beta_2$ spans $V,\ v \in span(\beta_1 \cup \beta_2)$

$\therefore\ v = a_1u_1 + +a_2u_2 + \cdots + b_1w_1 + b_2w_2 + \cdots \in W_1 + W_2$

$\therefore\ V = W_1 + W_2$

(ii) Since $\beta_1 \cap \beta_2 = \varnothing,\ W_1 \cap W_2 = \{0\}$

$\therefore\ V = W_1 \oplus W_2$

34. (a)

Let $\beta_1 = \{u_1, u_2, \cdots, u_k\}$ be a basis for $W_1$

and we can extend it to a basis for $V$, say $\beta$

Let $\beta = \{u_1, u_2, \cdots, u_k, u_{k+1}, \cdots, u_n\}$ and $\beta_2 = \{u_{k+1}, u_{k+2}, \cdots, u_n\}$

By the exercise 33, $V = W_1 \oplus W_2$

(b)

$V = W_1 \oplus W_2,\ W_1 = \{(a,0)|a \in R\}$, $W_2 = \{(0,b)|b \in R\}$

$V = W_1 \oplus W_{2'},\ W_1 = \{(a,0)|a \in R\}$, $W_{2'} = \{(-a,b)|a,b \in R\}$

35.

(a) $\beta' = \{u_{k+1} + W, u_{k+2} + W, \cdots, u_n + W\}$ is a basis for $V/W$. (p.23)

(i) $\forall \alpha \in V/W = \{v + W | v \in V\}$

$\alpha = (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + W$

$= (a_1u_1 + W) + (a_2u_2 + W) + \cdots + (a_nu_n + W)$

$= a_1(u_1 + W) + \cdots + a_k(u_k + W) + a_{k+1}(u_{k+1} + W) + \cdots + a_n(u_n + W)$

$= a_{k+1}(u_{k+1} + W) + a_{k+2}(u_{k+2} + W) + \cdots + a_n(u_n + W)$

$\therefore$ $\beta'$ spans $V/W$

(ii) Suppose $a_{k+1}(u_{k+1} + W) + a_{k+2}(u_{k+2} + W) + \cdots + a_n(u_n + W) = W, \ a_i \in F, \ i = k+1, \cdots, n$

then $a_{k+1}u_{k+1} + a_{k+2}u_{k+2} + \cdots + a_nu_n = 0$

Since $\{u_{k+1}, \cdots, u_n\}$ is linearly independent, $a_{k+1} = \cdots = a_n = 0$

$\therefore$ $\beta'$ is a basis for $V/W$.

(b) $\dim(V) = k + n, \ \dim(W) = k, \ \dim(V/W) = n - k$

$\therefore \ \dim(/WV) = \dim(V) - \dim(W)$

## 1.7. Maximal Linearly Independent Subsets

1. Label the following statement as true or false.

(a) Every family of sets contains a maximal element. (F)

($\because$) Let $\mathcal{F}$ be the family of all finite subsets of an infinite set $S$.

then $\mathcal{F}$ has no maximal element.

(b) Every chain contains a maximal element. (F)

($\because$) If $A$ : a partial ordered set and every chain ($\neq \varnothing$) of $A$ has an upper bound, then $A$ has a maximal element.

(ex) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

(c) If a family of sets has a maximal element, then that maximal element is unique. (F)

($\because$) Let $S = \{a = x^3 - 2x^2 - 5x - 3,\ b = 3x^3 - 5x^2 - 4x - 9,\ c = 2x^3 - 2x^2 + 12x - 6\}$

then $\{a, b\}, \{a, c\}, \{b, c\}$ are maximal linearly independent subsets of $S$

So maximal element need not be unique.

(d) If a chain of sets has a maximal element, then that maximal element is unique. (T)

(e) A basis for a vector space is a maximal linearly independent subset of that vector space. (T)

(f) A maximal linearly independent subset of a vector space is a basis for that vector space. (T)

2. Show that the set of convergent sequences is an infinite-dimensional subspace of the vector space of all sequences of real numbers. ( See Exercise 21 in Section 1.3.)

(i) By the exercise 21 of section 1.3, $W$ is a suspace of $V$

(ii) $\{1, 1, \cdots\}$ is a basis for $W$

3. Let $V$ be the set of real numbers regarded as a vector space over the field of rational numbers. Prove that $V$ is infinite-dimensional.

Let $V$ be a finite-dimensional vector space

$\beta = \{v_1, \cdots, v_n\}$

Since $\pi \in V,\ \pi = \alpha_1 v_1 + \cdots + \alpha_n v_n,\ \forall \alpha_i \in \mathbb{Q},\ i = 1, \cdots, n$

But $\pi$ is a transcendental number in $\mathbb{Q}$

It's a contrdiction

Thus $V$ is infinite dimensional

4. Let $W$ be a subspace of a (not necessarily finite-dimensional) vector space $V$. Prove that any basis for $W$ is a subset of a basis for $V$.

($\because$) Let $\beta_W$ be a basis for $W$, then $\beta_W \subseteq V$

(i) By theorem 1.13, $\exists\ \beta :$ a maximal linearly independent subset of $V$ that contains $\beta_W$.

(ii) We are going to show that $\beta$ is a basis for $V$ which contains $\beta_W$.

Since $\beta$ is linearly independent, so it suffices to show that $\beta$ spans $V$.

If $v \in V$ and $v$ is not contained in $\beta$ (i.e. $v$ is not in span($\beta$) ),

then $\beta \cup \{v\}$ is linearly independent.

This contradicts to the maximality of $\beta$.

$\therefore \forall v \in V, v \in span(\beta) \in V$

$\therefore span(\beta) = V$.

$\therefore \beta$ is a basis for $V$ which contains $\beta_W$.

5.

($\because$) ($\Rightarrow$) Let $\beta = \{v_1, v_2, \cdots\}$ is a basis for $V$.

If $v \in V$, then $v \in \text{span}V$

Thus $v$ is a linear combination of the vectors of $\beta$ (that is $v$ can be expressed by some finite vectors of $\beta$ and scalars in $F$)

Suppose that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $v = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ are two such representation of $v$. Then

$0 = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)u_n$

Since $\beta$ is linearly independent, it follows that $a_i = b_i$, where $1 \leq i \leq n$

($\Leftarrow$) Suppose each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$, then clearly $\beta$ spans $V$.

If $0 = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

And we also have $0 = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n$

By hypothesis, the representation of zero as a linear combination of the $u_i$ is unique. Hence each $a_i = 0$, and the $u_i$ are linearly independent.

6.

(i) Since $S_1 \in \mathcal{F},\ \mathcal{F} \neq \varnothing$

Let $C$ be a chain in $\mathcal{F}$ and $U = \cup\{A \mid A \in C\}$

Clearly $S_1 \subseteq U \subseteq S_2$

Let $a_1u_1 + \cdots + a_nu_n = 0$ s.t. $\forall u_i \in U,\ \forall a_i \in F,\ i = 1, \cdots, n$

Since $u_i \in A,\ \exists A_i \subseteq C$ s.t. $u_i \in A_i$

Since $C$ is a chain, $\exists A_k$ a.t. $\forall A_i \subseteq A_k$

Thus $\forall u_i \in A_k \in F$

$\therefore\ a_1 = \cdots = a_n = 0$

So $U$ is n upper bound of $C$

By the maximal principle, $F$ has a maximal element $\beta$

*i.e.* $\beta$ is a maximal linearly independent subset of $V$

(ii) $\forall v \in S_2$, if $v \in \beta$, then $v \in span(\beta)$

If $v \notin \beta,\ \beta \cup \{v\}$ is linearly independent

This contradicts to the maximality of $\beta$

$\therefore\ S_2 \subseteq span(\beta)$

Then $V = span(S_2) \subseteq span(\beta) \subseteq V$

$\therefore\ span(\beta) = V$

Therefore $\beta$ is a basis for $V$ s.t. $S_1 \subseteq \beta \subseteq S_2$

7. Let $\mathbb{S} = \{A \subseteq \beta \mid A \cap S = \varnothing,\ A \cup S$ is linearly independent $\}$

$\mathbb{S} \neq \varnothing$ ($\because\ A = \varnothing$)

Let $C$ be a nonempty chain in $\mathbb{S}$ and $B = \cup\{A \mid A \in C\}$

If $\exists x \in B \cap S$, then $x \in B$ and $x \in S$

Since $A \subseteq B,\ x \in A$ for some $A \in C$

thus $x \in A \cap S = \varnothing$

It's a contradiction, so $B \cap S = \varnothing$

If $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n = 0,\ \forall a_i, b_j \in F,\ u_i \in B, v_j \in S$

Since $B = \cup\{A \mid A \in C\},\ \exists A_1, \cdots, A_m \in C$ s.t. $\forall u_i \in A_i$

So we may assume that $u_1, \cdots, u_m \in A_m$

then $u_1, \cdots, u_m, v_1, \cdots, u_n \in A_m \cup S$ is linearly independent

$\therefore\ \forall a_i = 0,\ \forall b_j = 0$

$\therefore\ B \cup S$ is linearly independent

*i.e.* $B$ is an upper bound of $C$

By the maximal principle, $\mathcal{S}$ has a maximal element, say $S_1$

Clearly $S_1 \subseteq \beta,\ S_1 \cap S = \varnothing,\ S_1 \cup S_2$ is linearly independent

So we only need to show that either $\{\ S_1 \cup S$ is a maximal linearly independent subset of $V$ or $V\ \}$ or $\{\beta \subseteq span(S_1 \cup S)\}$

$\forall v \in \beta$, If $v \in S_1 \cup S$, then $v \in span(S_1 \cup S)$ If $v \notin S_1 \cup S$, then $S_1 \cup S \cup \{v\}$ is linearly independent

It's a contradiction, $\therefore\ \beta \subseteq span(S_1 \cup S)$

Therefore $S_1 \cup S$ is a basis for $V$

# §2. Linear Transformations and Matrices

## 2.1. Linear Transformations, Null Spaces, and Ranges

1. (a) T

(b) F ($\because$) If $\forall x, y \in V$ and $c \in F$, T$(x+y)$ =T$(x)$+T$(y)$ and T$(cx) = c$T$(x)$, then T is a linear transformation

(c) F ($\because$) T is linear and one-to-one if and only if

(d) T ($\because$) T$(0_V)$ =T$(0_V + 0_V)$ =T$(0_V)$+T$(0_V)$ $\therefore$ T$(0_V) = 0_W$

(e) F ($\because$) p.70 Theorem 2.3

(f) F ($\because$) T is linear and one to one, then

(g) T ($\because$) p.73 Corollary to Theorem 2.6

(h) F ($\because$) p.72 Theorem 2.6

2.

(1) $T((a_1, a_2, a_3) + (b_1, b_2, b_3)) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= ((a_1 + b_1) - (a_2 + b_2), 2(a_3 + b_3))$

$= (a_1 - a_2), 2a_3) + (b_1 - b_2), 2b_3)$

$= T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$T(c(a_1, a_2, a_3)) = T(ca_1, ca_2, ca_3) = (ca_1 - ca_2, 2ca_3) = c(a_1 - a_2, 2a_3) = cT(a_1, a_2, a_3)$

Thus, T is linear.

(2) N(T)=span$\{(1, 1, 0)\}$

R(T)=span$\{(1, 0), (0, 1)\}$

(3) Dim(V)=nullity(T) + rank(T)$= 1 + 2$

(4) T is not one-to-one ($\because$N(T)$\neq \{0\} = \{(0,0,0)\}$)

T is onto ($\because$ 2=rank(T)=dim(W)=2)

3.

(1) T is linear.

(2) N(T)=span$\{(0,0)\}$

R(T)=span$\{(1,0,0),(0,0,1)\}$

(3) Dim(V)=nullity(T) + rank(T)$= 0 + 2$

(4) T is one-to-one ($\because$N(T)$= \{0\}$)

T is not onto ($\because$ 2=rank(T)$\neq$ dim(W)=3)

4.

(1) T is linear.

(2) N(T)=span $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$

R(T)=span$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(3) Dim(V)=nullity(T) + rank(T)$= 4 + 2$

(4) T is not one-to-one ($\because$N(T)$\neq \{0\}$)

T is not onto ($\because$ 2=rank(T)$\neq$ dim(W)=4)

5. (1) T is linear.

(2) N(T)=span$\{0\}$

R(T)=span$\{1, x, x^2\}$

(3) Dim(V)=nullity(T) + rank(T)= 0 + 3

(4) T is one-to-one ($\because$N(T)= $\{0\}$)

T is not onto ($\because$ rank(T)$\neq$ dim(W))

6. (1) T is linear.

(2) N(T)=$\{A \mid T(A) = tr(A) = 0\}$=span$\{E_{ij}, E'_{ij}\}$,

where $E_{ij}$ denote the matrix whose $ij$-entry is 1 and zero elsewhere

and $E'_{ij}$ denote the matrix of $a_{11}$ component is 1, $a_{jj}$ component is $-1$ and others are all zero. $(2 \leq j \leq n)$

(*) $\beta = \{E_{ij} \mid i \neq j\} \cup \{E_{11} - E_{ii} \mid 2 \leq j \leq n\}$

R(T)=$\{T(A) \mid A \in Mat_{n \times n}\}$=span$\{1\}$

(3) $n^2$=Dim(V)=nullity(T) + rank(T)= $(n^2 - 1)$ + 1

* dim N(T)=$n^2 - 1$ (p.56 exercise 15, sec 1.6)

(4) T is not one-to-one but onto ($\because$ rank(T)= dim(W))

7.

(1) If $T$ is linear, then $T(0) = 0$

$T(0) = T(0 + 0) = T(0) + T(0)$

$\therefore\ T(0) = 0$

(2)

If $T$ is linear, $T(cx+y) = T(cx) + T(y) = cT(x) + T(y)$

Suppose $T(cx+y) = cT(x) + T(y)$, then

$c = 1,\ T(x+y) = T(x) + T(y)$

$j = 0,\ T(cx) = cT(x)$

(3) $0 = T(0) = T(x-x) = T(x) + T(-x)$

$\Rightarrow\ T(-x) = -T(x)\ \forall x$

So $T(x-y) = T(x) + T(-y) = T(x) - T(y)$

(4)

($\Rightarrow$) If $T$ is linear, then $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} T(a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$

($\Leftarrow$) clear

8. (a)

(i) $T_\theta((a_1, a_2) + (b_1, b_2)) = T_\theta(a_1 + b_1, a_2 + b_2)$

$= ((a_1 + b_1)\cos\theta - (a_2 + b_2)\sin\theta, (a_1 + b_1)\sin\theta + (a_2 + b_2)\cos\theta)$

$= (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta) + (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta)$

$= T_\theta(a_1, a_2) + T_\theta(b_1, b_2)$

(ii) $T_\theta c((a_1, a_2)) = T_\theta(ca_1, ca_2) = (ca_1\cos\theta - ca_2\sin\theta, ca_1\sin\theta + ca_2\cos\theta))$

$= c(a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta) = cT_\theta(a_1, a_2)$

(b)

(i) $T((a_1, a_2) + (b_1, b_2)) = T(a_1 + b_1, a_2 + b_2) = (a_1 + b_1, -a_2 - b_2) = (a_1 - a_2) +$

$(b_1 - b_2) = T(a_1 + a_2) + T(b_1 + b_2)$

(ii) $T(c(a_1, a_2)) = (ca_1, -ca_2) = c(a_1, -a_2) = cT(a_1, a_2)$

9. (a)

$T((a_1+a_2)+(b_1+b_2)) = (1, a_2+b_2)$

$T(a_1+a_2)+T(b_1,b_2) = (2, a_2+b_2)$

(b)

$T(c(a_1,a_2)) = (ca_1, c^2a_2^2)$ $cT(a_1,a_2) = (ca_1, ca_1^2)$

(c)

$T((a_1+a_2)+(b_1+b_2)) = (\sin(a_1+b_1), 0)$

$T(a_1+a_2)+T(b_1,b_2) = (\sin a_1+\sin b_1, 0)$

(d)

$T((a_1+a_2)+(b_1+b_2)) = (|a_1+b_1|, a_2+b_2)$

$T(a_1+a_2)+T(b_1,b_2) = (|a_1|+|b_1|, a_2+b_2)$

(e)

$T((a_1+a_2)+(b_1+b_2)) = (a_1+b_1+1, a_2+b_2)$

$T(a_1+a_2)+T(b_1,b_2) = (a_1+b_1+2, a_2+b_2)$

10.

(a) $T(2,3) = -1T(1,0)+3T(1,1) = (5,11)$

(b) T is one-to-one

$T(x,y) = xT(1,0)+yT(0,1) = xT(1,0)+y(T(1,1)-T(0,1)) = x(1,4)+y((2,5)-(1,4)) == (x+y, 4x+y) = (0,0)$

$\therefore\ (x,y) = (0,0)$

11. $T(8,11) = 2T(1,1) + 3T(2,3) = (5,-3,16)$

12. No $(T(2,0,6) \neq -2T(1,0,3))$

13.

S=$\{v_1, v_2, \cdots, v_k\}$ s.t $T(v_i) = w_i$, $i = 1, 2, \cdots, k$

Suppose $\sum_{i=1}^{k} a_i v_i = 0$ for some scalars $a_i \in F$,

then we need to show that all $a_i = 0$

$T(\sum_{i=1}^{k} a_i v_i) = \sum_{i=1}^{k} a_i T(v_i) = a_i \sum_{i=1}^{k} w_i = T(0) = 0$

Since $w_i$'s are linearly independent, so $a_i = 0$

$\therefore$ $S$ is linearly independent.

14. (a)

$(\Rightarrow)$ $\{v_1, v_2, \cdots, v_n\}$ is linearly independent subset of $V$

If $\sum a_i T(v_i) = 0$, then

$T(\sum a_i v_i) = 0$ and $\sum a_i v_i \in N(T) = \{0\}$

$\therefore$ $\forall a_i = 0$

$\therefore$ $\{T(v_1), \cdots, T(v_n)\}$ is linearly independent subset of $W$

$(\Leftarrow)$ Let $w_1 = \sum a_i T(v_i)$, $\forall a_i, b_i \in F$

If $w_1 = w_2$, then $\sum (a_i - b_i) T(v_i) = 0$

Since $T(v_i)'$s are linearly independent, $\forall a_i = b_i$

$\therefore\ \sum a_i v_i = \sum b_i v_i$

$\therefore\ T$ is one-to-one

(b)

($\Rightarrow$) By (a) , $T(S)$ is linearly independent

($\Leftarrow$) Let $T(S) = \{T(v_1), \cdots, T(v_n)\}$ " linearly independent

If $\sum a_i v_i = 0$, then $\sum a_i T(v_i) = T(0) = 0$

$\therefore\ \forall a_i = 0$

$\therefore\ S = \{v_1, \cdots, v_n\}$ is linearly independent

(c)

By (b), $T(\beta)$ is linearly independent

By the theorem 2.2, $R(T) = span(T(\beta))$

Since $T$ is onto, $span(T(\beta)) = W$

$\therefore\ T(\beta)$ is a basis for $W$

15. (a)

(i) $T(f(x)+g(x)) = \int_0^x (f(t)+g(t))dt = \int_0^x f(t)dt + \int_0^x g(t)dt = T(f(x))+T(g(x))$

(ii) $T(cf(x)) = \int_0^x cf(t)dt = c\int_0^x f(t)dt = cT(f(x))$

(b) If $T(f(x)) = T(g(x))$, then $\int_0^x f(t)dt = \int_0^x g(t)dt$

*i.e.* $\int_0^x (f(t) - g(t))dt = 0$

$\therefore\ f(x) = g(x)$

(c) $\{x, x^2, \cdots\}$ is a basis for $R(T)$

Since $spanR(T) \neq P(R),\ T$ is not onto

16. (a)

Let $f(x) = ax + b,\ g(x) = ax + d,\ b \neq d$ ($i.e.$ $f(x) \neq g(x)$)

But $T(f(x)) = T(g(x))$

(b) $\beta = \{1, x, x^2, \cdots\}$ is a standard basis for $P(R)$

It suffices to show that $\beta \subseteq R(T) = span(T(\beta))$

$\forall x^n \in \beta,\ x^n = T(\frac{1}{n+1}x^n) \in R(T)$

17. (a) By the theorem 2.3, $\dim R(T) \leq \dim V$

By the assumption, $\dim V < W$

$\therefore\ \dim R(T) < W$

$\therefore\ T$ can't be onto

(b) If $T$ is one-to-one $i.e.$ $nullity(T) = 0$, then $\dim V = rank(T) > \dim W$

It's a contradiction to $\dim R(T) \leq \dim W$

18. Let $\forall (a, b) \in R^2,\ T(a, b) = (0, a)$

Then $N(T) = \{(0, b) | b \in R\},\ R(T) = \{(0, a) | a \in R\}$

$\therefore\ N(T) = R(T)$

19. $\forall (a, b) \in R^2$, Let $T(a, b) = (0, a)$ and $U(a, b) = (0, 2a)$

Then $T \neq U,\ N(T) = \{(0, b) | b \in R\},\ N(U) = \{(0, b) | b \in R\}$

$R(T) = \{(0, a) | a \in R\},\ R(U) = \{(0, 2a) | a \in R\}$

20.

(a) Let $w_1, w_2 \in T(v_1),\ a \in F$

then $\exists v_1, v_2 \in V_1$ s.t. $w_i = T(v_i),\ i = 1, 2$

So $aw_1 + w_2 = aT(v_1) + T(v_2) = T(av_1 + v_2) \in T(v_1)$

$\therefore\ T(v_1)$ is a subspace of $W$

(b) Let $K = \{x \in V | T(x) \in W_1\}, a \in F$

Let $x_1, x_2 \imath K$ s.t. $T(x_1) = w_1, T(x_2) = w_2 \in W_1$

$T(ax_1 + x_2) = aT(x_1) + T(x_2) = aw_1 + w_2 \in W_1$

$ax_1 + x_2 \in K,\ a \in F$

$\therefore\ \{x \in V | T(x) \in W_1\}$ is a subspace of $V$

21. (a)

(i) $T(c(a_1, a_2, \cdots) + (b_1, b_2)) = T(ca_1 + b_1, ca_2 + b_2, \cdots) = (ca_2 + b_2, ca_3 + b_3, \cdots) = c(a_2, a_3 \cdots) + (b_2, b_3, \cdots) = cT(a_1, a_2, \cdots) + T(b_1, b_2, \cdots)$

(ii) $U(c(a_1, a_2, \cdots) + (b_1, b_2)) = U(ca_1 + b_1, ca_2 + b_2, \cdots) = (0, ca_1 + b_1, ca_2 + b_2, \cdots) = (0, ca_1, ca_2, \cdots) + (0, b_1, b_2, \cdots) = cU(a_1, a_2, \cdots) + U(b_1, b_2, \cdots)$

(b)

(i) $T$ is not one-to-one

$0 \neq (1, 0, 0, \cdots) \in N(T)$

(ii) $T$ is onto

$R(T) = \{(a_2, a_3, \cdots) | \forall a_i \in F\} = span\{e_1, e_2, \cdots\} = V$

(c)

(i) If $U(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots) = (0, 0, \cdots)$, then $\forall a_i = 0$

$\therefore\ N(U) = \{0\}$

(ii) $U$ is not onto

$R(U) = \{(0, a_1, a_2, \cdots) | \forall a_i \in F\} \neq V$

22.

(i) $T : R^3 \to R$ is linear

Let $\beta = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ a standard basis for $R^3$

Put $T(e_1) = a, T(e_2) = b, T(e_3) = c$

then $\forall (x, y, z) \in R^3$,

$T(x, y, z) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3) = ax + bycz$

(ii) $T : F^n \to F$ is linear

Let $\beta = \{e_1, e_2, \cdots, e_n\}$ a standard basis for $F^n$

Put $T(e_1) = a_1, T(e_2) = a_2, \cdots, T(e_n) = a_n$

then $T(x_1, x_2, \cdots, x_n) = a_1x_1 + \cdots + a_nx_n$

(iii) $T : F^n \to F^m$

For $\forall\ j\ (i \leq j \leq m),\ T(e_j) = \sum\limits_{i=1}^{m} a_{ij}w_i$

$T(x_1, x_2, \cdots, x_n) = T(\sum\limits_{j=1}^{n} x_je_j) = \sum\limits_{j=1}^{n} x_jT(e_j) = \sum\limits_{j=1}^{n} x_j(\sum\limits_{i=1}^{m} a_{ij}w_i) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m} a_{ij}x_jw_i = \sum\limits_{i=1}^{m}(\sum\limits_{j=1}^{n})a_{ij}x_jw_i = (\sum\limits_{j=1}^{n} a_{1j}x_j, \sum\limits_{j=1}^{n} a_{2j}x_j, \cdots, \sum\limits_{j=1}^{n} a_{mj}x_j)$

23. $T : R^3 \to R$ linear transformation

By the exercise 22, $\exists a, b, c \in R$ s.t. $T(x, y, z) = ax + by + cz$

$N(T) = \{(x, y, z) \in R^3 \mid T(x, y, z) = 0\}$

$= \{(x, y, z) \in R^3 \mid ax + by + cz)0\}$

$= R^3$ (where $a = b = c = 0$)

a plane through 0 (where $a^2 + b^2 + c^2 \neq 0$)

24.

(a) The projection on the $y$-axis along the $x$-axis

$T(a, b) = (0, b),\ \forall (a, b) \in R^2, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(b)

$T(a, b) = (a - b, a - b),\ \forall (a, b) \in R^2$

$(a, b) \mapsto (0, 0)$ if $a = b$

$(a - b, a - b)$ otherwise

25.

(a) Let $W_1 = \{(a, b, 0) | a, b \in R\},\ W_2 = \{(0, 0, c) | c \in R\}$

Then $R^3 = W_1 \oplus W_2$

$\forall x = (a, b, c) \in R^3$ s.t. $x_1 = (a, b, 0) \in W_1$ and $x_2 = (0, 0, c) \in W_2$

$T(x) = x_1$

$\therefore\ T$ is the projection on the $W_1$ along the $W_2$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $T(a,b,c) = (0,0,c)$

(c) Let $W_1 = \{(a,b,0)|a,b \in R\},\ W_2 = L = \{(a,0,a)|a \in R\}$

then $R^3 = W_1 + W_2$

For all $x = (a,b,c) \in R^3$ s.t. $x = x_1 + x_2,\ x_1 = (a-c,b,0) \in W_1,\ x_2 = (c,0,c) \in W_2$

$T(a,b,c) = (a-c,b,0) \in W_1$

$\therefore\ T : R^3 R^3$ is the projection on $W_1$ along the $W_2$, $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

26.

$\forall x \in V$ s.t $x = x_1 + x_2, x_1 \in W_1, x_2 \in W_2$ and $\mathrm{T}(x) = x_1$

(1) $T$ is linear

$\forall x, y \in$ V s.t. $x = x_1 + x_2$, $y = y_1 + y_2$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$

$T(x+y) = T((x_1+y_1)+(x_2+y_2)) = x_1 + y_1 = \mathrm{T}(x) + \mathrm{T}(y)$

$T(cx) = T(cx_1 + cx_2) = cx_1 = T(x)$

(2) $W_1 = \{x \in V\ |\mathrm{T}(x) = x\}$

($\subseteq$) If $x_1 \in W$, then $T(x_1) = x_1$

$\therefore x_1 \in \{x \in V \mid T(x) = x\}$

($\supseteq$) If $x \in V$ s.t $\mathrm{T}(x) = x$ and we have $T(x) = x_1$

($\therefore$) $x = x_1 \in W_1$

(b)

(1)$W_1 = R(T)$

($\supseteq$) $\forall\ T(x) \in R(T),\ T(x) = x_1 \in W_1 \therefore\ R(T) \subseteq W_1$

($\subseteq$) $\forall\ x_1 \in W_1,\ x_1 = T(x_1) \in R(T) \therefore\ W_1 \subseteq R(T)$

(2)$W_2 = N(T)$

$\forall x_2 \in W_2$, $T(x_2) = 0 \therefore x_2 \in N(T)$

$\therefore W_2 \subseteq N(T)$

($\subseteq$) $\forall x \in N(T), T(x) = x_1 = 0 \therefore x = x_1 + x_2 = 0 + x_2 = x_2 \in W_2$

($\supseteq$) $\therefore\ N(T) \subseteq W_2$

(c) Describe if $W_1 = V$

$T(x) = x,\ \forall x \in W_1 = V$

$\therefore\ T$ is the identity transformation($I_V$)

(d) Describe if $W_1 = 0$

$T(x) = x \Leftrightarrow x = 0$ ( or $(0) = W_1 = R(T)$ ) $\therefore\ T$ is the zero transformation($T_0$)

27.

Claim : $\forall W \leq V, \exists W' \leq V$ as subspace, $T = W + W'$ and $T : V \to V$, $T(V) = W$

(a) From the exercise 34 in Section 1.6(p.58),

If $W$ is any subspace of a finite dimensional vector space,

$\exists\ W'$ s.t $V = W \oplus W'$

so $\forall x \in V, \exists! x_1 \in W\ \ x_2 \in W' s.t x = x_1 + x_2$

Define $T : V \to V\ \ T(x) = x_1$ is a desired linear transformation

Then $T$ is a projection on $W$ along $W'$

* Say $\dim V = n \geq 2$

$\{v_1, v_2, \cdots, v_m\}$ : a basis for W

$\{v_1, v_2, \cdots, v_m, v_{m+1}, \cdots, v_n\}$ : a basis for V

Let $W' = span\{v_{m+1}, \cdots, v_n\}$ and $W'' = span\{v_{m+1} - v_1, v_{m+2} - v_3, \cdots, v_n - v_5\}$

then $V = W \oplus W' = W \oplus W''$

Clearly $W' \neq W''$

$(\because)$ If $W' = W''$, then $v_{m+1}, v_{m+1} - v_1 \in W''$

$\therefore\ v_1 \in W \cap W' = (0)$

It's a contradiction

(b) Example

(example 1) the projection on $W$ along $W_1'$

$\{(a, b)) = (0, b - \frac{1}{3}a) + (a, \frac{1}{3}a)\}$

(example 2) the projection on $W$ along $W_2'$

$(a, b) = (0, b) + (a, 0)$

28.

(1) $\{0\}$ is $T$-invariant

$\forall x \in \{0\}$, $T(x) = 0 \in \{0\}$ $(\because T$ is linear)

(2) $V$ is $T$-invariant (T(V)⊆V)

$\forall x \in V$, $T(x) \in V$ $(\because T : V \to V)$

(3) $R(T)$ is $T$-invariant (T(T(V))⊆T(V))

$\forall$, $T(x) \in R(T)$,

$T(T(x)) \in T(V) \subseteq R(T)$

(4) $N(T)$ is T-invariant

$\forall x \in N(T)$, T$(x) = 0 \in N(T)$ ($\because N(T) \leq V$ as subspace, so $N(T)$ has zero)

29.

If $T(W) \subseteq W$, $\forall x, y \in W$, and $c \in F$

$T_W(x+y) = T(x+y) = T(x) + T(y) = T_W(x) + T_W(y)$ (W is T-invariant and T is linear)

$T_W(cx) = T(cx) = cT(x) = cT_W(x)$

$\therefore$ $T_W$ is linear

30. $\forall x \in V,\ T(x) = x_1$ s.t $x = x_1 + x_2,\ x_1 \in W, x_2 \in W'$

(1) $W$ is $T$-invariant

$\forall x_1 \in W$, T$(x_1) = x_1 \in W$ $\therefore$ T$(W) \subseteq W$

(2) $T_W = I_W$

$T_W : W \to W,\ \forall x_1 \in W,\ T_W(x_1) = x_1$

$I_W : W \to W,\ \forall x_1 \in W,\ I_W(x_1) = x_1$

$\therefore$ $\forall x_1 \in W,\ T_W(x_1) = I_W(x_1)$

$\therefore$ $T_W = I_W$

31. $V = R(T) \oplus W$, W is T-invariant

(a)

$T(W) \subseteq T(V) \cap W = (0) \;\Rightarrow\; T(W) = 0$

$\therefore\; T(W) \subseteq N(T)$ (b)

Since $V = R(T) \oplus W$,

so $\dim V = \dim R(T) + \dim W \leq rank(T) + nullity(T) = \dim V$

Since $\dim V < \infty, \dim W = nullity(T)$

(c) (Example 1) Exercise 21, left shift

(Example 2) $\beta = \{v_1, v_2, \cdots\}$ for $V$

$T : V \to V,\; T(v_i) = 0$ if $i$ is odd

$\frac{i}{2}$ if $i$ is even

Then $R(T) = V,\; N(T) = span(\{v_1, v_3, v_5, \cdots\}),\; W = (0)$

$\therefore\; V = R(T) \oplus W$

(Example 3) $\dim V = \aleph_0$,

$\{v_1, v_2, v_3, v_4, v_5, v_6, \cdots\}$ : a basis for $V$

$\{v_1, v_2, v_3, v_5, v_6, v_7, v_9, v_{10}, \cdots\}$ : a basis for $R(T)$

$\{v_3, v_4, v_7, v_8, v_{11}, v_{12}, v_{15}, v_{16}, \cdots\}$ : a basis for $N(T)$

$W = span\{v_4, v_8, v_{12}, \cdots\}$

$i.e.\; V = R(T) \oplus W,\; W \subsetneq N(T)$

32.

(1) $N(T_W) = N(T) \cap W$

$(\subseteq)\; N(T_W) = \{x_1 \in W \mid T_W(x_1) = 0\}$

$\forall x_1 \in N(T_W),\ x_1 \in W$

$T_W(x_1) = T(x_1) = 0$ *i.e.* $x_1 \in N(T)$

$\therefore\ N(T_W) \subseteq N(T) \cap W$

($\supseteq$) If $x_1 \in N(T) \cap W$, then $x_1 \in W$ and $T_W(x_1) = T(x_1) = 0$ ($\because x_1 \in N(T)$)

$\therefore\ N(T) \cap W \subseteq N(T_W)$

(2) $R(T_W) = T(W),\ R(T_W) = \{T_W(x_1) \mid x_1 \in W\}$

($\subseteq$) $\forall x_1 \in W,\ T_W(x_1) = T(x_1) \in T(W)$

($\supseteq$) $\forall x_1 \in W,\ T(x_1) = T_W(x_1) \in R(T_W)$

33. Prove theorem 2.2 in case $\beta$ is infinite

Claim : $R(T) = span(T(\beta)) = span\{T(v_1), T(v_2), \cdots\}$

($\supseteq$) Clearly $T(v_i) \in R(T)$ for each $i$.

Since $R(T) \leq V$ as subspace,

$R(T) \supseteq span\{T(v_1), T(v_2), \cdots\} = span(T(\beta))$

($\subseteq$) Suppose that $w \in R(T)$, then $w =$T($v$) for some $v \in V$

Since $\beta$ is a basis for $V$,

each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$

It means there exist a finite number of vectors $v_1, v_2, \cdots, v_n$ in $\beta$ and scalars $a_1, a_2, \cdots, a_n$ in $F$ s.t $v = \sum_{i=1}^{n} a_i v_i$

Since $T$ is linear, $w = T(v) = \sum_{i=1}^{n} a_i T(v_i) \in spanT(\beta)$

$\therefore\ R(T) \subseteq span(T(\beta))$

34. Generalization of theorem 2.6

Claim : $\forall f : \beta \to W, \exists!\ T(x) = f(x),\ \forall x_i \in \beta$

Let $f : \beta \to W s.t f(x_i) = w_i$ where $\beta = \{x_i | i \in I\}$

Note that $\forall v \in V,\ v = a_i v_i$ in a unique way where $a_i \in F,\ x_i \in \beta$

Define $T : V \to W$ by $T(v) = \sum_i a_i f(x_i)$

(i) $T$ is well-defined and unique

(ii) $T$ is linear

(iii) $T(x) = f(x),\ \forall x \in \beta$

(*) Well-definess

(Example 1) $V = R^2,\ T : R^2 \to R$ by $v = a_1 v_1 + a_2 v_2 \mapsto (a_1 + a_2)$

$\beta = \{v_1 = (1,1), v_2 = (1,-1)\}$

$\forall v = (a,b) = (\frac{a+b}{2}, \frac{a+b}{2}) + (\frac{a-b}{2}, -\frac{a-b}{2}) = \frac{a+b}{2} v_1 + \frac{a-b}{2} v_2$

Then actually $T : (a,b) \mapsto a$

(Example 2) $V = R^2,\ T : R^2 \to R$ by $v = a_1 v_1 + a_2 v_2 + a_3 v_3 \mapsto (a_1 + a_2 + a_3)$

$\beta = \{v_1 = (1,1), v_2 = (1,-1), v_3 = (0,1)\}$

$\forall v = (a,b) = \frac{a+b}{2} v_1 + \frac{a-b}{2} v_2 + 0 \cdot v_3 \mapsto a$ and $v = (a,b) = a v_1 + 0 \cdot v_2 + (b-a) v_3 \mapsto b$

$\therefore\ T$ is not well-defined

35.

(a) Suppose $V = R(T) + N(T)$

Claim : $R(T) \cap N(T) = \{0\}$

($\supseteq$) Since $R(T), N(T) \leq V$as subspaces,

So $R(T)$ and $N(T)$ have $\{0\}$

$\therefore\ R(T) \cap N(T) \supseteq \{0\}$

($\subseteq$) If $v \in R(T) \cap N(T)$,

then $\exists x \in V$ s.t $T(x) = v$ and $T(v) = 0$

$v = T(x) = T^2(x) = T(v) = 0$

($\because\ T$ is a projection on $R(T)$ along $N(T)$, then $T^2 = T$)

$\therefore\ R(T) \cap N(T) = \{0\}$

(b) Suppose $R(T) \cap N(T) = \{0\}$

We are going to show that $V = R(T) + N(T)$

Let $\dim V = n,\ rank(T) = m,\ null(T) = k$ and $n = m + k$,

and let $\beta_1 = \{w_1, w_2, cdots, w_m\}$ a basis for $R(T)$

$\beta_2 = \{u_1, u_2, cdots, u_k\}$ a basis for $N(T)$

Claim : $\beta = \beta_1 + \beta_2 = \{w_1, w_2, cdots, w_m, u_1, u_2, cdots, u_k\}$ is linearly independent

If $a_1w_1 + \cdots + a_mw_m + b_1u_1 + \cdots + b_ku_k = 0, (\forall a_i, b_j \in F)$

then $a_1w_1 + \cdots + a_mw_m = -(b_1u_1 + \cdots + b_ku_k) \in R(T) \cap N(T) = \{0\}$

$\therefore\ a_1 = \cdots = a_m = b_1 \cdots = b_k = 0$

Since $\dim V = n,\ \beta$ is a basis for $V$,

$\forall v \in V,\ v = c_1w_1 + \cdots + c_mw_m + d_1u_1 + \cdots + d_ku_k = 0, (\forall c_i, d_j \in F)$

let $w = c_1w_1 + \cdots + c_mw_m$ and $u = d_1u_1 + \cdots + d_ku_k$

then $v = w + u \in R(T) + N(T)$

$\therefore\ V = R(T) \oplus N(T)$

36.

(a) We are going to show that if $V$ is infinite-dimensional, then $V$ doesn't hold the result of Exercise 35(a)

From Exercise 21, $\forall v = (a_1, a_2, \cdots) \in V$, $T : V \to V$ is left shift

then $N(T) = \{(a_1, 0, 0, \cdots) \mid a_1 \in F\}$

$R(T) = \{(a_2, a_3, a_4, \cdots) \mid \forall a_i \in F, i = 2, 3, \cdots\}$

$\therefore\ \forall v \in V,\ v = (a_1, a_2, a_3, \cdots) = (a_1, a_2, a_3, \cdots) + (0, 0, 0, \cdots) \in R(T) + N(T)$

$\therefore\ V = R(T) + N(T)$

But $R(T) \cap N(T) = \{(a, 0, 0, \cdots) \mid a \in F\} \neq \{0\}$

$\therefore\ V \neq R(T) \oplus N(T)$

(b) Find $T_1 : V \to V$ s.t $R(T_1) \cap N(T_1) = \{0\}$ but $V \neq R(T_1) \oplus N(T_1)$

$\forall v \in V,\ T_1(v) = T(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots)$

then $N(T_1) = \{0\}$

$R(T_1) = \{(0, a_1, a_2, \cdots) \mid a_i \in F\}$

$\therefore\ R(T_1) \cap N(T_1) = \{0\}$, but

for $0 \neq a \in F,\ \exists v = (a, 0, 0, \cdots)$ is not in $R(_1) + N(T_1)$

$\therefore\ R(T_1) + N(T_1) \subsetneq V$

$\therefore\ V \neq R(T_1) \oplus N(T_1)$

37. We are going to show that $T(\alpha x) = \alpha T(x),\ \alpha \in Q$

$\forall x, y \in V,\ T(x+y) = T(x) + T(y)$

Let $\alpha = \frac{b}{a}, a, b \in Z, a \neq 0$

$T(\alpha x) = T(\frac{b}{a}x) = T(\frac{b}{a}a \cdot \frac{x}{a}) = \frac{b}{a} \cdot aT(\frac{x}{a}) = \frac{b}{a} \cdot T(a\frac{x}{a}) = \frac{b}{a}T(x)$

$\therefore$ $T$ is linear

38.

$\forall x, y \in \mathbb{C}$ s.t $x = a + bi, y = c + di, \forall a, b, c, d \in \mathbb{R}, \alpha \in \mathbb{C}$

T$(x+y)$=T$((a+c)+(b+d)i) = (a+c)-(b+d)i = (a-bi)+(b-di)$=T$(x)$+T$(y)$

In case $\alpha = i$, T$(\alpha x)$=T$(-b+ai) = -b - ai$

but $\alpha$T$(x) = i(a - bi) = b - ai$

$\therefore$ T$(\alpha x) \neq \alpha$T$(x)$

$\therefore$ T is not linear

39.

Claim : $\exists$ T:$\mathbb{R} \to \mathbb{R}$ is an additive function that is not linear

let V be the set of real numbers regarded as a vector space over the field of rational numbers.

Since every vector space has a basis(p.61),so V has a basis $\beta$

For fixed $x, y \in \beta$ s.t $x \neq y$

Define $f : \beta \to V$ by $f(x) = y, f(y) = x$ and $f(z) = z$ otherwise

By Exercise 34, $\exists!$ T:V$\to$V : linear transformation over $\mathbb{Q}$ s.t T(u)=f(u), $\forall u \in \beta$

Then T is additive(from exercise 37) but T is not linear over $\mathbb{R}$

($\because$) In case $\alpha = \frac{y}{x}$,

T($\alpha x$)=T($\frac{y}{x}x$)=T($y$) $= f(y) = x$

$\alpha$T($x$)=$\frac{y}{x}$ T($x$) $= \frac{y}{x}f(x) = \frac{y}{x}y = \frac{y^2}{x}$

$\therefore$T($\alpha x$) $\neq \alpha$T($x$) ($\because x \neq y$)

$\therefore$ T is not linear

40. $\eta$: V$\rightarrow$V/W, $\eta(v) = v + W$ ($v + W = 0$ in V/W $\Leftrightarrow v \in W$)

(a) Prove $\eta$ is linear and N($\eta$)=W

$\forall v_1, v_2 \in V$, c$\in F$

(1) $\eta(cv_1 + v_2) = (cv_1 + v_2) + W = cv_1 + W + v_2 + W = c(v_1 + W) + (v_2 + W) = c\eta(v_1) + \eta(v_2)$

(2) $\forall v + W \in V/W$, $\exists v \in V$

(3) N($\eta$)=W

($\because$) ($\supseteq$) If $w \in W$, then $\eta(w) = w + W = 0 + W$

$\therefore w \in$N($\eta$)

($\subseteq$) If $v \in$N($\eta$), then $\eta(v) = 0 + W = w + W$ for some $w \in W$

$\therefore v \in$W

(b) Suppose V is finite-dimensional ($ker\eta = W, R(\eta) = V/W$)

dimN($\eta$)=dim(W), Rank($\eta$)=dim(V/W)=dimV-dimW (p.58 Sec1.6 Exercise35)

dimV=dimN($\eta$)+Rank($\eta$)=dim(W)+dim(V/W)

(c) same

## 2.2. The matrix representation of a linear transformation

1. $\beta = \{v_1, v_2, \cdots, v_n\}$, $\gamma = \{w_1, w_2, \cdots, w_m\}$ bases for V and W, respectively

(a) T (p.82 Theorem 2.7(a))

(b) T (p.73 The corollary to Theorem 2.6 and p.80)

(p.80) Let T:V→W is linear. Then for each $j$, $1 \leq j \leq n$, there exist unique scalars $a_{ij}, b_{ij} \in F, 1 \leq i \leq m$ s.t

T$(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, U$(v_j) = \sum_{i=1}^{m} b_{ij} w_i$ for $1 \leq j \leq n$

Suppose $[T]_\beta^\gamma = (a_{ij})_{m\times n}$,$[U]_\beta^\gamma = (b_{ij})_{m\times n}$

If $[T]_\beta^\gamma = [U]_\beta^\gamma$, then T$(v_j) = \sum_{i=1}^{m} a_{ij} w_i$=$\sum_{i=1}^{m} b_{ij} w_i$=U$(v_j)$ for all $a_{ij}, b_{ij} \in F, \forall v_j \in \beta$.

Hence T=U

(c) F ($[T]_\beta^\gamma$ is an $n \times m$ matrix )

(d) T (p.83 Theorem 2.8 (a))

(e) T ($0 \in \mathcal{L}(V, W)$) and Theorem 2.7 (a))

(f) F (p.104 $\mathcal{L}(V, W) \cong M_{m\times n}(F)$, $\mathcal{L}(W, V) \cong M_{n\times m}(F)$)

$(cf)$ ($\mathcal{L}(V, W) \cong \mathcal{L}(W, V)$) but ($\mathcal{L}(V, W) \neq \mathcal{L}(W, V)$)

2. Compute $[T]_\beta^\gamma$

(a) T(1,0)=(2,3,1)=$2w_1 + 3w_2 + 1w_3$, T(0,1)=$(-1, 4, 0) = -1w_1 + 4w_2 + 0w_3$

$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

(b) T(1,0,0)=(2,1), T(0,1,0)=(3,0), $T(0,0,1)=(-1,1)$
$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

(c) $[T]_\beta^\gamma = (1,0,-3)$
(d) $[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}$
(e) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$
(f) $[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$
(g) $[T]_\beta^\gamma = (1,0,\cdots,0,1)$

3.

(a) T(1,0)=(1,1,2)=$-\frac{1}{3}(1,1,0)+0(0,1,1)+\frac{2}{3}(2,2,3)$

T(0,1)=$(-1,0,1)$=$-1(1,1,0)+1(0,1,1)+0(2,2,3)$
$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$

(b) T(1,2)=$(-1,1,4)$=$-\frac{7}{3}(1,1,0)+2(0,1,1)+\frac{2}{3}(2,2,3)$

T(2,3)=$(-1,2,7)$=$-\frac{11}{3}(1,1,0)+3(0,1,1)+\frac{4}{3}(2,2,3)$

$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$

4. $\beta = \{e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$, $\gamma = \{1, x, x^2\}$

$\mathrm{T}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 0x^2$, $\mathrm{T}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 1x^2$,

$\mathrm{T}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 + 0x + 0x^2$, $\mathrm{T}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 2x + 0x^2$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

5.$\alpha = \{e_1, e_2, e_3, e_4\}$, $\beta = \{1, x, x^2\}$ , $\gamma = \{1\}$

(a)

$\mathrm{T}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1e_1 + 0e_2 + 0e_3 + 0e_4$,

$\mathrm{T}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0e_1 + 0e_2 + 1e_3 + 0e_4$,

$\mathrm{T}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0e_1 + 1e_2 + 0e_3 + 0e_4$,

$\mathrm{T}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0e_1 + 0e_2 + 0e_3 + 1e_4$

$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\mathrm{T}(1)=\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0e_1 + 2e_2 + 0e_3 + 0e_4$,

$\mathrm{T}(x)=\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1e_1 + 2e_2 + 0e_3 + 0e_4$, T($x^2$)=$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0e_1 + 2e_2 + 0e_3 + 2e_4$,

$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

(c) T($e_1$)=1, T($e_2$)=0, T($e_3$)=0, T($e_4$)=1

$[T]_\alpha^\gamma = (1, 0, 0, 1)$

(d) T(1)=1, T($x$)=2, T($x^2$)=4, T($f(x)$)=$f(2)$

$[T]_\beta^\gamma = (1, 2, 4)$

(e) A=$1e_1 + (-2)e_2 + 0e_3 + 4e_4$

$[A]_\alpha = (1, -2, 0, 4)^T$

(f) $f(x) = 3 \cdot 1 + (-6)x + 1x^2$

$[f(x)]_\beta = (3, -6, 1)^T$

(g)$\gamma = \{1\}$

$[a]_\gamma = a$

6. Theorem 2.7 (b)

$\mathcal{L}(V, W)$ is a vector space over F

(a) $T_0 \in \mathcal{L}(V, W)$, $T_0$ : the zero transformation

(b) $aT + U \in \mathcal{L}(V, W), \forall T, U \in \mathcal{L}(V, W), \forall a \in F$

7. Theorem 2.8 (b)

Let $\beta = \{v_1, v_2, \cdots, v_n\}$ and $\gamma = \{w_1, w_2, \cdots, w_m\}$ bases for V and W, respectively

and let $[T]_\beta^\gamma = (a_{ij})_{m\times n},\ a_{ij} \in F$

then $(aT)v_j = aT(v_j) = a\sum_{i=1}^{m} a_{ij}w_i = \sum_{i=1}^{m} a(a_{ij}w_i) = \sum_{i=1}^{m}(aa_{ij})w_i,\ \forall i$

So $[aT]_\beta^\gamma = (aa_{ij})_{m\times n} = a(a_{ij})_{m\times n} = a[T]_\beta^\gamma$

8. $\beta = \{v_1, \cdots, v_n\}$ : a basis for V

$\forall x, y \in V$ s.t $x = \sum_{i=1}^{n} a_i v_i,\ y = \sum_{i=1}^{n} b_i v_i \in V, a_i, b_i, c \in F$

T$(x) = [x]_\beta = (a_1, \cdots, a_n)^t$

and $cx + y = \sum_{i=1}^{n}(ca_i + b_i)v_i$, then

T$(cx + y)$=$[cx + y]_\beta$=$(ac_1 + b_1, \cdots, ca_n + b_n)^t = (ca_1, \cdots, ca_n)^t + (b_1, \cdots, b_n)^t =$

$c(a_1, \cdots, a_n)^t + (b_1, \cdots, b_n)^t = c[x]_\beta + [y]_\beta = c$T$(x)$+T$(y)$

$\therefore$ T is linear

(Indeed $T$ is an isomorphism)

9.

$\forall x = a + bi, y = c + di \in \mathcal{C},\ \alpha, a, b, c, d \in \mathcal{R}$

(a) T$(\alpha x + y)$=T$((\alpha a + c) + (\alpha b + d)i)$=$(\alpha a + c) - (\alpha b + d)i = (\alpha a - \alpha bi) + (c - di) =$

$\alpha(a - bi) + (c - d)i = \alpha$T$(a + bi)$+T$(c + di) = \alpha$T$(x)$+T$(y)$

$\therefore$ T is linear

(b) T(1)=1=1·1 $+ 0 \cdot i$ and T$(i) = -i = 0 \cdot 1 + (-1) \cdot i$

A=$[\mathrm{T}]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

10. Compute $[\mathrm{T}]_\beta$

$\mathrm{T}(v_1) = v_1 + v_0 = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$

$\mathrm{T}(v_2) = v_2 + v_1 = v_1 + v_2 = 1 \cdot v_1 + 1 \cdot v_2 + \cdots + 0 \cdot v_n$

$\mathrm{T}(v_3) = v_3 + v_2 = v_2 + v_3 = 0 \cdot v_1 + 1 \cdot v_2 + 1 \cdot v_3 + \cdots + 0 \cdot v_n$

$\vdots$

$\mathrm{T}(v_n) = v_n + v_{n-1} = v_{n-1} + v_n = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_{n-1} + 1 \cdot v_n$

$$\therefore [\mathrm{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

11.

We choose a basis $\{w_1, \cdots, w_k\}$ of W and extend it to a basis of V $(\{w_1, \cdots, w_k, v_1 \cdots, v_s\})$

Let dimV=n and n=k+s

$\mathrm{T}_W(w_1)$=$\mathrm{T}(w_1)$=$a_{11}w_1 + \cdots + a_{1k}w_k + 0 \cdot v_1 + \cdots + 0 \cdot v_s$

$\mathrm{T}_W(w_2)$=$\mathrm{T}(w_2)$=$a_{21}w_1 + \cdots + a_{2k}w_k + 0 \cdot v_1 + \cdots + 0 \cdot v_s$

$\vdots$

$\mathrm{T}_W(w_k)$=$\mathrm{T}(w_k)$=$a_{k1}w_1 + \cdots + a_{kk}w_k + 0 \cdot v_1 + \cdots + 0 \cdot v_s$

$\mathrm{T}(v_1)$=$b_{11}w_1 + \cdots + b_{1k}w_k + c_{11}v_1 + \cdots + c_{1s}v_s$

$\mathrm{T}(v_2)$=$b_{21}w_1 + \cdots + b_{2k}w_k + c_{21}v_1 + \cdots + c_{2s}v_s$

$\vdots$

$\mathrm{T}(v_s){=}b_{s1}w_1 + \cdots + b_{sk}w_k + c_{s1}v_1 + \cdots + c_{ss}v_s$

The matrix of T is the transpose of the matrix of coefficients in the above system of equations

$$\therefore [\mathrm{T}]_\beta = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & b_{11} & b_{12} & \cdots & b_{1s} \\ a_{21} & a_{22} & \cdots & a_{2k} & b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & b_{k1} & b_{k2} & \cdots & b_{ks} \\ 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1s} \\ 0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & c_{s1} & c_{s2} & \cdots & c_{ss} \end{pmatrix}$$

12. $T : V \to V$ : the projection on $W$ along $W'$

Let $\beta_1 = \{v_1, \cdots, v_k\}$, $\gamma_1 = \{v_{k+1}, \cdots, v_n\}$ bases for $W$ and $W'$, respectively

Since $V = W \oplus W'$, $\beta = \beta_1 \cup \gamma_1$ is an ordered basis for V

Claim : $[\mathrm{T}]_\beta$ is a diagonal matrix

Since $\beta = \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$, $\forall x \in V, x = \sum\limits_{i=1}^{n} a_i v_i, a_i \in F$

So $\mathrm{T}(x) = \sum\limits_{i=1}^{n} a_i \mathrm{T} v_i$

$\mathrm{T}(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$ $(\because v_1 \in W)$

$\vdots$

$\mathrm{T}(v_k) = v_k = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_k + \cdots + 0 \cdot v_n$ $(\because v_k \in W)$

$\mathrm{T}(v_{k+1}) = 0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$ $(\because v_{k+1} \in W')$

$\vdots$

$\mathrm{T}(v_n) = 0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$ $(\because v_n \in W')$

$$\therefore [\mathrm{T}]_\beta = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \begin{matrix} \\ \\ \\ \leftarrow k-th \\ \\ \\ \\ \end{matrix}$$

$\therefore [T]$ is a diagonal matrix

13. $\beta = \{v_1, v_2, \cdots\}$ : a basis for V

Claim: If $aT + bU = 0$, then $a = b = 0,\ a, b \in F$

Let $aT(v) + bU(v) = (aT + bU)(v) = 0(v) = 0$

$\Rightarrow\ aT(v) = -bU(v) \in R(T) \cap R(U) = 0$

$\Rightarrow\ (aT)(v) = 0,\ (bU)(v) = 0,\ \forall v \in V$

$\Rightarrow\ aT = 0,\ bU = 0$

$\Rightarrow^*\ a = 0,\ b = 0\ (\because\ T,\ U \neq 0)$

$\lceil$ $(^*)$

$a \in F,\ T \in \mathcal{L}(V, W)$

If $aT = 0\ \Rightarrow$either $a = 0$ or $T = 0$

$(\because)$ Assume $a \neq 0\ \Rightarrow\ \exists a^{-1} \in F$

$\Rightarrow\ T = 1 \cdot T = (a^{-1}a)T = a^{-1}(aT) = a^{-1}0 = 0$

or

Assume that $T \neq 0\ \Rightarrow$

$\exists v \in F$ s.t. $T(v) \neq 0$

$\Rightarrow\ a(T(v)) = (aT)(v) = T_0(v) = 0$

$\Rightarrow\ a = 0$

$(\because)\ aw = 0\ \Rightarrow\ a = 0$ or $w = 0$ ⌋

14.

Claim: If $(a_1T_1+a_2T_2+\cdots+a_nT_n)(x) = 0(x)$, then $a_1 = a_2 = \cdots = a_n = 0, a_i \in F$

Let $f(x) = k_0 + k_1x + k_2x^2 + \cdots + k_nx^n$ $\in$P(R)

$(a_1\mathrm{T}_1+a_2\mathrm{T}_2+\cdots+a_n\mathrm{T}_n)(f(x))= a_1\mathrm{T}_1(f(x))+a_2\mathrm{T}_2(f(x))+\cdots+a_n\mathrm{T}_n(f(x))=a_1f'(x)+$

$a_2f''(x) + \cdots + a_nf^{(n)}(x) = 0$

By Exercise 24. Sec1.6(p.56), we obtain $a_1 = a_2 = \cdots = a_n = 0$

$\therefore$ $\{\mathrm{T}_1,\mathrm{T}_2,\cdots,\mathrm{T}_n\}$ is a linearly independent subset of $\mathcal{L}(V)$, $\forall n \in Z^+$.

15. $S^0 = \{\mathrm{T}\in \mathcal{L}(V,W) \mid \mathrm{T}(x) = 0, \forall x \in S\}$

(a) $S^0$ is a subspace of $\mathcal{L}(V,W)$

(1) $\forall x \in V$, $\mathrm{T}_0 \in \mathcal{L}(V,W)$ s.t T$(x)$=0

$\therefore \forall x \in S$, $\mathrm{T}_0(x) = 0$

$\therefore$ $\mathrm{T}_0 \in S^0$

(2) If $\mathrm{T}_1$, $\mathrm{T}_2 \in S^0, \alpha \in F, \forall x \in S$

$(\alpha\mathrm{T}_1+\mathrm{T}_2)(x) = \alpha\mathrm{T}_1(x)+\mathrm{T}_2(x) = \alpha \cdot 0 + 0 = 0$

$\therefore \forall x \in S, (\alpha\mathrm{T}_1+\mathrm{T}_2)(x) = 0$

$\therefore \alpha\mathrm{T}_1+\mathrm{T}_2 \in S^0$

(b) $S_1^0 = \{\mathrm{T}\in \mathcal{L}(V,W) \mid \mathrm{T}(x) = 0, \forall x \in S_1\}$

and $S_2^0 = \{\mathrm{T}\in \mathcal{L}(V,W) \mid \mathrm{T}(x) = 0, \forall x \in S_2\}$

If T$\in S_2^0$, then $\forall x \in S_2$, $\mathrm{T}(x) = 0$

Since $S_1 \subseteq S_2$, $\forall x \in S_1$, $\mathrm{T}(x) = 0$

$\therefore$ T$\in S_1^0$

$\therefore S_2^0 \subseteq S_1^0$

(c) $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$

($\subseteq$) Since $V_1, V_2 \subseteq V_1 + V_2$ and (b)

$(V_1 + V_2)^0 \subseteq V_1^0$ and $(V_1 + V_2)^0 \subseteq V_2^0$

$\therefore (V_1 + V_2)^0 \subseteq V_1^0 \cap V_2^0$

($\supseteq$) $\forall$ T$\in V_1^0 \cap V_2^0$

Claim: T$\in (V_1+V_2)^0$ (i.e. $\forall x \in V_1+V_2$, $\mathrm{T}(x) = 0$ s.t $x = x_1+x_2, x_1 \in V_1, x_2 \in V_2$)

$\mathrm{T}(x)$ =T$(x_1 + x_2)$=T$(x_1)$+T$(x_2) = 0 + 0 = 0$ ($\because$ T is linear and T$\in V_1^0 \cap V_2^0$,)

$\therefore$ T$\in (V_1 + V_2)^0$

16.

Let $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{w_1, \cdots, w_n\}$ : bases for V and W, respectively

Claim : $[\mathrm{T}]_\beta$ is a diagonal matrix

Define $\mathrm{T}(v_i) = w_i$ for $i = 1, 2, \cdots, n$,

then $[\mathrm{T}]_\beta^\gamma = I_n$

$\therefore [\mathrm{T}]_\beta^\gamma$ is a diagonal matrix

## 2.3. Composition of Linear Transformations and Matrix Multiplication

1.

(a) F ($[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$) (p.88 Theorem 2.11)

(b) T (p.91 Theorem 2.14)

(c) F ($[U(w)]_\gamma = [U]_\beta^\gamma [w]_\beta$ for all $w \in W$)

(d) T (Since $\mathrm{I}_V(v_j) = v_j, 1 \leq i, j \leq n$ $[\mathrm{I}_V]_\alpha$=$\mathrm{I}_n$

(e) F

In case T:V→V, $[T^2]_\alpha = [T \cdot T]_\alpha = [T]_\alpha [T]_\alpha = ([T]_\alpha)^2$

(f) F

If $I \neq$A=$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then $\mathrm{A}^2$=I

(cf) A=$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(g) F

(T:$F^n \to F^m \Leftrightarrow$T=$L_A$ for some $A \in M_{m \times n}(F)$

(h) F

If A=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $\mathrm{A}^2 = 0$ even though A$\neq 0$

The cancelation property for multiplication in fields is not valid for matrices.

(i) T (p.93 Theorem 2.15(c))

(j) T (p.89 Definition)

2.

(a)

(1) A(2B+3C)= $\begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}$

(2) (AB)D= $\begin{pmatrix} 29 & -26 \end{pmatrix}$

(3) A(BD)= $\begin{pmatrix} 29 & -26 \end{pmatrix}$

(b)

(1) $\mathrm{A}^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$

(2) $\mathrm{A}^t\mathrm{B}$= $\begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}$

(3) $\mathrm{BC}^t$= $\begin{pmatrix} 12 \\ 16 \\ 19 \end{pmatrix}$

(4) CB= $\begin{pmatrix} 27 & 7 & 9 \end{pmatrix}$

(5) CA= $\begin{pmatrix} 20 & 26 \end{pmatrix}$

3.

(a)

(1) $[U]_\beta^\gamma$

$U(1) = U(1 + 0 \cdot x + 0 \cdot x^2) = (1, 0, 1)$

$U(x) = U(0 + 1 \cdot x + 0 \cdot x^2) = (1, 0, -1)$

$U(x^2) = U(0 + 0 \cdot x + 1 \cdot x^2) = (0, 1, 0)$

$\therefore [U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

(2) $[T]_\beta$

$T(1) = T(0 \cdot (3 + x) + 2 \cdot 1) = 2 = 2 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x) = T(1 \cdot (3+x) + 2 \cdot x) = 3x + 3 = 3 \cdot 1 + 3 \cdot x + 0 \cdot x^2$

$T(x^2) = T(2x \cdot (3+x) + 2 \cdot x^2) = 4x^2 + 6x = 0 \cdot 1 + 6 \cdot x + 4 \cdot x^2$

(3) $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$

$(UT)(1) = U(T(1)) = U(2 \cdot 1 + 0 \cdot x + 0 \cdot x^2) = (2, 0, 2)$

$(UT)(x) = U(T(x)) = U(3 \cdot 1 + 3 \cdot x + 0 \cdot x^2) = (6, 0, 0)$

$(UT)(x^2) = U(T(x^2)) = U(0 \cdot 1 + 6 \cdot x + 4 \cdot x^2) = (6, 4, -6)$

$\therefore [UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

And $[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

$\therefore [UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$

(b)

(1) $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

(2) $[U(h(x))]_\gamma = \begin{pmatrix} 1 & 1 & 5 \end{pmatrix}$

(3) $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \end{pmatrix}$

4.

(a) $[T(A)]_\alpha = [T]_\alpha [A]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$

(b) $[T(f(x))]_\alpha = [T]^\alpha_\beta[]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$

(c) $[T(A)]_\gamma = [T]^\gamma_\alpha[A]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \end{pmatrix}$

(d) $[T(f(x))]_\gamma = [T]^\gamma_\beta[A]_\beta = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \end{pmatrix}$

5. Theorem 2.12

(a) (D+E)A=DA+EA

$[(D+E)A]_{ij} = \sum_{k=1}^{m}(D+E)_{ik}A_{kj} = \sum_{k=1}^{m}(D_{ik}+E_{ik})A_{kj} = \sum_{k=1}^{m}(D_{ik}A_{kj}+E_{ik}A_{kj}) = \sum_{k=1}^{m} D_{ik}A_{kj} + \sum_{k=1}^{m} E_{ik}A_{kj} = (DA)_{ij} + (EA)_{ij} = [DA+EA]_{ij}$

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a

We have $[a(AB)]_{ij} = \sum_{k=1}^{n} a(A_{ik}B_{kj}) = \sum_{k=1}^{n}(aA_{ik})B_{kj} = [(aA)B]_{ij}$

and $[(aA)B]_{ij} = \sum_{k=1}^{n}(aA_{ik})B_{kj} = \sum_{k=1}^{n} A_{ik}(aB_{kj}) = [A(aB)]_{ij}$

$\therefore a(AB) = (aA)B = A(aB)$

(c) $A_{ij} = \sum_{k=1}^{n} A_{ik}\delta_{kj} = \sum_{k=1}^{n} A_{ik}(I_n)_{kj} = (A \cdot I_n)_{ij}$

(d) Let dimV=n and $\beta = \{v_1, \cdots, v_n\}$

$I_V : V \to V$ s.t $I_V(v_i) = v_i = 0 \cdot v_1 + \cdots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \cdots + v_n$

$$\therefore [I_V]_\beta = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n$$

Corollary.

(1) $A(\sum_{i=1}^{k} a_i B_i) = \sum_{k=1}^{k} AB_i$

$A(\sum_{i=1}^{k} a_i B_i) = A(a_1 B_1 + \cdots + a_k B_k) = A(a_1 B_1 + \cdots + A(a_k B_k) = a_1 AB_1 + \cdots + a_k AB_k = \sum_{i=1}^{k} a_i AB_i$

(2) $(\sum_{i=1}^{k} a_i C_i)A = (a_1 C_1 + \cdots + a_k C_k)A = (a_1 C_1)A + \cdots + (a_k C_k)A = \sum_{i=1}^{k} a_i C_i A$

6. Theorem 2.13

(b)

$$v_j = \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} = \begin{pmatrix} B_{11} & \cdots & B_{1j} & \cdots & B_{1n} \\ B_{21} & \cdots & B_{2j} & \cdots & B_{2n} \\ \vdots & & \vdots & & \vdots \\ B_{n1} & \cdots & B_{nj} & \cdots & B_{nn} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1(j-th) & 0 & \cdots & 0 \end{pmatrix}^t$$

$\therefore v_j = Be_j$

7. Theorem 2.15

(c) $\forall x \in F^n$

(1) $L_{A+B}(x) = (A+B)x = Ax + Bx = L_A(x) + L_B(x) = (L_A + L_B)(x)$

$\therefore L_{A+B} = L_A + L_B$

(2) $L_{aA}(x) = (aA)x = a(Ax) = (aL_A)(x)$

$\therefore L_{aA} = aL_A$

(f) $L_{I_n} = I_{F^n}$

$L_{I_n}(x) = I_n x = x$ and $I_{F^n}(x) = x\ \forall x \in F^n$

$\therefore L_{I_n} = I_{F^n}$

8.

(a)

(1) $(T(U_1 + U_2))(x) = T((U_1 + U_2)(x)) = T(U_1(x) + U_2(x)) = T(U_1(x)) + T(U_2(x)) = (TU_1)(x) + (TU_2)(x) = (TU_1 + TU_2)(x)$

$\therefore T(U_1 + U_2) = TU_1 + TU_2$

(2) $(U_1 + U_2)T(x) = U_1(T(x)) + U_2(T(x)) = (U_1T)(x) + (U_2T)(x) = (U_1T + U_2T)(x)$

$\therefore (U_1 + U_2)T = U_1T + U_2T$

(b)

$T(U_1U_2)(x) = T(U_1(U_2(x))) = (TU_1)(U_2(x)) = (TU_1)U_2(x)$

$\therefore T(U_1U_2) = (TU_1)U_2$ (c)

$TI(x) = T(I(x)) = T(x) \therefore TI = T$

$IT(x) = I(T(x)) = T(x) \therefore IT = T$

(d)

(1) $a(U_1U_2)(x) = aU_1(U_2(x)) = (aU_1)(U_2(x)) = ((aU_1)U_2)(x)$

$\therefore a(U_1U_2) = (aU_1)U_2$

(2) $U_1(aU_2)(x) = U_1(aU_2(x)) = aU_1(U_2(x)) = a(U_1U_2)(x)$

$\therefore U_1(aU_2) = a(U_1)U_2)$

More general result

Let V,U,W be vector spaces over K. Suppose the following mappings are linear

$F : V \to U, F' : V \to U$ and $G : U \to W, G' : U \to W$

Then for any scalars $k \in K$

(1) $G(F + F') = GF + GF'$

(2) $(G + G')F = GF + G'F$

(3) $k(GF) = (kG)F + G(kF)$

9.

(1) Let $U : F^2 \to F^2$ s.t $U(a, b) = (a + b, 0), \forall a, b \in F$ and

$T : F^2 \to F^2$ s.t $T(a, b) = (a, -a), \forall a, b \in F$

then $UT(a, b) = U(a, -a) = (0, 0)$ , i.e $UT = T_0$ and $TU(a, b) = T(a + b, 0) = (a + b, -a - b) \neq (0, 0)$, i.e $TU \neq T_0$

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

then $AB = 0$ but $BA \neq 0$

10.

($\supseteq$)Since $A_{ij} = \delta_{ij}A_{ij}$, if $i \neq j$, then $A_{ij} = 0$

thus A is a diagonal matrix

($\subseteq$) Since A is a diagonal matrix, if $i \neq j$, then $A_{ij} = 0$, thus $A_{ij} = \delta_{ij} A_{ij}$

and if $i = j$, then $\delta_{ij} = 1$, therefore $A_{ij} = \delta_{ij} A_{ij}$

11.

$(\subseteq)$ $\forall v \in V$, $T(V) \in R(T)$, since $T(T(V)) = \{0\}$, $T(V) \in N(T)$

$\therefore R(T) \subseteq N(T)$

$(\supseteq)$ $\forall v \in V$, $T(V) \subseteq R(T)$,

$\therefore R(T) \subseteq N(T)$

$\therefore T^2(v) \subseteq T(T(v) \subseteq T(N(T)) = 0$

$\therefore T^2 = T_0$

12.

(a) Assume $UT : V \to Z$ is one-to-one and let $T(V_1) = T(v_2)$ for $v_1, v_2 \in V$

Then $U(T(v_1)) = U(T(v_2))$ i.e $(UT)(v_1) = (UT)(v_2)$

Since $UT$ is one-to-one, $v_1 = v_2$

$\therefore$ T is one-to-one

(Example)

$V = R^2,\ W = Z = R^3$

$T : V \to W\ T(a, b) = (a, b, 0)$

$U : W \to Z\ U(a, b, c) = (a, b, 0)$

(b) Assume that $UT : V \to Z$ is onto and let $z \in Z$

$\exists v \in V$ s.t $(UT)(v) = z$

Let $w = T(v)$

Then $w \in W$ and $U(W) = U(T(v)) = (UT)(v) = z$

$\therefore U : W \to Z$ is onto

(Example)

$V = W = R^3,\ W = R^2$

$T : V \to W\ T(a, b, c) = (a, b, 0)$

$U : W \to Z\ U(a, b, c) = (a, b)$

$UT$ is one-to-one but $U$ is not

(c) (1) Let $v_1, v_2 \in V$ and assume $(UT)(v_1) = (UT)(v_2)$

If $U(T(v_1)) = U(T(v_2))$, then $T(v_1) = T(v)_2(\because$ U is one-to-one)

and $v_1 = v_2(\because)$ T is one-to-one

(Example)

$V = W = R^2,\ W = R^3$

$T : V \to W\ T(a, b) = (a, b, 0)$

$U : W \to Z\ U(a, b, c) = (a, b)$

$UT = i_{R^2}$ is one-to-one an onto

(2) Let $z \in Z$

Since U is onto, $\exists w \in W$ s.t $z = U(w)$

and T is onto, $\exists v \in V$ s.t $w = T(v))$

Thus $z = U(w) = U(T()v) = (UT)(w)$

$\therefore$ UT is onto

Therefore if U and T are one-to-one and onto, then so is UT.

13.

(a) $tr(AB) = \sum\limits_{i=1}^{n}(AB)_{ii} = \sum\limits_{i=1}\sum\limits_{j=1}^{n}(A)_{ij}(B)_{ji} = \sum\limits_{j=1}(\sum\limits_{i=1}^{n}(B)_{ji}(A)_{ij}) = \sum\limits_{j=1}^{n}(BA)_{jj} = tr(BA)$

(b) $tr(A) = \sum\limits_{i=1}^{n}(A)_{ii} = \sum\limits_{i=1}^{n}(A^t)_{ii} = tr(A^t)$

14.

(a) $z = (a_1, a_2, \cdots, a_p)^t = \sum\limits_{j=1}^{p} a_j e_j$

So $Bz = B(\sum\limits_{j=1}^{p} a_j e_j) = \sum\limits_{j=1}^{p} a_j(Be_j) = \sum\limits_{j=1}^{p} a_j v_j$

(b) $(AB)^j = (AB)e_j = A(Be_j) = Av_j = \sum\limits_{k=1}^{n} b_{kj}(u_k),\ v_j = (b_{1j}, \cdots, v_{nj})^t$

(c) $wA = (\sum\limits_{i=1}^{m} a_i e_i)A = \sum\limits_{i=1}^{m} a_i(e_i A) = \sum\limits_{i=1}^{m} a_i u_i$

(d) $(AB)_{(i)} = e_i(AB) = (e_i A)B = u_i B = \sum\limits_{k=1}^{n} b_{ik}(v_k),\ u_i = (b_{i1}, \cdots, v_{in})$

15.

$M = (\gamma_{ij})_{m\times n},\ A = (a_{jk})_{n\times p}$

If $A^{(j)} = \sum b_k A^{(k)} \ \Rightarrow\ MA^{(j)} = M(\sum b_k A^{(k)}) = \sum b_k MA^{(k)} = \sum b_k (MA)^{(k)}$

16.

(a) If $rank(T) = rank(T^2)$

$i.e.\ dimR(T) = dimR(T^2)\ \therefore\ dimN(T) = dimN(T^2)$

(i) But $dimR(T) \leq dimV < \infty \ \Rightarrow\ R(T) = R(T^2)$ ($\not\Rightarrow\ T = T^2$)

(ii) and $dimN(T^2) \leq dimV < \infty \ \Rightarrow\ N(T) = N(T^2)$

($\because$) If $v \in R(T^2)\ \Rightarrow\ v = T^2(z)$ for some $z \in V$

$\Rightarrow\ v = T(T(z)) \in R(T)$

$\therefore\ R(T^2) \subseteq R(T)$

(a) If $w \in R(T) \cap N(T)$

$\Rightarrow\ w = T(V)$ for some $v \in V$

Then $T^2(v) = T(w) = 0$

$\therefore\ v \in N(T^2) = N(T)$

$\therefore\ w = T(v) = 0$

(b)

Since $R(T) \supseteq R(T^2) \supseteq \cdots$

we have $dimV \geq rankT \geq rankT^2 \geq \cdots \geq 0$

So $\exists k(\geq 1),\ rank(T^k) = rank(T^{k-1})\ \Rightarrow^*\ R(T^k) = R(T^{k+1}) = \cdots = R(T^{2k})$

$\lceil(^*)$

If $w \in R(T^k)\ \Rightarrow\ w = T^k(v) = T^{k+}(v_1)$

and $T^{k+1}(v_1) = T(T^k(v_1)) = T(T^{k+1}(v_2)) = T^{k+2}(v_2)$

$\rfloor$

$i.e.\ R(T^k) = R(T^{2k})$

Let $u := T^k$

then $u^2 := T^{2k}$

$\therefore\ R(u) = R(u^2)$

Using the similar way to this, then $V = R(U) \oplus N(U)$

$\therefore\ V = R(T^k) \oplus N(T^k)$

17.

For every $x \in V$, $x = T(x) + (x - T(x))$ and

we are going to show that $V = \{y \mid T(y) = y\} \oplus N(T)$

Since $T^2(x) = T(T(x)) = T(x), T(x) \in \{y \mid T(y) = y\}$

and $T(T(x - T(x)) = T(x) - T^2(x) = T(x) - T(x) = 0$

$\therefore x - T(x) \in N(T)$

$\therefore V = \{y \mid T(y) = y\} + N(T)$

and if $\exists x \in \{y \mid T(y) = y\} \cap N(T)$, then $x = 0$

$(\because)$ Since $x \in \{y \mid T(y) = y\}, \therefore T(x) = x$

and $x \in N(T), T(x) = 0$

$\therefore x = 0$

$\therefore V = \{y \mid T(y) = y\} \oplus N(T)$

18.

Let A be an $m \times n$ matrix, B be an $n \times p$ matrix and C be an $p \times q$ matrix

We are going to show that (AB)C=A(BC), for $1 \leq i \leq m, 1 \leq j \leq q$

$((AB)C)_{ij} = (\sum_{k=1}^{p}(AB)_{ik}C_{kj}) = \sum_{k=1}^{p}(\sum_{l=1}^{n}(A)_{il}B_{lk})C_{kj} = \sum_{l=1}^{n}A_{il}(\sum_{k=1}^{p}B_{lk}C_{kj}) = \sum A_{il}(BC)_{lj} = (A(BC))_{ij}$

19.

$(B^3)_{kk} > 0 \;\Leftrightarrow\; B_{ki}B_{ij}B_{jk} = 1$ for some $i \neq j$ differ from $k$

$\Leftrightarrow\;$ $k$ belongs to a clique

20.

(a) Since $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$, $forall B^3_{ii} = 0$

$\therefore\ \nexists$ clique

(b) Since $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

$\therefore$ 1,3,4 belong to clique

21.

For convenience, let $A + A^2 = (b_{ij})_{n\times n}$ and for all $i(1 \leq i \leq n)$,

let $D(i) = \{j|a_{ij} = 1\}$

Choose a $k$ such that $D(k)$ is maximal in the set $\{D(i)|1 \leq i \leq n\}$

We will show that $b_{kj} = a_{kj} + \sum_{i=1}^{n} a_{ki}a_{ij} > 0$ for all $j \neq k$

For a fixed $j(\neq k)$, if $a_{kj} = 1$ then $b_{kj} > 0$

Now suppose $a_{kj} = 0$, then $a_{jk} = 1$

So $k \in D(j)$

If (for the case $a_{kj} = 0$) $a_{ij} = 0$ for all $i \in D(k)$, then

$a_{ji} = 1$ and hence $D(k) \subseteq D(j)$

But $k \in D(j),\ k \notin D(k)$

This is a contradiction to the choice of $k$

Thus $a_{ij} = 1$ for some $i \in D(k)$, and this proves the property $b_{kj} = a_{kj} + \sum_{i=1}^{n} a_{ki}a_{ij} > 0$

22.

1,2 and 3 dominate all the others in at most two stages, while 1,2, and 3 are dominated by all the others in at most two stages

23.

$n(n-1)/2$

## 2.4. Invertibility and Isomorphisms

1.

(a) F, $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$

(b) T

(c) F

$A \in Mat_{m\times n}(F),\ L_A : F^n \to F^m$

$\Rightarrow\ [L_A]_\alpha^\beta = A$ in case of $\alpha,\ \beta$ are the standard bases

(d) F, $dim(M_{2\times 3}(F)) \neq F^5$

(e) $P_n(F) \simeq P_m(F)$ iff n=m

($\Leftarrow$) clear

($\Rightarrow$)$\exists T : P_n(F) \to P_m(F)$ is isomorphic

Since T is one-to-one and onto,

$dim(P_n(F)) = rank(T) + nullity(T) = dim(P_m(F))$

$\therefore n + 1 = m + 1$

$\therefore n = m$

(f) F (In case A and B are $n \times n$ matrices, it's true)

(g) T

(h) T (Exercise 8)

(i) T

2. T is invertible iff $[T]_\beta^\gamma$

(a) $T : R^2 \to R^3$, $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$

For $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,0,0),(0,1,0),(0,0,1)\}$ bases for $R^2, R^3$, respectively

$T(1,0) = (1,0,3) = 1 \cdot (1,0,0) + 0 \cdot (0,1,0) + 3 \cdot (0,0,1)$

$T(0,1) = (-2,1,4) = -2 \cdot (1,0,0) + 1 \cdot (0,1,0) + 4 \cdot (0,0,1)$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}$ is not a aquare matrix

So $[T]_\beta^\gamma$ is not invertible

$\therefore$ T is not invertible

(b) $T : R^2 \to R^3$, $[T]_\beta^\gamma \in M_{3\times 2}(F)$ is not a square matrix

$\therefore$ T is not invertible

(c) $T : R^3 \to R^3$,

$T(1,0,0) = (3,0,3), T(0,1,0) = (0,1,4), T(0,0,1) = (-2,0,0)$

$\therefore [T]_\beta^\gamma = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$ is invertible

$\therefore$ T is invertible

$(cf)$ If $T(a_1, a_2, a_3) = (3a_1 - 2a_2, a_2, 3a_1 + 4a_2) = (0,0,0,)$

then $(a_1, a_2, a_3) = (0,0,0)$

$\therefore \; kerT = (0)$

(d) $T : P_3(R) \to P_2(R), T(p(x)) = p'(x)$

$[T]_\beta^\gamma \in M_{3\times 4}(R)$ is not a square matrix

$\therefore$ T is not invertible

(e) $T : M_{2\times 2}(R) \to P_2(R), T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$

$[T]_\beta^\gamma \in M_{3\times 4}(R)$ is not a square matrix

$\therefore$ T is not invertible

(f) $T : M_{2\times 2}(R) \to M_{2\times 2}(R), T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T\begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ is invertible

$\therefore$ T is invertible

$(cf)$ If $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $a = b = c = d = 0$

$\therefore\ kerT = (0)$

$\therefore\ T$ is invertible

3.

(a) $F^3 \ncong P_3(F)$ $(\because) dimF^3 \neq dimP_3(F)$

(b) $F^4 \cong P_3(F)$

(c) $M_{2\times 2}(R) \cong P_3(R)$

(d) $V = \{A \in M_{2\times 2}(R) \mid tr(A) = 0\} \ncong R^4$

$(\because) dimV = \mid \{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\} \mid = 3$

$dimR^4 = \mid \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\} \mid = 4$

$\therefore dimV \neq dimR^4$

4.

Let A and B $n \times n$ invertible matrix

Since $\exists A^{-1}$ and $B^{-1}$, $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

$\therefore B^{-1}A^{-1}$ is an inverse of $AB$

$\therefore AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

5.

Let $A$ is invertible

Since $\exists A^{-1}$ s.t $AA^{-1} = A^{-1}A = I_n$

$(A^{-1})^tA^t = (AA^{-1})^t = (I_n)^t = I_n$

$A^t(A^{-1})^t = (A^{-1}A)^t = (I_n)^t = I_n$

$\therefore (A^{-1})^t$ is an inverse of $A^t$

$\therefore (A^t)^{-1} = (A^{-1})^t$

6.

If A is invertible and AB=0, then $\exists A^{-1}$ s.t $A^{-1}A = AA^{-1} = I_n$

$\therefore B = IB = (A^{-1}A)B = A^{-1}0 = 0$

$\therefore B = 0$

7.

(a) Suppose $A^2 = 0$

Assume A is invertible, then $A^{-1}AA = A^{-1}0 \Rightarrow A = 0$

It's contradict to A is invertible

$\therefore$ A is not invertible

(b) Suppose AB=0 for some $0 \neq B \in M_{n\times n}(F)$

Assume A is invertible, then $A^{-1}AB = A^{-1}0 \Rightarrow B = 0$

It's contradict to $B \neq 0$

$\therefore$ A can't be invertible

8. (a) $[T]_\beta^\beta = [T]_\beta$

(b) $A = [L_A]_\beta$

($\Leftarrow$) $A = [L_A]_\beta \in M_{m\times n}(F)$, where $\beta$ the standard basis of $F^n$

Since $L_A$ is invertible, so $A$ is invertible

($\Rightarrow$) Since $A = [L_A]_\beta \in M_{m\times n}(F)$ is invertible

$\therefore$ $L_A$ is invertible

9.

Let $A, B \in M_{n\times n}(F)$ s.t AB is invertible

(a) Let $L_A, L_B, L_{AB} : F^n \to F^n$ be the left multiplication

Then clearly $L_A L_B = L_{AB}$ and $L_{AB}$ is invertible since AB is invertible

$\therefore$ $L_A$ is onto, $L_B$ is one-to-one

By theorem 2.5, $L_A$ and $L_B$ are both invertible

$\therefore$ A and B are invertible

(b) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 1 & 3 \\ -1 & 7 \end{pmatrix}$ is invertible

But A and B are not invertible

(Example) (b)

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$

but $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I$

10.

(a) $AB = I_n \Rightarrow A$ and $B$ are invertible by exercise 9

(b) $\exists C \in Mat_{n\times n}(F)$ s.t$CA = I_n$, so

$C = CI_n = C(AB) = (CA)B = I_nB = B$

that is, $BA = CA = I_n = AB$

$\therefore B = A^{-1}$ or $A + B^{-1}$

(c) Let V be of finite dimensional vector space and let $T : V \to V$ s.t $TR = I_V$

(1) T is invertible

$T,\ R : V \to V$ linear s.t. $TR = 1_V$

Since $TR = 1_V$, $T$ is onto

So $T$ is invertible

Similarly $R$ is ono-to-one

and hence $R$ is invertible

(2) $R = T^{-1}$

Since $T$ is invertible, $\exists\, T^{-1} : V \to V$

$\Rightarrow\ T^{-1} = T^{-1}1_V = T^{-1}(TR) = (T^{-1}T)R = IR = R$

$\therefore\ R = T^{-1}$

11. $T : P^3(R) \to M_{2\times 2}(R)$ is linear by $T(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$

We are going to show that $T(f) = 0 \Rightarrow f = 0$(the zero polynomial)

In this case, $f(1) = 0(i.e. f(c_0) = b_0), f(2) = 0, f(3) = 0, f(4) = 0$

$\therefore \forall f(c_i) = 0, i = 0, 1, 2, 3$

$\therefore f(x) = \sum_{i=0}^{3} b_i f_i(x) = \sum_{i=0}^{3} f(c_i) f_i(x) = 0$

$\therefore f$ is the zero polynomial

$\therefore$ T is one-to-one

12. $\phi_\beta(v_i) = [v_i]_\beta = e_i = (0, \cdots, 1, \cdots, 0)^t$

$[\phi_\beta]_\beta^\gamma = I_n$, when $\gamma$ is the standard basis for $F^n$

$\therefore\ \phi_\beta$ is an isomorphism

or

$\phi_\beta : V \to F^n$ is onto

$\Rightarrow\ \phi_\beta$ is an isomorphism because $dimV = dimF^n$

13. $\sim$ is an equivalence relation on the class of vector space over F

(i) $\sim$ is reflexive

$\forall V \in \mathcal{C}, V \sim V$

$(\because)$ $I_V : V \to V$ s.t $I_V = v, \forall v \in V$ is an isomorphism

(ii) $\sim$ is symmetric

If $V \sim W$, then $W \sim V$

$(\because)$ If $T : V \to W$ is isomorphic then $\exists T^{-1} : W \to V$ is isomorphic

$\therefore W \sim V$

(iii) $\sim$ is transitive

If $V \sim W$ and W$\sim Z$, then $V \sim Z$

$(\because)$ Let $T : V \to W$ and $U : W \to Z$ are isomorphic, then $UT$ is isomorphic

14.

Let $V = \{\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \mid a, b, c \in F\}$

$T : V \to F^3$ s.t $T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

For the basis for $V, \{v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

$\exists! T(v_i) = w_i$ is linear, $w_i \in F^3$ $i = 1, 2, 3$

and since $dimV = dimF^3$, $V$ is isomorphic to $F^3$

$\therefore$ T is an isomorphism from $V$ to $F^3$

15.

T is isomorphic iff $T(\beta)$ is a basis for W

$(\Rightarrow)$ Section 2.1 exercise 14 (c) (p75)

($\Leftarrow$) $T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$

$\forall w \in W$

$w = \sum a_i T(v_i) = T(\sum a_i v_i) = T(v)$, where $v = \sum a_i v_i \in V$

$\therefore$ $T$ is onto

$\therefore$ $T$ is invertible since $dimV = dimW = n$

16. $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$

$c \in M_{m\times n(F)},\ \exists A = B^{-1}CB \in M_{m\times n}(F)$ s.t. $\Phi(A) = C$

$\therefore$ $\Phi$ is onto

$\therefore$ $\Phi$ is an isomorphism

17. $V$ is finite dimensional and $T : V \to V$ is isomorphic

let $v_0 \in V$

(a) $T(V_0) \leq W$ as a subspace

(i) $T(v_1) + T(v_2) = T(v_1 + v_2) \in T(V_0)$

(ii) $T(av) = aT(v) \in T(V_0)$

(b) Since $T$ is an isomorphism, $rank(T) = dimW$ and $nullity(T) = 0$

therefore $nullity(T\mid_{V_0}) = 0$

$dim(V_0) = rank(T\mid_{V_0}) + nullity(T\mid_{V_0}) = dim(T(V_0))$

,where $T\mid_{V_0}: V_0 \to T(V_0)\ (\subseteq W)$

the restriction

18.

$$A = [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$L_A \pi_\beta(p(x)) = \pi_\gamma T(p(x)) = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

$$L_A \pi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

Since $T(p(x)) = p'(x) = 1 + 4x + 9x^2$

So $L_A \pi_\beta(p(x)) = \pi_\gamma T(p(x))$

19.

(a) $A = [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $L_A \pi_\beta(M) = A \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$

Since $T(M) = M^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

We have $\pi_\gamma T(M) = (1\ 3\ 2\ 4)^t$

20.

Let $A = (a_{ij})_{m \times n}$ be an incidence matrix associated with a dominance relation

Then $A + A^2$ has a row[column] in which every entry, except for the orthogonal, is positive

($\because$) For convenience, let $A + A^2 = (b_{ij})_{n\times n}$

and for $\forall i$ $(1 \le i \le n)$, let $D(i) = \{j \mid a_{ij} = 1\}$

Choose a $k$ such that $D(k)$ is maximal in the set $\{D(i) \mid 1 \le i \le n\}$

We will show that $b_{kj} = a_{kj} + \sum\limits_{i=1}^{n} a_{ki}a_{ij} > 0$ for all $j \ne k$

For a fixed $j(\ne k)$, if $a_{kj} = 1$ then $b_{kj} > 0$

Now suppose $a_{kj} = 0$, then $a_{ij} = 1 \ \Rightarrow \ k \in D(j)$

If (for the case $a_{kj} = 0$), $a_{ij} = 0$ for all $i \in D(k)$, then

$a_{ij} = 1$ and hence $D(k) \subseteq D(j)$

But $k \in D(j), \ k$ is not in $D(k)$

This is a contradiction to the choice of $k$

Thus $a_{ij} = 1$ for some $i \in D(k)$ and

this proves the property $b_{kj} = a_{kj} + \sum\limits_{i=1}^{n} a_{ki}a_{ij} > 0$

21. $\{T_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis for $dim\mathcal{L}(V,W)$

(a) Since $\dim \mathcal{L}(V,W) = mn$,

we need to show that the given set is linearly independent

Let $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij}T_{ij} = 0, a_{ij} \in F$

$\forall k(1 \le k \le n)$,

$0 = (\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij}T_{ij})(v_k) = \sum\limits_{i=1}^{m} a_{ik}w_i$

and $\xi = \{w_1, \cdots, w_n\}$ is a basis for W

$\therefore a_{ik} = 0, \forall i, k$

$\therefore \forall a_{ij} = 0$

$\therefore$ It's linearly independent

(b)

$\beta = \{v_1, v_2, \cdots, v_n\},\ \gamma = \{w_1, w_2, \cdots, w_n\}$

$\{T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ a basis for $\mathcal{L}(V, W)$ by (a)

$\forall k, T_{ij}(v_k)\ \ w_i = 0w_1 + \cdots + 1w_i + \cdots + 0w_n$ if $j = k$

$= 0w_1 + \cdots + 0w_n$ otherwise

$\therefore\ [T_{ij}]\gamma_\beta = M^{ij}$

(c) Now $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$ is defined by $\Phi(T_{ij}) = M^{ij}$ for all $i,\ j$

Since $\{M^{ij}\} = \Phi(T^{ij})$ is a basis of $M_{m\times n}(F)$

$\Phi$ is a n isomorphism by the exercise 15(p.108)

22.

(i) $T$ is well-defined and linear(check!)

$T(f + g) = T(f) + T(g)$

$T(\alpha f) = \alpha T(f)$

(ii) T is one-to-one

Suppose that $f \in ker(T)$ and let $f(x) = \sum_{i=0}^{n} b_i f_i(x)$,

where $f_i(x) = \frac{(x-c_0)\cdots(x-c_n)}{(c_i-c_0)\cdots(c_i-c_n)},\ b_i = f(c_i),\ \forall i$

Then $0 = T(f) = (f(c_0), f(c_1), \cdots, f(c_n)) = (b_0, b_1, \cdots, b_n)$

Since $\forall\ b_i = 0,\ f \equiv 0$

(iii) T is onto

$\forall a = (f(c_0), \cdots, f(c_n)) \in F^{n+1}, \exists f \in P_n(F)$ s.t $T(f) = a$

23. $T(\sigma) = T(\{a_n\}) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_m \neq 0,\ \forall m > n$

(i) T is well-defined

(ii) If $T(\sigma) = 0$, since $x_i's$ are linearly independent

$\therefore \forall a_i = 0$

$\therefore \sigma = \{0\}$

$\therefore$ T is one-to-one

(iii) T is onto

$\forall f(x) = \sum_{i=0}^{n} a_i x^i \in P_n(F)$,

$\exists \sigma = \{a_n\} = \{a_1, \cdots, a_n, 0, \cdots, 0\} \in V$ s.t $T(\sigma) = f(x)$

24. (a) $\overline{T}$ is well-defined

$v + N(T) = v' + N(T)$

$\Rightarrow\ v - v' \in N(T)$

$\Rightarrow\ T(v - v') = 0$

$\Rightarrow\ T(v) - T(v') = 0$

$\therefore\ T(v) = T(v')$

$\Rightarrow\ \overline{T}(v + N(T)) = T(v) = T(v') = \overline{T}(v' + N(T))$

(b) $\overline{T}$ is linear

$\overline{T}(\alpha(v+N(T))+(v'+N(T))) = \overline{T}(\alpha v+v'+N(T)) = T(\alpha v+v') = \alpha T(v)+T(v') =$

$\alpha\overline{T}(v + N(T)) + \overline{T}(v' + N(T))$

(c) $\overline{T}$ is an isomorphism

(i) If $\overline{T}(v + N(T)) = 0$, then $v + N(T) = N(T)$ i.e. $v \in N(T)$

($\because$) Since $0 = \overline{T}(v + N(T)) = T(v) \therefore v \in N(T)$

$\overline{T}(v + N(T)) = 0 \;\Rightarrow\; 0 = \overline{T}(v + N(T)) = T(V) \;\Rightarrow\; v \in N(T)$

$\therefore \overline{T}$ is one-to-one

(ii) Clearly $\overline{T}(V + N(T)) \subseteq T(V)$

If $v \in T(V)$, then $v = T(u)$ for some $u \in V$

and so $v = T(u) = \overline{T}(u + N(T)))$

$\therefore v \in \overline{T}(V + N(T))$

$\therefore \overline{T}$ is onto

(d) $T = \overline{T}\eta$

$T(v) = \overline{T}(v + N(T)) = \overline{T}(\eta(v)) = \overline{T}\eta(v)$

$\forall v \in V, \overline{T}\eta(v) = \overline{T}(v + N(T)) = T(v)$

$\therefore T = \overline{T}\eta$

25.

(i) $\Psi : \mathcal{C}(S, F) \to V$ is well-defined

(ii) $\Psi$ is onto

$\forall v \in V.\ v = \sum a_i s_i,\ s_i \in S$

Define $f(s_i) = a_i,\ \forall i$

(then $a_i = 0$ for all but a finite number of $i$)

then $\forall v,\ \exists f \in C(S,F)$ s.t. $\Psi(f) = \sum f(s_i)s_i = \sum a_i s_i,\ s_i = V$

(iii) $\Psi$ is one-to-one

$\Psi(f) = 0 \Rightarrow\ \sum f(s_i)s_i = 0,\ \forall s_i \in S\ \Rightarrow\ f \equiv 0$

## 2.5. The change of Coordinate Matrix

1.

(a) F ($[x'_j]_\beta$)

(b) T (Since $Q = [I_V]_{\beta'}^{\beta}$, $Q$ is an isomorphism)

(c) T

(d) F ($B = Q^{-1}AQ$)

(e) T ($[T]_\gamma = Q^{-1}[T]_\beta Q$)

2.$Q = [I_V]_{\beta'}^{\beta}$

(a) $I_V(a_1, a_2) = (a_1, a_2) = a_1e_1 + a_2e_2$

$I_V(b_1, b_2) = (b_1, b_2) = b_1e_1 + b_2e_2$

$\therefore Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

(c) $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

(d) $Q = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}$

3. $Q = [I_V]_{\beta'}^{\beta}$

(a) $I_V(a_2x^2 + a_1x + a_0) = a_2x^2 + a_1x + a_0$

$I_V(b_2x^2 + b_1x + b_0) = b_2x^2 + b_1x + b_0$

$I_V(c_2x^2 + c_1x + c_0) = c_2x^2 + c_1x + c_0$

$\therefore Q = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}$

(b) $Q = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$

(c) $Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 1 & 1 \end{pmatrix}$

(d) $Q = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$

(e) $Q = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

(f) $Q = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

4. $[T]_{\beta'} = Q^{-1}[T]_\beta Q$
$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$

$$[T]_{\beta'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$$

5. $[T]_{\beta'} = Q^{-1}[T]_\beta Q$
$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$[T]_{\beta'} = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 1/2 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

6. $[L_A]_\beta = Q^{-1}AQ$

(a) $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$[L_A]_\beta = Q^{-1}AQ = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ -2 & 4 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, Q^{-1} = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$[L_A]_\beta = Q^{-1}AQ = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$[L_A]_\beta = Q^{-1}AQ = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$

(d) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}, Q^{-1} = 6 \begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

$[L_A]_\beta = Q^{-1}AQ = 6 \begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} = 24 \begin{pmatrix} 6 & 0 & 5 \\ 0 & 18 & 0 \\ 0 & -6 & 31 \end{pmatrix}$

7. In $R^2$, let L be the line $y = mx, m \neq 0$

(a) T is the reflection of $R^2$ about L

$\beta' = \{\begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix}\}$ : an ordered basis for $R^2$

Since $T(1, m)^t = (1, m)^t = 1(1, m)^t + 0(-m, 1)^t$

$T(-m, 1)^t = (m, -1)^t = 0(1, m)^t + -1(-m, 1)^t$

$\therefore [T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let $\beta$ be the standard ordered basis for $R^2$ and Q be the matrix that changes $\beta'$-coordinates into $\beta$-coordinates

Then $Q = [I_V]^{\beta}_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$ and $Q^{-1}[T]_\beta Q = [T]_{\beta'}$

$\therefore [T]_\beta = Q[T]_{\beta'}Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} 1/(1+m^2)$

$= 1/(1+m^2) \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

$\therefore T \begin{pmatrix} x \\ y \end{pmatrix} = 1/(1+m^2) \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1/(1+m^2) \begin{pmatrix} (1-m^2)x + 2my \\ 2mx + (m^2-1)y \end{pmatrix}$

$(cf) \tan\theta = m \Rightarrow T = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

(b) T is the projection on L along the line perpendicular to L

$\beta' = \{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix} \}$

Since $T(1,m)^t = (1,m)^t = 1(1,m)^t) + 0(-m,1)^t$

$T(-m,1)^t = (0,0)^t = 0(1,m)^t) + 0(-m,1)^t$

$\therefore [T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} 1/(1+m^2) = 1/(1+m^2) \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$

$\therefore T \begin{pmatrix} x \\ y \end{pmatrix} = 1/(1+m^2) \begin{pmatrix} x + my \\ mx + m^2 y \end{pmatrix}$

$(cf) \tan\theta = m \Rightarrow T = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$

8. Let V and W be finite-dimensional vector spaces,

$T : V \to W$ be linear

$\beta$, $\beta'$ be ordered bases for V and $\gamma$, $\gamma'$ be ordered bases for W

Then $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$

$(\because)$ $[T]_{\beta'}^{\gamma'} = [I_W \cdot T \cdot I_V]_{\beta'}^{\gamma'} = [I_W]_{\gamma}^{\gamma'}[T]_{\beta}^{\gamma}[I_V]_{\beta'}^{\beta} = P^{-1}[T]_{\beta}^{\gamma}Q,$

where $P = [I_W]_{\gamma'}^{\gamma},\ Q = [I_V]_{\beta'}^{\beta}$

9. (i) $\forall A \in M_{n\times n}(F), A$ is similar to A

$(\because)\exists I$ s.t $A = I^{-1}AI$

(ii) If A is similar to B, then $\exists Q$s.t $A = Q^{-1}BQ$

$\therefore B = QAQ^{-1} = (Q^{-1})^{-1}A(Q^{-1})$

$\therefore$ B is similar to A

(iii) If A is similar to B and B is similar to C

then $\exists Q, P$ s.t $A = Q^{-1}BQ, B = P^{-1}CP$ $\therefore A = Q^{-1}BQ = Q^{-1})(P^{-1}CP)Q = (PC)^{-1}C(PQ)$

$\therefore$ A is similar to C

$\therefore$ "is similar to" is an equivalence relation on $M_{n\times n}(F)$

10. Since A is similar to B, $A = Q^{-1}BQ,$

$tr(A) = tr(QQ^{-1}B) = tr(B)$

11.

(a) Let $Q = [I_V]_{\alpha}^{\beta}, R = [I_V]_{\beta}^{\gamma}$

Then $RQ = [I_V]_{\beta}^{\gamma}[I_V]_{\alpha}^{\beta} = [I_V]_{\alpha}^{\gamma}$

(b) $Q^{-1} = ([I_V]_\alpha^\beta)^{-1} = [I_V^{-1}]_\beta^\alpha = [I_V]_\beta^\alpha$

$\therefore Q^{-1}$ changes $\beta'-$coordinates into $\beta-$coordinates

12.

$\beta$ : the standard ordered basis for $F^n$

$[L_A]_\gamma = [I_{F^n}]_\beta^\gamma [L_A]_\beta^\beta [I_{F^n}]_\gamma^\beta$

Let $[I_{F^n}]_\gamma^\beta = Q$, then $[L_A]_\gamma = Q^{-1}AQ$

13. $x_j' = \sum\limits_{i=1}^{n} Q_{ij}x_i, j = 1, \cdots, n$

(1)

By the theorem 2.6(p.72), there is a unique linear operator $T : V \to V$ s.t. $T(x_j) = x_j'$ for all $j = 1, 2, \cdots, n$

Clearly $[T]_{\beta'}^\beta = Q$

Since $Q$ is invertible, $T$ is an isomorphism by the theorem 2.8

So $\beta' = T(\beta)$ is a basis for $V$

(2)

Since $x_j' = \sum\limits_{i=1}^{n} Q_{ij}x_i (1 \leq j \leq n)$, $[x_j']_\beta$ is the j-th column of $Q$

$\therefore Q = [I_V]_{\beta'}^\beta$ changes $\beta'$-coordinates into $\beta$-coordinates

14.

If $A, B \in M_{m\times n}(F)$ and $P \in M_{m\times m}(F), Q \in M_{n\times n}(F)$ are invertible and $B = P^{-1}AQ$, then $\exists$ an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases $\beta$ and $\beta'$ for V and $\gamma$ and $\gamma'$ for W, and a linear transformation $T : V \to W$ s.t $A = [T]_\beta^\gamma, B = [T]_{\beta'}^{\gamma'}$

$(\because)$

Let $V = F^n, W = F^m, T = L_A$

$\beta = \{x_1, x_2, \cdots, x_n\}, \gamma = \{y_1, y_2, \cdots, y_m\}$ : the standard ordered bases for $F^n$ and $F^m$, respectively

Define $x_j' = \sum Q_{ij} e_j$ for $1 \le j \le n$

then the set $\beta' = \{x_1', \cdots, x_n'\}$ is a basis for V

and $Q = [I_{F^n}]_{\beta'}^{\beta}$

Define $w_j' = \sum P_{ij} e_i$ for $1 \le j \le m$

then the set $\gamma' = \{w_1', \cdots, w_m'\}$ is a basis for W

and $P = [I_{F^m}]_{\gamma'}^{\gamma}$

Now $[T]_\beta^\gamma = [L_A]_\beta^\gamma = A$ and

$[T]_{\beta'}^{\gamma'} = [I_{F^m}]_\gamma^{\gamma'} [T]_\beta^\gamma [I_{F^n}]_{\beta'}^{\beta} = P^{-1}AQ = B$

## 2.6. Dual Spaces

1.

(a) F

(linear transformation from V into its field of scalars F is called a linear functional)

(b) T

$(f : F \to F, [f] \in Mat_{1\times 1}(F))$

(c) T

$dimV^*$ =dim($\mathcal{L}(V, F)$)=dim$V$dim$F$=dim$V$

$\therefore V \simeq V^*$

(d) T

For a vector space $V$, we can define the dual space of $V$ i.e. $(\mathcal{L}(V, F)) = V^*$

Then $V$ is the dual space of $V^*$ $((V^*)^* = V)$

$\therefore$ Every vector space is the dual of some vector space

(e) (example) $V = \mathbb{R}^2, F = \mathbb{R}$

$\beta = \{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ :

So $V^* = \mathcal{L}(V, \mathbb{R})$ : $1 \times 2$ matrices

and $e_1^* = (1, 0), e_2^* = (0, 1)$ *i.e.* $\beta^* = \{e_1^* = (1, 0), e_2^* = (0, 1)\}$ Now if we define

$T : V \to V^*$ by $T(e_1) = (1, 1), T(e_2) = (1, -1)$

Since $\{T(e_1), T(e_2)\} = T(\beta)$ a basis of $V^*$, then clearly $T$ is an isomorphism

But $T(\beta) = \{T(e_1) = (1, 1) \neq e_1^*, T(e_2) = (1, -1) \neq e_2^*\} \neq \beta^*$

(example 2)

$V=\mathbb{R},\ F=\mathbb{R},\ f:V\to F\ \ i.e.V^*=F$ and $\beta^*=\{1^*\}$ ($\because\ \beta=\{1\}$)

But $T:R\to R$ is an isomorphism

$a\mapsto 2a\ \ T(\beta)=\{T(1)\}=\{2id\}\neq\beta^*=\{id\}$

(f) T

$T:V\to W, T^t:W^*\to V^*$ by $T^t(g)=gT$

$(T^t)^t:(V^*)^*\to(W^*)^*$

(g) T

$V\simeq W\Leftrightarrow T:V\to W$: an isomorphism $\Leftrightarrow\exists[T]_\beta^\gamma$: invertible $\Leftrightarrow$

$([T]_\beta^\gamma)^t=([T^t]_{\beta^*}^{\gamma^*})$ : invertible $\Leftrightarrow T^t:W^*\to V^*$ : an isomorphism $\Leftrightarrow V^*\simeq W^*$

(h) F

$f:D_n(\mathbb{R})\to\mathbb{R}$ by $f(g(x))=g'(x),\forall g(x)=D_n(\mathbb{R})$

but in case $g(x)=x^2, f(g(x))=g'(x)=2x$ is not in $\mathbb{R}$

$\therefore$ It's not a linear functional

2.

(a) $p(x),g(x)\in P(R),\alpha\in R$

$f(\alpha p(x)+g(x))=2(\alpha p'(0)+g'(0)+\alpha p''(1)+g''(1))=2\alpha p'(0)+\alpha p''(0)+2g'(0)+g''(1)=\alpha f(p(x))+f(g(x))$

$\therefore f$ is a linear functional

(b) $(x_1,y_1),(x_2,y_2)\in R^2,\alpha\in R$

$f(\alpha(x_1,y_1)+(x_2,y_2))=f(\alpha x_1+x_2,\alpha y_1+y_2)=2(\alpha x_1+x_2)+4(\alpha y_1+y_2)=$

$\alpha(2x_1 + 4y_1) + (2x_2 + 4y_2) = \alpha f(x_1, y_1) + f(x_2, y_2)$

(c) $A, B \in M_{2\times 2}(F), \alpha \in F$

$f(\alpha A + B) = tr(\alpha A + B) = tr(\alpha A) + tr(B) = \alpha tr(A) + tr(B) = \alpha f(A) + f(B)$

(d)

$f((x_1, y_1, z_1) + (x_2, y_2, z_2)) = f(x_1 + x_2, y_1 + y_2 + z_1 + z_2) = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2 \neq (x_1^2 + y_1^2 + z_1^2)^2 + (x_2^2 + y_2^2 + z_2^2)^2 = f(x_1, y_1, z_1) + f(x_2, y_2, z_2)$

$\therefore f$ is not a linear functional

(e)

$f(\alpha A + B) = \alpha A_{11} + B_{11} = \alpha f(A) + f(B)$

$\therefore f$ is a linear functional

3.

(a) $\beta = \{v_1 = (1, 0, 1), v_2 = (1, 2, 1), v_3 = (0, 0, 1)\}$

since $f_i(v_j) = \delta_{ij}$

$1 = f_1(v_1) = f_1(e_1 + e_2) = f_1(e_1) + f_1(e_3)$

$0 = f_1(v_2) = f_1(e_1 + 2e_2 + e_3) = f_1(e_1) + 2f_1(e_2) + f_1(e_3)$

$0 = f_1(v_3) = f_1(e_3)$

$\therefore f_1(e_3) = 0, f_1(e_1) = 1, f_1(e_2) = -1/2$

$\therefore f_1(x, y, z) = xf_1(e_1) + yf_1(e_2) + zf_1(e_3) = x - 1/2y$

$0 = f_2(v_1) = f_2(e_1) + f_2(e_3)$

$1 = f_2(v_2) = f_2(e_1 + 2f_2(e_2) + f_2(e_3)$

$0 = f_2(v_3) = f_2(e_3)$

$\therefore f_2(e_2) = f_2(e_3) = 0, f_2(e_1) = 0$

$\therefore f_2(x, y, z) = 1/2y$

$0 = f_3(v_1) = f_3(e_1) + f_3(e_3)$

$0 = f_3(v_2) = f_3(e_1 + 2f_3(e_2) + f_3(e_3)$

$1 = f_3(v_3) = f_3(e_3)$

$\therefore f_3(e_1) = -1, f_3(e_2) = 0, f_3(e_3) = 1$

$\therefore f_3(x, y, z) = -x + z$

(b) $\beta = \{1, x, x^2\}$

$f_1(a + bx + cx^2) = af_1(e_1) + bf_1(e_2) + cf_1(e_3) = a$

$f_2(a + bx + cx^2) = af_2(e_1) + bf_2(e_2) + cf_2(e_3) = b$

$f_3(a + bx + cx^2) = af_3(e_1) + bf_3(e_2) + cf_3(e_3) = c$

4. $\{f_1, f_2, f_3\}$ is linearly independent

$(af_1 + bf_2 + cf_3)(x, y, z) = 0(x, y, z)$

$af_1(x, y, z) + bf_2(x, y, z) + cf_3(x, y, z)$

$= a(x - 2y) + b(x + y + z) + c(y - 3z)$

$= (a + b)x + (-2a + b + c)y + (b - 3c)z = 0, \ \forall (x, y, z) \in V$

$\therefore \ a = b = c = 0$

$\therefore \ \{f_1, f_2, f_3\}$ is linearly independent in $V^*$

5.

(i) If $V_{col} \to V^*_{row}$

$V_{row} \to V_{col}^*$

$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 1 \times 1 (scalar)$

$V = P_1(\mathbb{R}) = \{a + bx \text{ or } \begin{pmatrix} a \\ b \end{pmatrix}\}$

$( \because \;\; \text{In } P_1(\mathbb{R}), 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\; \therefore \; a + bx \; \Rightarrow \;\; \begin{pmatrix} a \\ b \end{pmatrix})$

$f_1(a + bx) = (1, \frac{1}{2}) \begin{pmatrix} a \\ b \end{pmatrix} = a + \frac{1}{2} b$

$f_2(a + bx) = (2, 2) \begin{pmatrix} a \\ b \end{pmatrix} = 2a + 2b$

$\therefore \; af_1 + bf_2 = a(1, \frac{1}{2}) + b(2, 2) = (a + 2b, \frac{1}{2} a + 2b) = 0 = (0, 0)$

$\therefore \; a = b = 0$

7.

(a) $T^t(f) = g$, where $g(a + bx) = -3a - 4b$

(b) $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

(c) $[T]_\beta^\gamma = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

8.

Now let $\pi$ a plane in $\mathbb{R}^3$ through the origin

Then $\exists \; 0 \neq (a, b, c) \in \mathbb{R}^3 \;\; \Rightarrow \;\; \pi = \{(x, y, z) \mid ax + by + cz = 0\}$

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $(x, y, z) \mapsto ax + by + cz :$ linear functional

Then $f \in V^*$ and $ker f = \pi$

9.

$\beta = \{x_1, x_2, \cdots, x_n\}$

$\gamma = \{y_1, y_2, \cdots, y_n\}$ : standard bases of $F^n$ and $F^m$

10. $\{p_0 = 1, p_1 = (x - c_1), \cdots, p_n = (x - c_1)(x - c_2)\cdots(x - c_n)\}$

(a) Define $f_i(p(x)) = p(c_i)$

Suppose that $\sum_{i=0}^{n} \alpha_i f_i = 0$

then $\sum_{i=0}^{n} \alpha_i f_i(p_j(x)) = \sum_{i=0}^{n} \alpha_i(p_j(c_i)) = \alpha_i = 0$

$\therefore\ \alpha_i = 0,\ \forall i = 0, 1, \cdots, n$

$\therefore\ \{f_0, f_1, \cdots, f_n\}$ : linearly independent

Since $dimV = dimV^* = n + 1$,

$\{f_0, f_1, \cdots, f_n\}$ : a basis for $V^*$

(b) (i) By the corollary of theorem 2.26 and (a),

$\widehat{x_i}(f_j) = f_j(x_i) = \delta_{ij}$

then, $f_j(p_i(x)) = p_i(c_j) = \delta_{ij}$

(ii) Consider $\exists q_i$ s.t $f_j(q_i(x)) = q_i(c_j) = \delta_{ij},\ \forall i = 0, 1, \cdots, n,$

Let $R_i(x) = p_i(x) - q_i(x)$, then $R_i(c_j) = p_i(c_j) - q_i(c_j) = \delta_{ij} - \delta_{ij} = 0,\ \forall j = 0, 1, \cdots, n$

$\therefore R_i(x) = 0$

$(\because)\ dimR_i \leq n$ and $R_i(x)$ has $n + 1$ roots

$\therefore q_i = p_i$

(c) Assume that $\exists h(x) \in P_n(x)$ s.t. $h(c_i) = a_i,\ \forall i$

Since $\{P_0(x), P_1(x), \cdots, P_n(x)\}$ a basis for $P_n(x)$

$\therefore\ h(x) = \sum_{i=0}^{n} b_i P_i(x),\ (\forall b_i \in F)$

$a_j = h(c_j) = \sum_{i=0}^{n} b_i P_i(c_j) = b_j,\ \forall j$

$\therefore\ h(x) = \sum_{j=0}^{n} b_j P_j(x) = \sum_{j=0}^{n} a_i P_j(x) = q(x)$

(d) Let $c_0, \cdots, c_n$ be distinct scalars in $F$

The polynomials $p_0(x), \cdots, p_n(x)$ defined by

$$p_i(x) = \prod_{\substack{k=0 \\ k\neq i}} \frac{x - c_k}{c_i - c_k} \in P_n(F) \tag{1}$$

Since $p_i(c_j) = \delta_{ij}$ and $\{p_0, \cdots, p_n\}$ is linearly independent

$\{p_0, \cdots, p_n\}$ : a basis for $P_n(F)$

$p(x) = \sum_{i=0}^{n} aip_i(x),\ (a_i \in F)$

$p(c_j) = a_j,\ \forall j$

$\therefore\ p(x) = \sum_{i=0}^{n} aip_i(x) = p(x) = \sum_{i=0}^{n} p(c_i)p_i(x)$

(e) $\int_a^b p(t)dt = \sum_{i=0}^{n} p(c_i)d_i,\ d_i = \int_a^b p_i(t)dt$

$(\because) p(t) = p(c_0)p_0(t) + \cdots + p(c_n)p_n(t)$

$\int_a^b p(t)dt = \int_a^b (p(c_0)p_0(t) + \cdots + p(c_n)p_n(t))dt$

$= p(c_0)\int_a^b p_0(t)dt + \cdots + p(c_n)\int_a^b p_n(t)dt$

$= p(c_0)d_0 + \cdots + p(c_n)d_n$

$= \sum_{i=0}^{n} p(c_i)d_i$

Trapezoidal rule - $\int_a^b p(t)dt \approx (b-a)\frac{f(b)+f(a)}{2}$

Simpson's rule - $\int_a^b f(t)dt \approx \frac{(b-a)}{6}(f(a)+4f(\frac{a+b}{2})+f(b))$

11.

For $\forall x \in V,\ x \mapsto \psi_2 T(x) = \widehat{T(x)}$

$x \mapsto T^{tt}\psi_1(x) = T^{tt}(\widehat{x}) = \widehat{x}T^t$

To show commuting, $\widehat{T(x)} = \widehat{x}T^t$ in $W^{**} : W^* \to F$

$\forall g \in W^*,\ i.e.\ g : W \to F$ a linear functional

Show $(\widehat{x}T^t)(g) = \widehat{T(x)}(g)$

$\widehat{T(x)}(g) = g(T(x)) = (gT)(x) = widehatx(gT) = \widehat{x}(T^t g) = \widehat{x}T^t(g)$

$\therefore\ \psi_2 T = T^{tt}\psi_1$

12. $\psi(\beta) = \beta^{**}$

$\beta = \{x_1, x_2, \cdots, x_n\}$ a basis for $V$

$\Rightarrow\ \beta^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ a basis for $V^*$

, where $x_i^* : V \to F$ a linear functional s.t. $f_i(x_j) = \delta_{ij}$

Since $V^{**}$ is the dual space of $V^*$

$\exists\ \beta^{**} = \{x_1^{**}, x_2^{**}, \cdots, x_n^{**}\}$ a basis for $V^{**}$

s.t. $x_i^{**} : V^* \to F^*$ linear functional s.t. $x_i^{**}(x_j^*) = \delta_{ij}$

Show $x_i^{**} = \widehat{x_i},\ \forall i,\ \psi(\beta) = \{x_1, x_2, \cdots, x_n\}$

$\widehat{x}(x_i^*) = \delta_{ij} = x_i^{**}(x_j^*),\ \forall i, j$

$i.e.\widehat{x_i} = x_i^{**}$ on a basis $\beta^{**}$

$\Rightarrow\ \widehat{x_i} = x_i^{**},\ \forall i$

$\therefore\ \psi(\beta) = \beta^{**}$

13. $S^0 = \{f \in V^* | f(x) = 0, \forall x \in S\},\ S \subseteq V$

(a) $S^0$ is a subspace of $V^*$

(i) $0(x) = 0, \forall x \in S \quad \therefore 0 \in S0$

(ii) $\forall f, g \in S^0, \alpha \in F \Rightarrow \alpha f + g \in S^0$

$(\alpha f + g)(x) = \alpha f(x) + g(x) = \alpha 0 + 0 = 0, \forall x \in S$

$\therefore \alpha f + g \in S^0$

(b)

Let $\{x_1, x_2, \cdots, x_m\}$ be a basis of $W$

If $x$ is not in $W$, then $\{x_1, x_2, \cdots, x_m, x\} = \{x_1, x_2, \cdots, x_m\} \cup \{x\}$ is linearly independent

$\exists\ \beta = \{x_1, x_2, \cdots, x_m, x = x_{m+1}, \cdots, x_n\}$ a basis of $V$

$\Rightarrow\ \beta^* = \{f_1, f_2, \cdots, f_m, f_{m+1}, \cdots, f_n\}$ a basis of $V^*$

$f_{m+1}(W) = 0$ and $f_{m+1}(x) = f_{m+1}(x_{m+1}) = 1$

$\therefore\ f_{m+1} \in W^0,\ f_{m+1}(x) \neq 0$

(c) $(S^0)^0 =$span$\psi(S)$

$(\Leftarrow)$ If $\widehat{v}\ \in$span$\psi(S)$, then $\exists x_1, x_2, \cdots, x_n \in S$, s.t. $\widehat{v} = a_1\widehat{x_1} + a_2\widehat{x_2} + \cdots + a_n\widehat{x_n}$ for some $a_1, a_2, \cdots, a_n \in F$

Now $\forall f \in S^0$

$\widehat{v}(f) = (\sum_{i=1}^{n} a_i \widehat{x}_i)(f) = \sum_{i=1}^{n} a_i \widehat{x}_i(f) = a_i f(x_i) = \sum_{i=1}^{n} a_i 0 = 0$

So $\widehat{v} \in (S^0)^0$

$\therefore\ span\psi(S) \subseteq (S^0)^0$

$(\Rightarrow)$

(Step 1)

First note that for $\forall$ subset $S$ of $V$

$S^0 = (spanS)^0$

$\lceil$ For convenience, let $spanS = W$,

then clearly $S \subseteq W$, so $W^0 \subseteq S^0$

Now let $f \in S^0$ and $x \in W$ ; then

$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for some $x_i \in S,\ a_i \in F$

$\Rightarrow\ f(x) = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n) = 0$

$\therefore\ f \in W^0$

$\therefore\ S^0 \subseteq W^0\ \rfloor$

(Step 2)

Show if $W \leq V$ as a subspace, then $(W^0)^0 = \psi(W)$

$\lceil$ $(\subseteq)$ For$v \in V,\ \widehat{v} \in W^0)^0\ \Rightarrow\ \widehat{v}(f) = 0$ for all $f \in W^0$

$\Rightarrow\ f(V) = 0$

$\Rightarrow\ v \in W$ by the exercise 13(b)

$\Rightarrow\ \widehat{v} \in \psi(W)$

$(\supseteq)$ If $v \in W,\ \widehat{v} \in \psi(W)$

$\widehat{v}(f) = f(v) = 0$ for all $f \in W^0$

$\Rightarrow\ \widehat{v} \in (W^0)^0$

This implies that $\psi(W) \subseteq (W^0)^0$

(Step 3)

Clearly $\psi(W) = span\psi(S)$, where $W = span(S)$

$(\because)$ If $v \in W$

$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ $(\forall a_i \in F,\ x_i \in S)$

then $\psi(v) = \widehat{v} = a_1\widehat{x_1} + a_2\widehat{x_2} + \cdots + a_n\widehat{x_n} = \sum_{i=1}^{n} A_i\psi_i(x_i) \in span(\psi(S))$

$\therefore\ \widehat{v} \in \psi(S)$

$\widehat{v} \in span(\psi(S))$

Now $\psi(S) \subseteq \psi(W)$ and $\psi(W)$ is a subspace of $W^{**}$

$span(\psi(S)) \subseteq span(\psi(W)) = \psi(W)$

$\therefore\ \psi(W) = span(\psi(S))$

Finally $(S^0)^0 = (W^0)^0 = \psi(W) = span(\psi(S))$

(d) $W_1 = W_2 \Leftrightarrow W_1^0 = W_2^0$

$(\Rightarrow)$ clear

$(\Leftarrow)$ If $W_1 \neq W_2$, then $x \in W_2, x$ is not in $W_1$

by (b), $\exists\ f \in W_1^0,\ f(x) \neq 0$

*i.e.* $f$ is not in $W_2^0$

$\therefore W_1^0 \neq W_2^0$

(e)

$(\Rightarrow)$

$(W_1 + W_2)^0 \subseteq W_1^0, (W_1 + W_2)^0 \subseteq W_2^0$

$\therefore\ (W_1 + W_2)^0 \subseteq W_1^0 \cap W_2^0$

($\Leftarrow$) clear

14.

Let $\dim W = k$ and $\{x_1, \cdots, x_k\}$ : a basis for $W$

Extend it to $\{x_1, \cdots, x_k, x_{k+1}, \cdots, x_n\}$ : a basis for $V$

Let $\{f_1, \cdots, f_n\}$ be the basis for $V^*$

We are going to show that $\{f_{k+1}, \cdots, f_n\}$ is a basis for $W^0$

If $f \in W^0$ we have $f(x_i) = 0,\ i \leq k$

$\therefore\ f = \sum_{i=k+1}^{n} f(x_i) f_i$

$\therefore \{f_{k+1}, \cdots, f_n\}$ spans $W^0$

$\therefore$ Since $\dim W = k$ and $\dim V = n$, then $\dim W^0 = n - k$

15.

($\Rightarrow$) Suppose that $\phi \in N(T^t)$

*i.e.* $T^t(\phi) = \phi T = 0$

If $u \in R(T)$,then $u = T(v)$ for some $v \in V$

hence $\phi(u) = \phi(T(v)) = (\phi T)(v) = 0(v) = 0,\ \forall u \in R(T)$

$\therefore \phi \in (R(T))^0$

($\Leftarrow$)

If $\sigma \in (R(T))^0, \sigma(R(T)) = 0$

then, $\forall v \in V, (T^t(\sigma))(v) = (\sigma T)(v) = \sigma(T(v)) = 0(v) = 0$

$\therefore T^t(\sigma) = 0$

$\therefore \sigma \in N(T^t)$

16.

$rank(L_{A^t}) = rank(L_A)$

Let $dimV = n,\ dimW = m$ and $rank(T) = r$

by the exercise 14, $dim(R(T)) + dim(R(T))^\circ = dimW$

$\therefore dim(R(T))^\circ = m - r$

by the exercise 15 and the dimension theorem,

$N(T^t) = (R(T))^\circ,\ dim(W^*) = nullity(T^t) + rank(T^t)$

$\therefore\ rank(T^t) = dim(W^*) - nullity(T^t) = m - (m - r) = rank(T) - - - - (*)$

Since $rank(T) = rank(L_A)$ and $rank(T^t) = rank(L_{A^t}) - - - - (*)$

$rank(L_{A^t}) = rank(L_A)$

Q. $rank(T^t) = rank(L_{A^t})$

$(\because)$ When $T = L_A : F^n \to F^m$ left multiplication

$rank(L_A) = rank(T) = rank(T^t) = rank([T^t]_{\gamma^*}^{\beta^*})$

$= rank([T]_\beta^\gamma)^t = rankA^t = rankL_{A^t}$

17.

$(\Rightarrow)\ \forall \in W^\circ$

Since $T^t f = fT$ and $T(W) \subseteq W$,

$fT(W) \subseteq f(W) = 0$

$\therefore\ T^t f \in W^\circ$

$\therefore\ T^t(W^\circ) \subseteq W^\circ$

$\therefore\ W^\circ$ is $T^t$-invariant

($\Leftarrow$) If $T(W) \nsubseteq W,\ \exists\ w \in W$ s.t. $T(w) \in W$

by the exercise 13, $\exists f \in W^\circ$ s.t. $f(T(w)) \neq 0$

$T^t f(w) = fT(w) \neq 0$

$i.e.\ \exists f \in W^\circ$ s.t. $T^t f \notin W^\circ$

$\therefore\ W$ is $T$-invariant

18.

$\Phi : V^* \to \mathcal{L}(S, F)$

(Actually $\mathcal{L}(S, F) \equiv \mathcal{L}(V, F)$)

(i) Clearly $\mathcal{L}(S, F)$ is a vector space over $F$ and

(ii) $\Phi$ is a linear map

$\lceil\ f, g \in \mathcal{L}(S, F)$,

$\Phi(f + g) = (f + g) \mid_s = f \mid_s + g \mid_s = \Phi(f) + \Phi(g)$

$\Phi(\alpha f) = \alpha f \mid_s = \alpha f_s = \alpha\Phi(f)\ \rfloor$

(iii) $\forall f \in ker\Phi, \Phi(f) = f_s = 0$

$\therefore\ f \in S^\circ = (spanS)^\circ = V^\circ = \{0\}$

$\therefore\ ker\Phi = \{0\}$

$\therefore\ \Phi$ is one-to-one

(iv) By the exercise 34 in section 2.1

$(\forall f_s : S \to F,\ \exists!\ f : V \to F$ a linear map s.t. $f(x) = f_s(x),\ \forall x \in S)$

$\forall\ f_s \in \mathcal{L}(S, F),\quad \exists!\ f \in V^*$ s.t. $\Phi(f) = f_s$

$\therefore\ \Phi$ is onto

19.

(i) Choose $y \in V, y \notin W$ and let $\gamma$ : a basis of $W$

then $\gamma \cup \{y\}$: linearly independent

by (section 1.7 or) Maximal principle,

$\exists\ \beta :$ a basis of $V$ s.t. $\gamma \cup \{y\} \subseteq \beta$

Define a function : $g : \beta \to F$ s.t. $g(x) = 0\ \forall x \in \beta,\ x \neq y$

$g(y) = 1$

then by the exercise 18, $\exists\ f \in \mathcal{L}(V, F) = V^*$ s.t. $f\ |_\beta = g(\because\ \gamma \subseteq \beta)$

$i.e.$ $f$ is the function we desired.

20.

(a) $T : V \to W$: linear map $\Rightarrow\ T^t : W^* \to V^*$ given by $T^t(g) = gT,\ \forall g \in W^*$

$(\Rightarrow)$ Suppose that $T$ is onto

Let $g \in KerT^t \Rightarrow T^t(g) = 0 \Rightarrow gT = 0$ in $V^*$

$i.e.$ $gT(v) = (gT)(v) = 0,\ \forall v \in V$

Since $T$ is onto, $g(w) = gT(v) = 0$

$i.e.\ g(w) = 0(\because\ g \in KerT^t), \forall w \in W$

$\therefore\ g = 0$ in $V^*$

$\therefore\ T^t$ is one-to-one

($\Leftarrow$) Suppose that $T^t$ is one-to-one

Let $W_1 = R(T)$ : the range of $T$

If $W_1 \neq W\ \Rightarrow$ by the exercise 19, $\exists 0 \neq g \in W^*, g(W_1) = 0$ ($i.e.\ g \in W^\circ 1$)

$\Rightarrow\ (gT)(V) = g(T(V)) = g(W_1) = 0$

$\Rightarrow\ (T^t)g(V) = 0 \Rightarrow\ T^t g = 0$ in $W^*$

$\therefore\ g \in KerT^t$

Since $T^t$ is one-to-one, $g = 0$

$\therefore\ W_1 = W$

$\therefore\ T$ is onto

(b)

($\Leftarrow$) Suppose that $T$ is one-to-one

Let $f \in V^*$

Suppose that $T$ is one-to-one

$W = W_1 \oplus W_2$, where $W_1, W_2 \leq W$ and $W_1 = R(T) \cong V$

So the map $U : W_1 \oplus W_2 \to V,\ U(w_1 \oplus w_2) = v$, where $w_1 = T(v)$ is a well-defined linear map

$T(v) + w_2 \mapsto v$

Let $g = fU$, then $\exists\ g \in W^*$ and $T^t(g) = gT = f$

$\lceil(\because)\ (gT)(v) = g(T(v)) = (fU)(T(v)) = f(U(T(v))) = f(v),\ \forall v \in V$

$\therefore\ gT = f$

$i.e.\ T^t g = f \rfloor$

$\therefore\ T^t$ is onto

($\Rightarrow$) Suppose that $T^t : W^* \to V^*$ onto

show $T$ is one-to-one

Assume on the contrary $T$ is not one-to-one

$\Rightarrow\ \exists 0 \neq v \in V$ s.t. $T(v) = 0$

$\exists\ f \in V^*$ s.t. $f(v) = 1$ (by the exercise 18)

Now since $T^t$ is onto

$\exists g \in W^*$ s.t. $f = T^t(g) = gT$

$\Rightarrow\ 1 = f(v) = (gT)(v) = g(T(v)) = g(0) = 0$

$\therefore\ T$ must be one-to-one

## 2.7. Homogeneous Linear Differential Equations with Constant Coefficients

1.

(a) T (p.137 corollary to Theorem 2.32)

(b) T (p.132 Theorem 2.28)

(c) F

(d) F (Any solution is a linear combination of $e^{at}$ and $t^k e^{at}$)

(e) T

($\because$) If $x$ and $y$ are solutions of $p(D) = 0$,

then $p(D)(\alpha x + \beta y) = \alpha p(D)x + \beta p(D)y = 0 + 0 = 0, \alpha, \beta \in F$

$\therefore$ $\alpha x + \beta y$ is a solution of $p(D) = 0$

(f) F

($\because$) It's different with the multiplicity of $c_i$ (p.137 and 139, Theorem 2.33 and 2.34) (g) T (p.131)

2.

(a) F

Let $S = \{\frac{a}{1+t^2} \mid a \in R\}$ $\Rightarrow$ $S$: 1-dimensional subspace of $\mathcal{C}^\infty$

But there is no homogeneous linear differential equation with constant coefficients

(b) F

Let $\{t, t^2\}$ is the solution of $y'' + ay' + by = 0$

$0 + a + bt = 0$ $\Rightarrow$ $a = b = 0$

then $y'' + ay' + by = 0$ becomes $y'' = 0$

$(t^2)'' = 2 = 0$

(cf) $y''' = 0 \ \Rightarrow \ D^3 = 0 \ \Rightarrow \ t = 0$

$e^{0t}, te^{0t}, t^2e^{0t} \ i.e. 1, t, t^2$

$\exists \ y''' = 0$

(c) T

Let $x$ is a solution to the homogeneous linear differential equation with constant coefficients $P(D)y = 0$

Since $P(D)x = 0, \ P(D)x' = P(D)(Dx) = P(D)Dx = DP(D)x = D(0) = 0$

$\therefore \ x'$ is also a solution to the equation

(d)T

Let $p(D)x = 0$ and $q(D)y = 0$

$p(D)q(D)(x + y) = p(D)q(D)x + p(D)q(D)y = q(D)(p(D)x) + p(D)(q(D)y) = q(D)(0) + p(D)(0) = 0 + 0 = 0$

(e) F

Let $p(t) = t^2 + 2t + 1 = 0 \ \therefore \ p(D) = D^2 + 2D + 1$

$q(t) = t^3 - 1 \ \therefore \ q(D) = D^3 - 1$

$\{e^{-t}\}$ : a basis for the solution space of $p(D)$

$\{e^t\}$ : a basis for the solution space of $q(D)$

$p(D)q(D) = D^5 + 2D^4 + D^3 - D^2 - 2D - 1$

$p(D)q(D)y = y^{(5)} + 2y^{(4)} + y^{(3)} - y^{(2)} - 2y' - y$

$p(D)(e^{-t}) = 0,\ q(D)(e^{t}) = 0$

But $p(D)q(D)(e^{-t}e^{t}) \neq 0$

3.

(a) Given the differential equation is $y'' + 2y' + y = 0$ and its auxiliary polynomial is $p(t) = t^2 + 2t + 1 = (t+1)^2$

Hence, $e^{-t}$ and $te^{-t}$ are solutions to the differential equation because $c = -1$ is a zero of $p(t)$

$\therefore \{e^{-t}, -te^{-t}\}$ is a basis for the solution space

So any solution $y$ is to the given differential equation is of the form

$y(t) = b_1e^{-t} + b_2te^{-t}$ for unique $b_1$ and $b_2$

(b)

Since $y''' = y'$, the auxiliary polynomial is $t^3 - t = 0$

$\therefore t = 0, -1, 1$

$\therefore \{1, e^{-t}, e^{t}\}$

(c) $y^{(4)} - 2y^{(2)} + y = 0,$

the auxiliary polynomial is $t^4 - 2t^2 + 1 = (t^2-1)^2 = (t+1)^2(t-1)^2$

$\therefore t = -1, 1$

$\therefore \{e^{-t}, te^{-t}, e^{t}, te^{t}\}$

(d)=(a)

(e)

Since $y^{(3)} - y^{(2)} + 3y^{(1)} + 5y = 0,$

the auxiliary polynomial is $t^3 - t^2 + 3t + 5 = (t_1)(t^2 - 2t + 5)0$

$\therefore t = -1, 1 + 2i, 1 - 2i$

$\therefore \{e^{-t}, e^t e^{2it}, e^t e^{-2it}\}$

$\therefore \{e^{-t}, e^t cos2t, e^t sin2t\}$

4.

(a)

$p(t) = t^2 - t - 1 = 0$

$t = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$\therefore \{e^{\frac{(1+\sqrt{5})t}{2}}, e^{\frac{(1-\sqrt{5})t}{2}}\}$

(b)

$p(t) = t^3 - 3t^2 + 3t - 1 = (t - 1)^3 = 0$

$\therefore \{e^t, te^t, t^2 e^t\}$

(c)

$p(t) = t^3 + 6t^2 + 8t = t(t^2 + 6t + 8) = 0$

$\therefore t = 0, -2, -4$

$\therefore \{1, e^{-2t}, e^{-4t}\}$

5.

$\forall f, g \in C^\infty, \alpha \in F,$

$(\alpha f + g)^{(n)} = \alpha f^{(n)} + g^{(n)} \in C^\infty,\ \forall n$

$\therefore C^\infty$ is a subspace of $\mathcal{F}(R, C)$

6.

(a)

$\forall f, g \in C^\infty, \alpha \in F,$

$D(\alpha f + g) = (\alpha f + g)' = \alpha f' + g' = \alpha D(f) + D(g)$

$\therefore D : C^\infty \to C^\infty$ is a linear operator

(b)

$\forall f, g \in C^\infty, \alpha \in F,$

Define $L = p(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0 I$

$L(\alpha f + g) = a_n(\alpha f + g)^n + a_{n-1}(\alpha f + g)^{n-1} + \cdots + a_0 I$

$= a_n(\alpha f^{(n)} + g^{(n)}) + a_{n-1}(\alpha f^{(n-1)} + g^{(n-1)}) + \cdots + a_0 I$

$= \alpha(a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0) + (a_n g^{(n)} + a_{n-1} g^{(n-1)} + \cdots + a_1 g' + a_0)$

$= \alpha L(f) + L(g)$

$\therefore$ Any differential operator is a linear operator on $C^\infty$

(cf) If $D$ : a linear operator,

then $DD$ : a linear operator

$\vdots$

$D^n$ : a linear operator

$\therefore$ $p(D = D^n + a_{n-1} D^{n-1} + cdots + a_1 D + a_0 I)$ : a linear operator

7.

We only need to show that $\{\frac{x+y}{2}, \frac{x-y}{2i}\}$ is linearly independent

If $\alpha\frac{(x+y)}{2} + \beta\frac{(x-y)}{2i} = 0,\ (\alpha, \beta \in F)$

then $(\frac{\alpha}{2} + \frac{\beta}{2i})x + (\frac{\alpha}{2} - \frac{\beta}{2i})y = 0$

$\Rightarrow \frac{\alpha}{2} + \frac{\beta}{2i} = 0,\ \frac{\alpha}{2} - \frac{\beta}{2i} = 0$

$\therefore \alpha = \beta = 0$

$\therefore \{\frac{x+y}{2}, \frac{x-y}{2i}\}$ is a basis

8.

Let $t_1 = a + ib(a, b \in R, b \neq 0)$ is a zero of $p(t)$

then, $e^{(a+ib)t}$ is a solution of $p(D)$

Using exercise 7 in this chapter,

If $\{e^{(a+ib)t}, e^{(a-ib)t}\}$ is a basis, then so is $\{\frac{1}{2}(e^{(a+ib)t} + e^{(a-ib)t}), \frac{1}{2i}(e^{(a+ib)t} - e^{(a-ib)t})\}$

Since $\frac{1}{2}(e^{(a+ib)t} + e^{(a-ib)t}) = e^{at}cosbt, \frac{1}{2i}(e^{(a+ib)t} - e^{(a-ib)t}) = e^{at}sinbt$

$\therefore \{e^{at}cosbt, e^{at}sinbt\}$ is a basis

(cf) Let $p(t) = \alpha(t^2 - 2at + a^2 + b^2),\ t_1 = a + ib,\ t_2 = a - ib$

then $p(D)y = y'' - 2ay' + (a^2 + b^2)y = 0,\ y_1 = e^{at}cosbt,\ y_2 = e^{at}sinbt$

check $p(D)y_1 = 0$, and $p(D)y_2 = 0$

9.

$u \in N(U_i) \Rightarrow U_i(u) = 0$

So $U_1 \cdots U_n(u) = U_1 \cdots U_{i-1}U_{i+1} \cdots U_n U_i(u) = U_1 \cdots U_{i-1}U_{i+1} \cdots U_n(0) = 0$

$\therefore u \in N(U_1 \cdots U_n)$

10.

Suppose that $b_1 e^{c_1 t} + \cdots + b_n e^{c_n t} = 0, c_i$'s are distinct

Apply the mathematical induction on $n$,

If $n = 1$, then $b_1 e^{c_1 t} = 0 \therefore b_1 = 0$

Assume that this assertion is true for $n - 1$

We are going to prove that this is also true for $n$

$(D - c_n I)(b_1 e^{c_1 t} + \cdots + b_n e^{c_n t}) = 0$

$\Rightarrow b_1 c_1 e^{c_1 t} + \cdots + b_{n-1} c_{n-1} e^{c_{n-1} t} + b_n c_n e^{c_n t} - (b_1 c_n e^{c_1 t} + \cdots + b_{n-1} c_n e^{c_{n-1} t} + b_n c_n e^{c_n t}) = 0$

$\Rightarrow b_1(c_1 - c_n)e^{c_1 t} + b_2(c_2 - c_n)e^{c_2 t} + \cdots + b_{n-1}(c_{n-1} - c_n)e^{c_{n-1} t} = 0$

By the induction hypothesis and $c_i - c_n \neq 0$,

$\forall i = 1, \cdots, n-1, b_i = 0$

$\therefore \{e^{c_1 t}, \cdots, e^{c_n t}\}$ is linearly independent

Since the solution space is $n$-dimensional, the given set is a basis for the solution space of the differential equation

11.

Suppose that $\sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} c_{ij} t^j e^{c_i t} = 0$

Let $P_i(t) = \sum\limits_{j=0}^{n_i-1} c_{ij}t^j$

Then we have $P_1(t)e^{c_1t} + P_2(t)e^{c_2t} + \cdots + P_k(t)e^{c_kt} = 0$

Assume that not all $c_{ij}$ are zero, then $\exists P_i \neq 0$

Say, $P_k$

Divide the equation by $e^{c_1t}$

$P_1(t) + P_2(t)e^{(c_2-c_1)t} + \cdots + P_k(t)e^{(c_k-c_1)t} = 0 \cdots (1)$

Upon differentiating (1) sufficiently many times we can reduces $P_1(t)$ to 0

$Q_2(t)e^{(c_2-c_1)t} + \cdots + Q_k(t)e^{(c_k-c_1)t} = 0$, and deg$Q_i$=deg$P_i$

and $Q_k$ does not vanish identically

Continuing this process, $R_k(t)e^{(c_k-c_1)t} = 0$, and deg$R_k$=deg$P_k$

and $R_k$ does not vanish identically

But $R_k(t)e^{(c_k-c_1)t} = 0$ implies $R_k = 0$

It's a contradiction to $P_k \neq 0$

$\therefore P_k(t) = 0, \forall x \in I$

$\therefore$ All $c_{ij}$'s are zero

12.

(i) $g(D)(V) \subseteq N(h(D))$

(ii) $\dim N(h(D)) = \dim g(D)(V)$

Suppose $degg(t) = k,\ deg(h(t)) = m\ (n = k + m)$

Consider the linear map $g(D_V) : V \to V$

By the dimension theorem,

$\dim V = \dim R(g(D_V)) + \dim N(g(D_V))$

$= \dim R(g(D_V)) + \dim N(g(D)) \quad (\because N(g(D)) \subseteq V)$

$= \dim(g(D)(V)) + \dim N(g(D)) \quad (\because R(g(D_V)) = g(D)(V))$

$= \dim g(D)(V) + k$

$\therefore \ \dim g(D)(V) = n - k = m = \dim N(h(D))$

13.

(a) Ontoness of $P(D) : C^\infty \to C^\infty$

Since $\mathbb{C}$ is algebraically closed

$P(D) = \alpha(D - c_1)(D - c_2)\cdots(D - c_n)$, where $\alpha \neq 0, c_1, \cdots, c_1 \in \mathbb{C}$

Let $v \in C^\infty$ : by lemma 1, $\exists\ u_1 \in C^\infty$ s.t.$(D - c_1)u_1 = v$

and $\exists\ u_2 \in C^\infty$ $(D - c_2)u_2 = u_1 \cdots$ continuing this process,

we get $u_1, u_2, \cdots, u_n \in C^\infty$ s.t. $(D - c_i)u_i = u_{i-1}$ $(2 \leq i \leq n)$

Put $u = \frac{1}{\alpha}u_n : P(D)u = \alpha(D - c_1)(D - c_2)\cdots(D - c_{n-1})(D - c_n)(u)$

$= (D - c_1)(D - c_2)\cdots(D - c_{n-1})(D - c_n)u_n = v$

14.

By induction on $n$, $p(t)$: a polynomial of degree $n(\geq 1)$

A solution $x(t)$ of $p(D)y = 0 - - - (*)$

We may assume w.l.o.g that $p(t)$ monic

For $n = 1$, $(*)$ becomes $y' - ay = 0 \ \Rightarrow \ x(t) = ce^{at}(c \in \mathbb{C})$

if $x(t_0)=0 \Rightarrow ce^{at_0}=0 \Rightarrow c=0 \ \therefore \ x(t)=0$

Assume it is true for $n-1(n>1)$ and $degp(t)=n$

This case $p(t)=q(t)(t-c), \ q(t)$ of degree $n-1, \ c\in\mathbb{C}$

let $z=q(D)x$

then by $(*)$, we have $(D-cz=(D-c)q((D)))x=p(D)x=0$

$\therefore \ z$is a solution to $(D-c)y=0$

By hypothesis, $x(t_0)=x'(t_0)=\cdots=x^{n-2}(t_0)=x^{n-1}(t_0)=0$ for fixed $t_0\in\mathbb{R}$

$\Rightarrow \ \forall t\in\mathbb{R}, \ z(t)=x^{(n-1)}(t)+a_{n-1}x^{(n-2)}(t)+\cdots+a_1x'(t)+a_0x(t) \ \Rightarrow \ z(t_0)=0$

$\Rightarrow \ z(t_0)=0 \ \Rightarrow \ z'(t_0)=0 \ (\because \ z'(t_0)-cz(t_0)=0, \ z'(t_0)=0)$

$\Rightarrow \ z(t)=0, \ \forall t$

$\Rightarrow \ q(D)x=z=0$

15.

$\Phi: V\to\mathcal{C}^n, \Phi(x)=(x(t_0),x'(t_0),\cdots,x^{n-1}(t_0))^T, \forall x\in V$

(a)

(i) $\Phi$ is linear

$x,y\in V, \ \alpha\in F$

$$\Phi(x+y)=\begin{pmatrix}(x+y)(t_0)\\(x+y)'(t_0)\\\cdots\\(x+y)^{n-1}(t_0)\end{pmatrix}=\begin{pmatrix}x(t_0)+y(t_0)\\x'(t_0)+y'(t_0)\\\cdots\\x^{n-1}(t_0)+y^{n-1}(t_0)\end{pmatrix}=\Phi(x)+\Phi(y)$$

$$\Phi(\alpha x)=\begin{pmatrix}(\alpha x)(t_0)\\(\alpha x)'(t_0)\\\cdots\\(\alpha x)^{n-1}(t_0)\end{pmatrix}=\alpha\begin{pmatrix}x(t_0)\\x'(t_0)\\\cdots\\x^{n-1}(t_0)\end{pmatrix}=\alpha\Phi(x)$$

(ii) $\Phi(x) = 0 \Rightarrow x = \{0\}$

If $x(t_0) = x'(t_0) = \cdots = x^{n-1}(t_0) = 0$, then $x = 0$

Since $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} \mathbb{C}^n = n,\ \Phi$ is an isomorphism

(iii) Since $dim_{\mathcal{C}}^{V} = dim_{\mathcal{C}}^{\mathcal{C}^n} = n,\ \Phi$ is onto

(b)

Since $\Phi$ is an isomorphism,

Let $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{C}^n$ s.t $x(t_0) = c_0$ and $x^k(t_0) = c_k,\ k = 1, \cdots, n-1$

By (a),$\exists ! x \in V$

16.

(a) $\theta'' + \frac{g}{l}\theta = 0$

$t^2 + \frac{g}{l} = 0,\ t_1 = \sqrt{\frac{g}{l}}i, t_2 = -\sqrt{\frac{g}{l}}i$

$\therefore \theta = c_1 cos\frac{g}{l}t + c_2 sin\frac{g}{l}t$

(b) $\theta(0) = \theta_0 > 0, \theta'(0) = 0$

$\theta(0) = c_1 = \theta_0$

$\theta'(0) = c_2\sqrt{\frac{g}{l}} = 0$

$\therefore c_2 = 0$

$\therefore \theta = \theta_0 cos\sqrt{\frac{g}{l}}t$

(c) The period of the system is $\frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{g}{l}}$

17.

$y'' + \frac{k}{m}y = 0$

$y(t) = c_1 \cos\sqrt{\frac{k}{m}}t + ic_2 \sin\sqrt{\frac{k}{m}}t$

18.

(a) $my'' + ry' + ky = 0,\ r > 0$

Since the auxiliary polynomial is $p(t) = mt^2 + rt + k = 0$,

$\therefore t_1 = \frac{-r+\sqrt{r^2-4km}}{2m},\ t_2 = \frac{-r-\sqrt{r^2-4km}}{2m}$

$\therefore y(t) = c_1e^{t_1t} + c_2e^{t_2t}$

(b)

$y(0) = c_1 + c_2 = 0\ \therefore c_1 = -c_2$

$y'(0) = c_1t_1 + c_2t_2 = v_0$

$c_2 = \frac{v_0}{t_2-t_1}, c_1 = \frac{-v_0}{t_2-t_1}$

$\therefore y(t) = (\frac{-v_0}{t_2-t_1})e^{t_1t} + (\frac{v_0}{t_2-t_1})e^{t_2t},\ (t_2 - t_1 = \frac{\sqrt{r^2-4km}}{m})$

(c)

$y(t) = c_1e^{t_1t} + c_2e^{t_2}t = e^{\frac{-r}{2m}t}(c_1e^{\frac{\sqrt{r^2-4km}}{2m}t} + c_2e^{-\frac{\sqrt{r^2-4km}}{2m}t})$

$t \to \infty \Rightarrow e^{\frac{-r}{2m}t} \to 0$

$\therefore \lim_{t\to\infty} y(t) = 0$

19.

$\because \mathcal{C}$ is algebraically closed

20.

(a) Theorem 2.27

If $n = 1$, then $x' + a_0 x = 0 \ \Rightarrow \ x' = -a_0 x$

Since $x$ has a derivative $x'$, $x'$ must has a derivative $x^{''} = -a_0 x'$

Assume that this assertion is true for an $n-1$th-order homogeneous linear differential equation with constant coefficients

Then $x^{(n)} = x^{(n-1)}x = -a_0 x^{(n-2)}x = -a_0 x^{(n-1)}$

So $x^{(k)}$ exists for every positive integer $k$

(b)

$y_1 = e^{c+t}, \ y_2 = e^c e^t$ (for $\forall c \in \mathbb{R}$)

(i) Let $x = (e^{c+t} - e^c e^t) \ \Rightarrow \ x' = e^{c+t} - e^c e^t = x$

So $x$ is a solution to the equation $y' - y = 0$ with $x(0) = 0$

So by the exercise 14, $x(t) \equiv 0 \ \forall t$

$\therefore \ e^{c+t} = e^c e^t$ putting $t = d \in \mathbb{R}$

$e^{c+d} = e^c e^d$

(ii) $e^c e^{-c} = e^0 = 1 = e^c \frac{1}{e^c}$

$e^c(e^{-c} - \frac{1}{e^c}) = 0 \ \Rightarrow \ e^c \neq 0$

$\therefore \ e^{-c} = \frac{1}{e^c}$

Since $\mathbb{C}$ is algebraically closed

(c) Theorem 2.28

Any homogeneous linear differential equation with constant coefficients can be rewritten as $P(D)y = 0$, where $p(t)$ is the auxiliary polynomial associated with the equation.

Therefore the set of all solutions to a homogeneous linear differential equation with constant coefficients coincides with the null space of $P(D)$, where $p(t)$ is the auxiliary polynomial associated with the equation.

(d) Theorem 2.29

$f(t) = e^{ct} = e^{a+ibt} = e^{at}(cosbt + isinbt) = e^{at}cosbt + ie^{at}sinbt$

$f'(t) = ae^{at}cosbt - be^{at}sinbt + iae^{at}sinbt + ibe^{at}cosbt$

$= (a + ib)e^{at}cosbt + i(a + ib)e^{at}sinbt$

$= (a + ib)\{e^{at}(cosbt + isinbt)\}$

$= ce^{ct}$

(e)

$(xy)' = (u_1u_2 + i(u_1v_2 + u_2v_1) - v_1v_2)'$

$= \{(u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1)\}'$

$= (u_1'u_2 + u_1u_2' - v_1'v_2 - v_1v_2') + i(u_1'v_2 + u_1v_2' + u_2'v_1 + u_2v_1')$

$= \{u_1'u_2 + i(u_1'v_2 + u_2v_1') - v_1'v_2\} + \{u_1u_2' + i(u_1v_2' + u_2'v_1) - v_1v_2'\}$

$= (u_1' + iv_1')(u_2 + iv_2) + (u_1 + iv_1)(u_2' + iv_2')$

$= x'y + xy'$

(f)

Let $x = u + iv$

$x' = u' + iv' = 0$

$\therefore\ u' = 0, v' = 0$

$\therefore\ u, v$ is a constant function

$\therefore\ x = u + iv$ is a constant function

# §3. Elementary Matrix Operations and Systems of Linear Equations

## 3.1. Elementary Matrix Operations and Elementary Matrices

1. (a) T

(b) F

$(\because)$ $I_3 \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, by $-2 \times C_1 + C_3 \Rightarrow C_3$

(c) T

$(\because)$ $I_n \rightsquigarrow I_n$, by $1 \times C_1 \Rightarrow C_1$

(d) F

(e) T

$(\because)$ Theorem 3.2

(f) F

$(\because)$ Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

then, $E_1 + E_2$ is not an elementary matrix

(g) T

(h) F

(i) T

Let $A \rightsquigarrow B = EA$,

then $E$ is invertible and its inverse is also an elementary matrix

$\therefore\ B \rightsquigarrow A = E^{-1}B$

2.

(i) $A \rightsquigarrow B$, by $-2 \times C_1 + C_2 \to C_2$

(ii) $B \rightsquigarrow C$, by $-1 \times R_1 + R_2 \to R_2$

(iii) $C \rightsquigarrow I_3$, by

$\frac{1}{2} \times R_2 \to R_2$

$R_2 \leftrightarrow R_3$

$R_3 + R_2 \to R_2$

$\frac{1}{4} \times R_2 \to R_2$

$-1 \times R_2 + R_3 \to R_3$

$-3 \times R_3 + R_1 \to R_1$

3.

(a) $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$I_3 \Rightarrow (R_1 \leftrightarrow R_3) \Rightarrow E \Rightarrow (R_1 \leftrightarrow R_3) \Rightarrow I_3$

$\therefore\ E^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_3 \Rightarrow (3 \times R_2 \to R_2) \Rightarrow E \Rightarrow (\frac{1}{3} \times R_2 \to R_2) \Rightarrow I_3$

$\therefore\ E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$I_3 \Rightarrow (-2 \times R_1 + R_3 \to R_3) \Rightarrow E \Rightarrow (2 \times R_1 + R_3 \to R_3) \Rightarrow I_3$

$\therefore\ E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

4.

(i) $E$ is of type 1

$E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

$F = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n) = E$

(ii) $E$ is of type 2

$E = (e_1, \cdots, ae_j, \cdots, e_n)$

$F = (e_1, \cdots, ae_j, \cdots, e_n) = E$

(iii) $E$ is of type 3

$E = (e_1, \cdots, e_i, \cdots, e_j + ae_i, \cdots, e_n)$

$F = (e_1, \cdots, e_i + ae_j, \cdots, e_j, \cdots, e_n) = E^t$

5.

($\Rightarrow$)

(i) $E$ is of type 1

$E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

$E^t = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

(ii) $E$ is of type 2

$E = (e_1, \cdots, ae_j, \cdots, e_n)$

$E^t = (e_1, \cdots, ae_j, \cdots, e_n)$

(iii) $E$ is of type 3

$E = (e_1, \cdots, e_i, \cdots, e_j + ae_i, \cdots, e_n)$

$E^t = (e_1, \cdots, e_i + ae_j, \cdots, e_j, \cdots, e_n)$

($\Leftarrow$) Using the fact $(E^t)^t = E$, then it is clear

6.

(i) if $B = EA$, then $B^t = (EA)^t = A^t E^t$

(ii) if $B = AE$, then $B^t = (AE)^t = E^t A^t$

7.

(1) Elementary column operation

(i) $E$ is of type 1

$B = (A^{(1)}, \cdots, A^{(j)}, \cdots, A^{(i)}, \cdots, A^{(n)})$

$E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

$\Rightarrow \; AE = (Ae_1, \cdots, Ae_j, \cdots, Ae_i, \cdots, Ae_n) = B$

$\therefore \; B = AE$

(ii) $E$ is of type 2

$B = (A^{(1)}, \cdots, aA^{(j)}, \cdots, A^{(n)})$

$E = (e_1, \cdots, ae_j, \cdots, e_n)$

$\Rightarrow\ AE = (Ae_1, \cdots, aAe_j, \cdots, Ae_n) = B$

$\therefore\ B = AE$

(iii) $E$ is of type 3

$B = (A^{(1)}, \cdots, A^{(i)}, \cdots, A^{(j)} + aA^{(i)}, \cdots, A^{(n)})$

$E = (e_1, \cdots, e_i, \cdots, e_j + ae_i, \cdots, e_n)$

$\Rightarrow\ AE = (Ae_1, \cdots, Ae_i, \cdots, Ae_j + aAe_i, \cdots, Ae_n) = B$

$\therefore\ B = AE$

(2) Elementary row operation

(i) $E$ is of type 1

$B = (A_{(1)}, \cdots, A_{(j)}, \cdots, A_{(i)}, \cdots, A_{(m)})$

$E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_m)$

$\Rightarrow\ EA = (e_1A, \cdots, e_jA, \cdots, e_iA, \cdots, e_mA) = B$

$\therefore\ B = EA$

(ii) $E$ is of type 2

$B = (A_{(1)}, \cdots, aA_{(j)}, \cdots, A_{(m)})$

$E = (e_1, \cdots, ae_j, \cdots, e_m)$

$\Rightarrow\ EA = (e_1A, \cdots, ae_jA, \cdots, e_mA) = B$

$\therefore\ B = EA$

(iii) $E$ is of type 3

$B = (A_{(1)}, \cdots, A_{(i)}, \cdots, A_{(j)} + aA_{(i)}, \cdots, A_{(m)})$

$E = (e_1, \cdots, e_i, \cdots, e_j + ae_i, \cdots, e_m)$

$\Rightarrow \; EA = (e_1A, \cdots, e_iA, \cdots, e_jA + ae_iA, \cdots, e_mA) = B$

$\therefore \; B = EA$

8.

(i) $E$ is of type 1

$E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

$E^{-1} = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

(ii) $E$ is of type 2

$E = (e_1, \cdots, ae_j, \cdots, e_n)$

$E^{-1} = (e_1, \cdots, \frac{1}{a}e_j, \cdots, e_n)$

(iii) $E$ is of type 3

$E = (e_1, \cdots, e_i, \cdots, e_j + ae_i, \cdots, e_n)$

$E^{-1} = (e_1, \cdots, e_i, \cdots, e_j - ae_i, \cdots, e_n)$

If $P \rightsquigarrow Q = EP$, then $P = E^{-1}Q$

Let $E' = E^{-1}$

Since $E'$ is invertible and is an elementary matrix, $Q \rightsquigarrow P = E'Q$

9. $I_n \Rightarrow E_3', E_3'', E_3''', E_2 \Rightarrow E$

,where $E_3' : -1 \times R_i + R_j \Rightarrow R_j$ $i.e$ $E = (e_1, \cdots, e_i, \cdots, e_j - e_i, \cdots, e_n)$

$E_3'' : 1 \times R_j + R_i \Rightarrow R_i$ $i.e$ $E = (e_1, \cdots, e_i + (e_j - e_i), \cdots, e_j - e_i, \cdots, e_n)$

$E_3''' : -1 \times R_i + R_j \Rightarrow R_j$ $i.e$ $E = (e_1, \cdots, e_j, \cdots, -e_i, \cdots, e_n)$

$E_2 : -1 \times R_j \Rightarrow R_j$ $i.e$ $E = (e_1, \cdots, e_j, \cdots, e_i, \cdots, e_n)$

$\therefore$ $E$ is of type 1

10. $a$ is a nonzero scalar

$I_n \Rightarrow E_2 \Rightarrow E$

,where $E_2 : \frac{1}{a} \times R_i \rightarrow R_i$ $i.e$ $E = (e_1, \cdots, \frac{1}{a}e_i, \cdots, e_n)$

$\therefore$ $E$ is of type 2

11. $I_n \Rightarrow E_3 \Rightarrow E$

,where $E_3 : -a \times R_i + R_j \rightarrow R_j$ $i.e$ $E = (e_1, \cdots, e_i, \cdots, e_j - ae_i, \cdots, e_n)$

$\therefore$ $E$ is of type 3

12.

By induction on $n \geq 1$

If $n = 1$ o.k.

Assume that $n > 1$

If the first column of $A$ is zero, then $A = (O \mid B)$, where $B = ()_{m \times (n-1)}$

So by induction hypothesis, $B$ can be transformed by row operation of type 1 and 3

If the first column of $A$ is not zero, we may assume that $a_{11} \neq 0$

So $A \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{pmatrix}$, where $C = ()_{(m-1)\times(n-1)}$

By induction hypothesis, $C \rightsquigarrow$ U.T.M

$\therefore\ A \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & D & \\ 0 & & & \end{pmatrix}$ : U.T.M

## 3.2. The Rank of a Matrix and Matrix Inverse

1. (a) F (Theorem 3.5)

(b) F

($\because$) If $A \in M_{m\times n}(F),\ B \in M_{n\times n}(F)$ and $B$ is invertible,

then the rank(AB)=rank(A)

(Example)

$A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $rank(A) = rank(B) = 1$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $rank(AB) = 0$

(c) T

(d) T (p.153, Corollary to the Theorem 3.4)

(e) F (p.153, Corollary to the Theorem 3.4)

(f) T (p.153, Theorem 3.4 and Theorem 3.5)

(g) T (p.161)

(h) T

($\because$) $\forall A \in M_{m\times n}(F),\ rank(A) = dimR(L_A) \leq n$

(i) T

2.

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $rank(A)$=2

(b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $rank(A) = 3$

(c) $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$, $rank(A) = 2$

(d) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $rank(A) = 1$

(e) $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $rank(A) = 3$

(g) $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $rank(A) = 1$

3.

$\forall\ A \in M_{m\times n}(F),\ rank(A) = 0$ iff $A$ is the zero matrix

($\Leftarrow$) clear

($\Rightarrow$) let $A = (e_1, 0, \cdots, 0),\ e_1 \neq 0$

Since $rank(A) = 0,\ e_1$ is dependent

$\therefore\ \exists a \in F$s.t $e_1 = a0$

It's contradict to $e_1 \neq 0$

4.

(a) $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, 2

(b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 12 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, 2

5. $(A \mid I_n) \rightsquigarrow (I_n \mid B),\ B = A^{-1}$

(a) $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$

$\therefore\ A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},\ rank(A) = 2$

(b) $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$

$rank(A) = 1$, so $\nexists A^{-1}$

(c) $\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right)$

$rank(A) = 2$, so $\nexists A^{-1}$

(d) $\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & 15/9 & -1/3 \\ 0 & 1 & 0 & -5/18 & -4/9 & 2/9 \\ 0 & 0 & 1 & 1/9 & -2/9 & 1/9 \end{array}\right)$

$\therefore\ rank(A) = 3,\ A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{15}{9} & -\frac{1}{3} \\ -\frac{5}{18} & -\frac{4}{9} & \frac{2}{9} \\ \frac{1}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$

(e) $\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1/6 & 1/3 & 1/2 \end{array}\right)$

$\therefore\ rank(A) = 3,\ A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

(f) $\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & -1/2 & 1 \end{array}\right)$

$\therefore\ rank(A) = 2,$ , so $\nexists A^{-1}$

(g) $\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -51 & 15 & 7 & 2 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$

$\therefore\ rank(A) = 4,\ A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 2 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}$

(h) $\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3/2 & -1/10 & 13/10 & 1/10 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 & -7/5 & 1/2 \\ 0 & 0 & 1 & 0 & -3 & -1/5 & 16/10 & 1/5 \\ 0 & 0 & 0 & 1 & 1/2 & 1/10 & -3/10 & -1/10 \end{array}\right)$

$\therefore\ rank(A) = 4,\ A^{-1} = \begin{pmatrix} -3/2 & -1/10 & 13/10 & 1/10 \\ 1/2 & 1/2 & -7/5 & 1/2 \\ -3 & -1/5 & 16/10 & 1/5 \\ 1/2 & 1/10 & -3/10 & -1/10 \end{pmatrix}$

6.

(a) $[T]_\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix},\ rank[T]_\beta = 3$

$\therefore$T is invertible

$([T]_\beta)^{-1} = \begin{pmatrix} -1 & -2 & 10 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

$\therefore\ T^{-1}(ax^2 + bx + c) = -ax^2 - (4a + b)x - (10a + 2b + c)$

(b) $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},\ rank[T]_\beta = 2$

$\therefore$T is not invertible

(c) $[T]_\beta = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix},\ rank[T]_\beta = 3$

$\therefore$T is invertible

$$([T]_\beta)^{-1} = \begin{pmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/ \end{pmatrix}$$

$\therefore\ T^{-1}(a,b,c) = (\frac{1}{6}a - \frac{1}{3}b + \frac{1}{2}c, \frac{1}{2}a - \frac{1}{2}c, -\frac{1}{6}a + \frac{1}{3}b + \frac{1}{2}c)$

7.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \rightsquigarrow I_3 = E_6 \cdots E_1 A$$

$\therefore A = E_1^{-1} \cdots E_6^{-1}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8.

$cA = (cI_m)A$

Since $cI_m$ is invertible

$\therefore\ rank(cA) = rank((cI_m)A)$

9.

If $B$ is obtained from a matrix $A$ by an elementary column operation, then there exists an elementary matrix $E$ such that $B = AE$

Since $E$ is invertible and hence $rank(B) = rank(A)$

10. If $A$ is the zero matrix, then $r = 0$

$rank(A) = rank(D) = 0$

Suppose that $A$ is a nonzero matrix

By means of at most one type 1 row operation, at most one type 2 row operation,

and at most $(m-1)$ type 3 row operations

this matrix can be transformed into $(1, 0, \cdots, 0)^T$

$\therefore\ rank(A) = rank(D) = 1$

11. (By theorem 3.6)

$B' \ \rightsquigarrow\ D' = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix},\ rankB' = k$ and

$B \ \rightsquigarrow\ D = \left(\begin{array}{c|cc} 1 & O & \\ \hline O & I_k & O \\ & O & O \end{array}\right) = \begin{pmatrix} I_{k+1} & O \\ O & O \end{pmatrix}$

$\therefore\ rankB = k + 1 = r$

$\therefore\ rankB' = r - 1$

12. By induction on $n \geq 1$

If $n = 1$, it is clear

Assume $n > 1$

If the first column of $A$ is zero, then $A = \begin{pmatrix} \mathrm{O} & \mathrm{B} \end{pmatrix}$, where $B \in M_{m\times(n-1)}$

So by induction hypothesis, $B$ can be transformed by row operation of type 1 and 3

If the first column of $A$ is not zero,

we may assume that $a_{11} \neq 0$

So $A \rightsquigarrow \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & \text{C} & \\ 0 & & & \end{array}\right)$, where $C \in M_{(m-1)\times(n-1)}$

By by induction hypothesis, $C \rightsquigarrow U.T.M$

$\therefore\ A \rightsquigarrow U.T.M$

13. (b) and (c)

Since $rank(A) = rank(A^t)$, the maximal number of linearly independent columns of $A$ equals to the maximal number of linearly independent columns of $A^t$,

*i.e.* the maximal number of linearly independent rows of $A$

$\therefore\ colrank(A) = rowrank(A)$

$(cf)$ Let $S$ the solution space for $AX = 0$, then by the dimension theorem and $rank(L_A) = colrank(A)$,

$dimS = n - colrank(A)$

If $r = dim$(row space of $A$), then the solution space $S$ has a $(n - r)$ vectors,

*i.e.* $dimS = n - rowrank(A)$

$\therefore\ colrank(A) = rowrank(A)$

14. $T,\ U : V \to W$

(i) $w \in R(T + U)\ \Rightarrow\ \exists\ v \in V$ s.t. $w = (T + U)(v)$

$\Rightarrow\ w = T(v) + U(v) \in R(T) + R(U)$

(ii) $rank(T+U) \ \leq \ dim(R(T)+R(U))$

$\leq \ dim(R(T))+dim(R(U))$

$= \ rank(T)+rank(U)$

(c) By (b), $rank(A+B) = rank(L_{A+B})$

$= rank(L_A + L_B)$

$\leq \ rank(L_A)+rank(L_B)$

$= rank(A)+rank(B)$

15.

Let $A = (a_1, a_2, \cdots, a_p)_{n\times p}$ and $B = (b_1, b_2, \cdots, b_q)_{n\times q}$

$(A \mid B) = (a_1, a_2, \cdots, a_p, b_1, b_2, \cdots, b_q)_{n\times(p+q)}$

$M(A \mid B) = (Ma_1, Ma_2, \cdots, Ma_p, Mb_1, Mb_2, \cdots, Mb_q)_{n\times(p+q)}$

$(MA \mid MB)$

16. (Theorem 3.4 (b))

Let $V = R(L_A) = L_A(F^n)$

Then $V \leq F^m$ and $dimV = dimL_P(V)$

because $L_P$ is an isomorphism (by the exercise 17 in section 2.4)

$rank(A) = dimL_A(V) = dimV$

$= dimL_P(V)$

$= dimL_PL_A(F^n)$

$= dim L_{PA}(F^n)$

$rank(PA)$

17.

Let $PAQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$A = P^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Q^{-1}$

18.

Let $A = (a_1, a_2, \cdots, a_n)_{m \times n}$ where $a_k$ the $k-$th column of $A$ and let $A_k = (0, \cdots, a_k, \cdots, 0)_{m \times n}$

Thus $\forall k,\ rank(A_k B) \leq 1$ and $AB = A_1 B + \cdots + A_n B$

19.

By the theorem 3.7 $rank(AB) \leq rank(A)$ and

by the Sylvester inequality, $rank(AB) \geq rank(A)$

$\therefore\ rank(AB) = rank(A) = m$

$(cf)$the Sylvester inequality

$rank(A) + rank(B) \leq rank(AB) + n,$

where $n$ is the number of columns of $A$ and also the number of rows of $B$

20.

(a)

$A \rightsquigarrow D = EA = () :$ the reduced row echelon form of $A$

$\Rightarrow \ \{v_1 = (), v_2 = ()\}$ is a basis of $Null(L_A)$

Thus $AM = (0)_{5\times 5}$, where $M = (v_1, v_2, 0, 0, 0)$

(b)

Suppose $B = (b_1, b_2, \cdots, b_5)_{5\times 5}$ s.t. $AB = (0)_{4\times 5}$

$\Rightarrow \ b_1, b_2, \cdots, b_5 \in N(L_A)$ with null $L_A = 2$

Thus $rankB = dimspan(b_1, b_2, \cdots, b_5) \le dimNull(L_A) = 2$

21.

$A = (a_{ij})_{m\times n}, rank(A) = m$

$A \rightsquigarrow \ D = AQ = (I_m \mid O)_{m\times n}, \ Q \in M_{n\times n} :$ invertible matrix

Let $B = QM$, where $M = \begin{pmatrix} I_m \\ \hline O \end{pmatrix}_{n\times m}$

then $AB = (QM) = (AQ)M = (I_m \mid O) \begin{pmatrix} I_m \\ \hline O \end{pmatrix} = I_m$

22.

$B = (b_{ij})_{n\times m}, rank(B) = m$

$B \rightsquigarrow \ D = QB = \begin{pmatrix} I_m \\ \hline O \end{pmatrix}_{n\times m}, \ Q \in M_{n\times n} :$ invertible matrix

Let $A = MQ$, where $M = (I_m \mid O)m \times n$

then $AB = M(QB) = MD = I_m$

## 3.3. Systems of Linear equations - Theoretical aspects

1.

(a) F

p.170, Example 1 (c)

(b) F

(c) T

Any homogeneous system has at least one solution, namely, the zero vector

(d) F

p.174, Theorem 3.10

(e) F

(f) F

p.172, Theorem 3.9

(g) T

If $A$ is invertible, then $AX = 0$ has no nonzero solutions

(h) T

2. Let $K$ be the solution set of the given system and $A$ is the coefficient matrix of the system

(a) $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$

$rank(A) = 1,\ dim(K) = 2 - 1 = 1$

$\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ : a basis for $K$

(b) $A = \begin{pmatrix} 1 & 1 & -1 \\ 4 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$rank(A) = 2,\ dim(K) = 3 - 2 = 1$

Since $(1,2,3)^t$ is a solution to $AX = 0$,

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ : a basis for $K$

(c) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$rank(A) = 2,\ dim(K) = 3 - 2 = 1$

Since $(-1,1,1)^t$ is a solution to $AX = 0$,

$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ : a basis for $K$

(d) $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$rank(A) = 2,\ dim(K) = 3 - 2 = 1$

Since $(0,1,1)^t$ is a solution to $AX = 0$,

$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ : a basis for $K$

(e) $A = (1, 2, -3, 1) \rightsquigarrow (1, 0, 0, 0)$

$rank(A) = 1,\ dim(K) = 4 - 1 = 3$

$x_1 = -2x_2 + 3x_3 - 4x_4$

Note that $\{v_1, v_2, v_3\}$ is linearly independent vectors in $K$

$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ : a basis for $K$

(f) $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$rank(A) = 2,\ dim(K) = 2 - 2 = 0$

$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ : a basis for $K$

(So the given system is inconsistent)

(g) $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$rank(A) = 2,\ dim(K) = 4 - 2 = 2$

$\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ : a basis for $K$

3.

(a) $A = \left\{ \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} | t \in R \right\}$

(b) $A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} | t \in R \right\}$

(c) $A = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} | t \in R \right\}$

(d) $A = \left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} | t \in R \right\}$

(e) $A = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} | r, s, t \in R \right\}$

(f) $A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

(g) $A = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} | r, s \in R \right\}$

4.

(a)

(1) $A^{-1} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

(2) $x = A^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$

(b)

(1) $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix}$

(2) $x = A^{-1} \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

5.

$AX = B,\ A = (a_1, \cdots, a_n) \in M_{n\times n},\ a_i$ is the $n$th column of $A$

If $a_i$ is expressed by other column vectors of $A$, *i.e.* $a_i$'s are linearly independent,

then the given system has infinitely many solutions

(Example) $A = (a_1, \cdots, a_i, \cdots, ka_i, \cdots, a_n),\ a_j = ka_i,\ k \in F$

6.

$T^{-1}(\{(1, 11)\}) = \left\{ \begin{pmatrix} \frac{11}{2} \\ -\frac{9}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} | t \in R \right\}$

7. The systems in parts (b), (c), and (d) have solutions

(a) $rank(A) = 2,\ rank(A|b) = 3$

$A \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ (A|b) \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $rank(A) = rank(A|b) = 2$

(c) $rank(A) = rank(A|b) = 3$

(d) $rank(A) = rank(A|b) = 4$

(e) $rank(A) = 2,\ rank(A|b) = 3$

8.

(a) $v \in R(T)\ T(-2, 3, 0) = (1, 3, 2)$

(b) $v \in R(T)\ T(1, 1, 0) = (2, 1, 1)$

9.

$L_A : F^n \to F^m,\ A = (a_1, \cdots, a_n)$

Let $x = (x_1, \cdots, x_n)^t$ is a solution to $Ax = b$

$\Leftrightarrow\ b = Ax = a_1x_1 + \cdots + a_nx_n,\ x_i \in F$

$\Leftrightarrow\ b \in span(a_1, \cdots, a_n) = R(L_A)$

11.

$Ap = p \rightsquigarrow p = \begin{pmatrix} 1 \\ 0.75 \\ 1 \end{pmatrix}$

The farmer, tailor, and carpenter must have incomes in the proportions 4:3:4

12.

$0.60p_1 + 0.30p_2 = p_1$

$0.40p_1 - 0.30p_2 = 0$

$\therefore\ p = \begin{pmatrix} 0.75 \\ 1 \end{pmatrix}$

13.

There must be 7.8 units of the first commodity and 9.5 units of the second

$x = (I - A^{-1})d = \begin{pmatrix} 7.8 \\ 9.5 \end{pmatrix}$

## 3.4. Systems of Linear equations - Computational aspects

1.

(a) F (a finite row operations)

(b) T (P.182 Corollary)

(c) T (P.158 Corollary 1 to Theorem 3.6)

(d) T (p.187 Theorem 3.14)

(e) F

The system has a solution if and if only the echelon form of the augmented matrix M does not have a row of the form $(0, \cdots, 0, b)$ with $b \neq 0$

(f) T

$rank(A) = rank(A \mid b) \;\Leftrightarrow\; (A \mid b)$ is consistent

If the system $(A \mid b)$ is consistent and $rank(A) = r$, then the dimension of the solution set is $n - r$.

(g) T

Since $A$ is row equivalent to $A'$ $i.e. A' = EA$ $rank(A) = rank(EA') = rank(A')$

2.

(a) $\left\{ \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 9 \\ 4 \\ 0 \end{pmatrix} + r \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \mid r \in R \right\}$

(c) $\left\{\begin{pmatrix} 2 \\ 3 \\ -2 \\ 1 \end{pmatrix}\right\}$

(d) $\left\{\begin{pmatrix} 13 \\ 22 \\ -\frac{1}{26} \\ \frac{18}{13} \end{pmatrix} + r\begin{pmatrix} 9 \\ -15 \\ 0 \\ 1 \end{pmatrix} \mid r \in R\right\}$

(e) $\left\{\begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \mid r, s \in R\right\}$

(f) $\left\{\begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \mid r \in R\right\}$

(g) $\left\{\begin{pmatrix} -23 \\ 0 \\ 7 \\ 9 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -23 \\ 0 \\ 6 \\ 9 \\ 1 \end{pmatrix} \mid r, s \in R\right\}$

3.

(a)

($\Leftarrow$) If $(A' \mid b')$ has such a row, say $k-th$ row, then

the corresponding $k-th$ equation in the system $A'x = b'$ is

$0x_1 + 0x_2 + \cdots + 0x_n = c_k,\ c_k \neq 0$ ;

which has no solutions $i.e.$ $A'x = b'$ is inconsistent

$\therefore\ rank(A') \neq rank(A' \mid b')$

($\Rightarrow$) Assume that $(A' \mid b')$ has no such a row, then $b' \in R(L_{A'})$

$\therefore\ rank(A') = rank(A' \mid b')$

(b)

If two matrices are row equivalent, they have the same solution set

Since $(A' \mid b')$ is equivalent to $(A \mid b)$, so

$Ax = b$ is consistent

$\Leftrightarrow\ A'x = b'$ is consistent

$\Leftrightarrow\ rank(A') = rank(A' \mid b')$

By (a), $(A' \mid b')$ has no such a row

(cf) (a) $(\Rightarrow)$ $b' \in R(L_{A'})$ $(\because Theorem 3.16(b))$

4.

(a)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \text{-}1/2 & 4/3 \\ 0 & 1 & 0 & 1/2 & 1/3 \\ 0 & 0 & 1 & \text{-}1/2 & 0 \end{array}\right)$$

Since $(A' \mid b')$ contains no row in which the only nonzero entry lies in the last column, therefore $Ax = b$ is consistent

$$\left\{ \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 1/ \\ 1 \end{pmatrix} r \mid r \in R \right\} : \text{ the solution set}$$

$$\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\} : \text{ a basis for the solution set}$$

(b)

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & \text{-1/2} & 1 \\ 0 & 0 & 1 & \text{-1/2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since $rank(A') = rank(A' \mid b'),\ Ax = b$ is consistent

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} r + \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} s \mid r, s \in R \right\} : \text{ the solution set}$$

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} \right\} : \text{ a basis for the solution set}$$

(c)

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & \text{-1/2} & 7/4 \\ 0 & 0 & 1 & \text{-1/2} & 1/4 \\ 0 & 0 & 0 & 0 & \text{-3/4} \end{array}\right)$$

Since $rank(A') \neq rank(A' \mid b'),\ Ax = b$ is inconsistent

5.

$$B = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

Let $A = (a_1, a_2, \cdots, a_5)$ and $B = (b_1, b_2, \cdots, b_5)$

Since $b_3 = 2e_1 - 5e_2$, it follows that $a_3 = 2a_1 - 5a_2$

Moreover $b_5 = -2e_1 - 3e_2 + 6e_3$, the same result shows that $a_5 = -2a_1 - 3a_2 + 6a_4$

$$\therefore\ A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$

6.

$$B = \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $A = (a_1, a_2, \cdots, a_6)$ and $B = (b_1, b_2, \cdots, b_6)$

Since $b_2 = -3e_1$,

$b_4 = 4e_1 + 3e_2$,

$b_6 = 5e_1 + 2e_2 - e_3$

it follows that $a_2 = -3a_1$,

$a_4 = 4a_1 + 3a_3$,

$a_6 = 5a_1 + 2a_3 - a_5$

$$\therefore\ A = \begin{pmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{pmatrix}$$

7.
$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -4 & -4 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\therefore\ \{u_1, u_2, u_5\}$ is a basis for $W$

9.
$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 9 \\ 1 & -2 & -2 & 4 \\ -1 & 2 & 2 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}\right\}$$ : a basis for $W$

(cf)

$$\begin{pmatrix} 0 & \cdots & -1 \\ -1 & \cdots & 2 \\ -1 & \cdots & 2 \\ 1 & \cdots & -1 \end{pmatrix}$$

(a) (i) $S \subseteq V$

(ii) $S$ : linearly independent $A \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$\{u_1, u_2, \cdots, u_5\}$

10.

(a) It's a singleton set, so it's linearly independent

(b)

Since $x_1 = 2x_2 - 3x_3 + x_4 - 2x_5$, assign parametric values to $x_2, x_3, x_4$ and $x_5$

Let $x_2 = t_1, x_3 = t_2, x_4 = t_3$ and $x_5 = t_4$, then the vectors in $V$ have the form

$(x_1, x_2 \cdots, x_5) = t_1(2,1,0,0,0) + t_2(-3,0,1,0,0) + t_3(1,0,0,1,0) + t_4(-2,0,0,0,1)$

Hence

$\beta = \{(2,1,0,0,0), (-3,0,1,0,0), (1,0,0,1,0), (-2,0,0,0,1)\}$ is a basis for $V$

The matrix whose columns consist of the vectors in $S$ followed by those $\beta$ is

$$\begin{pmatrix} 0 & 2 & -3 & 1 & -2 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its reduced row echelon form is $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Thus $\{(0,1,1,1,0),(2,1,0,0,0),(-3,0,1,0,0),(-2,0,0,0,1)\}$ is a basis for $V$ containing $S$

11.

(a) It's a singleton set, so it's clear

(b)

The matrix whose columns consist of the vectors in $S$ followed by those $\beta$ is

$$\begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its reduced row echelon form is $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\{(1,2,1,0,0),(2,1,0,0,0),(1,0,0,1,0),(-2,0,0,0,1)\}$ is a basis for $V$ containing $S$

12.

(a) If $a(0,-1,0,1,1,0)+b(1,0,1,1,1,0)=(0,\cdots,0)$. $a,b\in F$

then $a=b=0$

$\therefore$ $S$ is linearly independent

(b)

Since $x_1=x_3-x_4+x_5-3x_6$

$x_2=x_3+x_4-2x_5-2x_6$

Let $x_3=t_1, x_4=t_2, x_5=t_3$ and $x_6=t_4$, then the vectors in $V$ have the form

$(x_1,x_2\cdots,x_6)=t_1(1,1,1,0,0,0)+t_2(-1,1,0,1,0,0)+t_3(1,-2,0,0,1,0)+t_4(-3,-2,0,0,0,1)$

Hence

$\beta=\{(2,1,0,0,0),(-3,0,1,0,0),(1,0,0,1,0),(-2,0,0,0,1)\}$ is a basis for $V$

The matrix whose columns consist of the vectors in $S$ followed by those $\beta$ is

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its reduced row echelon form is $\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\{(0,-1,0,1,1,0),(1,0,1,1,1,0),(-1,1,0,1,0,0),(-3,-2,0,0,0,1)\}$ is a basis for $V$ containing $S$

13.

(a) If $a(1,0,1,1,1,0)+b(0,2,1,1,0,0)=(0,\cdots,0)$. $a,b\in F$

then $a=b=0$

$\therefore$ $S$ is linearly independent

(b)

The matrix whose columns consist of the vectors in $S$ followed by those $\beta$ is

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 1 & -3 \\ 0 & 2 & 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its reduced row echelon form is $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\{(1,0,1,1,1,0),(0,2,1,1,0,0),(1,1,1,0,0,0),(-3,-2,0,0,0,1)\}$ is a basis for $V$ containing $S$

15.

Let $B$ is a reduced row echelon form to $A$, then $A$ is row equivalent to $B$,

*i.e.* $A \sim B$

If $C$ is an another reduced row echelon form to $A$, then $C \sim A \sim B$

Since row equivalent matrices have the same row space and they are finite dimensional, so must be identical

(cf) If $C$ is an another reduced row echelon form to $A$, then $C \sim A \sim B$

Since $\exists$ $E$ s.t. $C = EB$ and $E$ is invertible, we have $E^{-1}C = B$,

Comparing with their rank,

the row space $C$ and $B$ are the subspace of each other

So $C = B$

# §4. Determinants

## 4.1. Determinants of Order 2

1. (a) F $\det(A+B) \neq \det(A)+\det(B)$

(b) T (p.200. Theorem 4.1)

(c) F (p.201. Theorem 4.2)

(d) F

The area of the paralleogram determined by $u$ and $v$ equals $O\begin{pmatrix} u \\ v \end{pmatrix} \det\begin{pmatrix} u \\ v \end{pmatrix}$

(e) F (p.203)

2. (a) 30

(b) -17

(c) -8

3. (a) -10+15i

(b) 36+41i

(c) -24

4. (a) 4

(b) 10

(c) 14

(d) 26

5.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $B = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$

So $\det(B) = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -\det(A)$

6.

Let $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, and $B = \begin{pmatrix} b & b \\ a & a \end{pmatrix}$

By the exercise 15, $\det(B) = -\det(A)$

Since $\det(A) = \det(B)$

$\therefore\ \det(A) = 0$

7.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

$\det(A^t) = a_{11}a_{22} - a_{21}a_{12} = \det(A)$

$\therefore\ \det(A^t) = \det(A)$

8.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, then $\det(A) = a_{11}a_{22}$ : the product of the diagonal entries of $A$

9.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

then $AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

$\det(AB)$

$= (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22}) - (a_{11}b_{12}+a_{12}b_{22})(a_{21}b_{11}+a_{22}b_{21})$

$= (a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21})$

$= \det(A)\det(B)$

10.

$C = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

(a) $CA = AC = (\det A)I = \begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix}$

(b) $\det(C) = a_{22}a_{11} - a_{12}a_{21} = \det(A)$

(c) The classical adjoint of $A^t = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} = C^t$

(d) If $A$ is invertible, then $\det(A) \neq 0$

by (a), $A\{(\frac{1}{\det(A)})C\} = \{(\frac{1}{\det(A)})C\}A = I$

$\therefore\ A^{-1} = (\frac{1}{\det(A)})C$

11.

Let $A \in M_{2\times 2}(F)$

(1) If $A$ has *rank* less than 2, then by the assumption (ii) $\delta(A) = 0$

In this case $\det(A)$ equals to zero

$\therefore\ \delta(A) = \det(A)$

(2) If $A$ has *rank* 2, then $A$ is invertible

hence $A = E_k E_{k-1} \cdots E_1$ for some $k$

Since $\delta(I) = 1$,$\delta(E) = \det(E),\ \forall E$: an elementary matrix

Hence we have $\delta(A) = \delta(E_k E_{k-1} \cdots E_1)$

$= \delta(E_k)\delta(E_{k-1}) \cdots \delta(E_1)$

$= \det(E_k) \det(E_{k-1}) \cdots \det(E_1)$

$= \det(E_k E_{k-1} \cdots E_1)$

$= \det(A)$

12.

($\Leftarrow$) Let $u = (a_1, a_2)$, then $v = (a_1 \cos\theta - a_2 \sin\theta,\ a_1 \sin\theta + a_2 \cos\theta),\ 0 < \theta < \pi$

Since $\det \begin{pmatrix} u \\ v \end{pmatrix} = (a_1^2 + a_2^2) \sin\theta$ and $\det \begin{pmatrix} u \\ v \end{pmatrix} > 0$

$\therefore\ O \begin{pmatrix} u \\ v \end{pmatrix} = 1$ ($\Rightarrow$) Since $O \begin{pmatrix} u \\ v \end{pmatrix} = 1,\ \sin\theta\ > 0$

$\therefore\ 0 < \theta < \pi$

$\therefore\ \{u, v\}$ forms a right-handed coordinate system

## 4.2. Determinants of Order n

1.

(a) F

(b) T (Theorem 4.4)

(c) T (Corollary to Theorem 4.4)

(d) T (Theorem 4.5)

(e) F ($\det(B) = k\det(A)$)

(f) F ($\det(B) = \det(A)$)

(g) F (If $A \in M_{n\times n}(F)$ and $rankA = n \;\Rightarrow\; A$ : invertible$\Rightarrow\;\det A \neq 0$)

(h) T

2. $k = 3^3$

3. $k = 42$

4. $k = 2$

5. $\det(A) = -12$

6. $\det(A) = -13$

7. $\det(A) = -12$

8. $\det(A) = -13$

9. $\det(A) = 22$

10. $\det(A) = 4 + 2i$

11. $\det(A) = -3$

12. $\det(A) = 154$

13. $\det(A) = -8$

14. $\det(A) = -168$

15. $\det(A) = 0$

16. $\det(A) = 36$

17. $\det(A) = -49$

18. $\det(A) = 10$

19. $\det(A) = -28 - i$

20. $\det(A) = 17 - 3i$

21. $\det(A) = 95$

22. $\det(A) = 100$

23.
Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$
By expanding along the first column, we have
$\det A = a_{11} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ & a_{33} & \cdots & a_{3n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$
$= a_{11}a_{22} \det \begin{pmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ & a_{44} & \cdots & a_{4n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$
$\vdots$
$= a_{11}a_{22} \cdots a_{nn}$

24.

If $A$ has a row consisting entirely of zeros, then $\det(A) = 0$

Let $i-th$ row of $A$ is the zero row

Since $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}A_{ij}\det(\tilde{A_{ij}})$ and $\forall\ A_{ij} = 0,\ (j = 1, 2, \cdots, n)$

$\therefore\ \det(A) = 0$

25.

Let $A = (a_1, a_2, \cdots, a_n),\ a_i$ : rows of $A,\ i = 1, 2, \cdots, n$

If $A'$ is obtained by multiplying a row of $A$ by a nonzero scalar $k$,

then $\det(A') = k\det(A)$

$\det(kA) = \det(ka_1, ka_2, \cdots, ka_n)$

$= k\det(a_1, ka_2, \cdots, ka_n)$

$= k^2\det(a_1, a_2, ka_3, \cdots, a_n)$

$\vdots$

$= k^n\det(a_1, a_2, \cdots, a_n)$

$= k^n\det(A)$

26.

Since $\det(-A) = \det(-I_nA) = \det(-I_n)\det(A) = (-1)^n\det(A)$

$\therefore\ \det(-A) = \det(A),\ n = 2k,\ k \in N$

27.

Let $A = (a_1, \cdots, a_i, \cdots, a_j, \cdots, a_n)$ and $B = (a_1, \cdots, a_j, \cdots, a_i, \cdots, a_n)$

($a_i$ : columns of $A$ and $a_i = a_j$)

By the row-interchanging property, we have $\det(B) = -\det(A)$

Since $a_i = a_j$, $\det(B) = \det(A)$

$\therefore\ \det(A) = 0$

28.

(i) $E_1$ is of type 1

$\det(E_1) = -\det(I_n) = -1$

(ii) $E_2$ is of type 2

$\det(E_2) = k\det(I_n) = k$

(iii) $E_3$ is of type 1

$\det(E_3) = \det(I_n) = 1$

29.

(i) $E_1$ is of type 1

$\det(E_1) = \det(E_1^t) = -1$

(ii) $E_2$ is of type 2

$\det(E_2) = \det(E_2^t) = k$

(iii) $E_3$ is of type 1

$\det(E_3) = \det(E_3^t) = 1$

30.

$\det(B) = \det(a_n, a_{n-1}, \cdots, a_2, a_1)$

$= (-1)\det(a_1, a_{n-1}, \cdots, a_2, a_n)$

$= (-1)^2 \det(a_1, a_2, a_{n-2}, \cdots, a_3, a_{n-1}, a_n)$

$\vdots$

$= (-1)^{[n/2]} \det(a_1, a_2, \cdots, a_n)$

$= (-1)^{[n/2]} \det(A)$

## 4.3. Properties of Determinants

1.

(a) F (p.223)

(b) T (Theorem 4.7 p. 223)

(c) F

(d) T

(e) F (Theorem 4.8 p. 224)

(f) T

(g) F

(h) F

2.

(a) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Since $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, Cramer's rule applies

Using the notation of theorem 4.9, we have

$x_1 = \det(M_1)/\det(A) = (b_1a_{22} - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21})$

$x_2 = \det(M_2)/\det(A) = (b_2a_{11} - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21})$

$\therefore\ x = (x_1, x_2)$

$= ((b_1a_{22} - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}),\ (b_2a_{11} - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21}))$

3.

$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 10 \\ 0 \end{pmatrix}$, $\det(A) = -25$

$\det(M_1) = \det \begin{pmatrix} 5 & 1 & -3 \\ 10 & -2 & 1 \\ 0 & 4 & -2 \end{pmatrix}$, $\det(M_2) = \det \begin{pmatrix} 2 & 5 & -3 \\ 1 & 10 & 1 \\ 3 & 0 & -2 \end{pmatrix}$, $\det(M_3) = \det \begin{pmatrix} 2 & 1 & 5 \\ 1 & -2 & 10 \\ 3 & 4 & 0 \end{pmatrix}$

$x_1 = \det(M_1)/\det(A) = -100/-25 = 4$

$x_2 = \det(M_2)/\det(A) = 75/-25 = -3$

$x_3 = \det(M_3)/\det(A) = 0/-25 = 0$

$\therefore\ x = (4, -3, 0)$

5. $(4, -3, 0)$

7. $(0, -12, 16)$

8.

Since $\det(A^t) = \det(A)$,

from the theorem 4.3, $\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}^t = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}^t + k \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}^t$

that is,

$\det(a_1, \cdots, a_{r-1}, u + kv, a_{r+1}, \cdots, a_n)$

$= \det(a_1, \cdots, a_{r-1}, u, a_{r+1}, \cdots, a_n) + k \det(a_1, \cdots, a_{r-1}, v, a_{r+1}, \cdots, a_n)$

$, u, v, a_i's$ : column vectors in $F^n$

9.

($\Rightarrow$) SInce the determinant of an upper triangular matrix is the product of its diagonal entries

So $A$ is invertible, $\det(A) \neq 0$

thus its diagonal entries are nonzero

($\Leftarrow$) By hypothesis, $\det(A) \neq 0$

By the corollary to the theorem 4.7, $A$ is invertible

10.

$\det(M^k) = \det(0) = 0$

$\det(M^k) = \det(M \cdots M) = \det(M)^k = 0$

$\therefore \ \det(M) = 0$

11.

(i) $n$ is odd

$\det(M^t) = (-1)^n \det(M) = -\det(M)$

Since $\det(M^t) = \det(M), \ \det(M) = -\det(M)$

$\therefore \ \det(M) = 0$

(ii) $n$ is even

$\det(M^t) = (-1)^n \det(M) = \det(M)$

12.

$\det(QQ^t) = \det(Q)\det(Q^t) = 1$

Since $\det(Q^t) = \det(Q), \ \det(Q) = \pm 1$

13.

(a) $\det(\overline{M}) = \overline{\det(M)}$

Let $M = A + iB$, then

$\det(\overline{M}) = \det(A - iB) = \det(A) - i\det(B) = \overline{\det(M)}$

(b)

Since $Q$ is unitary, $\det(QQ^*) = \det(I) = 1$

$1 = \det(Q)\det(Q^*)$

$= \det(Q)\det(Q*)$

$= \det(Q)\det(\overline{Q^t})$

$= \det(Q)\overline{\det(Q^t)}$

$= \det(Q)\overline{\det(Q)}$

$=\mid \det(Q) \mid$

14.

(⇐) Since $\det(B) \neq 0,\ rank(B) = n$

Let $B'$ be the reduced row echelon form of $B$,

then $rank(B') = n$

Moreover $B' = I_n$

By the theorem 3.16, $b_i$'s consist of a basis for $F^n,\ i = 1, \cdots, n$

(⇒) Since $\beta$ is a basis, $B = I_n$

$\therefore\ \det B = 1 \neq 0$

15.

If $A,\ B$ are similar, then

$\exists\ Q \in M_{n\times n}(F)$ : invertible s.t. $A = Q^{-1}BQ$

$\therefore\ \det(A) = \det(Q^{-1}BQ)$

$= \det(Q^{-1})\det(B)\det(Q)$

$= \det(Q)^{-1}\det(B)\det(Q)$

$= \det(B)$

16.

Suppose that $\det(A)$ is not invertible

then $\det(A) = 0$

Since $1 = \det(AB) = \det(A)\det(B) = 0$

It's a contradiction

17.

Since $n$ is odd, $\det(-B) = (-1)^n \det(B) = -\det(B)$

Let $AB = -BA$

then $\det(A)\det(B) = \det(-B)\det(A) = -\det(B)\det(A)$

$\therefore\ 2\det(B) = 0$

Since $char(F) \neq 2,\ \det(B) = 0$

$\therefore\ B$ is not invertible

18.

(i) If $A$ is of type 2, then $\det(A) = k$

Since $AB$ is a matrix obtained by multiple of some row of $B$ by the nonzero scalar $k$,

$\det(AB) = k\det(B) = \det(A)\det(B)$

(ii) If $A$ is of type 3, then $\det(A) = 1$

Since $AB$ is a matrix obtained by adding a multiple of some row of $B$ to another row,

$\det(AB) = \det(B) = \det(A)\det(B)$

19.

Let $A = (a_{ij})$ be an $(2 \times 2)$ lower triangular matrix, then $\det(A) = a_{11}a_{22}$

Proceeding inductively, suppose that this assertion is true for any $(k \times k)$ lower triangular matrix, then

$$\det(A) = \begin{vmatrix} a_{11} & 0 & 0 & \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A', \text{ where } A' = \begin{vmatrix} a_{22} & 0 & 0 & \\ a_{32} & a_{33} & 0 & 0 \\ \vdots & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Since $\det(A') = a_{22}a_{33}\cdots a_{nn}$, so $\det(A) = a_{11}a_{22}\cdots a_{nn}$

20.

Reduce $C$ to upper triangular form with elementary column operations

21.

Reduce $C$ to upper triangular form with elementary row operations

then gain reduce $A$ to upper triangular form with elementary column operations

22.

(a) $[T]_\beta^\gamma = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^2 \\ 1 & c_1 & c_1^2 & \cdots & c_1^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^2 \end{pmatrix}$, $f = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$

(b) By the exercise 22 in section 2.4, $T$ is an isomorphism

so $M = [T]_\beta^\gamma$ is invertible

Thus $\det(M) \neq 0$

(c)

Proceed the following column operations ;

$-C_1 \times c_0 + C_2 \;\Rightarrow\; C_2$

$-\,C_2 \times c_0 + C_3 \;\Rightarrow\; C_3$

$\vdots$

$-C_n \times c_0 + C_{n+1} \Rightarrow C_{n+1}$

then we have

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (c_1 - c_0) & c_1(c_1 - c_0) & \cdots & c_1^{n-1}(c_1 - c_0) \\ 1 & (c_2 - c_0) & c_2(c_2 - c_0) & \cdots & c_2^{n-1}(c_2 - c_0) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (c_n - c_0) & c_n(c_n - c_0) & \cdots & c_n^{n-1}(c_n - c_0) \end{vmatrix}$$

$$= (c_1 - c_0)(c_2 - c_0)\cdots(c_n - c_0) \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{n-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^{n-1} \end{vmatrix}$$

$$= (c_1 - c_0)(c_2 - c_0)\cdots(c_n - c_0) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (c_2 - c_1) & c_2(c_2 - c_1) & \cdots & c_2^{n-2}(c_2 - c_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (c_n - c_1) & c_n(c_n - c_1) & \cdots & c_n^{n-2}(c_n - c_1) \end{vmatrix}$$

$$= (c_1 - c_0)\cdots(c_n - c_0)(c_2 - c_1)\cdots(c_n - c_1) \begin{vmatrix} 1 & c_2 & c_2^2 & \cdots & c_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^{n-2} \end{vmatrix}$$

$= \cdots$

$= \prod_{0 \le i < j \le n}(c_j - c_i)$

23.

a)

So $k$ is the largest number of linearly independent columns of $A$

thus $rankA(A) = k$

(b)

Since $rank(A) = k$, there exists $\{a_1, a_2, \cdots, a_k\}$ a linearly independent set of columns of $A$

So $A$ can be written as
$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & 0 & \cdots & 0 \end{pmatrix}$$
Since $dim(\text{row space } A) = dim(\text{column space of } A)$

Therefore
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} & \\ \vdots & & \vdots & O \\ a_{k1} & \cdots & a_{kk} & \\ & & & \\ & O & & O \\ & & & \end{pmatrix}$$

and $\det(A) \neq 0$

24.
$$A + tI = \begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t & a_{n+2} \\ 0 & 0 & 0 & \cdots & -1 & t + a_{n-1} \end{pmatrix}$$
Let $A$ be a $(2 \times 2)$ matrix, *i.e.* $n = 2$

by cofactor expansion along the first column,
$$\det(A + tI) = \begin{vmatrix} t & a_0 \\ -1 & t + a_1 \end{vmatrix} = t^2 + a_1 t + a_0$$
Assume that this assertion holds for $(n-1)$,

let $A \in M_{n\times n}(F)$

$$\det(A+tI) = t\begin{vmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & t & a_{n-2} \\ 0 & 0 & \cdots & -1 & t+a_{n-1} \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & t & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & t & 1 \\ a_0 & a_2 & a_3 & \cdots & a_{n-2} & t+a_{n-1} \end{vmatrix}$$

$= t(t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1) + a_0$

$= t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0$

25.

(a)

$\mid B \mid = \sum_{j=0}^{n} c_{jk}A_{jk}$

$c_{1k}A_{1k} + c_{2k}A_{2k} + \cdots + c_{jk} + \cdots + c_{nk}A_{nk}$

$= c_{1k}0 + c_{2k}0 + \cdots + c_{jk}1 + \cdots + c_{nk}0$

$= c_{jk}$

(b)

Apply $Cramer's rule$ to $Ax = e_j$

we have $\det(M_i) = a_{ji}$ and $x_i = \det(M_i)/\det(A)$ $i = 1, \cdots, n$

Since $A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = e_j$, so $A\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \det(A)e_j$

(c)

By (b),

$AC = A(c_1), c_2, \cdots, c_n)$

$= (Ac_1, Ac_2, \cdots, Ac_n)$

$(\det(A)e_1, \det(A)e_2, \cdots, \det(A)e_n)$

$\det(A)(e_1, e_2, \cdots, e_n)$

$\det(A)I$

(d)

Since $\det(B) = \det(r_1, \cdots, e_i, \cdots, r_n) = r_{ki}$

so $(r_{1i}, r_{2i}, \cdots, r_{ni})A = \det(A)e_i$

Let $CA = (r_1, r_2, \cdots, r_n)A$

$= (r_1A, r_2A, \cdots, r_nA)$

$= (\det(A)e_1, \det(A)e_2, \cdots, \det(A)en)$

$\det(A)I$

If $\det(A) \neq 0$,

by (b), $AC = \det(A)I$

by the above proof, $CA = \det(A)I$

$\therefore\ A^{-1} = \det(A)^{-1}C$

(d)

$A = ()_{n\times n},\ \det(adjA) = (\det A)^{n-1}$

$(proof)\ A = 0$ or $rankA = n\ o.k.$

$A \neq 0,\ rankA \neq n$

() AX=0

$= n - rankA\; eqn - 1$

$adjA \quad AX = 0$

$\Rightarrow\; adjA$

$\Rightarrow\;\; \det(adjA) = 0 = (\det A)^{n-1}$

In fact,

$(1) rankA = n \;\Rightarrow\; rank(adjA) = n$

$(2) rankA = n - 1 \;\Rightarrow\; rank(adjA) = 1$

$(3) rankA \leq n - 2 \;\Rightarrow\; rank(adjA) = 0\; i.e.\; adjA = (0) :$ zero matrix (?)

So $rankA - rank(adjA) \leq n - 2$

26.

(a)

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

(b)

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

(c)

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} 10 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -8 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 3 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 20 & -30 & 20 \\ 0 & 15 & -24 \\ 0 & 0 & 12 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1-i & 0 & 0 \\ 4 & 3i & 0 \\ 2i & 1+4i & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -3i & 0 & 0 \\ 4 & -1+i & 0 \\ 10+16i & -5-3i & 3+3i \end{pmatrix}$$

(f)

$$\begin{pmatrix} 6 & 22 & 12 \\ 12 & -2 & 24 \\ 21 & -39 & -27 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 18 & 28 & -6 \\ -20 & -21 & 37 \\ 48 & 14 & -16 \end{pmatrix}$$

(h)

$$\begin{pmatrix} -i & -8+i & -1+2i \\ 1-5i & 9-6i & -3i \\ -1+i & -6 & -3+i \end{pmatrix}$$

27.

$A \neq (0), \ \det(A) = 0$

$A = (a_{ij})_{m\times n}) = (A^1, \cdots, A^n)$

Since $\det(A) = 0, rank(A) < n$

$\exists\ k(1 \leq k \leq n), \ A^{(k)} = \sum_{j\neq k} b_j A^{(j)}, \ A^{(j)}$ is the $jth$ column of $A$

Without loss of generality, we may assume $k = n$

$(\widetilde{A}_{i(n-1)} = b_1 B_1 + \cdots + b_{n-2}B_{n-2} + b_{n-1}B_{n-1})$

$\widetilde{A}_{i(n-1)} = \sum_{j=1}^{n-2} b_j B_j + b_{n-1}\widetilde{A}_{in}$, where $B_j = ()_{(n-1)\times(n-1)}$ is the matrix obtained from $\widetilde{A}_{in}$ by replacing the last column with the $jth$ column $(1 \leq k \leq n-2)$

Actually $C_{i(n-1)} = (\det \widetilde{A}_{i(n-1)}) = \begin{pmatrix} c_{11} & \cdots & c_{1(n-1)} & c_{1n} \\ c_{11} & \cdots & c_{1(n-1)} & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{11} & \cdots & c_{1(n-1)} & c_{1n} \end{pmatrix}$

$= \begin{pmatrix} b_1c_{1n} & b_2c_{1n} & \cdots & b_{n-1}c_{1n} & c_{1n} \\ b_1c_{2n} & b_2c_{2n} & \cdots & b_{n-1}c_{2n} & c_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ b_1c_{nn} & b_2c_{nn} & \cdots & b_{n-1}c_{nn} & c_{nn} \end{pmatrix}$

$= (b_1c^n, \cdots, b_{n-1}c^n, c^n)$

$\therefore \;\; \det(adj(A)) = 0$

28.

(a) $T(y+z) = \det \begin{pmatrix} (y+z)(t) & y_1(t) & \cdots & y_n(t) \\ (y+z)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ (y+z)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$

$= \det \begin{pmatrix} (y)(t) & y_1(t) & \cdots & y_n(t) \\ (y)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ (y)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix} + \det \begin{pmatrix} (z)(t) & y_1(t) & \cdots & y_n(t) \\ (z)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ (z)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$

$T(ky)(t) = \det \begin{pmatrix} k(y)(t) & y_1(t) & \cdots & y_n(t) \\ k(y)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ k(y)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$

$= k \det \begin{pmatrix} (y)(t) & y_1(t) & \cdots & y_n(t) \\ (y)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ (y)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$

(b) Let $M(y) = \begin{pmatrix} (y)(t) & y_1(t) & \cdots & y_n(t) \\ (y)'(t) & y_1'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ (y)^{(n)}(t) & y_1^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$

$\Leftrightarrow\ T(y) = 0$

$\Leftrightarrow\ rank M(y) = m$

$\Leftrightarrow\ \begin{pmatrix} y_1 \\ y_1' \\ \vdots \\ y_1^{(n)} \end{pmatrix}\ \in\ span(\begin{pmatrix} y_2 \\ y_2' \\ \vdots \\ y_2^{(n)} \end{pmatrix},\ \cdots,\begin{pmatrix} y_n \\ y_n' \\ \vdots \\ y_n^{(n)} \end{pmatrix})$

$\Leftrightarrow\ y(t) \in span(y_1, y_2, \cdots, y_n)$

## 4.4. Summary-Important Facts about Determinants

1. (a) T

(b) T

(c) T

(d) F ($\det(B) = -\det(A)$)

(e) F ($\det(B) = k\det(A)$)

(f) T

(g) T

(h) F ($\det(A^t) = \det(A)$)

(i) T

(j) T

(k) T

2. (a) 22 (b) $-29$ (c) $2-4i$ (d) $-24+6i$

3. (a) $-12$ (b) $-13$ (c) $-12$ (d) $-13$ (e) 22 (f) $4+2i$ (g) $-2$ (h) 154

4. (a) 36 (b) $-100$ (c) $-49$ (d) $-10$ (e) $-28-i$ (f) $17-3i$ (g) 95

5 6. 20 and 21 in 4.3

## 4.5. A characterization of the Determinants

1. (a) F

(b) T

(c) T

(d) F ($\delta(B) = -\delta(A)$)

(e) F ($\delta(I) = 1$)

(f) T (p.238, 239)

2.

Determine all the $1 - linear$ functions $\delta : M_{1\times 1}(F) \to F$

Identity function

3. No

Let $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 + tv \end{pmatrix}$, $v = (b_1, b_2, b_3)$

$\delta(A) = k \neq k + tv = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\delta \begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

4. No

Let $A = \begin{pmatrix} a_1 \\ a_2 + kv \\ a_3 \end{pmatrix}$, $v = (b_1, b_2, b_3)$

$\delta(A) = a_2 \neq a_2 + ka_2 = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k\delta \begin{pmatrix} a_1 \\ v \\ a_3 \end{pmatrix}$

5. Yes

(i) $\delta \begin{pmatrix} a_1 + kv \\ a_2 \\ a_3 \end{pmatrix} = (A_{11} + kb_1)A_{23}A_{32} = A_{11}A_{23}A_{32} + k(b_1A_{23}A_{32}) = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k \begin{pmatrix} v \\ a_2 \\ a_3 \end{pmatrix}$

(ii) $\delta \begin{pmatrix} a_1 \\ a_2 + kv \\ a_3 \end{pmatrix} = A_{11}(A_{23} + kb_3)A_{32} = A_{11}A_{23}A_{32} + k(A_{11}b_3A_{32}) = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k \begin{pmatrix} a_1 \\ v \\ a_3 \end{pmatrix}$

(iii) $\delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11}A_{23}(A_{32} + kb_2) = A_{11}A_{23}A_{32} + k(A_{11}A_{23}b_2) = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k \begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

6. No

$\delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11} + A_{23} + (A_{32} + kb_2) \neq (A_{11} + A_{23} + A_{32}) + k(A_{11} + A_{23} + b_2) =$
$\delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k \begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

7. Yes

(i) $\delta \begin{pmatrix} a_1 + kv \\ a_2 \\ a_3 \end{pmatrix} = (A_{11} + kb_1)A_{21}A_{32} = A_{11}A_{21}A_{32} + k(b_1A_{21}A_{32}) = \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$

$k\begin{pmatrix} v \\ a_2 \\ a_3 \end{pmatrix}$

(ii) $\delta\begin{pmatrix} a_1 \\ a_2 + kv \\ a_3 \end{pmatrix} = A_{11}(A_{21} + kb_1)A_{32} = A_{11}A_{21}A_{32} + k(A_{11}b_1A_{32}) = \delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k\begin{pmatrix} a_1 \\ v \\ a_3 \end{pmatrix}$

(iii) $\delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11}A_{21}(A_{32} + kb_2) = A_{11}A_{21}A_{32} + k(A_{11}A_{21}b_2) = \delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k\begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

8. No
$\delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11}(A_{31}+kb_1)(A_{32}+kb_2) \neq A_{11}A_{31}A_{32}+k(A_{11}b_1b_2) = \delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k\begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

9. No
$\delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11}^2A_{22}^2(A_{33} + kb_3)^2 \neq A_{11}^2A_{22}^2A_{33}^2 + k(A_{11}^2A_{22}^2b_3^2) = \delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} +$
$k\begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

10. Yes

(i) $\delta \begin{pmatrix} a_1 + kv \\ a_2 \\ a_3 \end{pmatrix} = (A_{11} + kb_1)A_{22}A_{33} - (A_{11} + kb_1)A_{21}A_{32}$

$= A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32} + k(b_1A_{22}A_{33} - b_1A_{21}A_{32})$

$= \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k \begin{pmatrix} v \\ a_2 \\ a_3 \end{pmatrix}$

(ii) $\delta \begin{pmatrix} a_1 \\ a_2 + kv \\ a_3 \end{pmatrix} = A_{11}(A_{22} + kb_2)A_{33} - A_{11}(A_{21} + kb_1)A_{32}$

$= A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32} + k(A_{11}b_2A_{33} - A_{11}b_1A_{32})$

$= \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k \begin{pmatrix} a_1 \\ v \\ a_3 \end{pmatrix}$

(iii) $\delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 + kv \end{pmatrix} = A_{11}A_{22}(A_{33} + kb_3) - A_{11}A_{21}(A_{32} + kb_2)$

$= A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32} + k(A_{11}A_{22}b_3 - A_{11}A_{21}b_2)$

$= \delta \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k \begin{pmatrix} a_1 \\ a_2 \\ v \end{pmatrix}$

11.

(i) Corollary 2 to the theorem 4.10(p.241)

Let $M = (a_1, a_2, \cdots, a_n)^t$, $a_i's$ : rows of $M$

Since $rank(M) < n$,

some row of $M$, say $r$, is a linear combination of the other rows

That is, $\exists\ c_1, \cdots, c_{r-1}, c_{r+1}, \cdots, c_n \ \in F$ s.t.

$a_r = c_1a_1 + \cdots + c_{r-1}a_{r-1} + c_{r+1}a_{r+1} + \cdots + c_na_n$

If $M'$ is obtained from $M$ by adding $-c_i$ times row $i$ to row $r$ for each $i \neq r$, then row $r$ of $M'$ consists entirely of zeros, so $\delta(M') = 0$

But by the corollary 1 to the theorem 4.10, $\delta(M') = \delta(M)$

$\therefore\ \delta(M) = 0$

(i) Corollary 3 to the theorem 4.10

By the theorem 4.10 (a), $\delta(E_1) = -\delta(I)$

By the $n$-linearity, $\delta(E_2) = k\delta(I)$

By the corollary 1 to the theorem 4.10, $\delta(E_3) = \delta(I)$

12. Theorem 4.11

(i) $A = E_1 \ \Rightarrow\ \delta(E_1) = -1$

$\delta(AB) = -\delta(B) = \delta(A)\delta(B)$

(ii) $A = E_2 \ \Rightarrow\ \delta(E_2) = k$

$\delta(AB) = k\delta(B) = \delta(A)\delta(B)$

(ii) $A = E_3 \ \Rightarrow\ \delta(E_3) = 1$

$\delta(AB) = \delta(B) = \delta(A)\delta(B)$

13.

$\forall A \in M_{2\times 2}(F),\ A = (a_1, a_2),\ v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},\ a_i$ : columns of $A$

(i) $\det(a_1, a_2 + kv) = \det(a_1, a_2) + k\det(a_1, v)$

(ii) $\det(a_1 + kv, a_2) = \det(a_1, a_2) + k\det(v, a_2)$

14.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $a_i$ : rows of $A$, $v = (b_1, b_2)$

(i) $\delta \begin{pmatrix} a_1 + kv \\ a_2 \end{pmatrix} = \delta \begin{pmatrix} A_{11} + kb_1 & A_{12} + kb_2 \\ A_{21} & A_{22} \end{pmatrix} = (A_{11}+kb_1)A_{22}a+(A_{11}+kb_1)A_{21}b+(A_{12} + kb_2)A_{22}c + (A_{12} + kb_2)A_{21}d$

$\delta \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \delta \begin{pmatrix} v \\ a_2 \end{pmatrix} = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d + k(A_{22}b_1a + A_{21}b_1b + A_{22}b_2c + A_{21}b_1d)$

(ii) $\delta \begin{pmatrix} a_1 \\ a_2 + kv \end{pmatrix} = \delta \begin{pmatrix} A_{11} & A_{12} \\ A_{21} + kb_1 & A_{22} + kb_2 \end{pmatrix} = A_{11}(A_{22} + kb_2)a + A_{11}(A_{21} + kb_1)b + A_{12}(A_{22} + kb_2)c + A_{12}(A_{21} + kb_1)d$

$\delta \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ v \end{pmatrix} = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d + k(A_{11}b_2a + A_{11}b_1b + A_{12}b_2c + A_{12}b_1d)$

15.

($\Leftarrow$)Exercise 14

($\Rightarrow$) Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{2\times 2}(F)$

$\delta(A) = \delta \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} + \delta \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$

$= \delta \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} + \delta \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} + \delta \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}$

$= A_{11}A_{21}\delta \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + A_{12}A_{21}\delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + A_{11}A_{22}\delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A_{12}A_{22}\delta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Since $\delta$ is $2-linear$ function,

$\delta \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \delta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0,\ \delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1,\ \delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

Let $a = 1,\ b = 0,\ c = 0,\ d = -1$

then $\delta(A) = A_{11}A_{21} \cdot 0 + A_{12}A_{21} \cdot (-1) + A_{11}A_{22} \cdot 1 + A_{12}A_{22} \cdot 0$

16.

If $\delta(I) = t$, then

$\delta(E_1) = -t = t\det(E_1)$

$\delta(E_2) = kt = t\det(E_2)$

$\delta(E_3) = t = t\det(E_3)$

$\therefore\ \delta(E) = -t\det(E)$

17.

$$(a\delta_1 + b\delta_2)\begin{pmatrix} a_1 \\ \vdots \\ a_r + kv \\ \vdots \\ a_n \end{pmatrix} = (a\delta_1)\begin{pmatrix} a_1 \\ \vdots \\ a_r + kv \\ \vdots \\ a_n \end{pmatrix} + (b\delta_2)\begin{pmatrix} a_1 \\ \vdots \\ a_r + kv \\ \vdots \\ a_n \end{pmatrix}$$

$$= a(\delta_1\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + k\delta_1\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_n \end{pmatrix}) + b(\delta_2\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + k\delta_2\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_n \end{pmatrix})$$

$$= a\delta_1\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + b\delta_2\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + k(a\delta_1\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_n \end{pmatrix} + b\delta_2\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_n \end{pmatrix})$$

$$= (a\delta_1 + b\delta_2)\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + k(a\delta_1 + b\delta_2)\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_n \end{pmatrix}$$

18. $V$ : the set of all $m \times n$ matrices with entries from a field $F$ is a vector space, and $W \subseteq V$

(i) By the exercise 17,

$\forall\ \delta_1,\ \delta_2,\ \delta_1 + \delta_2 \in W$ and $c\delta_1 \in W$

(ii) By the example in p.238, $O \in W$

19.

Let $M = (a_1, a_2, \cdots, a_n)^t,\ a_i's$ : rows of $M$

Say $a_i$ and $a_j$ are identical rows in $M$

If $M'$ is obtained from $M$ by interchanging $a_i$ and $a_j$,

then $\delta(M') = -\delta(M)$

Since $a_i = a_j,\ \delta(M') = \delta(M)$

therefore $2\delta(M) = 0$

Since $char(F) \neq 2,\ \delta(M) = 0$

20.

In $\mathbb{Z}_2,\ 2\delta(M) = 0\ \not\Rightarrow\ \delta(M) = 0$