Gaurav Prakash
UID : 122250130

Data 601 - Homework 6

1. Consider our four acceptance sampling plans:

- A1 : sample $n = 10$, accept if # underweight bags ≤ 1
  A2 : sample $n = 10$, accept if # underweight bags ≤ 3
  A3 : sample $n = 15$, accept if # underweight bags ≤ 1
  A4 : sample $n = 15$, accept if # underweight bags ≤ 3

Assume • p : proportion of underweight bags in the batch.

We can model # underweight bags as a Binomial distribution as follows:

$$X \sim \text{Binomial}(n, p)$$

Acceptance probability:

$$A(p) = P(X \leq c) = \sum_{k=0}^{c} \binom{n}{k} p^k (1-p)^{n-k}$$

c: acceptance number

a) We can calculate A(p) as a function of p for four acceptance strategies

<u>A1</u>: $n=10, c=1$

$$A1(p) = \sum_{k=0}^{1} \binom{10}{k} p^k (1-p)^{10-k}$$

$$\boxed{A1(p) = (1-p)^{10} + 10p(1-p)^9}$$

<u>A2</u>: $n=10, c=3$

$$A2(p) = \sum_{k=0}^{3} \binom{10}{k} p^k (1-p)^{10-k}$$

$$\boxed{A2(p) = (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8 + 120p^3(1-p)^7}$$

<u>A3</u>: $n=15, c=1$

$$\boxed{A3(p) = (1-p)^{15} + 15p(1-p)^{14}}$$

<u>A4</u>: $n=15, c=3$

$$A4(p) = \sum_{k=0}^{3} \binom{15}{k} p^k (1-p)^{15-k}$$

$$\boxed{A4(p) = (1-p)^{15} + 15p(1-p)^{14} + 105p^2(1-p)^{13} + 455p^3(1-p)^{12}}$$

b) refer desmos graph

We can interpret the curves from an ROC standpoint to determine the ideal sampling strategy.

In acceptance sampling:
Ideally, $A(p) = 1$ when $p \leq 0.15$
and $A(p) = 0$ when $p > 0.15$

We would require,
High prob. of acceptance when $p \leq 0.15$ (good quality)
Low prob. of acceptance when $p > 0.15$ (bad quality)

If we compare curves:
A1($n=10$, $c=1$): Very strict, drops quickly for small p
A2($n=10$, $c=3$): More lenient, accepts even poor quality
A3($n=15$, $c=1$): Stricter than A1 as n is larger but $c=1$
A4($n=15$, $c=3$): More lenient than A3, but n is larger so better discrimination than A2

To choose the best one, we must lower bad quality for $p>0.15$, hence we want the steepest curve near $p=0.15$, which corresponds to $A_3(p)$

$\therefore$ $A_3(p)$ is the most conservative for the company, with low consumer risk

2. # of diodes, $n=200$
Probability of failure, $p=0.05$
We can assume independent failures of diodes.
Distribution, $X \sim \text{Binomial}(n=200, p=0.05)$

a) Expected # of diode failures

$$E[X] = np = 200 \times 0.05 = \underline{10}$$

$$Var(X) = np(1-p) = 200 \times 0.05 \times 0.95$$

$$\therefore \underline{Var(X) = 9.5}$$

$$\sigma_X = \sqrt{Var(X)} = \sqrt{9.5} \approx \underline{\underline{3.082}}$$

b) $P(X \geq 6) \Rightarrow$ at least 6 diodes fail

$$P(X \geq 6) = 1 - P(X \leq 5)$$

$$= 1 - \sum_{k=0}^{5} \binom{200}{k} (0.05)^k (0.95)^{200-k}$$

We ~~also~~ can simplify as:

$$P(X \geq 6) = 1 - P(X \leq 5)$$

refer after c)

c) ~~18~~ "≥ 180 diodes will not fail"

$\Rightarrow$ 180 working diodes atleast

$\Rightarrow$ at most $(200-180)$ failed

$\Rightarrow$ ≤ 20 failed diodes.

$$P(X\leq 20) = \sum_{k=0}^{20} \binom{200}{k} (0.05)^k (0.95)^{200-k}$$

We can simplify: b) and c) as follows:

b) $P(X\geq 6) = 1 - P(X\leq 5)$

$$P(X\leq 5) = \sum_{k=0}^{5} \binom{200}{k} (0.05)^k (0.95)^{200-k}$$

$$= (0.95)^{200}\left[1 + \binom{200}{1}\left(\frac{0.05}{0.95}\right) + \binom{200}{2}\left(\frac{0.05}{0.95}\right)^2 + \dots + \binom{200}{20}\left(\frac{0.05}{0.95}\right)^{5}\right]$$

Consider $r = \left(\frac{0.05}{0.95}\right) = \frac{1}{19}$

$$\therefore P(X\leq 5) = (0.95)^{200}\left[1 + \binom{200}{1} r^1 + \dots + \binom{200}{20} r^{5}\right]$$

This is the binomial expression of $(1+r)^n$ minus remaining terms.

$$\therefore P(X\leq 5) = (0.95)^{200}\left[(1+r)^{200} - \sum_{k=6}^{200} \binom{200}{k} r^k\right]$$

$$\therefore P(X\leq 5) = (0.95)^{200}\left[\left(\frac{20}{19}\right)^{200} - \sum_{k=6}^{200} \binom{200}{k} r^k\right]$$

$$= 1 - (0.95)^{200} \sum_{k=6}^{200} \binom{200}{k} (0.95)^k \quad -(1)$$

$\therefore P(X\geq 6) = 1 - P(X\leq 5)$

$$P(X\geq 6) = (0.95)^{200} \sum_{k=6}^{200} \binom{200}{k} (0.95)^k$$

Similarly, for c): (similar to (1))

$$P(X\leq 20) = 1 - (0.95)^{200} \sum_{k=21}^{200} \binom{200}{k} (0.95)^k$$

d) For large n, small p

$\lambda = np = 200 \times 0.05 = 10$

$\therefore X \approx \text{Poisson}(\lambda = 10)$

$P(X \geq 6) = 1 - P(X \leq 5)$

$P(X \leq 5)$ using Poisson:

$$P(X \leq 5) = e^{-10}\left(\frac{10^0}{0!} + \dots + \frac{10^5}{5!}\right)$$

$$= 4.54 \times 10^{-5}\left(1 + 10 + \frac{100}{2} + \frac{1000}{6} + \frac{10000}{24} + \frac{100000}{120}\right)$$

$$= 4.54 \times 10^{-5}(1 + 10 + 50 + 166.67 + 416.67 + 833.33)$$

$$= 4.54 \times 10^{-5} \times 1477.667$$

$P(X \leq 5) = \underline{0.067}$

$\therefore P(X \geq 6) = 1 - 0.067$

$\therefore \underline{P(X \geq 6) = 0.9329}$

e) We can consider $X = 0$ failures.

Prob. that single board works:

$P_{working} = (1 - 0.05)^{200} = (0.95)^{200}$

$$\boxed{P_{work} \approx 3.504 \times 10^{-5}}$$

We find # working out of 10 boards

$X \sim \text{Binomial}(10, P_{work})$

~~Let~~ Let Y = # of working boards

$P(Y \geq 8) = P(Y=8) + P(Y=9) + P(Y=10)$

Since P_{work} is negligible, $P(Y \geq 8) \approx P(Y=10)$, since others are much smaller

$\therefore P(Y \geq 8) = (3.504 \times 10^{-5})^{10}$

$= (3.504)^{10} \times 10^{-50}$

$\underline{P(Y \geq 8) = 2.79 \times 10^{-45}}$

Hence, it is practically zero as P_{work} is very small ($\approx 10^{-5}$)

3. $X \sim \text{Exponential}(\lambda = 1/3)$

$\therefore \text{pdf} = \lambda e^{-\lambda x}, \quad x \geq 0$

$\therefore \text{pdf} = \frac{1}{3} e^{-x/3}, \quad x \geq 0$

a) $p = P(X \geq 5) = e^{-5/3} \approx \underline{0.1889}$

Interpretation:

P(Random chosen bulb lasts 5y)

$n = 10$

Y = # bulbs lasting 5y

$Y \sim \text{Binomial}(n = 10, p)$

We want:

$$P(Y \geq 8) = \sum_{k=8}^{10} \binom{10}{k} p^k (1-p)^{10-k}$$

$$P(Y \geq 8) = \binom{10}{8}(0.1889)^8 (0.811)^2 + \binom{10}{9}(0.1889)^9 (0.811)$$

$$+ \binom{10}{10}(0.1889)^{10}$$

$$= 45 \times (2.278 \times 10^{-6}) \times 0.6579$$

$$+ 10 \times (4.3 \times 10^{-7}) \times 0.811$$

$$+ 8.13 \times 10^{-8}$$

$\therefore \underline{P(Y \geq 8) \approx 7.10 \times 10^{-5}}$

b) p: Probability of acceptable bulb

$N \sim \text{Binomial}(10, p)$

$$A(p) = P(N \geq 8) = \sum_{k=8}^{10} \binom{10}{k} p^k (1-p)^{10-k}$$

$$= \binom{10}{8} p^8 (1-p)^2 + \binom{10}{9} p^9 (1-p) + \binom{10}{10} p^{10}$$

$$\boxed{A(p) = 45p^8(1-p)^2 + 10p^9(1-p) + p^{10}}$$

c) refer desmos graph

We can draw an intersection of $y = 0.95$ on the graph to estimate p_0 for which $A(p_0) = 0.95$

From graph, using Desmos intersection tool:

$\underline{p_0 \approx 0.9127}$

Interpretation

91.2% of atleast 8 bulbs acceptable out of 10.

~~Atleast~~

d) $p_0 = P(X \geq x_0) = e^{-x_0/3}$

$\therefore \frac{-x_0}{3} = \ln p_0$

$$\boxed{x_0 = -3\ln p_0}$$

$\therefore x_0 = -3\ln(0.9127)$

$= -3 \times -0.091$

$\underline{x_0 = 0.274}$

ie, If we define acceptable as lifetime greater or equal to 0.274 years, then P(acceptable bulb) is 0.92

e) $A(p) = P(N \geq 3)$ and $N \sim \text{Binomial}(10, p)$

$A(p) = 1 - P(N \leq 2)$

$= 1 - \left[\binom{10}{0}(1-p)^{10} + \binom{10}{1}p(1-p)^9 + \binom{10}{2}p^2(1-p)^8\right]$

$$\boxed{\therefore A(p) = 1 - (1-p)^{10} - 10p(1-p)^9 - 45p^2(1-p)^8}$$

$A(p_0) = 0.95 \Rightarrow P(N \leq 2)$

Test $p = 0.5$:

$P(N \leq 2) \approx 0.054$, so $A(p) \approx 0.9453$

Test $p = 0.52$

$P(N \leq 2) \approx 0.0419$, so $A(p) \approx 0.958$ $> (0.95)$

$\therefore$ We can take $\underline{\underline{p_0 = 0.51}}$ (approx.)

$x_0 = -3\ln p_0 = -3 \times -0.672$

$\therefore \underline{x_0 \approx 2.018 \text{ years}}$

Comparison

Earlier $A(p) = P(N \geq 8)$

$p_0 \approx 0.9127$, $x_0 \approx 0.274$ y

Now $A(p) = P(N \geq 3)$

$p_0 \approx 0.5102$, $x_0 \approx 2.018$ y.

Interpretation

For $N \geq 8$, we need p_0 high, so lifetime threshold x_0 is small (achievable)

For $N \geq 3$, we can afford lower p_0, so the lifetime threshold is larger (harder to achieve).

4. $X \sim N(\mu_X = 69, \sigma_X^2 = 9)$ so $\sigma_X = 3$

a) $p = P(X \geq 72) = P\left(z \geq \frac{72-69}{3}\right)$

$\boxed{\therefore p = P(Z \geq 1)}$

$P(Z \geq 1) = 1 - P(Z \leq 1) = 1 - 0.8413$

$\therefore \underline{p = 0.1587}$

$n = 10$

Y = # of men taller than 72 in.

$Y \sim \text{Binomial}(n = 10, p = 0.1587)$

$$P(Y \geq 8) = \binom{10}{8}(0.1587)^8(0.8413)^2 + \binom{10}{9}(0.1587)^9(0.8413) + (0.1587)^{10}$$

$$= 8 \times 10^{-6} + 3.35 \times 10^{-7} + 6.32 \times 10^{-9}$$

$$\therefore \underline{P(Y \geq 8) \approx 8.34 \times 10^{-6}}$$

b) $N \sim \text{Binomial}(10, p)$

$$A(p) = P(N \geq 8) = \sum_{k=8}^{10} \binom{10}{k} p^k (1-p)^{10-k}$$

$$\boxed{A(p) = 45p^8(1-p)^2 + 10p^9(1-p) + p^{10}}$$

c) refer desmos graph

We computed $N \geq 8$ that
$p_0 = 0.9127$

<u>Interpretation</u>: If acceptable probability is 0.9127, then 95% chance that atleast 8 on 10 are acceptable.

d) $p_0 = P(X \geq x_0)$
$X \sim N(69, 9)$,

$$P(X \geq x_0) = P\left(Z \geq \frac{x_0 - 69}{3}\right) = 0.9127$$

$$\therefore P\left(Z \leq \frac{x_0 - 69}{3}\right) = 0.0873 \quad [1 - 0.9127]$$

From table

$$\frac{x_0 - 69}{3} = -1.36$$

$$x_0 = 69 - 3\times1.36$$

$$x_0 = 64.9 \text{ in}$$

Interpretation: if "acceptable" is height ≥ 64.92 in, then P(acceptable) is 0.9127 and $A(p_0) = 0.95$

e) $A(p) = P(N \geq 3)$

$= 1 - P(N \leq 2)$

$$A(p) = 1 - [(1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8]$$

We need $A(p) = 0.95$ or $P(N \leq 2) = 0.05$

From q3, $p_0 \approx 0.5102$

$$P(X \geq x_0) = 0.5102$$

$$P\left(Z \geq \frac{x_0 - 69}{3}\right) = 0.5102$$

$$P(Z \leq z_0) = 0.4898$$

From table, $\frac{x_0 - 69}{3} = -0.025$

$\therefore x_0 = 68.925$ in

Comparison

$N \geq 8$: $p_0 \approx 0.9127$, $x_0 \approx 64.92$ in

$N \geq 3$: $p_0 \approx 0.5102$, $x_0 \approx 68.93$ in

Interpretation

- Requiring ≥ 8 on 10 means the criterion will be easy to meet (short height)

- Requiring ≥ 3 on 10 means a stricter criterion (taller ≥ 68.93 in)

5. We can assume:
- $Z \geq 2$: exceptionally well ($\sim 2.5\%$)
- $1 \leq Z < 2$: moderate ($\sim 16\%$)
- $-1 < Z < 1$: middle $\sim 68\%$
- $-2 < Z \leq 1$: bottom 16%
- $Z \leq -2$: bottom $\sim 2.5\%$

a) $x = 50,\ \mu = 51,\ \sigma = 2$

$$z = \frac{50-51}{2} = \frac{-1}{2} = \underline{\underline{-0.5}}$$

$\therefore$ Average (below but under 1σ)

b) $x = 51,\ \mu = 50,\ \sigma = 2$

$$z = \frac{51-50}{2} = \frac{1}{2} = \underline{\underline{0.5}}$$

$\therefore$ Average (above but under 1σ)

c) $x = 50,\ \mu = 51,\ \sigma = 0.2$

$$z = \frac{50-51}{0.2} = \underline{\underline{-5}}$$

$\therefore$ Exceptionally poor (rare)

d) $x = 51,\ \mu = 50,\ \sigma = 0.2$

$$z = \frac{51-50}{0.2} = 5$$

$\therefore$ Exceptionally well

e) $x = 50,\ \mu = 51,\ \sigma = 0.7$

$$z = \frac{50-51}{0.7} \approx \underline{-1.428}$$

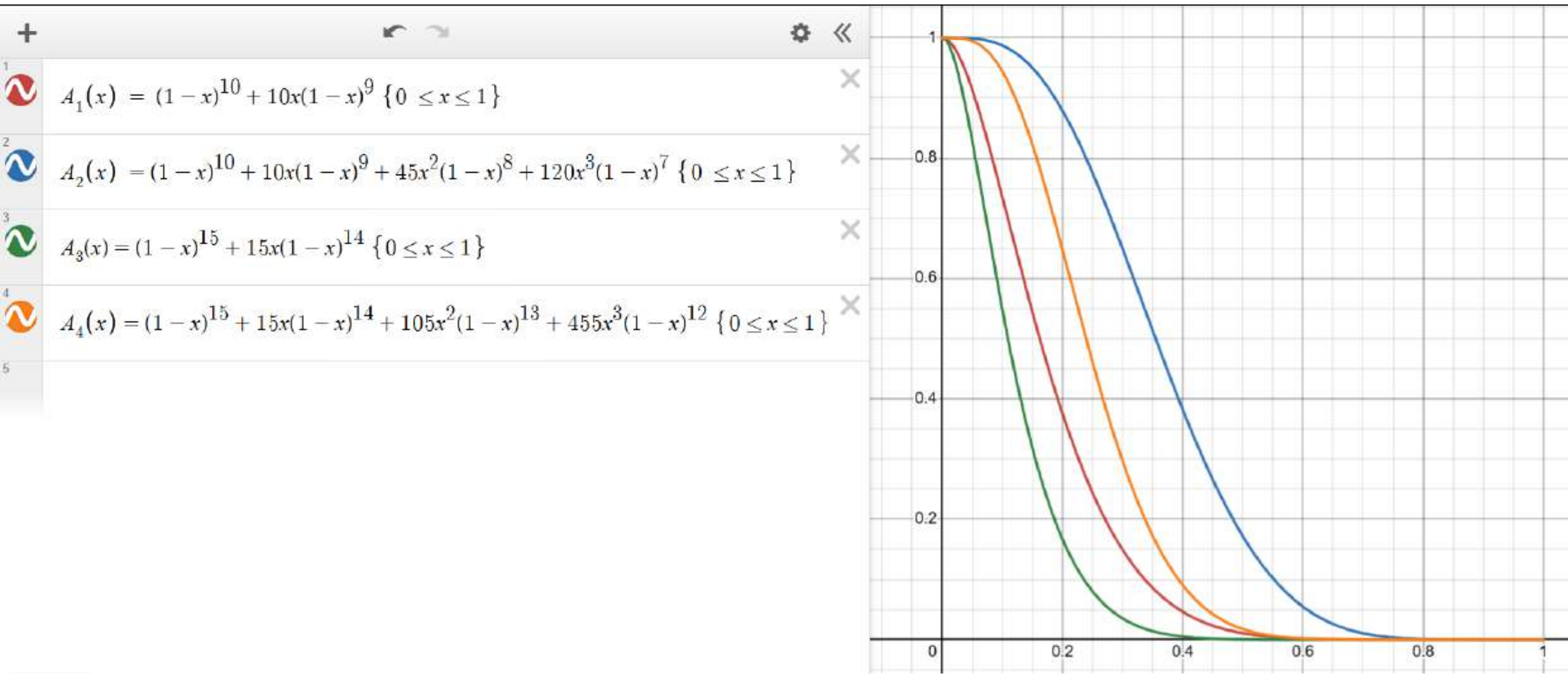
below

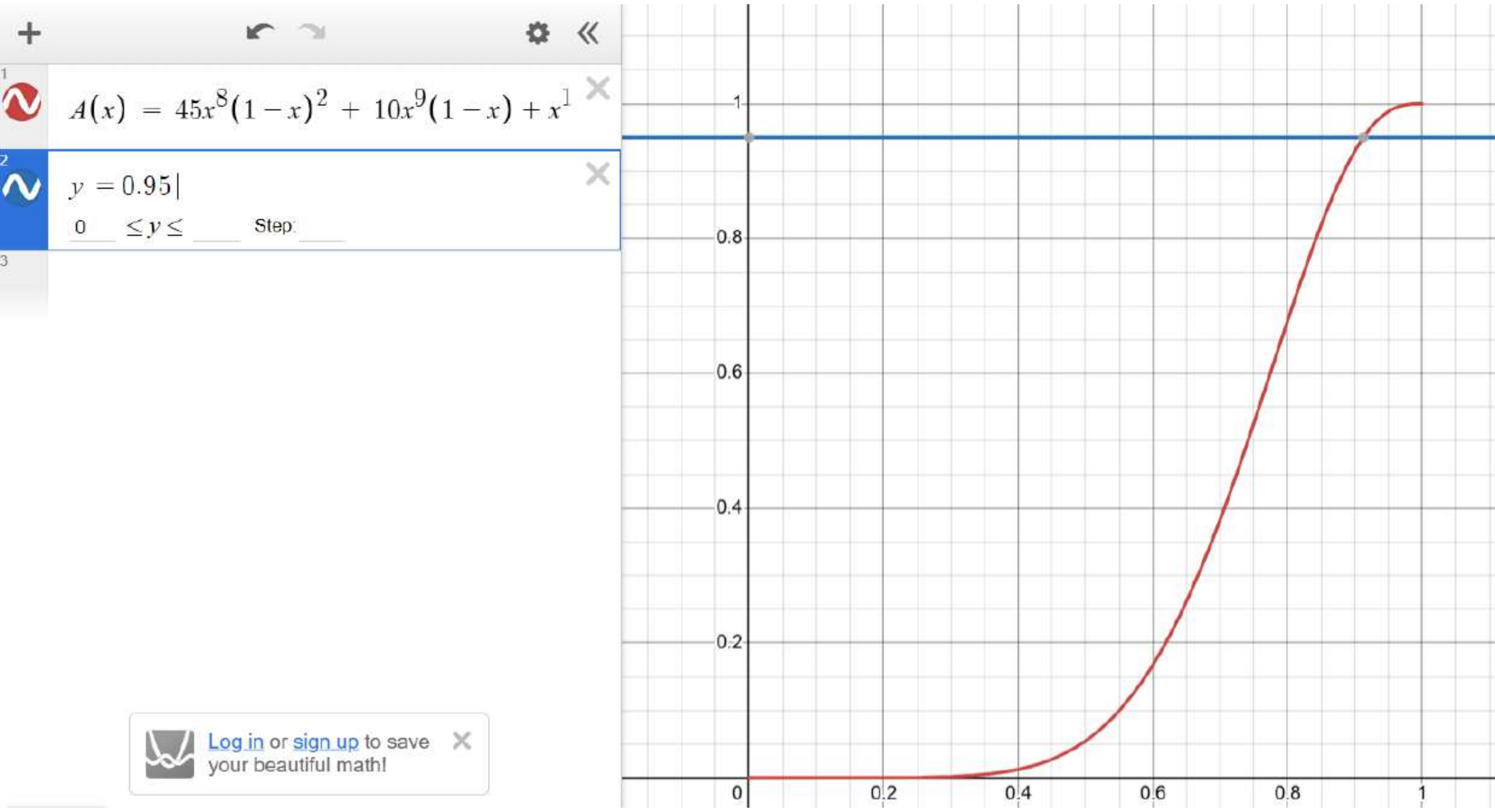
$\therefore$ Moderately poor (b/w 1σ and 2σ)

f) $x = 51,\ \mu = 50,\ \sigma = 0.7$

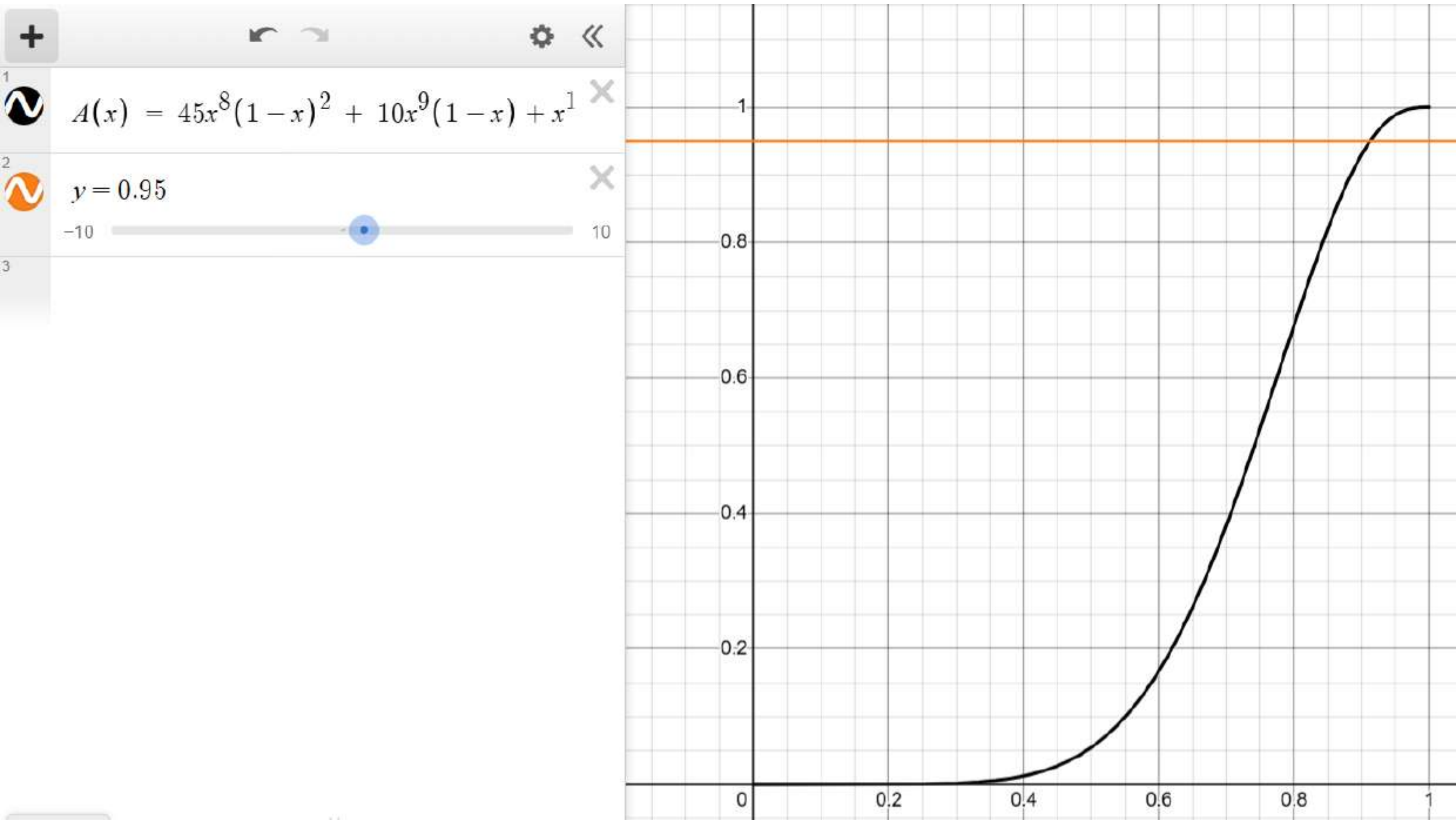
$$z = \frac{51-50}{0.7} \approx \underline{1.4286}$$

$\therefore$ Moderately well (b/w 1σ and 2σ)


$A_1(x) = (1-x)^{10} + 10x(1-x)^9 \{0 \le x \le 1\}$
$A_2(x) = (1-x)^{10} + 10x(1-x)^9 + 45x^2(1-x)^8 + 120x^3(1-x)^7 \{0 \le x \le 1\}$
$A_3(x) = (1-x)^{15} + 15x(1-x)^{14} \{0 \le x \le 1\}$
$A_4(x) = (1-x)^{15} + 15x(1-x)^{14} + 105x^2(1-x)^{13} + 455x^3(1-x)^{12} \{0 \le x \le 1\}$
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1



$A(x) = 45x^8(1-x)^2 + 10x^9(1-x) + x^1$
$y = 0.95$
0 $\le y \le$ Step:
Log in or sign up to save your beautiful math!
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1


$A(x) = 45x^8(1-x)^2 + 10x^9(1-x) + x^1$
$y = 0.95$
−10
10
0
0.2
0.4
0.6
0.8
1