# RandLib documentation

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## Contents

## Part I

# General information

## 1 Calculation of sample moments

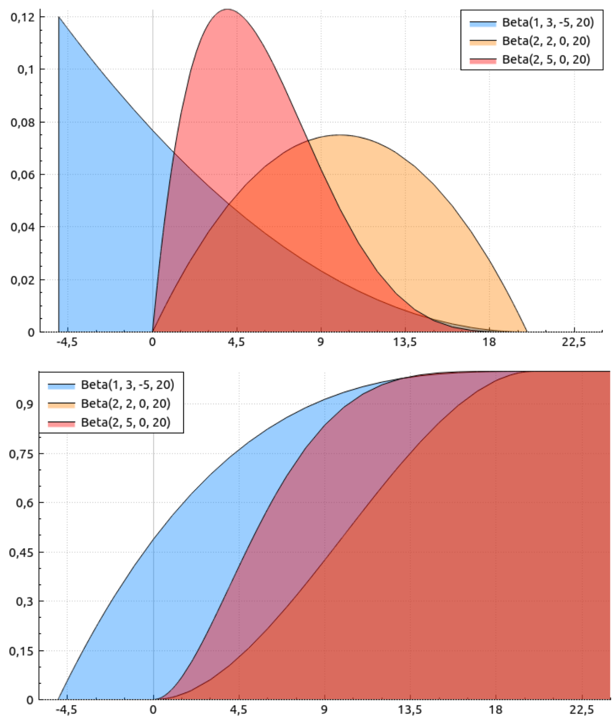
We use extension of Welford's method from Knuth. For every $n$-th element $x$ we have

$$
\begin{aligned}
\delta &= x - m_1, \\
m_1' &= m_1 + \frac{\delta}{n}, \\
m_2' &= m_2 + \delta^2 \frac{n-1}{n}, \\
m_3' &= m_3 + \delta^3 \frac{(n-1)(n-2)}{n^2} - 3\delta \frac{m_2}{n}, \\
m_4' &= m_4 + \delta^4 \frac{(n-1)(n^2-3n+3)}{n^3} + 6\delta^2 \frac{m_2}{n^2} - 4\delta \frac{m_3}{n}.
\end{aligned}
$$

Then $m_1'$, $\frac{m_2}{n}$, $\mathrm{Skew}(X) = \frac{\sqrt{n} m_3'}{m_2'^{3/2}}$ and $\mathrm{Kurt}(X) = \frac{n m_4'}{m_2'^2}$ (we return excess kurtosis).

# Part II
# Continuous univariate distributions

## 2 Beta distribution




| | |
|---|---|
| Notation | $X \sim \mathcal{B}(\alpha, \beta, a, b)$<br>$X \sim \mathcal{B}(\alpha, \beta)$ with $a = 0, b = 1$ |
| Parameters | $\alpha, \beta > 0,\ a, b \in \mathbb{R}$ |
| Domain | $x \in [a, b]$ |
| $f(x)$ | $\frac{y^{\alpha-1}(1-y)^{\beta-1}}{(b-a)B(\alpha,\beta)}$ with $y = \frac{x-a}{b-a}$ |
| $F(x)$ | $I_y(\alpha, \beta)$ for $y = \frac{x-a}{b-a}$ |
| $\mathbb{E}[X]$ | $a + (b-a)\frac{\alpha}{\alpha+\beta}$ |
| $\mathrm{Var}(X)$ | $(b-a)^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ |
| Median | Searched numerically |
| Mode | $a + (b-a)\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha, \beta > 1$. |
| $\phi(t)$ | Calculated numerically |

**Search of the median.** In general, the value of median is unkwnown and searched numerically with initial value:

$$m \approx a + (b-a)\frac{\alpha - \frac{1}{3}}{\alpha + \beta - \frac{2}{3}}$$

for $\alpha, \beta \geq 1$. However, there are analytical solutions for some particular values:

- $m = \frac{a+b}{2}$, for $\alpha = \beta$,
- $m = a + (b-a)(1 - 2^{-\frac{1}{\beta}})$, for $\alpha = 1$,
- $m = a + (b-a)2^{-\frac{1}{\alpha}}$, for $\beta = 1$.

**Calculation of characteristic function.** For $\alpha, \beta \geq 1$ we use numerical integration by definition

$$\phi(t) = \int_a^b \cos(tx) f(x) dx + i \int_a^b \sin(tx) f(x) dx.$$

For shape parameters $< 1$, $f(x)$ has singularity points at 0 or 1 or both of them, and numerical integration is impossible. Then we use the following technique: firstly, we can show that

$$\phi(t|a,b) = \mathbb{E}[e^{it(a+(b-a)X)}] = e^{ita}\phi(z|0,1)$$

with $z = (b-a)t$. Hence, w.l.o.g. we can consider standard case $a = 0$, $b = 1$. Then

$$\begin{aligned}
\Re(\phi(z)) &= \frac{1}{B(\alpha,\beta)} \int_0^1 \cos(zx) x^{\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 (\cos(zx)-1) x^{\alpha-1}(1-x)^{\beta-1} dx + 1 \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \frac{(\cos(zx)-1)x^{\alpha-1} - (\cos(z)-1)}{(1-x)^{1-\beta}} dx + 1 + \frac{\cos(z)-1}{bB(\alpha,\beta)}.
\end{aligned}$$

The integrand now doesn't have any singularities, neither for $\alpha < 1$, nor for $\beta < 1$. Analogously we transform the imaginary part:

$$\begin{aligned}
\Im(\phi(z)) &= \frac{1}{B(\alpha,\beta)} \int_0^1 \sin(zx) x^{\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \frac{\sin(zx)x^{\alpha-1} - \sin(z)}{(1-x)^{1-\beta}} dx + \frac{\sin(z)}{bB(\alpha,\beta)}.
\end{aligned}$$

**Estimation of shapes with known support.** Assume that $a = 0$, $b = 1$ and we have a sample $X = (X_1, \ldots, X_n)$. Then a log-likelihood function is

$$\begin{aligned}
\ln \mathcal{L}(\alpha,\beta|X) &= \sum_{i=1}^n \ln f(X_i;\alpha,\beta) \\
&= (\alpha-1)\sum_{i=1}^n \ln X_i + (\beta-1)\sum_{i=1}^n \ln(1-X_i) - n\ln B(\alpha,\beta).
\end{aligned} \tag{1}$$

Differentiating with respect to the shapes, we obtain

$$\frac{\partial \ln \mathcal{L}(\alpha,\beta|X)}{\partial \alpha} = \sum_{i=1}^n \ln X_i + n(\psi(\alpha+\beta) - \psi(\alpha)),$$

$$\frac{\partial \ln \mathcal{L}(\alpha,\beta|X)}{\partial \beta} = \sum_{i=1}^n \ln(1-X_i) + n(\psi(\alpha+\beta) - \psi(\beta)).$$

Differentiating again we get the Hessian matrix:

$$\mathrm{H}(\ln \mathcal{L}(\alpha, \beta|X)) = n \cdot \begin{pmatrix} \psi_1(\alpha+\beta) - \psi_1(\alpha) & \psi_1(\alpha+\beta) \\ \psi_1(\alpha+\beta) & \psi_1(\alpha+\beta) - \psi_1(\beta) \end{pmatrix}.$$

Then we can find the estimators numerically, using Newton's procedure. The initial values of estimators are found via method of moments:

$$\hat{\alpha}_0 = \overline{X}_n \left( \frac{\overline{X}_n(1-\overline{X}_n)}{\hat{s}_n^2} - 1 \right),$$

$$\hat{\beta}_0 = (1-\overline{X}_n) \left( \frac{\overline{X}_n(1-\overline{X}_n)}{\hat{s}_n^2} - 1 \right).$$

These values are applicable only if $\hat{s}_n^2 < \overline{X}_n(1-\overline{X}_n)$. If this condition is not satisfied, we set $\hat{\alpha}_0 = \hat{\beta}_0 = 0.001$.
In the general case, when $a \neq 0$ or $b \neq 1$, we use the following transformation:

$$Y_i = \frac{X_i - a}{b - a}$$

and estimate parameters, using sample $Y$.

**Exponential family parameterization.** Logarithm of probabilty density function:

$$\log f(x) = (\alpha - 1)\log y + (\beta - 1)\log(1-y) - \log(b-a) - \log B(\alpha, \beta)$$

with $y = \frac{x-a}{b-a}$. Therefore beta distribution with fixed $a$ and $b$ belongs to two-parameterized exponential family with sufficient statistics $T(x) = (\log y, \log(1-y))^T$, natural parameters $\theta = (\alpha - 1, \beta - 1)$, log-normalizer $F(\theta) = \log(b-a) + \log B(\theta_1 + 1, \theta_2 + 1)$ and carrier measure $k(x) = 0$. Gradient of log-normalizer: $\nabla F(\theta) = (\psi(\theta_1+1) - \psi(\theta_1+\theta_2+2), \psi(\theta_2+1) - \psi(\theta_1+\theta_2+2))^T$. Adjusted cross-entropy is

$$\begin{aligned} H_F(\theta_p \| \theta_q) &= F(\theta_q) - \langle \theta_q, \nabla F(\theta_p) \rangle \\ &= \log(b-a) + \log B(\theta_{q1} + 1, \theta_{p2} + 1) \\ &\quad - \theta_{q1}(\psi(\theta_{p1}+1) - \psi(\theta_{p1}+\theta_{p2}+2)) - \theta_{q2}(\psi(\theta_{p2}+1) - \psi(\theta_{p1}+\theta_{p2}+2)) \end{aligned}$$

Adjusted entropy is

$$\begin{aligned} H_F(\theta) = \log(b-a) &+ \log B(\alpha, \beta) \\ &- (\alpha-1)\psi(\alpha) - (\beta-1)\psi(\beta) + (\alpha+\beta-2)\psi(\alpha+\beta). \end{aligned}$$

And Kullback-Leibler divergence:

$$\begin{aligned} \mathrm{KL}(p\|q) &= H_F(\theta_p\|\theta_q) - H_F(\theta_p) \\ &= \log \frac{B(\alpha_q, \beta_q)}{B(\alpha_p, \beta_p)} - (\alpha_q - \alpha_p)\psi(\alpha_p) - (\beta_q - \beta_p)\psi(\beta_p) + (\alpha_q - \alpha_p + \beta_q - \beta_p)\psi(\alpha_p + \beta_p). \end{aligned}$$

## 2.1 Arcsine distribution

Notation:

$$X \sim \mathrm{Arcsine}(\alpha).$$

Relation to Beta distribution:

$$X \sim \mathcal{B}(1-\alpha, \alpha, a, b).$$

**Estimation of shape.** For Arcsine distribution log-likelihood function (1) turns into

$$\ln \mathcal{L}(\alpha|X) = -\alpha \sum_{i=1}^{n} \ln X_i + (\alpha - 1) \sum_{i=1}^{n} \ln(1 - X_i) - n \ln B(1-\alpha, \alpha).$$

Taking the derivative with respect to $\alpha$ we get

$$\frac{\partial \ln \mathcal{L}(\alpha|X)}{\partial \alpha} = \sum_{i=1}^{n} \ln \frac{1 - X_i}{X_i} + n\pi \cot(\pi\alpha).$$

Therefore, maximum-likelihood estimation is $\hat{\alpha} = \hat{\alpha}_0 + H(-\hat{\alpha}_0)$, where $\hat{\alpha}_0 = -\frac{1}{\pi} \operatorname{atan}\left(\frac{n\pi}{\sum_{i=1}^{n} \ln \frac{1-X_i}{X_i}}\right)$ and $H(\cdot)$ is a Heaviside step function.

## 2.2 Balding-Nichols distribution

Notation:

$$X \sim \text{Balding-Nichols}(p, F)$$

with $p, F \in (0, 1)$. Relation to Beta distribution:

$$X \sim \mathcal{B}(pF', (1-p)F')$$

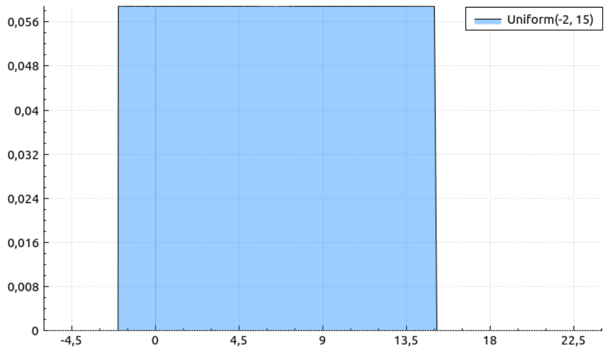
with $F' = (1 - F)/F$.

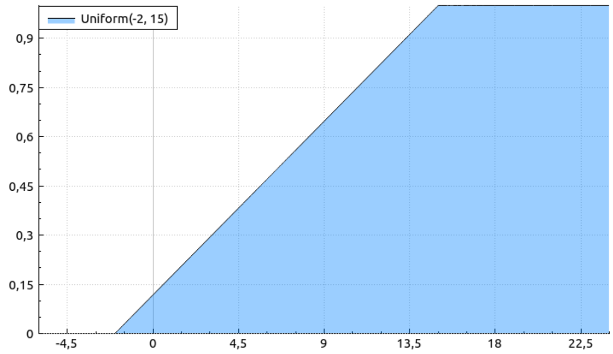
## 2.3 Uniform distribution

Relation to Beta distribution:

$$X \sim \mathcal{B}(1, 1, a, b).$$

**Estimation of support.**




| | |
|---|---|
| Notation | $X \sim \mathcal{U}(a, b)$ |
| Parameters | $a, b \in \mathbb{R}$ |
| Domain | $x \in [a, b]$ |
| $f(x)$ | $\frac{1}{b-a}$ |
| $F(x)$ | $\frac{x-a}{b-a}$ |
| $\mathbb{E}[X]$ | $\frac{a+b}{2}$ |
| $\mathrm{Var}(X)$ | $\frac{(b-a)^2}{12}$ |
| Median | $\frac{a+b}{2}$ |
| Mode | doesn't exist |
| $\phi(t)$ | $\frac{e^{itb}-e^{ita}}{it(b-a)}$ |

**Frequentist inference.** Likelihood function is

$$\mathcal{L}(a, b|X) = \frac{1}{(b-a)^n} \mathbf{1}_{\{X_i \in [a,b] \ \forall i=1,\dots,n\}}.$$

Therefore, $\mathcal{L}(a, b|X)$ is the largest for $\hat{b} = X_{(n)}$ and $\hat{a} = X_{(1)}$. However, using the fact that $X_{(k)} \sim B(k, n+1-k, a, b)$, these are biased estimators:

$$\mathbb{E}[X_{(1)}] = \frac{an + b}{n+1} \quad \text{and} \quad \mathbb{E}[X_{(n)}] = \frac{a + bn}{n+1}.$$

To get unbiased estimators we make the transformations:

$$\tilde{a} = \frac{nX_{(1)} - X_{(n)}}{n-1} \quad \text{and} \quad \tilde{b} = \frac{nX_{(n)} - X_{(1)}}{n-1}.$$

Then we get

$$\mathbb{E}[\tilde{a}] = \frac{n\mathbb{E}[X_{(1)}] - \mathbb{E}[X_{(n)}]}{n-1} = \frac{n(an+b) - (a+bn)}{n^2-1} = a.$$

Analogously, $\mathbb{E}[\tilde{b}] = b$.

**Bayesian inference.** Let us say, we try to estimate $\theta = b - a$ with known $a$. We set the prior distribution $\theta \sim \text{Pareto}(\alpha, \sigma)$:

$$h(\theta|\alpha, \sigma) = \frac{\alpha\sigma^\alpha}{\theta^{\alpha+1}} \mathbf{1}_{\{\theta \geq \sigma\}}.$$

The density of posterior distribution is

$$f(\theta|X) \propto \frac{\alpha\sigma^\alpha}{\theta^{\alpha+n+1}} \mathbf{1}_{\{\theta \geq \max(\sigma, X_{(n)} - a)\}} \sim \text{Pareto}(\alpha + n, \max(\sigma, X_{(n)} - a)).$$

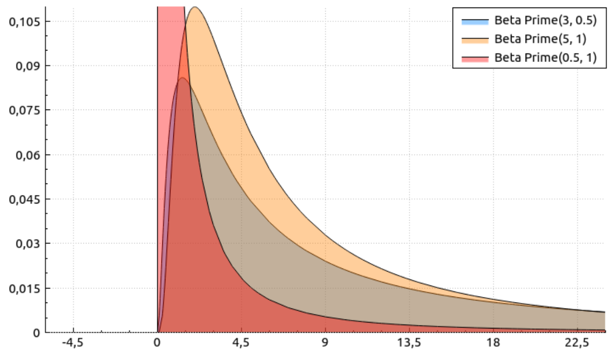
Hence, Bayesian estimator is

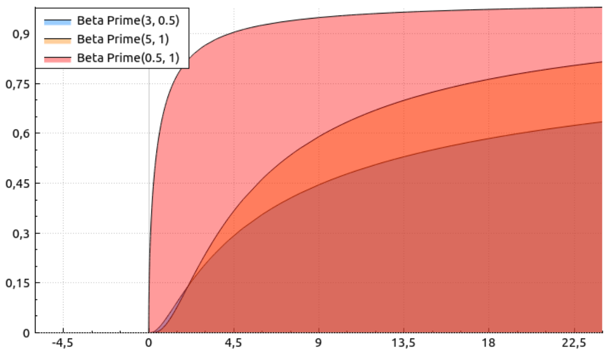
$$\mathbb{E}[\theta|X] = \frac{\alpha + n}{\alpha + n - 1} \max(\sigma, X_{(n)} - a)$$

and MAP estimator is

$$\theta_{MAP} = \max(\sigma, X_{(n)} - a).$$

## 3 Beta-prime distribution




| | |
|---|---|
| Notation | $X \sim \mathcal{B}'(\alpha, \beta)$ |
| Parameters | $\alpha, \beta > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$ |
| $F(x)$ | $I_{\frac{x}{1+x}}(\alpha, \beta)$ |
| $\mathbb{E}[X]$ | $\frac{\alpha}{\beta-1}\mathbf{1}_{\{\beta>1\}} + \infty\mathbf{1}_{\{\beta\leq 1\}}$ |
| $\mathrm{Var}(X)$ | $\frac{\alpha(\alpha+\beta-1)}{(\beta-2)(\beta-1)^2}$, if $\beta > 1$ |
| Median | Searched numerically |
| Mode | $\max\left(\frac{\alpha-1}{\beta+1}, 0\right)$. |
| $\phi(t)$ | Calculated numerically |

Relation to other distributions:

$$\frac{X}{1+X} \sim \mathcal{B}(\alpha, \beta),$$

$$\frac{\beta}{\alpha} X \sim F(2\alpha, 2\beta).$$

**Search of the median.** For $\alpha = \beta$ we have $m = 1$. Otherwise, we use the relation $m = \frac{m'}{1-m'}$, where $m'$ is the median of beta-distribution $\mathcal{B}(\alpha, \beta)$.

**Calculation of characteristic function.** For $\alpha \geq 1$ one can use numerical integration from section For $\alpha < 1$ we have $\lim_{x\to 0} f(x) \to \infty$ and $\int_0^\infty \cos(tx) f(x) dx$ is impossible to compute directly. Then we split the integral:

$$\int_0^\infty \cos(tx) f(x) dx = \int_0^\infty (\cos(tx) - 1) f(x) dx + 1.$$

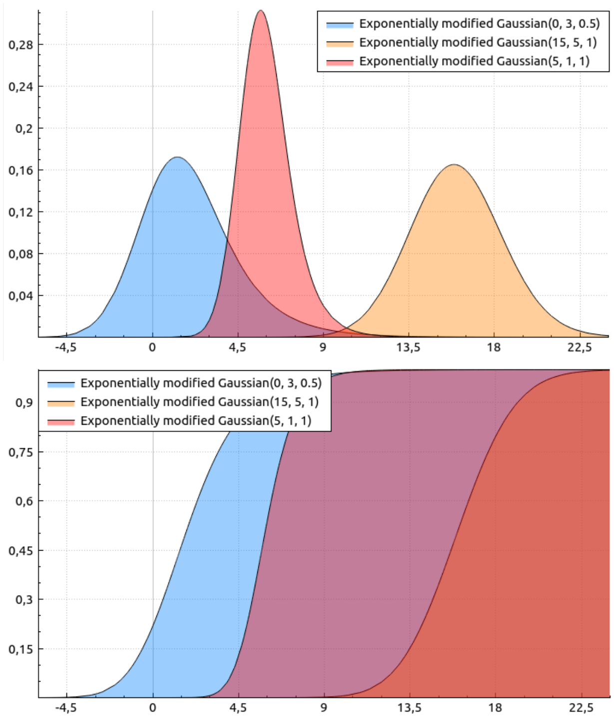
The limit of the integrand for $x \to 0$ is 0 now, regardless of the value of the shape $\alpha$.

**Estimation of shapes.** Using relationship with Beta distribution we transform the sample:

$$Y_i = \frac{X_i}{1 + X_i}, \quad 1 \le i \le N,$$

and run estimation for beta-distributed $Y$.

# 4 Exponentially-modified Gaussian distribution




| | |
|---|---|
| Notation | $X \sim \text{EMG}(\mu, \sigma, \lambda)$ |
| Parameters | $\mu \in \mathbb{R}, \sigma > 0, \lambda > 0$ |
| Domain | $x \in \mathbb{R}$ |
| $f(x)$ | $\frac{\lambda}{2} e^{\frac{\lambda}{2}(2\mu+\lambda\sigma^2-2x)} \operatorname{erfc}\left(\frac{\mu+\lambda\sigma^2-x}{\sqrt{2\sigma^2}}\right)$ |
| $F(x)$ | $\Phi(u, 0, v) - e^{-u+\frac{v^2}{2}+\log\Phi(u, v^2, v)}$, where $\Phi(x, \mu, \sigma)$ is Gaussian CDF, $u = \lambda(x - \mu)$, $v = \lambda\sigma$. |
| $\mathbb{E}[X]$ | $\mu + 1/\lambda$ |
| $\text{Var}(X)$ | $\sigma^2 + 1/\lambda^2$ |
| Median | Searched numerically |
| Mode | Searched numerically |
| $\phi(t)$ | $\left(1 - \frac{it}{\lambda}\right)^{-1} \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$ |

Relation to other distribution: if $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \text{Exp}(\lambda)$, then $X + Y \sim \text{EMG}(\mu, \sigma, \lambda)$.

## 5 F-distribution

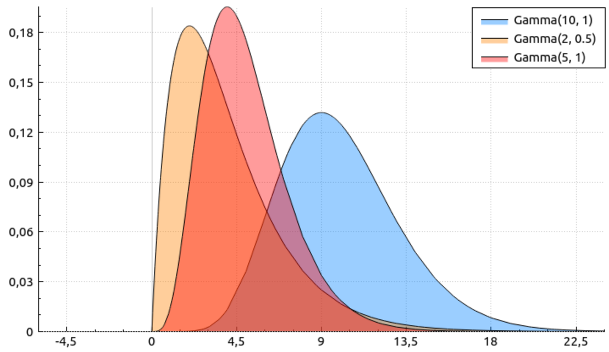
| | |
|---|---|
| Notation | $X \sim \mathrm{F}(d_1, d_2)$ |
| Parameters | $d_1, d_2 > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{\sqrt{\frac{(d_1 x)^{d_1} d_2{}^{d_2}}{(d_1 x + d_2)^{d_1+d_2}}}}{x B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$ |
| $F(x)$ | $I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right)$ |
| $\mathbb{E}[X]$ | $\frac{d_2}{d_2 - 2}$ for $d_2 > 2$ |
| $\mathrm{Var}(X)$ | $\frac{2 d_2^2 (d_1 + d_2 - 2)}{d_1 (d_2 - 2)^2 (d_2 - 4)}$ for $d_2 > 4$ |
| Median | Searched numerically |
| Mode | $\max\left(\frac{d_2(d_1 - 2)}{d_1(d_1 + 2)}, 0\right)$ |
| $\phi(t)$ | Calculated numerically |

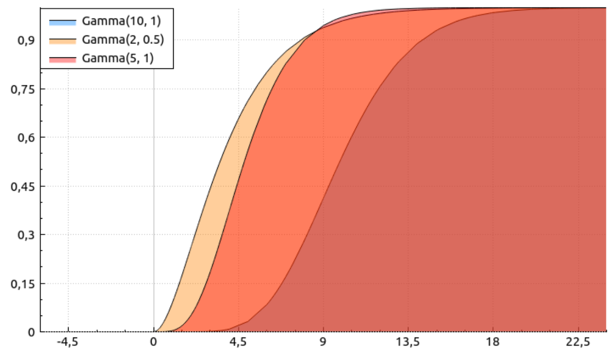
Relation to other distributions:

$$\frac{d_1 X}{d_2 + d_1 X} \sim \mathcal{B}\left(\frac{d_1}{2}, \frac{d_2}{2}\right),$$

$$\frac{d_1}{d_2} X \sim \mathcal{B}'\left(\frac{d_1}{2}, \frac{d_2}{2}\right).$$

# 6 Gamma distribution




| | |
|---|---|
| Notation | $X \sim \Gamma(\alpha, \beta)$ |
| Parameters | $\alpha > 0, \beta > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ |
| $F(x)$ | $P(\alpha, \beta x)$ |
| $\mathbb{E}[X]$ | $\frac{\alpha}{\beta}$ |
| $\text{Var}(X)$ | $\frac{\alpha}{\beta^2}$ |
| Median | Searched numerically |
| Mode | $\max\left(\frac{\alpha-1}{\beta}, 0\right)$ |
| $\phi(t)$ | $\left(1 - \frac{it}{\beta}\right)^{-\alpha}$ |

**More properties.**

- $\mathbb{E}[\ln X] = \psi(\alpha) - \ln(\beta)$, $\text{Var}(\ln X) = \psi^{(1)}(\alpha)$.
- $\mathbb{E}\left[\frac{1}{X}\right] = \frac{\beta}{\alpha - 1}$.
- Let $X_i \sim \Gamma(\alpha_i, \beta)$ for $i = 1, \ldots, n$. Then

$$\sum_{i=1}^n X_i \sim \Gamma\left(\sum_{i=1}^n \alpha_i, \beta\right).$$

**Estimation of parameters.**

**Frequentist inference.** Log-likelihood function:

$$\ln \mathcal{L}(\alpha, \beta | X) = n\alpha \ln \beta - n \ln \Gamma(\alpha) + (\alpha - 1) \sum_{i=1}^n \ln X_i - \beta \sum_{i=1}^n X_i.$$

Derivatives:

$$\frac{\partial \ln \mathcal{L}(\alpha, \beta | X)}{\partial \alpha} = n \ln \beta - n\psi(\alpha) + \sum_{i=1}^n \ln X_i,$$

$$\frac{\partial \ln \mathcal{L}(\alpha, \beta|X)}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} X_i.$$

While the solution for the second equation is analytic:

$$\hat{\beta} = \frac{\alpha}{\overline{X}_n},$$

the first equation is solved numerically, using second derivative:

$$\frac{\partial^2 \ln \mathcal{L}(\alpha, \beta|X)}{\partial \alpha^2} = -n\psi_1(\alpha),$$

or if $\beta$ is unknown:

$$\frac{\partial^2 \ln \mathcal{L}(\alpha, \beta|X)}{\partial \alpha^2} = -n\psi_1(\alpha) + \frac{n}{\alpha},$$

Moreover, the maximum-likelihood estimation of rate $\beta$ is biased:

$$\begin{aligned} \mathbb{E}[\hat{\beta}] &= \mathbb{E}\left[\frac{\alpha n}{\sum_{i=1}^{n} X_i}\right] \\ &= \frac{\alpha n \beta}{\alpha n - 1}. \end{aligned}$$

Unbiased estimator will be

$$\tilde{\beta} = \frac{\alpha}{\overline{X}_n}\Big(1 - \frac{1}{n}\Big).$$

**Bayesian inference.** We suppose that prior distribution of rate $\beta$ is $\Gamma(\kappa, \gamma)$:

$$h(\beta) = \frac{\gamma^{\kappa}}{\Gamma(\kappa)} \beta^{\kappa-1} e^{-\gamma\beta}.$$

Then

$$f(\beta|X) \propto \beta^{\alpha n} e^{-\beta \sum_{i=1}^{n} X_i} \cdot \beta^{\kappa-1} e^{-\gamma\beta} \sim \Gamma\left(\alpha n + \kappa, \gamma + \sum_{i=1}^{n} X_i\right).$$

Therefore, Bayesian estimator is

$$\mathbb{E}[\beta|X] = \frac{\alpha n + \kappa}{\gamma + \sum_{i=1}^{n} X_i},$$

and MAP estimator is

$$\beta_{MAP} = \frac{\alpha n + \kappa - 1}{\gamma + \sum_{i=1}^{n} X_i}.$$

**Exponential family parameterization** Logarithm of probability mass function:

$$\log \mathbb{P}(X = x) = \alpha \log \beta - \log \Gamma(\alpha) + (\alpha - 1) \log x - \beta x.$$

Therefore, sufficient statistics $T(x) = (\log x, x)^T$, natural parameters $\theta = (\alpha - 1, -\beta)$, log-normalizer $F(\theta) = \log \Gamma(\theta_1 + 1) - (\theta_1 + 1)\log(-\theta_2)$, carrier measure $k(x) = 0$. Gradient of log-normalizer is $\nabla F(\theta) = \left(\psi(\theta_1 + 1) - \log(-\theta_2), -\frac{\theta_1 + 1}{\theta_2}\right)^T$ We conclude that adjusted cross-entropy is

$$\begin{aligned}
H_F(\theta_p \| \theta_q) &= F(\theta_q) - \langle \theta_q, \nabla F(\theta_p) \rangle \\
&= \log \Gamma(\theta_{q1} + 1) - (\theta_{q1} + 1) \log(-\theta_{q2}) - \theta_{q1}(\psi(\theta_{p1} + 1) - \log(-\theta_{p2})) + \frac{\theta_{q2}(\theta_{p1} + 1)}{\theta_{p2}}.
\end{aligned}$$

Adjusted entropy is

$$\begin{aligned}
H_F(\theta) &= \log \Gamma(\theta_1 + 1) - \log(-\theta_2) - \theta_1 \psi(\theta_1 + 1) + \theta_1 + 1 \\
&= \log \Gamma(\alpha) - \log \beta - (\alpha - 1) \cdot \psi(\alpha) + \alpha.
\end{aligned}$$

And Kullback-Leibler divergence:

$$\begin{aligned}
\mathrm{KL}(p \| q) &= H_F(\theta_p \| \theta_q) - H_F(\theta_p) \\
&= \log \frac{\Gamma(\alpha_q)}{\Gamma(\alpha_p)} + \alpha_q \log \frac{\beta_p}{\beta_q} + (\alpha_p - \alpha_q)\psi(\alpha_p) + \alpha_p \left( \frac{\beta_q}{\beta_p} - 1 \right)
\end{aligned}$$

## 6.1 Chi-squared distribution

Notation:

$$X \sim \chi^2_k.$$

Relation to Gamma distribution:

$$X \sim \Gamma\left(\frac{k}{2}, \frac{1}{2}\right).$$

Kullback-Leibler divergence:

$$\mathrm{KL}(p \| q) = \log \frac{\Gamma(k_q/2)}{\Gamma(k_p/2)} + \frac{1}{2}(k_p - k_q)\psi(k_p/2).$$

Relation to other distributions: if $X_1, \ldots, X_k \sim \mathcal{N}(0, 1)$, then $\sum_{i=1}^k X_i^2 \sim \chi^2_k$.

## 6.2 Erlang distribution

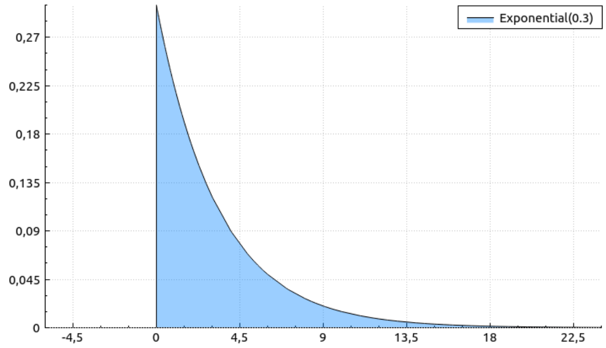
Notation:

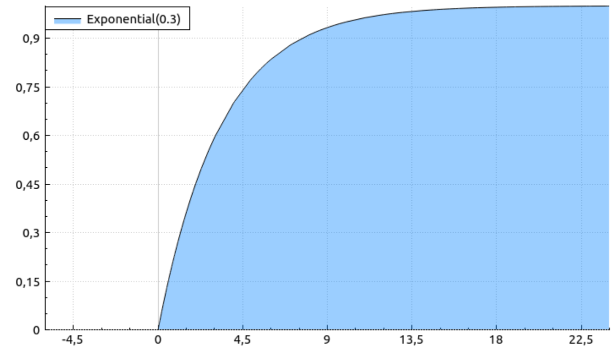
$$X \sim \mathrm{Erlang}(k, \beta).$$

The only difference between Gamma and Erlang distributions is that latter takes an integer number $k$ as a shape parameter. Relation to other distributions: if $X \sim \mathrm{Erlang}(k, \beta)$ and $Y \sim \mathrm{Po}(\beta x)$, then

$$\mathbb{P}(X < x) = P(k, \beta x) = \mathbb{P}(Y > k).$$

## 6.3 Exponential distribution




| | |
|---|---|
| Notation | $X \sim \text{Exp}(\lambda)$ |
| Parameters | $\lambda > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\lambda e^{-\lambda x}$ |
| $F(x)$ | $1 - e^{-\lambda x}$ |
| $\mathbb{E}[X]$ | $\frac{1}{\lambda}$ |
| $\text{Var}(X)$ | $\frac{1}{\lambda^2}$ |
| Median | $\frac{\ln(2)}{\lambda}$ |
| Mode | $0$ |
| $\phi(t)$ | $\frac{\lambda}{\lambda - it}$ |

Relation to Gamma distribution:

$$X \sim \Gamma(1, \lambda).$$

Hence, estimation of parameter $\lambda$ is the particular case of estimation of rate $\beta$ for Gamma distribution.
Adjusted cross-entropy:

$$H_F(\lambda_p \| \lambda_q) = \frac{\lambda_q}{\lambda_p} - \log \lambda_q.$$

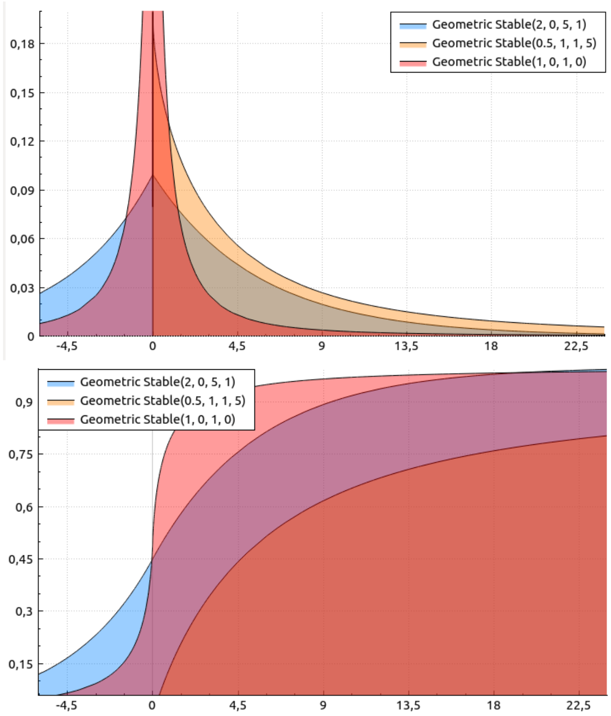
Thus adjusted entropy is

$$H_F(\lambda) = 1 - \log \lambda$$

and Kullback-Leibler divergence:

$$\text{KL}(p\|q) = \log \frac{\lambda_p}{\lambda_q} + \frac{\lambda_q}{\lambda_p} - 1.$$

# 7 Geometric Stable distribution




| | |
|---|---|
| Notation | $X \sim \mathrm{GS}_\alpha(\beta, \gamma, \mu)$ |
| Parameters | $\alpha \in (0, 2], \beta \in [-1, 1],$ $\gamma > 0, \mu \in \mathbb{R}$ |
| Domain | $x \in \ldots$ |
| $f(x)$ | Calculated numerically |
| $F(x)$ | Calculated numerically |
| $\mathbb{E}[X]$ | $k + \lambda$ |
| $\mathrm{Var}(X)$ | $2(k + 2\lambda)$ |
| Median | Searched numerically |
| Mode | Searched numerically |
| $\phi(t)$ | ... |

## 7.1 Asymmetric Laplace distribution

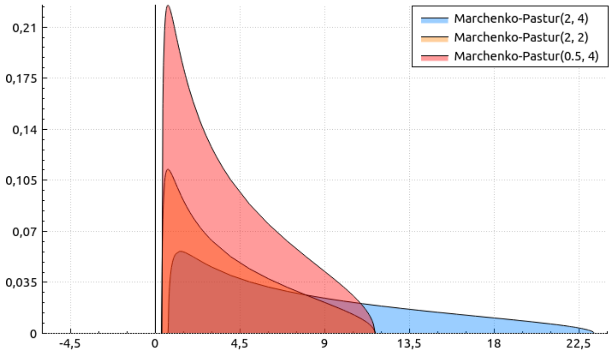
## 7.2 Laplace distribution

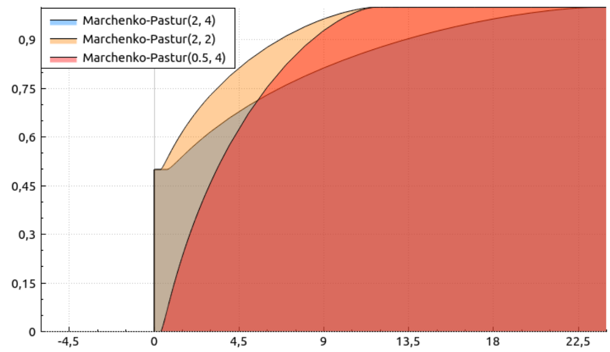
# 8 Kolmogorov-Smirnov distribution

# 9 Logistic distribution

## 10 Log-normal distribution

# 11 Marchenko-Pastur distribution




| | |
|---|---|
| Notation | $X \sim \mathcal{MP}(\lambda, \sigma^2)$ |
| Parameters | $\lambda, \sigma^2 > 0$ |
| Domain | $x \in [\sigma^2 a, \sigma^2 b]$, if $\lambda < 1$,<br>$x \in [\sigma^2 a, \sigma^2 b] \cup \{0\}$, otherwise,<br>where $a = (1 - \sqrt{\lambda})^2$ and $b = (1 + \sqrt{\lambda})^2$ |
| $f(x)$ | $\left(1 - \frac{1}{\lambda}\right)_+ \delta_0(x) + \frac{\sqrt{(\sigma^2 b - x)(x - \sigma^2 a)}}{2\pi\lambda\sigma^2 x}$ |
| $F(x)$ | Read text |
| $\mathbb{E}[X]$ | $\sigma^2$ |
| $\mathrm{Var}(X)$ | $\sigma^4 \lambda$ |
| Median | 0 if $\lambda > 2$,<br>otherwise searched numerically |
| Mode | $\frac{\sigma^2(\lambda-1)^2}{\lambda+1}$, if $\lambda < 1$,<br>0, otherwise |
| $\phi(t)$ | Calculated numerically |

**Calculation of cumulative distribution function.** ...

**Calculation of characteristic function.** For $\lambda > 1$ we use numerical integration by definition

$$\phi(t) = \int_{\sigma^2 a}^{\sigma^2 b} \cos(tx) f(x) dx + i \int_{\sigma^2 a}^{\sigma^2 b} \sin(tx) f(x) dx.$$

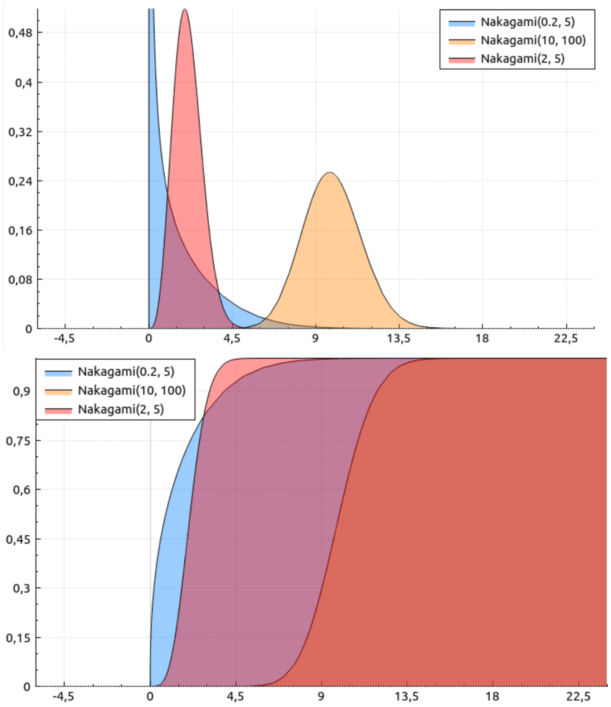
For $\lambda = 1$ we split the integrand for real part by $(\cos(tx) - 1) f(x)$ and $f(x)$:

$$\Re(\phi(t)) = \int_{\sigma^2 a}^{\sigma^2 b} (\cos(tx) - 1) f(x) dx + 1.$$

And for $\lambda < 1$ we calculate integral at point 0 separately:

$$\begin{aligned}\phi(t) &= \int_{\{0\} \cup [\sigma^2 a, \sigma^2 b]} \cos(tx) f(x) dx + i \int_{\{0\} \cup [\sigma^2 a, \sigma^2 b]} \sin(tx) f(x) dx \\ &= 1 - \frac{1}{\lambda} + \int_{\sigma^2 a}^{\sigma^2 b} \cos(tx) f(x) dx + i \int_{\sigma^2 a}^{\sigma^2 b} \sin(tx) f(x) dx.\end{aligned}$$

# 12 Nakagami distribution




| | |
|---|---|
| Notation | $X \sim \text{Nakagami}(\mu, \omega)$ |
| Parameters | $\mu, \omega > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{2\mu^\mu}{\Gamma(\mu)\omega^\mu} x^{2\mu-1} e^{-\frac{\mu}{\omega}x^2}$ |
| $F(x)$ | $P(\mu, \mu x^2/\omega)$ |
| $\mathbb{E}[X]$ | $\frac{\Gamma(\mu+1/2)}{\Gamma(\mu)} \sqrt{\frac{\omega}{\mu}}$ |
| $\text{Var}(X)$ | $\omega - (\mathbb{E}[X])^2$ |
| Median | Searched numerically |
| Mode | $\max\left(\sqrt{\frac{(2\mu-1)\omega}{2\mu}}, 0\right)$ |
| $\phi(t)$ | Calculated numerically |

Relation to other distributions: if $Y \sim \Gamma\big(\mu, \mu/\omega\big)$, then

$$X \sim \text{Nakagami}(\mu, \omega).$$

**Calculation of characteristic function.** For $\mu < 1$ $\lim_{x \to 0} f(x) \to \infty$. Then we use the following transformation for real part of characteristic function:

$$\begin{aligned}\Re(\phi(t)) &= \int_0^\infty \cos(tx) f(x) dx \\ &= \int_0^\infty (\cos(tx) - 1) f(x) + 1\end{aligned}$$

**Exponential family parameterization** Logarithm of probability mass function:

$$\log \mathbb{P}(X = x) = \mu \log(\mu/\omega) - \log(\Gamma(\mu)/2) + (2\mu - 1)\log x - \mu x^2/\omega.$$

Therefore, sufficient statistics $T(x) = (\log x, x^2)^T$, natural parameters

$$\theta = (2\mu - 1, -\mu/\omega),$$

log-normalizer

$$F(\theta) = \log \frac{\Gamma((\theta_1 + 1)/2)}{2} - \frac{\theta_1 + 1}{2} \log(-\theta_2),$$

carrier measure $k(x) = 0$. Gradient of log-normalizer is

$$\nabla F(\theta) = \left(\frac{1}{2}\psi\Big(\frac{\theta_1 + 1}{2}\Big), \frac{\theta_1 + 1}{2\theta_2}\right)^T$$

We conclude that adjusted cross-entropy is

$$\begin{aligned} H_F(\theta_p \| \theta_q) &= F(\theta_q) - \langle \theta_q, \nabla F(\theta_p) \rangle \\ &= \frac{1}{2}\bigg( \log\Gamma\Big(\frac{\theta_{q1} + 1}{2}\Big) - (\theta_{q1} + 1)\log(-\theta_{q2}) - \theta_{q1}\psi\Big(\frac{\theta_{p1} + 1}{2}\Big) - \frac{\theta_{q2}(\theta_{p1} + 1)}{\theta_{p2}}\bigg). \end{aligned}$$

Adjusted entropy is

$$\begin{aligned} H_F(\theta) &= \frac{1}{2}\bigg( \log\Gamma\Big(\frac{\theta_1 + 1}{2}\Big) - (\theta_1 + 1)\log(-\theta_2) - \theta_1\psi\Big(\frac{\theta_1 + 1}{2}\Big) - (\theta_1 + 1)\bigg) \\ &= \frac{\log\Gamma(\mu)}{2} - \mu\log(\mu/\omega) - \frac{2\mu - 1}{2}\psi(\mu) - \mu. \end{aligned}$$

And Kullback-Leibler divergence:

$$\begin{aligned} \mathrm{KL}(p\|q) &= H_F(\theta_p\|\theta_q) - H_F(\theta_p) \\ &= \frac{1}{2}\log\frac{\Gamma(\mu_q)}{\Gamma(\mu_p)} + \mu_p\log\frac{\mu_p}{\omega_p} - \mu_q\log\frac{\mu_q}{\omega_q} + (\mu_p - \mu_q)\psi(\mu_p) - \bigg(\frac{\mu_q/\omega_q}{\mu_p/\omega_p} - 1\bigg)\mu_p \end{aligned}$$

## 12.1 Chi distribution

Notation:

$$X \sim \chi_k$$

Relation to Nakagami distribution:

$$X \sim \text{Nakagami}\big(k/2, k\big).$$

Adjusted cross-entropy:

$$H_F(k_p\|k_q) = \frac{1}{2}\bigg( \log\Gamma(k_q/2) + k_q\log 2 - (2k_q - 1)\psi(k_p/2) - k_p\bigg).$$

Thus adjusted entropy is

$$H_F(k) = \frac{1}{2}\bigg( \log\Gamma(k/2) + k\log 2 - (2k - 1)\psi(k/2) - k\bigg)$$

and Kullback-Leibler divergence:

$$\mathrm{KL}(p\|q) = \frac{1}{2}\bigg( \log\frac{\Gamma(k_q/2)}{\Gamma(k_p/2)} + (\log 2 - 2\psi(k_p/2))(k_q - k_p)\bigg).$$

### 12.2 Maxwell-Bolzmann distribution

Notation:

$$X \sim \mathrm{MB}(\sigma)$$

Relation to Nakagami distribution:

$$X \sim \text{Nakagami}\left(3/2, \sigma^2\right).$$

Adjusted cross-entropy:

$$H_F(\sigma_p \| \sigma_q) = ...$$

Thus adjusted entropy is

$$H_F(\sigma) = ...$$

and Kullback-Leibler divergence:

$$\mathrm{KL}(p\|q) = ...$$

### 12.3 Rayleigh distribution

Notation:

$$X \sim \text{Rayleigh}(\sigma)$$

Relation to Nakagami distribution:

$$X \sim \text{Nakagami}\left(1, 2\sigma^2\right).$$

**Estimation of scale.** ...

Adjusted cross-entropy:

$$H_F(\sigma_p \| \sigma_q) = ...$$

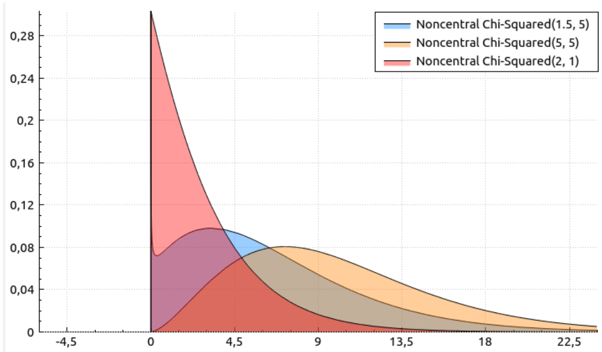
Thus adjusted entropy is

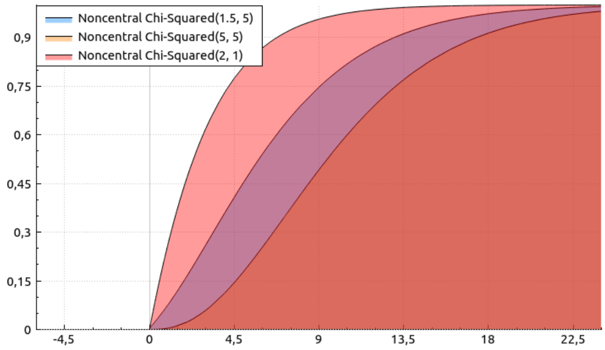
$$H_F(\sigma) = \log(2\sigma^2) + \gamma/2 - 1$$

and Kullback-Leibler divergence:

$$\mathrm{KL}(p\|q) = ...$$

# 13 Noncentral Chi-Squared distribution



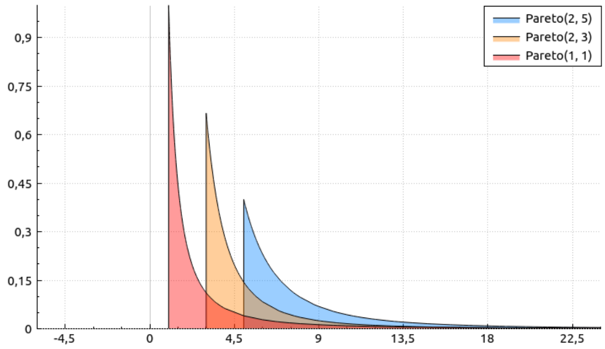



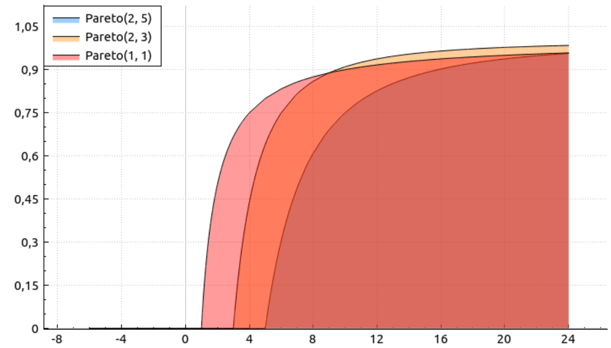

| | |
|---|---|
| Notation | $X \sim \chi_k'^2(\lambda)$ |
| Parameters | $k > 0, \lambda > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{1}{2}e^{-\frac{x+\lambda}{2}}\left(\frac{x}{\lambda}\right)^{\frac{k-2}{4}} I_{\frac{k}{2}-1}(\sqrt{\lambda x})$ |
| $F(x)$ | $\mathrm{MarcumP}_{\frac{k}{2}}(\frac{\lambda}{2}, \frac{x}{2})$ |
| $\mathbb{E}[X]$ | $k + \lambda$ |
| $\mathrm{Var}(X)$ | $2(k + 2\lambda)$ |
| Median | Searched numerically |
| Mode | Searched numerically for $k > 2$, 0, otherwise |
| $\phi(t)$ | $\frac{\exp \frac{it\lambda}{1-2it}}{(1-2it)^{k/2}}$ |

Relation to other distributions:

- Let $X_1, \dots, X_k$ be independent with $X_i \sim \mathcal{N}(\mu_i, 1)$, $i = 1, \dots, k$. Then
$$\sum_{i=1}^{k} X_i^2 \sim \chi_k'^2\big(\sum_{i=1}^{k} \mu_i^2\big).$$
- If $\lambda = 0$, then $X \sim \chi_k^2$.
- If $J \sim \mathrm{Po}(\lambda)$, then $\chi_{k+2J}^2 \sim \chi_k'^2(\lambda)$.

# 14 Pareto distribution




| | |
|---|---|
| Notation | $X \sim \text{Pareto}(\alpha, \sigma)$ |
| Parameters | $\alpha, \sigma > 0$ |
| Domain | $x \geq \sigma$ |
| $f(x)$ | $\frac{\alpha\sigma^\alpha}{x^{\alpha+1}}$ |
| $F(x)$ | $1 - \left(\frac{\sigma}{x}\right)^\alpha$ |
| $\mathbb{E}[X]$ | $\frac{\alpha\sigma}{\alpha-1}$ for $\alpha > 1$,<br>$\infty$ otherwise |
| $\text{Var}(X)$ | $\frac{\sigma^2\alpha}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$,<br>$\infty$ otherwise |
| Median | $\sigma 2^{1/\alpha}$ |
| Mode | $\sigma$ |
| $\phi(t)$ | Calculated numerically |

**Estimation of parameters.**

**Frequentist inference.** Log-likelihood function is

$$\ln \mathcal{L}(\alpha, \sigma | X) = n \ln \alpha + n\alpha \ln \sigma - (\alpha + 1) \sum_{i=1}^{n} \ln X_i.$$

We assume that $\sigma \leq X_{(1)}$, otherwise sample $X$ couldn't have been generated from such distribution. It is obvious, that $\ln \mathcal{L}(\alpha, \sigma | X)$ is an increasing function in terms of $\sigma$, therefore $\hat{\sigma} = X_{(1)}$ is an optimal estimator. Let's take derivative with respect to $\alpha$:

$$\frac{\partial \ln \mathcal{L}(\alpha, \sigma | X)}{\partial \alpha} = \frac{n}{\alpha} + n \ln \sigma - \sum_{i=1}^{n} \ln X_i.$$

From this we conclude that the maximum-likelihood estimator of shape is

$$\hat{\alpha} = \frac{1}{\frac{1}{n}(\sum_{i=1}^{n} \ln X_i) - \ln \hat{\sigma}}.$$

It is known that $\hat{\sigma} \sim \text{Pareto}(n\alpha, \sigma)$ and $\hat{\alpha} \sim \text{Inv-}\Gamma(n-1, n\alpha)$ and they are independent. Then

$$\mathbb{E}[\hat{\sigma}] = \frac{\sigma}{1 - \frac{1}{n\alpha}}$$

and

$$\mathbb{E}[\hat{\alpha}] = \frac{n\alpha}{n-2}.$$

Therefore, in order to get unbiased estimators we need to make the following transformations:

$$\tilde{\alpha} = \frac{n-2}{n}\hat{\alpha} \quad \text{and} \quad \tilde{\sigma} = \hat{\sigma}\left(1 - \frac{1}{(n-1)\hat{\alpha}}\right).$$

Note that if we estimate parameters separately, then $\hat{\alpha} \sim \text{Inv-}\Gamma(n, n\alpha)$ and transformations are different.

**Bayesian inference.** We now assume that $\sigma$ is known and prior distribution of $\alpha$ is $\Gamma(\kappa, \beta)$:

$$h(\alpha) = \frac{\beta^\kappa}{\Gamma(\kappa)}\alpha^{\kappa-1}e^{-\beta\alpha}.$$

The density of posterior distribution is

$$f(\alpha|X) \propto \prod_{i=1}^{n} \frac{\sigma^\alpha}{X_i^{\alpha-1}} \cdot \alpha^{\kappa+n-1}e^{-\beta\alpha} \propto \alpha^{\kappa+n-1}e^{-(\beta+\sum_{i=1}^{n}\ln(X_i/\sigma))\alpha}.$$

Therefore, $\alpha|X \sim \Gamma(\kappa+n, \beta + \sum_{i=1}^{n}\ln(X_i/\sigma))$ and Bayesian estimator is

$$\mathbb{E}[\alpha|X] = \frac{\kappa+n}{\beta + \sum_{i=1}^{n}\ln(X_i/\sigma)}.$$

MAP estimator is

$$\alpha_{MAP} = \frac{\kappa+n-1}{\beta + \sum_{i=1}^{n}\ln(X_i/\sigma)}.$$

*Note on fitting scale with Bayes:* let it be vice versa, $\alpha$ is known while $\sigma$ is not. Then we say that a priori $\sigma \sim \text{Pareto}(\kappa, \theta)$:

$$h(\sigma) = \frac{\kappa\theta^\kappa}{\sigma^{\kappa+1}}.$$

Then posterior distribution is:

$$f(\sigma|X) \propto \prod_{i=1}^{n} \frac{1}{X_i^{\alpha-1}} \cdot \sigma^{\alpha n-\kappa-1}\mathbf{1}_{\{\theta<\sigma<X_{(1)}\}} \sim \text{Bounded-Pareto}(\kappa - \alpha n, \theta, X_{(1)}).$$

This imposes the following additional constraints on the prior hyperparameters: $\kappa > \alpha n$ and $\theta < X_{(1)}$. Bayesian estimator:

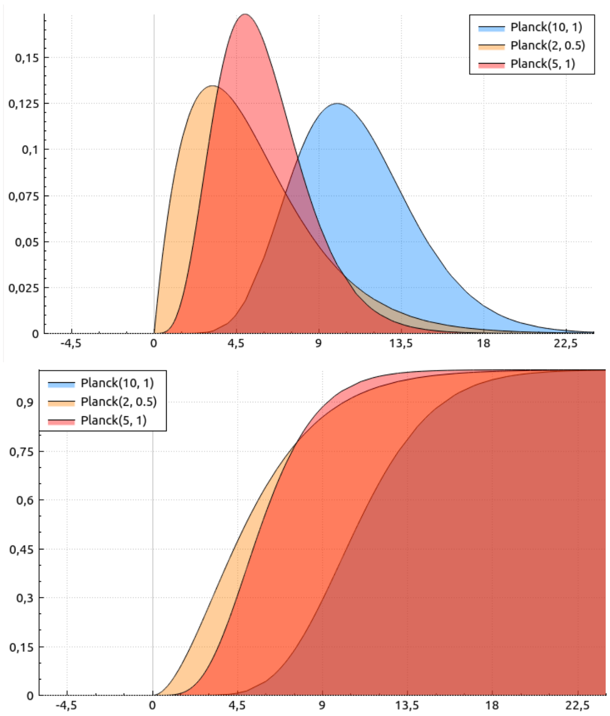
$$\mathbb{E}[\sigma|X] = \frac{\theta^{\alpha'}}{1 - \left(\frac{\theta}{X_{(1)}}\right)^{\alpha'}} \cdot \left(\frac{\alpha'}{\alpha'-1}\right) \cdot \left(\frac{1}{\theta^{\alpha'}} - \frac{1}{X_{(1)}^{\alpha'}}\right)$$

with $\alpha' = \kappa - \alpha n$. MAP estimator is just

$$\sigma_{MAP} = \theta.$$

However, Bounded-Pareto distribution is not yet supported in RandLib.

## 15 Planck distribution




| | |
|---|---|
| Notation | $X \sim \text{Planck}(a, b)$ |
| Parameters | $a, b > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{b^{a+1}}{\Gamma(a+1)\zeta(a+1)} \cdot \frac{x^a}{e^{bx}-1}$ |
| $F(x)$ | Calculated numerically |
| $\mathbb{E}[X]$ | $\frac{(a+1)\zeta(a+2)}{b\zeta(a+1)}$ |
| $\text{Var}(X)$ | $\frac{(a+1)(a+2)\zeta(a+3)}{b^2\zeta(a+1)} - (\mathbb{E}[X])^2$ |
| Median | Searched numerically |
| Mode | $\frac{W_0(-ae^{-a})+a}{b}$, if $a > 1$, otherwise 0 |
| $\phi(t)$ | Calculated numerically |

**Calculation of cumulative distribution function.** For $a \geq 1$ $F(x)$ can be calculated by straightforward numerical integration:

$$F(x) = \frac{b^{a+1}}{\Gamma(a+1)\zeta(a+1)} \int_0^x \frac{t^a}{e^{bt}-1} dt.$$

Note that for $\alpha < 1$ integrand has a singularity point at $t = 0$. In such case we define

$$h(t) = \frac{b^{a+2}t^{a+1}}{\Gamma(a+1)\zeta(a+1)} \cdot \left(\frac{1}{e^{bt}-1} - \frac{1}{bt}\right)$$

and then

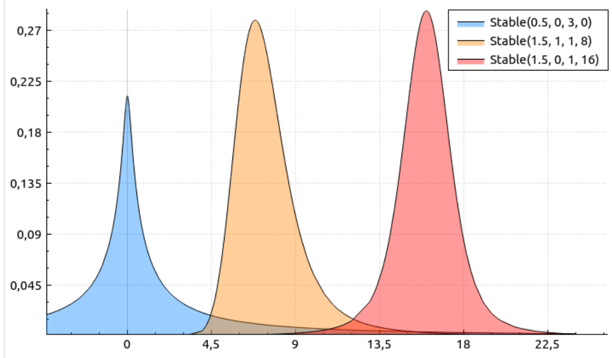
$$F(x) = \int_0^x h(t)dt + \frac{(bx)^a}{a\Gamma(a+1)\zeta(a+1)}.$$

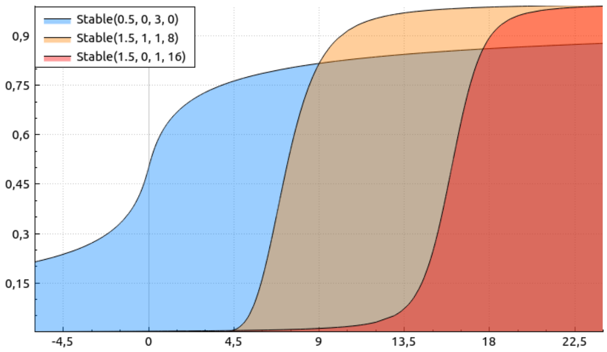
**Calculation of characteristic function.** The idea of calculations for $a < 1$ is near the same. We split the real part of $\phi(t)$ into 3 different integrals:

$$\Re(\phi(t)) = \int_0^1 \cos(tx)h(x)dx + \int_1^\infty \cos(tx)f(x)dx + \frac{b^a}{a\Gamma(a+1)\zeta(a+1)}\left(\cos(t) + t\int_0^1 \sin(tx)x^a dx\right).$$

All the indegrands now have no singularity points.

# 16 Stable distribution




| | |
|---|---|
| Notation | $X \sim S_\alpha(\beta, \gamma, \mu)$ |
| Parameters | $\alpha \in (0, 2], \beta \in [-1, 1]$,<br>$\gamma > 0, \mu \in \mathbb{R}$ |
| Domain | $x \in \mathbb{R}$, if $\beta \neq 1$,<br>$x \in [\mu, \infty)$, if $\beta = 1,\ \alpha < 2$,<br>$x \in (-\infty, \mu]$, if $\beta = -1,\ \alpha < 2$ |
| $f(x)$ | Calculated numerically |
| $F(x)$ | Calculated numerically |
| $\mathbb{E}[X]$ | $\mu$ for $\alpha > 1$,<br>otherwise undefined |
| $\mathrm{Var}(X)$ | $2\gamma^2 1_{\{\alpha=2\}} + \infty 1_{\{\alpha<2\}}$ |
| Median | $\mu$ for $\beta = 0$,<br>otherwise searched numerically |
| Mode | $\mu$, if $\beta = 0$ or $\alpha = 2$,<br>$\mu + \frac{\beta\gamma}{3}$, if $\|\beta\| = 1$ and $\alpha = \frac{1}{2}$,<br>otherwise searched numerically |
| $\phi(t)$ | ... |

**Calculation of p.d.f.**

**Calculation of c.d.f.**

## 16.1 Cauchy distribution

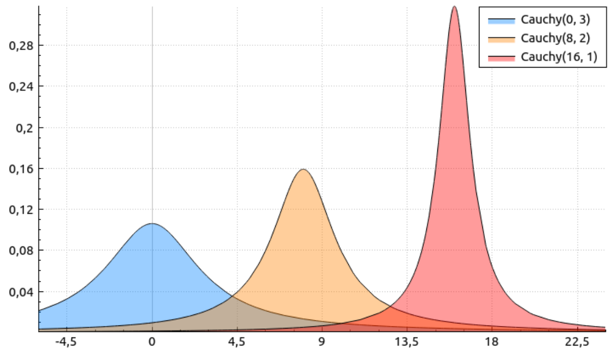
Relation to Stable distribution:

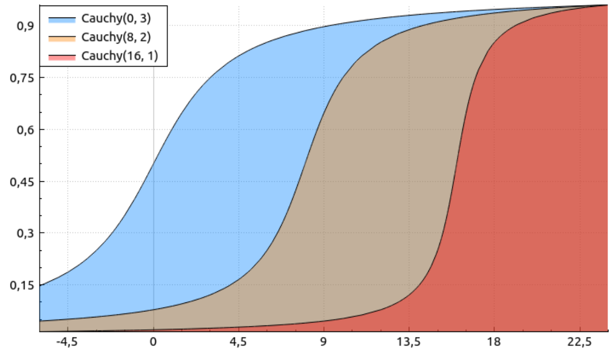
$$X \sim S_1(0, \gamma, \mu)$$

## 16.2 Levy distribution

Relation to Stable distribution:

$$X \sim S_{\frac{1}{2}}(1, \gamma, \mu)$$




| | |
|---|---|
| Notation | $X \sim \text{Cauchy}(\mu, \gamma)$ |
| Parameters | $\mu \in \mathbb{R}, \gamma^2 > 0$ |
| Domain | $x \in \mathbb{R}$ |
| $f(x)$ | $\frac{1}{\pi\gamma\left[1+\left(\frac{x-\mu}{\gamma}\right)^2\right]}$ |
| $F(x)$ | $\frac{1}{\pi}\,\text{atan}\left(\frac{x-\mu}{\gamma}\right) + \frac{1}{2}$ |
| $\mathbb{E}[X]$ | Undefined |
| $\text{Var}(X)$ | $\infty$ |
| Median | $\mu$ |
| Mode | $\mu$ |
| $\phi(t)$ | $e^{i\mu t - \gamma\lvert t\rvert}$ |

## 16.3 Normal distribution

Relation to Stable distribution:

$$X \sim S_2(\cdot, \sigma^2/2, \mu)$$

### Estimation of parameters

**Frequentist inference.** Maximum-likelihood estimators for Normal distribution are very well-known:

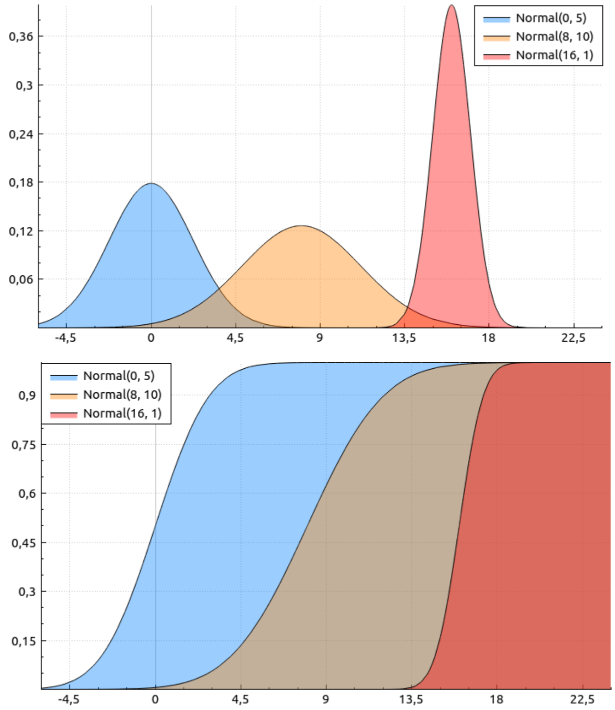
$$\hat{\mu} = \overline{X}_n \quad \text{and} \quad \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

However, for unknown $\mu$ the value of $\hat{\sigma^2} \sim \frac{\sigma^2}{n}\chi^2_{n-1}$. Therefore, unbiased estimator in this case would be

$$\widetilde{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

Moreover, if one is interested in estimating scale $\sigma$ with known $\mu$, then maximum likelihood estimator is

$$\hat{\sigma} = \sqrt{\hat{\sigma^2}} \sim \frac{\sigma}{\sqrt{n}}\chi_n$$




| | |
|---|---|
| Notation | $X \sim \mathcal{N}(\mu, \sigma^2)$ |
| Parameters | $\mu \in \mathbb{R}, \sigma^2 > 0$ |
| Domain | $x \in \mathbb{R}$ |
| $f(x)$ | $\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ |
| $F(x)$ | $\frac{1}{2}\operatorname{erfc}\left(\frac{\mu-x}{\sqrt{2\sigma^2}}\right)$ |
| $\mathbb{E}[X]$ | $\mu$ |
| $\mathrm{Var}(X)$ | $\sigma^2$ |
| Median | $\mu$ |
| Mode | $\mu$ |
| $\phi(t)$ | $e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$ |

and

$$\mathbb{E}[\hat{\sigma}] = \frac{\sigma}{\sqrt{n}}\sqrt{2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}.$$

Then unbiased estimator is

$$\widetilde{\sigma} = \hat{\sigma}\sqrt{\frac{n}{2}}\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

**Bayesian inference.** ...

## 16.4 Holtsmark distribution

Relation to Stable distribution:

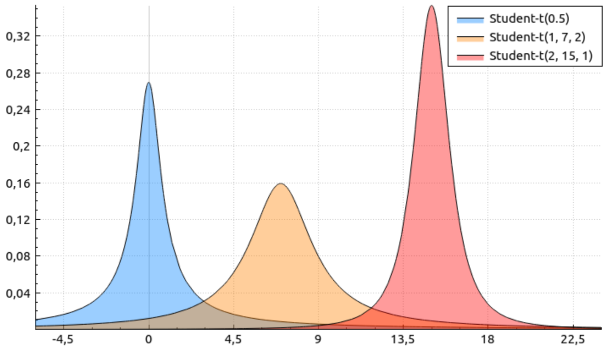
$$X \sim S_{\frac{3}{2}}(0, \gamma, \mu)$$

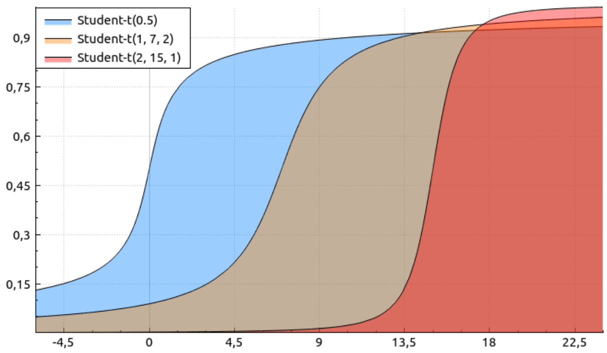
## 16.5 Landau distribution

Relation to Stable distribution:

$$X \sim S_1(1, \gamma, \mu)$$

# 17 Student's t-distribution



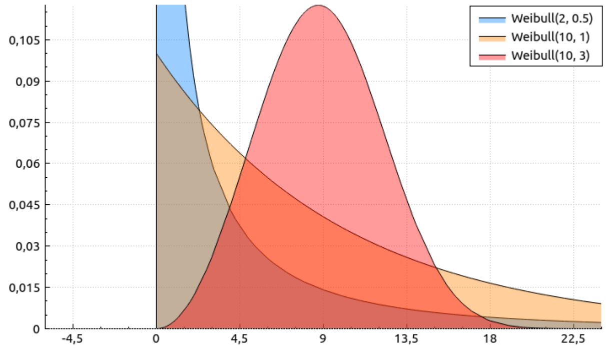



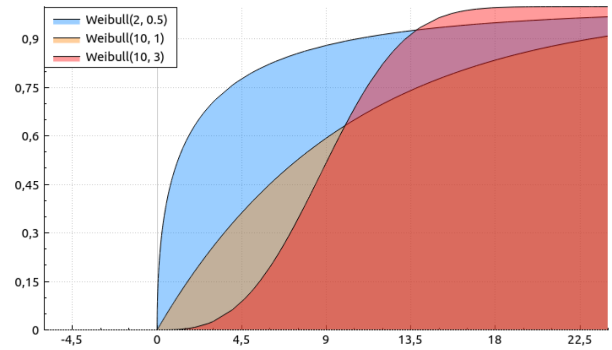

| | |
|---|---|
| Notation | $X \sim t(\nu, \mu, \sigma)$,<br>$X \sim t(\nu)$ with $\mu = 0$, $\sigma = 1$ |
| Parameters | $\nu, \sigma > 0, \mu \in \mathbb{R}$ |
| Domain | $x \in \mathbb{R}$ |
| $f(x)$ | $\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\sqrt{\nu\pi}\frac{\nu}{2}\right)}\left(1+\frac{(x-\mu)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$ |
| $F(x)$ | $h(y)$, $y < 0$,<br>$1 - h(y)$, otherwise<br>with $h(y) = \frac{1}{2} I_{\frac{\nu}{\nu+y^2}}\left(\frac{\nu}{2}, \frac{1}{2}\right)$<br>and $y = \frac{x-\mu}{\sigma}$ |
| $\mathbb{E}[X]$ | $\mu$ |
| $\mathrm{Var}(X)$ | $\sigma^2 \frac{\nu}{\nu+2}$ for $\nu > 2$,<br>$\infty$, otherwise. |
| Median | $\mu$ |
| Mode | $\mu$ |
| $\phi(t)$ | $\frac{K_{\nu/2}(\sqrt{\nu}\lvert t\rvert)\cdot(\sqrt{\nu}\lvert t\rvert)^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-1}}$ |

Relation to other distributions:

- If $X \sim t(\nu)$, then $\mu + \sigma X \sim t(\nu, \mu, \sigma)$.
- If $X \sim t(1, \mu, \sigma)$, then $X \sim \mathrm{Cauchy}(\mu, \sigma)$.
- If $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathrm{Nakagami}\left(\frac{\nu}{2}, 1\right)$, then $\frac{X}{Y} \sim t(\nu)$.
- If $X \sim t(\nu)$, then $X^2 \sim \mathrm{F}(1, \nu)$.

# 18 Weibull distribution




| | |
|---|---|
| Notation | $X \sim \text{Weibull}(\lambda, k)$ |
| Parameters | $\lambda, k > 0$ |
| Domain | $x \in \mathbb{R}^+$ |
| $f(x)$ | $\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp(-(x/\lambda)^k)$ |
| $F(x)$ | $1 - \exp(-(x/\lambda)^k)$ |
| $\mathbb{E}[X]$ | $\lambda\Gamma(1 + 1/k)$ |
| $\text{Var}(X)$ | $\lambda^2\Gamma(1 + 2/k) - (\mathbb{E}[X])^2$ |
| Median | $\lambda(\ln 2)^{\frac{1}{k}}$ |
| Mode | $\lambda\left(1 - \frac{1}{k}\right)^{\frac{1}{k}}$ |
| $\phi(t)$ | Calculated numerically |

**Estimation of scale** .

**Frequentist inference.** Log-likelihood function:

$$\ln \mathcal{L}(\lambda, k|X) = n(\ln k - \ln \lambda) + (k-1)\sum_{i=1}^{n}(\ln X_i - \ln \lambda) - \frac{1}{\lambda^k}\sum_{i=1}^{n} X_i^k.$$

The derivative with respect to scale:

$$\frac{\partial \ln \mathcal{L}(\lambda, k|X)}{\partial \lambda} = -\frac{nk}{\lambda} + \frac{k}{\lambda^{k+1}}\sum_{i=1}^{n} X_i^k = 0.$$

Therefore, maximum-likelihood estimation for $\lambda$ is

$$\hat{\lambda} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^k\right)^{\frac{1}{k}}.$$

**Bayesian inference.** Assume $k$ is known. Instead of estimating $\lambda$ we give an estimation for $\lambda^k$. Let's say that prior distribution of $\lambda^k$ is Inv-$\Gamma(\alpha, \beta)$:

$$h(\lambda^k) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{-k(\alpha+1)} e^{-\beta/\lambda^k}.$$

Posterior distribution then:

$$f(\lambda^k|X) \propto \lambda^{-k(\alpha+1+n)} e^{-\frac{1}{\lambda^k}(\beta + \sum_{i=1}^n X_i^k)} \sim \text{Inv-}\Gamma\left(\alpha+n, \beta + \sum_{i=1}^n X_i^k\right).$$

Bayesian estimator:

$$\mathbb{E}[\lambda^k|X] = \frac{\beta + \sum_{i=1}^n X_i^k}{\alpha + n - 1},$$

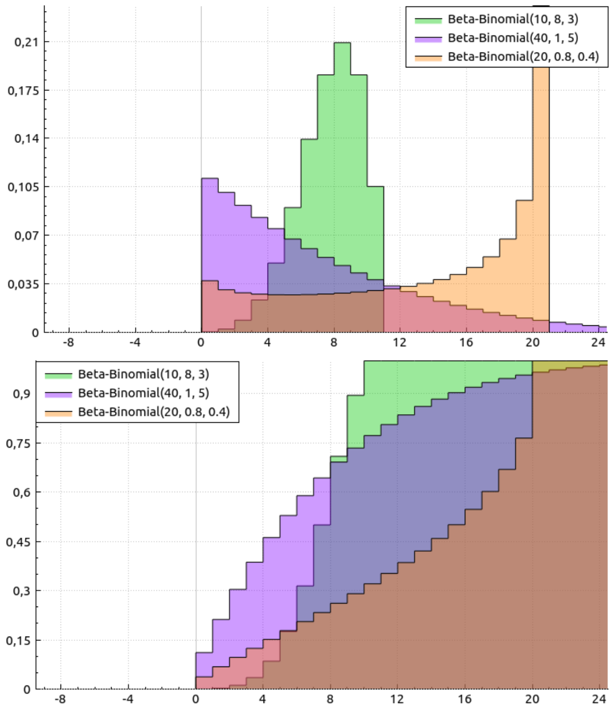
MAP estimator:

$$\lambda^k_{MAP} = \frac{\beta + \sum_{i=1}^n X_i^k}{\alpha + n + 1}.$$

Part III

# Discrete univariate distributions

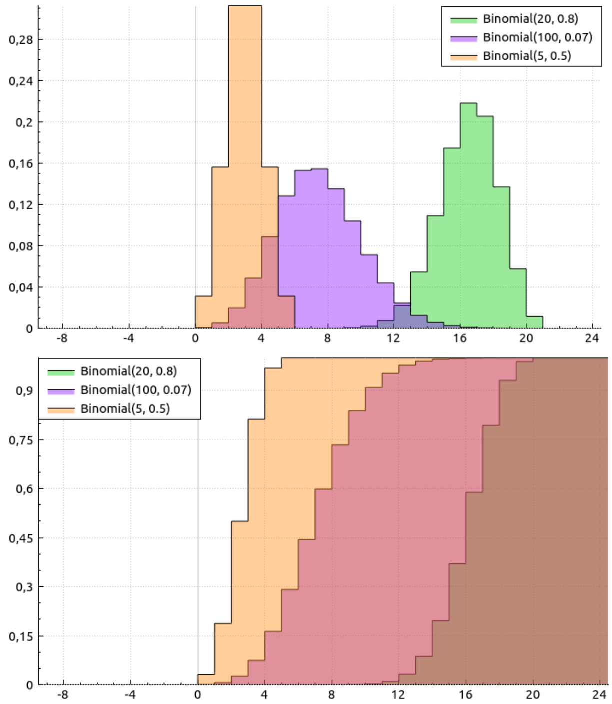
## 19 Beta-binomial distribution




| | |
|---|---|
| Notation | $X \sim \mathrm{BB}(n, \alpha, \beta)$ |
| Parameters | $n \in \mathbb{N}, \alpha, \beta > 0$ |
| Domain | $k \in \{0, \ldots, n\}$ |
| $\mathbb{P}(X = k)$ | $\binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha,\beta)}$ |
| $\mathbb{P}(X \leq k)$ | Calculated numerically |
| $\mathbb{E}[X]$ | $n \frac{\alpha}{\alpha+\beta}$ |
| $\mathrm{Var}(X)$ | $\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$ |
| Median | Searched numerically |
| Mode | Searched numerically |
| $\phi(t)$ | Calculated numerically |

Relation to other distributions: if $p \sim \mathcal{B}(\alpha, \beta)$, then $\mathrm{Bin}(n, p) \sim \mathrm{BB}(n, \alpha, \beta)$.

# 20 Binomial distribution




| | |
|---|---|
| Notation | $X \sim \text{Bin}(n, p)$ |
| Parameters | $n \in \mathbb{N}, p \in [0, 1]$ |
| Domain | $k \in \{0, \ldots, n\}$ |
| $\mathbb{P}(X = k)$ | $\binom{n}{k} p^k (1-p)^{n-k}$ |
| $\mathbb{P}(X \leq k)$ | $I_{1-p}(n-k, 1+k)$ |
| $\mathbb{E}[X]$ | $np$ |
| $\text{Var}(X)$ | $np(1-p)$ |
| Median | $[np]$ |
| Mode | $[(n+1)p]$ |
| $\phi(t)$ | $(1 - p + pe^{it})^n$ |

**Estimation of probability $p$ with known number $n$.**

**Frequentist inference.** Log-likelihood function:

$$\ln \mathcal{L}(p|X) \propto \sum_{i=1}^{k} \big(X_i \log p + (n - X_i) \log(1-p)\big)$$

The derivative with respect to $p$ is:

$$\frac{\partial \ln \mathcal{L}(p|X)}{\partial p} = \frac{\sum_{i=1}^{k} X_i}{p} - \frac{nk - \sum_{i=1}^{k} X_i}{1-p}.$$

Therefore we reach the maximum value of log-likelihood if

$$p = \frac{\overline{X}_k}{n}.$$

**Bayesian inference.** We set prior Beta distribution $\mathcal{B}(\alpha, \beta)$:

$$h(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}.$$

Then posterior is

$$f(p|X) \propto p^{\alpha-1+\sum_{i=1}^{k} X_i}(1-p)^{\beta-1+\sum_{i=1}^{k}(n-X_i)} \sim \mathcal{B}\Big(\alpha + \sum_{i=1}^{k} X_i, \beta + nk - \sum_{i=1}^{k} X_i\Big).$$

Thus Bayesian estimator is

$$\mathbb{E}[p|X] = \frac{\alpha + \sum_{i=1}^{k} X_i}{\alpha + \beta + nk}$$

and MAP estimator is

$$p_{MAP} = \frac{\alpha + \sum_{i=1}^{k} X_i - 1}{\alpha + \beta + nk - 2}.$$

Also, Minimax estimator is equal to Bayes estimator if $\alpha = \beta = \frac{1}{2}\sqrt{n}$.

**Exponential family parameterization.** Logarithm of probabilty mass function:

$$\log \mathbb{P}(X = x) = \log(n!) - \log(x!(n-x)!) + x \log \frac{p}{1-p} + n \log(1-p).$$

Therefore binomial distribution with fixed $n$ belongs to one-parameterized exponential family with sufficient statistics $T(x) = x$, natural parameter $\theta = \log \frac{p}{1-p}$, log-normalizer $F(\theta) = n\log(1 + \exp\theta) - \log(n!)$ and carrier measure $k(x) = -\log(x!(n-x)!)$. Gradient of log-normalizer: $\nabla F(\theta) = n\frac{\exp(\theta)}{1+\exp(\theta)}$. Adjusted cross-entropy is

$$\begin{aligned} H_F(\theta_1\|\theta_2) &= F(\theta_2) - \langle\theta_2, \nabla F(\theta_1)\rangle \\ &= n\log(1 + \exp\theta_2) - \log(n!) - \theta_2 n\frac{\exp(\theta_1)}{1+\exp(\theta_1)}. \end{aligned}$$

Adjusted entropy is

$$\begin{aligned} H_F(\theta) &= n\log(1 + \exp\theta) - \log(n!) - \theta n\frac{\exp(\theta)}{1+\exp(\theta)} \\ &= -n[(1-p)\log(1-p) + p\log p] - \log(n!). \end{aligned}$$

And Kullback-Leibler divergence:

$$\begin{aligned} \mathrm{KL}(p_1\|p_2) &= H_F(\theta_1\|\theta_2) - H_F(\theta_1) \\ &= n\log\frac{1+\exp\theta_2}{1+\exp\theta_1} - n(\theta_2 - \theta_1)\frac{\exp(\theta_1)}{1+\exp(\theta_1)} \\ &= n(1-p_1)\log\frac{1-p_1}{1-p_2} + np_1\log\frac{p_1}{p_2}. \end{aligned}$$

## 20.1 Bernoulli

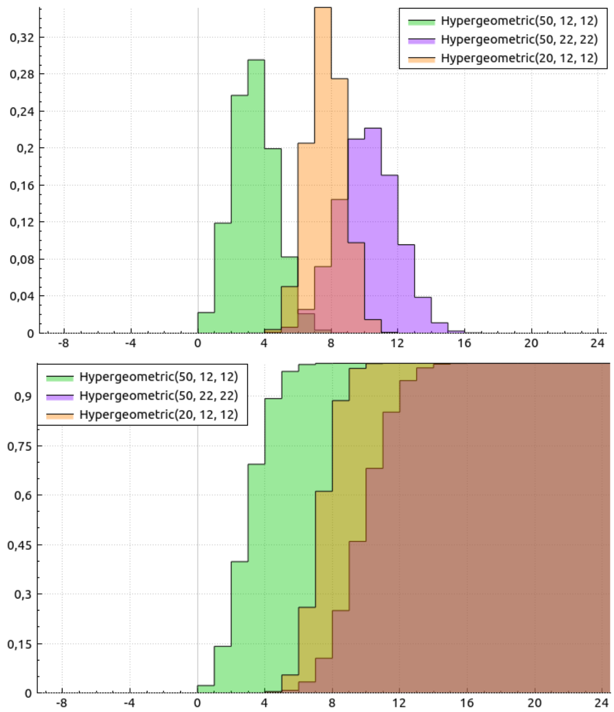
Notation:

$$X \sim \mathrm{Bernoulli}(p).$$

Relation to Binomial distribution:

$$X \sim \mathrm{Bin}(1, p).$$

# 21 Hypergeometric distribution




| | |
|---|---|
| Notation | $X \sim \mathrm{HG}(N, K, n)$ |
| Parameters | $N \in \mathbb{N}, K \in \{1, 2, \cdots, N\}$, $n \in \{1, 2, \cdots, N\}$ |
| Domain | $\max(0, n + K - N) \leq k \leq \min(n, K)$ |
| $\mathbb{P}(X = k)$ | $\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$ |
| $\mathbb{P}(X \leq k)$ | Calculated numerically |
| $\mathbb{E}[X]$ | $\frac{nK}{N}$ |
| $\mathrm{Var}(X)$ | $\frac{nK(N-K)(N-n)}{N^2(N-1)}$ |
| Median | Searched numerically |
| Mode | $\left\lfloor \frac{(n+1)(K+1)}{N+2} \right\rfloor$ |
| $\phi(t)$ | Calculated numerically |

**Estimation of number of target members of population K.**

**Bayesian inference.** Let prior distribution of $K$ be Beta-Binomial distribution $BB(N, \alpha, \beta)$:

$$h(K) = \binom{N}{K} \frac{B(K + \alpha, N - K + \beta)}{B(\alpha, \beta)}.$$

Then for one sample $X$:

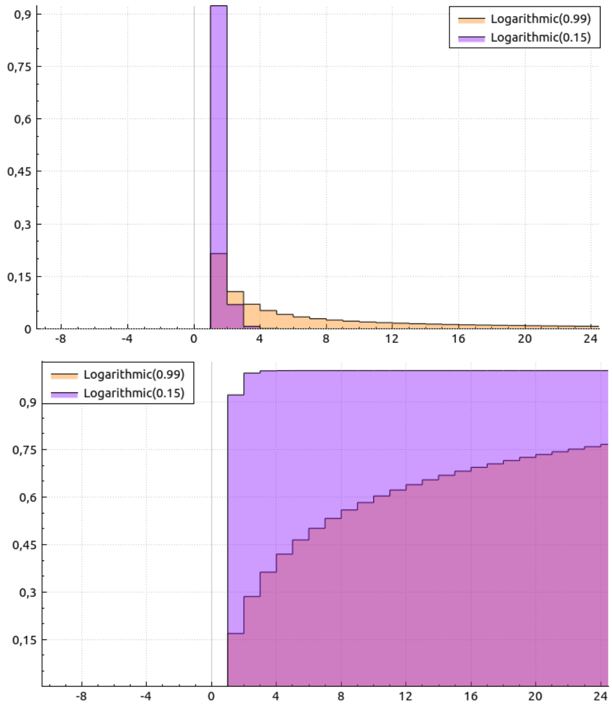
$$K - X \sim BB\big(N - n, \alpha + X, \beta + nk - X\big)$$

and therefore

$$\mathbb{E}[K|X] = X + (N - n)\frac{\alpha}{\alpha + \beta}.$$

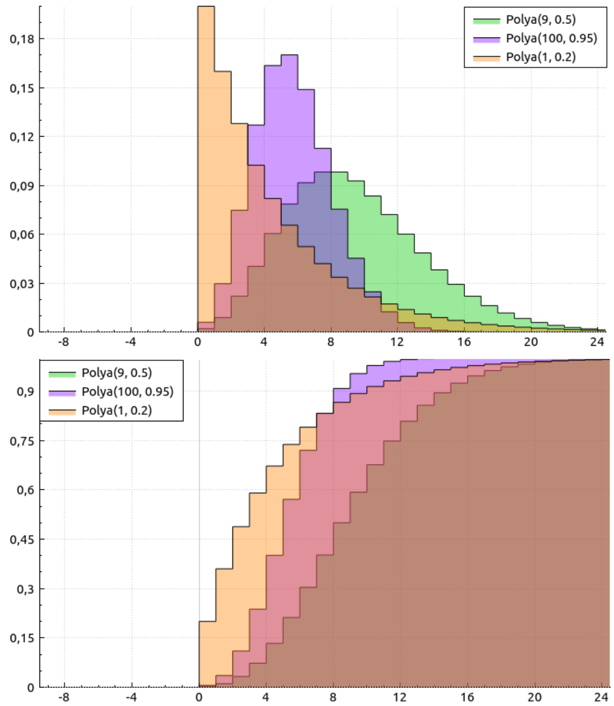
However, RandLib doesn't support Bayesian fitting for Hypergeometric distribution yet.

## 22 Logarithmic distribution




| | |
|---|---|
| Notation | $X \sim \text{Log}(p)$ |
| Parameters | $p \in (0,1)$ |
| Domain | $k \in \mathbb{N}$ |
| $\mathbb{P}(X = k)$ | $-\frac{p^k}{k\log(1-p)}$ |
| $\mathbb{P}(X \leq k)$ | $1 + \frac{B(p,k+1,0)}{\log(1-p)}$ |
| $\mathbb{E}[X]$ | $-\frac{p}{(1-p)\log(1-p)}r$ |
| $\text{Var}(X)$ | $-\frac{p(p+\log(1-p))}{(\log(1-p)(1-p))^2}$ |
| Median | Searched numerically |
| Mode | 1 |
| $\phi(t)$ | $\frac{\log(1-pe^{it})}{\log(1-p)}$ |

# 23 Negative-Binomial (Polya) distribution




| | |
|---|---|
| Notation | $X \sim \mathrm{NB}(r, p)$ |
| Parameters | $r > 0,\ p \in (0, 1)$ |
| Domain | $k \in \mathbb{N}_0$ |
| $\mathbb{P}(X = k)$ | $\binom{k+r-1}{k} p^r (1-p)^k$ |
| $\mathbb{P}(X \leq k)$ | $I_p(r, k+1)$ |
| $\mathbb{E}[X]$ | $\frac{1-p}{p} r$ |
| $\mathrm{Var}(X)$ | $\frac{1-p}{p^2} r$ |
| Median | Searched numerically |
| Mode | $\max\left(\left\lfloor \frac{(r-1)(1-p)}{p} \right\rfloor, 0\right)$ |
| $\phi(t)$ | $\left(\frac{p}{1-(1-p)e^{it}}\right)^r$ |

Relation to other distributions: if $\lambda \sim \mathrm{Gamma}\left(r, \frac{p}{1-p}\right)$, then $\mathrm{Po}(\lambda) \sim \mathrm{NB}(r, p)$.

## 23.1 Geometric distribution

Notation:

$$X \sim \mathrm{Geometric}(p).$$

Relation to Negative-Binomial distribution:

$$X \sim \mathrm{NB}(1, p).$$

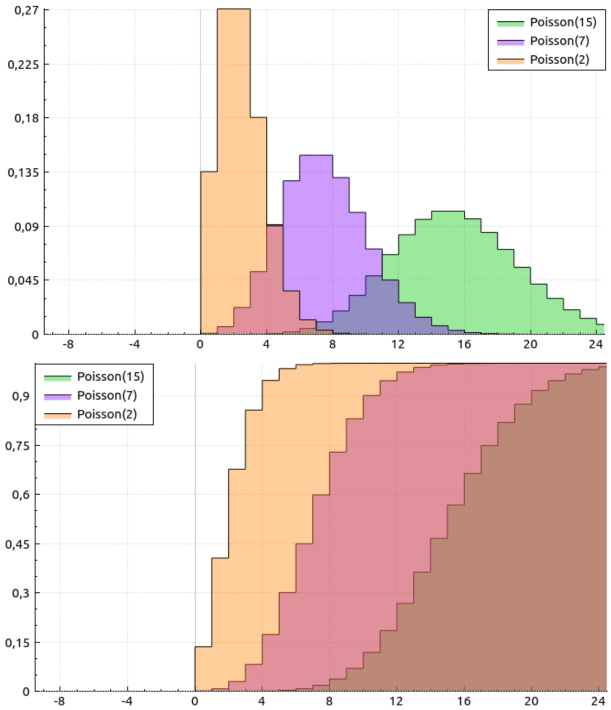
## 23.2 Pascal distribution

Notation:

$$X \sim \mathrm{Pascal}(r, p).$$

The only difference with Negative-Binomial distribution is that for Pascal distribution shape $r$ is an integer.

# 24 Poisson distribution




| | |
|---|---|
| Notation | $X \sim \mathrm{Po}(\lambda)$ |
| Parameters | $\lambda > 0$ |
| Domain | $k \in \mathbb{N}_0$ |
| $\mathbb{P}(X = k)$ | $\frac{\lambda^k e^{-\lambda}}{k!}$ |
| $\mathbb{P}(X \leq k)$ | $Q(k+1, \lambda)$ |
| $\mathbb{E}[X]$ | $\lambda$ |
| $\mathrm{Var}(X)$ | $\lambda$ |
| Median | $\sim \max\left([\lambda + \frac{1}{3} - \frac{0.02}{\lambda}], 0\right)$ |
| Mode | $[\lambda]$ |
| $\phi(t)$ | $\exp\{\lambda(e^{it} - 1)\}$ |

**Estimation of rate.**

**Frequentist inference.** Log-likelihood function:

$$\ln \mathcal{L}(\lambda | X) \propto -\lambda n + \sum_{i=1}^{n} X_i \log \lambda.$$

Setting the derivative w.r.t. rate to 0 we get the optimal value:

$$\lambda = \overline{X}_n.$$

**Bayesian inference.** Let set prior distribution of $\lambda \sim \Gamma(\alpha, \beta)$:

$$h(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha - 1} e^{-\beta\lambda}.$$

Posterior distribution:

$$f(\lambda | X) \propto e^{-\lambda(\beta + n)} \lambda^{\alpha - 1 + \sum_{i=1}^{n} X_i} \sim \Gamma(\alpha + \sum_{i=1}^{n} X_i, \beta + n).$$

Therefore, Bayesian estimator:

$$\mathbb{E}[\lambda|X] = \frac{\alpha + \sum_{i=1}^{n} X_i}{\beta + n}.$$

And MAP estimator:

$$\lambda_{MAP} = \max\left(\frac{\alpha + \sum_{i=1}^{n} X_i - 1}{\beta + n}, 0\right).$$

**Exponential family parameterization** Logarithm of probability mass function:

$$\log \mathbb{P}(X = x) = x \log \lambda - \lambda - \log(x!).$$

Therefore, sufficient statistics $T(x) = x$, natural parameter $\theta = \log \lambda$, log-normalizer $F(\theta) = \exp(\theta)$, carrier measure $k(x) = \log(x!)$. We conclude that adjusted cross-entropy is

$$\begin{aligned} H_F(\theta_p \| \theta_q) &= F(\theta_q) - \langle \theta_q, \nabla F(\theta_p) \rangle \\ &= \exp(\theta_q) - \theta_q \exp(\theta_p). \end{aligned}$$

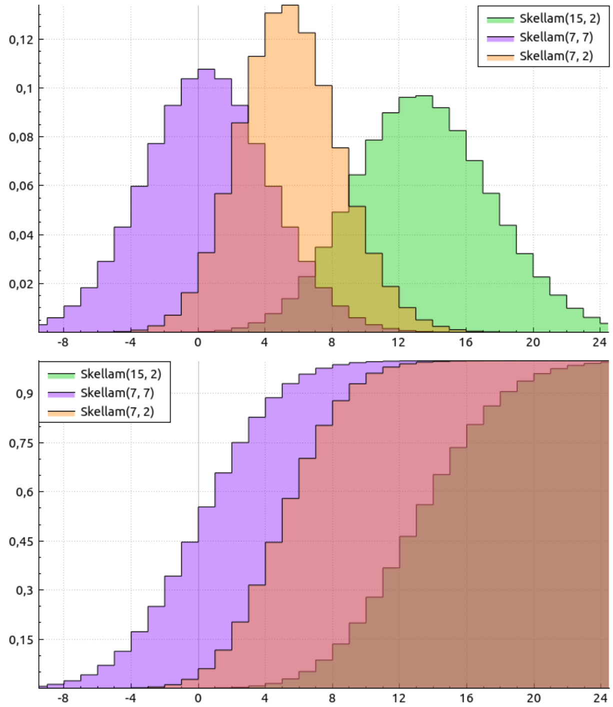
Adjusted entropy is

$$H_F(\theta) = \exp(\theta)(1 - \theta) = \lambda(1 - \log \lambda).$$

And Kullback-Leibler divergence:

$$\begin{aligned} \mathrm{KL}(p \| q) &= H_F(\theta_p \| \theta_q) - H_F(\theta_p) \\ &= \lambda_q - \lambda_p \left(1 + \log\left(\frac{\lambda_p}{\lambda_q}\right)\right). \end{aligned}$$

# 25 Skellam distribution

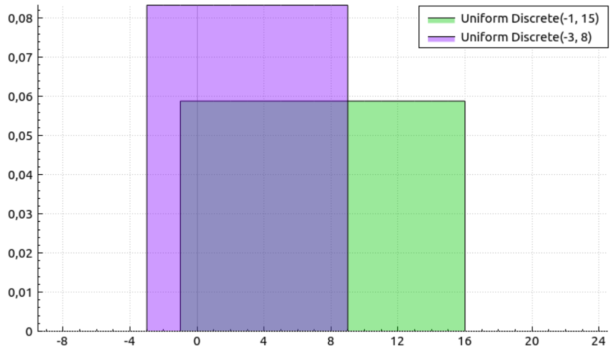


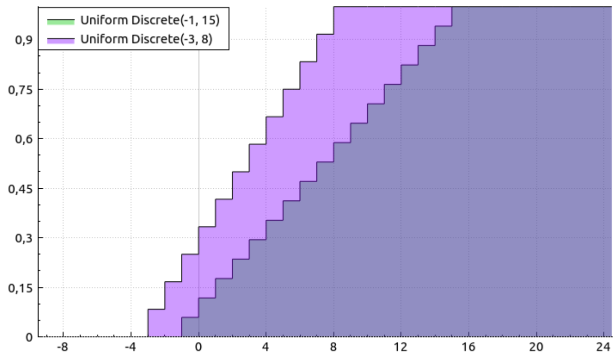

| | |
|---|---|
| Notation | $X \sim \text{Skellam}(\mu_1, \mu_2)$ |
| Parameters | $\mu_1, \mu_2 > 0$ |
| Domain | $k \in \mathbb{Z}$ |
| $\mathbb{P}(X = k)$. | $e^{-(\mu_1+\mu_2)} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{k}{2}} I_k(2\sqrt{\mu_1\mu_2})$ |
| $\mathbb{P}(X \leq k)$ | $\text{MarcumP}_{k+1}(\mu_2, \mu_1),\ k \geq 0$<br>$\text{MarcumQ}_{-k}(\mu_1, \mu_2),\ k < 0$ |
| $\mathbb{E}[X]$ | $\mu_1 - \mu_2$ |
| $\text{Var}(X)$ | $\mu_1 + \mu_2$ |
| Median | Searched numerically |
| Mode | $[\mu_1 - \mu_2]$ |
| $\phi(t)$ | $\exp\{\mu_1(e^{it} - 1) - \mu_2(e^{it} - 1)\}$ |

Relation to other distributions: if $Y \sim \text{Po}(\mu_1)$ and $Z \sim \text{Po}(\mu_2)$, then

$$Y - Z \sim \text{Skellam}(\mu_1, \mu_2).$$

# 26 Uniform discrete distribution

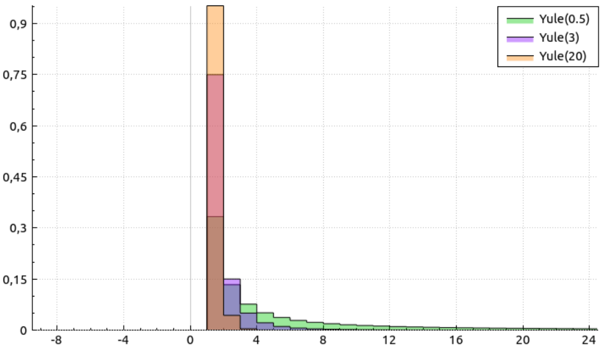


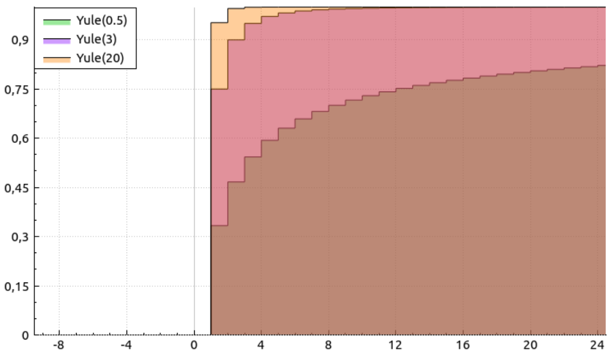




| | |
|---|---|
| Notation | $X \sim \mathcal{U}\{a, \ldots, b\}$ |
| Parameters | $a, b \in \mathbb{R}, a \leq b$ |
| Domain | $k \in \{a, \ldots, b\}$ |
| $\mathbb{P}(X = k)$ | $\frac{1}{n}$, where $n = b - a + 1$. |
| $\mathbb{P}(X \leq k)$ | $\frac{k-a+1}{n}$ |
| $\mathbb{E}[X]$ | $\frac{a+b}{2}$ |
| $\mathrm{Var}(X)$ | $\frac{(n-1)(n+1)}{12}$ |
| Median | $\frac{a+b}{2}$ |
| Mode | Any value between $a$ and $b$ |
| $\phi(t)$ | $\frac{e^{iat} - e^{i(b+1)t}}{n(1-e^{it})}$ |

Relation to other distributions: if $X \sim BB(n, 1, 1)$, then $X \sim \mathcal{U}\{0, \ldots, n\}$.

## 27 Yule distribution

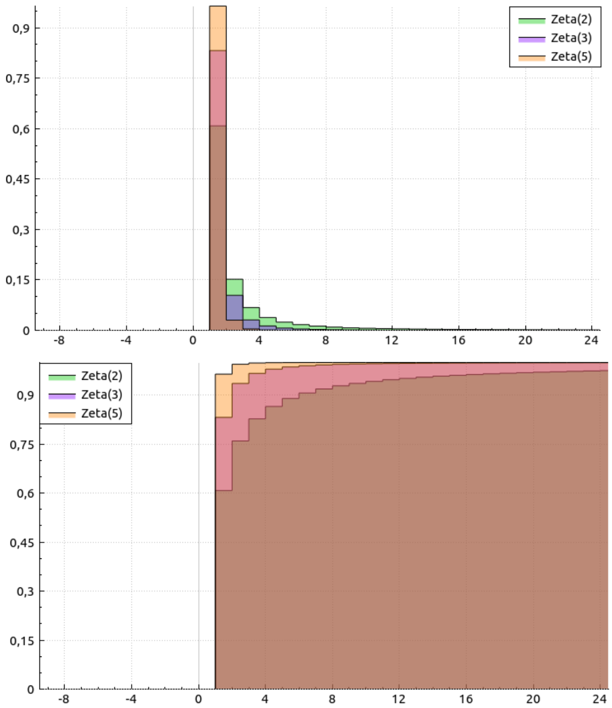



| | |
|---|---|
| Notation | $X \sim \text{Yule}(\rho)$ |
| Parameters | $\rho \in \mathbb{R}^+$ |
| Domain | $k \in \mathbb{N}$ |
| $\mathbb{P}(X = k)$ | $\rho \frac{\Gamma(1+\rho)(k-1)!}{\Gamma(k+\rho+1)}$ |
| $\mathbb{P}(X \leq k)$ | $1 - k \frac{\Gamma(1+\rho)(k-1)!}{\Gamma(k+\rho+1)}$ |
| $\mathbb{E}[X]$ | $\frac{\rho}{\rho-1}$, $\rho > 1$<br>$\infty$, otherwise |
| $\text{Var}(X)$ | $\frac{\rho^2}{(\rho-1)^2(\rho-2)}$, $\rho > 2$<br>$\infty$, otherwise |
| Median | Searched numerically |
| Mode | 1 |
| $\phi(t)$ | Calculated numerically |

Relation to other distributions: if $X \sim \text{Pareto}(\alpha, 1)$, then $\text{Geometric}(1/X) \sim \text{Yule}(\alpha)$.

# 28 Zeta distribution




| | |
|---|---|
| Notation | $X \sim \text{Zeta}(s)$ |
| Parameters | $s > 1$ |
| Domain | $k \in \mathbb{N}$ |
| $\mathbb{P}(X = k)$ | $\frac{1}{\zeta(s)k^s}$ |
| $\mathbb{P}(X \leq k)$ | $\frac{H(s,k)}{\zeta(s)}$ |
| $\mathbb{E}[X]$ | $\frac{\zeta(s-1)}{\zeta(s)}$, $s > 2$<br>$\infty$, otherwise |
| $\text{Var}(X)$ | $\frac{\zeta(s-2)}{\zeta(s)} - (\mathbb{E}[X])^2$, $\rho > 3$<br>$\infty$, otherwise |
| Median | Searched numerically |
| Mode | 1 |
| $\phi(t)$ | Calculated numerically |

## 29 Zipf distribution

Part IV

# Bivariate distributions

## 30 Bivariate Normal distribution

## 31 Normal-Inverse-Gamma distribution

## 32 Trinomial distribution

Part V

# Circular distributions

## 33 von Mises distribution

## 34 Wrapped Exponential distribution

# Part VI
# Singular distributions

## 35 Cantor distribution