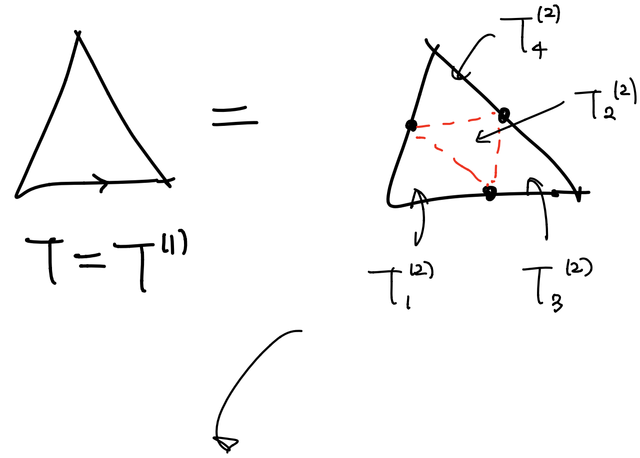
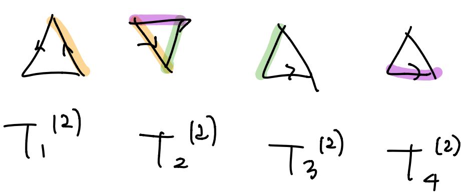
Theorem. Let D be an open set in $\mathbb{C}$, Let T be triangle such that T and its interior lie in D. If $f(z)$ is analytic in D, then

$$\oint_T f(z)\,dz = 0.$$

(pf)





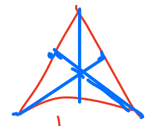
$$\int_{T^{(1)}} f(z)\,dz = \sum_{j}^{4} \int_{T_j^{(2)}} f(z)\,dz = (*)$$

There is j s.t.

$$\left| \int_{T^{(1)}} f(z)\,dz \right| \le 4 \left| \int_{T_j^{(2)}} f(z)\,dz \right|$$

Let $T^{(2)} = T_j^{(2)}$

Notation $d^{(1)}$ = diameter of $T^{(1)}$ ⟵ ex)



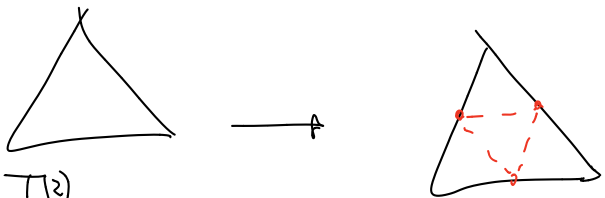
셋중 가장 긴 것이 diameter

$p^{(1)}$ = perimeter of $T^{(1)}$

둘레

$$d^{(2)} = \text{diameter of } T^{(2)} = \frac{1}{2} d^{(1)}$$

$$p^{(2)} = \text{perimeter of } T^{(2)} = \frac{1}{2} p^{(1)}$$



$T^{(2)}$

define $T_j^{(3)}$ $(j = 1, 2, 3, 4)$

Find j s.t. $\left\| \int_{T^{(2)}} f(z)\,dz \right\| \leq 4 \left| \int_{T_j^{(3)}} f(z)\,dz \right|$

Let $T^{(3)} = T_j^{(3)}$

$$\hookrightarrow \left| \int_{T^{(1)}} f(z)\,dz \right| \leq 4^2 \left| \int_{T^{(3)}} f(z)\,dz \right|$$

$$d^{(3)} = \frac{1}{2} d^{(2)} = \frac{1}{2^2} d^{(1)}$$

$$p^{(3)} = \cdots = \frac{1}{2^2} p^{(1)}$$

Find $T^{(1)}, T^{(2)}, \cdots, T^{(n)}$, then

① $T^{(n)}$ lies in $T^{(n-1)}$

② $d^{(n)} = \frac{1}{2} d^{(n-1)} = \frac{1}{2^{n-1}} d^{(1)}$

$p^{(n)} = \frac{1}{2} p^{(n-1)} = \frac{1}{2^{n-1}} p^{(1)}$

③ $\left| \int_{T^{(1)}} f(z)\,dz \right| \leq 4^{n-1} \left| \int_{T^{(n)}} f(z)\,dz \right|$

observe 1. $\bigcap_{n=1}^{\infty} T^{(n)} = \{z_0\}$ for some $z_0 \in D$

2. $\lim_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ ---- (*)

Let $\psi(z) = \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$, then

$$f(z) = \boxed{\underbrace{f(z_0)}_{\text{상수}} + \underbrace{f'(z_0)}_{\text{상수}}\underbrace{(z-z_0)}_{\text{polynomial}}} + \psi(z)(z-z_0)$$

$\downarrow$

primative function!

※ $\psi(z) \to 0$ as $z \to z_0$ by (*)

$$\left| \int_{T^{(1)}} f(z)\,dz \right| \leq 4^{n-1} \left| \int_{T^{(n)}} f(z)\,dz \right|$$

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} \boxed{\cancel{f(z_0) + f'(z_0)(z-z_0)}} + \psi(z)(z-z_0)\,dz$$

primative 이므로 적분 값이 0.

$$= \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$$

$$\left| \int_{T^{(n)}} f(z)\,dz \right| \le \Big(\overbrace{\max_{z\in T^{(n)}} \left| \eta(z) \right| \underbrace{\boxed{\max_{z\in T^{(n)}} (z - z_0)}}_{\parallel \atop d^{(n)}}}^{M} \Big) \Big(\overbrace{p^{(n)}}^{L} \Big)$$

$$= \frac{d^{(1)} p^{(1)}}{4^{n-1}} \max_{z\in T^{(n)}} \left| \eta(z) \right|$$

$\therefore$

$$\left\| \int_{T^{(1)}} f(z)\,dz \right\| \le 4^{n-1} \left\| \int_{T^{(n)}} f(z)\,dz \right\| \le 4^{n-1} \left(\frac{d^{(1)} p^{(1)}}{4^{n-1}} \max_{z\in T^{(n)}} \left| \eta(z) \right| \right)$$

$$\longrightarrow 0 \quad \text{as} \quad \underbrace{n \to \infty}$$

$$\downarrow$$

$$z \to z_0$$

$$\downarrow$$

$$\eta(z) \to 0$$

$$\downarrow$$

$$\max \left| \eta(z) \right| \to 0$$