Friedberg, Insel, and Spence
Linear algebra, 4th ed.

SOLUTIONS REFERENCE

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Preface

The aim of this document is to serve as a reference of problems and solutions from the fourth edition of "Linear Algebra" by Friedberg, Insel and Spence. Originally, I had intended the document to be used only by a student who was well-acquainted with linear algebra. However, as the document evolved, I found myself including an increasing number of problems. Therefore, I believe the document should be quite comprehensive once it is complete.

I do these problems because I am interested in mathematics and consider this kind of thing to be fun. I give no guarantee that any of my solutions are the "best" way to approach the corresponding problems. If you find any errors (regardless of subtlety) in the document, or you have different or more elegant ways to approach something, then I urge you to contact me at the e-mail address supplied above.

This document was started on July 4, 2010. By the end of August, I expect to have covered up to the end of Chapter 5, which corresponds to the end of MATH 146, "Linear Algebra 1 (Advanced Level)" at the University of Waterloo. **This document is currently a work in progress.**

Contents

1 Vector Spaces

1.1 Introduction

Section 1.1 consists of an introductory, geometrically intuitive treatment of *vectors* (more specifically, Euclidean vectors). The solutions to the exercises from this section are very basic and as such have not been included in this document.

1.2 Vector Spaces

Section 1.2 introduces an algebraic structure known as a *vector space* over a *field*, which is then used to provide a more abstract notion of a *vector* (namely, as an element of such an algebraic structure). Matrices and n-tuples are also introduced. Some elementary theorems are stated and proved, such as the Cancellation Law for Vector Addition (Theorem 1.1), in addition to a few uniqueness results concerning additive identities and additive inverses.

8. In any vector space V, show that $(a+b)(x+y) = ax + ay + bx + by$ for any $x, y \in \mathsf{V}$ and any $a, b \in F$.

   Solution. Noting that $(a+b) \in F$, we have

   $$\begin{aligned}(a+b)(x+y) &= (a+b)x + (a+b)y && \text{(by VS 7)}\\ &= ax + bx + ay + by && \text{(by VS 8)}\\ &= ax + ay + bx + by && \text{(by VS 1)}\end{aligned}$$

   as required.

9. Prove Corollaries 1 and 2 [uniqueness of additive identities and additive inverses] of Theorem 1.1 and Theorem 1.2(c) [$a0 = 0$ for each $a \in F$].

   Solution. First, let $0, 0' \in \mathsf{V}$ be additive identities, and let $x \in \mathsf{V}$. We have, by VS 3,

   $$x = x + 0 = x + 0'$$

   Whereupon VS 1 yields

   $$0 + x = 0' + x$$

   By Theorem 1.1 [Cancellation Law for Vector Addition], we obtain $0 = 0'$, proving that the additive identity is unique.

   Second, let $x \in \mathsf{V}$ and $y, y' \in \mathsf{V}$ be such that $x + y = x + y' = 0$. Then we obtain, by VS 1,

   $$y + x = y' + x$$

   By Theorem 1.1 [Cancellation Law for Vector Addition], we obtain $y = y'$, proving that each additive inverse is unique.

   Third, let $a \in F$. We have, by VS 3 and VS 7,

   $$a0 = a(0+0) = a0 + a0$$

   By Theorem 1.1 [Cancellation Law for Vector Addition], we obtain $a0 = 0$, as required.

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

    Solution. Let $f, g, h \in \mathsf{V}$ and $c, d, s \in R$. Due to rules taught in elementary calculus, we have $f + g \in \mathsf{V}$ as well as $cf \in \mathsf{V}$ for all $f, g \in \mathsf{V}$ and $c \in R$. Now, note that by the commutativity of addition in R, we have $f + g = f(s) + g(s) = g(s) + f(s) = g + f$ for all s. Therefore, V satisfies VS 1. Next, by the associativity of addition in R, we have $(f + g) + h = (f(s) + g(s)) + h(s) = f(s) + (g(s) + h(s)) = f + (g + h)$ for all s. Therefore, V satisfies VS 2. Next, let $0_{\mathsf{V}}(s) = 0_F$ for all s, where 0_F is the additive identity of R. Then $f + 0_{\mathsf{V}} = f(s) + 0_F = f(s) = f$ for all s. Therefore V satisfies VS 3. Now, for each s we have $f(s) \in R$ and so we are guaranteed an additive inverse, say $-f(s)$, such that $f(s) + (-f(s)) = 0_F$. Now, $0_{\mathsf{V}}(s) = 0_F$ for all s also, so we note $f + (-f) = 0_{\mathsf{V}}$ for all f. Therefore V satisfies VS 4. Next, let 1_F represent R's multiplicative identity. Then we have $1_F \cdot f = 1_F \cdot f(s) = f(s) = f$ for all s. Therefore, $1_F \cdot f = f$ for all f. So VS 5 is satisfied. Note also that $(cd)f = (cd) \cdot f(s) = c \cdot (df(s)) = c(df)$ for all s. This is because multiplication in R is associative. So VS 6 is satisfied. We also have $c(f+g) = c\cdot(f(s)+g(s)) = c\cdot f(s)+c\cdot g(s) = cf+cg$. This is because multiplication in R is distributive (over addition). So VS 7 is satisfied. Finally, note $(c + d)f = (c + d) \cdot f(s) = c \cdot f(s) + d \cdot f(s) = cf + df$. This is again because multiplication in R is distributive (over addition). So VS 8 is satisfied. This completes the proof that V, together with addition and scalar multiplication as defined in Example 3, is a vector space.

22. How many matrices are there in the vector space $\mathsf{M}_{m\times n}(Z_2)$?

    Solution. There are 2^{mn} vectors in this vector space.

1.3 Subspaces

Section 1.3 examines subsets of vector spaces, and defines a special type of subset, known as a *subspace*, which is a subset of a vector space that can be considered a vector space in its own right (only certain subsets satisfy this property). A certain theorem that is referred to as the *subspace test* is stated and proven, which provides a fast way of checking whether a given subset is a subspace. More concepts relating to matrices are introduced as well. The section closes with a proof that the intersection of two subspaces is itself a subspace.

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathsf{M}_{m\times n}(F)$ and any $a, b \in F$.

   Solution. We have $(aA)_{ij} = a \cdot A_{ij}$, $(bB)_{ij} = b \cdot B_{ij}$, so $(aA + bB)_{ij} = a \cdot A_{ij} + b \cdot B_{ij}$. So

   $$[(aA + bB)^t]_{ij} = a \cdot A_{ji} + b \cdot B_{ji}$$

   Now, $(A^t)_{ij} = A_{ji}$, $(B^t)_{ij} = B_{ji}$, $(aA^t)_{ij} = a \cdot A_{ji}$, $(bB^t)_{ij} = b \cdot B_{ji}$. So

   $$[aA^t + bB^t]_{ij} = a \cdot A_{ji} + b \cdot B_{ji}$$

   Therefore $(aA + bB)^t = aA^t + bB^t$ as required.

4. Prove that $(A^t)^t = A$ for each $A \in \mathsf{M}_{m\times n}(F)$.

Solution. We have $(A^t)_{ij} = A_{ji}$. Thus $[(A^t)^t]_{ij} = (A^t)_{ji} = A_{ij}$, so that $(A^t)^t = A$ as required.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

   Solution. We have, from exercises 3 and 4:

$$\begin{aligned}(A + A^t) &= A^t + A \\ &= A^t + (A^t)^t \\ &= (A + A^t)^t\end{aligned}$$

6. Prove that $\operatorname{tr}(aA + bB) = a \operatorname{tr}(A) + b \operatorname{tr}(B)$.

   Solution. We have

$$\begin{aligned}\operatorname{tr}(aA + bB) &= \sum_{i=1}^{n} (aA + bB)_{ii} \\ &= \sum_{i=1}^{n} (aA_{ii} + bB_{ii}) \\ &= a \sum_{i=1}^{n} A_{ii} + b \sum_{i=1}^{n} B_{ii} \\ &= a \cdot \operatorname{tr}(A) + b \cdot \operatorname{tr}(B)\end{aligned}$$

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace of V if and only if $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$.

    Solution. To prove the first direction, assume $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace of V. Assume also that $\mathsf{W}_1 \not\subseteq \mathsf{W}_2$. Then we are guaranteed some $a \in \mathsf{W}_1$ such that $a \notin \mathsf{W}_2$. Choose an arbitrary $b \in \mathsf{W}_2$, and consider the sum $a + b$. Clearly $(a + b) \in \mathsf{W}_1 \cup \mathsf{W}_2$, and so $(a + b) \in \mathsf{W}_1$ or $(a + b) \in \mathsf{W}_2$. Assume that $(a + b) \in \mathsf{W}_1$. Then, since W_1 and W_2 are subspaces, we have $(a + b) - a = b \in \mathsf{W}_1$. The other possibility is that $(a + b) \in \mathsf{W}_2$, in which case we obtain $(a + b) - b = a \in \mathsf{W}_2$. However, this is a contradiction, and so we conclude that $b \in \mathsf{W}_1$, implying $\mathsf{W}_2 \subseteq \mathsf{W}_1$.

    To prove the second direction, assume $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$. Then clearly the union $\mathsf{W}_1 \cup \mathsf{W}_2$ will be the larger of these two subspaces, and therefore $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace.

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1 w_1 + a_2 w_2 + \ldots + a_n w_n \in \mathsf{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

    Solution. For each i such that $1 \leq i \leq n$, we have $a_i w_i \in \mathsf{W}$ due to W's closure under scalar multiplication. We also know that $a_i w_i + a_{i+1} w_{i+1} \in \mathsf{W}$ for $1 \leq i \leq n - 1$, since W is closed under addition. An inductive argument can then be used to show that $a_1 w_1 + a_2 w_2 + \ldots + a_n w_n \in \mathsf{W}$.

Definition. If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.

Definition. A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$ and $\mathsf{W}_1 + \mathsf{W}_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

23. Let W_1 and W_2 be subspaces of a vector space V.

    (a) Prove that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V that contains both W_1 and W_2.

    Solution. We have that $\mathsf{W}_1 + \mathsf{W}_2 = \{x + y : x \in \mathsf{W}_1 \text{ and } y \in \mathsf{W}_2\}$. Since $\mathsf{W}_1, \mathsf{W}_2$ are subspaces, they both contain the additive identity 0 from V. Therefore, $0 + 0 = 0 \in (\mathsf{W}_1 + \mathsf{W}_2)$. Next, let $a, b \in (\mathsf{W}_1 + \mathsf{W}_2)$. Then we have $a = x_a + y_a$ and $b = x_b + y_b$ for $x_a, x_b \in \mathsf{W}_1$, $y_a, y_b \in \mathsf{W}_2$, and so $a + b = (x_a + y_a) + (x_b + y_b) = (x_a + x_b) + (y_a + y_b)$. Since $\mathsf{W}_1, \mathsf{W}_2$ are subspaces, we have $(x_a + x_b) \in \mathsf{W}_1$ and $(y_a + y_b) \in \mathsf{W}_2$, and therefore $a + b \in (W_1 + \mathsf{W}_2)$. Now let $c \in F$ and consider $ca = c(x_a + y_a) = (cx_a) + (cy_a)$. Since $\mathsf{W}_1, \mathsf{W}_2$ are subspaces, we again have $cx_a \in \mathsf{W}_1$ and $cy_a \in \mathsf{W}_2$, and therefore $ca \in (\mathsf{W}_1 + \mathsf{W}_2)$. We conclude that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace, by the subspace test. To show that $\mathsf{W}_1 \subseteq \mathsf{W}_1 + \mathsf{W}_2$, simply consider any element $u \in \mathsf{W}_1$. Since $0 \in \mathsf{W}_2$, we can write $u = u + 0 \in \mathsf{W}_1 + \mathsf{W}_2$ by the definition of $\mathsf{W}_1 + \mathsf{W}_2$, and so $\mathsf{W}_1 \subseteq \mathsf{W}_1 + \mathsf{W}_2$. To show that $\mathsf{W}_2 \subseteq \mathsf{W}_1 + \mathsf{W}_2$, note $0 \in \mathsf{W}_1$ also, and argue similarly.

    (b) Prove that any subspace that contains both W_1 and W_2 must also contain $\mathsf{W}_1 + \mathsf{W}_2$.

    Solution. Assume W is some subspace of V, and $\mathsf{W}_1 \subseteq \mathsf{W}$ and $\mathsf{W}_2 \subseteq \mathsf{W}$. Let $a \in (\mathsf{W}_1 + \mathsf{W}_2)$. Then $a = x + y$ for $x \in \mathsf{W}_1$ and $y \in \mathsf{W}_2$. However, because of the above, we obtain $x \in \mathsf{W}$ and $y \in \mathsf{W}$, and since W is a subspace and closed under addition, we obtain $a \in \mathsf{W}$. This shows that $\mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{W}$.

24. Show that F^n is the direct sum of the subspaces

$$\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n : a_n = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n : a_1 = a_2 = \ldots = a_{n-1} = 0\}.$$

Solution. First, let's examine $\mathsf{W}_1 \cap \mathsf{W}_2$. The definitions of W_1 and W_2 tell us that if $(a_1, a_2, \ldots, a_n) \in \mathsf{W}_1 \cap \mathsf{W}_2$, then $a_1 = a_2 = \ldots = a_n = 0$. This clearly only holds for the n-tuple $(0, 0, \ldots, 0)$. Next, let $x = (x_1, x_2, \ldots, x_n) \in \mathsf{F}^n$. Then we can clearly write $x = w_1 + w_2$ for some $w_1 \in \mathsf{W}_1$ and $w_2 \in \mathsf{W}_2$; more specifically, where $w_1 = (x_1, x_2, \ldots, x_{n-1}, 0)$ and $w_2 = (0, 0, \ldots, 0, x_n)$. So $\mathsf{F}^n \subseteq \mathsf{W}_1 + \mathsf{W}_2$. It is not hard to demonstrate that $\mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{F}^n$ to prove that $\mathsf{F}^n = \mathsf{W}_1 + \mathsf{W}_2$, proving that F^n is by definition the direct sum of W_1 and W_2.

31. Let W be a subspace of a vector space V over a field F. For any $v \in \mathsf{V}$ the set $\{v\} + \mathsf{W} = \{v + w : w \in \mathsf{W}\}$ is called the coset of W containing v. It is customary to denote this coset by $v + \mathsf{W}$ rather than $\{v\} + \mathsf{W}$.

    (a) Prove that $v + \mathsf{W}$ is a subspace of V if and only if $v \in \mathsf{W}$.

    Solution. To prove the first direction, assume $v + \mathsf{W}$ is a subspace of V. Clearly,

$v \in v + \mathsf{W}$ (since $0 \in \mathsf{W}$). Since $v + \mathsf{W}$ is a subspace of V, we also have $(1+1)v \in v + \mathsf{W}$. Assume, for the sake of contradiction, that $v \notin \mathsf{W}$. This implies $v + v = (1+1)v \notin v + \mathsf{W}$. This contradicts the fact that $v + \mathsf{W}$ is a subspace of V. Therefore $v \in \mathsf{W}$.

To prove the second direction, assume $v \in \mathsf{W}$. Since W is a vector space, $-v \in \mathsf{W}$ also. Therefore $v + (-v) = 0 \in v + \mathsf{W}$. Now, let $x, y \in v + \mathsf{W}$. Then $x = v + w_x$ for some $w_x \in \mathsf{W}$ and $y = v + w_y$ for some $w_y \in \mathsf{W}$. Consider $x + y = (v + w_x) + (v + w_y)$. This can be expressed as $x + y = v + (w_x + w_y + v)$. Since W is a vector space, it is closed under addition, and so $(w_x + w_y + v) \in \mathsf{W}$. The definition of $v + \mathsf{W}$ then yields that $(x+y) \in v + \mathsf{W}$, thereby proving $(v+\mathsf{W})$'s closure under addition. Now, let $c \in F$. Then $cx = c(v + w_x) = cv + cw_x$. By the closure of W under scalar multiplication, $cw_x \in \mathsf{W}$. We may write $cx = v + (cw_x + (c-1)v)$. We clearly have $cx \in v + \mathsf{W}$, thereby proving $(v + \mathsf{W})$'s closure under scalar multiplication. This proves $v + \mathsf{W}$ is a subspace of V by the subspace test.

(b) Prove that $v_1 + \mathsf{W} = v_2 + \mathsf{W}$ if and only if $v_1 - v_2 \in \mathsf{W}$.

Solution. To prove the first direction, assume $v_1 + \mathsf{W} = v_2 + \mathsf{W}$. Let $x \in v_1 + \mathsf{W}$. Then $x = v_1 + w_x$ for some $w_x \in \mathsf{W}$. Since $v_1 + \mathsf{W} = v_2 + \mathsf{W}$, there exists some $w_y \in \mathsf{W}$ such that $x = v_1 + w_x = v_2 + w_y$. Note that $v_1 - v_2 = w_y - w_x$. W is a subspace and so $(w_y - w_x) \in \mathsf{W}$, thereby proving that $(v_1 - v_2) \in \mathsf{W}$.

To prove the second direction, assume $v_1 - v_2 \in \mathsf{W}$. Let $x \in v_1 + \mathsf{W}$. Then $x = v_1 + w_x$ for some $w_x \in \mathsf{W}$. Let $y = v_2 + w_y$, where $w_y = (v_1 - v_2) + w_x$. Clearly $y = v_2 + w_y = v_2 + (v_1 - v_2) + w_x = v_1 + w_x = x$. So we have proved $x \in v_2 + \mathsf{W}$, thereby proving that $v_1 + \mathsf{W} \subseteq v_2 + \mathsf{W}$. A similar argument proves that $v_2 + \mathsf{W} \subseteq v_1 + \mathsf{W}$ (by using the fact that $v_2 - v_1 \in \mathsf{W}$ rather than $v_1 - v_2 \in \mathsf{W}$). Therefore $v_1 + \mathsf{W} = v_2 + \mathsf{W}$.

(c) Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + \mathsf{W} : v \in \mathsf{V}\}$ of all cosets of W as follows:

$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1 + v_2) + \mathsf{W}$$

for all $v_1, v_2 \in \mathsf{V}$ and

$$a(v + \mathsf{W}) = av + \mathsf{W}$$

for all $v \in \mathsf{V}$ and $a \in F$. Prove that the preceding operations are well defined; that is, show that if $v_1 + \mathsf{W} = v_1' + \mathsf{W}$ and $v_2 + \mathsf{W} = v_2' + \mathsf{W}$, then

$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1' + \mathsf{W}) + (v_2' + \mathsf{W})$$

and

$$a(v_1 + \mathsf{W}) = a(v_1' + \mathsf{W})$$

for all $a \in F$.

Solution. First, we prove the well-definedness of addition. Since $v_1 + \mathsf{W} = v_1' + \mathsf{W}$ and $v_2 + \mathsf{W} = v_2' + \mathsf{W}$, we have $v_1 - v_1' \in \mathsf{W}$, and $v_2 - v_2' \in \mathsf{W}$ by (c). Let $A = (v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1 + v_2) + \mathsf{W}$ and $B = (v_1' + \mathsf{W}) + (v_2' + \mathsf{W}) = (v_1' + v_2') + \mathsf{W}$. Let $a \in A$. Then $a = (v_1 + v_2) + w_a$ for some $w_a \in \mathsf{W}$. Let $w_b = w_a + (v_1 - v_1') + (v_2 - v_2')$. Since

W is a subspace and is closed under addition, we have $w_b \in \mathsf{W}$. Let $b = (v_1' + v_2') + w_b$. Now we have

$$\begin{aligned} b &= (v_1' + v_2') + w_b \\ &= (v_1' + v_2') + (w_a + (v_1 - v_1') + (v_2 - v_2')) \\ &= (v_1 + v_2) + (v_1' - v_1') + (v_2' - v_2') + w_a \\ &= (v_1 + v_2) + w_a \\ &= a \end{aligned}$$

Thus, $a = b$ and since $b \in B$, we have that $a \in B$ as well, proving $A \subseteq B$. A similar argument proves that $B \subseteq A$ (by using $v_1' - v_1 \in \mathsf{W}$ and $v_2' - v_2 \in \mathsf{W}$ rather than $v_1 - v_1' \in \mathsf{W}$ and $v_2 - v_2' \in \mathsf{W}$). Therefore $A = B$, and so addition is well-defined.

Second, we prove the well-definedness of scalar multiplication. Let $c \in F$. Since $v_1 + \mathsf{W} = v_1' + \mathsf{W}$, we have $v_1 - v_1' \in \mathsf{W}$ by (c). Let $A = c(v_1 + \mathsf{W}) = cv_1 + \mathsf{W}$ and $B = c(v_1' + \mathsf{W}) = cv_1' + \mathsf{W}$. Let $a \in A$. Then $a = cv_1 + w_a$ for some $w_a \in \mathsf{W}$. Since $v_1 - v_1' \in \mathsf{W}$, we have that $c(v_1 - v_1') = cv_1 - cv_1' \in \mathsf{W}$. Let $w_b = w_a + (cv_1 - cv_1')$. Since W is a subspace and is closed under addition, we have $w_b \in \mathsf{W}$. Let $b = cv_1' + w_b \in B$. Now we have

$$\begin{aligned} b &= cv_1' + w_b \\ &= cv_1' + (w_a + (cv_1 - cv_1')) \\ &= cv_1 + (cv_1' - cv_1') + w_a \\ &= cv_1 + w_a \\ &= a \end{aligned}$$

Thus, $a = b$ and since $b \in B$, we have that $a \in B$ as well, proving $A \subseteq B$. A similar argument proves that $B \subseteq A$ (by using $cv_1' - cv_1 \in \mathsf{W}$ rather than $cv_1 - cv_1' \in \mathsf{W}$). Therefore $A = B$, and so scalar multiplication is well-defined.

(d) Prove that S is a vector space with the operations defined in (c). This vector space is called the quotient space of V modulo W and is denoted by V/W.

Solution. To verify VS 1, let $x, y \in S$. Then $x = v_x + \mathsf{W}$, $y = v_y + \mathsf{W}$ for some $v_x, v_y \in \mathsf{V}$. We have $x + y = (v_x + \mathsf{W}) + (v_y + \mathsf{W}) = (v_x + v_y) + \mathsf{W}$, and $y + x = (v_y + \mathsf{W}) + (v_x + \mathsf{W}) = (v_y + v_x) + \mathsf{W}$. Since addition in V is commutative, however, we have $v_x + v_y = v_y + v_x$, thereby demonstrating that $x + y = y + x$.

To verify VS 2, let $x, y, z \in S$. Then $x = v_x + \mathsf{W}$, $y = v_y + \mathsf{W}$, $z = v_z + \mathsf{W}$ for some $v_x, v_y, v_z \in \mathsf{V}$. We have

$$\begin{aligned} (x + y) + z &= ((v_x + \mathsf{W}) + (v_y + \mathsf{W})) + (v_z + \mathsf{W}) \\ &= ((v_x + v_y) + \mathsf{W}) + (v_z + \mathsf{W}) \\ &= ((v_x + v_y) + v_z) + \mathsf{W} \\ &= (v_x + (v_y + v_z)) + \mathsf{W} \\ &= (v_x + \mathsf{W}) + ((v_y + \mathsf{W}) + (v_z + \mathsf{W})) \\ &= x + (y + z) \end{aligned}$$

This is due to the fact that addition in V is associative.

To verify VS 3, let $0' = (0 + \mathsf{W})$. Clearly $0' \in S$, since $0 \in \mathsf{V}$ (this must hold, since V is a vector space). Now, let $s \in S$. Then $s = v_s + \mathsf{W}$ for some $v_s \in \mathsf{V}$. We have

$$0' + s = (0 + \mathsf{W}) + (v_s + \mathsf{W}) = (0 + v_s) + \mathsf{W} = v_s + \mathsf{W} = s$$

as needed.

To verify VS 4, let $x \in S$. Then $x = v_x + \mathsf{W}$ for some $v_x \in \mathsf{V}$. Let $y = -v_x + \mathsf{W}$. Clearly, $-v_x \in \mathsf{V}$ since V is a vector space. Then $x + y = (v_x + \mathsf{W}) + (-v_x + \mathsf{W}) = (v_x - v_x) + \mathsf{W} = 0 + \mathsf{W}$, which is the zero vector of S described in VS 3.

To verify VS 5, let $s \in S$. Then $s = v_s + \mathsf{W}$ for some $v_s \in \mathsf{V}$. We have

$$\begin{aligned} 1s &= 1(v_s + \mathsf{W}) \\ &= (1v_s) + \mathsf{W} \\ &= v_s + \mathsf{W} \\ &= s \end{aligned}$$

This is due to the fact that since V is a vector space, $1v = v$ for all $v \in \mathsf{V}$.

To verify VS 6, let $a, b \in F$ and $s \in S$. Then $s = v_s + \mathsf{W}$ for some $v_s \in \mathsf{V}$. We have

$$\begin{aligned} (ab)s &= (ab)(v_s + \mathsf{W}) \\ &= (ab)v_s + \mathsf{W} \\ &= a(bv_s) + \mathsf{W} \\ &= a(bv_s + \mathsf{W}) \\ &= a(bs) \end{aligned}$$

This is due to the fact that as a vector space, V satisfies VS 6.

To verify VS 7, let $a \in F$ and $x, y \in S$. Then $x = v_x + \mathsf{W}$ and $y = v_y + \mathsf{W}$ for some $v_x, v_y \in \mathsf{V}$. We have

$$\begin{aligned} a(x + y) &= a[(v_x + \mathsf{W}) + (v_y + \mathsf{W})] \\ &= a[(v_x + v_y) + \mathsf{W}] \\ &= [a(v_x + v_y)] + \mathsf{W} \\ &= [av_x + av_y] + \mathsf{W} \\ &= (av_x + \mathsf{W}) + (av_y + \mathsf{W}) \\ &= a(v_x + \mathsf{W}) + a(v_y + \mathsf{W}) \\ &= ax + ay \end{aligned}$$

This is due to the fact that as a vector space, V satisfies VS 7.

To verify VS 8, let $a, b \in F$ and $s \in S$. Then $s = v_s + \mathsf{W}$ for some $v_s \in \mathsf{V}$. We have

$$\begin{aligned}(a+b)s &= (a+b)[v_s + \mathsf{W}] \\ &= [(a+b)v_s] + \mathsf{W} \\ &= [av_s + bv_s] + \mathsf{W} \\ &= (av_s + \mathsf{W}) + (bv_s + \mathsf{W}) \\ &= a(v_s + \mathsf{W}) + b(v_s + \mathsf{W}) \\ &= as + bs\end{aligned}$$

This is due to the fact that as a vector space, V satisfies VS 8.

Thus, $S = \mathsf{V}/\mathsf{W}$ is a vector space, since we have verified all the vector space axioms.

1.4 Linear Combinations and Systems of Linear Equations

Section 1.4 discusses the notion of *linear combination* and also provides a method for solving systems of linear equations which will later serve as the foundation for the row reduction of matrices. The *span* of a subset is introduced, in addition to the notion of a *generating subset*. Some geometric interpretations of the concepts are given.

2. Solve the following systems of linear equations by the method introduced in this section.

(a)

$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 \phantom{{}+5x_4} &= -2 \qquad (1.4.1) \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \qquad (1.4.2) \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \qquad (1.4.3) \end{aligned}$$

Solution. Begin by replacing equation (1.4.2) with (1.4.2) $- 3 \cdot$ (1.4.3) to obtain $4x_3 + 8x_4 = 16$. Divide equation (1.4.2) by 4 to obtain $x_3 + 2x_4 = 4$. Now replace equation (1.4.1) with (1.4.1) $- 2 \cdot$ (1.4.3) to obtain $x_3 + 2x_4 = 4$ also. Switch equations (1.4.3) and (1.4.1). We obtain

$$\begin{aligned} x_1 - x_2 - 2x_3 - x_4 &= -3 \\ x_3 + 2x_4 &= 4 \end{aligned}$$

Which gives us

$$\begin{aligned} x_3 &= 4 - 2x_4 \\ x_1 &= x_2 - 3x_4 + 5 \end{aligned}$$

Therefore, the solution set is the set

$$S = \{(x_2 - 3x_4 + 5, x_2, -2x_4 + 4, x_4) : x_2, x_4 \in R\}$$

Which can also be expressed, by letting $x_2 = s$ and $x_4 = t$, as

$$S = \{s(1, 1, 0, 0) + t(-3, 0, -2, 1) + (5, 0, 4, 0) : s, t \in R\}$$

simply by examining the coefficients of the different x_i, in addition to the constants, in the first representation of the set. This solution set represents a plane in R^3 because there are two parameters (s and t).

(b)

$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10 \quad (1.4.4) \\ x_1 - 2x_2 + x_3 &= 3 \quad (1.4.5) \\ 2x_1 - x_2 - 2x_3 &= 6 \quad (1.4.6) \end{aligned}$$

Solution. Begin by replacing equation (1.4.4) with (1.4.4) $- 3 \cdot$ (1.4.5) to obtain $-x_2 + x_3 = 1$. Next, replace equation (1.4.6) with (1.4.6) $- 2 \cdot$ (1.4.5) to obtain $3x_2 - 4x_3 = 0$. Now replace equation (1.4.6) with (1.4.6)$+3\cdot$(1.4.4) to obtain $-x_3 = 3$. Replace equation (1.4.6) with $-$(1.4.6) to obtain $x_3 = -3$. Replace equation (1.4.4) with (1.4.4) $-$ (1.4.6) to obtain $-x_2 = 4$. Replace equation (1.4.4) with $-$(1.4.4) to obtain $x_2 = -4$. Replace equation (1.4.5) with (1.4.5)$+2\cdot$(1.4.4) to obtain $x_1+x_3 = -5$. Replace equation (1.4.5) with (1.4.5) $-$ (1.4.6) to obtain $x_1 = -2$. Therefore, the solution set is the set

$$S = \{(-2, -4, -3)\}$$

which is a single point in R^3.

(c)

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \quad (1.4.7) \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \quad (1.4.8) \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \quad (1.4.9) \end{aligned}$$

Solution. Begin by replacing equation (1.4.8) with (1.4.8)$-$(1.4.7) to obtain $2x_2-2x_3-4x_4 = 1$. Next, replace equation (1.4.9) with (1.4.9)$- 2 \cdot$(1.4.7) obtain $-x_2+x_3+2x_4 = -2$. Now, replace equation (1.4.9) with (1.4.9) $+ \frac{1}{2} \cdot$ (1.4.8), to obtain $0 = -\frac{5}{2}$. This shows that the system is inconsistent and therefore our solution set is the empty set,

$$S = \emptyset$$

3. For each of the following lists of vectors in R^3, determine whether the first vector can be expressed as a linear combination of the other two.

(a) $(-2,0,3)$, $(1,3,0)$, $(2,4,-1)$

Solution. We need to verify that there exist $s, t \in R$ such that

$$(-2,0,3) = s(1,3,0) + t(2,4,-1) = (s+2t, 3s+4t, -t)$$

This yields the following system of equations:

$$\begin{aligned} s + 2t &= -2 && (1.4.10)\\ 3s + 4t &= 0 && (1.4.11)\\ -\ t &= 3 && (1.4.12) \end{aligned}$$

Applying the same procedure as previously, we obtain $s = 4$ and $t = -3$. The system therefore has solutions, and so we conclude $(-2,0,3)$ can be expressed as a linear combination of $(1,3,0)$ and $(2,4,-1)$.

4. For each list of polynomials in $\mathsf{P}_3(R)$, determine whether the first polynomial can be expressed as a linear combination of the other two.

(b) $4x^3 + 2x^2 - 6$, $x^3 - 2x^2 + 4x + 1$, $3x^3 - 6x^2 + x + 4$

Solution. We need to verify that there exist $s, t \in R$ such that

$$\begin{aligned} 4x^3 + 2x^2 - 6 &= s(x^3 - 2x^2 + 4x + 1) + t(3x^3 - 6x^2 + x + 4)\\ &= (s+3t)x^3 + (-2s-6t)x^2 + (4s+t)x + (s+4t) \end{aligned}$$

This yields the following system of equations:

$$\begin{aligned} s + 3t &= 4 && (1.4.13)\\ -2s - 6t &= 2 && (1.4.14)\\ 4s + t &= 0 && (1.4.15)\\ s + 4t &= -6 && (1.4.16) \end{aligned}$$

Applying the same procedure as previously, we obtain $-2s - 6t = -8$, which is inconsistent with (1.4.14). The system therefore has no solutions, and so we conclude that $4x^3 + 2x^2 - 6$ cannot be expressed as a linear combination of $x^3 - 2x^2 + 4x + 1$ and $3x^3 - 6x^2 + x + 4$.

10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Solution. Let A denote the set of all symmetric 2×2 matrices. First, let $a \in A$. Then we have, by the definition of a symmetric matrix,

$$a = \begin{pmatrix} x_{11} & n \\ n & x_{22} \end{pmatrix}$$

We see easily that

$$x_{11} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + n \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_{22} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & n \\ n & x_{22} \end{pmatrix} = a$$

and hence $a \in \text{span}(S)$ where we define $S = \{M_1, M_2, M_3\}$. This shows that $A \subseteq \text{span}(S)$. A similar argument demonstrates that $\text{span}(S) \subseteq A$, and this completes the proof.

11. Prove that $\text{span}(\{x\}) = \{ax : a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in R^3.

    Solution. Since $\{x\}$ contains only one vector, then for all $a \in \text{span}(\{x\})$ we can write, for some $c \in F$,

    $$a = cx$$

    which proves $\text{span}(\{x\}) \subseteq \{ax : a \in F\}$. Clearly, if $b \in \{ax : a \in F\}$ then b is a linear combination of the vectors in $\{x\}$, namely, a scalar multiple of x itself, so that $\{ax : a \in F\} \subseteq \text{span}(\{x\})$. This proves $\text{span}(\{x\}) = \{ax : a \in F\}$. In R^3 this result tells us that the span of any single vector will be a line passing through the origin.

12. Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(\mathsf{W}) = \mathsf{W}$.

    Solution. Assume $\mathsf{W} \subseteq \mathsf{V}$ such that $\text{span}(\mathsf{W}) = \mathsf{W}$. 0 is in the span of any subset of V due to the trivial representation. Now let $a, b \in \mathsf{W}$. Since $\text{span}(\mathsf{W}) = \mathsf{W}$, we have $a, b \in \text{span}(\mathsf{W})$, so we can write

    $$a + b = \left(\sum_{i=1}^{n} a_i v_i\right) + \left(\sum_{i=1}^{n} b_i v_i\right) = \sum_{i=1}^{n} (a_i + b_i) v_i$$

    for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ and $v_1, \ldots, v_n \in \mathsf{W}$, whereby we clearly see $a+b \in \text{span}(\mathsf{W})$, and hence $a + b \in \mathsf{W}$. This proves that W is closed under addition. Now consider, for some $c \in F$,

    $$ca = c\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} (ca_i) v_i$$

    which demonstrates that $ca \in \text{span}(\mathsf{W})$ and hence $ca \in \mathsf{W}$. This proves that W is closed under scalar multiplication. By the subspace test, we have that W is a subspace.

    Assume next that $\mathsf{W} \subseteq \mathsf{V}$ with W a subspace of V. Since W is a subspace of V, clearly $0 \in \mathsf{W}$, and W is closed under addition and scalar multiplication (in other words, linear combination). Any vector $a \in \mathsf{W}$ can be trivially represented as a linear combination of vectors in W (viz., itself multiplied by the scalar 1 from F) and so we have $a \in \text{span}(\mathsf{W})$, proving that $\mathsf{W} \subseteq \text{span}(\mathsf{W})$. Now, let $a \in \text{span}(\mathsf{W})$. Then, by the definition of $\text{span}(\mathsf{W})$, we see that a is a linear combination of vectors in W. By noting first that W is closed under scalar multiplication and then noting its closure under addition, we clearly have $a \in \mathsf{W}$, proving $\text{span}(\mathsf{W}) \subseteq \mathsf{W}$, and hence $\text{span}(\mathsf{W}) = \mathsf{W}$.

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = \mathsf{V}$, deduce that $\text{span}(S_2) = \mathsf{V}$.

Solution. Assume $a \in \text{span}(S_1)$. Then for $a_1, a_2, \ldots, a_n \in F$ and $v_1, v_2, \ldots, v_n \in S_1$ we can write

$$a = \sum_{i=1}^{n} a_i v_i$$

However, we have $S_1 \subseteq S_2$, which implies that for each i, we have $v_i \in S_2$ also. By the definition of the linear combination, then, we obtain that $a \in \text{span}(S_2)$, demonstrating that $\text{span}(S_1) \subseteq \text{span}(S_2)$. Now assume $S_1 \subseteq S_2$ and $\text{span}(S_1) = \mathsf{V}$. We obtain (from the proposition we just proved) that $\text{span}(S_1) = \mathsf{V} \subseteq \text{span}(S_2)$. However, each element in S_2 is also in V. Since V is closed under vector addition and scalar multiplication, surely every linear combination of vectors in S_2 must be in V. Therefore, $\text{span}(S_2) \subseteq \mathsf{V}$, which, since $\mathsf{V} \subseteq \text{span}(S_2)$ also, shows that $\text{span}(S_2) = \mathsf{V}$ in this case.

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sum of two subsets is defined in the exercises of Section 1.3.)

Solution. We will define

$$\begin{aligned} A &= \text{span}(S_1) + \text{span}(S_2) = \{(a+b) : a \in \text{span}(S_1), b \in \text{span}(S_2)\} \\ B &= \text{span}(S_1 \cup S_2) \end{aligned}$$

Assume $a \in A$. Then we have $a = u + v$ for some $u \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$. Therefore, we can write, for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in F$ and $x_1, x_2, \ldots, x_m \in S_1$ and $y_1, y_2, \ldots, y_n \in S_2$:

$$a = \left(\sum_{i=1}^{m} a_i x_i\right) + \left(\sum_{i=1}^{n} b_i y_i\right)$$

Since, for each i, we have $x_i \in S_1$ and hence $x_i \in (S_1 \cup S_2)$, and $y_i \in S_2$ and hence $y_i \in (S_1 \cup S_2)$, we clearly see that $u, v \in \text{span}(S_1 \cup S_2)$. Since by Theorem 1.5, the span of any subset of a vector space V is a subspace of V, it is closed under vector addition, implying $(u+v) = a \in B$, we have obtained $A \subseteq B$.

Now, assume $b \in B$. Then we can write, for some $a_1, a_2, \ldots, a_n \in F$ and $z_1, z_2, \ldots, z_n \in (S_1 \cup S_2)$, the following:

$$b = \sum_{i=1}^{n} a_i z_i$$

Therefore, for each i, we see that z_i satisfies either $z_i \in S_1$ or $z_i \in S_2$ (or both), for otherwise we have $z_i \notin (S_1 \cup S_2)$. Now, we partition the set of summands as follows:

$$\begin{aligned} P &= \{a_i z_i : z_i \in S_1, 0 \le i \le n\} \\ Q &= \{a_i z_i : z_i \in S_2, z_i \notin S_1, 0 \le i \le n\} \end{aligned}$$

It can easily be proved that $P \cup Q$ yields all the summands in the expression for b and that $P \cap Q = \emptyset$. Then we can write the decomposition

$$b = \sum_{i=1}^{n} a_i z_i = \sum_{x \in P} x + \sum_{x \in Q} x$$

We clearly have that $(\sum_{x \in P} x) \in \text{span}(S_1)$ and $(\sum_{x \in Q} x) \in \text{span}(S_2)$ which proves that $b \in A$, whereupon we obtain $B \subseteq A$. We have therefore proved that $A = B$, as required.

15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

    Solution. We will define

    $$A = \text{span}(S_1 \cap S_2)$$
    $$B = \text{span}(S_1) \cap \text{span}(S_2)$$

    Assume $a \in A$. Then we can write, for some $a_1, a_2, \ldots, a_n \in F$ and $v_1, v_2, \ldots, v_n$,

    $$a = \sum_{i=1}^{n} a_i v_i$$

    such that for each i, v_i satisfies both $v_i \in S_1$ and $v_i \in S_2$. Because of this, we see immediately that $a \in \text{span}(S_1)$ and $a \in \text{span}(S_2)$, proving that $a \in (\text{span}(S_1) \cap \text{span}(S_2))$. Therefore, $A \subseteq B$, as required.

    Now, let $\mathsf{V} = \mathsf{R}^2$ and $S_1 = \{(-1, 1)\}$, $S_2 = \{(1, 1)\}$. Then $S_1 \cap S_2 = \emptyset$, so that $A = \text{span}(S_1 \cap S_2) = \{0\}$ and $B = \text{span}(S_1) \cap \text{span}(S_2) = \{0\}$. This is a case in which $A = B$.

    Next, let $\mathsf{V} = \mathsf{R}^2$ and $S_1 = \{(2, 0)\}$ and $S_2 = \{(1, 0)\}$. Then $S_1 \cap S_2 = \emptyset$, yielding $A = \text{span}(S_1 \cap S_2) = \text{span}\,\emptyset = \{0\}$. Now,

    $$\text{span}(S_1) = \{k(2, 0) : k \in R\}$$
    $$\text{span}(S_2) = \{k(1, 0) : k \in R\}$$

    It is not hard to show that $\text{span}(S_1) = \text{span}(S_2)$, so we conclude $B = \text{span}(S_1) \cap \text{span}(S_2) = \text{span}(S_1)$. Clearly, $\text{span}(S_1) \neq \emptyset$, so this is a case in which $A \subsetneq B$.

16. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be uniquely written as a linear combination of vectors of S.

    Solution. Assume $v \in \text{span}(S)$, and assume also that we can represent v in two ways, like so, for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in F$ and $v_1, v_2, \ldots, v_n \in S$:

    $$v = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$$

    Then we clearly obtain

    $$0 = v - v = \left(\sum_{i=1}^{n} a_i v_i\right) - \left(\sum_{i=1}^{n} b_i v_i\right) = \sum_{i=1}^{n} (a_i - b_i) v_i$$

    At this point, we use the given property of S, and this gives us that for each i, we have

    $$a_i - b_i = 0$$

which implies $a_i = b_i$. This clearly proves that our two representations of v cannot be distinct, as required.

17. Let W be a subspace of a vector space V. Under what conditions are there only a finite number of distinct subsets S of W such that S generates W?

    Solution. I believe the following conditions would lead to there only being a finite number of distinct generating sets for W:

    - The case where $\mathsf{W} = \{0\}$, in which case there would only be the set $\{0\}$ and $\emptyset$ that generate W.
    - The case where F is a Galois field such as Z_2.

1.5 Linear Dependence and Linear Independence

Section 1.5 discusses the notion of *linear dependence*, first through the question "Can one of the vectors in this set be represented as a linear combination of the others?" and then reformulates this notion in terms of being able to represent the zero vector as a nontrivial linear combination of vectors in the set. Afterwards, some proofs concerning linear dependence and linear independence are given.

2. Determine whether the following sets are linearly dependent or linearly independent.

   (a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $\mathsf{M}_{2\times 2}(R)$

   Solution. We consider the equation

   $$a_1 \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} a_1 - 2a_2 & -3a_1 + 6a_2 \\ -2a_1 + 4a_2 & 4a_1 - 8a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   This yields the following system of equations:

   $$a_1 - 2a_2 = 0 \tag{1.5.1}$$
   $$-3a_1 + 6a_2 = 0 \tag{1.5.2}$$
   $$-2a_1 + 4a_2 = 0 \tag{1.5.3}$$
   $$4a_1 - 8a_2 = 0 \tag{1.5.4}$$

   Solving the system with the usual method, we obtain $a_1 = 2a_2$. Therefore there are infinitely many nontrivial solutions and so the set is linearly dependent in $\mathsf{M}_{2\times 2}(R)$.

   (b) $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\}$ in $\mathsf{M}_{2\times 2}(R)$

   Solution. We consider the equation

   $$a_1 \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & -2a_1 + a_2 \\ -a_1 + 2a_2 & 4a_1 - 4a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This yields the following system of equations:

$$
\begin{aligned}
a_1 - a_2 &= 0 &\quad (1.5.5)\\
-2a_1 + a_2 &= 0 &\quad (1.5.6)\\
-a_1 + 2a_2 &= 0 &\quad (1.5.7)\\
4a_1 - 4a_2 &= 0 &\quad (1.5.8)
\end{aligned}
$$

Solving the system with the usual method, we obtain $a_1 = 0$ and $a_2 = 0$ (which proves that only trivial solutions exist) and so the set is linearly independent in $\mathsf{M}_{2\times 2}(R)$.

4. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

   Solution. Upon writing out the system of equations, we see that the system immediately yields $a_i = 0$ for all i satisfying $1 \le i \le n$. Therefore, $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $\mathsf{M}_{2\times 2}(F)$ is a subspace. Find a linearly independent subset that generates this subspace.

   Solution. Since all diagonal matrices are in the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   We can simply use the set:

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

8. Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space F^3.

   (a) Prove that if $F = R$, then S is linearly independent.

   Solution. Assume $F = R$. Then we must prove there exist no nontrivial solutions $a_1, a_2, a_3 \in R$ to

$$a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) = (0,0,0)$$

   The equation above yields the following system:

$$
\begin{aligned}
a_1 + a_2 &= 0 &\quad (1.5.9)\\
a_1 + a_3 &= 0 &\quad (1.5.10)\\
a_2 + a_3 &= 0 &\quad (1.5.11)
\end{aligned}
$$

   Solving the system with the usual method, we obtain $a_1 = a_2 = a_3 = 0$.

9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

   Solution. First, assume without loss of generality that $v = cu$ for some $c \in F$. We assume $c \neq 0$, for otherwise $v = 0u = 0$ which implies that $\{u, v\}$ is linearly dependent. Also assume $u \neq 0$ for the same reason. Then we obtain $a_1 v + a_2 u = 0$, which becomes $a_1 cu + a_2 u =$

$(a_1 c + a_2)u = 0$ by VS 8. Then the equation will hold for any choice of $a_1, a_2 \in F$ such that $a_1 c + a_2 = 0$. This equation can clearly be satisfied by some choice of $a_1, a_2 \in F$, not both zero, because if we let $a_1 = 1$ and $a_2 = -c$ (these elements are guaranteed to be in F due to the field axioms), we obtain $a_1 c + a_2 = 1 \cdot c + (-c) = c + (-c) = 0$ by the field axioms. Therefore, $\{u, v\}$ is linearly dependent.

Now, assume $\{u, v\}$ is linearly dependent. Then there exist $a_1, a_2 \in F$, not all zero, such that $a_1 v + a_2 u = 0$. Since F is a field, division is possible, allowing us to write

$$v = -\frac{a_2}{a_1} u$$

which signifies v is a multiple of u, as required. Note that we have assumed above $a_1 \neq 0$, for otherwise we have $a_2 \neq 0$, which from the equation implies $u = 0$, and we know that 0 is (trivially) a multiple of any vector. This completes the proof.

11. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent subset of a vector space V over the field Z_2. How many vectors are there in $\text{span}(S)$? Justify your answer.

    Solution. The span of S will consist of every linear combination of vectors in S, or in some subset of S. The field Z_2 only admits two possible scalars, 0 and 1, therefore every subset of S can be regarded as a linear combination and vice versa. The power set of S (that is, the set of all subsets of S) will contain 2^n elements (by a theorem in set theory), since S contains n elements. There is therefore a bijection between the set of all subsets of S and the set of all linear combinations of vectors in S. For this reason, the cardinality of $\text{span}(S)$ must also be 2^n.

12. Prove Theorem 1.6 [supersets of a linearly dependent set are themselves linearly dependent] and its corollary [subsets of a linearly independent set are themselves linearly independent].

    Solution. Let V be a vector space and let $S_1 \subseteq S_2 \subseteq \mathsf{V}$. Assume S_1 is linearly dependent. We wish to show that S_2 is linearly dependent. Since S_1 is linearly dependent, there exist $a_1, a_2, \ldots, a_n \in F$, not all zero, and $v_1, v_2, \ldots, v_n \in S_1$ such that $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$. However, each v_i $(1 \leq i \leq n)$ is also in S_2 since $S_1 \subseteq S_2$. For this reason, S_2 is also linearly dependent.

    To prove the corollary, let V be a vector space and let $S_1 \subseteq S_2 \subseteq \mathsf{V}$. Assume S_2 is linearly independent. We wish to show that S_1 is linearly independent. To prove this, assume for the sake of contradiction that S_1 is linearly dependent. Then by the argument above, S_2 is linearly dependent. Since linear independence and linear dependence are mutually exclusive properties, this is a contradiction, and the corollary follows.

13. Let V be a vector space over a field of characteristic not equal to two.

    (a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

    Solution. First, assume $\{u, v\}$ is linearly independent. Then the equation $a_1 u + a_2 v = 0$, with $a_1, a_2 \in F$, holds only if $a_1 = a_2 = 0$. Assume, for the sake of contradiction, that $\{u + v, u - v\}$ is linearly dependent. Then there exist $b_1, b_2 \in F$, not all zero, such that $b_1(u + v) + b_2(u - v) = 0$. Note that this equation can be rewritten as

$(b_1+b_2)u+(b_1-b_2)v=0$. For the first case, assume $b_1+b_2=0$. Then $b_2=-b_1$ and so $b_1-b_2=b_1-(-b_1)=b_1+b_1\neq 0$, since the field's characteristic is not 2. For the second case, assume $b_1-b_2=0$. Then $b_1=b_2$ and so $b_1+b_2=b_1+b_1\neq 0$, since the field's characteristic is not 2. In either case we obtain two scalars $x, y\in F$, at least one of which is nonzero, that satisfy the equation $xu+yv=0$. This implies that $\{u,v\}$ is linearly dependent, which is a contradiction. Therefore, $\{u+v,u-v\}$ is linearly independent.

Now, assume $\{u+v,u-v\}$ is linearly independent. Then the equation $a_1(u+v)+a_2(u-v)=0$ is satisfied only when $a_1=a_2=0$. This equation can be rewritten as $(a_1+a_2)u+(a_1-a_2)v=0$. Assume, for the sake of contradiction, that $\{u,v\}$ is linearly dependent. Then there exist $b_1,b_2\in F$, not all zero, such that $b_1u+b_2v=0$. For the first case, assume $b_1\neq 0$. Then any scalars $a_1,a_2\in F$ such that $a_1+a_2=b_1$ will satisfy the equation $a_1(u+v)+a_2(u-v)=0$. Clearly these scalars a_1, a_2 are not both zero, since otherwise the equation $a_1+a_2=b_1$ could not hold. For the second case, assume $b_2\neq 0$. Then any scalars $a_1,a_2\in F$ such that $a_1-a_2=b_2$ will satisfy the equation $a_1(u+v)+a_2(u-v)=0$. Again, we see that at least one of these scalars is nonzero. In either case we obtain that $\{u+v,u-v\}$ is linearly dependent, which is a contradiction. Therefore, $\{u,v\}$ is linearly independent.

14. Prove that a set S is linearly dependent if and only if $S=\{0\}$ or if there exist distinct vectors $v,u_1,u_2,\ldots,u_n$ in S such that v is a linear combination of $u_1,u_2,\ldots,u_n$.

Solution. Assume that S is a linearly dependent subset of a vector space V over a field F. Then S is nonempty (for otherwise it cannot be linearly dependent) and contains vectors $v_1,\ldots,v_n$. By definition of linear dependence, the equation $a_1v_1+\ldots+a_nv_n=0$ is satisfied by some $a_1,\ldots,a_n\in F$, not all zero. If $n=1$, then the equation $a_1v_1=0$ for some nonzero $a_1\in F$ implies that $v_1=0$ and so $S=\{0\}$, so assume $n\geq 2$. Then there exists k satisfying $1\leq k\leq n$ such that $a_k\neq 0$. Then the equation $a_1v_1+\ldots+a_kv_k+\ldots+a_nv_n=0$ can be rearranged to yield $a_kv_k=-a_1v_1-\ldots-a_nv_n$, whereupon we divide both sides by a_k to obtain

$$v_k=-\left(\frac{a_1}{a_k}\right)v_1-\ldots-\left(\frac{a_n}{a_k}\right)v_n$$

which proves that v_k is a linear combination of some finite subset of vectors from S, as required.

Next, assume that either $S=\{0\}$ or there exist $v,u_1,u_2,\ldots,u_n\in S$ such that v is a linear combination of the vectors $u_1,\ldots,u_n$. If $S=\{0\}$, then S is clearly linearly dependent, since $k0=0$ for any $k\in F$ by Theorem 1.2(c) which was proved as exercise 9 of section 1.2. If there exist $v,u_1,\ldots,u_n\in S$ such that v is a linear combination of $u_1,\ldots,u_n$, then we obtain the equation

$$v=a_1u_1+\ldots+a_nu_n$$

and observing that the coefficient of v is 1, we rearrange the equation to obtain $v-a_1u_1-\ldots-a_nu_n=0$ which, being a nontrivial representation of 0 as a linear combination of the vectors in S, implies that S is linearly dependent as required.

15. Let $S=\{u_1,u_2,\ldots,u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1=0$ or $u_{k+1}\in\operatorname{span}(\{u_1,u_2,\ldots,u_k\})$ for some k $(1\leq k\leq n)$.

Solution. Assume S is linearly dependent. Then by exercise 14, either $S = \{0\}$ or there are $v, u_1, \ldots, u_n \in S$ such that v is a linear combination of S. If $S = \{0\}$ then $u_1 = 0$ as required. Otherwise, we can write $u_k = v$ so that $u_k = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ for some $a_1, \ldots, a_n \in F$. Either $k > n$ and the proof is complete, since the fact that we can write $u_k = a_1u_1 + a_2u_2 + \ldots + a_nu_n + 0u_{n+1} + \ldots + 0u_{k-1}$ clearly implies $u_k \in \operatorname{span}(\{u_1, \ldots, u_n, \ldots, u_{k-1}\})$ as required. Otherwise we can simply isolate u_n in the equation and use the same reasoning.

Now, assume either $S = \{0\}$ or there exist $u_{k+1} \in S$ such that $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k \leq n)$. If $S = \{0\}$, then S is linearly dependent by exercise 14. Otherwise, the fact that $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$ for some k satisfying $1 \leq k \leq n$ implies that there exists some vector $v \in S$ (namely, u_{k+1}) that is a linear combination of some other vectors in S (namely the set $\{u_1, u_2, \ldots, u_k\}$). Exercise 14 then implies that S is linearly dependent.

16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution. Assume S is linearly independent. For each finite subset $R \subseteq S$, R must be linearly independent by exercise 12, as required.

Next, assume that each finite subset $R \subseteq S$ is linearly independent. Assume, for the sake of contradiction, that S is linearly dependent. Then there exist vectors $v_1, \ldots, v_n \in S$, and scalars $a_1, \ldots, a_n \in F$, not all zero, such that $a_1v_1 + \ldots + a_nv_n = 0$. This implies that some finite subset $R \subseteq S$ is linearly dependent, which is contradictory to what we assumed. This completes the proof.

17. Let M be a square upper triangular matrix (as defined in Exercise 12 of Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

Solution. Clearly, $M_{i,j} = 0$ whenever $i > j$ and $M_{i,j} \neq 0$ whenever $i = j$. We wish to prove that the equation

$$a_1 \begin{pmatrix} M_{1,1} \\ \cdots \\ M_{n,1} \end{pmatrix} + a_2 \begin{pmatrix} M_{1,2} \\ \cdots \\ M_{n,2} \end{pmatrix} + \ldots + a_n \begin{pmatrix} M_{1,n} \\ \cdots \\ M_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}$$

only holds when $a_1 = \ldots = a_n = 0$. This, of course, yields the n equations:

$$\begin{aligned} a_1M_{1,1} + a_2M_{1,2} + \ldots + a_nM_{1,n} &= 0 \\ a_1M_{2,1} + a_2M_{2,2} + \ldots + a_nM_{2,n} &= 0 \\ &\vdots \\ a_1M_{n,1} + a_2M_{n,2} + \ldots + a_nM_{n,n} &= 0 \end{aligned}$$

We proceed by induction. Let $P(k)$ be the statement that $a_{n-k+1} = 0$ $(1 \leq k \leq n)$. Consider $P(1)$. We wish to show that $a_{n-1+1} = 0$, or in other words $a_n = 0$. Looking at the equation $n - 1 + 1$, or in other words equation n:

$$a_1M_{n,1} + a_2M_{n,2} + \ldots + a_nM_{n,n} = 0$$

we note that each $M_{n,i} = 0$ unless $i = n$, since this is the only case when $n > i$ does not hold (the matrix is upper triangular). This allows us to remove the first $n - 1$ summands from the equation, leaving us with

$$a_n M_{n,n} = 0$$

which immediately yields $a_n = 0$, since $M_{n,n} \neq 0$ (the entry lies on the diagonal, so it is nonzero). Therefore $P(1)$ holds. Now, assume $P(i)$ holds for all $1 \leq i \leq k$. We wish to show that $P(k+1)$ holds. In order to do this, we examine equation $n-(k+1)+1$, or in other words $n-k$:

$$a_1 M_{n-k,1} + a_2 M_{n-k,2} + \ldots + a_n M_{n-k,n} = 0$$

Since the matrix is upper triangular, $M_{n-k,i} = 0$ for all $n-k > i$. Then the equation becomes

$$a_{n-k} M_{n-k,n-k} + a_{n-k+1} M_{n-k,n-k+1} + a_{n-k+2} M_{n-k,n-k+2} + \ldots + a_n M_{n-k,n} = 0$$

We now use the induction hypothesis, which tells us that $a_{n-k+1} = a_{n-k+2} = \ldots = a_n = 0$. This causes the equation to reduce again, to

$$a_{n-k} M_{n-k,n-k} = 0$$

whereupon we note that $M_{n-k,n-k} \neq 0$ since it lies on the diagonal. This immediately yields that $a_{n-k} = 0$, proving that $P(k+1)$ is true. By strong induction, this proves that $a_1 = a_2 = \ldots = a_n = 0$ as required.

19. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $\mathsf{M}_{n\times n}(F)$, then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

Solution. Since $\{A_1, A_2, \ldots, A_k\}$ is linearly independent, the equation

$$a_1 A_1 + a_2 A_2 + \ldots + a_k A_k = 0$$

holds only for $a_1 = a_2 = \ldots = a_k = 0$. Taking the transpose of both sides and applying an earlier result we obtain

$$(a_1 A_1 + a_2 A_2 + \ldots + a_k A_k)^t = 0^t$$
$$a_1 A_1^t + a_2 A_2^t + \ldots + a_k A_k^t = 0$$

The linear independence of $\{A_1^t, A_2^t, \ldots, A_k^t\}$ follows.

1.6 Bases and Dimension

Section 1.6 presents, and gives examples of, the concept of a *basis*, that is, a linearly independent subset of a vector space that generates the vector space. Theorem 1.8 says that vectors in the vector space can be represented as unique linear combinations of vectors in a subset if and only if the subset is a basis. Theorem 1.9 says that every finite generating set for a vector space can be reduced to a basis for that vector space. Theorem 1.10 (the replacement theorem) gives a very important result: that a linearly independent subset in a vector space has no more elements in it than a set that generates that vector space, and that a subset of this generator can be unioned with the independent subset to form a generating set. Corollaries to this theorem are the fact that every basis in a finite-dimensional vector space contains the same number of elements, that any finite generating set for a vector space V contain at least $\dim(\mathsf{V})$ vectors, and that a generating set containing exactly $\dim(\mathsf{V})$ vectors is a basis. Additionally, any linearly independent subset of V with exactly $\dim(\mathsf{V})$ vectors is a basis, and every linearly independent subset of V can be extended to a basis for V.

2. Determine which of the following sets are bases for R^3.

   (a) $\{(1,0,-1),(2,5,1),(0,-4,3)\}$

   Solution. By a corollary to the replacement theorem, any linearly independent subset of 3 vectors generates R^3, and so is a basis for R^3. Therefore this problem is reduced simply to determining whether the set is linearly independent, which of course involves solving the usual homogeneous system of equations (that is, a system of equations where all the right-hand sides are equal to 0). This has been demonstrated previously in the document, and so will not be shown here.

4. Do the polynomials x^3-2x^2+1, $4x^2-x+3$, and $3x-2$ generate $\mathsf{P}_3(R)$? Justify your answer.

   Solution. No. This is a set of 3 polynomials, and the dimension of $\mathsf{P}_3(R)$ is 4. By a corollary to the replacement theorem, any set which generates a 4-dimensional vector space must contain at least 4 vectors.

5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of R^3? Justify your answer.

   Solution. No. This set contains 4 vectors, and R^3 is generated by three vectors: $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. If the set $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ was linearly independent, the replacement theorem would imply $4 \le 3$, a contradiction.

6. Give three different bases for F^2 and $\mathsf{M}_{2\times 2}(F)$.

   Solution. For F^2, we can use

$$\{(1,0),(0,1)\} \qquad \{(-1,1),(0,1)\} \qquad \{(1,-1),(1,0)\}$$

   For $\mathsf{M}_{2\times 2}(F)$, we can use

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \qquad \left\{\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \qquad \left\{\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}$$

7. The vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$, and $u_5=(-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for R^3.

   Solution. Start with the set $\{u_1\}$ and note that u_2 is not a multiple of u_1. Therefore $\{u_1,u_2\}$ will be linearly independent. Next, guess that u_5 is not a linear combination of the two vectors already in the set. To verify, solve the usual homogeneous system of linear equations to determine whether $\{u_1,u_2,u_5\}$ is linearly independent. As it turns out, it is, and therefore by a corollary to the replacement theorem, the set generates R^3, and so is a basis for R^3 also.

9. The vectors $u_1=(1,1,1,1)$, $u_2=(0,1,1,1)$, $u_3=(0,0,1,1)$, and $u_4=(0,0,0,1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1,a_2,a_3,a_4) in F^4 as a linear combination of u_1, u_2, u_3, and u_4.

Solution. We wish to find scalars $c_1, c_2, c_3, c_4 \in F$ such that

$$\begin{aligned} c_1 &= a_1 \\ c_1 + c_2 &= a_2 \\ c_1 + c_2 + c_3 &= a_3 \\ c_1 + c_2 + c_3 + c_4 &= a_4 \end{aligned}$$

We clearly obtain $(a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3)$ as our representation.

10. In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points.

(a) $(-2, -6)$, $(-1, 5)$, $(1, 3)$

Solution. Using the notation from the book, we set $c_0 = -2$, $c_1 = -1$, $c_2 = 1$, $b_0 = -6$, $b_1 = 5$, $b_2 = 3$. Formulating the Lagrange polynomials, we have:

$$\begin{aligned} f_0(x) &= \prod_{\substack{k=0 \\ k \neq 0}}^{2} \left(\frac{x - c_k}{c_0 - c_k} \right) = \left(\frac{x+1}{-1} \right) \left(\frac{x-1}{-3} \right) = \frac{1}{3}(x^2 - 1) \\ f_1(x) &= \prod_{\substack{k=0 \\ k \neq 1}}^{2} \left(\frac{x - c_k}{c_1 - c_k} \right) = \left(\frac{x+2}{1} \right) \left(\frac{x-1}{-2} \right) = -\frac{1}{2}(x^2 + x - 2) \\ f_2(x) &= \prod_{\substack{k=0 \\ k \neq 2}}^{2} \left(\frac{x - c_k}{c_2 - c_k} \right) = \left(\frac{x+2}{3} \right) \left(\frac{x+1}{2} \right) = \frac{1}{6}(x^2 + 3x + 2) \end{aligned}$$

We then use these polynomials in a linear combination, with the coefficient of each f_i equal to b_i:

$$\begin{aligned} g(x) &= \sum_{i=0}^{2} b_i f_i(x) \\ &= -6 \left[\frac{1}{3}(x^2 - 1) \right] + 5 \left[-\frac{1}{2}(x^2 + x - 2) \right] + 3 \left[\frac{1}{6}(x^2 + 3x + 2) \right] \\ &= \left(-2x^2 + 2 \right) + \left(-\frac{5}{2}x^2 - \frac{5}{2}x + 5 \right) + \left(\frac{1}{2}x^2 + \frac{3}{2}x + 1 \right) \\ &= -4x^2 - x + 8 \end{aligned}$$

(c) $(-2, 3)$, $(-1, -6)$, $(1, 0)$, $(3, -2)$

Solution. Using the notation from the book, we set $c_0 = -2$, $c_1 = -1$, $c_2 = 1$, $c_3 = 3$,

$b_0 = 3$, $b_1 = -6$, $b_2 = 0$, $b_3 = -2$. Formulating the Lagrange polynomials, we have:

$$f_0(x) = \prod_{\substack{k=0 \\ k\neq 0}}^{3} \left(\frac{x - c_k}{c_0 - c_k}\right) = \left(\frac{x+1}{-1}\right)\left(\frac{x-1}{-3}\right)\left(\frac{x-3}{-5}\right) = -\frac{1}{15}(x^3 - 3x^2 - x + 3)$$

$$f_1(x) = \prod_{\substack{k=0 \\ k\neq 1}}^{3} \left(\frac{x - c_k}{c_1 - c_k}\right) = \left(\frac{x+2}{1}\right)\left(\frac{x-1}{-2}\right)\left(\frac{x-3}{-4}\right) = \frac{1}{8}(x^3 - 2x^2 - 5x + 6)$$

$$f_2(x) = \prod_{\substack{k=0 \\ k\neq 2}}^{3} \left(\frac{x - c_k}{c_2 - c_k}\right) = \left(\frac{x+2}{3}\right)\left(\frac{x+1}{2}\right)\left(\frac{x-3}{-2}\right) = -\frac{1}{12}(x^3 - 7x + 6)$$

$$f_3(x) = \prod_{\substack{k=0 \\ k\neq 3}}^{3} \left(\frac{x - c_k}{c_3 - c_k}\right) = \left(\frac{x+2}{5}\right)\left(\frac{x+1}{4}\right)\left(\frac{x-1}{2}\right) = \frac{1}{40}(x^3 + 2x^2 - x - 2)$$

Note that there was no actual use in calculating $f_2(x)$ since $b_2 = 0$. We then use these polynomials in a linear combination, with the coefficient of each f_i equal to b_i:

$$\begin{aligned} g(x) &= \sum_{i=0}^{3} b_i f_i(x) \\ &= 3\left[-\frac{1}{15}(x^3 - 3x^2 - x + 3)\right] - 6\left[\frac{1}{8}(x^3 - 2x^2 - 5x + 6)\right] - 2\left[\frac{1}{40}(x^3 + 2x^2 - x - 2)\right] \\ &= -x^3 + 2x^2 + 4x - 5 \end{aligned}$$

11. Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b and nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for V.

Solution. Since $\{u, v\}$ is a basis for V, each vector $w \in \mathsf{V}$ can be written uniquely as $w = a_1u + a_2v$. Note

$$\begin{aligned} a_2(u + v) + \frac{a_1 - a_2}{a}(au) &= a_2u + a_2v + a_1u - a_2u \\ &= a_1 + a_2v \\ &= w \\ &= \frac{a_1}{a}(au) + \frac{a_2}{b}(bv) \end{aligned}$$

and therefore $\{u + v, au\}$ and $\{au, bv\}$ generate V. Now, since $\{u, v\}$ is a basis for V, the equation

$$a_1u + a_2v = 0$$

is satisfied only by $a_1 = a_2 = 0$. Therefore, the equation $a_1(u + v) + a_2(au) = 0$ can be rewritten as $(a_1 + a_2a)u + a_1v = 0$, whereby we obtain $a_1 = 0$ and $a_1 + a_2a = 0$ by the linear independence of $\{u, v\}$. This last equation yields $a_2a = 0$, however $a \neq 0$ as stated by the exercise, which implies that $a_2 = 0$. This proves the linear independence of $\{u+v, au\}$. Next, we examine $\{au, bv\}$, which gives the equation $a_1(au) + a_2(bv) = 0$. This can be rewritten as

$(a_1 a)u + (a_2 b)v = 0$. Again, $a \neq 0$ and $b \neq 0$, which gives $a_1 = a_2 = 0$, proving the linear independence of $\{au, bv\}$. Therefore both sets are linearly independent and generate V, so both are bases for V.

13. The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of R^3. Find a basis for this subspace.

Solution. Solving this system, we obtain that our solution set is the set

$$S = \{(x_1, x_2, x_3) : x_1 = x_2 = x_3\}$$

Therefore, a basis for this set could be $\{(1, 1, 1)\}$. The aforementioned set is linearly independent, since $a_1(1, 1, 1) = (0, 0, 0)$ clearly implies $a_1 = 0$. It can also be readily seen that any vector in S can be expressed in the form $a(1, 1, 1)$ where $a \in R$.

14. Find bases for the following subspaces of F^5:

$$\mathsf{W}_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}$$

What are the dimensions of W_1 and W_2?

Solution. For W_1, consider the set $\beta_1 = \{(0, 1, 0, 0, 0), (0, 0, 0, 0, 1), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0)\}$. For W_2, consider the set $\beta_2 = \{(0, 1, 1, 1, 0), (1, 0, 0, 0, -1)\}$. Let $v_1 \in \mathsf{W}_1$. Then $v_1 = (a_1, a_2, a_3, a_4, a_5)$ satisfying $a_1 - a_3 - a_4 = 0$, that is, $a_1 = a_3 + a_4$. Now, note

$$\begin{aligned} a_2(0, 1, 0, 0, 0) + a_5(0, 0, 0, 0, 1) + a_3(1, 0, 1, 0, 0) + a_4(1, 0, 0, 1, 0) &= (a_3 + a_4, a_2, a_3, a_4, a_5) \\ &= (a_1, a_2, a_3, a_4, a_5) \\ &= v_1 \end{aligned}$$

This proves that β_1 spans W_1. Next, let $v_2 \in \mathsf{W}_2$. Then $v_2 = (b_1, b_2, b_3, b_4, b_5)$ satisfying $b_2 = b_3 = b_4$ and $a_1 + a_5 = 0$. The latter equation implies $a_1 = -a_5$. Note

$$\begin{aligned} b_2(0, 1, 1, 1, 0) + b_1(1, 0, 0, 0, -1) &= (b_1, b_2, b_2, b_2, -b_1) \\ &= (b_1, b_2, b_3, b_4, b_5) \\ &= v_2 \end{aligned}$$

This proves that β_2 spans W_2. It is trivial to verify that both β_1 and β_2 are linearly independent. β_1 and β_2 therefore constitute bases for W_1 and W_2, respectively. Furthermore, because β_1 contains 4 vectors and β_2 contains 2 vectors, we conclude that W_1 and W_2 are of dimension 4 and 2, respectively.

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Example 4 of Section 1.3). Find a basis for W. What is the dimension of W?

Solution. For each i, j satisfying $1 \le i \le n$, $1 \le j \le n$, we define $S^{ij} \in \mathsf{M}_{n\times n}(F)$ in the following way:

$$S^{ij}_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$$

Next, for each i satisfying $2 \le i \le n$, define $D^i \in \mathsf{M}_{n\times n}(F)$ in the following way:

$$D^i_{kl} = \begin{cases} -1 & \text{if } k = l = 1 \\ 1 & \text{if } k = l = i \\ 0 & \text{otherwise} \end{cases}$$

Consider the set $\beta = \{S^{ij} : i \ne j\} \cup \{D^i : 2 \le i \le n\}$. We will prove this set is a basis for W. First, we show β is linearly independent. We need to show that the equation

$$\sum_{i=2}^{n} a_i D^i + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} b_{ij} S^{ij} = 0$$

holds only when, for each i satisfying $2 \le i \le n$, we have $a_i = 0$, and for each i, j satisfying $1 \le i \le n$, $1 \le j \le n$, and $i \ne j$, we have $b_{ij} = 0$. First, note that the matrix 0 has 0 for all of its entries. For each k satisfying $2 \le k \le n$, we obtain

$$0 = \sum_{i=2}^{n} a_i D^i_{kk} + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} b_{ij} S^{ij}_{kk} = \sum_{i=2}^{n} a_i D^i_{kk} + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} b_{ij} \cdot 0 = \sum_{i=2}^{n} a_i D^i_{kk}$$

To understand why this is, note that S^{ij}_{kk} is only nonzero when $k = i$ and $k = j$. However this can never be true in this case because the summation specifies $i \ne j$. Hence all the S^{ij}_{kk} are zero. Furthermore note that for all i satisfying $2 \le i \le n$, $D^i_{kk} = 1$ only when $i = k$. This permits us to simplify the equation even more:

$$\sum_{i=2}^{n} a_i D^i_{kk} = a_k \cdot 1 + \sum_{\substack{2\le i\le n \\ i\ne k}} a_i \cdot 0 = a_k$$

Therefore, $a_k = 0$ (for $2 \le k \le n$). We now wish to show that $b_{ij} = 0$ for all $1 \le i \le n$, $1 \le j \le n$, and $i \ne j$. For each k, l satisfying $1 \le k \le n$, $1 \le l \le n$, and $k \ne l$, we have

$$0 = \sum_{i=2}^{n} a_i D^i_{kl} + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} b_{ij} S^{ij}_{kl} = \sum_{i=2}^{n} a_i \cdot 0 + b_{kl} \cdot 1 + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j \\ (**)}} b_{ij} \cdot 0 = b_{kl}$$

where the (**) denotes the condition that the equations $i = k$ and $j = l$ do not hold simultaneously. To understand why this is, note that $D^i_{kl} = 0$ always (since $k \neq l$). Note also that $S^{ij}_{kl} = 0$ unless $i = k$ and $j = l$. We therefore pull out the unique term $b_{kl} \cdot S^{kl}_{kl}$ from the summation. $S^{kl}_{kl} = 1$ by definition, but all the other $S^{ij}_{kl} = 0$, and so we are left with $b_{kl} = 0$ (for $1 \le k \le n$, $1 \le l \le n$, and $k \neq l$). Therefore we have shown that the only representation of the zero matrix as a linear combination of the vectors within β is the trivial one, proving that β is linearly independent.

Now we need to show that $\text{span}(\beta) = \mathsf{W}$. Let $M \in \mathsf{W}$ and denote M's entry in row i, column j by M_{ij}. Then, since $\text{tr}(M) = 0$, we have

$$M_{11} = -M_{22} - \ldots - M_{nn}$$

We will define the matrix M' by

$$M' = \sum_{i=2}^{n} M_{ii} D^i + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} S^{ij}$$

and consider the entry in the upper-left corner

$$M'_{11} = \sum_{i=2}^{n} M_{ii} D^i_{11} + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} S^{ij}_{11} = \sum_{i=2}^{n} M_{ii} \cdot (-1) + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} \cdot 0 = \sum_{i=2}^{n} -M_{ii} = M_{11}$$

To make the above manipulations, we simply noted that $D^i_{11} = -1$ (for $2 \le i \le n$), and that $S^{ij}_{kk} = 0$ (for $1 \le i \le n$, $1 \le j \le n$, and $i \neq j$). With the same kind of reasoning, for $2 \le k \le n$, we obtain

$$M'_{kk} = \sum_{i=2}^{n} M_{ii} D^i_{kk} + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} S^{ij}_{kk} = M_{kk} \cdot 1 + \sum_{\substack{2 \le i \le n \\ i \neq k}} M_{ii} \cdot 0 + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} \cdot 0 = M_{kk}$$

It now remains to verify that $M'_{kl} = M_{kl}$ when $1 \le k \le n$, $1 \le l \le n$, and $k \neq l$. We obtain

$$M'_{kl} = \sum_{i=2}^{n} M_{ii} D^i_{kl} + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j}} M_{ij} S^{ij}_{kl} = \sum_{i=2}^{n} M_{ii} \cdot 0 + M_{kl} \cdot 1 + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i \neq j \\ (**)}} M_{ij} \cdot 0 = M_{kl}$$

where the (**) denotes the condition that the equations $i = k$ and $j = l$ do not hold simultaneously. This completes our proof that the entries of M' are exactly those of M, and so $M' = M$, which demonstrates $\mathsf{W} \subseteq \text{span}(\beta)$. Clearly we also have $\text{span}(\beta) \subseteq \mathsf{W}$, for otherwise, noting that each vector in β is also in W, W would not be closed under linear combination, contradicting the fact that it is a subspace, which was given in the exercise. Therefore $\text{span}(\beta) = \mathsf{W}$. This, as well as the linear independence of β, demonstrates that β

is indeed a basis for W, completing the proof. There are $n-1$ vectors in $\{D^i : 2 \leq i \leq n\}$ and n^2-n vectors in $\{S^{ij} : i \neq j\}$. Since β is the union of these two sets (note that the two sets are clearly disjoint as a result of the linear independence of β), β contains exactly $(n-1)+(n^2-n) = n^2-1$ vectors.

16. The set of all upper triangular $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Exercise 12 of Section 1.3). Find a basis for W. What is the dimension of W?

 Solution. For each i, j satisfying $1 \leq i \leq n$, $1 \leq j \leq n$, we define

$$S^{ij}_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$$

 and form the set $\beta = \{S^{ij} : i \leq j\}$. We will prove β is a basis for W. To verify the linear independence of β, we have to show that the equation

$$\sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j}} a_{ij} S^{ij} = 0$$

 holds only when $a_{ij} = 0$ for each i, j satisfying $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \leq j$. So consider the entry of this linear combination at row k, column l, for $1 \leq k \leq n$, $1 \leq l \leq n$, and $k \leq l$. We obtain

$$0 = \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j}} a_{ij} S^{ij} = a_{kl} \cdot 1 + \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j\\ (**)}} a_{ij} \cdot 0$$

 and thus $a_{kl} = 0$ for all $1 \leq k \leq n$, $1 \leq l \leq n$, and $k \leq l$. This proves that β is linearly independent. We now demonstrate that β generates W. Assume $M \in \mathsf{W}$, that is, M is an upper triangular matrix. We define the matrix $M' \in \mathsf{M}_{n\times n}(F)$ by

$$M' = \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j}} M_{ij} S^{ij}$$

 and examine the entries of this matrix

$$M'_{kl} = \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j}} M_{ij} S^{ij}_{kl} = M_{kl} \cdot 1 + \sum_{\substack{1\leq i\leq n\\ 1\leq j\leq n\\ i\leq j\\ (**)}} M_{ij} \cdot 0 = M_{kl}$$

 where the (**) denotes the condition that the equations $i = k$ and $j = l$ do not hold simultaneously. This is true because S^{ij}_{kl} is defined to be 0 unless $i = k$ and $j = l$. We have therefore demonstrated that $M' = M$. This shows that $\mathsf{W} \subseteq \operatorname{span}(\beta)$, and for the same reasons as in exercise 15, that $\operatorname{span}(\beta) = \mathsf{W}$. Therefore β is a basis for W. Since β contains $n+(n-1)+(n-2)+\ldots+(n-(n-1)) = n^2-(1+2+\ldots+(n-1)) = \frac{1}{2}(n^2+n)$ elements, we conclude $\dim(\mathsf{W}) = \frac{1}{2}(n^2+n)$.

17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $\mathsf{M}_{n \times n}(F)$ (see Exercise 28 of Section 1.3). Find a basis for W. What is the dimension of W?

    Solution. A skew-symmetric matrix is a matrix A such that $A^t = -A$. Since the diagonals of a matrix are invariant under the matrix transpose operation, special attention must be paid here, for in a field of characteristic 2, $a = -a$ does not imply $a = 0$, and every element is its own additive inverse. The definition of skew-symmetry is equivalent to that of symmetry for matrices with their entries in a field of characteristic 2 ($-A = A$ in such a field, so the equation $A^t = -A$ becomes $A^t = A$ which is precisely the definition of a symmetric matrix). We will assume F's characteristic is not 2.

    For each i, j satisfying $1 \le i \le n$, $1 \le j \le n$, we define

$$S^{ij}_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ -1 & \text{if } k = j \text{ and } l = i \\ 0 & \text{otherwise} \end{cases}$$

    and form the set $\beta = \{S^{ij} : i < j\}$. The proof that β is a basis is very similar to those given in exercises 15 and 16, so I will not repeat it. It is trivial to see that $\dim(\mathsf{W}) = \frac{1}{2}n(n-1)$.

19. Complete the proof of Theorem 1.8.

    Solution. We wish to prove that if V is a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ is a subset of V such that every $v \in \mathsf{V}$ can be uniquely expressed as a linear combination of vectors in β, then β is a basis for V.

    Note first that β is trivially a generating set for V, since we are told each $v \in \mathsf{V}$ is also in $\mathrm{span}(\beta)$. We know that if $v = a_1u_1 + \ldots + a_nu_n = b_1u_1 + \ldots + b_nu_n$, then $a_i = b_i$ for each i such that $1 \le i \le n$ for otherwise each v would not be uniquely expressible as a linear combination of vectors from β. This implies that $a_i - b_i = 0$ for each i in the equation, proving that β is linearly independent. β is therefore a basis for W.

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

    (a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)

    Solution. Since V is finite-dimensional (with dimension n), then some set $\beta = \{\beta_1, \ldots, \beta_n\}$ is a basis for V. Since $\beta \subseteq \mathsf{V}$ and S generates V, then for each i satisfying $1 \le i \le n$, there exists some positive integer c_i such that we can write

$$\beta_i = a^i_1 s^i_1 + \ldots + a^i_{c_i} s^i_{c_i}$$

    for some $a^i_1, \ldots, a^i_{c_i} \in F$ and $s^i_1, \ldots, s^i_{c_i} \in S$. For each i such that $1 \le i \le n$, define

$$S_i = \{s^i_1, \ldots, s^i_{c_i}\}$$

    and let

$$\gamma = \bigcup_{i=1}^{n} S_i$$

Clearly $\beta \subseteq \text{span}(\gamma)$ from above, and so by Theorem 1.5 we have $\text{span}(\beta) = \mathsf{V} \subseteq \text{span}(\gamma)$. Clearly, $\text{span}(\gamma) \subseteq \mathsf{V}$ since V is a vector space, and therefore $\text{span}(\gamma) = \mathsf{V}$. Theorem 1.9 implies that there exists some $\gamma' \subseteq \gamma$ such that γ' is a basis for V. This set γ' is clearly a subset of S, since γ itself is a subset of S.

(b) Prove that S contains at least n vectors.

Solution. Since γ' is a basis for V (which is of dimension n), γ' must contain n vectors, by Corollary 1 to the Replacement Theorem. The fact that $\gamma' \subseteq S$ then gives us that S contains at least n vectors.

1.7 Maximal Linearly Independent Subsets

Section 1.7 presents an alternative formulation of the concept of a basis. First, it defines the concept of a *chain* and a *maximal set* (with respect to set inclusion). Then, a principle known as the Maximal Principle (also known as the Hausdorff Maximal Principle) is presented, which is equivalent to the Axiom of Choice. *Maximal linearly independent subsets* are defined as sets that are linearly independent and are contained in no linearly independent subset with the exception of themselves. The equivalence between maximal linearly independent subsets and bases is proved. At the end of the section, it is shown that if S is a linearly independent subset of a vector space, there exists a maximal linearly independent subset of that vector space containing S, giving the obvious but important corollary that every vector space has a basis.

4. Let W be a subspace of a (not necessarily finite-dimensional) vector space V. Prove that any basis for W is a subest of a basis for V.

   Solution. Let β be a basis for W. β is clearly a linearly independent subset of W and hence of V. Therefore by Theorem 1.12, V possesses a maximal linearly independent subset (basis) containing β.

2 Linear Transformations and Matrices

2.1 Linear Transformations, Null spaces, and Ranges

In Section 2.1, *linear transformations* are introduced, as a special kind of function from one vector space to another. Some properties of linear transformations are discussed. Two special subspaces (of the domain and codomain, respectively), the *nullspace* and *range*, are presented, and their dimensions are called the *nullity* and *rank*, respectively. Furthermore, results are proven concerning the injectivity and surjectivity of linear transformations satisfying certain criteria.

For Exercises 2 through 6, prove that T is a linear transformation, and find bases for both $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

2. $\mathsf{T} : \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

   Solution. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Also let $c \in R$. Then $\mathsf{T}(ca + b) = (ca_1+b_1-ca_2-b_2, 2ca_3+2b_3) = (ca_1-ca_2, 2ca_3)+(b_1-b_2, 2b_3) = c(a_1-a_2, 2a_3)+(b_1-b_2, 2b_3) =$

$c\mathsf{T}(a) + \mathsf{T}(b)$. This proves that T is linear. Consider the set $\beta = \{(1,1,0)\}$. Let $v \in \text{span}(\beta)$. Then $v = (a,a,0)$ for some $a \in R$, so that $\mathsf{T}(v) = \mathsf{T}(a,a,0) = (a-a, 2\cdot 0) = (0,0)$. This proves $v \in \mathsf{N}(\mathsf{T})$ and hence $\text{span}(\beta) \subseteq \mathsf{N}(\mathsf{T})$. We also clearly have $\mathsf{N}(\mathsf{T}) \subseteq \text{span}(\beta)$, and so β spans the nullspace of T. β is clearly linearly independent since it consists of a single nonzero vector, so β is a basis for T's nullspace. Let $\beta' = \{(1,0,0),(0,1,0),(0,0,1)\}$. By Theorem 2.2, $\mathsf{R}^3 = \text{span}(\mathsf{T}(\beta')) = \text{span}(\{(1,0),(-1,0),(0,2)\}) = \text{span}(\{(1,0),(0,1)\})$. Since the set $\{(1,0),(0,1)\}$ is linearly independent, it constitutes a basis for the range of T. Thus, $\text{nullity}(\mathsf{T}) = 1$ and $\text{rank}(\mathsf{T}) = 2$. Therefore $1+2=3$, verifying the dimension theorem. Since $\text{nullity}(\mathsf{T}) \neq 0$, we see that $\mathsf{N}(\mathsf{T})$ contains some nonzero vector. By Theorem 2.4, then, T is not one-to-one. Note that $\text{rank}(\mathsf{T}) = \dim(\mathsf{R}^2)$, proving by Theorem 1.11 that $\mathsf{R}(\mathsf{T}) = \mathsf{W}$. Therefore T is onto.

6. $\mathsf{T} : \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \text{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Solution. Let $A, B \in \mathsf{M}_{n\times n}(F)$, and $c \in F$. Then $\mathsf{T}(cA+B) = \text{tr}(cA+B) = c\,\text{tr}(A) + \text{tr}(B) = c\mathsf{T}(A) + \mathsf{T}(B)$, by Exercise 6 of Section 1.3, proving the linearity of T. By exercise 15 of Section 1.6, a basis for $\mathsf{N}(\mathsf{T})$ is

$$\{S^{ij} : i \neq j\} \cup \{D^i : 2 \leq i \leq n\}$$

and the nullity of T is $n^2 - 1$, where S^{ij} and D^i are defined as in Exercise 15 of Section 1.6. It is trivial to see that $\text{rank}(\mathsf{T}) = 1$, and that $\{1\}$ is a basis for $\mathsf{R}(\mathsf{T})$. Therefore $\text{nullity}(\mathsf{T}) + \text{rank}(\mathsf{T}) = (n^2-1)+1 = n^2 = \dim(\mathsf{M}_{n\times n}(F))$, verifying the dimension theorem. Since $\text{rank}(\mathsf{T}) = 1 = \dim(F)$, we have $\mathsf{R}(\mathsf{T}) = F$ by Theorem 1.11, proving that T is onto. However, since $\mathsf{N}(\mathsf{T}) \neq \{0\}$ when $n > 2$, clearly, we have by Theorem 2.4 that T is not one-to-one unless $n = 1$ (since in this case, $\text{nullity}(\mathsf{T}) = 1^2 - 1 = 0$, implying $\mathsf{N}(\mathsf{T}) = \{0\}$).

7. Prove properties 1, 2, 3, and 4 on page 65.

Solution. Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear. Clearly $\mathsf{T}(0) = 0$, for otherwise by linearity $\mathsf{T}(0) = \mathsf{T}(x+(-x)) = \mathsf{T}(x) + \mathsf{T}(-x) = \mathsf{T}(x) + (-\mathsf{T}(x)) \neq 0$ which is absurd. Also note that since T is linear, $\mathsf{T}(cx+y) = \mathsf{T}(cx) + \mathsf{T}(y) = c\mathsf{T}(x) + \mathsf{T}(y)$, and $\mathsf{T}(x-y) = \mathsf{T}(x+(-y)) = \mathsf{T}(x) + \mathsf{T}(-y) = \mathsf{T}(x) - \mathsf{T}(y)$. To prove property 4, note that if T is linear, an inductive argument can be used to show that for all $x_1, \ldots, x_n \in \mathsf{V}$ and $a_1, \ldots, a_n \in F$, we have

$$\mathsf{T}\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} \mathsf{T}(a_i x_i) = \sum_{i=1}^{n} a_i \mathsf{T}(x_i)$$

Now, assume $\mathsf{T}(cx+y) = c\mathsf{T}(x) + \mathsf{T}(y)$ for all $x, y \in \mathsf{V}$ and $c \in F$. Let $x, y \in \mathsf{V}$. Then we obtain $\mathsf{T}(x+y) = \mathsf{T}(1x+y) = 1\cdot\mathsf{T}(x) + \mathsf{T}(y) = \mathsf{T}(x) + \mathsf{T}(y)$. Next, let $x \in \mathsf{V}$ and $c \in F$. Then we obtain $\mathsf{T}(cx) = \mathsf{T}(cx+0) = c\cdot\mathsf{T}(x) + \mathsf{T}(0) = c\cdot\mathsf{T}(x)$. This proves T is linear. The same type of reasoning can be used to show that if T satisfies

$$\mathsf{T}\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i \mathsf{T}(x_i)$$

then T must be linear.

10. Suppose that $\mathsf{T} : \mathsf{R}^2 \to \mathsf{R}^2$ is linear, $\mathsf{T}(1,0) = (1,4)$, and $\mathsf{T}(1,1) = (2,5)$. What is $\mathsf{T}(2,3)$? Is T one-to-one?

    Solution. By Theorem 2.6, T is defined by

    $$\mathsf{T}(ae_1 + be_2) = aw_1 + bw_2$$

    So that we obtain $w_1 = (1,4)$ and $w_2 = (2,5) - (1,4) = (1,1)$. Therefore we conclude $\mathsf{T}(2,3) = 2w_1 + 3w_2 = 2(1,4) + 3(1,1) = (2,8) + (3,3) = (5,11)$. Now, consider $x \in \mathsf{N}(\mathsf{T})$. Clearly, $x = ae_1 + be_2$, so $\mathsf{T}(x) = a\mathsf{T}(e_1) + b\mathsf{T}(e_2) = aw_1 + bw_2$ for some scalars a, b. However, $\{w_1, w_2\}$ is linearly independent, so that $a = b = 0$. This implies that $\mathsf{N}(\mathsf{T}) = \{0\}$, implying T is one-to-one by Theorem 2.4.

11. Prove that there exists a linear transformation $\mathsf{T} : \mathsf{R}^2 \to \mathsf{R}^3$ such that $\mathsf{T}(1,1) = (1,0,2)$ and $\mathsf{T}(2,3) = (1,-1,4)$. What is $\mathsf{T}(8,11)$?

    Solution. Since $\{(1,1),(2,3)\}$ is a basis for R^2, by Theorem 2.6, there exists a unique linear transformation such that $\mathsf{T}(1,1) = (1,0,2)$ and $\mathsf{T}(2,3) = (1,-1,4)$. This transformation is defined by

    $$\mathsf{T}(a(1,1) + b(2,3)) = a(1,0,2) + b(1,-1,4)$$

    To determine $\mathsf{T}(8,11)$, we write $(8,11) = 2(1,1) + 3(2,3)$. This gives $\mathsf{T}(8,11) = 2(1,0,2) + 3(1,-1,4) = (2,0,4) + (3,-3,12) = (5,-3,16)$.

12. Is there a linear transformation $\mathsf{T} : \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1,0,3) = (1,1)$ and $\mathsf{T}(-2,0,-6) = (2,1)$?

    Solution. There clearly exists no such linear transformation. Assume $\mathsf{T}(1,0,3) = (1,1)$. Then by the linearity of T we must have $\mathsf{T}(-2,0,-6) = -2\cdot\mathsf{T}(1,0,3) = -2(1,1) = (-2,-2) \neq (2,1)$.

13. Let V and W be vector spaces, let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $\mathsf{R}(\mathsf{T})$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathsf{T}(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

    Solution. Assume, for the sake of contradiction, that S is linearly dependent. Then there exist $a_1, \ldots, a_k \in F$, not all zero, such that

    $$a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0$$

    Then by the linearity of T, such scalars also satisfy

    $$a_1\mathsf{T}(v_1) + a_2\mathsf{T}(v_2) + \ldots + a_k\mathsf{T}(v_k) = 0$$

    Now recall that $\mathsf{T}(v_i) = w_i$ for $1 \leq i \leq k$. This implies that $\{w_1, w_2, \ldots, w_k\}$ is a linearly dependent subset of $\mathsf{R}(\mathsf{T})$, a contradiction. Therefore S must be linearly independent.

14. Let V and W be vector spaces and $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear.

    (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Solution. Assume T carries linearly independent subsets of V onto linearly independent subsets of W. Assume, for the sake of contradiction, that there exist distinct $x, y \in \mathsf{V}$ satisfying $\mathsf{T}(x) = \mathsf{T}(y)$. Then $\mathsf{T}(x) - \mathsf{T}(y) = 0$ which implies $x - y \in \mathsf{N}(\mathsf{T})$ by linearity. Let $S = \{x - y\}$. Since $x - y \neq 0$ (for otherwise $x = y$), we have that S is linearly independent. However, $\mathsf{T}(S) = \{0\}$, demonstrating the existence of a linearly independent subset of V that T carries onto a linearly dependent subset of W, which is impossible by assumption. Therefore no distinct x, y exist satisfying $\mathsf{T}(x) = \mathsf{T}(y)$. This proves that T is one-to-one.

Assume next that T is one-to-one and that S is a linearly independent subset of V. Assume, for the sake of contradiction, that $\mathsf{T}(S)$ is linearly dependent. Then there exist scalars $a_1, \ldots, a_n$, not all zero, and vectors $v_1, \ldots, v_n \in S$ such that

$$a_1 \mathsf{T}(v_1) + \ldots + a_n \mathsf{T}(v_n) = 0$$

However linearity implies that

$$0 = \mathsf{T}(0) = \mathsf{T}(a_1 v_1 + \ldots + a_n v_n)$$

Now, due to the linear independence of S, we see that $a_1 v_1 + \ldots + a_n v_n \neq 0$ unless all the a_i are zero, which we have assumed not to be the case. This contradicts that T is one-to-one. Therefore $\mathsf{T}(S)$ must be linearly independent, and so T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

Solution. Clearly, if S is linearly independent, then $\mathsf{T}(S)$ is linearly independent by (a). Therefore, assume $\mathsf{T}(S)$ is linearly independent, and for the sake of contradiction that S is linearly dependent. Then there exist scalars $a_1, \ldots, a_n$, not all zero, and vectors $v_1, \ldots, v_n \in S$ such that

$$a_1 v_1 + \ldots + a_n v_n = 0$$

Then the same scalars will, by linearity of T, satisfy

$$a_1 \mathsf{T}(v_1) + \ldots + a_n \mathsf{T}(v_n) = 0$$

This implies that $\mathsf{T}(S)$ is linearly dependent, a contradiction. Therefore S is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

Solution. Since T is one-to-one, (b) gives us that $\mathsf{T}(\beta)$ is linearly independent. Since β is a basis for V, Theorem 2.2 gives us that $\mathsf{R}(\mathsf{T})$ is generated by β. However, $\mathsf{R}(\mathsf{T}) = \mathsf{W}$ since T is onto. Therefore β is a basis for W.

17. Let V and W be finite-dimensional vector spaces and $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear.

(a) Prove that if $\dim(\mathsf{V}) < \dim(\mathsf{W})$, then T cannot be onto.

Lemma. Let V and W are vector spaces, $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear, and $S \subseteq \mathsf{V}$. If $\mathsf{T}(S)$ is linearly independent, then S is linearly independent.

Proof. Assume $\mathsf{T}(S)$ is linearly independent, and for the sake of contradiction, that S is linearly dependent. Then there exist scalars $a_1, \ldots, a_n$, not all zero, and vectors $s_1, \ldots, s_n \in S$ such that

$$a_1 s_1 + \ldots + a_n s_n = 0$$

However, $\mathsf{T}(0) = 0$, so by linearity we would obtain that $\mathsf{T}(S)$ is linearly dependent, a contradiction. Therefore S is linearly independent.

Solution. Let $\dim(\mathsf{V}) < \dim(\mathsf{W})$ and assume, for the sake of contradiction, that T is onto. Let β be a basis for W. Then β contains $\dim(\mathsf{W})$ elements. Since T is onto, $\beta = \mathsf{T}(S)$ for some subset $S \subseteq \mathsf{V}$. However, β is clearly linearly independent in W, so by the lemma we see S is linearly independent in V. Therefore V contains a linearly independent subset with $\dim(\mathsf{W})$ elements. Since $\dim(\mathsf{W}) > \dim(\mathsf{V})$, this contradicts the replacement theorem. Therefore T cannot be onto.

(b) Prove that if $\dim(\mathsf{V}) > \dim(\mathsf{W})$, then T cannot be one-to-one.

Solution. If T is one-to-one, then T carries linearly independent subsets of V onto linearly independent subsets of W. Let β be a basis for V. Then β contains $\dim(\mathsf{V})$ elements, and is linearly independent. So $\mathsf{T}(\beta)$ will be a linearly independent subset of W containing $\dim(\mathsf{V})$ elements, by (b) of Exercise 14. However, $\dim(\mathsf{W}) < \dim(\mathsf{V})$, so we obtain the same contradiction to the replacement theorem as in (a). Thus, T cannot be one-to-one.

2.2 The Matrix Representation of a Linear Transformation

Section 2.2 discusses a correspondence between linear transformations and matrices. It defines the matrix representation of a linear transformation in such a way that the columns of said matrix can be deduced simply by examining the image of each vector in the domain's basis, represented as a coordinate vector with respect to the basis of the codomain. The vector space of linear transformations from V to W, $\mathcal{L}(\mathsf{V}, \mathsf{W})$, is also defined. We also see that sums and scalar multiples of linear transformations have matrix representations as we would expect, i.e. the matrix representation of a sum of transformations is the sum of the matrix representations of the individual transformations.

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $\mathsf{T} : \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.

(a) $\mathsf{T} : \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

Solution. Recall that the j^{th} column of $[\mathsf{T}]_\beta^\gamma$ is the column matrix $[\mathsf{T}(\beta_j)]_\gamma$. Since we're working with the standard ordered bases here, this simplifies to $[\mathsf{T}(e_j)]_\gamma$. Notice $\mathsf{T}(e_1) = \mathsf{T}(1, 0) = (2 - 0, 3 + 0, 1) = (2, 3, 1) = 2e_1 + 3e_2 + 1e_3$, and $\mathsf{T}(e_2) = \mathsf{T}(0, 1) = (0 - 1, 0 + 4, 0) = (-1, 4, 0) = -1e_1 + 4e_2 + 0e_3$. Therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

(c) $\mathsf{T} : \mathsf{R}^3 \to R$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.

Solution. Applying the same method as before, we see $\mathsf{T}(1,0,0) = 2 = 2 \cdot 1$, $\mathsf{T}(0,1,0) = 1 = 1 \cdot 1$, and $\mathsf{T}(0,0,1) = -3 = -3 \cdot 1$. Therefore

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$$

(e) $\mathsf{T} : \mathsf{R}^n \to \mathsf{R}^n$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = (a_1, a_1, \ldots, a_1)$.

Solution. Let A be the $n \times n$ matrix defined by

$$A_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $[\mathsf{T}]_\beta^\gamma = A$.

3. Let $\mathsf{T} : \mathsf{R}^2 \to \mathsf{R}^3$ be defined by $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

Solution. We see that $\mathsf{T}(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$, and $\mathsf{T}(0,1) = (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3)$. Therefore, the coordinate vectors for $\mathsf{T}(e_1)$ and $\mathsf{T}(e_2)$ are $(-\frac{1}{3}, 0, \frac{2}{3})$ and $(-1,1,0)$ respectively. So we obtain

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

Next, we see that $\mathsf{T}(1,2) = (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$ and $\mathsf{T}(2,3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$. Therefore, the coordinate vectors for $\mathsf{T}(1,2)$ and $\mathsf{T}(2,3)$ are $(-\frac{7}{3}, 2, \frac{2}{3})$ and $(-\frac{11}{3}, 3, \frac{4}{3})$ respectively. So we obtain

$$[\mathsf{T}]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

Note that in both cases, a system of linear equations had to be solved to express each image vector as a combination of the vectors in the codomain's basis, although this was not shown.

8. Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T} : \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

Solution. Let $x, y \in \mathsf{V}$ and $c \in F$. Assume that $\beta = \{v_1, \ldots, v_n\}$ and that

$$x = \sum_{i=1}^n a_i v_i \qquad \text{and} \qquad y = \sum_{i=1}^n b_i v_i$$

for scalars $a_1, \ldots, a_n, b_1, \ldots, b_n$. Then we have

$$\mathsf{T}(cx + y) = [cx + y]_\beta = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = c[x]_\beta + [y]_\beta = c\mathsf{T}(x) + \mathsf{T}(y)$$

which proves T is linear.

2.3 Composition of Linear Transformations and Matrix Multiplication

Section 2.3 uses composition of linear transformations to motivate the usual definition of matrix multiplication. Some useful theorems are proved about compositions of transformations and matrix multiplication. An important relationship between matrices and linear transformations is demonstrated through the so-called "left-multiplication transformation", L_A.

13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.

Solution. Note that we have

$$\operatorname{tr}(BA) = \sum_{i=1}^{n}(BA)_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} B_{ik}A_{ki}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} A_{ki}B_{ik}\right) = \sum_{k=1}^{n}(AB)_{kk} = \operatorname{tr}(AB)$$

which proves the first proposition. To prove the second, simply note

$$\operatorname{tr}(A^t) = \sum_{i=1}^{n}(A^t)_{ii} = \sum_{i=1}^{n} A_{ii} = \operatorname{tr}(A).$$

15. Let M and A be matrices for which the product matrix MA is defined. If the jth column of A is a linear combination of a set of columns of A, prove that the jth column of MA is a linear combination of the corresponding columns of MA with the same corresponding coefficients.

Solution. Assume M is an $m \times n$ matrix, and A is an $n \times p$ matrix. Then for each k satisfying $1 \leq k \leq p$ and $k \neq j$, there exist a_k, with

$$A_{ij} = \sum_{\substack{k=1 \\ k\neq j}}^{p} a_k A_{ik}$$

for all i satisfying $1 \leq i \leq n$. Notice that

$$(MA)_{ij} = \sum_{z=1}^{n} M_{iz}A_{zj} = \sum_{z=1}^{n} M_{iz}\left(\sum_{\substack{k=1 \\ k\neq j}}^{p} a_k A_{zk}\right) = \sum_{\substack{k=1 \\ k\neq j}}^{p} a_k\left(\sum_{z=1}^{n} M_{iz}A_{zk}\right) = \sum_{\substack{k=1 \\ k\neq j}}^{p} a_k(MA)_{ik}$$

which proves the proposition.

18. Using only the definition of matrix multiplication, prove that multiplication of matrices is associative.

Solution. Assume A is an $m \times n$ matrix, B is an $n \times p$ matrix, and C is a $p \times q$ matrix. Then we have

$$[(AB)C]_{ij} = \sum_{z=1}^{p}(AB)_{iz}C_{zj} = \sum_{z=1}^{p}\left(\sum_{y=1}^{n} A_{iy}B_{yz}\right)C_{zj} = \sum_{y=1}^{n} A_{iy}\left(\sum_{z=1}^{p} B_{yz}C_{zj}\right) = [A(BC)]_{ij}.$$